Thomas Lorenz

First–order geometric evolutions and semilinear evolution equations A common mutational approach.

INAUGURAL – DISSERTATION

zur Erlangung der Doktorwürde der
Naturwissenschaftlich – Mathematischen
Gesamtfakultät der
Ruprecht – Karls – Universität Heidelberg

vorgelegt von
Diplom – Mathematiker
Thomas Lorenz
aus Heidelberg

Tag der Disputation : 28. Sept. 2004

First–order geometric evolutions and semilinear evolution equations A common mutational approach.

Gutachter : Prof. Willi Jäger (Heidelberg)
Prof. Jean–Pierre Aubin (Paris)

Zusammenfassung. Das zentrale Ziel dieser Dissertation besteht in einem einheitlichen Lösungsbegriff für verschiedenartige Evolutionsprobleme. Er soll die Grundlage schaffen, um Systeme zu lösen, deren Komponenten ihren Ursprung in völlig unterschiedlichen Anwendungen finden. Als analytischer Prüfstein für den allgemeinen Charakter des Lösungsbegriffs wird ein System herangezogen, bestehend aus

→ einer semilinearen Evolutionsgleichung in einem reflexiven Banachraum und
→ einer geometrischen Evolution 1. Ordnung.

Insbesondere bei geometrischen Evolutionen richtet sich das Augenmerk auf drei weitere Aspekte, die als Ausgangspunkt dienen sollen und einen Unterschied zu bekannten Ansätzen bedeuten :

→ Verallgemeinerung der zeitlichen Ableitung auf kompakte Teilmengen des $\mathbb{R}^N$ ohne a priori–Regularitätsbedingungen an den Rand. Insbesondere können sich topologische Eigenschaften des Randes im Laufe der Zeit ändern (z.B. Zusammenhangskomponenten verschwinden) .

→ Die Evolution des zeitabhängigen kompakten Teilmenge $K(t) \subset \mathbb{R}^N$ kann von nicht lokalen Eigenschaften von $K(t)$ und ihrer Normalkegel am Rande abhängen.

→ Keine Beschränkung auf das Inklusionsprinzip, d.h. ist eine Anfangsmenge in einer zweiten enthalten, so braucht diese Inklusion nicht für alle Zeiten erhalten bleiben.

Die Idee einer Ableitung als Approximation 1. Ordnung verlangt eine (verallgemeinerte) Abstandsfunktion. Sie ist im wesentlichen das einzige Mittel für einen abstrakten Ansatz außerhalb von Vektorräumen. Motiviert durch geometrische Evolutionen 1. Ordnung, werden sog. Scheinmetriken (ostensible metrics) betrachtet; sie erfüllen die Dreiecksungleichung und haben den Wert 0 für den Abstand eines Punktes von sich selbst, aber brauchen nicht symmetrisch zu sein.

Hier werden 2 Konzepte vorgestellt. Beide verallgemeinern die sog. Mutationsgleichungen von Jean–Pierre Aubin (in metrischen Räumen) auf Mengen mit einer abzählbaren Familie von Scheinmetriken. Eine analytische Schwierigkeit ergibt sich dabei aus dem Verzicht auf Symmetrie, denn dadurch ist der Abstand zwischen glatten Kurven im Allgemeinen nicht mehr stetig.

Das erste Konzept stützt sich auf den Vergleich der Zustände zu den Zeiten t und $t+h$ (für $h \downarrow 0$) und wird daher als "nach vorn gerichtet" bezeichnet. Es erweitert die Grundidee von Distributionen, indem eine wichtige Eigenschaft herausgegriffen wird und nur noch von den Elementen einer vorgegebenen "Testmenge" erfüllt werden muss. Daraus ergeben sich Existenz und Stabilität für milde Lösungen und zeitabhängige Kompakta in obigem Modellproblem.

Das zweite ("zurück gerichtete") Konzept berücksichtigt die Zustände zu den Zeiten $t-h$ und t (für $h \downarrow 0$). Zwar kann es (geometrische) Anwendungen nicht im gleichen Umfang erfassen, aber es bietet eine Alternative im Umgang mit (zeitlichen) Halbstetigkeiten.

Abstract. The primary aim of this Ph.D. thesis is to unify the definition of "solution" for completely different types of evolutions. Such a common approach is to lay the foundations for solving systems whose components have their origins in diverse applications. The analytical touchstone of the general character consists of

→ a semilinear evolution equation in a reflexive Banach space

→ a first–order geometric evolution in $\mathbb{R}^N$.

In regard to geometric evolutions, the concept is to fulfill 3 substantial conditions :

→ Extending the notion of derivative to compact subsets of $\mathbb{R}^N$ without a priori restrictions on the regularity of the boundary. In particular, topological changes of the boundary are not excluded with the course of time (e.g. connected components might disappear).

→ The evolution of time–dependent compact subset $K(t) \subset \mathbb{R}^N$ might depend on nonlocal properties of $K(t)$ and its limiting normal cones at the boundary.

→ No inclusion principle in general, i.e. if an initial set contains another one, then this inclusion need not be preserved while evolving.

Taking up the widespread idea of derivatives as first–order approximations, distance functions (maybe in a generalized sense) are required and essentially the only tool to use for a general approach beyond vector spaces. Motivated by first–order geometric evolutions, we consider so–called *ostensible metrics* that satisfy the triangle inequality and are equal to 0 for identical arguments, but need not be symmetric.

Here two concepts are presented, both of which are based on generalizing the mutational equations of Jean–Pierre Aubin (in metric spaces) to a set with a countable family of ostensible metrics. A main analytical difficulty results from dispensing with symmetry. Indeed, the distances between smooth curves might be only semicontinuous with respect to time whereas in metric spaces, their continuity is obvious.

The first approach considers the evolution in "forward" time direction (i.e. it compares the states at time t and $t+h$ for $h \downarrow 0$) and extends the basic idea of distributions in the figurative sense that an important property has to be satisfied merely by the elements of a given "test set" (instead of all elements). It provides existence and stability for mild solutions and time–dependent compact sets solving the above–mentioned model system.

The second approach takes the states at time $t-h$ and t (for $h \downarrow 0$) into account and thus is called as "backward". It is based on another idea of dealing with semicontinuities in time, but is of rather less interest to (geometric) applications.

Acknowledgments. This Ph.D. thesis would not have been elaborated if I had not benefited from the harmony and the support in my vicinity. Both the scientific and the private aspect are closely related in this context.

Prof. Dr. Dr. h.c. mult. Willi Jäger has been my academic teacher since my very first semester at Heidelberg University. Infected by the “virus” of analysis, I have followed his courses for gaining insight into mathematical relations. In particular, his regular emphases on conceptual backgrounds have shaped my personal interest in very general approaches. As a part of his scientific support, he drew my attention to set–valued maps quite early before my diploma thesis and gave me the opportunity to gain some experience by using them for image segmentation.
In regard to my Ph.D. thesis, Prof. Jäger has allowed me great latitude so that I have been able to investigate an approach that other professors had evaluated in a rather sceptical way. It is also a feature of this scientific latitude that I have been independent from “local boundary conditions” with respect to analytical topics. This has enabled me to work autonomously. So I would like to express my deep gratitude to Prof. Jäger.

Moreover, I am deeply indebted to Prof. Dr. Dr. h.c. Jean–Pierre Aubin and Dr. Hélène Frankowska. Their mathematical influence on me started quite early — as a consequence of their monographs (particularly those about set–valued analysis and mutational analysis).
Considering set–valued maps instead of single–valued functions exemplified my personal interest in more general approaches (i.e. in short, how to cover more applications with less assumptions). From my mathematical point of view, they have complemented each other perfectly. On the one hand I have taken up many conceptual ideas of Jean–Pierre Aubin (particularly about mutational analysis) and, on the other hand I have found many sophisticated results of Hélène Frankowska as indispensable tools of control theory and nonsmooth analysis.
During two stays at CREA in Paris, I benefited from collaborating with them and, their personal support has enabled me to finish this Ph.D. thesis efficiently.

Working on my Ph.D. thesis has been focused on two cities, i.e. Heidelberg and Paris. As a consequence, there are also two financial supports that I would like to thank : Sonderforschungsbereich 359 “Reactive flow, diffusion and transport” of the German Research Foundation (DFG) and — in short–term periods — the Research Training Network “Evolution Equations for Deterministic and Stochastic Systems” (HPRN–CT–2002–00281) of the European Community.

Finally, I would like to express my deep gratitude to my family. My parents have always supported me and have provided the harmonic vicinity so that I have been able to concentrate on my studies. Surely I would not have reached my current situation without them as a permanent pillar. Meanwhile my wife Irina Surovtsova is at my side. In a figurative sense, it is also thanks to Prof. Jäger since Irina and I met each other as colleagues in the same office. I have always been enjoying the chats with her about currently engaging topics. Since our basic attitudes coincide completely, it is usually easier to help each other without being emotionally involved to the same extent. I have always trusted her to give me optimal advice and, so she has often enabled me to overcome obstacles — both in everyday life and in science.

Mathematics as an abstract art. The basic character of mathematics has been evaluated in several ways — particularly in regard to the benefit. I consider mathematics to be an abstract art appealing due to its motivations of modeling. However the application in describing nature always has limits as Goethe realized in a figurative sense.

Natur und Kunst sie scheinen sich zu fliehen,
Und haben sich, eh' man es denkt, gefunden;
Der Widerwille ist auch mir verschwunden,
Und beide scheinen gleich mich anzuziehen.

Es gilt wohl ein redliches Bemühen !
Und wenn wir erst in abgemess'nen Stunden
Mit Geist und Fleiß uns an die Kunst gebunden,
Mag frei Natur im Herzen wieder glühen.

So ist's mit aller Bildung auch beschaffen :
Vergebens werden ungebundne Geister
Nach der Vollendung reiner Höhe streben.

Wer Großes will muß sich zusammenraffen;
In der Beschränkung zeigt sich erst der Meister,
Und das Gesetz nur kann uns Freiheit geben.

Johann Wolfgang von Goethe

Contents

Chapter 0

Overview

0.1 Diverse evolutions come together under the same roof

Many applications consist of diverse components so that their mathematical description as functions often starts with long preliminaries (like basic assumptions about regularity). However, *shapes and images are basically sets, not even smooth* (Jean–Pierre Aubin [2]). So the question is posed how to specify models in which both real– or vector–valued functions and shapes are involved. Usually the components depend on time and have a huge amount of influence over each other. Consider, for example :

→ A bacterial colony is growing in a nonhomogenous nutrient broth. For the bacteria, both speed and direction of expansion are depending on the nutrient concentration close to the boundary. On the other hand, the concentration of nutrient is changing due to consumption and diffusion.
(Further applications of set–valued flows in biological modeling are presented in [29, Demongeot, Kulesa, Murray 97].)

→ A chemical reaction in a liquid is endothermic and depends strongly on the dissolved catalyst. However, this catalyst is forming crystals due to temperature decreasing.

→ In image segmentation, a computer is to detect the region belonging one and the same object. The example of a so–called region growing method (presented in [45, Lorenz 2001]) is based on constructing time–dependent compact segments so that an error functional is decreasing in the course of time. So far, smoothing effects on the image within the current segment are not taken into account.
Basically speaking, it is an example how to extend Lyapunov method to shape optimization. (Other examples are in [30, Demongeot, Leitner 96], [31, Doyen 95].)

→ In dynamic economic theory, the results of control theory form the mathematical basis for important conclusions ([3, Aubin 97]). Coalitions of economic agents, technological progress and social effects due to migration, however, have an important impact on the dynamic process that is difficult to quantify by vector–valued functions. So some parameters ought to be described as sets of permissible values and, these subsets might depend on current and former states.

