Dissertation

submitted to the
Combined Faculties for the Natural Sciences and for Mathematics
of the Ruperto–Carola University of Heidelberg, Germany
for the degree of
Doctor of Natural Sciences

presented by

Diplom–Physiker Jörg Jäckel
born in Darmstadt

Oral examination: May 14, 2003

Effective Actions for Strongly Interacting Fermionic Systems

Referees: Prof. Dr. Christof Wetterich
Prof. Dr. Michael G. Schmidt

Effektive Wirkungen für stark wechselwirkende fermionische Systeme

Zusammenfassung

Wir vergleichen verschiedene nicht-störungstheoretische Methoden zur Beschreibung fermionischer Systeme, die gebundene bosonische Zustände (BBS) und spontane Symmetriebrechung (SSB) aufweisen. In einer rein fermionischen Sprache erfordert das Eindringen in die SSB Phase Techniken jenseits von Störungstheorie und Renormierungsgruppengleichungen. Dazu ist eine Beschreibung, die BBS und elementare Teilchen gleichberechtigt behandelt, besser geeignet. Die "Partielle Bosonisierung" führt aber zu einer Willkür in der Wahl der BBS Felder, da diese durch die klassische Wirkung nicht eindeutig festgelegt ist. Die Ergebnisse approximativer Rechnungen, z.B. mean field theory, können aber von dieser Wahl abhängen. Dies beschränkt die quantitative Aussagekraft. Am Beispiel des Nambu–Jona-Lasinio-Modells zeigen wir, wie diese Abhängigkeit durch geeignet gewählte Approximationen reduziert und manchmal sogar zum Verschwinden gebracht werden kann.

Schwinger-Dyson-Gleichungen (SDE) erlauben eine Beschreibung von SSB ohne Hilfsfelder. Die 2PI-Wirkung ermöglicht es uns, verschiedene Lösungen der SDE zu vergleichen und so die stabile zu finden. Diese Methode wenden wir auf eine sechs-Fermion Wechselwirkung an, die der drei-Flavor-Instantonwechselwirkung der QCD ähnelt. Wir finden einen Phasenübergang erster Ordnung in die chiral gebrochene Phase, aber keine stabile Phase mit gebrochener color-Symmetrie.

Die Existenz eines elementaren skalaren bosonischen Teilchens im Standardmodell – dem Higgs – führt zu mehreren Fragen. Die mit fundamentalen Skalen ($\sim M_{\mathrm{GUT}}$) verglichen kleine Masse erfordert ein extremes Maß an Finetuning. Außerdem ist das ϕ^4-Potential möglicherweise nicht renormierbar im strengen Sinne. Im Hinblick darauf diskutieren wir die Möglichkeit, daß das Higgs ein BBS aus Fermionen ist.

Effective Actions for Strongly Interacting Fermionic Systems

Abstract

We compare different non-perturbative methods for calculating the effective action for fermionic systems featuring bosonic bound states (BBS) and spontaneous symmetry breaking (SSB). In a purely fermionic language proceeding into the SSB phase requires techniques beyond perturbation theory and renormalization group equations. Improvement comes from a description with BBS fields and elementary fields treated on equal footing. Yet, "partial bosonization" introduces an arbitrariness as the choice for the composite fields is usually not completely determined by the classical action. Results of approximate calculations, e.g. mean field theory, may depend strongly on this choice, thus limiting their quantitative reliability. Using the Nambu–Jona-Lasinio model as an example we demonstrate how this dependence can be reduced, sometimes even be eliminated by suitably chosen approximations.

Schwinger-Dyson equations (SDE) allow for a description of SSB without auxiliary fields. The 2PI effective action enables us to compare different solutions of the SDE and find the stable one. We apply this method to a six-fermion interaction resembling the three-flavor instanton interaction in QCD. We find a first order chiral phase transition but no stable phase with broken color symmetry.

The existence of an elementary scalar boson in the Standard Model – the Higgs – raises several questions. The smallness of its mass compared to some fundamental scale ($\sim M_{\mathrm{GUT}}$) requires an extreme amount of fine-tuning. Moreover, its ϕ^4-potential may not be renormalizable in a strict sense. In view of this we discuss the possibility of a Higgs as BBS of fermions.

Contents

Chapter 1

Introduction

There are no eternal facts, as there are no absolute truths.

Friedrich Nietzsche

1.1 Bosons Made up of Fermions

Everything is Made of Fermions?

The search for "fundamental" particles is one of the most ambitious enterprizes in physical research. While it has been very successful in terms of uncovering new particles it has also led us to question over and over again what "fundamental" really means. The notion "fundamental particle" has changed with time. First it was atoms then it was electrons and nuclei and later the latter ones were split into protons and neutrons. Today, we are quite sure that even neutrons and protons are made up of the more fundamental quarks. What once had been fundamental particles became bound states.

Looking at Fig. 1.1 we can see that this evolution is simply like turning up the resolution of a microscope. At a low resolution we see nothing but a point (particle) while at a higher resolution it exhibits structure, i.e. we can see that it is composed of other particles. In particle physics the "microscope" is a scattering experiment and the resolution improves with smaller wavelength $\lambda \sim \frac{\hbar c}{E}$, and therefore higher energy E of the scattering particle. Consequently, with progress in accelerator physics, it might turn out that some (or even all) of the particles of the Standard Model are not fundamental, but bound states.

What has all that to do with fermions? At the moment we are in a very peculiar situation. Classifying particles by spin (S) and statistics all particles of the

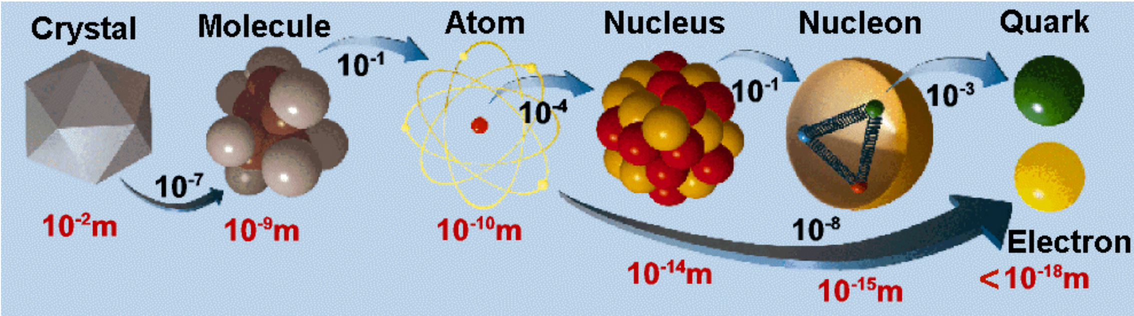


Figure 1.1: With the exploration of smaller and smaller scales it often became apparent that particles which were thought to be fundamental are instead composed of even smaller particles. Nevertheless, for a description at a given scale it is often useful to treat bound states as "fundamental" particles, e.g. for a first description of water vapor it is a very good approximation to treat the water molecules as fundamental, i.e. we use an "effective theory" in which the water molecules are pointlike and have some kind of effective interaction. However, if we want to describe the absorption and emission of electromagnetic waves it becomes necessary to consider the water molecule as composed of atoms (infrared) and eventually the latter ones to be composed of electrons and a nucleus (visible, ultraviolet). Corresponding to the scale we probe, we have to consider different effective theories. This kind of scale dependent description is one of the main ideas behind the renormalization group (RG) (cf. Sect. 2.1.2). The RG provides us with a means to calculate the couplings of an effective theory at a given scale from the ones of an effective theory valid at a smaller length scale, and ultimately from the fundamental one.

"Standard Model" are either vector bosons with $S = 1$ or fermions with $S = \frac{1}{2}$. The only exception is the Higgs as it is supposed to be a scalar ($S = 0$) boson. Although, recent experiments [1–5] at the final runs of LEP may have detected a Standard Model Higgs (or a supersymmetric extension of it), the evidence is still quite shaky, leaving room for speculations. Being provocative we could state that so far we have not yet observed any fundamental scalar boson. Hence, one could conjecture that the Higgs might be a bound state of fermions, more explicitly a top-antitop bound state [6–10]. In addition to this phenomenological aspect, the idea of "top-quark condensation" might also be a way to circumvent some technical problems, like the triviality of ϕ^4-theory (Higgs potential!) [11–15], paving the way to a "renormalizable" Standard Model.

Spontaneous Symmetry Breaking

The phenomenon of spontaneous symmetry breaking (SSB) and the formation of bosonic bound states in strongly interacting fermionic systems are tightly connected.

First of all, according to Goldstone's theorem [16; 17] it is inevitable to have massless Spin-0 particles, i.e. scalar bosons, if we spontaneously break a continuous global symmetry. In a purely fermionic theory these must be bound states. An example for these bosons are the pseudoscalar mesons of QCD, which are the Goldstone bosons of chiral symmetry breaking.

Secondly, it is impossible for a fermionic (Grassmannian) field to acquire a non-vanishing vacuum expectation value. Therefore, the simplest possible symmetry-breaking term is a bosonic operator made up of two fermions. Thus, even for purely fermionic theories, the phenomenon of SSB is characterized by bosons.

Physically, we can imagine a Mexican-hat-type potential for the bosonic composite operator. Depending on whether we have a local or a global symmetry the Goldstone bosons, corresponding to the angular excitations, may or may not be eaten up by gauge bosons. But, in addition to those we always have the radial excitations, corresponding to (usually) massive bosons. E.g. in the top condensation model this would be the Higgs boson. So, at this level, there is really no difference to a model with elementary bosons.

Yet, there is a slight difference in the way the Mexican-hat is generated. For elementary bosons we quite often simply choose the potential to be a Mexican-hat, whereas for fermions the generic case is that the non-trivial minima are generated dynamically by quantum fluctuations. That is why it is often referred to as dynamical symmetry breaking.

Finally, we note that the case where the described radial boson becomes massless corresponds to a second order phase transition. With features like universality these special cases are especially interesting.

Last but not least, of course, there can be bound states not directly linked to SSB, e.g. the hydrogen atom or positronium.

A lot of Bound States – Some Models

Of course, a speculative model of the Higgs and chiral symmetry breaking are not the only situations where we encounter the mechanisms described above.

Color superconductivity: At very high density it is expected that the QCD ground state is a color superconducting phase [18–26]. Depending on the specifics of temperature, number of flavors etc. there are several different phases. Let us just mention the color flavor locking phase characterized by a non-vanishing expectation value of[1] $\langle\psi^{a}_{\mathrm{L}i}\psi^{b}_{\mathrm{L}j}\rangle \sim (\delta_{ia}\delta_{bj} - \frac{1}{\mathrm{N_c}}\delta_{ij}\delta_{ab})$ as an example. In this phase the $SU(3)_{\mathrm{color}} \times SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}}$ is broken down to a vectorlike $SU(3)_{V}$. Breaking global (chiral)

[1] L refers to left handed, R to right handed.

as well as local (color) symmetries, we have Goldstone bosons as well as massive gauge bosons (gluons).

Color symmetry breaking in the vacuum: It was conjectured [27; 28] that the QCD ground state at zero density might also have a broken color symmetry. In this case the order parameter is $\langle \overline{\psi}^{a}_{\mathrm{L}i} \psi^{b}_{\mathrm{R}j} \rangle$ with the same type of expectation value and symmetry breaking pattern as for the color flavor locking phase in color superconductivity. It is worth noting that the massive gauge bosons can then be associated to the vector mesons of QCD. We will investigate this possibility briefly in Chap. 6.

Chiral symmetry breaking: This is probably one of the most studied cases of SSB [29]. A vacuum expectation value $\langle \overline{\psi}^{a}_{\mathrm{L}i} \psi^{b}_{\mathrm{R}i} \rangle \sim \delta_{ab}$ gives a mass to the (nearly) massless quarks. The pseudoscalar mesons (pions etc.) bear physical witness of this process.

Superconductivity: Yeah, just plain old ordinary superconductivity [30] in ordinary matter like metals at temperatures of some Kelvin. It comes about due to the formation of Cooper pairs and the condensation a non-vanishing $\langle \psi\psi \rangle$ where ψ is now an ordinary electron field. This gives us an example where we are at a scale of some meV instead of several hundred MeV (chiral SSB) or even GeV (Higgs model).

1.2 Describing Bound States

As there are plenty of systems featuring bosonic bound states we better start looking for ways to calculate something useful. Interesting quantities are, of course, masses and couplings of the bosons, and a potential vacuum expectation value.

A very useful tool to study such quantities is the effective action [17; 31; 32], replacing the action of classical field theory, it allows for a simple description of SSB, yet it includes quantum effects. Hence, this is the object we would like to calculate.

The NJL model

Many of the problems associated with the description of bound states can be studied already in a very simple NJL[2]-type model (for only one fermion species) with a

[2]NJL stands for Nambu and Jona-Lasinio who used this model to study chiral symmetry breaking [29]. Due to its simplicity this and similar models are still very popular, e.g. a sizable part of the studies on color superconductivity is based on this model [18–22; 24; 26].

chirally invariant pointlike four-fermion interaction:

$$\mathrm{S_F} = \int d^4x \; \Big\{ \; \bar{\psi} i\partial\!\!\!/ \psi \tag{1.1}$$
$$+ \; \frac{1}{2}\lambda_\sigma[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2] - \frac{1}{2}\lambda_V[(\bar{\psi}\gamma^\mu\psi)^2] - \frac{1}{2}\lambda_A[(\bar{\psi}\gamma^\mu\gamma^5\psi)^2]\Big\}.$$

Depending on the value of $\overrightarrow{\lambda} = (\lambda_\sigma, \lambda_V, \lambda_A)$ we are in a symmetric phase or in a phase with broken chiral symmetry. As it turns out, the critical $\overrightarrow{\lambda}_{\mathrm{crit}}$ separating these two phases is an interesting but relatively easy to calculate quantity.

Arbitrary Parameters – Fierz Ambiguity

The simplest calculation which comes into ones mind is probably a mean field calculation. We will apply this method (well-known from statistical physics) to the model Eq. (1.1) at the beginning of Chap. 3. We find that there is a basic ambiguity connected to the possibility to perform Fierz transformations (FT) on the initial Lagrangian – we will refer to it as Fierz ambiguity. This Fierz ambiguity can influence the value of the critical coupling quite dramatically, severly limiting the applicability of MFT [33; 34].

The origin of the Fierz ambiguity can be understood quite well when looking at the model of Eq. (1.1). Due to the Fierz identity (s. App. B for our conventions on γ-matrices)

$$\left[(\bar{\psi}\gamma^\mu\psi)^2 - (\bar{\psi}\gamma^\mu\gamma^5\psi)^2\right] + 2\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2\right] = 0 \tag{1.2}$$

only two of the quartic couplings are independent and we write

$$\lambda_\sigma = \bar{\lambda}_\sigma + 2\gamma\bar{\lambda}_V, \quad \lambda_V = (1-\gamma)\bar{\lambda}_V, \quad \lambda_A = \gamma\bar{\lambda}_V. \tag{1.3}$$

Where γ parametrizes the "symmetry", and the $\bar{\lambda}$ are "invariant" couplings.

In a naive way one would like to combine the fermions in the four-fermion interaction of Eq. (1.1) into pairs and interpret those as bosons, e.g. one would like to take the term multiplying λ_A pair it into two $\bar{\psi}\gamma^\mu\gamma^5\psi \sim A^\mu$ and hence interpret this term as an interaction (mass term) for "axial vector bosons". However, using Eq. (1.2) we can now transform this term to zero, eliminating the "axial vector bosons". As this pairing is the basic idea of MFT, an ambiguity seems inevitable.

Auxiliary Fields

There are several standard methods (perturbation theory, RG equations, Schwinger-Dyson equations (SDE)) which allow us to calculate the critical coupling without any reference to a pairing into bosons. The results are then naturally unambiguous. Yet, these descriptions have their problems, too. Perturbation theory is unable to describe SSB at all, the RG calculation cannot be extended into the broken phase without considerable calculational difficulty and SDE's beyond the simplest approximation become quite difficult to solve, too. At least part of the problem is that we lack an intuitive understanding of the momentum dependence of complicated fermionic operators.

Looking at the example of mesons again it is known that an effective theory with bosonic meson fields interacting with themselves and the quarks works quite well without having very complicated terms in the effective Lagrangian. In addition, bosonic fields allow for a good understanding of SSB. Thus, it seems a reasonable step to introduce auxiliary fields to describe the bound states. Formalizing the naive pairing procedure, partial bosonization [35–39] (cf. Chap. 4) leads to a model with massive bosonic fields and Yukawa-type interactions, but no four-fermion interactions,

$$\begin{aligned}\mathrm{S_B} = \int d^4x \quad \Big\{ \quad & i\overline{\psi}\partial\!\!\!/\psi + \mu_\sigma^2 \phi^\star\phi + \frac{\mu_V^2}{2} V_\mu V^\mu + \frac{\mu_A^2}{2} A_\mu A^\mu \qquad (1.4)\\ & + h_\sigma \left[\overline{\psi} \left(\frac{1+\gamma^5}{2} \right) \phi\psi - \overline{\psi} \left(\frac{1-\gamma^5}{2} \right) \phi^\star \psi \right] - h_V \overline{\psi}\gamma_\mu V^\mu \psi - h_A \overline{\psi}\gamma_\mu\gamma^5 A^\mu \psi \Big\}.\end{aligned}$$

The identification

$$\mu_\sigma^2 = \frac{h_\sigma^2}{2\lambda_\sigma}, \quad \mu_V^2 = \frac{h_V^2}{\lambda_V}, \quad \mu_A^2 = \frac{h_A^2}{\lambda_A} \qquad (1.5)$$

makes this model equivalent to the NJL-type model (1.1).

However, due to the Fierz identity (1.2) the couplings in (1.4) are not unique, bringing back the ambiguity of MFT. Indeed, MFT appears as a simple approximation to this model, neglecting all bosonic fluctuations.

This is not the only situation in physics where an arbitrary parameter (in our case γ) appears in calculations. Prominent examples are the renormalization point μ or the gauge fixing α in gauge theories. Ultimately, in an exact calculation physical quantities should not depend on such a parameter. Nevertheless, approximate calculations usually do. Improvement in the approximation often tends to reduce the dependence on such parameters. Sometimes, there are even special approximation schemes where we can achieve independence of such parameters, e.g. every order of perturbation theory provides such a scheme for the gauge fixing parameter. Beside the practical advantages, finding such an approximation scheme also shows that there is nothing fundamentally wrong with the method in question. From another point of view, considering an approximation which is not independent of the arbitrary parameter, we can say that the spread of the results under a variation of the arbitrary parameter gives us an estimate of the minimal uncertainty of a given approximation.

We will find a very simple example of an approximation which is independent of the Fierz parameter [40; 41]. Interestingly, it turns out that this introduces the concept of scale dependent degrees of freedom. Technically we use the scale dependence of our auxiliary fields to keep the form of the (effective action) simple [42]. However, on a deeper level we would like to interpret this simplicity as a first step to a description with the "right" degrees of freedom at every scale.

Unfortunately, there is a huge number of possible field redefinitions and in simple approximations, e.g. so called local potential approximation (LPA) [43–45], it is a priori not clear which is the "correct" one. A criterium can be obtained only by the consideration of terms with derivatives of the fields. In the end this leaves us to choose between high algebraic (and/or numerical) difficulty and a "physical guess".

2PI Effective Action

In view of the many problems connected with auxiliary fields it seems prudent to look for alternatives. One possibility is the 2PI effective action [46–48]. After the introduction of sources for the composite operators bilinear in the fields we can omit the introduction of auxiliary fields and directly perform an additional Legendre transformation with respect to the sources of the composite operators. For purely fermionic theories it turns out that this description is redundant and the dependence on the propagators is sufficient. Hence, we can omit the dependence on the fields, leaving us with a description completely in terms of bosonic variables (the propagators). Nevertheless, this method is naturally not bothered by the Fierz ambiguity as we do not introduce auxiliary fields. In general the sources in question do not need to be local. However, for an interpretation as a potential this is useful, but even with this restrictions the 2PI effective action has its own problems. Simple approximations are often unbounded from below raising serious questions about the interpretation of the 2PI effective action.

Nevertheless, we do not want to leave this model without at least one not completely trivial application. We will investigate an interaction resembling the three-flavor three-color instanton interaction of QCD. This six-fermion interaction provides for a mechanism to have a first order chiral phase transition. Furthermore, it was conjectured to have a color-symmetry breaking vacuum [27; 28; 49].

Outline

As the effective action is the central object of our interest, we will briefly review its basic definitions and some simple ways to calculate it in Chap. 2. In Chap. 3 we will present some very simple calculations of the critical coupling of the model Eq. (1.1). In particular, we will perform a MFT calculation and encounter the Fierz ambiguity. Chap. 4 introduces the concept of partial bosonization, and clarifies the origin of the Fierz ambiguity. A description with scale-dependent bosonic degrees of freedom allow us to cure the Fierz ambiguity for the RG calculation in Chap. 5. The following Chap. 6 turns to the concept of the 2PI effective action to circumvent the problems of partial bosonization. As an application of the 2PI effective action we investigate an instanton-like interaction. In particular, we focus on chiral symmetry breaking and a possible breaking of color symmetry by a non-vanishing color-octet condensate. Finally, in Chap. 7 we will return to the question of a possible composite Higgs. In particular, we are interested in the possibility of a non-perturbative renormalizable "Standard Model". We will only sketch some of the ideas and possible problems, accordingly this will be more like an outlook. Chap. 8 summarizes and concludes this work.

Chapter 2

1PI Effective action

The effective action Γ [17; 31; 32] is a very useful tool in quantum field theory (QFT). It allows us to calculate interesting quantities like vacuum expectation values, propagators and correlation functions more or less by simply taking (functional) derivatives. Indeed, we can promote a classical equation to full quantum status by replacing the action by the effective action $S \to \Gamma$ and the fields by their expectation values $\phi \to \langle\phi\rangle$. E.g. the equation of motion becomes

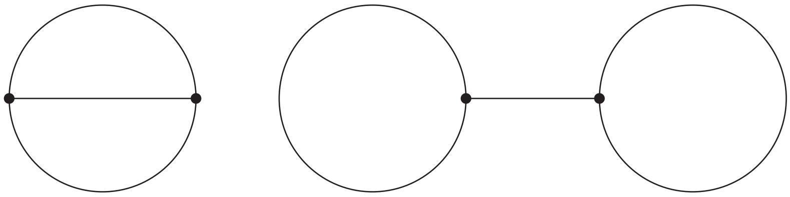
$$\frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle} = 0. \tag{2.1}$$

Knowledge of the effective action is equivalent to knowledge of the full quantum theory. From this one can already deduce that calculating the effective action is a quite difficult task and can usually be done only approximately.

Before going into more detail let us briefly review the definition of the (1PI) effective action[1], and some of its basic properties. In the following we will write $\tilde{\phi}$ for the fluctuating quantum field and $\phi = \langle\tilde{\phi}\rangle$. We suppress all indices. Indeed, ϕ might also contain fermionic degrees of freedom. A typical ϕ might therefore look like $(\sigma, \sigma^\star, V^\mu, A^\mu, \ldots, \psi_i, \overline{\psi}_j \ldots)$ with several bosonic and fermionic species. If we keep track of the order of fields and differential operators, no problems arise from this notation.

Moreover, we work in Euclidean space. That is why we have a minus sign in the path integrals instead of an i in front of the action. The transition to Euclidean time is usually done via a Wick rotation. We do not want to go into detail, here, however, for fermions there are some slight difficulties because the action is no longer necessarily Hermitian [50; 51].

[1] 1PI abbreviates one particle irreducible (cf. Fig. 2.1).



(a) (b)

Figure 2.1: An example of a diagram which is 1PI 2.1(a) and one which is not 2.1(b). The latter one can be split into two by cutting the line between the two bubbles.

The generating functional (or partition function if one prefers the statistical mechanics language) of a quantum field theory is defined by the following functional integral

$$Z[j] = \int \mathcal{D}\tilde{\phi} \exp(-S[\tilde{\phi}] + j\tilde{\phi}). \tag{2.2}$$

Here, $S[\tilde{\phi}]$ is the classical action and j is an external source. We recall that in our matrix notation $j\tilde{\phi} = \int d^dx j(x)\tilde{\phi}(x)$.

Using the generating functional, expectation values of fields (and products of fields) can be calculated by taking derivatives with respect to j

$$\phi[j](q) = \langle\tilde{\phi}(q)\rangle = \frac{\int \mathcal{D}\tilde{\phi}\, \tilde{\phi}(q) \exp(-S[\tilde{\phi}] + j\tilde{\phi})}{\int \mathcal{D}\tilde{\phi} \exp(-S[\tilde{\phi}] + j\tilde{\phi})} = \frac{1}{Z[j]}\frac{\delta Z[j]}{\delta j(q)} = \frac{\delta W[j]}{\delta j(q)}, \tag{2.3}$$

with

$$W[j] = \ln(Z[j]). \tag{2.4}$$

Physical values are obtained at vanishing external sources e.g. $\phi[0]$.

The (1PI) effective action is now the Legendre transform of W,

$$\Gamma[\phi] = -W[j[\phi]] + j[\phi]\phi \tag{2.5}$$

and depends on the expectation value of the field. Combining (2.2), (2.4), (2.5) and shifting the integration variable to $\hat{\phi} = \tilde{\phi} - \phi$ we obtain the following very useful formula

$$\Gamma[\phi] = -\ln \int \mathcal{D}\hat{\phi} \exp\left(-S[\phi + \hat{\phi}] + \frac{\delta\Gamma[\phi]}{\delta\phi}\hat{\phi}\right), \tag{2.6}$$

since

$$\frac{\delta\Gamma[\phi]}{\delta\phi} = j. \tag{2.7}$$

We note that due to the shift of the integration variable $\langle\hat{\phi}\rangle = 0$. Finally, we would like to comment on the notion of one particle irreducibility (1PI). This is

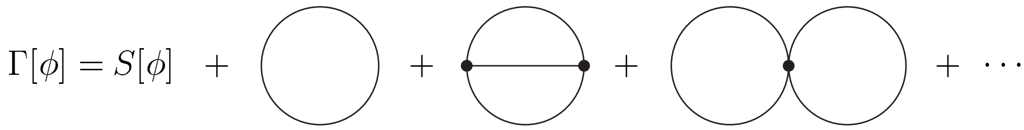


Figure 2.2: Perturbative expansion of the effective action. To be explicit we choose a theory which has a quartic interaction. The propagators are propagators in a background field ϕ. For the example of a theory with a $\frac{\lambda}{12}\tilde{\phi}^4$ interaction the propagator in the background field would be $(p^2 + m^2 + \lambda\phi^2)^{-1}$.

explained most easily in terms of Feynman diagrams. A diagram is one particle irreducible if it is impossible to split it into two parts by cutting an internal line (cf. Fig 2.1). The effective action is now the generating functional of the 1PI diagrams. We stress this point because later in Chap. 6.1 we will encounter a 2PI effective action. For a proof of this statement and further details about the 1PI effective action we refer to textbooks as e.g. [51–54].

2.1 Calculating the Effective Action

In the following we will shortly present some of the standard methods to calculate the effective action. Before we start let us mention that in this section we will not explain mean field theory as it will later (Chap. 4) be shown to be a one-loop approximation of a modified theory with auxiliary fields. Instead, we will explain by example a common method to obtain it in the next chapter and postpone a somewhat more thorough discussion until we have introduced partial bosonization in Chap. 4.

2.1.1 Loop Expansion

The loop expansion is a perturbative technique to calculate the effective action. It can be shown that the effective action is nothing but the sum of all 1PI vacuum diagrams in a background field ϕ as depicted in Fig. 2.2 An easy way to obtain this expansion is to make a saddle-point approximation of Eq. (2.6). In lowest non-trivial order one obtains,

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\mathrm{STr}\ln(S^{(2)}[\phi]) + \cdots, \tag{2.8}$$

where

$$S^{(2)}[\phi] = \frac{\overrightarrow{\delta}}{\delta\phi^{\mathrm{T}}} S[\phi] \frac{\overleftarrow{\delta}}{\delta\phi}. \tag{2.9}$$

In the diagrammatic language (Fig. 2.2) the second term of Eq. (2.8) corresponds to the one-loop diagram, and we have omitted the higher loop diagrams in Eq. (2.8). The supertrace (STr) comes around due to our notation where bosonic as well as fermionic degrees of freedom are contained in ϕ. However, its effect is very simple as it only provides a minus sign in the fermionic sector of the matrix.

An advantage of perturbation theory is that it can usually be constructed to preserve symmetries order by order in the expansion. However, as we will see in the next chapter it has severe shortcomings if we want to go into the non-perturbative domain (hence the name) where the coupling is not small. Indeed, we will see that it is unable to describe the interesting phenomenon of SSB.

2.1.2 Renormalization Group Equations

Originally devised as a tool to hide the infinities of quantum field theory the renormalization group has shown to give us a much deeper insight into the scale dependence of physics. Especially, the understanding of critical phenomena has profited immensely. In addition, in the last fifteen years or so the renormalization group has been established as a powerful tool for doing actual calculations in the non-perturbative regime. It is the latter aspect on which we want to focus.

The basic idea of renormalization group equations is to introduce scale dependent effective couplings. Roughly speaking we look at the physical system of interest with a microscope with varying resolution. E.g. consider a lattice where a particle with spin sits at each lattice site. With a very high resolution we can see every particle and can measure each spin independently. However, shifting to a lower resolution things become a little bit blurry and we can only resolve regions which already contain several particles. What we then measure is something like an effective spin (more or less the sum of the individual spins) of several particles combined. Indeed, the first papers [55–58] on renormalization group equations considered spin models like the Ising model.

There is a wide variety of means for putting this intuitive picture into a mathematical form. A very convenient picture to do this is the path integral formulation. For a theory with an UV cutoff Λ one can write the partition function as

$$Z = \prod_{p\leq\Lambda} \int d\phi(p) \exp(-S[\phi]), \tag{2.10}$$

where the symbolic notation shall indicate that we integrate only over momentum modes with $p \leq \Lambda$.

We can now implement the idea of decreasing the resolution of our microscope by integrating out modes in a small momentum shell $[\Lambda' = \Lambda - \Delta\Lambda, \Lambda]$. This procedure averages over the small length scales we do not want to see,

$$Z = \prod_{p \leq \Lambda'} \int d\phi(p) \exp(-S'[\phi]) \tag{2.11}$$

with

$$\exp(-S'[\phi]) = \prod_{\Lambda' \leq p \leq \Lambda} \int d\phi(p) \exp(-S[\phi]). \tag{2.12}$$

Where the so-called Wilson effective action S' is now integrated only over a smaller range of momenta. Considering a Λ' which is infinitesimally close to Λ one can derive an evolution equation for S' in the form of a differential equation.

Following this general procedure (often followed by a re-scaling $p \to \frac{\Lambda}{\Lambda'}p$ to recover the initial momentum range) allows to derive a variety of equations for different physical quantities like the Hamilton operator, correlation functions, coupling constants etc.. In addition, we do not need to restrict ourselves to the sharp momentum cutoff indicated above. Indeed, a smooth momentum cutoff is often much more convenient, as the sharp one has a tendency to introduce non-localities in position space.

All equations derived in this way are exact, i.e. if they could be solved exactly they would give the same results for physical quantities. Hence, they are called exact renormalization group equations (ERGE). In principle they are all equivalent. Nevertheless, in practical computations where we have to use approximations they usually differ. For a review and a comparison s. [59].

Now, let us go directly to a formulation for the effective action or, more precisely, the effective average action as introduced in [60–65]. Let us begin by noting that for the effective action the running cutoff Λ' becomes an infrared cutoff. This is due to the fact that the effective action contains the modes which are already integrated out, thus those in the range $[\Lambda', \Lambda]$. Λ' becomes a lower bound for the modes included in $\Gamma_{\Lambda'}$ and therefore an infrared cutoff. We remark that $\Gamma_{\Lambda'}$ depends both on the IR cutoff Λ' as well as on the physical UV cutoff Λ. The first is quite clear since Λ' simply measures how far we have progressed in integrating out modes. The dependence on the UV cutoff is actually more physical. E.g. if we want to describe interacting particles on a lattice a continuum description might be quite good for length scales larger than the lattice spacing. However, roughly speaking this model does not contain any momentum modes higher than the inverse lattice spacing. Placing the same particles on lattices with different spacing will obviously produce different results. Thus, we have a dependence on the UV cutoff (for some more details s. also App. C). Indeed, only in very special theories is it possible to send the UV cutoff to infinity and still obtain finite results. These are, of course, the renormalizable theories. In this case we can have $\Lambda = \infty$.

A big advantage of the formulation in terms of the effective action is that $\Gamma_{\Lambda'}$ has a direct physical interpretation. Having included all quantum fluctuations above Λ' we can now view $\Gamma_{\Lambda'}$ as the "microscopic action" on the scale Λ' where we have averaged over volumes of a size $\sim \Lambda'^{(-d)}$. Hence, the name effective average action or coarse-grained effective action. In an ideal description we would be able to observe the change of the relevant degrees of freedom from one scale to another, e.g. at very small scales we would have quark and gluon degrees of freedom while at a larger scale we observe mesons and nucleons, and at even larger scales we would have atoms or molecules, putting the basic idea of the renormalization group into full effect.

Let us now get started and derive an explicit equation. To establish more clearly its function as an IR cutoff we write k instead of Λ' or use the convenient $t = \ln(k)$.

In order to achieve a suppression of the low-momentum modes in the functional integral we add an effective momentum-dependent mass term

$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi^{\mathrm{T}}(-p)R_k(p)\phi(p) \tag{2.13}$$

to the initial action. The idea is to add a high mass to the momentum modes $p \leq k$ and a small or zero mass to those with $p \gg k$, therefore effectively removing (or at least suppressing) the low-momentum modes in the functional integral. To render this more precise we demand the following constraints for the function $R_k(p)$,

$$\mathbf{1}.\ \lim_{p^2/k^2\to 0} R_k(p) > 0, \quad \mathbf{2}.\ \lim_{p^2/k^2\to\infty} R_k(p) = 0, \quad \mathbf{3}.\ \lim_{k\to\infty} R_k(p) \to \infty. \tag{2.14}$$

The first condition is the statement that we want to suppress the low momentum modes by an additional mass term. Having a mass term for the zero-momentum modes has another very nice effect as it removes all IR divergences produced by massless particles. The second condition ensures that the high-momentum modes (high compared to k) are not suppressed and that the cutoff is removed in the limit $k \to 0$. As we will later see when we have the explicit expression for the flow equation it is useful to choose a cutoff that vanishes sufficiently fast (e.g. exponentially) in the UV to avoid UV divergences. The third condition ensures that at $k \to \infty$ no modes are integrated out. However, a comment is in order as we want to send k to infinity. Originally, we wanted to suppress all modes with $p < k$. But, if we have a finite physical cutoff Λ there are no modes with $p > \Lambda$. Thus, one might want to rewrite condition 3 as

$$\mathbf{3'}.\ \lim_{k\to\Lambda} R_k(p) \to \infty,$$

and indeed this is a quite common form given and used e.g. in [66–68]. However, both conditions are equivalent as we can use every bijective function $k'(k)$ which maps $[0,\Lambda]$ into $[0,\infty]$ to re-scale k such that either 3 or $3'$ is fulfilled. Using a smooth cutoff the notion of modes being included or not included is anyway somewhat blurry. With a little bit of discretion in the choice of $k'(k)$ we do not distort the picture of integrating out modes down to the scale k very much for $k < \Lambda$, making the choice between 3 and 3′ a matter of convenience[2]. In this work we will choose the former since it makes some of the analytical expressions easier. More important for the interpretation of k as the coarse graining with modes $p \lesssim k$ not yet being integrated out is that the shift between small values of R_k and large values of R_k occurs roughly at $p \approx k$. This is not a necessary condition to define the flow equation but useful for the interpretation. We leave it at the somewhat rough statement

$$\mathbf{4}.\ R_k(p) \text{ large for } p < k, \quad R_k(p) \text{ small for } p > k. \tag{2.15}$$

Having talked so much about the cutoff let us finally give an explicit example of a very convenient one introduced in [69],

$$R_k(p) = Z_k(1 - p^2)\Theta\left(1 - \frac{p^2}{k^2}\right). \tag{2.16}$$

Here we have included a factor of the wave function renormalization Z_k. This choice allows us to write

$$Z_k p^2 + R_k(p) = Z_k P(p), \tag{2.17}$$

and guarantees that the reparametrization invariance of physical quantities $\phi \to \alpha\phi$ is respected.

As discussed in the last few paragraphs the term (2.13) acts more or less like an additional mass term. In consequence we might get concerned about violating symmetries like chiral or gauge symmetry. In principle there are two possible ways to tackle this problem. The first is to construct a cutoff function $R_k(p)$ which fulfills Eq. (2.14) but does not violate the symmetry. This is relatively simple for chiral fermions [61; 70–72]. An example is

$$R_k(p) = Z_k \not{p}\left(\sqrt{\frac{k^2}{p^2}} - 1\right)\Theta(1 - \frac{p^2}{k^2}), \tag{2.18}$$

which is more or less (2.16) adapted to chiral fermions. The second strategy (usually used for gauge theories) is to accept the fact that during the flow the symmetry might be violated by the cutoff and is completely symmetric only at the endpoint when the cutoff vanishes. Nevertheless, the symmetry is only hidden during the flow and reveals itself in modified Ward-Takashi identities [73–77]. Taking these into account it is possible to construct an invariant flow.

Adding the cutoff to the initial action we get a set of *different* actions parametrized by the scale k,

$$S_k[\phi] = S[\phi] + \Delta S_k[\phi]. \tag{2.19}$$

[2]Some additional details concerning different possible UV regularizations are provided in App. C.

Correspondingly, we obtain

$$W_k[j] = \ln(Z_k[j]) = \int \mathcal{D}\tilde{\phi} \exp(-S_k[\tilde{\phi}] + j\tilde{\phi}) \tag{2.20}$$

and

$$\phi = \langle\tilde{\phi}\rangle = \frac{\overrightarrow{\delta}}{\delta j} W_k[j]. \tag{2.21}$$

Now, we want to introduce the effective average action by a modified Legendre transform

$$\Gamma_k[\phi] = -W_k[j[\phi]] + j[\phi]\phi - \Delta S_k[\phi], \tag{2.22}$$

where we have substracted $\Delta S_k[\phi]$ in order to remove the cutoff effects from Γ_k. This is particularly clear in the case $k \to \infty$. Consider the formula analogous to Eq. (2.6),

$$\Gamma_k = -\ln \int \mathcal{D}\hat{\phi} \exp\left(-S[\phi + \hat{\phi}] + \frac{\delta\Gamma_k[\phi]}{\delta\phi}\hat{\phi} - \Delta S_k[\hat{\phi}]\right). \tag{2.23}$$

Due to the third condition in (2.14) the additional term $\exp(-\Delta S_k[\hat{\phi}])$ in the functional integral acts like a functional $\delta(\hat{\phi})$-function for $k \to \infty$ and we obtain

$$\Gamma_\infty[\phi] = S[\phi], \tag{2.24}$$

the microscopic action without the cutoff.

Since the cutoff vanishes for $k \to 0$ (second condition) we have in addition

$$\Gamma_0[\phi] = \Gamma[\phi]. \tag{2.25}$$

Thus, the effective average action interpolates between the classical or bare action and the full effective action (cf. Fig. 2.4).

