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Wrapped Floer Homology in the Circular Restricted Three-Body Problem

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Abstract

Applications of symplectic geometry to the Three-Body Problem have slowly begun appearing in the past few years, allowing one to approach the problem with tools from Floer theory.

In this thesis, we introduce a new variant of Floer theory: Local Wrapped Floer Homology, which generalises the previously-existing 'Wrapped Floer Homology' to degenerate settings. We use this new machinery to prove a generalisation of the famous Poincaré-Birkhoff theorem to open-ended paths with exact Lagrangian ends in a Liouville domain, assuming a twist condition first stated in [MK22a].

We then proceed to improve the applicability of our theorem to real-world problems, by replacing the constraining 'twist condition' mentioned above by a 'Weakened Twist Condition', and by adapting the setup to degenerate Liouville domains.

Finally, we deduce applications to the Three-Body Problem: first to prove existence of infinitely many trajectories of collision, and then trajectories bi-normal to the xz-plane; under the assumption of the Weakened Twist Condition.

Zusammenfassung

Anwendungen der symplektischen Geometrie auf das *Three-Body Problem* (Drei-Körper-Problem) sind in den letzten Jahren allmählich in der Literatur erschienen und ermöglichen es uns, das Problem mit Methoden der Floer-Theorie anzugehen.

In dieser Dissertation führen wir eine neue Variante der Floer-Theorie ein: Die *Local Wrapped Floer Homology*, die bisher bekannte *Wrapped Floer Homology* auf degenerierte Situationen verallgemeinert. Mit diesem neuen Werkzeug beweisen wir eine Verallgemeinerung des berühmten Satzes von Poincaré-Birkhoff für nicht geschlossene Pfade mit exakten Lagrange-Enden in einer Liouville-Domäne, unter Annahme einer twist-Bedingung, die erstmals in [MK22a] formuliert wurde.

Anschließend verbessern wir die Anwendbarkeit unseres Satzes auf reale Probleme, indem wir die die oben-genannte twist-Bedingung durch eine Weakened-Twist-Bedingung ersetzen und unseren Satz von Poincaré-Birkhoff auf degenerierte Liouville-Domänen anpassen.

Schließlich leiten wir Anwendungen auf das *Three-Body Problem* her: Zunächst zum Nachweis der Existenz unendlich vieler Kollisionsbahnen und anschließend von Bahnen, die binormal zur xz-Ebene verlaufen; jeweils unter der Annahme der *Weakened Twist Condition*.

Acknowledgements

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Contents

Notation and sign conventions

List of abbreviations

2BP	**2-Bo**dy **P**roblem (or Kepler Problem)
3BP	**3-Bo**dy-**P**roblem
PCR3BP	**P**lanar **C**ircular **R**estricted **3-Bo**dy-**P**roblem
SCR3BP	**S**patial **C**ircular **R**estricted **3-Bo**dy-**P**roblem
RKP	**R**otating **K**epler **P**roblem (2BP in a rotating frame)

List of symbols

$\pitchfork$	transverse intersection
$\bar{\mathcal{U}}$	topological closure of the set $\mathcal{U}$
$\mathcal{U} \subset\subset \mathcal{V}$	$\mathcal{U}$ is a pre-compact subset of $\mathcal{V}$ (i.e $\mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{V}$)
$M \natural N$	boundary connected sum of the manifolds M and N

Symplectic sign conventions

Object	Expression
Euclidean coordinates	$\mathbb{R}^{2n} = \{(q_1, p_1, \ldots, q_n, p_n)\}$
Standard complex structure (on $\mathbb{R}^2$)	$J_0 = i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Symplectic form associated to a Riemannian metric g	$\omega = g(J\cdot, \cdot)$
Standard symplectic form	$\omega_0 = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$
Standard Liouville form ($\omega_0 = -\mathrm{d}\lambda_0$)	$\lambda_0 = \sum_i p_i \mathrm{d}q_i$
Hamiltonian vector field	$X_H \lrcorner\, \omega = \omega(X_H, \cdot) = -\mathrm{d}H$
Hamiltonian vector field (also)	$X_H = J\nabla H$

« - C'est pas compliqué comme question !
- Non, c'est compliqué comme réponse. »

Alexandre Astier, *Kaamelott Livre II: "Le Tourment II"*, 2005.

Part I

Floer Theory

Chapter 1

Introduction

1.1 First words

This thesis lies at an odd intersection: that of an old, unyielding physics problem stated almost three and a half centuries ago, the *Three-Body Problem*, and of a much more recent field of mathematics born in the twentieth century, *symplectic geometry*.

• **Definition.** The **Three-Body Problem** is the study of the motion of three bodies in space (planets, stars, spacecrafts,...), according to Newton's classical laws of physics.

• **Definition.** **Symplectic geometry** is a branch of differential geometry which provides a rigorous framework for classical physics, on any 'geometric space' (manifold).

Early on the symplectic community started showing interest in the Three-Body Problem. People like Conley and Moser pioneered the development of symplectic geometry, motivated by problems from classical mechanics and astrodynamics. Conley contributed to defining the Conley-Zehnder index, an essential tool in symplectic geometry which we shall use all throughout this thesis. Meanwhile, he was also the first to propose using the Restricted Three-Body Problem to find low-energy transfer orbits to the Moon in the sixties, back when engineers were still using two-body models. Such techniques were notably used in Belbruno (a student of Moser) and Miller's rescue of Japan's Hiten mission in 1990-1991, after a failure of one of the transmitters – see [FK18, §1.4] for a history.

The real breakthrough however, the one which turned symplectic geometry from a useful framework to study the equations of motion into a perennial bridge between topology and physics, happened in the eighties. Although the story which led to it is rich with great insights and contributions, by Arnold, Gromov, Taubes, Rabinowitz,... [Hof21] this breakthrough can mostly be traced back to one name: Andreas Floer (1956 - 1991).

Floer laid down a recipe for re-formulating the famous « Principle of Least Action » from classical physics into algebraic topological terms, thus relating solutions of the equations of motion to the topology of the underlying spaces these equations were written on.

✧ **Example.** In Chapter 9, we will be studying trajectories of collision in the Circular Restricted Three-Body Problem. We will see that such trajectories can be modelled as paths with ends in a Lagrangian L_{col}, and that we can arrange them into an algebraic object

$$HW(L_{\text{col}})$$

called the Wrapped Floer Homology of L_{col}. The Lagrangian L_{col} has a well-known topology (it is a cotangent fibre of the sphere). In particular, an off-the-shelf theorem from [AS04] will yield $HW(L_{\text{col}}) \cong H(\Omega\mathbb{S}^2)$, where H denotes singular homology, and $\Omega\mathbb{S}^2$ the based loop space of the sphere. Standard algebraic topology then tells us that

$$\dim H(\Omega\mathbb{S}^2) = \infty,$$

which will help us conclude **existence of infinitely many trajectories of collision** in the Spatial Circular Restricted Three-Body Problem (Theorem C), under the assumption of a twist condition.

Of course, we skipped quite a few steps in the above "proof". Indeed, while Floer-theoretical tools flourished and matured in the decades following their introduction, the gap to bridge in order to be able to apply them to the Three-Body Problem remained wide, and thus our story only really began in the early 2010s, with [Alb+12a; Alb+12b].

The above papers opened the door for the use of symplectic geometry in the study of the Circular Restricted Three-Body Problem (CR3BP), by showing that low-energy level sets were contact. This was another milestone. It told us that while the governing equations could become very hard to track, we could appeal to a whole new range of tools from the symplectic world to study their solutions (pseudo-holomorphic curves, Floer theory,...)

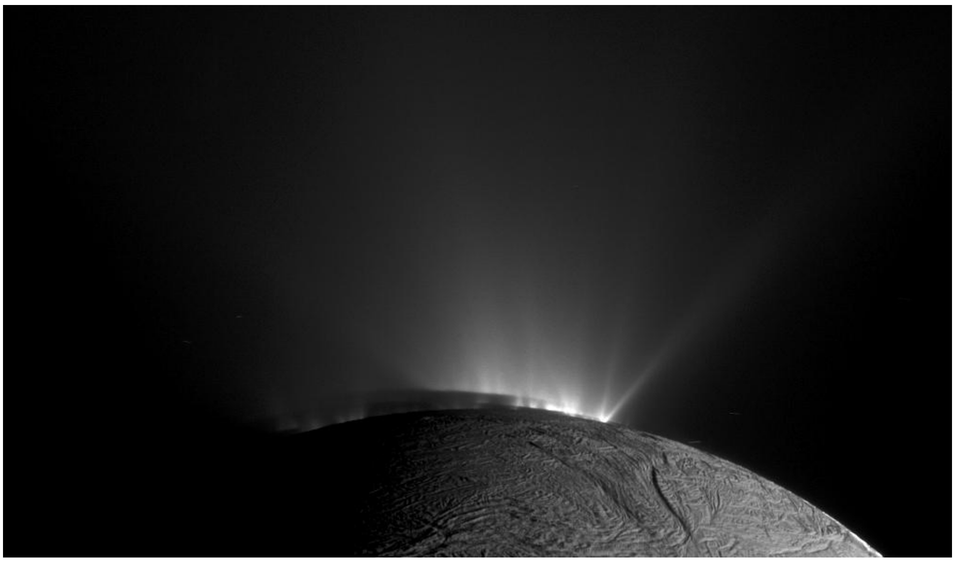
Still, even after [Alb+12a; Alb+12b], it took near a decade for such techniques to actually start being applied to the CR3BP, mostly under the impulse of the research groups of Frauenfelder, Moreno, and van Koert. This ten-year gap is not so surprising, when one considers the complexity of the matter at hand. The equations for the Three-Body Problem are notoriously hard, and there exists a plethora of different Floer-theoretical models, all equally complicated, so that it was not *a priori* obvious which was the right one to use[(1)].

In this thesis, we begin from a model developed by Moreno & van Koert in [MK22a; MK22b], combining Floer theory with a strategy discovered by Poincaré [Poi12] for finding trajectories in the CR3BP. We introduce new theoretical tools to study open-ended trajectories, and then improve Moreno & van Koert's model by significantly relaxing most of the assumptions. Finally, we use this improved model to study certain types of trajectories in the CR3BP: trajectories of collision, and trajectories bi-normal to the xz-plane.

➤ Why do we still care about the Three-Body Problem?

The Three-Body Problem is *old*. As such it has been studied thoroughly, and we now know that the equations are not integrable [Poi90; Yag24], which essentially means that we will never be able to write down a complete set of solutions. However, the amount of numerical work which has been done – especially since the advent of the computer – is colossal, and soon one might start wondering whether there is still sense in trying to refine our models. For a comparison, knowing the number π up to about 40 decimals is enough to measure the length of the observable universe up to the size of a hydrogen atom (physicists' notion of a small ε). So drawing inspiration, couldn't we just call it quits, and content ourselves with our current three hundred years of approximations for the Three-Body Problem?

To answer this, let us look at an example of high interest right now.



Picture: Encroaching Shadow (PIA17184), NASA, Cassini, 2010.

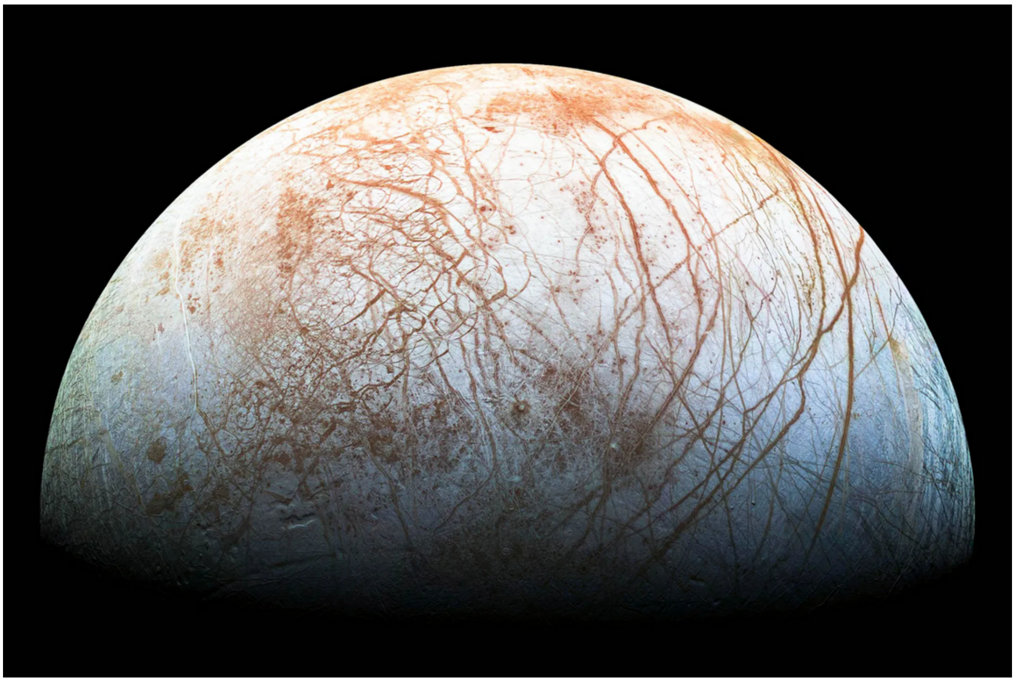
(1) It still isn't, to some extent. Work done independently from this thesis in [Ruc23] recovers a slightly different version of Theorem C, from the example, using a different model (Rabinowitz Floer Homology).

Above is a picture of Enceladus, Saturn's sixth largest moon. The white rays emanating from it are actually *plumes of liquid water* shooting out from cracks in the surface, at an upward velocity of about 500 to 1000 metres per second [Dor+24]. These plumes were first observed in 2005 by NASA's Cassini satellite, which happened to fly by the icy moon while on an unrelated mission to study Saturn's rings.

This discovery surprised the scientific world. Enceladus had garnered *some* interest in the eighties, after the Voyager I and II fly-bys had hinted at the presence of geothermal activity. Already then, a model proposed by Cook and Terrile [Smi+81] suggested the presence of a subsurface ocean of liquid water to explain these observations. However, evidence remained scarce, and the title of Saturn's most popular moon remained Titan's by a long shot.

That is, until Cassini's 2005 observations tipped the balance. The mission was eventually extended and re-routed, flying by Enceladus twenty-two times in the following decade. Subsequent studies [al06; Tho+16; TMT20; Hao+22] confirmed the presence of a *geothermally active* ocean of *liquid water* under the crust of Enceladus, rich in *salt* and *phosphorus*, with an estimated depth of about 30 kilometres. This turned the moon into one of the prime candidates for habitability and for the search for extra-terrestrial life in our solar system.

It is not the only candidate though. Europa – Jupiter's fourth largest moon, twice as close to Earth as Enceladus – is heavily suspected to harbour a similar subsurface ocean. ESA's *Juice* and NASA's *Europa Clipper* missions, respectively launched in April 2023 and October 2024 as this thesis was being written, are currently on their way to study it.



Picture: Europa's Stunning Surface (PIA19048), NASA, 2014

Europa Clipper is expected to reach Europa by 2030, and proceed to no less than 49 fly-bys of the icy moon, as close as 25 kilometres from the surface. One way to generate such close fly-by trajectories is by slightly perturbing collision trajectories, like the ones in Chapter 9.

Planning such missions requires high precision and trust in the mathematical models (let us keep in mind the orders of magnitude: Europa is some 600 million kilometres from us). This makes the need for long-term, rigorous forecasting ever increasing, motivating the search for ever more precise qualitative and quantitative techniques.

Without further ado, let us present the layout and main results of this thesis. I am aware that, no matter how much I tried to attenuate it, the difficulty curve in this manuscript is quite steep. For the reader who is not yet convinced of the complexity of the Three-Body Problem, I hope the next few pages can be of enlightenment.

1.2 Outline of the thesis

This thesis is organised around three main axes:

Part I: Introducing the **Floer-theoretical machinery**:

Chapter 1	Introduction	
Chapter 2	Floer theory pre-requisites	Literature review
Chapter 3		
Chapter 4	Defining Local Wrapped Floer theory	New results

Part II: Deriving a few variants of the **Poincaré-Birkhoff theorem**:

Chapter 5	Relative Poincaré--Birkhoff theorem	New results
Chapter 6	The Weakened Twist Condition	
Chapter 7	Degenerate Relative Poincaré-Birkhoff theorem	

Part III: Applying our results to the **Three-Body Problem**:

Chapter 8	Symplectic geometry in the Three-Body Problem	Literature review
Chapter 9	Collision trajectories	New results
Chapter 10	Bi-normal trajectories	
Chapter 11	Conclusion	

In this whole thesis, we work with the **C**ircular **R**estricted Three-**B**ody **P**roblem (**CR3BP**). This means that we study the motion of three bodies, which we call the Earth (E), the Moon (M), and a Satellite (S), in three-dimensional space, assuming:

Assumption (Circular)**.** The Moon moves in a circle around the Earth.

Assumption (Restricted)**.** The satellite has mass $m = 0$.

The CR3BP is a reasonable model for a number of systems of interest in the space community: Earth-Moon-Spacecraft, Jupiter-Europa-Spacecraft, Jupiter-Ganymede-Spacecraft, Saturn-Enceladus-Spacecraft,... as we shall see in Chapter 8.

➤ This thesis gave birth to three research papers: [ML24; ML25; LM25]. [ML24] corresponds to Chapters 4, 5, and 9 of this manuscript, [LM25] corresponds to Chapters 6 and 7, and [ML25] corresponds to Chapter 10.

➤ Before we formally state the main results, let us briefly explain what each of them does:

- Theorems A1, A2: Local Wrapped Floer Cohomology is well-defined, and one can build a spectral sequence from it which converges to the global Wrapped Floer cohomology of the Lagrangian.
- Theorems B1, B2, B3: Different variants of a Poincaré-Birkhoff theorem for chords with Lagrangian ends in a Liouville domain. Respectively: assuming Moreno & van Koert's twist condition, assuming a weakened twist condition, stated for the first time in this thesis, and assuming a weakened twist condition on a degenerate Liouville domain.
- Theorem C: Existence of infinitely many trajectories of collision in the CR3BP, assuming the Weakened Twist Condition.
- Theorem D: Existence of infinitely many trajectories bi-normal to the xz-plane in the CR3BP, assuming the Weakened Twist Condition.

1.3 Main results

1.3.1 Local Wrapped Floer theory

In Chapter 4, we introduce the notion of *Local Wrapped Floer Cohomology*. This is an adaptation of the already-existing *Wrapped Floer Cohomology*, whose construction we recall in Chapter 3. By definition, Wrapped Floer Cohomology associates cohomology groups $HW^*(L)$ to any "nice" Lagrangian in a Liouville domain W (exact, spin, with Legendrian boundary), by choosing a Hamiltonian $H_t : W \to \mathbb{R}$ and studying flow lines with ends in L, or chords. However, the way this cohomology theory is constructed is by essence *global*: it deals with every chord at once.

The new feature of *Local Wrapped Floer Cohomology* is that it allows to assign a cohomology $HW^*_{\mathrm{loc}}(x)$ to a single chord $x : [0,1] \to W$ with ends in L. Precisely:

Theorem A1. *Let $(W, \omega = \mathrm{d}\lambda)$ be a Liouville domain, $L \subset W$ an exact spin Lagrangian with Legendrian boundary, and $H_t : W \to \mathbb{R}$ a possibly degenerate Hamiltonian. Then, for any chord x of H with ends in L, there is a notion of* Local Wrapped Floer Cohomology*:*

$$HW^*_{\mathrm{loc}}(x)$$

defined as an invariant of the chord x (in the sense explained in §4.3).

The upshot of this result is that while standard Wrapped Floer Cohomology cannot be defined with degenerate Hamiltonians, this version can. Indeed, if a Hamiltonian $H_t : W \to \mathbb{R}$ is degenerate, we can still construct global Wrapped Floer Cohomology by collecting all of its local cohomologies, and arranging them in a spectral sequence. Formally:

Theorem A2. *Let $(\widehat{W}, \hat{\omega} = \mathrm{d}\lambda)$ be the completion of $(W, \omega = \mathrm{d}\lambda)$, and $H_t : \widehat{W} \to \mathbb{R}$ be a Hamiltonian which is strongly convex at infinity (Assumption 3.68), and whose chords are isolated. Denote by $\{A_k\}_{k\in\mathbb{N}}$ their actions. There exists a spectral sequence with first page:*

$$E_1^{p,q} = \begin{cases} \bigoplus_{\mathcal{A}_H(x)=A_k} HW^*_{\mathrm{loc}}(x) & p = 2k \\ 0 & p = 2k+1 \end{cases}$$

and which converges to $HW^*(L)$, the global wrapped Floer cohomology of L.

➢ The last two sections of Chapter 4, §4.3.1-§4.3.2, are then aimed towards applications of the theory towards the Circular Restricted Three-Body Problem. We explain how one can extract a **numerically-computable invariant** from $HW^*_{\mathrm{loc}}(x)$ called the Floer number, and how this invariant can be used in the numerical continuation of trajectories. This builds up on work by Aydin, van Koert, Frauenfelder, Koh and Moreno ([Ayd23b; FKM23]) done for periodic orbits, where they instead used local symplectic cohomology SH^*_{loc}.

1.3.2 Poincaré-Birkhoff theorems

Part II is entirely dedicated to generalisations of the Poincaré-Birkhoff theorem [Poi12; Bir13]. In particular, we prove three closely-related results:

Theorem B1. *Let $(W, \omega = \mathrm{d}\lambda)$ be a connected Liouville domain, $L \subset W$ be an exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ be an exact symplectomorphism. Further assume that:*

- **(Wrapped Floer Cohomology).** $\dim HW^\bullet(L) = \infty$;
- ***(Index growth).*** *if $\dim W \geq 4$, then $c_1(W) = 0$, and $(\partial W, \alpha := \lambda|_{\partial W})$ is strongly index-definite (Assumption 5.7). In particular, the contact structure $\xi = \ker \alpha$ must be globally trivialisable;*
- ***(Twist condition)*** *f is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$ whose fixed points are isolated, and which satisfies the twist condition, i.e.*

$$X_{H_t} = h_t \mathcal{R}_\alpha \quad \text{for } h_t > 0 \text{ smooth.}$$

Then f admits infinitely many interior chords with respect to L, of arbitrarily large order, and which are not sub-chords of any periodic chord.

Theorem B2. *Let $(W, \omega = \mathrm{d}\lambda)$ be a connected Liouville domain, $L \subset W$ be an exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ be an exact symplectomorphism. Further assume that:*

- ***(Wrapped Floer Cohomology).*** *$HW^*(L) \neq 0$ in infinitely many degrees;*
- ***(Chern class).*** *if $\dim W \geq 4$, then $c_1(W) = 0$.*
- ***(Quantitative Weakened Twist Condition)*** *f is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$ whose fixed points are isolated, and which satisfies the quantitative weakened twist condition, i.e. $H_t|_{\partial W} > 0$, and*

$$\partial_r H_t|_{\partial W} > \max_{\partial W} H_t$$

Then f admits infinitely many interior chords with respect to L, of arbitrarily large order, and which are not sub-chords of any periodic chord.

Theorem B3. *Let (W, λ, α) be a connected **degenerate** Liouville domain, $L \subset W$ be an exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ be an exact symplectomorphism. Further assume that:*

- ***(Wrapped Floer Cohomology).*** *$HW^*(L) \neq 0$ in infinitely many degrees;*
- ***(Chern class).*** *if $\dim W \geq 4$, then $c_1(W) = 0$;*
- ***(Weakened Twist Condition)*** *f is generated by a Hamiltonian $H_t : W \to \mathbb{R}$ whose fixed points are isolated, which is $\mathcal{C}^2$ on the interior of W but does **not** $\mathcal{C}^1$-extend to the boundary, and such that:*

$$\partial_r H_t > 0$$

  in a neighbourhood of ∂W.

Then f admits infinitely many interior chords with respect to L, of arbitrarily large order, and which are not sub-chords of any periodic chord.

Let us briefly comment on each of these results:

- Theorem B1 is based on the model and assumptions of Moreno & van Koert [MK22a]. In particular, it adapts their main theorem from symplectic to Wrapped Floer homology, hence allowing one to study open-ended trajectories with prescribed boundary conditions, instead of periodic orbits. However, it has the same built-in issues as Moreno & van Koert's model: it relies on a very constraining twist condition, on a strong index-definiteness assumption (which requires $\xi|_{\partial W} = \ker \alpha$ to be trivialisable), and it is not applicable to the Circular Restricted Three-Body Problem because the latter requires us to work on a *degenerate* Liouville domain.

- Theorem B2 improves Theorem B1 by relaxing the twist condition, and dropping the strong index-definiteness assumption. These already constitute major improvements toward applicability, as the Weakened Twist Condition is an open condition (whereas the twist condition wasn't), and we no longer need to assume trivialisability of ∂W. However, we still cannot apply Theorem B2 to the Circular Restricted Three-Body Problem, because it cannot deal with degenerate Liouville domains.

- Theorem B3 adapts Theorem B2 to degenerate Liouville domains. As such, it is the right model for the Circular Restricted Three-Body Problem, modulo the Weakened Twist Condition. We can rephrase this theorem by using the dictionary between "$\mathcal{C}^2$-Hamiltonian twist maps on a degenerate Liouville domain" and "$\mathcal{C}^0$-Hamiltonian twist maps on a (non-degenerate) Liouville domain" explained in Chapter 7, and thus obtain:

Theorem B3 bis. *Let $(W, \omega = d\lambda)$ be a connected Liouville domain, $L \subset W$ be an exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ be an exact symplectomorphism. Further assume that:*

- ***(Wrapped Floer Cohomology).*** $HW^*(L) \neq 0$ *in infinitely many degrees;*
- ***(Chern class).*** *if* $\dim W \geq 4$, *then* $c_1(W) = 0$*;*
- ***(Weakened Twist Condition)*** f *is generated by a $\mathcal{C}^2$-Hamiltonian H_t on int(W), whose fixed points are isolated. We further assume that both f and H_t admit $\mathcal{C}^0$-extensions to the boundary, such that*

$$h_t := \alpha(X_{H_t}) > 0, \tag{1.1}$$

near ∂W, and $h_t \to \infty$ as we approach ∂W.

Then f admits infinitely many interior chords with respect to L, of arbitrarily large order, and which are not sub-chords of any periodic chord.

Each of the above theorems admits a natural adaptation to the closed orbit case, instead of open-ended trajectories, by simply replacing Wrapped Floer Cohomology by Symplectic Cohomology (see [MK22a] for Theorem B1 and [LM25] for Theorems B2 & B3).

◇ **Remark** (Weakening the assumption on HW^*)**.** In each of these three theorems, we assume that HW^* is non-vanishing in infinitely many degrees (except in Theorem B1, where we simply assume $\dim HW^* = \infty$). One could actually make these results slightly stronger, by adapting Ginzburg's arguments using symplectically degenerate maxima from his proof of the Conley conjecture [Gin10][2], and actually only require that HW^* be non-zero. It is still conjectured, at the time of writing this thesis, that $HW^* \neq 0 \iff \dim HW^* = \infty$.

The rest of the thesis is then dedicated to applying these techniques to the Circular Restricted Three-Body Problem, to try and study specific types of trajectories.

[2] From which our results are inspired.

1.3.3 Collision trajectories in the Three-Body Problem

In Chapter 9, we prove:

Theorem C. *If the Circular Restricted Three-Body Problem satisfies the Weakened Twist Condition, then for every energy $c < H(L_1) + \varepsilon$, there exist infinitely many trajectories of spatial (consecutive) collision near the Earth and Moon, of arbitrarily large length.*

The proof consists in reducing the problem to a situation in which we can apply Theorem B3. More precisely, in the case $c < H(L_1)$, we show that collision trajectories can be modelled as Hamiltonian chords with ends in an appropriate Lagrangian L_{col} inside a degenerate Liouville domain $P_{\pi/2} \cong \mathbb{D}^\star\mathbb{S}^2$ ($P_{\pi/2}$ is a page of the CR3BP open book [MK22b]). As we shall see, L_{col} is nothing but a cotangent fibre in $\mathbb{D}^\star\mathbb{S}^2$, so that by [AS04] we have

$$HW^*(L_{\text{col}}) \cong H^*(\Omega\mathbb{S}^2),$$

which we can easily show to satisfy the assumptions of Theorem B3. The proof in the case $H(L_1) < c < H(L_1) + \varepsilon$ is exactly the same, except $P_{\pi/2} \cong \mathbb{D}^\star\mathbb{S}^2 \natural \mathbb{D}^\star\mathbb{S}^2$.

1.3.4 Bi-normal trajectories in the Three-Body Problem

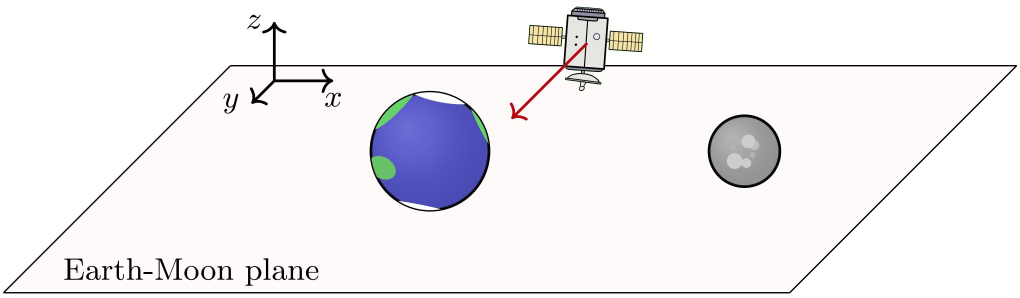
In Chapter 10, we prove:

Theorem D. *If the Circular Restricted Three-Body Problem satisfies the Weakened Twist Condition, then for every energy $c < H(L_1) + \varepsilon$, there exist infinitely many trajectories bi-normal to the xz-plane near the Earth and Moon, of arbitrarily large length.*

We call a trajectory $\overrightarrow{x}(t)$ *bi-normal to the xz-plane* (where x is the Earth-Moon axis, and the xy-plane the Earth-Moon plane) if there exist times $t_0 \neq t_1$ such that $\overrightarrow{x}(t_0)$ and $\overrightarrow{x}(t_1)$ are normal to the xz-plane, *i.e.*

$$y(t_j) = 0 = \dot{x}(t_j) = \dot{z}(t_j), \quad j = 0, 1$$

where the trajectory is given by $\overrightarrow{x}(t) = \big(x, y, z, \dot{x}, \dot{y}, \dot{z}\big)(t)$. In other words, the trajectory of the satellite is normal to the xz-plane at two distinct points in time.



Picture from [ML24].

The proof strategy for Theorem D is very similar to that of Theorem C: we find an appropriate Lagrangian in $P_{\pi/2}$, and show that it satisfies the assumptions of Theorem B3.

We will conclude Chapter 10 with a discussion on:

Conjecture 1. *Assuming the Weakened Twist Condition or a variation thereof, there exist infinitely many trajectories bi-normal to the x-axis in the Spatial Circular Restricted Three-Body Problem, in the low-energy range and near the primaries.*

This is the analogue to Theorem D for Reeb chords with Legendrian ends, though beyond the scope of current Floer-theoretical machinery.

1.4 From classical mechanics to symplectic geometry

Symplectic geometry appeared in the later half of the twentieth century, out of a desire to develop a rigorous mathematical framework for the equations of classical mechanics. It allowed to rewrite these equations in a concise way, on any manifold.

• **Definition.** A manifold M is a geometric space on which one can still do calculus. More precisely, it is a second-countable, Hausdorff space such that around every point $p \in M$, one can find an open neighbourhood $\mathcal{U}$ of p, and a diffeomorphism

$$\phi : \mathcal{U} \xrightarrow{\cong} \mathcal{V}$$

where $\mathcal{V}$ is an open set in $\mathbb{R}^n$. This collection of charts $(\mathcal{U}, \phi)$, consisting of open neighbourhoods and diffeomorphisms (which we ask to be compatible wherever they overlap) allows one to pull back the notions of derivatives, integrals,... from $\mathbb{R}^n$ to M.

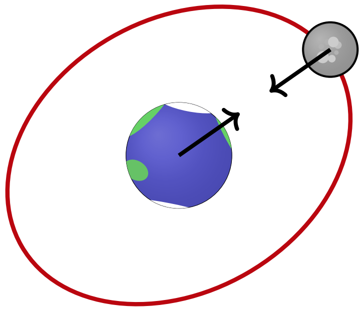
The goal of symplectic geometry is to provide a *dictionary* between physics and geometry: enabling us to translate notions from classical physics into rigorous mathematical concepts.

1.4.1 The Two-Body Problem

✧ **Example 1.1.** Assume we have two objects, say the Earth and Moon (figure not to scale), which we represent by points in three-dimensional space, $\mathbb{R}^3$.

Without loss of generality, assume the Earth is centred at 0 and has mass 1, while the Moon has position and momentum $q, p \in \mathbb{R}^3$, and mass m.

Define the angular momentum $L = q \times p$ of the Moon. One can easily show that L is conserved along the motion. In particular, the Earth and Moon always move in a fixed plane (the **ecliptic**), and we can hence reduce our study to the plane $\mathbb{R}^2$.



There are three standard, equivalent ways to go about this problem in classical mechanics: the Newtonian and Hamiltonian formulations, which we will explore now, and the Lagrangian formulation, which we will discuss in Chapter 2, as it is the starting point of Floer theory.

A) (Newtonian formulation) Using Newton's original theory [New87], we can write down the force applied by the Earth on the Moon as:

$$F = -G\frac{m}{|q|^3}q = -G\frac{m}{|q|^2}\hat{q} \qquad (\hat{q} := q/|q|), \tag{1.2}$$

where $G \approx 6.674 \cdot 10^{-11} kg^{-1} \cdot m^3 \cdot s^{-2}$ is a universal constant of the universe.

To solve for the orbit of the Moon, one then simply integrates Newton's second law:

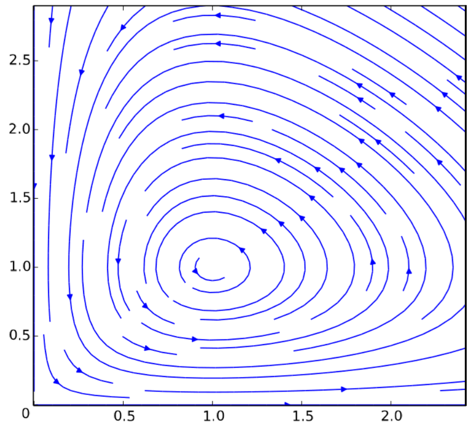
$$F = m\ddot{q}, \tag{1.3}$$

which yields conical orbits (ellipses, parabolas, or hyperbolas). Formally, one has to solve a second-order ODE whose solutions will be linear combinations of sines and cosines.

B) (Hamiltonian formulation) Some one hundred and fifty years later, Hamilton reformulated Newton's laws into a system of PDEs. We will directly present the more modern approach of his work, since it is the foundation for symplectic geometry. First, one definition:

• **Definition 1.2.** Given M a manifold, a flow $\phi : \mathbb{R} \times M \to M$ is an $\mathbb{R}$-family of diffeomorphisms $\phi^t : M \to M$, such that $\phi^0 = \mathrm{id}$, and $\forall s, t \in \mathbb{R} : \phi^s \circ \phi^t = \phi^{s+t}$

The idea of Hamilton's formulation of classical mechanics is to represent the physical evolution of the object as a flow on phase space.



• **Definition 1.3.** Given an object moving in position space $\mathbb{R}^n$, we define the phase space:

$$M = \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$$
$$(q\ ,\ p)$$

where q ought to be thought of as position, and p as momentum. One can then write down the total energy of our system

$$H := \text{Kinetic Energy} + \text{Potential Energy}$$

as a function $H : M \to \mathbb{R}$, which we call the **Hamiltonian**.

Kinetic energy is given by $K = \frac{1}{2}\|p\|^2$, while the potential is a function $V = V(q)$ satisfying $F = -\nabla V$, where F is a Newtonian force. In the case of the Two-Body Problem, where F is given by (1.2), we get:

$$H : \mathbb{R}^4 \backslash \{q = 0\} \longrightarrow \mathbb{R} : (q, p) \longmapsto \frac{1}{2}\|p\|^2 - G\frac{m}{|q|}. \tag{1.4}$$

Note that $\partial_{q_i} H = \partial_{q_i} V = -\partial^2_{q_i} F$, and $\partial_{p_i} H = p_i$, so that we can rewrite Newton's second law $F = m\ddot{q}$ as a system of PDEs:

$$\begin{cases} \dot{q}_i &= \partial_{p_i} H \\ \dot{p}_i &= -\partial_{q_i} H \end{cases} \tag{1.5}$$

called the Hamiltonian equations of motion. Given any initial conditions $(q^0, p^0) \in M$ in phase space, we can consider the trajectory

$$\gamma : t \mapsto (q_i(t), p_i(t)),\ \ \gamma(0) = (q^0, p^0)$$

determined by the equations (1.5). This describes the trajectory of the physical object in phase space. Following all these trajectories defines a flow ϕ^t on M, formally:

• **Definition 1.4.** The **Hamiltonian flow** is the flow $\phi^t : M \to M$ defined by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^t = \begin{pmatrix} \partial_{p_i} H \\ -\partial_{q_i} H \end{pmatrix}.$$

Its trajectories, or flow lines, are by definition solutions to the Hamiltonian equations of motion – *i.e.* physical trajectories of our system.

In geometric language, we say that the Hamiltonian flow is generated by the vector field

$$X_H := \begin{pmatrix} \partial_{p_i} H \\ -\partial_{q_i} H \end{pmatrix}.$$

X_H, thus defined, is called the **Hamiltonian vector field**. As it happens, we can simplify its expression to make the equations more appealing. Indeed, let us introduce the standard almost complex structure (Definition B.1) J_0 on $M = \mathbb{R}^4$ by:

$$J_0\partial_{q_i} = \partial_{p_i},\ \ J_0\partial_{p_i} = -\partial_{q_i}. \tag{1.6}$$

Formally, it can be defined as the pullback of $i \oplus i$ under the obvious diffeomorphism $\mathbb{R}^4 \to \mathbb{C}^2$. Then, observe that:

$$X_H = \begin{pmatrix} \partial_{p_i} H \\ -\partial_{q_i} H \end{pmatrix} = -J_0 \nabla H,$$

so that we can rewrite the classical equations of motion (1.5) as:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\phi^t = X_H = -J_0 \nabla H} \tag{1.7}$$

which is essentially a system of partial differential equations on phase space $M = \mathbb{R}^4 \backslash \{q = 0\}$. Let us now set aside the example of the Two-Body Problem for a while, and explain how the above intuition can be generalised to a much wider class of spaces: *symplectic manifolds*, which should be viewed as generalisations of phase space.

1.4.2 Symplectic formalism

• **Definition 1.5.** An even-dimensional manifold M is called **symplectic** if it can be endowed with a closed, non-degenerate 2-form ω, *i.e.* $\mathrm{d}\omega = 0$ and ω induces a bundle isomorphism

$$TM \xrightarrow{\sim} T^\star M$$

between the tangent and cotangent bundles of M.

• **Definition 1.6.** Let (M, ω) be symplectic. A **Hamiltonian** is a function $H : M \to \mathbb{R}$.

• **Definition 1.7.** Given a $\mathcal{C}^1$ Hamiltonian H, its **Hamiltonian vector field** is defined as the unique vector field X_H satisfying $X_H \lrcorner\, \omega := \omega(X_H, \cdot) = -\mathrm{d}H$.

Now, an almost complex structure J on M is a bundle endomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$ on every tangent space, so that it essentially mimics the action of the complex structure $i = \sqrt{-1}$. Such a J is called compatible (Definition B.3) with the symplectic form ω if $J^\star\omega = \omega$, and $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric on M.

One can easily show (Lemma B.4) that every symplectic manifold (M, ω) admits a compatible almost complex structure, allowing us to re-write X_H as:

Corollary 1.8. *If (M, ω) is equipped with a compatible almost complex structure J, then:*

$$X_H = J\nabla H,$$

where ∇ denotes the gradient associated to the metric $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$.

◇ **Remark 1.9.** Note that this expression differs from the one derived in the previous subsection by a sign, since we previously had $X_H = -J\nabla H$. Technically, this means we are now working with the *backwards* Hamiltonian flow. We choose this sign convention for the rest of the thesis, as it will simplify our calculations.

As we saw in Example 1.1, the flow of X_H is called the **Hamiltonian flow**. Let us write it as ϕ^t, and state and prove one famous physical principle:

Proposition 1.10 (Conservation of energy)**.** *Energy remains constant along the motion. i.e. for every $c \in \mathbb{R}$, the energy level set $H^{-1}(c)$ is invariant under the flow ϕ^t.*

Proof. This follows from the fact that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H \circ \phi^t\right)|_{t=0} = \mathrm{d}H\left(\left.\frac{\mathrm{d}\phi^t}{\mathrm{d}t}\right|_{t=0}\right) = \mathrm{d}H(X_H) = -\omega(X_H, X_H) = 0. \qquad \square$$

• **Definition 1.11.** Let $(M, \omega), (W, \Omega)$ be symplectic manifolds. A **symplectomorphism**

$$f : (M, \omega) \longrightarrow (W, \Omega)$$

is a diffeomorphism $f : M \xrightarrow{\cong} W$ such that $f^\star\Omega = \omega$.

✧ **Example 1.12.** Let $H : M \to \mathbb{R}$ be a Hamiltonian, and ϕ^t its flow. Then $\varphi := \phi^{t=1}$ is a symplectomorphism. It is often called the Hamiltonian diffeomorphism associated to H.

We could cite many more theorems in elementary symplectic geometry. For example, Darboux's theorem tells us that all symplectic manifolds of the same dimension are locally symplectomorphic to each other. This implies that there exist no local symplectic invariants, and that every symplectic manifold locally looks the same. Standard examples are:

✧ **Example 1.13.** The prototypical symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is defined as $\omega_0(\cdot,\cdot) = g(J_0\cdot,\cdot)$, with g the standard Riemannian metric on $\mathbb{R}^{2n}$, and J_0 the standard almost complex structure (1.6). Choosing coordinates $\{q_i, p_i\}$ for $\mathbb{R}^{2n}$, one can easily compute:

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

✧ **Example 1.14** (Cotangent bundles)**.** Let Q be any smooth manifold with local coordinates $q_1, \ldots, q_n$. Consider its cotangent bundle $T^\star Q$, with coordinates (q, p), and define:

$$\lambda := -\sum_i p_i \, \mathrm{d}q_i,$$

a 1-form on $T^\star Q$ which we call the Liouville form. Then,

$$\omega := \mathrm{d}\lambda = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$$

defines an exact symplectic form on $T^\star Q$. Cotangent bundles are the natural place where to do classical mechanics. Indeed, if we assume that a physical object is moving in position space Q, then $T^\star Q$ can be viewed as its phase space (interpreting q as position and p as momentum), so that one can write down the Hamiltonian equations of motion on $T^\star Q$, as we did in the previous subsection with $Q = \mathbb{R}^n, T^\star Q \cong \mathbb{R}^{2n}$.

This concludes our brief overview of Symplectic Geometry. We refer to §A.1 for further discussion on Hamiltonian dynamics, and for a brief introduction to contact geometry (sometimes referred to as the 'odd-dimensional cousin of symplectic geometry'). We will now proceed to introduce the cornerstone of this thesis: Floer theory.

Chapter 2

Foundations of Floer theory

2.1 Short history and motivation

The words « Floer theory » have now become an umbrella term for many different constructions, all tracing back to foundational work of Andreas Floer in the 1980s. The first half of the twentieth century had already seen the birth of *Morse theory*, the ancestor of Floer theory, which gave a topological interpretation of critical points of functions.

• **Definition 2.1.** Let M be a manifold and $f : M \to \mathbb{R}$ be a differentiable function. A point $x \in M$ is called a **critical point** of f if $f'(x) = 0$.

The goal of Morse theory is to extract **algebraic invariants** from critical points of f. These algebraic invariants come in the form of groups, and they turn out to be independent of the choice of f, only carrying information about the topology of M. As such if we understand the function f well, then we can leverage knowledge about its critical points to study M, and *vice versa*. Let us draw a parallel with physics.

✧ **Continuation of Example 1.1.** When discussing the Two-Body Problem, we mentioned that there were three standard ways of solving a problem in classical mechanics: the Newtonian formulation, the Hamiltonian formulation, and the Lagrangian formulation. We discussed the first two in Chapter 1, let us now address the third.

Consider a physical object moving in position space Q, and let $M = T^\star Q$ denote phase space. Given a path $x : [a, b] \to TQ$, we define its **action**:

$$\mathcal{A}(x) = \int_x K - V = \int_a^b \big(K - V\big)(x(t))\,\mathrm{d}t, \tag{2.1}$$

where K denotes kinetic energy and V potential energy. Then, we have:

Principle 2.2 (Principle of Least Action)**.** *Physical trajectories of our system are those that extremise the action,* i.e. *the paths x such that* $\mathrm{d}\mathcal{A}(x) = 0$.

◇ **Remark.** Despite the name of the principle, which stuck since its original formulation by Maupertuis, we are not only interested in *minima* of the action, but indeed in every possible extremum. To state it more simply: we are looking for critical points of $\mathcal{A}$.

The action functional $\mathcal{A}$ should be viewed as the *energetical-cost* it would for take an object to travel along a given trajectory. Therefore, the Principle of Least Action should be viewed as the universe trying to optimise the energetical cost of anything it does.

➢ The reason we drew this parallel between Morse theory and the Principle of Least Action is because in both cases we are interested in **critical points**. The "only" difference is that in Morse theory, we study functions on finite-dimensional manifolds, whereas the action functional $\mathcal{A}$ is defined on the space of paths on M, $\mathcal{C}^\infty([a, b], M)$, which is infinite-dimensional.

The idea of adapting the construction and defining a 'Morse theory for action functionals' was already being discussed in the late 1970s, when Andreas Floer was still but an undergrad. However, the technical difficulties one needed to overcome to establish such a theory were deemed too great, and the task hopeless.

> « [T]his variational principle is very degenerate, [...] and is certainly not suitable for an existence proof. »
>
> Moser, *Periodic orbits near an equilibrium and a theorem by Alan Weinstein* [Mos76]

At that time, there was large interest in the symplectic community for the Arnold conjecture, which hypothesised a relation between periodic orbits of a physical system and the topology of the underlying phase space:

Conjecture (Arnold)**.** *Let* (M, ω) *be symplectic, and* $H : M \to \mathbb{R}$ *be a Hamiltonian. Then:*

$$\#\{\text{1-periodic orbits of } H\} \geq \sum_{i=0}^{\dim M} \dim H_i(M),$$

where $H_*(M)$ *denotes the singular homology of* M*, if* H *is non-degenerate. If* H *is degenerate, then we replace* $H_*(M)$ *by the Morse homology of a Morse function on* M*.*

The work of Arnold can in many ways be viewed as the foundation of symplectic geometry, and this conjecture (still far from being solved in full generality[(1)]) as one of its pillars. The interest for it was high in the 1970s for a different reason however. In 1978, Eliashberg, then living in the USSR and unable to communicate with mathematicians outside the country, found a proof of the Arnold conjecture on surfaces. With the help of Katok, he managed to smuggle out an incomplete draft of his proof to Gromov, who brought it to the attention of the Parisian symplectic community. (For a detailed and fascinating account of the events leading up to the discovery of Floer theory, we refer to Hofer's talk [Hof21]).

A few years later, as he was finishing his graduate studies, Floer believed he could prove the Arnold conjecture for a wide class of manifolds, namely any compact symplectic manifold, as long as he could rule out a « bubbling off » phenomenon. His construction was highly technical, relying on an extensive use of pseudo-holomorphic curves, a tool barely just introduced by Gromov in [Gro85], and on a number of very technical arguments from topology and functional analysis (work by Taubes, Uhlenbeck, Conley, Zehnder,...)

Floer gained instant recognition in the mathematical world for his construction. The community very quickly recognised that his work did far more than proving the Arnold conjecture: it provided a general recipe for translating variational problems into topological ones. Before Floer's original *Floer theory* was even written up (this came later, in a series of papers [Flo88a; Flo88b; Flo89a; Flo89b]) everyone in the geometrical world was already dreaming of adapting it to their own favourite problem. Unsurprisingly, the next two decades saw the birth of many a flavour of Floer theory. To list but a few:

Theory	**Objects of study**
Hamiltonian Floer homology *Floer (1980s)*	Periodic orbits in a closed symplectically aspherical manifold.
Lagrangian Floer homology *Floer (1980s)*	Lagrangian intersections in a closed symplectically aspherical manifold.
Embedded contact homology (ECH) *Hutchings & Taubes (2000s)*	Reeb dynamics in the symplectisation of a 3-fold.
Heegaard-Floer homology *Ozsváth & Szabó (2000s)*	Knots and 3-folds.
Symplectic homology *Hofer & Wysocki & Floer (1990s)*	Periodic orbits in a Liouville domain.
Wrapped Floer homology *Abouzaid & Seidel, Abbondandolo & Schwarz (2000s)*	Lagrangian intersections in a Liouville domain.

(1) The version for non-degenerate Hamiltonians is widely considered to have been solved, thanks to work by Fukaya-Oh-Ohta-Ono over $\mathbb{Q}$, and Abouzaid-Blumberg & Bai-Xu over $\mathbb{Z}$.

In this thesis, we will mainly be concerned with the last two entries in the table: symplectic and Wrapped Floer homologies, developed in the 1990s and 2000s. These adapt Floer's original work (which essentially took care of the first two entries) to *Liouville domains*, a specific type of compact symplectic manifolds with boundary.

Before we start introducing them though, we will start with the "easy" cases: Hamiltonian and Lagrangian Floer homologies. And before we even do that, we will formally introduce Morse homology, since it serves as the prototype for any Floer theory.

2.2 Finite-dimensional prototype

2.2.1 Homology and cohomology

For completeness, let us briefly recall what a homology theory is.

• **Definition 2.3** (Homology)**.** Let R be a commutative ring, and (C_n) a sequence of R-modules (*e.g.* vector spaces, abelian groups,...) endowed with morphisms:

$$\dots \xrightarrow{\partial_{n+3}} C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \tag{2.2}$$

Then (C_*, ∂) is called a **chain complex** if, at every step of the way, $\partial^2 = 0$. In other words, $\forall n : \mathrm{im}(\partial_n) \subseteq \ker(\partial_{n-1})$. This motivates the definition:

$$H_n(C_*, \partial) := \frac{\ker\{\partial_n : C_n \longrightarrow C_{n-1}\}}{\mathrm{im}\{\partial_{n+1} : C_{n+1} \longrightarrow C_n\}}.$$

We call H_n the **n-th homology group** of the chain complex (C_*, ∂). We often use the asterisk * as a placeholder for the index, and hence write H_* for simplicity.

∂ is called the **differential** of the chain complex. Then, an element $[x]$ in H_* will be the equivalence class of an element x in C_* such that $\partial x = 0$. We call such an $x \in C_*$ closed.

$[x]$ being 0 in H_* corresponds to there existing $y \in C_{*+1}$ such that $\partial y = x$, in which case we call $x \in C_*$ a boundary.

◇ **Remark.** It is sometimes more practical to work with the dual picture, *co*homology. Define $C^* := \mathrm{Hom}_R(C_*, R)$, the dual of the module C_*. We then obtain:

$$\dots \xrightarrow{\mathrm{d}} C^{n-1} \xrightarrow{\mathrm{d}} C^n \xrightarrow{\mathrm{d}} C^{n+1} \xrightarrow{\mathrm{d}} C^{n+2} \xrightarrow{\mathrm{d}} \dots \tag{2.3}$$

which still satisfies $\mathrm{d}^2 = 0$, where d is the dual of ∂. We can define:

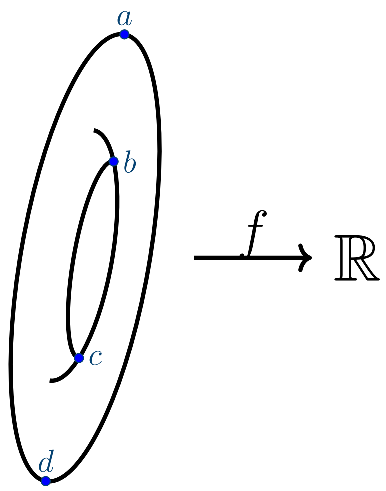
$$H^*(C^*, \mathrm{d}) := \frac{\ker\left\{\mathrm{d} : C^n \longrightarrow C^{n+1}\right\}}{\mathrm{im}\left\{\mathrm{d} : C^{n-1} \longrightarrow C^n\right\}},$$

called the **cohomology** of (C^*, d). In practice, when constructing a (co)homology theory from actual problems, the theory may present in either form. For example, singular/simplicial theory, relating to triangulations of topological spaces, naturally arises as a homology theory [Hat02]. Meanwhile de Rham theory, relating to integration of differential forms, rather lends itself to cohomology [Lee12]. The bridge between homology and cohomology is given by the now standard Universal Coefficients Theorem [Hat02], and it is no issue passing from one to the other. In this thesis, we will work exclusively with **co**homology, by convention.

2.2.2 Morse cohomology theory

We do not prove any results in this subsection, and refer to [AD13, Ch. 1-4], or most any Morse theory textbook for proofs and a rigorous exposition.

Let M be a smooth manifold, and $f \in \mathcal{C}^\infty(M, \mathbb{R})$. We call $x \in M$ a **critical point** of f if $f'(x) = 0$, and we call this critical point **degenerate** if $\mathrm{Hess}_x f$ is singular. (See Example A.28 for the definition of the Hessian on a manifold). We call the function f **Morse** if it has no degenerate critical points, which is a generic condition.



The main idea behind Morse theory is that given any Morse function $f : M \to \mathbb{R}$, the critical points of f record qualitative changes in the topology of M.

For example, if $M = \mathbb{T}^2$ is a torus, and $f : M \to \mathbb{R}$ the height function, then f has four critical points, which we denote a, b, c, d from top to bottom.

a is the maximum of f, d its minimum, and b and c saddle points.

Morse theory constructs a cohomology theory from these critical points, and shows that it is isomorphic to singular cohomology. In other words, critical points of a generic function f are determined by the topology of M, and *vice versa*.

➤ We briefly sketch the construction of Morse theory. For simplicity, let us work over the coefficient ring $R = \mathbb{Z}$. Since f is Morse, all its critical points are non-degenerate. Hence, given $x \in \mathrm{Crit}(f)$, we can define its **Morse index** $\mu(x) \in \mathbb{N}$ as the number of negative eigenvalues of $\mathrm{Hess}_x f$. We then define, for $n \in \mathbb{N} = \{0, 1, \dots\}$:

$$CM^n(M, f) := \bigoplus_{\substack{x \in \mathrm{Crit}(f) \\ \mu(x) = n}} \mathbb{Z}\langle x \rangle .$$

i.e. $CM^n(M, f)$ is defined as the free $\mathbb{Z}$-module (abelian group) generated by critical points of f of index n. In order to define a cohomology theory though, we still need a differential $\mathrm{d} : CM^{n-1} \to CM^n$, *i.e.* a map from formal sums of critical points of degree $n-1$ to formal sums of critical points of degree n.

We construct such a map by following flow lines of $-\nabla f$. Indeed:

Lemma 2.4. *Let $f : M \to \mathbb{R}$ be Morse*[(2)]*, and $u : \mathbb{R} \to M$ be a flow line of $-\nabla f$, i.e.*

$$\frac{\mathrm{d}u}{\mathrm{d}s} = -\nabla f(u(s)).$$

Then u ends in critical points of f with distinct indices, i.e.

$$\lim_{s \to -\infty} u(s) = x, \quad \lim_{s \to +\infty} u(s) = y,$$

where $x, y \in Crit(f)$, and $\mu(x) \neq \mu(y)$.

◇ **Remark 2.5.** The way one should imagine this, in the case where $f : M \to \mathbb{R}$ is the height function of the torus is that we are pouring water onto our manifold from above, and that flow lines of $-\nabla f$ correspond to trajectories of steepest descent of the water.

The reason the torus on the picture is slightly tilted, and not purely vertical, is to avoid flow lines between points of same index. Indeed, tilting the torus ensures that there are no trajectories from one saddle point to the other. This is, in crude terms, why we impose the Morse-Smale condition mentioned in the footmark.

➢ Given two critical points x and y of f, we can define the moduli space:

$$\widehat{\mathcal{M}}(x, y) := \left\{ u : \mathbb{R} \to M \;\middle|\; \begin{array}{l} u \text{ is a trajectory of } -\nabla f \text{ and} \\ \lim_{s \to -\infty} u(s) = x, \quad \lim_{s \to +\infty} u(s) = y \end{array} \right\},$$

which can be shown to be a smooth manifold of dimension $\mu(x) - \mu(y)$. Elements in this moduli space naturally carry a parametrisation (they are maps $\mathbb{R} \to M : s \mapsto u(s)$), which induces a free action $\mathbb{R} \circlearrowleft \widehat{\mathcal{M}}(x, y)$. Quotienting by it yields

[(1)] Morse-Smale actually, which is a slightly stronger, but still generic condition; see Remark 2.5.

$$\mathcal{M}(x,y) := \widehat{\mathcal{M}}(x,y)/\mathbb{R},$$

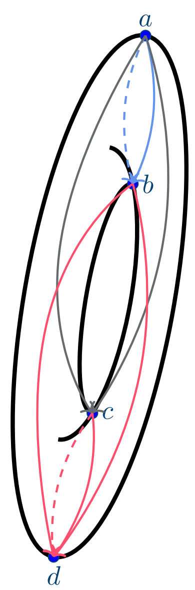
whose points are in one-to-one correspondence with Morse trajectories between x and y; and which is now a manifold of dimension $\mu(x) - \mu(y) - 1$. In particular, if x and y are critical points of consecutive indices, $\mathcal{M}(x,y)$ is a 0-dimensional manifold. One can show it to be compact, so that there are finitely many Morse trajectories between x and y.

• **Definition 2.6.** The **Morse differential** is defined, for every n, as the map:

$$\begin{aligned} \mathrm{d} : CM^{n-1}(M,f) &\longrightarrow CM^n(M,f) \\ y &\longmapsto \sum_{x \in CM^n(M,f)} \varepsilon(x,y)x \end{aligned} \tag{2.4}$$

where $\varepsilon(x,y)$ is the oriented count of trajectories in $\mathcal{M}(x,y)$. One needs to be a bit careful in order to define this orientation (see [AD13]), but for simplicity, one may choose to work over the field of coefficients $R = \mathbb{Z}/2\mathbb{Z}$, so that $\varepsilon(x,y)$ simply becomes the mod 2 count of trajectories between x and y, and orientation becomes irrelevant.

✧ **Example 2.7.** To return to our visual example of the height function f on the torus: the maximum a has index $\mu(a) = 2$, while $\mu(d) = 0$, and $\mu(b) = \mu(c) = 1$.



On the picture, we have drawn trajectories of the flow $-\nabla f$ between critical points of consecutive indices (in blue trajectories flowing into b, in gray trajectories flowing into c, and in pink trajectories flowing into d).

Now say that we want to construct the Morse cohomology of $f : M \to \mathbb{R}$, with coefficients in $R = \mathbb{Z}/2\mathbb{Z}$. By (2.4), to define the Morse differential d of a point it suffices to count incoming trajectories into this point modulo 2. Therefore:

$$\begin{cases} \mathrm{d}a = 0 \\ \mathrm{d}b = \mathrm{d}c = 2a = 0 \\ \mathrm{d}d = 2b + 2c = 0 \end{cases}$$

Therefore, the Morse complex

$$0 \xrightarrow{\mathrm{d}_0} CM^0(M,f) \xrightarrow{\mathrm{d}_1} CM^1(M,f) \xrightarrow{\mathrm{d}_2} CM^2(M,f) \xrightarrow{\mathrm{d}_3} 0$$

actually becomes

$$0 \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} 0,$$

thus yielding cohomology groups:

$$\begin{aligned} H^0(M,f) &= \frac{\ker \mathrm{d}_0}{\operatorname{im} \mathrm{d}_{-1}} = \frac{\mathbb{Z}/2\mathbb{Z}}{0} = \mathbb{Z}/2\mathbb{Z}, \\ H^1(M,f) &= \frac{\ker \mathrm{d}_1}{\operatorname{im} \mathrm{d}_0} = \frac{(\mathbb{Z}/2\mathbb{Z})^2}{0} = (\mathbb{Z}/2\mathbb{Z})^2, \\ H^2(M,f) &= \frac{\ker \mathrm{d}_2}{\operatorname{im} \mathrm{d}_3} = \frac{\mathbb{Z}/2\mathbb{Z}}{0} = \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

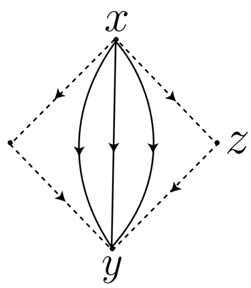
This concludes the example.

➢ In the above example, we trivially had $\mathrm{d}^2 = 0$, which remember from Definition 2.3 is essential for constructing a cohomology theory. However, in general, one has to do quite some work to achieve this result. Indeed:

We mentioned earlier that if x and y are points of consecutive indices, $\mathcal{M}(x, y)$ is a compact 0-dimensional manifold, and hence finite. However, when $\mu(x) - \mu(y) > 1$, this manifold is no longer compact. This is fixed by the following proposition:

Proposition 2.8 (Compactification of moduli spaces)**.** *For $\mu(x) - \mu(x) > 1$, $\mathcal{M}(x, y)$ can be compactified by adding broken trajectories.*

Let us sketch the intuition behind this result. Say x and y are such that $\mu(x) - \mu(y) = 2$. Then $\mathcal{M}(x, y)/\mathbb{R}$ has dimension 1, meaning that we have a 1-dimensional family of trajectories connecting x and y.



Visually, the claim is that this family $\theta \mapsto u_\theta$ of trajectories will converge, to either side ($\theta \to \pm\infty$), to a broken trajectory; *i.e.* the concatenation of two smooth trajectories passing through a point of intermediary index.

More formally, say that $\mu(x) = k + 1$ and $\mu(y) = k - 1$, then:

$$\partial\mathcal{M}(x, y) = \coprod_{\mu(z)=k} \mathcal{M}(x, z) \times \mathcal{M}(z, y),$$

which is what we mean by « we compactify $\mathcal{M}(x, y)$ by adding broken trajectories ». In the setup of Floer theory, this will be referred to as Gromov compactness (Lemma 3.41). The picture above was borrowed from Chapter 3 of [AD13], to which we refer for proofs.

Corollary 2.9. $\mathrm{d}^2 = 0$.

Proof. Let c be a critical point of our Morse function. Unfolding the definitions gives:

$$\mathrm{d}^2 c = \sum_{\mu(a)=\mu(c)=+2} \left(\sum_{\mu(b)=\mu(c)+1} \varepsilon(a, b)\varepsilon(b, c) \right) a.$$

Therefore, to show that $\mathrm{d}^2 c = 0$, it suffices to show that for every a with index 2 higher than c, we have

$$\sum_{\mu(b)=\mu(c)+1} \varepsilon(a, b)\varepsilon(b, c) = 0. \tag{2.5}$$

This corresponds to the cardinality of $\coprod_{\mu(b)=\mu(a)-1} \mathcal{M}(a, b) \times \mathcal{M}(b, c)$, which by Proposition 2.8 is nothing but the boundary of $\mathcal{M}(a, c)$. However, $\dim \mathcal{M}(a, c) = \mu(a) - \mu(c) - 1 = 1$. Compact 1-dimensional manifolds are completely classified: $\mathcal{M}(a, c)$ must be a disjoint union of intervals and circles. It is an easy exercise to show that the oriented count of boundary points of such a manifold is zero.

□

Thanks to this corollary, we can now define:

• **Definition 2.10.** **Morse cohomology** $HM^*(M, f)$ is defined as the cohomology of the complex (CM^*, d), for any Morse-Smale function $f : M \to \mathbb{R}$.

One of the most marvellous results is then that Morse homology captures the changes in topology of the manifold M. More precisely:

Theorem 2.11 (Isomorphism with singular homology)**.** *If M is compact and f is Morse-Smale, then $HM^*(M, f) \cong H^*(M)$, where H^* denotes singular cohomology.*

➤ Before we leave Morse theory aside to focus on infinite-dimensional generalisations, let us briefly summarise the recipe we just used to construct it:

1. Choose a generic function $f : M \to \mathbb{R}$, and define an index $\mu : \mathrm{Crit}(f) \to \mathbb{Z}$.
2. Study trajectories of $\mathrm{d}u/\mathrm{d}s = -\nabla f\big(u(s)\big)$. Define the moduli spaces of solutions, and show that these are smooth manifolds, which compactify nicely.
3. The previous point shows that if we define a co-chain complex

$$CM^*(M,f) := \{x \in \mathrm{Crit}(f) \mid \mu(x) = *\},$$

along with a differential d counting anti-gradient trajectories, then $\mathrm{d}^2 = 0$, allowing us to define the cohomology groups

$$HM^*(M,f) := \frac{\ker \mathrm{d}^*}{\mathrm{im}\,\mathrm{d}^{*-1}}.$$

4. By compactness of M, we can show that these modules are independent of f generic.

2.3 Elementary Floer theories

The recipe for a Floer theory is philosophically the same as for a Morse theory. We will, in the rest of this chapter, expose the main ideas behind Hamiltonian and Lagrangian Floer theories. Both of these are standardly defined assuming:

Assumption 2.12. (M,ω) is a closed symplectically aspherical manifold, meaning that M is compact without boundary, and that every 2-sphere has symplectic area zero, *i.e.* :

$$\forall u : \mathbb{S}^2 \to M : \int_{\mathbb{S}^2} u^\star\omega = 0.$$

2.3.1 Hamiltonian Floer theory

We had already motivated this theory at the beginning of the chapter, as a way of topologically reformulating the Principle of Least Action in physics. More precisely:

➢ Let (M,ω) be a manifold satisfying Assumption 2.12, and $H_t : M \to \mathbb{R}$ a Hamiltonian. Say that we are looking for periodic orbits of this system.

Then we can write down an action functional:

$$\mathcal{A}_H : \mathcal{C}^\infty(\mathbb{S}^1, M) \longrightarrow \mathbb{R} : x \longmapsto -\int_{\mathbb{D}^2} \tilde{x}^\star\omega + \int_{\mathbb{S}^1} H_t\,(x(t))\,\mathrm{d}t, \tag{2.6}$$

where $\tilde{x} : \mathbb{D}^2 \to M$ is a capping disk of x, *i.e.* $\partial\mathrm{im}(\tilde{x}) = \mathrm{im}(x)$. This generalises the standard action from classical physics, from (2.1). By the principle of least action (Principle 2.2), physical trajectories are exactly the critical points of $\mathcal{A}_H$.

In other words, periodic orbits of the system are the same as critical points of

$$\mathcal{A}_H : \mathcal{C}^\infty(\mathbb{S}^1, M) \longrightarrow \mathbb{R}.$$

Let us therefore mimic the constructions we did for Morse theory, and try to construct a cohomology theory from these critical points. The recipe will be exactly the same:

Principle 2.13 (General recipe for a Floer theory)**.**

1. Analytically study critical points of $\mathcal{A}_H$, and define an index $\mu : \text{Crit}\mathcal{A}_H \to \mathbb{Z}$.
2. Study trajectories of $\partial_s u = -\nabla \mathcal{A}_H\big(u(s)\big)$. Define the moduli spaces of solutions, and show that those are manifolds which compactify in a nice way.
3. Define the co-chain complex

$$CF^*(M,H) := \left\{ x \in \text{Crit}\mathcal{A}_H \,\middle|\, \mu(x) = * \right\}, \tag{2.7}$$

with a Morse-like differential. Then the previous point shows that $\mathrm{d}^2 = 0$, and so we can define a cohomology theory

$$HF^*(M,H) := H^*(CF^*, \mathrm{d}),$$

called **Hamiltonian Floer Cohomology**, which detects 1-periodic orbits of our physical system.

4. It remains to show, like in Morse theory, that we have some sort of invariance statement. In particular, using closedness of M, one can show that $HF^*(M,H)$ is independent of the choice of H. Furthermore, if H is sufficiently $\mathcal{C}^2$-small, then Hamiltonian Floer cohomology is isomorphic to Morse cohomology.

We, of course, sweep quite a bit of technicalities under the rug. We will not rigorously go through the construction of Hamiltonian Floer homology, as we will essentially reproduce the same recipe in Chapter 3 in full details, in the case of Wrapped Floer Homology. However, let us briefly expand on each of these four steps, in an objective to hopefully enlighten the reader's intuition. For a thorough exposition, see [AD13, Ch.6-11].

1. The action functional is a map $\mathcal{A}_H : \mathcal{C}^\infty(\mathbb{S}^1, M) \longrightarrow \mathbb{R}$. Hence, its critical points are period 1 loops $x : \mathbb{S}^1 \to M$, which will be the elements of our homology theory. It is a relatively easy exercise in analysis to compute $\mathrm{d}\mathcal{A}_H, \nabla\mathcal{A}_H$, and $\text{Hess}_x\mathcal{A}_H$ (we carry out these calculations in Chapter 3, for a very similar $\mathcal{A}_H$), from which one can deduce analytical criteria for an orbit to be critical, or degenerate.

   • **Definition 2.14.** A periodic orbit x is called degenerate if $\text{Hess}_x\mathcal{A}_H$ is singular. Equivalently, if the linearisation $D\phi^1_H|_{x(0)}$ has 1 as an eigenvalue.

   Assumption 2.15. Assume that H has no degenerate 1-periodic orbits.

   One can then assign to these orbits an index $\mu \in \mathbb{Z}$ called the Maslov index, which we use as a substitute for the Morse index (see §3.2.4).

2. We now understand critical points of $\mathcal{A}_H$. Let us see how to connect them with anti-gradient trajectories. As mentioned above, we can explicitly compute that:

$$\nabla\mathcal{A}_H(x) = J_t\big(\dot{x}(t) - X_H(x(t))\big),$$

where J_t is a choice of compatible almost-complex structure on (M,ω), and $X_H = J\nabla H$ is the Hamiltonian vector field (Definition 1.7). Hence, the equation for anti-gradient trajectories $\partial_s u = -\nabla\mathcal{A}_H$ becomes:

$$\frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} + \nabla H = 0, \tag{2.8}$$

and is called the **Floer equation**. Its solutions will be maps

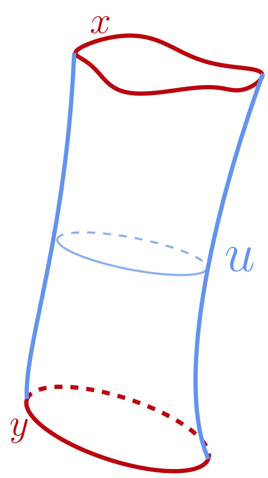
$$u : \mathbb{R} \times \mathbb{S}^1 \longrightarrow M : (s,t) \longmapsto u(s,t),$$

i.e. cylinders in M. We will see, in Proposition 2.24, that these cylinders actually become pseudo-holomorphic after a change of coordinates.

Just like in Morse theory, we want these anti-gradient trajectories to connect critical points of $\mathcal{A}_H$. Hence we are looking for solutions u such that:

- u solves the Floer equation (2.8),
- $\lim_{s \to -\infty} u(s, \cdot) = x(\cdot)$,
- $\lim_{s \to +\infty} u(s, \cdot) = y(\cdot)$,

where x and y are 1-periodic orbits of our Hamiltonian. We call these solutions Floer cylinders, or trajectories.



Now that these are defined, we can define the moduli space $\widehat{\mathcal{M}}(x, y)$ of Floer cylinders joining two orbits x and y, like in Morse theory, and show that it is a smooth manifold, as well as study how it compactifies. This is where the symplectic asphericity assumption comes in (see Lemma 3.41).

3. Then, in the same way as for Morse theory, we define:

$$\begin{aligned} \mathrm{d} : CF^*(M, H) &\longrightarrow CF^{*+1}(M, H) \\ y &\longmapsto \sum_{\mu(x)-\mu(y)=1} \varepsilon(x, y)x \end{aligned}$$

and we prove that $\mathrm{d}^2 = 0$, allowing us to define **Hamiltonian Floer cohomology**:

$$HF^*(M, H) := H^*(CF^*, \mathrm{d}).$$

4. The final step is then a standard argument, which can be found in full details in [AD13, Ch. 11], and which we adapt in §3.4.3 to Wrapped Floer theory. We do not further mention the isomorphism between Hamiltonian Floer and Morse homologies (for $\mathcal{C}^2$-small Hamiltonians), for it will not be of particular relevance to us in this thesis, and we refer the interested reader to [AD13, Ch. 10].

2.3.2 Lagrangian Floer theory

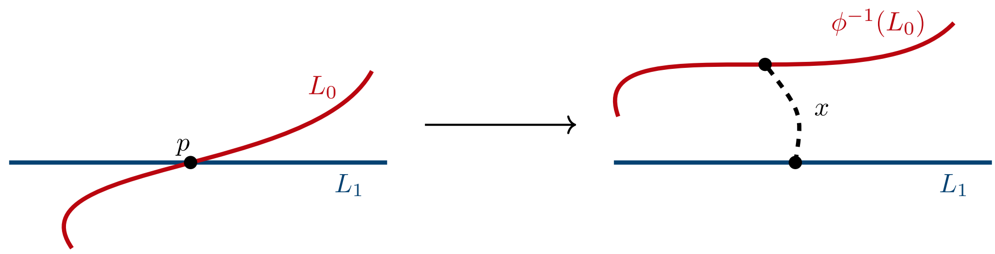
We just constructed *Hamiltonian* Floer cohomology, to study periodic orbits of a Hamiltonian $H_t : M \to \mathbb{R}$. However, this turns out to only be a special case of a more general construction: *Lagrangian Floer cohomology*. Let us make a few definitions. Let (M, ω) be a symplectic manifold.

• **Definition 2.16.** A submanifold L of (M, ω) is said to be **Lagrangian** if $\omega|_L \equiv 0$, and L is of maximal dimension (by non-degeneracy of ω, this means $\dim L = \dim M/2$).

✧ **Example 2.17.** Let Q be a smooth manifold and $T^\star Q$ its cotangent bundle. We embed $Q \hookrightarrow T^\star Q : q \mapsto (q, 0)$ as the zero section. It is an easy exercise to show that the image of this embedding is a Lagrangian submanifold of $T^\star Q$ with its canonical symplectic structure. The graph of a 1-form $\alpha \in \Omega^1(L) = \Gamma(T^\star L)$ is a Lagrangian submanifold of $T^\star L$ iff $\mathrm{d}\alpha = 0$.

➤ **Setup:** Let (M, ω) be a manifold satisfying Assumption 2.12, and $L_0, L_1 \subset M$ be compact Lagrangian submanifolds. The set $L_0 \cap L_1$ is finite. Therefore, we can try and construct a homology theory whose objects will be these intersection points.

To achieve this, we need a way of viewing intersection points between L_0 and L_1 as critical points of some functional. It is for this easier to adopt a dynamical viewpoint, and choose some Hamiltonian $H_t : M \to \mathbb{R}$ with which we perturb one of the Lagrangians. For example:



where $\phi = \phi_H^{t=1}$ is the time-1 map of H. With this process, a point $p \in L_0 \cap L_1$ gets associated to a trajectory $x : [0,1] \to M$ of the Hamiltonian H with ends in $\phi^{-1}(L_0)$ and L_1.

• **Definition 2.18.** Let Λ_0, Λ_1 be Lagrangians in M, and $H_t : M \to \mathbb{R}$ a Hamiltonian. A trajectory $x : [0,T] \to M$ of X_H such that

$$x(0) \in \Lambda_0, \ \ x(1) \in \Lambda_1$$

is called a **Hamiltonian chord** of H_t of length T.

As we have just argued:

Lemma 2.19. *Given any $H_t : M \to \mathbb{R}$, there is a bijection between $L_0 \cap L_1$ and the set of Hamiltonian chords of length 1 between $\Lambda_0 := \phi^{-1}(L_0)$ and $\Lambda_1 := L_1$.*

The question now becomes to find an action functional whose critical points are exactly these Hamiltonian chords of length 1. For this, we can use a very similar action functional to the one used for Hamiltonian Floer theory in the previous subsection. The only difference is that we are working with paths $[0,1] \to M$ instead of loops. Then, we can unfold the same recipe and define **Lagrangian Floer cohomology** $HL^*(L_0, L_1; H)$, whose objects will be cohomology classes of Hamiltonian chords (alternatively, intersection points between Lagrangians). Provided that M is closed and symplectically aspherical, this construction actually turns out to be independent of the choice of Hamiltonian $H_t : M \to \mathbb{R}$.

We do not elaborate further on the construction of Lagrangian Floer theory, as our construction of Wrapped Floer theory in Chapter 3 will be very similar.

✧ **Example 2.20** (Hamiltonian Floer theory is a Lagrangian Floer theory)**.** The Hamiltonian Floer theory we defined in the previous section can actually be seen as a special case of Lagrangian Floer theory. Indeed, recall that by construction, $HF^*(H)$ detected 1-periodic orbits of the Hamiltonian H. These are in bijection with fixed points of the Hamiltonian diffeomorphism

$$\phi := \phi_H^{t=1} : M \longrightarrow M,$$

or alternatively with intersection points of:

$$\mathrm{Graph}(\phi) := \left\{ \big(x, \phi(x)\big) \mid x \in M \right\} \subset M \times M, \tag{2.9}$$

$$\Delta := \left\{ (x, x) \mid x \in M \right\} \subset M \times M. \tag{2.10}$$

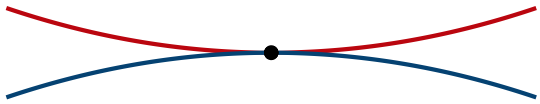
(2.9) and (2.10) can easily be shown to be Lagrangians in $\big(M \times M, \omega \oplus -\omega\big)$, and so Hamiltonian Floer theory is but a special case of Lagrangian Floer theory. □

Let us make one final note on Lagrangian Floer theory. Formally, it is only defined if all Hamiltonian chords of H are *non-degenerate*, in the sense:

• **Definition 2.21.** A Hamiltonian chord $x : [0,1] \to M$ is degenerate if $\mathrm{Hess}_x \mathcal{A}_H$ is singular.

This can be interpreted visually, looking back at our Lagrangian intersection picture.

Lemma 2.22. *In the correspondence between Hamiltonian chords and Lagrangian intersections, a Hamiltonian chord is degenerate iff the corresponding intersection is non-transverse.*



We will prove this formally in the next chapter (see Lemma 3.27). However, one thing we can easily do now is show that this new definition of degeneracy is compatible with the one we had given for Hamiltonian Floer theory (Definition 2.14), when viewing the latter as a special case of Lagrangian Floer theory.