The primary aim of this thesis is to unify the definition of "solution" for completely different types of evolutions. Here the following model problem is the touchstone :

For each point of time $t \in [0, T[$, we consider a pair $(u(t), K(t))$ whose first component $u(t)$ is an element of a reflexive Banach space X whereas the second component $K(t)$ is a nonempty compact subset of $I\!R^N$.
Roughly speaking, the "rate of change with respect to time" of each component depends on time t, the vector $u(t) \in X$ and the compact set $K(t) \subset I\!R^N$ (including its limiting normal cones $N_{K(t)}(\cdot)$ that will be defined later). In particular, the topological boundary of $K(t)$ might have an explicit influence on the evolution.

For a vector–valued function $u(\cdot)$ of time, specifying the "rate of change" has a long tradition leading to several versions of the term "derivative". So to be more precise, here $u(\cdot)$ is to satisfy a semilinear evolution equation

$$\partial_t\, u(t) \;=\; A\, u(t) \;+\; f\left(t,\; u(t),\; K(t),\; N_{K(t)}\right)$$

with the infinitesimal generator A of a strongly continuous semigroup on X.

Considering the second component $K(t)$, it is not directly evident how to define the "rate of change" for a compact subset of $I\!R^N$. The widespread idea of prescribing the normal velocity, for example, has the disadvantage that much preparation is usually required for generalizing the speed in normal direction to arbitrary compact subsets (see e.g. [22, Chen, Giga, Goto 91], [60, Soner 93]. [9, Barles, Soner, Souganidis 93] [10, Barles, Souganidis 98], [1, Ambrosio 2000], [19, Cardaliaguet 2000], [18, Cardaliaguet 2001]). Many widespread concepts start with basic assumptions that restrict applications to local effects on deformation.
So the aspect of geometric evolutions poses three additional challenges. They require new ideas in comparison with previous approaches and provide the main starting points for generalizing.

- *Extending the notion of derivative to time–dependent compact subsets $K(t) \subset I\!R^N$ without any regularity conditions on its topological boundary $\partial K(t)$.*

  Anticipating the overview of the next sections for a moment, the derivative of $K(\cdot)$ at time t is described by a set $\overset{\circ}{K}(t)$ of continuous maps of deformation
$$\begin{array}{rcl} [0,1] \times I\!R^N & \longrightarrow & I\!R^N \\ (\text{time } h, \ \text{ initial point } x) & \longmapsto & \text{end point of } x \text{ at time } h \end{array}$$
that induce a first–order approximation of $K(t + \cdot)$ each. So a distance between compact subsets (maybe in a generalized sense) is essential.
This thesis is to differ from many other concepts of geometric evolution in two respects. Firstly, no regularity conditions on the topological boundaries are supposed a priori and secondly, no subsets of the boundaries have to be neglected as in geometric measure theory, for example (see [36, Federer 69], [15, Brakke 78]).

- *Evolution of $K(t)$ depending on nonlocal properties "up to first order".*

  In regard to the model problem, a given map of deformation (on $I\!R^N$) depending on time t, the vector $u(t) \in X$ and the compact set $K(t) \subset I\!R^N$ (including its normal cones at the boundary) is to be contained in the set $\overset{\circ}{K}(t)$:
$$\overset{\circ}{K}(t) \ \ni \ g\left(t, \ u(t), \ K(t), \ N_{K(t)}(\cdot)|_{\partial K(t)}\right)$$
So on the one hand, we exclude boundary properties of second order (like mean curvature), but on the other hand nonlocal features of both $K(t)$ and the graph of normal cones $N_{K(t)}(\cdot)$ can be taken into consideration.
In this respect, the concept here differs from many approaches, especially from most papers using the level set method (see [1, Ambrosio 2000] for a general survey).

- *No restricting to geometric evolutions with inclusion principle.*

  If a compact initial set is contained in another one, then the so–called *inclusion principle* states that this inclusion is preserved while the sets are evolving.
Several approaches use it as a geometric starting point for extending analytical tools to nonsmooth subsets. An excellent example is De Giorgi's theory of barriers formulated in [27, De Giorgi 94] and elaborated in [14, Bellettini, Novaga 97], [13, Bellettini, Novaga 98]. Another widespread concept is based on the level set method using viscosity solutions. There the inclusion principle is closely related with the corresponding partial differential equation being degenerate parabolic and thus, it can be regarded as a geometric counterpart of the maximum principle (see [10, Barles, Souganidis 98], [1, Ambrosio 2000], for example).

An elegant approach to front propagation problems with nonlocal terms has been presented in [19, Cardaliaguet 2000], [18, Cardaliaguet 2001], [20, Cardaliaguet, Pasquignon 2001]. The inclusion principle again is the key for generalizing the evolution from $C^{1,1}$ submanifolds with boundary to nonsmooth subsets of $I\!R^N$.

As mentioned before, the primary aim of this thesis consists in a unified concept for completely different types of evolutions and, geometric evolutions represent just a typical example. So we prefer another starting point for generalizing definitions. Basically, we use only the properties of compact subsets with respect to a given generalized distance function (as presented in § 0.5).
In comparison with the preceding approaches, it has the advantage of covering the very easy example that the normal velocity at the boundary is a given positive nonincreasing function of the set diameter.

0.2 A (very) brief outline

This chapter summarizes the main steps on our way unifying the term "solution". In the beginning, previous approaches are sketched in § 0.3 : C^0 semigroups have been a very successful concept for evolution equations in Banach spaces, but the two main pillars (i.e. exponential series and Cauchy integral formula) cannot be used beyond vector spaces.

The mutational equations of Jean–Pierre Aubin extend ordinary differential equations even to metric spaces and thus provide our starting point for combining diverse types of evolutions. In [2, Aubin 99], the primary geometric example is the set $\mathcal{K}(I\!R^N)$ of all nonempty compact subsets of $I\!R^N$ supplied with the Pompeiu–Hausdorff distance $d\!l$.

There is a link between mild solutions of semilinear evolution equations and mutational equations — presented in § 0.4. Indeed, considering the weak topology instead of the norm topology has the analytical interpretation that the metric is replaced by a family of pseudo–metrics. Then appropriate assumptions about the reflexive Banach space X and the infinitesimal generator of the semigroup imply the existence of solutions for systems in both X and $(\mathcal{K}(I\!R^N), d\!l)$ (see Proposition 0.4.3).

However, *first–order* geometric evolutions have not been covered so far because the topological boundary and its normal cones are not taken into account. In § 0.5, the two main obstacles due to normal cones are sketched. They motivate both the definition of "ostensible metric" and extending the basic idea of distributions (in the figurative sense that an important property has to be satisfied merely by the elements of a given "test set" instead of all elements).

Then in § 0.6, this notion is formulated for a nonempty set with a countable family of ostensible metrics. It generalizes the mutational analysis of Jean–Pierre Aubin and provides results about existence, stability and uniqueness of so–called *right–hand forward solutions*. They prove to be a special case of so–called *timed* right–hand forward solutions sketched in § 0.8 and, the details are presented in chapter 2 (following the topological preliminaries of chapter 1).

In § 0.7, right–hand forward solutions demonstrate how useful they are for first–order geometric evolutions in $\mathcal{K}(I\!\!R^N)$. They apply some results about reachable sets of differential inclusions (like the regularity at the boundary). All these tools are collected in Appendix A and, the details are presented in chapter 4.
Theorem 0.7.14 states the existence of solutions for the model problem (consisting of a semilinear evolution equation and a first–order geometric evolution).

In § 0.8, right–hand forward solutions are extended to their final general form. In particular, the time direction is now taken into consideration, i.e. roughly speaking, a "later" element is always compared with an "earlier" one or — to be more precise — the arguments of ostensible metrics are always sorted by time. So the triangle inequality can be replaced by the weaker condition called *timed* triangle inequality.
Chapter 2 contains all the details.

Finally, the second concept is sketched in § 0.9 and presented in chapter 3. Although we cannot overcome the second obstacle mentioned in § 0.5, we pursue this "backward" idea for two reasons : Firstly, the semicontinuity of the distance between two curves is handled in a completely different way and secondly, the existence of solutions is also proven as a consequence of completeness (and not just sequential compactness).

0.3 Previous approaches : C^0 semigroups and the mutational equations of J.–P. Aubin

Roughly speaking, many evolution problems have in common that the derivative of a time–dependent state is prescribed as a function of time and current state. The ordinary differential equation $\frac{d}{dt}\, x(t) = f(t, x(t))$ for a differentiable function $x : [0,\infty[\longrightarrow I\!R^N$ represents a typical example and, its results about existence and uniqueness of solutions have a very long tradition indeed. In fact, the same basic idea has proved to be very useful in situations that are more general than finite dimensional vector spaces.

An important concept has been developed for Banach spaces since the beginning of the 20th century. For a given linear operator A on a Banach space X, the search for a solution $u(\cdot) : [0,\infty[\longrightarrow X$ of the linear initial value problem

$$\wedge \begin{cases} \frac{d}{dt}\, u(t) &= A\; u(t) \\ u(0) &= u_0 \in X \end{cases}$$

leads to a so–called (*one–parameter*) *semigroup* $(S(t))_{t\,\geq 0}$ on X, i.e. a family of bounded linear operators $S(t) : X \longrightarrow X \quad (t \geq 0)$ satisfying the functional equations

$$\wedge \begin{cases} S(t_1 + t_2) &= S(t_1) \circ S(t_2) \qquad \text{for all } t_1, t_2 \geq 0, \\ S(0) &= \mathrm{Id}_X \end{cases}$$

(see [51, Pazy 83], [34, Engel, Nagel 2000], for example). Indeed, the wanted solution is $u(t) = S(t)\, u_0$ for all $t \geq 0$ and, the semigroup $(S(t))_{t\,\geq 0}$ is related to A according to

$$A\, u_0 = \lim_{h\downarrow 0} \; \tfrac{1}{h} \cdot \Big(S(h)\, u_0 - u_0\Big)$$

For this reason, A is called *infinitesimal generator* of the semigroup.
Proving the existence of $(S(t))_{t\,\geq 0}$ for a given generator A is based on two methods : The first approach starts with the exponential series for bounded linear $A : X \longrightarrow X$

$$S(t) := \sum_{j=0}^{\infty} \tfrac{t^j}{j!}\, A^j \;:\; X \longrightarrow X$$

and then uses approximations for generalizing as stated by the Theorem of Hille–Yosida, for example. The second pillar of semigroup theory is Cauchy integral formula (applied to so–called sectorial operators $A : D_A \longrightarrow X,\;\; D_A \subset X$).
Obviously, both methods fail beyond vector spaces because addition is required.

An approach to evolution problems in metric spaces is the *mutational analysis* of Jean–Pierre Aubin (presented in [4, Aubin 93], [2, Aubin 99]). It proves to be the more general background of "shape derivatives" introduced by Jean Céa and Jean–Paul Zolésio and has similarities to "quasidifferential equations" of Panasyuk (e.g. [50, Panasyuk 85]).

Roughly speaking, the starting point consists in extending the terms "direction" and "velocity" from vector spaces to metric spaces. Then the basic idea of first–order approximation leads to a definition of derivative for curves in a metric space and step by step, we can follow the same track as for ordinary differential equations.