Now, let us come to the final piece, the ERGE which governs the evolution from $k = \infty$ to $k = 0$. Taking a derivative with respect to k,

$$\begin{aligned}
\frac{\partial}{\partial k}(\Gamma_k[\phi]) &= -(\partial_k W_k)[j] - (\partial_k j)\frac{\overrightarrow{\delta}}{\delta j}W_k[j] + (\partial_k j)\phi - \partial_k \Delta S_k[\phi] \qquad (2.26)\\
&= \langle \partial_k \Delta S_k[\tilde{\phi}]\rangle - \partial_k \Delta S_k[\phi]\\
&= \frac{1}{2}\left(\langle\tilde{\phi}^{\mathrm{T}}(\partial_k R_k)\tilde{\phi}\rangle - \langle\tilde{\phi}^{\mathrm{T}}\rangle(\partial_k R_k)\langle\tilde{\phi}\rangle\right)\\
&= \frac{1}{2}\left(\mathrm{STr}\left[(\langle\tilde{\phi}\tilde{\phi}^{\mathrm{T}}\rangle - \langle\tilde{\phi}\rangle\langle\tilde{\phi}^{\mathrm{T}}\rangle)\partial_k R_k\right]\right).
\end{aligned}$$

Expressing this with

$$W_k^{(2)} = \frac{\overrightarrow{\delta}}{\delta j}\frac{\overrightarrow{\delta}}{\delta j^{\mathrm{T}}} W_k = \langle\tilde{\phi}\tilde{\phi}^{\mathrm{T}}\rangle - \langle\tilde{\phi}\rangle\langle\tilde{\phi}^{\mathrm{T}}\rangle. \tag{2.27}$$

gives

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{STr}(W_k^{(2)} \partial_k R_k). \tag{2.28}$$

Since

$$j_k = (\Gamma_k + \Delta S_k)\frac{\overleftarrow{\delta}}{\delta\phi} \tag{2.29}$$

we have

$$\mathbf{1} = \left(\frac{\overrightarrow{\delta}}{\delta j}\phi^{\mathrm{T}}\right)\left(\frac{\overrightarrow{\delta}}{\delta\phi^{\mathrm{T}}}j\right) = W_k^{(2)}(\Gamma_k^{(2)} + R_k) \tag{2.30}$$

with

$$\Gamma_k^{(2)} = \frac{\overrightarrow{\delta}}{\delta\phi^{\mathrm{T}}}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\phi}. \tag{2.31}$$

Inserting this into Eq. (2.28) yields, finally, the flow equation

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{STr}\left\{(\Gamma_k^{(2)} + R_k)^{-1}\partial_k R_k\right\}. \tag{2.32}$$

Defining the operator

$$\tilde{\partial}_t = (\partial_t R_k)\frac{\partial}{\partial R_k} \tag{2.33}$$

we can obtain an even more compact form

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{STr}[\tilde{\partial}_t \ln(\Gamma_k^{(2)} + R_k)]. \tag{2.34}$$

This looks quite similar to Eq. (2.8), and indeed if we neglect the change of Γ_k on the right hand side, integrate and use (2.24) at $k = \infty$, $t = -\infty$, we recover (2.8).

In short, substituting $\Gamma_k^{(2)} + R_k$ for $S^{(2)}$ and writing it in a differential form turns the one-loop expression (2.8) into an exact equation. The one-loop form of Eqs. (2.32), (2.34) is depicted in Fig. 2.3.

In addition to being exact, Eq. (2.32) has two more nice features. First, due to the presence of R_k the expression is IR finite. Second, for R_k decreasing sufficiently fast in the UV, $\partial_k R_k$ provides an UV regularization. The flow equation is therefore completely finite. Of course, those divergences must still be included in some way. While the IR divergences might reappear in the integration of the flow equation for $k \to 0$, the UV divergences have been absorbed in the initial condition for some finite k. Specifying the initial conditions of the flow at some finite k gives a special regularization scheme called the ERGE scheme (cf. App. C.3).

Having a flow equation is only part of the game. Since it is a functional differential equation, it is in most cases impossible to solve it analytically, and we remember once again that this would amount to solving the quantum field theory in question.

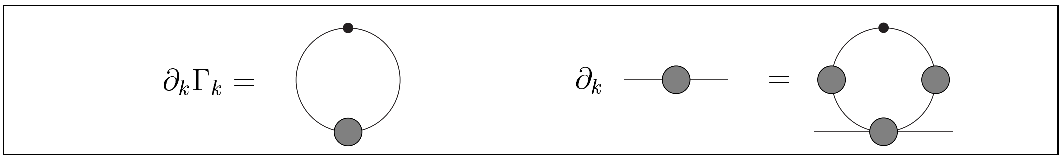


Figure 2.3: Depiction of the flow equation (2.32). The line with the shaded circle is the full field dependent and IR regularized propagator $\Gamma_k^{(2)} + R_k$. The dot denotes the insertion of $\partial_k R_k$. Taking functional derivatives with respect to the field ϕ adds external legs, i.e. we obtain flow equations for the propagators and vertices. An example for the flow of the propagator is shown on the right side. The shaded circle denotes the full k-dependent vertex.

So, it will be impossible or at least difficult to avoid using approximations in most of the physically relevant cases.

A consistent and systematic approach is the use of truncations. In a truncation we restrict the space of all possible actions (Γ_k), spanned by all possible combinations of field operators compatible with the symmetries to a (very often finite dimensional) subspace given by a subset of operators. The approximate flow equation now is the projection of the flow onto this subspace (cf. Fig. 2.4). From this we can calculate flow equations for the coefficients (generalized couplings) in front of the operators. We stress that the approximate flow is only driven by the operators in the subset. An easy, nevertheless usually quite tedious, way to improve the approximation and to check for errors is to enlarge the subspace. Doing this successively we may find a "convergence" of the results, and we may be tempted to interpret this as the approach to the right result. Still, we should be careful with this as it may well be, that we have indeed convergence, but convergence to the wrong result. This is usually the case when we have missed a relevant operator. As the number of all possible linearly independent operators is infinite we can always add operators but still miss the relevant one.

From a physics point of view it is clear that a good approximation should include all relevant degrees of freedom, i.e. the corresponding operators. However, as we discussed in the introduction the relevant degrees of freedom can change with scale, making it necessary to adapt the description during the flow. For our case of interest a step in this direction was taken in [42] and we will discuss it at length in Chap. 5.

Finally, let us come to the role the cutoff plays in approximations of the flow equation. By construction, an exact solution has a $\Gamma_0 = \Gamma$ independent of the cutoff. But, the trajectory Γ_k is not independent of the cutoff. As depicted in Fig. 2.4 it may well be that a certain choice of IR cutoff may bring the real trajectory closer to the subspace defining our truncation, therefore usually improving the approximation. A systematic study to exploit this possibility has been put forward in [68; 69; 78–81].

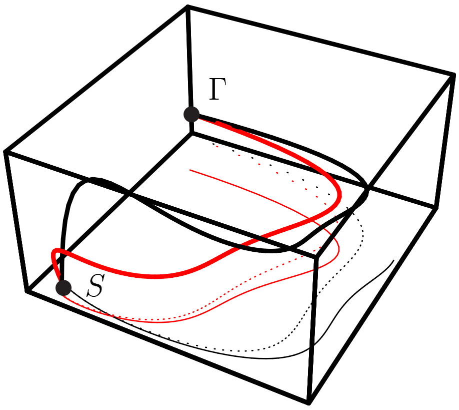


Figure 2.4: RG flow in the space of all action functionals. The thick line is the exact flow in the (here 3-dimensional) space, the thin line is the approximate flow in the 2-dimensional truncation. We point out that the approximate flow does, of course, not coincide with the projection of the exact flow on the subspace of the truncation (dotted line), as the latter is driven also by the operator in the third direction. The red set of lines, shows the same but for an "optimized" IR regulator. The exact flows coincide only at the start- and endpoint. The optimized flow is generally closer to the plane of the truncation, and the approximate flow is improved.

A more traditional approach would be to use the IR cutoff dependence as a measure of uncertainty, s. e.g. [82].

2.1.3 Schwinger-Dyson Equations

Schwinger-Dyson equations (SDE) [83; 84] (for a review and some applications s. [85–87]) were one of the first really non-perturbative tools in quantum field theory. For a theory with polynomial interactions up to ϕ^m they provide an (infinite) hierarchy of equations which connect a 1PI Greens function of order n (nth derivative of the effective action) on the one hand with a set of 1PI Greens functions up to order $(n+m)$ on the other hand.

In principle they are a consequence of the fact, that the functional integral over a total derivative vanishes, if the functional vanishes at the boundary (like in normal calculus),

$$\begin{aligned} 0 &= \int \mathcal{D}\phi \frac{\overrightarrow{\delta}}{\delta\phi} \exp(-S[\phi] + j\phi) \qquad (2.35) \\ &= \int \mathcal{D}\phi \left(-\frac{\overrightarrow{\delta} S[\phi]}{\delta\phi} \pm j \right) \exp(-S[\phi] + j\phi) \end{aligned}$$

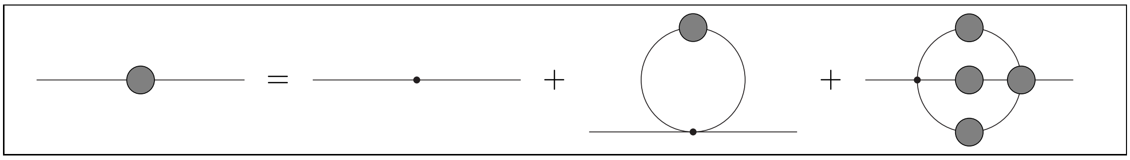


Figure 2.5: SDE for the propagator (in ϕ^4-theory). Full propagators and vertices are depicted with a shaded circle while bare quantities are represented by a dot. This equation is exact. However, it involves the full 4-point function which is given by another SDE. A simple approximation would be to neglect the last diagram on the right hand side giving us a closed equation.

$$= \left(\frac{\overrightarrow{\delta} S}{\delta \phi}[\frac{\delta}{\delta j}] \pm j \right) Z[j],$$

where $+$ is for bosons and $-$ is for fermions, respectively. Examining the last line in Eq. (2.35) it becomes clear that the SDE's are the Euler Lagrange equations of quantum field theory.

Due to the appearance of S on the right hand side of Eq. (2.35) we have always exactly one bare vertex in every expression contributing to the right hand side. A typical SDE therefore looks like in Fig. 2.5.

In its basic form the SDE allow us to calculate only the derivatives of the effective action. At first sight this may not seem like a major weakness, but in situations where we have multiple solutions for the SDE we have no way to compare them without knowledge of the value of the effective action. However, the case of multiple solutions is one of the most interesting ones, as it usually signals the possibility for spontaneous symmetry breaking. In Chap. 6 we will discuss the 2PI effective action [46–48] as a remedy to this problem.

As in the case of the ERGE most of the time we are unable to solve the complete set of SDE. A popular approximation scheme is to neglect all 1PI Greens functions starting from a certain order. This gives a closed set of integral equations. More generally, similar to a truncation for an ERGE, one can restrict the space of all possible Γ and hence its derivatives to a subspace.

On the exact level the RG and SD approaches are equivalent in the sense that the propagator and higher N-point functions calculated using the flow equation (2.32) are also solutions of the SDE [74; 88]. Nevertheless, once approximations are used the results will, in general, differ.

Chapter 3

A Simple Example: The NJL Model

In this chapter we want to get a grasp of the problems associated with the introduction of bosonic composite fields by studying the model, Eq. (1.1) in some very simple approximations. In particular, we use this as an opportunity to introduce MFT.

3.1 Critical Couplings from Mean Field Theory

A mean-field calculation treats the fermionic fluctuations in a homogenous background of fermion bilinears $\tilde{\phi} = \langle\overline{\psi}\left(\frac{1-\gamma^5}{2}\right)\psi\rangle$, $\tilde{\phi}^\star = -\langle\overline{\psi}\left(\frac{1+\gamma^5}{2}\right)\psi\rangle$, $\tilde{V}_\mu = \langle\overline{\psi}\gamma_\mu\psi\rangle$ and $\tilde{A}_\mu = \langle\overline{\psi}\gamma_\mu\gamma^5\psi\rangle$. It seems straightforward to replace in the four-fermion interaction in Eq. (1.1) one factor by the bosonic mean field, i.e.

$$
\begin{aligned}
(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma^5\psi)^2 &\rightarrow 2\tilde{\phi}\overline{\psi}(1+\gamma^5)\psi - 2\tilde{\phi}^\star\overline{\psi}(1-\gamma^5)\psi, \\
(\overline{\psi}\gamma_\mu\psi)^2 &\rightarrow 2\tilde{V}_\mu\overline{\psi}\gamma^\mu\psi, \\
(\overline{\psi}\gamma_\mu\gamma^5\psi)^2 &\rightarrow 2\tilde{A}_\mu\overline{\psi}\gamma^\mu\gamma^5\psi.
\end{aligned}
\tag{3.1}
$$

The partition function becomes then a functional of $\tilde{\phi}$, $\tilde{V}_\mu$, $\tilde{A}_\mu$,

$$
Z[\tilde{\phi},\tilde{V},\tilde{A}] = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left(-\mathrm{S}[\overline{\psi},\psi,\tilde{\phi},\tilde{V},\tilde{A}]\right), \tag{3.2}
$$

where S is given by (1.1), with the replacements (3.1). Self consistency for the expectation values of the fermion bilinears requires

$$\begin{aligned} \tilde{\phi} &= \frac{1}{2}\langle\overline{\psi}(1-\gamma^5)\psi\rangle = \frac{1}{2}\lambda_\sigma^{-1}\frac{\partial}{\partial\tilde{\phi}^\star}\ln Z, \\ \tilde{V}_\mu &= \langle\overline{\psi}\gamma_\mu\psi\rangle = \lambda_V^{-1}\frac{\partial}{\partial\tilde{V}^\mu}\ln Z, \end{aligned} \tag{3.3}$$

and similar for the other bilinear $\tilde{A}_\mu$. Chiral symmetry breaking by a nonzero $\tilde{\phi}$ requires that the "field equation" (3.3) has a nontrivial solution. We note that $Z[\tilde{\phi},\tilde{V},\tilde{A}]$ corresponds to a one-loop expression for the fermionic fluctuations in a bosonic background. With $\Gamma_1^{(\mathrm{F})} = -\ln Z$ the field equation is equivalent to an extremum of

$$\Gamma^{(\mathrm{F})} = \int d^4x \left\{2\lambda_\sigma\tilde{\phi}^\star\tilde{\phi} + \frac{1}{2}\lambda_V\tilde{V}_\mu\tilde{V}^\mu + \frac{1}{2}\lambda_A\tilde{A}_\mu\tilde{A}^\mu\right\} + \Gamma_1^{(\mathrm{F})}. \tag{3.4}$$

A discussion of spontaneous symmetry breaking in MFT amounts therefore to a calculation of the minima of $\Gamma^{(\mathrm{F})}$.

As we already noted in the introduction this calculation can be done equivalently in the Yukawa theory (1.4), (1.5). The mapping of the bosonic fields reads $\phi = (h_\sigma/\mu_\sigma^2)\,\tilde{\phi}$, $V_\mu = (h_V/\mu_V^2)\,\tilde{V}_\mu$, $A_\mu = (h_A/\mu_A^2)\,\tilde{A}_\mu$. Keeping the bosonic fields fixed and performing the remaining Gaussian fermionic functional integral yields precisely Eq. (3.4). Mean field theory therefore corresponds precisely to an evaluation of the effective action in the partially bosonized Yukawa model in a limit where the bosonic fluctuations are neglected.

We want to compute here the critical couplings (more precisely, the critical line in the plane of couplings $\overline{\lambda}_\sigma$, $\overline{\lambda}_V$) for which a nonzero expectation value $\phi \neq 0$ indicates the onset of spontaneous symmetry breaking. For this purpose we calculate the mass term $\sim \phi^\star\phi$ in $\Gamma^{(\mathrm{F})}$ and look when it turns negative. This defines the critical couplings. We assume here a situation where the expectation values of other bosonic fields like V_μ or A_μ vanish in the relevant range of couplings. It is then sufficient to evaluate $\Gamma^{(\mathrm{F})}$ for $V_\mu = A_\mu = 0$.

In a diagrammatic language Gaussian integration over the fermionic variables corresponds to evaluating the diagram of Fig. 3.1. We define our model with a fixed ultraviolet momentum cutoff $q^2 < \Lambda^2$, such that the MFT result becomes ($v_4 = 1/(32\pi^2)$, $x = q^2$):

$$\Gamma_1^{(\mathrm{F})} = -4v_4\int_0^{\Lambda^2} dx\, x\ln(x + h_\sigma^2\phi^\star\phi). \tag{3.5}$$

From this one finds the mean field effective action

$$\begin{aligned} \Gamma^{(\mathrm{F})} &= \Gamma_0^{(\mathrm{F})} + \Gamma_1^{(\mathrm{F})} \\ &= \left(\mu_\sigma^2 - 4v_4h_\sigma^2\Lambda^2\right)\phi^\star\phi + \mathrm{const} + \mathcal{O}\left((\phi^\star\phi)^2\right), \end{aligned} \tag{3.6}$$

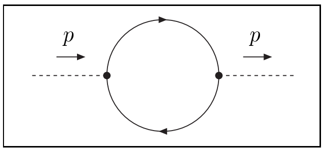


Figure 3.1: Bosonic mass correction due to fermion fluctuations. Fermionic lines are solid with an arrow, bosonic or "mean field lines" $\sim \langle \overline{\psi} \left(\frac{1-\gamma^5}{2} \right) \psi \rangle$ are dashed. For use in Chap. 5 we have indicated an external momentum p. A MFT calculation corresponds to an evaluation for $p = 0$.

where we have expanded in powers of ϕ. The mass term turns negative if

$$\frac{2\mu_\sigma^2}{h_\sigma^2 \Lambda^2} < 8v_4, \tag{3.7}$$

As it should be this result only depends on the ratio $h_\sigma^2/\mu_\sigma^2 = 2\lambda_\sigma$.

We now want to determine the critical line in the plane of invariant couplings $\overline{\lambda}_\sigma$, $\overline{\lambda}_V$ from the condition (3.7), i.e.

$$\lambda_\sigma^{\mathrm{crit}} = \frac{1}{8v_4\Lambda^2}. \tag{3.8}$$

Using the relation (1.3) we infer a linear dependence of $\overline{\lambda}_\sigma^{\mathrm{crit}}$ on the arbitrary unphysical parameter γ whenever $\overline{\lambda}_V \neq 0$

$$\overline{\lambda}_\sigma^{\mathrm{crit}} = \frac{1}{8v_4\Lambda^2} - 2\gamma\overline{\lambda}_V. \tag{3.9}$$

(For numerical values see Tabs. 3.1 and 3.2). This dependence is a major shortcoming of MFT. We will refer to it as "Fierz ambiguity". The Fierz ambiguity does not only affect the critical couplings but also influences the values of masses, effective couplings etc..

The origin of the Fierz ambiguity can be traced back to the treatment of fluctuations. A FT of the type (1.2) changes the effective mean field. In a symbolic language a FT maps $(\overline{\psi}_a\psi_a)(\overline{\psi}_b\psi_b) \to (\overline{\psi}_a\psi_b)(\overline{\psi}_b\psi_a)$ where the brackets denote contraction over spinor indices and matrices $\sim \gamma_\mu$ or $\sim \gamma^5$ are omitted. A mean field $\overline{\psi}_a\psi_a$, appears after the FT as $\overline{\psi}_a\psi_b$. From the viewpoint of the fluctuations one integrates out different fluctuating fields before and after the FT. It is therefore no surprise that all results depend on γ.

Approximation	Chap.	$\gamma = 0$	0.25	0.5	0.75	1
MFT	3.1	39.48	38.48	37.48	36.48	35.48
Ferm. RG	3.3	41.54	41.54	41.54	41.54	41.54
Bos. RG	4.2	36.83	36.88	36.95	37.02	37.12
Adapted Bos. RG	5.1	41.54	41.54	41.54	41.54	41.54
SD	3.4	37.48	37.48	37.48	37.48	37.48

Table 3.1: Critical values $\overline{\lambda}_\sigma^{\text{crit}}$ for $\overline{\lambda}_V = 2$ and for various values of the unphysical parameter γ (with $\Lambda = 1$). In anticipation of Chaps. 4, 5 we give also results for the (adapted) bosonic RG. Progressing from MFT to the bosonic RG and adapted bosonic RG the dependence on γ decreases as more and more diagrams are included. The Schwinger-Dyson result is independent of γ but contains no vertex corrections in contrast to the RG calculations.

Approximation	Chap.	$\gamma = 0$	0.25	0.5	0.75	1
MFT	3.1	39.48	29.48	19.48	9.48	-0.52
Ferm. RG	3.3	14.62	14.62	14.62	14.62	14.62
Bos. RG	4.2	15.44	13.39	13.45	15.55	19.46
Adapted Bos. RG	5.1	14.62	14.62	14.62	14.62	14.62
SD	3.4	19.48	19.48	19.48	19.48	19.48

Table 3.2: The same[1] as in Tab. 3.1 but with $\overline{\lambda}_V = 20$.

3.2 Perturbation Theory

In order to cure the unpleasant dependence of the MFT result on γ we will include part of the bosonic fluctuations in Chaps. 4 and 5. Some guidance for the level of approximations needed can be gained from perturbation theory in the fermionic language. Since the four-fermion vertex is uniquely characterized by $\overline{\lambda}_\sigma$ and $\overline{\lambda}_V$ the perturbative result must be independent of γ at any given loop order. The lowest-order corrections to the four-fermion couplings are obtained by expanding the one-loop expression for the effective action Eq. (2.8)[2]

$$\Delta\Gamma^{(\text{1-loop})} = \frac{1}{2}\mathrm{STr}\left[\ln\left(\mathrm{S}^{(2)}\right)\right] = -\mathrm{Tr}\left[\ln\left(\mathrm{S}^{(2)}_{\overline{F}F}\right)\right] \tag{3.10}$$

[1] The negative sign for the critical coupling at $\gamma = 1$ in the MFT calculation means that the system is in the broken phase for any positive value of $\overline{\lambda}_\sigma$ in this calculation.

[2] We remember that in the full $\mathrm{S}^{(2)}$ matrix we have a term from the $\delta^2/\delta\overline{\psi}\delta\psi$ derivative ($\mathrm{S}^{(2)}_{\overline{F}F}$) and a term from $\delta^2/\delta\psi\delta\overline{\psi}$, accounting for a factor of 2 in the language with the normal trace. Moreover, the trace includes momentum integration and summation over internal indices.

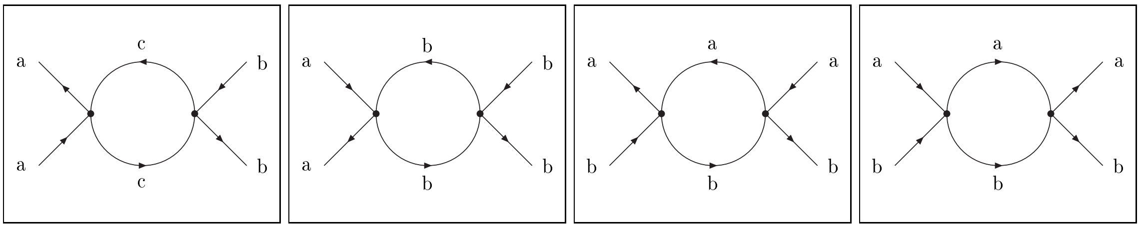


Figure 3.2: Perturbative correction to the four-fermion interaction. Solid lines with an arrow denote fermionic lines. The letters in the diagrams are given to visualize the ways in which the fermionic operators are contracted, e.g. the first diagram in the second row results from a term $[(\overline{\psi}_a\psi_a)(\overline{\psi}_c\psi_c)][(\overline{\psi}_c\psi_c)(\overline{\psi}_b\psi_b)]$. Evaluating the diagrams for k-dependent "full" vertices and IR regularized propagators the above set of diagrams gives us the flow equation of Sect. 3.3.

up to order $(\overline{\psi}\psi)^2$. For this it is useful to decompose $S^{(2)}$ according to

$$S^{(2)} = \mathcal{P} + \mathcal{F}, \tag{3.11}$$

into a field-independent part $\mathcal{P}$ (inverse popagator) and a field-dependent part $\mathcal{F}$. The RHS can then be expanded as follows,

$$\begin{aligned} \Delta\Gamma = \frac{1}{2}\mathrm{STr}\Big\{\Big(\frac{1}{\mathcal{P}}\mathcal{F}\Big)\Big\} - \frac{1}{4}\mathrm{STr}\Big\{\Big(\frac{1}{\mathcal{P}}\mathcal{F}\Big)^2\Big\} &+ \frac{1}{6}\mathrm{STr}\Big\{\Big(\frac{1}{\mathcal{P}}\mathcal{F}\Big)^3\Big\} \\ &- \frac{1}{8}\mathrm{STr}\Big\{\Big(\frac{1}{\mathcal{P}}\mathcal{F}\Big)^4\Big\} + \cdots . \end{aligned} \tag{3.12}$$

This amounts to an expansion in powers of fields and we can compare the coefficients of the four-fermion terms with the couplings specified by Eq. (1.1). Only the second term on the RHS contributes to order $(\overline{\psi}\psi)^2$. The corresponding graphs with two vertices are shown in Fig. 3.2. From

$$\begin{aligned} \Delta\Gamma^{(\text{1-loop})} = v_4\Lambda^2 \Big\{ & [4\lambda_\sigma^2 - 4\lambda_\sigma(\lambda_A - 2\lambda_V)]\Big[\big(\overline{\psi}\psi\big)^2 - \big(\overline{\psi}\gamma^5\psi\big)^2\Big] \\ +\; & [-2\lambda_\sigma\lambda_V + 4(\lambda_A - \lambda_V)\lambda_V]\Big[\big(\overline{\psi}\gamma^\mu\psi\big)^2\Big] \\ +\; & \big[-\lambda_\sigma^2 + 2\lambda_\sigma\lambda_A + 3\lambda_V^2 - 2\lambda_A\lambda_V - \lambda_A^2\big]\Big[\big(\overline{\psi}\gamma^\mu\gamma^5\psi\big)^2\Big]\Big\}. \end{aligned} \tag{3.13}$$

we can read off the corrections $\Delta\lambda_\sigma$, $\Delta\lambda_V$ and $\Delta\lambda_A$ to the coupling constants. In order to establish that our result is independent of γ we use the freedom of FT to bring our results into a standard form, such that $\frac{\Delta\lambda_A}{\Delta\lambda_V} = \frac{\gamma}{1-\gamma}$. Inserting next the invariant variables (1.3) leads us to:

$$\begin{aligned}\Delta\overline{\lambda}_\sigma &= 4v_4\Lambda^2(\overline{\lambda}_\sigma^2 + 4\overline{\lambda}_\sigma\overline{\lambda}_V + 3\overline{\lambda}_V^2),\\ \Delta\overline{\lambda}_V &= 2v_4\Lambda^2(\overline{\lambda}_\sigma + \overline{\lambda}_V)^2.\end{aligned} \tag{3.14}$$

In contrast to MFT the result does not depend on γ.

The perturbative result, Eq. (3.14), always leads to finite corrections to the coupling constants. Remembering that in the fermionic language the onset of SSB is marked by a divergence of the coupling constants, it becomes clear that we will never get SSB in perturbation theory. No critical couplings can be calculated. This is a severe shortcoming of perturbation theory which cannot be overcome by calculating higher loop orders. Only an infinite number of loops can give SSB. In the next section we establish how a renormalization group treatment can overcome this difficulty without encountering the Fierz ambiguity of MFT. A calculation of the critical coupling becomes feasible. Nevertheless, even this RG treatment has its limitations once the couplings diverge. In particular, it does not allow us to penetrate the phase with SSB. In Chaps. 4 and 5 this shortcoming will be cured by a RG treatment in the partially bosonized language. In particular, we will see in Chap. 5 which diagrams are needed in order to maintain the independence of results on γ in analogy to perturbation theory.

3.3 Renormalization Group for Fermionic Interactions

Let us now turn to an RG equation. More explicitly the ERGE for the effective average action discussed in Chap. 2. Neglecting all change on the RHS leads to the perturbative result of the previous section. Consequently, in this approximation we cannot observe SSB. For a better approximation we restrict Γ_k to the terms specified in Eq. (1.1) but take all couplings to be explicitly k-dependent. In the action (1.1) we have only local interactions. Expressed in momentum space the four-fermion interactions have no momentum dependence. This is often referred to as the local potential approximation (LPA) [43–45].

Performing the decomposition into a field-dependent part and a field-independent part as in the previous section,

$$\Gamma_k^{(2)} + R_k = \mathcal{P}_k + \mathcal{F}_k \tag{3.15}$$

we obtain an expansion of the flow equation (2.34)

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{STr}\left\{\tilde{\partial}_t\left(\frac{1}{\mathcal{P}_k}\mathcal{F}_k\right)\right\} - \frac{1}{4}\mathrm{STr}\left\{\tilde{\partial}_t\left(\frac{1}{\mathcal{P}_k}\mathcal{F}_k\right)^2\right\} + \frac{1}{6}\mathrm{STr}\left\{\tilde{\partial}_t\left(\frac{1}{\mathcal{P}_k}\mathcal{F}_k\right)^3\right\} - \frac{1}{8}\mathrm{STr}\left\{\tilde{\partial}_t\left(\frac{1}{\mathcal{P}_k}\mathcal{F}_k\right)^4\right\} + \cdots . \tag{3.16}$$

This is in complete analogy to the previous section, only written in a differential form and with k-dependent vertices. We obtain a set of ordinary differential equations for the couplings:

$$\begin{aligned}\partial_t\overline{\lambda}_{\sigma,k} &= -8v_4 l_1^{(F),4}(s)k^2(\overline{\lambda}^2_{\sigma,k} + 4\overline{\lambda}_{\sigma,k}\overline{\lambda}_{V,k} + 3\overline{\lambda}^2_{V,k}),\\ \partial_t\overline{\lambda}_{V,k} &= -4v_4 l_1^{(F),4}(s)k^2(\overline{\lambda}_{\sigma,k} + \overline{\lambda}_{V,k})^2,\end{aligned} \tag{3.17}$$

in agreement with [82] where the same model has been studied. The threshold functions $l_1^{(F),4}$ (for our conventions cf. App. C.2 or [67]) originate from the momentum space integration over the IR regulated propagators and replace the factor $\frac{\Lambda^2}{2}$ in Eq. (3.14). For our actual calculation we use a linear cutoff[3] [78] and adapt the threshold functions to our setting with fixed momentum cutoff $q^2 < \Lambda^2$ in App. C.2. The dependence on $s = k^2/\Lambda^2$ becomes relevant only for $k > \Lambda$ whereas for $k < \Lambda$ one has constants $l_1^{(F),4} = 1/2$. Although useful for the comparison of the RG with MFT or perturbation theory, the explicit k-dependence of the threshold functions makes the fixed momentum cutoff somewhat cumbersome. An alternative is to use the so called ERGE scheme for the UV regularization. The basic idea (for details s. App. C.3) is to use standard threshold functions without a UV cutoff in the momentum integral and implement the UV regularization by specifying the initial conditions for Γ_k at some *finite* $k = \Lambda$. This has the advantage that threshold functions without a finite UV cutoff are not explicitly k-dependent, greatly simplifying numerical calculations. The price to pay is that it is in general impossible to compare non-universal quantities like critical couplings for different choices of the IR cutoff function $R_k(p)$.

The fermionic flow equations[4] (3.17) do not depend on γ. In a diagrammatic language we again have evaluated the diagrams of Fig. 5.2 but now with k-dependent vertices. In the RG formulation we only go a tiny step Δk, and reinsert the resulting couplings (one-loop diagrams) before we do the next step. This leads to a resummation of loops. Since Eq. (3.17) is now nonlinear (quadratic terms on the RHS) the couplings can and do diverge for a finite k if the initial couplings are large enough. Therefore we observe the onset of SSB and find a critical coupling. Since

[3]The threshold functions depend on the precise choice of the cutoff. For the very simple truncation used in this section this dependence can actually be absorbed by a suitable rescaling of k, cf. App. C.2.

[4]As discussed above, the perturbative result Eq. (3.14) can be recovered from Eq. (3.17) if we neglect the k-dependence of the couplings on the RHS and perform the t-integration.

Eq. (3.17) is invariant this critical coupling does not depend on γ! Values for the critical coupling obtained by numerically solving Eq. (3.17) can be found in Tabs. 3.1 and 3.2.

The next step in improving this calculation in the fermionic language would be to take the momentum dependence of the couplings into account (e.g. [89]) or to include higher orders of the fermionic fields into the truncation. This seems quite complicated and at first sight we have no physical guess of what is relevant. The renormalization group treatment of the bosonic formulation in Chap. 4 seems much more promising in this respect.

3.4 Gap Equation

Let us finally turn to the SDE as the last method discussed in Chap. 2. For the model Eq. (1.1) the SDE, approximated to lowest order, is depicted in Fig. 3.3. It is a closed equation since only the bare four-fermion vertex appears. (Only higher order terms involve the full four-fermion vertex.) We write the full fermionic propagator G_{F} as

$$G_{\mathrm{F}}^{-1}(p) = G_{\mathrm{F0}}^{-1}(p) + \Sigma_{\mathrm{F}}(p) \tag{3.18}$$

with the free propagator G_{F0} and self energy Σ_{F}. Using this one obtains a gap equation for the self energy which can be solved self consistently. To simplify the discussion we make an ansatz for the self energy:

$$\Sigma_{\mathrm{F}} = M_{\mathrm{F}}\gamma^5, \tag{3.19}$$

where the effective fermion mass M_{F} obeys the gap equation

$$M_{\mathrm{F}} = 8v_4 \left[\overline{\lambda}_\sigma + \overline{\lambda}_V\right] \int_0^{\Lambda^2} dx\, x \frac{M_{\mathrm{F}}}{x + M_{\mathrm{F}}^2}. \tag{3.20}$$

The onset for nontrivial solutions determines the critical couplings:

$$\left[\overline{\lambda}_\sigma + \overline{\lambda}_V\right]_{\mathrm{crit}} = \frac{1}{8v_4\Lambda^2}. \tag{3.21}$$

This result is shown in Tabs. 3.1, 3.2 and does not depend on γ, as expected for a fermionic calculation. We observe that the MFT result for the $\overline{\lambda}_\sigma^{\mathrm{crit}}$ coincides with the SD approach for a particular choice $\gamma = 1/2$. However, in general MFT is not equivalent to the lowest-order SDE. This can be seen by computing also the critical coupling for the onset of SSB in the vector channel. The MFT and SD results do not coincide for the choice $\gamma = 1/2$. This becomes evident if we use a vectorlike ansatz instead of Eq. (3.19) for the self-energy:

$$\Sigma_{\mathrm{F}} = \not{V}. \tag{3.22}$$

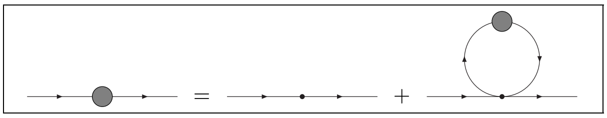


Figure 3.3: Diagrammatic representation of the lowest order Schwinger-Dyson equations for the fermionic model Eq. (1.1). The shaded circles depict the full propagator.

Using this ansatz for the self energy we find for the onset of non trivial solutions

$$[\overline{\lambda}_\sigma + 3\overline{\lambda}_V]_V^{\mathrm{crit}} = \frac{1}{2v_4\Lambda^2}. \tag{3.23}$$

This is different from the MFT result

$$[\overline{\lambda}_V]_V^{\mathrm{crit,MFT}} = \frac{1}{4v_4\Lambda^2(1-\gamma)} \tag{3.24}$$

obtained from Eq. (3.3) by setting $\tilde{\phi} = \tilde{A}^\mu = 0$. Again it would be possible to find a choice for γ which makes both results equal. But, in general, this will not be $\gamma = 1/2$. We note that in this channel the dependence of the MFT result is even worse than in the scalar channel.

At last, let us note that MFT and the SDE gap equation share a nice feature, both allow us to proceed into the region of broken symmetry. The self consistency conditions (3.3) and (3.20) are valid for non-trivial solutions and hence finite values of the bosonic condensate, whereas the simple flow equation (3.17) breaks down (the couplings become infinite) when we approach the phase with broken symmetry.

Chapter 4

Partial Bosonization I: Basic Idea

The MFT calculation introduces "mean fields" composed of fermion - antifermion (or fermion - fermion) bilinears. This is motivated by the fact that in many physical systems the fermions are not the only relevant degrees of freedom at low energies. Bosonic bound states become important and may condense. Examples are Cooper pairs in superconductivity or mesons in QCD. For a detailed description of the interplay between fermionic and composite bosonic fluctuations it seems appropriate to treat both on equal footing by introducing explicit fields for the relevant composite bosons. This will also shed more light on the status of MFT.

4.1 Calculating the Bosonized Action

A method for introducing the desired composite fields is partial bosonization [35–39], sometimes also referred to as a Hubbard Stratonovich transformation. Regardless of the name, in principle it is nothing else but the insertion of a nicely written factor of unity in the functional integral for the partition function.

4.1.1 A Toy Model Bosonization ($d = 0$)

Before going into the details, let us demonstrate the idea on a 0-dimensional toy model. In 0 dimensions the field variable $\phi(p)$ is replaced by a simple c- or Grassmann number x depending on whether we deal with fermions or bosons[1]. To be explicit let us consider an "action"

$$S(x) = \frac{1}{2}x^{\mathrm{T}}Mx + \frac{\lambda}{2}(x^{\mathrm{T}}Sx)^2 \tag{4.1}$$

[1]Of course, if x is a Grassmann number it is necessary to have several different components x_i in order to have non-trivial interactions.

where M and S are matrices simulating the mass and kinetic terms (M), and the spin structure of the interactions (S). Since we live in 0 dimensions the momentum or position space integral is trivial, i.e. absent. This action models a massive field with an inverse propagator M and a quartic interaction with coupling strength λ. In the following we want to introduce an auxiliary "composite" field y for the operator,

$$O_S(x) = -\frac{h}{m^2} x^{\mathrm{T}} S x, \tag{4.2}$$

where we have introduced the for the moment arbitrary constants h, m for later convenience. To study a composite operator $O_S(x)$ it is useful to introduce a source term $kO_S(x)$ for this composite operator in addition to the ordinary source term jx. The functional integral over the field variables becomes an ordinary integral over x, and the partition function reads,

$$Z(j,k) = \int dx \exp(-S(x) + jx + kO_S(x) + \frac{a}{2}k^2), \tag{4.3}$$

where we have used the freedom to add a field independent term quadratic in k to the action.

Using the translation invariance of the integral we can obtain the following rather trivial identity,

$$1 = N \int dy \exp(-\frac{m^2}{2} y^2) = N \int dy \exp(-\frac{m^2}{2}(y - O_S(x) + d)^2), \tag{4.4}$$

where N is nothing but a normalization constant $N = \left(\int dy \exp(-\frac{m^2}{2}y^2)\right)^{-1}$ and d is for the moment arbitrary, but will be determined later. Inserting this under the integral in (4.3) yields,

$$\begin{aligned} Z(j,\hat{k}) &= N \int dxdy \exp(-\hat{S}(x,y) + jx + \hat{k}y), \qquad (4.5)\\ \hat{S}(x,y) &= \frac{1}{2}x^{\mathrm{T}} M x + \frac{\lambda}{2}(x^{\mathrm{T}} S x)^2 + k\frac{h}{m^2} x^{\mathrm{T}} S x + \frac{a}{2}k^2 + \frac{m^2}{2} y^2 \\ &\quad + hyx^{\mathrm{T}} S x + \frac{1}{2}\frac{h^2}{m^2}(x^{\mathrm{T}} S x)^2 + m^2 dy + hdx^{\mathrm{T}} S x + \frac{m^2 d^2}{2} + \hat{k}y. \end{aligned}$$

The first line looks promising, as it is the partition function for a theory with two "fields" x, y and an action $\hat{S}(x,y)$. The second line is still a mess which over and above depends explicitly on the sources k and $\hat{k}$. However, remembering that we have introduced several arbitrary parameters we can choose those to our convenience,

$$a = -\frac{1}{m^2}, \quad d = -\frac{k}{m^2}, \quad \hat{k} = k, \tag{4.6}$$

simplifying

$$\hat{S}(x,y) = \frac{1}{2}x^{\mathrm{T}}Mx + \frac{1}{2}\left(\lambda + \frac{h^2}{m^2}\right)(x^{\mathrm{T}}Sx)^2 + \frac{m^2}{2}y^2 + hyx^{\mathrm{T}}Sx. \tag{4.7}$$

Finally, employing the choice

$$\frac{h^2}{m^2} = -\lambda, \tag{4.8}$$

cancels all quartic interactions of x, leaving us with a mass term for the "composite" field y, a Yukawa-type interaction between y and x in addition to the propagator term $x^{\mathrm{T}}Mx$ for the "elementary" field x,

$$\hat{S}(x,y) = \frac{1}{2}x^{\mathrm{T}}Mx + \frac{m^2}{2}y^2 + hyx^{\mathrm{T}}Sx. \tag{4.9}$$

This also explains why we have introduced the constants h and m in the normalization of $O_S[x]$, Eq. (4.2).