✧ **Example 2.23.** In Hamiltonian Floer theory, a 1-periodic orbit $\gamma : \mathbb{S}^1 \to M$ of a Hamiltonian H is called **degenerate** if $D\phi|_{\gamma(0)}$ has 1 as an eigenvalue. We want to show that, under the correspondence given in Example 2.20, degeneracy corresponds to non-transversality of the intersection. So we want to show that

$$1 \text{ is an eigenvalue of } D\phi|_{\gamma(0)} \iff \text{Graph}(\phi) \cap \Delta \text{ is non-transverse.}$$

Since both Lagrangians have dimension $\frac{1}{2}\dim M$, their intersection being non-transverse at $p \in M$ reduces to:

$$T_p\text{Graph}(\phi) \cap T_p\Delta \neq \emptyset.$$

It is enough to work in a local neighbourhood, so let us identify M with $\mathbb{R}^{2n}$ and $M \times M$ with $\mathbb{R}^{4n}$. Then Graph(ϕ) is locally the zero set of the function:

$$\Xi : M \times M \longrightarrow M : (x, y) \longmapsto y - \phi(x),$$

which means that at the point $p \in \text{Graph}(\phi)$, we have:

$$T_p\text{Graph}(\phi) = \ker D\Xi|_p = \ker\left(-D\phi \mid \text{id}\right)|_p = \left\{ \begin{pmatrix} Z \\ D\phi|_p Z \end{pmatrix} \mid Z \in \mathbb{R}^{2n} \right\},$$

while the diagonal Δ is the zero-set of

$$\Theta : M \times M \longrightarrow M : (x, y) \longmapsto y - x,$$

which implies that its tangent space at p is:

$$T_p\Delta = \ker D\Theta|_p = \ker\left(\text{id} \mid -\text{id}\right) = \left\{ \begin{pmatrix} Z \\ Z \end{pmatrix} \mid Z \in \mathbb{R}^{2n} \right\}.$$

Hence, if $p \in \text{Graph}(\phi) \cap \Delta$, then :

$$T_p\text{Graph}(\phi) \cap T_p\Delta = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{4n} \mid Y = X = \text{D}\phi|_p X \right\}. \tag{2.11}$$

Therefore

$$\begin{aligned} \text{The intersection is non-transverse} &\iff T_p\text{Graph}(\phi) \cap T_p\Delta \neq \emptyset \\ &\iff \exists X \in \mathbb{R}^{2n} \text{ such that } X - D\phi|_{\gamma(0)}X = 0 \\ &\iff 1 \text{ is an eigenvalue of } D\phi|_{\gamma(0)} \end{aligned}$$

□

2.3.3 Floer theories and pseudo-holomorphic curves

One of the main tools used by Floer in his original construction was pseudo-holomorphic curve theory, then just introduced by Gromov in [Gro85]. We recall basic definitions on pseudo-holomorphic curves in §B.1.2, but to summarise in a few words:

While a symplectic manifold (M, ω) will rarely be complex, it will always support an almost complex structure J compatible with the symplectic form (Lemma B.4), allowing us to define an appropriate notion of pseudo-holomorphicity for curves $u : (\Sigma, i) \to (M, J)$, where (Σ, i) is an arbitrary Riemann surface. This condition reduces to asking that

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0.$$

The equation above is often called the Cauchy-Riemann equation.

When sketching the construction of Hamiltonian Floer theory, we looked at Floer cylinders. These were solutions

$$u : \mathbb{R} \times \mathbb{S}^1 \longrightarrow M$$

of the gradient-flow equation of $\mathcal{A}_H$:

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H = 0. \tag{2.12}$$

Let us show that, up to reparametrisation, these Floer cylinders can be viewed as pseudo-holomorphic curves.

Proposition 2.24. *The Floer equation can be reparametrised into a Cauchy-Riemann equation. Hence, its solutions can be viewed as pseudo-holomorphic cylinders.*

Proof. Let ϕ^t be the flow of our Hamiltonian H. For a curve $u : \mathbb{R} \times \mathbb{S}^1 \to M$, let $\tilde{u}(s,t) := (\phi^t)^{-1} u(s,t)$. Or alternatively, $\phi^t\big(\tilde{u}(s,t)\big) = u(s,t)$. Apply $\partial/\partial t$:

$$\begin{aligned}
\frac{\partial}{\partial t}(\phi^t \circ \tilde{u})(s,t) = \left(\frac{\partial}{\partial t}\phi^t\right) \circ \tilde{u}(s,t) + \phi^t_\star \frac{\partial \tilde{u}}{\partial t} &= \frac{\partial u}{\partial t} \\
\implies X_H(\tilde{u}) + \phi^t_\star \frac{\partial \tilde{u}}{\partial t} &= \frac{\partial u}{\partial t} \\
\implies \frac{\partial \tilde{u}}{\partial t} &= (\phi^t)^\star \left(\frac{\partial u}{\partial t} - X_H\right)
\end{aligned}$$

Now, equation (2.12) can be rewritten $\left(\frac{\partial u}{\partial t} - X_H\right) = -J_t \frac{\partial u}{\partial s}$. Hence:

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t} = -(\phi^t)^\star \left(J_t \frac{\partial u}{\partial s}\right) \\
\implies &\frac{\partial \tilde{u}}{\partial t} = -\tilde{J}_t \frac{\partial u}{\partial s} \qquad \text{by setting } \tilde{J}_t := (\phi^t)^\star J_t \\
\implies &\frac{\partial u}{\partial s} + \tilde{J}_t \frac{\partial \tilde{u}}{\partial t} = 0 \\
\implies &\frac{\partial \tilde{u}}{\partial s} + \tilde{J}_t \frac{\partial \tilde{u}}{\partial t} = 0
\end{aligned}$$

because $(\partial/\partial s)\tilde{u} = (\partial/\partial s)u$, since ϕ^t does not depend on s. So the Floer equation can indeed be reparametrised into a Cauchy-Riemann equation. ☐

This means that one can use all the machinery from pseudo-holomorphic curve theory to study solutions of the Floer equation: bubbling off analysis (§B.1.4), asymptotical analysis, Siefring intersection theory (in dimension 4),... which have all become essential in Floer theory.

Chapter 3

Wrapped Floer Cohomology

The two first variants of Floer theory which we introduced in the previous chapter, though extremely powerful at *studying periodic orbits* and *Lagrangian intersections* in closed symplectic manifolds, fail when (M,ω) is non-compact, or admits a boundary. As it happens, there is a way of adapting these constructions to a wider class of symplectic manifolds, with a nice boundary. Paradoxically, the way we get rid of one problem (the boundary) is by turning it into an other (non-compactness). This "adaptation" will yield two new theories:

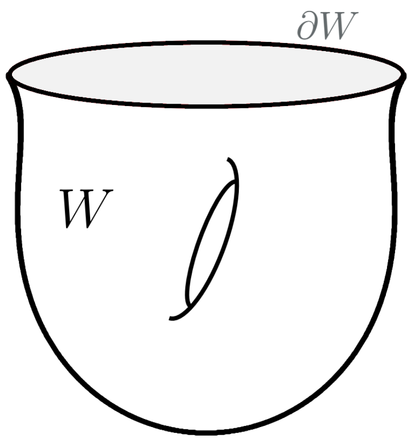
	Closed symplectically aspherical manifolds	Liouville domains
Periodic orbits	Hamiltonian Floer Cohomology	Symplectic Cohomology
Lagrangian intersections	Lagrangian Floer Cohomology	Wrapped Floer Cohomology

Wrapped Floer Cohomology will be our main interest in this thesis. However, Symplectic Cohomology works extremely similarly – and one can adapt every statement/proof in this chapter by removing all mentions of the Lagrangians, and considering periodic orbits instead of Hamiltonian chords; see [Rit13] for example.

3.1 Geometric setup

3.1.1 Liouville domains and Liouville manifolds

• **Definition 3.1.** A **Liouville domain** is a compact, exact symplectic manifold $(W, \omega = \mathrm{d}\lambda)$ with restricted contact-type boundary (*i.e.* $\alpha := \lambda|_{\partial W}$ is a contact form).



◇ **Remark 3.2.** By Stokes' theorem, ω being exact automatically implies that the boundary must be non-empty. It also implies that any sphere $\mathbb{S}^2 \to W$ has symplectic area zero, which means that we no longer need to assume symplectic asphericity, like we did in Chapter 2 (Assumption 2.12), since it now comes for free.

By Proposition A.20, we can re-write the definition as:

• **Definition 3.1 (bis).** A **Liouville domain** is a compact, exact symplectic manifold $(W, \omega = \mathrm{d}\lambda)$ such that there locally exists a Liouville vector field near ∂W which is positively transverse to it.

This definition is easier to work with in practice, and provides us with examples:

✧ **Example 3.3.** $\left(\mathbb{D}^{2n}, \omega = \frac{1}{2}\mathrm{d}\sum_i q_i \mathrm{d}p_i - p_i \mathrm{d}q_i\right)$ is a Liouville domain with boundary $\mathbb{S}^{2n-1}$.

Proof. $\mathbb{D}^{2n}$ is indeed a compact, exact symplectic manifold with this symplectic structure, and its boundary is indeed $\mathbb{S}^{2n-1}$. The radial vector field $V = \sum q_i \partial_{q_i} + p_i \partial_{p_i}$ is then a Liouville vector field (see Example A.21). □

✧ **Example 3.4.** Let Q be a manifold, and $(T^\star Q, \omega = -\mathrm{d}\lambda)$ its cotangent bundle (Example 1.14), endowed with a compatible almost complex structure (which exists by Lemma B.4). Let $\|\cdot\|$ denote the induced norm on M, and *a fortiori* $T^\star M$. We define:

• **Definition 3.5.** The **disc cotangent bundle** $\mathbb{D}^\star Q$ and **unit cotangent bundle** $\mathbb{S}^\star Q$ of Q are defined as:

$$\mathbb{D}^\star Q := \{(x, \xi) \in T^\star Q \mid ||\xi|| \leq 1\},$$
$$\mathbb{S}^\star Q := \{(x, \xi) \in T^\star Q \mid ||\xi|| = 1\}.$$

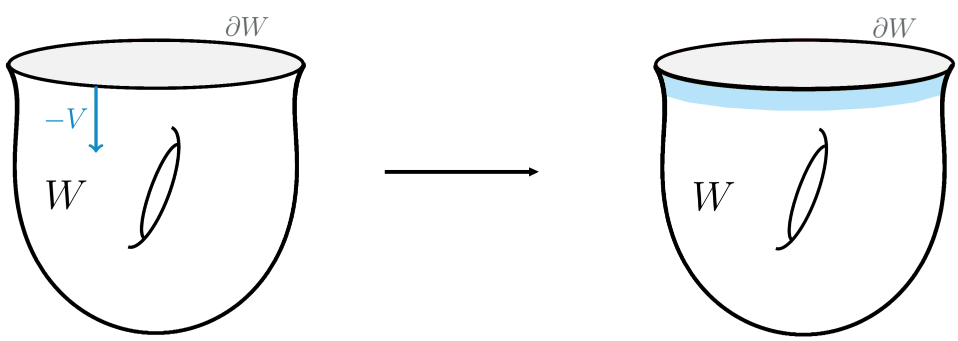
Then for any manifold Q, $(\mathbb{D}^\star Q, \omega = \mathrm{d}\lambda)$ is a Liouville domain, with boundary $\mathbb{S}^\star Q$, where

$$\lambda := -\sum_i p_i \mathrm{d}q_i.$$

Proof. Heuristically, this is proved by finding an atlas for $\mathbb{D}^\star Q$, and applying the previous example in each chart. We do this formally in Calculation C.1. □

➤ At first glance, the main obstruction to defining a Floer theory for $(W, \omega = \mathrm{d}\lambda)$ seems to be its boundary, which usual Floer theory is not very well equipped to deal with it. As a matter of fact, it turns out to be easier to drop the compactness assumption, and instead work with a « non-compact manifold without boundary » than to work with a « compact manifold with boundary ». For these purposes, we describe a process, called Liouville completion, which allows to get rid of the boundary of a Liouville domain.

By definition, the Liouville vector field V is positively transverse to the boundary, so that by flowing backwards along it, we can parametrise a collar neighbourhood $(-\epsilon, 0] \times \partial W$:



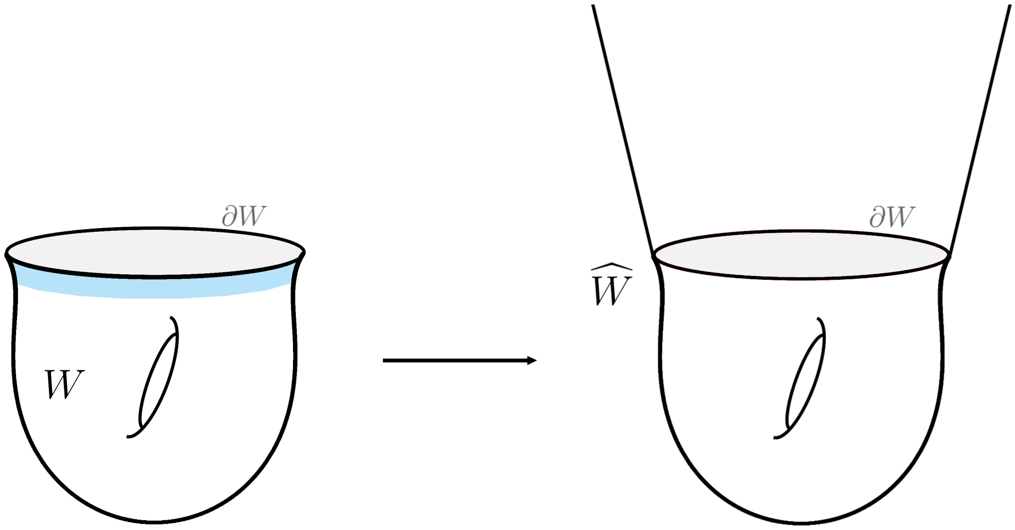
By §A.1.3, this collar neighbourhood $(-\epsilon, 0] \times \partial W$ can be endowed with the symplectic form

$$\tilde{\omega} = \mathrm{d}(e^t \alpha),$$

where $\alpha := \lambda|_{\partial W}$, and t is the coordinate on $(-\epsilon, 0]$, and a Liouville vector field is given by $V = \partial_t$. For later convenience, we rescale by setting $r = e^t$. The collar neighbourhood now becomes $(1-\varepsilon, 1] \times \partial W$, and by Remark A.19, we have $V = r\partial_r$.

• **Definition 3.6.** The **Liouville completion** $\widehat{W}$ of $(W, \omega = \mathrm{d}\lambda)$ is given by half symplectising $(\partial W, \alpha)$ in the Liouville direction. In other words, we glue a half-clylinder $[1, +\infty) \times \partial W$ along the boundary, defining:

$$\widehat{W} := W \cup_{\partial W} [1, +\infty) \times \partial W.$$



We can endow this extension with an exact symplectic form by setting:

$$\hat{\lambda} := r\lambda,$$
$$\hat{\omega} := \mathrm{d}\hat{\lambda}.$$

This smoothly extends the symplectic structure from $(W, \omega = \mathrm{d}\lambda)$ to $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$, and ensures that $V = r\partial_r$ remains the Liouville vector field on the extension $[1, +\infty) \times \partial W$.

Lemma 3.7 ([CE12], Prop. 11.8)**.** *Up to symplectomorphism, the completion of* $(W, \omega = \mathrm{d}\lambda)$ *only depends on the homotopy class of* λ*.*

The Wrapped Floer Cohomology of a Liouville domain $(W, \omega = \mathrm{d}\lambda)$ is defined by working on this Liouville completion $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$. Hence, two ingredients we need are:

- a Hamiltonian on $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$;
- an almost complex structure on $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$.

In Chapter 5, we will explicitly see how to extend a Hamiltonian on $(W, \omega = \mathrm{d}\lambda)$ to one on $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$, which is important when working with concrete problems. For now though, since we are just laying the theory, let us directly pick a Hamiltonian $H_t : \widehat{W} \longrightarrow \mathbb{R}$.

Now, for our choice of almost complex structure:

• **Definition 3.8.** An almost complex structure J on $\widehat{W}$ is said to be of **contact type** (or sometimes, SFT-like) if:

- $J|_\xi$ is compatible with $\mathrm{d}\alpha$, where $\xi = \ker \alpha$ is the contact structure on $(\partial W, \alpha)$. In other words, $\mathrm{d}\alpha(\cdot, J\cdot)$ defines a Riemannian metric on ξ.

- On $[1, +\infty) \times \partial W$, we have:

$$J\partial_r = \mathcal{R}_\alpha, \tag{3.1}$$

where $\mathcal{R}_\alpha$ is the Reeb vector field on $(\partial W, \alpha)$. In other words, J maps the Liouville direction to the Reeb one. (Recall that $V = r\partial_r$ is the Liouville vector field).

◇ **Remark 3.9** (Comments on J)**.** Let us clarify what we mean by (3.1), since while ∂_r is defined on the whole of $[1, +\infty) \times \partial W$, the Reeb vector field $\mathcal{R}_\alpha$ is *a priori* only defined along the boundary. However, notice that every slice $\{r_0\} \times \partial W$ parallel to the boundary is also of contact type with the form

$$\alpha_{r_0} := \hat{\lambda}|_{\{r_0\}\times\partial W} = r_0\alpha$$

In particular, the associated Reeb vector field

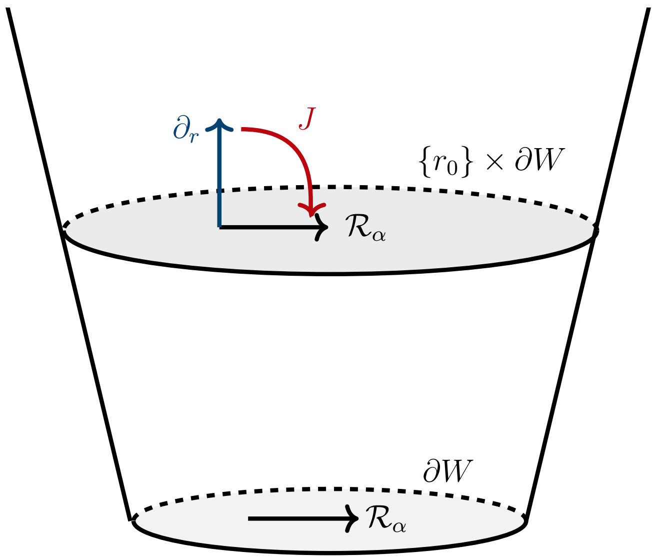
$$\mathcal{R}_{\alpha_0} = \frac{1}{r_0}\mathcal{R}_\alpha$$

is but a rescaling of the original Reeb vector field on ∂W, translated to the slice $\{r_0\} \times \partial W$.

So formally, $\mathcal{R}_\alpha$ does live on every slice $\{r_0\} \times \partial W$, and therefore it makes sense to ask that $J\partial_r = \mathcal{R}_\alpha$ everywhere.

Alternatively, we could rewrite $J\partial_r = \mathcal{R}_\alpha$ as:

$$J^\star\hat{\lambda} = r\mathrm{d}r. \tag{3.2}$$



3.1.2 Hamiltonians dynamics on the extension

Say we have a Hamiltonian $H_t : \widehat{W} \to \mathbb{R}$, on which we impose:

Assumption 3.10. There exists $R_0 \geq 1$, and a smooth function $h_t : [R_0, +\infty) \to \mathbb{R}$ such that, for $r \geq R_0$, $H_t = h_t(r)$.

In practice, this is not too wild an assumption since most of the time we construct the extension of H to $[1, +\infty) \times \partial W$ ourselves (see Chapter 5). Therefore, we can enforce prescribed behaviours on it – we will see that we can make it polynomial at infinity.

Now, by definitions of the Hamiltonian vector field X_H and of J, we have:

$$\begin{aligned} X_{H_t} &= J\nabla H_t \\ &= J\left(h_t'(r)\partial_r\right) \\ &= h_t'(r)\mathcal{R}_\alpha. \end{aligned}$$

Corollary 3.11. *On $[R_0, +\infty) \times \partial W$, the Hamiltonian vector field is a reparametrisation of the Reeb vector field. In particular, it is constrained to slices $\{r\} \times \partial W$.*

Let us now bring Lagrangians into the mix. Observe that, just like we completed:

$$W \longrightarrow \widehat{W} := W \cup_{\partial W} [1, +\infty) \times \partial W,$$

a Lagrangian $L \subset W$ which intersects ∂W can also be completed:

$$L \longrightarrow \widehat{L} := L \cup_{\partial L} [1, +\infty) \times \partial W.$$

• **Definition 3.12.** A Lagrangian L is said to be **exact** if there exists $f : L \to \mathbb{R}$ such that $\lambda|_L \equiv \mathrm{d}f$. It is said to be **admissible** if it is exact and has Legendrian intersection with the boundary ∂W. This means that, if we write $\partial L := L \cap \partial W$, and $\alpha = \lambda|_{\partial W}$, then:

$$\alpha|_{\partial L} \equiv 0.$$

◇ **Remark 3.13.** The reason we assume exactness of L is to ensure that every disc in W with boundary in L has symplectic area zero, by Stokes' theorem. This is very similar in essence to the requirement of symplectic asphericity, and indeed both of these conditions are used to avoid the bubbling off of pseudo-holomorphic spheres or discs (see §B.1.4), in the proof of compactification statements for moduli spaces.

Lemma 3.14. *Let L be admissible. Then its completion to $\widehat{W}$ is also admissible at infinity (on $[R_0, +\infty) \times \partial W$). In particular, the function f is constant at infinity.*

Proof. First, observe that ∂L is Legendrian in ∂W, so that $\alpha|_{\partial L} \equiv 0$. In particular,

$$\hat{\lambda}|_{[R_0,+\infty)\times\partial W} = r\alpha|_{\partial W} \equiv 0.$$

Therefore, $f \equiv$ constant is a primitive for $\hat{\lambda}|_{\widehat{L}}$ on $[R_0, \infty) \times \partial W$. Then, one needs to verify that the completion of L still intersects the boundary transversely, which is a standard argument (Lemma 2.2 of [ML24]). □

Assumption 3.15. Let $\Lambda_0, \Lambda_1 \subset \widehat{W}$ be two Lagrangians on $\widehat{W}$ which are of the form $\Lambda_i \cong \widehat{L}_i$ for some compact admissible Lagrangians $L_i \subset W$.

Just like in Definition 2.18, one can define **Hamiltonian chords** as trajectories of the Hamiltonian vector field with ends in Λ_0 and Λ_1 (which for the exact same reasons are in bijection with intersection points in $\phi(\Lambda_0) \cap \Lambda_1$). We can also define:

• **Definition 3.16.** Consider the Legendrians $\partial\Lambda_i := \Lambda_i \cap \partial W$. A trajectory $y : [0, T] \to \partial W$ of $\mathcal{R}_\alpha$, such that $y(0) \in \partial\Lambda_0$ and $y(T) \in \partial\Lambda_1$ is called a **Reeb chord** of length T.

We have:

Proposition 3.17. *Hamiltonian chords of H_t on $[R_0, +\infty) \times \partial W$ with ends in Λ_0, Λ_1 are in bijection with Reeb chords on $(\partial W, \alpha)$ with ends in $\partial\Lambda_0, \partial\Lambda_1$.*

Proof. By Assumption 3.10, $H_t = h_t(r)$ on $[R_0, +\infty) \times \partial W$. By Corollary 3.11, on the slice $\{r\} \times \partial W$ we have $X_{H_t} = h_t'(r)\mathcal{R}_\alpha$. So a chord x of X_{H_t} can be reparametrised into $y(t) := x(t/h_t'(r))$, which we can view as a Reeb chord in $(\partial W, \alpha)$. In particular, if x has length τ, then y has length $\tau h_t'(r)$. □

3.2 Analysis of the action functional

The ground is now set for Wrapped Floer Cohomology. For the rest of this chapter, we fix $(W, \omega = \mathrm{d}\lambda)$ our Liouville domain, with completion $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$, and $\Lambda_0, \Lambda_1 \subset \widehat{W}$ our admissible Lagrangians.

3.2.1 Definition of $\mathcal{A}_H$

Choose a Hamiltonian $H_t : \widehat{W} \to \mathbb{R}$ on which we enforce for now no restrictions, besides $\mathcal{C}^\ell$ regularity for some $\ell \geq 2$. We also choose an almost complex structure like in Definition 3.8, which we also take to be $\mathcal{C}^\ell$. We refer to (J, H) as our pair of Floer data.

• **Definition 3.18.** The space of candidates for Hamiltonian chords is:

$$\mathscr{P} := \left\{ x \in \mathbf{W}^{1,p}([0,1], W) \;\middle|\; \begin{matrix} x(0) \in \Lambda_0 \\ x(1) \in \Lambda_1 \end{matrix}, x \text{ is contractible} \right\}, \tag{3.3}$$

for $p \geq 2$, and where x is said to be contractible if $[\hat{x}] = 0 \in \pi_1(M, L)$ (where $\hat{x}$ is the loop $x \# x^{-1}$). As a subspace of $\mathbf{W}^{1,p}$, $\mathscr{P}$ naturally carries a Banach manifold structure (see §6.8 of [AD13]), *i.e.* it is separable, Hausdorff, and locally homeomorphic to a(n infinite-dimensional) Banach space. Hence, given $x \in \mathscr{P}$, one can speak of the tangent space:

$$T_x\mathscr{P} := \left\{ \zeta \in \mathbf{W}^{1,p}\left([0,1], x^\star T\widehat{W}\right) \,\middle|\, \begin{matrix} \zeta(0) \in T_{x(0)}\Lambda_0 \\ \zeta(1) \in T_{x(1)}\Lambda_1 \end{matrix} \right\}. \tag{3.4}$$

This essentially describes the space of vector fields in $\widehat{W}$ along the path x.

◇ **Remark.** Recall that our Lagrangians are exact, *i.e.* $\exists f_i \in \mathcal{C}^\infty(\Lambda_i)$ s.t $\lambda|_{\Lambda_i} = \mathrm{d}f_i$ $(i = 0, 1)$.

• **Definition 3.19.** Let $H_t : W \to \mathbb{R}$ be a Hamiltonian. The associated **Wrapped Floer action functional** $\mathcal{A}_H$ is given by:

$$\mathcal{A}_{H_t} : \mathscr{P} \longrightarrow \mathbb{R} : x \longmapsto f_1(x(1)) - f_0(x(0)) - \int_0^1 x^\star\hat{\lambda} + \int_0^1 H_t(x(t))\,\mathrm{d}t. \tag{3.5}$$

✧ **Example 3.20.** If, like in the previous subsection, we assume that H_t only depends on r at infinity (*i.e.* $\exists R_0$ s.t $\forall r \geq R_0 : H_t = h_t(r)$), then we have:

$$\mathcal{A}_{H_t}(x) = f_1\big(x(1)\big) - f_0\big(x(0)\big) - r h_t'(r) + h_t(r). \tag{3.6}$$

The proof is a routine calculation (see Calculation C.2).

◇ **Remark.** Note that while H is allowed to be time-dependent, we shall hide this from the equations, to make notation lighter. Technically, we will only state the definitions in the time-independent case, but every statement/proof carries out to the time-dependent case without change (see, for example, [KK16, §8]).

➢ Now, we claim that $\mathcal{A}_H$ is indeed an appropriate functional for our problem, *i.e.* that critical points of $\mathcal{A}_H$ correspond to Hamiltonian chords between Λ_0 and Λ_1, as defined in Definition 2.18. We now verify this.

Proposition 3.21. *The differential* $\mathrm{d}\mathcal{A}_H$ *of the above functional is given by:*

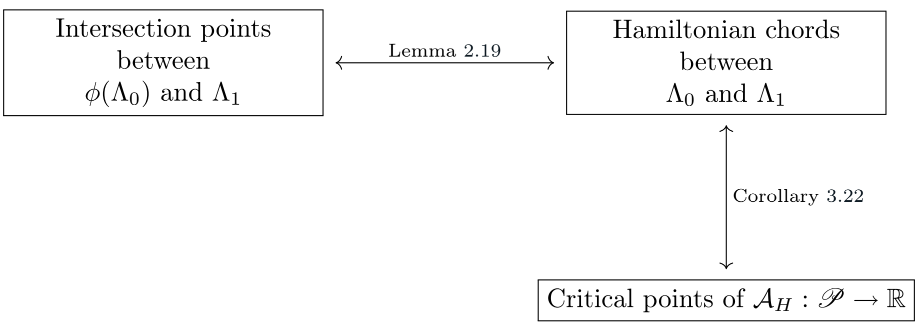
$$\mathrm{d}\mathcal{A}_H(x) : T_x\mathscr{P} \longrightarrow \mathbb{R} : \zeta \longmapsto \int_0^1 \mathrm{d}\hat{\lambda}\big(\dot{x}(t) - X_H(x(t)), \zeta(t)\big)\mathrm{d}t. \tag{3.7}$$

Proof. See Computational Appendix, Calculation C.3. The proof essentially boils down to calculus of variations.

Corollary 3.22. *Critical points of* $\mathcal{A}_H$ *are the same thing as length* 1 *Hamiltonian chords between* Λ_0 *and* Λ_1.

Proof. For x to be a critical point of $\mathcal{A}_H$, we'd need $\mathrm{d}\mathcal{A}_H(x) \equiv 0$, which would mean that the integral in (3.7) is zero for any choice of vector $\zeta \in T_x\mathscr{P}$. This implies that the integrand $\mathrm{d}\lambda\left(\dot{x}(t) - X_H(x(t)), \zeta(t)\right)$ is itself 0. Since $\zeta(t)$ is arbitrary, this is only possible when $\dot{x}(t) = X_H(x(t))$, which is the defining equation for a Hamiltonian chord. So x is a Hamiltonian chord. (The boundary conditions are automatically satisfied, since we require them of any element in $\mathscr{P}$). The converse implication is trivial. □

Hence, we now have the double dictionary:



These objects will be our main focus in the Floer theory we define. More precisely, they will be the *generators of our chain complex*. To get there though, we first need to define a notion of index for our chords, and before that we need to introduce a notion of degeneracy, for which we need to define $\nabla\mathcal{A}_H$ and $\mathrm{Hess}\mathcal{A}_H$.

3.2.2 Gradient and Hessian of $\mathcal{A}_H$

The Hessian of a function is standardly defined as the linearisation of its gradient at critical points. Hence, we first need to define $\nabla\mathcal{A}_H$; which requires having a metric on $T\mathscr{P}$.

For generality, we may assume without hassle that our complex structure $J = J_t$ is time-*dependent*, and define the associated Riemannian metrics $g_t := \hat{\omega}(\cdot, J_t\cdot)$ on $\widehat{W}$. Then, we can define an L^2-metric on $T\mathscr{P}$ by setting:

$$\forall \zeta_1, \zeta_2 \in T_x\mathscr{P} : \ \langle \zeta_1, \zeta_2 \rangle := \int_0^1 g_t\big(\zeta_1(t), \zeta_2(t)\big)\mathrm{d}t = \int_0^1 \hat{\omega}\big(\zeta_1(t), J_t\zeta_2(t)\big)\mathrm{d}t. \tag{3.8}$$

This turns $\mathscr{P}$ into a Hilbert manifold, allowing us to define:

• **Definition 3.23.** $\nabla\mathcal{A}_H \in \Gamma(T\mathscr{P})$ is defined as the unique vector field such that, at every point $x \in \mathscr{P}$, we have $\langle \nabla\mathcal{A}_H, \cdot\rangle = \mathrm{d}\mathcal{A}_H(\cdot)$ (this is an equality of 1-forms on $\mathscr{P}$, *i.e.* elements of $\Gamma(T^\star\mathscr{P})$). It is called the gradient of $\mathcal{A}_H$.

Lemma 3.24. *We have, at any point $x \in \mathscr{P}$:*

$$\nabla\mathcal{A}_H(x) = J_t(x)\big(\dot{x}(t) - X_H(x(t))\big) \tag{3.9}$$

Proof. By definition, for any $\zeta \in T_x\mathscr{P}$, $\langle \nabla\mathcal{A}_H, \zeta\rangle = \int_0^1 \hat{\omega}\big(\nabla\mathcal{A}_H(x), J_t\zeta(t)\big)\mathrm{d}t$.
We want this to equal $\mathrm{d}\mathcal{A}_H(\zeta)$ which, as we saw in Proposition 3.21, is given by

$$\int_0^1 \hat{\omega}\big(\dot{x}(t) - X_H(x(t)), \zeta(t)\big)\mathrm{d}t.$$

Since we want this equality to be true for all ζ, and $\hat{\omega}$ is non-degenerate, this implies that $-J_t\nabla\mathcal{A}_H(x) = \dot{x}(t) - X_H(x(t))$, and hence that $\nabla\mathcal{A}_H(x)$ has the desired form. □

◇ **Remark.** By Definition 3.23, $\nabla\mathcal{A}_H = 0 \iff \mathrm{d}\mathcal{A}_H = 0$. So $x \in \mathscr{P}$ is a critical point iff $\nabla\mathcal{A}_H(x) = 0$. Write $\mathrm{Crit}\mathcal{A}_H$ the set of such critical points.

➤ We can now define the Hessian of $\mathcal{A}_H$, which is defined as the linearisation of $\nabla\mathcal{A}_H$ at its zeroes (hence, at critical points of $\mathcal{A}_H$). The process of linearising sections at their zeroes is explained in §A.2.2 of the appendix.

Lemma 3.25. *The* ***Hessian*** *of $\mathcal{A}_H$ at $x \in \mathrm{Crit}(\mathcal{A}_H) \subset \mathscr{P}$ is given by:*

$$\mathrm{Hess}_x\mathcal{A}_H : T_x\mathscr{P} \longrightarrow L^2(x^\star(TM)) : \zeta \longmapsto J_t(x)\big(\nabla_t\zeta - \nabla_\zeta X_H(x(t))\big),$$

where ∇ denotes the Levi-Civita connection associated to the metric g_t on $\widehat{W}$, and ∇_t denotes the covariant derivative along the curve $t \mapsto x(t)$ (i.e. $\nabla_t := \nabla_{\dot{x}(t)}$).

Proof. See Computational Appendix, Calculation C.5. The proof relies on the fact, explained in §A.2.2, that the linearisation of a section at its zeroes is independent of the choice of connection. This allows us to pick a particularly nice connection (the Levi-Civita one), in which the computation can be carried out explicitly.

3.2.3 On degeneracy of chords

Recall that a Hamiltonian chord can simply be viewed as a critical point of $\mathcal{A}_H : \mathscr{P} \to \mathbb{R}$.

• **Definition 3.26.** Just like in Morse theory, we call a Hamiltonian chord x **degenerate** if $\mathrm{Hess}_x\mathcal{A}_H$ is singular (*i.e.* if $\ker\mathrm{Hess}_x\mathcal{A}_H \neq \{0\}$).

In Chapter 2, we stated that this was equivalent to asking that the intersection between the corresponding Lagrangians be non-transverse, but only proved it in a special case. We remedy this:

Lemma 3.27. *A Hamiltonian chord between Λ_0 and Λ_1 is degenerate iff the corresponding intersection between $\phi(\Lambda_0)$ and Λ_1 is non-transverse.*

Proof. See Calculation C.6 of the Computational Appendix.

We also mention an essential result, whose proof is the content of §8 of [AS10], in which Wrapped Floer theory was first formally defined:

Theorem 3.28. *A generic Hamiltonian H has no degenerate chords. Moreover, for a generic H, the end point of a chord is never the starting point of another chord.*

This concludes our discussion on degeneracy. We can add it to our double dictionary:

	Analytical picture	*Dynamical picture*	*Geometric picture*
Chord	Critical point of $\mathcal{A}_H$	Hamiltonian path between Λ_0 and Λ_1	Point in $\phi(\Lambda_0)$ and Λ_1
Degeneracy	$\ker \operatorname{Hess}_x \mathcal{A}_H \neq \{0\}$	$D\phi\|_x^{t=1}$ does not have 1 as an eigenvalue	$\phi(\Lambda_0) \cap \Lambda_1$ is non-transverse

◇ **Remark 3.29.** Since the beginning of the chapter, we have been assuming our Hamiltonian chords to have $W^{1,p}$ regularity, for $p \geq 2$. However, since we have $\dot{x} = J\nabla H$, then by an elliptic bootstrapping argument (see Theorem. B.17), x has regularity at least as high as the pair (J, H). In other words, if (J, H) is $\mathcal{C}^\ell$, then so is x.

3.2.4 On the grading of chords

We now have objects for our cohomology theory: Hamiltonian chords. One last thing we need though, to construct a Floer theory, is a notion of index like the Morse index, associating an integer to each chord and allowing us to arrange them in a graded cochain complex.

We first briefly explain how to define such an index in $\mathbb{R}^{2n}$. The original construction is due to Maslov, and it relies on noticing that $\mathbf{\Lambda}$, the space of Lagrangians in $(\mathbb{R}^{2n}, \omega_0)$, is diffeomorphic to U_n/O_n. Hence the map:

$$\rho : \mathbf{\Lambda} \cong U_n/O_n \longrightarrow \mathbb{S}^1 : U \longmapsto \det U^2$$

is well-defined; and one can easily show that it descends to an isomorphism on the fundamental groups (by studying the long exact sequence of homotopy groups). Therefore:

• **Definition 3.30.** To every loop x of Lagrangian subspaces in $\mathbb{R}^{2n}$, one can associate an integer $\rho(x)$ which uniquely determines it up to homotopy. We call this integer the **Maslov index** of x.

This construction was generalised by Conley & Zehnder, and Robbin & Salamon to paths of symplectic matrices (*i.e.* paths in the symplectic linear group $Sp(2n)$) which start at the identity. For a modern exposition, refer to Chapter 7 of [AD13]. The bottomline is that it allows us to associate an index to chord by considering the linearisations

$$t \longmapsto D\phi|_x^t, \quad t \in [0, 1],$$

which induce a path of symplectic matrices.

Now, on our Liouville manifold, provided that we can find local trivialisations around each chord, then we can associate such a Conley-Zehnder index μ_{CZ} to every chord. In particular:

Proposition 3.31. *Say that $2c_1(TW) = 0$, where c_1 is the first Chern class. Then, the Conley-Zehnder index induces a $\mathbb{Z}$-grading on chords.*

For which we refer to §9 of [AS10] or §3.6 of [Kim18]. The condition on c_1 is here to ensure the existence of trivialisations.

3.3 The Floer equation and trajectories

We follow the prototypical recipe for a Morse/Floer theory that we laid out in Principle 2.13. We now have an action functional $\mathcal{A}_H : \mathscr{P} \to \mathbb{R}$, whose critical points (Hamiltonian chords) we understand well; and we want to find anti-gradient trajectories to connect them together.

3.3.1 The Floer equation

Our functional is $\mathcal{A}_H$, therefore the equation for the anti-gradient flow is given by:

$$\frac{\partial u}{\partial s} = -\nabla \mathcal{A}_H(u). \tag{3.10}$$

Using the formula for $\nabla \mathcal{A}_H$ from Prop. 3.9, this equation can be rewritten as:

$$\frac{\partial u}{\partial s} + J_t\big(\frac{\partial u}{\partial t} - X_H(u)\big) = 0, \tag{3.11}$$

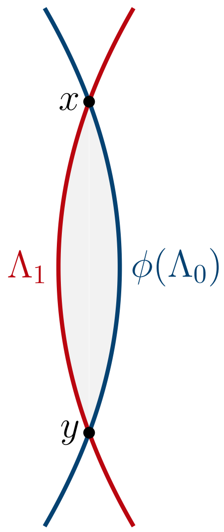
or again, using the fact that $X_H = J_t \nabla H$:

$$\frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} + \nabla H(u) = 0. \tag{3.12}$$

In either of these three forms, this is called the **Floer equation**. Its solutions will be maps $u : \mathbb{R} \times [0,1] \to M$, which we shall study in the next subsection. In the same way as in Proposition 2.24, the Floer equation can be reparametrised into a Cauchy-Riemann equation (by composing J_t with the Hamiltonian flow), so that solutions to the Floer equation can be viewed as pseudo-holomorphic curves (§B.1.2). Let us study these solutions further.

3.3.2 Floer trajectories

To understand solutions of the Floer equation more visually, it becomes useful to revert to the geometric picture, and view critical points of $\mathcal{A}_H$ as intersection points between the Lagrangians $\phi(\Lambda_0)$ and Λ_1. Then:



We are interested in solutions $u : \mathbb{R} \times [0,1] \to \widehat{W}$ of the Floer equation (3.10), which satisfy the boundary conditions:

- $\lim_{s \to -\infty} u(s,t) = x(t)$, $\lim_{s \to +\infty} u(s,t) = y(t)$
- $u(s,0) \in \phi(\Lambda_0)\ \forall s \in \mathbb{R}$
- $u(s,1) \in \Lambda_1\ \forall s \in \mathbb{R}$

where x and y are intersection points.
These solutions u are called Floer trajectories, or strips. We ask for them to be at least $\mathbf{W}^{1,p}$, for $p > 2$, so that by elliptic bootstrapping (Theorem B.17), they have the same regularity as (J, H).

Note that, since we assume $H : M \to \mathbb{R}$ to be non-degenerate, all these intersections are transverse.

• **Definition 3.32** (Energy of a Floer trajectory)**.** The **energy** of u is defined as:

$$E(u) := \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial u}{\partial s}\right|^2 \mathrm{d}s \wedge \mathrm{d}t.$$

Lemma 3.33. *$E(u) \geq 0$, with equality iff $u \equiv x$, where x is a Hamiltonian chord of H.*

Proof.

$$\begin{aligned} E(u) = 0 &\iff \frac{\partial u}{\partial s} = 0 \text{ almost everywhere} \\ &\iff \frac{\partial u}{\partial s} \equiv 0 \text{ (by continuity)} \\ &\iff \left| \frac{\partial u}{\partial t} - X_H \right| \equiv 0 \text{ (by (3.11))} \\ &\iff u \text{ is equal to a Hamiltonian chord of } H. \end{aligned}$$

□

A more interesting result, however, is the following:

Lemma 3.34. *Let u be a Floer trajectory between two Hamiltonian chords x and y of (J, H); i.e. $\lim_{s \to -\infty} u(s, \cdot) = x$, $\lim_{s \to \infty} u(s, \cdot) = y$. Then, we have:*

$$E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y).$$

Proof. From Definition 3.32, we have:

$$\begin{aligned} E(u) &= \int_{\mathbb{R}\times[0,1]} \left| \frac{\partial u}{\partial s} \right|^2 \mathrm{d}s \wedge \mathrm{d}t \\ &= \int_{\mathbb{R}\times[0,1]} g_t \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s} \right) \mathrm{d}s \wedge \mathrm{d}t \\ &= \int_{\mathbb{R}\times[0,1]} \hat{\omega} \left(\frac{\partial u}{\partial s}, J_t \frac{\partial u}{\partial s} \right) \mathrm{d}s \wedge \mathrm{d}t \\ &= \int_{\mathbb{R}\times[0,1]} \hat{\omega} \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_H \right) \mathrm{d}s \wedge \mathrm{d}t. \end{aligned}$$

Recall from Proposition 3.21 that the differential of $\mathcal{A}_H$ is given by:

$$\mathrm{d}\mathcal{A}_H(x) : T_x\mathscr{P} \longrightarrow \mathbb{R} : \zeta \longmapsto \int_0^1 \mathrm{d}\hat{\lambda}\big(\dot{x}(t) - X_H(x(t)), \zeta(t)\big)\mathrm{d}t.$$

So

$$\begin{aligned} E(u) &= \int_{\mathbb{R}\times[0,1]} \hat{\omega} \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_H \right) \mathrm{d}s \wedge \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \int_0^1 \mathrm{d}\hat{\lambda} \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_H \right) \mathrm{d}t \, \mathrm{d}s \\ &= - \int_{-\infty}^{\infty} \mathrm{d}\mathcal{A}_H \left(\frac{\partial u}{\partial s} \right) \mathrm{d}s \\ &= - \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{A}_H \left(u(s, \cdot) \right) \mathrm{d}s \\ &= \lim_{s\to-\infty} \mathcal{A}_H\big(u(s,\cdot)\big) - \lim_{s\to+\infty} \mathcal{A}_H\left(u(s,\cdot)\right) = \mathcal{A}_H(x) - \mathcal{A}_H(y) \end{aligned}$$

□

◇ **Remark 3.35** (On the orientation of trajectories)**.** Like in Morse theory, one needs to be able to define an orientation on Floer trajectories, in order to define a Floer differential.

This can be avoided by working over the coefficient ring $R = \mathbb{Z}/2\mathbb{Z}$, in which case we simply need to be able to count trajectories modulo 2. In general though, orienting trajectories is non trivial. A sufficient condition for it to be possible is that the Lagrangians Λ_1 and Λ_2 are **spin** (Definition A.25). We refer to §9 of [Rit13] for more details on orientation of trajectories, and choose to leave it as a black box – since it does not influence any of our discussions.

3.3.3 Linear Hamiltonians and the maximum principle

We have now defined Floer trajectories, which are anti-gradient trajectories connecting critical points of $\mathcal{A}_H$. We can define moduli spaces of such trajectories. To then obtain a well-defined cohomology theory, we would need some kind of compactification statement for these moduli spaces. However, the proofs of such results, now standard in Floer theories, all rely on some compactness result for the underlying manifold.

Our manifold $\widehat{W}$ is non-compact. So before we can proceed with the standard recipe for a Floer theory, we need to impose some further assumptions. Namely:

Assumption 3.36. Our Hamiltonian $H : \widehat{W} \to \mathbb{R}$ is non-degenerate and **linear at** ∞, *i.e.*

$$\exists R_0 \geq 1, \exists\, a, b \in \mathbb{R} \text{ such that for } r \geq R_0 : H = h(r) = ar + b,$$

where $a > 0$ and it is not the length of any Reeb chord on $(\partial W, \alpha)$ (which we write $a \notin \operatorname{spec} \alpha$).

Lemma 3.37. *If $H : \widehat{W} \to \mathbb{R}$ satisfies Assumption 3.36, then all its Hamiltonian chords of length 1 lie in the compact region $\widehat{W} \cap \{r \leq R_0\}$.*

Proof. By Proposition 3.17, length 1 Hamiltonian chords of H on $[R_0, +\infty) \times \partial W$ are in bijection with length a Reeb chords on $(\partial W, \alpha)$. However, since we assume $a \notin \operatorname{spec}\alpha$, there can exist no such Reeb chords. □

Lemma 3.38 (Maximum principle)**.** *This is also true for (non-trivial) Floer trajectories. More precisely, let u be a Floer trajectory whose image intersects $[R_0, +\infty) \times \partial W$. Then, the coordinate $r \circ u$ cannot have local maxima, unless it is constant. Hence, a Floer trajectory joining two chords x and y is contained in $\widehat{W} \cap \{r \leq \max\{r(x), r(y), R_0\}\}$.*

Proof sketch. See Lemma 19.1 of [Rit13]. Essentially, this consists in using the contact-type condition (3.1) and our expression for X_H in terms of $\mathcal{R}_\alpha$ in order to simplify the expression of the Laplacian $\Delta(r \circ u)$; then reducing the problem to the standard maximum principle for an elliptic operator.

◇ **Remark 3.39.** These two lemmas ensure that, even though our manifold is non-compact, we will be able to feign compactness in the next subsection. Indeed, there we will sketch the proofs of moduli space compactification results, which are standard and common to any Floer theory. Usually these are proved assuming compactness of the underlying manifold, for two reasons:

- to derive lower/upper bounds;
- to extract converging subsequences from sequences of chords/trajectories.

Lemmas 3.37 and 3.38 guarantee that we can still do this, even though $\widehat{W}$ is non-compact. Indeed, since all Hamiltonian chords/Floer trajectories are contained in a compact subset of $\widehat{W}$, then we can find bounds/extract converging subsequences there with no issue.

3.3.4 Moduli spaces of trajectories

Our manifold $\widehat{W}$ and Lagrangians Λ_0, Λ_1 are like earlier, but we now impose that our Hamiltonian $H : \widehat{W} \to \mathbb{R}$ be linear at infinity (Assumption 3.36).

• **Definition 3.40.** Let x and y be Hamiltonian chords of (J, H) . Define $\widehat{\mathcal{M}}_{(J,H)}(x, y)$ to be the space of Floer trajectories joining them, as defined in §3.3.2. Since we, for now, make no assumptions on x and y, this may *a priori* be empty. We also define:

$$\forall\, x, y : \mathcal{M}_{(J,H)}(x, y) := \widehat{\mathcal{M}}_{(J,H)}(x, y)/\mathbb{R}$$

to be the moduli space of *unparametrised* Floer trajectories, where we quotient by the $\mathbb{R}$-action induced by translation in the s-variable (*i.e.* we identify $u(s, t) \sim u(s + s_0, t)\, \forall s_0 \in \mathbb{R}$). Note that, often, we will drop the (J, H) from the notation and simply write $\mathcal{M}(x, y)$, when the choice of Floer data is non-ambiguous.

The goal of this subsection will be to sketch two standard results on these moduli spaces: that they are smooth manifolds, and that they compactify in a nice way, by adding broken trajectories. Let us start with an intermediary result:

Lemma 3.41 (Gromov compactness)**.** *Let $\mathcal{M} := \bigcup \mathcal{M}(x, y)$, where the union ranges over all pairs of chords x, y in M. Then $\mathcal{M}$ is compact.*

◇ **Remark.** The complete proof is standard, and can be found in §6.6 of [AD13]. We here sketch the main argument, and point out by the symbol ⚠ the two instances where compactness of the underlying manifold is used in the proof. These are *a priori* obstacles, since we work on a non-compact manifold $\widehat{W}$, but they are easily lifted by Remark 3.39.

Note that this statement usually also requires symplectic asphericity of $\widehat{W}$ – but this is already follows from the fact that $\widehat{W}$ is symplectic exact, as was already pointed out in Remark 3.2.

Proof sketch. Let (u_n) be a sequence of Floer trajectories in $\mathcal{M}$. By compactness of the underlying manifold (⚠), and Lemma 3.34, there is a uniform bound for $E(u_n)$. Hence, by a bubbling off argument (Prop. B.22), one can find a uniform $\mathcal{C}^0$ bound for $\|\nabla u_n\|$. By Arzelà-Ascoli, one can then extract a convergent subsequence, which is shown to be $\mathcal{C}^\ell_{\text{loc}}$ by elliptic regularity arguments. Then, we again use compactness of the underlying manifold (⚠) to argue that $\mathcal{C}^\ell_{\text{loc}}$-convergence is the same thing as $\mathcal{C}^\ell$-convergence. □

Corollary 3.42. *Any solution u of the Floer equation (3.10) ends in Hamiltonian chords, i.e. there exist $x, y \in \mathit{Crit}\mathcal{A}_H$ such that $\lim_{s\to-\infty} u(s,t) = x$, $\lim_{s\to+\infty} u(s,t) = y$.*

Proof sketch. This proof is contained in Theorem 6.5.6 of [AD13]. Though it is done for Hamiltonian Floer theory, the argument is exactly the same (one simply needs to replace orbits by chords). ⚠ This again makes us of the compactness of the underlying manifold, because at some point in the proof one needs to derive an upper bound for the Hamiltonian vector field. However, since all chords and trajectories are contained in a compact region, this upper bound is well-defined, as already explained in Remark 3.39.

➢ So we have seen that $\mathcal{M} = \bigcup \mathcal{M}(x, y)$ is compact. However, the individual moduli spaces $\mathcal{M}(x, y)$ might not be. Before we see how to compactify them, let us investigate their differentiable structure. Recall from §3.2.4 that we have a grading on chords, given by the Conley-Zehnder index $\mu = \mu_{\text{CZ}} \in \mathbb{Z}$.

Proposition 3.43. *For a generic pair of Floer data (J, H), and for any two chords x, y, the moduli spaces $\mathcal{M}(x, y)$ are smooth manifolds of dimension $\mu(x) - \mu(y) - 1$. We will call such a pair (J, H) regular.*

Proof sketch. A complete proof can be found in Lemma 2.7 of [Gao17], or §3.5 of [Kim18]. This is virtually the same transversality argument as is always used to show that moduli spaces of pseudo-holomorphic curves are smooth manifolds. Other good sources (besides the original 1994 paper by Floer-Hofer-Salamon) for such transversality arguments are §8 of [AD13], or §15 of [FK18], although those last two are not for Wrapped Floer theory.

The proof broadly decomposes in three steps:

1. To show that $\widehat{\mathcal{M}}(x, y)$ is a smooth manifold, the idea is to realise it as the zero set of a particular function $\mathscr{F} : \mathscr{B} \to \mathscr{E}$, and then use an implicit function theorem.

   We take $\mathscr{B}$ to be the Banach manifold of candidates for Floer trajectories, *i.e.* it consists of all the trajectories of a given regularity class (say $W^{1,2}$) with ends x and y and boundary in the Lagrangians. We let $\mathscr{E}_u := L^p(u^\star T\widehat{W})$ be the fibre above $u \in \mathscr{B}$. We can then realise $\widehat{\mathcal{M}}(x, y)$ as $\mathscr{F}^{-1}(0)$ where:

$$\begin{aligned} \mathscr{F} : \mathscr{B} &\longrightarrow \mathscr{E} \\ u &\longmapsto \frac{\partial u}{\partial s} + J_t\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) \in \mathscr{E}_u. \end{aligned}$$

   $\mathscr{F}$ is standardly called the **Floer operator**. Since it is a section of the bundle $\mathscr{E} \twoheadrightarrow \mathscr{B}$, one can linearise it (§A.2.2) along any Floer solution u to get:

$$D_u\mathscr{F} : T_u\mathscr{B} \longrightarrow T_{(u,0)}\mathscr{E}_u.$$

The proofs in [Gao17] and [Kim18] then rely on standard arguments of functional analysis to show that $D_u\mathscr{F}$ is a surjective Fredholm operator for a generic choice of data (J, H). This allows us to apply an implicit function theorem, hence showing that $\widehat{\mathcal{M}} = \mathscr{F}^{-1}(0)$ is a smooth manifold whose dimension is the Fredholm index $\text{ind}D_u\mathscr{F}$.

2. Let us sketch the proof that $\text{ind}D_u\mathscr{F} = \mu(x) - \mu(y)$. This is formally proved in Prop. 3.6.9 of [Kim18], or in Theorem 8.8.1 of [AD13].

   Recall the trivialisation we used in §3.2.4 in order to define Conley-Zehnder indices; call it ϵ. In this trivialisation, $D_u\mathscr{F}$ can be shown to look like:

$$D_u^\epsilon\mathscr{F} = \overline{\partial} + S, \tag{3.13}$$

   where $S \in \mathcal{C}^\infty(\mathbb{R}\times[0,1], Sp(2n))$. In other words, the linearisation of $\mathscr{F}$ is a perturbed Cauchy-Riemann operator. Furthermore, $S_{\pm\infty} := \lim_{s\to\pm\infty} S(s,t)$ are *symmetric* matrices for every t (and this convergence is uniform in t).

   Note that, as $s \longrightarrow \pm\infty$, the Floer equation converges to:

$$\frac{\partial u}{\partial s} + J_t\big(\frac{\partial u}{\partial t} - X_H(u)\big) = 0 \underset{s\to\pm\infty}{\longrightarrow} \frac{\partial u}{\partial t} = X_H(u),$$

   which is none other than the defining equation for a Hamiltonian chord. Therefore, taking (3.13) to $\pm\infty$ gives:

$$\lim_{s\to\pm\infty} D_u^\epsilon\mathscr{F} = \overline{\partial} + S_{\pm\infty},$$

   which can be interpreted as the **linearisation of the Hamiltonian flow** at the chords x or y. This implies that the Conley-Zehnder indices associated to the paths of symmetric matrices[(1)] $S_{\pm\infty}$ naturally correspond to $\mu(x)$ and $\mu(y)$.

   Now, a standard result from Fredholm theory shows that the index of $D_u^\epsilon\mathscr{F}$ is entirely determined by the indices at $\pm\infty$, giving us:

$$\text{ind}D_u^\epsilon\mathscr{F} = \mu(x) - \mu(y),$$

   and that this is in independent of the choice of trivialisation ϵ. This is an explicit computation on the dimension of the operator's kernel, and we refer to §8.8 of [AD13] for full details. For now, let us note that combining this with step 1, we get that:

$$\widehat{\mathcal{M}}(x,y) \text{ is a smooth manifold of dimension } \mu(x) - \mu(y)$$

3. If, moreover, none of the solutions $u \in \widehat{\mathcal{M}}(x,y)$ is constant, the action $\mathbb{R} \circlearrowleft \widehat{W}$ is free. One can then show, following the arguments of §15.7 in [FK18], that

$$\mathcal{M}(x,y) = \widehat{\mathcal{M}}(x,y)/\mathbb{R}$$

   is also a smooth manifold, of dimension one lower. □

Corollary 3.44. *If $\mu(x) = \mu(y)$ then either $x = y$, in which case $\mathcal{M}(x,y) = \{x\}$, or $x \neq y$ and $\mathcal{M}(x,y) = \emptyset$.*

Corollary 3.45. *If $\mu(x) = \mu(y)+1$, then $\mathcal{M}(x,y)$ is compact. In particular, there are finitely many Hamiltonian chords, and Floer trajectories joining them.*

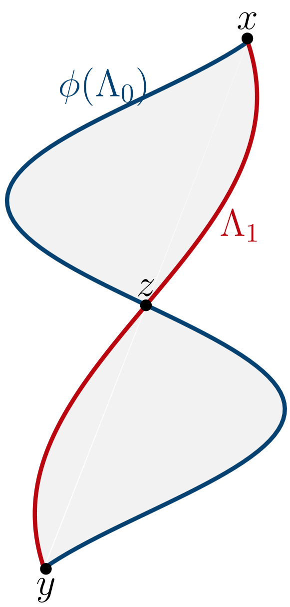
(1) To every path of symmetric matrices $t \mapsto S(t)$ one can associate a path of symplectic matrices (and vice versa); see §7.2 of [AD13]. This is why it makes sense to talk of the Conley-Zehnder index of a path of symmetric matrices.

Proof. $\mathcal{M}(x,y)$ is 0-dimensional, hence a discrete subset of the manifold $\mathcal{M}$, which is compact by Lemma 3.41. Hence $\mathcal{M}(x,y)$ is also compact. This implies that there must be finitely many Floer trajectories joining points of consecutive indices, and *a fortiori*, Hamiltonian chords. □

As we are about to see though, this is no longer true if $\mu(x) > \mu(y)+1$. However, since we have shown that $\mathcal{M} = \bigcup \mathcal{M}(x,y)$ is compact, then we know that the closure of $\mathcal{M}(x,y)$ is still contained in $\mathcal{M}$; which gives us hope to understand it.

Proposition 3.46 (Compactification by broken trajectories)**.** *If $\mu(x) > \mu(y)+1$, then $\mathcal{M}(x,y)$ compactifies by adding broken trajectories through points of intermediary indices, i.e. :*

$$\partial\bar{\mathcal{M}}(x,y) = \coprod_{\substack{\mu(x)>\mu(z_i)>\mu(y)\\ \mu(z_{i+1})>\mu(z_i)}} \mathcal{M}(x,z_k)\times\mathcal{M}(z_k,z_{k-1})\times\ldots\mathcal{M}(z_2,z_1)\times\mathcal{M}(z_1,y).$$



More visually, say for example, that $\mu(x) = \mu(y)+2$. Then, a sequence of trajectories $u_n \in \mathcal{M}(x,y)$ may converge to what we call a **broken trajectory**, as represented on the left, which is be piecewise composed of trajectories between points of intermediate indices.

On the picture, we have:

$$\begin{cases}\mu(x) = \mu(z)+1\\ \mu(z) = \mu(y)+1\end{cases}$$

The idea is the same as for broken trajectories in Morse theory; see Proposition 2.8.

Proof of Proposition 3.46. See §3.7 of [Kim18].

3.4 Wrapped Floer Cohomology with linear Hamiltonians

3.4.1 Intuition

We have everything we need to write down a first, naïve definition of Wrapped Floer Cohomology. We choose, like in this whole chapter, a Liouville manifold $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$, along with two admissible Lagrangians Λ_0, Λ_1 (Assumption 3.15), a generic contact-type almost complex structure (Definition 3.8). We now know that if $H : \widehat{W} \to \mathbb{R}$ is non-degenerate and linear at infinity (Assumption 3.36), then moduli spaces of its Floer trajectories are smooth manifolds which compactify by adding broken trajectories.

Therefore, we can now follow the recipe for a Floer theory laid down in Principle 2.13.

- **Definition 3.47.** Let R be a ring. We define the Wrapped Floer modules:

$$CW^*(J,H) := \bigoplus_{\substack{x\in\mathrm{Crit}\mathcal{A}_H\\ \mu(x)=n}} R\langle x\rangle\,, \tag{3.14}$$

where $\mu_{\mathrm{CZ}} : \mathrm{Crit}\mathcal{A}_H \to \mathbb{Z}$ denotes the Conley-Zehnder index (§3.2.4).

- **Definition 3.48.** We define a differential d which counts *incoming* trajectories, *ie*:

$$\begin{aligned} \mathrm{d} : CW^*(J,H) &\longrightarrow CW^{*+1}(J,H) \\ y &\longmapsto \sum_{\mu(x)=\mu(y)+1} \varepsilon(x,y)x \end{aligned}$$

where $\varepsilon(x, y)$ is the oriented count of trajectories in $\mathcal{M}_{(J,H)}(x, y)$. Often, we decide to work with coefficients in $R = \mathbb{Z}/2\mathbb{Z}$, so as not to worry with orientation. Then $\varepsilon(x, y)$ is simply the mod 2 count of trajectories between x and y.

Lemma 3.49. $\mathrm{d}^2 = 0$.

Proof. Pick an arbitrary $y \in CW^*$. Then, unfolding the definitions, we have:

$$\begin{aligned} \mathrm{d}y = \sum_{\mu(x)=*+1} \varepsilon(x,y)x, \ \mathrm{d}^2 y &= \sum_{\mu(x)=*+1} \sum_{\mu(z)=*+2} \varepsilon(z,x)\varepsilon(x,y)z \\ &= \sum_{\mu(z)=*+2} \left(\sum_{\mu(x)=*+1} \varepsilon(z,x)\varepsilon(x,y) \right) z \end{aligned}$$

To show that $\mathrm{d}^2 y = 0$, it suffices to show that:

$$\forall \text{ chord } z \in CW^{*+2} : \sum_{\mu(x)=*+1} \varepsilon(z,x)\varepsilon(x,y) = 0.$$

By proposition 3.46, this quantity corresponds exactly to $\#\left(\partial\bar{\mathcal{M}}(z,y)\right)$. Observe that, by Proposition 3.43, $\bar{\mathcal{M}}(z, y)$ is 1-dimensional. Hence, to show that $\mathrm{d}^2 = 0$, it suffices to show that the oriented count of boundary points of a 1-dimensional manifold is zero. Up to homeomorphism, a 1-manifold will be a disjoint union of circles $\mathbb{S}^1$ and intervals $[0, 1]$. Since $\partial\mathbb{S}^1 = \emptyset$, the problem reduces to showing that, given our choice of orientation, each point in $\partial[0, 1]$ has a different orientation, which is an easy exercise. ☐

• **Definition 3.50.** Lemma 3.49 proves that (CW^*, d) is a co-chain complex, allowing us to define its cohomology $HW^*(\Lambda_0, \Lambda_1; J, H) := H^*(CW^*(J, H), \mathrm{d})$. We call this the **Wrapped Floer Cohomology** of the Lagrangians Λ_0, Λ_1, with respect to (J, H).

Hence, given a non-degenerate linear Hamiltonian $H : \widehat{W} \to \mathbb{R}$, we have managed to define a cohomology theory $HW^*(\Lambda_0, \Lambda_1; J, H)$, which records Hamiltonian chords of H between Λ_0 and Λ_1 (or, alternatively, intersections between $\phi(\Lambda_0)$ and Λ_1).

However, while in Lagrangian Floer theory we can show that our construction does not depend on the choice of Hamiltonian H, the proof heavily depends on compactness, and does not carry over to Liouville domains. Hence there is no reason to assume that two different linear Hamiltonians yield the same cohomology (and in general, they won't).

The rest of this section will be dedicated to fixing this problem. To do this, we will choose a family (H_j) of linear Hamiltonians with slopes increasing to $+\infty$, and we will show that, for $j \leq j'$, we can construct maps

$$HW^*(\Lambda_0, \Lambda_1; J, H_j) \longrightarrow HW^*(\Lambda_0, \Lambda_1; J, H_{j'}). \tag{3.15}$$

These maps define a directed system, allowing us to take a limit:

$$HW^*(\Lambda_0, \Lambda_1) = \varinjlim_{j} HW^*(\Lambda_0, \Lambda_1; J, H_j).$$

This will yield a construction which does not depend on any choice of Floer data (J, H), and which we will call the **Wrapped Floer Cohomology of the pair** (Λ_0, Λ_1).

◇ **Remark 3.51.** As we mentioned at the very start of this chapter, the process of Liouville extending our manifold $W \to \widehat{W}$ had the effect of dilating the Reeb dynamics going on at the boundary. Indeed we saw in Corollary 3.11 that a length 1 Hamiltonian chord on the extension corresponded to a length slope(H) Reeb chord on the boundary.

Therefore, the issue we have when working with a fixed linear Hamiltonian with slope $a \in \mathbb{R}_+$, is that $HW^*(\Lambda_0, \Lambda_1; J, H)$ only counts Reeb chords on $(\partial W, \alpha)$ up to length a. This is why, by essence, two Hamiltonians of different slopes may yield different cohomologies. However, as one can imagine: if we keep raising the slope of our Hamiltonian, then new Reeb chords will appear on $(\partial W, \alpha)$, which is the intuitive reason why the maps (3.15) exist. One should simply think of these maps as inclusions, at the cochain level. Therefore, if one wants to define a cohomology theory which contains all of them, they should take a direct limit over this sequence of inclusions.

3.4.2 Floer equation with parameters

Let $\mathscr{J} \times \mathscr{H}$ be our space of Floer data, *i.e.* of $\mathcal{C}^\ell$ pairs (J, H). Pick two pairs (J_0, H_0), $(J_1, H_1) \in \mathscr{J} \times \mathscr{H}$, and assume they are regular (so that their wrapped Floer cohomology is well-defined), and that H_0, H_1 are linear at infinity. Then our goal in this subsection is to find out under which conditions we can construct a map

$$HW^*(\Lambda_0, \Lambda_1; J_0, H_0) \longrightarrow HW^*(\Lambda_0, \Lambda_1; J_1, H_1).$$

➢ To do this, find a homotopy between (J_0, H_0) and (J_1, H_1). This can always be done since $\mathscr{J}$ is standardly known to be contractible (see [Wen15]), and H_0 and H_1 are both linear outside of a compact region. So we have a path $s \mapsto \Gamma_s \in \mathscr{J} \times \mathscr{H}$ such that:

$$\Gamma_s = (J_s, H_s) = \begin{cases} (J_0, H_0), s \leq 0 \\ (J_1, H_1), s \geq 1 \end{cases}$$

The idea is that we will now define an action functional with parameters, which varies along the homotopy:

$$\mathcal{A}_{H_s}(x) := f_1\big(x(1)\big) - f_0\big(x(0)\big) - \int_0^1 x^\star \lambda + \int_0^1 H_s\big(x(t)\big) \mathrm{d}t. \tag{3.16}$$

At $s = 0$, critical points of $\mathcal{A}_{H_0}$ are Hamiltonian chords of (J_0, H_0), and at $s = 1$, critical points are chords of (J_1, H_1). We can now connect these by using, like before, anti-gradient trajectories. The equation

$$\frac{\partial u}{\partial s} = -\nabla \mathcal{A}_{H_s}$$

becomes

$$\frac{\partial u}{\partial s} + J_{s,t}\big(\frac{\partial u}{\partial t} - X_{H_s}(u)\big) = 0, \tag{3.17}$$

which we call the **Floer equation with parameters**.

Now, the next natural thing to do would be: given x_0 a Hamiltonian chord of H_0 and x_1 a Hamiltonian chord of H_1, we would like to find solutions of (3.17) connecting them.

- **Definition 3.52.** A Floer trajectory with parameters is a map $u : \mathbb{R} \times [0, 1] \to \widehat{W}$ s.t:
  - u is a $\mathbf{W}^{1,p}$ solution of (3.17) (for $p > 2$)
  - $u(0, \cdot) \in \phi_{H_s}(\Lambda_0), u(1, \cdot) \in \Lambda_1$

Just like in §3.3.4: given x_0 a chord of (J_0, H_0), and x_1 a chord of (J_1, H_1), we denote by $\mathcal{M}_\Gamma(x_0, x_1)$ the moduli space of trajectories u such that:

$$\lim_{s \to -\infty} u(s, \cdot) = x_0, \lim_{s \to +\infty} u(s, \cdot) = x_1.$$

However, we can no longer quotient by the s variable, since now our Floer data also varies along it. Then like in §3.3.2, we can define a notion of energy, and we have the following result:

Lemma 3.53 (*A priori* energy estimate). *Given a homotopy Γ_s between (J_0, H_0) and (J_1, H_1), and a solution u of the Floer equation with parameters (3.17) with end chords x_0, x_1, we have:*

$$E(u) = \mathcal{A}_{H_0}(x_0) - \mathcal{A}_{H_1}(x_1) + \int_{\mathbb{R}\times[0,1]} (\partial_s H_s)(u)\mathrm{d}s \wedge \mathrm{d}t. \tag{3.18}$$

Proof. This is proved in Calculation C.10 of the Computational Appendix. The proof is virtually equivalent to the proof of Lemma 3.34, we just pick up a new term because of the s-dependency.

From this lemma, we deduce that if we define the moduli spaces:

$$\mathcal{M}_\Gamma(x_0, y_0) := \left\{ u \text{ is a solution of (3.17)} \mid \lim_{s\to-\infty} u(s,\cdot) = x_0, \lim_{s\to\infty} u(s,\cdot) = x_1 \right\},$$
$$\mathcal{M}_\Gamma := \bigcup_{x_0,y_0} \mathcal{M}_\Gamma(x_0, y_0).$$

Then $\mathcal{M}_\Gamma$ is compact, for philosophically the same reasons as in Lemma 3.41. The full details of this proof are carried out in Chapter 11 of [AD13]; the only differences being that:

- in [AD13], they work with periodic orbits instead of Hamiltonian chords. This makes virtually no difference in the analysis;
- they work on a compact manifold, whereas we work on $\widehat{W}$. However, since H_0 and H_1 are linear at infinity, then for the exact same reasons as in Remark 3.39, we can ensure that this does not pose any issue when carrying over to the non-compact setting. Indeed, Hamiltonian chords of H_0 and H_1 are both contained in a compact region of $\widehat{W}$, therefore we only need to ensure that the Floer trajectories connecting them are as well. This requires an additional assumption.

Assumption 3.54. We assume that the homotopy $\Gamma_s = (J_s, H_s)$ is **monotone**, *i.e.* :

$$\exists R_0 \geq 1 \text{ s.t } H_s = h_s(r),$$

and:

$$\exists R_1 \geq R_0 \text{ s.t } \partial_s h_s'(r) \leq 0 \text{ for } r \geq R_1.$$

For simplicity, we set $R_0 := \max(R_0, R_1)$.

◇ **Remark 3.55.** This assumption automatically implies that the expression (3.18) is bounded from above, since the integrand of the third term will become non-positive for s large enough; hence yielding our desired $\mathcal{C}^0$-bound on the gradient of trajectories, and therefore on the energy, by the exact same proof as in Lemma 3.41.

Meanwhile, we claim that Assumption 3.54 also forces Floer trajectories with parameters to be contained in a compact region of $\widehat{W}$:

Lemma 3.56 (Maximum principle with parameters). *Let $\Gamma_s = (J_s, H_s)$ be a monotone homotopy of Floer data such that, for every s, $h_s'(r) \leq 0$. Then for any solution of (3.17) intersecting $\widehat{W} \cap \{r \geq R_0\}$, the coordinate $r \circ u$ cannot have local maxima, unless it is constant. Hence, all trajectories joining chords x_0, x_1 are contained in $\widehat{W} \cap \{r \leq \max\{r(x_0), r(x_1), R_0\}\}$.*

Proof. Lemma 19.1 of [Rit13].

Corollary 3.57. *The moduli spaces $\mathcal{M}_\Gamma(x_0, y_0)$ of the Floer equation with parameters are smooth manifolds of dimension $\mu(x_0) - \mu(y_0)$, which compactify by adding broken trajectories.*

Proof. The proof of this corollary is exactly the same as in §3.3.4. Once again, we refer to Chapter 11 of [AD13] for the explicit proof; where once again, all the issues arising from passing from a compact setting to a non-compact one are addressed by Remark 3.39, and the newly-found maximum principle with parameters. □

3.4.3 Continuation maps in Wrapped Floer Cohomology

Recall what was our objective in considering the Floer equation with parameters: given two pairs of Floer data $(J_0, H_0), (J_1, H_1)$, we want to find a way of constructing a map between their respective Wrapped Floer Cohomology theories.

Recall that by assumption, H_0 and H_1 are linear at infinity. Clearly, we have:

Lemma 3.58. *There exists a monotone homotopy from* (J_0, H_0) *to* (J_1, H_1) *iff*

$$\text{slope}(H_1) \leq \text{slope}(H_0).$$

Let us assume this is the case. Then, at a chain-level, we define:

• **Definition 3.59** (Continuation maps)**.**

$$\begin{aligned} f_\Gamma : CW^*(J_1, H_1) &\longrightarrow CW^*(J_0, H_0) \\ x_1 &\longmapsto \sum_{\mu(x_0)=\mu(x_1)} \varepsilon(x_0, x_1) x_0 \end{aligned}$$

where $\varepsilon(x_0, x_1)$ is the oriented count of trajectories in $\mathcal{M}_\Gamma(x_0, x_1)$. So f_Γ is essentially defined just like a standard Morse/Floer differential: it maps a Hamiltonian chord x_1 of H_1 to all the Hamiltonian chords of H_0 that flow into it, along the Floer equation with parameters.

Then, Lemma 3.57 tells us that f_Γ is a chain morphism, in much the same fashion as we proved that $\mathrm{d}^2 = 0$ in Lemma 3.49 (see the discussion right after Prop. 11.1.14 of [AD13]). Therefore, f_Γ descends to a map on cohomology:

$$f_\Gamma : HW^*(J_1, H_1) \longrightarrow HW^*(J_0, H_0),$$

which we call a **continuation map**. In summary, we have sketched the proof that:

Corollary 3.60. *If* (J_a, H_a) *and* (J_b, H_b) *are such that* $\text{slope}(H_a) \leq \text{slope}(H_b)$*, then there exists a continuation map* $f_\Gamma^{ab} : HW^*(\Lambda_0, \Lambda_1; J_a, H_a) \to HW^*(\Lambda_0, \Lambda_1; J_b, H_b)$*.*

However, we still have something important to prove:

Proposition 3.61. *Given two such pairs* $(J_a, H_a), (J_b, H_b)$*, the continuation map*

$$f^{ab} : HW^*(\Lambda_0, \Lambda_1; J_a, H_a) \to HW^*(\Lambda_0, \Lambda_1; J_b, H_b)$$

does not depend on the choice of monotone homotopy between them.

Proof sketch. The strategy is basically « re-doing everything one more time ». One chooses a homotopy of homotopies of Floer data, $\lambda \mapsto \Gamma_s^\lambda$. Then, one can once again derive a Floer equation (now with three parameters, t, s, and λ) and study its solutions. The proofs that moduli spaces are manifolds, and of the compactification will be yet more technical, but in the end, they allow us to define a chain homotopy $f_{\Gamma^0} \to f_{\Gamma^1}$, in the same way we defined both the Floer differential, and the continuation maps.

Once again, full explicit details can be found in Chapter 11 of [AD13], where the compactness issues are once again resolved by Remark 3.39. □

➢ This is all we needed to prove for the purposes of defining Wrapped Floer Cohomology. However, let us mention a few properties of these continuation maps, which may come in handy later in the thesis.

Lemma 3.62. *Say* $(J_a, H_a) = (J_b, H_b)$ *and* $\Gamma \equiv id$*. Then,* $f_\Gamma \equiv id$*.*

Proof. At a chain-level we have, for any chord x:

$$f_\Gamma(x) = \sum_{\mu(y)=\mu(x)} \varepsilon(y, x) y$$

Since $\mu(y) = \mu(x)$, we have $\mathcal{M}(y, x) = \begin{cases} \{x\} \text{ if } x = y \\ \emptyset \text{ else} \end{cases}$, so that $f_\Gamma(x) = x$. □

Proposition 3.63. *Assume that there exist monotone homotopies in both directions:*

$$(J_a, H_a) \underset{\Gamma'}{\overset{\Gamma}{\rightleftarrows}} (J_b, H_b)$$

Then, the induced continuation maps are isomorphisms.

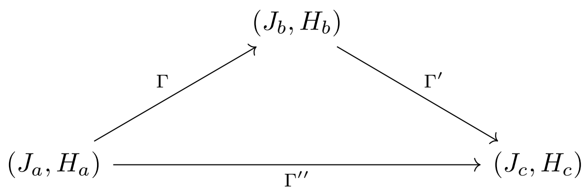
Before we prove this, let us explore one easy corollary:

Corollary 3.64. *If H_a and H_b are linear at infinity, and of the same slope, then*

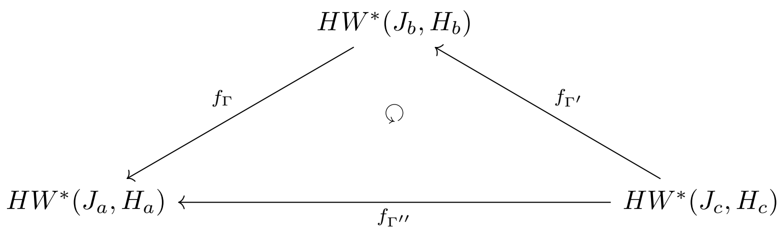
$$HW^*(\Lambda_0, \Lambda_1; J_a, H_a) \cong HW^*(\Lambda_0, \Lambda_1; J_b, H_b).$$

Proof of Corollary 3.64. Construct a homotopy between H_a and H_b which leaves the slope untouched, and alters only the constant term. Then, both this homotopy and its backwards homotopy (obtained by reversing the s-direction) are monotone, so that we can define continuation maps in both directions, and conclude by Proposition 3.63. □

Proof strategy of Proposition 3.63. We only sketch the proof and refer to §11.4-11.5 of [AD13] for the full explicit argument. Once again, the only differences are that we replace "orbits" by "chords", as well as the compactness issues, addressed in Remark 3.39. The technical part of the argument lies in showing that a triangle of homotopies:



induces a *commutative* triangle of continuation maps:



Once we know this, it suffices to take $(J_a, H_a) = (J_c, H_c)$ and $\Gamma'' = \mathrm{id}$ to show that our two homotopies Γ and Γ' induce inverses on cohomology. □

3.4.4 Wrapped Floer Cohomology: First definition

We are now finally in a position to define Wrapped Floer Cohomology.

Let us recall the setup. We work on a Liouville manifold $\widehat{W}$, and have two Lagrangians Λ_0, Λ_1 satisfying Assumption 3.15, as well as a generic contact-type almost complex structure J (Definition 3.8). We have shown that, given a non-degenerate Hamiltonian H which is linear at infinity, we could define $HW^*(\Lambda_0, \Lambda_1; J, H)$, and that given two such Hamiltonians H_a, H_b with $\mathrm{slope}(H_a) \leq \mathrm{slope}(H_b)$, we could define a continuation map:

$$f^{ab} : HW^*(\Lambda_0, \Lambda_1; J_a, H_a) \longrightarrow HW^*(\Lambda_0, \Lambda_1; J_b, H_b).$$

We now define:

• **Definition 3.65** (Wrapped Floer Cohomology)**.** The **Wrapped Floer Cohomology** of the pair of Lagrangians (Λ_0, Λ_1) in $\widehat{W}$ is defined as:

$$HW^*(\Lambda_0, \Lambda_1) := \varinjlim_j HW^*(\Lambda_0, \Lambda_1; J, H_j),$$

where (H_j) is a family of non-degenerate Hamiltonians with increasing slopes going to infinity. There, $\varinjlim$ denotes the direct limit of this system, as defined in Definition B.26. If $\Lambda_0, \Lambda_1 = \widehat{L}_0, \widehat{L}_1$, then we often simply write $HW^*(L_0, L_1)$, since Λ_0, Λ_1 are unique up to symplectomorphism (Lemma 3.7).