Let us now describe the mutational approach in more detail : In a vector space like $\mathbb{R}^N$, a velocity is usually represented by a vector $v \neq 0$. Seizing the idea of translations, each initial point $x \in \mathbb{R}^N$ moves in direction v for some time $h > 0$ and reaches the point $x + h\,v$. Strictly speaking, it is a continuous function

$$[0,\infty[\, \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad (h,x) \longmapsto x + h\,v$$

mapping the time and the initial point to its final point — very similar to the topological notion of a homotopy between paths. This concept does not really require addition or scalar multiplication and thus can be applied to every metric space (M,d) instead : According to [2, Aubin 99], a map $\vartheta : [0,1] \times M \longrightarrow M$ is called *transition* on (M,d) if it satisfies

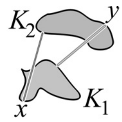
1. $\vartheta(0,x) = x \qquad \forall\ x \in M,$
2. $\limsup_{h\downarrow 0}\ \frac{1}{h} \cdot d\Big(\vartheta(h,\ \vartheta(t,x)),\ \ \vartheta(t+h,\ x)\Big) = 0 \qquad \forall\ x \in M,\ t \in [0,1[,$
3. $\alpha(\vartheta) := \sup_{x \neq y}\ \limsup_{h\downarrow 0}\ \left(\dfrac{d(\vartheta(h,\ x),\ \ \vartheta(h,\ y)) - d(x,y)}{h\ \ d(x,y)}\right)^+ < \infty,$
4. $\beta(\vartheta) := \sup_{x \in M}\ \limsup_{h\downarrow 0}\ \dfrac{d(x,\ \ \vartheta(h,\ x))}{h} < \infty$

with the abbreviation $(r)^+ := \max(0,r)$ for $r \in \mathbb{R}$.
Condition (1.) guarantees that the second argument x represents the initial point at time $t = 0$. Moreover condition (2.) can be regarded as a weakened form of the semigroup property. Finally the parameters $\alpha(\vartheta)$, $\beta(\vartheta)$ imply the continuity of ϑ with respect to both arguments. In particular, condition (4.) together with Gronwall's Lemma ensures the uniform Lipschitz continuity of ϑ with respect to time :

$$d\Big(\vartheta(s,x),\ \ \vartheta(t,x)\Big) \le \beta(\vartheta) \cdot |t-s| \qquad \text{for all } s,t \in [0,1],\ x \in M.$$

Obviously the function $[0,1] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N,\ (h,x) \longmapsto x + h\,v$ mentioned before fulfills the conditions on a transition on $(\mathbb{R}^N, |\cdot|)$. Let us give some further examples :

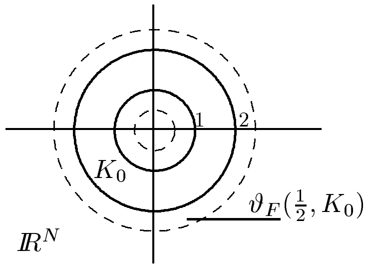
1. The constant velocity v of translation is replaced by a vector field, i.e. for a given bounded Lipschitz function $f : I\!R^N \longrightarrow I\!R^N$, every initial point $x_0 \in I\!R^N$ is moving along the trajectory $x(\cdot) : [0, \infty[\longrightarrow I\!R^N$ of $\frac{d}{dt}\, x(t) = f\,(x(t))$.
So, $\vartheta_f(t, x_0) := x(t)$ with the solution $x(\cdot) \in C^1([0,t], I\!R^N)$ of the initial value problem $\frac{d}{dt}\, x(t) = f\,(x(t))$, $x(0) = x_0$. The well–known Theorem of Cauchy–Lipschitz implies that $\vartheta_f : [0,1] \times I\!R^N \longrightarrow I\!R^N$ satisfies the conditions (1.)–(4.).

2. Leaving now the familiar field of points in $I\!R^N$, we consider the set $\mathcal{K}(I\!R^N)$ of all nonempty compact subsets of $I\!R^N$. The so–called *Pompeiu–Hausdorff distance* between two sets $K_1, K_2 \in \mathcal{K}(I\!R^N)$ is defined as
$$d\!l(K_1, K_2) := \max \Big\{ \sup_{x \in K_1} \operatorname{dist}(x, K_2), \quad \sup_{y \in K_2} \operatorname{dist}(y, K_1) \Big\}$$


It is a metric on $\mathcal{K}(I\!R^N)$ and has the advantage that $(\mathcal{K}(I\!R^N), d\!l)$ is compact (see e.g. [2, Aubin 99] or [55, Rockafellar, Wets 98]).
Correspondingly to the preceding example, suppose $f : I\!R^N \longrightarrow I\!R^N$ to be bounded and Lipschitz. Now the initial points $x_0 \in I\!R^N$ are replaced by subsets $K_0 \in \mathcal{K}(I\!R^N)$, i.e.
$$\begin{aligned} \vartheta_f : \; [0,1] \times \mathcal{K}(I\!R^N) &\longrightarrow \mathcal{K}(I\!R^N) \\ (t,\; K_0) &\longmapsto \Big\{ x(t) \;\Big|\; \exists\; x(\cdot) \in C^1([0,t], I\!R^N) : \\ &\qquad \tfrac{d}{dt}\, x(\cdot) = f(x(\cdot)), \qquad x(0) \in K_0 \Big\}. \end{aligned}$$
$\vartheta_f(t, K_0)$ is called *reachable set* of the vector field f and the initial set K_0 at time t. The Theorem of Cauchy–Lipschitz about ordinary differential equations ensures that ϑ_f is a transition on $(\mathcal{K}(I\!R^N), d\!l)$ and, $\alpha(\vartheta_f) \le \operatorname{Lip} f$, $\beta(\vartheta_f) \le \|f\|_{L^\infty}$ (see [2, Aubin 99], Proposition 3.5.2).

3. Our next step is to admit more than one velocity at every point of $I\!R^N$, i.e. strictly speaking, the vector field $f : I\!R^N \longrightarrow I\!R^N$ is replaced by a set–valued map $F : I\!R^N \leadsto I\!R^N$ and, we consider the differential inclusion $\frac{d}{dt}\, x(\cdot) \in F(x(\cdot))$ instead of the differential equation $\frac{d}{dt}\, x(\cdot) = f(x(\cdot))$.
For every bounded Lipschitz map $F : I\!R^N \leadsto I\!R^N$ with convex values in $\mathcal{K}(I\!R^N)$,
$$\begin{aligned} \vartheta_F : \; [0,1] \times \mathcal{K}(I\!R^N) &\longrightarrow \mathcal{K}(I\!R^N) \\ (t,\; K_0) &\longmapsto \Big\{ x(t) \;\Big|\; \exists\; x(\cdot) \in AC([0,t], I\!R^N) : \\ &\qquad \tfrac{d}{dt}\, x(\cdot) \in F(x(\cdot)) \text{ a.e.}, \quad x(0) \in K_0 \Big\} \end{aligned}$$
is a transition on $(\mathcal{K}(I\!R^N), d\!l)$ — as a consequence of Filippov's Theorem A.1.2 (see [2, Aubin 99], Proposition 3.7.3). For any $\lambda > 0$, $\mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$ abbreviates the set of bounded λ–Lipschitz maps $F : I\!R^N \leadsto I\!R^N$ with compact convex values.

In contrast to example (2.), the reachable set $\vartheta_F(t, K_0)$ of a set–valued map F might change its topological properties : The simple case $F(\cdot) := \mathbb{B}_1 \stackrel{\text{Def.}}{=} \{v \mid |v| \le 1\}$ leads to the expansion with constant speed 1 in all directions. Roughly speaking, the "hole" of the set $K_0 := \{x \mid 1 \le |x| \le 2\} \subset \mathbb{R}^N$ close to 0 disappears at time 1.



This phenomenon cannot occur in the examples of ordinary differential equations (with Lipschitz right–hand side) since their evolutions are reversible in time.

Now this generalized form of a direction is laying the foundations for defining the derivative of a curve $x(\cdot) : [0, T[\longrightarrow M$. A transition $\vartheta : [0,1] \times M \longrightarrow M$ provides a first–order approximation of $x(\cdot)$ at time $t \in [0, T[$ if

$$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\Big(\vartheta(h, \, x(t)), \quad x(t+h)\Big) \ = \ 0.$$

Naturally ϑ need not be unique in general and so, all transitions fulfilling this condition form the so–called *mutation* of $x(\cdot)$ at time t, abbreviated as $\overset{\circ}{x}(t)$.
A *mutational equation* is based on a given function f of time $t \in [0, T[$ and state $x \in M$ whose values are transitions on (M, d), i.e. $f : M \times [0, T[\longrightarrow \Theta(M, d)$, $(x, t) \longmapsto f(x, t)$, and we look for a Lipschitz curve $x(\cdot) : [0, T[\longrightarrow (M, d)$ such that $f(x(t), t)$ belongs to its mutation $\overset{\circ}{x}(t)$ for almost every time $t \in [0, T[$ (see [2, Aubin 99], Definition 1.3.1).

The Theorem of Cauchy–Lipschitz and its proof suggest Euler method for constructing solutions of mutational equations. In this context we need an upper estimate of the distance between two points while evolving along two (different) transitions.
First of all, a distance between two transitions $\vartheta, \tau : [0,1] \times M \longrightarrow M$ has to be defined and, it is based on comparing the evolution of one and the same initial point

$$D(\vartheta, \tau) \ := \ \sup_{x \in M} \ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d\Big(\vartheta(h, \, x), \ \ \tau(h, \, x)\Big)$$

(see [2, Aubin 99], Definition 1.1.2). Consider the preceding example of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and reachable sets $\vartheta_F(\cdot, \cdot)$, $\vartheta_G(\cdot, \cdot)$ of bounded Lipschitz maps $F, G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. Then Filippov's Theorem implies $D(\vartheta_F, \vartheta_G) \le \sup_{x \in \mathbb{R}^N} d\!\!l(F(x), G(x))$ (see [2, Aubin 99], Proposition 3.7.3).

These definitions lead to the substantial estimate

$$d\Big(\vartheta(h, \, x), \ \ \tau(h, \, y)\Big) \ \le \ d(x, y) \cdot e^{\alpha(\vartheta)\, h} \ + \ h \ D(\vartheta, \tau) \cdot \tfrac{e^{\alpha(\vartheta)\, h} - 1}{\alpha(\vartheta)\ h} \qquad (*)$$

for arbitrary points $x, y \in M$ and time $h \in [0, 1[$ (see [2, Aubin 99], Lemma 1.1.3).

The proof of this inequality provides an excellent insight into the basic technique for drawing global conclusions from local properties : Due to the definition of transitions, the distance $\psi: \ [0,1] \longrightarrow [0,\infty[, \quad h \longmapsto d\Big(\vartheta(h,x),\ \tau(h,y)\Big)$
is a Lipschitz continuous function of time and satisfies

$$\begin{aligned}
&\lim_{h\downarrow 0} \tfrac{\psi(t+h)-\psi(t)}{h} = \\
&= \lim_{h\downarrow 0} \tfrac{1}{h} \cdot \Big(d\Big(\vartheta(t+h,x),\ \tau(t+h,y)\Big) - d\Big(\vartheta(t,x),\ \tau(t,y)\Big)\Big) \\
&\le \limsup_{h\downarrow 0} \tfrac{1}{h} \cdot \Big(d\Big(\vartheta(t+h,x),\ \vartheta(h,\vartheta(t,x))\Big) + \\
&\qquad d\Big(\vartheta(h,\vartheta(t,x)),\ \vartheta(h,\tau(t,y))\Big) - d\Big(\vartheta(t,x),\ \tau(t,y)\Big) + \\
&\qquad d\Big(\vartheta(h,\tau(t,y)),\ \tau(h,\tau(t,y))\Big) + \\
&\qquad d\Big(\tau(h,\tau(t,y)),\ \tau(t+h,\ y)\Big)\Big) \\
&\le 0 + \alpha(\vartheta)\cdot\psi(t) + D(\vartheta,\tau) + 0
\end{aligned}$$

for almost every $t \in [0,1[$ (i.e. every t at which the limit on the left–hand side exists). So the estimate results from well–known Gronwall's Lemma about Lipschitz continuous functions. In fact, Gronwall's Lemma proves to be the key analytical tool for all these conclusions of mutational analysis and, its integral version holds even for continuous functions (see [2, Aubin 99], Lemma 8.3.1).

Considering now mutational equations, estimate $(*)$ is laying the foundations for proving the convergence of Euler method. It leads to the following mutational counterpart of the Theorem of Cauchy–Lipschitz (quoted from Theorem 1.4.2 in [2, Aubin 99]).