Having accomplished the "partial bosonization" of our 0-dimensional model we would like to comment on some rather technical points:

1. Physically it is clear that with respect to the symmetries y should have the same transformation properties as the composite operator $O_S(x)$. From a more technical point of view this is necessary as we would otherwise be unable to perform the shift in the integration variable in Eq. (4.4).

2. In the derivation given above we did not specify if x is bosonic or fermionic. We can use the same procedure to introduce composites made up of fermions or bosons. However, we should be careful. If the integral in (4.3) is fermionic, it is convergent for all possible choices of λ, because of the rules of Grassmannian integration. The integral over the auxiliary field y, Eq. (4.4), is only convergent for $m^2 > 0$. This gives us the condition that $\lambda < 0$, in order to render everything finite. For bosons, however, a $\lambda < 0$ leads to a divergence in Eq. (4.3). So, naively our bosonization procedure works only for fermions and a certain region of the coupling constant. Although it is possible to circumvent these naive arguments by an integration along the complex axis, any interpretation of y as a bound state is still awkward. Therefore, we will restrict ourselves to stable potentials of the composite field (s. below), i.e. the integration over the composite is convergent.

3. It is not necessary that the integration over the auxiliary field is Gaussian as in Eq. (4.4). Indeed, we can replace the term $-\frac{m^2}{2}y^2$ by any potential $-V(y)$ as long as,

$$V(y) > c, \quad \lim_{|y|\to\infty} \frac{V(y)}{(\ln(y))^2} \to \infty, \tag{4.10}$$

i.e. $V(y)$ is bounded from below and grows sufficiently fast for $y \to \infty$. This allows us to absorb also higher order interaction as e.g. a term $-\kappa(x^{\mathrm{T}}Sx)^4$ by a term $\kappa\frac{m^8}{h^4}y^4$ in $V(y)$. In general,

$$-F(O_S(x)) \to -F(O_S(x)) + F(O_S(x) - y), \tag{4.11}$$

such that the purely fermionic terms are cancelled. Of course, this leaves us with non-linear Yukawa couplings, as e.g. $\sim y^3x^2$ and other complicated interactions $\sim y^2x^4$ or $\sim yx^6$.

4. As can be seen from (4.11) we can also treat terms linear in the composite operator $O_S(x)$, removing e.g. all parts $\sim x^{\mathrm{T}}Sx$ from $\frac{1}{2}x^{\mathrm{T}}Mx$. A typical example for this would be the translation of a fermionic mass term $m\overline{\psi}\psi$ into a source term $j\phi$ for a boson corresponding to $\overline{\psi}\psi$.

5. Although it is the most common case, it is not necessary that the composite operators that we want to bosonize are made up of exactly two field operators. In principle, they can contain an arbitrary number of fields. The composites can even be fermionic.

6. Using the translation invariance of the integral as in Eq. (4.4) is the simplest but not the only possible way to obtain an identity useful for the introduction of composite fields. In general, any identity

$$1 = N\exp(F(O_S(x)))\int dy\,\exp(-V(x,y)), \tag{4.12}$$

can be used to cancel a part $F(O_S(x))$ in the initial action. However, the direct interpretation $y \sim O_S(x)$ will usually be lost. Of course the $V(x,y)$ in Eq. (4.12) is far from unique. One possibility is always $V(x,y) = F(O_S(x) - y)$ as obtained in (4.11). In practice it quite difficult to find a $V(x,y)$ with a suitably simple form like $V(x,y) \sim xy + V'(y)$.

7. Sometimes, it might seem useful to add some form of interaction, e.g. $\sim y^4$, between the composite fields to the bosonized action (4.9). We can then use the argument of 6 in a backward way to determine what kind of (higher order) interactions this would introduce into the initial unbosonized action.

8. We can recover the initial action by performing the integration over the auxiliary field y,

$$\exp(-S(x)) = \int dy\,\exp(-\hat{S}(x,y)). \tag{4.13}$$

4.1.2 The Fierz Ambiguity

Already in this simple model we can get a grasp how the Fierz ambiguity arises. Let us take a look at the "four-fermion interaction" in Eq. (4.1),

$$\begin{aligned}\frac{\lambda}{2}(x^{\mathrm{T}}Sx)^2 = \frac{\lambda}{2}S_{i_1 i_2}S_{i_3 i_4}x_{i_1}x_{i_2}x_{i_3}x_{i_4} &= \frac{\lambda}{2}(S\otimes S)_{i_1 i_2 i_3 i_4}x_{i_1}x_{i_2}x_{i_3}x_{i_4} \\ &= \frac{1}{2}\Lambda_{i_1 i_2 i_3 i_4}x_{i_1}x_{i_2}x_{i_3}x_{i_4},\end{aligned} \tag{4.14}$$

where the LHS defines Λ. We can now permute the x_i, e.g. let us exchange x_{i_2} and x_{i_4},

$$\Lambda_{i_1 i_2 i_3 i_4}x_{i_1}x_{i_2}x_{i_3}x_{i_4} = \pm\Lambda_{i_1 i_2 i_3 i_4}x_{i_1}x_{i_4}x_{i_3}x_{i_2} = \Lambda'_{i_1 i_2 i_3 i_4}x_{i_1}x_{i_2}x_{i_3}x_{i_4}, \tag{4.15}$$

where

$$\Lambda'_{i_1 i_2 i_3 i_4} = \pm\Lambda_{i_1 i_4 i_3 i_2}, \tag{4.16}$$

and the sign is + for bosons and − for fermions, respectively. Now, let us assume that Λ' has a decomposition (this assumption is nearly always fulfilled),

$$\Lambda' = \lambda_{TT'}(T\otimes T') \neq \lambda(S\otimes S). \tag{4.17}$$

Accordingly, we would bosonize the RHS of Eq. (4.15) with fields corresponding to the operators $O_T(x)$ and the coupling strengths $\lambda_{TT'}$. Hence, we obtain a different set of composite fields and coupling strengths for the *identical* action. In general, we can perform an arbitrary permutation of the x_i, and we can obtain not only two but several different bosonized actions. This is even worse for higher order (e.g. x^6) interactions.

Comparing Eq. (4.17) with the Fierz identity Eq. (B.5) it becomes clear that an exchange of fields like in Eq. (4.15) is a Fierz transformation. Since the bosonized action may look quite different, it is no big surprise that simple approximations might yield different results. This is what we call "Fierz Ambiguity".

One might wonder about the fact that different Λ, Λ' describe the same (unbosonized) action. For fermions this is quite easy to understand. Due to the Grassmann identity $x_i x_j = -x_j x_i$ only the completely antisymmetric parts of Λ give non-vanishing contributions to the action. Hence, all

$$\Lambda = \hat{\Lambda} + \Sigma \tag{4.18}$$

yield identical action, as long as Σ is a sum of terms which are symmetric in at least two indices. If we want, we can choose $\hat{\Lambda}$ to be completely antisymmetric. Any Fierz transformation described above can be obtained by adding a suitable chosen Σ. The problem is that a non-vanishing Σ usually does *not* give a vanishing contribution in the partially bosonized action. As there is great freedom in choosing Σ we can get a nearly arbitrary bosonized action. For bosons the story is essentially the same, only one has to replace symmetric by antisymmetric and vice versa.

4.1.3 The Case $d > 0$, MFT Revisited

It is straightforward to generalize the procedure described in the previous section to the case of $d > 0$. Indeed, the change is more or less only a matter of semantics, as we replace functions by functionals and integration by functional integration,

$$F(\ ) \to F[\ ], \quad d \to \mathcal{D}. \tag{4.19}$$

To demonstrate this let us repeat the procedure for the action (1.1)[2]. Introducing bosonic fields for the composite operators corresponding to scalar, vector and axial vector bosons,

$$O_\phi[\psi] = \frac{h_\sigma}{2\mu_\sigma^2}\overline{\psi}(1-\gamma^5)\psi = O^\dagger_{\phi^\star}[\psi], \quad O_V[\psi] = \frac{h_V}{\mu_V^2}\overline{\psi}\gamma^\mu\psi, \quad O_A[\psi] = \frac{h_A}{\mu_A^2}\overline{\psi}\gamma^\mu\gamma^5\psi \tag{4.20}$$

we obtain,

$$Z = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left(-\mathrm{S}[\psi]\right) = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}\phi\mathcal{D}V^\mu\mathcal{D}A^\mu\mathcal{N}_\phi\mathcal{N}_V\mathcal{N}_A \exp\left(-\mathrm{S}[\psi]\right) \tag{4.21}$$

with

$$\begin{aligned}
\mathcal{N}_\phi &= \exp\left[-\int_x \mu_\sigma^2\left(\phi^\star + \frac{h_\sigma}{2\mu_\sigma^2}\overline{\psi}(1+\gamma^5)\psi\right)\left(\phi - \frac{h_\sigma}{2\mu_\sigma^2}\overline{\psi}(1-\gamma^5)\psi\right)\right],\\
\mathcal{N}_V &= \exp\left[-\int_x \frac{\mu_V^2}{2}\left(V^\mu - \frac{h_V}{\mu_V^2}\overline{\psi}\gamma^\mu\psi\right)\left(V_\mu - \frac{h_V}{\mu_V^2}\overline{\psi}\gamma_\mu\psi\right)\right],\\
\mathcal{N}_A &= \exp\left[-\int_x \frac{\mu_A^2}{2}\left(A^\mu - \frac{h_A}{\mu_A^2}\overline{\psi}\gamma^\mu\gamma^5\psi\right)\left(A_\mu - \frac{h_A}{\mu_A^2}\overline{\psi}\gamma_\mu\gamma^5\psi\right)\right].
\end{aligned} \tag{4.22}$$

Collecting the terms in the exponentials and using Eq. (1.5) as the equivalent to Eq. (4.8), the four-fermion interaction is cancelled. As expected, it is now replaced by mass terms for the bosons and Yukawa couplings between bosons and fermions as given by the expression (1.4). We note, that the bosons do not yet have a non-trivial kinetic term and the propagator is simply $\frac{1}{\mu^2}$.

Having arrived at the partially bosonized action (1.4) for our model (1.1) we can use it to gain new insight into MFT. The action Eq. (1.4) is quadratic in the fermionic fields, hence the functional integral over the fermionic degrees of freedom is Gaussian and can be done in one step. As we have seen in the previous chapter this leads exactly to the MFT results. More precisely, we understand now that for different choices of γ the MFT treatment leaves out different bosonic fluctuations.

In this context we note that the condition (4.10) restricts the possible couplings to $\lambda_\sigma, \lambda_V, \lambda_A > 0$. In the invariant variables this restriction translates to $\overline{\lambda}_\sigma, \overline{\lambda}_V > 0$ and for γ it implies $0 < \gamma < 1$.

[2]For simplicity we skip the introduction of bosonic sources.

Incidentally, we note that in the case of a four-fermion interaction we have an alternative to Eq. (4.13) to recover the initial action from the partially bosonized one. Instead of integrating over the bosonic auxiliary fields we can simply solve the classical field equations for the bosonic fields in terms of the fermionic fields and reinsert them in the partially bosonized action. Starting from Eq. (1.4) this returns us to Eq. (1.1).

The crucial advantage of the bosonic formulation is that it can easily be generalized. For example, the bosonic bound states become dynamical fields if we allow for appropriate kinetic terms in the truncation, i.e.

$$\Delta\Gamma_{\mathrm{kin}} = \int d^4x \Big\{ Z_\phi \partial_\mu \phi^\star \partial^\mu \phi + \frac{Z_V}{4} V_{\mu\nu} V^{\mu\nu} + \frac{Z_A}{4} A_{\mu\nu} A^{\mu\nu} + \frac{Z_V}{2\alpha_V} (\partial_\mu V^\mu)^2 + \frac{Z_A}{2\alpha_A} (\partial_\mu A^\mu)^2 \Big\} \tag{4.23}$$

with

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu, \quad A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.24}$$

Also spontaneous symmetry breaking can be explicitly studied if we replace $\mu_\sigma^2 \phi^\star \phi$ by an effective potential $U(\phi^\star\phi)$ which may have a minimum for $\phi \neq 0$. This approach has been followed in previous studies [90–93]. We remark that for those terms to be present in the effective action it is not necessary for them to be present in the initial (bosonized) action. They naturally receive non-vanishing corrections by loop diagrams. E.g. the kinetic terms (4.23) get a correction from the diagram depicted in Fig. 3.1 with non-zero external momentum. Nevertheless, it is instructive to investigate how such terms would look like in the unbosonized language. For the potential terms this has already been discussed in Sect. 4.1.1, i.e. Eqs. (4.11), (4.12), for the kinetic (derivative) terms we will do this in the next section.

4.1.4 Beyond Pointlike Interactions[0]

So far our bosonization procedure seems relatively simple, and it is. However, we should mention that above we have bosonized only the very special case of a *pointlike*, i.e. *local* four-fermion interaction[3],

$$\int_x \lambda\, \Psi(x)\Psi(x)\Psi(x)\Psi(x) = \int_{p_1,p_2,p_3,p_4} \lambda\, \Psi(p_1)\Psi(p_2)\Psi(p_4)\Psi(p_3)\delta(p_1+p_2+p_3+p_4). \tag{4.25}$$

[0] This section discusses some details needed in Sect. 5.4, and can also be read then.

[3] In this section we suppress all internal indices. In particular indices distinguishing between $\overline{\psi}$ and ψ. λ is a matrix with four such indices. All problems connected with the internal indices are completely analogous to the previous sections. Therefore, we allow ourselves to be somewhat sloppy concerning the internal indices, simplifying the notation.

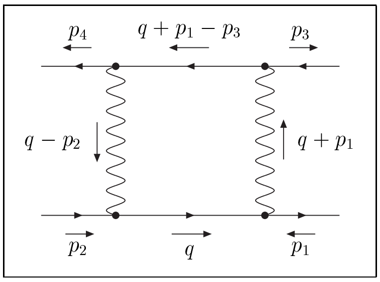


Figure 4.1: Typical diagram contributing to an effective four-fermion interaction like in Eq. (4.25) in QCD. The solid lines denote fermions with propagator P_F^{-1}, the wiggled lines gluons with propagator P_B^{-1}. The labelled arrows denote the momentum flow. The diagram suggests that the contribution $\sim \int_q P_F^{-1}(q)P_F^{-1}(q+p_1-p_3)P_B^{-1}(q+p_1)P_B^{-1}(q-p_2)$ to the effective four-fermion interactions is not a constant but depends, at least on some combination of the external momenta p_1, p_2, p_3, p_4.

This is by no means the most general form of a four-fermion interaction. Giving up locality, λ can become an arbitrary[4] function of the four momentum variables,

$$\lambda \to \lambda(p_1, p_2, p_3, p_4). \tag{4.26}$$

At first, giving up locality sounds like a big step not to be treated lightly. However, we should remember, that we frequently use those four- and multi-fermion interactions not as a fundamental interaction but to model an effective interaction at some intermediate scale. An example are the four-fermion interactions used to model QCD at low energies. As an example, a diagram contributing to lowest order is depicted in Fig. 4.1.

So, what can we do about the bosonization of those awfully complicated interactions? In the previous section we have considered local operators of the form (C is a constant matrix in the space of internal indices),

$$O[\Psi](x) = \Psi^{\mathrm{T}}(x)C\Psi(x) = \int_{x,y} \Psi^{\mathrm{T}}(y)C\Psi(z)\delta(x-y)\delta(x-z) = \int_p O[\Psi](p)\exp(ipx), \tag{4.27}$$

with

$$O[\Psi](p) = \int_{q_1,q_2} \Psi^{\mathrm{T}}(q_1)C\Psi(q_2)\delta(q_1+q_2-p). \tag{4.28}$$

Keeping in mind our physical picture of a bound state, we find that (4.27) is very restrictive. Indeed, for a physical bound state we would expect that the particles of which the bound state is composed are usually not at the same

[4] Of course, it must have the right transformation properties under Lorentz transformations.

place, but smeared out over a certain region of space. Therefore, we replace $\delta(x-y)\delta(x-z) \to \tilde{g}(x-y, x-z)$ in Eq. (4.27), i.e. the "elementary particles" need no longer be located exactly at x but they can be somewhat distributed around x. In a momentum space formulation we find,

$$O[\Psi](p) = hG(p) \int_{p_1,p_2} \Psi^{\mathrm{T}}(p_1) C \Psi(p_2) g(p_1,p_2) \delta(p_1+p_2-p). \tag{4.29}$$

establishing that the momentum p of the composite operator is the sum, $p = p_1 + p_2$, of the momenta of the "elementary particles", as it must be for a bound state. The function $g(p_1, p_2)$ is the so called (amputated) Bethe-Salpeter wave function [66; 94]. The function $hG(p)$ is a generalization of the factor $\frac{h}{m^2}$ in Eq. (4.2), and it serves the same purpose namely, to obtain a simple form with Yukawa coupling $\sim h$ while keeping the direct relation $\phi \hat{=} O[\Psi]$ between the bosonic fields ϕ and the composite operator $O[\Psi]$, as we will see below.

Proceeding along the lines of the previous sections we insert a functional integral

$$1 = \mathcal{N} \int \mathcal{D}\phi \exp(-\int_p \frac{1}{2} \phi^{\mathrm{T}}(-p) G^{-1}(p^2) \phi(p)). \tag{4.30}$$

Shifting the functional integral by the operator (4.29) we find that we can absorb a four-fermion interaction of the following form,

$$\lambda(p_1,p_2,p_3,p_4) \sim h^2 g(p_1,p_2) G(s) g(-p_3,-p_4) C^{\mathrm{T}} \otimes C, \quad s = (p_1+p_2)^2 = (p_3+p_4)^2, \tag{4.31}$$

in a contribution

$$S_\lambda[\Psi,\phi] = \int_{p_1,p_2} \left[hg(p_1,p_2)\Psi^{\mathrm{T}}(p_1) C \phi^{\mathrm{T}}(-p_1-p_2)\Psi(p_2) + \phi^{\mathrm{T}}(p_1) G^{-1}(p_1^2)\phi(p_2)\delta(p_1+p_2) \right], \tag{4.32}$$

to the partially bosonized action. It is clear that $G(s)$ plays the role of the bosonic propagator. Hence, $G(s)$ should have an appropriate pole structure in the complex s-plane, e.g.

$$G(s) \sim \frac{1}{m^2+s}. \tag{4.33}$$

Let us summarize this in the following properties:

1. Eq. (4.31) with the pole structure given by (4.33) is the most general momentum structure for the four-fermion interaction $\sim \lambda$ we can absorb in a single bosonic field and an action quadratic in the bosons. We can only bosonize four-fermion interactions *factorizing* into two pairs of momenta. Usually, contributions to λ like the one depicted in Fig. 4.1 do not factorize completely, therefore bosonization is usually only an approximation. On the other hand factorization of the four-fermion interaction signals the onset of physical bound states and can be checked numerically [64].

2. Permutation of the fields (corresponding to Fierz transformations) permute the momentum variables on the RHS of Eq. (4.31). This allows us to absorb momentum structures with poles in the t- respectively u-channels ($t = (p_1 - p_3)^2 = (p_2 - p_4)^2$, $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$, s, t, u are the Mandelstam variables).

3. Turning the argument of 2. around, we can determine the "correct" Fierz transformation by an examination of the momentum structure (poles!).

Since it might help us to resolve the whole mess of the Fierz ambiguity, let us illustrate the third point by calculating an example. In addition this will also demonstrate how a momentum dependence of the four-fermion interaction and a wave function renormalization for the composite bosons are connected.

Starting from the action (1.1) extended by the kinetic terms (4.23) let us calculate the corresponding purely fermionic action. For simplicity we take $Z_V = Z_A = h_V = h_A = 0$, i.e. we have no vector and axial vector bosons. In momentum space we then have,

$$\begin{aligned}
S_{\mathrm{B}} = \int_{q_1,q_2} \delta(q_1+q_2) \Bigg[& (\mu_\sigma^2 + Z_\sigma q_1^2)\phi^\star(q_1)\phi(q_2) \\
& + \int_{p_1,p_2} h_\sigma \overline{\psi}(-p_1)\left(\frac{1+\gamma^5}{2}\right)\psi(p_2)\phi(p_2)\delta(q_1-p_1-p_2) \\
& - \int_{p_3,p_4} h_\sigma \phi^\star(q_1)\overline{\psi}(p_4)\left(\frac{1-\gamma^5}{2}\right)\psi(-p_3)\delta(q_2-p_3-p_4)\Bigg]
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
&= \int_{q_1,q_2} \delta(q_1+q_2)\ (\mu_\sigma^2 + Z_\sigma q_1^2) \\
&\quad \left(\phi^\star(q_1) + \int_{p_1,p_2}\frac{h_\sigma}{\mu_\sigma^2 + Z_\sigma q_1^2}\overline{\psi}(-p_1)\left(\frac{1+\gamma^5}{2}\right)\psi(p_2)\delta(q_1-p_1-p_2)\right) \\
&\quad \times \left(\phi(q_2) - \int_{p_3,p_4}\frac{h_\sigma}{\mu_\sigma^2 + Z_\sigma q_2^2}\overline{\psi}(p_4)\left(\frac{1-\gamma^5}{2}\right)\psi(-p_3)\delta(q_2-p_3-p_4)\right) \\
&+\frac{1}{2}\int_{p_1,p_2,p_3,p_4} \delta(p_1+p_2-p_3-p_4)\frac{h_\sigma^2}{2(\mu_\sigma^2 + Z_\sigma(p_1+p_2)^2)} \\
&\qquad \times \left[\overline{\psi}(-p_1)\psi(p_2)\overline{\psi}(p_4)\psi(-p_3) - \overline{\psi}(-p_1)\gamma^5\psi(p_2)\overline{\psi}(p_4)\gamma^5\psi(-p_3)\right].
\end{aligned}$$

After the usual shift in the integration variable we can perform the Gaussian integration over the bosonic fields, removing the first term on the RHS of Eq. (4.34). From the second term we can read of the four-fermion interaction,

$$\lambda_\sigma(p_1,p_2,p_3,p_4) = \frac{h_\sigma^2}{2(\mu_\sigma^2 + Z_\sigma(p_1+p_2)^2)} = \frac{h_\sigma^2}{2(\mu_\sigma^2 + Z_\sigma(p_3+p_4)^2)} = \frac{h_\sigma^2}{2(\mu_\sigma^2 + Z_\sigma s)}. \tag{4.35}$$

In particular, the four-fermion interaction λ only depends on the Mandelstam variable s, while it is constant in t and u.

An FT permutes $-p_2$ and p_3. After relabelling the integration indices we find,

$$\begin{aligned}\lambda_\sigma(p_1,p_2,p_3,p_4) &= 0, \\ \lambda_V(p_1,p_2,p_3,p_4) &= -\lambda_A(p_1,p_2,p_3,p_4) = -\frac{h_\sigma^2}{4(\mu_\sigma^2 + Z_\sigma(p_1-p_3)^2)} = -\frac{h_\sigma^2}{4(\mu_\sigma^2 + Z_\sigma t)}.\end{aligned} \tag{4.36}$$

First of all these coupling do not factorize in functions depending only on p_1, p_2 and p_3, p_4 respectively. Secondly, they have a pole in t which cannot be directly absorbed by bosonization. Therefore, this is not the "right" FT. A similar calculation for the vector and axial vector bosons would have resulted in four-fermion interactions depending on s in the vector and axial vector channels, respectively. While after an FT we would have interactions in all channels, scalar, vector and axial vector, but again with the "wrong" dependence on t which cannot be absorbed into bosons. Roughly speaking we have the following recipe, if the four-fermion interaction depends on s, bosonize, else if it depends on t FT exactly once and then bosonize.

4.2 Bosonic RG flow

Having talked at length about how we can obtain the partially bosonized action Eq. (1.4) it is time to put it to some use. Let us start with an RG calculation in a very simple truncation. The flow equations in the bosonic language are obtained in complete analogy with the fermionic formulation. In this section we restrict the discussion to a "pointlike" truncation as given by Eq. (1.4) with k-dependent couplings. We will see (Chap. 5) that we reproduce the result of the last section in this approximation if we take care of the fact that new fermionic interactions are generated by the flow and have to be absorbed by an appropriate k-dependent redefinition of the bosonic fields.

It is instructive to neglect in a first step all bosonic fluctuations by setting all bosonic entries in the propagator matrix $\mathcal{P}^{-1}$ equal to zero. This removes all diagrams with internal bosonic lines. Among other things this neglects the vertex correction Fig. 4.2 and therefore the running of the Yukawa couplings. Indeed, Fig. 3.1 is the only contributing diagram and we recover MFT. One obtains the flow equations

$$\begin{aligned}\partial_t \mu_{\sigma,k}^2 &= 8h_{\sigma,k}^2 v_4 k^2 l_1^{(F),4}(s), \quad \partial_t \mu_{V,k}^2 = 8h_{V,k}^2 v_4 k^2 l_1^{(F),4}(s), \\ \partial_t \mu_{A,k}^2 &= 8h_{A,k}^2 v_4 k^2 l_1^{(F),4}(s), \\ \partial_t h_{\sigma,k} &= 0, \quad \partial_t h_{V,k} = 0, \quad \partial_t h_{A,k} = 0.\end{aligned} \tag{4.37}$$

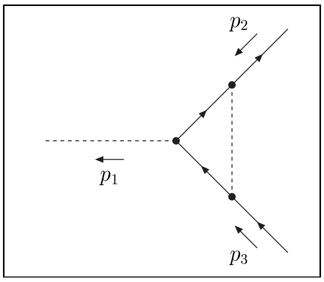


Figure 4.2: Vertex correction diagram in the bosonized model. Solid lines are fermions, dashed lines are bosons. There exist several diagrams of this type since we have different species of bosons. The momentum configuration indicated by the arrows is such that it gives a contribution $\sim \phi(-p_1)\overline{\psi}(-p_2)\psi(p_3)$. The pointlike limit employed in this section corresponds to an evaluation for $p_i = 0$.

As long as we do not consider the wave function renormalization (4.23) for the bosons, the flow can completely be described in terms of the dimensionless combinations

$$\tilde{\epsilon}_{\sigma,k} = \frac{\mu^2_{\sigma,k}}{h^2_{\sigma,k}k^2}, \quad \tilde{\epsilon}_{V,k} = \frac{\mu^2_{V,k}}{h^2_{V,k}k^2}, \quad \tilde{\epsilon}_{A,k} = \frac{\mu^2_{A,k}}{h^2_{A,k}k^2}. \tag{4.38}$$

Due to the constant Yukawa couplings we can integrate Eq. (4.37). We find critical couplings:

$$\frac{\mu^2_\sigma}{h^2_\sigma \Lambda^2}\mid_{\mathrm{crit}} = 4v_4, \quad \frac{\mu^2_V}{h^2_V \Lambda^2}\mid_{\mathrm{crit}} = 4v_4, \quad \frac{\mu^2_A}{h^2_A \Lambda^2}\mid_{\mathrm{crit}} = 4v_4. \tag{4.39}$$

These are, of course, the results of MFT, Eq. (3.7). We note that in Eq. (4.37) the equations for the different species of bosons are completely decoupled. The mass terms do not turn negative at the same scale for the different species. Indeed it is possible that the mass of one boson species turns negative while the others do not. Such a behavior is expected for the full theory, whereas for the fermionic RG of Sect. 3.3 all couplings diverge simultaneously due to their mutual coupling. However, no real conclusion can be taken from Eq. (4.39) because of the strong dependence on the unphysical parameter γ.

Now, let us also take into account the bosonic fluctuations. This includes the vertex correction Fig. 4.2 and the flow of the Yukawa couplings does not vanish anymore

$$\begin{aligned}
\partial_t h^2_{\sigma,k} &= -32 v_4 l_1^{(F),4}(s) k^2 h^2_{\sigma,k}\left[\frac{h^2_{V,k}}{\mu^2_{V,k}} - \frac{h^2_{A,k}}{\mu^2_{A,k}}\right], \qquad (4.40)\\
\partial_t h^2_{V,k} &= -4 v_4 l_1^{(F),4}(s) k^2 h^2_{V,k}\left[\frac{h^2_{\sigma,k}}{\mu^2_{\sigma,k}} + 2\left(\frac{h^2_{V,k}}{\mu^2_{V,k}} + \frac{h^2_{A,k}}{\mu^2_{A,k}}\right)\right],\\
\partial_t h^2_{A,k} &= -4 v_4 l_1^{(F),4}(s) k^2 h^2_{A,k}\left[-\frac{h^2_{\sigma,k}}{\mu^2_{\sigma,k}} + 2\left(\frac{h^2_{V,k}}{\mu^2_{V,k}} + \frac{h^2_{A,k}}{\mu^2_{A,k}}\right)\right].
\end{aligned}$$

Using the dimensionless $\tilde{\epsilon}$'s we now find:

$$\begin{aligned}
\partial_t \tilde{\epsilon}_{\sigma,k} &= -2\tilde{\epsilon}_{\sigma,k} + 8\left[1 + 4\tilde{\epsilon}_{\sigma,k}\left(\frac{1}{\tilde{\epsilon}_{V,k}} - \frac{1}{\tilde{\epsilon}_{A,k}}\right)\right] v_4 l_1^{(F),4}(s), \qquad (4.41)\\
\partial_t \tilde{\epsilon}_{V,k} &= -2\tilde{\epsilon}_{V,k} + 8\left[2 + \tilde{\epsilon}_{V,k}\left(\frac{1}{2\tilde{\epsilon}_{\sigma,k}} + \frac{1}{\tilde{\epsilon}_{A,k}}\right)\right] v_4 l_1^{(F),4}(s),\\
\partial_t \tilde{\epsilon}_{A,k} &= -2\tilde{\epsilon}_{A,k} + 8\left[2 - \tilde{\epsilon}_{A,k}\left(\frac{1}{2\tilde{\epsilon}_{\sigma,k}} - \frac{1}{\tilde{\epsilon}_{V,k}}\right)\right] v_4 l_1^{(F),4}(s).
\end{aligned}$$

The onset of spontaneous symmetry breaking is indicated by a vanishing of $\tilde{\epsilon}$ for at least one species of bosons. Large $\tilde{\epsilon}$ means that the corresponding bosonic species becomes very massive and therefore effectively drops out of the flow.

For initial couplings larger than the critical values (see Tabs. 3.1 and 3.2) both $\tilde{\epsilon}_{\sigma,k}$ and $\tilde{\epsilon}_{V,k}$ reach zero for finite t. Due to the coupling between the different channels they reach zero at the same t. At this point $\tilde{\epsilon}_{A,k}$ reaches infinity and drops out of the flow. This is quite different from the flow without the bosonic fluctuations where the flow equations for the different species were decoupled. The breakdown of all equations at one point resembles[5] now the case of the fermionic model discussed in Sect. 3.3. The γ-dependence of the critical couplings is reduced considerably, as compared to MFT. This shows that the inclusion of the bosonic fluctuations is crucial for any quantitatively reliable result. Nevertheless, the difference between the bosonic and the fermionic flow remains of the order of 10%.

4.3 Gap equation in the Bosonized Language

Next, we turn to the SDE for the bosonized model (1.4). They are depicted in Fig. 4.3. We will make here two further approximations by replacing in the last graph of Fig. 4.3 the full fermion-fermion-boson vertex by the classical Yukawa coupling and the full bosonic propagator by μ_{B}^{-2}. We remain with two coupled equations.

[5] This is an artefact of the pointlike approximation.

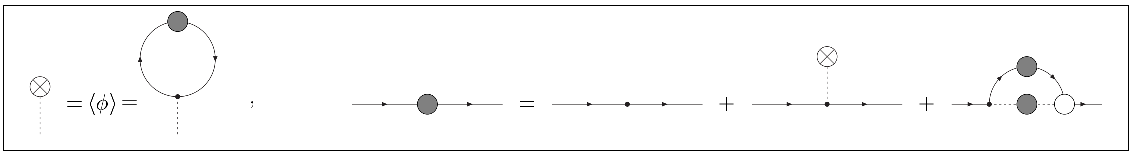


Figure 4.3: Diagrammatic representation of the lowest-order SDE for the partially bosonized model (Eq. (1.4)). The shaded circles depict the full propagator, the circle with the cross is the expectation value of the bosonic field and the empty circle is the full Yukawa vertex.

In a first step we approximate this equations even further by neglecting the last diagram in Fig. 4.3 altogether. Then no fermionic propagator appears on the right hand side of the equation for the fermionic propagator which only receives a mass correction for $\langle\phi\rangle \neq 0$. Without loss of generality we take ϕ real such that $M_{\mathrm{F}} = h_\sigma \phi$ and

$$G_{\mathrm{F}}^{-1}(q) = -\not{q} + h_\sigma \phi \gamma^5. \tag{4.42}$$

Inserting this into the equation for the expectation value ϕ we find

$$\phi = \frac{4v_4}{\mu_\sigma^2} \int_0^{\Lambda^2} dx\, x \frac{h_\sigma^2 \phi}{x + h_\sigma^2 \phi^2}. \tag{4.43}$$

For the onset of nontrivial solutions we now find the critical value

$$\left[\frac{h_\sigma^2}{2\mu_\sigma^2}\right]_{\mathrm{crit}} = \frac{1}{8v_4\Lambda^2} = \left[\overline{\lambda}_\sigma + 2\gamma\overline{\lambda}_V\right]_{\mathrm{crit}} \tag{4.44}$$

which is the (ambiguous) result from MFT given in Eqs. (3.7) and (3.9). This is not surprising since this *exactly is* MFT from the viewpoint of Schwinger-Dyson equations. Indeed, Eq. (4.43) is precisely the field equation which follows by differentiation of the MFT effective action (6.19) with respect to ϕ.

$$\Gamma^{(\mathrm{F})} = \mu_\sigma^2 \phi^2 - 4v_4 \int_0^{\Lambda^2} dx\, x \ln(x + h_\sigma^2 \phi^2). \tag{4.45}$$

In a next step we improve our approximation and include the full set of diagrams shown in Fig. 4.3. Using the same ansatz as before, the self-energy Σ_{F} now has two contributions

$$\Sigma_{\mathrm{F}} = M_{\mathrm{F}}\gamma^5 = h_\sigma \phi \gamma^5 + \Delta m_{\mathrm{F}} \gamma^5. \tag{4.46}$$

The first one is the contribution from the expectation value of the bosonic field whereas Δm_{F} is the contribution from the last diagram in Fig. 4.3, given by an integral which depends on M_{F}. Both in the equation for $\langle\phi\rangle$ and in the equation for the fermionic propagator only M_{F} appears on the RHS. Inserting $\langle\phi\rangle$ in the graph Fig. 4.3 one finds a gap equation which determines M_{F}:

$$M_{\mathrm{F}} = 8v_4 \left[\frac{h_\sigma^2}{2\mu_\sigma^2} + \frac{h_V^2}{\mu_V^2} - \frac{h_A^2}{\mu_A^2}\right] \int_0^{\Lambda^2} dx\, x \frac{M_{\mathrm{F}}}{x + M_{\mathrm{F}}^2}. \tag{4.47}$$

Once more, this can be expressed in terms of the invariant couplings and again we arrive at Eq. (3.20).

We point out that, in order to recover the result of the fermionic SDE we have started with MFT and added diagrams. Therefore, the fermionic SDE (or the bosonic SDE with the extra contribution from the fermionic mass shift diagram) sums over a larger class of diagrams which contains MFT as a subset. This is evident in the language of statistical physics: MFT is the Hartree approximation, while the fermionic SDE is Hartree-Fock.

Looking more closely at the two contributions to M_{F} we find that alone neither the contribution $\sim \phi$ (which amounts to MFT as we have discussed above) nor the "fermionic contribution" Δm_{F} are invariant under FT's. Only the combination M_{F}, which is the fermion mass and therefore a physical quantity, is invariant. Indeed, changing the FT amounts to a redefinition of the bosonic fields. This allows us to choose bosonic fields such that $\Delta m_{\mathrm{F}} = 0$. Taking $\gamma = 1/2$ gives us such a choice of the bosonic fields. This explains why MFT gives the the same result as the purely fermionic calculation in this special case.

In the next chapter we want to adapt this idea of a redefinition of the bosonic fields to the RG calculation, i.e. we want to do it continuously during the flow.

Chapter 5

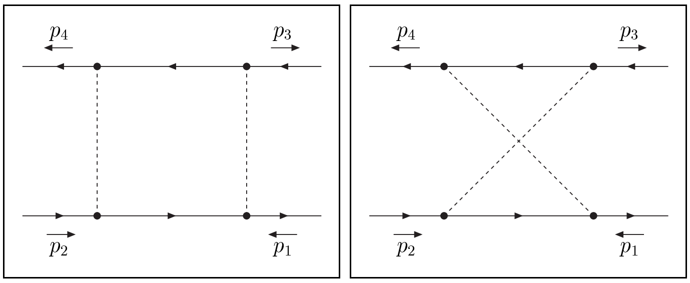
Partial Bosonization II: Scale Dependent Degrees of Freedom

In the last Sect. 4.3 we found that in the SDE formulation we can complete MFT by adding the mass-shift diagram for the fermions (cf. Fig. 4.3). The mass-shift diagram is a contribution to the purely fermionic part of the effective action. Therefore, it makes sense to look for purely fermionic contributions in the RG, too.

5.1 New Four-Fermion Interactions

In our truncation of Chap. 4 the bosonic propagators are approximated by constants μ_k^{-2}. The exchange of bosons therefore produces effective pointlike four-fermion interactions. One would therefore suspect that this approximation should contain the same information as the fermionic formulation with pointlike four-fermion interactions. An inspection of the results in Tabs. 3.1, 3.2 shows, however, that this is not the case for the formulation in the RG context. In particular, in contrast to the fermionic language the results of the bosonic flow equations still depend on the unphysical parameter γ.