◇ **Remark.** This definition is now completely independent of the choice of Hamiltonian: it only depends on the Lagrangians $\Lambda_0, \Lambda_1 \subset \widehat{W}$, because we have explicitly gotten rid of the dependencies. Also, note that here, (H_j) is taken to be a *family* of Hamiltonians; but in practice, any sequence (H_n) such that $\text{slope}(H_n) \to \infty$ will do the job.

➢ Now, since the beginning of this chapter, we have been discussing Lagrangian intersection theory of two Lagrangians. However, our construction could also be used to study Hamiltonian chords from a single Lagrangian to itself. Indeed:

• **Definition 3.66.** Let Λ be an admissible Lagrangian in $\widehat{W}$. We define a Hamiltonian chord on Λ as a path $x : [0,1] \to \widehat{W}$ such that $x(0), x(1) \in \Lambda$. Then, we can define exactly the same cohomology theory for Λ, and write it $HW^*(\Lambda)$. We call it the **Wrapped Floer Cohomology** of Λ.

Proposition 3.67. *There exists a module morphism:*

$$H^*(L) \longrightarrow HW^*(L),$$

over the appropriate ring, where H^* denotes singular cohomology.

Proof. Let H be linear at infinity, *i.e.* $H = ar + b$ for r above some R_0. By Lemma 3.37 and Lemma 3.38, all chords and Floer trajectories are contained in $\widehat{W}_0 := \widehat{W} \cap \{r \leq R_0\}$. So from a Floer point of view, nothing of interest happens above R_0. Hence, we have:

$$HW^*(\widehat{L} \cap \widehat{W}_0; J, H) = HW^*(\widehat{L}; J, H),$$

which is an equality, not just an isomorphism, since we already have an equality at chain-level, both of $CW^*(J, H)$ and of d.

Now, it is a standard fact from Floer theory (see [Oh96] for HL^*) that, given a generic J, and some H_ε sufficiently $\mathcal{C}^2$-small on $\widehat{W}_0$, there exists an isomorphism between Floer and Morse cohomologies, on compact manifolds. Therefore:

$$\begin{aligned} HW^*(\widehat{L} \cap \widehat{W}_0; J, H_\varepsilon) &\cong HL^*(\widehat{L} \cap \widehat{W}_0; J, H_\varepsilon) \\ &\cong HM^*(\widehat{L} \cap \widehat{W}_0; H_\varepsilon) \\ &\cong H^*(\widehat{L} \cap \widehat{W}_0). \end{aligned}$$

From the way we constructed our cylindrical end, $\widehat{W}_0$ deformation retracts onto W, and thus $\widehat{L} \cap \widehat{W}_0$ onto L, giving us $HL^*(\widehat{L} \cap \widehat{W}_0; J, H_\varepsilon) \cong H^*(L)$.

Now, since H_ε is non-degenerate, we can fit $HW^*(\widehat{L} \cap \widehat{W}_0; J, H_\varepsilon)$ in our directed system with increasing slopes:

$$\begin{array}{ccccccc} HW^*(\widehat{L} \cap \widehat{W}_0; J, H_\varepsilon) & \longrightarrow \cdots \longrightarrow & HW^*(\widehat{L} \cap \widehat{W}_0; J, H_j) & \longrightarrow & HW^*(\widehat{L} \cap \widehat{W}_0; J, H_{j'}) & \longrightarrow & \ldots \\ \updownarrow \cong & & & & & & \\ H^*(L) & & & & & & \end{array}$$

The isomorphism $H^*(L) \xrightarrow{\cong} HW^*(\widehat{L} \cap \widehat{W}_0; J, H_\varepsilon)$ will most likely not survive the direct limit process – however, the map

$$H^*(L) \to \varinjlim HW^*(\widehat{L} \cap \widehat{W}_0; J, H) = HW^*(L)$$

will, yielding our desired module homomorphism. ☐

3.5 Wrapped Floer Cohomology with strongly convex Hamiltonians

3.5.1 Intuition

We just gave a definition of wrapped Floer cohomology, allowing us to associate a cohomology $HW^*(\Lambda_0, \Lambda_1)$ to two admissible Lagrangians in a Liouville manifold. We went to great lengths to make sure that this construction did not depend on any choice of Hamiltonian, taking a direct limit over a large family of Hamiltonians whose slopes blew up.

Alternatively, we could avoid having to take a direct limit altogether by choosing a Hamiltonian extension H with a faster growth. Indeed, recall from Remark 3.51 that the point of taking a direct limit over a sequence of Hamiltonians, with slope going to ∞, was that in doing so we counted Reeb chords of higher and higher length on $(\partial W, \alpha)$, and made sure not to miss any. However, in theory, such a result could also be achieved by choosing a single Hamiltonian H such that at infinity, $H = h(r)$ where $h'(r) \to \infty$.

Then, Hamiltonian chords appearing higher and higher in our extension will correspond to Reeb chords of higher and higher lengths, so that philosophically, we should directly obtain an isomorphism $HW^*(\Lambda_0, \Lambda_1; J, H) \cong HW^*(\Lambda_0, \Lambda_1)$, without having to take a limit.

This definition looks easier to construct than the one with linear Hamiltonians. The main issue – which makes it less popular – lies in the fact that it is much harder to ensure that H is non-degenerate, since it may have infinitely many chords shooting up to ∞, so that one would need to take global perturbations of (J, H). We will resolve this issue in the next chapter by defining Local Wrapped Floer Cohomology. For now, let us just assume H is non-degenerate.

Assumption 3.68. In general, in order to define wrapped Floer cohomology with a single, non-degenerate Hamiltonian $H : \widehat{W} \to \mathbb{R}$, we assume:

- H has infinitely many chords, but they are isolated, so that there are countably many.
- The chords of H shoot off to ∞ in the collar $\widehat{W} \backslash \mathrm{int}(W)$.
- $H = h(r)$ at infinity, where h is **strongly convex** ($\exists \delta > 0$ such that $h''(r) \geq \delta$).

◇ **Remark 3.69.** The first and second assumptions are harmless. Indeed, if H has finitely many chords, or if they don't shoot off to ∞, then they are all contained in a compact region, so that we can use our previous constructions to define $HW^*(\Lambda_0, \Lambda_1; J, H)$.

✧ **Example 3.70.** A few examples of Hamiltonians satisfying Assumption 3.68 are:

- H grows like a quadratic at infinity. *i.e.* $h(r) \sim \frac{1}{2} a r^2$ for large r.
- H grows faster than a quadratic at infinity. *i.e.* $h(r)/r^2 \to \infty$ as $r \to \infty$.

Corollary 3.71. *Let H satisfy Assumption 3.68. Write $\{x_k\}$ the set of Hamiltonian chords, which we order by height in the collar (i.e, by the r coordinate). Write $A_k := \mathcal{A}_H(x_k)$. Then, for $k \gg 1$, A_k becomes strictly decreasing, and goes to $-\infty$.*

Proof. By Lemma 3.20, if $H = h(r)$ at infinity, then:

$$\mathcal{A}_H(x) = \mathcal{A}_H(r) = -rh'(r) + h(r),$$

where r is the height of the chord x (by Corollary 3.11, Hamiltonian chords high enough in the collar are constrained to slices of constant height). Differentiating, we get:

$$\partial_r \mathcal{A}_H(r) = -rh''(r).$$

Since h is strongly convex at infinity, this becomes strictly negative for $r \gg 1$, so that $\mathcal{A}_H(r)$ becomes strictly decreasing, and goes to $-\infty$ by the mean value inequality. □

3.5.2 The action filtration

To define wrapped Floer cohomology with a single Hamiltonian, we shall need one essential tool: the action filtration.

• **Definition 3.72.** Let J be generic and $H : \widehat{W} \to \mathbb{R}$ be a Hamiltonian satisfying Assumption 3.68. We define the Floer cochain complex:

$$\begin{aligned} CW^*(J,H) :=&\{\text{Hamiltonian chords of } H \text{ of index } *\} \\ =&\{x \in \mathrm{Crit}(\mathcal{A}_H) \mid \mu_{\mathrm{CZ}} = *\}, \end{aligned}$$

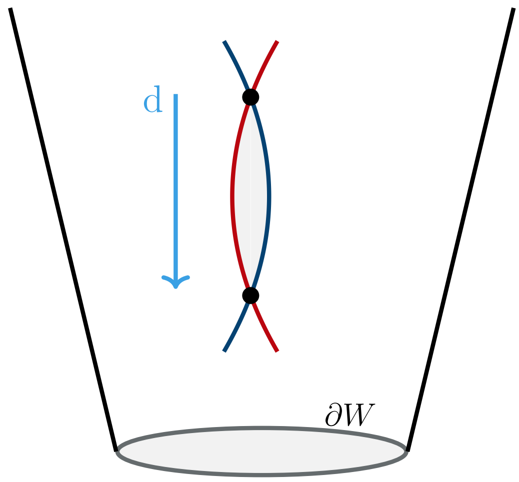
and the Floer differential d as in Definition 3.48. At this point, we don't know whether the cohomology will be well-defined, but the complex for sure is.

Lemma 3.73. *The Floer differential* d *is action-increasing.*

Proof. Let y be a chord. Recall that $\mathrm{d}y$, as defined in Definition 3.48, counts incoming Floer trajectories into y. Recall from Lemma 3.34 that, given a trajectory from some chord x to y, we have $E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y) \geq 0$. Therefore, $\mathrm{d}y$ is a formal sum of chords x, each of which must have action $\mathcal{A}_H(x) \geq \mathcal{A}_H(y)$. □

Corollary 3.74. *The Floer differential goes downwards, when high enough in the collar* $\widehat{W}\backslash\mathrm{int}(W)$.

Proof. If we go high enough in the collar, the action decreases as r increases, by Corollary 3.71. However, d is action-increasing by Lemma 3.73, therefore it must necessarily go downwards. □



Because of the above results, we can choose an increasing sequence $r_p \to \infty$, where $p \in \mathbb{N}$, and a corresponding sequence $a_p \to -\infty$ such that:

$$\mathcal{A}_H(x) \geq a_p \iff \mathrm{im}(x) \subset \widehat{W} \cap \{r \leq r_p\}.$$

For brevity, let us write $\widehat{W}_p := \widehat{W} \cap \{r \leq r_p\}$. Note that this manifold is compact (it consists of the truncation of $\widehat{W}$ at height $r = r_p$).

• **Definition 3.75** (Action filtration)**.** Given our chain complex $CW^*(J,H)$, we define the action filtration:

$$F_pCW^*(J,H) = \{x \in CW^*(J,H) \mid \mathcal{A}_H(x) \geq a_p\}, p \in \mathbb{N}. \tag{3.19}$$

By the discussion above, we can also write:

$$F_pCW^*(J,H) = \left\{x \in CW^*(J,H) \mid \mathrm{im}(x) \subset \widehat{W}_p\right\}. \tag{3.20}$$

By Corollary 3.71, this filtration is increasing, *i.e.* :

$$F_pCW^*(J,H) \subset F_{p+1}CW^*(J,H),$$

and by Lemma 3.73, the Floer differential respects the action filtration, *i.e.*

$$\mathrm{d}\left(F_pCW^*\right) \subset F_pCW^{*+1},$$

which implies that we can write F_*CW as a filtration on the *total* complex $CW = \bigoplus CW^*$.

• **Definition 3.76.** We write

$$HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H) := H^*(F_p CW^*(J, H)),$$

and call $HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H)$ the **filtered Wrapped Floer Cohomology** of the pair (Λ_0, Λ_1), with respect to J and H.

Since the total complex $CW^*(J, H)$ contains all chords, its cohomology is by definition the total wrapped Floer cohomology:

$$H^*(CW^*(J, H), \mathrm{d}) =: HW^*(\Lambda_0, \Lambda_1; J, H) \tag{3.21}$$

Note that, *a priori*, there is no reason to assume that either $HW^*_{\geq a_p}$ or HW^* is well-defined. We will prove this formally in §3.5.3. Before that, let us prove:

Proposition 3.77. *Assuming that both sides of the equation below are well-defined, then we have the isomorphism:*

$$HW^*(\Lambda_0, \Lambda_1; J, H) \cong \varinjlim_{p} HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H).$$

Proof. We will prove this from first principles, by showing that the left-hand side satisfies the universal property of a direct limit (Defn. B.26). First, we construct continuation maps:

$$f_p : HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H) \longrightarrow HW^*_{\geq a_{p+1}}(\Lambda_0, \Lambda_1; J, H), \tag{3.22}$$

which compose well (§3.4.3), and such that:

$$f_p([x]) = 0 \text{ if } x \text{ is a boundary in } F_{p+1}CW^*(J, H), \;\; f_p([x]) = [x] \text{ else}^{(2)}. \tag{3.23}$$

Similarly, we define:

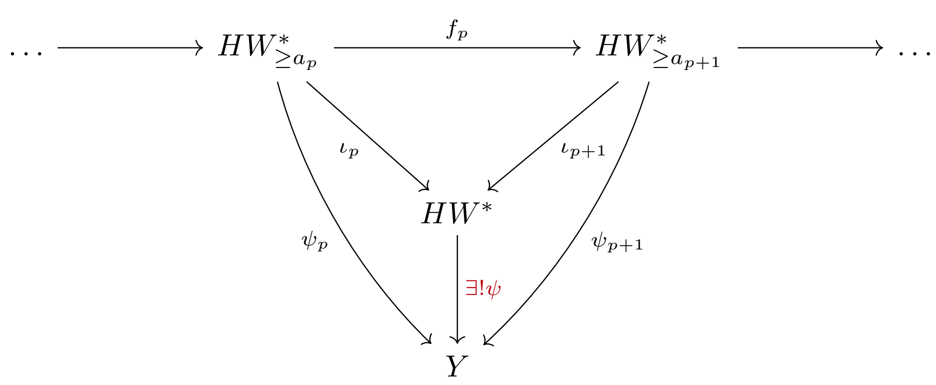
$$\iota_p : HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H) \longrightarrow H^*(C^*(J, H), \mathrm{d}) \tag{3.24}$$

where $\iota_p([x]) := 0$ if x eventually becomes a boundary as we increase p, and $\iota_p([x]) := [x] \in H^*(C^*(J, H), \mathrm{d})$ else.

Note that, despite their intuitive construction, it is not *a priori* clear that f_p and ι_p are well-defined (but, note that we haven't even shown that $HW^*_{\geq a_p}$ or HW^* are, for that matter). All of this will be formally proved after this proposition, hence let us assume for now that all these maps exist. We want to show that $HW^*(\Lambda_0, \Lambda_1; J, H)$ is the direct limit of this system. Going back to first principles (Definition B.26), we need to show that given any object Y (a module over the same ring as our cohomologies), and collection of morphisms:

$$\psi_p : HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H) \longrightarrow Y \tag{3.25}$$

such that $\psi_{p+1} \circ f_p = \psi_p$, then there exists a unique $\psi : HW^*(\Lambda_0, \Lambda_1; J, H) \to Y$ making the following diagram commute:



(2) For every p such that $\mathcal{A}_H(x) \geq a_p$, else taking $f_p([x])$ does not make sense.

We can construct this ψ explicitly. Let $[x] \in HW^*$, and let $[x]_p$ be its representative in $HW^*_{\geq a_p}$. Pick any p_0 (3), and set $\psi([x]) := \psi_{p_0}([x]_{p_0})$.

This is well-defined, indeed: if $[x] \neq 0$, then x never becomes a boundary as we increase the filtration window. Hence, our continuation map f_p acts as the identity for every p. Since at every step we have $\psi_p = \psi_{p+1}(f)$, the quantity is $\psi_{p_0}([x]_{p_0})$ is independent of the choice of p_0.

Meanwhile, if $[x] = 0$, then the choice of p_0 does not matter either. If it is such that $[x]_{p_0} = 0$, then we directly have $\psi([x]) := \psi(0)$. Else, we know that $\exists p$ such that $[x]_p = 0$, and since all the maps ψ_p commute with the f_p's, the value of $\psi_p([x]_p)$ must also be $\psi(0)$.

Therefore, we can indeed define a map $\psi : HW^*(\Lambda_0, \Lambda_1; J, H)$, verifying that:

$$HW^*(\Lambda_0, \Lambda_1; J, H) \cong \varinjlim_p HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H).$$

□

In conclusion, provided that both quantities are well-defined, then the total cohomology of the complex is isomorphic to the direct limit over all the filtered cohomologies. This will become essential in the next section.

3.5.3 Wrapped Floer Cohomology: Second definition

Given our choice of Floer data (J, H), where H satisfies Assumption 3.68, we just constructed a cochain complex $CW^*(J, H)$, as well as an increasing action filtration F_*CW^* on it. Furthermore, we showed that:

$$HW^*(\Lambda_0, \Lambda_1; J, H) \cong \varinjlim_p HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H),$$

provided that both sides were-well defined. Let us first prove that the right-hand side is.

Proposition 3.78. *One can find a Hamiltonian $H_p : \widehat{W} \to \mathbb{R}$, which is linear at infinity, and such that:*

$$F_pCW^*(J, H) = CW^*(J, H_p),$$

hence ensuring, by §3.4, that $HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H)$ is well-defined, and that we have:

$$HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H) = HW^*(\Lambda_0, \Lambda_1; J, H_p),$$

where this is an equality, not just an isomorphism.

Proof. (From [Rit13]) Let $F_pCW^*(J, H)$ be the action filtration from the previous section. Recall that we had defined:

$$HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H) := H^*(F_pCW^*(J, H), \mathrm{d}),$$

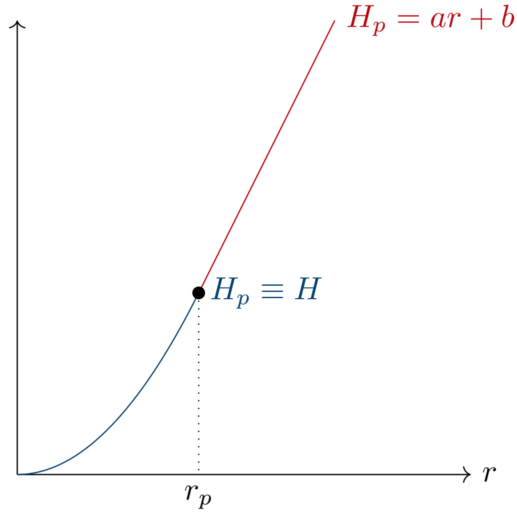
though with no guarantee that this was well-defined. Besides, recall that we had an increasing sequence r_p in $[1, +\infty)$ such that, writing $a_p := \mathcal{A}_H(r_p)$, we have:

$$\forall x \in CW^*(J, H) : \mathcal{A}_H(x) \geq a_p \iff \mathrm{im}(x) \subset \widehat{W}_p := \widehat{W} \cap \{r \leq r_p\}.$$

Assume that H has no Hamiltonian chords on the slice $\{r_p\} \times \partial W$ (this is always possible, since our chords are isolated), and construct a new Hamiltonian H_p in the following way:

➢ On $\widehat{W}_p$, we set $H_p \equiv H$. We then chop off the rest of H, and replace it by a linear extension at infinity (choosing the coefficients so that H_p is $\mathcal{C}^1$ at r_p).

(3) Such that $\mathcal{A}_H(x) \geq a_{p_0}$, for the same reason as in the previous footnote.



By construction, H and H_p will have the same Hamiltonian chords on $\widehat{W}_p$, and by Lemma 3.37 H_p has no chords outside of $\widehat{W}_p$. Since by Lemma 3.38 Floer trajectories do not leave $\widehat{W}_p$ either, we have:

$$CW^*(\Lambda_0, \Lambda_1; J, H_p) = F_p CW^*(\Lambda_0, \Lambda_1; J, H).$$

In particular, since the cohomology of the left-hand side is well-defined by §3.4, then so is it on the right-hand side, and we have:

$$HW^*(\Lambda_0, \Lambda_1; J, H_p) = H^*(F_p CW^*, \mathrm{d}) =: HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H),$$

where these are equalities, not just isomorphisms. □

Corollary 3.79. *If it is well-defined, then $HW^*(\Lambda_0, \Lambda_1; J, H) \cong HW^*(\Lambda_0, \Lambda_1)$.*

Proof. This follows directly from the previous results. We showed in Proposition 3.77 that, if it was well-defined, then $HW^*(\Lambda_0, \Lambda_1; J, H)$ was isomorphic to the direct limit over the action filtration $HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H)$. Then we showed that each of these modules was isomorphic to the homology of a linear Hamiltonian H_p.

As we take $a_p \to -\infty$, the slope of the Hamiltonians H_p goes to ∞, so that the direct limit over the action filtration recovers the Wrapped Floer Cohomology $HW^*(\Lambda_0, \Lambda_1)$ defined in the §3.4. (Note that the existence of continuation maps from the proof of Proposition 3.77 is now obvious, since they are the same maps as in §3.4.3). □

∴ In summary, we have shown that **if** $HW^*(\Lambda_0, \Lambda_1; J, H)$ is well-defined, then it equals the direct limit over all the filtered Floer cohomologies. We have also shown that, at each step of the filtration, cohomology is well-defined, and that so is the direct limit. Actually, not only is it well-defined, but it is isomorphic to standard Wrapped Floer Cohomology $HW^*(\Lambda_0, \Lambda_1)$. Therefore, we can take as a definition:

• **Definition 3.80** (**Wrapped Floer Cohomology**)**.** Let (J, H) be a regular pair of Floer data, where H satisfies Assumption 3.68. Then, we can assign to it:

$$HW^*(\Lambda_0, \Lambda_1; J, H) := \varinjlim_p HW^*_{\geq a_p}(\Lambda_0, \Lambda_1; J, H) \cong HW^*(\Lambda_0, \Lambda_1).$$

◇ **Remark 3.81.** Instead of taking this as a definition, one could first independently show that $HW^*(\Lambda_0, \Lambda_1; J, H)$ is well-defined, and *then* prove the isomorphism with $HW^*(\Lambda_0, \Lambda_1)$. The isomorphims are proved as above; however, the argument to show that the first quantity is well-defined is quite technical (see §18 of [Rit13]).

This is why we do this "trick" of taking the above isomorphism as a definition, instead of making it a theorem.

➤ To summarise this chapter, we now have two ways of defining Floer cohomology, given admissible Lagrangians in a Liouville domain W:

1. by choosing a sequence of non-degenerate Hamiltonians $H_k : \widehat{W} \to \mathbb{R}$ which are linear at infinity, and with increasing slopes going to $+\infty$, and defining wrapped Floer cohomology as the direct limit over the Lagrangian Floer cohomologies of all these Hamiltonians;

2. by choosing a single non-degenerate Hamiltonian $H : \widehat{W} \to \mathbb{R}$, which is strongly convex at infinity (see Assumption 3.68), and defining wrapped Floer cohomology as a direct limit over its action filtration.

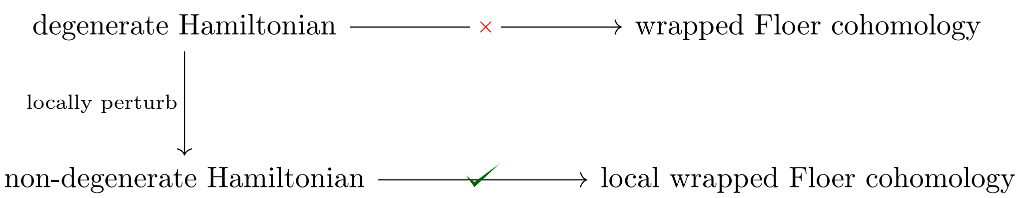
Both of these constructions are equivalent, and yield the same algebraic invariants $HW^*(\Lambda_0, \Lambda_1)$ of the pair of Lagrangians $(\Lambda_0, \Lambda_1) \subset W$.

Chapter 4

Local Wrapped Floer Cohomology

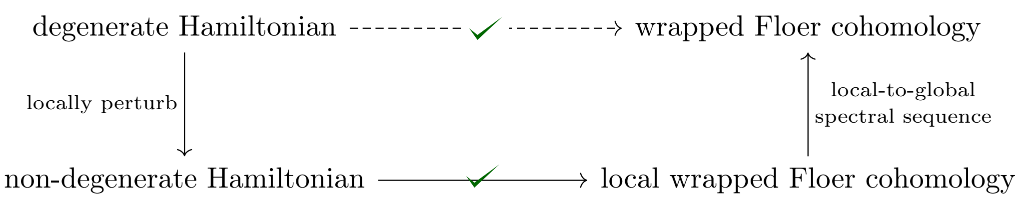
In this chapter, we propose a notion of *Local Wrapped Floer cohomology.* This will allow us to assign wrapped Floer cohomology to Hamiltonians with *degenerate* Hamiltonian chords. Indeed, given a degenerate Hamiltonian, the Floer-theoretical constructions from Chapter 3 fail; most saliently because we no longer have a well-defined Conley-Zehnder index, and therefore notion of grading, as per §3.2.4.

To fix this, the rough strategy we will employ is the following:



More precisely, given a degenerate chord x of our Hamiltonian, we will zoom in on a small neighbourhood $\mathcal{U}$ of x, perturb our Hamiltonian so that it becomes non-degenerate on $\mathcal{U}$, and show that we can then define a local wrapped Floer cohomology $HW^*_{\text{loc}}(x)$ which only records local information, and is independent of the perturbation. This will be the content of Theorem A1.

Then, Theorem A2 will give us a way to complete this diagram, by means of a « local-to-global spectral sequence », which will recover the global cohomology from all the local ones:



This was first introduced by the author and Agustin Moreno in [ML24], in order to prove a Poincaré-Birkhoff type theorem for wrapped Floer cohomology (see Chapter 5). It was inspired by similar constructions already done for other types of Floer theories, like for example in [GG10] (for local Hamiltonian Floer theory), [Oh96] (for local Lagrangian Floer theory), or [KK16] (for local symplectic homology).

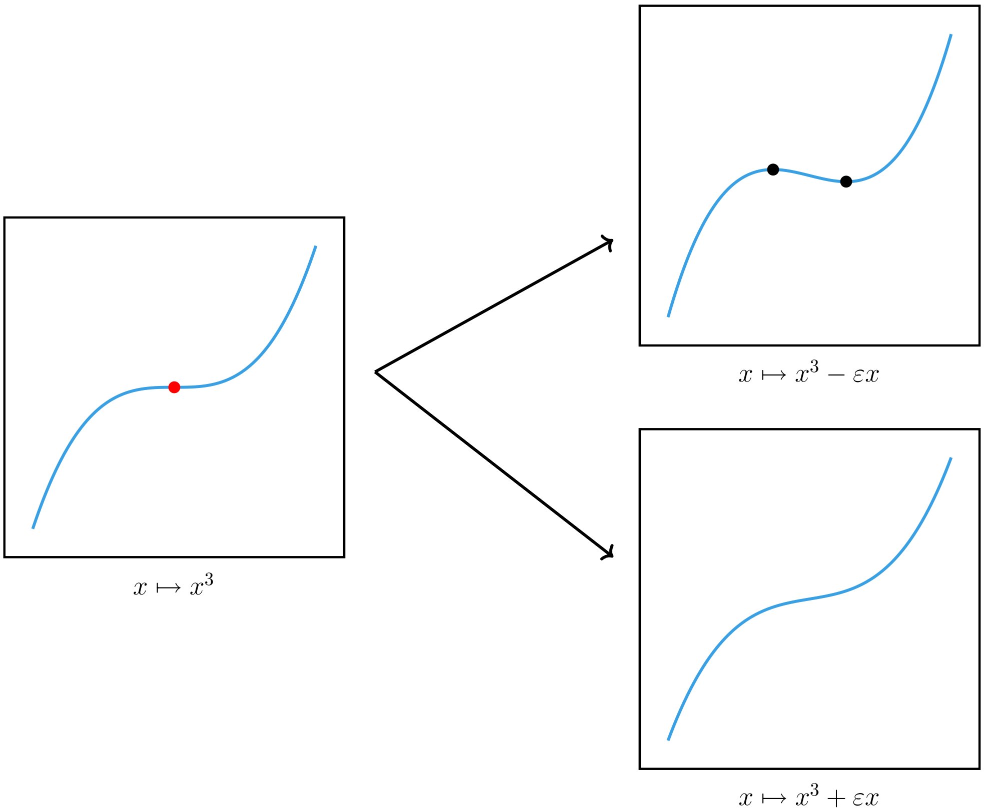
We here go through the construction from [ML24], while adding some motivation, a few technical details, and properties of HW^*_{loc} which were not included in the paper.

4.1 Definition of Local Wrapped Floer Cohomology

4.1.1 What is a *local* cohomology theory?

If our Hamiltonian H is degenerate, then it is no longer possible to associate a Floer theory to it. To understand why, let us go back to the prototype of Floer theory: Morse theory.

✧ **Example 4.1.** Consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$. It has a degeneracy at $x = 0$, since $f'(0) = f''(0) = 0$. What this means is that, under small deformations of f, the critical point at $x = 0$ will *bifurcate*. In this case, it undergoes a « birth-death bifurcation ».



• **Definition 4.2.** A **deformation** of f is a continuous family $s \mapsto f_s$, where $f_0 = f$.
A **perturbation** $\tilde{f}$ of $f : x \mapsto x^3$ is a function which is close to f in one's desired topology. Therefore, a $\mathcal{C}^\ell$-deformation is a family of $\mathcal{C}^\ell$-perturbations.

In our case, notice that while the cohomology of f is not well-defined because of the degeneracy at 0, the cohomology of any close enough perturbation $\tilde{f}$ is. Actually:

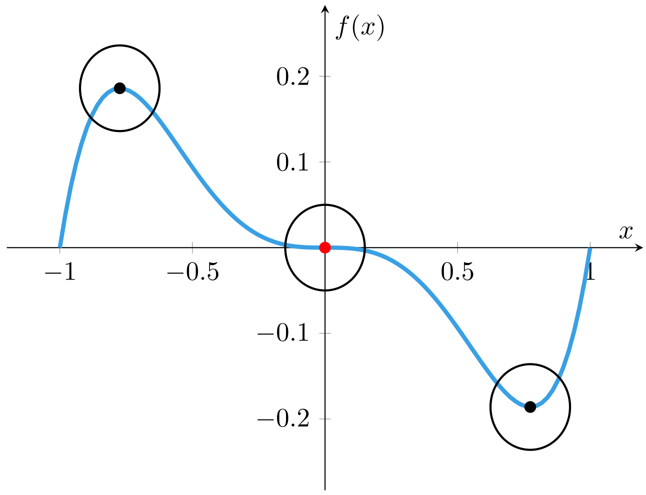
- if $\tilde{f}(x) = x^3 + \varepsilon x$ $(\varepsilon > 0)$, then it has no critical points, so that $HM^*(\tilde{f}) = 0$, where HM^* denotes Morse cohomology (§2.2.2).
- if $\tilde{f}(x) = x^3 - \varepsilon x$, then $\tilde{f}$ has two critical points, a and b. One can easily show that they have distinct, consecutive Morse indices, so that from elementary Morse theory, we must have $\mathrm{d}a = b$ without loss of generality. In particular, a is not closed, so that it does not count toward cohomology, and b is a boundary, so that it is zero in cohomology. Hence $HM^*(\tilde{f}) = 0$.
- by Thom's classification of small-order bifurcations (see, for example, [Gol78]), any generic deformation of $f : x \mapsto x^3$ can be written in the form $\varepsilon \mapsto x^3 + \varepsilon x$. Hence, we have covered all the cases.

Therefore, a generic perturbation $\tilde{f}$ of f is non-degenerate, and has zero Morse cohomology. This motivates us to define the **local Morse cohomology** of the function f, at the critical point $x = 0$, to be:

$$HM^*_{\text{loc}}(f, 0) := 0$$

We just did this with a function which only has one critical point, but if f had more than one, then we would need to enforce that they are isolated from each other.

Then, around each critical point x, one would choose a small neighbourhood $\mathcal{U}$, perturb f on $\mathcal{U}$, and thus define local Morse cohomology $HM^*_{\text{loc}}(x)$ as the cohomology of this perturbation:



We end up with many different local Morse cohomologies: one for each critical point. Then, from all these local cohomologies, one can recover the global Morse cohomology of the underlying manifold, by means of a spectral sequence. We will not show this in the context of Morse theory, for we will give the same proof for Wrapped Floer cohomology, in Theorem A2.

4.1.2 Analytical estimates

The proof that Local Wrapped Floer cohomology is well-defined, as well as the construction of the aforementioned local-to-global spectral sequence, will mainly rely on two lemmas. We claim that these two lemmas are necessary and sufficient ingredients for constructing *any local Floer theory*, and its local-to-global spectral sequence.

Like in the previous chapter, Λ_0 and Λ_1 are two admissible Lagrangians (Assumption 3.15) in a Liouville manifold $(\widehat{W}, \hat{\omega} = d\hat{\lambda})$, and J is a generic contact-type almost complex structure. We choose an at least $\mathcal{C}^2$ Hamiltonian $H : \widehat{W} \to \mathbb{R}$, on which we enforce no restrictions (in particular, it could be degenerate), except:

Assumption 4.3. Hamiltonian chords of H are isolated.

• **Definition 4.4.** Let x be a Hamiltonian chord of H between the Lagrangians Λ_0 and Λ_1. We call an open set $\mathcal{U}$ an **isolating neighbourhood** of x if it contains x, and its closure intersects no other chords of H.

Then, the first ingredient for defining a local cohomology theory is:

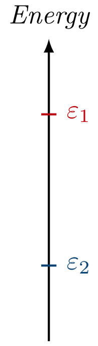
Lemma 4.5 ([Cie+96])**.** *Let x be a chord of H and $\mathcal{U}$ an isolating neighbourhood. Then, for every open set $\mathcal{V} \subset\subset \mathcal{U}$, there exists a $\mathcal{C}^\ell$ neighbourhood $\mathfrak{U}$ of (J, H) (for $\ell \geq 1$ any integer) such that, for any $(\tilde{J}, \tilde{H}) \in \mathfrak{U}$, we have:*

- *all chords of $(\tilde{J}, \tilde{H})$ contained in $\mathcal{U}$ are already contained in $\mathcal{V}$;*
- *all Floer trajectories of $(\tilde{J}, \tilde{H})$ contained in $\mathcal{U}$ are already contained in $\mathcal{V}$.*

We do not include the proof since it is the same as in [Cie+96]. This is essentially a proof by contradiction, which is very similar to the one given in **2.** of the next lemma.

The second lemma is an analytical estimate, inspired by a similar estimate from [Oh96]. However, we insist on giving it a name, the « Energy Separation Property », for this property is a cornerstone both in the construction of local Floer (co)homologies, and of their associated local-to-global spectral sequences.

Lemma 4.6 (Energy Separation Property)**.** *Let x be a chord of H. For any $\varepsilon_1 > 0$, there exist an isolating neighbourhood $\mathcal{U}$ of x, as well as a $\mathcal{C}^1$-neighbourhood $\mathfrak{U}$ of (J, H), and $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $(\tilde{J}, \tilde{H}) \in \mathfrak{U}$, and for any Floer trajectory u of $(\tilde{J}, \tilde{H})$ intersecting $\mathcal{U}$:*



1. $im(u) \subset \mathcal{U} \implies E(u) < \varepsilon_1$,
2. $E(u) < \varepsilon_2 \implies im(u) \subset \mathcal{U}$,

where $E(u)$ is the energy of u (Definition 3.32).

Proof. First, notice that if $\varepsilon_2 > 0$ exists, then without loss of generality $\varepsilon_2 \leq \varepsilon_1$. Indeed, if we had $\varepsilon_2 > \varepsilon_1$, then there could exist no Floer trajectory u with energy in $[\varepsilon_1, \varepsilon_2)$, because then we would have the contradiction:

$$E(u) < \varepsilon_2 \overset{2.}{\implies} \mathrm{im}(u) \subset \mathcal{U} \overset{1.}{\implies} E(u) < \varepsilon_1.$$

Hence, we may always set $\varepsilon_2 \leq \varepsilon_1$. Now let us prove 1. and 2.

1. Fix $\varepsilon_1 > 0$. The first implication relies on the following analytical estimate:

Lemma 4.7. *Let x be a (possibly degenerate) Hamiltonian chord of (J, H), with isolating neighbourhood $\mathcal{U}$. Then, there exists $C > 0$ such that, for every $\mathcal{C}^1$-close enough perturbation $(\tilde{J}, \tilde{H})$ of (J, H), and chord $\tilde{x}$ of $(\tilde{J}, \tilde{H})$:*

$$\left|\mathcal{A}_H(x) - \mathcal{A}_{\tilde{H}}(\tilde{x})\right| < C \left\|x - \tilde{x}\right\|_{\mathcal{C}^1} + |H - \tilde{H}|_{\mathcal{C}^0},$$

*where C does **not** depend on the choice of perturbation.*

Proof. We prove this in Calculation C.11 of the Computational Appendix. Essentially, this relies on finding a $\mathcal{C}^1$-Lipschitz estimate for each term of $\mathcal{A}_H$.

Now, notice that since $\dot{x} = X_H = J\nabla H$, the $\mathcal{C}^1$-distance between x and $\tilde{x}$ is governed by both the $\mathcal{C}^1$-distance between (J, H) and $(\tilde{J}, \tilde{H})$, and the $\mathcal{C}^0$ distance between x and $\tilde{x}$; which is in turn bounded from above by $\mathrm{diam}\,\mathcal{U}$.(1) In particular, we can rewrite:

$$\left|\mathcal{A}_{\tilde{H}}(\tilde{x}) - \mathcal{A}_H(x)\right| < \tilde{C}\left(\mathrm{diam}\,\mathcal{U} + \left\|(J, H) - (\tilde{J}, \tilde{H})\right\|_{\mathcal{C}^1}\right). \tag{4.1}$$

Hence, by shrinking $\mathcal{U}$, and choosing a perturbation $(\tilde{J}, \tilde{H})$ which is very $\mathcal{C}^1$-close to (J, H), we can make the left-hand side smaller than $\varepsilon_1/2$

Let u be a Floer trajectory contained in $\mathcal{U}$. *A fortiori*, so are its end chords, which we write y and z. By Lemma 3.34, we have $E(u) = \mathcal{A}_{\tilde{H}}(y) - \mathcal{A}_{\tilde{H}}(z)$. Hence, by the triangle inequality:

(1) Note that since $\mathcal{U}$ needs to contain the chord x, it cannot be any small neighbourhood, like a ball. One can take, for example, a slightly enlarged tubular neighbourhood of x, in which case, when we say $\mathrm{diam}\,\mathcal{U}$, we mean its diameter in the normal direction.

$$E(u) = \mathcal{A}_{\tilde{H}}(y) - \mathcal{A}_{\tilde{H}}(z) \leq \left|\mathcal{A}_{\tilde{H}}(y) - \mathcal{A}_H(x)\right| + \left|\mathcal{A}_{\tilde{H}}(z) - \mathcal{A}_H(x)\right|$$
$$< \varepsilon_1/2 + \varepsilon_1/2 = \varepsilon_1$$

which concludes the proof of **1.**

2. Let $\mathcal{U}$ be the isolating neighbourhood and $\mathfrak{U}$ the $\mathcal{C}^1$ neighbourhood of (J, H) in $\mathscr{J} \times \mathscr{H}$ from Step **1.**. Assume **2.** doesn't hold, *i.e.* there exists a sequence $(J_n, H_n) \to (J, H)$, as well as a sequence (u_n) of (J_n, H_n) trajectories, and $\varepsilon_n \to 0$ such that:

$$E(u_n) < \varepsilon_n,$$
$$\mathrm{im}(u_n) \notin \mathcal{U}.$$

By elliptic regularity (Theorem B.19), we extract a converging subsequence with limit u_∞. Since $E(u_\infty)$ is necessarily 0, u_∞ must be equal to a Hamiltonian chord, by Lemma 3.33. Since $\mathcal{U}$ is isolating, x is the only chord that intersects it, telling us that necessarily $u_\infty \equiv x$, up to parametrisation. However, since we assumed that $\mathrm{im}(u_n) \notin \mathcal{U}$ $\forall n$, then there must exist $t_\infty \in [0,1]$ such that $u_\infty(t_\infty) \notin \mathcal{U}$, which is a contradiction. □

We claim that these two lemmas are the two essential ingredients to defining a local cohomology theory, as well as its local-to-global spectral sequence. This will be made clear in the case of Wrapped Floer Cohomology, by Theorems A1 and A2.

4.1.3 Local Wrapped Floer Cohomology: definition

Let $(W, \omega = \mathrm{d}\lambda)$ be a Liouville domain, which we complete to $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$, and let Λ_0, Λ_1 be admissible Lagrangians in $\widehat{W}$ (Assumption 3.15). We choose an almost complex structure J of contact type (*i.e.* $J\partial_r = \mathcal{R}_\alpha$, where $V = r\partial_r$ is the Liouville vector field, and $\alpha := \lambda|_{\partial W}$), and a Hamiltonian $H : \widehat{W} \to \mathbb{R}$ on which we enforce no assumptions.

In particular, it may have degenerate chords. We only ask that these chords be isolated.

• **Definition 4.8.** Let x be a Hamiltonian chord of H. We define its **Local Wrapped Floer Cohomology** $HW^*_{\mathbf{loc}}(x)$ in the following way:

- choose an isolating neighbourhood $\mathcal{U}$ of x, and a generic $\mathcal{C}^1$-close perturbation $(\tilde{J}, \tilde{H})$ of (J, H), which we choose to be non-degenerate;
- since $(\tilde{J}, \tilde{H})$ is non-degenerate, we can locally define its Lagrangian Floer cohomology on $\mathcal{U}$, as the cohomology of the complex of chords, and set:

$$HW^*_{\mathrm{loc}}(x) := HL^*(\mathcal{U}, \Lambda_0, \Lambda_1; \tilde{J}, \tilde{H}). \tag{4.2}$$

We call this the local wrapped Floer cohomology of the chord x.

Theorem A1. *Local wrapped Floer cohomology $HW^*_{loc}(x)$ is well-defined, for any chord x.*

The proof of this will constitute the remainder of this section, and will consist in three steps:

1. First, we must show that given two Hamiltonian chords of $\tilde{H}$ in the isolating neighbourhood $\mathcal{U}$, Floer trajectories connecting them stay in $\mathcal{U}$; which is essential if we want to define our local cohomology. Once this is done, we shall still need to argue that $HL^*(\mathcal{U})$ is well-defined, which is not automatic since $\mathcal{U}$ is open, and HL^* is usually defined for closed manifolds.
2. Then, we show that the right-hand side of (4.2) is independent of the choice of perturbation $(\tilde{J}, \tilde{H})$.

3. Then, we show that it does not depend on the choice of neighbourhood $\mathcal{U}$ either.

Let us proceed.

Step 1 (Locality). We show:

Proposition 4.9. *Let x be a (potentially degenerate) Hamiltonian chord. One can find an isolating neighbourhood $\mathcal{V}$ of x such that, for $(\tilde{J}, \tilde{H})$ sufficiently $\mathcal{C}^1$-close to (J, H), and y and z any two chords of $(\tilde{J}, \tilde{H})$ in $\mathcal{V}$, Floer trajectories connecting y and z do not exit $\mathcal{V}$.*

Proof. This follows from Lemmas 4.5 and 4.6. Indeed, fix $\varepsilon_1 > 0$. The Energy Separation Property gives us an isolating neighbourhood $\mathcal{U}$ of x, as well as an $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$E(u) < \varepsilon_2 \implies \operatorname{im}(u) \subset \mathcal{U} \tag{4.3}$$

for any Floer trajectory u of $(\tilde{J}, \tilde{H})$ intersecting $\mathcal{U}$. By the action estimate (4.1), one can find a smaller isolating neighbourhood $\mathcal{V} \subset\subset \mathcal{U}$ such that $E(u) < \varepsilon_2$ holds for any trajectory u of $(\tilde{J}, \tilde{H})$ with ends in $\mathcal{V}$. By (4.3), such a trajectory will be contained in $\mathcal{U}$. By Lemma 4.5, $\operatorname{im}(u) \subset \mathcal{U} \implies \operatorname{im}(u) \subset \mathcal{V}$, which concludes the proof. □

This was the main technical step (which seems trivial now since we had already proved Lemmas 4.5 and 4.6), ensuring that trajectories between local chords stay local. Recall that the reason we wanted to prove this was so that we could define something like

$$\text{"}HW^*_{\text{loc}}(x) := HL^*(\mathcal{V}, \Lambda_0, \Lambda_1; \tilde{J}, \tilde{H})\text{"}.$$

Now, this right-hand side is simply defined as the cohomology of the complex:

$$\begin{aligned} CF^*(\mathcal{V}; \tilde{J}, \tilde{H}) :=& \big\{x \in CF^*(\tilde{J}, \tilde{H}) \mid \operatorname{im}(x) \subset \mathcal{V}\big\} \\ =& \big\{\text{Hamiltonian chords of } \tilde{H} \text{ in } \mathcal{V}\big\}. \end{aligned}$$

It is not automatic that its cohomology, $HL^*(CF^*(\mathcal{V}; J, H))$ is well-defined, since HL^* is usually defined on a *closed* manifold (*i.e.* compact without boundary), and $\mathcal{V}$ is an open neighbourhood.

However, we have already seen one way to go about this issue: in Remark 3.39, and all through the rest of Chapter 3, we were working on a non-compact symplectic manifold, but every chord/trajectory was contained in a compact region, which allowed us both to extract bounds and converging subsequences. We can make a similar argument here:

Recall that we had originally started from an isolating neighbourhood $\mathcal{U}$ of x, and produced $\mathcal{V} \subset\subset \mathcal{U}$ such that for small enough perturbations of (J, H), all chords/trajectories from $\mathcal{U}$ were contained in $\mathcal{V}$. In particular, if we take $\mathcal{U}$ to be our new manifold of interest, then we are back in the situation of Remark 3.39: all chords/trajectories from $\mathcal{U}$ are contained in the compact subset $\overline{\mathcal{V}}$. Hence, we can use bounds from this region to bound objects in $\mathcal{U}$, and extract converging subsequences to any sequences of chords/trajectories. Therefore:

$$HL^*(\mathcal{U}, \Lambda_0, \Lambda_1; \tilde{J}, \tilde{H}) := H^*\big((CF^*(\mathcal{U}; \tilde{J}, \tilde{H}), \mathrm{d}\big) = H^*\big((CF^*(\mathcal{V}; \tilde{J}, \tilde{H}), \mathrm{d}\big)$$

is well-defined, and we can formally define:

$$\boxed{HW^*_{\text{loc}}(x) := HL^*(\mathcal{U}, \Lambda_0, \Lambda_1; \tilde{J}, \tilde{H})} \tag{4.4}$$

which concludes this first step.

Step 2 (Perturbation invariance). Let us now show that the right-hand side of (4.4) is independent of the choice of perturbation $(\tilde{J}, \tilde{H})$, as long as it is sufficiently $\mathcal{C}^1$ close to (J, H).

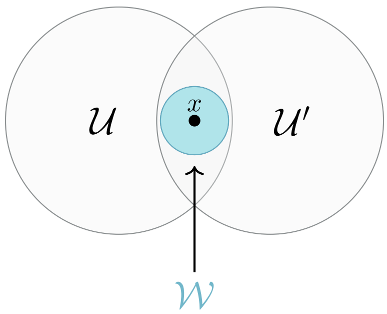
This is the exact same argument as in §3.4.3, and in particular Proposition 3.63. Indeed, since we only work locally, we need not worry about the maximum principle, or trajectories escaping to infinity. We can ensure that trajectories stay inside $\mathcal{U}$ thanks to the *a priori* energy estimate (Lemma 3.53) and Energy Separation Property (Lemma 4.6).

Hence, given any two pairs of regular Floer data (J_a, H_a), (J_b, H_b), we can construct a homotopy and its inverse, and then reproduce the argument from Proposition 3.63 to show that they induce inverses in cohomology:

$$HL^*(\mathcal{U}; \Lambda_0, \Lambda_1; J_a, H_a) \cong HL^*(\mathcal{U}, \Lambda_0, \Lambda_1; J_b, H_b).$$

Once again, one needs to be careful since $\mathcal{U}$ is an open neighbourhood, and not a closed manifold, but this issue is easily addressed in the same way as in Step 1.

Step 3 (Invariance on $\mathcal{U}$) Let us now show that our construction is independent of the choice of isolating neighbourhood $\mathcal{U}$. In other words, let $\mathcal{U}, \mathcal{U}'$ be two isolating neighbourhoods containing the chord x.



Choose an open neighbourhood $\mathcal{W} \subset \mathcal{U} \cap \mathcal{U}'$ of x (*a fortiori*, also isolating). Then, following the recipe from Steps **1.** and **2.**, one can find perturbations

$$\begin{cases} (\tilde{J}, \tilde{H}) \\ (\tilde{J}', \tilde{H}') \end{cases}$$

of (J, H), on $\mathcal{U}$ and $\mathcal{U}'$ respectively, and which allow us to define:

$$\begin{cases} HL^*(\mathcal{U}, \Lambda_0, \Lambda_1; \tilde{J}, \tilde{H}) \\ HL^*(\mathcal{U}', \Lambda_0, \Lambda_1; \tilde{J}', \tilde{H}') \end{cases}$$

Note that actually, by Step **2.**, neither of these quantities depend on the perturbation used to define them, so that we can drop it from the notation.

Lemma 4.10. $HL^*(\mathcal{U}, \Lambda_0, \Lambda_1) \cong HL^*(\mathcal{W}, \Lambda_0, \Lambda_1)$.

Proof. Choose $K \subset \mathcal{W}$ compact, and a bump function $\psi : \mathcal{U} \to [0, 1]$ such that

$$\psi|_K \equiv 1, \;\; \psi|_{\mathcal{U} \setminus \mathcal{W}} \equiv 0$$

Now define $(\tilde{J}_\star, \tilde{H}_\star) := (\tilde{J}, \tilde{H}) \circ \psi$. Then, $(\tilde{J}_\star, \tilde{H}_\star)$ is a non-degenerate pair of Floer data on $\mathcal{W}$, so that it allows us to construct $HL^*(\mathcal{W}, \Lambda_0, \Lambda_1)$. We can construct a homotopy between $(\tilde{J}, \tilde{H})$ and $(\tilde{J}_\star, \tilde{H}_\star)$, so that by Step **2.** the induced cohomologies are isomorphic. □

Since we can reproduce the exact same argument for $\mathcal{U}'$, we get that:

$$HL^*(\mathcal{U}, \Lambda_0, \Lambda_1) \cong HL^*(\mathcal{U}', \Lambda_0, \Lambda_1)$$

which concludes the proof of Step **3.**, and hence of Theorem A1. □

➤ **Summary.** Given any Hamiltonian chord x of (J, H), potentially degenerate, we can construct its local Wrapped Floer Cohomology by finding an isolating neighbourhood $\mathcal{U}$ of x and a close enough perturbation $(\tilde{J}, \tilde{H})$ of (J, H) – as imposed by Proposition 4.9 –, and then defining:

$$HW^*_{\mathrm{loc}}(x) := HL^*(\mathcal{U}, \Lambda_0, \Lambda_1; \tilde{J}, \tilde{H}).$$

This cohomology theory is independent of the choice isolating neighbourhood $\mathcal{U}$ and perturbation $(\tilde{J}, \tilde{H})$, and is therefore an invariant of the Hamiltonian chord x.

4.2 The local-to-global spectral sequence

4.2.1 Statement of the theorem

As before, we work with two admissible Lagrangians Λ_0, Λ_1 in a Liouville manifold $\widehat{W}$, and a pair of Floer data (J, H). In the previous section, we saw how one could associate a local cohomology to every Hamiltonian chord x of H, even degenerate ones.

Now let us show how one can recover the global cohomology $HW^*(\Lambda_0, \Lambda_1)$ from all these local cohomologies.

Theorem A2. *Let $(W, \omega = \mathrm{d}\lambda)$ be a Liouville domain with completion $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$, and $\Lambda_0, \Lambda_1 \subset \widehat{W}$ be admissible Lagrangians. Let $H : \widehat{W} \to \mathbb{R}$ be a Hamiltonian which is strongly convex at infinity (Assumption 3.68). Write $\{x_k\}_{k\in\mathbb{N}}$ the set of chords of H on $\widehat{L}$, and $A_k := \mathcal{A}_H(x_k)$ the sequence of actions (from which we discard the repeated values). Then there exists a spectral sequence $(E_n^{*,*})$ such that:*

$$E_1^{p,q} = \left.\begin{array}{ll} \displaystyle\bigoplus_{\mathcal{A}_H(x)=A_k} HW^*_{loc}(x) & p = 2k \\ 0 & p \ \ odd \end{array}\right\} \Longrightarrow HW^*(\Lambda_0, \Lambda_1)$$

which we call the ***local-to-global spectral sequence****.*

For example, if we assume that all chords x_k have distinct action, then E_1 looks like:

⋮	⋮	⋮	⋮	⋮	⋮	⋮
0	$HW^3_{\mathrm{loc}}(x_{k-1})$	0	$HW^3_{\mathrm{loc}}(x_k)$	0	$HW^3_{\mathrm{loc}}(x_{k+1})$	0
0	$HW^2_{\mathrm{loc}}(x_{k-1})$	0	$HW^2_{\mathrm{loc}}(x_k)$	0	$HW^2_{\mathrm{loc}}(x_{k+1})$	0
0	$HW^1_{\mathrm{loc}}(x_{k-1})$	0	$HW^1_{\mathrm{loc}}(x_k)$	0	$HW^1_{\mathrm{loc}}(x_{k+1})$	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮ → p

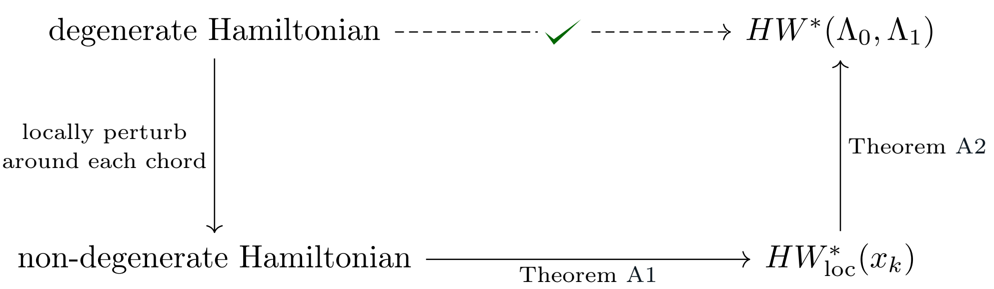
Let us point out that we can order the chords in a countable sequence because they are isolated, as per Assumption 3.68. While there is no reason to assume that all their actions are distinct, we still have:

Lemma 4.11. *Under Assumption 3.68, the sequence (A_k) has no accumulation points.*

Proof. This follows directly from the fact that for high values of k, (A_k) becomes strictly decreasing, and $A_k \xrightarrow{k\to\infty} -\infty$ (Corollary 3.71). $\square$

Hence for low (even) values of p, the p-th column of the E_1-page contains the local cohomology of finitely many chords, while for high values of p, each column corresponds to exactly one chord.

◇ **Remark 4.12.** The main point of Theorem A2 is that it allows us to associate a wrapped Floer cohomology theory to a degenerate Hamiltonian, by following the heuristic diagram presented at the start of the chapter:



Hence, not only can we associate a wrapped Floer cohomology theory to a degenerate Hamiltonian, but it will be isomorphic to the standard $HW^*(\Lambda_0, \Lambda_1)$, as defined in Chapter 3.

This is the **bottomline of this chapter**: even if we start from a degenerate Hamiltonian system (*e.g.* from physics), we can still recover knowledge about its dynamics from $HW^*(\Lambda_0, \Lambda_1)$, which is a purely topological invariant of Λ_0 and Λ_1.

Before we prove Theorem A2, let us state one modified, simpler version:

Theorem 4.13. *Let $H : \widehat{W} \to \mathbb{R}$ be a Hamiltonian which is linear at infinity, and whose chords are isolated. Then, there exists a spectral sequence $(E_n^{*,*})$ such that:*

$$E_1^{p,q} = \left.\begin{matrix} \bigoplus\limits_{\mathcal{A}_H(x)=A_k} HW^*_{loc}(x) & p = 2k \\ 0 & p \ \ odd \end{matrix}\right\} \implies HW^*(\Lambda_0, \Lambda_1; J, \widehat{H})$$

Proof. We will prove this after Theorem A2, in Calculation 4.16.

◇ **Remark 4.14.** Heuristically, the reason this second spectral sequence only converges to $HW^*(\Lambda_0, \Lambda_1; J, H)$, instead of the whole $HW^*(\Lambda_0, \Lambda_1)$, is for the same reason as in §3.5.1. Indeed, recall from Prop. 3.17 that a Hamiltonian chord of H at height r corresponds to a Reeb chord of period $h'(r)$ on the boundary. Hence, if we choose a Hamiltonian H with slope a, then our spectral sequence can by construction only count Reeb chords with period $\leq a$, so that it cannot recover the whole cohomology.

Meanwhile, if we choose $\widehat{H}$ to satisfy Assumption 3.68, then $H = h(r)$ at infinity, with $h'(r) \to \infty$. Therefore, all chords will appear in the spectral sequence.

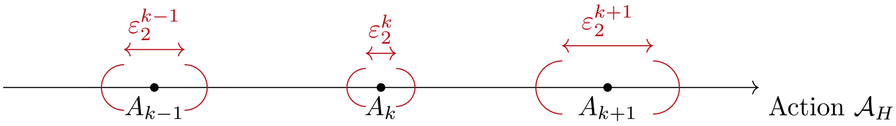
4.2.2 Construction of the spectral sequence

Let us construct the spectral sequence from Theorem A2. Recall that we have a sequence of actions $A_k := \mathcal{A}_H(x_k)$, with no accumulation points, no repeated values, and which eventually becomes strictly decreasing. Re-order it so that (A_k) is stricly decreasing for every k.

Now fix $\varepsilon_1 > 0$. The Energy Separation Property (Lemma 4.6) gives us, for every chord x_k, an isolating neighbourhood $\mathcal{U}_k$, as well as some $\varepsilon_2^k \in (0, \varepsilon_1]$ such that: given any close enough perturbation $(\tilde{J}, \tilde{H})$ of (J, H) with Floer trajectory u intersecting $\mathcal{U}_k$:

$$\mathrm{im}(u) \subset \mathcal{U}_k \implies E(u) < \varepsilon_1$$
$$E(u) < \varepsilon_2^k \implies \mathrm{im}(u) \subset \mathcal{U}_k$$

Now, construct ε_2^k-neighbourhoods centred around each A_k:



which we can choose not to overlap (if they do, try again with a smaller ε_1. Since (A_k) has no accumulation points, $\min_k |A_k - A_{k+1}| > 0$, so that this is possible). Then, define a new sequence $(a_p), p \in \mathbb{N}$, by taking the bounds of these intervals, *ie*:

$$\text{For } p = 2k : \begin{cases} a_p & := A_k - \frac{1}{2}\varepsilon_2^k \\ a_{p+1} & := A_k + \frac{1}{2}\varepsilon_2^k. \end{cases} \tag{4.5}$$

Pick $(\tilde{J}, \tilde{H})$ a local, non-degenerate perturbation of (J, H), and consider the chain complex $CF^*(\tilde{J}, \tilde{H})$. There, define the action-filtration:

$$F_p CF^*(\tilde{J}, \tilde{H}) := \left\{ x \in CF^*(\tilde{J}, \tilde{H}) \mid \mathcal{A}_H(x) \geq a_p \right\}, p \in \mathbb{N} \tag{4.6}$$

where the grading $*$ is given by the Conley-Zehnder index (§3.2.4). The latter is well-defined since $(\tilde{J}, \tilde{H})$ is non-degenerate.

By standard arguments (see for example, §14 of [BT82]), such a filtration induces a spectral sequence with first page:

$$E_1^{p,*} = H^*(F_{p+1}CF^*(\tilde{J}, \tilde{H})/F_p CF^*(\tilde{J}, \tilde{H}), \mathrm{d}^*)$$

where d^* descends from the standard Floer differential.

With our definitions, $F_{p+1}CF^*(\tilde{J}, \tilde{H})/F_p CF^*(\tilde{J}, \tilde{H})$ consists of Hamiltonian chords with action in $(a_p, a_{p+1}]$. Therefore:

- if p is even, then $(a_p, a_{p+1}] = (A_k - \frac{1}{2}\varepsilon_2^k, A_k + \frac{1}{2}\varepsilon_2^k]$. Take two chords y and z of $(\tilde{J}, \tilde{H})$ with actions in this interval. If y and z are both in $\mathcal{U}_k$, then

  $$|\mathcal{A}_{\tilde{H}}(y) - \mathcal{A}_{\tilde{H}}(z)| < \varepsilon_2^k$$

  so that Floer trajectories connecting y and z are in $\mathcal{U}_k$, by the Energy Separation Property. Therefore: $E_1^{p,*} \supset \oplus HW^*_{\mathrm{loc}}(x)$, where the direct sum is taken over all chords with action A_k.

  This is actually all of it. Indeed, by the Energy Separation Property, Floer trajectories exiting $\mathcal{U}_k$ have energy strictly greater than ε_2^k. This cannot happen since y and z both have action in an interval of amplitude $< \varepsilon_2^k$. Hence, for p even:

  $$E_1^{p,*} = \bigoplus_{\mathcal{A}_H(x) = A_{p/2}} HW^*_{\mathrm{loc}}(x)$$

- if p is odd, then $(a_p, a_{p+1}] = (A_k + \frac{1}{2}\varepsilon_2^k, A_{k+1} - \frac{1}{2}\varepsilon_2^{k+1}]$. Hence, Floer trajectories would try to connect chords with actions *outside* of the ε_2^k intervals. Since we have chosen those ε_2^k neighbourhoods to not overlap, then by the action estimate (4.1), there exist no such chords. Hence, $H^*(F_{p+1}CF^*(\tilde{J}, \tilde{H}), F_p CF^*(\tilde{J}, \tilde{H}); \mathrm{d}^*) = 0$.

Therefore, we have a spectral sequence whose first page is given by:

$$E_1^{p,q} = \begin{cases} \bigoplus_{\mathcal{A}_H(x)=A_{p/2}} HW^q_{\text{loc}}(x), & \text{for } p \in 2\mathbb{N} \\ 0 & \text{for } p \in 2\mathbb{N}+1 \end{cases}, \quad q \in \mathbb{Z}$$

which is our candidate for the spectral sequence of Theorem A2.

4.2.3 Convergence of the spectral sequence

While our spectral sequence $(E_n^{*,*})$ has the desired first page, it is *a priori* unbounded (since H has infinitely many chords), so that its convergence is not obvious. To prove it, we will construct a sequence of intermediary spectral sequences, and force them to converge to (E_n). Namely, we prove the intermediary proposition:

Proposition 4.15 (Filtered spectral sequence)**.** *Let (a_j) be our sequence of interval bounds from the previous section. For every j, there exists a spectral sequence $({}^jE_n^{**})$ such that:*

$$\left.{}^jE_1^{p,q} = \begin{matrix} \bigoplus_{\mathcal{A}_H(x)=A_k \geq a_j} HW^*_{loc}(x) & p = 2k \\ 0 & p \ \ odd \end{matrix}\right\} \implies HW^*_{\geq a_j}(\Lambda_0, \Lambda_1; J, H)$$

Proof. The E_1 page of the spectral sequence is constructed exactly as in §4.2.2, giving it its desired form. The only difference is that now, this spectral sequence is bounded (it has finitely many columns, since chords of action $\geq a_j$ are contained in a compact region). Hence, by a standard theorem on filtration spectral sequences (§14 of [BT82]), $({}^jE_n^{**})$ converges to the cohomology of the total complex:

$$F_jCF^*(\tilde{J}, \tilde{H}).$$

By Proposition 3.78, $H^*(F_jCF^*(\tilde{J}, \tilde{H}), \mathrm{d})$ is well-defined and isomorphic to the filtered cohomology $HW_{\geq a_j}(\Lambda_0, \Lambda_1; \tilde{J}, \tilde{H})$, concluding the proof. □

Hence, we have a sequence $({}^jE)_j$ of spectral sequences, each of which satisfying

$${}^jE \implies HW^*_{a_j}(\Lambda_0, \Lambda_1; \tilde{J}, \tilde{H}).$$

Now, since $a_j > a_{j+1}$, recall by §3.4.3 that there exist continuations maps $HW^*_{a_j} \to HW^*_{a_{j+1}}$. *A fortiori*, there exist continuation maps ${}^jE \to {}^{j+1}E$. On the first page, this continuation map is just an inclusion:

$${}^jE_1 \hookrightarrow {}^{j+1}E_1$$

which, depending on whether j is odd or even, is either the identity, or the addition of a new column. Taking a direct limit over this sequence of inclusions, we get:

$$E_1 = \varinjlim_j {}^jE_1 \tag{4.7}$$

where E_1 is the first page of the global spectral sequence constructed in the previous section. Hence, there is an isomorphism between the first page of the sequences (E_n) and $(\varinjlim_j {}^jE_n)$. By the comparison theorem for spectral sequences [Wei94, Thm 5.2.12], this isomorphism will survive in every page, ensuring in particular that:

$$E_\infty = \varinjlim_j {}^jE_\infty \tag{4.8}$$

Now, by Proposition 4.15, $({}^jE_n)$ converges to $HW^*_{\geq a_j}(\Lambda_0, \Lambda_1; \tilde{J}, \tilde{H})$. Therefore, (E_n) converges to:

$$\varinjlim_j HW^*_{\geq a_j}(\Lambda_0, \Lambda_1; \tilde{J}, \tilde{H})$$

which is isomorphic to $HW^*(\Lambda_0, \Lambda_1)$ by Proposition 3.80, hence concluding the proof of Theorem A2. □

Calculation 4.16 (Proof of Corollary 4.13)**.** Assume we have a Hamiltonian $H : \widehat{W} \to \mathbb{R}$ which is linear at infinity. Then, the action of chords no longer goes to $-\infty$, however we can still order them decreasingly and construct the same action-filtration and spectral sequence. By Corollary 3.45, H has finitely many Hamiltonian chords, so that this spectral sequence is bounded. Hence, by the same argument as above, it converges to:

$$H^*(CF^*(\tilde{J}, H), \mathrm{d}^*) = HW^*(\Lambda_0, \Lambda_1; \tilde{J}, H)$$

which is what we wanted to prove. □

4.3 Applications to dynamics and mission design

The bottomline of this section is the following:

Theorem 4.17. *Given $H_t : W \to \mathbb{R}$ a Hamiltonian on a Liouville domain, then to any Hamiltonian chord x of H between admissible Lagrangians, one can assign a* numerically computable *integer $\chi(x)$ which stays invariant under deformations of H.*

This is a very concrete result, with direct applications to the numerical continuation of families of trajectories. By computing this integer at different stages of the process, we can directly know whether our numerical algorithms have missed some solutions or not. This strategy is not new: it has already been experimented with for periodic orbits in [FKM23; Ayd+24a] as part of a collaboration between Heidelberg, Augsburg, Seoul, and NASA's JPL.

The work in this chapter allows us to easily adapt their scheme to study *open-ended trajectories with Lagrangian ends* instead of periodic orbits. In Part III, we will give two examples of such trajectories in the Circular Restricted Three-Body Problem: trajectories of collision, and trajectories bi-normal to the xz-plane; though our methods are abstract, and carry over to any trajectory with appropriate boundary conditions.

4.3.1 The Floer number

Recall that given a cohomology theory $(H^*)_{*\in\mathbb{Z}}$, its Euler number is defined as:

$$\chi := \sum_{n\in\mathbb{Z}} (-1)^n \dim H^n.$$

• **Definition 4.18.** Given any Hamiltonian chord x, we define its **Floer number** as the Euler number of $HW^*_{\mathrm{loc}}(x)$:

$$\chi(x) := \sum_{n\in\mathbb{Z}} (-1)^n \dim HW^n_{\mathrm{loc}}(x). \tag{4.9}$$

Lemma 4.19. *$\chi(x)$ is invariant under deformation, i.e. given a smooth family (x_s) of trajectories, $\chi(x_s)$ is constant. Furthermore, if a chord x is non-degenerate, then $\chi(x) = \pm 1$.*

Proof. Invariance of $\chi(x)$ follows from invariance of $HW^*_{\mathrm{loc}}(x)$ under perturbation, which we argued in Step 2 of §4.1.3. If a chord x is non-degenerate, then $HW^*_{\mathrm{loc}}(x)$ only contains the element $[x]$ so that $\chi(x) = (-1)^m$ for some $m \in \mathbb{Z}$. □

We can yet be more precise. By §3.2.4, the grading on Wrapped Floer Cohomology is given by the *Conley-Zehnder index* μ_{CZ}.

Lemma 4.20. *Let x be a (potentially degenerate) Hamiltonian chord. Then:*

$$\chi(x) = \sum_{y\in HW^\bullet_{loc}(x)} (-1)^{CZ(y)}.$$

Proof. This is simply re-arranging the terms in the definition of $\chi(x)$. Indeed, write $d_n := \dim HW^n_{\text{loc}}(x)$, and note that by definition:

$$\forall y \in HW^n_{\text{loc}}(x) : \mu_{\text{CZ}}(y) = n.$$

Then, we can re-write:

$$\begin{aligned} \chi(x) &= \sum_{n\in\mathbb{Z}} (-1)^n \dim HW^n_{\text{loc}}(x) \\ &= \sum_{n\in\mathbb{Z}} (-1)^n \underbrace{(1+\cdots+1)}_{d_n \text{ times}} \\ &= \sum_{y\in HW^\bullet_{\text{loc}}(x)} (-1)^{\mu_{\text{CZ}}(y)} \end{aligned}$$

□

In particular, this implies:

Corollary 4.21. $\chi(x)$ *is* **numerically computable**.

Proof. The Conley-Zehnder of a trajectory can be explicitly computed. See the work of C. Aydin ([Ayd23a; Ayd23b; Ayd+24b], who developed a deformation technique. He studied all possible jumps in the Conley-Zehnder index which may occur under bifurcation, and implemented (in Python) a scheme which computes μ_{CZ} as we continue a family of trajectories. This is in particular powerful when we study perturbations of known trajectories (say, from the planar problem, or the Rotating Kepler problem).

More recently, in [Ayd+24a], O. van Koert gave a full Python implementation of an algorithm computing the Conley-Zehnder index from scratch, following the original definition. This has recently been reproduced in Matlab by B. Kumar, and can readily import their codes and use them right out of the box to compute the Floer number. □

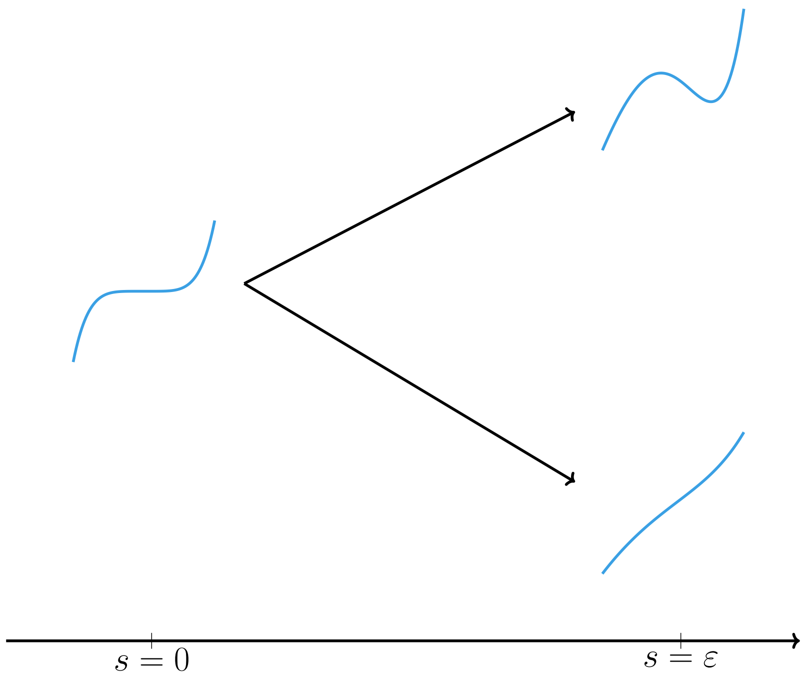
So $\chi(x)$ is perturbation-invariant and numerically computable. This makes it possible to implement it concretely for trajectory design (see [FKM23; Ayd23b; Ayd+24b; AB24]). There, it becomes useful in the numerical continuation of trajectories. Let us see how.

4.3.2 A sanity check to refine data-bases of trajectories

The previous subsection provided us with an invariant χ of Hamiltonian chords in a Liouville domain. The bottomline of Part III of this thesis is that some trajectories in the Circular Restricted Three-Body Problem (CR3BP) can be viewed as Hamiltonian chords in a Liouville domain, where the Lagrangians correspond to physical boundary conditions.

Let x be such a physical trajectory (*e.g.* collision trajectory or halo orbit, as in Part III), and $s \mapsto x_s$ a continuous deformation such that $x_0 = x$. By the previous section, $\chi(x_s)$ remains constant along s. Let us give an example of how we can use this fact in practice.

➢ Say that we are numerically continuing a family of trajectories. In practice, this is done by discretising time and using numerical ODE solvers at each time increment. Assume that while following this routine, we observe that our family of trajectories has undergone bifurcation between times $s = 0$ and $s = \varepsilon$.

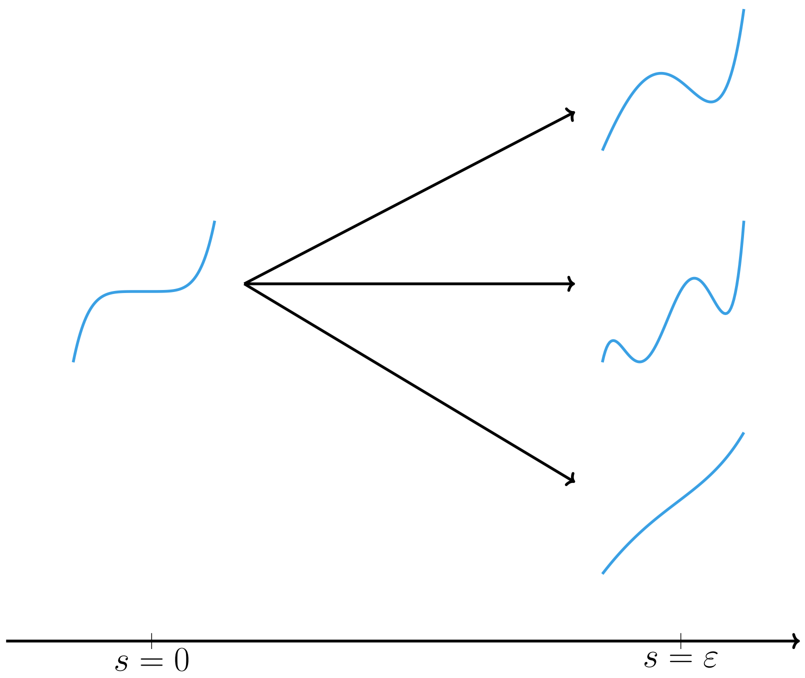


Since our search is numerical by essence, and relies on ODE solvers, we may have « missed » some solutions. However, the Floer number allows us to perform a **sanity check**. Indeed, write x_0 the original trajectory at $s = 0$, and $x_\varepsilon, x'_\varepsilon$ the ones at $s = \varepsilon$. By Corollary 4.21, we can numerically compute the Conley-Zehnder indices of each of these trajectories.

Invariance of χ (Lemma 4.19) then tells us that we should have:

$$(-1)^{\mu_{\text{CZ}}(x_0)} = (-1)^{\mu_{\text{CZ}}(x_\varepsilon)} + (-1)^{\mu_{\text{CZ}}(x'_\varepsilon)}.$$

There is no way this equality can hold. This means we must have missed a trajectory:



After searching more, say our ODE solvers found a third trajectory. We check again if:

$$\chi(x_0) = \chi(x_\varepsilon) + \chi(x'_\varepsilon) + \chi(x''_\varepsilon). \tag{4.10}$$

If not, we keep searching. If the equality *does* hold, then we may reasonably tell our solver to have a rest – unless we have strong cause to believe that there may be more trajectories. Indeed, it might be that both sides of (4.10) agree, but that we are still missing trajectories, for example a birth-death pair, whose Conley-Zehnder indices have different parity.

Hence this algorithm remains imperfect. However it already gives us a first sanity check, which we can concretely implement to refine our data-bases of periodic orbits/trajectories. Consequent work has been done in this direction by C. Aydin in the past few years, especially in the Earth-Moon problem [Ayd23b; AB24]. We also refer to [FKM23; Ayd+24a] for other concrete examples in mission design, as well as other tools from symplectic geometry with applications to astrodynamics (B-signature, GIT sequence,...) not mentioned in this thesis.

Chapter 5

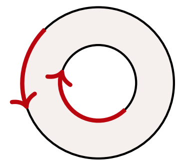
A relative Poincaré-Birkhoff theorem

5.1 Poincaré's last geometric theorem

5.1.1 Statement of the theorem(s)

The whole second part of this thesis will be dedicated to different generalisations of the Poincaré-Birkhoff theorem. In its original form, the theorem reads:

Theorem 5.1 (Poincaré-Birkhoff, 1912-1913)**.** *Let f be an area-preserving self-homeomorphism of the annulus which satisfies the **twist condition**, i.e. it rotates both its boundary components in opposite directions. Then, f has infinitely many interior periodic points (points x such that $f^k(x) = x$ for $k \in \mathbb{N}$), of arbitrarily large period.*



This theorem was stated and partially proved by Poincaré in [Poi12], a few months before his demise – conferring it the title of 'Poincaré's last geometric theorem'. It was then proved in full generality by Birkhoff in [Bir13]. The theorem takes its source in the study of the Three-Body Problem, and was the key element in constructing periodic orbits in the PCR3BP (Chapter 8, §A.3). The goal of Part II of this thesis is to generalise the theorem to higher dimensions, in an attempt to apply it to the Spatial Circular Restricted Three-Body Problem.

The first step in this direction was made by Moreno & van Koert, a century after the original Poincaré-Birkhoff theorem:

Theorem 5.2 (Generalised Poincaré-Birkhoff theorem [MK22a])**.** *Let $(W, \omega = \mathrm{d}\lambda)$ be a connected Liouville domain, and $f : W \to W$ an exact symplectomorphism such that:*

- ***(Symplectic cohomology).*** $\dim SH^\bullet(W) = \infty$.
- ***(Index growth).*** *if $\dim W \geq 4$, then $c_1(W) = 0$, and $(\partial W, \alpha := \lambda|_{\partial W})$ is strongly index-definite (Assumption 5.7). In particular, it is globally trivialisable;*
- ***(Twist condition)*** *f is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$ satisfying the twist condition (Assumption 5.5), and whose fixed points are isolated;*

Then f admits infinitely many interior periodic points, of arbitrarily large order.

We will not address here the physics behind the statement, and why it should apply to the Spatial CR3BP – we leave the physical intuition for Part III. However, we remark that:

◇ **Remark 5.3.** In the case of the Spatial Circular Restricted Three-Body Problem, our Liouville domain of interest will be $W = \mathbb{D}^\star\mathbb{S}^2$, and we readily have $\dim SH^*(W) = \infty$ by [AS04]. Similarly, it is shown in [MK22a] that the index growth assumption holds in a certain range of {mass, energy} values called the *convexity range*.

However, at the time of writing this thesis, Moreno & van Koert's twist condition has still not been verified in the Spatial Circular Restricted Three-Body Problem. Actually, it has not even been proved in its simplest limit case: the Rotating Kepler Problem!

This will motivate us, in Chapters 6 and 7, to find a weakening of the twist condition, and show how we can also get rid of the other technical assumptions, thus significantly improving the applicability of Moreno & van Koert's theorem.