Theorem 0.3.1 *Assume that the closed bounded balls of the metric space (M,d) are compact. Let f be a function from M to a set of transitions on (M,d) satisfying*

1. $\exists\ \lambda > 0: \qquad D(f(x),\ f(y)) \ \le\ \lambda\cdot d(x,y) \qquad \forall\ x,y\in M$
2. $A := \sup_{x\in M} \alpha(f(x)) < \infty.$

Moreover suppose for $y(\cdot): [0,T[\longrightarrow M$ that its mutation $\overset{\circ}{y}(t)$ is nonempty for each t. Then for every initial value $x_0 \in M$, there exists a unique solution $x(\cdot): [0,T[\longrightarrow M$ of the mutational equation $\overset{\circ}{x}(t) \ni f(x(t))$, i.e. for almost every $t\in[0,T[$,

$$\limsup_{h\downarrow 0} \tfrac{1}{h}\cdot d\Big(x(t+h),\ f(x(t))\,(h,x(t))\Big) = 0,$$

satisfying $x(0)=x_0$ and the inequality (for every $t\in[0,T[$)

$$d\Big(x(t),\ y(t)\Big) \le d(x_0,\ y(0))\cdot e^{(A+\lambda)\,t} + \int_0^t e^{(A+\lambda)\,(t-s)} \cdot \inf_{\vartheta\in\overset{\circ}{y}(s)} D\Big(f(y(s)),\ \vartheta\Big)\ ds.$$

□

0.4 Linking semilinear evolution equations to mutational equations

The two preceding concepts — semigroups on the one hand and mutational equations of J.–P. Aubin on the other hand — can be linked to each other. Indeed, so–called *strongly continuous semigroups* on reflexive Banach spaces induce an interesting example of transitions in a slightly generalized sense. The basic idea here is to replace the metric by a family of distance functions :

Let $A : D_A \longrightarrow X$ $(D_A \subset X)$ be a closed linear operator on a Banach space X generating a semigroup $(S(t))_{t \geq 0}$. Then for every $w \in X$ and initial point $u_0 \in X$, the inhomogeneous equation $\frac{d}{dt} u(t) = A\, u(t) + w$ has a unique solution $u : [0, \infty[\longrightarrow X$ with $u(0) = u_0$, namely

$$\Sigma_w(t,\ u_0) \ := \ u(t) \ = \ S(t)\, u_0 \ + \int_0^t S(t-s) \ w \ ds.$$

In particular, we obtain $\Sigma_w(t_1, u_0) - \Sigma_w(t_2, u_0) \ = \ S(t_1) u_0 - S(t_2) u_0$ for every $t_1, t_2 \geq 0$ and fixed $u_0,\ w \in X$. If $\Sigma_w(\cdot, \cdot)$ is a transition on $(X,\ \|\cdot\|_X)$, then the condition

$$\beta(\Sigma_w) \ \stackrel{\text{Def.}}{=} \ \sup_{u_0 \in X} \ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot \Big\| u_0 \ - \ \Sigma_w(h,\ u_0) \Big\|_X \ < \ \infty$$

implies that the infinitesimal generator $A : X \longrightarrow X$ is bounded and so many important examples of semigroup theory are excluded. Their applications often lead to only *strongly continuous semigroups* or C^0 *semigroups*, i.e. in addition to the preceding functional equation, every vector $x \in X$ induces a continuous function of time

$$[0, \infty[\longrightarrow X, \quad t \longmapsto S(t)\, x.$$

According to the Theorems of Hille–Yosida and Feller–Miyadera–Phillips, the generator of a C^0 semigroup is closed, but need not be bounded.

For applying the mutational approach to C^0 semigroups, we prefer the weak topology on X to the norm $\|\cdot\|_X$ and define

$$q_{v'} : X \times X \longrightarrow [0, \infty[, \qquad (x, y) \ \longmapsto \ |\langle x - y,\ v' \rangle|$$

for every linear form $v' \in X'$ with $\|v'\|_{X'} \leq 1$. Each $q_{v'}$ is a so–called *pseudo–metric*, i.e. it is reflexive ($q_{v'}(x, x) = 0$ for all x), symmetric ($q_{v'}(x, y) = q_{v'}(y, x)$ for all x, y) and satisfies the triangle inequality. The family $\{q_{v'}\}$ induces the weak topology on X.

From now on, we suppose the Banach space X to be reflexive. This additional assumption has two advantages : Firstly, closed bounded balls of X are weakly compact (see [68, Yosida 78], for example). So any bounded sequence in X has a subsequence converging with respect to every $q_{v'}$ simultaneously.

Secondly, the reflexivity of X guarantees that the adjoint operators $S(t)' : X' \longrightarrow X'$ $(t \geq 0)$ form a C^0 semigroup on X' with the infinitesimal generator A' (see Lemma 4.5.4, quoted from [34, Engel,Nagel 2000], Prop. I.5.14). This useful consequence opens the possibility that $\Sigma_w(\cdot,\cdot)$ fulfills (slightly weakened) continuity conditions on transitions with respect to each $q_{v'}$ for $v' \in X'$ fixed : In regard to time, we obtain

$$\begin{aligned} q_{v'}\Big(\Sigma_w(t_1,u_0),\ \Sigma_w(t_2,u_0)\Big) &= \Big|\Big\langle S(t_1)\,u_0 - S(t_2)\,u_0,\quad v'\Big\rangle\Big| \\ &= \Big|\Big\langle u_0,\ \Big(S(t_1)' - S(t_2)'\Big)\,v'\Big\rangle\Big| \\ &\longrightarrow 0 \qquad\qquad \text{for } t_2 - t_1 \longrightarrow 0 \end{aligned}$$

uniformly for all $u_0 \in X$, $\|u_0\|_X \leq 1$. So for every $\rho > 0$ and each $v' \in D_{A'} \subset X'$,

$$\sup_{\substack{\|u_0\|_X \,\leq\, \rho \\ 0 \,\leq\, t \,\leq\, 1}} \ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot q_{v'}\Big(\Sigma_w(t,u_0),\quad \Sigma_w(t+h,u_0)\Big) \ \leq\ \rho\ \|A'\,v'\|_{X'},$$

i.e. restricting ourselves to a priori bounded subsets of X, we can follow the steps of mutational analysis using a finite parameter $\beta(\Sigma_w)$ with respect to $q_{v'}$.

Similarly, all initial points $u_0, u_1 \in X$ and every linear form $v' \in D_{A'} \subset X'$ satisfy

$$\begin{aligned} q_{v'}\Big(\Sigma_w(h,u_0),\ \Sigma_w(h,u_1)\Big) - q_{v'}(u_0,\ u_1) &= \Big|\Big\langle S(h)\,u_0 - S(h)\,u_1,\ \ v'\Big\rangle\Big| - |\langle u_0 - u_1,\ \ v'\rangle| \\ &= \Big|\Big\langle u_0 - u_1,\quad S(h)'\,v'\Big\rangle\Big| \qquad - |\langle u_0 - u_1,\ \ v'\rangle| \\ &\leq \Big|\Big\langle u_0 - u_1,\ \Big(S(h)' - \mathrm{Id}_{X'}\Big)\,v'\Big\rangle\Big| \\ \limsup_{h\downarrow 0}\ \frac{q_{v'}\big(\Sigma_w(h,u_0),\ \Sigma_w(h,u_1)\big) - q_{v'}(u_0,\,u_1)}{h} &\leq \Big|\Big\langle u_0 - u_1,\quad A'\ \ v'\Big\rangle\Big|. \end{aligned}$$

If additionally $v' \in D_{A'}$ is an eigenvector of A' (and λ its eigenvalue), then it provides an upper estimate of the parameter $\alpha(\Sigma_w)$ with respect to $q_{v'}$

$$\limsup_{h\downarrow 0}\ \frac{q_{v'}\big(\Sigma_w(h,u_0),\ \Sigma_w(h,u_1)\big) - q_{v'}(u_0,\,u_1)}{h\qquad q_{v'}(u_0,\,u_1)} \ \leq\ |\lambda| \qquad \text{for all } u_0,\ u_1 \in X,\ \ q_{v'}(u_0,u_1) > 0.$$

These preliminaries form the basis for proving the existence of so–called *mild solutions* by means of mutational analysis :

Proposition 0.4.1 *Suppose :*

1. X *is a reflexive Banach space.*
2. *The linear operator* A *generates a* C^0 *semigroup* $(S(t))_{t\,\geq\,0}$ *on* X.
3. *The dual operator* A' *of* A *has a countable family of eigenvectors* $\{v_j'\}_{j\in\mathcal{J}}$ $(\|v_j'\|_{X'} = 1)$ *spanning the dual space* X', *i.e.* $X' = \overline{\sum_{j\in\mathcal{J}} I\!R\, v_j'}$.
   λ_j *abbreviates the eigenvalue of* A' *belonging to* v_j' *and,* $q_j := q_{v_j'}$.

4. *Let* $f : X \times [0,T] \longrightarrow X$ *satisfy* $\|f\|_{L^\infty} < \infty$ *and for each* $j \in \mathcal{J}$,
$q_j\Big(f(x_1,t_1),\ f(x_2,t_2)\Big) \ \le\ \omega_j\Big(q_j(x_1,x_2) + |t_2 - t_1|\Big)$ *for all* $x_1, x_2,\ t_1, t_2$
with a modulus $\omega_j(\cdot)$ *of continuity.*

For each $x_0 \in X$, *there exists a mild solution* $x : [0,T[\longrightarrow X$ *of the initial value problem*

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &=& A\, x(t) \;+\; f(x(t),t) \\ \quad x(0) &=& x_0 \end{cases}$$

i.e.
$$x(t) \;=\; S(t)\, x_0 \;+\; \int_0^t S(t-s)\ f(x(s),\, s)\ ds \qquad \text{(by definition).}$$

Proof. Due to $f \in L^\infty$, there is an obvious a priori bound for Euler approximations starting in a given point $x_0 \in X$. Proposition 4.5.8 states the existence of $x : [0,T[\longrightarrow X$ that is uniformly continuous with respect to q_j for each $j \in \mathcal{J}$ and fulfills

$$\limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot q_j\Big(\Sigma_{f(x(t),t)}\,(h, x(t)),\ \ x(t+h)\Big) \;=\; 0.$$

Assumption (4.) ensures the continuity of each $[0,T[\longrightarrow (X, q_j),\ \ t \longmapsto f(x(t),\, t)$. So following the same steps as in Lemma 4.5.11 (4.), we obtain $f(x(\cdot),\cdot) \in L^\infty([0,T[, X)$. Finally Lemma 4.5.12 (quoted from [8, Ball 77]) guarantees that the mild solution $z(\cdot) : [0,T[\longrightarrow X$ of

$$\wedge \begin{cases} \frac{d}{dt}\, z(t) &=& A\, z(t) \;+\; f(x(t),t) \\ \quad z(0) &=& x_0 \end{cases}$$

is also a weak solution and thus, it must be identical to $x(\cdot)$ since

$$\langle\, x(t) - z(t),\ v_j' \,\rangle \;=\; \int_0^t \langle\, x(s) - z(s),\ \ A'\, v_j' \,\rangle\ ds \;=\; \lambda_j \cdot \int_0^t \langle\, x(s) - z(s),\ v_j' \,\rangle\ ds. \ \square$$

Assumptions (1.)–(3.) are formulated in a quite general way for pointing out the key features. Basic results of functional analysis provide interesting examples like

- a compact symmetric operator $A : X \longrightarrow X$ on a separable Hilbert space X, e.g. some integral operators of Hilbert–Schmidt type on $L^2(O)$ ($O \subset I\!\!R^N$ open),
- an infinitesimal generator $A : D_A \longrightarrow X$ of a C^0 semigroup on a Hilbert space X whose resolvent is compact and normal, e.g. a strongly elliptic differential operator (of second order) in divergence form with smooth autonomous coefficients.