In fact, even for small couplings λ the bosonic flow equations of sect. 4.2 do not reproduce the perturbative result. The reason is that at the one-loop level new quartic fermion interactions are generated by the box diagrams shown in Fig. 5.1. A straightforward inspection shows that they contribute to the same order λ^2 as the diagrams in Figs. 3.1 and 4.2. Even if we start from vanishing quartic couplings after partial bosonization, such couplings are generated by the flow. The diagrams



(a) (b)

Figure 5.1: Box diagrams for the bosonized model. Again, solid lines are fermions, dashed lines bosons and vertices are marked with a dot. The diagrams generate new four-fermion $\sim \overline{\psi}(-p_1)\psi(p_2)\overline{\psi}(p_4)\psi(-p_3)$ interactions even for the model (1.4) without direct four-fermion interactions.

in Fig. 5.1 yield

$$
\begin{aligned}
\partial_t \lambda_{\sigma,k} &= \beta_{\lambda_\sigma} = -8 v_4 l_1^{(F),4}(s) k^2 \frac{h_{\sigma,k}^2}{\mu_{\sigma,k}^2}\frac{h_{A,k}^2}{\mu_{A,k}^2} + 4k^{-2}\tilde{\gamma}(k), \qquad (5.1)\\
\partial_t \lambda_{V,k} &= \beta_{\lambda_V} = 24 v_4 l_1^{(F),4}(s) k^2 \frac{h_{V,k}^2}{\mu_{V,k}^2}\frac{h_{A,k}^2}{\mu_{A,k}^2} - 2k^{-2}\tilde{\gamma}(k),\\
\partial_t \lambda_{A,k} &= \beta_{\lambda_A} = -v_4 l_1^{(F),4}(s) k^2 \left[\frac{h_{\sigma,k}^4}{\mu_{\sigma,k}^4} - 12\frac{h_{V,k}^4}{\mu_{V,k}^4} - 12\frac{h_{A,k}^4}{\mu_{A,k}^4}\right] + 2k^{-2}\tilde{\gamma}(k).
\end{aligned}
$$

Here, $\tilde{\gamma}(k)$ is an in principle arbitrary function of scale determining the choice of FT for the generated four-fermion interactions. In other words, $\tilde{\gamma}(k)$ allows for the fact that we can choose a different Fierz representation at every scale. We will make a special choice of this function (similar to the one made in sects. 3.2 and 3.3) namely we require

$$
\frac{\tilde{\epsilon}_{V,k}}{\tilde{\epsilon}_{A,k}} = \frac{\gamma}{1-\gamma} \quad \forall\, k, \qquad (5.2)
$$

with $\tilde{\epsilon}$ given in Eq. (4.38). The resulting equation $\partial_t(\tilde{\epsilon}_{V,k}/\tilde{\epsilon}_{A,k}) = 0$ fixes $\tilde{\gamma}(k)$. An improved choice of $\tilde{\gamma}(k)$ can be obtained once the momentum dependence of vertices is considered more carefully (cf. [42] and Sects. 4.1.4, 5.4).

5.2 Adapted Flow Equation: Solving the Ambiguity

An inclusion of the couplings λ_k into the truncation of the effective average action does not seem very attractive. Despite the partial bosonization we would still have to deal with multi-fermion interactions and the bosonic formulation would be of even higher algebraic complexity than the fermionic formulation. A way out of this has been proposed in [42]. There, it has been shown that it is possible to reabsorb all four-fermion interactions generated during the flow by a redefinition of the bosonic fields. In the following brief description of this method we use a very symbolic notation[1]. In Sect. 5.4 we will add some details on the momentum dependence of field redefinitions.

Introducing an explicit k-dependence for the definition of the bosonic fields in terms of fermion bilinears, the flow equation Eq. (2.32) is modified[2]:

$$\partial_t \Gamma_k = \partial_t \Gamma_k|_{\phi_k} + \frac{\delta \Gamma_k}{\delta \phi_k} \partial_t \phi_k. \tag{5.3}$$

Here $\partial_t \Gamma_k|_{\phi_k} \equiv \partial_t \Gamma_k|$ is the flow of the effective average action at fixed fields. Shifting ϕ by

$$\partial_t \phi_k = \left(\overline{\psi}\psi\right) \partial_t \omega_k \tag{5.4}$$

we find

$$\partial_t \mu^2 = \partial_t \mu^2|, \quad \partial_t h = \partial_t h| + \mu^2 \partial_t \omega_k, \quad \partial_t \lambda = \partial_t \lambda| - h \partial_t \omega_k \tag{5.5}$$

and we can choose ω_k to establish:

$$\partial_t \lambda = 0. \tag{5.6}$$

Instead of including running four-fermion couplings explicitly we therefore have to use only adapted flow equations for the couplings contained in Eq. (1.4).

Let us now apply this method explicitly to our model. Shifting

$$\begin{aligned} \partial_t \phi &= -\overline{\psi} \left(\frac{1-\gamma^5}{2}\right) \psi \partial_t \omega_{\sigma,k}, \quad \partial_t \phi^\star = \overline{\psi} \left(\frac{1+\gamma^5}{2}\right) \psi \partial_t \omega_{\sigma,k}, \\ \partial_t V^\mu &= -\overline{\psi} \gamma^\mu \psi \partial_t \omega_{V,k}, \quad \partial_t A^\mu = -\overline{\psi} \gamma^\mu \gamma^5 \psi \partial_t \omega_{A,k} \end{aligned} \tag{5.7}$$

[1] We refrain from explicitly returning to the 0-dimensional toy model of Sect. 4.1.1, as we hope it is clear from Sect. 4.1.3 that for pointlike interactions the case $d > 0$ involves no additional difficulties. Nevertheless, let us note for completeness, that the replacements $\delta \to \partial$, $\phi_k \to y_k$, $\overline{\psi}\psi \to x^{\mathrm{T}} S x$ would bring us back to the toy model.

[2] It has been pointed out by Jan Pawlowski that after the appropriate modification of the infrared cutoff for the scale-dependent fields [42], the flow equation Eq. (2.32) does not give the exact flow for $\partial_t \Gamma_k|$. However, in the simple approximation of this section the bosonic fields do not yet have an infrared cutoff. Therefore, we can still use (2.32).

we have

$$
\begin{aligned}
\partial_t\lambda_{\sigma,k} &= \partial_t\lambda_{\sigma,k}| - h_{\sigma,k}\partial_t\omega_{\sigma,k}, \\
\partial_t\lambda_{V,k} &= \partial_t\lambda_{V,k}| - 2h_{V,k}\partial_t\omega_{V,k}, \quad \partial_t\lambda_{A,k} = \partial_t\lambda_{A,k}| - 2h_{A,k}\partial_t\omega_{A,k}.
\end{aligned} \tag{5.8}
$$

Requiring $\partial_t\lambda = 0$ for all λ's we can determine the functions ω:

$$
\partial_t\omega_{\sigma,k} = \frac{\beta_{\lambda_\sigma}}{h_{\sigma,k}}, \quad \partial_t\omega_{V,k} = \frac{\beta_{\lambda_V}}{2h_{V,k}}, \quad \partial_t\omega_{A,k} = \frac{\beta_{\lambda_A}}{2h_{A,k}} \tag{5.9}
$$

with the β-functions given in Eq. (5.1). This yields the adapted flow equations for the Yukawa couplings

$$
\begin{aligned}
\partial_t h_{\sigma,k} &= \partial_t h_{\sigma,k}| + \mu^2_{\sigma,k}\partial_t\omega_{\sigma,k}, \\
\partial_t h_{V,k} &= \partial_t h_{V,k}| + \mu^2_{V,k}\partial_t\omega_{V,k}, \quad \partial_t h_{A,k} = \partial_t h_{A,k}| + \mu^2_{A,k}\partial_t\omega_{A,k}.
\end{aligned} \tag{5.10}
$$

Combining Eqs. (4.41), (5.1), (5.2), (5.9), (5.10) determines $\tilde{\gamma}(k)$

$$
\tilde{\gamma}(k) = 2v_4 l_1^{(F),4}(s)\left[-\frac{3}{\tilde{\epsilon}^2_{V,k}} + \frac{1}{\tilde{\epsilon}_{V,k}\tilde{\epsilon}_{A,k}} + \frac{(4\tilde{\epsilon}_{\sigma,k} - \tilde{\epsilon}_{A,k})^2}{4\tilde{\epsilon}^2_{\sigma,k}\tilde{\epsilon}_{A,k}(\tilde{\epsilon}_{V,k} + \tilde{\epsilon}_{A,k})}\right]. \tag{5.11}
$$

Having fixed the ratio between $\tilde{\epsilon}_{V,k}$ and $\tilde{\epsilon}_{A,k}$ we only need two equations to describe the flow. We will use the ones for $\tilde{\epsilon}_{\sigma,k}$ and $\overline{\epsilon}_{V,k} = (1-\gamma)\tilde{\epsilon}_{V,k}$

$$
\begin{aligned}
\partial_t\tilde{\epsilon}_{\sigma,k} &= -2\tilde{\epsilon}_{\sigma,k} + 4\left[(1+\gamma) - 4(-2+\gamma+2\gamma^2)\frac{\tilde{\epsilon}_{\sigma,k}}{\overline{\epsilon}_{V,k}} + 4(3-7\gamma+4\gamma^3)\frac{\tilde{\epsilon}^2_{\sigma,k}}{\overline{\epsilon}^2_{V,k}}\right] l_1^{(F),4}(s)v_4, \\
\partial_t\overline{\epsilon}_{V,k} &= -2\overline{\epsilon}_{V,k} + 4\left[\frac{\overline{\epsilon}_{V,k}}{2\tilde{\epsilon}_{\sigma,k}} - (2\gamma-1)\right]^2 l_1^{(F),4}(s)v_4.
\end{aligned} \tag{5.12}
$$

These equations are completely equivalent to the fermionic flow Eq. (3.17). In order to see this we recall that the simple truncation of the form (1.4) is at most quadratic in the bosonic fields. We can therefore easily solve the bosonic field equations as a functional of the fermion fields. Reinserting the solution into the effective average action we obtain the form (1.1) with the k-dependent quartic couplings

$$
\overline{\lambda}_{\sigma,k} = \frac{1}{2k^2\tilde{\epsilon}_{\sigma,k}} - 2\gamma\frac{1}{k^2\overline{\epsilon}_{V,k}}, \quad \overline{\lambda}_{V,k} = \frac{1}{k^2\overline{\epsilon}_{V,k}}. \tag{5.13}
$$

Inserting this into Eq. (5.12) we find Eq. (3.17), establishing both the exact equivalence to the fermionic model and the γ-independence of physical quantities. On a numerical level, we can see the equivalence from the critical couplings listed in Tabs. 3.1, 3.2.

On this level of truncation the equivalence between the fermionic and the adapted bosonic flow can also be seen on a diagrammatic level. As long as we do not have a kinetic term for the bosons the internal bosonic lines shrink to points. On the one-loop level we find an exact correspondence between the diagrams for the bosonized and the purely fermionic model summarized in Fig. 5.2. This demonstrates again that one-loop accuracy cannot be obtained without adaption of the flow.

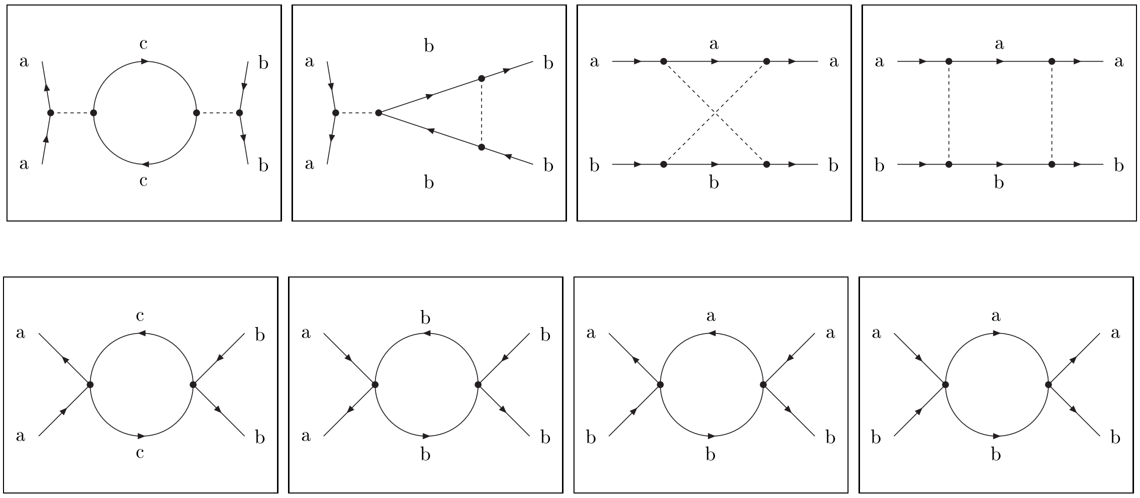


Figure 5.2: Summary of all diagrams encountered in the previous sections. There is a one to one correspondence between the diagrams of the bosonized model (first row) and the purely fermionic model (second row). Solid lines with an arrow denote fermionic lines. The letters in the diagrams are given for visualizing the ways in which the fermionic operators are contracted, e.g. the first diagram in the second row results from a term $[(\overline{\psi}_a\psi_a)(\overline{\psi}_c\psi_c)][(\overline{\psi}_c\psi_c)(\overline{\psi}_b\psi_b)]$. Shrinking bosonic lines (dashed) to points maps the diagrams in the first row to the second row. In the approximations of sect. 4.2 only the first or the first two diagrams are taken into account.

5.3 Trouble With the LPA, an Example

So far, everything seems quite satisfactory. In our simple truncation we have been able to solve the problem of the Fierz ambiguity for the partially bosonized language. Moreover, the adaption of the bosonic flow is quite intuitive as it implements the idea of scale dependent degrees of freedom. But, not everything is as rosy as it seems. Looking a little bit more closely we notice, that although the critical coupling is independent of γ, the values of the individual bosonic couplings $\tilde{\epsilon}_\sigma$, $\tilde{\epsilon}_V$, $\tilde{\epsilon}_A$ are *not*. Only two of them are fixed by the flow equation (5.12), while the third one can be chosen freely. In particular, we cannot determine from this truncation which type of boson will condense. This is completely analogous to the purely fermionic description, and in view of the equivalence of both descriptions not too surprising. It seems evident that this is a shortcoming of our present truncation.

Thinking about enlarging the truncation, two possibilities come to mind immediately, a more complicated bosonic potential and a non-zero kinetic terms for the bosons. In this section we will consider the first possibility. As it turns out this does not solve the problem, i.e. we still cannot decide which type of bosons will condense. Hence, we turn to the alternative of kinetic terms for the bosons in Sect. 5.4.

5.3.1 The Gross-Neveu Model

To get a first impression it is instructive to study an even simpler model than Eq. (1.1), the so called $N = 1$[3] Gross Neveu model [95] in three spacetime dimensions,

$$S_{\mathrm{F}}[\overline{\psi}, \psi] = \int d^3x \left\{ i\overline{\psi}\partial\!\!\!/\psi + \frac{G}{2}(\overline{\psi}\psi)^2 \right\} \tag{5.14}$$

Actually, it is indeed the same model, only the number of spacetime dimensions has been reduced by one. The simplification as compared to Eq. (1.1) lies in the fact that in three dimensions we can use spinors with only two components (s. App. B).

Despite its simplicity it is still interesting in its own right. In particular it has a parity-like symmetry $\psi(x) \to -\psi(-x)$, $\overline{\psi}(x) \to \overline{\psi}(-x)$ which is expected to be spontaneously broken by a non-vanishing vacuum expectation value $\langle\overline{\psi}\psi\rangle$ for large enough G [96].

Following the lines of Chap. 4 it is straightforward to obtain the equivalent partially bosonized action,

$$S_{\mathrm{B}}[\overline{\psi}, \psi] = \int d^3x \left\{ i\overline{\psi}\partial\!\!\!/\psi + ih\phi\overline{\psi}\psi + \frac{m^2}{2}\phi^2 \right\}, \quad G = \frac{h^2}{m^2}. \tag{5.15}$$

We now want to study this model in a truncation which includes an arbitrary local potential $V(\phi)$ but no kinetic term $\sim \partial_\mu\phi\partial^\mu\phi$ (in the following we will suppress the integration over the spacetime coordinates),

$$\Gamma_k = i\overline{\psi}\partial\!\!\!/\psi + ih_k\phi\overline{\psi}\psi + V_k(\phi). \tag{5.16}$$

The parity like symmetry translates into $\phi(x) \to -\phi(-x)$ in the bosonic language, restricting any bosonic potential $V(\phi)$ to even powers of ϕ.

Let us, for the moment, assume that during the flow we generate fermionic $\sim (\overline{\psi}\psi)^n$, bosonic $\sim \phi^n$ and mixed $\sim \phi^n(\overline{\psi}\psi)^m$ interactions. We neglect all other contributions as they lie outside of our present truncation. Hence, we write for the flow at fixed fields,

$$\partial_t\Gamma_k| \equiv \partial_t U_k(\phi_k, \overline{\psi}\psi) = \sum_n a_k^n(\phi_k)\left(\overline{\psi}\psi + \frac{V_k'(\phi_k)}{ih_k}\right)^n, \tag{5.17}$$

where we have expanded the RHS in powers of $\overline{\psi}\psi$ about the point $-\frac{V'(\phi)}{ih}$, and

[3]N is the number of different fermion species.

$V_k'(\phi_k) = \frac{\partial V_k(\phi_k)}{\partial \phi_k}$. Allowing for a scale dependence of the field ϕ_k we obtain,

$$\begin{aligned}
\partial_t \Gamma_k &= \partial_t \Gamma_k| + \frac{\delta \Gamma_k}{\delta \phi_k} \partial_t \phi_k \\
&= \partial_t U_k(\phi_k, \overline{\psi}\psi) + (ih_k \overline{\psi}\psi + V_k'(\phi_k))\partial_t \phi_k \\
&= a_k^0(\phi_k) + \sum_{n\geq 1} a_k^n(\phi_k) \left(\overline{\psi}\psi + \frac{V_k'(\phi_k)}{ih_k}\right)^n + (ih_k \overline{\psi}\psi + V_k'(\phi_k))\partial_t \phi_k.
\end{aligned} \tag{5.18}$$

Now, let us make a non-linear field redefinition[4],

$$\partial_t \phi_k = -\frac{1}{ih_k} \sum_{n\geq 1} a_k^n(\phi_k) \left(\overline{\psi}\psi + \frac{V_k'(\phi_k)}{ih_k}\right)^{n-1}. \tag{5.19}$$

Inserting this into Eq. (5.18) yields a simplified flow which only affects the potential in ϕ_k,

$$\partial_t \Gamma_k = a_k^0(\phi_k) = \partial_t U_k \left(\phi_k, i\frac{V_k'(\phi_k)}{h_k}\right). \tag{5.20}$$

At this point one might wonder why h_k does not receive any corrections as in (5.10). The reason for this is that we have used the additional freedom to rescale ϕ with a contribution $\sim \phi$ in $\partial_t \phi$, transforming any change in h_k into a change of $V(\phi)$.

So far this looks quite appealing, as we succeeded in including all the complicated interactions contained in $\partial_t U_k(\phi, \overline{\psi}\psi)$ into a simple truncation which includes a potential depending only on ϕ. However, let us now show that we can simplify our result even further by adding a suitably written 0 to $\partial_t U_k(\phi_k, \overline{\psi}\psi)$. Indeed, starting from the action (5.15) it is possible to reduce $V_k(\phi_k)$ to a mass term $\frac{m_k^2}{2}\phi_k^2$. If $V_k(\phi_k) = \frac{m_k^2}{2}\phi_k^2$ at a given point, we have $V'(\phi_k) = m_k^2 \phi_k$, and the flow of the potential reads,

$$\partial_t V_k(\phi_k) = a_k^0(\phi_k) = \partial_t U_k \left(\phi_k, i\frac{m_k^2}{h_k}\phi_k\right) = \sum_n b_k^n \phi^n. \tag{5.21}$$

Since we do not consider gravity, field independent terms in the effective action are of no physical significance. Dropping those and remembering that $V(\phi)$ can contain only even powers of ϕ, it is no restriction to write

$$\partial_t U_k \left(\phi, i\frac{m_k^2}{h}\phi_k\right) = \sum_{n\geq 1} b_k^{2n} \phi^{2n}. \tag{5.22}$$

[4] One may wonder if such a field redefinition has the right symmetry properties. A thorough inspection tells us that the given field redefinition is of the type $\partial_t \phi_k = \frac{\partial X(\phi_k, \overline{\psi}\psi, \ldots)}{\partial \phi_k}$, where X is a singlet under all symmetries (the dots denote any type of additional fields). A derivative of this type belongs to the conjugate representation of ϕ_k and thereby to the same as ϕ_k since ϕ_k is self-conjugate. This also provides us with a recipe how we can obtain field redefinitions respecting the symmetries for more complicated fields, e.g. vector fields.

Now, let us use that the spinors have only two components, i.e. at any given point x we have only four different Grassmann variables, $\overline{\psi}_{1,2}(x)$, $\psi_{1,2}(x)$. Consequently, we obtain

$$(\overline{\psi}(x)\psi(x))^n = 0, \quad \forall n \geq 3, \tag{5.23}$$

by use of the anticommutation relations. In particular, we have,

$$\partial_t \hat{U}_k(\phi_k, \overline{\psi}\psi) = \partial_t U_k(\phi_k, \overline{\psi}\psi) - \sum_{n\geq 2} b_k^{2n} \left(\frac{ih}{m_k^2}\right)^{2n} (\overline{\psi}\psi)^{2n} = \partial_t U_k(\phi_k, \overline{\psi}\psi). \tag{5.24}$$

Performing field redefinitions as above, but for $\partial_t \hat{U}$ instead of $\partial_t U$ yields,

$$\partial_t V_k(\phi_k) = \partial_t \hat{U}\left(\phi_k, i\frac{m_k^2}{h_k}\phi_k\right) = b_k^2 \phi_k^2, \tag{5.25}$$

i.e. our potential remains a mass term for all k as we have claimed above.

At first, this seems very strange. Yet, as we know from Chap. 4 we can, at least formally, remove the bosonic fields by performing the appropriate functional integral. Since we have no kinetic term for the bosons, and $V(\phi)$ is local, this results in a completely local fermionic interaction. However, Eq. (5.23) tells us that the highest order local and purely fermionic interaction is $(\overline{\psi}\psi)^2$. Therefore, any action of the form (5.16) is equivalent to (5.14) and in consequence also to Eq. (5.15), as long as we choose G and $\frac{h^2}{m^2}$ correctly. In other words, if we do not consider a non-vanishing kinetic term for the bosons, an inclusion of a full bosonic potential does not give us any more physical information than a simple truncation to a mass term or a purely fermionic calculation with a local four-fermion interaction. In particular, we cannot proceed into the SSB regime.

5.3.2 General Discussion

It is straightforward to extend Eq. (5.21) to more general cases with several different bosonic composite operators and corresponding bosonic fields, e.g., induced by fermions with more components. In fact, using our symbolic notation of Chap. 2 the generalization looks like Eq. (5.21). The discussion leading to Eq. (5.25) can be generalized, too, but it is somewhat more complicated, as we have to use the inverse function V'^{-1} of V' to write down $\partial_t \hat{U}$. Assuming the existence of V'^{-1} we find,

$$\partial_t \hat{U}_k(\phi, \overline{\psi}O\psi) = \partial_t U_k(\phi, \overline{\psi}O\psi) - \sum_{|n|>N} b_k^n \left[V'^{-1}(-ih_k\overline{\psi}O\psi)\right]^n, \tag{5.26}$$

where, for simplicity, we have employed a symbolic notation with the components ϕ_i corresponding to the operators O_i. The b's are defined as in Eq. (5.24), but

$n = (n_1, \ldots)$ is now a multi index. N is the number of components of the spinor ψ and determines the highest-order monomial in ψ which does not vanish by the anticommutation relations. E.g. for a Dirac fermion in four dimensions (four spin indices) with three colors we have $N = 4 \times 3 = 12$. Using $\partial_t \hat{U}$ to define the field redefinitions we can force all contributions ϕ^n, $|n| > N$ to vanish as in Eq. (5.25).

Typically, a potential up to order ϕ^4 is sufficient to describe at least the basic features of SSB. As most models have $N \geq 4$ we might be tempted to conclude that, with the exception of some special cases, the LPA works. However, this is not the case. We have already seen that we can modify $\partial_t V_k(\phi)$ by adding a conveniently written zero to $\partial_t U(\phi, \overline{\psi}\psi)$. But, Eq. (5.23) and its generalizations to the case of more spinor components are not the only way we can write a zero. Fierz identities like Eq. (1.2) provide another one. As we can see from the example of Eq. (1.2) they allow us to find non-vanishing c^n such that

$$\sum_{|n|=m} c^n (\overline{\psi} O \psi)^n = 0, \quad m \leq N. \tag{5.27}$$

An addition of this to $\partial_t U(\phi, \overline{\psi} O \psi)$ can be used to eliminate terms with ϕ^n and $|n| \leq N$ in the potential. The c^n in Eq. (5.27) are not all independent, but typically we can eliminate at least one species of bosons completely from the potential. In our model and truncation this freedom is reflected by the arbitrariness of $\tilde{\gamma}(k)$ in Eq. (5.1) and γ in Eq. (5.12), respectively, e.g. choosing $\gamma = 0$ in Eq. (5.12) yields $h_A = 0$ and effectively removes the axial vector bosons.

Physically, our findings in this section tell us that in the LPA without any kinetic terms for the bosons we simply cannot decide which type of boson will condense. To do this we need additional information. Therefore, we will investigate (simple) momentum dependent terms in the effective action in the next section.

5.4 Going beyond the LPA

In Sect. 4.1.4 we have seen that a simple kinetic term in the bosonized action gives a momentum dependent four-fermion interaction in the purely fermionic language. Moreover, we could absorb this momentum dependent four-fermion interaction into a boson only if we chose a certain Fierz transformation. This provided us with the information to decide which FT is the "right" one. Therefore, let us add the kinetic terms specified in Eq. (4.23) to our pointlike action (1.4). Finally, let us impose for simplicity one more restriction, $\alpha_V = \alpha_A = 1$ on our effective action. This simplifies the expressions for the vector-boson propagators, similar to Feynman gauge in gauge theories.

Let us first outline the following complicated calculation, using a symbolic notation. In a next step we then discuss some points hidden in the notation and some details of the employed approximations. Finally, we comment on a few properties and give numerical results, while the whole set of equations and a more explicit and step-by-step calculation is given in App. D.

5.4.1 Adapting the Flow

We restrict our truncation to lowest non-trivial order in p^2, and expand all couplings up to this order. We write,

$$\begin{aligned} \mu^2(p) &= \mu^2 + p^2 Z + \mathcal{O}(p^4), & \partial_t \mu^2(p) &= \partial_t \mu^2 + p^2 \partial_t Z + \ldots, \\ h(p) &= h + p^2 h^{(2)} + \mathcal{O}(p^4), & \partial_t h(p) &= \beta_h^{(0)} + p^2 k^{-2} \beta_h^{(2)} + \ldots, \\ \lambda(p) &= \lambda^{(0)} + p^2 \lambda^{(2)} + \mathcal{O}(p^4), & \partial_t \lambda(p) &= k^{-2} \beta_\lambda^{(0)} + p^2 k^{-4} \beta_\lambda^{(2)} + \ldots. \end{aligned} \tag{5.28}$$

At first sight, in the partially bosonized language it seems reasonable to take $\mu^2(p)$ as in (5.28) but restrict $h(p) = h$ and $\lambda(p) = 0$. However, in Sect. 5.2 we have seen that at least the latter is not a good approximation because the term $\lambda^{(0)}$ is crucial for restoring the invariance under FT of the initial (pointlike) interaction. Furthermore, we found that the flows of μ^2, h and $\lambda^{(0)}$ all contribute to the same order to the effective four-fermion interaction (after integrating out the bosons). It seems natural that this is also true for the terms of order p^2: Z, $h^{(2)}$, $\lambda^{(2)}$. Hence, we consider all these terms on equal footing.

Having chosen our truncation, we can calculate the flow equations. Since we did not add higher powers of field operators, the remaining task is to evaluate the diagrams depicted in Figs. 3.1, 4.2, 5.1. The only difference to our previous calculations is that we have to consider non-trivial external momenta.

As in Sect. 5.2 we want to keep the desired simple form of the effective action, i.e. $h(p) = h$ and $\lambda(p) = 0$, by choosing appropriate field redefinitions and neglecting terms of order $\mathcal{O}(p^4)$. To do so, we shift

$$\partial_t \phi_k(q) = (\overline{\psi}\psi)(q) \partial_t \omega_k(q) + \partial_t \alpha_k(q) \phi_k(q), \tag{5.29}$$

and, as in Sect. 5.2, we employ[5],

$$\partial_t \Gamma_k = \partial_t \Gamma_k| + \int_q \frac{\delta \Gamma_k}{\delta \phi_k(q)} \partial_t \phi_k(q). \tag{5.30}$$

[5]As mentioned in Sect. 5.2, after the appropriate modification of the cutoff, the flow equation (2.32) does not give the exact flow $\partial_t \Gamma_k|$. Since we now have a cutoff for the bosonic fields, the use of Eq. (2.32) is really an approximation. However, numerical tests performed in the appendix of [42] for a very similar case suggest that it is a very good approximation. So we will use it without further comment.

This results in the following changes of the flow equations,

$$\begin{aligned} \partial_t \mu^2(q) &= \partial_t \mu^2(q)| + 2\mu^2(q)\partial_t \alpha_k(q), \\ \partial_t h(q) &= \partial_t h(q)| + h(q)\partial_t \alpha_k(q) + \mu^2(q)\partial_t \omega_k(q), \\ \partial_t \lambda(q) &= \partial_t \lambda(q)| - h(q)\partial_t \omega_k(q). \end{aligned} \tag{5.31}$$

We can now use the freedom in choosing the field redefinitions to enforce,

$$\partial_t \lambda(q) = 0, \quad \partial_t h(q) = \partial_t h^{(0)}, \quad \partial_t \mu^2(q) = \partial_t \mu^2 + \mathcal{O}(p^4). \tag{5.32}$$

Roughly speaking we absorb the four-fermion interactions in the masses and Yukawa-couplings, and the momentum dependence of the latter ones in the wave function renormalizations for the bosonic fields ($\partial_t Z = 0$). In particular, we keep the simple form with a momentum independent Yukawa coupling and no four-fermion interaction.

Using the fact that we start with a constant Yukawa coupling and $\lambda = 0$ we can solve the equations (5.32) and find,

$$\eta \equiv 2\partial_t \alpha(0) = -\frac{\partial_t Z|}{Z} + \frac{2\mu^2}{h}\left[\partial_t h^{(2)}| + \mu^2 \partial_t \lambda^{(2)}| + Z\partial_t \lambda^{(0)}|\right]. \tag{5.33}$$

We use this equation to define the anomalous dimension. From Eq. (5.29) it is clear that $\partial_t \alpha(0)$ modifies the overall normalization of ϕ, hence the wave function renormalization. Moreover, setting $\partial_t h^{(2)}| = 0$, $\partial_t \lambda(q)| = 0$, it coincides with the original definition $\eta = -\frac{\partial_t Z}{Z}$. Using this, we get pretty much the standard equations for the flow of the mass and Yukawa coupling,

$$\begin{aligned} \partial_t \mu^2 &= \eta \mu^2 + \partial_t \mu^2|, \\ \partial_t h &= \frac{1}{2}\eta h + \partial_t h| + \frac{\mu^2}{h}\partial_t \lambda^{(0)}|. \end{aligned} \tag{5.34}$$

We remark that together with the initial condition $Z = 1$ the conditions (5.32) automatically guarantee that the couplings are renormalized.

5.4.2 Choosing the Momentum Configurations

So far everything seemed relatively straightforward. However, looking more closely, we soon find that $\partial_t h(p)$ can actually depend on two and $\partial_t \lambda(p)$ even on three momentum variables. This is in contrast to $\mu^2(p)$ which depends only on p^2. As we ultimately want to absorb those momentum dependencies in $\partial_t \mu^2(p)$ it is clear that we have to make an approximation such that $\partial_t h(p)$ and $\partial_t \lambda(p)$ depend only on one momentum squared. To decide which of the possible momenta to choose, we look at our example of Sect. 4.1.4, in particular at Eq. (4.34). From this we can read of, that the bosons have the form,

$$\phi(q) = f(q) \int_{p_1,p_2} \overline{\psi}(p_1)\psi(-p_2)\delta(q-p_1-p_2). \tag{5.35}$$

Since we want to keep the simple form of the effective action, it is clear that the bosons have to keep this form, too.

$$\partial_t \phi_k(q) = \partial_t \alpha(q)\phi_k(q) + \partial_t \omega_k(q) \int_{p_1,p_2} \overline{\psi}(p_1)\psi(-p_2)\delta(q-p_1-p_2), \tag{5.36}$$

is the most general form that accomplishes this. With Eq. (5.36) we can give meaning to our notation,

$$(\overline{\psi}\psi)(q) = \int_{p_1,p_2} \overline{\psi}(p_1)\psi(-p_2)\delta(q-p_1-p_2). \tag{5.37}$$

Inserting this into Eq. (5.30) we find that the most general structures we can absorb are,

$$\int_{p_1,p_2,p_3} F(p_1^2)\ \phi(-p_1)\overline{\psi}(p_2)\psi(-p_3)\delta(p_1+p_2+p_3), \tag{5.38}$$

$$\int_{p_1,p_2,p_3,p_4} G((p_1+p_2)^2)\ \overline{\psi}(-p_1)\psi(p_2)\overline{\psi}(p_3)\psi(-p_4)\delta(p_1+p_2-p_3-p_4), \tag{5.39}$$

where F and G are arbitrary functions which can then be expressed in terms of $\partial_t \alpha_k$ and $\partial_t \omega_k$.

Comparing the vertex correction $\sim \phi(-p_1)\overline{\psi}(-p_2)\psi(p_3)$ depicted in Fig. 4.2 with (5.38) it is clear that we have to restrict the momentum dependence to p_1. A suitable configuration for the evaluation then is, $(p_1, p_2, p_3) = (p, \frac{1}{2}p, \frac{1}{2}p)$.

Recalling that a Fierz transformation for the four-fermion interaction exchanges p_2 and $-p_3$ we can absorb either a dependence on $s = (p_1 + p_2)^2$ or one on $t = (p_1 - p_3)^2$ (corresponding momentum configurations would be e.g. $(p_1, p_2, p_3, p_4) = \frac{1}{2}(p, p, p, p)$ and $(p_1, p_2, p_3, p_4) = \frac{1}{2}(p, -p, -p, p)$, respectively, cf. Fig. 5.1). Therefore, we have to ask which gives the better approximation. In principle, we would have to calculate $\partial_t \lambda(s,t)$ (or even better $\partial_t \lambda(p_1,p_2,p_3,p_4)$) and test at every scale whether $\partial_t \lambda(s,t) \approx \partial_t \lambda(t)$ or $\partial_t \lambda(s,t) \approx \partial_t \lambda(s)$ is a better approximation. Analytically as well as numerically this is rather complicated. Therefore, we have adapted a much simpler scheme: we have always absorbed the dependence on t, i.e. we have evaluated the diagrams in Fig. 5.1 and Fierz transformed the resulting interaction once. There are two reasons why we believe that this is a reasonable approximation. First of all, in the pointlike limit (at the beginning of the flow), i.e.

$$\mu^2 \to \infty, \quad h^2 \to \infty, \quad \frac{h^2}{\mu^2} = \text{const}, \tag{5.40}$$

one finds $\partial_t \lambda(s,t) = \partial_t \lambda(t)$ exactly. Secondly, we have checked for various combinations of h and μ that in the vicinity of $(s,t) = (0,0)$ the dependence on t is usually (but not always) stronger than the dependence on s.

5.4.3 Initial Flow and Numerical Results

To get an impression of what we have achieved by all this let us take a look at the effective four-fermion interaction at the beginning of the flow[6].

As we want to start with action (1.1) where the four-fermion interaction is pointlike, the kinetic terms in the partially bosonized action vanish and the renormalized couplings obey Eq. (5.40) where the constants are given by Eq. (1.5).

In Sect. 4.1.4 we have already calculated,

$$\lambda(p) = \frac{h^2}{\mu^2 + p^2} = \frac{h^2}{\mu^2} - \frac{h^2}{\mu^4} p^2 + \cdots , \tag{5.41}$$

where μ^2 and h^2 are constants in momentum space. Using our flow equations given in App. D and using the properties of the threshold functions given in C.2.3 we find (after some algebra) Eq. (5.12) for the flow of the p^0-terms. For the flow of the p^2-term in λ_σ we find,

$$\partial_t \left(\frac{h_\sigma^2}{\mu_\sigma^4} \right) = 16 v_4 \gamma_2^{(F),4}(0) (\overline{\lambda}_\sigma + \overline{\lambda}_V)^2, \tag{5.42}$$

and similar expressions for λ_V and λ_A which are invariant under Fierz transformations, too. This shows that at least at the beginning we have no Fierz ambiguity up to order p^2.

Now, let us come to the numerical results. Numerically it is impossible to employ the pointlike limit exactly, therefore we have started with large values of $\mu^2 \sim 10^5$. In addition to the results of the full set of flow equations we have given some results for more simple approximations in Tabs. 5.1, 5.2. The first approximation, $(1) + (3)$, corresponds to the naive approach to the bosonized model, where all contributions from the four-fermion interactions and the momentum dependence of the Yukawa coupling are neglected. In the next step, $(1) - (3)$, we have included the box diagrams, but only its constant parts, not the terms of order p^2. However, indirectly we have included some knowledge of the momentum dependence as we have chosen the same FT as for the truncation $(1) - (4)$ where we have included all terms of order p^2.

[6] Actually, in this section we make one more approximation: we have ignored the anomalous dimensions in the arguments of the threshold functions. In the pointlike limit this neglects terms of order λ^3. The flow equations in App. D include those terms, but they make the evaluation much more difficult.

Approximation	Chap.	$\gamma = 0.1$	0.25	0.5	0.75	0.9
MFT	3.1	78.56	77.96	76.96	75.96	75.36
SD	3.4	76.96	76.96	76.96	76.96	76.96
(1)	4.2	76.32	76.36	76.42	76.49	75.54
Ferm. RG = (1) + (2)	3.3	86.15	86.15	86.15	86.15	86.15
(1) + (3)	5.4	76.43	76.45	76.49	76.55	76.58
(1) – (3)	5.4	83.97	83.95	83.92	83.89	83.87
(1) – (4)	5.4	86.17	86.18	86.20	86.21	86.22

Table 5.1: Critical values $\overline{\lambda}_\sigma^{\mathrm{crit}}$ for $\overline{\lambda}_V = 2$ and for various values of the unphysical parameter γ (with $\Lambda = 1$). To keep the table of manageable size we have abbreviated: (1) the pointlike contibutions to the mass and the Yukawa coupling (Figs. 3.1, 4.2), (2) the pointlike contributions from the box diagrams (Fig. 5.1), (3) the contribution to the WFR from the purely bosonic diagram (Fig. 3.1) and (4) the contribution to the WFR from the momentum dependence of the diagrams 4.2 and 5.1. We point out that differing from Tabs. 3.1, 3.2 we have employed a UV regularization by the ERGE scheme (cf. App. C.3) with the linear cutoff Eq. (C.3) which is better suited for numerical computations. This is why the values for the critical coupling are roughly twice of those given in Tabs. 3.1, 3.2, since the critical coupling is not a universal quantity, and therefore scheme dependent.

Approximation	Chap.	$\gamma = 0.1$	0.25	0.5	0.75	0.9
MFT	3.1	74.96	68.96	58.96	48.96	42.96
SD	3.4	58.96	58.96	58.96	58.96	58.96
(1)	4.2	53.16	52.93	53.32	54.64	55.88
Ferm. RG = (1) + (2)	3.3	58.83	58.83	58.83	58.83	58.83
(1) + (3)	5.4	53.89	53.66	54.00	55.23	56.37
(1) – (3)	5.4	58.14	58.04	57.88	57.73	57.64
(1) – (4)	5.4	61.60	61.69	61.82	61.91	61.94

Table 5.2: The same as in Tab. 5.1 but with $\overline{\lambda}_V = 20$.

Moreover, we notice that the values for the critical coupling in the pointlike approximations are roughly twice of those given in Tabs. 3.1, 3.2. This is due to a difference in the UV regularization. In this section we have employed the ERGE scheme described in App. C.3. Non-universal quantities can and do depend on the choice of UV regularization. In the pointlike approximation this yields exactly a factor of two in our case (for the pair of couplings $(\overline{\lambda}_\sigma, \overline{\lambda}_V)$). In the more involved approximations this is not necessarily so, but the factor will still be somewhere around two.