Before we proceed to weaken the assumptions in Moreno & van Koert's model though, let us generalise Theorem 5.2 to a different setting. Assume we have a Lagrangian L in our Liouville domain $(W, \omega = \mathrm{d}\lambda)$, satisfying the assumptions of Part I. Instead of periodic orbits of $H_t : W \to \mathbb{R}$ we will consider Hamiltonian chords, *i.e.* paths $x : [0,1] \to W$ such that

$$\dot{x}(t) = X_H\big(x(t)\big), \quad x(0), x(1) \in L. \tag{5.1}$$

Then, we shall prove the following theorem, which we first stated in [ML24]:

Theorem B1. *Let $(W, \omega = \mathrm{d}\lambda)$ be a connected Liouville domain, $L \subset W$ be an exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ be an exact symplectomorphism. Further assume that:*

- ***(Wrapped Floer Homology).*** $\dim HW^\bullet(L) = \infty$*;*
- ***(Index growth).*** *if $\dim W \geq 4$, then $c_1(W) = 0$, and $(\partial W, \alpha := \lambda|_{\partial W})$ is strongly index-definite (Assumption 5.7). In particular, it is globally trivialisable;*
- ***(Twist condition)*** *f is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$ satisfying the twist condition (Assumption 5.5), and whose fixed points are isolated.*

Then f admits infinitely many interior chords with respect to L, of arbitrarily large order, and which are not sub-chords of any periodic chord.

◇ **Remark** (Degeneracy)**.** Notice that this theorem does not make any assumptions on non-degeneracy. In particular, the Hamiltonian H_t generating f may very well be degenerate. This is the reason why we developed Local Wrapped Floer Cohomology in Chapter 4: so we would be able to take care of such cases.

In Part III of this thesis we will see how we can apply Theorem B1 (or rather, its improvements Theorem B2 and B3) to the Circular Restricted Three-Body Problem and deduce concrete statements about the physics. In the meantime though, before we prove Theorem B1, let us clarify on what we mean by the 'order of chords'.

Note that a length-1 Hamiltonian chord, as defined in (5.1), can alternatively be viewed as a point $p := x(0) \in L$ such that $f(p) \in L$. We call such a point a chord of order 1.

• **Definition 5.4.** A **chord of order** $m \in \mathbb{N}$ is a point $p \in L$ such that $f^m(p) \in L$. The chord is called periodic if $f^k(p) = p$ for some $k \in \mathbb{N}$.

We respectively call minimal order and minimal period the smallest such m and k.

5.1.2 Initial assumptions: Moreno & van Koert's model

In Theorem B1, we make the same assumptions as Moreno & van Koert in Theorem 5.2, namely:

Assumption 5.5 (**Twist Condition**)**.** $f : W \to W$ is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$ such that:

$$X_{H_t}|_{\partial W} = h_t \mathcal{R}_\alpha,$$

where $h_t > 0$ is a smooth function and $\mathcal{R}_\alpha$ is the Reeb vector field on $(\partial W, \alpha)$. In other words, we ask that the Hamiltonian and Reeb dynamics on the boundary be positive reparametrisations of each other.

◇ **Remark 5.6.** This is indeed a generalisation of Poincaré's original twist condition. Indeed, let $W = \mathbb{D}^\star\mathbb{S}^1$ be the annulus with its standard orientations. Then the boundary circles have opposite orientations, so that $X_{H_t}|_{\partial W} = h_t \mathcal{R}_\alpha$ with $h_t > 0 \iff f$ rotates the boundary components in opposite directions.

Assumption 5.7. $(\partial W, \alpha)$ is called **strongly index-definite**, in the sense of [MK22a], if there exists a global trivialisation ϵ of $T\partial W$, and constants $c > 0, d \in \mathbb{R}$ such that for any Reeb arc $x : [0,T] \to \partial W$:

$$|\mu_{\mathrm{CZ}}(x;\epsilon)| > cT + d,$$

where μ_{CZ} denotes the Conley-Zehnder index (see §3.2.4).

Then, with these two assumptions, we will have:

Proof sketch of Theorem B1 (Full proof in §5.3).

1. First, we need to extend H to a Hamiltonian $\widehat{H}$ on the Liouville completion $\widehat{W}$ (§5.2.1).
2. By studying the Hamiltonian chords of $\widehat{H}$, we can define $HW^*(L)$, the Wrapped Floer Cohomology of L. If $\widehat{H}$ is non-degenerate, then this is done using the standard Wrapped Floer constructions from Chapter 3. If H is *degenerate*, then this can be done using our Local Wrapped Floer theory from Chapter 4; see Remark 4.12.
3. We argue, by an ***index growth*** argument (§5.2.2), that we can ignore chords on $\widehat{W}\backslash\mathrm{int}(W)$ in our cohomology $HW^*(L)$. This is where Assumptions 5.5 and 5.7 are used.
4. Therefore, the assumption that $\dim HW^*(L) = \infty$ implies that there are infinitely many (homologically distinct) interior chords. □

We will combine all of these elements together, and carry out this proof rigorously in §5.3. First, we need to introduce the necessary ingredients. Hence without further ado:

5.2 Index growth

5.2.1 Extending our Hamiltonian from W to $\widehat{W}$

The first thing we need to do, if we are to assign a Wrapped Floer Cohomology to our Hamiltonian $H_t : W \to \mathbb{R}$, is to extend it to the Liouville completion $\widehat{W}$.

First, recall how $\widehat{W}$ is constructed (§3.1.1):
W being a Liouville domain means that ∂W is of restricted contact type, *i.e.* that it is transverse to the Liouville vector field. Hence, by flowing backwards along the latter, one can parametrise a neighbourhood $(1-\varepsilon, 1] \times \partial W$ of ∂W, with coordinate $r \in (1-\varepsilon, 1]$. Then, $\widehat{W}$ is constructed by smoothly gluing $[1, +\infty) \times \partial W$ along the boundary.

➢ Let us now construct a polynomial extension $\widehat{H}$ of H. First, note that in $(1-\varepsilon, 1] \times \partial W$, we can write $H = H(r,b,t)$ where b is the coordinate in ∂W.

Now Taylor expand H in the r-direction:

$$\text{For } r \in (1-\varepsilon, 1] : H(r,b,t) = H|_{\partial W} + (r-1)(\partial_r H)|_{\partial W} + \frac{(r-1)^2}{2!}(\partial_r^2 H)|_{\partial W} + \dots \quad (5.2)$$

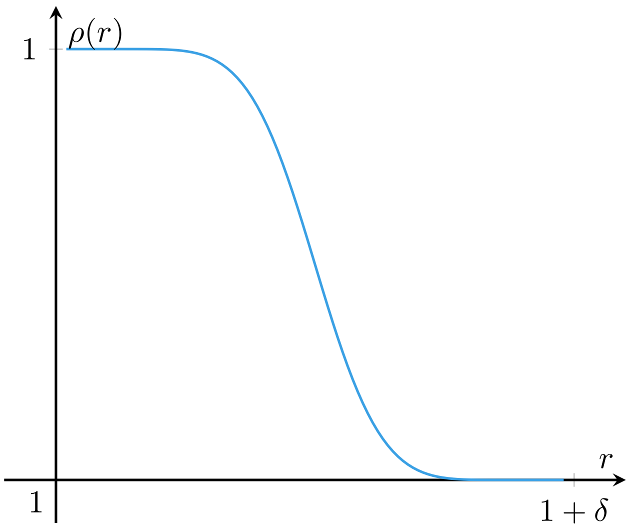
More concisely, write $H_i := (\partial_r^i H)|_{\partial W}$. Then

$$H(r,b,t) = \sum_{i=0}^{m} \frac{(r-1)^i}{i!} H_i(b,t) \ + R_m(r,b,t), \quad (5.3)$$

where the remainder R_m is $o\big((r-1)^m\big)$ for $r \sim 1$, *i.e.* :

$$\lim_{r \to 1} \frac{R_m(r,b,t)}{(r-1)^m} = 0.$$

Now choose a smooth cut-off function $\rho : [1, +\infty) \to \mathbb{R}$ such that $\begin{cases} \rho(1) = 1 \\ \rho(r) = 0, \text{ for } r \geq 1 + \delta > 1 \end{cases}$



An example of such a ρ is given in §6.2. Define:

- $\widehat{H}_i(r, b, t) := \rho(r) H_i + \big(1 - \rho(r)\big) C_i \quad$ for some $C_i \geq \max_{\partial W} H_i$,
- $\widehat{R}_m(r, b, t) := \rho(r) R_m(r)$.

◇ **Remark 5.8.** Formally, before we write $\widehat{R}_m = \rho(r) R_m$, we need to extend R_m to the collar $[1, +\infty) \times \partial W$. A method for doing this is given in [See64].

• **Definition 5.9.** For any $m \in \mathbb{R}$, we define $\widehat{H}$ such that:

$$\begin{cases} \text{On } W: & \widehat{H} \equiv H \\ \text{On } [1, +\infty) \times \partial W: & \widehat{H}(r, b, t) = \displaystyle\sum_{i=0}^{m} \frac{(r-1)^i}{i!} \widehat{H}_i + \widehat{R}_m \end{cases}$$

This defines a smooth extension of $H : W \to \mathbb{R}$ to the whole of $\widehat{W}$, with the feature that $\widehat{H}$ is polynomial at infinity. Indeed:

$$\forall r \geq 1 + \delta : H(r, b, t) = C_0 + C_1(r-1) + \cdots + C_m(r-1)^m.$$

Note that we can choose the coefficients C_i to be anything we want, as long as $C_i \geq \max H_i$, and we can control the speed at which $\widehat{H}$ becomes polynomial by modifying δ; two features we will exploit quite a lot in the next few chapters.

➤ In summary, we have found a way to artificially extend our original Hamiltonian H on W (which may come from a physical problem, say the CR3BP) to a Hamiltonian $\widehat{H}$ on $\widehat{W}$. This allows us to now construct Symplectic or Wrapped Floer Cohomology.

⚠ **Problem:** when constructing our extension $\widehat{H}$, we made *choices* (in particular, the cut-off function ρ). These choices may cause the appearance of *undesirable Hamiltonian chords* on $[1, \infty) \times \partial W$. We call them undesirable because these chords have no physical relevance, and are simply artifacts of our extension process.

As of now, we do not know of a method for distinguishing physically relevant trajectories from undesirable ones. This is the reason why Theorem B1 is only concerned with interior chords: so far, our only way about the issue is to ignore all chords on $[1, +\infty) \times \partial W$ altogether.

5.2.2 Index growth

As discussed at the very end of the last subsection, our extension $\widehat{H}_t : \widehat{W} \to \mathbb{R}$ presents one issue: it may have undesirable chords on the extension $[1, +\infty) \times \partial W$. To fix this, we will impose additional assumptions (the twist condition + index growth) to ensure that our homological tools ignore the dynamics on $[1, +\infty) \times \partial W$, and only see interior dynamics.

Namely, we prove:

Proposition 5.10 (Index growth [MK22a]). *Let $(W, \omega = d\lambda)$ be a Liouville domain whose boundary $(\partial W, \alpha)$ is strongly index-definite, and $H_t : W \to \mathbb{R}$ be a Hamiltonian satisfying the twist condition. Then, the flow of the polynomial extension $\widehat{H}$ from Definition 5.9 is strongly index-definite on $[1, 1+\delta) \times \partial W$, for any degree m, after potentially shrinking $\delta > 0$.*

This was originally proved in [MK22a], for $m = 1$. We will show in this section that their proof also applies to polynomial extensions of degree $m > 1$. Note that if $m = 1$, then the flow being strongly index-definite on $[1, 1+\delta) \times \partial W$ means that it is on the whole of $[1, +\infty) \times \partial W$, since by Lemma 3.37 there are no chords on $[1+\delta, +\infty) \times \partial W$.

Before we proceed with the proof, let us show why it does the job that we intended:

Corollary 5.11. *Using Proposition 5.10, we can homologically separate interior chords from the ones on $[1, +\infty) \times \partial W$, and thus ignore the latter.*

Proof sketch. This is only a proof sketch, because this argument is the core of the proof of Theorem B1 (see §5.3). Essentially, the idea is that since the flow of $\widehat{H}$ is strongly index-definite on $[1, +\infty) \times \partial W$, we have

$$|\mu_{CZ}(x; \epsilon)| > cT + d, \tag{5.4}$$

for every Hamiltonian arc $x : [0, T] \to [1, +\infty) \times \partial W$; for some $c > 0, d \in \mathbb{R}$.

Meanwhile, recall that Wrapped Floer Cohomology $HW^*(L)$ is defined as

$$HW^*(L) = \varinjlim_n HW^*(L; J, H_n), \tag{5.5}$$

where (H_n) is any sequence of Hamiltonians which are linear at infinity, with increasing slopes going to ∞. In particular, we can define

$$HW^*(L) = \varinjlim_n HW^*(L; J, \widehat{H}^{\#n}),$$

where $\widehat{H}^{\#n}$ is the n-th iterate of $\widehat{H}$ (Definition A.3). Hamiltonian chords of length 1 of $\widehat{H}^{\#n}$ correspond to Hamiltonian chords of length n of $\widehat{H}$. In particular, by (5.4), any such chord x will satisfy

$$\mu_{CZ}(x; \epsilon) > cn + d.$$

As we travel further and further along the direct limit, the index of chords on the extension $[1, +\infty) \times \partial W$ hence blows up to infinity, and so these chords appear extremely high in the grading. This will allow us, in Theorem B1, to ignore them, and show that $\dim HW^*(L)$ being infinite implies the existence of infinitely many *interior* chords. □

This phenomenon, first observed in [MK22a] in the case of symplectic homology, is called **index growth**, because it relies on the fact that the indices of chords blow up to infinity. As we shall see, its proof requires both the twist condition (as stated by Moreno & van Koert), and the index growth assumption for $(\partial W, \alpha)$.

◇ **Remark 5.12.** In Chapter 6 of this thesis, we show that the index growth argument can be replaced by an « *action growth* » argument – actions of chords on the extension blow up to ∞. This argument will actually rely on weaker assumptions: we will no longer need index growth of $(\partial W, \alpha)$, and we will be able to replace the twist condition by the *Weakened Twist Condition*. Moreover, this new setup will be better suited to the study of degenerate Liouville domains (Definition 7.1), making it more applicable to concrete examples like the CR3BP.

Without further ado, let us proceed with the proof of Proposition 5.10. It relies on one linear algebra lemma, from Appendix E of [MK22a]:

Lemma 5.13 ([MK22a]). *Let $\dot{\psi}(t) = A_0(t)\psi(t)$ be a strongly index-definite system of ODEs, where $A_0(t) \in \mathfrak{sp}_{2n-2}(\mathbb{R})$. Let $A(t) \in \mathfrak{sp}_{2n}(\mathbb{R})$ be of the form form:*

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{A_0(t)} & J\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & a(t) & 0 \\ U^t(t) & V^t(t) & b(t) & -a(t) \end{array}\right).$$

Then, the system $\dot{\psi}(t) = A(t)\psi(t)$ is also strongly index-definite.

<u>*Proof of Lemma 5.13.*</u> See Lemma E.2 of [MK22a].

The proof of Proposition 5.10 then relies on studying the system $\dot{\psi}(t) = \nabla X_{\widehat{H}}\psi(t)$, which is the linearisation of the Hamiltonian flow. We will show that, asymptotically near the boundary, $\nabla X_{\widehat{H}}$ can be put in the appropriate form for Lemma 5.13, thanks to the twist condition. In particular, the top-left block will correspond to the Reeb flow on $(\partial W, \alpha)$, which is strongly index-definite by assumption, allowing us to conclude.

<u>*Proof of Proposition 5.10.*</u> The proof contains five steps:

1. Computing $X_{\widehat{H}}$.
2. Computing $\nabla X_{\widehat{H}}$.
3. Showing that the matrix $\nabla X_{\widehat{H}}$ can be put in the form $L_0 + (r-1)L_1$.
   Consequently, if $\delta \ll 1$, then $\nabla X_{\widehat{H}} \sim L_0$.
4. Proving that $L_0, L_1 \in \mathfrak{sp}_{2n}$, and that L_0 is in the right form for Lemma 5.13.
5. Using Lemma 5.13 to show that the system of ODEs $\dot{\psi} = L_0\psi$ is strongly index-definite. Since the Conley-Zehnder index measures a crossing number for solutions of $\dot{\psi}(t) = A(t)\psi(t)$, they remain close-by given two matrices $A(t)$ and $A'(t)$ asymptotically close (see Lemma 2.2.9 of [Ust99] for a proof), which will conclude.

Steps $1-3$ are purely computational, and we therefore relegate them to Calculation C.12 of the Computational appendix. They yield:

$$X_{\widehat{H}} = F\mathcal{R}_\alpha - (r-1)G\partial_r + (r-1)X^\xi \tag{5.6}$$

$$\nabla X_{\widehat{H}} = \mathrm{d}F \otimes \mathcal{R}_\alpha + F\nabla\mathcal{R}_\alpha + \mathrm{d}r \otimes (X^\xi - G\partial_r) + (r-1)\left(\nabla X^\xi - \mathrm{d}G \otimes \partial_r\right) \tag{5.7}$$

$$= \underbrace{\left(\begin{array}{c|cc} F\nabla^\xi\mathcal{R}_\alpha & X^\xi & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 \quad 0 & a & 0 \\ \mathrm{d}F|_\xi & b & c \end{array}\right)}_{L_0} + (r-1) \underbrace{\left(\begin{array}{c|cc} \nabla^\xi X^\xi & \begin{matrix} 0 \\ 0 \end{matrix} & \nabla_{\mathcal{R}_\alpha} X^\xi|_\xi \\ \hline -\mathrm{d}^\xi G & 0 & a' \\ 0 \quad 0 & 0 & 0 \end{array}\right)}_{L_1} \tag{5.8}$$

where the matrix expressions are given in the frame $\hat{\xi} \cup \{\mathcal{R}_\alpha, \partial_r\}$, with $\hat{\xi}$ a frame for the contact structure ξ; and where:

$$F = \partial_r \widehat{H}_0 + \left(\sum_{i=1}^{m} \frac{(r-1)^{i-1}}{(i-1)!} \widehat{H}_i + \frac{(r-1)^i}{i!} \partial_r \widehat{H}_i \right) + \frac{(r-1)^m}{m!} \widehat{R} + \frac{(r-1)^{m+1}}{(m+1)!} \partial_r \widehat{R}$$
$$G := \sum_{i=1}^{m} \frac{(r-1)^{i-1}}{i!} \mathrm{d}\widehat{H}_i(\mathcal{R}_\alpha) + \frac{(r-1)^m}{(m+1)!} \mathrm{d}\widehat{R}(\mathcal{R}_\alpha)$$
$$X^\xi := \sum_{i=1}^{m} \frac{(r-1)^{i-1}}{i!} X^\xi_{\widehat{H}_i} + \frac{(r-1)^m}{(m+1)!} X^\xi_{\widehat{R}}$$

◇ **Remark 5.14.** Note that Step 1 of the proof (see Calculation C.12) is the only place where we make explicit use of the twist condition; in order to show that G has no term containing $\mathrm{d}\widehat{H}_0$. This use of the twist condition is marked by the symbol ⚠.

Meanwhile, the strong index-definiteness assumption for $(\partial W, \alpha)$ only comes in in Step 5, when we use the recursive argument from Lemma 5.13.

Step 4. The proof of Step 4, also relegated to Calculation C.12 of the Computational Appendix, relies on heavy but elementary linear algebra. We first show that $L_1 \in \mathfrak{sp}_{2n}$, which is equivalent to showing that JL_1 is symmetric. The tricky part of this is showing that $-\mathrm{d}^\xi G$ and $\nabla_{\mathcal{R}_\alpha} X^\xi|_\xi$ are dual to each other. This implicitly relies on the **twist condition**, since it uses the expression for G found in Step 1, and the fact that there is no term in $\mathrm{d}\widehat{H}_0$ (see Remark 5.14).

Then, we argue that since $\nabla X_{\widehat{H}}$ is the linearisation of a Hamiltonian flow we have $\nabla X_{\widehat{H}} \in \mathfrak{sp}_{2n}$, from which we deduce that

$$L_0 = \nabla X_{\widehat{H}} - (r-1)L_1 \in \mathfrak{sp}_{2n}.$$

In particular, this implies that JL_0 is symmetric, so that L_0 must have the form:

$$L_0 = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{F\nabla^\xi \mathcal{R}_\alpha} & J\begin{pmatrix} U \\ V \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & a & 0 \\ U^t & V^t & b & -a \end{array} \right).$$

Step 5 (Conclusion). Recall that we want to show that the Hamiltonian flow of $\widehat{H}$ on $[1, +\infty) \times \partial W$, whose linearisation is given by

$$\dot{\psi}(t) = \nabla X_{\widehat{H}}\, \psi(t),$$

is strongly index-definite. From Step 3 we have $\nabla X_{\widehat{H}} = L_0 + (r-1)L_1$. If we choose the cut-off function ρ from Definition 5.9 to be fast enough (*i.e.* $\delta \ll 1$), then we have $\nabla X_{\widehat{H}} \sim L_0$ on $[1, 1+\delta) \times \partial W$.

Meanwhile, the top-left block of L_0 is strongly index-definite (it corresponds to the linearisation of the Reeb flow, which is strongly index-definite by assumption), so that by Lemma 5.13, the system $\dot{\psi}(t) = L_0 \psi(t)$ is also strongly index-definite.

Since $\nabla X_{\widehat{H}}$ and L_0 are asymptotically close to each other, then so are the Conley-Zehnder indices of the systems

$$\dot{\psi}(t) = \nabla X_{\widehat{H}} \psi(t),$$
$$\dot{\psi}(t) = L_0 \psi(t),$$

because the Conley-Zehnder index is continuous (see Lemma 2.2.9 of [Ust99]). Therefore, the flow of $\widehat{H}$ on $[1, 1+\delta) \times \partial W$ is also strongly index-definite. □

This concludes the proof of Proposition 5.10, and hence of our discussion on the « index growth phenomenon ». Let us now introduce the last ingredients for the proof of Theorem B1.

5.2.3 The support and the mean index

Before we proceed with the proof of Theorem B1, we need two last ingredients. First:

• **Definition 5.15.** The **support** $HW^*(L)$ is defined as the discrete subset:

$$\operatorname{supp} HW^*(L) := \{n \mid HW^n(L) \neq 0\} \subset \mathbb{Z}.$$

In other words, it is the set of degrees in which $HW^*(L)$ is non-trivial.

Lemma 5.16. *Assume $HW^*(L)$ is infinite-dimensional, and $(\partial W, \alpha)$ is strongly index-definite. Then, $|\operatorname{supp} HW^*(L)| = \infty$.*

Proof. Recall that $HW^*(L)$ can be defined as a direct limit over a sequence of linear Hamiltonians (§3.4). Pick $H : \widehat{W} \to \mathbb{R}$ linear at infinity and non-degenerate, and define $H^{\#n}$ to be its n-th iterate. Then, we can write:

$$HW^*(L) = \varinjlim_n HL^*(L; H^{\#n}).$$

Since $H^{\#n}$ is linear at infinity, then by Corollary 3.45 it has finitely many chords, so that there are finitely many generators in $CF^*(L; H^{\#n})$. Since $\dim HW^*(L) = \infty$, then that means new chords appear arbitrarily late in the direct limit. However, since chords $[0,1] \to \widehat{W}$ of $H^{\#n}$ are the same thing as trajectories $[0,n] \to \widehat{W}$ of H, then, by the strong index-definiteness condition, these chords have indices:

$$|\mu_{\mathrm{CZ}}| \geq c \cdot n + d.$$

Since μ_{CZ} gives the grading on wrapped Floer cohomology, this implies that $HW^*(L)$ has Floer generators of arbitrarily large degree, which verifies the claim. □

➢ Now, notice that we can also define the support of *local* cohomology of a chord x:

$$\operatorname{supp}\ HW_{\mathrm{loc}}(x) := \{i \in \mathbb{Z} \mid HW^i_{\mathrm{loc}}(x) \neq 0\}.$$

In particular, the last lemma we will need before the proof of Theorem B1 will be that, for any x, $\operatorname{supp} HW^*_{\mathrm{loc}}(x)$ is bounded. Even better: we can show that all the non-vanishing degrees of $HW^*_{\mathrm{loc}}(x)$ are concentrated around a specific value: the **mean index** of x.

◇ **Remark 5.17.** Note that $\dim HW^i_{\mathrm{loc}}(x)$ being non-zero implies that there exist arbitrarily small perturbations of x with index i. So this lemma will show that, under small perturbations of the Hamiltonian, the Conley-Zehnder index of a chord does not stray too far.

The mean index is a different type of index than the Conley-Zehnder one (which is, by definition, a homotopy invariant of paths in $Sp(2n)$; see §3.2.4). We follow the introduction of the mean index by Ginzburg and Gurel in [Gin10; GG15].

• **Definition 5.18.** A map $F : G \to G'$ between Lie groups is said to be a quasimorphism if

$$\forall\, \psi, \phi \in G : \ |F(\psi\phi) - F(\psi) - F(\phi)| < C,$$

where C is a constant that depends only on G and G'.

• **Definition 5.19.** Let $\widetilde{Sp(2n)}$ be the universal cover of the symplectic group $Sp(2n)$. Then, one can prove ([GG15]) that there exists a unique quasimorphism $\Delta : \widetilde{Sp(2n)} \longrightarrow \mathbb{R}$ satisfying:

1. Let Φ be a symplectic path. For any close enough non-degenerate perturbation $\tilde{\Phi}$ of Φ, we have $|\Delta(\Phi) - \mu_{\mathrm{CZ}}(\tilde{\Phi})| \leq n$.

2. Assume Φ is non-degenerate. Then $\lim_{T\to\infty} \frac{\mu_{\mathrm{CZ}}(\Phi|_{[0,T]})}{T} = \Delta(\Phi)$.

3. Assume $\Phi = \gamma$ is a *loop*, and write γ^k its k-iteration. Then, $\Delta(\gamma^k) = k\Delta(\gamma)$.

We call Δ the **mean index**.

We refer to §3 of [GG15] for a more thorough exposition, and interpretation of the mean index. We can now prove our claim that the local Wrapped Floer cohomology of a chord is concentrated around its mean index:

Lemma 5.20. *Let H be some $\mathcal{C}^2$ Hamiltonian, and x a Hamiltonian chord. Then:*

$$\operatorname{supp} HW_{loc}(x) \subset [\Delta(x) - n, \Delta(x) + n],$$

where $2n$ is the dimension of the ambient manifold.

Proof. This directly follows from property (1) of Definition 5.19. □

5.3 Proof of Theorem B1

For the rest of this chapter, we fix $(W, \omega = \mathrm{d}\lambda)$ our connected Liouville domain (with strongly index-definite boundary – Assumption 5.7), $L \subset W$ our exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ our exact symplectomorphism, generated by a Hamiltonian H satisfying the twist condition (Assumption 5.5). We construct a linear extension $\widehat{H}$ of H like in Definition 5.9, *i.e.*

$$\exists \delta > 0 \text{ such that, for } r \geq 1 + \delta, H = H(r) = ar + b \quad (a > 0, b \in \mathbb{R}).$$

Given $p \in \mathbb{N}\backslash\{0\}$, we write $\widehat{H}^{\#p}$ its p-th iteration, *i.e.* the Hamiltonian s.t. $\phi^{t=1}_{\widehat{H}^{\#p}} = \phi^{t=p}_{\widehat{H}}$.

◇ **Remark 5.21.** While a Hamiltonian chord $\gamma : [0,1] \to \widehat{W}$ of $\widehat{H}^{\#p}$ is by definition a trajectory $\tilde{\gamma} : [0,p] \to \widehat{W}$ of $\widehat{H}$ with ends in L, there is no reason to assume that the latter is constituted of consecutive 1-chords of $\widehat{H}$. In fact, Lemma 8.2 of [AS10] tells us that for a generic Hamiltonian, end points of chords are never the starting points of other chords. This shows that generically, chords of f do not have sub-chords; and that their minimal order is always equal to their order, unless they are periodic.

However, recall once again that our inspiration is the Three-Body Problem, where our Hamiltonian might very well be degenerate. Therefore we cannot afford to make genericity assumptions.

Proof of Theorem B1. By assumption, f has finitely many interior periodic chords. Write:

- $c_1, \ldots, c_k$ the interior periodic chords of period 1.
- $d_1, \ldots, d_q$ the interior periodic chords of minimal periods $n_1, \ldots, n_q$, with $n_i > 1$.

Assume for a contradiction that f has finitely many interior chords which are not periodic chords, nor sub-chords of periodic chords. Write them $x_1, \ldots, x_\ell$, and let $m_1, \ldots, m_\ell$ denote their minimal orders.

Let H_t be the Hamiltonian generating f (satisfying the twist condition), and $\widehat{H}$ be a linear extension of H_t to $\widehat{W}$ (as provided by Definition 5.9). Then, recall that we have:

$$HW^*(L) = \varinjlim_i HW^*(L; \widehat{H}^{\#p_i}), \tag{5.9}$$

where we choose (p_i) to be a sequence of primes going to ∞, such that $p_i > \max_{j,k}\{n_j, m_k\}$; and where $\widehat{H}^{\#p_i}$ denotes the p_i-th iterate of $\widehat{H}$.

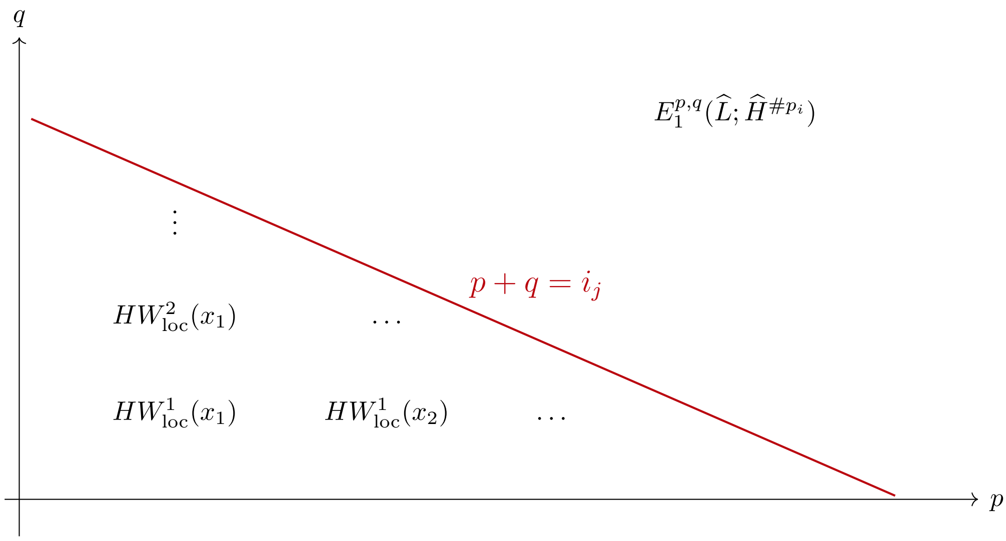
Indeed, all these cohomologies are well-defined, because even if $\widehat{H}$ (or one of its iterates) were degenerate, we could appeal to the results from Chapter 4 (*i.e.* local Floer cohomologies, and the local-to-global spectral sequence) to ensure that each $HW^*(L;\widehat{H}^{\#p_i})$, and *a fortiori* $HW^*(L)$, are well-defined.

Pick $N > 2nk$ (where $2n = \dim W$). By Lemma 5.16, $HW^*(L)$ is non-zero in infinitely many degrees. Therefore we can find $i_1, \dots, i_N$, ordered by increasing absolute value, and such that $HW^{i_j}(\widehat{L}) \neq 0$. Combining this with Proposition 5.10, we may choose i sufficiently large such that the following hold:

(1) Each chord of $\widehat{H}^{\#p_i}$ that is contained in $\widehat{W} \setminus \mathrm{int}(W)$ has a Conley-Zehnder index whose absolute value is larger than $|i_N| + 2n$;

(2) the Floer cohomology groups $HL^{i_j}(\widehat{L}, \widehat{H}^{\#p_i})$ are non-trivial for $j = 1, \dots, N$.

Now look at the E_1 page of the spectral sequence associated to $\widehat{H}^{\#p_i}$, from Theorem 4.13.



From (2) we deduce that there must be non-trivial summands with $p + q = i_j$. From (1) we know that no chord in $\widehat{W} \setminus \mathrm{int}(W)$ can contribute to local Floer homology of degree i_j, since their Conley-Zehnder indices are too large. Therefore, each of these non-trivial summands must come from the local wrapped Floer homology of a chord in $\mathrm{int}(W)$.

By assumption though, there are only finitely many such interior chords:

- $c_1, \dots, c_k$, interior periodic chords of period 1;
- $d_1, \dots, d_q$, interior periodic chords of periods $n_1, \dots, n_q > 1$;
- $x_1, \dots, x_\ell$, interior chords which are neither periodic, nor sub-chords of periodic chords, with minimal orders $m_1, \dots, m_\ell$.

We have chosen the sequence of primes (p_i) such that $p_i > \max_{j,k}\{n_j, m_k\}$. Therefore, the chords $d_1, \dots, d_q, x_1, \dots, x_\ell$ cannot contribute to the cohomology $HW^{i_j}(\widehat{L};\widehat{H}^{\#p_i})$, and nor can the iterates of the d_i's (since $n_j \not| p_i$).

Therefore, the chords contributing towards $HW^{i_j}(\widehat{L};\widehat{H}^{\#p_i})$ must necessarily be iterates $c_j^{p_i}$ of interior periodic chords c_j of period 1. However, by Lemma 5.20, we get:

$$\begin{aligned}\operatorname{supp} HW^*_{\mathrm{loc}}(c_j^{p_i}) &\subset [\Delta(c_j^{p_i}) - n, \Delta(c_j^{p_i}) + n] \\ &= [p_i\Delta(c_j) - n, p_i\Delta(c_j) + n] \quad \text{by 3. of Defn. 5.19.}\end{aligned}$$

For each chord c_j, this support ranges over $2n$ degrees, and there are k periodic chords of period 1 in total. Therefore, this covers at most $2nk$ different degrees, leaving some of the degrees $i_1, \dots, i_N$ uncovered since we had chosen $N > 2nk$. This yields a contradiction.

Therefore, there must exist infinitely many interior chords which are not sub-chords of any periodic chord; and they must have arbitrarily large degrees. $\square$

Chapter 6

The Weakened Twist Condition

6.1 Weakening the twist condition

6.1.1 Why weaken the twist condition?

The twist condition is the main assumption in Moreno & van Koert's model, and as such remains the main obstruction to applying Theorems 5.2 and B1 to the Three-Body Problem. Indeed, as pointed out in Remark 5.3, the twist condition still hasn't been verified – not even in the Rotating Kepler Problem. In a way, this is not so surprising. Indeed, consider a connected Liouville domain $(W, \omega = \mathrm{d}\lambda)$. The twist condition asks that we find a Hamiltonian $H_t : W \to \mathbb{R}$ whose Hamiltonian vector field satisfies

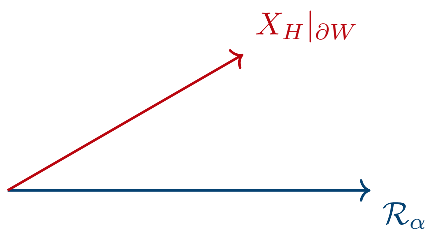
$$X_{H_t} = h_t \mathcal{R}_\alpha \quad (h_t > 0 \text{ smooth}),$$

where $\mathcal{R}_\alpha$ is the Reeb vector field on $(\partial W, \alpha := \lambda|_{\partial W})$. This is an incredibly constraining assumption on the Hamiltonian; and therefore it is little surprise that we have not been able to verify it in practice.

In the next two chapters, we shall prove that we can replace the twist condition by the *Weakened Twist Condition*, which only asks that

$$\langle X_{H_t}, \mathcal{R}_\alpha \rangle > 0.$$

In other words, instead of trying to find a Hamiltonian whose vector field is collinear with the Reeb vector field along the boundary, we only need to ensure that the two vector fields "point roughly in the same direction"[(1)].



However, we are jumping ahead. This version of the Weakened Twist Condition will only be the content of Theorem B3. For now, let us prove the intermediary result:

Theorem B2. *Let $(W, \omega = \mathrm{d}\lambda)$ be a connected Liouville domain, $L \subset W$ be an exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ be an exact symplectomorphism. Further assume that:*

- ***(Wrapped Floer Homology).*** *$HW^*(L) \neq 0$ in infinitely many degrees;*
- ***(Chern class).*** *if $\dim W \geq 4$, then $c_1(W) = 0$.*

[(1)]The metric $\langle \cdot, \cdot \rangle$ used here is not arbitrary. It is the one given by $\mathrm{d}\alpha(\cdot, J\dot{}) + \alpha \otimes \alpha$.

- ***(Quantitative Weakened Twist Condition)*** *f is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$ satisfying the quantitative weakened twist condition (Assumption 6.2), and whose fixed points are isolated;*

Then f admits infinitely many interior chords with respect to L, of arbitrarily large order, and which are not sub-chords of any periodic chord.

◇ **Remark 6.1** (On assumptions)**.** This intermediary statement relies on a 'Quantitative Weakened Twist Condition', stated right below. It is not the definitive version we are looking for, which we can apply to the Three-Body Problem. However, it already constitutes a significant improvement of Moreno & van Koert's twist condition.

Besides weakening the twist condition (which is now open), observe that we no longer need to assume the strong index-definiteness assumption, and therefore that the contact structure on ∂W is trivialisable, which is a major improvement for applicability of this theorem. We still require $\mathcal{C}^2$ regularity of our Hamiltonian because we need to have a well-defined grading.

The mild trade-off is that we now need to assume that $HW^*(L)$ is non-zero in infinitely many degrees, whereas Theorem B1 only assumed that it was infinite-dimensional. In all our cases of interest from Part III, this won't matter.

Assumption 6.2 (**Quantitative Weakened Twist Condition**)**.** Let $(W, \omega = \mathrm{d}\lambda)$ be a connected Liouville domain and $f : W \to W$ a diffeomorphism. f is said to satisfy the Weakened Twist Condition if it can be generated as the time 1-map of a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$ such that:

(1) $H_t|_{\partial W} > 0$;

(2) $\min_{\partial W} \partial_r H_t > \max_{\partial W} H_t$.

Note that, because of our choice of complex structure J (such that $J\partial_r = \mathcal{R}_\alpha$), we have $(\partial_r H_t)|_{\partial W} = \alpha(X_{H_t}) = \langle X_{H_t}, \mathcal{R}_\alpha \rangle$, so that we can rewrite:

(2) $\min_{\partial W} \langle X_H|_{\partial W}, \mathcal{R}_\alpha \rangle > \max_{\partial W} H$.

Note that (1) can always be achieved by shifting H_t by a constant, so that (2) is the main part of the assumption. So in other words, we no longer require that X_{H_t} and $\mathcal{R}_\alpha$ be collinear, but that their inner product be large enough.

6.1.2 *How* to weaken the twist condition?

So in short, our goal is to rewrite Theorems 5.2 and B1, but replacing both the twist condition and strong index-definiteness assumption by the Quantitative Weakened Twist Condition. We will present the argument for adapting Theorem B1 here, and refer to [LM25] for Theorem 5.2 (the proofs are almost exactly the same, replacing Wrapped Floer by Symplectic Cohomology).

➢ **Question:** where in the proof of Theorem B1 were Assumptions 5.5 and 5.7 used?

Short answer: in the « index growth » argument (§5.2.2). This allowed to separate chords/orbits in the interior of W from chords/orbits on the extension $[1, +\infty) \times \partial W$, in order to ignore the latter in our local-to-global spectral sequence. The reason we did this is because there may be undesirable chords in $[1, +\infty) \times \partial W$ which are solely artifacts of our extension process, and have no physical relevance.

Long answer: The twist condition was used in Step 1 of the proof of Proposition 5.10 (the Index Growth proposition), when computing the r-derivative of H at the boundary.

Indeed, recall from Definition 3.8 that all along this thesis, we choose an almost complex structure J on $\widehat{W}$ such that $J\partial_r = \mathcal{R}_\alpha$, where $V = r\partial_r$ is the Liouville vector field near $(\partial W, \alpha)$. By definition, $X_H = J\nabla H$, so that the twist condition

$$X_H|_{\partial W} = h_t \mathcal{R}_\alpha, \;\; h_t > 0 \text{ smooth,}$$

can be rewritten

$$\nabla H \equiv (\partial_r H)|_{\partial W}\partial_r, \ \ (\partial_r H)|_{\partial W} > 0 \text{ smooth.}$$

This is the form in which we actually use the twist condition in the proof of the index growth argument (Proposition 5.10), and it is essential to the proof: because the ultimate goal of the argument is to put the linearised Hamiltonian flow $\dot{\psi}(t) = \nabla X_{\widehat{H}}\psi(t)$ of an extension $\widehat{H}$ of H in a nice form. The twist condition allows to kill many off-diagonal terms in $\nabla X_{\widehat{H}}$.

Then, once this matrix has been put in a nice form, one uses the strong index-definiteness assumption along with an recursive linear algebra argument to conclude the proof.

➤ So in summary, the index growth argument was what the reason we needed the twist condition and strong index-definiteness assumption in B1. In a nutshell, we ignored trajectories on the collar $[1, +\infty) \times \partial W$ was by showing they satisfied

$$\mu > cT + d,$$

where μ the Conley-Zehnder index, T the period of a trajectory, and $c > 0, d \in \mathbb{R}$ constants independent of the trajectory.

Let us now see how we could replace this index growth argument.

6.2 Action growth

We propose, instead of separating interior chords from those on the completion via index, to do it via *action*. In other words, we show:

Theorem 6.3 (Action growth)**.** *Let $(W, \omega = \mathrm{d}\lambda)$ be a connected Liouville domain, and $H_t : W \to \mathbb{R}$ satisfy the Quantitative Weakened Twist Condition (Assumption 6.2). Then we can construct an extension $\widehat{H}$ of H such that, at infinity, $\widehat{H}$ looks like:*

$$\widehat{H} = ar - \varepsilon, \tag{6.1}$$

for $a, \varepsilon > 0$ satisfying (6.3) and (6.4), and there exist constants $c > 0, d \in \mathbb{R}$ such that for every trajectory $x : [0, T] \to [1, +\infty) \times \partial W$ of the flow of $\widehat{H}$, we have:

$$\mathcal{A}_{\widehat{H}}(x) < -c \cdot T + d. \tag{6.2}$$

◇ **Remark 6.4** (On a and ε)**.** We have quite a decent amount of control over the constants a and ε in the expression of $\widehat{H}$. The only constraints are that a must be sufficiently big, and ε sufficiently small. More precisely, need:

$$a \geq \max_{\partial W} \partial_r H \tag{6.3}$$

$$0 < \varepsilon < \min_{\partial W} \partial_r H - \max_{\partial W} H \tag{6.4}$$

In particular, we can choose $\widehat{H}$ to be arbitrarily close to the Hamiltonian ar – though then, as we shall see in the proof, the action growth phenomenon will become arbitrarily slow.

◇ **Remark 6.5.** As we shall see in the proof of Theorem B2, this action growth argument completely replaces the index growth argument from Theorem B1. Indeed, instead of ignoring chords on $[1, +\infty) \times \partial W$ because their indices blow up to infinity, we will ignore them because their *actions* blow up to $-\infty$.

Proof of Theorem 6.3. The proof will essentially consist in brute-forcing some asymptotical estimates out of the action functional $\mathcal{A}_{\widehat{H}}$.

Step 1 (Simplifying the statement).
Recall that we are working with Hamiltonian chords with ends in an exact Lagrangian $L \subset W$ (exactness means that $\lambda|_L = \mathrm{d}f$ for some function f). We complete W to a Liouville manifold $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$, with $\hat{\lambda} = r\alpha = r\lambda|_{\partial W}$ on the completion $[1, +\infty) \times \partial W$, and we also complete L to $\widehat{L}$ (which is still exact Lagrangian by Lemma 9.4).

Then, from Chapter 3, the action functional of $\widehat{H}$ on $\widehat{W}$ is defined as

$$\mathcal{A}_{\widehat{H}}(x) := f(x(1)) - f(x(0)) - \int_0^T x^\star \hat{\lambda} + \int_0^T \widehat{H}(x(t))\mathrm{d}t$$

for a Hamiltonian chord $x : [0, T] \to \widehat{W}$. By our extension process $H \to \widehat{H}$ (which we recall in the next step), we have that outside of $W \cup_{\partial W} [1, 1+\delta) \times \partial W$, the Hamiltonian $\widehat{H}$ is linear. Therefore, by Lemma 3.37, it has no Hamiltonian chords there, and hence it suffices to prove action growth in $[1, 1+\delta) \times \partial W$.

In this window, the function f is bounded, so that it actually suffices to prove action growth for the functional defined by:

$$\mathcal{A}(x) := -\int_0^T x^\star \hat{\lambda} + \int_0^T \widehat{H}(x(t))\mathrm{d}t.$$

In particular, observe that $\mathcal{A}$ is exactly the action functional for symplectic cohomology (up to replacing $[0, T]$ by $\mathbb{R}/T\mathbb{Z}$), which means that our proof in the Wrapped Floer cohomology case directly carries over to symplectic cohomology.

Step 2 (Expanding $\mathcal{A}$).
Let us now expand the expression for $\mathcal{A}$. Recall from Definition 5.9 that the linear extension $\widehat{H}$ of H is defined on $[1, +\infty) \times W$ as:

$$\forall r \geq 1 : \widehat{H}(r, b, t) := \widehat{H}_0(r, b, t) + (r-1)\widehat{H}_1(r, b, t) + \frac{(r-1)^2}{2!}\widehat{R}_1(r, b, t), \tag{6.5}$$

where b is the coordinate on ∂W, and $H_i := (\partial_r^i H)|_{\partial W}$. We then choose a small $\delta > 0$, and a cut-off function ρ such that

$$\begin{cases} \rho(1) = 1 \\ \rho(r) = 0 \quad \text{for } r \geq 1 + \delta \end{cases}$$

and we define:

$$\begin{aligned} \widehat{H}_i(r, b, t) &:= \rho(r)H_i(b, t) + (1 - \rho(r))C_i, \\ \widehat{R}(r, b, t) &:= \rho(r)\sum_{i \geq 2} \frac{(r-1)^i}{i!}\widehat{H}_i, \end{aligned}$$

where the numbers $C_i > 0$ are constants. The point is that, for $r \geq 1 + \delta$, we have:

$$\widehat{H} = C_0 + C_1(r-1).$$

For now, we impose no assumption on the C_i's, except that $C_i \geq \max_{\partial W} H_i$, like in Definition 5.9, and that $C_1 \notin \operatorname{spec}\alpha$ (which can always be achieved by a perturbation), so that $\widehat{H}$ has no Hamiltonian chords on $[1+\delta, \infty) \times \partial W$, by Lemma 3.37.

From Step 1, we want to prove action growth for the associated functional

$$\mathcal{A} : x \mapsto -\int_0^T x^\star \hat{\lambda} + \int_0^T \widehat{H}(x(t))\mathrm{d}t,$$

for Hamiltonian arcs $x : [0, T] \to [1, 1+\delta) \times \partial W$. Such arcs must satisfy $\dot{x}(t) = X_{\widehat{H}}(x(t))$, by definition; and since our almost complex structure satisfies $J\partial_r = \mathcal{R}_\alpha$, we must have:

$$\begin{aligned} x^\star \hat{\lambda} &= \hat{\lambda}\big(x(t)\big)\mathrm{d}t \\ &= \hat{\lambda}(X_{\widehat{H}})\mathrm{d}t \\ &= r\alpha(X_{\widehat{H}})\mathrm{d}t \quad (\text{Because } \hat{\lambda} = r\alpha = r\lambda|_{\partial W} \text{ on } [1, +\infty) \times \partial W]) \\ &= r\alpha(J\nabla \widehat{H})\mathrm{d}t \\ &= r\partial_r \widehat{H}\, \mathrm{d}t \quad (\text{Because } J\partial_r = \mathcal{R}_\alpha) \end{aligned}$$

Now, just like in Calculation C.12, we can explicitly compute:

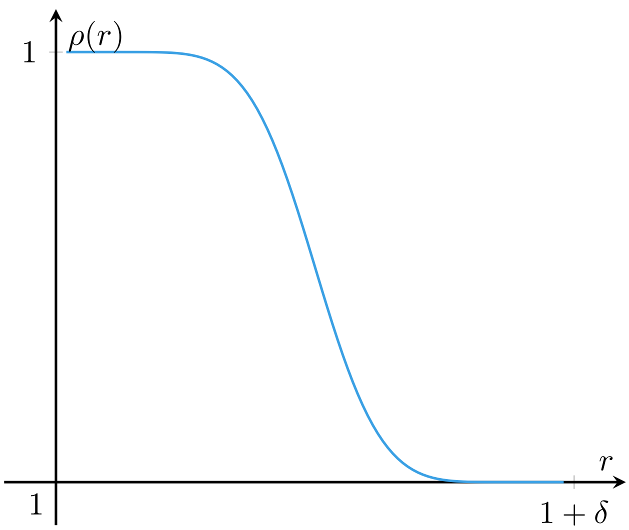
$$\begin{aligned} \partial_r \widehat{H} &= \frac{\partial}{\partial r}\left(\widehat{H}_0 + (r-1)\widehat{H}_1 + \frac{(r-1)^2}{2!}\widehat{R}\right) \\ &= \partial_r \widehat{H}_0 + \widehat{H}_1 + (r-1)\partial_r \widehat{H}_1 + (r-1)\widehat{R} + \frac{(r-1)^2}{2!}\partial_r \widehat{R} =: F. \end{aligned}$$

With this new notation, we can write:

$$\mathcal{A}(x) = -\int_0^T rF\mathrm{d}t + \int_0^T \widehat{H}\big(x(t)\big)\mathrm{d}t.$$

Step 3 (Asymptotically approximating $\mathcal{A}$).

To construct our extension $\widehat{H}$ in the previous step, we had to choose a cut-off function $\rho : [1, 1+\delta] \to [0, 1]$, which looked something like:



For example, we have here drawn the cut-off function $\rho(r) := \phi(r-1)$, with:

$$\phi : (0, \delta) \longrightarrow (0, 1) : s \longmapsto \frac{\exp\left(\frac{1}{s}\right)}{\exp\left(\frac{1}{s}\right) + \exp\left(\frac{1}{\delta - s}\right)},$$

which is then continuously extended to $[0, \delta]$.

Let us choose $\delta > 0$ extremely small, so that our neighbourhood $[1, 1+\delta] \times \partial W$ is extremely thin, and we always have $r \sim 1$. In particular, we can ignore terms of order ≥ 1, and asymptotically approximate:

$$\begin{aligned}\widehat{H} &\simeq \widehat{H}_0,\\ F &\simeq \partial_r \widehat{H}_0 + \widehat{H}_1.\end{aligned}$$

Therefore, the functional $\mathcal{A}$ asymptotically becomes:

$$\begin{aligned}\mathcal{A}(x) &= -\int_0^T rF\,\mathrm{d}t + \int_0^T \widehat{H}\big(x(t)\big)\mathrm{d}t\\ &\simeq -\int_0^T (\partial_r \widehat{H}_0 + \widehat{H}_1)\mathrm{d}t + \int_0^T \widehat{H}_0 \mathrm{d}t\\ &\simeq \int_0^T (\widehat{H}_0 - \widehat{H}_1)\mathrm{d}t - \int_0^T \partial_r \widehat{H}_0\,\mathrm{d}t\end{aligned}$$

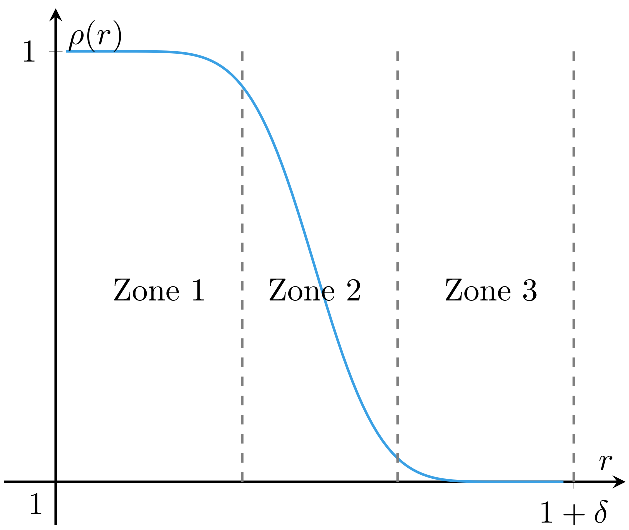
on $[1, 1+\delta) \times \partial W$. Now, from the expressions of $\widehat{H}_0$ and $\widehat{H}_1$, we can simplify:

$$\begin{aligned}\widehat{H}_0 - \widehat{H}_1 &= \rho(r)\Big((H_0 - H_1) - (C_0 - C_1)\Big) + (C_0 - C_1),\\ \partial_r \widehat{H}_0 &= \rho'(r)\big(H_0 - C_0\big),\end{aligned}$$

so that our functional $\mathcal{A}$ becomes:

$$\mathcal{A}(x) \simeq \int_0^T \rho(r)\Big((H_0 - H_1) - (C_0 - C_1)\Big)\mathrm{d}t + \int_0^T (C_0 - C_1)\mathrm{d}t - \int_0^T \rho'(r)\big(H_0 - C_0\big)\,\mathrm{d}t. \tag{6.6}$$

Step 4 (Zone-appropriate asymptotics).
Recall that we want to prove an action growth statement for $\mathcal{A}$. We will not attack the expression (6.6) head-on; instead, we will make use of the particularly nice form of the function ρ to approximate $\mathcal{A}$ in different zones:



We can roughly delimit the interval $[1, 1+\delta]$ into three zones, depending on the behaviour of the cut-off function $\rho : [1, 1+\delta] \to [0, 1]$. Indeed:

- in Zone 1, $\rho(r) \simeq 1$, $\rho'(r) \simeq 0$;
- in Zone 2 ρ decreases rapidly, *i.e.* $\rho'(r) < -\epsilon$ for some $\epsilon > 0$;
- in Zone 3, $\rho(r) \simeq 0, \rho'(r) \simeq 0$.

We choose these zones to slightly overlap so that they cover the whole interval $[1, 1+\delta]$. We can then asymptotically approximate $\mathcal{A}$ in each of these zones:

Zone 1. We have $\rho \simeq 1, \rho' \simeq 0$, therefore:

$$\begin{aligned}\mathcal{A}(x) &\simeq \int_0^T \Big((H_0 - H_1) - (C_0 - C_1)\Big)\mathrm{d}t + \int_0^T (C_0 - C_1)\mathrm{d}t \\ &= \int_0^T (H_0 - H_1)\mathrm{d}t.\end{aligned}$$

However, by the Quantitative Weakened Twist Condition (Assumption 6.2), we have

$$\min_{\partial W} \partial_r H > \max_{\partial W} H,$$

or in other words

$$\min_{\partial W} H_1 > \max_{\partial W} H_0.$$

Define $c_1 := \min_{\partial W} H_1 - \max_{\partial W} H_0 > 0$. Then, we must have:

$$\boxed{\mathcal{A}(x) \lesssim -c_1 \cdot T}$$

so that $\mathcal{A}$ does satisfy action growth for chords in Zone 1.

Zone 3. Zone 3 is even simpler than Zone 1. We have $\rho \simeq 0 \simeq \rho'$, so that $\mathcal{A}$ simplifies to

$$\mathcal{A}(x) \simeq \int_0^T \big(C_0 - C_1\big)\mathrm{d}t.$$

Recall that the only constraints on the constants C_0 and C_1, so far, are that $C_i \geq \max H_i$. In particular, if we choose $C_1 > C_0$, then there exists $c_3 > 0$ such that

$$\boxed{\mathcal{A}(x) \lesssim -c_3 \cdot T}$$

Zone 2. ρ is rapidly decreasing from 1 to 0, and we have $\partial_r \rho < -\epsilon$, for some $\epsilon > 0$.

First, let us look at the last term of $\mathcal{A}$:

$$-\int_0^T \rho'(r)\big(H_0 - C_0\big)\mathrm{d}t.$$

Since $C_0 \geq \max H_0$ by construction, the above expression is negative, and we must have:

$$-\int_0^T \rho'(r)\big(H_0 - C_0\big)\mathrm{d}t < -\varepsilon(C_0 - \max H_0) \cdot T.$$

➤ Let us enforce $C_0 > \max H_0$ (which we can do, since we have control over the coefficients C_0 and C_1). Then, $\exists\, c_2'' > 0$ such that:

$$-\int_0^T \rho'(r)\big(H_0 - C_0\big)\mathrm{d}t < -c_2'' \cdot T.$$

Now let us look at the first term of $\mathcal{A}$:

$$\int_0^T \rho(r)\Big((H_0 - H_1) - (C_0 - C_1)\Big)\mathrm{d}t.$$

We would also like it to satisfy an "action growth" result. First note that, on Zone 2, $\min \rho > \epsilon > 0$. For the second term to be negative, we would need:

$$C_1 - C_0 < H_1 - H_0. \tag{6.7}$$

Recall that $H_1 > H_0$ by the Weakened Twist Condition (actually, $\min H_1 > \max H_0$). In particular, we must have

$$\min H_1 - \max H_0 \leq \min \big(H_1 - H_0\big),$$

since H_0 and H_1 are real-valued functions.

➤ Therefore, let us choose C_0 and C_1 such that $0 < C_1 - C_0 < \min H_1 - \max H_0$. Then:

$$(C_1 - C_0) < \min_{\partial W} H_1 - \max_{\partial W} H_0 \leq \min_{\partial W}(H_1 - H_0),$$

so that there exists a constant $c_2' > 0$ such that:

$$\min_{\partial W}(H_1 - H_0) - (C_1 - C_0) > c_2',$$

and hence:

$$\int_0^T \rho(r)\Big((H_0 - H_1) - (C_0 - C_1)\Big)\mathrm{d}t < -c_2' \cdot T.$$

Therefore, we can asymptotically bound $\mathcal{A}$:

$$\mathcal{A}(x) \simeq \underbrace{\int_0^T \rho(r)\Big((H_0 - H_1) - (C_0 - C_1)\Big)\mathrm{d}t}_{\lesssim -c_2' \cdot T} + \underbrace{\int_0^T (C_0 - C_1)\mathrm{d}t}_{\lesssim -b_3 \cdot T} - \underbrace{\int_0^T \rho'(r)\big(H_0 - C_0\big)\,\mathrm{d}t}_{\lesssim -c_2'' \cdot T}.$$

where $b_3 = C_0 - C_1 < 0$. Then, if we set $c_2 = c_2' + c_2'' + c_3$, we get:

$$\boxed{\mathcal{A}(x) \lesssim -c_2 \cdot T}$$

for every chord x of length T in Zone 2.

Step 5 (Patching the zones together).

To summarise Step 4:

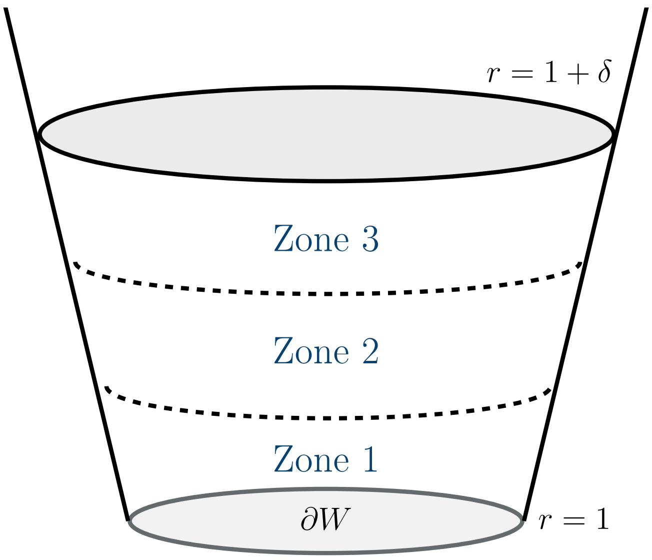
We delimited the neighbourhood $[1, 1+\delta] \times \partial W$ into three Zones $1, 2, 3$, and found constants $c_i > 0$ so that for every Hamiltonian path

$$x : [0, T] \to \text{Zone } i,$$

we had

$$\mathcal{A}(x) \lesssim -c_i \cdot T.$$

Of course, in general, a path x will not lie exclusively in one of the three zones, but it may travel freely across them. To account for this, we can partition $[0, T]$ into sub-intervals:



$$[0, T] = [t_0, t_1] \cup \cdots \cup [t_j, t_{j+1}] \cup \cdots \cup [t_{N-1}, t_N],$$

such that on each $[t_j, t_{j+1}]$, x lies constantly in one of the zones. (Such partitions exist because we have chosen our three zones to overlap). In particular, even if x a chord x in $[1, 1+\delta) \times \partial W$

travels freely between the zones, we still have:

$$\mathcal{A}(x) \lesssim -c \cdot T$$

for $c := \min(c_1, c_2, c_3) > 0$.

Step 6 (Concluding).
So we have shown that, for any Hamiltonian trajectory $x : [0, T] \to [1, 1 + \delta) \times \partial W$ of our extension $\widehat{H}$, we have

$$\mathcal{A}(x) \lesssim -c \cdot T,$$

where the symbol $\lesssim$ means "less than, up to a small error term", so that there exists some small constant $d \in \mathbb{R}$ such that

$$\mathcal{A}(x) < -c \cdot T + d.$$

By Step 1, this concludes the proof of action growth for $\mathcal{A}_{\widehat{H}}$ on $[1, +\infty) \times \partial W$. Let us briefly recall the assumptions that we made on the constants C_i along the way:

- by construction of the extension $\widehat{H}$ (Definition 5.9), we had $C_i \geq \max H_i$.
- in Step 4, for the purposes of studying Zone 2, we assumed that $C_0 > \max H_0$.
- also for the purposes of studying Zone 2 (and 3), we assumed that $C_1 = C_0 + \varepsilon$, where $\varepsilon < \min H_1 - \max H_0$. In particular, this implies that we must have $C_0 \geq \max H_1 - \varepsilon$.

Therefore, our extension $\widehat{H}$ must look like, at infinity:

$$\begin{aligned} \widehat{H} &= C_0 + (C_0 + \varepsilon)(r - 1) \\ &= (C_0 + \varepsilon)r - \varepsilon \\ &= ar - \varepsilon \qquad \text{(Setting } a = C_0 + \varepsilon) \end{aligned}$$

where $a \geq \max H_1$, and $\varepsilon < \min H_1 - \max H_0$. □

◇ **Remark 6.6.** If we're interested in a specific chord x, and want to refine the bound on its action, then we can do this by using our partition from above. Indeed, for $i \in \{1, 2, 3\}$, let us define $T_i < T$ to be the time that the chord x spends in Zone 1. In other words:

$$T_i := \sum_j (t_{j+1} - t_j),$$

where the sum is taken over all indices j such that

$$x|_{[t_j, t_{j+1}]}$$

lies in Zone i. Then, we have:

$$\mathcal{A}(x) \lesssim -c_1 \cdot T_1 - c_2 \cdot T_2 - c_3 \cdot T_3,$$

where we explicitly determined c_1, c_2, and c_3 in Step 4.

◇ **Remark 6.7.** The Weakened Twist Condition was key in proving the action growth of chords *extremely close* to the boundary ∂W (in Zone 1). It was also important in Zone 2 (especially at "low values of r"), to make sure the first term satisfied action growth, but the more we progressed along Zone 2, the more $\mathcal{A}_{\widehat{H}}$ became dominated by the integral in $-\rho'(r)$, because of ρ's rapid decrease.

Finally, in Zone 3, we didn't need the Weakened Twist Condition at all, and simply achieved action growth thanks through our choice of constants C_0, C_1.

6.3 Proof of Theorem B2

We now have all the ingredients necessary to prove Theorem B2. The proof will be almost exactly the same as that of Theorem B1, in Chapter 5, except we replace index growth by action growth.

Proof of Theorem B2. By assumption, f has finitely many interior periodic chords on L. Let us write:

- $c_1, \dots, c_k$ the period-1 periodic chords of f on L.
- $d_1, \dots, d_q$ the interior periodic chords of minimal periods $n_1, \dots, n_q$, with $n_i > 1$.

Assume, for a contradiction, that f has finitely many interior chords which are not periodic chords, nor sub-chords of periodic chords, and let $m_1, \dots, m_\ell$ denote their (minimal) orders. Choose a sequence of primes (p_i) going to ∞, and such that $p_i > \max_{j,k}\{n_j, m_k\}$. Then, pick $N > 2nk$ (where $2n = \dim W$). Since $HW^*(L)$ is non-zero in infinitely many degrees (by assumption), we can find $i_1, \dots, i_N$ such that $HW^{i_j}(L) \neq 0$. Recall that, by definition,

$$HW^*(L) := \varinjlim_{p_i} HL^*(\widehat{L}, \widehat{H}^{\#p_i}).$$

This tells us that:

$$HW^{i_j}(L) \neq 0 \implies HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i}) \neq 0 \quad \text{for } p_i \text{ large enough.}$$

Indeed, if $HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i})$ were 0 for infinitely many values of p_i, then $HW^{i_j}(L) = \varinjlim HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i})$ would be zero. Note that by action growth (Theorem 6.3), we have that for every chord x of $\widehat{H}^{\#p_i}$ on the collar $[1, +\infty) \times \partial W$:

$$\mathcal{A}_{\widehat{H}}(x) \leq -c \cdot p_i + d. \tag{6.8}$$

Then, define the co-chain complex:

$$CW^*(\widehat{H}) := \left\{ x \text{ is a chord of degree } * \text{ of } \widehat{H}^{\#p_j}, \text{ for some } j \right\}.$$

as well as the associated action-filtration by $\mathcal{A}_{\widehat{H}}$ (see §3.5.2). By Proposition 3.78, there exists a spectral sequence whose E_1 page consists of the local cohomology of chords with action $\geq -c \cdot p_i$, and such that:

$$E_1 \implies HL^*(\widehat{L}; \widehat{H}^{\#p_i}).$$

As argued above, $HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i}) \neq 0$ for our specific values $i_1, \dots, i_N$. Recall that convergence of spectral sequences means that:

$$HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i}) \cong \bigoplus_{p+q=i_j} E_\infty^{p,q}.$$

Therefore, there must be non-zero elements on the diagonals $p + q = i_j$ of E_∞. Pick p_i very large, so that $p_i \gg |i_j| \ \forall j$. Then, by (6.8), chords of $\widehat{H}^{\#p_i}$ on $[1, +\infty) \times \partial W$ will have action going to $-\infty$, and hence will appear very late in the action-filtration (on columns very far away, with $p \gg 1$). Hence, such chords cannot count towards the diagonals $p + q = i_j$, and *a fortiori* towards $HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i})$, so that only interior chords contribute to it.

Also, recall that we had assumed p prime, so that $n_j \nmid p_i$, ensuring that none of the chords contributing towards our cohomology can be the iterate of a non-fixed interior periodic chord. Hence, the only chords which can count towards $HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i})$ are iterates $c_j^{p_i}$ of the fixed chords $c_1, \dots, c_k$. However, by Lemma 5.20:

$$\forall j : \operatorname{supp} HW^*_{\mathrm{loc}}(c_j^{p_i}) \subset [\Delta(c_j^{p_i}) - n, \Delta(c_j^{p_i}) + n]$$

Since we have k fixed chords, the $E_1 = \bigoplus_{i,j} HW_{\mathrm{loc}}(x_j^{p_i})$ covers at most $2nk$ degrees in cohomology. However, we know that $HL^{i_j}(\widehat{L}; \widehat{H}^{\#p_i})$ must be non-zero in at least N degrees, where $N > 2nk$, yielding a contradiction. Hence, there must exist infinitely many interior chords, with arbitrary large lengths. ☐

Chapter 7

A relative Poincaré-Birkhoff theorem on degenerate Liouville domains

Let us now conclude the trilogy of Poincaré-Birkhoff theorems we began in Chapter 5. Recall: Theorem B1 showed existence of infinitely many interior chords with Lagrangian ends, in the model of Moreno & van Koert. Theorem B2 then generalised said model by weakening the assumptions. We will now adapt Theorem B2 to a different class of manifolds: *degenerate Liouville domains* (Definition 7.1).

All the results in this chapter were originally published in [LM25].

Theorem B3. *Let (W, λ, α) be a connected degenerate Liouville domain, $L \subset W$ be an exact spin Lagrangian with Legendrian boundary, and $f : W \to W$ be an exact symplectomorphism. Further assume that:*

- ***(Wrapped Floer Homology).*** *$HW^*(L) \neq 0$ in infinitely many degrees;*
- ***(Chern class).*** *if $\dim W \geq 4$, then $c_1(W) = 0$;*
- ***(Weakened Twist Condition)*** *f is generated by a Hamiltonian $H_t : W \to \mathbb{R}$ whose fixed points are isolated, which is $\mathcal{C}^2$ on the interior of W but does **not** $\mathcal{C}^1$-extend to the boundary, and such that:*

$$\partial_r H_t > 0$$

in a neighbourhood of ∂W.

Then f admits infinitely many interior chords with respect to L, of arbitrarily large order, which are not sub-chords of any periodic chord.

The main idea behind the proof of this theorem is that we will use a process called non-degeneration, turning (W, λ, α) into a non-degenerate Liouville domain, and our $\mathcal{C}^2$ twist Hamiltonian into a $\mathcal{C}^0$ Hamiltonian. We will then approximate this new $\mathcal{C}^0$ Hamiltonian by a sequence of $\mathcal{C}^2$ Hamiltonians (on the non-degeneration of (W, λ, α)), and show that far along enough in this sequence, the Quantitative Weakened Twist Condition from the previous chapter will be satisfied; allowing us to replicate the argument of Theorem B2.

We give a more precise proof sketch in §7.1.3, after explaining the non-degeneration process. Note that, in the above theorem, the assumption that H_t does **not** $\mathcal{C}^1$-extend to ∂W is crucial, as without it, the Quantitative Weakened Twist Condition could not hold in the approximating sequence – a fact we explore in Remark 7.6.

◇ **Remark.** Since, by essence, the proof of Theorem B3 relies on a non-degeneration process, we could rephrase the theorem. Instead of a theorem for $\mathcal{C}^2$ Hamiltonian twist maps on degenerate Liouville domains, we could state a theorem for $\mathcal{C}^0$ Hamiltonian twist maps on non-degenerate Liouville domains, which we did at the end of §1.3.2.

7.1 Bypassing the degeneracy

7.1.1 Degenerate Liouville domains

• **Definition 7.1.** A **degenerate Liouville domain** is a triplet (W, λ, α) where:

- W is a smooth, compact manifold with non-empty boundary ∂W;
- α is a contact form on ∂W.
- λ is a 1-form on W such that $\omega := \mathrm{d}\lambda$ is symplectic in the interior, but degenerates along the boundary, i.e
$$\exists X \text{ such that } (X \lrcorner\, \omega)|_{\partial W} \equiv 0.$$

The last condition actually implies that $\lambda = A(r)\alpha$ on some collar neighbourhood $(1-\varepsilon, 1] \times \partial W$, where
$$\partial_r A|_r > 0 \text{ for } r \neq 1, \;\; \partial_r A|_{r=1} = 0, \;\; A(1) = 0.$$

So in short, a degenerate Liouville domain is a Liouville domain whose symplectic form degenerates along the boundary.

✧ **Example 7.2.** We will see explicit examples in Chapter 8. Most notably, we will see that when an open book is adapted to a Reeb flow in the sense of Giroux, its pages are degenerate Liouville domains. In particular, since the CR3BP exhibits an open book decomposition for low energies [MK22b], then we can find physically relevant degenerate Liouville domains (see Lemma 8.25), most notably, by studying the open book associated to the Circular Restricted Three-Body Problem (see Chapter 8). Another example is that of billiards, which are currently of high interest in Hamiltonian dynamics [Mor24, Ch. 4, §6].

7.1.2 Degeneration and non-degeneration

As already explored in [Mor24, Ch. 4], we can turn a degenerate Liouville manifold into a standard, non-degenerate one, and *vice versa*. Namely:

Lemma 7.3. *Let (W, λ, α) be a degenerate Liouville domain. There exists a homeomorphism $Q : W \to W$, smooth in the interior but only continuous along the boundary, such that if we set $\lambda_Q := Q^\star\lambda$, then $(W, \mathrm{d}\lambda_Q)$ is a non-degenerate Liouville domain.*

Lemma 7.4. *Given $f : W \to W$ smooth, then $f_Q := Q^{-1} \circ f \circ Q$ is smooth in the interior, but only continuous along the boundary.*

We will make use of this bridge between degenerate and non-degenerate Liouville domains all throughout this chapter.

Proof sketch of Lemma 7.3. Let b denote the coordinate on the boundary, and consider a collar neighbourhood $(-\varepsilon, 0] \times \partial W$, with $s \in (-\varepsilon, 0]$. There, the Liouville form λ can be written $\lambda = A(s)\alpha$, where $A(0) = 1, \partial_s A|_{s=0} = 0$ and $\partial_s A > 0$ for every $s \neq 0$. Then, define a map
$$Q : (s, b) \mapsto (\varphi(s), b),$$
where φ is a solution of the ODE
$$\varphi'(s) = -\frac{1}{\partial_s A|_s} > 0. \tag{7.1}$$
By integration, we get $(A \circ \varphi)(s) = 1 - s$. Writing $r := 1 - s \in (1-\varepsilon, 1] \times \partial W$, and defining the form $\lambda_Q := Q^\star\lambda$, we see that
$$\mathrm{d}\lambda_Q = \mathrm{d}(A(\varphi(s))\alpha) = \mathrm{d}(r\alpha),$$
which is the standard symplectic form, and *a fortiori* non-degenerate. □

Proof of Lemma 7.4. This directly follows from the fact that Q is a diffeomorphism on $\text{int}(W)$, but only continuous along ∂W; which itself follows from the fact that (7.1) only has one solution satisfying $\varphi(0) = 0$, which is only continuous at $s = 0$. □

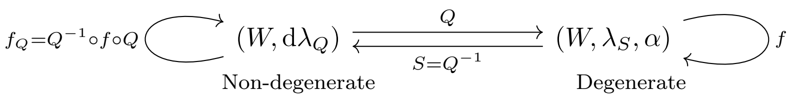
We refer to §2 of [LM25] or Chapter 4 of [Mor24] for more details on this non-degeneration technique.

What the above lemmas tell us is that we can think of Q as a "squareroot map" near the boundary. We can explicitly compute an inverse S, which turns a standard Liouville domain into a degenerate one. Indeed, given $(W, \omega = d\lambda)$ a "true" Liouville domain, then by setting

$$
\begin{aligned}
S : (s, b) \mapsto& (s^2, b), \\
\lambda_S := S^\star \lambda =& (1 - s^2)\alpha,
\end{aligned}
$$

we obtain a degenerate Liouville domain $(W, \lambda_S, \alpha := \lambda_S|_{\partial W})$.

In summary, we have homeomorphisms Q and S such that Q turns a non-degenerate Liouville domain into a degenerate one, and S does the reverse. In short:



◇ **Remark 7.5.** Let us make one final observation, about our construction of the last two lemmas. Near the boundary, we have $Q = (F, \text{id})$ with $F(r) = 1 - \varphi(1 - r)$, where $F, F' > 0$ on the interior of W, $F|_{\partial W} \equiv 1$, and F' blows up to infinity along $\partial W = \{r = 1\}$.

7.1.3 Intuition behind Theorem B3, and some definitions

Proof sketch of Theorem B3.

1. Let (W, λ, α) be degenerate, and say we have a diffeomorphism $f : W \to W$ satisfying the Weakened Twist Condition, *i.e.* it is generated by a $\mathcal{C}^2$ Hamiltonian H_t such that

$$(\partial_r H_t)|_{\partial W} > 0.$$

2. Use the non-degeneration process described above (pull back by Q), to obtain a non-degenerate Liouville domain (W, ω), and a map f_Q which is now only $\mathcal{C}^0$ at the boundary.

3. Approximate f_Q by a sequence of $\mathcal{C}^2$ maps f_ϵ such that

$$f_\epsilon \xrightarrow{\mathcal{C}^0} f_Q \quad \text{as } \epsilon \to 0.$$

   We can construct this sequence in such a way that f_ϵ satisfies the Quantitative Weakened Twist Condition from Chapter 6 for $\varepsilon \ll 1$. We call this process a smoothing.

4. Therefore, by re-working slightly the proof of Theorem B2, we can assign a Wrapped Floer Cohomology to f_Q (*a fortiori*, f) which only detects its interior chords.

◇ **Remark 7.6** (Why is H_t forbidden from $\mathcal{C}^1$-extending to ∂W?)**.** There remains to explain one subtlety: why do we enforce that H_t does **not** $\mathcal{C}^1$-extend to the boundary in the statement of Theorem B3?

This is because, on the degenerate Liouville domain from Theorem B3, the symplectic form near the boundary is given by $\omega_S = d(A(r)\alpha)$, where $A(r) \equiv 1$ along ∂W (in particular, $\partial_r A(r)|_{\partial W} \equiv 0$). Assume that the Hamiltonian H_t *did* $\mathcal{C}^1$-extend to the boundary, implying that X_{H_t} was well-defined. Then, by a routine calculation, we would have:

$$
\begin{aligned}
\partial_r A \cdot \alpha(X_{H_t}) &= \omega_S(\partial_r, X_{H_t}) \\
&= (\partial_r A \, dr \wedge \alpha + A d\alpha)(\partial_r, X_{H_t}) \\
&= \partial_r H_t.
\end{aligned}
$$

In particular, we would then have

$$\frac{\partial_r H_t}{\partial_r A} = \alpha(X_{H_t}) < \infty.$$

However, since $\partial_r A \equiv 0$ along ∂W, then this would necessarily imply that $\partial_r H_t \equiv 0$ too. This would make it impossible for the Weakened Twist Condition to hold, and *a fortiori*, for the Quantitative Weakened Twist Condition to hold for the smoothing sequence approximating the non-degeneration of H_t. Therefore, we must necessarily ask that X_{H_t} does not $\mathcal{C}^0$-extend to ∂W in the statement of Theorem B3.

➢ As explained in the proof sketch, the first step in the proof corresponds to non-degenerating our original setup. In particular, if we take our $\mathcal{C}^2$ Hamiltonian twist map through the non-degeneration process, we get:

• **Definition 7.7. ($\mathcal{C}^0$-Hamiltonian twist maps)** Let $f : (W, \omega) \to (W, \omega)$ be a map on a Liouville domain, and let α be the contact form along ∂W. We say that f is a $\mathcal{C}^0$*-Hamiltonian twist map* if the following conditions hold:

- **(Hamiltonian)** $f|_{\text{int}(W)} = \phi^1_H$ is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : \text{int}(W) \to \mathbb{R}$;
- **(Extension)** Both f and the Hamiltonian H_t admit $\mathcal{C}^0$ extensions to the boundary, but not necessarily $\mathcal{C}^2$ extensions; and
- **(Weakened Twist Condition)** Near the boundary ∂W, the generating Hamiltonian satisfies $\alpha(X_{H_t}) = \partial_r H_t > 0$, and $\alpha(X_{H_t}) = \partial_r H_t \to +\infty$ as we approach ∂W.

We say that the isotopy H_t is *infinitely strictly wrapping* or *infinitely positively wrapping*, or simply *infinitely wrapping*.

As explained in the main idea, we would like to approximate such a $\mathcal{C}^0$-Hamiltonian twist map by a sequence of $\mathcal{C}^2$-Hamiltonian maps, with which we can use our Floer-theoretical arguments from Chapter 6. For this purpose, we also make the definition:

• **Definition 7.8. ($\mathcal{C}^2$-Hamiltonian twist map)** Let $f : (W, \omega) \to (W, \omega)$ be a map on a Liouville domain. We say that it is a $\mathcal{C}^2$*-Hamiltonian twist map*, if

- **(Hamiltonian)** $f = \phi^1_H$ is generated by a $\mathcal{C}^2$ Hamiltonian $H_t : W \to \mathbb{R}$;
- **(Weakened Twist Condition)** H_t satisfies $(\partial_r H_t)|_{\partial W} > 0$.