Assumption (4.) of Proposition 0.4.1 is very restrictive because $f : X \times [0,T] \longrightarrow X$ has to be continuous with respect to each linear form v_j' separately. Even easy examples of rotation might fail to satisfy this condition. Thus, we take more than one linear form v_j' ($j \in \mathcal{J} = \{j_1, j_2, j_3 \ldots\}$) into consideration simultaneously (see Proposition 4.5.10) :

Proposition 0.4.2 *In addition to assumptions (1.)–(3.) of Proposition 0.4.1, let* $f : X \times [0,T] \longrightarrow X$ *fulfill* $\|f\|_{L^\infty} < \infty$ *and*

$$\sum_{k=1}^{\infty} 2^{-k} \; q_{j_k}\Big(f(x,t_1),\;\; f(y,t_2)\Big) \;\le\; \widehat{\omega}\Big(\sum_{k=1}^{\infty} 2^{-k} \; \tfrac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)} \;+\; |t_2-t_1|\Big)$$

for all $x, y \in X$ *and* $t_1, t_2 \in [0,T]$ *with a modulus* $\widehat{\omega}(\cdot)$ *of continuity.*
For each $x_0 \in X$, *there exists a mild solution* $x : [0,T[\longrightarrow X$ *of the semilinear equation*

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &=& A\, x(t) \;+\; f(x(t), t) \\ \quad x(0) &=& x_0 \end{cases}$$

i.e.

$$x(t) \;=\; S(t)\, x_0 \;+\; \int_0^t S(t-s)\;\, f(x(s),\, s)\;\, ds \qquad \textit{(by definition).}$$

After replacing the metric by a family of distance functions, the main steps of mutational analysis have not changed so far. So in principle, we can already deal with systems of semilinear evolution equations in reflexive Banach spaces and mutational equations in $(\mathcal{K}(I\!\!R^N),\; d\!\!l)$. Using the abbreviations for $x, y \in X$

$$p_\infty(x,y) \;:=\; \sum_{k=1}^{\infty} 2^{-k}\; \tfrac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)}\;, \qquad P_\infty(x,y) \;:=\; \sum_{k=1}^{\infty} 2^{-k}\;\; q_{j_k}(x,y),$$

Proposition 0.4.3
In addition to assumptions (1.)–(3.) of Proposition 0.4.1, suppose for

$$\begin{aligned} f:&\;\; X \times \mathcal{K}(I\!\!R^N) \times [0,T] \;\longrightarrow\; X \\ g:&\;\; X \times \mathcal{K}(I\!\!R^N) \times [0,T] \;\longrightarrow\; \mathrm{LIP}_\Lambda(I\!\!R^N, I\!\!R^N)\;: \end{aligned}$$

4. $\|f\|_{L^\infty} \;<\; \infty, \qquad \Lambda < \infty$

5. $P_\infty\Big(f(x_1, K_1, t_1),\;\; f(x_2, K_2, t_2)\Big) \;\le\; \omega\Big(p_\infty(x_1,x_2) + d\!\!l(K_1,K_2) + t_2 - t_1\Big)$

6. $\sup_{z \in I\!\!R^N} \; d\!\!l\Big(g(x_1,K_1,t_1)\,(z),\;\; g(x_2,K_2,t_2)\,(z)\Big) \;\le\; \omega\Big(p_\infty(x_1,x_2) + d\!\!l(K_1,K_2) + t_2 - t_1\Big)$

for all $x_1, x_2 \in X$, $K_1, K_2 \in \mathcal{K}(I\!\!R^N)$, $0 \le t_1 \le t_2 \le T$ *with a modulus* $\omega(\cdot)$ *of continuity.*

Then for every initial elements $x_0 \in X$ *and* $K_0 \in \mathcal{K}(I\!\!R^N)$, *there exists a solution* $(x, K) \;:\; [0,T[\;\longrightarrow\; X \times \mathcal{K}(I\!\!R^N)$ *of the following problem :*

a) $x : [0,T[\longrightarrow X$ *is a mild solution of the initial value problem*

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &=& A\, x(t) \;+\; f(x(t),\; K(t),\; t) \\ \quad x(0) &=& x_0 \end{cases}$$

b) $K(\cdot) : [0,T[\leadsto \mathcal{K}(I\!\!R^N)$ *is Lipschitz with respect to* $d\!\!l$ *and,* $K(0) = K_0$.

c) $\limsup_{h \downarrow 0} \; \frac{1}{h} \cdot d\!\!l\Big(\vartheta_{g(x(t),\, K(t),\, t)}\,(h,\; K(t)),\;\; K(t+h)\Big) \;=\; 0$ *for almost every* $t \in [0,T[$.

Proof results from Proposition 2.4.6 in the same way as Proposition 4.6.1.

0.5 Obstacles due to normal cones

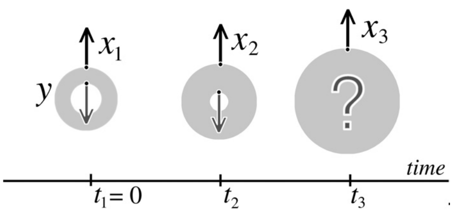
Semilinear evolution equations have just exemplified the first notion of generalizing Aubin's mutational equations, i.e. replacing the metric (of a set) by a countable family of distance functions. Following the same steps as before, we approach the model problem described in § 0.1. In Proposition 0.4.3 however, the normal cones at the topological boundary are not taken into consideration explicitly.

Applying the mutational analysis of Jean–Pierre Aubin to a metric space (M, d), obstacles are mostly related to the continuity parameters of a transition ϑ

$$\alpha(\vartheta) \stackrel{\text{Def.}}{=} \sup_{x \neq y} \ \limsup_{h \downarrow 0} \ \left(\frac{d\big(\vartheta(h,x),\ \vartheta(h,y)\big) \ - \ d(x,y)}{h \quad d(x,y)}\right)^+ < \infty,$$

$$\beta(\vartheta) \stackrel{\text{Def.}}{=} \sup_{x \in M} \ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d(x,\ \vartheta(h,\ x)) \qquad < \infty.$$

In regard to first–order geometric evolutions, these difficulties arise when incorporating normal cones into a distance function of compact subsets. We are going to use reachable sets $\vartheta_F(\cdot,\cdot)$ of differential inclusions $\dot{x}(\cdot) \in F(x(\cdot))$ a.e. as candidates for transitions on $\mathcal{K}(I\!\!R^N)$. So the topological properties of $\vartheta_F(t, K)$ may change in the course of time (as mentioned in § 0.3). Roughly speaking, "holes" might disappear, but they cannot occur suddenly.



Let us consider first the consequences of the boundary for the continuity of $\vartheta_F : [0,1] \times \mathcal{K}(I\!\!R^N) \longrightarrow \mathcal{K}(I\!\!R^N)$ with respect to time.
The example in the left figure starts with an annulus $K_{\circledcirc}$ expanding isotropically at a constant speed. After a positive finite time t_3, the "hole" in the center has disappeared of course.

Every boundary point x_3 at time t_3 has close counterparts at earlier sets. To be more precise, $x_3 \in \partial\,\vartheta_F(t_3, K_{\circledcirc})$ is final point of a trajectory $x(\cdot) : [0, t_3] \longrightarrow I\!\!R^2$ of $F(\cdot) := I\!\!B_1$ and, each $x(t)$ belongs to the boundary of $\vartheta_F(t, K_{\circledcirc})$. The well–known Hamilton condition (quoted in Proposition A.3.1) guarantees even the existence of an adjoint arc connecting a normal vector at x_3 to a normal vector at $x(t)$ (for each time $t \in [0, t_3[$). However, this tool results from a necessary condition in control theory and thus only provides boundary points in backward time direction. In particular, starting at a point $y \in \partial K_{\circledcirc}$ of the "hole", there is no trajectory that belongs to the boundary of each $\vartheta_F(t, K_{\circledcirc})$ up to time t_3.

This easy example illustrates that in general, the topological boundary of the reachable set $\vartheta_F(\cdot, K) : [0, \infty[\rightsquigarrow I\!\!R^N$ (with $K \in \mathcal{K}(I\!\!R^N)$) is not continuous with respect to $d\!l$. Furthermore, it shows that roughly speaking, the normals of *later* sets find close counterparts among the normals of *earlier* sets, but usually not vice versa.

In short, the wanted distance function on $\mathcal{K}(I\!\!R^N)$ is to combine two properties : On the one hand, normal cones at the boundary are taken into consideration explicitly and on the other hand, reachable sets $\vartheta_F(\cdot, K) : [0, \infty[\rightsquigarrow I\!\!R^N$ of differential inclusions are continuous.

For this purpose, we dispense with the symmetry condition on a metric.
So in comparison with mutational analysis, the metric on a given set E is replaced by a generalized distance function $q : E \times E \longrightarrow [0, \infty[$. Naturally the triangle inequality is essential for estimating the distance between two points by means of a third element. Moreover the distance from a point to itself ought to be zero. This so–called *reflexivity* is a weaker condition than the positive definiteness of a metric. The main feature here is that q need not be symmetric.
In literature on topology, this generalized form of distance is sometimes called *quasi–pseudo–metric* (see e.g. [42, Kelly 63], [43, Künzi 92]), but just for linguistic reasons we prefer the adjective "ostensible".

Definition 0.5.1 *Let E be a nonempty set.*
$q : E \times E \longrightarrow [0, \infty[$ *is called* ostensible metric *on E if it satisfies the conditions :*

1. $\forall\ x \in E : \quad q(x, x) = 0$ (*reflexive*)
2. $\forall\ x, y, z \in E : \quad q(x, z) \le q(x, y) + q(y, z)$ (*triangle inequality*).

Then (E, q) *is called* ostensible metric space.

In regard to the first–order geometric evolution, we suggest the ostensible metric

$$q_{\mathcal{K},N} : \mathcal{K}(I\!\!R^N) \times \mathcal{K}(I\!\!R^N) \longrightarrow [0, \infty[$$
$$(K_1,\ K_2) \longmapsto d\!l(K_1, K_2) + \operatorname{dist}\Big(\operatorname{Graph}\ {}^\flat N_{K_2},\ \operatorname{Graph}\ {}^\flat N_{K_1}\Big)$$

with $N_K(x)$ denoting the limiting normal cone of $K \subset I\!\!R^N$ at $x \in \partial K$ (Def. 4.1.4),
${}^\flat N_K(x) := N_K(x) \cap I\!\!B_1$.
So, $q_{\mathcal{K},N}(K_1,\ K_2) \ge 0$ takes the graphical distance from the limiting normal vectors ${}^\flat N_{K_2} \subset I\!\!B_1$ to ${}^\flat N_{K_1} \subset I\!\!B_1$ into account. Correspondingly to the example of an annulus $K_{\circledcirc}$ expanding isotropically, the first argument K_1 can be regarded as *earlier* set whereas the second argument K_2 represents the *later* set. In particular,
$q_{\mathcal{K},N}(\vartheta_F(s, K_{\circledcirc}),\ \vartheta_F(t, K_{\circledcirc})) \le \text{const} \cdot (t-s)$ for all $s \le t \le 1$ (due to Lemma 4.4.23).