Aside from this, there is nothing new in the first four lines of Tabs. 5.1, 5.2. Comparing the pointlike truncations for the RG with the improved approximations of this section we find that the effect is of the order of 10%. Moreover, comparing the different non-pointlike approximations we find that the differences between them are of the order of 5% – 10%, too. While most of the Fierz ambiguity (more important for large values of $\overline{\lambda}_V$, Tab. 5.2) is eliminated by including the pointlike contributions of the boxes (in the "right" FT), (2), for the absolute values of the critical coupling, the momentum dependence of the Yukawa coupling and the box diagrams is not negligible.

On the more qualitative side we have checked for various values that for all truncations which include kinetic terms (last three lines in the tables) and values of $\overline{\lambda}_\sigma$ slightly larger than the critical $\overline{\lambda}_\sigma^{\mathrm{crit}}$ only the renormalized scalar boson mass turns negative, while the renormalized vector and axial vector boson masses remain positive. This allows the conclusion that the scalar boson will condense first, and we have a phase where only chiral symmetry is broken (at least in our approximation).

Chapter 6

Bosonic Effective Action (2PI)

In the last two chapters we have mainly worked on improving the RG description in the partially bosonized language. It turned out, that it is necessary to include a wave function renormalization (WFR) for the bosons. Without a WFR we cannot determine the type of the bosonic condensate (e.g. if it is a vector or scalar condensate). Yet, consistent inclusion of a WFR leads to high algebraic complexity. This might be appropriate for a quantitative description. However, if we want to get a first, more qualitative, overview of a physical system this seems to be a little bit excessive. For this purposes the SDE or MFT approaches seem much more suitable.

Both methods allow for a computation of the order parameter in systems which exhibit spontaneous symmetry breaking (SSB). However, while the SDE approach leads directly to the gap equation the MFT approach provides naturally a free-energy functional for the bosonic composite degrees of freedom introduced by partial bosonization via a Hubbard Stratonovich transformation (s. Chap. 4). The field equation for this functional corresponds to the gap equation. Knowledge of the free-energy functional becomes necessary if the gap (or field) equation allows for solutions with different order parameters and the free energy for the different solutions has to be compared. The reconstruction of the free-energy functional from the gap equation is not trivial and the method used in [97] for the case of color superconductivity may not always work.

From this it seems that MFT is superior to SDE. Unfortunately, as we have seen in Chaps. 3, 4, it has a severe disadvantage: partial bosonization is not unique and the results of the MFT calculation depend strongly on the choice of the mean field. Moreover, MFT only includes a subset of the SDE diagrams.

Hence, we want to find a functional which has the SDE as its equation of motion, and which can be interpreted as a free energy. Such a functional is given by the 2PI effective action [46–48].

In general, the 2PI effective action is a functional of fields and propagators $\Gamma^{(2PI)}[\phi, G]$. However, for a purely fermionic system, all the information is already contained in $\Gamma^{(2PI)}[0, G]$. $\Gamma^{(2PI)}[0, G]$ depends only on the bosonic variable G, and therefore we will call it Bosonic Effective Action (BEA) [98].

In this chapter, we want to calculate a simple approximation of the BEA for a general local multi-fermion interaction. Already for a four-fermion interaction the lowest non-trivial contribution to the BEA is of two-loop order. For a general n-fermion interaction we have an $\frac{n}{2}$-loop structure. However, we will show that this can be reduced to a one-loop expression at the solution of the SDE, allowing for a comparison to MFT.

As an application we want to study an interaction resembling the six-fermion interaction generated by instantons in the case of three flavors and three colors [99–103]. In QCD this interaction is of special interest as it is $U(1)$-anomalous and solves the famous $U(1)$-problem [104]. In the simpler case of two flavors instantons mediate a four-fermion interaction which has been investigated in works on chiral symmetry breaking [105–107] and color superconductivity e.g. [19; 20].

The effective interaction generated by the instantons does not only lead to interactions between color singlet effective quark-antiquark degrees of freedom ($\to$ chiral symmetry breaking) but also between octets leading to the possibility of octet condensation and spontaneous color symmetry breaking [27; 28; 49]. In the following we will consider both possibilities.

6.1 Bosonic Effective Action (2PI)

To simplify the presentation we summarize all indices of the fermionic field in $\tilde{\psi}_\alpha$. The index alpha contains all internal indices (spin, color, flavor etc.) as well as position or momentum. Furthermore it also differentiates between ψ and $\overline{\psi}$.

The partition function reads

$$Z[\eta, j] = \int \mathcal{D}\tilde{\psi} \exp(\eta_\alpha \tilde{\psi}_\alpha + \frac{1}{2} j_{\alpha\beta} \tilde{\psi}_\alpha \tilde{\psi}_\beta - S_{\text{int}}[\tilde{\psi}]) \tag{6.1}$$

where we treat all quadratic terms as a bosonic source term.

We specify the interaction as

$$S_{\text{int}}[\tilde{\psi}] = \sum_n \frac{1}{n!} \lambda^{(n)}_{\alpha_1 \dots \alpha_n} \tilde{\psi}_{\alpha_1} \cdots \tilde{\psi}_{\alpha_n}. \tag{6.2}$$

The usual generating functional of 1PI Greens functions in presence of the bosonic sources j is defined by a Legendre transform with respect to the fermionic source term η:

$$\Gamma_F[\psi, j] = -W[\eta, j] + \eta_\alpha \psi_\alpha \tag{6.3}$$

where

$$W = \ln Z[\eta, j], \quad \psi_\alpha = \langle \tilde{\psi}_\alpha \rangle = \frac{\partial W}{\partial \eta_\alpha}. \tag{6.4}$$

Γ_F can also be obtained by the following functional integral[1]:

$$\begin{aligned} \Gamma_F[\psi, j] &= -\ln \int \mathcal{D}\tilde{\psi} \exp(\eta_\alpha \tilde{\psi}_\alpha - S_j[\tilde{\psi} + \psi]), \\ S_j[\tilde{\psi}] &= -\frac{1}{2} j_{\alpha\beta} \tilde{\psi}_\alpha \tilde{\psi}_\beta + S_{\text{int}}[\tilde{\psi}]. \end{aligned} \tag{6.5}$$

This form is especially useful to derive the SDE. Taking a derivative with respect to ψ we find

$$\begin{aligned} \frac{\partial \Gamma_F}{\partial \psi_\beta} &= -j_{\beta\alpha_2} \psi_{\alpha_2} \\ &\quad + \sum_n \frac{\lambda^{(n)}_{\beta\alpha_2 \ldots \alpha_n}}{F_n} \psi_{\alpha_2} \Big\{ (\Gamma_F^{(2)})^{-1}_{\alpha_3\alpha_4} \cdots (\Gamma_F^{(2)})^{-1}_{\alpha_{n-1}\alpha_n} + Z_{\alpha_3 \ldots \alpha_n} + \mathcal{O}(\psi^2) \Big\} \end{aligned} \tag{6.6}$$

where

$$F_n = (n-2)(n-4) \cdots 2 \tag{6.7}$$

and Z summarizes all terms containing third and higher derivatives of Γ. These are terms which have at least two vertices. Taking another derivative with respect to ψ_α and evaluating at $\psi = 0$ we find the SDE:

$$(\Gamma_F^{(2)})_{\alpha\beta} = -j_{\alpha\beta} + \sum_n \frac{\lambda^{(n)}_{\alpha\beta\alpha_3 \ldots \alpha_n}}{F_n} \Big\{ (\Gamma_F^{(2)})^{-1}_{\alpha_3\alpha_4} \cdots (\Gamma_F^{(2)})^{-1}_{\alpha_{n-1}\alpha_n} + Z'_{\alpha_3 \ldots \alpha_n} \Big\}. \tag{6.8}$$

In this chapter we are only interested in the lowest order. Therefore, from now on, we neglect Z, i.e. terms with at least two vertices.

The "Bosonic Effective Action" (BEA) [98], is defined by another Legendre transform with respect to j:

$$\Gamma_B[G] = -W[0, j] + jG, \tag{6.9}$$

$$G_{\alpha\beta} = \frac{\partial W}{\partial j_{\alpha\beta}} = (\Gamma_F^{(2)})^{-1}_{\alpha\beta}, \quad \frac{\partial \Gamma_B}{\partial G_{\alpha\beta}} = j_{\alpha\beta}. \tag{6.10}$$

[1]Note that in this formula $\tilde{\psi}$ is shifted such that $\langle \tilde{\psi} \rangle = 0$.

Since Γ_F is an even functional of ψ the BEA contains the same information as Γ_F. Indeed it is related to Γ_F by means of functional differential equations like Eq. (6.10). Using this relation we can conveniently write the SDE (6.8) as

$$G^{-1}_{\alpha\beta} = -j_{\alpha\beta} + \sum_n \frac{\lambda^{(n)}_{\alpha\beta\alpha_3\ldots\alpha_n}}{F_n} G_{\alpha_3\alpha_4}\cdots G_{\alpha_{n-1}\alpha_n}. \tag{6.11}$$

Using (6.10) we obtain a differential equation for Γ_B

$$\frac{\partial\Gamma_B}{\partial G_{\alpha\beta}} = -G^{-1}_{\alpha\beta} + \sum_n \frac{\lambda^{(n)}_{\alpha\beta\alpha_3\ldots\alpha_n}}{F_n} G_{\alpha_3\alpha_4}\cdots G_{\alpha_{n-1}\alpha_n}. \tag{6.12}$$

which we can integrate[2] to obtain

$$\Gamma_B = \frac{1}{2}\mathrm{Tr}\ln G + \sum_n \frac{\lambda^{(n)}_{\alpha_1\ldots\alpha_n}}{nF_n} G_{\alpha_1\alpha_2}\cdots G_{\alpha_{n-1}\alpha_n}, \tag{6.13}$$

the BEA at "one-vertex order".

It is sometimes convenient to introduce an auxiliary effective action

$$\hat{\Gamma}[G,j] = \Gamma_B - \frac{1}{2} j_{\alpha\beta}G_{\alpha\beta} \tag{6.14}$$

such that the physical propagator corresponds to the minimum of $\hat{\Gamma}$ (cf. Eq. (6.10)).

6.2 BEA for Local Interactions

In the following we want to consider local interactions. For clarity we now write x (or momentum p) explicitly and use latin letters for the remaining indices. The standard procedure would be the insertion of the ansatz $G^{-1}_{ab}(x,y) = -j_{ab}(x,y) + \Delta_{ab}(x)\delta(x-y)$ into Eq. (6.11) to obtain the SDE for the local gap Δ. Since the BEA Eq. (6.13) is related to the SDE (6.11) by differentiation with respect to G it is not clear that an effective action functional depending on Δ can be obtained by integration with respect to Δ. Instead we want to follow the construction presented in [98] and start directly from the approximate BEA Eq. (6.13). With

$$g_{ab}(x) = G_{ab}(x,x) \tag{6.15}$$

[2]Note that in our notation $\frac{\partial G_{\alpha\beta}}{\partial G_{\gamma\delta}} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$.

we have

$$\hat{\Gamma} = \frac{1}{2}\mathrm{Tr}\ln G + \frac{1}{2}\mathrm{Tr}(Gj) + \int_x \sum_n \frac{\lambda^{(n)}_{a_1\ldots a_n}}{nF_n} g_{a_1a_2}(x)\cdots g_{a_{n-1}a_n}(x). \tag{6.16}$$

For this relation it is essential that the interaction is strictly local. Furthermore, we can use the locality of the interaction to write (6.11) in the form of a local gap equation

$$G^{-1}_{ab}(x,y) = -j_{ab}(x,y) + \Delta_{ab}(x)\delta(x-y). \tag{6.17}$$

We will evaluate the functional $\Gamma[G]$ for $G_{\alpha\beta}$ corresponding to Eq. (6.17). This is actually a restriction to a subspace of all possible G. However, locality tells us that the extremum (solution of the SDE) is contained in this subspace.

Using $j = -G^{-1} + \Delta$ we find (up to a shift in the irrelevant constant and using $\Delta_{ab}(x,y) = \Delta_{ab}(x)\delta(x-y)$)

$$\begin{aligned}\hat{\Gamma}[g,\Delta] &= -\frac{1}{2}\mathrm{Tr}\ln(-j+\Delta) - \frac{1}{2}\int_x \Delta_{ab}(x)g_{ab}(x) \\ &\quad + \int_x \sum_n \frac{\lambda^{(n)}_{a_1\ldots a_n}}{nF_n} g_{a_1a_2}(x)\cdots g_{a_{n-1}a_n}(x).\end{aligned} \tag{6.18}$$

For the search of extrema of $\hat{\Gamma}$ it is actually convenient to treat Δ and g as independent variables. The extremum of $\hat{\Gamma}[g,\Delta]$ then obeys

$$\frac{\partial\hat{\Gamma}[g,\Delta]}{\partial\Delta} = 0, \quad \frac{\partial\hat{\Gamma}[g,\Delta]}{\partial g} = 0. \tag{6.19}$$

Evaluating the derivative with respect to Δ we recover the inverse of Eq. (6.17) for $x = y$,

$$g_{ab}(x) = (-j+\Delta)^{-1}_{ab}(x,x) = g[\Delta(x)] \tag{6.20}$$

Inserting this functional relation into Eq. (6.11) leads to a gap equation for Δ. In case of a six-fermion interaction this takes, however, the form of a two-loop equation.

For n-fermion interactions with $n > 4$ it is more appropriate to go the other way around and first take a derivative with respect to g. We obtain

$$\begin{aligned}\Delta_{ab}(x) &= \sum_n \frac{\lambda^{(n)}_{aba_3\ldots a_n}}{F_n} g_{a_3a_4}(x)\cdots g_{a_{n-1}a_n}(x) \\ &= \Delta_{ab}[g(x)],\end{aligned} \tag{6.21}$$

which is precisely the value of the gap in Eq. (6.11). Inserting $\Delta[g]$ into (6.18) we find the effective action depending on g

$$\begin{aligned}\hat{\Gamma}[g] &= -\frac{1}{2}\operatorname{Tr}\ln(-j+\Delta[g]) - \frac{1}{2}\int_x \Delta_{ab}[g](x) g_{ab}(x) \\ &\quad + \int_x \sum_n \frac{\lambda^{(n)}_{a_1\ldots a_n}}{nF_n} g_{a_1a_2}(x)\cdots g_{a_{n-1}a_n}(x). \end{aligned} \tag{6.22}$$

Searching for an extremum yields

$$\frac{\partial\hat{\Gamma}[g]}{\partial g} = \left\{(-j+\Delta[g])^{-1} - g\right\}\frac{d\Delta[g]}{dg} = 0. \tag{6.23}$$

For $\frac{d\Delta}{dg} \neq 0$ Eq. (6.23) indeed corresponds to the SDE (6.11), i.e.

$$g_{ab}(x) = (-j+\Delta[g])^{-1}_{ab}(x,x) \tag{6.24}$$

This will be our central gap equation. We should point out that possible extrema of $\hat{\Gamma}[g]$ corresponding to $\frac{d\Delta}{dg} = 0$ are not solutions of the gap equation (6.11) and should be discarded. Finally, we also have

$$\begin{aligned}\frac{d\hat{\Gamma}[g]}{dg} &= \frac{d\hat{\Gamma}[g,\Delta[g]]}{dg} = \frac{\partial\Gamma[g,\Delta[g]]}{\partial g} + \frac{\partial\hat{\Gamma}[g,\Delta[g]]}{\partial\Delta}\frac{d\Delta[g]}{dg} \\ &= \frac{\partial\hat{\Gamma}[g,\Delta[g]]}{\partial\Delta}\frac{d\Delta[g]}{dg}. \end{aligned} \tag{6.25}$$

Only as long as $\frac{d\Delta[g]}{dg} \neq 0$ is fulfilled we can conclude that a solution of (6.23) fulfills both extremum conditions (6.19).

Our procedure is quite powerful if $\operatorname{Tr}\ln(-j+\Delta)$ can be explicitly evaluated as a functional of Δ. Then $\hat{\Gamma}[g]$ allows not only a search for the extremum (discarding those with $\frac{d\Delta}{dg} = 0$) but also a simple direct comparison of the relative free energy of different local extrema. This is crucial for the determination of the ground state in the case of several "competing gaps".

This "one-loop" form of the equation of motion but also of the effective action itself (6.22) is very close to what we would expect from MFT (cf. also the next section). In contrast to the standard SDE, which is an equation of motion, we can use Eq. (6.22) to compare the values for the effective action at different solutions of the equation of motion (6.23), providing us with information about the stability.

Nevertheless, we have to be careful when considering Eq. (6.22) at points which are not solutions of Eq. (6.23). Going step by step through the procedure above, we find that if we are not at a solution of (6.23) we do not necessarily fulfill the ansatz (6.17). Therefore, at these points we are mathematically not allowed to insert the ansatz into Eq. (6.13). So, strictly speaking (6.22) only gives the value of the effective action at the solution of the equation of motion[3]. Although, this is already more than we get from the standard SDE we would like to interpret (6.22) as a reasonable approximation in a small neighborhood of the solution to the equation of motion. Remembering $g(x) = \langle \tilde{\psi}(x)\tilde{\psi}(x)\rangle$ it is suggestive to interpret g as a bosonic field. Eq. (6.21) gives the (non-linear) "Yukawa coupling" of g to the fermions, i.e. the relation between the gap and the bosonic field. The Tr ln is the contribution from the fermionic loop in a background field g. The remaining terms can then be interpreted as the cost in energy to generate the background field g. This interpretation allows us to use (6.22) to calculate the mass and the couplings of the bosonic field g.

6.3 Comparison with MFT

From Chap. 4 we know that partial bosonization is not restricted to four-fermion interactions. In particular Eq. (4.11) provides us with the means to calculate a partially bosonized action for an arbitrary local multi-fermion interaction. Associating $\phi_{\alpha\beta}(x) = \langle \tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x)\rangle$, the partially bosonized form of Eq. (6.2) becomes

$$\begin{aligned} S_{\mathrm{int}}[\phi,\tilde{\psi}] &= \int_x \sum_n m^{(n)}_{a_1 b_1 \dots a_{\frac{n}{2}} b_{\frac{n}{2}}} \Big\{ \phi_{a_1 b_1}(x) \cdots \phi_{a_{\frac{n}{2}} b_{\frac{n}{2}}}(x) \qquad (6.26) \\ &+ \frac{1}{(\frac{n}{2}-1)!} \left[\psi_{a_1}(x)\psi_{b_1}(x)\phi_{a_2 b_2}(x) \cdots \phi_{a_{\frac{n}{2}} b_{\frac{n}{2}}}(x) + \text{perm. } \{1, \dots, \tfrac{n}{2}\} \right] + \mathcal{O}(\tilde{\psi}^3) \Big\}, \end{aligned}$$

$$m^{(n)}_{a_1 \dots a_n} = -\frac{\lambda^{(n)}_{a_1 \dots a_n}}{n!} + \Sigma_{a_1 \dots a_n}. \qquad (6.27)$$

Where Σ is a sum of terms which are symmetric in at least one pair of indices (cf. Sect. 4.1.2). The condition (6.27) ensures that the partially bosonized action is equivalent to the original fermionic one.

Neglecting the terms $\mathcal{O}(\tilde{\psi}^3)$ and performing the functional integral over the fermions provides us with the MF effective action:

$$\begin{aligned} \Gamma^{\mathrm{MF}}[\phi] &= -\frac{1}{2}\mathrm{Tr}\ln(-\jmath + g[\phi]) + \int_x \sum_n m^{(n)}_{a_1 \dots b_{\frac{n}{2}}} \phi_{a_1 b_1} \cdots \phi_{a_{\frac{n}{2}} b_{\frac{n}{2}}}, \qquad (6.28) \\ g[\phi]_{a_1 b_1} &= \frac{1}{(\frac{n}{2}-1)!} \left[m^{(n)}_{a_1 \dots b_{\frac{n}{2}}} \phi_{a_2 b_2} \cdots \phi_{a_{\frac{n}{2}} b_{\frac{n}{2}}} + \text{perm. } \{1, \dots, \tfrac{n}{2}\} \right]. \end{aligned}$$

[3] An alternative would be to choose the gap Δ as the "bosonic field". Inserting Eq. (6.17) into Eq. (6.13) we could calculate a functional $\Gamma[\Delta]$. However, as one can check there are two drawbacks. First, even for four-fermion interactions, $\Gamma[\Delta]$ is usually unbounded from below when considering $\Delta \to \infty$. Second, in the case of a large four-fermion coupling the "stable" solution of the field equation is usually a local maximum.

By construction this is a one-loop result. Moreover, it is strikingly similar to Eq. (6.22). However, the coefficients differ. As discussed in Sect. 4.1.2 the Fierz ambiguity is reflected by the presence of a nearly arbitrary Σ in Eq. (6.27). Results usually depend on the choice of Σ. Of course, further considerations as e.g. the stability of the initial bosonic potential might reduce the freedom of Σ somewhat. But, sometimes this is not even enough to get qualitatively unambiguous results [33; 34].

Eqs. (6.18), (6.22), (6.23) do not suffer from such an ambiguity since in the derivation of the SDE (6.8) the coefficients become antisymmetrized and symmetric terms drop out. In Sect. 4.3 we demonstrated that the inclusions of certain diagrams cures the Fierz ambiguity for four-fermion interactions and leads to the SD result. We believe that this holds for higher fermion interactions, too. Thus, we propose (6.22) as a natural generalization of (6.28).

Finally, let us stress the similarity of both approaches by noting that $g(x) = \langle\tilde{\psi}(x)\tilde{\psi}(x)\rangle$ is exactly what we had in mind as a "mean field".

6.4 Wave Function Renormalization

In the partially bosonized language (cf. sect. 6.3) it is possible to calculate a wave function renormalization for the bosons. Allowing not only for constant but also for a slightly varying ϕ (p small)

$$\phi(x) = \phi(0) + \delta\phi \exp(ipx). \tag{6.29}$$

Using this ϕ it is still possible to perform the fermionic integral. Expanding in powers of the momentum up to the p^2-term we can read off the wave function renormalization.

The same can be done for Eq. (6.18)

$$g(x) = g(0) + \delta g \exp(ipx). \tag{6.30}$$

Expanding again in powers of momentum we interpret the p^2-term as the wave function renormalization for the boson corresponding to the field g. As in MFT the only contribution to the wave function renormalization comes from the $\mathrm{Tr}\ln$ and therefore from a simple one-loop expression.

Knowledge of the wave function renormalization together with the second derivative of the effective action for constant fields allows us to compute the mass of the boson.

Again, this calculation is unambiguous. This is in contrast to a calculation in the partially bosonized language where we again have the problems with the Fierz ambiguity. On the other hand we do not want to hide the fact that by considering values of g which do not coincide with the solution of the SDE we have left the solid ground of a direct computation from the BEA given in Eq. (6.16).

6.5 Chiral Symmetry Breaking from a 3-Flavor Instanton Interaction

In this section we want to use the method described above to study chiral symmetry breaking in an NJL-type model with a six-fermion interaction modelling the QCD-instanton interaction with three colors and three flavors [99–103]. The three flavor instanton vertex can be written in the following convenient form [49]

$$
\begin{aligned}
S_{\text{inst}}[\psi] = -\frac{\zeta}{6}\int_x \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}
&\Big\{ \Big[(\overline{\psi}_L^{a_1}\psi_R^{b_1})(\overline{\psi}_L^{a_2}\psi_R^{b_2})(\overline{\psi}_L^{a_3}\psi_R^{b_3}) - \frac{1}{8}(\overline{\psi}_L^{a_1}\lambda^z\psi_R^{b_1})(\overline{\psi}_L^{a_2}\lambda^z\psi_R^{b_2})(\overline{\psi}_L^{a_3}\psi_R^{b_3}) \\
&- \frac{1}{8}(\overline{\psi}_L^{a_1}\lambda^z\psi_R^{b_1})(\overline{\psi}_L^{a_2}\psi_R^{b_2})(\overline{\psi}_L^{a_3}\lambda^z\psi_R^{b_3}) - \frac{1}{8}(\overline{\psi}_L^{a_1}\psi_R^{b_1})(\overline{\psi}_L^{a_2}\lambda^z\psi_R^{b_2})(\overline{\psi}_L^{a_3}\lambda^z\psi_R^{b_3})\Big] \\
&-(R\leftrightarrow L)\Big\}
\end{aligned}
\tag{6.31}
$$

where λ^z are the Gell-Mann matrices corresponding to the $SU(3)_\mathrm{c}$ color group and the brackets ( ) indicate contractions over color and spinor indices.

The coupling constant ζ can be calculated in terms of the gauge coupling. However, it involves an IR divergent integral over the instanton size. Therefore, one needs to provide a physical cutoff mechanism. To avoid this difficulty we treat ζ as a parameter.

Inspection of (6.31) tells us that this interaction is $U(1)$ anomalous with a residual $\mathcal{Z}_3$-symmetry. This is important because we cannot restrict ourselves to real condensates from the start.

In order to extract the interaction matrix λ we have to antisymmetrize over flavor indices ($a = 1\dots3$), color indices ($i = 1\dots3$) Weyl spinor indices ($\alpha = 1, 2$), chirality indices ($\chi = 1, 2 = L, R$) and the indices distinguishing between ψ and $\overline{\psi}$

$(s = 1, 2)$.

$$\begin{aligned}
\lambda_{m_1 \ldots m_6} &= P\{\tilde{\lambda}_{m_1 \ldots m_6}\}, & (6.32)\\
\tilde{\lambda}_{m_1 \ldots m_6} &= \frac{\zeta}{12}\epsilon_{a_1 a_2 a_3}\epsilon_{a_4 a_5 a_6}\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_5}\delta_{\alpha_3\alpha_6} \\
&\quad \times[\delta_{i_1 i_4}\delta_{i_2 i_5}\delta_{i_3 i_6} - \frac{1}{8}\lambda^z_{i_1 i_4}\lambda^z_{i_2 i_5}\delta_{i_3 i_6} - \frac{1}{8}\lambda^z_{i_1 i_4}\delta_{i_2 i_5}\lambda^z_{i_3 i_6} - \frac{1}{8}\delta_{i_1 i_4}\lambda^z_{i_2 i_5}\lambda^z_{i_3 i_6}] \\
&\quad \times[\delta_{\chi_1 1}\delta_{\chi_2 1}\delta_{\chi_3 1}\delta_{\chi_4 2}\delta_{\chi_5 2}\delta_{\chi_6 2} - \delta_{\chi_1 2}\delta_{\chi_2 2}\delta_{\chi_3 2}\delta_{\chi_4 1}\delta_{\chi_5 1}\delta_{\chi_6 1}] \\
&\quad \times[\delta_{s_1 2}\delta_{s_2 2}\delta_{s_3 2}\delta_{s_4 1}\delta_{s_5 1}\delta_{s_6 1} + \delta_{s_1 1}\delta_{s_2 1}\delta_{s_3 1}\delta_{s_4 2}\delta_{s_5 2}\delta_{s_6 2}].
\end{aligned}$$

Here P denotes the sum over all 6! permutations of the multiindices $m_j = (a_j, i_j, \alpha_j, \chi_j, s_j)$, $j = 1 \ldots 6$, with minus signs appropriate for total antisymmetrization.

As a first example we consider a flavor singlet, color singlet scalar chiral bilinear ($\sigma = \frac{1}{3}\overline{\psi}^a_L\psi^a_R$, $\sigma^\star = -\frac{1}{3}\overline{\psi}^a_R\psi^a_L$)

$$\begin{aligned}
g_{mn} &= g_{ai\alpha\chi s, bj\beta\tau t} & (6.33)\\
&= \frac{1}{6}\delta_{ab}\delta_{ij}\delta_{\alpha\beta}\Big[\sigma(\delta_{\chi 1}\delta_{\tau 2}\delta_{s2}\delta_{t1} - \delta_{\chi 2}\delta_{\tau 1}\delta_{s1}\delta_{t2}) - \sigma^\star(\delta_{\chi 2}\delta_{\tau 1}\delta_{s2}\delta_{t1} - \delta_{\chi 1}\delta_{\tau 2}\delta_{s1}\delta_{t2})\Big].
\end{aligned}$$

We evaluate

$$\begin{aligned}
\Delta[g]_{mn} &= -\frac{\lambda^{(n)}_{m m_2 m_3 n m_5 m_6}}{8} g_{m_2 m_5} g_{m_3 m_6} & (6.34)\\
&= -6\Big[\tilde{\lambda}^{(n)}_{m m_2 m_3 n m_5 m_6} - \tilde{\lambda}^{(n)}_{m m_2 m_3 m_5 n m_6} + \tilde{\lambda}^{(n)}_{m m_2 m_3 m_5 m_6 n}\Big] \\
&\qquad \times\Big[g_{m_2 m_5} g_{m_3 m_6} - g_{m_2 m_6} g_{m_3 m_5}\Big]
\end{aligned}$$

and

$$\begin{aligned}
U &= -\frac{1}{2}\Delta[g]_{mn} g_{mn} + \frac{\lambda_{m_1 \ldots m_6}}{48} g_{m_1 m_2} g_{m_3 m_4} g_{m_5 m_6} \\
&= -\frac{1}{24}\lambda_{m_1 \ldots m_6} g_{m_1 m_2} g_{m_3 m_4} g_{m_5} g_{m_6} \\
&= -\frac{1}{3}\Delta[g]_{mn} g_{mn}, & (6.35)
\end{aligned}$$

where we have used the fact that $\tilde{\lambda}$ is symmetric under permutations of the three $\overline{\psi}\psi$ bilinears. Exploiting the flavor, spin and color structure for (6.34) yields

$$\begin{aligned}
\Delta[g]_{mn} &= -\frac{10}{9}\zeta\delta_{ab}\delta_{ij}\delta_{\alpha\beta}\Big[\sigma^2(\delta_{\chi 1}\delta_{\tau 2}\delta_{s2}\delta_{t1} - \delta_{\chi 2}\delta_{\tau 1}\delta_{s1}\delta_{t2}) \\
&\qquad -\sigma^{\star 2}(\delta_{\chi 2}\delta_{\tau 1}\delta_{s2}\delta_{t1} - \delta_{\chi 1}\delta_{\tau 2}\delta_{s1}\delta_{\tau 2})\Big] & (6.36)
\end{aligned}$$

and

$$U[\sigma] = \frac{20}{9}\zeta(\sigma^3 + \sigma^{\star 3}). \tag{6.37}$$

Evaluating for a σ constant in space and pulling out a volume factor we obtain the effective potential and the relation between condensate and fermion mass,

$$\begin{aligned} \hat{\Gamma}[\sigma] &= -36v_4 \int dx\; x[\ln(x + |m|^2_\sigma)] + U[\sigma], \\ m_\sigma &= \frac{10}{9}\zeta\sigma^2 + m^0_\sigma. \end{aligned} \tag{6.38}$$

Here m^0_σ is a current quark mass which we take to be equal for all quarks. The integral in (6.38) is, of course, divergent. Our UV regularization is simply to cut it off at Λ^2. Measuring all quantities in units of Λ we can put $\Lambda = 1$.

6.5.1 The chiral limit $m^0_\sigma = 0$

Let us now look at the field equation or equivalently search for extrema of $\hat{\Gamma}[\sigma]$. Since $\frac{d\Delta[\sigma]}{d\sigma} \neq 0$ for all $\sigma \neq 0$ we do not need to worry for non-trivial solutions to be spurious. In addition $\sigma = 0$ is always a solution in the chiral limit.

Inspection of $\hat{\Gamma}[\sigma]$ tells us that it is invariant under the combined operation $\zeta \to -\zeta$, $\sigma \to -\sigma$. This allows us to restrict our analysis to positive ζ.

In the chiral limit it is useful to parametrize

$$\sigma = |\sigma| \exp(i\alpha). \tag{6.39}$$

Form this one finds

$$\hat{\Gamma}[|\sigma|, \alpha] = \frac{40}{9}\zeta|\sigma|^3\cos(3\alpha) + f(|\sigma|), \tag{6.40}$$

where f is a function determined by the integral in Eq. (6.38). We can see that the only α-dependence comes from $\cos(3\alpha)$ which is the explicit manifestation of the $\mathcal{Z}_3$-symmetry.

It is clear that extrema can only occur at $\alpha = \frac{n\pi}{3}$, $n \in \mathbb{Z}$. Using the $\mathcal{Z}_3$-symmetry we can restrict ourselves to $\alpha = 0, \pi$ or restrict ourselves simply to real σ.

Taking all this into account we find up to three solutions (cf. Fig. 6.1). As already mentioned $\sigma = 0$ is a solution for all values of the coupling. Going to larger couplings we encounter a point ζ_{crit} where we have two solutions. For even larger couplings there are three solutions $0 = \sigma_0 < \sigma_1 \leq \sigma_2$. We know that $\hat{\Gamma}[\sigma_1] > \hat{\Gamma}[\sigma_0 = 0]$

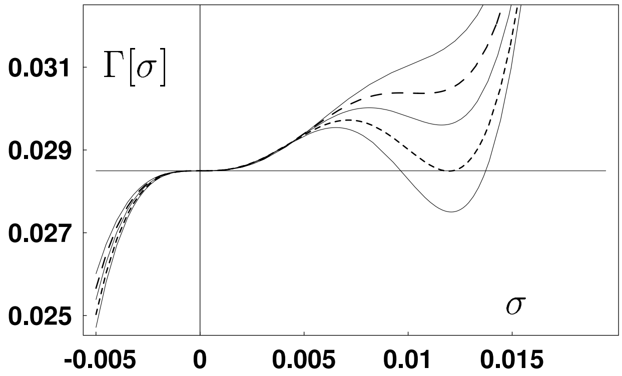


Figure 6.1: Plot of the BEA for various values of the coupling constant increasing from the topmost line $\zeta = 3000$ to the lowest line $\zeta = 4200$. The second line (long dashed) is for $\zeta_{\rm crit} \approx 3350$, the critical coupling for the onset of non-vanishing solutions. The third is for $\zeta = 3600$ while the fourth (short dashed) is the for the onset of SSB $\tilde{\zeta}_{\rm SSB} \approx 3900$. The horizontal line indicates the value of $\Gamma[\sigma]$ at the trivial solution $\sigma = 0$.

therefore σ_1 is not the stable solution. As can be seen from Figs. 6.1, 6.2 there is a $\zeta_{\rm crit} \leq \zeta \leq \zeta_{\rm SSB}$ where there exist non-trivial solutions to the SDE but there is still no SSB because $\hat{\Gamma}[\sigma_2] \geq \hat{\Gamma}[\sigma_0 = 0]$. We point out that in order to calculate $\zeta_{\rm SSB}$ we need to know the value of $\hat{\Gamma}$, i.e. information beyond the SDE.

In Fig. 6.2 we have plotted the mass gap versus the six-fermion coupling strength. Looking at Fig. 6.2 we observe a first order phase transition. From Eq. (6.22) we actually expect this quite generically as long as we have only n-fermion interactions with $n \geq 6$. However, this might also be an artifact of the "one-vertex" approximation.

Finally, we would like to remark that in general it is not enough to simply integrate the SDE with respect to the gap Δ reconstruct the effective action functional. This functional, let us call it $\tilde{\Gamma}[\Delta]$ is in general *not* equal to the the BEA $\hat{\Gamma}[G]$, not even at solutions of the SDE. Indeed, as can be seen from Fig. 6.2, the results for physical quantities like the effective fermion mass can differ. The underlying reason for this is that the gap Δ is not the correct integration variable. The SDE is obtained by a G-derivative of the BEA functional $\hat{\Gamma}[G]$. Therefore, in order to reconstruct $\hat{\Gamma}[G]$, we have to integrate with respect to G. As can be seen from Eq. (6.21) Δ is, in general, not even a linear function of G. Simple integration with respect to Δ therefore neglects the Jacobi matrix, which is a non-trivial function of Δ for interactions more complicated than a four-fermion interaction.

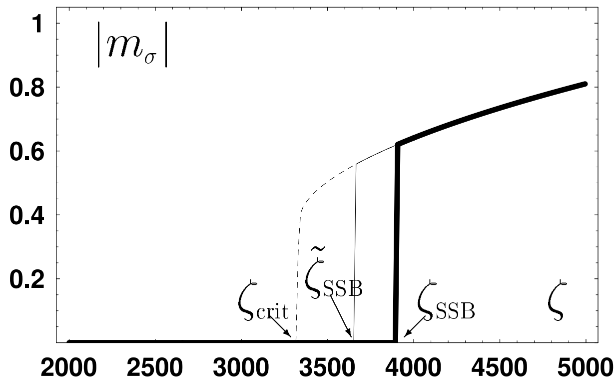


Figure 6.2: Fermion mass $|m|_\sigma$ (gap) versus the strength of the six-fermion interaction ζ in the chiral limit $m_\sigma^0 = 0$. The thick line corresponds to the solution with smallest action. The dashed line is the largest non-trivial solution. Finally, the thin line is obtained by minimizing an "effective action" $\tilde{\Gamma}[\Delta]$ obtained by direct integration of the SDE with respect to the gap Δ. We find three special couplings, $\zeta_{\rm crit}$ for the onset of non-trivial solutions to the SDE, $\zeta_{\rm SSB}$ for the onset of SSB, i.e. a non-trivial solution has lower action than the trivial solution and $\tilde{\zeta}_{\rm SSB}$ where the lowest extremum of $\tilde{\Gamma}[\Delta]$ becomes non-trivial. We point out that these three "critical" couplings differ. Moreover, to calculate $\zeta_{\rm SSB}$ we need to know the effective action. In our approximation we obtain a first order phase transition.

6.5.2 Non-vanishing current quark masses $m_\sigma^0 \neq 0$

The non-vanishing current quark mass explicitly breaks the residual $\mathcal{Z}_3$-symmetry. $\hat{\Gamma}[|\sigma|, \alpha]$ does no longer depend on $\cos(3\alpha)$ only, and we have to look at the complete complex σ-plane for possible extrema.

Moreover, for $m_\sigma^0 = 0$, $\hat{\Gamma}[\sigma]$ is completely symmetric under $\zeta \to -\zeta, \sigma \to -\sigma$. Therefore, we could restrict ourselves to $\zeta \geq 0$. For $m_\sigma^0 \neq 0$ we need to add the transformation $m_\sigma^0 \to -m_\sigma^0$. We can still restrict ourselves to positive ζ but we need to consider both positive and negative m_σ^0.

In the case of $m_\sigma^0 \neq 0$ we still encounter an extremum of $\hat{\Gamma}[\sigma]$ at $\sigma = 0$. However, in this case it is not a solution of the SDE. It is a spurious solution due to $\frac{d\Delta[\sigma]}{d\sigma} = 0$. The difference to the chiral limit is that the derivative of the effective potential $\hat{\Gamma}[\sigma]$ now has only a simple zero while in the chiral limit it is a twofold zero. After dividing the field equation by $\frac{d\Delta[\sigma]}{d\sigma}$ a simple zero remains, giving a solution of the SDE in the chiral limit.

Although, chiral symmetry is now broken explicitly we can still observe a first-order phase transition signaled by a jump in the fermion mass. The critical coupling

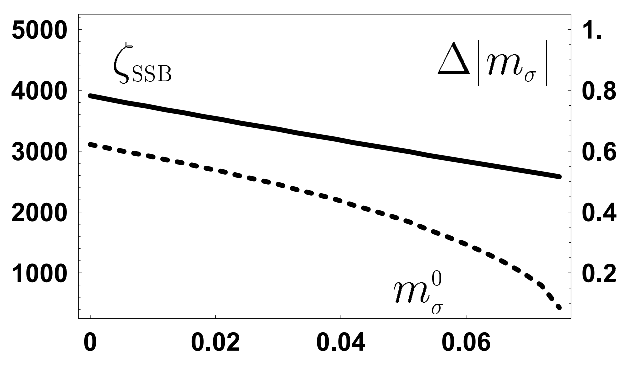


Figure 6.3: Dependence of the critical coupling on the current quark mass m_σ^0 (thick solid line). Jump in the fermion mass at the phase transition (thick dashed line). We observe that there exists a critical $m_\sigma^0 \approx 0.076$ above which there is no phase transition.

for the phase transition depends on m_σ^0 as depicted in Fig. 6.3. The critical line ends at $m_\sigma^0 = m_{\sigma,\mathrm{crit}}^0 \approx 0.076$, i.e. for $m_\sigma^0 > m_{\sigma,\mathrm{crit}}^0$ we have no first order phase transition in our approximation.