In this case, we say that the isotopy H_t is *strictly wrapping* or *positively wrapping*.

7.1.4 $\mathcal{C}^2$-approximating a $\mathcal{C}^0$-Hamiltonian twist map (Smoothing)

The goal of this subsection is to provide a scheme for constructing a $\mathcal{C}^2$-approximating sequence (f_ϵ) of a $\mathcal{C}^0$-twist map $f^{(1)}$, on a standard Liouville domain (W, ω).

Theorem 7.9 ([LM25]). *Let f be a $\mathcal{C}^0$-Hamiltonian twist map. For $\epsilon \geq 0$, there exists a family of $\mathcal{C}^2$-Hamiltonian twist maps f_ϵ on a Liouville domain (W, ω_ϵ), such that:*

- *f_ϵ is generated by a $\mathcal{C}^2$ Hamiltonian $H_\epsilon = H_{t,\epsilon}$ which converges in $\mathcal{C}^0$ to H_t as $\epsilon \to 0$.*
- *Along B, the function $h_{t,\epsilon} = \alpha(X_{H_{t,\epsilon}}) = \partial_r H_{t,\epsilon}$ diverges uniformly and monotonically as $\epsilon \to 0$, but all derivatives of $H_{t,\epsilon}$ in directions tangent to B remain uniformly bounded.*
- *As $\epsilon \to 0$, ω_ϵ converges to ω in $\mathcal{C}^\infty$.*
- *The completion $\widehat{\omega}_\epsilon$ on $\widehat{W}$ is independent of $\epsilon > 0$, i.e. it is symplectomorphic to $\widehat{\omega}$.*

(1) Note that in the proof sketch laid out in the previous section, our current f was actually called f_Q.

- *The slope of the extension $\widehat{H}_\epsilon$ on $\widehat{W}$ is bounded below by C/ϵ with $C > 0$, and hence monotonically diverges as $\epsilon \to 0$.*

◇ **Remark 7.10.** This proof is one of the main results in [LM25], and we therefore only include a proof sketch. The main idea can be traced back to an observation done in [Mor24, Ch. 5, §5], where both the calculations sketched below, and the non-degeneration process were explained. The novelty is that now that we have access to the Quantitative Weakened Twist Condition, and to Theorem B2, we can combine these ideas in such a way as to force the Quantitative Weakened Twist Condition to hold along the smoothing sequence.

Proof sketch of Theorem 7.9 [LM25]. Let (W, ω) be a Liouville domain and f a $\mathcal{C}^0$-Hamiltonian twist map, with generating Hamiltonian H_t. Let E_t be the degeneration of H_t, *i.e.* E_t is the Hamiltonian on the degeneration (W, ω_s) such that $H_t = E_t \circ Q$.

$$(W,\omega) \xrightarrow{\;Q\;} (W,\omega_S) \xrightarrow{\;E\;} \mathbb{R}, \qquad H : (W,\omega) \to \mathbb{R}$$

Then one can compute (see [Mor24, Ch. 5]):

$$X_{H_t} = [((\partial_r E_t)\circ Q)\cdot F'(r)]\, R_\alpha + \frac{1}{F(r)}\left[\left(X^{\xi}_{E_t} - \mathrm{d}E_t(R_\alpha)V\right)\circ Q\right], \tag{7.2}$$

which in particular has a pole at $r = 1$ because F' does, by Remark 7.5. Now recall that Q was defined by $Q = (F, \mathrm{id})$ on $(1-\varepsilon, 1] \times \partial W$, where

$$F(r) = 1 - \varphi(1-r),$$

and where $\varphi = \varphi(s)$ solves the ODE (7.1), for $s := 1 - r$. Set $g := \varphi$, and define a smooth sequence (g_ϵ) such that $g_\epsilon(0) = 1/\epsilon$, and $g_\epsilon \equiv \mathrm{id}$ on $[\epsilon, 1]$. Then, define:

$$\varphi_\epsilon(s) = \int_0^s g_\epsilon(x)\mathrm{d}x,$$

and $Q_\epsilon := (\varphi_\epsilon, \mathrm{id})$. Going back to the coordinate $r = 1 - s$, this yields

$$Q_\epsilon(r, b) = \big(1 - \varphi_\epsilon(1-r), b\big).$$

(Q_ϵ) is hence a sequence of truncations, equal to Q away from $(1-\epsilon, 1] \times \partial W$, and whose r-derivatives blow up to ∞ along ∂W. We can now construct our desired sequence of Hamiltonians:

$$H_{t,\epsilon} := E_t \circ Q_\epsilon.$$

Then, for every ϵ we have:

$$X_{H_{t,\epsilon}} = [((\partial_r E_t)\circ Q_\epsilon)\cdot g_\epsilon(1-r)]\, R_\alpha + \frac{1}{F_\epsilon(r)}\left[\left(X^{\xi}_{H_t} - \mathrm{d}H_t(R_\alpha)Y\right)\circ Q_\epsilon\right], \tag{7.3}$$

which is now smooth, and no longer blows up at 0 (it will be at most of the order of $1/\epsilon$). Formally, we have

$$h_t := \alpha(X_{H_{t,\epsilon}}) = (\partial_r H_{t,\epsilon})|_B = \frac{1}{\epsilon}(\partial_r H_t)|_B,$$

so that $(\partial_r H_{t,\epsilon})|_B > C/\epsilon$ where $C := \max_B(\partial_r H_t)$. This quantity is positive, by the Weakened Twist Condition. In particular, the extension $\widehat{H}_\epsilon$ we construct, as per Chapter 5, will have slope $\geq C/\epsilon$. □

Corollary 7.11. *Given an admissible Lagrangian $L \subset (W, \omega = \mathrm{d}\lambda)$, we have*

$$HW^*(L) \cong \varinjlim_{\epsilon \to 0} HW^*(\widehat{L}, \widehat{H}_\epsilon).$$

Proof. By construction, H_ϵ is $\mathcal{C}^2$, and its extension $\widehat{H}_\epsilon$ is linear at infinity with slope $\geq C/\epsilon$. Therefore, the above isomorphism holds by Chapter 3. □

Corollary 7.12. *Given an increasing sequence of integers $m_\epsilon \to \infty$ as $\epsilon \to 0$, we also have*

$$HW^*(L) \cong HW^*(\widehat{L}, \widehat{H}_\epsilon^{\#m_\epsilon}).$$

where $\widehat{H}_\epsilon^{\#m_\epsilon}$ denotes the m_ϵ-th iteration of $\widehat{H}$.

Proof. This converges to $HW^*(L)$ for the same reason as in the previous corollary – the only difference being that it converges faster. Indeed, recall that a Hamiltonian of slope a counts Reeb chords on the boundary up to length a. Therefore, the difference is that in Corollary 7.11 we counted Reeb chords on ∂W up to length slope$(\widehat{H}_\epsilon)$ at each step in the limit, whereas we now count them up to length

$$\text{slope}(\widehat{H}_\epsilon^{\#m_\epsilon}) = m_\epsilon \cdot \text{slope}(\widehat{H}_\epsilon).$$

□

7.2 Proof of Theorem B3

7.2.1 Intuition

Let us take a step back and think about what it is we actually want to prove:

We want to prove a Poincaré-Birkhoff theorem for maps satisfying the Weakened Twist Condition on a degenerate Liouville domain – or, alternatively, for $\mathcal{C}^0$-Hamiltonian twist maps on a non-degenerate Liouville domain.

We have already proved two Poincaré-Birkhoff theorems in Chapters 5 and 6: first for $\mathcal{C}^2$ Hamiltonian twist maps satisfying a twist condition, and then for $\mathcal{C}^2$ Hamiltonian twist maps satisfying a quantitative weakened twist condition. In both cases, we first constructed a linear extension of our Hamiltonian, and then took a direct limit over a sequence of iterations.

Our proof for $\mathcal{C}^0$-Hamiltonian maps will be extremely similar, except since we can't directly work with our Hamiltonian H_t, we will work with the smoothing sequence (H_ϵ), and take a direct limit both over smoothing and iterations, to make sure that we count chords of arbitrarily large order, arbitrarily close to the boundary. In other words, we will construct Wrapped Floer Cohomology as

$$HW^*(L) \cong \varinjlim_{\epsilon \to 0} HW^*(\widehat{L}; \widehat{H}_\epsilon^{p_\epsilon}),$$

where $p_\epsilon \to \infty$ as $\epsilon \to 0$, and then reproduce the exact same argument as in the proof of Theorem B2.

7.2.2 Proof of the theorem

Proof. By assumption, f has finitely many interior periodic chords on L. Let us write:

- $c, \ldots, c_k$ the period-1 periodic chords of f on L.
- $d_1, \ldots, d_q$ the interior periodic chords of minimal periods $n_1, \ldots, n_q$, with $n_i > 1$.

Assume for a contradiction that f has finitely many interior chords $x_1, \ldots, x_\ell$ which are not periodic chords, nor sub-chords of periodic chords, with minimal orders $m_1, \ldots, m_\ell$.

For $\epsilon \geq 0$, denote by H_ϵ a smoothing of H as given in §7.1.4, and by $\widehat{H}_\epsilon$ the corresponding admissible extension. We assume that ϵ has been chosen small enough so that the c_j, d_j, and x_i chords of f all lie far enough away from the boundary, so that they are also chords of the smoothing H_ϵ.

Choose a sequence of primes $\{p_i\}$ going to $+\infty$, and such that $p_i > \max_{j,l} m_j, d_l$. Let $\epsilon_i = 1/p_i > 0$. By Corollary 7.12, we have:

$$HW^*(L) := \varinjlim_{p_i} HW^*(\widehat{L}; \widehat{H}_{\epsilon_i}^{\#p_i}),$$

where $\widehat{H}_{\epsilon_i}^{\#p_i}$ is the p_i-th iterate of $\widehat{H}_{\epsilon_i}$.

Now pick $N > 2nk$, where $2n = \dim W$. Recall that by assumption, $HW^\bullet(L)$ is non-zero in infinitely many degrees, so that we can find $i_1, \ldots, i_N$ such that $HW^{i_j}(W) \neq 0$. Therefore

$$HW^{i_j}(\widehat{L}; \widehat{H}_{\epsilon_i}^{\#p_i}) \neq 0 \text{ for } i \text{ large enough and for all } j = 1, \ldots, N.$$

By action growth (Theorem 6.3), we have, for any length-1 path x of $\widehat{H}_{\epsilon_i}^{\#p_i}$ on the collar $[1, +\infty) \times \partial W$:

$$\mathcal{A}_{\widehat{H}_{\epsilon_i}^{\#p_i}}(x) \leq -c_i \cdot p_i + d, \tag{7.4}$$

with $c_i \to +\infty$ as $i \to +\infty$. Moreover, by Theorem 4.13, for every i there exists a local-to-global spectral sequence whose E_1 page is

$$E_1^{pq}(\widehat{H}_{\epsilon_i}^{\#p_i}) = \bigoplus_x HW_{loc}^{p+q}(x; \widehat{H}_{\epsilon_i}^{\#p_i}),$$

made up of the local Floer cohomologies of length-1 chords x of $\widehat{H}_{\epsilon_i}^{\#p_i}$. By Theorem A2, this spectral sequence converges to the Floer homology:

$$E_1(\widehat{H}_{\epsilon_i}^{\#p_i}) \implies HW^*(\widehat{L}; \widehat{H}_{\epsilon_i}^{\#p_i}), \text{ i.e. } HW^{i_j}(\widehat{L}; \widehat{H}_{\epsilon_i}^{\#p_i}) \cong \bigoplus_{p+q=i_j} E_\infty^{p,q}.$$

Since $HW^{i_j}(\widehat{H}_{\epsilon_i}^{\#p_i}) \neq 0$ for all j, there must be non-zero elements on the diagonals $p+q = i_j$ of E_∞. If $p_i \gg \max_j |i_j|$, then by (7.4), chords of $\widehat{H}_{\epsilon_i}^{\#p_i}$ on $[1, +\infty) \times \partial W$ will have action escaping to $-\infty$, and hence appear on columns with $p \gg 1$. Therefore such chords cannot contribute to the diagonals $p + q = i_j$, and *a fortiori* towards $HW^{i_j}(\widehat{L}; \widehat{H}_{\epsilon_i}^{\#p_i})$, so that only interior chords contribute.

Now recall that p_i is prime, so that $n_1, \ldots, n_q \nmid p_i$, ensuring that none of $x_1, \ldots, x_\ell, d_1, \ldots, d_q$, or the iterates of the d_i's contribute to $HW^{i_j}(\widehat{L}; \widehat{H}_{\epsilon_i}^{\#p_i})$. Then the chords we have found contributing to $HW^{i_j}(\widehat{H}_{\epsilon_i}^{\#p_i})$ must necessarily be iterates $c_j^{p_i}$ of one of the period-1 chords c_j. However, by Lemma 5.20:

$$\operatorname{supp} HW_{\mathrm{loc}}^\bullet(c_j^{p_i}; \widehat{H}_{\epsilon_i}^{\#p_i}) \subset [\Delta(c_j^{p_i}) - n, \Delta(c_j^{p_i}) + n] \text{ for all } j,$$

where the *support* denotes the range of degrees in which the local cohomology is non-vanishing. Hence, each chord $\gamma_j^{p_i}$ can contribute to at most $2n$ degrees in cohomology, so that by counting all of them, we have covered $2nk$ degrees. However, we had found $N > 2nk$ values for the degree in which the cohomology is non-zero. This yields a contradiction.

Therefore, there must exist infinitely many interior chords which are not sub-chords of any periodic chord, and of arbitrarily large order. □

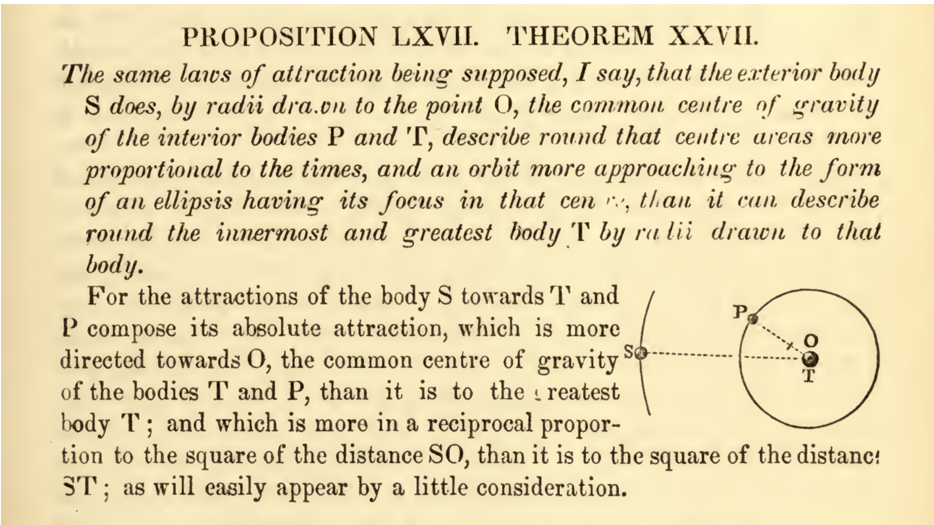
Chapter 8

Symplectic introduction to the Three-Body Problem

8.1 Introduction to the problem

The first mention of the Three-Body Problem can be traced all the way back to Newton's *Principia Mathematica* ([New87]). After establishing his rules of calculus and stating his three laws of motion, Newton employed them to try and understand the Solar system. In the first few chapters, he initially focused on the motion of bodies all under the influence of one large fixed attractor, and which did not influence it back or each other. Though this model provided a good enough approximation of the solar system, given how massive the Sun is, Newton himself recognised it was not representative of reality.

In [New87, Sec. XI], he proceeded to allow bodies to mutually attract each other. For two bodies, the system is completely solvable, as we saw in Example 1.1, and as Newton explored in a few propositions[(1)]. However, he quickly realised that as soon as he added a third body, the problem became virtually unsolvable. Though he stayed optimistic, managing to scrape a few approximations through elementary, albeit ingenious geometry, he achieved no general solution.



PROPOSITION LXVII. THEOREM XXVII.

The same laws of attraction being supposed, I say, that the exterior body S does, by radii drawn to the point O, the common centre of gravity of the interior bodies P and T, describe round that centre areas more proportional to the times, and an orbit more approaching to the form of an ellipsis having its focus in that centre, than it can describe round the innermost and greatest body T by radii drawn to that body.

For the attractions of the body S towards T and P compose its absolute attraction, which is more directed towards O, the common centre of gravity of the bodies T and P, than it is to the greatest body T; and which is more in a reciprocal proportion to the square of the distance SO, than it is to the square of the distance ST; as will easily appear by a little consideration.

Excerpt from Newton's *Principia Mathematica*, Book 1, Section XI,
Translated from Latin to English by Andrew Motte.

The following centuries saw contributions from many great mathematicians: Euler, Lagrange, Hill, Poincaré,... It was the latter who finally proved, in the late nineteenth century, that the equations were non-integrable, and that the problem exhibited chaotic behaviour.

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(1) His study is nowadays a bit difficult to read, because the modern language for calculus hadn't yet been developed when Newton wrote his *Principia Mathematica*, and hence everything was stated in terms of elementary geometry. However, in the commentary to her French translation, the Marquise Émilie du Châtelet carried out all of Newton's calculations (and more) in modern calculus language. In particular, she gave an explicit solution of the Two-Body Problem in Proposition XII, Problèmes VII-VIII of the first section of *Solution analytique des principaux problèmes qui concernent le système du monde.*

Henri Poincaré had started working on the Three-Body Problem in the 1880s, two centuries after Newton. In his *Méthodes Nouvelles de la Mécanique Céleste* (1892), he developed much of the theory we still use to this day for the qualitative study of dynamical systems. He pioneered the use of geometric and topological methods in dynamics, for the very purpose of understanding three-body motion.

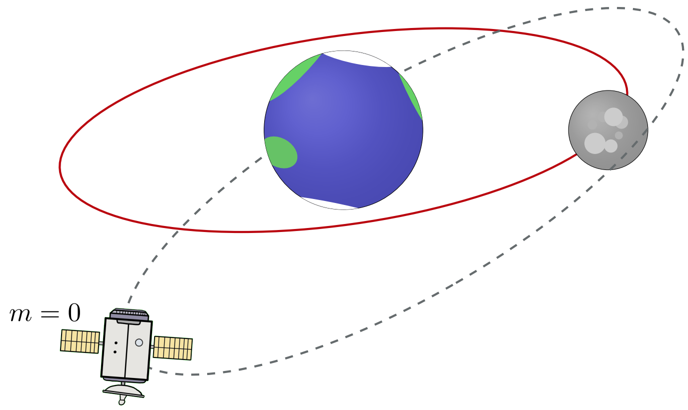
Near the end of his life, he imagined a scheme to find periodic orbits, relying on the Poincaré-Birkhoff theorem (Theorem 5.1). While he did not have time to complete the proof, his ideas gave birth to a set of tools dynamicists still use extensively, and which we will explore in the next chapters. First though, let us formally introduce the Three-Body Problem.

8.1.1 The Circular Restricted Three-Body Problem

Consider three bodies, which we call the Earth (E), the Moon (M) and a satellite (S), moving in standard three-dimensional space $\mathbb{R}^3$, under the influence of Newtonian gravity, as defined in Example 1.1. We make two assumptions:

Assumption 8.1 (Circular)**.** The Earth and Moon move in circles around their common centre of mass.

Assumption 8.2 (Restricted)**.** The satellite has mass zero.



This problem is called the Circular Restricted Three-Body Problem **(CR3BP)**, or sometimes the Spatial Circular Restricted Three-Body Problem **(SCR3BP)**, to emphasise the fact that the satellite is allowed to move in all of $\mathbb{R}^3$. By opposition, if we enforce the extra assumption that the satellite is constrained to the Earth-Moon plane (the ecliptic), then we speak of the Planar Circular Restricted Three-Body Problem **(PCR3BP)**. Throughout this thesis, CR3BP is understood to mean SCR3BP.

◇ **Remark 8.3.** The restricted assumption tells us about scenarios where a small object (e.g. a spacecraft) is orbiting two large bodies. Indeed, the mass of a human-made satellite is negligible compared to the masses of planets or moons. Meanwhile, most systems of interest in astrodynamics at the moment satisfy the circular assumption, due to the second body's orbit having very low eccentricity around the largest one (see the table below).

Despite our notation, we are of course not just interested in the Earth and Moon, but in any two bodies with arbitrary masses m_E, m_M. Since the satellite has mass zero, we can re-normalise the problem so that the masses m_E and m_M sum up to 1. By convention, this is done by defining the ratio:

$$\mu = \frac{m_M}{m_E + m_M}. \tag{8.1}$$

System	Mass ratio	Eccentricity
Earth-Moon	$\mu \approx 1.21 \cdot 10^{-2}$	$e \approx 0.0549$
Jupiter-Europa	$\mu \approx 2.53 \cdot 10^{-5}$	$e \approx 0.009$
Saturn-Enceladus	$\mu \approx 1.90 \cdot 10^{-7}$	$e \approx 0.0047$

➤ Let $\mathbb{R}^3$ denote position space with coordinate q, and $T^\star\mathbb{R}^3$ denote its phase space with coordinates (q, p). The Hamiltonian of the CR3BP is given by:

$$\begin{aligned} H(q,p,t) &= \text{Kinetic energy} + \text{Potential energy} \\ &= \frac{1}{2}\|p\|^2 - \frac{1-\mu}{\left\|q - \overrightarrow{E}(t)\right\|} - \frac{\mu}{\left\|q - \overrightarrow{M}(t)\right\|}, \end{aligned}$$

where $\overrightarrow{E}(t)$ and $\overrightarrow{M}(t)$ are the respective positions of the Earth and Moon. Now observe that we can get rid of the time-dependency thanks to Assumption 8.1. Indeed, since the Moon moves in a circle about the Earth, we can choose a rotating frame in which they are both fixed with $\overrightarrow{E} = (\mu, 0, 0)$, $\overrightarrow{M} = (\mu - 1, 0, 0)$. This only causes the appearance of an angular momentum term $L = q_1p_2 - q_2p_1$, giving us the new, autonomous Hamiltonian:

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{1-\mu}{\left\|q - \overrightarrow{E}\right\|} - \frac{\mu}{\left\|q - \overrightarrow{M}\right\|} + q_1p_2 - q_2p_1. \tag{8.2}$$

Formally, this Hamiltonian is a map $H : T^\star(\mathbb{R}^3 \backslash \{\overrightarrow{E}, \overrightarrow{M}\}) \longrightarrow \mathbb{R}$.

✧ **Example 8.4** (Rotating Kepler problem)**.** Say we set $\mu = 1$, or $\mu = 0$. Then, H virtually describes the Two-Body Problem in a rotating frame. It is as if we had removed one of the large masses, but we remembered its rotational effect on the coordinates. We call this limit the **Rotating Kepler Problem**.

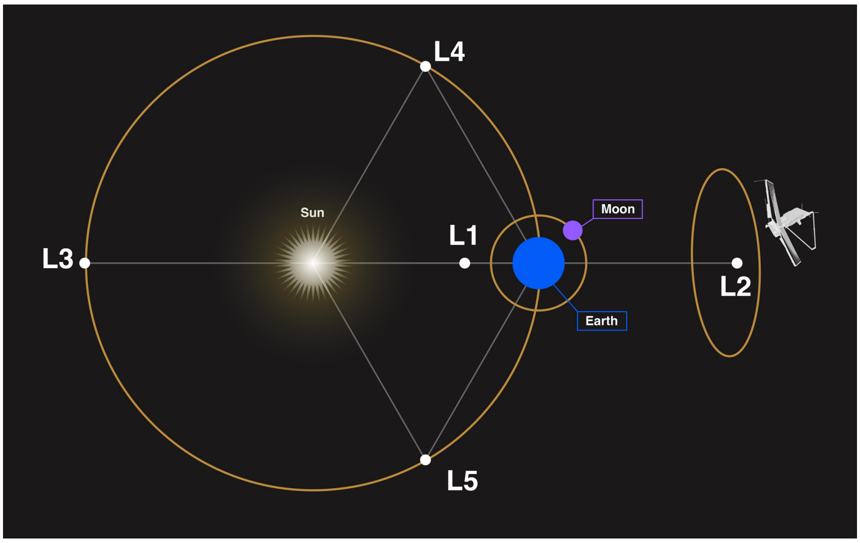
8.1.2 The Lagrange points and the Hill regions

• **Definition 8.5.** The **Lagrange points** L_i are the equilibrium configurations of the CR3BP. In other words, they are critical points of H $(H'(L_i) = 0)$.

One can show that there are exactly five Lagrange points [FK18], which we order by:

$$H(L_1) \leq H(L_2) \leq H(L_3) \leq H(L_4) \leq H(L_5).$$

✧ **Example 8.6.** For instance, in the Sun-Earth-Moon three-body system:



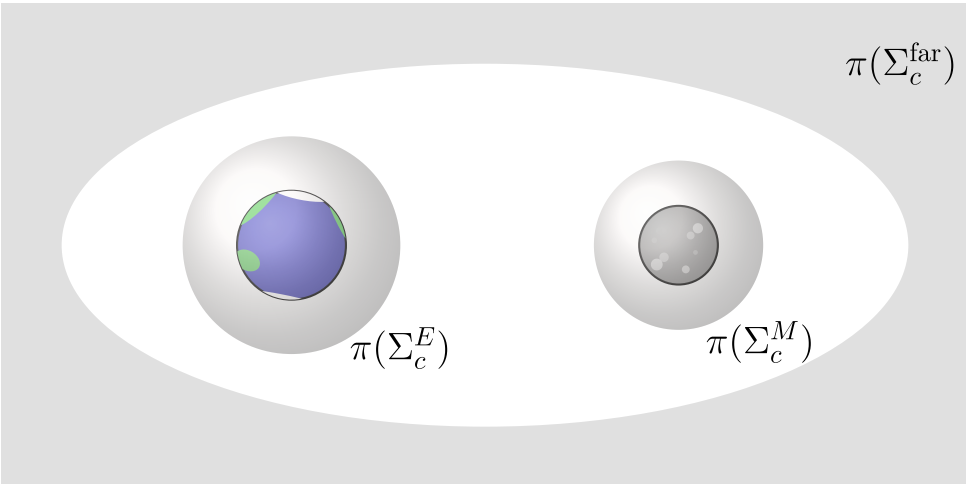
Picture: Webb's Orbit at Sun-Earth Lagrange Point 2 (L2), NASA, STScI, CSA, 2021.

The first three, L_1, L_2, L_3, originally found by Euler, all lie on the Earth-Moon axis. They are unstable, which means that the least perturbation of the satellite would send it flying away. Meanwhile, the points L_4 and L_5, discovered by Lagrange, are stable. This is good news since, if we look for example at the Sun-Jupiter system, then the L_4 and L_5 points accommodate the Trojan asteroids – a number of which range hundreds of kilometres in diameter.

We refer to [FK18, Ch. 5] for more details on these critical points/equilibrium configurations, we simply recall the following fact:

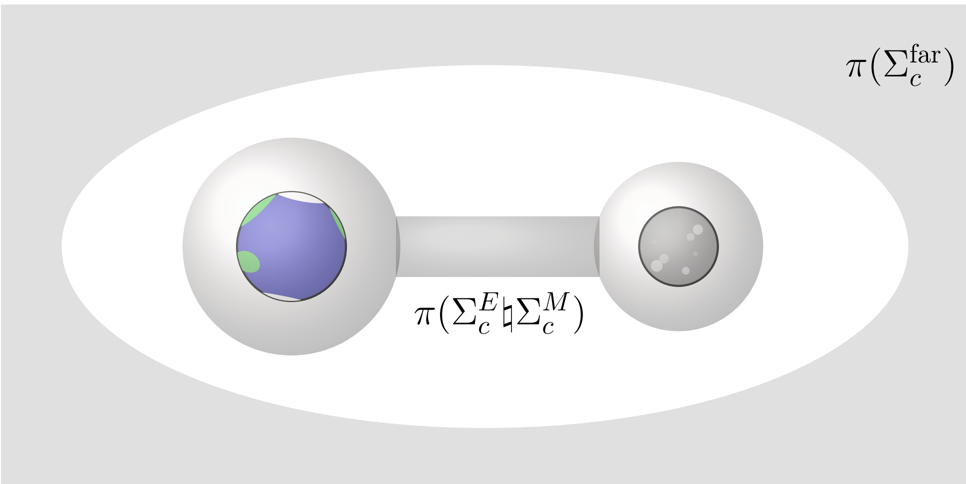
Lemma 8.7. *$H(L_1) \leq -3/2$.*

Pick an energy $c < H(L_1)$, and define the level set $\Sigma_c := H^{-1}(c) \subset T^\star\mathbb{R}^3$. It has 3 connected components: one near the Earth, one near the Moon, and one far away. Their projections to position space, under the map $\pi : T^\star\mathbb{R}^3 \twoheadrightarrow \mathbb{R}^3$ are called the **Hill regions**. They represent the regions accessible by a satellite with energy $H = c$.



For example, a satellite starting near the Earth with energy $c < H(L_1)$ will be constrained to the gray ball around it; likewise for the Moon. Meanwhile, an object starting far away from Earth with energy c will stay far away from it, and the region in white is forbidden.

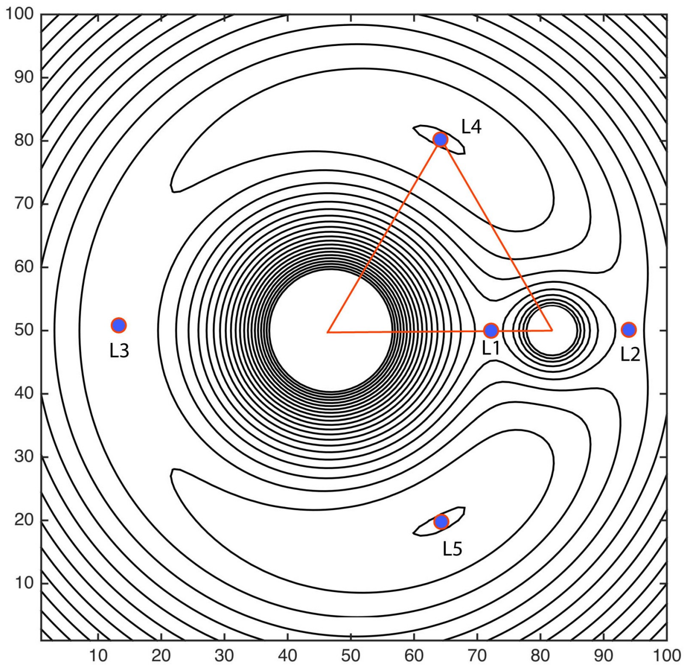
➤ Now let us ask ourselves what happens if we take our energy to be slightly above the first Lagrange energy, *i.e.* $H(L_1) \leq c < H(L_1) + \varepsilon$. From Morse theory, passing the critical point L_1 corresponds to adding a topological handle to the level set $H^{-1}(c)$. Therefore, projected to position space, the new level set will look something like:



In other words, by crossing the critical energy $H(L_1)$ we add a **transfer window** between the Earth and the Moon, allowing us to send the satellite from one to the other.

Hence, Σ_c now only consists of two connected components: the boundary connected sum $\Sigma_c^{E,M} \cong \Sigma_c^E \natural \Sigma_c^M$, comprising the Earth and Moon, and Σ_c^{far}, far away at infinity.

◇ **Remark.** A more physical way to observe these Hill regions is by directly looking at the contour lines of the Three-Body Problem potential:



Picture: Effective potential for the planar three-body problem, [Nol19], 2019.

8.1.3 Regularising at collisions

Now assume that our satellite of mass zero is on course to collide with one of the two large bodies; without loss of generality, the Earth. Assume that the satellite has subcritical energy $c < H(L_1)$, so that we can restrict our attention to the connected component Σ_c^E.

Lemma 8.8. *As the satellite collides with the Earth, momentum p blows up to infinity.*

Proof. If the satellite collides with the Earth (*i.e.* $q \to \vec{E}$), the second term of

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{1-\mu}{\left\|q-\vec{E}\right\|} - \frac{\mu}{\left\|q-\vec{M}\right\|} + (q_1p_2 - q_2p_1)$$

blows up to infinity. However, conservation of energy says that $H \equiv c$, which implies that momentum p must necessarily go to infinity to counter-act this blow up. □

One way to rephrase this is by saying that the connected component Σ_c^E of $\Sigma_c = H^{-1}(c)$ is **non-compact**. More precisely, since Σ_c^E is contained in phase space $T^\star(\mathbb{R}^3 \backslash \{\vec{E}, \vec{M}\})$, with coordinates (q,p), then we can observe that:

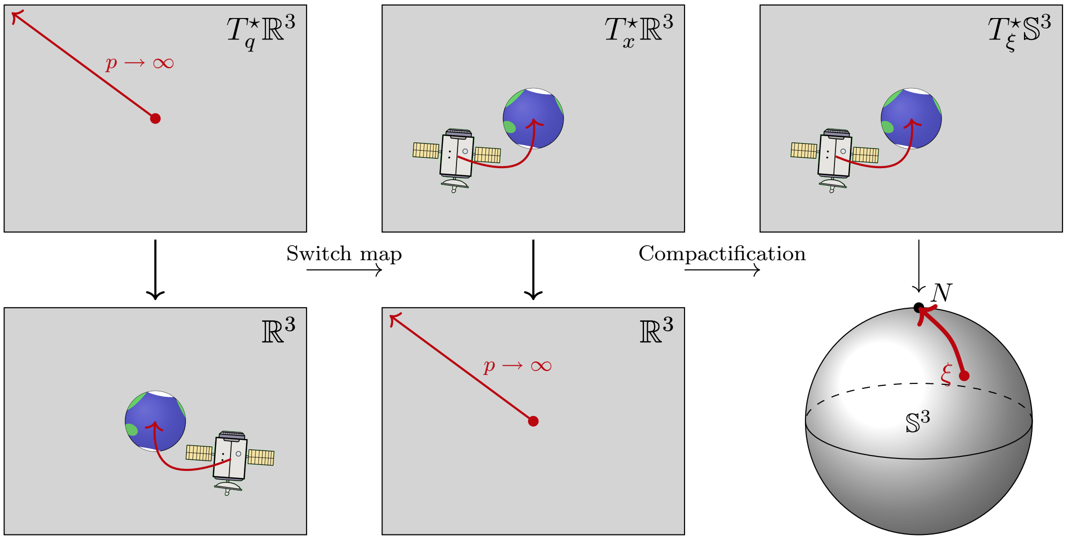
As $q \to \vec{E}$ *in the base space*, $p \to \infty$ *in the fibres.*

The fact that this blow up happens in the fibres is not very convenient. Indeed, it would be easier to compactify the base space once and for all than each fibre individually. For this reason, we swap the q and p coordinates, by applying a simple switch map:

$$\text{switch} : T^\star\mathbb{R}^3 \longrightarrow T^\star\mathbb{R}^3 : (q,p) \longmapsto (p,-q) =: (x,y). \tag{8.3}$$

We add a minus sign so that switch preserves the symplectic form. Ambient space is still $T^\star\mathbb{R}^3$, except base space is now momentum space, and the fibres are minus position space. This is much easier to compactify, because it suffices to compactify the base space $\mathbb{R}^3$ into $\mathbb{S}^3$, by adding a point N at $p = \infty$. This process is called Moser regularisation.

Schematic visualisation of Moser regularisation:



Formally, the compactification step is done via inverse stereographic projection. Write:

$$\begin{aligned} T^\star\mathbb{S}^3 &:= \left\{(\xi,\eta) \in T^\star\mathbb{R}^4 \mid \|\xi\|^2 = 1\right\} \\ &\cong \left\{(\xi,\eta) \in \mathbb{R}^4 \oplus \mathbb{R}^4 \mid \|\xi\|^2 = 1, \langle \xi,\eta\rangle = 0\right\}, \end{aligned} \tag{8.4}$$

with $N = (1,0,0,0) \in \mathbb{S}^3$ be the North pole. Then, the standard stereographic projection is defined as the map $T^\star\mathbb{S}^3\backslash\{N\} \to \mathbb{T}^\star\mathbb{R}^3 : (\xi,\eta) \mapsto (x,y)$ such that:

$$x = \frac{\xi}{1-\xi_0}, \quad y = \eta_0\xi + (1-\xi_0)\eta. \tag{8.5}$$

Its inverse Ξ is given by:

$$\begin{aligned} \xi_0 &= \frac{\|x\|^2 - 1}{\|x\|^2 + 1} \\ \xi_i &= \frac{2x_i}{\|x\|^2 + 1} \quad \text{for i = 1,2,3} \\ \eta_0 &= \langle x, y\rangle \\ \eta_i &= \frac{\|x\|^2 + 1}{2} y_i - \langle x,y\rangle\, x_i \quad \text{for i = 1,2,3} \end{aligned} \tag{8.6}$$

Proposition 8.9 ([Alb+12b; CJK20]). *In these regularised coordinates, the Spatial Circular Restricted Three-Body Problem is described by the Hamiltonian:*

$$Q(\xi,\eta) = \frac{1}{2} f(\xi,\eta)^2 \|\eta\|^2, \tag{8.7}$$

where:

$$f(\xi,\eta) := 1 + (1-\xi_0)b(\xi,\eta) + M(\xi,\eta), \tag{8.8}$$

$$b(\xi,\eta) := -(c+1/2) - \frac{1-\mu}{\left\|\overrightarrow{\eta}(1-\xi_0) + \overrightarrow{\xi}\eta_0 + \overrightarrow{m} - \overrightarrow{e}\right\|}, \tag{8.9}$$

$$M(\xi,\eta) := (1-\xi_0)(\xi_2\eta_1 - \xi_1\eta_2) - \xi_2(1-\mu), \tag{8.10}$$

where $\overrightarrow{e}$ and $\overrightarrow{m}$ are the coordinates of the Earth and Moon after regularisation. In particular, the regularisation of the energy hypersurface $H^{-1}(c)$ is given by $Q^{-1}\left(\frac{1}{2}\mu^2\right)$.

Proof sketch. (For full proof, see §6 of [CJK20]). For energies $c < H(L_1)$, consider the connected component Σ_c^E above the Earth $\overrightarrow{E}$. We saw in Lemma 8.8 that the singularity in the Hamiltonian arises as $q \to \overrightarrow{E}$, in which case the term

$$\frac{1-\mu}{\|q-E\|}$$

blows up to ∞ in the expression for $H(q,p)$. The first, naïve guess to get rid of this singularity is to consider:

$$H'(q,p) = H(q,p) \cdot \left\|q - \overrightarrow{E}\right\|,$$

which is now continuous at $q = \overrightarrow{E}$. Implicitly, defining this new Hamiltonian corresponds to reparametrising the time-variable by

$$\mathrm{d}t' = \int \frac{1}{\left\|q - \overrightarrow{E}\right\|}\mathrm{d}t.$$

We explain this time reparametrisation in Remark A.30. Now, if we apply the Moser change of coordinates

$$(q,p) \xmapsto{\text{switch}} (x,y) \xrightarrow{\Xi} (\xi,\eta)$$

to H', then we get:

$$\begin{aligned}\tilde{H}'(\xi,\eta) &= \|\eta\| \left(1 - \frac{(1-\mu)(1-\xi_0)}{\|(1-\xi_0)\eta + \eta_0\xi + \overrightarrow{m} - \overrightarrow{e}\|} + (1-\xi_0)(\xi_2\eta_1 - \xi_1\eta_2) + \xi_2 m_1 - \big(c+1/2\big)(1-\xi_0)\right) - \mu \\ &= \|\eta\|\, f(\xi,\eta) - \mu.\end{aligned}$$

$\therefore$ In particular, the level set $H'^{-1}(0)$ describes the regularisation of $H^{-1}(c)$.

However, it has the undesirable property that $\tilde{H}'$ is not smooth at $\eta = 0$, since the norm $\|\cdot\|$ isn't. To fix this, we consider instead the Hamiltonian:

$$Q(\xi,\eta) := \frac{1}{2}\left(\tilde{H}'(\xi,\eta) + \mu\right)^2 = \frac{1}{2}f(\xi,\eta)^2\,\|\eta\|^2,$$

which is now smooth, and has the same dynamics as $\tilde{H}'$ up to a time reparametrisation. In particular, we have $\tilde{H}'^{-1}(0) = Q^{-1}\left(\frac{1}{2}\mu^2\right)$. □

To formally prove that $\tilde{H}'^{-1}(0) = Q^{-1}\left(\frac{1}{2}\mu^2\right)$, at the end of the previous proof, we technically need to ensure that f does not change signs along it, namely:

Lemma 8.10. *f is positive along $Q^{-1}\left(\frac{1}{2}\mu^2\right)$. Actually, $f(\xi,\eta) > \mu/2$.*

Proof. In §6.2 [CJK20], they explicitly derive the estimate

$$|f(\xi,\eta)| > \mu/2$$

essentially by using the triangle inequality on the expression of f, (8.8). Therefore, f does not change signs. Meanwhile, notice that $\tilde{H}'^{-1}(0) \subset Q^{-1}\left(\frac{1}{2}\mu^2\right)$. Indeed,

$$f(\xi,\eta)\,\|\eta\| = \mu \implies f(\xi,\eta)^2\,\|\eta\|^2 = \mu^2$$

On the level set $\tilde{H}' = f(\xi,\eta)\,\|\eta\| - \mu = 0$ we must have $f > 0$. Therefore, we have $f > 0$ on all of $Q^{-1}(\frac{1}{2}\mu^2)$. □

➤ So in summary, the regularised flow of the Spatial Circular Restricted Three-Body Problem, for low energies and near the Earth and Moon, can be expressed as the flow of:

$$\begin{aligned} Q : T^\star\mathbb{S}^3 &\longrightarrow \mathbb{R} \\ (\xi,\eta) &\longmapsto \frac{1}{2} f(\xi,\eta)^2\,\|\eta\|^2\,, \end{aligned}$$

which is a deformation of the free-particle Hamiltonian $Q_0(\xi,\eta) = \frac{1}{2}\|\eta\|^2$ on $\mathbb{S}^3$.

✧ **Example 8.11.** The Hamiltonian $Q_0(\xi,\eta) = \frac{1}{2}\|\eta\|^2$ generates the standard geodesic flow on $T^\star\mathbb{S}^3$. Indeed, it corresponds to the total energy of a particle moving in $\mathbb{S}^3$, on which no external forces are applied. Its trajectories project down to geodesics of the 3-sphere $\mathbb{S}^3$ – in other words, great circles. Hypersurfaces of this flow are given by:

$$\tilde{\Sigma}_c := Q^{-1}\left(\frac{1}{2}\right) = \left\{(\xi,\eta) \in T^\star\mathbb{S}^3 \;\middle|\; \|\eta\| = 1\right\} = \mathbb{S}^\star\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{S}^2. \tag{8.11}$$

In the CR3BP, we do not have such a nice expression for $\tilde{\Sigma}_c = Q^{-1}\left(\frac{1}{2}\mu^2\right)$. However, one can still construct a diffeomorphism $\tilde{\Sigma}_c \cong S^\star\mathbb{S}^3$ (see §2 of [Kum82]). Hence, the regularised flow of the CR3BP can be viewed, up to a diffeomorphism, as a flow on $\mathbb{S}^\star\mathbb{S}^3$.

∴ **Take-away:** In general, for energies $c < H(L_1)$, the regularised connected component $\tilde{\Sigma}_c^E$ near the Earth is diffeomorphic to $\mathbb{S}^\star\mathbb{S}^3$. Similarly, $\tilde{\Sigma}_c^M \cong \mathbb{S}^\star\mathbb{S}^3$ near the Moon.

◇ **Remark 8.12.** All the computations we did were for energies $c < H(L_1)$. If c is, instead, between $H(L_1)$ and $H(L_2)$, then we have:

$$\tilde{\Sigma}_c^{M,E} \cong \tilde{\Sigma}_c^M \natural \tilde{\Sigma}_c^E \cong \mathbb{S}^\star\mathbb{S}^3 \natural \mathbb{S}^\star \natural \mathbb{S}^3,$$

where $\natural$ denotes the boundary connected sum.

◇ **Remark 8.13.** Ultimately, this change of coordinates we performed had the effect of regularising two-body collisions into *elastic collisions*. Indeed, in these new coordinates, it looks as if the satellite slowed down to speed 0 when colliding with the Earth and then bounced back where it came from (see Remark A.30). This bouncing back phenomenon is more easily observed in the planar case (Remark A.31), where our change of coordinates is simply the cotangent lift of $z \mapsto z^2$. For a great, general survey on how to regularise central force problems via elastic bouncing, from a symplectic point-of-view, see [FZ20].

◇ **Remark 8.14** (On other regularisation processes)**.** As we have already mentioned, there are more than one regularisation techniques. The one we described, Moser regularisation, provides a regularisation of a singular flow on $T^\star\mathbb{R}^n$ to $T^\star\mathbb{S}^n$. In the case of the SCR3BP, regularised energy hypersurfaces become $\mathbb{S}^\star\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{S}^2$; and in the case of the PCR3BP, they become $\mathbb{S}^\star\mathbb{S}^2 \cong \mathbb{RP}^3$.

This was not the first regularisation technique to be discovered though. The first one, in the case of the Kepler and Three-Body Problems, was due to Levi-Civita. It regularised hypersurfaces of the PCR3BP flow to $\mathbb{S}^3$, which is nothing but the double cover of $\mathbb{RP}^3$. In particular, one can show that Levi-Civita regularisation is a double-cover of Moser regularisation in dimension $n = 2$. We explain Levi-Civita regularisation in §A.3.1.

There are many other types of regularisation techniques in classical physics/celestial mechanics; each achieving a different purpose. In the presence of N bodies, for N a large number, we also need to talk about three-body collision, four-body collision, etc... For a (non-exhaustive) survey of regularisation techniques in N-body problems, see [Int21]. In this thesis, we will be exclusively interested in Moser regularisation.

8.2 Early symplectic steps in space

8.2.1 Contact geometry of the Three-Body Problem

The first steps towards marrying modern symplectic geometry and the Three-Body Problem were taken in the 2010s. The theorem which opened the door for the whole alley of research we follow in this thesis can concisely be written:

Theorem 8.15 ([Alb+12b; CJK20]). *For subcritical energies $c < H(L_1)$, the connected components $\tilde{\Sigma}_c^E$ and $\tilde{\Sigma}_c^M$ of the regularised energy hypersurface, endowed with the CR3BP flow, are contactomorphic to $(\mathbb{S}^\star\mathbb{S}^3, \alpha_{std})$. In particular, the regularised flow of the CR3BP is proportional of the standard Reeb flow on $\mathbb{S}^\star\mathbb{S}^3$.*

This is still true for energies $H(L_1) \leq c < H(L_1) + \varepsilon$, for small enough ε, except we now have the contactomorphism $\tilde{\Sigma}_c^{E,M} \cong (\mathbb{S}^\star\mathbb{S}^3 \natural \mathbb{S}^\star\mathbb{S}^3, \alpha_{\text{std}})$.

This was proved in two stages. First, in [Alb+12b], Albers, Frauenfelder, Paternain, and van Koert proved it for the *planar* circular restricted three-body problem (PCR3BP). Hence, they showed that energy hypersurfaces of the PCR3BP were *contactomorphic* to $\mathbb{S}^\star\mathbb{S}^2 \cong \mathbb{RP}^3$ with its standard contact structure.

Less than a decade later, the result was generalised to the spatial problem (SCR3BP), which is the form in which we stated it, by Cho, Jung, and Kim.

Proof sketch of Theorem 8.15. The planar and spatial proofs ([Alb+12b] and [CJK20]) are extremely similar to each other. The idea, for energies $c < H(L_1)$, is to pick the connected component near the Earth or Moon (say the Earth, which we centre at zero). Then, after regularisation to $T^\star\mathbb{S}^3$, the Liouville vector field is given by

$$V = \sum_i \eta_i \frac{\partial}{\partial \eta_i}.$$

One can explicitly show that $V(H) > 0$, by using the estimate from Lemma 8.10. See §6 of [CJK20] for a complete proof. ☐

Corollary 8.16. *Consider, for example, the connected component $\tilde{\Sigma}_c^E$, in the SCR3BP. Write $(\tilde{\Sigma}_c^E)^P$ its equivalent in the PCR3BP, i.e. we enforce $q_3 = p_3 = 0$ before regularising at collisions. Let α^P be the contact form obtained on $(\tilde{\Sigma}_c^E)^P$ by Theorem 8.15, and α the one on $\tilde{\Sigma}_c^E$. Then:*

$$\alpha^P \equiv \alpha|_{(\tilde{\Sigma}_c^E)^P}.$$

In other words, if we restrict the contact form generating the SCR3BP flow to the planar problem, then it generates the PCR3BP flow.

Proof. Both cases [Alb+12b; CJK20] are proved in the exact same way, the only difference is that when restricting from the spatial to the planar case, we drop the last term in the expression of V. In particular, the contact structure induced on Σ^P is simply the restriction of the one induced on Σ. ☐

Theorem 8.15 is the foundation of our research. It opens the door for the use of all sorts of tools from symplectic and contact geometry in the study of the Three-Body Problem: Conley-Zehnder indices, pseudo-holomorphic curves, Floer theory,...

In particular, our hope now is to use our Poincaré-Birkhoff theorems from Part II to find trajectories in the CR3BP. For this purpose, we need to find Liouville domains.

8.2.2 Poincaré's last theorem and the search for periodic orbits

In this section, we explain Poincaré's original idea to find periodic orbits in the Three-Body Problem. We start with the Planar Circular Restricted Three-Body Problem (PCR3BP), since that was the setup Poincaré used; and we will then generalise to the spatial one.

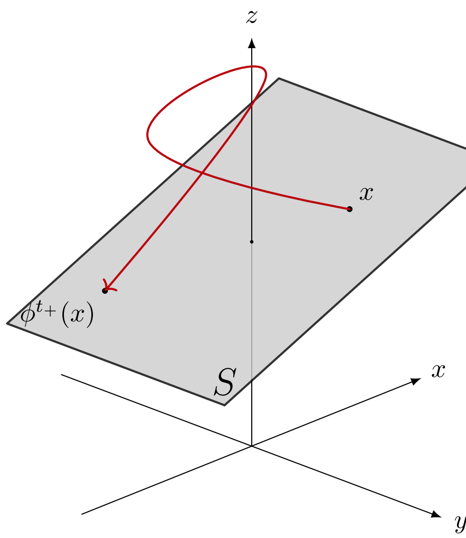
In the PCR3BP, unregularised phase space is given by $T^\star(\mathbb{R}^2\backslash\{\overrightarrow{E},\overrightarrow{M}\})$, which after Moser regularisation becomes $T^\star\mathbb{S}^2$. For energy $c < H(L_1)$, the bounded connected components of the regularised energy hypersurface are then diffeomorphic to $\mathbb{S}^\star\mathbb{S}^2 \cong \mathbb{RP}^3$.

• **Definition 8.17.** Let ϕ^t be a flow on a manifold M. A **global hypersurface of section** is a compact, oriented, codimension 1 submanifold $S \subset M$ such that:

(i) ∂S is invariant under ϕ^t (it could be empty);

(ii) the flow ϕ^t is positively transverse to the interior of S;

(iii) for every $x \in \text{int}(S)$, there exist $t_+ > 0$ and $t_- < 0$ such that $\phi^{t_\pm}(x) \in S$. In other words, for any x in the interior of Σ, the flow of x returns to S both in the future and in the past.



The most important property being the third one. It allows us to define a map

$$\tau : \text{int}(S) \longrightarrow \text{int}(S) : x \longmapsto \phi^{t_+}(x),$$

where $t_+ > 0$ is taken to be the first $t_+ > 0$ such that $\phi^{t_+}(x) \in S$. $\tau : S \to S$ is often referred to as the **Poincaré return map**.

Poincaré's insight was then the following: a periodic orbit of the flow is the same thing as a periodic point of the map $\tau : S \to S$, *i.e.* a point x such that $\tau^k(x) = x$ for some $k \in \mathbb{N}\backslash\{0\}$. Therefore, if he could find a global hypersurface of section for the PCR3BP, and prove a fixed point theorem for its return map, then he had come up with a scheme to find periodic orbits.

In a perturbative setting, Poincaré and Birkhoff showed [Poi12; Bir13] that one could find such a global hypersurface of section, in the form of an annulus called the **Birkhoff annulus**.

As it happens, Poincaré *did* find such hypersurface of section for the PCR3BP, in the form of an annulus, whose construction we explicit in §A.3.2 of the appendix. His construction was actually carried out in the Two-Body Problem, and as such only generalises to perturbative cases of the PCR3BP. Its general existence is still an open problem, encapsulated in:

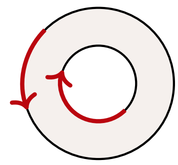
Conjecture (Birkhoff conjecture)**.** *For every energy $c < H(L_1) + \varepsilon$, there exists an annulus-like global surface of section for the PCR3BP flow.*

At the time of writing this thesis, the Birkhoff conjecture is known to hold in the convexity range, a specific range of parameters (μ, c) (see Remark 5.3). For a state-of-the-art on this conjecture, see [FK18, Chapter 1].

➤ Hence, finding periodic orbits of period 1 of the Planar Circular Restricted Three-Body Problem corresponds to finding fixed points of a map τ on the annulus (when it exists). One can easily show that this map is a symplectomorphism ([FK18]), and we observe in §A.3.2 that the boundary circles of this annulus are the direct and retrograde orbits of the moon – so that τ rotates them in opposite direction.

In particular, we are now in a position to apply the Poincaré-Birkhoff theorem, which we already stated in Chapter 5:

Theorem (Poincaré-Birkhoff, 1912-1913)**.** *Let τ be an area-preserving self-homeomorphism of the annulus which satisfies the* ***twist condition****, i.e. it rotates both its boundary components in opposite directions. Then, τ has infinitely interior periodic points, of arbitrarily large order.*



Corollary 8.18. *For every pair (μ, c) in the convexity range, where μ is the mass ratio and c the energy, there exist infinitely many periodic orbits in the Planar Circular Restricted Three-Body Problem, of arbitrarily large period.*

In [MK22a; MK22b], Moreno and van Koert reproduced roughly the same scheme as Poincaré, but for the *Spatial* Circular Restricted Three-Body Problem (SCR3BP). The first step was, hence, to find a global hypersurface of section.

8.3 The Three-Body Problem open book

8.3.1 What is an open book?

In [MK22b], Moreno and van Koert constructed a global hypersurface of section for the SCR3BP. Actually, they constructed a whole $\mathbb{S}^1$-family of them, arranged in an *open book*.

• **Definition 8.19.** Let M be a manifold of dimension 3 or more, and B be a codimension 2 submanifold. An **open book decomposition** is a fibration $\pi : M \backslash B \to \mathbb{S}^1$ such that, in a tubular neighbourhood of B, π agrees with an angular coordinate θ. B is called the binding, and the fibres are called pages, written P.

The reason for this name is because locally, this decomposition looks just as if we had taken an actual book and glued its front cover to its back, so that the pages spread out in a circle. Then B corresponds to the actual binding of the book, and the pages to actual pages.

Picture: An open book in $\mathbb{R}^3$ (Credit: Microsoft Copilot AI).

Every page P of the open book shares the binding B as a boundary, and every point in $M\backslash B$ belongs to exactly one page of the open book.

For convenience, we assume that M is oriented, so that this orientation, along with the standard one on $\mathbb{S}^1$, induces orientations on every page P and on the binding B.

✧ **Example 8.20.** Let $M = \mathbb{R}^3 \cong \mathbb{R}^2 \oplus \mathbb{R}$, with coordinates (r, θ, z), and let $B = \{z\text{-axis}\}$. Then a natural open book decomposition is given by $\pi : M\backslash B : (r, \theta, z) \mapsto \theta$; with pages $P = \{\theta = \text{cst}\}$.

• **Definition 8.21.** A flow ϕ^t on M is said to be **adapted to the open book** if every page of the open book is a global hypersurface of section for ϕ^t, and the binding B is ϕ^t-invariant.

Actually, if the flow ϕ^t is a Reeb flow, we can make this definition stronger:

• **Definition 8.22.** Let (M, ξ) be a contact manifold. Then an open book on M is adapted to ξ in the sense of Giroux if there exists a contact form α supporting ξ such that

- $\alpha|_B$ is of positive contact type, where B is the binding of the open book.
- $d\alpha$ is positive, and induces a symplectic structure on the interior of every page.

We can invoke a standard lemma, of which a proof can be found in [Koe17]:

Lemma 8.23. *A contact form α is adapted to an open book in the sense of Giroux iff the binding B is invariant, and the Reeb vector field $\mathcal{R}_\alpha$ is positively transverse to the interior of every page.*

✧ **Example 8.24** (Geodesic open book on $\mathbb{S}^\star\mathbb{S}^n$)**.** Consider the standard Reeb flow on $(\mathbb{S}^\star\mathbb{S}^3, \alpha_{\text{std}})$. One can easily show that this is the same thing as the geodesic flow on $\mathbb{S}^3$, generated by the free-particle Hamiltonian $Q_0(\xi, \eta) = \frac{1}{2}\|\eta\|^2$ (see Example 8.11). This geodesic flow formally lives $T^\star\mathbb{S}^3$, but in practice it suffices to work on hypersurfaces of the form:

$$Q_0^{-1}\left(\frac{1}{2}\right) = \left\{(\xi, \eta) \in T^\star\mathbb{S}^n \mid \|\eta\|^2 = 1\right\} = \mathbb{S}^\star\mathbb{S}^n.$$

Notice that $B := \{\xi_n = \eta_n = 0\}$ is invariant under the flow of Q_0, so that we can define:

$$\pi_g : \mathbb{S}^\star\mathbb{S}^n \backslash B \longrightarrow \mathbb{S}^1 : (\xi_0, \dots, \xi_n, \eta_0, \dots, \eta_n) \longmapsto \frac{\xi_n + i\eta_n}{\|\xi_n + i\eta_n\|}. \tag{8.12}$$

A short calculation (see §6.2 of [MK22b]) ensures that the Hamiltonian vector field, X_{Q_0}, is positively transverse to the interiorvery page, verifying (ii) of Definition 8.17. The geodesic flow is well-known to be periodic (its orbits are great circles), therefore the flow returns to every page[(2)], verifying (iii). Hence, π_g does define an open book on $\mathbb{S}^\star\mathbb{S}^n$, which we call the **geodesic open book**. By Lemma 8.23, this open book is adapted to the geodesic flow in the sense of Giroux.

➤ We can explicitly study the symplectic structure of the pages:

Lemma 8.25. *Let (Σ, α) be a contact manifold, with an adapted open book decomposition in the sense of Giroux. Then every page P is a degenerate Liouville domain (Definition 7.1).*

Proof. $\lambda := \alpha|_P$. Then by assumption, $\mathrm{d}\lambda$ defines a symplectic form on $\mathrm{int}(P)$. However, it degenerates along $B := \partial P$ because $\lambda|_B \equiv \alpha|_B$ which is of contact type. Therefore, $\mathrm{d}\lambda(\mathcal{R}_\alpha, \cdot) \equiv 0$, so that $\mathrm{d}\lambda$ degenerates along the boundary. □

8.3.2 The open book of the SCR3BP

The upshot of this subsection is that for low energies in the SCR3BP, **there exists an open book** adapted to the regularised flow, whose **binding is the PCR3BP**.

To make sense of this last statement: let us go back to unregularised coordinates (q, p). In position space, the Planar Circular Restricted Three-Body Problem (PCR3BP) is obtained by constraining the satellite to the plane $\{q_3 = p_3 = 0\}$.

Therefore, in phase space $T^\star(\mathbb{R}^3 \backslash \{\overrightarrow{E}, \overrightarrow{M}\})$, the PCR3BP can be viewed as the subspace:

$$\left\{(q,p) \in T^\star(\mathbb{R}^3 \backslash \{\overrightarrow{E}, \overrightarrow{M}\}) \mid q_3 = p_3 = 0\right\} \cong T^\star(\mathbb{R}^2 \backslash \{\overrightarrow{E}, \overrightarrow{M}\}),$$

which, after Moser regularisation (§8.1.3), becomes

$$B := \left\{(\xi, \eta) \in T^\star\mathbb{S}^3 \mid \xi_3 = \eta_3 = 0\right\}. \tag{8.13}$$

Then, the main result that interests us is:

Theorem 8.26 (Moreno, van Koert [MK22b])**.** *Fix an energy $c < H(L_1)$, and any mass ratio $\mu \in [0,1]$ in the **Spatial** Circular Restricted Three-Body Problem. Then the connected components $\tilde{\Sigma}_c^E$ and $\tilde{\Sigma}_c^M$ containing the Earth and Moon admit an adapted open book decomposition, in the sense of Giroux, whose binding is $B = \{\xi_3 = \eta_3 = 0\} \cong \mathbb{S}^\star\mathbb{S}^2$.*

In other words, energy hypersurfaces of the SCR3BP admit open book decompositions, whose bindings are the corresponding energy hypersurfaces for the PCR3BP.

Proof sketch of Theorem 8.26 (see [MK22b]). By §8.1.3, the regularised Hamiltonian of the SCR3BP on $T^\star\mathbb{S}^3$ is given by $Q(\xi, \eta) = \frac{1}{2} f(\xi, \eta)^2 \|\eta\|^2$. If f were constantly equal to 1, the connected component $\tilde{\Sigma}_c^E$ of $Q^{-1}\left(\frac{1}{2}\mu^2\right)$ containing the Earth would be **equal** to $\mathbb{S}^\star\mathbb{S}^3$; in which case we could use the geodesic open book from Example 8.24, given by:

$$\pi_g : \tilde{\Sigma}_c^E \backslash B \longrightarrow \mathbb{S}^1 : (\xi_0, \dots, \xi_3, \eta_0, \dots, \eta_3) \longmapsto \frac{\xi_3 + i\eta_3}{\|\xi_n + i\eta_n\|}. \tag{8.14}$$

In the general case though, $f \not\equiv 1$, and actually has quite a complicated expression (8.8). However, we still have a diffeomorphism $\tilde{\Sigma}_c^E \cong \mathbb{S}^\star\mathbb{S}^3$ (and actually, a contactomorphism, by Theorem 8.15), and the map (8.14) still defines an $\mathbb{S}^1$-fibration. Therefore, we can still define the geodesic open book as in Example 8.24.

(2) The return map of the flow is hence nothing but the identity.

The issue arises when trying to show that this open book is adapted to the contact flow (the tricky point being to show that X_Q is transverse to the pages). In [MK22b], Moreno and van Koert manage to do this in a neighbourhood of the collision locus $\{\xi_0 = 1\}$. However, their proof does not allow to conclude away from the collision locus.

To palliate this, they use a different open book π_p, the **physical open book**, defined right after this proof sketch, and for which one can prove that X_Q is transverse to the pages *away from the collision locus*. Then, in [MK22b, §8], they interpolate between the two open books, so as to construct one global open book, which is everywhere adapted to the contact flow, by Lemma 8.23. ☐

• **Definition 8.27** (Physical open book)**.** To define the physical open book on $\tilde{\Sigma}_c^E$, let us first revert to unregularised coordinates (q, p), in $T^\star(\mathbb{R}^3 \backslash \{\vec{E}, \vec{M}\})$. There, in much the same fashion as in Example 8.24, we can define an $\mathbb{S}^1$-fibration:

$$\begin{aligned} \pi_p : T^\star(\mathbb{R}^3 \backslash \{\vec{E}, \vec{M}\}) \backslash B &\longrightarrow \mathbb{S}^1 \\ (q,p) &\longmapsto \frac{q_3 + ip_3}{\|q_3 + ip_3\|}, \end{aligned} \tag{8.15}$$

where $B = \{(q,p) \mid q_3 = p_3 = 0\}$ is the phase space of the *Planar* CR3BP. After Moser regularisation (§8.1.3), the map π_p becomes

$$\pi_p : T^\star \mathbb{S}^3 \longrightarrow \mathbb{S}^1 \backslash B : (\xi, \eta) \longmapsto \frac{\Theta_p(\xi,\eta)}{\|\Theta_p(\xi,\eta)\|}, \tag{8.16}$$

where

$$\Theta_p(\xi,\eta) = \xi_3 + i(1-\xi_0)\left(\eta_0 \xi_3 + (1-\xi_0)\eta_3\right). \tag{8.17}$$

The induced fibration $\pi_p : \tilde{\Sigma}_c^E \backslash B \longrightarrow \mathbb{S}^1$ is called the **physical open book**, where now $B = \{\xi_3 = \eta_3 = 0\}$.

The computation of Θ, as well as the proof that this open book is adapted to the SCR3BP flow on $\tilde{\Sigma}_c^E$ away from the collision locus are carried out in [MK22b, §6-8].

Theorem 8.28 ([MK22b])**.** *Let $\tau : P_{\pi/2} \to P_{\pi/2}$ be the return map of the SCR3BP flow (Definition 8.17). Then τ is an exact symplectomorphism, i.e it can be generated as the time-1 map of a Hamiltonian $H_t : P_{\pi/2} \to \mathbb{R}$. Moreover, it extends smoothly to $\partial P_{\pi/2}$.*

➤ In summary: for sub-critical energies, Moreno & van Koert proved the existence of an open book adapted to the regularised CR3BP flow, and that the return map on the pages was an exact symplectomorphism. The upshot of Part III of this thesis is that we want to use Wrapped Floer theory on pages of this open book.

8.3.3 A particularly nice page of the open book

Let us single out a particularly nice page of the CR3BP open book, which we will use to do Floer theory.

Recall from §8.1.3 that for energies $c < H(L_1)$, the connected component of the energy hypersurface near the Earth (or Moon) is given by:

$$Q^{-1}\left(\frac{1}{2}\mu^2\right) = \left\{(\xi,\eta) \in T^\star\mathbb{R}^4 \mid \|\xi\|^2 = 1, \langle \xi, \eta \rangle = 0,\ Q(\xi,\eta) = \frac{1}{2}\mu^2\right\}.$$

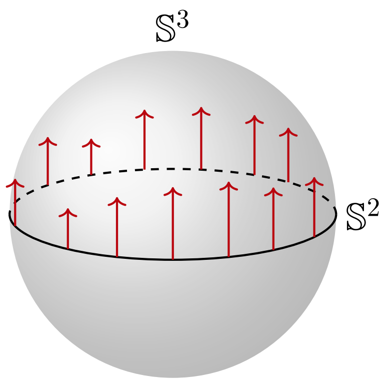
There, let us define the hypersurface:

$$P_{\pi/2} = \left\{(\xi,\eta) \in Q^{-1}\left(\frac{1}{2}\mu^2\right) \mid \xi_3 = 0, \eta_3 \geq 0\right\}. \tag{8.18}$$

◇ **Remark 8.29.** In unregularised coordinates (q, p), $P_{\pi/2}$ translates to $\{q_3 \leq 0, p_3 = 0\}$. Since $p = \dot{q}$ along physical trajectories, by the Hamiltonian equations of motion, $P_{\pi/2}$ effectively captures when trajectories $x(t) = (q(t), p(t))$ reach their minimum height $q_3 \leq 0$.

Back to regularised coordinates. Observe that $P_{\pi/2}$ can be re-written as:

$$\begin{aligned} P_{\pi/2} &= \left\{(\xi_0, \xi_1, \xi_2, 0, \eta_0, \eta_1, \eta_2, \eta_3) \in T^\star\mathbb{R}^4 \mid \|\xi\|^2 = 1,\, Q(\xi, \eta) = \mu^2/2,\, \eta_3 \geq 0\right\} \\ &= \left\{(\xi_0, \xi_1, \xi_2, 0, \eta_0, \eta_1, \eta_2, \eta_3) \in T^\star\mathbb{R}^3 \mid \|\xi\|^2 = 1,\, f(\xi, \eta)^2 \|\eta\|^2 = \mu^2, \eta_3 \geq 0\right\}. \end{aligned}$$



In other words, viewing $\mathbb{S}^2$ as the equator in $\mathbb{S}^3$, then $P_{\pi/2} \subset T^\star\mathbb{S}^3$ consists of all the vectors pointing away from it and into the upper hemisphere, with length $\mu/f(\xi, \eta)$.

Lemma 8.30. *$P_{\pi/2}$ is diffeomorphic to the disc cotangent bundle $\mathbb{D}^\star\mathbb{S}^2$ (Definition 3.5).*

Proof. This is because $1/f$ is smooth (due to f being positive, Lemma 8.10), so that we can smoothly transform the condition $\|\tilde{\eta}\|^2 \leq \mu^2/f^2$ into $\|\tilde{\eta}\|^2 \leq 1$. □

Proposition 8.31 ([MK22b])**.** *$P_{\pi/2}$ is a page both for the geodesic open book π_g (Example 8.24), and for the physical open book π_p (Defn. 8.27). Therefore, it is a page for the SCR3BP open book.*

Proof sketch. Near the collision locus, we use the geodesic open book:

$$\pi_g(\xi, \eta) = \frac{\xi_3 + i\eta_3}{\|\xi_3 + i\eta_3\|},$$

and away from it, we use the physical open book, given by:

$$\pi_p(\xi, \eta) = \frac{\Theta_p(\xi, \eta)}{\|\Theta_p(\xi, \eta)\|}, \quad \Theta_p(\xi, \eta) = \xi_3 + i(1 - \xi_0)\left(\eta_0\xi_3 + (1 - \xi_0)\eta_3\right).$$

We can easily observe that $P_{\pi/2} = \pi_g^{-1}(i)$ near the collision locus, and $P_{\pi/2} = \pi_p^{-1}(i)$ away from it. *A fortiori*, for a sensible interpolation π between π_g and π_p (see [MK22b, §6-8]), we get $P_{\pi/2} = \pi^{-1}(i)$ globally. □

◇ **Remark 8.32.** Note that $\pi^{-1}(-i) = \{\xi_3 = 0, \eta_3 \leq 0\}$ is also a page of the open book. Visually, it lies diametrically opposite to $P_{\pi/2}$.

✧ **Example 8.33** (Truncated coordinates on $P_{\pi/2}$)**.** Since $P_{\pi/2}$ is given by

$$P_{\pi/2} = Q^{-1}\left(\frac{1}{2}\mu^2\right) \cap \left\{(\xi, \eta) \mid \xi_3 = 0, \eta_3 \geq 0\right\},$$

we can get rid of two of the coordinates, and describe it using coordinates on $T^\star\mathbb{S}^2$. Indeed, let us write $\tilde{\xi} := (\xi_0, \xi_1, \xi_2)$ and $\tilde{\eta} := (\eta_0, \eta_1, \eta_2)$. Then, since $\eta_3 \geq 0$ on $P_{\pi/2}$, we can recover:

$$\eta_3 = \sqrt{\frac{\mu^2}{f^2} - \|\tilde{\eta}\|^2}.$$

In particular, if we decide to forget about the ξ_3 and η_3 coordinates, and write:

$$f(\tilde{\xi}, \tilde{\eta}) := f\left(\tilde{\xi}, 0, \tilde{\eta}, \sqrt{\frac{\mu^2}{f^2} - \|\tilde{\eta}\|^2}\right), \tag{8.19}$$

then we can rewrite:

$$P_{\pi/2} = \left\{(\tilde{\xi}, \tilde{\eta}) \in T^\star\mathbb{S}^2 \mid \frac{1}{2} f(\tilde{\xi}, \tilde{\eta})^2 \|\eta\|^2 = \frac{1}{2}\mu^2\right\}.$$

Since, by construction, $\|\tilde{\eta}\|^2 \leq \|\eta\|^2$, with equality iff $\eta_3 = 0$ (*i.e.* we lie on the boundary $\partial P_{\pi/2}$), then we can rewrite:

$$P_{\pi/2} = \left\{(\tilde{\xi}, \tilde{\eta}) \in T^\star\mathbb{S}^2 \mid \frac{1}{2} f(\tilde{\xi}, \tilde{\eta})^2 \|\tilde{\eta}\|^2 \leq \frac{1}{2}\mu^2\right\}, \tag{8.20}$$

$$\partial P_{\pi/2} = \left\{(\tilde{\xi}, \tilde{\eta}) \in T^\star\mathbb{S}^2 \mid \frac{1}{2} f(\tilde{\xi}, \tilde{\eta})^2 \|\tilde{\eta}\|^2 = \frac{1}{2}\mu^2\right\}. \tag{8.21}$$

For simplicity, write $F(\tilde{\xi}, \tilde{\eta}) := f(\tilde{\xi}, \tilde{\eta}) \|\tilde{\eta}\|$. Since, by Lemma 8.10, the function f is positive, we can rewrite:

$$P_{\pi/2} = \left\{(\tilde{\xi}, \tilde{\eta}) \in T^\star\mathbb{S}^2 \mid F(\tilde{\xi}, \tilde{\eta}) \leq \mu\right\}, \tag{8.22}$$

$$\partial P_{\pi/2} = \left\{(\tilde{\xi}, \tilde{\eta}) \in T^\star\mathbb{S}^2 \mid F(\tilde{\xi}, \tilde{\eta}) = \mu\right\}. \tag{8.23}$$

8.4 The Floer-theoretical model

In [MK22a; MK22b], Moreno & van Koert proved a first Poincaré-Birkhoff theorem in the Spatial CR3BP. However, its applicability remained limited due to a number of technicalities:

Assumption (Moreno & van Koert's CR3BP model)**.**

- $P_{\pi/2}$ is a non-degenerate Liouville domain. ✗ False.
- $(\partial P_{\pi/2}, \alpha)$ is strongly index-definite. ✓ True, but only in a certain range of mass and energy parameters, called the *convexity range*.
- $\tau : P_{\pi/2} \to P_{\pi/2}$ is an exact symplectomorphism. ✓ True.
- $\tau : P_{\pi/2} \to P_{\pi/2}$ satisfies the twist condition. (?) Unknown.

The work from Part II of this thesis, which culminated in Theorem B3, allows us to replace their model by the improved:

Assumption (Improved CR3BP model)**.**

- $P_{\pi/2}$ is a degenerate Liouville domain. ✓ True.
- $\tau : P_{\pi/2} \to P_{\pi/2}$ is an exact symplectomorphism. ✓ True.
- $\tau : P_{\pi/2} \to P_{\pi/2}$ satisfies the Weakened Twist Condition. (?) Unknown.

In Chapters 9 and 10 we will apply this model to concrete examples in the CR3BP, assuming the veracity of the Weakened Twist Condition – which remains an open problem, at least away from a perturbative setting.

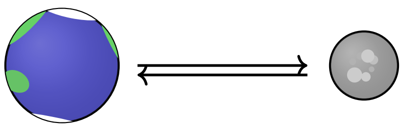
Chapter 9

Collision trajectories in the Circular Restricted Three-Body Problem

This chapter presents a first application of our machinery from Part II to the Circular Restricted Three-Body Problem. This result was first presented in [ML24].

Theorem C. *If the Circular Restricted Three-Body Problem satisfies the Weakened Twist Condition, then for every energy $c < H(L_1) + \varepsilon$, there exist infinitely many trajectories of spatial (consecutive) collision near the Earth and Moon, of arbitrarily large length.*

◇ **Remark 9.1.** The terminology « consecutive collision » may be confusing from a physics point-of-view. It is a by-product of the fact that we *regularised* at collisions (§8.1.3), which made it so that the dynamics continued after the satellite collided with the Earth or Moon, effectively as if it bounced back (see Remark 8.13).



There are many reasons why one may care about collision trajectories. Avoiding them seems like a valid one, for example. More interestingly, by perturbing a trajectory of collision, one obtains a trajectory of *close fly-by*. Those are for instance interesting for *gravitational assist*: using the pull of a large body to propel a spacecraft far into space.

◇ **Remark.** A similar theorem to Theorem C had already been derived in the Planar case, in [FZ19]. More recently, at the time of writing this thesis, Theorem C was obtained independently by Ruck in [Ruc23], using Rabinowitz Floer instead of Wrapped Floer Cohomology.

9.1 The collision Lagrangian

We will prove Theorem C as a direct application of Theorem B3. We will first find a Lagrangian $L_{\text{col}} \subset P_{\pi/2}$ describing configurations of collision in the SCR3BP, where $P_{\pi/2}$ is our page of the open book from §8.3.3; and we will show that $HW^*(L_{\text{col}})$ is non-zero in infinitely many degrees. Then, since $P_{\pi/2}$ is a degenerate Liouville domain, we will conclude that if the SCR3BP satisfies the Weakened Twist Condition, then there exist infinitely many trajectories of collision, by Theorem B3.

Assumption 9.2. $c < H(L_1)$. Without loss of generality, the satellite starts near the Earth.

Recall the scheme that we used to regularise at collisions in §8.1.3. We were faced with the issue that $p \to \infty$ as $q \to \overrightarrow{E}$. To fix the blow-up at collision, first we had swapped coordinates:

$$T^\star \mathbb{R}^3 \ni (q, p) \longmapsto (p, -q) \in T^\star \mathbb{R}^3,$$

so that the blow up now happened in the base space (instead of the fibres), and then we had compactified the base space by adding a point at $p = \infty$, obtaining:

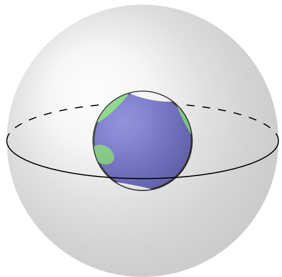
$$T^\star\mathbb{R}^3 \longrightarrow T^\star\mathbb{S}^3.$$

We call the $\{p = \infty\}$ the North pole $N \in \mathbb{S}^3$. Formally, the compactification above is given by the inverse stereographic projection (see §8.1.3).

➢ In unregularised coordinates (q, p), we define:

• **Definition 9.3.** The **collision locus** is the set of normalised directions along which the satellite can come crashing into the Earth.

Topologically, it simply corresponds to a 2-sphere $\mathbb{S}^2$ around the Earth.

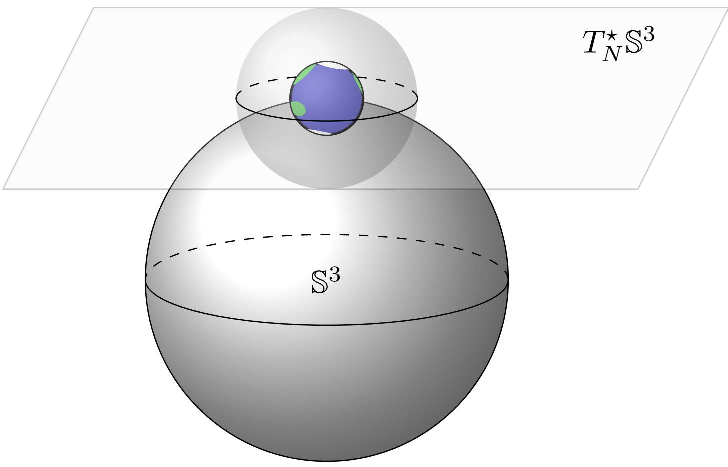


Heuristically, in unregularised phase space, this collision locus can be written:

$$\mathbb{S}^2 \times \{p = \infty\}. \tag{9.1}$$

After regularisation this becomes a spherical fibre in $T^\star\mathbb{S}^3$ above the point $N = \{p = \infty\}$. In other words:

$$\mathcal{L}_{\text{col}} \cong \mathbb{S}^\star_N\mathbb{S}^3 := \left\{(\xi, \eta) \in T^\star\mathbb{S}^3 \mid \|\eta\| = 1\right\}. \tag{9.2}$$



In other words, in regularised coordinates, the collision locus corresponds to the fibre above the North pole in $\mathbb{S}^\star\mathbb{S}^3$. This is nothing surprising: recall that by definition the North pole corresponds to trajectories with infinite momentum.