Applying now the steps of mutational analysis to an ostensible metric space (E, q), we encounter analytical obstacles soon. In § 0.3, for example, Gronwall's Lemma is mentioned as key tool for proving estimate $(*)$

$$d\Big(\vartheta(h,\, x),\;\; \tau(h,\, y)\Big) \;\;\le\;\; d(x,y)\,\cdot\, e^{\alpha(\vartheta)\,h} \;\;+\;\; h\;\; D(\vartheta,\tau)\,\cdot\, \tfrac{e^{\alpha(\vartheta)\,h}\,-\,1}{\alpha(\vartheta)\;\;h}, \qquad (*)$$

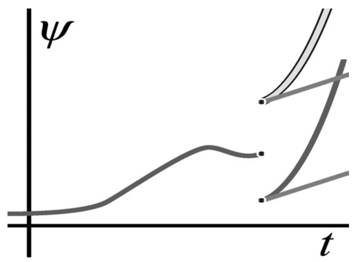
but the well–known versions of Gronwall's Lemma hold only for continuous functions. The ostensible metric space $(\mathcal{K}(I\!\!R^N),\, q_{\mathcal{K},N})$ exemplifies that the generalized distance

$$[0,1] \;\longrightarrow\; [0,\infty[, \quad t \;\longmapsto q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\;\; \vartheta_F(t,K_2)\Big)$$

is not continuous in general. For example, the isotropic expansion at a speed of 1 (i.e. $F(\cdot) := I\!\!B_1$) and initial sets $K_1 := I\!\!B_2,\;\; K_2 := \;\{1 \le |x| \le 2\} \subset I\!\!R^N$ satisfy

$$q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\;\vartheta_F(t,K_2)\Big) \;\begin{cases} \ge\; 1 & \text{for}\;\; 0 \,\le\, t \,<\, 1 \\ =\; 0 & \text{for}\;\; \qquad t \,\ge\, 1 \end{cases}.$$

So we cannot apply the proof of estimate $(*)$ (given in § 0.3) to ostensible metric spaces immediately. A more general form of Gronwall's Lemma is needed instead — without supposing continuity. Strictly speaking, the generalized distance between transitions leads to a function $\psi : [0,1] \longrightarrow [0,\infty[$ fulfilling



$$\limsup_{h\downarrow 0}\; \tfrac{\psi(t+h)\,-\,\psi(t)}{h} \;\;\le\;\; a\cdot\psi(t) \,+\, b \;\;<\;\; \infty$$

for every $t \in [0,1[$ with constants $a, b \ge 0$.

Obviously, $\psi(t) \;\ge\; \limsup_{h\downarrow 0}\; \psi(t+h)$, but it does not provide any information about $\psi(t-h)$ for $h \downarrow 0$.

Thus, we need the additional assumption $\psi(t) \;\le\; \limsup_{h\downarrow 0}\; \psi(t-h)$ for excluding a discontinuity of ψ in upward direction. It provides a more general version of Gronwall's Lemma for semicontinuous functions (proven later in Lemma 1.5.1).

Lemma 0.5.2 (Lemma of Gronwall : Subdifferential version)
Let $\psi : [a,b] \longrightarrow I\!\!R,\;\; f, g \in C^0([a,b[, I\!\!R)$ *satisfy* $f(\cdot) \ge 0$ *and*

$$\begin{aligned} \psi(t) \;&\le\; \limsup_{h\downarrow 0}\;\psi(t-h), && \forall\; t \in \,]a,\,b], \\ \psi(t) \;&\ge\; \limsup_{h\downarrow 0}\;\psi(t+h), && \forall\; t \in [a,\,b[, \\ \limsup_{h\downarrow 0}\;\tfrac{\psi(t+h)-\psi(t)}{h} \;&\le\; f(t)\,\cdot\,\limsup_{h\downarrow 0}\;\psi(t-h) \,+\, g(t) && \forall\; t \in \,]a,\,b[. \end{aligned}$$

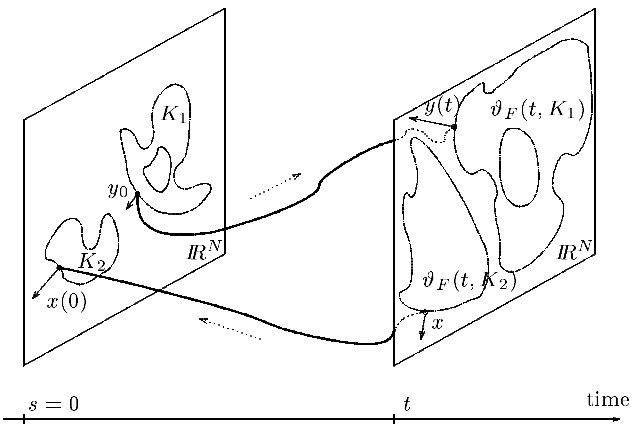
Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t) \;\;\le\;\; \psi(a)\,\cdot\, e^{\mu(t)} \;\;+\;\; \int_a^t e^{\mu(t)-\mu(s)}\; g(s)\; ds$$

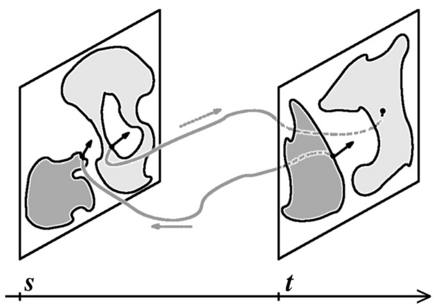
with $\mu(t) := \displaystyle\int_a^t f(s)\; ds.$

When extending estimate $(*)$ to transitions ϑ, τ on an ostensible metric space (E, q), the required semicontinuity of the distance $t \longmapsto q(\vartheta(t,x),\ \tau(t,y))$ will be guaranteed by a further condition on generalized transitions.

Now we consider the consequences of the topological boundary for the continuity of the reachable set $\vartheta_F : [0,1] \times \mathcal{K}(I\!R^N) \longrightarrow \mathcal{K}(I\!R^N)$ with respect to the second argument.
For any initial sets $K_1, K_2 \in \mathcal{K}(I\!R^N)$ and a given map $F \in \mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$, the reachable sets of the differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ a.e. at time t are compared with respect to $q_{\mathcal{K},N}$.
In particular, we need an estimate of the distance from any $x \in \partial\,\vartheta_F(t, K_2)$ to the boundary of $\vartheta_F(t, K_1)$.
Due to the definition of reachable sets, there exists an absolutely continuous trajectory $x(\cdot)$ of F linking K_2 to x. Then, $x(0)$ is in the boundary of K_2.



Furthermore, every normal vector to $\vartheta_F(t, K_2)$ at x is connected to some $p_0 \in N_{K_2}(x(0))$ by an adjoint arc (according to the extended Hamilton condition of Proposition A.3.1). The graphical distance from ${}^\flat N_{K_2}(\cdot)|_{\partial K_2}$ to ${}^\flat N_{K_1}(\cdot)|_{\partial K_1}$ is bounded by $q_{\mathcal{K},N}(K_1, K_2)$. Thus, we can find the closest counterpart $(y_0, q_0) \in \text{Graph } N_{K_1}|_{\partial K_1}$ of $(x(0), p_0)$ and estimate their distance.



Filippov's Theorem A.1.2 states the existence of a trajectory $y(\cdot) \in AC([0,t], I\!R^N)$ of F with $y(0) = y_0$ and $|y(t) - x(t)| \le |y(0) - x(0)| \cdot e^{\lambda\, t}$. (That is the basis for estimating the parameter $\alpha(\vartheta_F)$ with respect to $d\!l$ in the mutational analysis of Aubin.)
However we cannot guarantee that such a trajectory stays in the boundary of $\vartheta_F(s, K_1)$ up to time t. Roughly speaking, y_0 might belong to a "hole" of K_1 disappearing with the course of time.
For excluding this phenomenon, additional assumptions about K_1 are needed. Suitable conditions on F guarantee, for example, that compact sets with $C^{1,1}$ boundary preserve this regularity for short times (at least) and this evolution is reversible in time (see Corollary A.5.2). So their topological properties cannot change within this short period.

Assuming further conditions on one of the sets $K_1, K_2 \in \mathcal{K}(I\!\!R^N)$ prevents us from using the mutational analysis of Jean–Pierre Aubin directly. Thus, we use the basic idea of distributions.

In an ostensible metric space, there are no obvious generalizations of linear forms or partial integration and so, distributions in their widespread sense cannot be introduced. More generally speaking, *their basic idea is to select an important property and demand it for all elements of a given "test set".*

This notion is rather easy to apply to an ostensible metric space like $(\mathcal{K}(I\!\!R^N),\ q_{\mathcal{K},N})$: In the mutational analysis of a metric space (M,d), the preceding estimate $(*)$ represents the probably most important tool for constructing solutions by means of Euler method. So it is our starting point for overcoming the recent obstacle in $(\mathcal{K}(I\!\!R^N), q_{\mathcal{K},N})$. Following the basic idea of distributions, we are interested in how to realize the formal estimate

$$q_{\mathcal{K},N}\Big(\vartheta_F(h,\,K_1),\ \ \vartheta_G(h,\,K_2)\Big) \ \le\ \Big(q_{\mathcal{K},N}(K_1,K_2)\ +\ h\ Q^{\mapsto}(\vartheta_F,\ \vartheta_G)\Big)\ \cdot\ e^{\alpha^{\mapsto}\,h} \quad (**)$$

for all $K_1, K_2 \in \mathcal{K}(I\!\!R^N)$, generalized transitions $\vartheta_F, \vartheta_G : [0,1]\times\mathcal{K}(I\!\!R^N) \longrightarrow \mathcal{K}(I\!\!R^N)$ and every time $h\in[0,1]$ such that $\vartheta_F(s,K_1)$ belongs to a fixed "test subset" of $\mathcal{K}(I\!\!R^N)$ for all $s\in[0,h]$.

Consider a differential inclusion $\dot{x}(\cdot)\in F(x(\cdot))$ and two initial sets $K_1,K_2\in\mathcal{K}(I\!\!R^N)$. In regard to estimate $(**)$, we are looking for a parameter $\alpha^{\mapsto} = \alpha^{\mapsto}(\vartheta_F)\ge 0$ with

$$q_{\mathcal{K},N}\Big(\vartheta_F(h,K_1),\ \ \vartheta_F(h,K_2)\Big)\ \le\ q_{\mathcal{K},N}(K_1,\ K_2)\ \cdot\ e^{\alpha^{\mapsto}(\vartheta_F)\,\cdot\, h}$$

for sufficiently small $h\ge 0$. Suitable conditions on $F: I\!\!R^N \rightsquigarrow I\!\!R^N$ make such an upper bound of $q_{\mathcal{K},N}(\vartheta_F(t,K_1),\vartheta_F(t,K_2))$ available for every $K_1\in\mathcal{K}(I\!\!R^N)$ with $C^{1,1}$ boundary (as we sketch in § 0.7). So the set $\mathcal{K}_{C^{1,1}}(I\!\!R^N)$ of all nonempty compact N–dimensional $C^{1,1}$ submanifolds of $I\!\!R^N$ with boundary is chosen as "test set" of $\mathcal{K}(I\!\!R^N)$.

Similarly to Aubin's definition of the parameter $\alpha(\cdot)$ in § 0.3, we want $\alpha^{\mapsto}(\vartheta_F)$ to satisfy

$$\sup_{\substack{K_1\,\in\,\mathcal{K}_{C^{1,1}}(I\!\!R^N),\\ K_2\,\in\,\mathcal{K}(I\!\!R^N)}}\ \ \limsup_{h\downarrow 0}\ \left(\frac{q_{\mathcal{K},N}(\vartheta_F(h,\,K_1),\ \ \vartheta_F(h,\,K_2))\ -\ q_{\mathcal{K},N}(K_1,K_2)}{h\qquad q_{\mathcal{K},N}(K_1,K_2)}\right)^{+}\ \le\ \alpha^{\mapsto}(\vartheta_F).$$

In the special case of $(\mathcal{K}(I\!\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!\!R^N),\ q_{\mathcal{K},N})$, Lemma 4.4.25 provides sufficient conditions on $F: I\!\!R^N \rightsquigarrow I\!\!R^N$ for ensuring the existence of $\alpha^{\mapsto}(\vartheta_F)$.