6.6 Color-Octet Condensation

In the last section we have considered only one direction in the space of all possible g resulting in a phase diagram for chiral symmetry breaking. Let us now consider the more general case where we also allow for a non-vanishing expectation value in the color-octet channel, more explicitly in the color-flavor locking direction.

$$\begin{aligned} g_{mn} &= g_{ai\alpha\chi s,bj\beta\tau t} \qquad (6.41)\\ &= \frac{1}{6}\delta_{ab}\delta_{ij}\delta_{\alpha\beta}\Big[\sigma(\delta_{\chi 1}\delta_{\tau 2}\delta_{s2}\delta_{t1} - \delta_{\chi 2}\delta_{\tau 1}\delta_{s1}\delta_{t2}) - \sigma^\star(\delta_{\chi 2}\delta_{\tau 1}\delta_{s2}\delta_{t1} - \delta_{\chi 1}\delta_{\tau 2}\delta_{s1}\delta_{t2})\Big]\\ &+ \frac{1}{12}\lambda^z_{ab}\lambda^z_{ij}\delta_{\alpha\beta}\Big[\chi(\delta_{\chi 1}\delta_{\tau 2}\delta_{s2}\delta_{t1} - \delta_{\chi 2}\delta_{\tau 1}\delta_{s1}\delta_{t2}) - \chi^\star(\delta_{\chi 2}\delta_{\tau 1}\delta_{s2}\delta_{t1} - \delta_{\chi 1}\delta_{\tau 2}\delta_{s1}\delta_{t2})\Big] \end{aligned}$$

Following the outline of the previous section we obtain

$$\begin{aligned} \Delta[g]_{mn} &= -\zeta\left[\frac{5}{1296}(288\sigma^2 + 6\sigma\chi - 7\chi^2)\delta_{ab}\delta_{ij} - \frac{5}{432}\chi(6\sigma+\chi)\delta_{ai}\delta_{bj}\right]\\ &\qquad \times\delta_{\alpha\beta}\Big[\delta_{\chi 1}\delta_{\tau 2}\delta_{s2}\delta_{t1} - \delta_{\chi 2}\delta_{\tau 1}\delta_{s1}\delta_{t2}\Big]\\ &\quad -(\chi \leftrightarrow \tau, \sigma \to \sigma^\star, \chi \to \chi^\star) \end{aligned}$$

and

$$U[\sigma,\chi] = \zeta\left[\frac{20}{9}\sigma^3 - \frac{5}{27}\sigma\chi^2 - \frac{5}{243}\chi^3 + c.c.\right]. \tag{6.42}$$

$$\begin{aligned}
\hat{\Gamma}[\sigma,\chi] &= -4v_4\int_x dx\; x[8\ln(x+|m|^2_\sigma)+\ln(x+|m|^2_\chi)] + U[\sigma,\chi] \\
m_\sigma &= \frac{5}{1296}\zeta(288\sigma^2+6\sigma\chi-7\chi^2)+m^0_\sigma \\
m_\chi &= \frac{5}{81}\zeta(18\sigma^2-3\sigma\chi-\chi^2)+m^0_\sigma.
\end{aligned} \tag{6.43}$$

In the chiral limit every point on the line $\chi = -6\sigma$ ($m_\sigma = 0$ and $m_\chi = 0$) has the same value of $\hat{\Gamma}$, and both derivatives with respect to σ and χ vanish. But, in this direction the derivative of Δ with respect to the condensates vanishes, too, and is indeed the null function in this direction. Therefore, on this line only the point $(\sigma,\chi) = (0,0)$ is a true solution to the SDE.

Restricting both σ and χ to be real we have not found a solution with $\chi \neq 0$. Thus, we have not identified a solution which breaks color symmetry but not parity.

For the most general case of complex σ and χ things are considerably more difficult since we now have to search for an extremum of a potential which depends on four real parameters. We checked several values of the coupling constant. So far we have not found a solution which has lower action than the lowest one with $\chi = 0$.

Still, we would like to point out that the potential is unbounded from below. In various directions including those with $\chi \neq 0$, $\hat{\Gamma} \to -\infty$. Therefore, a physical cutoff mechanism like the one discussed in [49] or a better approximation which makes the potential bounded from below may provide additional solutions.

Chapter 7

Outlook: Quest for a Renormalizable Standard Model[0]

The "Standard Model" (SM) of particle physics is probably one of the most widely studied physical theories. It describes a wide range of physical situations with a satisfactory amount of accuracy. Yet, there are still open questions. In particular, we want to concentrate on "Renormalizability" and the "Hierarchy Problem", as those are problems tightly connected to the existence of an elementary scalar boson – the Higgs – in the SM. This returns us to the speculation of a composite Higgs which we mentioned in the introduction.

Before going into details, let us briefly outline those two problems.

Renormalizability: If asked whether the SM is renormalizable many physicists[1] would answer this question positively. So, why search for something we have already found? Well, while this answer is certainly correct for most practical purposes, i.e. when the UV-cutoff scale is of a reasonable size $\Lambda \lesssim 1\mathrm{TeV}$, it is not certain that this is so if we send the cutoff to infinity.

To get a grasp of the problem let us look at the example of ϕ^4-theory (a model for the Higgs potential). Roughly speaking renormalizability means that the physics at short distances does not really matter, and therefore we can send the cutoff to infinity without changing the results. Yet, a straightforward calculation gives the (1-loop) running four-boson coupling as follows,

$$\lambda_{\mathrm{eff}}(q^2) = \frac{\lambda}{1 - \frac{3\lambda}{4\pi^2}\ln\left(\frac{q^2}{\mu^2}\right)}, \tag{7.1}$$

[0]This is work in progress in collaboration with Holger Gies and Christof Wetterich.

[1]A (not representative) survey in the "Graduiertenkolleg: Physical Systems with many Degrees of Freedom" of the university of Heidelberg resulted in $\approx 70\%$ positive answers.

where q^2 is the (Euclidean) momentum scale and λ is the coupling at the scale μ^2. From this we can read off, that the coupling constant grows with increasing momentum scale. It is plausible that this does not fit into our picture that long distance physics (small momenta) decouples from short distance physics (large momenta), as the coupling is strong in the latter regime.

This gives us an intuitive understanding that problems might arise when we want to send the UV cutoff to infinity. Those theories are not renormalizable in a strict sense. Requiring that a "fundamental" theory should be renormalizable such theories cannot be "fundamental", i.e. valid at all scales. They are valid only as effective theories up to a certain cutoff scale Λ where "new" physics will set in.

This [11–15] and a similar problem in QED (Landau pole!) [108; 109], hint to serious trouble in the Higgs respectively $U(1)$-sector of the SM. This is often referred to as "triviality" because when starting with some bare interactions and sending the cutoff to infinity, the renormalized coupling will be strictly zero.

Hierarchy Problem: In contrast to chiral fermions, where chiral symmetry prevents the fermion mass from acquiring large quantum correction, the mass of a scalar boson is not protected against such corrections. Thus, assuming that the RG flow of the SM is "released" at, say, the GUT[2] scale, enormous fine-tuning of the scalar initial conditions is required to separate the electroweak scale from the GUT scale by many orders of magnitude.

To illustrate this let us do a very simplified calculation. In a crude approximation the Higgs mass runs as follows ($c^2 = \text{const.} = \mathcal{O}(1)$),

$$m_H^2(q_1^2) - m_H^2(q_2^2) = c^2(q_1^2 - q_2^2). \tag{7.2}$$

Defining the dimensionless Higgs mass $\mu^2(q^2) = \frac{m_H^2(q^2)}{q^2}$ this can be rewritten as,

$$\mu^2(q_1^2) = c^2 - \frac{q_2^2}{q_1^2}\left(c^2 + \mu^2(q_2^2)\right). \tag{7.3}$$

So far so good, but let us now consider two very different scales, e.g. the GUT scale $q_1^2 = M_{\mathrm{GUT}}^2 \sim 10^{30}(\mathrm{GeV})^2$ and the electroweak symmetry breaking scale $q_2^2 \sim 10^4(\mathrm{GeV})^2$. Inserting values for the Higgs mass at the electroweak scale we find that the brackets on the RHS are of order $\mathcal{O}(1)$. Hence,

$$\mu^2(M_{\mathrm{P}}) = c^2 - \mathcal{O}(10^{-26}), \tag{7.4}$$

the dimensionless mass at the GUT scale must be fine-tuned to c^2 by an incredible amount. Although not excluded, such an amount of fine-tuning seems unnatural.

[2]Grand Unified Theory.

7.1 UV Fixed Points and Renormalizability

Let us now address the problem of renormalizability on a more formal level.

7.1.1 Non-perturbative Renormalizability

Commonly, theories are considered to be renormalizable if they have a (small coupling) perturbative expansion and we can absorb the infinities by a finite number of counterterms. In consequence, they only have a finite number of couplings (masses etc.) since it requires physical information to fix a counterterm unambiguously. For instance, the value of a coupling or mass at some scale cannot be determined in the theory itself, but must be measured. All other quantities can be calculated from these couplings. From this renormalizable theories derive their predictive power. In particular, the renormalization procedure allows us to take the continuum limit, i.e. to send the UV cutoff to infinity. In this sense we can consider renormalizable theories as fundamental. A prominent example of such a theory is QCD.

It comes as no surprise that not all theories have this property of "perturbative renormalizability" (PR). Gravity is probably the most well known example, but there are many more: as theories which contain a coupling of negative mass dimension are usually not PR. The NJL model (1.1) has a coupling $\sim (mass)^{-2}$ and consequently is not PR, either. Moreover, naively renormalizable theories with dimensionless coupling constants may also face difficulties spoiling PR, as discussed at the beginning of this chapter. Usually, theories which are not PR are thought to be effective field theories valid only up to a finite UV scale Λ.

Yet, PR is not the end of the game. Using a little bit of imagination it is obvious that a logical alternative is a "non-perturbative" renormalizable (NPR) [110–116] theory. So far so good, but does such a thing exist and what is it? Fortunately, the answer to the first question is yes. Among the examples are various versions of the Gross-Neveu model. More recently, it has been proposed that gravity is NPR, too [117; 118]. But now, let us find out what hides behind NPR[3].

The space of all possible action functionals can be parametrized by an infinite number of dimensionless couplings (if necessary we use a suitable rescaling with the scale k). As discussed in Sect. 2.1.2 the RG describes a trajectory in this space (it is parametrized by the scale $t = \ln(k/k_0)$),

$$\partial_t g^i_t = \beta^i(g^1, g^2, \ldots). \tag{7.5}$$

Starting point for the construction of a NPR is the existence of a non-Gaussian fixed point (FP) $g_\star = (g^1_\star, g^2_\star, \ldots)$ with at least one $g^i_\star \neq 0$, in the RG flow,

$$\beta^i(g^1_\star, g^2_\star, \ldots) = 0 \quad \forall i. \tag{7.6}$$

[3]In this brief description of NPR we follow [118].

In the setting of statistical physics such a FP is exactly what we would associate with a second order phase transition. The dimensionless constants do not change with the scale (typically the scale can be associated with the difference from the critical temperature $\sim |T - T_c|$), accordingly the dimensionful quantities are simply powers of the scale[4]. This gives the typical power laws of the critical behavior near a second order phase transition. Moreover, we would like to remark that in this context a PR is just the special case of a Gaussian, i.e. $g_\star = 0$, FP.

In the vicinity of this FP we can linearize the RG equations,

$$\partial_t g^i = \sum_j B^{ij}(g^j_\star - g^j_t) \tag{7.7}$$

where

$$B^{ij} = \left.\frac{\partial \beta^i}{\partial g^j}\right|_{g_\star} \tag{7.8}$$

is the Jacobi matrix of the β-functions. This is a linear differential equation and the general solution reads,

$$g^i_t = g^i_\star + \sum_I C^I V^I_i \left(\frac{k_0}{k}\right)^{\Theta^I}, \tag{7.9}$$

where the V^I are right eigenvectors of B with corresponding eigenvalues[5] $-\Theta^I$, $BV^I = -\Theta^I V^I$, and the C^I are constants determined by the initial conditions. Setting $C^I = 0$ if the corresponding eigenvalue[6] $\Theta^I < 0$ we can safely take the UV limit $k \to \infty$. In other words this gives us an RG trajectory which ends in $g_\star$ for $k \to \infty$. Any such trajectory defines a theory with a meaningful UV limit. The space of all such trajectories is a submanifold $\mathcal{S}$ of dimensionality Δ, given by the number of eigenvalues $\Theta^I > 0$. We can specify a trajectory in this space by giving the values of integration constants C^I. Thus our theory has Δ "renormalizable couplings", which have to be taken from experiment. In particular, Δ must be *finite* (and preferably small) in order for our theory to have predictive power.

Stated differently, all trajectories in the submanifold $\mathcal{S}$ are *attracted* toward $g_\star$ for *increasing* scale k, consequently, for *decreasing* k they are *repelled*. Therefore, the Δ parameters specifying the trajectory are the *relevant* parameters to describe

[4]It is often useful to include the wave function renormalization into the couplings. This modifies the naive power laws by anomalous dimensions.

[5]The Θ^I are not necessarily real as B is not necessarily symmetric. For simplicity let us pretend that they are real, anyway. The general case can be recovered by replacing, $\Theta > 0 \to Re(\Theta) > 0$ etc. in our argumentation.

[6]For $\Theta^I = 0$ it depends on the details if the UV limit is finite or not. Correspondingly, it might or might not be necessary to set $C^I = 0$. Prominent examples for both cases are the ϕ^4-theory where the UV limit is not finite, and QCD where the limit is finite and which is therefore strictly renormalizable.

physics in the fixed point regime. The remaining "non-renormalizable couplings" corresponding to $\Theta^I < 0$ are *irrelevant* in the sense that starting with a finite value at some scale $k < \infty$ they are attracted toward the submanifold $\mathcal{S}$ for decreasing k. Thus it does not really matter if we give them some finite value at a large scale[7] k.

Finally, let us give a naive argument why the submanifold $\mathcal{S}$ should be finite dimensional. To obtain the dimensionless coupling we have to rescale the dimensionful coupling constant G^i by an appropriate power of k, $g^i = k^{-d^i}G$. The β-function has now the form $\beta^i = -d^i g^i + \cdots$ where the dots denote the loop corrections. In the Jacobi matrix this gives a contribution $B^{ij} = -d^i\delta^{ij} + \cdots$. The eigenvalues get a contribution $\Theta^i = d^i + \cdots$. Constructing a higher order operator, we usually add derivatives or field operators, thus decreasing d^i. Therefore, we should only have a finite number of $\Theta^i < 0$. Yet, this argument is based on the assumption that the loop corrections are small. Hence, it might be not valid in the non-perturbative regime with strong coupling and large loop corrections. Still, it seems reasonable to first look for a possible FP and establish that it is not an artifact of the approximation, and then worry about the dimensionality of $\mathcal{S}$.

7.1.2 A Toy Model

Let us illustrate this idea for a simple NJL model in d-dimensions, a four-fermion interaction with dimensionless (positive) coupling constant $\hat{\lambda} = k^{d-2}\lambda$, and the flow equation (which, for the moment, is assumed to be exact),

$$\partial_t\hat{\lambda} = (d-2)\hat{\lambda} - C\hat{\lambda}^2. \tag{7.10}$$

For $d > 2$ the Gaussian FP $\hat{\lambda} = 0$ is UV repulsive, i.e. $\hat{\lambda}$ is an eigenvector with eigenvalue $-\Theta = (d-2) > 0$. The model is not PR for a non-vanishing $\hat{\lambda}$. Moreover, for any value $\hat{\lambda} < \frac{d-2}{C}$ we approach the Gaussian FP in the IR, $\hat{\lambda}$ is "irrelevant". By contrast, for $\hat{\lambda} \geq \frac{d-2}{C}$ the coupling $\hat{\lambda}$ grows and eventually diverges. This typically signals some kind of symmetry breaking, $\hat{\lambda}$ is not so irrelevant any more. Looking a little bit more closely we notice that $\hat{\lambda}_c = \frac{d-2}{C}$ is a FP. The eigenvector $(\hat{\lambda} - \hat{\lambda}_c)$ has an eigenvalue $-\Theta' = -(d-2) < 0$, i.e. it is a relevant parameter in the fixed point regime. On the other hand, the FP is UV attractive and we have a meaningful UV limit even for a (small) non-vanishing $(\hat{\lambda} - \hat{\lambda}_c)$. The theory with this parameter is NPR.

Finally, let us remark that the ad hoc flow equation of our toy model resembled a typical flow equation for an NJL-type model in a truncation to four-fermion interactions, e.g. with $C = 4(N-2)v_3 l_1^{(\mathrm{F}),4}$ and $d = 3$ the flow equation for the Gross-Neveu model in three dimensions.

[7] We will see below that this statement should be taken with some care. More precisely, this is true only if we are close enough to the fixed point.

7.1.3 Manifestation of the Hierarchy Problem

In this setting we also have the tools available for discussing the hierarchy problem. So far we have already seen that a renormalizable coupling is relevant in the infrared, i.e. starting with a small deviation from the fixed point in the far UV this deviation soon grows large. This is quantified in Eq. (7.9). Taking k_0 to be some large UV scale $k_0 = \Lambda \sim M_{\mathrm{P}}$ and $\Theta \sim \mathcal{O}(1)$, the deviation grows very fast with $(\frac{\Lambda}{k})^{\Theta}$ when k is lowered. Turning the argument around, we have to fine tune the initial conditions (choose the initial C very small) at the scale Λ in order to achieve a value comparable to some scale $k \ll \Lambda$ for the coupling at the scale k.

From this it may seem that renormalizability and a solution to the hierarchy problem more or less exclude each other. However, there is a way out, we simply need to make Θ small or zero. At first this may sound simply like another type of fine-tuning, but it is not necessarily so, as the eigenvalues are a prediction of our fixed point solution and not a parameter. To understand this we can once more look at our simple toy model. For $d = 2$ we have exactly the case of a vanishing eigenvalue. Solving the flow equation (7.10), we find

$$\hat{\lambda}(t = \ln(k)) = \frac{\hat{\lambda}_0}{1 + C\hat{\lambda}_0 \ln\frac{k}{\Lambda}}, \tag{7.11}$$

where $\hat{\lambda}_0 = \hat{\lambda}(t_0 = \ln(\Lambda))$. As $\hat{\lambda}$ depends only logarithmic on the scale we can have very different scales without fine-tuning.

Finally, let us remark, that Eq. (7.11) also gives us an understanding of what can happen when $\Theta = 0$. Only for $C > 0$ we approach the fixed point $\hat{\lambda}^{\star} = 0$ in the far ultraviolet. Only in this case our simple toy model is renormalizable. Moreover, our $d = 2$ flow equation for our toy model has the same form as the lowest order terms of the $d = 4$ flow in QED and QCD around their Gaussian fixed points. In this language QCD has $C > 0$ and is renormalizable while QED has $C < 0$ and is not strictly renormalizable.

7.2 One more NJL Model

Inspired by our toy model let us once again investigate an NJL model but now with $\mathrm{N_f}$ fermion species and an $SU(\mathrm{N_c})$-gauge interaction.

7.2.1 Truncation and Flow Equations

A simple truncation including all possible pointlike the four-fermion interaction reads,

$$\Gamma_k = \int \bar{\psi} \mathrm{i} \not{D} \psi + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \Big[\lambda_\sigma \, (\text{S–P}) + \lambda_- \, (\text{V–A}) + \lambda_+ \, (\text{V+A}) \\ + \lambda_\sigma^{\mathrm{f}} \, (\text{S–P})^{\mathrm{N}} + \lambda_\sigma^{\mathrm{c}} \, (\text{S–P})_{\mathrm{N}} + \lambda_{\mathrm{VA}} \, (\text{V–A})_{\mathrm{N}} \Big], \tag{7.12}$$

with the covariant derivative,

$$\not{D} = \not{\partial} - ig\not{A}. \tag{7.13}$$

The color and flavor singlets are

$$\begin{aligned}
(\text{S–P}) &= (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2, \\
(\text{V–A}) &= (\bar{\psi}\gamma_\mu\psi)^2 + (\bar{\psi}\gamma_\mu\gamma_5\psi)^2, \\
(\text{V+A}) &= (\bar{\psi}\gamma_\mu\psi)^2 - (\bar{\psi}\gamma_\mu\gamma_5\psi)^2,
\end{aligned}$$

where color $(i, j, \dots)$ and flavor $(a, b, \dots)$ indices are pairwise contracted, e.g., $(\bar{\psi}\psi) \equiv (\bar{\psi}_i^a \psi_i^a)$. The operators of non-trivial color or flavor structure are denoted by,

$$\begin{aligned}
(\text{S–P})_{\mathrm{N}} &= (\bar{\psi}_i\psi_j)^2 - (\bar{\psi}_i\gamma_5\psi_j)^2 \equiv (\bar{\psi}_i^a\psi_j^a)^2 - (\bar{\psi}_i^a\gamma_5\psi_j^a)^2, \\
(\text{S–P})^{\mathrm{N}} &= (\bar{\psi}^a\psi^b)^2 - (\bar{\psi}^a\gamma_5\psi^b)^2 \equiv (\bar{\psi}_i^a\psi_i^b)^2 - (\bar{\psi}_i^a\gamma_5\psi_i^b)^2, \\
(\text{V–A})_{\mathrm{N}} &= (\bar{\psi}_i\gamma_\mu\psi_j)^2 + (\bar{\psi}_i\gamma_\mu\gamma_5\psi_j)^2 \equiv (\text{V–A})^{\mathrm{N}}.
\end{aligned}$$

The last equation holds because of a Dirac Fierz identity (cf. App. B.3). The truncation (7.12) is symmetric under a simultaneous exchange of $\mathrm{N_c} \leftrightarrow \mathrm{N_f}$, $\lambda_\sigma^{\mathrm{f}} \leftrightarrow \lambda_\sigma^{\mathrm{c}}$, $(\dots)_{\mathrm{N}} \leftrightarrow (\dots)^{\mathrm{N}}$. However, it is not invariant under $SU(\mathrm{N_f})_{\mathrm{L}} \times SU(\mathrm{N_f})_{\mathrm{R}}$. We can obtain the part invariant under this additional symmetry by setting $\hat{\lambda}_\sigma = \hat{\lambda}_\sigma^{\mathrm{c}} = 0$. The full action Eq. (7.12) is invariant only under the subgroup $SU(\mathrm{N_f})_{\mathrm{V}}$ of simultaneous right and left handed rotations.

Following along the lines of Sect. 3.3, in particular using Eq. (3.16), the calculation of the flow equations is straightforward. Using the dimensionless coupling constants $\hat\lambda = k^2\lambda$ we find,

$$\begin{aligned}
\partial_t\hat\lambda_\sigma &= \left(2\hat\lambda_\sigma - 12g^2\left(\frac{\mathrm{N_c^2}-1}{\mathrm{N_c}}\hat\lambda_\sigma + \hat\lambda_\sigma^{\mathrm{c}}\right)v_4\right) \\
&- 8v_4 l_1^{(\mathrm{F}),4}\Big[2\mathrm{N_c N_f}\hat\lambda_\sigma^2 - 2\hat\lambda_\sigma\big[\hat\lambda_- + 3\hat\lambda_+ - 2(\mathrm{N_f}\hat\lambda_\sigma^{\mathrm{c}} + \mathrm{N_c}\hat\lambda_\sigma^{\mathrm{f}})\big] + 3\hat\lambda_\sigma^{\mathrm{c}}\hat\lambda_\sigma^{\mathrm{f}}\Big], \\
\partial_t\hat\lambda_- &= -\frac{1}{8\mathrm{N_c^2}}\left(12+\mathrm{N_c^2}(3+\alpha)^2\right)g^4 v_4 l_{1,2}^{(\mathrm{FB}),4} + \left(2\hat\lambda_- - \frac{12}{\mathrm{N_c}}g^2\left(\hat\lambda_- - \mathrm{N_c}\hat\lambda_{\mathrm{VA}}\right)v_4\right) \\
&- 8v_4 l_1^{(\mathrm{F}),4}\Big[-\mathrm{N_f N_c}(\hat\lambda_-^2+\hat\lambda_+^2) + \hat\lambda_-^2 - 2(\mathrm{N_c}+\mathrm{N_f})\hat\lambda_-\hat\lambda_{\mathrm{VA}} \\
&\qquad + \hat\lambda_+(\hat\lambda_\sigma + \mathrm{N_c}\hat\lambda_\sigma^{\mathrm{c}} + \mathrm{N_f}\hat\lambda_\sigma^{\mathrm{f}}) + 2\hat\lambda_{\mathrm{VA}}^2 - \tfrac14\hat\lambda_\sigma^2 - \tfrac12\hat\lambda_\sigma^{\mathrm{c}}\hat\lambda_\sigma^{\mathrm{f}}\Big], \\
\partial_t\hat\lambda_+ &= \frac{1}{8\mathrm{N_c^2}}\left(12+\mathrm{N_c^2}(3-\alpha(6+\alpha))\right)g^4 v_4 l_{1,2}^{(\mathrm{FB}),4} + \left(2+\frac{12}{\mathrm{N_c}}g^2 v_4\right)\hat\lambda_+ \\
&- 8v_4 l_1^{(\mathrm{F}),4}\Big[-3\hat\lambda_+^2 - 2\mathrm{N_c N_f}\hat\lambda_-\hat\lambda_+ - 2\hat\lambda_+(\hat\lambda_- + (\mathrm{N_c}+\mathrm{N_f})\hat\lambda_{\mathrm{VA}}) \\
&\qquad + \hat\lambda_-(\hat\lambda_\sigma + \mathrm{N_c}\hat\lambda_\sigma^{\mathrm{c}} + \mathrm{N_f}\hat\lambda_\sigma^{\mathrm{f}}) + \hat\lambda_{\mathrm{VA}}(\hat\lambda_\sigma^{\mathrm{c}} + \hat\lambda_\sigma^{\mathrm{f}}) + \tfrac14(\hat\lambda_\sigma^2 + \hat\lambda_\sigma^{\mathrm{c}\,2} + \hat\lambda_\sigma^{\mathrm{f}\,2})\Big], \\
\partial_t\hat\lambda_\sigma^{\mathrm{c}} &= \left(2+\frac{12}{\mathrm{N_c}}g^2 v_4\right)\hat\lambda_\sigma^{\mathrm{c}} \\
&- 8v_4 l_1^{(\mathrm{F}),4}\Big[2\mathrm{N_f}\hat\lambda_\sigma^{\mathrm{c}\,2} - 2\hat\lambda_\sigma\hat\lambda_{\mathrm{VA}} - 2\hat\lambda_-\hat\lambda_\sigma^{\mathrm{c}} - 2\mathrm{N_c}\hat\lambda_\sigma^{\mathrm{c}}\hat\lambda_{\mathrm{VA}} - 6\hat\lambda_+\hat\lambda_\sigma^{\mathrm{c}} - \hat\lambda_\sigma\hat\lambda_\sigma^{\mathrm{f}}\Big], \\
\partial_t\hat\lambda_\sigma^{\mathrm{f}} &= \frac{1}{4\mathrm{N_c}}\left(24-\mathrm{N_c^2}(3+\alpha)^2\right)g^4 v_4 l_{1,2}^{(\mathrm{FB}),4} + \left(2\hat\lambda_\sigma^{\mathrm{f}} + 24g^2\left(\hat\lambda_+ - \frac{\mathrm{N_c^2}-1}{2\mathrm{N_c}}\hat\lambda_\sigma^{\mathrm{f}}\right)v_4\right) \\
&- 8v_4 l_1^{(\mathrm{F}),4}\Big[2\mathrm{N_c}\hat\lambda_\sigma^{\mathrm{f}\,2} - 2\hat\lambda_\sigma\hat\lambda_{\mathrm{VA}} - 2\hat\lambda_-\hat\lambda_\sigma^{\mathrm{f}} - 2\mathrm{N_f}\hat\lambda_\sigma^{\mathrm{f}}\hat\lambda_{\mathrm{VA}} - 6\hat\lambda_+\hat\lambda_\sigma^{\mathrm{f}} - \hat\lambda_\sigma\hat\lambda_\sigma^{\mathrm{c}}\Big], \\
\partial_t\hat\lambda_{\mathrm{VA}} &= \frac{1}{8\mathrm{N_c}}\left(24-\mathrm{N_c^2}(3-\alpha(6+\alpha))\right)g^4 v_4 l_{1,2}^{(\mathrm{FB}),4} + \left(2\hat\lambda_{\mathrm{VA}} - \frac{12}{\mathrm{N_c}}(\hat\lambda_{\mathrm{VA}} - \mathrm{N_c}\hat\lambda_-)g^2 v_4\right) \\
&- 8v_4 l_1^{(\mathrm{F}),4}\Big[-(\mathrm{N_c}+\mathrm{N_f})\hat\lambda_{\mathrm{VA}}^2 + 4\hat\lambda_-\hat\lambda_{\mathrm{VA}} - \tfrac14(\mathrm{N_c}\hat\lambda_\sigma^{\mathrm{c}\,2} + \mathrm{N_f}\hat\lambda_\sigma^{\mathrm{f}\,2}) - \tfrac12\hat\lambda_\sigma(\hat\lambda_\sigma^{\mathrm{c}} + \hat\lambda_\sigma^{\mathrm{f}})\Big],
\end{aligned} \tag{7.14}$$

where the gauge fixing parameter is denoted by α.

7.2.2 Many Fixed Points but no Solution to the Hierarchy Problem – the Case of Vanishing Gauge Coupling

To get a first insight we have numerically solved the FP equation,

$$\partial_t\hat\lambda = 0, \tag{7.15}$$

for vanishing gauge coupling g and $\mathrm{N_c} = \mathrm{N_f} = 3$. It turns out that not only do we find a solution, but quite a few, $64 = 2^6$ to be exact (one is always the Gaussian FP). Looking more closely, only 44 are real.

Let us, for the moment postpone the question, if the FP are physical or an artifact of the truncation. Having found so many FP we could become kind of greedy and ask if among all those 44 FP there is one which in addition to providing us with a renormalizable theory, could solve the hierarchy problem. However, calculating the eigenvalues of the stability matrix we find, that the largest eigenvalue $\Theta^{\max} \geq 2$ for all non-trivial fixed points. Or, stated more physically, one direction is at least as unstable as a scalar boson mass.

Is this a very special property of our choice of $\mathrm{N_f}$ and $\mathrm{N_c}$? No, but it is a property of our truncation to pointlike four-fermion interactions and vanishing gauge coupling. More precisely, in this truncation there is always an eigenvalue $\Theta = (d-2)$. This can be seen by the following argument. In a four-fermion truncation we can write the flow equations in the form,

$$\partial_t \lambda_i = (d-2)\lambda_i + \lambda_k A^i_{kl} \lambda_l, \tag{7.16}$$

where A^i is a symmetric matrix i.e. $A^i_{kl} = A^i_{lk}$. The stability matrix is then given by,

$$B_{ij} = \frac{\partial(\partial_t \lambda_i)}{\partial \lambda_j} = (d-2)\delta_{ij} + 2\lambda_k A^i_{kj}. \tag{7.17}$$

Let $\lambda^\star$ be a solution of the fixed point equation,

$$\partial_t \lambda^\star_i = (d-2)\lambda^\star_i + \lambda^\star_k A^i_{kl} \lambda^\star_l = 0 \quad \forall i. \tag{7.18}$$

Acting with $B_{ij}\mid_{\lambda^\star}$ on $\lambda^\star_j \neq 0$ we have,

$$\begin{aligned} B_{ij}\mid_{\lambda^\star} \lambda^\star_j &= (d-2)\lambda^\star_i + 2\lambda^\star_j A^i_{jk} \lambda^\star_k \\ &= -(d-2)\lambda^\star_i + 2(d-2)\lambda^\star_i + 2\lambda^\star_j A^i_{jk}\lambda^\star_k \\ &= -(d-2)\lambda^\star_i + 2((d-2)\lambda^\star_i + \lambda^\star_j A^i_{jk}\lambda^\star_k) \\ &= -(d-2)\lambda^\star_i, \end{aligned} \tag{7.19}$$

where we have used the fixed point equation (7.18) in the last step.

This shows that $\lambda^\star$ itself is an eigenvector of the stability matrix with the eigenvalue $-(d-2)$, hence $\Theta = (d-2)$. Therefore, in this truncation there cannot be an infrared stable fixed point beside $\lambda = 0$.

7.2.3 Non-vanishing Gauge Coupling

Having not found the desired properties for the eigenvalues of the fixed point, let us look if a non-vanishing gauge coupling can stabilize the fixed point.

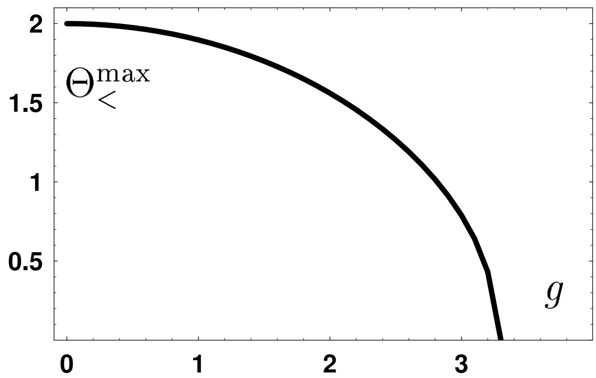


Figure 7.1: Largest eigenvalue $\Theta_<^{\max}$ for the most "stable" fixed point (cf. Eq. (7.20)) depending on the gauge coupling. At a realistic value $g \approx 1$ for the gauge coupling $\Theta_<^{\max} > 1.5$. Giving us no solution to the hierarchy problem.

A measure for the stability of a fixed point $\hat{\lambda}_x^\star$, $x = 1 \ldots 44$ is the size of its largest eigenvalue $\Theta_x^{\max}$. When $\Theta_x^{\max}$ is smaller the fixed point $\hat{\lambda}_x^\star$ is more stable. Thus, we have searched for the smallest,

$$\Theta_<^{\max} = \min_x \Theta_x^{\max}. \tag{7.20}$$

We have plotted this eigenvalue as a function of the gauge coupling in Fig. 7.1. To get an impression of a reasonable value of the gauge coupling, let us take the strong gauge coupling. At a scale of ~ 100GeV we have $g_s \approx 1$. At larger scales the gauge coupling is even smaller. As can be seen from Fig. 7.1 $\Theta_<^{\max} > 1.5$ in this range, bringing us nowhere near the desired $\Theta \approx 0$.

7.3 The Future

So far the results of our toy model can be summarized as follows. It looks as if there are many possibilities to have fixed points, but it seems difficult to find one which has very small positive eigenvalues Θ. But, as we have already seen in Sect. 7.2.2 this may be an artifact of our truncation to four-fermion interactions. Therefore, a next step is certainly to enlarge the truncation.

However, there is another interesting direction we can take. So far, in our simple approximations we have neglected the influence of the four-fermion interaction on the running of the gauge coupling. Instead we have treated the gauge coupling as coming from the outside. Yet, at a strong-coupling non-Gaussian fixed point the flow of the gauge-boson–fermion vertex is certainly influenced by the presence of the

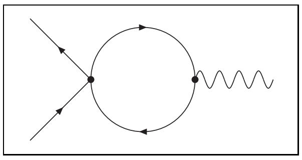


Figure 7.2: Correction to the gauge-boson–fermion vertex (fermions solid with arrow, gauge boson wiggled) in the presence of a four-fermion interaction.

four-fermion interaction. The lowest order contribution is depicted in Fig. 7.2. If we define the gauge coupling by this vertex, the running is modified by a contribution $\sim g^2\hat{\lambda}$,

$$\partial_t g^2 \;=\; -8v_4 g^2\left[l_1^{(\mathrm{F}),4}(\lambda_\sigma - 2\hat{\lambda}_- + \mathrm{N_f}\hat{\lambda}_\sigma^{\mathrm{f}} - 2\mathrm{N_f}\hat{\lambda}_{\mathrm{VA}}) + \beta_0 g^2\right], \qquad (7.21)$$

with

$$\beta_0 = \frac{1}{2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right). \qquad (7.22)$$

First of all this may have interesting consequences for properties like asymptotic freedom, because at a non-Gaussian fixed point the terms $\sim g^2\hat{\lambda}$ will dominate for very small gauge coupling. Therefore, the fixed point in $\hat{\lambda}$ determines if the gauge interaction is asymptotically free or not[8].

Secondly, in Fig. 7.2 we have concentrated on the gauge-boson–fermion vertex. But, in non-abelian gauge theories alternative definitions of the gauge coupling are by the three- and four-gauge-boson vertices. Some thought reveals that those do not get a direct contribution from the four-fermion interaction. This raises the question how this can be reconciled with the fact that, at least naively, gauge invariance tells us that both definitions agree. These interesting possibilities are subject to future investigations.

[8]As $\beta_0 > 0$ it is sufficient for asymptotic freedom that the sum of the four-fermion couplings in the brackets is positive. First numerical checks indicate that regardless of which of the 44 fixed points ($\mathrm{N_c} = \mathrm{N_f} = 3$) we choose, asymptotic freedom is ensured. This even seems to apply to various other combinations of $\mathrm{N_f}$, $\mathrm{N_c}$.

Chapter 8

Summary and Conclusions

We encounter strongly interacting fermions in many situations, ranging from color superconductivity and chiral symmetry breaking ($\sim$ 100MeV) to ordinary superconductivity ($\sim$ meV). Typical features of such systems are the formation of bosonic bound states and spontaneous symmetry breaking (SSB). Non-perturbative techniques are essential as SSB cannot be described by (standard) perturbation theory in these systems (cf. Sect. 3.2). Therefore, all methods discussed in the following correspond to non-perturbative resummations of perturbative diagrams.

We have started this work with a comparison of various standard methods used for non-perturbative calculations in this kind of systems. We have calculated the critical coupling for the onset of spontaneous chiral symmetry breaking in a simple NJL model. In particular, we have used mean field theory (MFT), a renormalization group (RG) equation with a truncation to pointlike four-fermion interactions (from now on referred to as fermionic RG) and the lowest-order Schwinger-Dyson equation (SDE). The results for the critical coupling $\overline{\lambda}_{\sigma}^{\text{crit}}$ and two different values of $\overline{\lambda}_{V}$ can be found in Tabs. 5.1, 5.2. Since the most characteristic features and problems of the different methods are most clearly seen when the couplings $\overline{\lambda}_{\sigma}$ and $\overline{\lambda}_{V}$ are of similar size we concentrate on Tab. 5.2.

Both MFT and the lowest order SDE sum only over fermionic fluctuations in presence of a bosonic background. They include, in principle the same type of diagrams, Fig. 3.1. The MFT result depends strongly on the choice of the background field. This "Fierz ambiguity" (FA) is expressed by the dependence on the unphysical parameter γ in the tables. No such ambiguity appears in the SDE approach which therefore seems more reliable. We note that for a particular choice of γ the MFT and the SD approaches give identical results - in our case $\gamma = 1/2$. This has led to widespread belief that MFT and SD are equivalent if the basis for the Fierz ordering is appropriately chosen. However, this is not the case, as can be seen by calculating also the critical coupling where spontaneous symmetry breaking sets in in the vector channel (in absence of other order parameters). There is again a value $\gamma = -(\overline{\lambda}_\sigma + \overline{\lambda}_V)/(2\overline{\lambda}_V)$ where MFT and SD give identical results, but it differs from $\gamma = 1/2$ as encountered in the scalar channel[1]. We conclude that there is no possible choice of γ where *both* critical couplings for SSB in the scalar and vector channels are identical in the MFT and SD approaches.