Now let $P_{\pi/2} \cong \mathbb{D}^\star\mathbb{S}^2$ be our page of the open book from §8.3.3, given in regularised coordinates by:

$$P_{\pi/2} = \left\{(\xi, \eta) \in T^\star\mathbb{S}^3 \mid \xi_3 = 0, \eta_3 \geq 0, \|\eta\|^2 \leq \mu^2/f^2\right\}. \tag{9.3}$$

In this page, we have:

Lemma 9.4. *$L_{col} := \mathcal{L}_{col} \cap P_{\pi/2}$ is diffeomorphic to the fibre $\mathbb{D}^\star_N\mathbb{S}^2$, and it is an exact Lagrangian in $P_{\pi/2}$. We call it the* ***collision Lagrangian****.*

Proof. Since the page $P_{\pi/2}$ agrees near the collision locus with the set of covectors along the equator pointing into the upper-hemisphere, then we directly have that $W \cap T^\star_N\mathbb{S}^3$ is identified with $\mathbb{D}^\star_N\mathbb{S}^2$, which is a disc under the diffeomorphism $W \cong \mathbb{D}^*S^2$. Visually, one can view it as the upper-hemisphere of the collision locus $\mathbb{S}^2$ (the fiber of Σ_c over N), whose boundary is the planar collision locus $\mathbb{S}^1$, embedded as the equator. In particular, L_{col} is exact Lagrangian because this is true of any fibre in a cotangent bundle. □

◇ **Remark 9.5.** In the case where our satellite has energy $c \in [H(L_1), H(L_1)+\varepsilon)$ (so that, by §8.1.1, there is a transfer window between the Earth and the Moon), then energy hypersurfaces are given by $Q^{-1}(1/2) \cong \mathbb{S}^\star\mathbb{S}^3 \natural \mathbb{S}^\star\mathbb{S}^3$, while $P_{\pi/2} \cong \mathbb{D}^\star\mathbb{S}^2 \natural \mathbb{D}^\star\mathbb{S}^2$. The collision Lagrangian is given by $L_{\text{col}} \cong \mathbb{D}^\star_N\mathbb{S}^2 \natural \mathbb{D}^\star_N\mathbb{S}^2$.

9.2 Floer cohomology of the collision Lagrangian

Lemma 9.4 tells us that L_{col} is exact. Observe that it is also orientable (it is a disc), and thus spin. Therefore, L_{col} is admissible for wrapped Floer cohomology, meaning that $HW^*(L_{\text{col}})$ is well-defined. Let us compute it.

This computation relies on work which even predated wrapped Floer theory, and which we owe to Viterbo, Salamon-Weber, Abbondandolo-Schwarz,... In its current formulation, the theorem we will use is:

Theorem 9.6 ([AS04])**.** *Let M be a compact, connected, orientable smooth manifold, and ΩM be its based loop space (Definition B.37). Let $pt \in M$ be a point. There is an isomorphism:*

$$HW^*(T^\star_{pt}M) \cong H^*(\Omega M)$$

where H^* denotes singular cohomology.

Corollary 9.7. *$HW^*(L_{col})$ is non-zero in infinitely many degrees.*

Proof. $L_{\text{col}} \subset P_{\pi/2}$ is diffeomorphic to $\mathbb{D}^\star_N\mathbb{S}^2$. Therefore, its completion in $\widehat{P_{\pi/2}} \cong T^\star\mathbb{S}^2$ is diffeomorphic to the fibre $T^\star_N\mathbb{S}^2$. Hence, by Theorem 9.6:

$$HW^*(L_{\text{col}}) \cong HW^*(T^\star_N\mathbb{S}^2) \cong H^*(\Omega\mathbb{S}^2).$$

From standard algebraic topology (see Lemma B.41), $H^*(\Omega\mathbb{S}^2)$ is non-zero in infinitely many degrees. □

Therefore, L_{col} is an exact spin Lagrangian in $P_{\pi/2} \cong \mathbb{D}^\star\mathbb{S}^2$, and $HW^*(L_{\text{col}}) \neq 0$ in infinitely many degrees. So we are exactly in the position to use Theorem B3, which readily yields:

Theorem C. *If the Circular Restricted Three-Body Problem satisfies the Weakened Twist Condition, then for every energy $c < H(L_1) + \varepsilon$, there exist infinitely many trajectories of spatial (consecutive) collision near the Earth and Moon, of arbitrarily large length.*

Chapter 10

Bi-normal trajectories in the Circular Restricted Three-Body Problem

This chapter contains a second application of Theorem B3, this time from the paper [ML25]. We study elementary symmetries of the SCR3BP and observe that, in our preferred page of the open book, the fixed point sets of these symmetries give, in turn: the planar problem, a Legendrian, and a Lagrangian satisfying the hypotheses of Theorem B3.

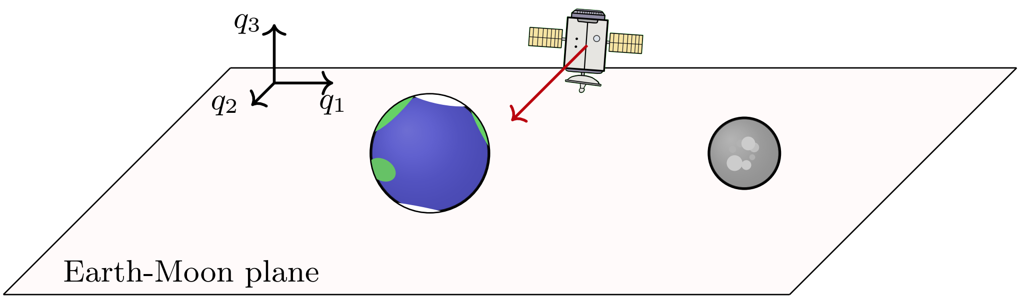
In particular, applying our machinery to the aforementioned Lagrangian yields:

Theorem D. *If the Circular Restricted Three-Body Problem satisfies the Weakened Twist Condition, then for every energy $c < H(L_1) + \varepsilon$, there exist infinitely many trajectories bi-normal to the xz-plane near the Earth and Moon, of arbitrarily large length.*

• **Definition 10.1.** We call a trajectory **bi-normal to the xz-plane** (or $q_1 q_3$ -plane) if there exists times $t_0 < t_1$ such that the trajectory is normal the the plane at time t_0, *i.e.*

$$q_2(t_0) = \dot{q}_1(t_0) = \dot{q}_3(t_0) = 0$$

and then again at time $t_1 > t_0$.



In our coordinates, the xy (or $q_1 q_2$)-plane is the Earth-Moon plane (the ecliptic), and the x-axis in the Earth-Moon axis. In our symplectic language, trajectories correspond to chords (hence, "open-ended trajectories", which do not close up on themselves).

✧ **Example 10.2** (Example of such trajectories)**.** One of the most famous examples of trajectories bi-normal to the xz-plane is what the mission design community calls *halo orbits.*

The notion of halo orbit was defined in the 1960s by Farquhar, as a family of orbits bifurcating from Lyapunov orbits of the planar problem, around some of the Lagrange points L_1, L_2, L_3. In particular, the first studied were halo orbits around L_2 in the Earth-Moon system (L_2 is situated on the far side of the Moon). Farquhar proposed that such orbits would yield, with minimum fuel usage, uninterrupted communication with the far side of the Moon, in particular the lunar south pole (see [Far66; GS19]).

This was experimented with by the China National Space Administration (CNSA)'s *Queqiao-1* satellite, a communication relay satellite placed on a halo orbit near L_2 in 2018. Other famous examples of famous satellites on halo orbits include NASA's James Webb and ESA's Euclid telescopes, or ISRO's Aditya-L1 satellite, launched as this thesis was being written.

Since halo orbits are constructed as bifurcations from planar orbits, one often makes the simplifying assumption of an xz-symmetry when numerically looking for them [GS19; YSM13]. Therefore, most of the halo orbits we know of are *by construction* bi-normal to the xz-plane. The fact that we can approach them with Floer homology, as we will see in this chapter, tells us that we can use the tools from §4.3 to study their bifurcation behaviour.

10.1 Symmetries of the spatial problem

10.1.1 The group of symmetries

In unregularised coordinates, the Hamiltonian of the CR3BP is given by:

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{m_E}{\left\|q - \overrightarrow{E}\right\|} - \frac{m_M}{\left\|q - \overrightarrow{M}\right\|} + q_1p_2 - q_2p_1, \tag{10.1}$$

where $\overrightarrow{E} = (\mu, 0, 0)$ and $\overrightarrow{M} = (-1+\mu, 0, 0)$, $\mu = m_M/(m_E + m_M)$, and $L = q_1p_2 - q_2p_1$.

From looking at (10.1), we can directly observe a few symmetries of the Hamiltonian:

- $r : \mathbb{R}^6 \to \mathbb{R}^6 : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, q_2, -q_3, p_1, p_2, -p_3)$. Its fixed point set is

$$\operatorname{Fix} r = \{(q,p) \in \mathbb{R}^6 \mid q_3 = p_3 = 0\}, \tag{10.2}$$

  which is the Earth-Moon plane (the ecliptic). Hence, the symmetry r is simply reflection about the ecliptic.

- $\rho_1 : \mathbb{R}^6 \to \mathbb{R}^6 : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, -q_3, -p_1, p_2, p_3)$. Its fixed point set is:

$$\operatorname{Fix}\rho_1 = \{(q,p) \in \mathbb{R}^6 \mid q_2 = q_3 = p_1 = 0\}, \tag{10.3}$$

- $\rho_2 : \mathbb{R}^6 \to \mathbb{R}^6 : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, q_3, -p_1, p_2, -p_3)$. Its fixed point set is:

$$\operatorname{Fix}\rho_2 = \{(q,p) \in \mathbb{R}^6 \mid q_2 = p_1 = p_3 = 0\}. \tag{10.4}$$

Indeed, we can easily check that $H \circ r = H \circ \rho_1 = H \circ \rho_2 = H$, so that all of these are symmetries of the CR3BP. Furthermore, they are all involutions, and r preserves the symplectic form, while ρ_1 and ρ_2 invert it.

Let $\mathbb{1} : \mathbb{R}^6 \to \mathbb{R}^6$ be the identity, and define $G := \{\mathbb{1}, r, \rho_1, \rho_2\}$. Then, we can easily see that:

$$r \circ \rho_1 = \rho_1 \circ r = \rho_2, r \circ \rho_2 = \rho_2 \circ r = \rho_1, \rho_1 \circ \rho_2 = \rho_2 \circ \rho_1 = r,$$

thus explicitly verifying that $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the Klein 4-group. We call G the **group of symmetries** of the (spatial) circular restricted three-body problem.

10.1.2 The symmetry Lagrangian and Legendrian

Let us now look at what the symmetries r, ρ_1, and ρ_2 become after regularising at collisions. Write Ξ the inverse stereographic projection, whose expression is given in (8.6). We restrict ourselves to low energies $c < H(L_1)$, so that we are working in a neighbourhood of the Earth or the Moon, like in §8.1.3.

The image under Ξ of the fixed point sets $\operatorname{Fix}(r), \operatorname{Fix}(\rho_1), \operatorname{Fix}(\rho_2)$ is given by:

$$\tilde{F}_0 := \Xi\big(\operatorname{Fix}(r)\big) = \left\{(\xi,\eta) \in T^\star\mathbb{S}^3 \mid \xi_3 = \eta_3 = 0\right\}, \tag{10.5}$$

$$\tilde{F}_1 := \Xi\big(\operatorname{Fix}(\rho_1)\big) = \left\{(\xi,\eta) \in T^\star\mathbb{S}^3 \mid \xi_1 = \eta_0 = \eta_2 = \eta_3 = 0\right\}, \tag{10.6}$$

$$\tilde{F}_2 := \Xi\big(\operatorname{Fix}(\rho_2)\big) = \left\{(\xi,\eta) \in T^\star\mathbb{S}^3 \mid \xi_1 = \xi_3 = \eta_0 = \eta_2 = 0\right\}. \tag{10.7}$$

We have already studied the first of these in Chapter 8. Indeed, $\tilde{F}_0 \cong \mathbb{S}^\star\mathbb{S}^2$ is a regularised energy hypersurface of the PCR3BP, for low energies. There, we had called it B, and used it as a binding for the SCR3BP open book (Theorem 8.26).

Let us focus our attention on the other two, $\tilde{F}_1$, $\tilde{F}_2$, *i.e.* the regularised fixed point sets of the symmetries ρ_1 and ρ_2. Alternatively, we can write them:

$$\tilde{F}_1 = \left\{(\xi, \eta) \in T^\star\mathbb{R}^4 \mid \|\xi\|^2 = 1, \xi_1 = \eta_0 = \eta_2 = \eta_3 = 0\right\}, \tag{10.8}$$

$$\tilde{F}_2 = \left\{(\xi, \eta) \in T^\star\mathbb{R}^4 \mid \|\xi\|^2 = 1, \xi_1 = \xi_3 = \eta_0 = \eta_2 = 0\right\}. \tag{10.9}$$

Since energy is conserved along the motion, we are only interested in $\tilde{F}_i \cap \tilde{\Sigma}$, where $\tilde{\Sigma}$ is our regularised energy hypersurface. By §8.1.3, we can write $\tilde{\Sigma} = Q^{-1}(1/2)$, so that:

$$\tilde{F}_1 \cap \tilde{\Sigma} = \left\{(\xi, \eta) \in T^\star\mathbb{R}^4 \mid Q(\xi, \eta) = \frac{1}{2}, \|\xi\|^2 = 1, \xi_1 = \eta_0 = \eta_2 = \eta_3 = 0\right\}, \tag{10.10}$$

$$\tilde{F}_2 \cap \tilde{\Sigma} = \left\{(\xi, \eta) \in T^\star\mathbb{R}^4 \mid Q(\xi, \eta) = \frac{1}{2}, \|\xi\|^2 = 1, \xi_1 = \xi_3 = \eta_0 = \eta_2 = 0\right\}. \tag{10.11}$$

This is better, but we can still drop down one dimension. Like in Chapters 8 and 9, we use the SCR3BP open book from [MK22b], and we restrict our attention to the particularly nice page we derived in §8.3.3, given by $P_{\pi/2} = \{(\xi, \eta) \mid Q(\xi, \eta) = \frac{1}{2}, \xi_3 = 0, \eta_3 \geq 0\}$. Then:

$$\tilde{F}_1 \cap \tilde{\Sigma} \cap P_{\pi/2} = \left\{(\xi, \eta) \in T^\star\mathbb{R}^4 \mid Q(\xi, \eta) = \frac{1}{2}, \|\xi\|^2 = 1, \xi_1 = \xi_3 = \eta_0 = \eta_2 = \eta_3 = 0\right\}, \tag{10.12}$$

$$\tilde{F}_2 \cap \tilde{\Sigma} \cap P_{\pi/2} = \left\{(\xi, \eta) \in T^\star\mathbb{R}^4 \mid Q(\xi, \eta) = \frac{1}{2}, \|\xi\|^2 = 1, \xi_1 = \xi_3 = \eta_0 = \eta_2 = 0, \eta_3 \geq 0\right\}. \tag{10.13}$$

By the implicit function theorem, the first of these two submanifolds is only one-dimensional:

$$\tilde{F}_1 \cap \tilde{\Sigma} \cap P_{\pi/2} = \left\{(\xi, \eta) = (\xi_0, 0, \xi_2, 0, 0, \eta_1, 0, 0, 0) \mid Q(\xi, \eta) = \frac{1}{2}, \xi_0^2 + \xi_2^2 = 1\right\} \tag{10.14}$$

Meanwhile, the second one is a surface in W, which can be written:

$$\tilde{F}_2 \cap \tilde{\Sigma} \cap P_{\pi/2} = \left\{(\xi, \eta) = (\xi_0, 0, \xi_2, 0, 0, \eta_1, 0, \eta_3) \mid Q(\xi, \eta) = \frac{1}{2}, \xi_0^2 + \xi_2^2 = 1, \eta_3 \geq 0\right\} \tag{10.15}$$

• **Definition 10.3.** Write $\Lambda_1 := \tilde{F}_1 \cap \tilde{\Sigma} \cap P_{\pi/2}$, and $L_2 := \tilde{F}_2 \cap \tilde{\Sigma} \cap P_{\pi/2}$.

Observe that $\Lambda_1 = \partial L_2$. Actually, we can prove:

Lemma 10.4. *L_2 is an exact Lagrangian submanifold of $(P_{\pi/2}, \omega = \mathrm{d}\lambda)$, and $\Lambda_1 = \partial L_2$ is its Legendrian boundary in $(\partial P_{\pi/2}, \alpha = \lambda|_{\partial P_{\pi/2}})$.*

Proof. $P_{\pi/2}$ inherits its symplectic form from the contact form λ on $Q^{-1}(\mu^2/2)$, which by the proof of Theorem 8.15 is the standard $\alpha = -\sum_i \eta_i \mathrm{d}\xi_i$.

$$\omega|_{P_{\pi/2}} = \sum_{i=0}^{2} \mathrm{d}\xi_i \wedge \mathrm{d}\eta_i = \mathrm{d}\left(-\sum_{i=0}^{2} \eta_i \mathrm{d}\xi_i\right) = \mathrm{d}\lambda.$$

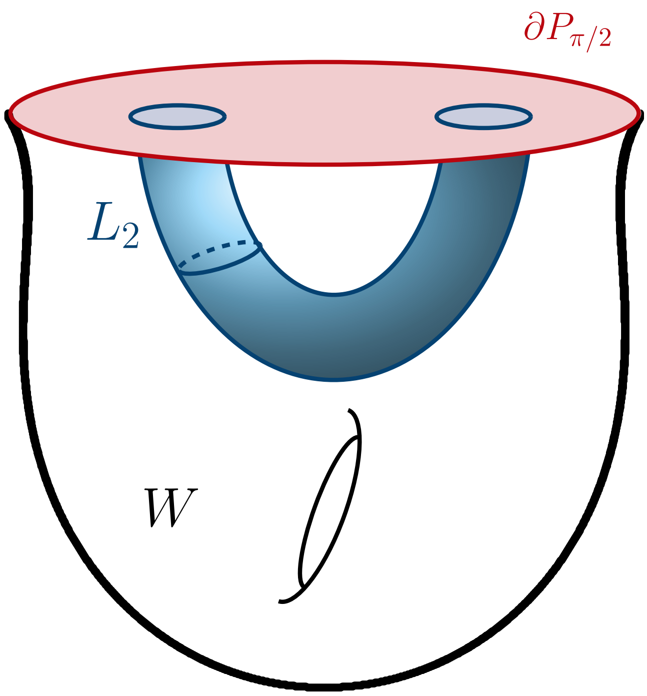
Clearly from (10.15) λ vanishes on L_2 ($\supset \Lambda_1$), which proves both claims. □

We are getting closer and closer to being able to attack this problem with Floer theory. Before we do that though, let us try and visualise what these regularised symmetry sets look like.

✧ **Example 10.5.** Let us first consider the simplifying case $f \equiv 1$. Then, $Q(\xi, \eta) = \frac{1}{2} \|\eta\|^2$ induces the standard geodesic flow on $\mathbb{S}^3$ (the regularised flow of the two-body problem), and the symmetry Lagrangian L_2 is given by:

$$L_2 = \left\{ (\xi_0, 0, \xi_2, 0, 0, \eta_1, 0, \eta_3) \mid \xi_0^2 + \xi_2^2 = 1, \eta_1^2 + \eta_3^2 = 1, \eta_3 \geq 0 \right\}.$$

This describes a half-torus in $P_{\pi/2} \cong \mathbb{D}^\star \mathbb{S}^2$, whose boundary is a disjoint union of circles in $\partial P_{\pi/2}$, *i.e.* $\Lambda_1 \cong \mathbb{S}^1 \sqcup \mathbb{S}^1 \subset \partial P_{\pi/2}$.



Topologically, a half-torus is the same thing as an annulus, *i.e.* $L_2 \cong \mathbb{D}^\star \mathbb{S}^1$.

In the general SCR3BP case, where $f \not\equiv 1$, the expression for L_2 will not be as easy. However, as the topology of the open book does not change (from the Two-Body to the Spatial Circular Restricted Three-Body Problem), one can pull back L_2 through a global diffeomorphism of the pages, and thus still obtain that L_2 is an annulus.

10.2 Floer cohomology of the symmetry fixed point sets

10.2.1 Floer cohomology of L_2 and bi-normal trajectories

Let us focus on the Lagrangian $L_2 \subset P_{\pi/2}$ defined in the previous section.

We have seen that it was exact (*i.e.* $\exists f : L_2 \to \mathbb{R}$ s.t $\lambda|_L = \mathrm{d}f$), and from its coordinate expression (10.15) we see that it is an orientable surface, so that it is spin.

Therefore, its wrapped Floer cohomology $HW^*(L)$ is well-defined, by Chapter 3. Given a Hamiltonian $H_t : P_{\pi/2} \to \mathbb{R}$ satisfying the twist condition, we can use the relative Poincaré-Birkhoff theorem from Chapter 5 to prove the existence of infinitely many Hamiltonian chords on L_2 – provided we can show that $HW^*(L_2)$ is supported in infinitely many degrees.

We will do this by showing that L_2 can be expressed as a conormal bundle in $P_{\pi/2} \cong \mathbb{D}^\star \mathbb{S}^2$, allowing us to explicitly compute this cohomology, using off-the-shelf results from [APS08].

• **Definition 10.6.** Let R be a submanifold in some ambient Riemannian manifold (M, g). The normal bundle of R is defined as the complementary of TR in TM, with respect to g. We write it $NR \twoheadrightarrow R$. By definition, it is a subbundle of TM.

The **conormal bundle** $N^\star R \twoheadrightarrow R$ is the fibrewise dual of $NR \twoheadrightarrow R$. By definition, it is a subbundle of $T^\star M$.

With this definition in mind, we call to the result:

Proposition 10.7 ([APS08], Prop. 2.1)**.** *Let M be a manifold, and L a submanifold of $T^\star M$ on which the Liouville form λ vanishes identically. Then the intersection of L with the zero section of $T^\star M$ is a submanifold R. Furthermore, if L is a closed subset of $T^\star M$, then $L = N^\star R$.*

This directly tells us that L can be viewed as the conormal bundle of some $R \subset \mathbb{S}^2$. However, we can be more specific:

Proposition 10.8. *$L_2 \subset P_{\pi/2}$ can be viewed as the conormal bundle of the equator:*

$$R := \left\{(\xi_0, 0, \xi_2) \in \mathbb{R}^3 \mid \xi_0^2 + \xi_2^2 = 1\right\} \cong \mathbb{S}^1 \tag{10.16}$$

Proof.[1] The tangent bundle of R is given by:

$$TR := \left\{(\xi_0, 0, \xi_2, v_0, 0, v_2) \mid \xi_0^2 + \xi_2^2 = 1, \begin{pmatrix}\xi_0\\ \xi_2\end{pmatrix} \cdot \begin{pmatrix}v_0\\ v_2\end{pmatrix} = 0\right\} \hookrightarrow T^\star\mathbb{S}^2.$$

Thus, its normal bundle (in ambient space $T^\star\mathbb{S}^2$) is given by:

$$\begin{aligned} NR :&= \left\{(\xi, v) = (\xi_0, 0, \xi_2, v_0, v_1, v_2) \mid \xi_0^2 + \xi_2^2 = 1, v \cdot w = 0 \ \forall w \in T^\star\mathbb{S}^2\right\} \\ &= \left\{(\xi, v) = (\xi_0, 0, \xi_2, 0, v_1, 0) \mid \xi_0^2 + \xi_2^2 = 1, v_1 \in \mathbb{R}\right\}. \end{aligned}$$

The conormal bundle is then obtained by fibrewise dualising NR, so that:

$$N^\star R = \left\{(\xi_0, 0, \xi_2, 0, \eta_1, 0) \mid \xi_0^2 + \xi_2^2 = 1, \eta_1 \in \mathrm{End}(T_{(\xi_0,\xi_2)}R, \mathbb{R})\right\}.$$

We can identify $\mathrm{End}(T_{(\xi_0,\xi_2)}R, \mathbb{R}) \cong \mathbb{R}$. Hence, we get:

$$N^\star R = \left\{(\xi_0, 0, \xi_2, 0, \eta_1, 0) \mid \xi_0^2 + \xi_2^2 = 1, \eta_1 \in \mathbb{R}\right\},$$

and thus:

$$N^\star R \cap P_{\pi/2} = \left\{(\xi_0, 0, \xi_2, 0, \eta_1, 0) \mid \xi_0^2 + \xi_2^2 = 1, \eta_1^2 \leq \mu^2/f^2\right\},$$

which is exactly our expression for L_2 from (10.15). □

So topologically, L_2 is the conormal bundle of a circle, intersected with our page $P_{\pi/2} \cong \mathbb{D}^\star\mathbb{S}^2$. We claim that this determines the wrapped Floer cohomology of L_2. Indeed:

Theorem 10.9 ([APS08])**.** *Let M be a manifold, and $L \subset T^\star M$ be a Lagrangian such that $L = N^\star R$ for some submanifold $R \subset M$. Then:*

$$HW^*(L) \cong H^*(\mathcal{P}_R M),$$

where $\mathcal{P}_R M$ is the space of paths in M with endpoints in R (Definition B.43), and H^* denotes singular homology.

Corollary 10.10. *$HW^*(L_2)$ is non-zero in infinitely many degrees.*

Proof. By Theorem 10.9, $HW^*(L_2) \cong H^*(\mathcal{P}_{\mathbb{S}^1}\mathbb{S}^2)$. To show that this is infinite-dimensional, we first invoke the fact that there exists a path space fibration:

$$\Omega\mathbb{S}^2 \longrightarrow \mathcal{P}_{\mathbb{S}^1}\mathbb{S}^2 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1,$$

as shown in Lemma B.45 of the appendix, where $\Omega\mathbb{S}^2$ denotes the loop space of $\mathbb{S}^2$. By using Proposition B.36 of the appendix, such a fibration induces a spectral sequence:

$$E_2^{p,q} = H^p(\mathbb{S}^1 \times \mathbb{S}^1; H^q(\Omega\mathbb{S}^2)) \implies H^{p+q}(\mathcal{P}_{\mathbb{S}^1}\mathbb{S}^2), \tag{10.17}$$

[1] We check this in regularised coordinates for completeness. However, this result is obvious from the fact that, in unregularised coordinates, the fixed point set of ρ_2 is the conormal bundle of the xz-plane.

and from standard singular cohomology, given M a module of coefficients we have:

$$\begin{cases} H^p(\mathbb{S}^1 \times \mathbb{S}^1; M) \cong M \quad \text{for } p \in \{0,2\}, \\ H^p(\mathbb{S}^1 \times \mathbb{S}^1; M) \cong M^2, \ \text{for } p = 1, \\ H^p(\mathbb{S}^1 \times \mathbb{S}^1; M) = 0, \ \text{for } p \notin \{0,1,2\}. \end{cases}$$

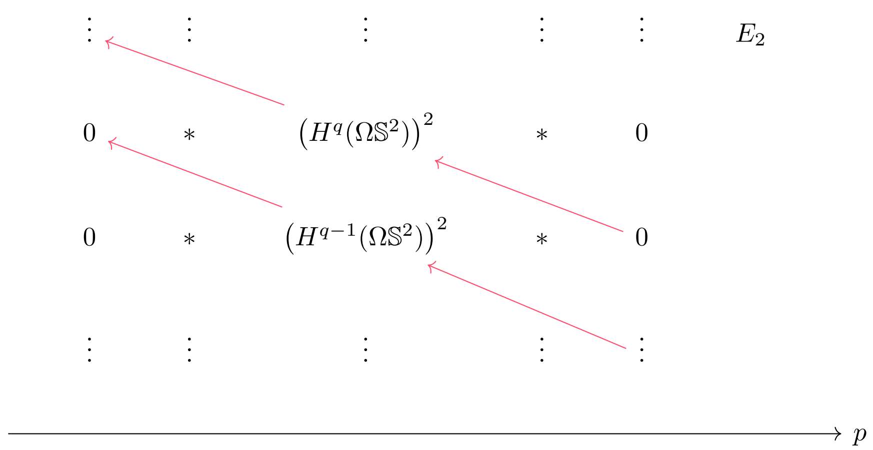
Hence, we have

$$\begin{cases} E_2^{p,q} \cong H^q(\Omega\mathbb{S}^2) \quad \text{for } p \in \{0,2\}, \\ E_2^{p,q} = H^q(\Omega\mathbb{S}^2)^2, \ \text{for } p = 1, \\ E_2^{p,q} = 0, \ \text{for } p \notin \{0,1,2\}. \end{cases}$$

In particular, since the spectral sequence differential on the E_2 page goes from degrees::

$$\mathrm{d}_2 : E_r^{p,q} \to E_r^{p-2,q-1},$$

then derivatives from and towards the column $\{p = 1\}$ are all zero.



In particular, the column $p = 1$ will survive unchanged to the next page of the spectral sequence, since

$$E_3^{p,q} = \frac{\ker\{\mathrm{d}_2 : E_2^{p,q} \to E_2^{p-2,q+1}\}}{\mathrm{im}\{\mathrm{d}_2 : E_2^{p+2,q-1} \to E_2^{p,q}\}} = \frac{\left(H^q(\Omega\mathbb{S}^2)\right)^2}{0} = \left(H^q(\Omega\mathbb{S}^2)\right)^2.$$

By induction, the column $\{p = 1\}$ will survive all the way to E_∞. Now recall from Lemma B.41 that $H^q(\Omega\mathbb{S}^2)$ is non-zero in infinitely many degrees. Since $E_2^{p,q} \implies H^*(\mathcal{P}_{\mathbb{S}^1}\mathbb{S}^2)$, we have:

$$\forall n : H^n(\mathcal{P}_{\mathbb{S}^1}\mathbb{S}^2) = \bigoplus_{p+q=n} E_\infty^{p,q},$$

which implies that $H^*(\mathcal{P}_{\mathbb{S}^1}\mathbb{S}^2)$ is non-zero in infinitely many degrees, concluding the proof. □

Hence, L_2 satisfies all the conditions necessary for Theorem B3, yielding:

Theorem D. *If the Circular Restricted Three-Body Problem satisfies the Weakened Twist Condition, then for every energy $c < H(L_1) + \varepsilon$, there exist infinitely many trajectories bi-normal to the xz-plane near the Earth and Moon, of arbitrarily large length.*

Proof. By definition, a bi-normal trajectory in phase space $T^\star(\mathbb{R}^3 \backslash \{\overrightarrow{E}, \overrightarrow{M}\})$ (unregularised coordinates) is a path $\gamma(t)$ such that $\gamma(0), \gamma(1) \in \mathrm{Fix}\rho_2$, where ρ_2 is the symmetry of the

SCR3BP defined in (10.4). After regularising at collisions, and restricting to our page $P_{\pi/2}$ of the open book, we see that Theorem D reduces to Theorem B3. □

10.2.2 Conjecture for trajectories bi-normal to the x-axis

At the beginning of the chapter, we defined the group of symmetries

$$G = \{\mathbb{1}, r, \rho_1, \rho_2\},$$

of the CR3BP. By studying the fixed point set of ρ_2, and assuming the Weakened Twist Condition, we deduced existence of infinitely many trajectories bi-normal to the xz-plane. However, the anti-symplectic involution

$$\rho_1 : \mathbb{R}^6 \to \mathbb{R}^6 : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, -q_3, -p_1, p_2, p_3)$$

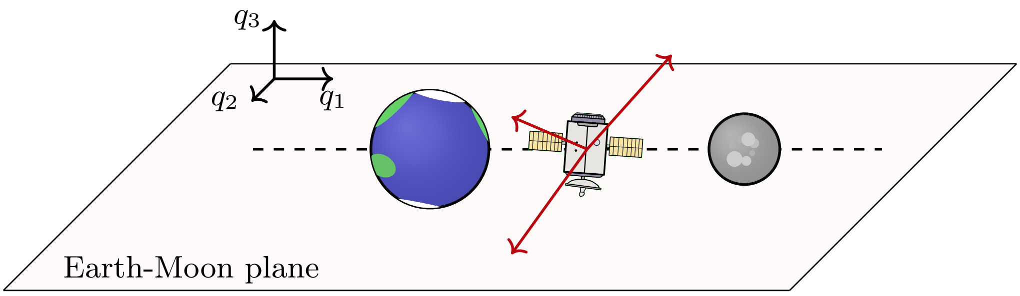
also has an interesting fixed point set. Indeed, we saw in Lemma 10.4 that it corresponds to the Legendrian boundary of L_2, in the binding $(\partial P_{\pi/2}, \alpha)$.

Hence, *if we could prove a Poincaré-Birkhoff type theorem for Legendrian contact homology*, instead of the ones we already have for symplectic homology ([MK22a]) and wrapped Floer homology (Chapter 5), then we would get the following result:

Conjecture 1. *Assuming the Weakened Twist Condition or a variation thereof, there exist infinitely many trajectories bi-normal to the x-axis in the Circular Restricted Three-Body Problem, in the low-energy range and near the Earth and Moon. These are trajectories $x = (q_i, \dot{q}_i)(t)$ such that there exist times $t_0 \neq t_1$ with:*

$$q_2(t_j) = q_3(t_j) = \dot{q}_1(t_j) = 0,$$

for $j = 0, 1$. In other words, the trajectory starts on the Earth-Moon axis, but with velocity pointing strictly outward, and comes back to satisfy the same condition after finite time.



Recent work done by Broćić, Cant, and Shelukhin [BCS24] shows that, given $R \subset M$ closed manifolds, and $\Lambda \subset \mathbb{S}^\star M$ a Legendrian which is isotopic to $\partial N^\star R$, then *the chord conjecture holds*; meaning that there exists at least one Reeb chord on $(\partial P_{\pi/2}, \alpha)$ with ends in Λ. This gives existence of *at least* one trajectory bi-normal to the x-axis.

In particular, since their techniques do not require completing $P_{\pi/2}$ to $\widehat{P_{\pi/2}}$, they do not need a way of ignoring chords on $[1, +\infty) \times \partial P_{\pi/2}$, so that in particular they don't need to assume any variation of the twist condition.

Chapter 11

Conclusion

11.1 Final state-of-the-art

Our goal, all throughout this thesis, has been to apply Floer-theoretical tools to the Three-Body Problem.

We began with a model by Moreno & van Koert [MK22a; MK22b], in which we proved a Poincaré-Birkhoff theorem for Wrapped Floer cohomology. We then improved the model by relaxing most of the technical assumptions, and we derived applications to the Spatial Circular Restricted Three-Body Problem (SCR3BP) – under the assumption of a Weakened Twist Condition.

Therefore, the current state-of-the-art Floer-theoretical model for the SCR3BP is:

- For low energies, the Moser-regularised flow of the SCR3BP is a Reeb flow on a contact 5-fold (Σ, α), contactomorphic to $(\mathbb{S}^\star\mathbb{S}^3, \alpha_{\text{std}})$.

  ✓[Alb+12b; CJK20].

- (Σ, α) admits an adapted open book decomposition, whose every page P is a degenerate Liouville domain with diffeomorphism type $\mathbb{D}^\star\mathbb{S}^2$.

  ✓[MK22b].

- The Poincaré return map $\tau : P \to P$ of the regularised flow is an exact symplectomorphism, which extends smoothly to the (degenerate) boundary.

  ✓[MK22b].

- There are two Poincaré-Birkhoff theorems, Theorem A of [LM25] and Theorem B3, which respectively give the existence of infinitely many periodic orbits and Hamiltonian chords in $\text{int}(P)$, assuming a Weakened Twist Condition.

  ✓[LM25], Chapter 7.

- Assuming the Weakened Twist Condition, the above theorems can explicitly be used to study trajectories in the Spatial Circular Restricted Three-Body Problem.

  ✓Chapters 4, 9, 10.

This journey led us to formulate two conjectures.

11.2 Open problems and conjectures

11.2.1 On trajectories bi-normal to the x-axis

In Chapter 10 we studied symmetries of the Spatial Circular Restricted Three-Body Problem. We saw that two of them, ρ_1 and ρ_2, were anti-symplectic involutions. Denoting by Λ_1 and L_2 their fixed point sets in the page P of our open book, we saw that L_2 was a Lagrangian admissible for Wrapped Floer Cohomology, and that Λ_1 was its Legendrian boundary.

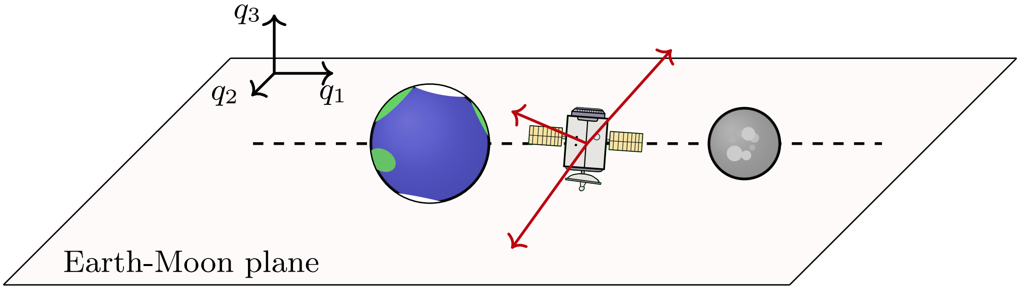
By applying our theorems from Part II to L_2 we proved Theorem D, showing existence of infinitely many trajectories bi-normal to the xz-plane in the Circular Restricted Three-Body Problem, under the assumption of the Weakened Twist Condition.

If we could prove a similar result for the Legendrian $\Lambda_1 \cong N^\star \mathbb{S}^1$, we would get:

Conjecture 1. *Assuming the weakened twist condition or a variation thereof, there exist infinitely many trajectories bi-normal to the x-axis in the SCR3BP, in the low-energy range and near the primaries. These correspond to trajectories $x(t) = (q_i, \dot{q}_i)(t)$ such that there exist times $t_0 \neq t_1$ with:*

$$q_2(t_j) = q_3(t_j) = \dot{q}_1(t_j) = 0 \text{ for } j = 0, 1.$$

In other words, the trajectory starts on the Earth-Moon axis but with velocity pointing strictly outwards, and comes back to satisfy the same condition after finite time.



Showing this though would require different machinery than the one developed in Part II. One way to prove this conjecture would be by proving a Poincaré-Birkhoff type theorem for Legendrian Contact Cohomology – which has not been done yet. Another way would be to stick with our current machinery, but try and find a way of distinguishing physically relevant chords on the collar $[1, +\infty) \times \partial P$ from the undesirable ones.

So far we have no way of doing this. This is why we used an index growth argument in Chapter 5, or an action growth one in Chapter 6: so we could ignore chords on $[1, +\infty) \times \partial P$ altogether, and care only about the dynamics in $\text{int}(P)$. If we could refine this part of the argument and find a way of efficiently distinguishing between physical and undesirable chords on the collar, then we may gain a lot of understanding about Reeb chords on $(\partial P, \alpha)$.

Recent work conducted in [BCS24] showed that the Reeb chord conjecture held for Legendrian submanifolds isotopic to the boundary of a co-normal bundle. This yields:

Corollary. *There exists at least one Reeb chord on $(\partial P, \alpha)$ with ends in Λ_1. In particular, there exists at least one trajectory bi-normal to the x-axis in the SCR3BP, for low energies and near the primaries.*

The argument of Broćić, Cant, and Shelukhin does not require any form of twist condition, because they do not need to complete their Liouville domain. Instead, they prove existence of the chord directly, using tools from homotopy theory (a version of the Hurewicz theorem). However it seems like a stretch to generalise these methods to showing existence of infinitely many Reeb chords, without making further assumptions.

11.2.2 The Weakened Twist Condition: the final obstacle

As recalled at the beginning of this chapter, the results in this thesis allowed us to significantly refine the Floer-theoretical model of the Spatial Circular Restricted Three-Body Problem: by getting rid of most of the technical assumptions from the work of Moreno-van-Koert, and considerably weakening their twist condition. However, we are still left with one obstacle:

Conjecture 2. *The Weakened Twist Condition, or a variation thereof, holds in the SCR3BP, for every energy below or slightly above the first Lagrange energy, near the primaries.*

At this stage, the universe splits into two categories.

An optimist would say that so much has already been done: getting to this weakening. The new condition (a derivative being positive) is so much simpler than the original twist condition (getting two unrelated vectors to be collinear). And it *is* possible to partially generate the SCR3BP return map in a way that satisfies the Weakened Twist Condition. Indeed, working on the page of the open book of interest, if we choose to use the regularised *planar* Hamiltonian instead of the regularised spatial Hamiltonian, then we can generate a map which is also a rotation, and satisfies the Weakened Twist Condition.(1) This rotation does not coincide exactly with the SCR3BP return map, due to an error in angle, however one could try to correct this angle in a way that leaves the Weakened Twist Condition unchanged – which is theoretically possible because the Weakened Twist Condition is an open condition, whereas the original twist condition was not.(2)

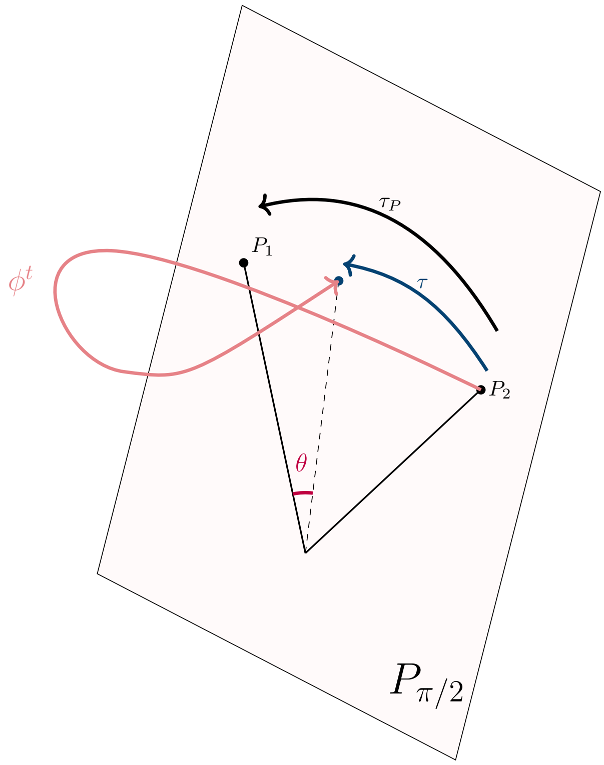


FIGURE: ϕ^t represents the flow of the SCR3BP, τ its return map on the page $P_{\pi/2}$, and τ_P represents the map we can currently generate satisfying the Weakened Twist Condition, thanks to work of Dr. Connor Jackman and the author. θ is the angle error (or overshoot) which one has to palliate to prove Conjecture 2.

(1) The calculations leading to that result remain partial, and are therefore not included in this thesis. They, however, constitute work in progress which the author is quite optimistic about.

(2) It seems that, using a slow composition trick, one could manage to generate the error in angle very slowly, so as not to affect the positivity of the derivatives, and hence not affect the Weakened Twist Condition. This would imply that the Weakened Twist Condition can be satisfied in very slow time, instead of time 1, and for extremely low energies. This is not yet the desired statement, but it constitutes a good case for the optimist.

On the other hand, a pessimist would say that too much has already been done. Getting to such a weakening might have been the final push. Indeed, to prove the Weakened Twist Condition, one would need to find a very specific Hamiltonian on the page; one that is $\mathcal{C}^2$ on the interior but does not $\mathcal{C}^1$ extend to the boundary, all while generating the return map of the SCR3BP. Any attempt which seems to get close to generating the Weakened Twist Condition, like the partial calculations mentioned in the previous paragraph, seem to lead to a blow up in time, or other fundamental issues which seem, at the moment, impossible to circumvent.

Even if we decide to adopt a pessimistic viewpoint though, no matter how fundamental some of the obstructions we face may seem, it is not all bad news. Our Weakened Twist Condition was a refinement of the one in [MK22a], which was itself a generalisation of the one in [Poi12]. It is more than likely that the story is not over, and that we need yet a different version of the twist condition, or more advanced machinery – in the form of a Floer theory for time-∞ maps for example.

What this thesis tells us is that the Weakened Twist Condition is the **final obstacle**, and that all the other technical obstructions (present in the previous models) have been taken care of. Once we overtake this final obstacle, we will be in full power of applying our Floer-theoretical machinery to the Spatial Circular Restricted Three-Body Problem.

11.3 Closing words

It is still a bit early to observe major impacts of Floer theory on space mission design. However, as effort in this direction gets increasingly coordinated, the timeline is shrinking fast. Work done in [FKM23; Ayd+24a] between the departments of Heidelberg (Germany), Augsburg (Germany), Seoul (Korea), and NASA's Jet Propulsion Laboratory (USA) established a state-of-the-art of the current symplectic toolkit which could be applied to mission analysis, to study *periodic orbits*:

- Local Floer invariants;
- Maslov-type indices;
- B-signatures;
- the GIT sequence (a refinement of the Broucke stability diagram).

The work done in this thesis generalises these tools to *open-ended trajectories* with nice boundary conditions (*e.g.* collision trajectories, halo orbits, etc...) instead of periodic orbits. This enterprise, however, is far from over.

The marriage between symplectic geometry and the Three-Body Problem is still burgeoning, and many more tools and techniques need to be investigated at every stage: from the theory to the engineering. Bi-lateral feedback between the two is essential, as has been demonstrated time and again in recent years. The frontier is becoming increasingly porous, as theoretical tools start getting applied to engineering (*e.g.* the Conley-Zehnder index), and conversely some are being developed for the very purpose of mission design (*e.g.* Local Wrapped Floer Cohomology from Part I, which we introduced specifically for the SCR3BP). Another salient example of this dialogue is that of the Broucke stability diagram, known to the engineering community since the 1960s, but recently re-discovered and generalised by geometers, though under a different name (the GIT sequence) [Ayd+24a].

➤ So what now?

In a hundred and twenty pages or so we introduced one new algebraic topological invariant, proved three versions of one fixed point theorem, and then two existence theorems in the Circular Restricted Three-Body Problem. We improved a previously-existing model by lifting three assumptions, but ended up with one final obstruction: the Weakened Twist Condition.

Let us take a few steps back.

In its three and a half centuries of existence the Three-Body Problem has observed, sometimes even prompted the birth of many mathematical theories, and survived them all. Likely it will keep inspiring new mathematical, physical, engineering techniques for quite some time; and survive many more. Floer theory is but one of the many heads of the hydra: while it gives us powerful heuristics, and invaluable insights into how physical motion is constrained by loop space homology, it comes with its own drawbacks. Despite its elegance, the theory seems bound to forever remain somewhat niche, due to its sheer difficulty.

It is likely that the methods discussed in this thesis eventually get generalised, and our Floer-theoretical model for the SCR3BP superseded by finer models. This is the natural way of things: this never-ending cycle is woven into the history of the Three-Body Problem.

But even when a head of the hydra is chopped off, echoes of it remain. The insights we gained in this manuscript cannot be forgotten, nor can the bridge between physics and topology unveiled by Floer theory.

The Three-Body Problem is analytical, numerical, geometric, topological. It is old, complicated, deceitful, humbling. It captivates far beyond science, and has by now well seeped into pop culture. We can only hope that it keeps inspiring movies, TV shows, books aimed at a general audience (like this manuscript), and new generations of people fascinated by space; for the story is far from over. Despite its disconcerting simplicity, the Three-Body Problem may never be within the grasp of our mathematics. It is fascinating that, no matter how elaborate our tools and models become, they cannot help but fall short in one way or another. Likely we shall have to content ourselves with what we have been doing for more than three centuries: slowly chipping away at the surface of the problem, decades at a time.

> "There's this emperor, and he asks this shepherd's boy 'How many seconds in eternity?' And the shepherd's boy says, 'There's this mountain of pure diamond. It takes an hour to climb it and an hour to go around it. Every hundred years a little bird comes and sharpens its beak on the diamond mountain. When the entire mountain is chiselled away, the first second of eternity will have passed.' You may think that's a hell of a long time. Personally, I think that's a hell of a bird."

Peter Capaldi, *Doctor Who: "Heaven Sent"*, written by Steven Moffat, BBC, 2015.
Based on a fairy tale by the Brothers Grimm.

Appendices

Appendix A

Geometric appendix

A.1 More symplectic and contact geometry

A.1.1 Hamiltonian dynamics

The goal of this section is to complement the brief introduction to Symplectic Geometry we gave at the beginning of this thesis (see §1.4). In particular, let us mention a few basic definitions/facts about the dynamics of Hamiltonians on symplectic manifolds.

Periodic Hamiltonians.

Let (M, ω) be a symplectic manifold, and $H : \mathbb{R} \times M \to \mathbb{R}$ a time-dependent Hamiltonian. Our first claim is that, without loss of generality, we may assume that H is periodic.

Lemma A.1. *Let $H : \mathbb{R} \times M \longrightarrow \mathbb{R}$ be a time-dependent Hamiltonian, with flow ϕ_H^t. Then, we can find a periodic Hamiltonian $K : \mathbb{S}^1 \times M \to \mathbb{R}$ such that the time-1 maps $\phi_H^{t=1}$ and $\phi_K^{t=1}$ have the same fixed points.*

Proof. Choose a smooth map $\alpha : [0,1] \to [0,1]$ such that $\alpha(0) = 0, \alpha(1) = 1$, and α is flat near 0 and 1, and replace ϕ_H^t by $\phi_H^{\alpha(t)}$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_H^{\alpha(t)}(p) = \frac{\mathrm{d}\alpha}{\mathrm{d}t} X_{H_{\alpha(t)}}\big(\phi_H^{\alpha(t)}(p)\big) = X_{\alpha'(t)H_{\alpha(t)}}\big(\phi_H^{\alpha(t)}(p)\big).$$

In particular, the flow $t \mapsto \phi_H^{\alpha(t)}$ is generated by the Hamiltonian $K_t := \alpha'(t)H_t$. By construction, $\phi_H^{t=1}$ and $\phi_K^{t=1} = \phi_H^{\alpha(1)}$ have the same fixed points, since α is flat near 1. Since α is also flat near zero, then we can artificially turn it into a periodic function $\alpha : \mathbb{S}^1 \to [0,1]$, and therefore make K periodic. □

Let us now define an important notion, very heavily used in Chapters 5 and 6: that of composing, and iterating Hamiltonians.

• **Definition A.2.** Let K and H be two 1-periodic Hamiltonians. We define their composition $K \# H$ by:

$$(K\#H)_t := K_t + H_t \circ (\phi_{K_t}^t)^{-1}.$$

This is defined so that $\phi_{K\#H} = \phi_K \circ \phi_H$.

• **Definition A.3.** For $k \in \mathbb{N}$, we define the k-iteration $H^{\#k}$ of H by:

$$H^{\#k} := \underbrace{H\#\dots\#H}_{k \text{ times}}.$$

Then, $\phi_{H^{\#k}}^t = \phi_H^t \circ \dots \circ \phi_H^t = \phi_H^{kt}$.

Corollary A.4. *A time 1 trajectory $x : [0,1] \to M$ of the Hamiltonian $H^{\#k}$ corresponds to a time k trajectory $x : [0,k] \to M$ of H.*

Which follows directly from the line above. In particular, using Definition A.2, we can show by an easy induction that

$$H^{\#k} = H + \sum_{i=1}^{k-1} H_t \circ (\phi_H^{it})^{-1}.$$

Liouville vector fields. Let us now define an important object in symplectic (and later on, contact) dynamics: Liouville vector fields.

• **Definition A.5.** A **Liouville vector field** V on a symplectic manifold (M, ω) is a vector field such that $\mathcal{L}_V\omega = \omega$, where $\mathcal{L}$ denotes the Lie derivative.

Lemma A.6. *The existence of Liouville vector fields on (M, ω) is equivalent to ω being exact; and a choice of Liouville vector field V corresponds to a choice of primitive λ for ω (also called a Liouville form).*

Proof. Assume such a V exists, and define $\lambda := V \lrcorner\, \omega$. Then, by Cartan's magic formula:

$$\mathrm{d}\lambda = \mathrm{d}\big(V \lrcorner\, \omega\big) = \mathcal{L}_V\omega - V \lrcorner\, \mathrm{d}\omega = \omega, \tag{A.1}$$

and vice versa: if λ is already a primitive for ω, then we can explicitly construct a V satisfying (A.1), as is shown in the Example below. □

✧ **Example A.7.** Let Q be a smooth manifold, and $T^\star Q$ be its cotangent bundle, with coordinates (q_i, p_i), and symplectic form $\omega = \mathrm{d}\lambda_0$, with

$$\lambda_0 = -\sum_i p_i \mathrm{d}q_i.$$

Write $V = a_i\partial_{q_i} + b_i\partial_{p_i}$ for some coefficients $a_i, b_i \in \mathbb{R}$, and plug V into the equation $V \lrcorner\, \omega = \lambda$. Comparing coefficients, we get:

$$V_0 = \sum_i p_i \partial_{p_i}. \tag{A.2}$$

✧ **Example A.8.** Say that, instead of $-\lambda_0$, we pick the following primitive of ω:

$$\lambda = \frac{1}{2}\sum_i q_i \mathrm{d}p_i - p_i \mathrm{d}q_i.$$

Then, a short calculation gives:

$$V = \sum_i q_i\partial_{q_i} + p_i\partial_{p_i}. \tag{A.3}$$

A.1.2 Contact geometry

Contact geometry is often introduced as the odd cousin of symplectic geometry. Definitions are stated in the same language, tools from one field are frequently applicable in the other, and often results from the symplectic and contact worlds turn out to be intricately related, sometimes in quite beautiful ways. To start from the basics:

• **Definition A.9.** Let N^{2n-1} be an odd-dimensional manifold. A **contact form** on N is a 1-form α such that:

$$\alpha \wedge (\mathrm{d}\alpha)^{n-1} \quad \text{is a volume form.}$$

In other words, $\alpha \wedge (\mathrm{d}\alpha)^{n-1}$ never vanishes. In particular, it has constant sign, and we call α positive (resp. negative) if $\alpha \wedge (\mathrm{d}\alpha)^{n-1} > 0$ (resp. < 0).

• **Definition A.10.** The hyperplane distribution defined by $\xi := \ker\alpha$ is called the **contact structure** induced by α. It is a rank $2n-2$ bundle over N.

Lemma A.11. *$(\xi, \mathrm{d}\alpha)$ is a symplectic vector bundle, i.e. $\mathrm{d}\alpha$ is closed and non-degenerate.*

Proof. By assumption, $\alpha \wedge (\mathrm{d}\alpha)^{n-1}$ is never zero. Hence, $\mathrm{d}\alpha$ cannot vanish on $\xi = \ker \alpha$, and actually neither can $(\mathrm{d}\alpha)^{n-1}$. By standard linear algebra, this implies that $\mathrm{d}\alpha$ is non-degenerate. It is trivially closed since $\mathrm{d}^2 = 0$. □

◇ **Remark A.12.** By the Frobenius theorem (see [Lee12], Chapter 21), the vanishing of the form $(\alpha \wedge \mathrm{d}\alpha)^{n-1}$ is related to the integrability of the distribution $\xi = \ker \alpha$. Therefore, it never vanishing is equivalent to saying that the distribution ξ is **maximally non-integrable** (*i.e.* it can never be viewed, not even locally, as the tangent bundle of a submanifold of N).

Contact structures are somehow a more canonical piece of structure than contact forms. Indeed, if α' and α are proportional to each other ($\alpha' = f\alpha$, for $f \neq 0$), then they induce the same ξ. This motivates the following definitions:

• **Definition A.13.** A **contact manifold** (N, ξ) is an odd-dimensional manifold, endowed with a maximally non-integrable hyperplane distribution.

• **Definition A.14.** A **strict** contact manifold $(N, \xi = \ker \alpha)$ is a contact manifold, along with a choice of contact form generating ξ.

Choosing a contact form on a contact manifold is not an innocent choice. It can be akin to choosing a Hamiltonian H on a symplectic manifold. Indeed:

> Any choice of contact form α on (N, ξ) induces some specific dynamics on N.

Indeed, consider the 2-form $\mathrm{d}\alpha$ on N. By Lemma A.11, it is non-degenerate on ξn which is a $2n-2$ bundle over N (in other words, $\xi \subset TN$ has codimension 1, and $\xi_x \subset T_xN$ $\forall x \in N$). However, there remains one direction to be covered in the tangent bundle. One can easily show that a 2-form on an odd-dimensional (vector) space must necessarily degenerate in at least one direction, so that we have $\mathrm{d}\alpha \equiv 0$ in $TN \backslash \xi$.

• **Definition A.15.** The **Reeb vector field** R_α on $(N, \xi = \ker \alpha)$ is the unique vector field on N satisfying $\mathrm{d}\alpha(R_\alpha, \cdot) \equiv 0$ and $\alpha(R_\alpha) = 1$.

Existence of $\mathcal{R}_\alpha$ follows from the discussion right above, while unicity is ensured by the normalisation condition $\alpha(\mathcal{R}_\alpha) = 1$. We call **Reeb flow** the flow of $\mathcal{R}_\alpha$, and refer to its dynamics as the **Reeb dynamics**.

✧ **Example A.16.** The standard contact manifold is given by $(\mathbb{R}^{2n-1}, \xi = \ker \alpha)$, where

$$\alpha = \mathrm{d}z + \sum_{i=1}^{n-1} \mathrm{d}x_i \wedge \mathrm{d}y_i,$$

and where $(x_1, y_1, \ldots, x_n, y_n, z)$ are coordinates for $\mathbb{R}^{2n-1}$.

We hold off on more interesting examples of contact manifolds, and Reeb flows for now, to focus on one specific construction.

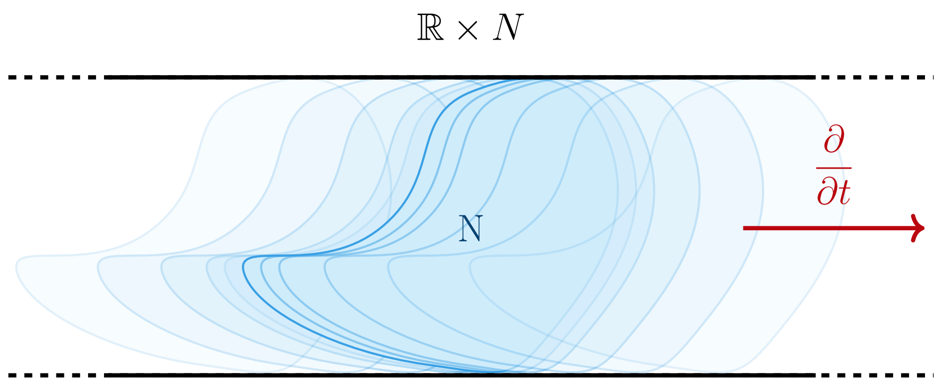
A.1.3 The symplectisation of a contact manifold

The concept of symplectisation can be summarised concisely:

> Every (strict) contact manifold can be embedded in a symplectic manifold.

The idea is quite simple:

• **Definition A.17.** Let $(N, \xi = \ker \alpha)$ be a strict contact manifold. We define its **symplectisation** as $M = \mathbb{R} \times N$:



where we write t the coordinate on $\mathbb{R}$. We can endow $M = \mathbb{R} \times N$ with the 2-form

$$\omega = \mathrm{d}(e^t\alpha),$$

which turns it into a symplectic manifold.

Lemma A.18. *Let $\big(M, \omega = \mathrm{d}(e^t\alpha)\big)$ be the symplectisation of $(N, \xi = \ker\alpha)$. Then, the associated Liouville vector field (Definition A.5) is given by $V = \partial_t$.*

Proof. First, notice that if we set $V = \partial_t$, then we do indeed have

$$V \lrcorner\, \omega = V \lrcorner \left(e^t \mathrm{d}t \wedge \alpha + e^t \mathrm{d}\alpha\right) = e^t\alpha,$$

and we can explicitly check that V is Liouville. Indeed, by Cartan's magic formula:

$$\begin{aligned} \mathcal{L}_V\omega &= V \lrcorner\, \mathrm{d}\omega + \mathrm{d}(V \lrcorner\, \omega) \\ &= 0 + \mathrm{d}(V \lrcorner\, \omega). \end{aligned}$$

Now, by Leibniz: $\omega = \mathrm{d}(\mathrm{e}^t\alpha) = \mathrm{e}^t\mathrm{d}t \wedge \alpha + \mathrm{e}^t\mathrm{d}\alpha$. Therefore:

$$V \lrcorner\, \omega = e^t\alpha - e^t\mathrm{d}t \wedge (V \lrcorner\, \alpha) + \mathrm{e}^t(V \lrcorner\, \mathrm{d}\alpha),$$

but since α (and hence $\mathrm{d}\alpha$) are only defined on N, and that $V = \partial_t$ is orthogonal to it, $V \lrcorner\, \alpha = V \lrcorner\, \mathrm{d}\alpha = 0$. So:

$$\mathscr{L}_V\omega = \mathrm{d}(V \lrcorner\, \omega) = \mathrm{d}(\mathrm{e}^t\alpha - 0 + 0) = \mathrm{d}(\mathrm{e}^t\alpha) = \omega,$$

which is what we wanted to prove. □

◇ **Remark A.19.** In Chapter 3, we will consider a special case of symplectisation, called *Liouville completion*, in which we only symplectise our contact manifold in one direction, *i.e.* we work on $\mathbb{R}_+ \times N$, instead of $\mathbb{R} \times N$. For convenience, we will work with the coordinate $r = e^t \in [1, +\infty)$ instead of t. Then, the Liouville vector field will be given by:

$$V = r\frac{\partial}{\partial_r}. \tag{A.4}$$

Proof. $V = \frac{\partial}{\partial t} = \frac{\partial r}{\partial t}\frac{\partial}{\partial r} = e^t\frac{\partial}{\partial r} = r\frac{\partial}{\partial r}$. □

A.1.4 Examples

The reason we defined the process of symplectisation before actually giving examples of contact manifolds is the following:

Proposition A.20. *Let (M, ω) be a symplectic manifold. Then a hypersurface $S \subset M$ is of contact type $\iff$ there exists a Liouville vector field, defined locally around S, and which is transverse to it.*

Proof. This is a standard result in contact geometry. Our proof follows from the presentation in [Wen15].

($\implies$) By the tubular neighbourhood theorem, we can find a neighbourhood $(-\varepsilon, \varepsilon) \times S$ of S in M. Since we know S to be contact, then we can treat this neighbourhood as a symplectisation, and define a symplectic form $\omega = \mathrm{d}(e^t \alpha)$, and *a fortiori* a Liouville vector field.

($\impliedby$) Assume that there is a neighbourhood $\mathcal{U}$ of S in M in which we can find a Liouville vector field V. Define $\lambda := V \lrcorner\, \omega$, and recall that by Lemma A.6, we have $\omega = \mathrm{d}\lambda$.

Claim. $\alpha := \lambda|_S$ is a contact form on S.

Indeed,
$$\begin{aligned} \alpha \wedge (\mathrm{d}\alpha)^{n-1} &= \left(\lambda \wedge (\mathrm{d}\lambda)^{n-1}\right)|_S \\ &= \left((V \lrcorner\, \omega) \wedge \omega^{n-1}\right)|_S \\ &\propto (V \lrcorner\, \omega^n)|_S \end{aligned}$$

Now, ω^n is a volume form on M, and V is transverse to S, so $(V \lrcorner\, \omega^n)|_S$ never vanishes. Therefore, $\alpha := \lambda|_S$ is a contact form on S. □

✧ **Example A.21.** Consider $(\mathbb{R}^{2n}, \omega_0)$ with its standard symplectic form
$$\omega_0 = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i,$$
but consider the primitive:
$$\lambda = \frac{1}{2} \sum_i \mathrm{q}_i \mathrm{d}p_i - p_i \mathrm{d}q_i.$$
Then $(\mathbb{S}^{2n-1}, \lambda)$ is a contact manifold.

Proof. We showed in Example A.8 that the associated Liouville vector field is given by $V = \sum_i q_i \partial_{q_i} + p_i \partial_{p_i}$. This points radially outwards, and is therefore clearly transverse to the sphere. □

We see more examples of contact manifolds at the beginning of Chapter 3, where they are not only hypersurfaces in symplectic manifolds, but boundaries.

A.2 Fibre bundles

A.2.1 Spin structures

• **Definition A.22.** Let G be a Lie group. A fibre bundle $\pi : E \to B$ is called a **principal G-bundle** if there exists a free, transitive right-action of G on E.

✧ **Example A.23.** Let M be a manifold, and $\mathrm{Fr}(M) \to M$ its unit frame bundle, *i.e.* the fibre above a point $x \in M$ consists of all orthonormal bases of $T_x M$.

Then, $\mathrm{Fr}(M) \to M$ is a principal O_n-bundle, because O_n acts freely and transitively on each fibre.

If we define $\mathrm{Fr}^+(M)$ as the space of *oriented* orthonormal frames, then $\mathrm{Fr}^+(M) \to M$ is a principal SO_n-bundle.

Let us focus a bit more on SO_n, the special orthonormal (Lie) group. From standard algebraic topology, $\pi_1(SO_n) = \mathbb{Z}/2\mathbb{Z}$, which tells us that its universal cover is a double cover. We define:

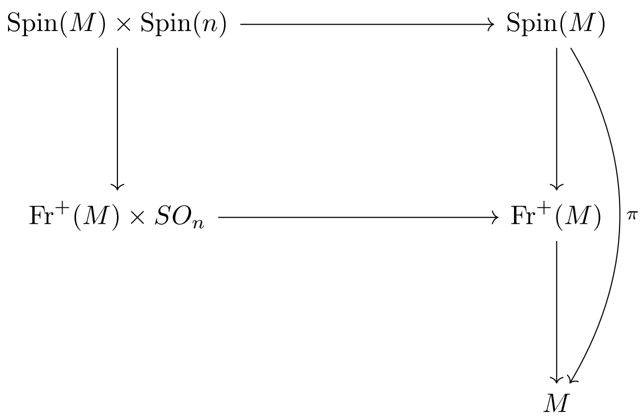
• **Definition A.24.** For $n > 2$, the **Spin group Spin(n)** is defined as the double cover of SO_n.

Since it is a double cover, the bundle $\mathrm{Spin}(n) \to SO_n$ is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle.

This allows us to make one definition:

• **Definition A.25** (Spin structures)**.** Let M be an oriented, Riemannian manifold of dimension $n \geq 3$. Then it is said to be **spin** if there exists an equivariant lift of its frame bundle $\mathrm{Fr}^+(M) \to M$ with respect to the double-cover $\mathrm{Spin}(n) \to SO_n$.

In other words, there exists a principal $\mathrm{Spin}(n)$-bundle $\pi : \mathrm{Spin}(M) \to M$, along with a map $\mathrm{Spin}(M) \to \mathrm{Fr}^+(M)$ making the following diagram commute:



The bundle $\pi : \mathrm{Spin}(M) \to M$ is often called a **spin-structure** on M.

A.2.2 Linearisation of a section at zero

We assume the reader is familiar with the different definitions of connections in Riemannian geometry, and we will move freely between them.

Consider a vector bundle $\pi : E \to M$, and a section $s : M \to E$. We will explain what it means to *linearise* s at 0. More precisely, we will show that, at any point $x \in M$ such that $s(x) = 0$, its total covariant derivative $\nabla s(x) : T_xM \longrightarrow V_{s(x)}E$ does *not* depend on the choice of connection ∇ – it will be what we call the **linearisation** of s at x.

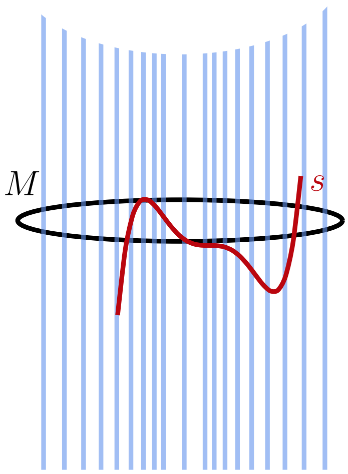
◇ **Remark A.26.** Consider the standard covariant derivative

$$\nabla_X : \Gamma(E) \longrightarrow \Gamma(E)$$

along a given vector field X, from Riemannian geometry, which allows us to differentiate sections. The **total** covariant derivative ∇ is defined as the map:

$$\nabla : \mathfrak{X}(M) \otimes \Gamma(E) \longrightarrow \Gamma(E)$$

which, when contracted with any $X \in \mathfrak{X}(M)$, becomes ∇_X.



Since $E \to M$ is a bundle, we can naturally include M into E as the zero section $(M, 0) \hookrightarrow E$.

Along this zero section, there is a canonical splitting of TE into horizontal and vertical subspaces:

$$T_xE = T_xM \oplus E_x.$$

Where the vertical subspaces are simply the fibres E_x (vertical blue lines, on the picture), and the horizontal distribution is the tangent bundle of our original manifold, M.

From standard Riemannian geometry, the connection ∇ on E can also be viewed as a choice of horizontal distribution HE such that $TE = HE \oplus VE$.

Alternatively, one could see it as a *fibrewise-linear* map $K : TE \to VE$ which restricts to the identity on VE. Indeed, when provided with such a map, it suffices to set $HE := \ker K$ to get the horizontal distribution. With this formalism, one can prove (see [Wen16])

$$\nabla_X s = K \circ Ds(x). \tag{A.5}$$

In other words, the covariant derivative is the vertical part of the standard, geometric derivative.

Proposition A.27. *Given a section $s : M \to E$ and a point x such that $s(x) = 0$ (by which we mean, $s(x)$ lies in the zero section $M \hookrightarrow E$); then the covariant derivative $\nabla s(x) : T_x M \to V_{s(x)}E$ does **not** depend on the choice of connection ∇. It is denoted $Ds(x)$, and called the **linearisation of s at x**.*

Proof. As we saw, along the zero section $M \overset{0}{\hookrightarrow} E$, there is a canonical splitting:

$$T_{s(x)}E = T_x M \oplus E_x, \tag{A.6}$$

which holds true at any x satisfying $s(x) = 0$.

Hence, at such an x, the connection map $K|_x : T_{s(x)}E \longrightarrow E_x$ must be the projection $K : T_{s(x)}E \twoheadrightarrow E_x$ onto the second factor of the splitting. (There is no other possible choice for the map, since we require that $K|_{VE} = \mathrm{id}$).

Therefore, the map K at x is **independent** of the choice of connection, and the covariant derivative, given by A.5, reduces to the usual derivative:

$$\nabla_X s(x) = K \circ Ds(x) \equiv \mathrm{id}|_{E_x} \circ Ds(x) = Ds(x).$$

□

✧ **Example A.28.** (Hessian of a function) Let M be a manifold endowed with a Riemannian metric g. Recall that the gradient of f is defined as the unique vector field $\vec{\nabla} f$ which is dual to the differential $\mathrm{d}f$ through g, *i.e.* :

$$\mathrm{d}f = g(\vec{\nabla} f, \cdot\,) \quad \text{everywhere.}$$

The gradient is a vector field, so in other words, a section of the tangent bundle $TM \to M$. As such, it can be linearised at the points where $\vec{\nabla} f = 0$ (otherwise known as the critical points of f).

• **Definition A.29.** The **Hessian** of f at a critical point x, which we denote $\mathrm{Hess}_x f$, is the linearisation $D\vec{\nabla} f$ of the section $\vec{\nabla} f$.

So, choosing *any* connection ∇ on TM, we can compute $\mathrm{Hess}_x f$ as $\nabla \vec{\nabla} f$; and the result will not depend on ∇ by Proposition A.27. Working in local coordinates $\{x_i\}_i$ for M, we have:

$$\begin{aligned} \vec{\nabla} f : \mathbb{R}^n &\longrightarrow T\mathbb{R}^n = \mathbb{R}^n \oplus \mathbb{R}^n \\ x &\longmapsto \left(x, \left(\frac{\partial f}{\partial x_1}\Big|_x, \dots, \frac{\partial f}{\partial x_n}\Big|_x\right)\right). \end{aligned}$$

Now, as we saw in A.5, the linearisation of $\vec{\nabla} f$ is the vertical part of its usual geometric derivative, so it is the symmetric operator given by:

$$\mathrm{Hess}_x f = \mathrm{pr}_2 \circ D(\vec{\nabla} f)_x = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_x \right)_{i,j}. \tag{A.7}$$

◇Notice that in $\mathbb{R}^n$, we usually define this Hessian matrix *globally*; which is possible because the canonical splitting of $T\mathbb{R}^n$ into horizontal/vertical subspaces exists everywhere. On an arbitrary Riemannian manifold however, the Hessian is *only defined at critical points*.

A.3 The Two-Body Problem

A.3.1 Levi-Civita regularisation

In §8.1.3, we described a scheme to regularise the Circular Restricted Three-Body Problem at collisions. Let $n = 2$ or 3 (2 for the planar problem, 3 for the spatial problem), then this scheme regularised the low-energy dynamics from $T^\star(\mathbb{R}^n \backslash \{\overrightarrow{E}, \overrightarrow{M}\})$ to $T^\star\mathbb{S}^n$, thus effectively compactifying the energy hypersurfaces near the Earth and Moon to $\mathbb{S}^\star\mathbb{S}^n$.

In the *planar* case (PCR3BP), the regularisation is given by $\mathbb{S}^\star\mathbb{S}^2 \cong \mathbb{RP}^3$. Let us describe a different regularisation scheme which, historically, came earlier, and which instead compactifies the dynamics of the planar problem to $\mathbb{S}^3$, the double-cover of $\mathbb{RP}^3$.

This section is entirely based on §2 of [Che85], and §4-5 of [Bla62]. Since both sources are in French, we thought pertinent to dress a summary of the main ideas.

Since we only really care about two-body collisions, we simplify our model to that of the Two-Body Problem (or Kepler problem). The PCR3BP is then but a perturbation of that model, and every result we discuss holds likewise (see the sources we cited, or [FZ20]).

Setup. Consider the Two-Body Problem (Example 1.1), with Hamiltonian:

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{1}{\|q\|}, \tag{A.8}$$

where H is defined on phase space $T^\star(\mathbb{R}^2\backslash\{0\}) = \{(q,p) \mid q,p \in \mathbb{R}^2\}$; q being position, and p momentum. Instead of vectors in $\mathbb{R}^2$, we choose to view q and p as complex numbers:

$$q = q_1 + iq_2, \;\; p = p_1 + ip_2,$$

and thus view H as a Hamiltonian on $T^\star(\mathbb{C}\backslash\{0\})$ instead.

Fix a negative value c of the energy, which we decide to write $c = -1/\varepsilon^2$ for some $\varepsilon > 0$, following [Che85]. We are interested in the energy hypersurface $\Sigma_c := H^{-1}(c)$.

Σ_c is non-compact. Indeed, by Lemma 8.8, p must blow up to infinity as $q \to 0$, by conservation of energy. Hence, we need to find a way to regularise at collisions. Levi-Civita proposes the change of coordinates given by:

$$(q,p,t) \longmapsto (z,w,t') \text{ where } \begin{cases} q &= z^2 \\ p &= \dfrac{w}{\varepsilon \bar{z}} \\ \mathrm{d}t &= 2\varepsilon \|q\| \,\mathrm{d}t' \end{cases} \tag{A.9}$$

which is but the cotangent lift of the map $\mathbb{C} \to \mathbb{C} : z \mapsto z^2$ (see §3.3 of [FZ20]).

◇ **Remark A.30.** The change in the time coordinate can be re-written:

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{1}{2\varepsilon\|q\|},$$

or yet

$$\mathrm{d}t' = \int \frac{1}{2\varepsilon\|q\|}\mathrm{d}t.$$

This reparametrisation serves to ensure that the speed no longer goes to infinity at collision. Heuristically: by conservation of energy ($H \equiv -1/\varepsilon^2$), $\|p\|^2 \to \infty$ as $q \to 0$. In particular, $\|p\|^2$ must grow to infinity as fast as $1/\|q\|$, so that:

$$|p| \sim \frac{1}{\sqrt{\|q\|}} \quad \text{as } q \sim 0. \tag{A.10}$$

Consider a trajectory $(q(t), p(t) = \dot{q}(t))$ satisfying the equations of motion, and such that $q(t) \to 0$. Then, in unregularised time, we have:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = p(t) \longrightarrow \infty \;\text{ as } q \to 0.$$

However, in the t' coordinate, we have:

$$\frac{dq}{dt'} = \frac{dq}{dt/(2\varepsilon \|q\|)} = 2\varepsilon \|q\| \frac{dq}{dt} = 2\varepsilon \|q\| p(t)$$
$$\sim 2\varepsilon \frac{\|q\|}{\sqrt{\|q\|}} \text{ by (A.10)}$$
$$\sim 0 \text{ for } q \sim 0.$$

◇ **Remark A.31** (Regularisation via elastic bouncing)**.** With the time change we did above, speed becomes zero at collision. This is a symptom of the type of regularisation technique that we use: by doing a $z \mapsto z^2$ change of coordinates, we are essentially replacing the original collision with *elastic bouncing.*

In other words, the colliding body bounces back, returning whence it came with no loss in kinetic energy. For more on the symplectic approach to regularising central force problems through elastic bouncing, we refer to the great survey [FZ20]. In particular, §3.2-3.4 for the Moser and Levi-Civita regularisations.

In these new coordinates (z, w, t'), we define the regularised Hamiltonian:

$$K : T^\star(\mathbb{C}\backslash\{0\}) \to \mathbb{R} : (z, w) \mapsto \varepsilon^2 |z|^2 \left(H(z^2, \frac{w}{\varepsilon \overline{z}}) + \frac{1}{\varepsilon^2} \right), \tag{A.11}$$

which, by a simple calculation, reduces to:

$$K(z, w) = \frac{1}{2}|w|^2 + |z|^2 - \varepsilon^2 \tag{A.12}$$
$$= \frac{1}{2} w\overline{w} + z\overline{z} - \varepsilon^2. \tag{A.13}$$

In particular, the Hamiltonian equations of motion are given by

$$\dot{z} = \frac{\partial K}{\partial \overline{w}} = w/2, \quad \dot{w} = -\frac{\partial K}{\partial \overline{z}} = -z. \tag{A.14}$$

This gives us the regularised equations of motion on the hypersurface $\tilde{\Sigma}_c := K^{-1}(0)$. In particular, from (A.12), it is clear that $\tilde{\Sigma}_c$ is a 3-sphere of radius ε. In summary:

Corollary A.32 (Levi-Civita regularisation)**.** *For low energies $c = 1/\varepsilon^2$, the energy hypersurface $\Sigma_c = H^{-1}(c)$ compactifies to $\tilde{\Sigma}_c \cong \mathbb{S}^3$, and the Hamiltonian equations of motion can be regularised into the Hamiltonian flow of (A.14) on $\mathbb{S}^3$.*

As we have already mentioned in Remark 8.14, Levi-Civita regularisation is the double-cover of Moser regularisation (§8.1.3) in dimension $n = 2$; so in particular, for the PCR3BP. We will use this fact in the next section when trying to construct a global hypersurface of section.

A.3.2 The Birkhoff annulus, direct, and retrograde orbits

Just like in the previous section, we here use the Two-Body Problem (or Kepler Problem) as a toy model, but all the constructions we discuss hold, and are by now standard in the Planar Circular Restricted Three-Body Problem (PCR3BP), see for example [Che85]; with the exception of the Birkhoff annulus, as we shall explain.