Moreover, the formal estimate $(**)$ indicates how to generalize Aubin's definition of $D(\vartheta,\tau)$ for transitions ϑ,τ on a metric space (M,d), i.e.

$$D(\vartheta,\tau)\ \stackrel{\text{Def.}}{=}\ \sup_{x\,\in\,M}\ \ \limsup_{h\downarrow 0}\ \ \tfrac{1}{h}\ \cdot\ d\Big(\vartheta(h,\,x),\ \ \tau(h,\,x)\Big).$$

Indeed, the first argument of $q_{\mathcal{K},N}(\vartheta_F(t,K_1),\ \vartheta_F(t,K_2))$ in estimate $(**)$ is always supposed to be an element of the "test set" $\mathcal{K}_{C^{1,1}}(I\!R^N) \subset \mathcal{K}(I\!R^N)$ and so, $Q^{\mapsto}(\vartheta_F, \vartheta_G)$ cannot be found just by setting $K_1 = K_2$. We suggest the following definition instead

$$Q^{\mapsto}(\vartheta_F,\vartheta_G) \;:=\; \sup_{\substack{K_1 \,\in\, \mathcal{K}_{C^{1,1}}(I\!R^N),\\ K_2 \,\in\, \mathcal{K}(I\!R^N)}} \;\limsup_{h\downarrow 0}\; \left(\frac{q_{\mathcal{K},N}\big(\vartheta_F(h,K_1),\ \vartheta_G(h,K_2)\big) \;-\; q_{\mathcal{K},N}(K_1,K_2)\cdot e^{\alpha^{\mapsto}\cdot\, h}}{h}\right)^{+}$$

with a parameter $\alpha^{\mapsto} \geq 0$ depending merely on ϑ_F or ϑ_G. Under adequate assumptions about $F, G : I\!R^N \leadsto I\!R^N$, Proposition 4.4.26 states a relation between $Q^{\mapsto}(\vartheta_F, \vartheta_G)$ and the Hamiltonian functions of F, G: $Q^{\mapsto}(\vartheta_F,\vartheta_G) \;\leq\; 4\,N\;\|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N\times\,\partial I\!B_1)}$.
Here these results are just to indicate that our expectations can be realized indeed. The gist of their proofs is presented in § 0.7 later.

This example of $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$ is the starting point for a concept generalizing mutational analysis to ostensible metric spaces. It has the substantial advantage that it is not based on geometric properties like the inclusion principle, but uses only the ostensible metric $q_{\mathcal{K},N}$ and the fixed "test set" $\mathcal{K}_{C^{1,1}}(I\!R^N)$.

0.6 Generalized mutational equations: Right–hand forward solutions.

Now we specify the approach of the preceding example $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$ for the more general situation of a nonempty set E (instead of $\mathcal{K}(I\!R^N)$).
Let $D \subset E$ denote a fixed nonempty "test set" — corresponding to $\mathcal{K}_{C^{1,1}}(I\!R^N)$ before.

As a consequence of § 0.4 (about linking semilinear evolution equations to mutational equations), we consider more than one distance function on E. Thus, suppose $(q_\varepsilon)_{\varepsilon\in\mathcal{J}}$ to be a countable family of ostensible metrics on E.
Assuming $\mathcal{J}$ to be countable makes the Cantor diagonal construction available for proofs of existence. Indeed, selecting converging subsequences for each q_ε (one after the other) then leads to a subsequence converging with respect to all q_ε (simultaneously).

The above–mentioned example of $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$ is now generalized leading to so–called *forward transitions*. Here the term "forward" and the symbol $\mapsto$ indicate that we usually compare the state at time t with the element at time $t+h$ for $h \downarrow 0$. All the definitions and results of this section are special cases of their counterparts in chapter 2.

Definition 0.6.1 *Assume for* $\vartheta : [0,1] \times E \longrightarrow E$ *and each index* $\varepsilon \in \mathcal{J}$

1. $\vartheta(0,\cdot) \;=\; \mathrm{Id}_E,$

2. $\limsup_{h \downarrow 0} \; \frac{1}{h} \cdot q_\varepsilon\Big(\vartheta(h,\; \vartheta(t,x)), \quad \vartheta(t+h,\; x)\Big) \;=\; 0 \qquad \forall\; x \in E,\; t \in [0,1[,$
   $\limsup_{h \downarrow 0} \; \frac{1}{h} \cdot q_\varepsilon\Big(\vartheta(t+h,\; x), \quad \vartheta(h,\; \vartheta(t,x))\Big) \;=\; 0 \qquad \forall\; x \in E,\; t \in [0,1[,$

3. $\exists\; \alpha_\varepsilon^{\mapsto}(\vartheta) < \infty : \quad \sup_{x \in D,\, y \in E} \; \limsup_{h \downarrow 0} \left(\frac{q_\varepsilon\big(\vartheta(h,x),\, \vartheta(h,y)\big) \;-\; q_\varepsilon(x,y)}{h \quad q_\varepsilon(x,y)} \right)^{+} \;\leq\; \alpha_\varepsilon^{\mapsto}(\vartheta)$

4. $\exists\; \beta_\varepsilon(\vartheta) :\;]0,1] \;\longrightarrow [0,\infty[: \quad \beta_\varepsilon(\vartheta)(\cdot)$ *nondecreasing,* $\quad \limsup_{h \downarrow 0} \; \beta_\varepsilon(\vartheta)(h) \;=\; 0,$
   $q_\varepsilon\Big(\vartheta(s,x), \;\; \vartheta(t,x)\Big) \;\leq\; \beta_\varepsilon(\vartheta)(t-s) \qquad \forall\; s < t \leq 1,\;\; x \in E,$

5. $\forall\; x \in D \qquad \exists\; \mathcal{T}_\Theta = \mathcal{T}_\Theta(\vartheta,x) \in \,]0,1] : \quad \vartheta(t,x) \in D \qquad \forall\; t \in [0, \mathcal{T}_\Theta],$

6. $\limsup_{h \downarrow 0} \; q_\varepsilon\Big(\vartheta(t-h,\; x),\; y\Big) \;\geq\; q_\varepsilon\Big(\vartheta(t,x),\; y\Big) \qquad \forall\; x \in D,\; y \in E,\; t \in \,]0, \mathcal{T}_\Theta]$

Then $\vartheta(\cdot,\cdot)$ *is a so–called* forward transition *on* $(E, D, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$.

Conditions (1.)–(4.) are quite similar to the properties of Aubin's transitions on metric spaces (see § 0.3). Indeed, condition (1.) states that x is the initial value of $[0,1] \longrightarrow E$, $t \longmapsto \vartheta(t,x)$ and, condition (2.) can again be regarded as a weakened form of the semigroup property. It consists of two demands as q_ε need not be symmetric any longer. Condition (3.) differs from its earlier counterpart in two respects : The first argument is restricted to elements x of the "test set" D and, $\alpha_\varepsilon^{\mapsto}(\vartheta)$ may be chosen larger than necessary. Thus, it is easier to define $\alpha_\varepsilon^{\mapsto}(\cdot) < \infty$ uniformly in some applications like the geometric example $(\mathcal{K}(I\!R^N),\, \mathcal{K}_{C^{1,1}}(I\!R^N),\, q_{\mathcal{K},N})$ (discussed in § 4.4, particularly § 4.4.4). In condition (4.), the Lipschitz continuity of Aubin's transitions is replaced by equi–continuity with respect to time as this detail is used only for technical reasons in proofs.

Condition (5.) guarantees that every element $x \in D$ stays in the "test set" D for short times at least. Roughly speaking, it means in the preceding geometric example that smooth sets stay smooth shortly. This assumption is required because estimates using the parameter $\alpha^{\mapsto}(\cdot)$ can be ensured only within this period. Further conditions on $\mathcal{T}_\Theta(\vartheta,\cdot) > 0$ are avoidable for proving existence of solutions, but they are used for proving uniqueness (see §§ 2.3.2, 2.3.3).

Condition (6.) forms the basis for applying (generalized) Gronwall's Lemma 0.5.2. Indeed, every function $y : [0,1] \longrightarrow E$ with $q_\varepsilon(y(t-h),\, y(t)) \longrightarrow 0$ (for $h \downarrow 0$, each t) satisfies
$$q_\varepsilon\Big(\vartheta(t,x),\; y(t)\Big) \;\leq\; \limsup_{h \downarrow 0} \; q_\varepsilon\Big(\vartheta(t-h,x), \;\; y(t-h)\Big).$$
for all elements $x \in D$ and times $t \in \;]\,0,\; \mathcal{T}_\Theta(\vartheta,x)]$ (due to Lemma 2.1.3).

In the preceding section, we mentioned the formal estimate (∗∗) as starting point for the example $(\mathcal{K}(I\!\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!\!R^N),\ q_{\mathcal{K},N})$. Its general counterpart in $(E,\ D,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is

$$q_\varepsilon\Big(\vartheta(h,\ x),\ \ \tau(h,\ y)\Big)\ \le\ \Big(q_\varepsilon(x,y)\ +\ h\ Q_\varepsilon^{\mapsto}(\vartheta,\ \tau)\Big)\ \cdot\ e^{\text{const}\cdot h}$$

for all $x\in D,\ y\in E,\ \varepsilon\in\mathcal{J}$ and small $t>0$. For realizing this formal inequality, we specify the distance between forward transitions on $(E,D,(q_\varepsilon))$ in the following way :

Definition 0.6.2

$\Theta^{\mapsto}(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *denotes a set of forward transitions on* $(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *supposing*

$$Q_\varepsilon^{\mapsto}(\vartheta,\tau)\ :=\ \sup_{x\in D,\,y\in E}\ \limsup_{h\downarrow 0}\ \left(\frac{q_\varepsilon(\vartheta(h,x),\ \tau(h,y))\ -\ q_\varepsilon(x,y)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\tau)\,h}}{h}\right)^+\ <\ \infty$$

for all $\vartheta,\tau\in\Theta^{\mapsto}(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}}),\ \ \varepsilon\in\mathcal{J}$.

Using here the parameter $\alpha_\varepsilon^{\mapsto}(\tau)$ of the second argument τ (instead of ϑ) is just for technical reasons. Indeed, it ensures the triangle inequality of $Q_\varepsilon^{\mapsto}$ immediately, i.e.

$$Q_\varepsilon^{\mapsto}(\vartheta_1,\ \vartheta_3)\ \le\ Q_\varepsilon^{\mapsto}(\vartheta_1,\ \vartheta_2)\ +\ Q_\varepsilon^{\mapsto}(\vartheta_2,\ \vartheta_3)$$

for any transitions $\vartheta_1,\vartheta_2,\vartheta_3$ on $(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ because for all $x\in D,\ y\in E,\ t\in[0,1]$, we conclude from $q_\varepsilon(x,x)=0$ and the triangle inequality of q_ε

$$\begin{aligned} & q_\varepsilon\Big(\vartheta_1(h,x),\ \ \vartheta_3(h,y)\Big)\ -\ q_\varepsilon(x,y)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\vartheta_3)\,h}\\ \le\ \ & q_\varepsilon\Big(\vartheta_1(h,x),\ \ \vartheta_2(h,x)\Big)\ -\ q_\varepsilon(x,x)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\vartheta_2)\,h}\\ +\ & q_\varepsilon\Big(\vartheta_2(h,x),\ \ \vartheta_3(h,y)\Big)\ -\ q_\varepsilon(x,y)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\vartheta_3)\,h}. \end{aligned}$$

Moreover, it usually does not impose serious restrictions on applications since the parameter $\alpha_\varepsilon^{\mapsto}(\vartheta)$ is often chosen as a global constant (as the examples of chapter 4 show).

These definitions are laying the foundations for proving the wanted estimate in detail :

Proposition 0.6.3 *Let* $\vartheta,\tau\in\Theta^{\mapsto}(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *be forward transitions,* $\varepsilon\in\mathcal{J}$, $x\in D,\ \ y\in E$ *and* $0\le h<\mathcal{T}_\Theta(\vartheta,x)$.