For a better understanding of the FA of MFT it is instructive to consider MFT on a more formal basis as a simple approximation, taking into account only the fermionic fluctuations, in a "partially bosonized" language. Partial bosonization is a powerful tool for an understanding of strongly interacting fermionic systems beyond the level of MFT or SDE. It allows us to treat the bosonic fluctuations in an explicit manner, treating them on equal footing with those of the elementary particles, and provides for a rather simple framework for the discussion of SSB. Most importantly, it permits the direct exploration of the ordered phase which is, in practice, almost inaccessible for the fermionic RG. Yet, already on the level of the classical action we can get a grasp of the origin of the FA. Partial bosonization is not unique. From one fermionic action we can obtain a whole family of equivalent bosonized actions, related to each other by redefinitions of the bosonic fields.

In order to permit a simple comparison with the fermionic RG we have used a rather crude approximation for the purely bosonic sector by retaining only a mass term and neglecting bosonic interactions as well as the momentum dependence of the bosonic propagator. In this approximation the effect of the boson exchange between fermions does not go beyond pointlike fermionic interactions. Taking into account only the running of the Yukawa couplings (Fig. 4.2) in the bosonic RG of Sect. 4.2, we observe already a very substantial improvement as compared to MFT. The dependence on γ is greatly reduced and the numerical value of the critical coupling comes already close to the result of the fermionic RG. These features can be compared to the inclusion of higher loop effects in perturbation theory in particle physics: they often reduce the dependence of the results on unphysical parameters, like the choice of the renormalization scale.

Using SDE in a partially bosonized language we fared even better. Again, MFT appears as the approximation which includes only the fermionic fluctuations in a bosonic background. Adding the mass-shift diagram (cf. Fig. 4.3) we recover the unambiguous result of the fermionic SDE – MFT sums only over a subset of the diagrams contained in the fermionic SDE.

The mass-shift diagram in the partially bosonized language has only fermionic external legs. This has prompted us to look for similar purely fermionic contributions in our partially bosonized RG description. The box diagrams (Fig. 5.1) generate new four-fermion interactions and contribute to the same order as the mass-shift and vertex corrections. Those four-fermion interactions are included in the adapted

[1]Actually, γ is negative and therefore outside the range of strict validity of MFT.

bosonic RG discussed in Sect. 5.2. Here the relation between the bosonic composite fields and the fermion bilinears becomes scale dependent. This formulation is well adapted to the basic idea of renormalization where only effective degrees of freedom at a certain scale k and their effective couplings should matter for physics associated to momenta $q^2 \lesssim k^2$. The system should loose all memory of the detailed microscopic physics. In particular, the choice of an optimal bosonic field for the long distance physics should not involve the parameters of the microscopic theory, but rather the renormalized parameters at the scale k. In this formulation it has also become apparent that the distinction between "fundamental degrees of freedom" and "bound states" becomes a matter of scale [42]. The adapted bosonic RG reproduces in our crude approximation the results of the fermionic RG. We argue that for precision estimates in the partially bosonized approach the "adaption" of the definition of the composite field seems mandatory.

For an improvement of the truncation two possibilities come to mind immediately. One is to enlarge the bosonic potential beyond the mass term, the other is to add kinetic terms for the bosons. Yet, our discussion of Sect. 5.3 shows that including only a bosonic potential without kinetic terms for the bosons does not help us in deciding which type of boson will condense in the SSB phase. The basic reason for this is, that the corresponding interactions in the fermionic language are still pointlike and can be Fierz transformed in many ways. Kinetic terms correspond to a momentum dependence of the interactions in the fermionic language, greatly reducing the freedom to Fierz transform. For this reason we have turned to the second possibility. To cut a long story short, for a consequent inclusion of all terms with up to two derivatives on the bosonic fields, we have to take into account not only the momentum dependence of the mass-shift diagram (cf. Fig. 3.1), but in addition part of the momentum dependence of the vertex correction and the box diagrams (cf. Figs. 4.2, 5.1).

Comparing the numerical values (cf. Tabs. 5.1, 5.2) with those of the pointlike approximations, we find that the inclusion of the kinetic terms affects the critical coupling on a 10% level. Moreover, the contributions from the different diagrams are of similar size, confirming once more that an adaption of the flow is necessary to achieve a high precision. The γ dependence is small, as we would expect for a systematic enlargement of the truncation. Nevertheless, it does not completely vanish since we have been forced to make approximations when including the momentum dependence of the vertex corrections and box diagrams. Moreover, if the "right" FT is known, inclusion of the pointlike contributions seems already sufficient for reducing the γ dependence (cf. the second to last line in Tab. 5.2). In view of the high algebraic complexity involved with including the full momentum dependence we would like to suggest this approximation for more moderate demands on accuracy. Finally, let us point out the following important feature (shared by all of the three considered approximations which include kinetic terms): with kinetic terms for the bosons we can decide which boson will condense. For values $\overline{\lambda}_\sigma$ (slightly) larger than $\overline{\lambda}_\sigma^{\rm crit}$, only the renormalized mass of the scalar boson turns negative, while the renormalized masses of the vector and axial vector bosons remain positive. This confirms that we have a phase where chiral symmetry (and nothing else) is broken.

In summary, the FA can be used as one possible test of approximation errors for some of the methods proposed to deal with strongly interacting fermionic systems. The spread of the results within an acceptable range of the unphysical parameter γ should be considered as a lower bound for the systematic uncertainty within a given approximation. We find that MFT can have a very substantial ambiguity which should then be reduced by systematic improvements. On the other hand, the FA is, of course, not the only source of error. Several methods such as SDE or the fermionic RG have no such ambiguity by construction. This holds similarly for the adapted bosonic RG (with or without kinetic terms) which is constructed to reduce the Fierz ambiguity. Hence, in this cases the uncertainty due to the Fierz ambiguity is small compared to other uncertainties which can roughly be estimated from the spread of the results between the different approximations. However, improving the truncation we should keep an eye on the FA as it is likely to increase, when our "improvement" neglects something essential. Comparing the RG and SD approaches the adapted bosonic RG sums over a larger class of diagrams and therefore seems more reliable. Moreover, the RG accounts for the fact that physics at the scale k should involve only renormalized parameters at this scale, while the SDE involves "bare" couplings. We think that with the adapted bosonic RG we have reached a promising starting point for future investigations along the lines of [39; 90]. In particular, a more elaborate bosonic potential would allow us to proceed into the SSB phase.

Removing the FA in a partially bosonized setting has turned out to be quite tedious. For a first investigation of the phase diagram, lowest order SDE seem to be a viable alternative which allow for a description of the SSB phase without auxiliary fields. However, at first sight there is one drawback: SDE determine only the derivatives of the action, not the action itself, making it difficult to compare different solutions with regard to stability. This shortcoming is cured by the use of the 2PI effective action, or its simplified form at vanishing fermionic sources, the bosonic effective action (BEA). In this context the SDE is the field equation of the BEA.

Integrating the lowest-order SDE for a multi-fermion interaction we have obtained the bosonic effective action at "one-vertex" level. Within this approximation we computed a simple "one-loop" expression for the BEA and the SDE. In this form the BEA appears very similar to MFT but does not suffer from its ambiguities and, as we have already seen, sums over a larger class of diagrams.

We have applied the BEA at one-vertex level to a six-fermion interaction resembling the instanton vertex for three colors and flavors. For vanishing current quark masses we find a first-order phase transition. It turns out that the value ζ_{crit} for the onset of non-trivial solutions for the SDE is not necessarily equal to the value ζ_{SSB} for the onset of SSB. To calculate the latter we need the value of the effective action in addition to the solution of the SDE. For nonzero current quark masses and positive a coupling constant we find a phase transition only for current quark masses below a $m^0_{\mathrm{crit}} \approx 0.076$. At this point the gap in the effective mass vanishes and we expect a second-order phase transition. We have also searched for solutions with a non-vanishing color-octet condensate. Although they are not excluded by symmetry we have not found a stable solution of the field equations with non-vanishing octet condensate $\chi \neq 0$. Nevertheless, we would like to point out that in our simple approximation the BEA is unbounded from below in several directions including some with $\chi \neq 0$. Hence, some sort of cutoff mechanism which renders the potential bounded might yield additional solutions. Therefore, a more detailed investigation of the instanton interaction in a color-octet background would be of great value.

Leaving aside the more technical aspects of this work, we have turned to an intriguing speculation, namely that the Higgs is not an elementary scalar boson, but a bound state of fermions, more precisely a top-antitop bound state [6–9]. To motivate a thorough investigation we have briefly reviewed the “Hierarchy Problem” and the “Triviality” of the ϕ^4-theory which cause trouble for the Standard Model (SM) with its *elementary* scalar boson. In particular, the latter problem may prevent the SM from being renormalizable in a strict sense. In a model with a bound state Higgs, renormalizability may be provided by a non-Gaussian fixed point in the RG. A toy model with $\mathrm{N_f}$ fermions interacting via a four-fermion interaction and a $SU(\mathrm{N_c})$ gauge interaction implied that there might be plenty of those. On the other hand it became apparent that a solution to the hierarchy problem is beyond a simple approximation to a pointlike four-fermion interaction, as in this setting the required fine-tuning to achieve a separation of scales is more or less as bad as for an elementary scalar. However, already in this simple truncation another interesting effect turned up: at a non-Gaussian fixed point, the flow of the gauge coupling might be crucially influenced by the four-fermion-interactions. In particular, this aspect seems to be interesting for future investigations.

It may well be that there are fundamental scalar fields, most candidates for unifying theories have plenty of them, but so far not even one has been detected with certainty. Hence, it might also be that nature has simulated once more an elementary scalar with a bound state. Upcoming experiments (e.g. at Tevatron and LHC) will help deciding this issue. But, independent of this, bound states of fermions are still abundant in nature and we hope that our formalism with scale-dependent degrees of freedom may be of help in understanding some of them.

Appendix A

Conventions, Abbreviations and Symbols

A.1 Conventions

- We use Euclidean conventions in flat spacetime, i.e. the metric is the d dimensional unit matrix.

- Greek indices $\mu, \nu \ldots = 0, \ldots, d$ denote spacetime indices.

- Latin indices $a, b \ldots = 1, \ldots, \mathrm{N_f}$ are flavor indices, $i, j, \ldots = 1, \ldots, \mathrm{N_c}$ color indices.

- Our conventions for the Fourier transform are

$$
\begin{aligned}
\phi(x) &= \int \frac{d^d q}{(2\pi)^d} \phi(q) \exp(ipx) \qquad \text{(A.1)} \\
\psi(x) &= \int \frac{d^d q}{(2\pi)^d} \psi(q) \exp(ipx), \quad \overline{\psi}(x) = \int \frac{d^d q}{(2\pi)^d} \overline{\psi}(q) \exp(-ipx).
\end{aligned}
$$

A.2 Mathematical Symbols

$\sim$	Proportional to
$\approx$	approximately equal to
$\otimes$	Tensor product
$\dagger$	Hermitian conjugation
C	Charge Conjugation (operator)
P	Parity (operator)
R	Reflection (operator) in Euclidean spacetime
T	Time Reversal (operator)

A.3 Abbreviations

Ad.	Adapted
BBS	Bosonic Bound State
BEA	Bosonic Effective Action
cf.	confer
Chap.	Chapter
ERGE	Exact Renormalization Group Equation(s)
Eq.	Equation
FA	Fierz Ambiguity
FP	Fixed Point
FT	Fierz Transformation
IR	Infrared
LHS	Left Hand Side
LPA	Local Potential Approximation
MF	Mean Field
MFT	Mean Field Theory
RG	Renormalization Group
RHS	Right Hand Side
s.	see
SD	Schwinger-Dyson
SDE	Schwinger-Dyson Equation(s)
Sect.	Section
SSB	Spontaneous Symmetry Breaking
UV	Ultraviolet
WFR	Wave Function Renormalization(s)

Appendix B

Fermion Conventions, Fierz Identities

B.1 Dirac-Algebra in 4 Dimensions

Throughout this work we use an Euclidean metric, $g^{\mu\nu} = \delta^{\mu\nu}$ and $|x|^2 = x_0^2 + x_1^2 + \cdots + x_{d-1}^2$. With the exception of one tiny excursion in Chap. 5 the number of spacetime dimensions will be $d = 4$.

Accordingly the Dirac-algebra is

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}\mathbf{1}, \\ (\gamma^\mu)^\dagger &= \gamma^\mu, \\ \gamma^5 &= \gamma^1\gamma^2\gamma^3\gamma^0, \\ \sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \end{aligned} \tag{B.1}$$

We use a chiral basis $\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}$, $\overline{\psi} = (\overline{\psi}_{\mathrm{R}}, \overline{\psi}_{\mathrm{L}})$, with the projection operators $P_{\mathrm{L,R}} = \frac{1 \pm \gamma^5}{2}$ on the chiral components. An explicit representation is then given by,

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\sigma^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{B.2}$$

with $\sigma^\mu = (i\mathbf{1}, \sigma^i)$. σ^i are the standard Pauli-matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & o \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{B.3}$$

and $\mathbf{1}$ is here the 2×2 unit matrix.

Using Eq. (B.1) it is quite easy to derive relations to simplify expressions containing several γ matrices, e.g. $\gamma^\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$. This can be automated and we use the *Tracer*-package [119] for *Mathematica* to do this.

B.2 Dirac-Algebra in 3 Dimensions

In Chap. 5 we use the Gross-Neveu model in 3 dimensions to demonstrate a shortcoming of partial bosonization. The Dirac-algebra in odd dimensions is somewhat different from the case of even dimensions.

Nevertheless, it is quite easy to find an explicit realization as the Pauli matrices already fulfill the requirements for the Dirac-algebra,

$$\left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij}. \tag{B.4}$$

Consequently, we can use spinors with only two components. Since we do not need any more subtle properties, let us leave it at that and return to the case of four dimensions.

B.3 Fierz Identities

Defining $O_S = \mathbf{1}$, $O_P = \gamma^5$, $O_V = \gamma^\mu$, $O_A = \gamma^\mu\gamma^5$ and $O_T = \sigma^{\mu\nu}$ we obtain the following Fierz identities,

$$(\overline{\psi}_a O_X \psi_d)(\overline{\psi}_c O_X \psi_a) = \sum_Y C_{XY} (\overline{\psi}_a O_Y \psi_b)(\overline{\psi}_c O_Y \psi_d), \tag{B.5}$$

with

$$C_{XY} = \frac{1}{4}\begin{pmatrix} -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & -1 \\ -4 & 4 & 2 & 2 & 0 \\ 4 & -4 & 2 & 2 & 0 \\ -6 & -6 & 0 & 0 & 2 \end{pmatrix}. \tag{B.6}$$

From the indices a, b, c, d which combine all but spin-indices we can see that we have done nothing but exchanged the two ψ's appearing in the four-fermion term.

For the special case of only one fermionic species we find that the structure $(\overline{\psi} O_V \psi)^2 + (\overline{\psi} O_A \psi)^2$ is invariant. Moreover, we can use Eqs. (B.5), (B.6) to obtain the identity (actually this is exactly Eq. (1.2)),

$$(\overline{\psi} O_V \psi)^2 - (\overline{\psi} O_A \psi)^2 = -2[(\overline{\psi} O_S \psi)^2 - (\overline{\psi} O_P \psi)], \tag{B.7}$$

which allows us to transform $(\overline{\psi}O_V\psi)^2 - (\overline{\psi}O_A\psi)^2$ completely into scalar and pseudoscalar channels.

If we have more than one fermion species, e.g. several flavors and/or colors, the Fierz transformations turns singlets into non-singlets and vice versa. This can be used to reduce the number of possible couplings as we do in Chap. 7.

Finally, let us mention two possible generalizations of the above. First, the same idea of permuting the ψ's can, of course, also be applied to higher order interactions like a 6- or 8-fermion interaction. Second, the four-fermion operators considered above are invariant under the discrete transformations C, P and T (charge conjugation, parity and time reversal). However, as we know the weak interactions violate parity. Therefore, we might also want to consider interactions which are only invariant under CP and T. The set of parity violating operators can be obtained by multiplying one O_X in the four-fermion operator by γ^5. This yields $(\overline{\psi}O_X\psi)(\overline{\psi}O_X\gamma^5\psi)$. Of course there is a set of equations similar to Eqs. (B.5), (B.6).

Appendix C

Infrared and Ultraviolet Regularization

One of the central building blocks of the flow equation Eqs. (2.32), (2.34) is the IR cutoff R_k. In this appendix we want to give explicit examples of the cutoff functions and define the threshold functions which appear in explicit calculations as a consequence of the trace over momentum space. Furthermore, we briefly comment on some technical points concerning the correspondence between UV and IR regularization.

C.1 Cutoff Functions

To derive the flow equation (2.34) in Chap. 2 we added an additional term to the effective action providing for an IR regularization,

$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi^{\mathrm{T}}(-p)R_k(p)\phi(p). \tag{C.1}$$

The inverse massless average (i.e. regularized) propagator P_B for bosons and the corresponding squared quantity P_F for fermions are given by (cf. Sect. 2.1.2, Eqs. (2.16), (2.18)),

$$\begin{aligned} P_B &= q^2 + Z_{\phi,k}^{-1}R_k(q) = q^2(1+r_B(q^2)), \\ P_F &= q^2(1+r_F(q^2))^2, \end{aligned} \tag{C.2}$$

where r_B and r_F reflect the presence of the IR cutoff. The dimensionless functions r_B and r_F depend only on $y = q^2/k^2$.

Expressed in terms of the r's the linear cutoff, Eqs. (2.16), (2.18)) is as follows,

$$\begin{aligned} r_B(y) &= \left(\frac{1}{y} - 1\right)\Theta(1-y), \\ r_F(y) &= \left(\frac{1}{\sqrt{y}} - 1\right)\Theta(1-y). \end{aligned} \tag{C.3}$$

Other typical examples are the sharp momentum cutoff,

$$\begin{aligned} r_B(y) &= \frac{\Theta(1-y)}{1-\Theta(1-y)}, \\ r_F(y) &= \frac{\Theta(1-y)}{1-\Theta(1-y)}, \end{aligned} \tag{C.4}$$

and the popular exponential cutoff

$$\begin{aligned} r_B(y) &= \frac{1}{1-\exp(-y)} - 1, \\ r_F(y) &= \sqrt{\frac{1}{1-\exp(-y)}} - 1. \end{aligned} \tag{C.5}$$

C.2 Threshold Functions

The (super-)trace in the flow equations (2.32), (2.34) includes a momentum space integral, reminiscent of a one-loop expression. Typically, the integral kernels are products of the IR regularized propagators and their derivatives. In most parts of this work, we use standard threshold functions as defined in [67], which we evaluate below explicitly for a finite UV cutoff Λ and for the linear cutoff (C.3). In Sect. 5.4 we need several additional threshold functions, not defined in the standard literature. To facilitate the notation we define a new, enlarged set of threshold functions in Sect. C.2.3, and label the threshold functions somewhat differently.

C.2.1 Evaluation with Finite UV Cutoff

To adapt the IR regulator to a finite UV cutoff one can modify the cutoff functions by a term which becomes infinite for all $y \geq \frac{\Lambda^2}{k^2}$. For our purposes it is simpler to restrict the range of x to $[0, \Lambda^2]$. This has the same effect. In absence of mass terms the threshold functions can only depend on the ratio $s = k^2/\Lambda^2$. With

$$\tilde{\partial}_t = \frac{q^2}{Z_{\phi,k}} \frac{\partial [Z_{\phi,k} r_B]}{\partial t} \frac{\partial}{\partial P_B} + \frac{2}{Z_{\psi,k}} \frac{P_F}{1+r_F} \frac{\partial [Z_{\psi,k} r_F]}{\partial t} \frac{\partial}{\partial P_F} \tag{C.6}$$

we find for bosons ($x = q^2$)

$$\begin{aligned} l_0^{(B),d}(\omega,\eta_\phi,s) &= \frac{1}{2}k^{-d}\int_0^{\Lambda^2} dx\, x^{\frac{d}{2}-1}\tilde{\partial}_t \ln(P_B(x)+\omega k^2) \\ &= \frac{2}{d}\left[1-\frac{\eta_\phi}{d+2}\right]\frac{1}{1+\omega}\Theta(1-s) \\ &\quad +\frac{2}{d}s^{-\frac{d}{2}}\left[1-\frac{(2+d(1-s^{-1}))\,\eta_\phi}{2(d+2)}\right]\frac{1}{1+\omega}\Theta(s-1) \end{aligned} \tag{C.7}$$

and for fermions

$$\begin{aligned} l_0^{(F),d}(\omega,\eta_\psi,s) &= \frac{1}{2}k^{-d}\int_0^{\Lambda^2} dx\, x^{\frac{d}{2}-1}\tilde{\partial}_t \ln(P_F(x)+\omega k^2) \\ &= \frac{2}{d}\left[1-\frac{\eta_\psi}{d+1}\right]\frac{1}{1+\omega}\Theta(1-s) \\ &\quad +\frac{2}{d}s^{-\frac{d}{2}}\left[1-\frac{(d+1-ds^{-1})\eta_\psi}{d+1}\right]\frac{1}{1+\omega}\Theta(s-1). \end{aligned} \tag{C.8}$$

Higher threshold functions can be obtained simply by differentiating with respect to ω:

$$l_{n+1}^d(\omega,\eta,s) = -\frac{1}{n+\delta_{n,0}}\frac{d}{d\omega}l_n^d(\omega,\eta,s). \tag{C.9}$$

For a finite value of the UV cutoff Λ the threshold functions are explicitly s- and therefore k-dependent. Taking $\Lambda \to \infty$ we have $s = 0$ for any value of k. This renders the threshold functions k-independent.

In the pointlike truncations of Chaps. 3-5 we neglect the anomalous dimensions η_ϕ, η_ψ and effectively only consider a fermionic cutoff since $Z_{\phi,k} = 0$. Moreover, the fermions are massless and we abbreviate for $\omega = 0$

$$l_n^d(0,0,s) = l_n^d(s). \tag{C.10}$$

This yields explicitly

$$l_1^{(F),4}(s) = \frac{1}{2}\left[\Theta(1-s)+s^{-2}\Theta(s-1)\right]. \tag{C.11}$$

To obtain the perturbative result from the fermionic RG equation we used

$$\int_{-\infty}^{\infty} dt k^2 l_1^{(F),4}(s) = \int_0^{\infty} dk\, k l_1^{(F),4}(s) = \frac{\Lambda^2}{2}. \tag{C.12}$$

As long as we keep the sharp momentum cutoff at $q = \Lambda$ this integral is universal, i.e. it does not depend on the precise choice of the IR cutoff. Indeed the universality is necessary to reproduce perturbation theory for every choice of the IR cutoff.

C.2.2 Cutoff Independence for Pointlike Truncations

We have also used other cutoff functions R_k different from the linear cutoff. Within the local interaction approximation we have found that the value of the critical coupling comes out independent of the choice of R_k. The basic reason is that a multiplicative change of $l_1^{(F),4}$ due to the use of another threshold function can be compensated by a rescaling of k (cf. Eq. (3.17)). The rescaling is simply multiplicative for $s < 1$, with a suitable generalization for $s > 1$. Critical values of the flow which are defined for $k \to \infty$ are not affected by the rescaling. Let us demonstrate this for $\overline{\lambda}_V$. Writing Eq. (3.17) in the scale variable k we have

$$\partial_k \overline{\lambda}_{V,k} = -4 v_4 l_1^{(F),4}(s) k (\overline{\lambda}_{\sigma,k} + \overline{\lambda}_{V,k})^2. \tag{C.13}$$

Rescaling to

$$\tilde{k}(k) = \int_0^k dk\, k l_1^{(F),4}(s) \tag{C.14}$$

we find

$$\partial_{\tilde{k}} \overline{\lambda}_{V,\tilde{k}} = -4 v_4 (\overline{\lambda}_{\sigma,\tilde{k}} + \overline{\lambda}_{V,\tilde{k}})^2. \tag{C.15}$$

Due to the universality of Eq. (C.12) the domain for k, $[0, \infty]$, is now mapped to $[0, \frac{\Lambda^2}{2}]$, giving the domain for $\tilde{k}$ independent of the IR cutoff. Having obtained identical differential equations for every choice of IR cutoff without any rescaling of $\overline{\lambda}$ establishes the above claim for the critical couplings.

Note however, that this would not hold if we would start the integration of the flow equation at $k = \Lambda$. In this case the domain $[0, \Lambda]$ for k is mapped into an interval for $\tilde{k}$ that depends on the threshold function and therefore on R_k. Actually, the R_k dependence in this case is not very surprising because different IR cutoffs then correspond to different UV regularizations. Since our model is naively non-renormalizable results can depend on the choice of UV regularization (cf. Sect. C.3).

C.2.3 Threshold Functions for Sect. 5.4

Similar to Eq. (C.7), (C.8) we define our modified threshold functions by integrals over $x = q^2$. Threshold functions with derivatives of the fermion propagator are denoted by a Greek letter, all others by Latin letters. With

$$F^{-1}(x) = \frac{1 + r_F(x)}{P_F(x)}, \tag{C.16}$$

and suppressing the arguments $(\omega_F, \eta_F, \omega_1, \eta_1, \omega_2, \eta_2) = (F, 1, 2)$ of the threshold functions and the argument (x) of the inverse propagators we write,

$$a^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2-1)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}} F^{-n} P_1^{-m_1} P_2^{-m_2}, \tag{C.17}$$

$$\beta^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}} \dot{F} F^{-(n+1)} P_1^{-m_1} P_2^{-m_2}, \tag{C.18}$$

$$b^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}} F^{-n} \dot{P}_1 P_1^{-(m_1+1)} P_2^{-m_2}, \tag{C.19}$$

$$\gamma^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}+1} (\dot{F})^2 F^{-(n+2)} P_1^{-m_1} P_2^{-m_2}, \tag{C.20}$$

$$c^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}+1} F^{-n} (\dot{P}_1)^2 P_1^{-(m_1+2)} P_2^{-m_2}, \tag{C.21}$$

$$\delta^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}+1} \dot{F} F^{-(n+1)} \dot{P}_1 P_1^{-(m_1+1)} P_2^{-m_2}, \tag{C.22}$$

$$d^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}+1} F^{-n} \dot{P}_1 P_1^{-(m_1+1)} \dot{P}_2 P_2^{-(m_2+1)}, \tag{C.23}$$

$$\epsilon^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}+1} \ddot{F} F^{-(n+1)} P_1^{-m_1} P_2^{-m_2}, \tag{C.24}$$

$$e^{(FBB),d}_{n,m_1,m_2} = -\frac{1}{2} k^{2(n+m_1+m_2)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}+1} F^{-n} \ddot{P}_1 P_1^{-(m_1+1)} P_2^{-m_2}. \tag{C.25}$$

If possible, i.e. if in the threshold function t no x-derivatives act on P_2, respectively P_1 and P_2, we abbreviate

$$t^{(FBB),d}_{n,m,0} = t^{(FB),d}_{n,m} \quad \text{and} \quad t^{(FB),d}_{n,0} = t^{(F),d}_{n}. \tag{C.26}$$

For the linear cutoff Eq. (C.3) the integrals can be done explicitly as in Sect. C.2.1. This is useful for numerical computations but not very enlightening.

Some of the threshold functions defined above are straightforwardly related to the standard threshold functions used in the literature. Particularly noteworthy are the following relations,

$$\begin{aligned} l^{(F),d}_n(0,\eta_F) &= a^{(F),d+2(n-1)}_{2n}(0,\eta_F) \\ l^{(B),d}_n(\omega,\eta) &= a^{(FB),d-2}_{0,n}(0,0,\omega,\eta) \\ l^{(FB),d}_{n,m}(0,\eta_F,\omega,\eta) &= a^{(FB),d+2(n-1)}_{n,m}(0,\eta_F,\omega,\eta) \\ m^{(F),d}_4(\omega_F,\eta_F) &= \gamma^{(F),d}_2(\omega_F,\eta_F). \end{aligned} \tag{C.27}$$

The threshold functions defined above are not mutually independent. By partial integration one can obtain relations linking some of the threshold functions above.

Let us content ourselves with two examples which are useful for a comparison with results in the literature,

$$\beta^{(FBB),d}_{n,m_1,m_2}(F,1,2) = \frac{d}{2n} a^{(FBB),d-2}_{n,m_1,m_2}(F,1,2) - \frac{m_1}{n} b^{(FBB),d}_{n,m_1,m_2}(F,1,2) - \frac{m_2}{n} b^{(FBB),d}_{n,m_2,m_1}(F,2,1). \tag{C.28}$$

In the special case $m_1 = m_2 = 0$, $d = 2n = 4$ this reduces to

$$\beta^{(F),4}_2(F) = a^{(F),2}_2(F). \tag{C.29}$$

Moreover,

$$\begin{aligned}\epsilon^{(FBB),d}_{n,m_1,m_2}(F,1,2) = &- \frac{d+2}{2}\beta^{(FBB),d}_{n,m_1,m_2}(F,1,2) + (n+1)\gamma^{(FBB),d}_{n,m_1,m_2}(F,1,2) \\ &+ m_1\delta^{(FBB),d}_{n,m_1,m_2}(F,1,2) + m_2\delta^{(FBB),d}_{n,m_1,m_2}(F,2,1),\end{aligned} \tag{C.30}$$

considering the same case as above and using relation (C.29) we find

$$\epsilon^{(F),d}_2(F) + x a^{(F),2}_2(F) + y\beta^{(F),4}_2(F) = 3\gamma^{(F),4}_2(F) = 3m^{(F),4}_4(F), \quad \text{for } (x+y) = 3. \tag{C.31}$$

Finally, let us come to a property we need when studying the pointlike limit for the bosons. For large boson masses the inverse propagator acts like ω^2. This gives us the asymptotic properties[1],

$$\begin{aligned}a^{(FBB),d}_{n,m_1,m_2} &= -\frac{1}{2}k^{2(n+m_1+m_2-1)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}} F^{-n}(x)\omega_1^{-2m_1}(x)\omega_2^{-2m_2}(x) \\ &= \frac{1}{\omega_1^{2m_1}\omega_2^{2m_2}} a^{(F),d}_n \\ b^{(FBB),d}_{n,m_1,m_2} &= -\frac{1}{2}k^{2(n+m_1+m_2-1)-d}\,\tilde{\partial}_t \int dx x^{\frac{d}{2}} F^{-n}(x)\dot{P}_1(x)\omega_1^{-2(m_1+1)}(x)\omega_2^{-2m_2}(x) \\ &= \frac{1}{\omega_1^{2(m_1+1)}\omega_2^{2m_2}} b^{(F),d}_n,\end{aligned} \tag{C.32}$$

and similar for all other threshold functions.

C.3 UV Regularization – ERGE Scheme

C.3.1 Effect of UV Regularization

UV regularization can be implemented by a sharp cutoff in all integrals over momentum space[2]. Yet, this is not the only possibility. Indeed, it is often not the most

[1] This argument and the corresponding properties are only valid at fixed anomalous dimension.

[2] When choosing a UV regularization one has to be careful to take one compatible with the symmetries of the theory in question. E.g. for gauge theories the sharp momentum cutoff violates gauge symmetry and is therefore not a suitable choice.

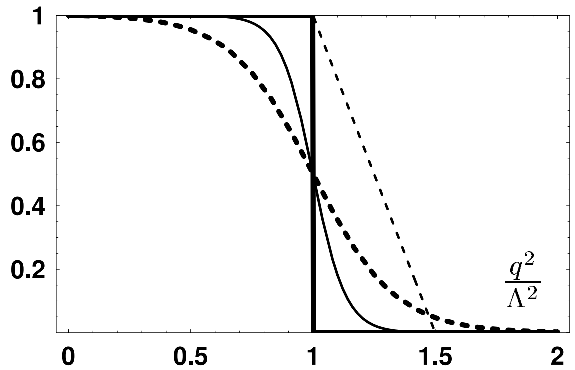


Figure C.1: In a UV regularized theory not all modes contribute completely. This plot schematically depicts "how much" each mode contributes. The thick line is for the sharp momentum cutoff. All modes with $q^2 \leq \Lambda^2$ are included completely. Other UV regularizations (dashed, thin dashed and thin solid line) typically not only include a small fraction of the high momentum modes, $q^2 > \Lambda^2$, but in addition leave out a small fraction of the low momentum modes.

practical regularization. E.g. in perturbation theory dimensional regularization is often much more convenient. But, even at fixed spacetime dimension there are other regularization methods. Prominent examples are the Pauli-Villars and Schwinger proper time regularization (for details s. e.g. [120]). More or less any modification (compatible with the symmetries) of the short distance behavior of the propagators which renders all Feynman diagrams finite, can be called a UV regularization. In general, the modes with $q^2 > \Lambda^2$ are not completely left out, only suppressed, as depicted in Fig. C.1.

From another point of view, a modification of the propagator can be implemented in the action. This gives then an "UV regularized classical action". The function describing the suppression of the UV modes (cf. Fig. C.1) in some way appears in this UV regularized action, e.g. the sharp cutoff limit can be implemented by multiplying all terms in the action by appropriate $\Theta(1 - \frac{q^2}{\Lambda^2})$-functions. Hence, different UV regularizations usually correspond to different "classical actions". Therefore, it is no surprise, that different UV regularizations give different results. In particular, this is true for the critical coupling in our NJL model Eq. (1.1), as one can see by comparing Tabs. 3.1, 3.2 with Tabs. 5.1, 5.2, calculated using a sharp UV cutoff at Λ and a UV regularization by the ERGE scheme (s. below) with the linear cutoff Eq. (C.3), respectively. Only in renormalizable theories[3] we can remove the cutoff and obtain results independent of the specifics of the UV regularization.

[3]This includes critical behavior at second-order phase transitions (cf. Chap. 7), in particular critical exponents and other universal quantities.

C.3.2 ERGE Scheme

In Sect. C.2.1 we have evaluated the threshold functions for a theory which is UV regularized by a sharp cutoff in momentum space. It turned out that the threshold functions depended on the ratio $s = \frac{k^2}{\Lambda^2}$ in a rather complicated way. Constant threshold functions would be desirable, among other things to simplify numerical calculations.

We already noted, that for $\Lambda = \infty$ the cutoff functions are indeed constant for all k, because $s = 0$ for all finite k. Moreover, for an IR cutoff decreasing sufficiently fast in the UV all threshold functions are finite, even for $\Lambda = \infty$. The flow equation (2.32), (2.34) is UV finite. Now, let us remember that we have chosen the IR cutoff such that only the modes around $q^2 = k^2$ effectively contribute to the flow (Eq. (2.15)). Starting the flow at $k_0 = \Lambda$ and integrating to $k = 0$ we have included only modes with $q^2 \lesssim \Lambda^2$. This is exactly what we expect for an UV regularization. More precisely, the UV regularization is now implemented in the finite choice for the initial conditions at $k = k_0 = \Lambda$. This is the so called ERGE scheme for the UV regularization.

As flow equations are much simpler with constant threshold functions, this is the method of choice to implement UV regularization in ERGE. Nevertheless, this is bought at the prize that we cannot any longer compare non-universal quantities between different IR regularizations, as they automatically lead to different UV regularizations.

Although it is usually not the simplest method, we can invoke UV regularization by the ERGE scheme also in the context of perturbation theory or SDE. This follows along the lines indicated in Sect. 2.1.2 for perturbation theory. Typically any expression can be written in terms of inverse propagators P, internal momenta q we integrate over, and external momenta p we do not integrate over,

$$\int_q F(P, q, p). \tag{C.33}$$

A specific ERGE scheme is specified by the choice of the IR regulator R_k. Replacing the inverse propagator P by the IR regularized inverse propagator $P + R_k$ we can calculate the contribution from each scale k, $k^{-1}\tilde{\partial}_t F(P + R_k, q, p)$. Integrating over all scales from $k_0 = \Lambda$ to $k = 0$ we obtain the UV regularized expression,

$$\int_{k_0=\Lambda}^{0} dk\, k^{-1}\tilde{\partial}_t \left[\int_q F(P + R_k, q, p) \right]. \tag{C.34}$$

Appendix D

Flow Equations for Sect. 5.4

In this appendix we list the flow equations for the effective action (1.4) generalized to include kinetic terms (4.23). Moreover, we calculate the (momentum-dependent) field redefinitions necessary to keep a simple form of the action with Yukawa couplings constant in momentum space and no four-fermion interactions.

D.1 Flow Equations at Fixed Fields

The first diagram that we have to evaluate is Fig. 3.1, giving the self-energy $\mu^2(p)$. Momentum conservation implies that it depends only on one momentum variable p. Expanding for small values of p^2 and abbreviating the arguments of the threshold functions (cf. App. C.2.3) as $F = (\omega_F, \eta_F)$ we find,

$$\begin{aligned} \partial_t \mu_\sigma^2 &= 8 v_4 k^2 h_\sigma^2 a_2^{(F),4}(F), \\ \partial_t \mu_V^2 &= 8 v_4 k^2 h_V^2 a_2^{(F),4}(F), \quad \partial_t \mu_A^2 = 8 v_4 k^2 h_A^2 a_2^{(F),4}(F) \end{aligned} \tag{D.1}$$

and

$$\begin{aligned} \frac{\partial_t Z_\sigma}{Z_\sigma} &= -4 v_4 h_\sigma^2 \gamma_2^{(F),4}(F), \\ \frac{\partial_t Z_V}{Z_V} &= -\frac{16}{3} v_4 h_V^2 \gamma_2^{(F),4}(F), \quad \frac{\partial_t Z_A}{Z_A} = -\frac{16}{3} v_4 h_A^2 \gamma_2^{(F),4}(F). \end{aligned} \tag{D.2}$$

In general, the vertex correction depicted in Fig. 4.2 depends on two momentum variables. As discussed in Sect. 5.4.2 we perform the evaluation for the configuration

$(p_1, p_2, p_3) = (p, \frac{1}{2}p, \frac{1}{2}p)$. With $\sigma = (\omega_\sigma, \eta_\sigma)$, $V = (\omega_V, \eta_V)$, $A = (\omega_A, \eta_A)$ this yields,

$$\begin{aligned}
\beta_{h_\sigma}^{(0)} &= -16 v_4 h_\sigma \Big[h_V^2 a_{2,1}^{(FB),4}(F,V) - h_A^2 a_{2,1}^{(FB),4}(F,A) \Big], \\
\beta_{h_V}^{(0)} &= -2 v_4 h_V \Big[h_\sigma^2 a_{2,1}^{(FB),4}(F,\sigma) + 2h_V^2 a_{2,1}^{(FB),4}(F,V) + 2h_A^2 a_{2,1}^{(FB),4}(F,A) \Big], \\
\beta_{h_A}^{(0)} &= 2 v_4 h_A \Big[h_\sigma^2 a_{2,1}^{(FB),4}(F,\sigma) - 2h_V^2 a_{2,1}^{(FB),4}(F,V) - 2h_A^2 a_{2,1}^{(FB),4}(F,A) \Big],
\end{aligned} \tag{D.3}$$

and

$$\begin{aligned}
\beta_{h_\sigma}^{(2)} &= 4 v_4 h_\sigma \Big[h_V^2 (a_{2,1}^{(FB),2}(F,V) + 2\beta_{2,1}^{(FB),4}(F,V) - \gamma_{2,1}^{(FB),4}(F,V) + \epsilon_{2,1}^{(FB),4}(F,V)) \\
&\qquad - (V \to A) \Big], \\
\beta_{h_V}^{(2)} &= \frac{1}{3} v_4 h_V \Big[h_\sigma^2 (3a_{2,1}^{(FB),2}(F,\sigma) + 3\beta_{2,1}^{(FB),4}(F,\sigma) - 2\gamma_{2,1}^{(FB),4}(F,\sigma) + 2\epsilon_{2,1}^{(FB),4}(F,\sigma)) \\
&\qquad + 2h_V^2 (3a_{2,1}^{(FB),2}(F,V) + 3\beta_{2,1}^{(FB),4}(F,V) - 2\gamma_{2,1}^{(FB),4}(F,V) + 2\epsilon_{2,1}^{(FB),4}(F,V)) \\
&\qquad + (V \to A) \Big], \\
\beta_{h_A}^{(2)} &= \frac{1}{3} v_4 h_A \Big[- h_\sigma^2 (3a_{2,1}^{(FB),2}(F,\sigma) + 3\beta_{2,1}^{(FB),4}(F,\sigma) - 2\gamma_{2,1}^{(FB),4}(F,\sigma) + 2\epsilon_{2,1}^{(FB),4}(F,\sigma)) \\
&\qquad + 2h_V^2 (3a_{2,1}^{(FB),2}(F,V) + 3\beta_{2,1}^{(FB),4}(F,V) - 2\gamma_{2,1}^{(FB),4}(F,V) + 2\epsilon_{2,1}^{(FB),4}(F,V))) \\
&\qquad + (V \to A) \Big].
\end{aligned} \tag{D.4}$$

In order to compare the momentum dependence on s and t in the vicinity of $(s,t) = (0,0)$ we have evaluated both momentum configurations discussed in Sect. 5.4.2.