Before we define our global surface of section, we fully solve the Two-Body Problem – as this will help us in our construction. Recall, from the previous section, that we have a regularised flow on $\mathbb{S}^3$. First, let us once again change coordinates:

$$u_1 := w + iz, u_2 := \overline{w} + i\overline{z}.$$

Then, observe that we can rewrite our regularised hypersurface

$$K^{-1}(0) = \left\{(z, w) \mid |z|^2 + |w|^2 = \varepsilon^2\right\}$$
$$= \left\{(u_1, u_2) \mid |u_1|^2 + |u_2|^2 = 2\varepsilon^2\right\}.$$

In these new coordinates, the equations of motion (A.14) turn into $\begin{cases} \dot{u}_1 &= iu_1 \\ \dot{u}_2 &= iu_2 \end{cases}$

where the dot denotes the time derivative with respect to regularised time, t'. Therefore, an easy integration shows that integral curves of the flow have expression:

$$(u_1(t'), u_2(t')) = \left(c_1 e^{i(t'+s_1)}, c_2 e^{i(t'+s_2)}\right) \text{ where } c_1^2 + c_2^2 = 2\varepsilon^2, \tag{A.15}$$

where s_1 and s_2 are initial times for both variables, and we can assume without loss of generality that $c_j \in \mathbb{R}$ (if they were complex, then we could write $c_j = c_j' e^{i\varphi_j}$, where $c_j' \in \mathbb{R}$, and then the argument φ_j would get absorbed into $s_j \in [0, 2\pi)$).

Now that we have solved the Two-Body Problem in regularised coordinates (u_1, u_2), let us go back to our original, unregularised coordinates (q, p), which we can recover by:

$$q = 2z^2 = -\frac{1}{4}(u_1 - \overline{u}_2)^2, \quad p = \frac{w}{\varepsilon \overline{z}} = \frac{1}{i\varepsilon} \frac{u_1 + \overline{u}_2}{\overline{u}_1 - u_2}.$$

Then, in position space $\mathbb{C}\backslash\{0\}$, solutions to the Two-Body Problem are given by:

$$q = -\frac{1}{2} e^{i\varphi} \left((c_1^2 + c_2^2) \cos\chi - 2c_1 c_2 + i(c_1^2 - c_2^2) \sin\chi \right). \tag{A.16}$$

where

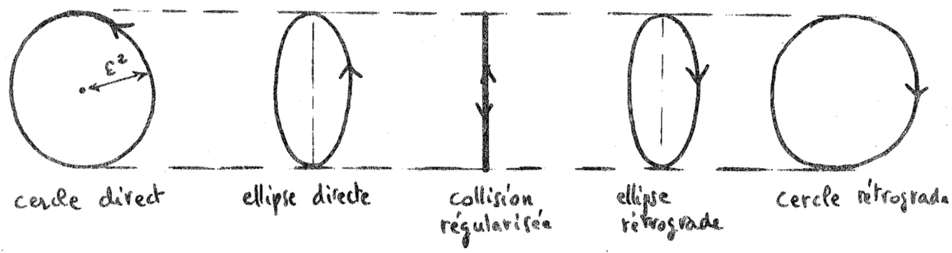
$$\varphi := \arg(u_1) - \arg(u_2) \bmod 2\pi, \tag{A.17}$$

$$\chi := \arg(u_2) + \arg(u_2) \bmod 2\pi. \tag{A.18}$$

From (A.15), we get $\varphi = s_1 - s_2 \equiv \text{cst}$ along every fixed trajectory q, and $\chi = s_1 + s_2 + 2t'$.

The trajectory q from (A.16) draws out an ellipse in $\mathbb{C}\backslash\{0\} \cong \mathbb{R}^2\backslash\{0\}$, with inclination φ from the q-axis, and whose axes have lengths $c_1^2 + c_2^2 = 2\varepsilon^2$, and $|c_1^2 - c_2^2|$; see [Che85, p.12].

➢ The limit cases of our ellipse are obtained by setting $c_1 = 0$, or $c_2 = 0$, in which case the expression (A.16) reduces to the equation of a circle. In general, we borrow the following picture from [Che85], showing what happens as we vary from $c_2 = 0$ to $c_1 = 0$:



Formally, we define:

• **Definition A.33.** The **direct (circular) orbit** is defined as the orbit obtained by setting $c_2 = 0$, *i.e.* it is the circle $|u_1| = \sqrt{2}\varepsilon$.

The **retrograde (circular) orbit** is defined as the orbit obtained by setting $c_1 = 0$, *i.e.* it is the circle $|u_2| = \sqrt{2}\varepsilon$.

The names direct and retrograde have a long history − taking their root in stargazing. Say, for example, that we are studying the Earth-Moon system. Then the Moon's direct orbit is its one circular orbit going in the same direction as Earth's spin on itself, and its retrograde orbit is the circular orbit which goes in the opposite direction.

We may sometimes call an ellipse direct or retrograde when it goes in the same direction as the direct or retrograde (circular) orbits. However, when we say **the** direct or retrograde orbits, we mean the circular ones, *i.e.* the limit cases of the above diagram, at $c_j = 0$.

➢ Now let us define our surface of section, discovered by Poincaré, and first studied in [Poi12; Bir13; Bir15].

Recall from (A.17) that $\varphi \equiv \text{cst}$ along a solution, and that $\chi = s_1 + s_2 + 2t$, where $s_1 + s_2 \equiv \text{cst}$. Therefore, every trajectory q can be parametrised by χ (with the subtlety that this reparametrisation doubles the speed). Now define the set:

$$\text{int}(A) := \left\{ (u_1, u_2) \in \mathbb{S}^3 \mid \chi = \arg(u_1) + \arg(u_2) \equiv 0 \bmod 2\pi \right\}, \tag{A.19}$$

and denote by A its closure.

Proposition A.34. *A is an annulus in $\mathbb{S}^3$, with boundary the direct and retrograde circular orbits. Moreover, it is a global surface of section (Definition 8.17) for the regularised flow of the two-body problem.*

Proof. First, notice that $(\arg(u_1), c_1^2 - c_2^2)$ define coordinates on $\text{int}(A)$ (since we know that $c_1^2 + c_2^2 = 2\varepsilon^2$ already). Write:

$$\theta := \arg(u_1) \in [0, 2\pi), \tag{A.20}$$

and:

$$\rho := c_1^2 - c_2^2 \in [-2\varepsilon^2, 2\varepsilon^2], \tag{A.21}$$

so that:

$$A \cong \left\{ (u_1, u_2) \in \mathbb{C}^2 \mid c_1^2 + c_2^2 = 2\varepsilon^2, \theta \in [0, 2\pi), \rho \in [-2\varepsilon^2, 2\varepsilon^2] \right\}.$$

This is indeed diffeomorphic to the annulus $(\varepsilon\sqrt{2})\mathbb{S}^1 \times [-2\varepsilon^2, 2\varepsilon^2]$ in $\mathbb{S}^3$. The two boundary components are given by setting $\rho = -2\varepsilon^2$ or $\rho = 2\varepsilon^2$. These respectively correspond to $c_2 = 0$ and $c_1 = 0$, *i.e.* the direct and retrograde orbits.

Let us now show that A is a global surface of section. First, it clearly is compact, oriented, and has codimension 1 in $\mathbb{S}^3$. Then, notice that its boundary is indeed flow-invariant, since it consists of the circular and direct orbits, which are both periodic orbits of the 2BP flow.

Since solutions of the two-body problem (A.16) are periodic in χ with period 2π, and since

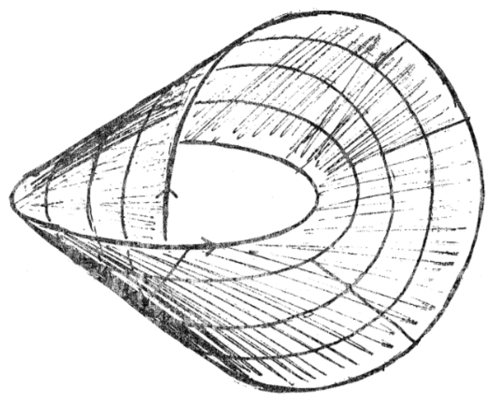
$$\chi = \arg(u_1) + \arg(u_2) \equiv 0 \bmod 2\pi,$$

$\text{int}(A)$ picks up, for every orbit, a point every 2π. Therefore, A satisfies that every point in the interior returns to the surface both in the future and in the past.

The last thing to show now is that the flow is transverse to the interior of A. By definition, $(u_1, u_2) \in \mathbb{S}^3$ belongs to $\text{int}(A)$ iff $\chi \equiv 0 \bmod 2\pi$, where $\chi = \arg(u_1) + \arg(u_2)$. Also recall that $\chi = s_1 + s_2 + 2t'$, so that $\dot{\chi} = 2$. In particular, along $\text{int}(A)$, $\dot{\chi} > 0$, so that the flow is indeed transverse to the interior.

Therefore, A is a global surface of section for the regularised 2BP flow. □

Visually, if we imagine $\mathbb{S}^3$ as compactified $\mathbb{R}^3$, the Birkhoff annulus looks something like:



where this picture is again borrowed from [Che85]. We can clearly see that the two boundary components, the direct and retrograde orbits, rotate in different directions.

◇ **Remark A.35.** One thing which may appear clear from the figure is that the two boundary components of the annulus above form a Hopf link. We can state this formally: in our Levi-Civita regularised coordinates (u_1, u_2), the map

$$\mathbb{S}^3 \ni (u_1, u_2) \longmapsto [u_1, u_2] \in \mathbb{CP}^1 \cong \mathbb{S}^2$$

is the standard Hopf fibration (see [Che85]). Actually, the converse also holds from from Prop. 1.15 of [HSW22]: any pair of Reeb orbits in a Hopf link on the standard contact 3-sphere, and with positive Conley-Zehnder indices, bounds an annulus-like global surface of section (which follows from more general existence results from [HSW22]).

◇ **Remark A.36.** Unlike all other constructions in this appendix, the aforementioned annulus does not readily carry away from the Two-Body Problem to the Planar Circular Restricted Three-Body Problem, as explained in §8.2.2. Indeed, the existence of this annulus in the PCR3BP has so far only been proved in the convexity range

➤ As claimed at the beginning of the section, all this work we carried out for the Two-Body Problem also holds in the Planar Circular Restricted Three-Body Problem. Hence, in summary, we have described a way to regularise collisions in the PCR3BP, by changing coordinates to $\mathbb{S}^3$. As we had claimed in Remark 8.14, Levi-Civita regularisation actually turns out to be the double-cover of Moser regularisation, in dimension $n = 2$ (see [FZ20]). Recall that Moser regularisation (§8.1.3) provided a change of coordinates to $\mathbb{S}^\star\mathbb{S}^2 \cong \mathbb{RP}^3$, for the PCR3BP.

In the next lemma, we show (for the Two-Body Problem) that if we take the surface of section $A \subset \mathbb{S}^3$ we have found through the quotient

$$\mathbb{S}^3 \twoheadrightarrow \mathbb{S}^3/\mathbb{Z}_2 \cong \mathbb{RP}^3,$$

then it is still a surface of section in $\mathbb{RP}^3$, and still diffeomorphic to an annulus.

Lemma A.37. *Consider the map $\sigma : \mathbb{C}^2 \to \mathbb{C}^2 : (z, w) \mapsto (-z, -w)$, and its restriction $\sigma : \mathbb{S}^3 \to \mathbb{S}^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$. Then, the image of A under σ is still an annulus, and it is a global surface of section for the regularised two-body problem flow on $\mathbb{RP}^3$.*

Proof. In (u_1, u_2) coordinates, σ also reads $(u_1, u_2) \mapsto (-u_1, -u_2)$. Now, note that the expression (A.19) for $\text{int}(A)$ is preserved by σ, since the norms $|u_i|^2$ are unaffected, and we have that $\arg \circ \sigma = \arg - \pi$. Hence, the condition

$$\arg(u_1) + \arg(u_2) \equiv 0 \bmod 2\pi$$

is unchanged. As for the boundary components, *i.e.* the circular curves $|u_i| = 0$, these also project down to curves which are still flow-invariant.

Then, it remains to show that the flow of the 2BP is transverse to the interior of our new annulus, and that the flow always comes back to it in the past and the future, no matter where we start. These hold for the exact same reasons as in Lemma A.34. □

Appendix B

Topological appendix

B.1 Pseudo-holomorphic curves

B.1.1 Almost complex structures

Let M be a smooth even-dimensional manifold, and TM its tangent bundle.

• **Definition B.1.** An **almost complex structure** on M is a linear bundle endomorphism $J : TM \to TM$ such that, on every tangent space T_pM, $J^2 = -\mathbb{1}$.

So J « emulates » the action of the standard complex structure $i = \sqrt{-1} \in \mathbb{C}$ on every tangent space. If M admits an almost complex structure J, then we call (M, J) an **almost complex manifold**.

✧ **Example B.2.** On $\mathbb{R}^{2n}$, the standard (almost) complex structure is given by:

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{B.1}$$

which is simply the pullback of $i \oplus \cdots \oplus i$ under the standard isomorphism $\mathbb{R}^{2n} \xrightarrow{\cong} \mathbb{C}^n$.

• **Definition B.3.** Let (M, ω) be a symplectic manifold, and J an almost complex structure. J is said to be **compatible** with ω if $J^\star\omega = \omega$, and $\omega(\cdot, J\cdot)$ defines a Riemannian metric.

Lemma B.4. *Every symplectic manifold (M, ω) admits an almost complex structure compatible with ω.*

Proof. It suffices to locally construct a symplectic frame $\{u_i, v_i\}$ (satisfying $\omega(u_i, v_j) = \delta_i^j$, and $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$), and define J such that $Ju_i = v_i, Jv_i = -u_i$. Then, it is a simple exercise to show that $g := \omega(\cdot, J\cdot)$ is a Riemannian metric. □

Given a symplectic manifold (M, ω), let us denote by $\mathscr{J}(M, \omega)$ the space of almost complex structures compatible with J. We endow it with the $\mathcal{C}^\infty_{\text{loc}}$-topology, *i.e.* a sequence (J_n) is said to converge if it converges on every compact subset of M.

We just showed that $\mathscr{J}(M, \omega) \neq \emptyset$. However, from a homotopical point-of-view, we have:

Proposition B.5 (Gromov)**.** *$\mathscr{J}(M, \omega)$ is contractible.*

Proof. See §2.2 of [Wen15].

This is a standard, but very important fact, which we use in Chapters 3 and 4, because it implies that any two compatible almost complex structures can be connected by a homotopy.

We then state one last definition:

• **Definition B.6.** We call an almost complex structure J on M **integrable** if it arises from holomorphic coordinates on M. In other words, we can find a complex atlas for M (turning it into a complex manifold), such that in every chart, J reduces to the standard complex structure $i \oplus \cdots \oplus i$.

✧ **Example B.7.** It is standard knowledge that in dimension 2, every complex structure is integrable (see [Wen15], Theorem 2.1.4). Therefore, any almost complex surface (Σ, J) (*i.e.* $\dim_{\mathbb{R}} \Sigma = 2$) is actually a complex manifold of complex dimension 1 – a Riemann surface.

Therefore, when dealing with Riemann surfaces, we often simply assume that $J = i$, and write (Σ, i).

B.1.2 Pseudo-holomorphic curves

Defining pseudo-holomorphicity is quite straightforward: we reproduce the usual definition of holomorphicity, but with an *almost* complex structure, instead of i. The real importance of pseudo-holomorphic curves lies in their *power* to probe the topology of a symplectic manifold.

They were first introduced by Gromov, in [Gro85], to prove his famous « non-squeezing theorem », which states one cannot symplectically embed a ball of radius r into a cylinder of smaller radius $r' < r$ (in other words, symplectic maps cannot « squeeze/shrink » stuff).

The theory of pseudo-holomorphic curves was then thoroughly developed in the next two decades, by names like Hofer, Wysocki, Zehnder, Siefring, Wendl,... and we now have a good understanding of their behaviour, especially in dimension 4.

Most importantly for us: the main objects of Floer theory (solutions of the Floer equation) turn out to be pseudo-holomorphic curves, as we shall see in Proposition 2.24. So let us formally define what a pseudo-holomorphic curve is. First, recall:

• **Definition B.8.** Let M be a complex manifold. Then a function $f : M \to \mathbb{C}$ is said to be **holomorphic** if

$$Df \circ i = i \circ \mathrm{D}f.$$

By choosing coordinates $(z_1, \ldots, z_m) = (x_1 + iy_1, \ldots, x_m + iy_m)$ on M, we can rewrite the above criterion as

$$\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) f = 0 \quad \forall i,$$

or, more concisely, as

$$\frac{\partial}{\partial \overline{z}} f = 0, \tag{B.2}$$

where $\dfrac{\partial}{\partial \overline{z}_j} = \dfrac{\partial}{\partial x_j} + i \dfrac{\partial}{\partial y_j}$, and $\dfrac{\partial}{\partial \overline{z}} = \left(\dfrac{\partial}{\partial \overline{z}_1}, \ldots, \dfrac{\partial}{\partial \overline{z}_m} \right)^t$.

Equation (B.2) is called the **Cauchy-Riemann equation**.[(1)] We then generalise this definition to maps $f : M \to N$ between complex manifolds by decomposing $f = (f_1, \ldots, f_n)$, and calling it holomorphic iff every f_j is.

• **Definition B.9.** Let (X, j) and (M, J) be *almost complex manifolds*. A map $f : X \to M$ is said to be **pseudo-holomorphic** if, on every tangent space:

$$\mathrm{D}f \circ j = J \circ \mathrm{D}f.$$

[(1)] If we write $f = u + iv$, with u and v real-valued functions, then Equation (B.2) reduces to

$$\begin{cases} \dfrac{\partial u}{\partial x_j} = \dfrac{\partial v}{\partial y_j} \\ \dfrac{\partial u}{\partial y_j} = -\dfrac{\partial v}{\partial x_j} \end{cases}$$

which is the form in which it is commonly stated.

• **Definition B.10.** Let (Σ, i) be a Riemann surface and (M, J) an almost complex manifold. Then, a pseudo-holomorphic map

$$u : (\Sigma, i) \longrightarrow (M, J)$$

is called a **pseudo-holomorphic curve**.

Writing $z = s + it$ the coordinate on Σ, we reformulate the condition of u being a pseudo-holomorphic curve as:

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0, \tag{B.3}$$

which is again nothing but the Cauchy-Riemann equation.

We do not address in this thesis all the applications of pseudo-holomorphic curves to symplectic and contact topology; we refer to [Wen15; AH19; MS04] for starting points in the theory. The main reason we care about them is because of their importance in Floer theory; and we will hence only introduce the relevant machinery.

First, just like in §3.3.2, we can define a notion of energy for pseudo-holomorphic curves:

• **Definition B.11.** Let $u : (\Sigma, i) \to (M, \omega, J)$ be a pseudo-holomorphic curve in a symplectic manifold. Then we define its (ω-)**energy**:

$$E(u) := \int_\Sigma u^\star\omega.$$

Lemma B.12. *Assume that (M, ω, J) is a symplectic manifold, and J is compatible with ω. Then, we have:*

$$E(u) = \int_\Sigma |\partial_s u|^2 \,\mathrm{d}s \wedge \mathrm{d}t = \int_\Sigma |\partial_t u|^2 \mathrm{d}s \wedge \mathrm{d}t,$$

where $g := \omega(\cdot, J\cdot)$ is the Riemannian metric induced by ω and J.

Proof. It is an easy exercise in linear algebra to show that

$$u^\star\omega = \omega(\partial_s u, \partial_t u)\,\mathrm{d}s \wedge \mathrm{d}t.$$

Hence, we have:

$$\begin{aligned}
E(u) &= \int_\Sigma u^\star\omega \\
&= \int_\Sigma \omega(\partial_s u, \partial_t u)\,\mathrm{d}s \wedge \mathrm{d}t \\
&= \int_\Sigma g(\partial_s u, -J\partial_t u)\,\mathrm{d}s \wedge \mathrm{d}t \quad \text{(By compatibility)} \\
&= \int_\Sigma g(\partial_s u, \partial_s u)\,\mathrm{d}s \wedge \mathrm{d}t \quad \text{(Since } \partial_s u + J\partial_t u = 0\text{)} \\
&= \int_\Sigma |\partial_s u|^2\,\mathrm{d}s \wedge \mathrm{d}t \\
&= \int_\Sigma |\partial_t u|^2\,\mathrm{d}s \wedge \mathrm{d}t \quad \text{(Since } J \text{ is an isometry w.r.t } g\text{)}.
\end{aligned}$$

□

Corollary B.13. *$E(u) = 0$ iff u is constant.*

➤ For the rest of this section, we will be concerned with sequences of pseudo-holomorphic curves, and their convergence properties. For these purposes, let us take a small step aside, and introduce some standard elements functional analysis.

B.1.3 Arzelà-Ascoli and elliptic regularity

We will state, in their simplest form, some standard results about convergence of sequences of functions. First, let us recall a few definitions.

➤ Fix two metric spaces X and Y, with respective metrics d_X and d_Y.

• **Definition B.14.** Let (f_n) be a sequence of functions $f : X \to Y$. The sequence (f_n) is said to **converge uniformly** to f_∞ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$\forall\, n \geq N, \forall\, x \in X : \mathrm{d}_Y\left(f_n(x), f(x)\right) < \varepsilon.$$

• **Definition B.15.** A family of functions $F \subset \mathcal{C}(X, Y)$ is said to be **equicontinuous** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\forall f \in F, \forall x_1, x_2 \in X : (\mathrm{d}_X(x_1, x_2) < \delta) \implies (\mathrm{d}_Y(f(x_1), f(x_2)) < \varepsilon)\,.$$

Note that this is attained if, for example, the derivatives of functions in F are uniformly bounded.

Theorem B.16 (Arzelà-Ascoli)**.** *Let $f_n : \mathbb{R}^k \to \mathbb{R}^d$ be a sequence of functions which is uniformly bounded and equicontinuous. Then (f_n) admits a convergent subsequence (in the $\mathcal{C}^0$ topology).*

◇ *Note:* there exist many statements of the Arzelà-Ascoli theorem, including some quite more general than this one, but this is the form that we will mostly use throughout this thesis.

We now state the two main elliptic results, which are standard in symplectic geometry, and we refer to Appendix B.4 of [MS04] (from which we borrowed these statements), or Chap. 12 of [AD13] for a more detailed exposition, and proofs.

Theorem B.17 (Elliptic bootstrapping)**.** *Let Σ be a Riemann surface with a smooth complex structure j, and M a manifold with a $\mathcal{C}^\ell$ almost complex structure J, for some integer $\ell \geq 2$. Assume that*

$$u : (\Sigma, j) \longrightarrow (M, J)$$

is a pseudo-holomorphic curve (i.e. $J \circ \mathrm{D}u = \mathrm{D}u \circ j$) of class $\mathbf{W}^{1,p}$, for some real $p > 2$.

Corollary B.18. *Let $u \in \mathbf{W}^{1,p}$ for $p > 2$ and J be $\mathcal{C}^\ell$. Then u is $\mathcal{C}^\ell$. In particular, if J is smooth, then so is u.*

Theorem B.19 (Elliptic compactness)**.** *Let (Σ, j) and (M, J) be as before. Assume we have a sequence (j_n) of complex structures on Σ such that $j_n \underset{\mathcal{C}^\infty}{\longrightarrow} j$, and a sequence of $\mathcal{C}^\ell$ almost complex structures on M such that $J_n \underset{\mathcal{C}^\ell}{\longrightarrow} J$, where $\ell \geq 2$.*
Now, let (U_n) be an increasing sequence of open sets which exhausts Σ, and write ∂U_n the boundary $U_n \cap \partial\Sigma$. Furthermore, fix L a totally real subspace of (M, J).

Assume we have a sequence of (j_n, J_n)-holomorphic curves:

$$u_n : (U_n, \partial U_n) \longrightarrow (M, L)$$

of class $\mathbf{W}^{1,p}$, whose first derivatives are uniformly (L^p-)bounded on every compact set.

Then, (u_n) ***admits a converging subsequence*** *in the $\mathcal{C}^{\ell-1}_{loc}$ norm; i.e. it converges in the $\mathcal{C}^{\ell-1}$ norm on every compact subset of Σ.*

This theorem is pretty bulky to state, but it can be summarised very concisely:

$$\boxed{\begin{array}{c}\text{First derivatives are}\\ \text{uniformly bounded}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{There exists a}\\ \text{converging subsequence}\end{array}} \tag{B.4}$$

Let us see how we can ensure this.

B.1.4 Bubbling off

In light of the last theorem, we ask ourselves:

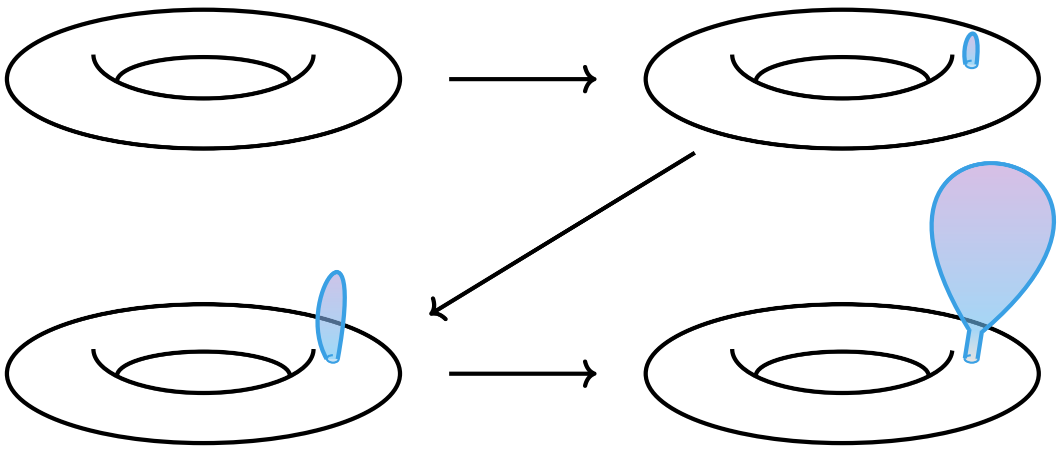
➢ Let (u_n) be a sequence of pseudo-holomorphic curves. What is a sufficient condition for all its first derivatives to be uniformly bounded?

A good candidate answer would be the following:

$$\boxed{\begin{array}{c}\text{Energies } E(u_n) \text{ are} \\ \text{uniformly bounded}\end{array}} \overset{?}{\Longrightarrow} \boxed{\begin{array}{c}\text{First derivatives are} \\ \text{uniformly bounded}\end{array}} \tag{B.5}$$

Main idea. We will soon see that an obstruction to having this implication is the appearance of spherical **bubbles**. Visually, one should imagine that as we progress along the sequence (u_n), we see the appearance of a sphere « bubbling off of the surface of our curves »; as if one had attached a balloon to it, and started inflating it.

✧ **Example B.20** (A sphere bubbling off of a sequence of pseudo-holomorphic tori)**.**



Instead of trying to stop these « bubbles » from appearing, we take another approach: making sure that if they exist, they have energy zero, so that they do not mess with our calculations. More precisely:

• **Definition B.21.** A symplectic manifold (M, ω) is said to be **symplectically aspherical** if every continuous 2-sphere in M has energy zero. In other words:

$$\forall v : \mathbb{S}^2 \longrightarrow M : \int_{\mathbb{S}^2} v^\star \omega = 0.$$

With this assumption, we can now prove the main result:

Proposition B.22 (Bubbling off)**.** *Let (M, ω) be a symplectically aspherical manifold endowed with a smooth, compatible almost complex structure J. Let (J_n) be a sequence such that $J_n \longrightarrow J$ in $\mathcal{C}^\infty$, and (u_n) a sequence of J_n-holomorphic curves with uniformly bounded energy; i.e. there exists $A > 0$ such that*

$$E(u_n) \leq A \text{ for every } n.$$

Then, the gradients of the u_n's are uniformly bounded, i.e. $\exists\, B > 0$ such that:

$$\|\nabla u_n\|_{\mathcal{C}^0} \leq B \text{ for every } n.$$

This argument, which we will review in the proof below, is what is commonly referred to as the « bubbling off argument ». For other expositions, see §6.6 of [AD13] (which is the one we follow), §4 of [MS04], §5.3 of [AH19], §5.3 of [Wen15]...

The main take-away is:

Corollary B.23. *Let (M,ω) be a symplectically aspherical manifold, and (u_n) a sequence of pseudo-holomorphic curves with uniformly bounded energies. Then (u_n) admits a converging subsequence.*

Proof of Proposition B.22. Assume for a contradiction that the ∇u_n's ar not uniformly bounded, so that there exists a sequence of points $z_n \in \Sigma$ (the domain of our holomorphic curves) such that $R_n := |\nabla u_k(z_n)| \longrightarrow \infty$. We appeal to the following lemma by Hofer:

Lemma B.24 (Half-maximum lemma)**.** *Consider a complete metric space (X,d), and a continuous function $g : X \to \mathbb{R}_+$. Then for any $x_0 \in X$, and $\varepsilon_0 > 0$, there exists $y \in X$ and $\varepsilon \in (0,\varepsilon_0]$ such that:*

$$\begin{cases} \mathrm{d}(y,x_0) \leq 2\varepsilon \\ \varepsilon g(y) \geq \varepsilon_0 g(x_0) \\ \forall x \in B_\varepsilon(y) : g(x) \leq 2g(y) \end{cases}$$

where $B_\varepsilon(y)$ is the ball of radius ε around y.

Proof. See Lemma 6.6.3 of [AD13]. This is an easy recursion argument.

In light of this, choose a sequence $\varepsilon_n \longrightarrow 0$, such that we still have $R_n\varepsilon_n \longrightarrow \infty$. Then, a conformal rescaling of the holomorphic curves:

$$v_n(z) := u_k(\frac{z}{R_n} + z_n)$$

gives us a sequence (v_n) such that, for every n: $\begin{cases} \|\nabla v_n(0)\| = 1 \\ \|\nabla v_n(\cdot)\| \leq 2 \text{ on } B_{\varepsilon_n R_n}(0) \end{cases}$.

By elliptic regularity (Theorem B.19), we can extract a $\mathcal{C}^\infty_{\mathrm{loc}}$-converging subsequence, and hence a limit v, such that $\begin{cases} \|\nabla v(0)\| = 1 \\ \|\nabla v\|_{\mathcal{C}^0} \leq 2 \end{cases}$.

We will derive a contradition by showing that this limit v must necessarily contain a bubble of non-zero energy, contradicting the symplectic asphericity assumption. First:

Lemma B.25 (Lemma 6.6.5 of [AD13])**.** *We can find an increasing sequence of balls $B_{r_n}(0)$, with $r_n \to \infty$, and such that:*

$$\mathrm{length}\left(v(\partial B_{r_n}(0))\right) \xrightarrow{n\to\infty} 0.$$

Proof. Since v is pseudo-holomorphic, $v^\star\omega$ is a symplectic form on Σ, allowing us to write $v^\star\omega = f(\rho,\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\rho$, for some $f > 0$. Then, compatibility of ω and J gives us a Riemannian metric, and hence allows us to talk about lengths. We define:

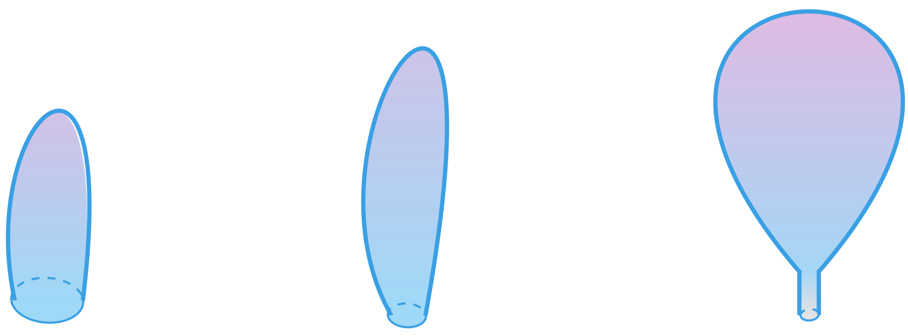
$$\ell(r) := \mathrm{length}\big(v(B_r(0))\big) = r\int_0^{2\pi} \sqrt{f(r,\theta)}\,\mathrm{d}\theta,$$

$$E(r) := E(v|_{B_r(0)}) = \int_{B_r(0)} v^\star\omega = \int_0^{2\pi}\int_0^r f(\rho,\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

A short calculation of $E'(r)$, followed by the Cauchy-Schwarz inequality, gives:

$$\ell(r)^2 \leq 2\pi r E'(r).$$

Since E is bounded and differentiable, it is now an easy exercise in analysis (see Lemma 6.6.5 of [AD13]) to show that there exists a sequence of radii $r_n \to \infty$ such that $r_n E'(r_n) \to 0$. □



$$v(B_{r_n}(0)) \longrightarrow 0$$

In other words, as $r_n \to \infty$, the boundaries of the balls $B_{r_n}(0)$ get squished into a point. This corresponds to the formation of what we call a **bubble**.

However, since eventually the $B_{r_n}(0)$ exhaust all of Σ, we have:

$$\lim_{n\to\infty} \int_{B_{r_n}(0)} v^\star\omega = \int_\Sigma v^\star\omega = E(v),$$

which is non-zero since v is non-constant (by construction, its gradient at 0 is non-zero).

We now work to derive a contradiction:

Pick $r_n \gg 1$, so that $v(\partial B_{r_n}(0))$ is contained in a small neighbourhood $\mathcal{U} \cong \mathbb{R}^{2n}$ in M. Then, in this neighbourhood, the symplectic form is exact, *i.e.* there exists a 1-form λ such that $\omega|_{\mathcal{U}} = \mathrm{d}\lambda$.

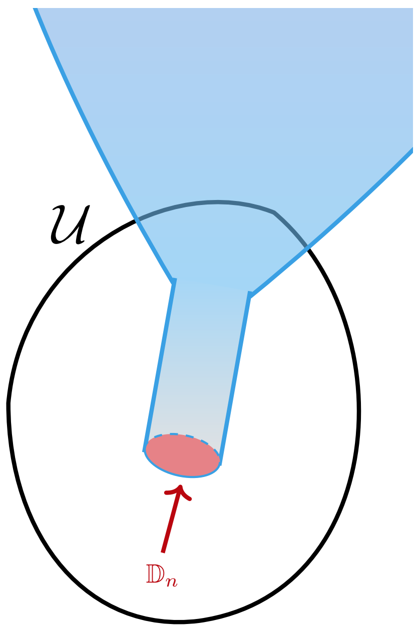
Topologically, $v(\partial B_r(0))$ is a circle so that we can find a disc filling $\mathbb{D}_n \cong \mathbb{D}^2$ of it (in red on the figure).

The union:

$$S_r := v(B_r(0)) \cup \mathbb{D}_n$$

gives us a 2-sphere in M, whose energy is given by:

$$\int_{S_r} \omega = \lim_{n\to\infty} \int_{v(B_{r_n}(0))} \omega + \int_{\mathbb{D}_n} \omega.$$



We have already established that

$$\int_{v(B_{r_n}(0))} \omega = \int_{B_{r_n}(0)} v^\star\omega \xrightarrow{n\to\infty} E(v) \neq 0.$$

Meanwhile, the second term goes to 0 as $n \to \infty$, which we can show by using successively Stokes' theorem, and the ML-inequality:

$$\left|\int_{\mathbb{D}_n} \omega|_{\mathcal{U}}\right| = \left|\int_{\mathbb{D}_n} \mathrm{d}\lambda\right| = \left|\int_{v(\partial B_r(0))} \lambda\right|$$
$$\leq \operatorname{length}\left(v(B_{r_n}(0))\right) \cdot \sup_{\mathcal{U}}(|\lambda|)$$
$$\xrightarrow[n\to\infty]{} 0.$$

Therefore, $\int_{S_r} \omega = E(v) \neq 0$. This contradicts the symplectic asphericity assumption, which asks that every 2-sphere in M have energy zero.

In other words, **the existence of such a bubble is prohibited**.

$\therefore$ Our assumption that there existed a sequence $r_n = |\nabla u_k(z_n)| \longrightarrow \infty$, which allowed us to rescale our holomorphic curves, and produce this bubble, must necessarily be false, concluding the proof. □

➤ This closes our section on pseudo-holomorphic curves. For the rest of this appendix, we will forget about complex geometry, and dive a little into algebraic topology, to derive results that will help us compute Floer homology, in Chapters 9 and 10.

B.2 A bit of homotopy theory

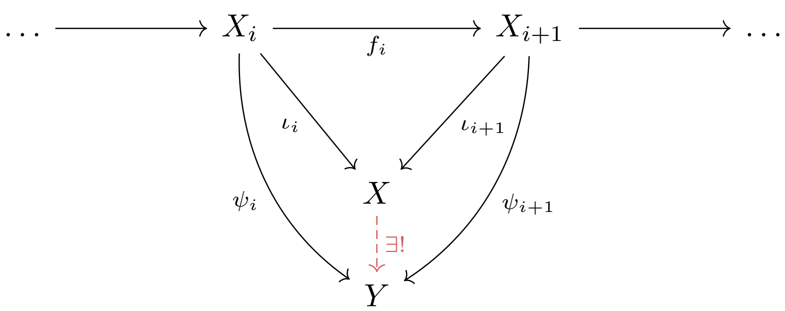
We assume the reader is familiar with the definitions of categories and functors, which can be found in most any textbook on the topic.

B.2.1 Direct and inverse limits

• **Definition B.26.** Let (X_i, f_i) be a directed system, *ie* a sequence of objects X_i with morphisms $f_i : X_i \to X_{i+1}$; which we furthermore assume to be index over a *small* category $\mathcal{I}$ (*ie*, Ob$\mathcal{I}$ is a set).

Then, its **direct limit** (or **colimit**), denoted $\varinjlim X_I =: X$, is the object X, along with morphisms $\iota_i : X_i \longrightarrow X$ such that:

For every object Y and collection of morphisms $\psi_i : X_i \longrightarrow Y$, the morphisms ψ_i factor uniquely through X. In other words, the following diagram commutes:



If such an X exists, then any other object in the category satisfying this (universal) property is isomorphic to X.

✧ **Example B.27.** The most natural example is that of a directed system of inclusions. If $X_i \hookrightarrow X_{i+1}\ \forall i$, then $\varinjlim X_i = \bigcup_i X_i$ is the natural limit of the sequence. Indeed, we can easily check that the union *does* satisfy the universal property that we want, then invoke uniqueness.

◇ **Remark B.28.** Though the construction applies to much more general situations than sequences of inclusions, the overall idea is essentially the same:

From the definition we gave, two elements in $X = \varinjlim X_i$ are identified if they eventually become equal in the sequence (X_i, f_i). In other words, the underlying set of X is:

$$(\bigsqcup X_i)/\sim \ \text{(where } X_i \ni x_i \sim x_j \in X_j \text{ if } \exists k > i,j \text{ s.t } f_{ik}(x_i) = f_{jk}(x_j) \in X_k)$$

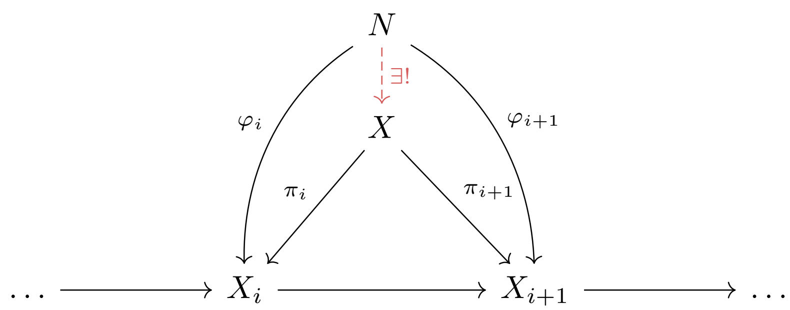
where f_{ab} denotes the composition of all the maps from X_a to X_b.

Hence, the limit $\varinjlim X_i$ consists of all these elements that survive all throughout the direct system (X_i, f_i).

➢ By standard methods in category theory, we can prove that if the direct limit of a system exists, then it is **unique** (up to isomorphism). We can also define the concept of inverse limit in a very similar fashion:

• **Definition B.29.** The **inverse limit** (or **limit**) of the directed system (X_i), denoted $\varprojlim$ if it exists, is an object X together with maps $\pi_i : X \to X_i$ such that:

For every cone N above (X_i) (*ie* an object N together with morphisms $\varphi_i : N \longrightarrow X_i$), the morphisms φ_i factor uniquely through X. In other words:



✧ **Example B.30.** Let us look at one of the easiest examples of inverse limits. Consider R a commutative ring, and let $I \triangleleft R$ be an ideal.

We can endow R with a topology by taking $\{I^m\}_{m\geq 0}$ to be a system of open neighbourhoods of 0; where recall that the product of two ideals I and $J \triangleleft R$ is given by

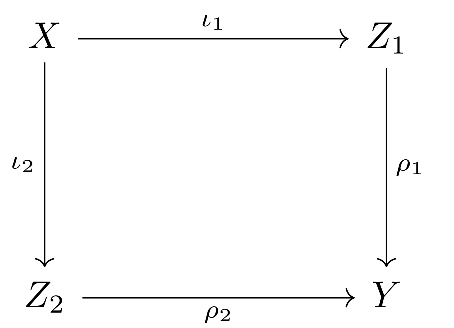
$$I \cdot J := \{\sum_{k<\infty} i_k j_k \mid i_k \in I, j_k \in J\}.$$

Now, observe that we have natural surjections $R/I^{n+1} \twoheadrightarrow R/I^n$, and hence a system of R-modules. The inverse limit: $\widehat{R} := \varprojlim R/I^m$ is called the **I-adic completion** $\widehat{R}$ of R.

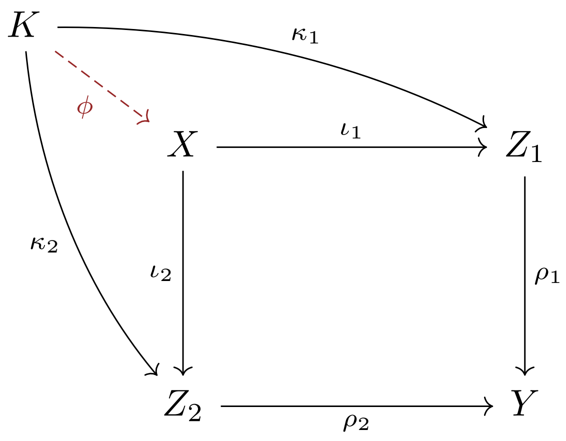
B.2.2 Fibrations

In this section, we work in the smooth category, so that all objects are topological spaces, and all maps are taken to be continuous. We write I the unit interval $[0, 1]$.

• **Definition B.31.** A commutative square

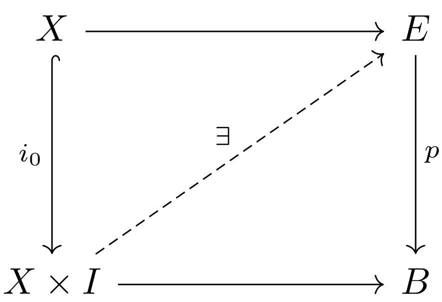


is called a **pullback square** if for any space K and pair of maps $\kappa_i : K \to Z_i$ commuting with everything, there exists a unique map $\phi : K \to X$ such that the following diagram:



still commutes. We will see some examples of this notion in §B.2.4.

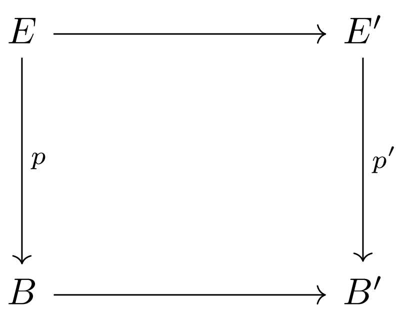
• **Definition B.32.** Let $p : E \to B$ be a map, and X be a space. p is said to satisfy the **(right) homotopy lifting property** w.r.t X if: given a homotopy $X \times I \to B$ such that $X \times \{0\} \to B$ lifts to E, then the whole homotopy lifts to E. More concisely:



• **Definition B.33.** $p : E \to B$ is a called a Hurewicz fibration, or simply fibration, if it satisfies the right homotopy lifting property w.r.t every space X.

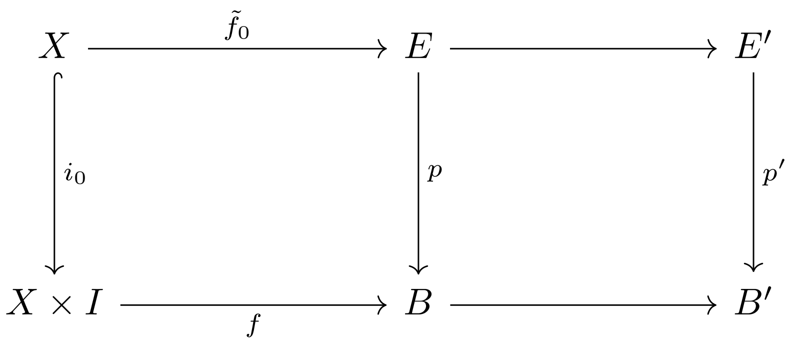
It is called a Serre fibration, or weak fibration, if it satisfies it w.r.t every CW complex X (which can be reduced to showing it satisfies it for every disc $\mathbb{D}^n$).

Lemma B.34. *Let $p' : E' \to F'$ be a (Hurewicz/Serre) fibration, and*

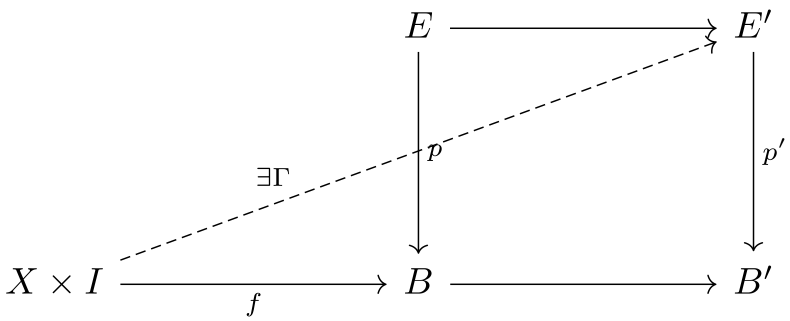


be a pullback square. Then $p : E \to B$ is also a (Hurewicz/Serre) fibration.

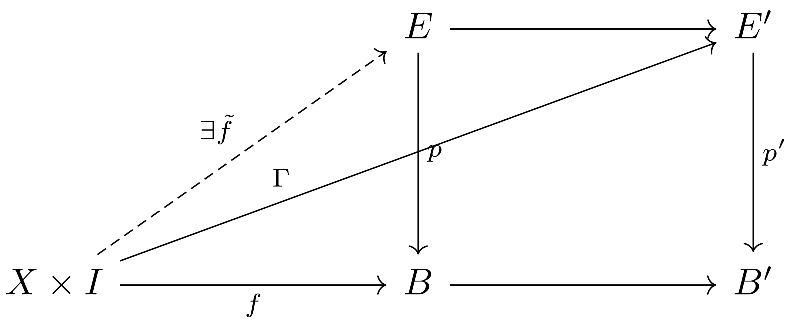
Proof. Being a Hurewicz/Serre fibration means satisfying the homotopy lifting property for a certain class of spaces. Let X be in said class, and assume we have a homotopy $f : X \times I \to B$ such that $f|_{X \times \{0\}} : X \to B$ lifts to $\tilde{f}_0 : X \to E$. *i.e.* :



First, notice that since $p : E' \to B'$ is a fibration, then the bottom composition lifts to a map $\Gamma : X \times I \to E'$. Hence, forgetting the top left element of the diagram, we have:



Then, recall that the square on the right is a pullback square, so that by definition, the existence of f and Γ implies the existence of a map:



which is our desired lifting. □

➤ Now, the reason we study fibrations is mostly for homotopical/homological purposes:

Proposition B.35. *Given a fibration, all fibres have the same homotopy-type.*

A proof can be found, for instance, in [Hat02, Prop. 4.65]. In particular, it suffices to know the fibre at one point to know it everywhere, up to homotopy. It is also standard knowledge that fibrations induce long exact sequences in homotopy (see [Mit01], for example). Actually, looking at homology, we can obtain an even stronger result:

Proposition B.36. *Let X be a manifold. A Serre fibration $F \to E \to X$ induces a spectral sequence called the* ***Leray-Serre spectral sequence****, such that:*

$$E_2^{p,q} = H^p(X; H^q(F)) \implies H^{p+q}(E).$$

Furthermore, if X is simply connected, and $H^\bullet(F)$ is finite-dimensional, then the second page decomposes:

$$E_2^{p,q} = H^p(X) \otimes H^q(F).$$

Proof. See §14 of [BT82], which is dedicated to this construction.

B.2.3 Homology of loop spaces

• **Definition B.37.** Let X be a topological space. Its **free loop space** is defined as:

$$\mathcal{L}X := \mathrm{Map}(\mathbb{S}^1, X). \tag{B.6}$$

We can add a base point $pt \in X$, and ask that all the loops start at pt, thus defining the **based loop space**:

$$\Omega_{pt}X := \mathrm{Map}\Big((\mathbb{S}^1, 1), (X, pt)\Big). \tag{B.7}$$

Both of these are topological spaces, endowed with the compact-open topology.

Letting $\Omega X := \bigsqcup_{pt} \Omega_{pt}X$, then we have a natural fibration, called the loop space fibration.

$$\begin{array}{ccc} \Omega X & \longrightarrow & \mathcal{L}X \\ & & \downarrow \\ & & X \end{array}$$

In this thesis, since our inspiration comes from physical examples, we will mostly be interested in the loop and path spaces of spheres (mostly for the purposes of Chapters 9 and 10). There, we can actually uncover more structure:

Lemma B.38. *$X = G$ is a topological group. Then the short exact sequence $\Omega G \to \mathcal{L}G \to G$ splits. i.e. $\mathcal{L}G = G \times \Omega G$ (where this decomposition holds as spaces, not just groups).*

Proof. See [Jai21].

✧ **Example B.39.** Let $X = \mathbb{S}^1$. From standard algebraic topology, $\Omega\mathbb{S}^1 \simeq \mathbb{Z}$ up to homotopy. Since $\mathbb{S}^1$ can be viewed as a topological group, the previous lemma then gives us $\mathcal{L}\mathbb{S}^1 \simeq \mathbb{Z}\times\mathbb{S}^1$. In particular, $H^*(\Omega\mathbb{S}^1)$ and $H^*(\mathcal{L}\mathbb{S}^1)$ are both infinite-dimensional, where H^* denotes singular cohomology.

Lemma B.40 ([Zil77])**.** *Let $n \geq 2$ be even. Then, for every $m \geq 1$, we have:*

$$H_{2m(n-1)}(\mathcal{L}\mathbb{S}^n, \mathbb{Z}) \cong \mathbb{Z}/2.$$

In particular, $H^\bullet(\mathcal{L}\mathbb{S}^n, \mathbb{Z})$ is infinite-dimensional, and non-zero in infinitely many degrees.

Actually, we can deduce more:

Lemma B.41. *Let $n \geq 2$ be even. Then $H^\bullet(\Omega\mathbb{S}^n; \mathbb{Z})$ is also infinite-dimensional.*

Proof. By Proposition B.36, the fibration $\Omega\mathbb{S}^n \to \mathcal{L}\mathbb{S}^n \to \mathbb{S}^n$ induces a spectral sequence converging to the total cohomology of $\mathcal{L}\mathbb{S}^n$ (which we now know to be infinite-dimensional). Since $n \geq 2$, $\mathbb{S}^n$ is simply connected, and if we assume $\Omega\mathbb{S}^n$ to have finite-dimensional cohomology, then the second page decomposes:

$$E_2^{p,q} = H^p(\mathbb{S}^n) \otimes H^q(\Omega\mathbb{S}^n)$$

This is a contradiction, since $E_2^{p,q}$ now only has finitely many non-zero elements, while it should converge to something infinite-dimensional. □

◇ **Remark B.42.** The (co)homology of the based/free loop spaces of spheres is actually very well-understood. For a brief survey on the homology of $\Omega\mathbb{S}^n$, we refer to [Dev16], which fully explicits its algebraic structure. For more general work on the cohomology of loop spaces, we refer to [Zil77], for example. We have only stated a few special results because for the purposes of this thesis, we are only concerned with specific cases.

B.2.4 Homology of path spaces

For the purposes of Chapter 10, we are also interested in the *path spaces* of spheres. Formally:

• **Definition B.43.** Let X be a topological space. Its **free path space** is defined as:

$$\mathcal{P}X := \mathrm{Map}([0,1], X). \tag{B.8}$$

If we add a base point $pt \in X$, then we can define its **based path space**:

$$\mathcal{P}_{pt}X := \{\gamma : [0,1] \mid \gamma(0) = pt\}\,. \tag{B.9}$$

Actually, we can go further than that. Given $Q \subset X$ a subspace, we can define the space of paths with ends in Q:

$$\mathcal{P}_Q X := \{\gamma : [0,1] \to X \mid \gamma(0), \gamma(1) \in Q\}\,. \tag{B.10}$$

Lemma B.44 (Loop-path fibration)**.** *There is a fibration:*

$$\Omega X \longrightarrow \mathcal{P}X \xrightarrow{\pi} X \times X, \tag{B.11}$$

where $\pi = (\mathrm{ev}_0, \mathrm{ev}_1) : \mathcal{P}X \to X \times X$ maps a path to its endpoints.

Proof. To show $\pi : \mathcal{P}X \to X \times X$ is a fibration, we show it satisfies the homotopy lifting property. One can explicitly write down the lift of any homotopy to $X \times X$; which is done in [Swi02, Prop. 4.3]. Hence, we have a fibration $F \to \mathcal{P}X \to X \times X$.

By Proposition B.35, the homotopy type of fibres is unique, so that it suffices to determine F above one point. Pick $(x, x) \in X \times X$ in the diagonal. Then clearly, $\pi^{-1}(x) = \Omega_x X$. Therefore, in general, $F \cong \Omega_{pt} X$. □

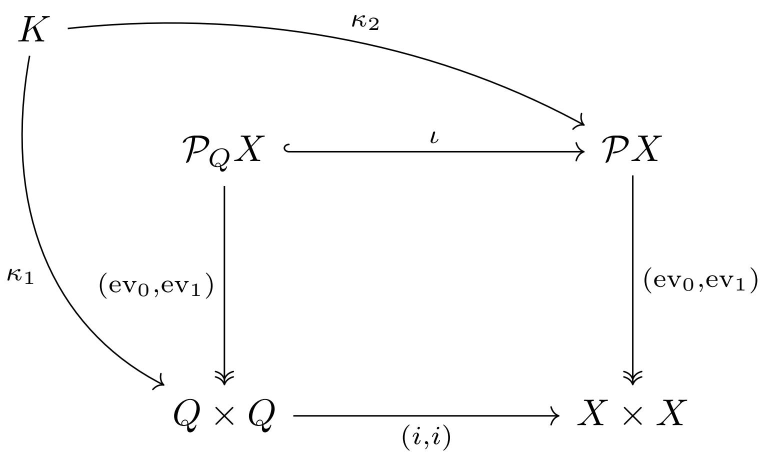
Lemma B.45. *Let $Q \subset X$. Then, the diagram:*

$$\begin{array}{ccc} \mathcal{P}_Q X & \xhookrightarrow{\iota} & \mathcal{P}X \\ {\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ Q \times Q & \xrightarrow[(i,i)]{} & X \times X \end{array}$$

is a pullback square (Definition B.31). Moreover, we have a fibration:

$$\Omega X \longrightarrow \mathcal{P}_Q X \longrightarrow Q \times Q. \tag{B.12}$$

Proof. 1) First, let us prove that the square above is a pullback square. Assume we have a topological space K, along with maps κ_1, κ_2 such that the diagram



commutes. This necessarily implies that, for any $k \in K$, $\kappa_2(k) \in \mathcal{P}X$ is a path with ends in Q, since we must have:

$$(\mathrm{ev}_0, \mathrm{ev}_1) \circ \kappa_2 = (i, i) \circ \kappa_1.$$

Hence, there exists a natural map $K \to \mathcal{P}_Q X : k \mapsto \kappa_2(k)$, which verifies that the original square was indeed a pullback square.

2) Now that we know we have a pullback square, we can invoke Lemma B.34, which tells us that, since $\mathcal{P}X \to X \times X$ is a fibration, then $\mathcal{P}_Q X \to Q \times Q$ must also be a fibration. Hence it only remains to determine its fibre.

Like in the previous Lemma, we appeal to Prop. B.35, which tells us that all fibres have the same homotopy-type, so that it suffices to look at one fibre. In particular, above a point (q, q) in the diagonal of $Q \times Q$, the fibre will be homotopic to $\Omega_q X$, so that all fibres are, concluding the proof. □

✧ **Example B.46.** An example of application of these arguments is the proof that $H^*(\mathcal{P}_{\mathbb{S}^1}\mathbb{S}^2)$ is infinite-dimensional, which we carry out in Corollary 10.10.

Appendix C

Computational appendix

This appendix contains all the computations/proofs which did not make it into the main body, often to avoid breaking the flow of the discussions with lengthy calculations.

Calculation C.1. Let us prove the fact (from Example 3.4) that for any manifold Q, $\mathbb{D}^\star Q$ is a Liouville domain, with boundary $\mathbb{S}^\star Q$. First, we need to show that $\mathbb{S}^\star Q$ is indeed the manifold boundary of $\mathbb{D}^\star Q$.

We will accept, without proof, the following standard fact:

Fact. $\mathbb{S}^{n-1}$ is the manifold boundary of $\mathbb{D}^n$. In particular, we can find an atlas $\{(U_i, \psi_i)\}$ of $\mathbb{D}^n$ consisting of both interior charts

$$\psi_i : U_i \to V_i \subset \mathbb{R}^n,$$

and boundary charts

$$\psi_i : U_i \to V_i \subset \mathbb{R}^{n-1} \times \mathbb{R}_+.$$

Now, take $n = \dim Q$. Then $\mathbb{D}^\star Q$ is by definition a $\mathbb{D}^n$-bundle over Q, so that we can find local trivialisations

$$\begin{array}{ccc} \mathbb{D}^\star \mathcal{U}_i & \xrightarrow{\;\cong\;} & \mathcal{V}_i \times \mathbb{D}^n \\ \downarrow & \swarrow & \\ \mathcal{U}_i & & \end{array}$$

where each open set $\mathcal{U}_i \subset Q$ supports a chart $\phi_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{V}_i \subset \mathbb{R}^n$.

➢ Then we can define an atlas $(\mathcal{U}_i \times U_j, \phi_i \oplus \psi_j)$ for $\mathbb{D}^\star Q$, such that $\phi_i \oplus \psi_j$ is a boundary (resp. interior) chart whenever ψ_j is. In particular, in every local trivialisation, the boundary of $\mathbb{D}^\star Q$ will look like $\mathcal{V}_i \times \mathbb{S}^{n-1}$, which we can identify with $\mathbb{S}^\star Q$, by definition.

➢ It remains to prove that this boundary is of restricted contact type, with respect to the natural symplectic form on $\mathbb{D}^\star Q$. As we saw, this is equivalent to asking that the associated Liouville vector field V (such that $V \lrcorner\, \omega = \lambda$) be positively transverse to the boundary $\mathbb{S}^\star Q$.

Now recall that we are working with the contact form

$$\lambda = -\sum_i p_i \mathrm{d}q_i,$$

where the (q_i, p_i) are the coordinates on our local trivialisations. Then, from the condition $V \lrcorner\, \omega = \lambda$, we can easily deduce that we have:

$$V = \sum_i p_i \frac{\partial}{\partial p_i}$$

on $\mathcal{V}_\rangle \times \mathbb{D}^n$. In particular, V points radially outwards in the fibre $\mathbb{D}^n$, so that it is indeed transverse to $\mathbb{S}^{n-1}$. □

Calculation C.2. We prove the statement from Example 3.20, *ie* given a Hamiltonian on $(\widehat{W}, \hat{\omega} = \mathrm{d}\hat{\lambda})$ satisfying $H = h(r)$ for $r \geq R_0$, we have that every Hamiltonian chord on the collar $[R_0, +\infty) \times \partial W$ has action:

$$\mathcal{A}_H(x) = f_1\big(x(1)\big) - f_0\big(x(0)\big) - rh'(r) + h(r).$$

Recall from (3.5) that:

$$\mathcal{A}_H : \mathscr{P} \longrightarrow \mathbb{R} : x \longmapsto f_1\left(x(1)\right) - f_0\left(x(0)\right) - \int_0^1 x^\star\hat{\lambda} + \int_0^1 H\left(x(t)\right)\mathrm{d}t.$$

We focus on the last two terms.

- First, notice that $\int_0^1 x^\star\hat{\lambda}$ is standardly the $\hat{\lambda}$-period of the chord x (where 'period' is to be understood as Reeb-length). Recall that $\hat{\lambda} = r\alpha$, where $\alpha = \lambda|_{\partial W}$, the contact form on the boundary. Since $H = h(r)$ on the collar end, we have $X_H = h'(r)R_\alpha$, as shown in Corollary 3.11.

  In particular, Hamiltonian chords x are restrained to slices $\{r\}\times\partial W$, and can be written $x(t) = y(h'(r)t)$ where y is a Reeb chord on ∂W. Therefore:

$$\begin{aligned}\int_0^1 x^\star\hat{\lambda} &= \int_0^{h'(r)} y^\star\hat{\lambda}\\ &= \int_0^{h'(r)} y^\star(r\alpha)\\ &= r\int_0^{h'(r)} y^\star\alpha = rh'(r),\end{aligned}$$

  since $\int_0^{h'(r)} y^\star\alpha$ is the Reeb-length of y on ∂W.

- For the last term, recall that for $r \geq R_0$, $H = h(r)$ only depends on the r-coordinate. Furthermore, as shown in Cor. 3.11, Hamiltonian chords above r_0 are restrained to horizontal slices $\{r\} \times \partial W$. Therefore:

$$\int_0^1 H\big(x(t)\big)\mathrm{d}t = \int_0^1 h(r)\mathrm{d}t = h(r).$$

  This completes the calculation. □

Calculation C.3 (Proof of Proposition 3.21)**.** Let $\mathcal{A}_H : \mathscr{P} \longrightarrow \mathbb{R}$ be the action functional from Wrapped Floer homology, as defined in Defn 3.19. Then, its derivative at a point $x \in \mathscr{P}$ is given by:

$$\mathrm{d}\mathcal{A}_H(x) : T_x\mathscr{P} \longrightarrow \mathbb{R} : \zeta \longmapsto \int_0^1 \mathrm{d}\hat{\lambda}\big(\dot{x}(t) - X_H(x(t)), \zeta(t)\big)\mathrm{d}t.$$

Proof. (From [Kim18], but we rewrite it because we use a different sign convention - and add details):

Step 1. We want to compute the derivative at a point $x \in \mathscr{P}$, where $\mathscr{P}$ is the space of $\mathbf{W}^{1,2}$ Hamiltonian chords (trajectories) which start in $\hat{\lambda}_0$ and end in $\hat{\lambda}_1$.

Recall that $T_x\mathscr{P} = \left\{\zeta \in \mathbf{W}^{1,2}\left([0,1], x^\star T\widehat{W}\right) \mid \zeta(0) \in T_{x(0)}\hat{\lambda}_0, \zeta(1) \in T_{x(1)}\hat{\lambda}_1\right\}$, whose elements are vector fields along the path x.

Pick an arbitrary $\zeta \in T_x\mathscr{P}$. Then, we can find a curve (x_s) $\big(s \in (-\epsilon, \epsilon)\big)$ in $\mathscr{P}$ (so essentially, a 1-parameter family of paths in $\widehat{W}$), such that

$$\frac{\mathrm{d}}{\mathrm{d}s} x_s|_{s=0} = \zeta.$$

We can extend ζ to a neighbourhood of x in $\widehat{W}$, while ensuring that $\zeta = (\mathrm{d}/\mathrm{d}s)x_s$ along the whole family (x_s). (On the other dimensions of $\widehat{W}$, we extend it arbitrarily; which will not matter in the end since we will always work along the family $\big\{(s,t) \longmapsto x_s(t)\big\}$, which yields an embedded surface in $\widehat{W}$).

Now let us get to the calculations. Since $\zeta = (\mathrm{d}/\mathrm{d}s)x_s|_{s=0}$, then by definition:

$$\mathrm{d}\mathcal{A}_H(x)\zeta = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}_H(x_s)|_{s=0}.$$

Now, recall that $\mathcal{A}_H(x) := f_1\big(x(1)\big) - f_0\big(x(0)\big) - \int_0^1 x^\star\hat{\lambda} + \int_0^1 H\big(x(t)\big)\mathrm{d}t$, where f_i is the primitive of $\hat{\lambda}$ along $\widehat{L}_i$ (since we assume the Lagrangians to be exact). So:

$$\begin{aligned}\mathrm{d}\mathcal{A}_H(x)\zeta &= \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}_H(x_s)|_{s=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big(f_1\big(x_s(1)\big) - f_0\big(x_s(0)\big) - \int_0^1 x_s^\star\hat{\lambda} + \int_0^1 H\big(x_s(t)\big)\mathrm{d}t\Big).\end{aligned}$$

Let us differentiate this expression term by term:

- $\frac{\mathrm{d}}{\mathrm{d}s} f_j\big(x_s(j)\big) = \mathrm{d}f_j\big(\zeta(x(j))\big)$ by chain rule. We can rewrite this as $\mathrm{d}f_j\big(\zeta\big)(x(j))$; and recall that since the Lagrangians are exact, we have $\hat{\lambda}|_{\widehat{L}_j} = \mathrm{d}f_j$, and hence $\mathrm{d}f_j\big(\zeta\big)(x(j)) = \hat{\lambda}\big(\zeta\big)(x(j))$.
- $\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int_0^1 x_s^\star\hat{\lambda} = \int_0^1 x^\star\mathscr{L}_\zeta\hat{\lambda}$, where $\mathscr{L}_\zeta$ denotes the Lie derivative with respect to the vector field ζ. The proof of this equality in itself is a bit involved, so we move it to calculation C.4.
- $\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int_0^1 H\big(x_s(t)\big)\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}H\big(x_s(t)\big)\Big|_{s=0}\mathrm{d}t = \int_0^1 \mathrm{d}H\big(\zeta(t)\big)\mathrm{d}t$, where we can differentiate under the integral sign thanks to the Leibniz integral rule.

Therefore:

$$\mathrm{d}\mathcal{A}_H(x)\zeta = \hat{\lambda}\big(\zeta\big)(x(1)) - \hat{\lambda}\big(\zeta\big)(x(0)) - \int_0^1 x^\star\mathscr{L}_\zeta\hat{\lambda} + \int_0^1 \mathrm{d}H\big(\zeta(t)\big)\mathrm{d}t.$$

Step 2. We can simplify this expression further. Indeed, recall that Cartan's magic formula says: $\mathscr{L}_\zeta\hat{\lambda} = \mathrm{d}\big(\zeta \lrcorner \hat{\lambda}\big) + \zeta \lrcorner \mathrm{d}\hat{\lambda}$, where $\lrcorner$ denotes the interior product. So the third term of the expression above becomes:

$$\begin{aligned}\int_0^1 x^\star\mathscr{L}_\zeta\hat{\lambda} &= \int_0^1 x^\star\big(\mathrm{d}(\zeta \lrcorner \hat{\lambda}) + \zeta \lrcorner \mathrm{d}\hat{\lambda}\big)\\ &= \int_0^1 \mathrm{d}\big(\hat{\lambda}(\zeta)\big)(x(t))\mathrm{d}t + \int_0^1 x^\star\big(\mathrm{d}\hat{\lambda}(\zeta,\cdot)\big)\\ &= \hat{\lambda}\big(\zeta\big)(x(1)) - \hat{\lambda}\big(\zeta\big)(x(0)) + \int_0^1 x^\star\big(\mathrm{d}\hat{\lambda}(\zeta,\cdot)\big).\end{aligned}$$

Therefore:

$$\begin{aligned}
\mathrm{d}\mathcal{A}_H(x)\zeta =& \big(\hat{\lambda}\big(\zeta\big)(x(1)) - \hat{\lambda}\big(\zeta\big)(x(0))\big) - \big(\hat{\lambda}\big(\zeta\big)(x(1)) - \hat{\lambda}\big(\zeta\big)(x(0))\big) \\
& - \int_0^1 x^\star\big(\mathrm{d}\hat{\lambda}(\zeta,\cdot)\big) + \int_0^1 \mathrm{d}H\big(\zeta(t)\big)\mathrm{d}t \\
=& - \int_0^1 x^\star\big(\mathrm{d}\hat{\lambda}(\zeta,\cdot)\big) + \int_0^1 \mathrm{d}H\big(\zeta(t)\big)\mathrm{d}t.
\end{aligned}$$

Now, it is a standard exercise in linear algebra to show that $x^\star\big(\mathrm{d}\hat{\lambda}(\zeta,\cdot)\big) = \mathrm{d}\hat{\lambda}\big(\zeta(t),\dot{x}(t)\big)\mathrm{d}t$ (this is true of any 1-form). Recall that by definition, we have $-\mathrm{d}H = X_H \lrcorner\, \mathrm{d}\hat{\lambda}$, where X_H represents the Hamiltonian vector field of H. So we have:

$$\begin{aligned}
\mathrm{d}\mathcal{A}_H(x)\zeta &= - \int_0^1 x^\star\big(\mathrm{d}\hat{\lambda}(\zeta,\cdot)\big) + \int_0^1 \mathrm{d}H\big(\zeta(t)\big)\mathrm{d}t \\
&= - \int_0^1 \mathrm{d}\hat{\lambda}\big(\zeta(t),\dot{x}(t)\big)\mathrm{d}t - \int_0^1 \mathrm{d}\hat{\lambda}\big(X_H(x(t)),\zeta(t)\big)\mathrm{d}t \\
&= \int_0^1 \mathrm{d}\hat{\lambda}\big(\dot{x}(t),\zeta(t)\big)\mathrm{d}t - \int_0^1 \mathrm{d}\hat{\lambda}\big(X_H(x(t)),\zeta(t)\big)\mathrm{d}t \\
&= \int_0^1 \mathrm{d}\hat{\lambda}\big(\dot{x}(t) - X_H(x(t)),\zeta(t)\big)\mathrm{d}t.
\end{aligned}$$

□

Calculation C.4. In the previous calculation, we stated a fact which we did not prove:

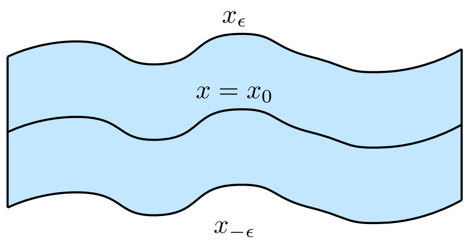
$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \int_0^1 x_s^\star\hat{\lambda} = \int_0^1 x^\star \mathscr{L}_\zeta \hat{\lambda},$$

where $\mathscr{L}_\zeta$ denotes the Lie derivative with respect to the vector field ζ.
We can differentiate under the integral sign, hence we really want to show that

$$\frac{\mathrm{d}}{\mathrm{d}s}(x_s^\star\hat{\lambda})|_{s=0} = x^\star \mathcal{L}_\zeta \hat{\lambda}.$$

Let $(-\epsilon,\epsilon) \ni s \mapsto x_s$ be our family of paths (*i.e.* for every s, $x_s : [0,1] \to \widehat{W}$), and define:

$$\zeta_s := \frac{\mathrm{d}}{\mathrm{d}s}x_s. \tag{C.1}$$



So $\zeta_0 = \zeta$, our previously-defined vector field along $x = x_0$. Define the surface:

$$S := \{x_s(t) \mid s \in (-\epsilon,\epsilon), t \in [0,1]\} \subset \widehat{W}.$$

Then, given $t_0 \in [0,1]$, the path $s \mapsto x_s(t_0)$ can be interpreted as a flow line of the flow of ζ_s, by definition. Since $\zeta_s = (\mathrm{d}/\mathrm{d}s)x_s \in TS$, this flow line is contained is S (actually, S is foliated by such flow lines, by definition).

Hence, there is a flow $\psi_s : S \to S$, with infinitesimal generator $\zeta_s \in TS$. By construction, we must have:

$$\psi_s \circ x_0 = x_s \ \ \forall s \in (-\epsilon,\epsilon). \tag{C.2}$$

Now, since $\zeta \in TS$, and $\mathrm{im}(x_s) \subset S$ by definition, we can make sense of the expression $\mathcal{L}_\zeta\hat{\lambda}$ on S (and actually, even if we extended ζ to a neighbourhood $\mathcal{U}$ of S in $\widehat{W}$, we would have $(\mathcal{L}_\zeta\hat{\lambda})|_\mathcal{U} \equiv (\mathcal{L}_\zeta\hat{\lambda})|_S$). We compute:

$$\begin{aligned}
x^\star \mathcal{L}_\zeta \hat{\lambda} = x_0^\star \mathcal{L}_\zeta \hat{\lambda} = x_0^\star \Big(\lim_{s\to 0} \frac{1}{s}\Big(\psi_s^\star \hat{\lambda} - \hat{\lambda}\Big)\Big) \\
&= \lim_{s\to 0} \frac{1}{s}\Big(x_0^\star \psi_s^\star \hat{\lambda} - x_0^\star \hat{\lambda}\Big) \\
&= \lim_{s\to 0} \frac{1}{s}\Big((\psi_s \circ x_0)^\star \hat{\lambda} - x_0^\star \hat{\lambda}\Big) \\
&= \lim_{s\to 0} \frac{1}{s}\Big(x_s^\star \hat{\lambda} - x_0^\star \hat{\lambda}\Big) \quad \text{by (C.2)} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}(x_s^\star \hat{\lambda})|_{s_0}.
\end{aligned}$$

□

Calculation C.5 (Proof or proposition 3.25)**.** We want to show that the Hessian of the action functional that we have been studying is given by:

$$\mathrm{Hess}_x \mathcal{A}_H : T_x\mathscr{P} \longrightarrow L^2(x^\star(TM)) : \zeta \longmapsto J_t(x)\big(\nabla_t \zeta - \nabla_\zeta X_H(x(t))\big).$$

Recall from A.28 that we can define the Hessian of $\mathcal{A}_H$ as the linearisation of the section $\nabla \mathcal{A}_H$. This process (which is described in subsection A.27 of the appendix) can only be done at points where $\nabla \mathcal{A}_H = 0$; hence, at critical points.

As proved in Proposition A.2.2 of the appendix, this linearization is independent of the choice of connection we make. So we need only choose a connection ∇, and compute $\nabla(\nabla \mathcal{A}_H))$. Let us pick $\nabla = \nabla^{\mathrm{LC}}$ the Levi-Civita connection on M.(1)

Now pick a vector $\zeta \in T_x\mathscr{P}$, and let (x_s) be a curve in $\mathscr{P}$ generating it (*ie* $\frac{\mathrm{d}x_s}{\mathrm{d}s}|_0 = \zeta$). We then have:

$$\begin{aligned}
\mathrm{Hess}_x \mathcal{A}_H \zeta : &= D_x(\nabla \mathcal{A}_H)\zeta \\
&= \nabla^{\mathrm{LC}} \nabla \mathcal{A}_H(x)\zeta \\
&= \nabla_s \nabla \mathcal{A}_H(x_s)|_{s=0} \\
&= \nabla_s \big(J_t(x_s)\big(\dot{x}_s(t) - X_H(x_s)\big)\big)|_{s=0} \\
&= \nabla_s J_t(x_s)\big(\dot{x}_s(t) - X_H(x_s)\big)|_{s=0} + J_t(x_s)|_{s=0}\big(\nabla_s \frac{\partial x_s}{\partial t} - \nabla_s X_H(x_s)\big)|_{s=0} \\
&= \nabla_\zeta J_t(x)\big(\dot{x}(t) - X_H(x)\big) + J_t(x)\big(\nabla_t \frac{\partial x_s}{\partial s}|_{s=0} - \nabla_\zeta X_H(x)\big);
\end{aligned}$$

where, to pass to the last line, we took $s \to 0$ wherever we could; and made use of the equality $\nabla_s \partial_t = \nabla_t \partial_s$, which is a standard fact from Riemannian geometry.

Now, the first term above becomes 0, since $\dot{x}(t) = X_H\big(x(t)\big)$. To simplify the second term, we make use of the fact that $\frac{\mathrm{d}x_s}{\mathrm{d}s}|_{s=0} = \zeta$. So we get:

$$\mathrm{Hess}_x \mathcal{A}_H(x)\,\zeta = J_t(x)\big(\nabla_t \zeta - \nabla_\zeta X_H(x)\big).$$

□

Calculation C.6 (Proof of Lemma 3.27)**.** We want to prove the following equivalence:

$$\ker \mathrm{Hess}_x \mathcal{A}_H \neq \{0\} \iff D\phi(T_{x(0)}\Lambda_0) \cap T_{x(1)}\Lambda_1 \text{ is not transverse.}$$

(1) Technically, this connection varies with t. Indeed, recall that we work with a family (J_t) of almost complex structures ($0 \leq t \leq 1$), and the associated Riemannian metrics $g_t := \omega(\cdot, J\cdot)$. We then take ∇ to be the Levi-Civita connection associated to g_t (not expliciting clearly the dependency on t; for it is not important in the calculations).

• ($\Longleftarrow$) First, let us assume that the intersection $D\phi(T_{x(0)}\Lambda_0) \cap T_{x(1)}\Lambda_1$ is non-transverse. For dimensional reasons, this means there exists a non-zero $v^1 \in D\phi(T_{x(0)}\Lambda_0) \cap T_{x(1)}\Lambda_1$. In other words, $\exists v^0 \in T_{x(0)}\Lambda_0$ such that $v^1 = D\phi(v^0) \in T_{x(1)}\Lambda_1$.

Now, our goal is to prove that there exists a non-zero $\zeta \in \ker \mathrm{Hess}_x\mathcal{A}_H$. Recall, from Lemma 3.25, that:

$$\begin{aligned} \mathrm{Hess}_x\mathcal{A}_H : T_x\mathscr{P} &\longrightarrow L^2(\Gamma(x^\star T\widehat{W})) \\ \zeta &\longmapsto J_t\big(\nabla_t\zeta - \nabla_\zeta X_H\big). \end{aligned}$$

where $T_x\mathscr{P} = \big\{\zeta \in \Gamma(x^\star TM) \mid \zeta(0) \in T_{x(0)}\Lambda_0, \zeta(1) \in T_{x(1)}\Lambda_1\big\}$.

So essentially, we want to find a vector field in $\Gamma(x^\star T\widehat{W})$, with Lagrangian ends, and which is in the kernel of $\mathrm{Hess}_x\mathcal{A}_H$. From the discussion at the start, a natural candidate for such a vector field would be one that "connects" v^0 and v^1.

Indeed, since $\phi = \phi^1_H$ is the time-1 map of the Hamiltonian flow, we can define:

$$\begin{aligned} v : [0,1] &\longrightarrow x^\star T\widehat{W} \\ t &\longmapsto v^t = \phi^t_\star v^0 = D\phi^t_H(v^0) \in T_{x(t)}\widehat{W}. \end{aligned}$$

which is, by construction, a non-zero vector field in $T_x\mathscr{P}$. So let us see whether or not $\mathrm{Hess}_x\mathcal{A}_H v = 0$.

In the proof of Lemma 3.25, we chose ∇ to be the Levi-Civita connection, which is *torsion-free*. What this implies, in particular, is that for any two vector fields X, Y:

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where $[\cdot,\cdot]$ is the standard Lie bracket on the tangent bundle.

In particular, it is a standard result from differential geometry that this Lie bracket is zero if and only if the flows of X and Y commute.

So since $\mathrm{Hess}_x\mathcal{A}_H\zeta = J_t[X_H, \zeta]$, we want to prove that for the vector field v we defined above, $[X_H, v] = 0$. Since $v^t = \phi^t_\star v^0$ by definition, the flows of v and X_H are the same. Hence they clearly commute. So $[X_H, v] = 0$.

•($\Longrightarrow$) Let us now prove the converse. So assume we have a non-zero vector field $v \in \ker \mathrm{Hess}_x\mathcal{A}_H$, and let us try and prove $D\phi(T_{x(0)}\Lambda_0) \cap T_{x(1)}\Lambda_1 \neq \{0\}$. So we need to find a non-zero vector in this intersection.

Like earlier, there is an immediate candidate for what this vector could be: perhaps we could choose $v^{t=1}$. This is clearly in $T_{x(1)}\Lambda_1$, but there is *a priori* no guarantee that it would belong to $D\phi\big(T_{x(0)}\Lambda_0\big)$. However, one can prove that it does using the following lemma:

Lemma C.7. *If $v \in \ker \mathrm{Hess}_x\mathcal{A}_H$, then we have $v^t = \phi^t_\star v^0$; where $\phi^t = \phi^t_H$.*

(This construction is completely inspired from the ($\Longleftarrow$) direction of the proof.)

Proof. Let $\tilde{v}$ be the vector field defined by $\tilde{v}^t = \phi^t_\star v^0$. We want to show that $\tilde{v} = v$. We will do this via an Existence/Uniqueness theorem for differential equations.

Indeed, v and $\tilde{v}$ satisfy $\begin{cases} \nabla_{X_H} v - \nabla_v X_H = 0 \\ \nabla_{X_H}\tilde{v} - \nabla_{\tilde{v}} X_H = 0 \end{cases}$.(2)

The second equation holds for the same reason as in the ($\Longleftarrow$) direction.

So both v and $\tilde{v}$ satisfy the same first-order differential equation. Hence, to show that they are equal, it suffices to show that two (independent) initial conditions agree. Hence, it would be enough to show that:

(2) These are the equations $(\mathrm{Hess}_x\mathcal{A}_H)v = 0$ and $(\mathrm{Hess}_x\mathcal{A}_H)\tilde{v} = 0$. We have removed the J_t factor, since it does not change whether the result is 0 or not.

$$\begin{cases} v^0 = \tilde{v}^0 & \text{(C.3.a)} \\ (\nabla_{X_H} v)|_{t=0} = (\nabla_{X_H} \tilde{v})|_{t=0} & \text{(C.3.b)} \end{cases}$$

.

(C.3.a) is directly satisfied by definition of $\tilde{v}$. Meanwhile, using the differential equation which both v and $\tilde{v}$ satisfy, (C.3.b) can be rewritten as the condition:

$$(\nabla_v X_H)|_{t=0} = (\nabla_{\tilde{v}} X_H)|_{t=0}. \tag{C.4}$$

Notice that the expression $(\nabla_v X_H)|_{t=0}$ only depends on v at the point $t = 0$; in other words, $(\nabla_v X_H)|_{t=0} = \nabla_{v^0} X_H$.

Hence, condition C.4 (which we want to prove holds) reduces to:

$$\nabla_{v^0} X_H = \nabla_{\tilde{v}^0} X_H \tag{C.5}$$

(which is an equality at the point $x(0) \in \widehat{W}$). This trivially follows from the fact that $v^0 = \tilde{v}^0$; hence concluding the proof of the lemma.

$\therefore$ Therefore, we have seen that for any vector field $v \in T_x\mathscr{P}$ in the kernel of $\mathrm{Hess}_x\mathcal{A}_H$, we have that $v^t = \phi^t_\star v^0$ along the chord x. Thus, $v^1 \in D\phi(T_{x(0)}\Lambda_0) \cap T_{x(1)}\Lambda_1$.

Hence, the existence of a non-zero vector $v \in \ker \mathrm{Hess}_x\mathcal{A}_H$ would imply that the intersection $D\phi(T_{x(0)}\Lambda_0) \cap T_{x(1)}\Lambda_1$ is non-transverse. □

The goal of the next lemma and calculation is to set up for Calculation C.10, in which we prove a *a priori energy estimate* for solutions of the Floer equation with parameters (3.17).

Lemma C.8. *Assume $H = H_s$ is s-dependent. Then, we have:*

$$\mathrm{d}H_s(\frac{\partial u}{\partial s}) = \partial_s(H_s \circ u) - (\partial_s H_s)(u).$$

This is basically a fancy application of the chain rule; but let us prove it formally, because we shall use it in the next calculation.

Proof. Recall that u is a map $\mathbb{R} \times [0,1] \to \widehat{W}$ (where s is the $\mathbb{R}$-coordinate). Consider $\tilde{H} : \mathbb{R} \times \widehat{W} \longrightarrow \mathbb{R} : (s, \cdot) \longmapsto H_s$. Let $\tilde{u} = (s, u(s,t))$.

The point of this abstract mumbo-jumbo is so we can write $\partial_s(H_s \circ u) = \partial_s(\tilde{H} \circ \tilde{u})$. Let us now choose local coordinates $(y_1, \ldots, y_{2n+1}) = (s, x_1, \ldots, x_{2n})$ on $\mathbb{R} \times \widehat{W}$. Then:

$$\begin{aligned} \frac{\partial}{\partial s}\big(\tilde{H} \circ \tilde{u}\big) &= \sum_{k=1}^{2n+1} \frac{\partial \tilde{H}}{\partial y_k} \frac{\partial \tilde{u}^k}{\partial s} \\ &= \frac{\partial \tilde{H}}{\partial y_1}(\tilde{u}) \frac{\partial \tilde{u}^1}{\partial s} + \sum_{k=2}^{2n+1} \frac{\partial \tilde{H}}{\partial y_k} \frac{\partial \tilde{u}^k}{\partial s} \\ &= \frac{\partial \tilde{H}}{\partial s}(\tilde{u}) \frac{\partial s}{\partial s} + \sum_{j=1}^{2n} \frac{\partial \tilde{H}}{\partial x_j} \frac{\partial u^j}{\partial s} \\ &= \frac{\partial \tilde{H}}{\partial s}(\tilde{u}) + \mathrm{d}H_s(\frac{\partial u}{\partial s}) \\ &= \frac{\partial H_s}{\partial s}(u) + \mathrm{d}H_s(\frac{\partial u}{\partial s}) \end{aligned}$$

Hence, $\mathrm{d}H_s(\frac{\partial u}{\partial s}) = \partial_s(H_s \circ u) - (\partial_s H_s)(u)$. □

Calculation C.9. Here, we compute the differential of the perturbed action functional (3.16), for the purposes of the next calculation. Recall that:

$$\mathcal{A}_{H_s}(x) := f_1\big(x(1)\big) - f_0\big(x(0)\big) - \int_0^1 x^\star\lambda + \int_0^1 H_s\big(x(t)\big)\mathrm{d}t, \tag{C.6}$$

where $s \longmapsto (J_s, H_s)$ is a homotopy of Floer data.

The first three terms are the same as for the non-s-dependent action functional, and hence their derivative is the same (Calculation C.3). The last term, however, picks up an s dependency. Let us see how it affects $\mathrm{d}\mathcal{A}_{H_s}$.

Recall from Chapter 3 that $\mathcal{A}_{H_s}$ is defined on a space $\mathscr{P}$ of paths $[0,1] \to M$, and hence, $\mathrm{d}\mathcal{A}_{H_s}(x) : T_x\mathscr{P} \to \mathbb{R}$. Pick a vector $\zeta \in T_x\mathscr{P}$, and a path (x_s) in $\mathscr{P}$ s.t $(\mathrm{d}/\mathrm{d}s)x_s = \zeta$.

Then, $\mathrm{d}\mathcal{A}_{H_s}(\zeta) = \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{d}\mathcal{A}_{H_s}(x_s)$. In particular, the last term in (C.6) becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^1 H_s(x_s(t))\mathrm{d}t &= \int_0^1 \partial_s\big(H_s(x(t)\big)\mathrm{d}t \\ &= \int_0^1 \mathrm{d}H_s(\zeta)\mathrm{d}t + \int_0^1 (\partial_s H_s)(x(t))\mathrm{d}t \text{ (By Lemma C.8).}\end{aligned}$$

Combining this with Step 1 of calculation C.3 (where we compute the derivative of the first three terms), we get:

$$\mathrm{d}\mathcal{A}_{H_s}(x)\zeta = \lambda\big(\zeta\big)(x(1)) - \lambda\big(\zeta\big)(x(0)) - \int_0^1 x^\star\mathscr{L}_\zeta\lambda + \int_0^1 \mathrm{d}H_s\big(\zeta(t)\big)\mathrm{d}t + \int_0^1 (\partial_s H_s)(x(t))\mathrm{d}t.$$

In the same way as in Step 2 of Calculation C.3, one can identify the first four terms of this expression with:

$$\int_0^1 \mathrm{d}\lambda\left(\dot{x}(t) - X_{H_s}(x(t)), \zeta(t)\right)\mathrm{d}t.$$

(All the calculations in Step 2. work the same when H depends on s). Therefore, we have:

$$\mathrm{d}\mathcal{A}_{H_s}(x)\zeta = \int_0^1 \mathrm{d}\lambda\left(\dot{x}(t) - X_{H_s}(x(t)), \zeta(t)\right)\mathrm{d}t + \int_0^1 (\partial_s H_s)(x(t))\mathrm{d}t. \tag{C.7}$$

Calculation C.10. We here prove the *a priori* energy estimate for solutions of the Floer equation with parameters (Proposition 3.53). From expression (C.7), we have:

$$\int_0^1 \mathrm{d}\lambda\left(\zeta, \dot{x}(t) - X_{H_s}(x(t)\right)\mathrm{d}t = -\mathrm{d}\mathcal{A}_{H_s}(x)\zeta + \int_0^1 (\partial_s H_s)(x(t))\mathrm{d}t, \tag{C.8}$$

where $(x : [0,1] \to M) \in \mathscr{P}$, and $\zeta \in T_x\mathscr{P}$ is a tangent vector.

In particular, consider a Floer strip $u : \mathbb{R} \times [0,1] \to M$ (which is a solution of the perturbed Floer equation, (3.17)). Then for every $s \in \mathbb{R}$, we have a path $u(s, \cdot) \in \mathscr{P}$. We consider the tangent vector $\zeta = (\partial/\partial s)u$. This gives us:

$$\int_0^1 \mathrm{d}\lambda\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_{H_s}(u)\right)\mathrm{d}t = -\mathrm{d}\mathcal{A}_{H_s}(u)\frac{\partial u}{\partial s} + \int_0^1 (\partial_s H_s)(u)\mathrm{d}t. \tag{C.9}$$

Now, we can expand the left-hand side with:

$$\begin{aligned}\int_0^1 \mathrm{d}\lambda\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_{H_s}(u)\right)\mathrm{d}t &= \int_0^1 \omega\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_{H_s}(u)\right)\mathrm{d}t \\ &= \int_0^1 \omega\left((\frac{\partial u}{\partial s}, J_{s,t}\frac{\partial u}{\partial s}\right))\mathrm{d}t \\ &= \int_0^1 g_{s,t}\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s}\right)\mathrm{d}t \\ &= \int_0^1 \left|\frac{\partial u}{\partial s}\right|^2 \mathrm{d}t.\end{aligned}$$

Notice that the energy $E(u)$ is then the integral of this quantity with respect to s. So, integrating both sides of (C.9), we get:

$$\begin{aligned}E(u) &= -\int_{-\infty}^{\infty} \mathrm{d}\mathcal{A}_{H_s}(u)\frac{\partial u}{\partial s}\mathrm{d}s + \int_{\mathbb{R}\times[0,1]} (\partial_s H_s)(u)\mathrm{d}s\wedge\mathrm{d}t \\ &= -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}_{H_s}(u) + \int_{\mathbb{R}\times[0,1]} (\partial_s H_s)(u)\mathrm{d}s\wedge\mathrm{d}t \\ &= \lim_{s\to-\infty}\mathcal{A}_{H_s}(u(s,t)) - \lim_{s\to+\infty}\mathcal{A}_{H_s}(u(s,t)) + \int_{\mathbb{R}\times[0,1]} (\partial_s H_s)(u)\mathrm{d}s\wedge\mathrm{d}t \\ &= \mathcal{A}_{H_0}(x_0) - \mathcal{A}_{H_1}(x_1) + \int_{\mathbb{R}\times[0,1]} (\partial_s H_s)(u)\mathrm{d}s\wedge\mathrm{d}t,\end{aligned}$$

which concludes the proof of the *a priori* energy estimate. □

Calculation C.11. [Proof of Lemma 4.7] Let x be a chord of (J, H) with isolating neighbourhood $\mathcal{U}$, and $(\tilde{J}, \tilde{H})$ be a perturbation of our Floer data, which has a chord $\tilde{x}$ in $\mathcal{U}$. By definition:

$$\begin{cases}\mathcal{A}_H(x) = f_1(x(1)) - f_0(x(0)) - \int_0^1 x^\star\hat{\lambda} + \int_0^1 H(x(t))\,\mathrm{d}t \\ \mathcal{A}_{\tilde{H}}(\tilde{x}) = f_1(\tilde{x}(1)) - f_0(\tilde{x}(0)) - \int_0^1 \tilde{x}^\star\hat{\lambda} + \int_0^1 \tilde{H}(\tilde{x}(t))\,\mathrm{d}t\end{cases}$$

◇ *Note:* the functions f_i are the same in both expressions. Indeed, recall that they are defined (Assumption 3.15) so that $\hat{\lambda}_{\Lambda_i} = \mathrm{d}f_i$, and the Lagrangians are not affected by perturbations of our Floer data (J, H), so these f_i's are the same in both expressions.

Hence, we have:

$$\mathcal{A}_{\tilde{H}}(\tilde{x}) - \mathcal{A}_H(x) = (f_1(\tilde{x}(1)) - f_0(\tilde{x}(0))) - (f_1(x(1)) - f_0(x(0))) \tag{C.10}$$

$$- \int_0^1 \tilde{x}^\star\hat{\lambda} + \int_0^1 x^\star\hat{\lambda} \tag{C.11}$$

$$+ \int_0^1 \tilde{H}(\tilde{x}(t))\mathrm{d}t - \int_0^1 H(x(t))\mathrm{d}t \tag{C.12}$$

We have numbered these lines for we will bound each of them separately. Since x and $\tilde{x}$ are both contained in $\mathcal{U}$, we will restrict our attention to this neighbourhood. Let us proceed:

Step 1. Let us find an upper bound for the first line, (C.10). We can rewrite it as:

$$(f_1(\tilde{x}(1)) - f_1(x(1))) - (f_0(\tilde{x}(0)) - f_0(x(0)))\,. \tag{C.13}$$

For $i = 0, 1$, the function f_i is $\mathcal{C}^1$ and hence Lipschitz-continuous on $\mathcal{U}$. Indeed, by the mean-value inequality:

$$\forall p, q \in \mathcal{U} : |f_i(p) - f_i(q)| \leq |f_i'(\xi_i)| \cdot \|p - q\|,$$

for some ξ_i in the segment $[p, q] \subset \Lambda_i$. Since the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$ is compact, $|f_i'(\xi_i)|$ reaches a maximum $M_i \geq 0$, and therefore:

$$\||f_i(p) - f_i(q)|\| \leq M_i \|p - q\|.$$

Hence, we can bound (C.13):

$$\begin{aligned}\big|\,(f_1(\tilde{x}(1)) - f_1(x(1))) - (f_0(\tilde{x}(0)) - f_0(x(0)))\,\big| &\leq M_1 \|\tilde{x}(1) - x(1)\| + M_0 \|\tilde{x}(0) - x(0)\| \\ &\leq (M_0 + M_1) \sup_{t \in [0,1]} \|\tilde{x}(t) - x(t)\| \\ &\leq (M_0 + M_1) \|\tilde{x} - x\|_{\mathcal{C}^0}\end{aligned}$$

Step 2. To bound the third line, (C.12), notice that:

$$\forall t : |\tilde{H}(\tilde{x}(t)) - H(x(t))| \leq \sup_{p,q \in \mathcal{U}} |\tilde{H}(p) - H(q)|.$$

The supremum is finite, since $\overline{\mathcal{U}} \times \overline{\mathcal{U}}$ is compact. The right-hand side can be simplified by the triangle inequality:

$$\begin{aligned}|\tilde{H}(p) - H(q)| &= |\tilde{H}(p) - H(p) + H(p) - H(q)| \\ &\leq |\tilde{H}(p) - H(p)| + |H(p) - H(q)|.\end{aligned}$$

Now, from Lipschitz continuity, $\exists B > 0$ s.t $|H(p) - H(q)| \leq B \|p - q\|$. As for the other term, we have $|\tilde{H}(p) - H(p)| \leq \sup_{p \in \mathcal{U}} |\tilde{H}(p) - H(p)| =: |\tilde{H} - H|_{\mathcal{C}^0}$

Therefore:

$$\begin{aligned}\int_0^1 \tilde{H}(\tilde{x}(t))\mathrm{d}t - \int_0^1 H(x(t))\mathrm{d}t &= \int_0^1 \Big(\tilde{H}(\tilde{x}(t)) - H(x(t))\Big)\,\mathrm{d}t \\ &\leq \int_0^1 \Big(B \|\tilde{x}(t) - x(t)\| + |\tilde{H} - H|_{\mathcal{C}^0}\Big)\,\mathrm{d}t \\ &\leq B \|x - \tilde{x}\|_{\mathcal{C}^0} + |\tilde{H} - H|_{\mathcal{C}^0}.\end{aligned}$$

Step 3. Finally, for the second line, (C.11), we have:

$$\int_0^1 x^\star \hat{\lambda} - \int_0^1 \tilde{x}^\star \hat{\lambda} = \int_0^1 \Big(\hat{\lambda}_{x(t)}(\dot{x}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))\Big)\,\mathrm{d}t,$$

from Step 2 of Calculation C.4, and where $\dot{x}(t)$ is the tangent vector:

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{2n}(t) \end{pmatrix} = \sum_i \dot{x}_i(t)\partial_{x_i} \in T_{x(t)}\mathcal{U},$$

where we assume $\mathcal{U}$ to be small enough that $\mathcal{U} \subset \mathbb{R}^{2n}$, with coordinates $\{x_1, \ldots, x_{2n}\}$.

Note that we can't abbreviate the last integrand into $\hat{\lambda}(\dot{x}(t) - \dot{\tilde{x}}(t))\mathrm{d}t$, since terms in the integrand are the 1-form $\hat{\lambda}$ evaluated on different tangent spaces. However:

$$\begin{aligned}\Big(\hat{\lambda}_{x(t)}(\dot{x}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))\Big) &= \hat{\lambda}_{x(t)}(\dot{x}(t)) - \hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t)) + \hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t)) \\ &= \Big(\hat{\lambda}_{x(t)}(\dot{x}(t)) - \hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t))\Big) + \Big(\hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))\Big).\end{aligned}$$

We proceed term by term.

- Since we assume $\mathcal{U} \subset \mathbb{R}^{2n}$, $\hat{\lambda}_{x(t)}$ can be viewed as a linear map $\mathbb{R}^{2n} \to \mathbb{R}$. Hence, by a Lipschitz continuity argument, we can find $D > 0$ such that:

$$\forall t : \left|\hat{\lambda}_{x(t)}(\dot{x}(t)) - \hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t))\right| \leq D \left\|\dot{x} - \dot{\tilde{x}}\right\|_{\mathcal{C}^0}.$$

  This gives us a bound on the first term.

- In coordinates on $\mathcal{U}$, we can write $\hat{\lambda}$ as:

$$\hat{\lambda}_p = \sum_{i=1}^{2n} \lambda_i(p)\mathrm{d}x_i,$$

  where $\lambda_i \in \mathcal{C}^\infty(\mathcal{U})$. Then, the second term can be rewritten as:

$$\left(\hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))\right) = \sum_i \left(\lambda_i(x(t)) - \lambda_i(\tilde{x}(t))\right) \dot{\tilde{x}}_i(t). \tag{C.14}$$

  Since each individual λ_i is smooth, we can find $C_i > 0$ s.t:

$$\forall t : |\lambda_i(x(t)) - \lambda_i(\tilde{x}(t))| \leq C_i \|x - \tilde{x}\|_{\mathcal{C}^0}$$

  Then, taking the absolute value of (C.14), we get:

$$\begin{aligned}
\left|\hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))\right| &= \left|\sum_i \left(\lambda_i(x(t)) - \lambda_i(\tilde{x}(t))\right) \dot{\tilde{x}}_i(t)\right| \\
&\leq \sum_i |\lambda_i(x(t)) - \lambda_i(\tilde{x}(t))| \, |\dot{\tilde{x}}_i(t)| \\
&\leq 2n \max_i |\lambda_i\left(x(t)\right) - \lambda_i\left(\tilde{x}(t)\right)| |\dot{\tilde{x}}_i(t)| \\
&\leq 2n \left(\max_i C_i\right) \left(\max_i |\dot{\tilde{x}}(t)|\right) \|x - \tilde{x}\|_{\mathcal{C}^0}.
\end{aligned}$$

  Now, $\max_i |\dot{\tilde{x}}_i(t)|$ is known as the maximum, or infinity norm, denoted $\left\|\dot{\tilde{x}}(t)\right\|_\infty$. Since all norms on a finite-dimensional vector space are equivalent, there is some $K > 0$ such that $\left\|\dot{\tilde{x}}(t)\right\|_\infty \leq K \left\|\dot{\tilde{x}}(t)\right\|$. Therefore, writing $C := \max_i C_i$, and taking suprema, we get:

$$|\hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))| \leq \left(2KnC \left\|\dot{\tilde{x}}\right\|_{\mathcal{C}^0}\right) \|x - \tilde{x}\|_{\mathcal{C}^0}. \tag{C.15}$$

  So we have found an upper bound. The only problem is that the multiplicative constant in front of $\|x - \tilde{x}\|_{\mathcal{C}^0}$ depends on the perturbed chord $\tilde{x}$, which is undesirable. However, this is easily fixed. Indeed, since

$$\begin{cases} \dot{x} &= X_H = J\nabla H \\ \dot{\tilde{x}} &= X_{\tilde{H}} = \tilde{J}\nabla\tilde{H} \end{cases},$$

  Then for every $\varepsilon > 0$, we can choose (J, H) and $(\tilde{J}, \tilde{H})$ close enough, in the $\mathcal{C}^1$ norm, such that $\left\|\dot{x} - \dot{\tilde{x}}\right\|_{\mathcal{C}^0} < \varepsilon$. Note that this is the only place in the proof where we make use of the "$\mathcal{C}^1$-close" assumption.

  In particular, choose $\varepsilon < \|\dot{x}\|_{\mathcal{C}^0}$, so that, by triangle inequality, $\left\|\dot{\tilde{x}}\right\|_{\mathcal{C}^0} < 2\|\dot{x}\|_{\mathcal{C}^0}$. Then, setting $\tilde{C} := 4KnC\|\dot{x}\|_{\mathcal{C}^0}$, (C.15) becomes:

$$|\hat{\lambda}_{x(t)}(\dot{\tilde{x}}(t) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))| < \tilde{C} \|x - \tilde{x}\|_{\mathcal{C}^0}.$$

  The constant $\tilde{C}$ no longer depends on the perturbed chord $\tilde{x}$, and is therefore universal for any small perturbation of (J, H).

- Combining these two substeps, we have:

$$\begin{aligned}\left|\int_0^1 x^\star\hat{\lambda} - \int_0^1 \tilde{x}^\star\hat{\lambda}\right| &= \left|\int_0^1 \left(\hat{\lambda}_{x(t)}(\dot{x}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))\right)\right| \\ &\le \left|\left(\hat{\lambda}_{x(t)}(\dot{x}(t)) - \hat{\lambda}_{\tilde{x}(t)}(\dot{\tilde{x}}(t))\right)\right| \\ &< \tilde{C}\,\|x-\tilde{x}\|_{\mathcal{C}^0} + D\,\left\|\dot{x}-\dot{\tilde{x}}\right\|_{\mathcal{C}^0}.\end{aligned}$$

Step 4. Combining the results from Steps 1, 2, and 3, we now get:

$$\left|\mathcal{A}_{\tilde{H}}(\tilde{x}) - \mathcal{A}_H(x)\right| < (M_0 + M_1 + B + \tilde{C})\,\|x-\tilde{x}\|_{\mathcal{C}^0} + D\,\left\|\dot{x}-\dot{\tilde{x}}\right\|_{\mathcal{C}^0} + |\tilde{H} - H|_{\mathcal{C}^0}.$$

The $\mathcal{C}^1$-norm is defined as:

$$\begin{aligned}\|x-\tilde{x}\|_{\mathcal{C}^1} &:= \sup_{t\in[0,1]} \|x(t)-\tilde{x}(t)\| + \sup_{t\in[0,1]} \left\|\dot{x}(t)-\dot{\tilde{x}}(t)\right\| \\ &= \|x-\tilde{x}\|_{\mathcal{C}^0} + \left\|\dot{x}-\dot{\tilde{x}}\right\|_{\mathcal{C}^0}.\end{aligned}$$

Therefore, setting $C := \max\{M_0 + M_1 + B + \tilde{C}, D\}$, we have:

$$\boxed{|\mathcal{A}_{\tilde{H}}(\tilde{x}) - \mathcal{A}_H(x)| < C\,\|x-\tilde{x}\|_{\mathcal{C}^1} + |\tilde{H} - H|_{\mathcal{C}^0}}$$

which is the desired inequality. All we have left to prove now is that this constant C doesn't depend on the choice of perturbation.

Step 5. Showing that C is universal.

We want to show that C does not depend on the choice of perturbation (J_s, H_s), or of perturbed chord $\tilde{x}$. We will show that this is the case for all the intermediary constants we derived:

1. The constants M_0, M_1 from Step 1 can be set to: $M_i := \sup_{\xi_i\in\mathcal{U}\cap\Lambda_i} |f_i'(\xi_i)|$. These only depend on Λ_i, f_i, and $\mathcal{U}$. So this works for any perturbation.
2. Similarly, the constant B from Step 2 can be taken to be:

   $$B := \sup_{p\in\mathcal{U}} |H'(p)|,$$

   which only depends on H and $\mathcal{U}$.
3. In Step 3, we derive two constants: $\tilde{C}$ and D. We have already made sure in Step 3 that $\tilde{C}$ did not depend on the perturbation.

Meanwhile, D is the Lipschitz-continuity constant for $\hat{\lambda}_{x(t)}$, and hence only depends on $\hat{\lambda}$ and x.

$\therefore$ In conclusion, $C = \max\{M_0 + M_1 + B + \tilde{C}, D\}$ only depends on $\mathcal{U}$, Λ_i, f_i, H and x. In particular, it works for any perturbation of (J, H) whose chords stay in $\mathcal{U}$. Hence, C is universal.

Therefore, though we constructed C after choosing a specific perturbation, this choice will work for any perturbation $(\tilde{J}, \tilde{H})$, which is $\mathcal{C}^1$-close enough to (J, H).

This concludes the proof of the lemma. □

Calculation C.12 (Steps 1-4 of Proposition 5.10)**.**

Step 1 (Computing $X_{\widehat{H}}$). Recall that, on $[1,+\infty) \times \partial W$:

$$\widehat{H}(r,b,t) = \sum_{i=0}^{m} \frac{(r-1)^i}{i!}\widehat{H}_i + \widehat{R}_m. \tag{C.16}$$

First, note that we can rewrite $\widehat{R}_m = (r-1)^{m+1}\widehat{R}$, for a function $\widehat{R} : [1,+\infty) \times \partial W \to \mathbb{R}$. We rewrite it in this way, for later convenience.

Now, our goal is to compute $X_{\widehat{H}} = J\nabla\widehat{H}$. We will work in a frame $\hat{\xi} \cup \{\mathcal{R}_\alpha, \partial_r\}$ where $\hat{\xi}$ is a frame for the contact structure $\xi = \ker\alpha$, $\mathcal{R}_\alpha$ is the Reeb vector field on ∂W, and ∂_r is the derivative of the coordinate on the collar. Recall from Chapter 3 that $r\partial_r$ is the Liouville vector field, and that $J\partial_r = \mathcal{R}_\alpha$.

By definition, we have:

$$\nabla\widehat{H} = (\partial_r\widehat{H})\partial_r + (\mathcal{R}_\alpha \cdot \widehat{H})\mathcal{R}_\alpha + \nabla^\xi\widehat{H}, \tag{C.17}$$

where ∇^ξ is the restriction of the gradient operator ∇ to ξ. We can easily compute the first term using our expression (C.16) for H:

$$\partial_r\widehat{H} = \partial_r\widehat{H}_0 + \left(\sum_{i=1}^{m} \frac{(r-1)^{i-1}}{(i-1)!}\widehat{H}_i + \frac{(r-1)^i}{i!}\partial_r\widehat{H}_i\right) + \frac{(r-1)^m}{m!}\widehat{R} + \frac{(r-1)^{m+1}}{(m+1)!}\partial_r\widehat{R} \tag{C.18}$$

$$=: F(r,b), \tag{C.19}$$

where b is the coordinate on the boundary ∂W. Now let us look at the second term in the expression (C.17). By definition, $\mathcal{R}_\alpha \cdot \widehat{H} = \mathrm{d}\widehat{H}(\mathcal{R}_\alpha)$, and:

$$\mathrm{d}\widehat{H} = \mathrm{d}\widehat{H}_0 + \left(\sum_{i=1}^{m} \frac{(r-1)^{i-1}}{(i-1)!}\widehat{H}_i\mathrm{d}r + \frac{(r-1)^i}{i!}\mathrm{d}\widehat{H}_i\right) + \frac{(r-1)^m}{m!}\widehat{R}\mathrm{d}r + \frac{(r-1)^{m+1}}{(m+1)!}\mathrm{d}\widehat{R}.$$

Since $\mathrm{d}r(\mathcal{R}_\alpha) = 0$, we have:

$$\mathrm{d}\widehat{H}(\mathcal{R}_\alpha) = \mathrm{d}\widehat{H}_0(\mathcal{R}_\alpha) + \sum_{i=1}^{m} \frac{(r-1)^i}{i!}\mathrm{d}\widehat{H}_i(\mathcal{R}_\alpha) + \frac{(r-1)^{m+1}}{(m+1)!}\mathrm{d}\widehat{R}(\mathcal{R}_\alpha). \tag{C.20}$$

make Let us further show that the first term is 0. **This will use the twist condition** (see ⚠).

By definition, $\mathrm{d}\widehat{H}_0$ is the dual of the gradient $\nabla\widehat{H}_0$ through the metric, and recall that $\widehat{H}_0 := \rho(r)H_0 + (1-\rho(r))C_0$, where $C_0 \geq \max_{\partial W} H_0$. Therefore:

$$\nabla\widehat{H}_0 = \rho'(r)\,(H_0 - C_0)\,\partial_r + \rho(r)\nabla H_0.$$

Let us argue that $\nabla H_0 = 0$. By definition, we have $H_0 = H|_{\partial W}$. Let $\hat{\xi} := \{\partial_i\}$ be a frame for ξ, so that $\{\partial_i, \mathcal{R}_\alpha\}$ is a frame for $T\partial W = \xi \oplus \langle\mathcal{R}_\alpha\rangle$. Then:

$$\begin{aligned}\nabla H_0 &= \sum_{i=1}^{2n-2} \partial_i H_0 \partial_i + \mathrm{d}H_0(\mathcal{R}_\alpha)\mathcal{R}_\alpha \\ &= \sum_i (\partial_i H)|_{r=1}\partial_i + \mathrm{d}H(\mathcal{R}_\alpha)|_{r=1}\mathcal{R}_\alpha.\end{aligned}$$

Meanwhile:

$$\begin{aligned}(\nabla H)|_{\partial W} &= (\partial_r H)|_{r=1}\partial_r + \sum_{i=1}^{2n-2} (\partial_i H)|_{r=1}\partial_i + \mathrm{d}H(\mathcal{R}_\alpha)\mathcal{R}_\alpha \\ &= H_1\partial_r + \nabla H_0.\end{aligned} \tag{C.21}$$

⚠By the **twist condition**, we have $X_H|_{\partial W} = J\nabla H = h_t\mathcal{R}_\alpha$, for some function $h_t > 0$. Therefore, we must have:

$$\boxed{H_1 \equiv h_t, \nabla H_0 \equiv 0}.$$

In particular, the first term in (C.20) is zero[(3)]. Hence, we can write:

$$\mathrm{d}\widehat{H}(\mathcal{R}_\alpha) = (r-1)\left(\sum_{i=1}^{m}\frac{(r-1)^{i-1}}{i!}\mathrm{d}\widehat{H}_i(\mathcal{R}_\alpha) + \frac{(r-1)^m}{(m+1)!}\mathrm{d}\widehat{R}(\mathcal{R}_\alpha)\right).$$

For convenience, let us define the function:

$$G = \sum_{i=1}^{m}\frac{(r-1)^{i-1}}{i!}\mathrm{d}\widehat{H}_i(\mathcal{R}_\alpha) + \frac{(r-1)^m}{(m+1)!}\mathrm{d}\widehat{R}(\mathcal{R}_\alpha), \tag{C.22}$$

on $\widehat{W}\backslash\mathrm{int}(W)$. This allows us to write:

$$\nabla\widehat{H} = F\partial_r + (r-1)G\mathcal{R}_\alpha + \nabla^\xi\widehat{H}.$$

Finally, let us look at the last term of that expression. By definition:

$$\nabla^\xi\widehat{H} = \sum_{i=0}^{m}\frac{(r-1)^i}{i!}\nabla^\xi\widehat{H}_i + \frac{(r-1)^{m+1}}{(m+1)!}\nabla^\xi\widehat{R}.$$

Indeed, note that we have $\nabla^\xi r = 0$, since the coordinate r is independent from the contact structure, and we proved, a few lines above, that $\nabla\widehat{H}_0 \propto \mathcal{R}_\alpha$, so that $\nabla^\xi\widehat{H}_0 = 0$. Therefore, we have:

$$\nabla^\xi\widehat{H} = (r-1)\left(\sum_{i=1}^{m}\frac{(r-1)^i}{i!}\nabla^\xi\widehat{H}_i + \frac{(r-1)^m}{(m+1)!}\widehat{R}\right)$$

Then, recall that we want to compute the Hamiltonian vector field $X_{\widehat{H}} = J\nabla\widehat{H}$. Let us set:

$$X^\xi := \sum_{i=1}^{m}\frac{(r-1)^i}{i!}J\nabla^\xi\widehat{H}_i + \frac{(r-1)^m}{(m+1)!}J\nabla^\xi\widehat{R}, \tag{C.23}$$

so that $J\nabla^\xi\widehat{H} = (r-1)X^\xi$. Then, we have:

$$\begin{aligned} X_{\widehat{H}} = J\nabla\widehat{H} &= J\left(F\partial_r + (r-1)G\mathcal{R}_\alpha + \nabla^\xi\widehat{H}\right) \\ &= F\mathcal{R}_\alpha - (r-1)G\partial_r + (r-1)X^\xi. \end{aligned}$$

Where the expressions for F, G and X^ξ are given respectively in (C.18), (C.22), and (C.23). This concludes step 1.

Step 2 (Computing $\nabla X_{\widehat{H}}$).
Let us now linearise the vector field $X_{\widehat{H}}$. This first requires a choice of connection. Define the metric[(4)]:

$$\eta = \mathrm{d}\alpha(\cdot, J\cdot) + \mathrm{d}r\otimes\mathrm{d}r + \alpha\otimes\alpha$$

on $T(\widehat{W}\backslash\mathrm{int}(W))$, and choose the associated Levi-Civita connection ∇. We can easily show (see Calculation C.14), that it has the following properties:

1. $\nabla\partial_r = 0$;

[(3)] Essentially, we proved that $H_0 = H|_{\partial W} \equiv \mathrm{cst}$. This reproves the statement that a regular energy hypersurface of a Hamiltonian is contact iff the Hamiltonian and Reeb vector field are reparametrizations of each other (see, for *e.g.* [Wen15]).

[(4)] Note that in the original proof, in Lemma 4.5 of [MK22a], the authors have a factor of $\frac{1}{r^2}$ in front of the $\mathrm{d}r\otimes\mathrm{d}r$ coefficient; because they choose to work with the Liouville vector field $r\partial_r$ in their frame for $T\widehat{W}$; while we pick ∂_r. (Of course, the computations are purely equivalent, but note that some factors may differ from our computation to theirs).

2. $\nabla_{\mathcal{R}_\alpha}\mathcal{R}_\alpha = 0$ and $\nabla_{\partial_r}\mathcal{R}_\alpha = 0$;
3. $\forall Z \in \xi \, : \nabla_Z \mathcal{R}_\alpha \in \xi$;
4. $\forall Z \in \xi \, : \nabla_{\partial_r} Z = 0$;
5. by assumption, $(\xi, \mathrm{d}\alpha)$ is symplectically trivial. Let $\{e_i\}_{i \leq 2n-2}$ be a trivialising frame. Then, $\nabla_{\mathcal{R}_\alpha} e_i = 0$.

With the above properties, we can proceed with the computation of $\nabla X_{\widehat{H}}$. Recall that this notation denotes the *total* covariant derivative of $X_{\widehat{H}}$, *ie* the 1-form such that for a vector field Z, $(\nabla X_{\widehat{H}})(Z) = \nabla_Z X_{\widehat{H}}$. Hence, to compute it, pick some $Z \in T(\widehat{W}\backslash \mathrm{int}(W))$:

$$\begin{aligned}\nabla_Z X_{\widehat{H}} &= \nabla_Z \left(F\mathcal{R}_\alpha - (r-1)G\partial_r + (r-1)X^\xi\right) \\ &= \mathrm{d}F(Z)\mathcal{R}_\alpha + F\nabla_Z \mathcal{R}_\alpha - \mathrm{d}r(Z)G\partial_r - (r-1)\mathrm{d}G(Z)\partial_r - (r-1)G\nabla_Z\partial_r \\ &\quad + \mathrm{d}r(Z)X^\xi + (r-1)\nabla_Z X^\xi \\ &= \mathrm{d}F(Z)\mathcal{R}_\alpha + F\nabla_Z\mathcal{R}_\alpha + \mathrm{d}r(Z)\left(X^\xi - G\partial_r\right) + (r-1)\left(\nabla_Z X^\xi - \mathrm{d}G(Z)\partial_r\right)\end{aligned}$$

Hence, we have:

$$\nabla X_{\widehat{H}} = \mathrm{d}F \otimes \mathcal{R}_\alpha + F\nabla\mathcal{R}_\alpha + \mathrm{d}r \otimes (X^\xi - G\partial_r) + (r-1)\left(\nabla X^\xi - \mathrm{d}G \otimes \partial_r\right), \qquad \text{(C.24)}$$

which concludes Step 2.

Step 3 (Decomposing $\nabla X_{\widehat{H}}$).
Let us see what can still be simplified in the expression (C.24). First, expand:

$$\nabla X^\xi = \mathrm{d}r \otimes \nabla_{\partial_r} X^\xi + \alpha \otimes \nabla_{\mathcal{R}_\alpha} X^\xi + \nabla^\xi X^\xi.$$

Observe that the first term is zero, by property 4 of Step 2.

The term $\mathrm{d}G$ in (C.24) can also be expanded: $\mathrm{d}G = \mathrm{dG}(\partial_r)\mathrm{d}r + \mathcal{R}_\alpha(G)\alpha + \mathrm{d}^\xi G$, where d^ξ is the exterior derivative on ξ. Therefore, we can now write:

$$\begin{aligned}\nabla X_{\widehat{H}} &= \mathrm{d}F \otimes \mathcal{R}_\alpha + F\nabla R_\alpha + \mathrm{d}r \otimes (X^\xi - G\partial_r) \\ &\quad + (r-1)\left(\alpha \otimes \nabla_{\mathcal{R}_\alpha} X^\xi + \nabla^\xi X^\xi - \left(\mathrm{dG}(\partial_r)\mathrm{d}r + \mathcal{R}_\alpha(G)\alpha + \mathrm{d}^\xi G\right) \otimes \partial_r\right) \\ &= L_0 + (r-1)L_1,\end{aligned}$$

where we have set:

$$L_0 := \mathrm{d}F \otimes \mathcal{R}_\alpha + F\nabla R_\alpha + \mathrm{d}r \otimes (X^\xi - G\partial_r) - (r-1)\mathrm{d}G(\partial_r)\mathrm{d}r \otimes \partial_r \qquad \text{(C.25)}$$

$$L_1 := \alpha \otimes \nabla_{\mathcal{R}_\alpha} X^\xi + \nabla^\xi X^\xi - R_\alpha(G)\alpha \otimes \partial_r - \mathrm{d}^\xi G \otimes \partial_r. \qquad \text{(C.26)}$$

Let us express L_0 and L_1 in matrix form, in the frame $\hat{\xi} \oplus \{\partial_r, \mathcal{R}_\alpha\}$ of $T(\widehat{W}\backslash \mathrm{int}(W))$. Write this frame $\{e_1, \ldots, e_{2n}\}$. Then, both matrices will look like:

$$L_i = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{L_i^ξ}} & \star & \star \\ & & \star & \star \\ \hline \star & \star & a & b \\ \star & \star & c & d \end{array}\right),$$

where the top-left block is $L_i|_\xi \cdot \hat{\xi}$, and is hence $(2n-2) \times (2n-2)$; the bottom-right block is $L_i|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} \cdot \{\partial_r, \mathcal{R}_\alpha\}$, and is 2×2. Meanwhile, the off-diagonal blocks are $(2n-2) \times 2$ and $2 \times (2n-2)$, and respectively correspond to $L_i|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} \cdot \hat{\xi}$ and $L_i|_\xi \cdot \{\partial_r, \mathcal{R}_\alpha\}$.

➢ Let us first look at L_0.

- First, to determine the two left blocks, we look at $L_0|_\xi$. The last two terms of L_0 are zero, since $\mathrm{d}r|_\xi = 0$. Hence:

$$L_0|_\xi = \mathrm{d}F|_\xi \otimes \mathcal{R}_\alpha + F\nabla^\xi \mathcal{R}_\alpha.$$

(Note that by the properties of ∇ from Step 2, $\nabla\mathcal{R}_\alpha = \nabla^\xi\mathcal{R}_\alpha$).

Hence, the top-left block is given by $F\nabla^\xi\mathcal{R}_\alpha$, and the bottom-left block by

$$\begin{pmatrix} 0 \quad 0 \\ \mathrm{d}F|_\xi \end{pmatrix}.$$

- Meawnhile, $L_0|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} = \mathrm{d}F|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} \otimes \mathcal{R}_\alpha + \mathrm{d}r \otimes (X^\xi - G\partial_r) - (r-1)\mathrm{d}G(\partial_r)\mathrm{d}r \otimes \partial_r$.

Hence, the top-right block is given by:

$$L_0|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} \cdot \hat{\xi} = \mathrm{d}r \otimes X^\xi = \begin{pmatrix} X^\xi & \begin{matrix} 0 \\ 0 \end{matrix} \end{pmatrix},$$

and the bottom-right block by:

$$L_0|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} \cdot \{\partial_r, \mathcal{R}_\alpha\} = \begin{pmatrix} -G-(r-1)\mathrm{d}G(\partial_r) & 0 \\ \mathrm{d}F(\partial_r) & \mathrm{d}F(\mathcal{R}_\alpha) \end{pmatrix} =: \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

Finally, this gives us:

$$L_0 = \left(\begin{array}{c|cc} F\nabla^\xi\mathcal{R}_\alpha & X^\xi & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 \quad 0 \\ \mathrm{d}F|_\xi \end{matrix} & \begin{matrix} a \\ b \end{matrix} & \begin{matrix} 0 \\ c \end{matrix} \end{array}\right).$$

➢ Let us now look at L_1.

- first, $L_1|_\xi = \nabla^\xi X^\xi - \mathrm{d}^\xi G \otimes \partial_r$, so that the top-left block is given by $\nabla^\xi X^\xi$, and the bottom-left block by:

$$\begin{pmatrix} -\mathrm{d}^\xi G \\ 0 \quad 0 \end{pmatrix}.$$

- $L_1|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} = \alpha \otimes \nabla_{\mathcal{R}_\alpha} X^\xi - \mathcal{R}_\alpha(G)\alpha \otimes \partial_r$. Applied to ∂_r, it gives 0, which tells us that the column before last is zero; moreover, we see that the top-right block is given by:

$$L_1|_{\langle \partial_r, \mathcal{R}_\alpha\rangle} \cdot \{\partial_r, \mathcal{R}_\alpha\} = \alpha\nabla_{\mathcal{R}_\alpha} X^\xi|_\xi = \begin{pmatrix} \begin{matrix} 0 \\ 0 \end{matrix} & \nabla_{\mathcal{R}_\alpha} X^\xi|_\xi \end{pmatrix}.$$

Furthermore, notice that $\langle \nabla_{\mathcal{R}_\alpha} X^\xi, \mathcal{R}_\alpha\rangle = 0$. Indeed, with our choice of metric, $\mathcal{R}_\alpha$ is orthogonal to ξ, so that:

$$0 = \nabla_{\mathcal{R}_\alpha} \langle X^\xi, \mathcal{R}_\alpha\rangle = \langle \nabla_{\mathcal{R}_\alpha} X^\xi, \mathcal{R}_\alpha\rangle + \langle X^\xi, \nabla_{\mathcal{R}_\alpha}\mathcal{R}_\alpha\rangle.$$

Hence the only non-zero coefficient in the bottom-right block is $\langle \nabla_{\mathcal{R}_\alpha} X^\xi, \partial_r\rangle =: a'$.

This finally gives us $L_1 = \left(\begin{array}{c|cc} \nabla^\xi X^\xi & \begin{matrix} 0 \\ 0 \end{matrix} & \nabla_{\mathcal{R}_\alpha} X^\xi|_\xi \\ \hline \begin{matrix} -\mathrm{d}^\xi G \\ 0 \quad 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} a' \\ 0 \end{matrix} \end{array}\right).$

∴ In conclusion, we have:

$$\nabla X_{\widehat{H}} = L_0 + (r-1)L_1$$
$$= \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{F\nabla^\xi \mathcal{R}_\alpha} & X^\xi & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & a & 0 \\ \multicolumn{2}{c|}{\mathrm{d}F|_\xi} & b & c \end{array}\right) + (r-1)\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\nabla^\xi X^\xi} & \begin{matrix}0\\0\end{matrix} & \nabla_{\mathcal{R}_\alpha} X^\xi|_\xi \\ \hline \multicolumn{2}{c|}{-\mathrm{d}^\xi G} & 0 & a' \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Step 4 (Proof that $L_0, L_1 \in \mathfrak{sp}_{2n}$).

Recall that we want to use Lemma 5.13, for which we need to show that $L_0, L_1 \in \mathfrak{sp}_{2n}$. This requires showing that JL_0 and JL_1 are symmetric. We start with JL_1.

A. i) First, note that since $J = i \oplus \ldots i = J|_\xi \oplus i$, the top-left block of L_1 simply gets taken to $J|_\xi \nabla^\xi X^\xi$. So it suffices to show that this is symmetric.

This follows from the fact that X^ξ is made-up of Hamiltonian vector fields on ξ. Indeed, recall that:

$$X^\xi = \sum_{i=1}^{m} \frac{(r-1)^i}{i!} J\nabla^\xi \widehat{H}_i + \frac{(r-1)^m}{(m+1)!} J\nabla^\xi \widehat{R}$$
$$= \sum_{i=1}^{m} \frac{(r-1)^i}{i!} JX^\xi_{\widehat{H}_i} + \frac{(r-1)^m}{(m+1)!} X^\xi_{\widehat{R}}.$$

Since $\nabla^\xi r = 0$, it suffices to show that for a Hamiltonian $\tilde{H}$, the matrix $J|_\xi \nabla^\xi X^\xi_{\tilde{H}}$ is symmetric. This is a straightforward computation. Indeed, first compute:

$$\nabla^\xi X^\xi_{\tilde{H}} = \nabla^\xi J|_\xi \nabla^\xi \tilde{H}$$
$$= J|_\xi \nabla^\xi \left(\nabla^\xi \tilde{H}\right) \text{ (since } J\nabla = \nabla J \text{ on a Kähler manifold)}$$
$$= J|_\xi \begin{pmatrix} & \cdots & \\ & \cdots & \\ \nabla_1 \nabla^\xi \tilde{H} & \cdots & \nabla_{2n-2}\nabla^\xi \tilde{H} \\ & \cdots & \\ & \cdots & \end{pmatrix} = J|_\xi \begin{pmatrix} \nabla_1 \partial_1 \tilde{H} & \nabla_2 \partial_1 \tilde{H} & \cdots \\ \nabla_1 \partial_2 \tilde{H} & \nabla_2 \partial_2 \tilde{H} & \cdots \\ \vdots & \vdots & \vdots \\ \nabla_1 \partial_{2n-2} \tilde{H} & \nabla_2 \partial_{2n-2} \tilde{H} & \ddots \end{pmatrix}.$$

We want to show that $J\nabla^\xi X^\xi_{\tilde{H}} = (J\nabla^\xi X^\xi_{\tilde{H}})^t$, which is equivalent to showing that the matrix $\left(\nabla_i \partial_j \tilde{H}\right)_{ij}$ is symmetric. Note that, since $\partial_j \tilde{H}$ is a function, $\nabla_i \partial_j \tilde{H} = \partial_i \partial_j \tilde{H}$. This expression is symmetric in (i,j) since we assume $\tilde{H}$ to be at least $\mathcal{C}^2$.

$\therefore$ So the top-left block of L_1 belongs to $\mathfrak{sp}_{2n-2}$.

A. ii) The bottom-right block clearly belongs to $\mathfrak{sp}_2$, since it is of the form $\begin{pmatrix} 0 & a' \\ 0 & 0 \end{pmatrix}$, which gives $\begin{pmatrix} 0 & 0 \\ 0 & a' \end{pmatrix}$ when multiplied on the left by J.

A. iii) Finally, let us look at the off-diagonal blocks. Recall that $L_1 = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\nabla^\xi X^\xi} & \begin{matrix}0\\0\end{matrix} & \nabla_{\mathcal{R}_\alpha} X^\xi|_\xi \\ \hline \multicolumn{2}{c|}{-\mathrm{d}^\xi G} & 0 & a' \\ 0 & 0 & 0 & 0 \end{array}\right)$.

For simplicity, let us write $\nabla_{\mathcal{R}_\alpha} X^\xi|_\xi =: \binom{U'}{V'}$, and $-\mathrm{d}^\xi G = (W' \, Z')$, where U and V are $(n-1)$-column vectors, and W and Z are $(n-1)$-row vectors. Then:

$$JL_1 = J \left(\begin{array}{cc|cc} & & 0 & U' \\ & & 0 & V' \\ \hline W' & Z' & & \\ 0 & 0 & & \end{array} \right) = \left(\begin{array}{cc|c} & & \begin{matrix} 0 \\ 0 \end{matrix} \; J\binom{U'}{V'} \\ \hline 0 & 0 & \\ W' & Z' & \end{array} \right),$$

so that $(JL_1)^t = \left(\begin{array}{c|cc} & 0 & (W')^t \\ & 0 & (Z')^t \\ \hline \begin{matrix} 0 & 0 \end{matrix} & & \\ J\binom{U'}{V'} & & \end{array} \right)$.

Hence, for JL_1 to be symmetric, we require that $J\binom{U'}{V'} = (W'\, Z')^t$. Or, in other words, $J\nabla_{\mathcal{R}_\alpha} X^\xi|_\xi = (-\mathrm{d}^\xi G)^t$.

Since the matrix transpose corresponds to taking the dual under the metric, we want:

$$\langle J\nabla_{\mathcal{R}_\alpha} X^\xi|_\xi, \cdot\rangle = -\mathrm{d}^\xi G,$$

where $\langle\cdot,\cdot\rangle = \mathrm{d}\alpha(\cdot, J\cdot)$ on ξ. Recall that:

$$X^\xi := \sum_{i=1}^{m} \frac{(r-1)^{i-1}}{i!} X^\xi_{\widehat{H}_i} + \frac{(r-1)^m}{(m+1)!} X^\xi_{\widehat{R}},$$
$$G := \sum_{i=1}^{m} \frac{(r-1)^{i-1}}{i!} \mathrm{d}\widehat{H}_i(\mathcal{R}_\alpha) + \frac{(r-1)^m}{(m+1)!} \mathrm{d}\widehat{R}(\mathcal{R}_\alpha).$$

◇ **Remark C.13.** ⚠ The following argument implicitly relies on the twist condition, because the latter implies why G looks like this (it has no term in $\mathrm{d}\widehat{H}_0$).

Since $\nabla_{\mathcal{R}_\alpha} r = \mathrm{d}^\xi r = 0$, we can treat the terms in r as linear. Therefore, without loss of generality, it suffices to show that:

$$\langle J\nabla_{\mathcal{R}_\alpha} X^\xi_{\tilde{H}}|_\xi, \cdot\rangle = -\mathrm{d}^\xi \tilde{G}, \tag{C.27}$$

where $\tilde{H}$ is any $\mathcal{C}^2$ Hamiltonian with vector field $X^\xi_{\tilde{H}}$ on ξ; and where $\tilde{G} := \mathrm{d}\tilde{H}(\mathcal{R}_\alpha)$.

Then, recall that since $(\widehat{W}, \hat{\omega}, J)$ is Kähler, we have $J\nabla = \nabla J$, and hence:

$$\begin{aligned} J\nabla_{\mathcal{R}_\alpha} X^\xi_{\tilde{H}} &= -\nabla_{\mathcal{R}_\alpha} \nabla^\xi \tilde{H} \\ &= -\nabla_{\mathcal{R}_\alpha} \left(\sum_{i=1}^{2n-2} \partial_i \tilde{H} \partial_i \right) \\ &= -\sum_{i=1}^{2n-2} \partial_\alpha \partial_i \tilde{H} \partial_i - \sum_{i=1}^{2n-2} \partial_i \tilde{H} \nabla_{\mathcal{R}_\alpha} \partial_i, \end{aligned}$$

where $\partial_i = e_i$ is our frame for ξ; and we write $\partial_\alpha := \mathcal{R}_\alpha$. The second sum in the last line is zero by property 5 from Step 2 (note that this property uses our assumption that $(\xi, \mathrm{d}\alpha)$ is symplectically trivial).

$$\begin{aligned} \text{Meanwhile,} \quad -\mathrm{d}^\xi \tilde{G} &= -\sum_{i=1}^{2n-2} \partial_i \tilde{G} \partial_i \\ &= -\sum_{i=1}^{2n-2} \partial_i(\mathrm{d}\tilde{H}(\mathcal{R}_\alpha))\partial_i \\ &= -\sum_{i=1}^{2n-2} \partial_i \partial_\alpha \tilde{H} \partial_i = -\sum_{i=1}^{2n-2} \partial_\alpha \partial_i \tilde{H} \partial_i. \end{aligned}$$

by using the fact that $\tilde{H}$ is $\mathcal{C}^2$ in the last line, in order to commute the partial derivatives. This verifies that $J\nabla_{\mathcal{R}_\alpha} X^\xi$ and $-\mathrm{d}^\xi G$ are dual under the metric $\langle\cdot,\cdot\rangle$, which is what we wanted to prove.

$\therefore$ This concludes the proof that JL_1 is symmetric, and thus that $L_1 \in \mathfrak{sp}_{2n}$.

B. The proof that $L_0 \in \mathfrak{sp}_{2n}$ is then quite straightforward. Indeed, $L_0 = \nabla X_{\widehat{H}} - (r-1)L_1$, and we have already argued in **A. i)** above that matrices of the form $\nabla X_{\widehat{H}}$ belonged to $\mathfrak{sp}_{2n}$. Since the latter is a vector space, we automatically have $L_0 \in \mathfrak{sp}_{2n}$.

We can still make one more observation. Recall that L_0 has the form:

$$L_0 = \left(\begin{array}{c|cc} {}^F\nabla^\xi \mathcal{R}_\alpha & X^\xi & \begin{matrix}0\\0\end{matrix} \\ \hline 0 \quad 0 & a & 0 \\ \mathrm{d}F|_\xi & b & c \end{array}\right).$$

First, for the same reason as above, we observe that the top-left block belongs to $\mathfrak{sp}_{2n-2}$. Moreover, since we now know that JL_0 is symmetric, we can write it in simplified form:

$$L_0 = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{{}^F\nabla^\xi \mathcal{R}_\alpha} & J\begin{pmatrix}U\\V\end{pmatrix} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & a & 0 \\ U^t & V^t & b & -a \end{array}\right).$$

Step 5 (Conclusion of the proof).

The conclusion is given in the proof of Proposition 5.10, in §5.2.2. □

Calculation C.14. We prove the properties of ∇, from Step 2 of Calculation C.12. Recall that we use a metric η with $\eta|_\xi = \mathrm{d}\alpha(\cdot, J\cdot)$, and $\eta_{rr} = 1$ and $\eta_{\alpha\alpha} = 1$. In Einstein notation, the Christoffel symbols are given by:

$$\Gamma^k_{ij} = \frac{1}{2}\eta^{ka}\left(\partial_i \eta_{ja} + \partial_j \eta_{ia} - \partial_a \eta_{ij}\right).$$

Recall that, since ∇ is the Levi-Civita connection associated to η, we by definition have $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$, where

$$\{e_1, \ldots, e_{2n}\} = \{\partial_1, \ldots, \partial_{2n-2}, \partial_r, \mathcal{R}_\alpha\}$$

is a frame for $T(\widehat{W} \backslash \mathrm{int}(W))$ (with $\{\partial_i\}$ a frame for ξ).

1. We want to show that $\nabla \partial_r = 0$. In other words, we want to show that $\Gamma^k_{ir} = 0$ for all indices r, k. We have:

$$\begin{aligned} \Gamma^k_{ir} = \frac{1}{2}\Gamma^k_{ij} &= \frac{1}{2}\eta^{ka}\left(\partial_i \eta_{ra} + \partial_r \eta_{ia} - \partial_a \eta_{ir}\right) \\ &= \frac{1}{2}\eta^{kk}\left(\partial_i \eta_{rk} + \partial_r \eta_{ik} - \partial_k \eta_{ir}\right). \end{aligned}$$

   If $k = r$, then this is 0 for all i, since $\eta_{rr} = 1$. If $k = \alpha$, then it is also 0, since $\eta_{\alpha\alpha}$ does not depend on r, and η_{rr} does not depend on α. Finally, if $k \leq 2n-2$, then $\Gamma^k_{ij} = 0$, since $\eta|_\xi$ does not depend on r, and η_{rr} does not depend on ξ. So $\nabla \partial_r = 0$.

2. Similarly, let us look at $\Gamma^k_{\alpha\alpha} = \frac{1}{2}g^{kk}\left(2\partial_\alpha \eta_{\alpha k} - \partial_k \eta \alpha\alpha\right)$. Since $\eta_{\alpha\alpha} = 1$, this is always zero, ensuring that $\nabla_{\mathcal{R}\alpha} \mathcal{R}_\alpha = 0$. We have:

$$\Gamma^k_{r\alpha} = \frac{1}{2}\eta^{kk}\left(\partial_r \eta_{\alpha k} + \partial_\alpha \eta_{rk} - \partial_k \eta_{r\alpha}\right).$$

   which is always zero since $\eta_{rr} = \eta_{\alpha\alpha} = 1$. So $\nabla_{\partial_r} \mathcal{R}_\alpha$.

3. We want to show that if Z is a vector in ξ, then $\nabla_Z \mathcal{R}_\alpha \in \xi$. In other words, we want to show that if $i \leq 2n-2$, and $k > 2n-2$, then $\Gamma^k_{i\alpha} = 0$ (indeed, recall that $\{e_i\}$ for $i \leq 2n-2$ is a frame for ξ).

First, observe that

$$\Gamma^r_{i\alpha} = \frac{1}{2}\eta^{rr}\left(\partial_i \eta_{\alpha r} + \partial_\alpha \eta_{ir} - \partial_r \eta_{i\alpha}\right) = 0,$$

and then:

$$\Gamma^\alpha_{i\alpha} = \frac{1}{2}\eta^{\alpha\alpha}\left(\partial_i \eta_{\alpha\alpha} + \partial_\alpha \eta_{\alpha i} - \partial_\alpha \eta_{\alpha i}\right) = 0,$$

since $\eta_{\alpha\alpha} = 1$; which proves the statement.

4. Finally, we want to show that $\nabla_{\partial_r} Z = 0$ for every $Z \in \xi$. This would require us showing that $\Gamma^k_{ri} = 0$ for every k (and fixing $i \leq 2n - 2$). However, we have already shown, in step 1, that $\Gamma^k_{ir} = 0$ for every i and k. Since the Levi-Civita connection is symmetric, we are done.

5. Finally, we want to show that $\nabla_{\mathcal{R}_\alpha} e_i$ is 0, where $\{e_i\}$ is a trivialising frame for $(\xi, \mathrm{d}\alpha)$. Recall that this implies that the metric $\eta := \mathrm{d}\alpha(\cdot, J\cdot)$ is the standard metric $\eta = e^i \otimes e^i$, where e^i is the dual of e_i. Then:

$$\Gamma^k_{\alpha i} = \frac{1}{2}\eta^{kk}\left(\partial_\alpha \eta_{ik} + \partial_i \eta_{\alpha k} - \partial_k \eta_{\alpha i}\right).$$

If $k = \alpha$, then this is $\frac{1}{2}\eta^{\alpha\alpha}\partial_i \eta_{\alpha\alpha} = 0$, and if $k = i$, then this is $\frac{1}{2}\eta^{ii}\partial_\alpha \eta_{ii} = 0$, by assumption. □

Bibliography

[AB24] Cengiz Aydin and Alexander Batkhin. *Studying network of symmetric periodic orbit families of the Hill problem via symplectic invariants*. 2024. arXiv: 2410.21245 [math.DS].

[AD13] Michèle Audin and Mihai Damian. *Morse Theory and Floer Homology*. Universitext. Springer London, 2013. ISBN: 978-1-4471-5495-2. DOI: https://doi.org/10.1007/978-1-4471-5496-9.

[AH19] Casim Abbas and Helmut Hofer. *Holomorphic Curves and Global Questions in Contact Geometry*. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser Cham, 2019. ISBN: 978-3-030-11802-0. DOI: https://doi.org/10.1007/978-3-030-11803-7.

[al06] C.C. Porco et al. "Cassini Observes the Active South Pole of Enceladus". In: *Science* 311 (2006), pp. 1393–1401.

[Alb+12a] Peter Albers, Joel W. Fish, Urs Frauenfelder, Helmut Hofer, and Otto van Koert. "Global Surfaces of Section in the Planar Restricted 3-Body Problem". In: *Arch. Rational Mech. Anal.* 204 (April 2012 2012), pp. 273–284. DOI: 10.1007/s00205-011-0475-2. URL: https://doi.org/10.1007/s00205-011-0475-2.

[Alb+12b] Peter Albers, Urs Frauenfelder, Otto Van Koert, and Gabriel P. Paternain. "Contact geometry of the restricted three-body problem". In: *Communications on Pure and Applied Mathematics* 65 (2 2012). First published: 20 October 2011, pp. 229–263. DOI: 10.1002/cpa.21380.

[APS08] A. Abbondandolo, A. Portaluri, and M. Schwarz. "The homology of path spaces and Floer homology with conormal boundary conditions". In: *J. Fixed Point Theory Appl.* 4 (2008), pp. 263–293. DOI: https://doi.org/10.1007/s11784-008-0097-y.

[AS04] Alberto Abbondandolo and Matthias Schwarz. "On the Floer Homology of Cotangent Bundles". In: *Communications on Pure and Applied Mathematics* 67 (2004), pp. 670–691.

[AS10] Mohammed Abouzaid and Paul Seidel. "An Open String Analogue of Viterbo Functoriality". In: *Geometry Topology* 14.2 (2010), pp. 627–718. DOI: http://dx.doi.org/10.2140/gt.2010.14.627.

[Ayd+24a] Cengiz Aydin, Urs Frauenfelder, Otto van Koert, Dayung Koh, and Agustin Moreno. *Symplectic geometry and space mission design*. 2024. arXiv: 2308.03391 [math.SG].

[Ayd+24b] Cengiz Aydin, Agustin Moreno, Otto van Koert, Urs Frauenfelder, and Dayung Koh. "Bifurcation Graphs for the CR3BP via Symplectic Methods: On the Jupiter–Europa and Saturn–Enceladus Systems". In: *The Journal of the Astronautical Sciences* 71.51 (2024). DOI: 10.1007/s40295-024-00462-7. URL: https://doi.org/10.1007/s40295-024-00462-7.

[Ayd23a] Cengiz Aydin. "A Study of the Hill Three-Body Problem by Modern Symplectic Geometry". PhD Thesis. Université Augsburg, Deutschland: Institut de mathématiques, 2023, p. 98.

[Ayd23b] Cengiz Aydin. "From Babylonian lunar observations to Floquet multipliers and Conley–Zehnder indices". In: *J. Math. Phys.* 64 (2023), p. 082902. DOI: https://doi.org/10.1063/5.0156959.

[BCS24] Filip Broćić, Dylan Cant, and Egor Shelukhin. *The chord conjecture for conormal bundles*. 2024. arXiv: `2401.08842 [math.SG]`.

[Bir13] George D. Birkhoff. "Proof of Poincaré's Geometric Theorem". In: *Transactions of the American Mathematical Society* 14.1 (1913), pp. 14–22. DOI: `https://doi.org/10.2307/1988766`.

[Bir15] G. D. Birkhoff. "The Restricted Problem of Three Bodies". In: *Rendiconti del Circolo Matematico di Palermo* 39 (1915), pp. 265–334. DOI: `10.1007/BF03015982`. URL: `https://doi.org/10.1007/BF03015982`.

[Bla62] J-C. Blaive. "Le dernier théorème de Poincaré et le problème restreint des trois corps". In: *Séminaire Janet. Mécanique analytique et mécanique céleste* (1962). talk:14.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in Algebraic Topology*. Springer-Verlag, 1982. DOI: `https://doi.org/10.1007/978-1-4757-3951-0`.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds*. American Mathematical Society, 2012. URL: `https://opus.bibliothek.uni-augsburg.de/opus4/frontdoor/deliver/index/docId/37288/file/stein105.pdf`.

[Che85] A. Chenciner. *Le problème de la lune et la théorie des systèmes dynamiques*. Unpublished notes, Université Paris 7. 1985.

[Cie+96] Kai Cielebak, Andreas Floer, Helmut Hofer, and Kris Wysocki. "Applications of symplectic homology II: Stability of the action spectrum." In: *Mathematische Zeitschrift* 223.1 (1996), pp. 27–46. URL: `http://eudml.org/doc/174909`.

[CJK20] W. Cho, H. Jung, and G. Kim. "The contact geometry of the spatial circular restricted 3-body problem". In: *Abh. Math. Semin. Univ. Hambg.* 90 (2020). First published on Oct. 13 2018, pp. 161–181. DOI: `https://doi.org/10.1007/s12188-020-00222-y`.

[Dev16] S. Devalapurkar. *The (co)homology of the loop space of a n-sphere*. 2016. URL: `https://erdosninth.wordpress.com/2016/12/02/the-cohomology-of-the-loop-space-of-a-n-sphere/`.

[Dor+24] P.T. Doran et al. "The COSPAR planetary protection policy for missions to Icy Worlds: A review of history, current scientific knowledge, and future directions". In: *Life Sciences in Space Research* 41 (May 2024), pp. 86–99.

[Far66] Robert W. Farquhar. "Station-Keeping in the Vicinity of Collinear Libration Points with an Application to a Lunar Communications Problem". In: *AAS Science and Technology Series: Space Flight Mechanics Specialist Symposium*. Vol. 11. 1966, pp. 519–535.

[FK18] Urs Frauenfelder and Otto van Koert. *The Restricted Three-Body Problem and Holomorphic Curves*. Pathways in Mathematics. Birkhäuser Cham, 2018. ISBN: 978-3-319-72277-1. DOI: `https://doi.org/10.1007/978-3-319-72278-8`.

[FKM23] Urs Frauenfelder, Dayung Koh, and Agustin Moreno. "Symplectic Methods in the Numerical Search of Orbits in Real-Life Planetary Systems". In: *SIAM Journal on Applied Dynamical Systems* 22.4 (2023). DOI: `10.1137/22M1500459`. URL: `https://doi.org/10.1137/22M1500459`.

[Flo88a] A. Floer. "Morse theory for Lagrangian intersections". In: *Journal of Differential Geometry* (1988). DOI: `10.4310/JDG/1214442477`.

[Flo88b] Andreas Floer. "The unregularized gradient flow of the symplectic action". In: *Communications on Pure and Applied Mathematics* 41 (6 1988), pp. 775–813. DOI: `https://doi.org/10.1002/cpa.3160410603`.

[Flo89a] Andreas Floer. "Symplectic fixed points and holomorphic spheres". In: *Comm. Math. Phys.* 120.4 (1989), pp. 575–611.

[Flo89b] Andreas Floer. “Witten’s complex and infinite-dimensional Morse theory”. In: *J. Differential Geom.* 30.1 (1989), pp. 207–221. DOI: 10.4310/jdg/1214443291.

[FZ19] U. Frauenfelder and L. Zhao. “Existence of either a periodic collisional orbit or infinitely many consecutive collision orbits in the planar circular restricted three-body problem”. In: *Mathematische Zeitschrift* 291 (2019), pp. 215–225. DOI: 10.1007/s00209-018-2080-7.

[FZ20] Urs Frauenfelder and Lei Zhao. “Transverse Regularizations of Central Force Problems by Hamiltonian Structure”. In: *Proceedings of the International Consortium of Chinese Mathematicians, Second Annual Meeting, 27–29 December 2018, Taipei, Taiwan.* Ed. by Lizhen Ji and Shing-Tung Yau. International Press of Boston, 2020, pp. 539–552.

[Gao17] Yuan Gao. *Wrapped Floer cohomology and Lagrangian correspondences.* 2017. arXiv: 1703.04032 [math.SG].

[GG10] Viktor L. Ginzburg and Basak Z. Gurel. “Local Floer Homology and the Action Gap”. In: *Journal of Symplectic Geometry* 8 (3 2010), pp. 1–34. DOI: 10.4310/JSG.2010.v8.n3.a4.

[GG15] V.L. Ginzburg and B.Z. Gürel. “The Conley Conjecture and Beyond”. In: *Arnold Math J.* 1 (2015), pp. 299–337. DOI: 10.1007/s40598-015-0017-3.

[Gin10] Viktor L. Ginzburg. “The Conley conjecture”. In: *Annals of Mathematics* 172.2 (2010), pp. 1127–1180.

[Gol78] Martin Golubitsky. “An Introduction to Catastrophe Theory and its Applications”. In: *SIAM Review* 20.2 (1978). 36 pages, Published: April 1978, pp. 352–387.

[Gro85] M. Gromov. “Pseudo holomorphic curves in symplectic manifolds”. In: *Inventiones Mathematicae* 82 (1985), pp. 307–347. DOI: 10.1007/BF01388806. URL: https://doi.org/10.1007/BF01388806.

[GS19] S. Ghotekar and R.K. Sharma. “Halo Orbits in the Photo-Gravitational Restricted Three-Body Problem”. In: *International Journal of Astronomy and Astrophysics* 9 (2019), pp. 274–291. DOI: 10.4236/ijaa.2019.93020. URL: https://doi.org/10.4236/ijaa.2019.93020.

[Hao+22] Jihua Hao, Christopher R. Glein, Fang Huang, and et al. “Abundant phosphorus expected for possible life in Enceladus’s ocean”. In: *Proceedings of the National Academy of Sciences* 119.39 (2022). Edited by Norman Sleep, Stanford University, Stanford, CA, e2201388119. DOI: 10.1073/pnas.2201388119.

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[Hof21] Helmut Hofer. *The Floer Jungle: 35 years of Floer theory.* Talk presented at the Joint IAS/Princeton/Montreal/Paris/Tel-Aviv Symplectic Geometry Seminar. July 16, 2021. 2021.

[HSW22] Umberto L. Hryniewicz, Pedro A. S. Salomão, and Krzysztof Wysocki. “Genus zero global surfaces of section for Reeb flows and a result of Birkhoff”. In: *Journal of the European Mathematical Society* 25.9 (2022). DOI: 10.4171/JEMS/1220.

[Int21] M. Introna. “A Study on the Regularization of Collisions in the n-Body Problem (Master’s thesis)”. In: (2021).

[Jai21] P. Jaine. “Splitting free loop spaces”. In: (2021). URL: https://math.berkeley.edu/~phaine/files/Splitting_free_loops.pdf.

[Kim18] Joontae Kim. “Wrapped Floer Homology of Real Lagrangians and Volume Growth of Fibered Twists”. PhD thesis. Seoul National University, 2018. URL: https://s-space.snu.ac.kr/bitstream/10371/141151/1/000000150867.pdf.

[KK16] Myeonggi Kwon and Otto van Koert. “Brieskorn manifolds in contact topology”. In: *Bulletin of the London Mathematical Society* 48.2 (2016), pp. 173–241. DOI: 10.1112/blms/bdv088.

[Koe17] Otto van Koert. “Lecture notes on stabilization of contact open books”. In: *Münster Journal of Mathematics* 10 (2017). URN: urn:nbn:de:hbz:6-70299611015, pp. 425–455. DOI: 10.17879/70299609615. URL: https://www.uni-muenster.de/MathematicsMuenster/journal/.

[Kum82] M. Kummer. “On the regularization of the Kepler problem”. In: *Communications in Mathematical Physics* 84 (1982), pp. 133–152. DOI: 10.1007/BF01208375.

[Lee12] John M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, 2012. DOI: https://doi.org/10.1007/978-1-4419-9982-5.

[LM25] Arthur Limoge and Agustin Moreno. *A Poincaré–Birkhoff theorem for C^0-Hamiltonian maps*. 2025. arXiv: 2506.10545 [math.SG].

[Mit01] S. Mitchell. “Notes on Serre fibrations”. In: (2001). URL: https://sites.math.washington.edu/~mitchell/Atopc/serre.pdf.

[MK22a] Agustin Moreno and Otto van Koert. “A generalized Poincaré–Birkhoff theorem”. In: *Journal of Fixed Point Theory and Applications* 24 (2022). Published: 08 April 2022. DOI: 10.1007/s11784-022-00937-6.

[MK22b] Agustin Moreno and Otto van Koert. “Global hypersurfaces of section in the spatial restricted three-body problem”. In: *Nonlinearity* 35.6 (2022). Published: 27 May 2022, p. 2920. DOI: 10.1088/1361-6544/ac692b.

[ML24] Agustin Moreno and Arthur Limoge. *A Relative Poincaré-Birkhoff theorem*. 2024. arXiv: 2408.06919 [math.SG].

[ML25] Agustin Moreno and Arthur Limoge. “Bi-normal trajectories in the circular restricted three-body problem”. In: *Journal of Fixed Point Theory and Applications* 27 (2025). DOI: 10.1007/s11784-025-01203-5. URL: https://doi.org/10.1007/s11784-025-01203-5.

[Mor24] Agustin Moreno. *The symplectic geometry of the three-body problem*. 2024. arXiv: 2101.04438 [math.SG].

[Mos76] J. Moser. “Periodic orbits near an equilibrium and a theorem by Alan Weinstein”. In: *Communications on Pure and Applied Mathematics* 29 (1976), pp. 724–747.

[MS04] D. McDuff and D. Salamon. *J-Holomorphic Curves and Symplectic Topology*. Vol. 52. American Mathematical Society Colloquium Publications. Providence: American Mathematical Society, 2004. DOI: https://doi.org/10.1090/coll/052.

[New87] Isaac Newton. *Philosophiæ Naturalis Principia Mathematica*. Auctore Js. Newton, 1687. DOI: 10.3931/e-rara-440.

[Nol19] David D. Nolte. *The Three-Body Problem, Longitude at Sea, and Lagrange’s Points*. Blog post on *Galileo Unbound*. Accessed via https://galileo-unbound.blog/2019/07/05/the-three-body-problem-longitude-at-sea-and-lagranges-points/. July 2019.

[Oh96] Yong-Geun Oh. “Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings”. In: *International Mathematics Research Notices* 1996 (7 1996), pp. 305–346. DOI: 10.1155/S1073792896000219.

[Poi12] H. Poincaré. “Sur un théorème de géométrie”. In: *Rend. Circ. Matem. Palermo* 33 (1912), pp. 375–407. DOI: https://doi.org/10.1007/BF03015314.

[Poi90] Henri Poincaré. “Sur le problème des trois corps et les équations de la dynamique”. In: *Acta Mathematica* 13.1-2 (1890).

[Rit13] Alexander F. Ritter. “Topological Quantum Field Theory Structure on Symplectic Cohomology”. In: *Journal of Topology* (2013). DOI: https://doi.org/10.1112/jtopol/jts038.

[Ruc23] Kevin Ruck. “Lagrangian Rabinowitz Floer Homology and its Application to Powered Flyby Orbits in the Restricted Three Body Problem”. PhD thesis. Augsburg: Universität Augsburg, 2023.

[See64] Robert Thomas Seeley. "Extension of C^∞ functions defined in a half space". In: *Proceedings of the American Mathematical Society* 15 (1964), pp. 625–626.

[Smi+81] B. A. Smith, L. Soderblom, R. F. Beebe, and et al. "Encounter with Saturn: Voyager 1 Imaging Science Results". In: *Science* 212.4491 (1981), pp. 163–191. DOI: 10.1126/science.212.4491.163.

[Swi02] Robert M. Switzer. *Algebraic Topology - Homotopy and Homology*. Classics in Mathematics. Springer Berlin, Heidelberg, 2002. ISBN: 978-3-540-42750-6. DOI: https://doi.org/10.1007/978-3-642-61923-6.

[Tho+16] P. C. Thomas et al. "Enceladus's Measured Physical Libration Requires a Global Subsurface Ocean". In: *Icarus* 264 (2016), pp. 37–47. DOI: 10.1016/j.icarus.2015.08.037. URL: https://doi.org/10.1016/j.icarus.2015.08.037.

[TMT20] J. Tjoa, M. Mueller, and F. F. S. van der Tak. "The subsurface habitability of small, icy exomoons". In: *Astronomy Astrophysics* 642 (2020), A38. DOI: 10.1051/0004-6361/201937035. URL: https://doi.org/10.1051/0004-6361/201937035.

[Ust99] Ilya Ustilovsky. "Contact homology and contact structures on S^{4m+1}". PhD thesis. Stanford University, 1999. URL: https://s-space.snu.ac.kr/bitstream/10371/141151/1/000000150867.pdf.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994. DOI: https://doi.org/10.1017/CBO9781139644136.

[Wen15] C. Wendl. *Lectures on Holomorphic Curves in Symplectic and Contact Geometry*. 2015. URL: https://www.mathematik.hu-berlin.de/~wendl/pub/jhol_bookv33.pdf.

[Wen16] C. Wendl. *Lecture notes on Differential Geometry*. 2016.

[Yag24] Kazuyuki Yagasaki. "Non-integrability of the restricted three-body problem". In: *Ergodic Theory and Dynamical Systems* 44.10 (2024), pp. 3012–3040. DOI: 10.1017/etds.2024.4. URL: https://doi.org/10.1017/etds.2024.4.

[YSM13] D. G. Yárnoz, J. P. Sánchez, and Colin R. McInnes. "Easily Retrievable Objects among the NEO Population". In: *Celestial Mechanics and Dynamical Astronomy* 116 (2013), pp. 367–388. DOI: 10.1007/s10569-013-9495-6. URL: https://doi.org/10.1007/s10569-013-9495-6.

[Zil77] W. Ziller. "The free loop space of globally symmetric spaces". In: *Invent Math* 41 (1977), pp. 1–22. DOI: https://doi.org/10.1007/BF01390161.