For the configuration $(p_1, p_2, p_3, p_4) = \frac{1}{2}(p,p,p,p)$ with $t = 0$ and $s = p^2$ we find

$$\begin{aligned}
\beta_{\lambda_\sigma}^{(0)} &= -8 v_4 h_\sigma^2 h_A^2 a_{2,1,1}^{(FBB),4}(F,\sigma,V), \\
\beta_{\lambda_V}^{(0)} &= 24 v_4 h_V h_A a_{2,1,1}^{(FBB),4}(F,V,A), \\
\beta_{\lambda_A}^{(0)} &= -v_4 [h_\sigma^4 a_{2,2}^{(FB),4}(F,\sigma) - 12 h_V^4 a_{2,2}^{(FB),4}(F,V) - 12 h_A^4 a_{2,2}^{(FB),4}(F,A)],
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
\beta^{(2)}_{\lambda_\sigma} &= v_4 h_\sigma^2 h_A^2 \Big[2b^{(FBB),4}_{2,1,1}(F,\sigma,A) - 2c^{(FBB),4}_{2,1,1}(F,\sigma,A) + d^{(FBB),4}_{2,1,1}(F,\sigma,a) \\
&\qquad + e^{(FBB),4}_{2,1,1}(F,\sigma,A) + (\sigma \leftrightarrow A) \Big], \\
\beta^{(2)}_{\lambda_V} &= -\frac{2}{3} v_4 h_V^2 h_A^2 \Big[9b^{(FBB),4}_{2,1,1}(F,V,A) - 10c^{(FBB),4}_{2,1,1}(F,V,A) + 5d^{(FBB),4}_{2,1,1}(F,V,A) \\
&\qquad + e^{(FBB),4}_{2,1,1}(F,V,A) + (V \leftrightarrow A) \Big], \\
\beta^{(2)}_{\lambda_A} &= -\frac{1}{6} v_4 \Big[4h_V^4 (9b^{(FB),4}_{2,2}(F,V) - 5c^{(FB),4}_{2,2}(F,V) + 5e^{(FB),4}_{2,2}(F,V)) + (V \to A) \\
&\qquad + h_\sigma^4 (-3b^{(FB),4}_{2,2}(F,\sigma) + c^{(FB),4}_{2,2}(F,\sigma) - e^{(FB),4}_{2,2}(F,\sigma) \Big].
\end{aligned} \tag{D.6}$$

After the appropriate Fierz transformation the configuration $(p_1, p_2, p_3, p_4) = \frac{1}{2}(p, -p, -p, p)$ with $s = 0$ and $t = p^2$ yields,

$$\begin{aligned}
\beta^{(0)}_{\lambda_\sigma} &= v_4 \Big[h_\sigma^4 a^{(FB),4}_{2,2}(F,\sigma) + 24 h_V^2 h_A^2 a^{(FBB),4}_{2,1,1}(F,V,A) - 12 h_V^4 a^{(FB),4}_{2,2}(F,V) \\
&\qquad - 12 h_A^4 a^{(FB),4}_{2,2}(F,A) \Big], \\
\beta^{(0)}_{\lambda_V} &= -v_4 \Big[\frac{1}{2} h_\sigma^4 a^{(FB),4}_{2,2}(F,\sigma) + 4 h_\sigma^2 h_A^2 a^{(FBB),4}_{2,1,1}(F,\sigma,A) - 12 h_V^2 h_A^2 a^{(FBB),4}_{2,1,1}(F,V,A) \\
&\qquad - 6 h_V^4 a^{(FB),4}_{2,2}(F,V) - 6 h_A^4 a^{(FB),4}_{2,2}(F,A) \Big], \\
\beta^{(0)}_{\lambda_A} &= -v_4 \Big[\frac{1}{2} h_\sigma^4 a^{(FB),4}_{2,2}(F,\sigma) - 4 h_\sigma^2 h_A^2 a^{(FBB),4}_{2,1,1}(F,\sigma,A) - 12 h_V^2 h_A^2 a^{(FBB),4}_{2,1,1}(F,V,A) \\
&\qquad - 6 h_V^4 a^{(FB),4}_{2,2}(F,V) - 6 h_A^4 a^{(FB),4}_{2,2}(F,A) \Big],
\end{aligned} \tag{D.7}$$

$$\begin{aligned}
\beta^{(2)}_{\lambda_\sigma} &= \frac{1}{4} v_4 \Big[h_\sigma^4 (a^{(FB),2}_{2,2}(F,\sigma) - 4\beta^{(FB),4}_{2,2}(F,\sigma) + 3\gamma^{(FB),4}_{2,2}(F,\sigma) - \epsilon^{(FB),4}_{2,2}(F,\sigma)) \\
&\qquad - 8 h_V^2 h_A^2 (5a^{(FBB),2}_{2,1,1}(F,V,A) + 4\beta^{(FBB),4}_{2,1,1}(F,V,A) - \gamma^{(FBB),4}_{2,1,1}(F,V,A) \\
&\qquad + 3\epsilon^{(FBB),4}_{2,1,1}(F,V,A)) - \frac{1}{2}(A \to V) - \frac{1}{2}(V \to A) \Big], \\
\beta^{(2)}_{\lambda_V} &= \frac{1}{12} v_4 \Big[h_\sigma^4 (3a^{(FB),2}_{2,2}(F,\sigma) + 3\beta^{(FB),4}_{2,2}(F,\sigma) - 2\gamma^{(FB),4}_{2,2}(F,\sigma) + 2\epsilon^{(FB),4}_{2,2}(F,\sigma)) \\
&\qquad + 4 h_\sigma^2 h_V^2 (3a^{(FBB),2}_{2,1,1}(F,\sigma,V) + \gamma^{(FBB),4}_{2,1,1}(F,\sigma,V) + \epsilon^{(FBB),4}_{2,1,1}(F,\sigma,V)) \\
&\qquad + 4 h_\sigma^2 h_A^2 (3a^{(FBB),2}_{2,1,1}(F,\sigma,A) + 6\beta^{(FBB),4}_{2,1,1}(F,\sigma,A) - 5\gamma^{(FBB),4}_{2,1,1}(F,\sigma,A)
\end{aligned} \tag{D.8}$$

$$
\begin{aligned}
&\qquad\qquad +3\epsilon^{(FBB),4}_{2,1,1}(F,\sigma,A))\\
&\quad +8h_V^2h_A^2(9a^{(FBB),2}_{2,1,1}(F,V,A)-21\beta^{(FBB),4}_{2,1,1}(F,V,A)+16\gamma^{(FBB),4}_{2,1,1}(F,V,A)\\
&\qquad\qquad -4\epsilon^{(FBB),4}_{2,1,1}(F,V,A))+\frac{1}{2}(A\to V)+\frac{1}{2}(V\to A)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\beta^{(2)}_{\lambda_A} &= \frac{1}{12}v_4\Big[h_\sigma^4(3a^{(FB),2}_{2,2}(F,\sigma)+3\beta^{(FB),4}_{2,2}(F,\sigma)-2\gamma^{(FB),4}_{2,2}(F,\sigma)+2\epsilon^{(FB),4}_{2,2}(F,\sigma))\\
&\quad -4h_\sigma^2h_V^2(3a^{(FBB),2}_{2,1,1}(F,\sigma,V)+\gamma^{(FBB),4}_{2,1,1}(F,\sigma,V)+\epsilon^{(FBB),4}_{2,1,1}(F,\sigma,V))\\
&\quad -4h_\sigma^2h_A^2(3a^{(FBB),2}_{2,1,1}(F,\sigma,A)+6\beta^{(FBB),4}_{2,1,1}(F,\sigma,A)-5\gamma^{(FBB),4}_{2,1,1}(F,\sigma,A)\\
&\qquad\qquad +3\epsilon^{(FBB),4}_{2,1,1}(F,\sigma,A))\\
&\quad +8h_V^2h_A^2(9a^{(FBB),2}_{2,1,1}(F,V,A)-21\beta^{(FBB),4}_{2,1,1}(F,V,A)+16\gamma^{(FBB),4}_{2,1,1}(F,V,A)\\
&\qquad\qquad -4\epsilon^{(FBB),4}_{2,1,1}(F,V,A))+\frac{1}{2}(A\to V)+\frac{1}{2}(V\to A)\Big].
\end{aligned}
$$

D.2 Field Redefinitions

Before we start, let us remark that in this section we write down explicit factors of the wave function renormalization Z. To obtain the expressions in the renormalized couplings we simply have to set $Z=1$. To keep the form of the effective action simple (more precisely to retain Yukawa couplings constant in momentum space and $\lambda(p)=0$) we allow for momentum-dependent field redefinitions,

$$
\begin{aligned}
\partial_t\phi(q) &= -\left(\overline{\psi}\left(\frac{1-\gamma^5}{2}\right)\psi\right)(q)\partial_t\omega_\sigma(q)+\phi(q)\partial_t\alpha_\sigma(q), \qquad \text{(D.9)}\\
\partial_t\phi^\star(q) &= \left(\overline{\psi}\left(\frac{1+\gamma^5}{2}\right)\psi\right)(q)\partial_t\omega_\sigma(q)+\phi^\star(q)\partial_t\alpha_\sigma(q),\\
\partial_tV^\mu(q) &= (\overline{\psi}\gamma^\mu\psi)(q)\partial_t\omega_V(q)+V^\mu(q)\partial_t\alpha_V(q),\\
\partial_tA^\mu(q) &= (\overline{\psi}\gamma^\mu\gamma^5\psi)(q)\partial_t\omega_A(q)+A^\mu(q)\partial_t\alpha_A(q).
\end{aligned}
$$

Evaluating

$$
\partial_t\Gamma=\partial_t\gamma\mid+\int_q\frac{\delta\Gamma}{\delta\phi(q)}\partial_t\phi(q)+\frac{\delta\Gamma}{\delta\phi^\star(q)}\partial_t\phi^\star(q)+\frac{\delta\Gamma}{\delta V^\mu(q)}\partial_tV^\mu(q)+\frac{\delta\Gamma}{\delta A^\mu(q)}\partial_tA^\mu(q) \qquad \text{(D.10)}
$$

we find

$$
\begin{aligned}
\partial_t\lambda_\sigma(q) &= \partial_t\lambda_\sigma(q)\mid -h_\sigma\partial_t\omega_\sigma(q), \qquad \text{(D.11)}\\
\partial_th_\sigma(q) &= \partial_th_\sigma(q)\mid +(\mu_\sigma^2+Z_\sigma q^2)\partial_t\omega_\sigma(q)+h_\sigma\partial_t\alpha_\sigma(q),\\
\partial_t\mu_\sigma^2 &= \partial_t\mu_\sigma^2\mid +2\mu_\sigma^2\partial_t\alpha_\sigma(0),\\
\partial_tZ_\sigma(q) &= \partial_tZ_\sigma(q)\mid +2\mu_\sigma^2\frac{\partial_t\alpha_\sigma(q)-\partial_t\alpha_\sigma(0)}{q^2}+2Z_\sigma\partial_t\alpha_\sigma(q),
\end{aligned}
$$

$$\begin{aligned}
\partial_t \lambda_V(q) &= \partial_t \lambda_V(q)\mid + 2h_V \partial_t \omega_V(q), \\
\partial_t h_V(q) &= \partial_t h_V(q)\mid - (\mu_V^2 + Z_V q^2)\partial_t \omega_V(q) + h_V \partial_t \alpha_V(q), \\
\partial_t \mu_V^2 &= \partial_t \mu_V^2 \mid + 2\mu_V^2 \partial_t \alpha_V(q), \\
\partial_t Z_V(q) &= \partial_t Z_V(q)\mid + 2\mu_V^2 \frac{\partial_t \alpha_V(q) - \partial_t \alpha_V(0)}{q^2} + 2Z_V \partial_t \alpha_V(q),
\end{aligned} \tag{D.12}$$

and an analogous equation with $(V \to A)$. As the expressions for the axial vector boson can always be obtained by this replacement, we write in the following only the expression for the vector boson.

Imposing the condition

$$\partial_t \lambda(q) = 0, \tag{D.13}$$

determines the functions $\partial_t \omega$,

$$\partial_t \omega_\sigma(q) = \frac{\partial_t \lambda_\sigma(q)}{h_\sigma}, \quad \partial_t \omega_V(q) = -\frac{\partial_t \lambda_V(q)}{2h_V}. \tag{D.14}$$

The requirement for a constant Yukawa coupling reads,

$$\partial_t \left(\frac{h_\sigma(q) - h_\sigma(0)}{q^2} \right) = 0, \quad \partial_t \left(\frac{h_V(q) - h_V(0)}{q^2} \right) = 0. \tag{D.15}$$

This fixes the functions $\partial_t \alpha(q)$ up to a constant in q,

$$\begin{aligned}
\frac{\partial_t \alpha_\sigma(q) - \partial_t \alpha_\sigma(0)}{q^2} &= -\frac{1}{h_\sigma}\Bigg[\partial_t \left(\frac{h_\sigma(q) - h_\sigma(0)}{q^2} \right) \Big| \\
&\quad + \frac{\mu_\sigma^2}{h_\sigma} \partial_t \left(\frac{\lambda_\sigma(q) - \lambda_\sigma(0)}{q^2} \right) \Big| + \frac{Z_\sigma}{h_\sigma} \partial_t \lambda_\sigma(q) \Big| \Bigg], \\
\frac{\partial_t \alpha_V(q) - \partial_t \alpha_V(0)}{q^2} &= -\frac{1}{h_V}\Bigg[\partial_t \left(\frac{h_V(q) - h_V(0)}{q^2} \right) \Big| \\
&\quad + \frac{\mu_V^2}{2h_V} \partial_t \left(\frac{\lambda_V(q) - \lambda_V(0)}{q^2} \right) \Big| + \frac{Z_V}{2h_V} \partial_t \lambda_V(q) \Big| \Bigg].
\end{aligned} \tag{D.16}$$

The remaining constant can be fixed by requiring that our fields are always renormalized if they are so at the beginning, i.e.

$$\partial_t Z(0) = 0. \tag{D.17}$$

Using our definition of the anomalous dimension, $\eta = 2\partial_t \alpha^{(0)}$, we find,

$$\begin{aligned}
\eta_\sigma = 2\partial_t \alpha_\sigma^{(0)} &= -\frac{\partial_t Z_\sigma(0)\mid}{Z_\sigma} + \frac{2\mu_\sigma^2}{Z_\sigma h_\sigma}\left[\partial_t h_\sigma^{(2)} \mid + \frac{\mu_\sigma^2}{h_\sigma} \partial_t \lambda_\sigma^{(2)} \mid + \frac{Z_\sigma}{h_\sigma} \partial_t \lambda_\sigma^{(0)} \mid \right], \\
\eta_V = 2\partial_t \alpha_V^{(0)} &= -\frac{\partial_t Z_V(0)\mid}{Z_V} + \frac{2\mu_V^2}{Z_V h_V}\left[\partial_t h_V^{(2)} \mid + \frac{\mu_V^2}{2h_V} \partial_t \lambda_V^{(2)} \mid + \frac{Z_V}{2h_V} \partial_t \lambda_V^{(0)} \mid \right].
\end{aligned} \tag{D.18}$$

Inserting this into Eqs. (D.11), (D.12) we obtain our final form for the flow equations,

$$\begin{aligned} \partial_t h_\sigma^2 &= \eta_\sigma h_\sigma^2 + 2h_\sigma \partial_t h_\sigma| + 2\mu_\sigma^2 \partial_t \lambda_\sigma^{(0)}|, \\ \partial_t h_V^2 &= \eta_V h_V^2 + 2h_V \partial_t h_V| + \mu_V^2 \partial_t \lambda_V^{(0)}|, \end{aligned} \tag{D.19}$$

$$\begin{aligned} \partial_t \mu_\sigma^2 &= \eta_\sigma \mu_\sigma^2 + \partial_t \mu_\sigma^2|, \\ \partial_t \mu_V^2 &= \eta_V \mu_V^2 + \partial_t \mu_V^2|. \end{aligned} \tag{D.20}$$

Bibliography

[1] R. Barate *et al.* [ALEPH Collaboration], "Observation of an Excess in the Search for the Standard Model Higgs Boson at ALEPH," Phys. Lett. B **495** (2000) 1 [hep-ex/0011045].

[2] M. Acciarri *et al.* [L3 Collaboration], "Higgs Candidates in e+ e- Interactions at s**(1/2) = 206.6-GeV," Phys. Lett. B **495** (2000) 18 [hep-ex/0011043].

[3] G. Abbiendi *et al.* [OPAL Collaboration], "Search for the Standard Model Higgs Boson in e+ e- collisions at s**(1/2) = 192-GeV - 209-GeV," Phys. Lett. B **499** (2001) 38 [hep-ex/0101014].

[4] P. Abreu *et al.* [DELPHI Collaboration], "Search for the Standard Model Higgs Boson at LEP in the Year 2000," Phys. Lett. B **499** (2001) 23 [hep-ex/0102036].

[5] [LEP Higgs Working Group for Higgs Boson Searches Collaboration], "Search for the Standard Model Higgs Boson at LEP," hep-ex/0107029.

[6] W. A. Bardeen, C. T. Hill and M. Lindner, "Minimal Dynamical Symmetry Breaking of the Standard Model," Phys. Rev. D **41** (1990) 1647.

[7] V. A. Miransky, M. Tanabashi and K. Yamawaki, "Dynamical Electroweak Symmetry Breaking with Large Anomalous Dimension and T Quark Condensate," Phys. Lett. B **221** (1989) 177.

[8] V. A. Miransky, M. Tanabashi and K. Yamawaki, "Is the T Quark Responsible for the Mass of W and Z Bosons?," Mod. Phys. Lett. A **4** (1989) 1043.

[9] Y. Nambu, "BCS Mechanism, Quasi-Supersymmetry, and Fermion Masses," In *Kazimierz 1988, Proceedings, New theories in physics* 1-10.

[10] G. Cvetic, "Top Quark Condensation: A Review," Rev. Mod. Phys. **71** (1999) 513 [hep-ph/9702381].

[11] M. Aizenman, "Proof of the Triviality of Phi**4 in D-Dimensions Field Theory and some Mean Field Features of Ising Models for $D > 4$," Phys. Rev. Lett. **47** (1981) 1.

[12] J. Frohlich, "On the Triviality of Lambda (Phi**4) in D-Dimensions Theories and the Approach to the Critical Point in $D >=$ Four-Dimensions," Nucl. Phys. B **200** (1982) 281.

[13] M. Luscher and P. Weisz, Nucl. Phys. B **290** (1987) 25.

[14] M. Lindner, "Implications of Triviality for the Standard Model," Z. Phys. C **31** (1986) 295.

[15] D. J. Callaway, "Triviality Pursuit: Can Elementary Scalar Particles Exist?," Phys. Rept. **167** (1988) 241.

[16] J. Goldstone, "Field Theories with 'Superconductor' Solutions," Nuovo Cim. **19** (1961) 154.

[17] J. Goldstone, A. Salam and S. Weinberg, "Broken Symmetries," Phys. Rev. **127** (1962) 965.

[18] D. Bailin and A. Love, "Superfluidity and Superconductivity in Relativistic Fermion Systems," Phys. Rept. **107** (1984) 325.

[19] M. G. Alford, K. Rajagopal and F. Wilczek, "QCD at Finite Baryon Density: Nucleon Droplets and Color Superconductivity," Phys. Lett. B **422** (1998) 247 [hep-ph/9711395].

[20] J. Berges and K. Rajagopal, "Color Superconductivity and Chiral Symmetry Restoration at nonzero Baryon Density and Temperature," Nucl. Phys. B **538** (1999) 215 [hep-ph/9804233].

[21] M. G. Alford, K. Rajagopal and F. Wilczek, "Color-Flavor Locking and Chiral Symmetry Breaking in High Density QCD," Nucl. Phys. B **537** (1999) 443 [hep-ph/9804403].

[22] T. Schaefer and F. Wilczek, "Continuity of Quark and Hadron Matter," Phys. Rev. Lett. **82** (1999) 3956 [hep-ph/9811473].

[23] D. T. Son, "Superconductivity by Long-range Color Magnetic Interaction in High-density Quark Matter," Phys. Rev. D **59** (1999) 094019 [hep-ph/9812287].

[24] M. G. Alford, J. Berges and K. Rajagopal, "Unlocking Color andFlavor in Superconducting Strange Quark Matter," Nucl. Phys. B **558** (1999) 219 [hep-ph/9903502].

[25] T. Schafer and F. Wilczek, "Superconductivity from Perturbative One-gluon Exchange in High Density Quark Matter," Phys. Rev. D **60** (1999) 114033 [hep-ph/9906512].

[26] M. Oertel and M. Buballa, "Color-Flavor (Un-)locking," hep-ph/0202098.

[27] C. Wetterich, "Gluon-Meson Duality," Phys. Lett. B **462** (1999) 164 [hep-th/9906062].

[28] C. Wetterich, "Spontaneously Broken Color," Phys. Rev. D **64** (2001) 036003 [hep-ph/0008150].

[29] Y. Nambu and G. Jona-Lasinio, "Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I," Phys. Rev. **122** (1961) 345; Phys. Rev. **124** (1961) 246.

[30] J. Bardeen, L. N. Cooper and J. R. Schrieffer, "Theory of Superconductivity," Phys. Rev. **108** (1957) 1175.

[31] G. Jona-Lasinio, "Relativistic Field Theories with Symmetry Breaking Solutions," Nuovo Cim. **34** (1964) 1790.

[32] B. De Witt, *Relativity, Groups and Topology - Lectures delivered at Les Houches during the 1963 Session of the Summer School of Theoretical Physics*, C. De Witt and B. De Witt, eds., Gordon and Breach, New York, 1964

[33] T. Baier, E. Bick and C. Wetterich, "Spontaneous Symmetry Breaking in the Colored Hubbard Model," Phys. Rev. B **62**, (2000) 15471 [cond-mat/0005218].

[34] T. Baier, "A Renormalization Group Approach to the Hubbard Model", Ph.D. thesis, Heidelberg (2002).

[35] R. Stratonovich, Dokl. Akad. Nauk. SSSR **115**, 1097 (1957).

[36] J. Hubbard, "Calculation of Partition Functions," Phys. Rev. Lett. **3** (1959) 77.

[37] S. P. Klevansky, "The Nambu-Jona-Lasinio Model of Quantum Chromodynamics," Rev. Mod. Phys. **64** (1992) 649.

[38] R. Alkofer, H. Reinhardt and H. Weigel, "Baryons as Chiral Solitons in the Nambu-Jona-Lasinio Model," Phys. Rept. **265** (1996) 139 [hep-ph/9501213].

[39] J. Berges, D. U. Jungnickel and C. Wetterich, "Two Flavor Chiral Phase Transition from Nonperturbative Flow Equations," Phys. Rev. D **59** (1999) 034010 [hep-ph/9705474].

[40] J. Jaeckel, "Understanding the Fierz Ambiguity of Partially Bosonized Theories," hep-ph/0205154.

[41] J. Jaeckel and C. Wetterich, "Solving Mean Field Ambiguity by Flow Equations," hep-ph/0207094.

[42] H. Gies and C. Wetterich, "Renormalization Flow of Bound States," Phys. Rev. D **65** (2002) 065001 [hep-th/0107221].

[43] A. Hasenfratz and P. Hasenfratz, "Renormalization Group Study of Scalar Field Theories," Nucl. Phys. B **270** (1986) 687.

[44] N. Tetradis and C. Wetterich, "Critical Exponents from Effective Average Action," Nucl. Phys. B **422** (1994) 541 [hep-ph/9308214].

[45] T. R. Morris, "On Truncations of the Exact Renormalization Group," Phys. Lett. B **334** (1994) 355 [hep-th/9405190].

[46] G. Baym, and L. P. Kadanoff, "Conservation Laws and Correlation Functions," Phys. Rev. **124** (1961) 287.

[47] G. Baym, "Self-Consistent Approximations in Many-Body Systems," Phys. Rev. **127** (1962) 1391.

[48] J. M. Cornwall, R. Jackiw and E. Tomboulis, "Effective Action for Composite Operators," Phys. Rev. D **10** (1974) 2428.

[49] C. Wetterich, "Instantons and Spontaneous Color Symmetry Breaking," Phys. Lett. B **525** (2002) 277 [hep-ph/0011076].

[50] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's Functions," Commun. Math. Phys. **31** (1973) 83.

[51] J.Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford Science Publications, USA (1995)

[52] M. E. Peskin and D. V. Schroeder, *Quantum Field Theory*, Perseus Books, Cambridge MA, 1995

[53] Steven Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, Cambridge UK, 1996

[54] Brian Hatfield, *Quantum Field Theory of Point Particles and Strings*, Addison-Wesley, Reading MA, 1992

[55] L. P. Kadanoff, "Scaling Laws for Ising Models near T(C)," Physics **2** (1966) 263.

[56] K. G. Wilson, "Renormalization Group and Critical Phenomena. 1. Renormalization Group and the Kadanoff Scaling Picture," Phys. Rev. B **4** (1971) 3174.

[57] F. J. Wegner, “Renormalization Group for Critical Phenomena.” Phys. Rev. A **8** (1973) 401.

[58] K. G. Wilson and J. B. Kogut, “The Renormalization Group and the Epsilon Expansion,” Phys. Rept. **12** (1974) 75.

[59] C. Bagnuls and C. Bervillier, “Exact Renormalization Group Equations: An Introductory Review,” Phys. Rept. **348** (2001) 91 [hep-th/0002034].

[60] C. Wetterich, “Average Action and the Renormalization Group Equations,” Nucl. Phys. B **352** (1991) 529.

[61] C. Wetterich, “Quadratic Renormalization of the Average Potential and the Naturalness of Quadratic Mass Relations for the Top Quark,” Z. Phys. C **48** (1990) 693.

[62] C. Wetterich, “Exact Evolution Equation for the Effective Potential,” Phys. Lett. B **301** (1993) 90.

[63] M. Bonini, M. D’Attanasio and G. Marchesini, “Perturbative Renormalization and Infrared Finiteness in the Wilson Renormalization Group: The Massless Scalar Case,” Nucl. Phys. B **409** (1993) 441 [hep-th/9301114].

[64] U. Ellwanger, “Flow Equations for N Point Functions and Bound States,” Z. Phys. C **62** (1994) 503 [hep-ph/9308260].

[65] C. Wetterich, “The Average Action for Scalar Fields near Phase Transitions,” Phys. Lett. B **301** (1993) 90; Z. Phys. C **57** (1993) 451.

[66] U. Ellwanger and C. Wetterich, “Evolution Equations for the Quark - Meson Transition,” Nucl. Phys. B **423** (1994) 137 [hep-ph/9402221].

[67] J. Berges, N. Tetradis and C. Wetterich, “Non-perturbative Renormalization Flow in Quantum Field Theory and Statistical Physics,” Phys. Rept. **363** (2002) 223 [hep-ph/0005122].

[68] H. Gies, “Running Coupling in Yang-Mills Theory: A flow Equation Study,” Phys. Rev. D **66** (2002) 025006 [hep-th/0202207].

[69] D. F. Litim, “Optimised Renormalisation Group Flows,” Phys. Rev. D **64** (2001) 105007 [hep-th/0103195].

[70] S. Bornholdt and C. Wetterich, “Selforganizing Criticality, Large Anomalous Mass Dimension and the Gauge Hierarchy Problem,” Phys. Lett. B **282** (1992) 399.

[71] S. Bornholdt and C. Wetterich, "Average Action for Models with Fermions," Z. Phys. C **58** (1993) 585.

[72] J. Comellas, Y. Kubyshin and E. Moreno, "Exact Renormalization Group Study of Fermionic Theories," Nucl. Phys. B **490** (1997) 653 [hep-th/9512086].

[73] U. Ellwanger, M. Hirsch and A. Weber, "Flow Equations for the Relevant Part of the Pure Yang-Mills Action," Z. Phys. C **69** (1996) 687 [hep-th/9506019].

[74] U. Ellwanger, M. Hirsch and A. Weber, "The Heavy Quark Potential from Wilson's Exact Renormalization Group," Eur. Phys. J. C **1** (1998) 563 [hep-ph/9606468].

[75] M. Bonini, M. D'Attanasio and G. Marchesini, "BRS Symmetry for Yang-Mills Theory with Exact Renormalization Group," Nucl. Phys. B **437** (1995) 163 [hep-th/9410138].

[76] M. Bonini, M. D'Attanasio and G. Marchesini, "BRS Symmetry from Renormalization Group Flow," Phys. Lett. B **346** (1995) 87 [hep-th/9412195].

[77] M. D'Attanasio and T. R. Morris, "Gauge Invariance, the Quantum Action Principle, and the Renormalization Group," Phys. Lett. B **378** (1996) 213 [hep-th/9602156].

[78] D. F. Litim, "Optimisation of the Exact Renormalisation Group," Phys. Lett. B **486** (2000) 92 [hep-th/0005245].

[79] D. F. Litim, "Derivative Expansion and Renormalisation Group Flows," JHEP **0111** (2001) 059 [hep-th/0111159].

[80] D. F. Litim, "Critical Exponents from Optimised Renormalisation Group Flows," Nucl. Phys. B **631** (2002) 128 [hep-th/0203006].

[81] L. Canet, B. Delamotte, D. Mouhanna and J. Vidal, "Optimization of the Derivative Expansion in the Nonperturbative Renormalization Group," hep-th/0211055.

[82] K. I. Aoki, K. I. Morikawa, J. I. Sumi, H. Terao and M. Tomoyose, "Non-perturbative Renormalization Group Analysis of the Chiral Critical Behaviors in QED," Prog. Theor. Phys. **97** (1997) 479 [hep-ph/9612459].

[83] F. J. Dyson, "The S Matrix in Quantum Electrodynamics," Phys. Rev. **75** (1949) 1736.

[84] J. S. Schwinger, "On the Green's Functions of Quantized Fields. 1," Proc. Nat. Acad. Sci. **37** (1951) 452.

[85] C. D. Roberts and A. G. Williams, "Dyson-Schwinger Equations and their Application to Hadronic Physics," Prog. Part. Nucl. Phys. **33** (1994) 477 [hep-ph/9403224].

[86] C. D. Roberts and S. M. Schmidt, "Dyson-Schwinger Equations: Density, Temperature and Continuum Strong QCD," Prog. Part. Nucl. Phys. **45** (2000) S1 [nucl-th/0005064].

[87] R. Alkofer and L. von Smekal, "The Infrared Behavior of QCD Green's Functions: Confinement, Dynamical Symmetry Breaking, and Hadrons as Relativistic Bound States," Phys. Rept. **353** (2001) 281 [hep-ph/0007355].

[88] H. Terao, "ERG and Schwinger-Dyson Equations - Comparison in Formulations and Applications," Int. J. Mod. Phys. A **16** (2001) 1913 [hep-ph/0101107].

[89] E. Meggiolaro and C. Wetterich, "Evolution Equations for the Effective Four-quark Interactions in QCD," Nucl. Phys. B **606** (2001) 337 [hep-ph/0012081].

[90] D. U. Jungnickel and C. Wetterich, "Effective Action for the Chiral Quark-Meson Model," Phys. Rev. D **53** (1996) 5142 [hep-ph/9505267].

[91] B. J. Schaefer and H. J. Pirner, "The Equation of State of Quarks and Mesons in a Renormalization Group Flow Picture," Nucl. Phys. A **660** (1999) 439 [nucl-th/9903003].

[92] B. Bergerhoff, J. Manus and J. Reingruber, "The Thermal Renormalization Group for Fermions, Universality, and the Chiral Phase Transition," Phys. Rev. D **61** (2000) 125005 [hep-ph/9912474].

[93] H. Kodama and J. I. Sumi, "Application of Non-perturbative Renormalization Group to Nambu-Jona-Lasinio/Gross-Neveu Model at Finite Temperature and Chemical Potential," Prog. Theor. Phys. **103** (2000) 393 [hep-th/9912215].

[94] R. W. Haymaker, "Variational Methods for Composite Operators," Riv. Nuovo Cim. **14N8** (1991) 1.

[95] D. J. Gross and A. Neveu, "Dynamical Symmetry Breaking in Asymptotically Free Field Theories," Phys. Rev. D **10** (1974) 3235.

[96] F. Hoefling, C. Nowak and C. Wetterich, "Phase Transition and Critical Behaviour of the d = 3 Gross-Neveu Model," Phys. Rev. B **66** (2002) 205111 [cond-mat/0203588].

[97] J. A. Bowers and K. Rajagopal, "The Crystallography of Color Superconductivity," Phys. Rev. D **66** (2002) 065002 [hep-ph/0204079].

[98] C. Wetterich, “Bosonic Effective Action for Interacting Fermions,” cond-mat/0208361.

[99] G. ’t Hooft, “Computation of the Quantum Effects due to a Four-Dimensional Pseudoparticle,” Phys. Rev. D **14** (1976) 3432 [Erratum-ibid. D **18** (1978) 2199].

[100] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, “Instanton Density in a Theory with Massless Quarks,” Nucl. Phys. B **163** (1980) 46.

[101] C. G. Callan, R. F. Dashen and D. J. Gross, “Toward a Theory of the Strong Interactions,” Phys. Rev. D **17** (1978) 2717.

[102] E. V. Shuryak, “The Role of Instantons in Quantum Chromodynamics. 3. Quark - Gluon Plasma,” Nucl. Phys. B **203** (1982) 140.

[103] T. Schafer and E. V. Shuryak, “Instantons in QCD,” Rev. Mod. Phys. **70** (1998) 323 [hep-ph/9610451].

[104] G. ’t Hooft, “How Instantons Solve the U(1) Problem,” Phys. Rept. **142** (1986) 357.

[105] D. Diakonov and V. Y. Petrov, “Chiral Condensate in the Instanton Vacuum,” Phys. Lett. B **147** (1984) 351.

[106] D. Diakonov and V. Y. Petrov, “A Theory of Light Quarks in the Instanton Vacuum,” Nucl. Phys. B **272** (1986) 457.

[107] G. W. Carter and D. Diakonov, “Chiral Symmetry Breaking and Color Superconductivity in the Instanton Picture,” hep-ph/9905465.

[108] M. Gockeler, R. Horsley, E. Laermann, U. J. Wiese, P. Rakow, G. Schierholz and R. Sommer, “The Continuum Limit of QED: Renormalization Group Analysis and the Question of Triviality,” Phys. Lett. B **251** (1990) 567 [Erratum-ibid. B **256** (1991) 562].

[109] B. Broda, “Magnetic Random Walk Representation for Scalar QED and Triviality Problem,” Phys. Rev. Lett. **63** (1989) 2709.

[110] G. Parisi, “The Theory of Nonrenormalizable Interactions. 1. the Large N Expansion,” Nucl. Phys. B **100** (1975) 368.

[111] S. Weinberg, “Critical Phenomena For Field Theorists,” HUTP-76/160 *Lectures presented at Int. School of Subnuclear Physics, Ettore Majorana, Erice, Sicily, Jul 23 - Aug 8, 1976*

[112] K. Gawedzki and A. Kupiainen, “Exact Renormalization for the Gross-Neveu Model of Quantum Fields,” Phys. Rev. Lett. **54** (1985) 2191.

[113] K. Gawedzki and A. Kupiainen, "Renormalizing the Nonrenormalizable," Phys. Rev. Lett. **55** (1985) 363.

[114] K. Gawedzki and A. Kupiainen, "Renormalization of a Nonrenormalizable Quantum Field Theory," Nucl. Phys. B **262** (1985) 33.

[115] B. Rosenstein, B. Warr and S. H. Park, "Dynamical Symmetry Breaking in Four Fermi Interaction Models," Phys. Rept. **205** (1991) 59.

[116] C. de Calan, P. A. Faria da Veiga, J. Magnen and R. Seneor, "Constructing the Three-Dimensional Gross-Neveu Model with a Large Number of Flavor Components," Phys. Rev. Lett. **66** (1991) 3233.

[117] O. Lauscher and M. Reuter, Phys. Rev. D **65** (2002) 025013 [arXiv:hep-th/0108040].

[118] O. Lauscher and M. Reuter, "Is Quantum Einstein Gravity Nonperturbatively Renormalizable?," Class. Quant. Grav. **19** (2002) 483 [hep-th/0110021].

[119] M. Jamin and M. E. Lautenbacher, "TRACER: Version 1.1: A Mathematica Package for gamma Algebra in Arbitrary Dimensions," Comput. Phys. Commun. **74** (1993) 265.

[120] J. Zinn-Justin, "Chiral Anomalies and Topology," hep-th/0201220.

[121] M. Doran and J. Jaeckel, "Loop Corrections to Scalar Quintessence Potentials," Phys. Rev. D **66** (2002) 043519 [astro-ph/0203018].

[122] M. Doran and J. Jaeckel, "Loop Corrections to Scalar Quintessence Potentials," astro-ph/0205206.

Thanks...

First of all, to Prof. Dr. Christof Wetterich, for the kind and encouraging supervision of this thesis and his ongoing optimism. For answering a nearly infinite amount of questions, while leaving room for my own ideas.

To my second supervisor, Prof. Dr. Michael G. Schmidt, for the friendly interest in my work and his willingness to referee this thesis.

I am deeply indebted to Dr. Holger Gies for many fruitful discussions, collaboration on wild ideas ("Renormalizable Standard Model"), for proofreading and many suggestions.

Special thanks go to Dr. Michael Doran for collaboration on quantum corrections to Quintessence potentials resulting in my first paper [121; 122] (yeah, that's the fine art of highly unnecessary self-citation), a nice trip to Les Arcs ("Jacques, you know I don't want to bother you, but...") and many interesting discussions.

I am very grateful to Dr. Tobias Baier for many discussions on the "Fierz ambiguity" and for reading this manuscript.

To Dr. Jürgen Berges and Prof. Dr. Berthold Stech for illuminating explanations.

Of course, thanks to the rest of the Les Arcs gang, Lala Adueva, Jan Schwindt, Dr. Frank Steffen (thanks for the nice workshop!), and all the others, Juliane Behrend, Dr. Eike Bick, Sebastian Diehl, Dietrich Foethke, Felix Höfling, Björn O. Lange, Christian M. Müller, Markus M. Müller, Christian Nowak, Gregor Schäfer and Kai Schwenzer, who were always ready to discuss matters in and out of physics – in short, it was fun.

Finally, to Dr. Christiane Jäckel, Bengt Jäckel and Kurt Fippinger for being family.