Then, $q_\varepsilon\Big(\vartheta(h,x),\ \tau(h,y)\Big)\ \le\ q_\varepsilon(x,\ y)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\tilde\tau)\ h}\ +\ h\ \ Q_\varepsilon^{\mapsto}(\vartheta,\tau)\ \ \frac{e^{\alpha_\varepsilon^{\mapsto}(\tilde\tau)\,h}-1}{\alpha_\varepsilon^{\mapsto}(\tilde\tau)\,h}$.

Proof results from the generalized version of Gronwall's Lemma 0.5.2 applied to $\varphi_\varepsilon : h\longmapsto q_\varepsilon(\vartheta(h,x),\ \tau(h,y))$. It is a special case of Proposition 2.1.5.

The next step is to generalize the term “mutation” to $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$. Considering a curve $x(\cdot) : [0,T[\longrightarrow M$ in a metric space (M,d), its mutation $\overset{\circ}{x}(t)$ at time $t \in [0,T[$ consists of all transitions ϑ on (M,d) satisfying

$$\limsup_{h\downarrow 0} \; \tfrac{1}{h} \cdot d\Big(\vartheta(h,\, x(t)),\;\; x(t+h)\Big) \;=\; 0$$

according to the definition of Jean–Pierre Aubin ([2, Aubin 99], § 1.2). It reflects the idea of first–order approximation that most concepts of “derivative” start with. For $(E,\, D,\, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ however, we prefer adapting the criterion to the key estimate of Prop. 0.6.3. So firstly, we want to use only elements of D in the first argument of q_ε and secondly, a first–order approximation is to have the same effect, roughly speaking, as if the factor $Q^{\mapsto}(\cdot,\cdot)$ was 0.
Thus, a forward transition ϑ on $(E,\, D,\, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is regarded as a generalized derivative of a curve $x(\cdot) : [0,T[\longrightarrow E$ at time t if for each $\varepsilon \in \mathcal{J}$, there is a parameter $\widehat{\alpha}_\varepsilon^{\mapsto}(t) \ge 0$ with

$$\limsup_{h\downarrow 0} \; \tfrac{1}{h} \; \Big(q_\varepsilon\Big(\vartheta(h,y),\;\; x(t+h)\Big) \;-\; q_\varepsilon(y,\; x(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \Big) \;\le\; 0$$

for all “test elements” $y \in D$. To minimize the risk of confusion over Aubin’s concept and its generalization here, we dispense with a new definition of “mutation” and introduce the term “primitive” instead (in accordance with the more general Def. 2.2.1).

Definition 0.6.4 *Let* $\vartheta(\cdot) : [0,T[\longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *be a given function and, suppose for* $x(\cdot) : [0,T[\longrightarrow (E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$

1. $\forall \;\; t \in [0,T[, \;\; \varepsilon \in \mathcal{J} \qquad \exists \;\; \widehat{\alpha}_\varepsilon^{\mapsto}(t) = \widehat{\alpha}_\varepsilon^{\mapsto}(t, x(\cdot), \vartheta(\cdot)) < \infty$:
$$\limsup_{h\downarrow 0} \; \tfrac{1}{h} \; \Big(q_\varepsilon\Big(\vartheta(t)\,(h,y),\;\; x(t+h)\Big) \;-\; q_\varepsilon(y,\; x(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \Big) \;\le\; 0,$$
for all $y \in D$ *and* $\widehat{\alpha}_\varepsilon^{\mapsto}(t) \;\ge\; \alpha_\varepsilon^{\mapsto}(\vartheta(t)) \;\ge\; 0$,
2. $x(\cdot)$ *is uniformly continuous in time direction with respect to each* q_ε, *i.e. there is* $\omega_\varepsilon(x,\cdot) :]0,T[\longrightarrow [0,\infty[$ *such that* $\limsup_{h\downarrow 0} \; \omega_\varepsilon(x,h) \;=\; 0$ *and* $q_\varepsilon\Big(x(s),\; x(t)\Big) \;\le\; \omega_\varepsilon(x,\; t-s)$ *for* $0 \le s < t < T$.

Then $x(\cdot)$ *is a so–called* right–hand forward primitive *of* $\vartheta(\cdot)$, *abbreviated to* $\overset{\circ}{x}(\cdot) \ni \vartheta(\cdot)$.

In particular, the limit superior of first–order approximation in condition (1.) uses the information at the current time t and at a later point of time $t+h$ with $h \downarrow 0$. This feature again motivates the term “forward” and is symbolized by $\mapsto$ (representing the time axis). Furthermore the expression “right–hand” indicates that $x(\cdot)$ appears in the second argument of the vanishing distances q_ε $(\varepsilon \in \mathcal{J})$.

Forward transitions induce their own primitives. To be more precise, every constant function $\vartheta(\cdot) : [0,1[\longrightarrow \Theta^{\mapsto}(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ with $\vartheta(\cdot) = \vartheta_0$ has the right–hand forward primitives $[0,1[\longrightarrow E, \quad t \longmapsto \vartheta_0(t,x)$ with any $x \in E$ — as a consequence of Proposition 0.6.3 in a slightly generalized form

$$q_\varepsilon\Big(\vartheta(t_1+h,x),\ \tau(t_2+h,y)\Big) \ \leq\ \Big(q_\varepsilon\Big(\vartheta(t_1,x),\ \tau(t_2,y)\Big) \ +\ h\ Q_\varepsilon^{\mapsto}(\vartheta,\tau)\Big) \cdot e^{\alpha_\varepsilon^{\mapsto}(\tilde{\tau})\ h}$$

(see Proposition 2.1.5). This property is easy to extend to piecewise constant functions $[0,T[\longrightarrow \Theta^{\mapsto}(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ and so it will be the basis for Euler approximations later.

Let us apply now this concept to mutational equations in a generalized form. Correspondingly to ordinary differential equations, the definition of "solution" can be formulated by means of "primitives".

Definition 0.6.5 *For $f : E\times[0,T[\longrightarrow \Theta^{\mapsto}(E,D,(q_\varepsilon))$ given, a map $x : [0,T[\longrightarrow E$ is a so–called* right–hand forward solution *of the generalized mutational equation*

$$\overset{\circ}{x}(\cdot) \ \ni\ f(x(\cdot),\cdot)$$

if $x(\cdot)$ is a right–hand forward primitive of $f(x(\cdot),\cdot) : [0,T[\longrightarrow \Theta^{\mapsto}(E,\ D,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$, i.e. for each $\varepsilon \in \mathcal{J}$,

1. $\forall\ \ t\in[0,T[\qquad \exists\ \widehat{\alpha}_\varepsilon^{\mapsto}(t) \geq \alpha_\varepsilon^{\mapsto}(f(x(t),t)) \ : \qquad \forall\ y\in D$
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\ \Big(q_\varepsilon\Big(f(x(t),t)\,(h,y),\ x(t+h)\Big) \ -\ q_\varepsilon(y,\ x(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h}\Big) \ \leq\ 0,$$
2. *$x(\cdot)$ is uniformly continuous in time direction with respect to each q_ε, i.e. there is $\omega_\varepsilon(x,\cdot) :]0,T[\longrightarrow [0,\infty[$ such that $\limsup_{h\downarrow 0}\ \omega_\varepsilon(x,h) = 0$ and $q_\varepsilon\Big(x(s),\ x(t)\Big) \leq \omega_\varepsilon(x,\ t-s)$ for $0\leq s<t<T$.*

This collection of definitions is put to the test of well–posed problems, i.e. we are interested in sufficient conditions with regard to existence, uniqueness and stability of right–hand forward solutions.
Adapting the notions of ordinary differential equations, we are going to construct these solutions by means of Euler methods. So we have to realize first which kind of convergence preserves the solution property. An answer follows from the next special case of Convergence Theorem 2.3.2.

Theorem 0.6.6 (of Convergence)
Suppose the following properties of

$$\begin{array}{llcl} f_m,\ \ f: & E\times[0,T[& \longrightarrow & \Theta^{\mapsto}(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}}) \qquad (m\in I\!N) \\ x_m,\ \ x: & [0,T[& \longrightarrow & E\ : \end{array}$$

1. $M_\varepsilon \;:=\; \sup_{m,t,z} \{\, \alpha_\varepsilon^{\mapsto}(f_m(z,t)) \,\} \;<\; \infty,$
2. $Q_\varepsilon^{\mapsto}\Big(f_m(z_1,t_1),\;\, f_m(z_2,t_2)\Big) \longrightarrow 0 \quad$ *for* $m \to \infty, \quad t_2 - t_1 \downarrow 0, \quad q_\varepsilon(z_1,z_2) \to 0$
3. $\overset{\circ}{x}_m(\cdot) \;\ni\; f_m(x_m(\cdot),\,\cdot) \qquad$ *in* $[0,T[$ *(according to Definition 0.6.5)*
4. $\widehat{\omega}_\varepsilon(h) \;:=\; \sup_m \; \omega_\varepsilon(x_m,h) \;<\; \infty$ *(moduli of continuity w.r.t.* q_ε*)* $\quad \forall\; h \in \,]0,T],$
   $\limsup_{h\downarrow 0}\; \widehat{\omega}_\varepsilon(h) \;=\; 0,$
5. $\forall\; t_1,t_2 \in [0,T[, \quad t_3 \in \,]0,T[\quad \exists\; (m_j)_{j\in I\!N}$ *with* $m_j \nearrow \infty$ *and*
   (i) $\limsup_{j\longrightarrow\infty}\; Q_\varepsilon^{\mapsto}\Big(f(x(t_1),t_1),\; f_{m_j}(x(t_1),t_1)\Big) \;=\; 0,$
   (ii) $\exists\; (\delta_j')_{j\in I\!N}$ *in* $[0,1[\;:\qquad q_\varepsilon\Big(x(t_2),\;\, x_{m_j}(t_2+\delta_j')\Big) \longrightarrow 0, \qquad \delta_j' \longrightarrow 0,$
   (iii) $\exists\; (\delta_j)_{j\in I\!N}$ *in* $[0,t_3[\,:\qquad q_\varepsilon\Big(x_{m_j}(t_3-\delta_j),\;\, x(t_3)\Big) \longrightarrow 0, \qquad \delta_j \longrightarrow 0$

for each $\varepsilon \in \mathcal{J}$.

Then, $x(\cdot)$ *is a right–hand forward solution of* $\overset{\circ}{x}(\cdot) \;\ni\; f(x(\cdot),\,\cdot)$ *in* $[0,T[$.

In short, $(x_m(\cdot))_{m\in I\!N}$ is a sequence of right–hand forward solutions for $(f_m(\cdot,\cdot))_{m\in I\!N}$, respectively. Assumptions (1.), (2.), (4.) exemplify common bounds of the parameter $\alpha_\varepsilon^{\mapsto}(\cdot)$ and moduli of continuity for each $\varepsilon \in \mathcal{J}$. Finally condition (5.) links the sequences $(x_m(\cdot))_m$, $(f_m(\cdot,\cdot))_m$ to their “limit functions” $x(\cdot)$, $f(\cdot,\cdot)$.
It is quite obvious that $x(\cdot)$ has to appear in both arguments of q_ε for each $\varepsilon \in \mathcal{J}$ since its uniform continuity in time direction has to be concluded from the equicontinuity of $(x_m(\cdot))_{m\in I\!N}$.
Assumptions (5.ii), (5.iii) can be interpreted as graphical convergence in time direction. We again recognize the basic idea that the first argument of q_ε refers to the earlier point whereas the second argument of q_ε represents the later element. This notion led us to dispensing with the symmetry of distances in § 0.5.

Constructing solutions of ordinary differential equations is usually based on one of the following principles : Firstly, we suppose the right–hand side to be smooth enough so that the solution is fixed point of a contracting map. Then Banach's Fixed Point Theorem requires the assumption of completeness. The second method uses a sequence of approximations in combination with (sequential) compactness so that the solution is obtained as limit function.
We prefer the latter approach since the available estimates for transitions on $(E,D,(q_\varepsilon))$ hold only for elements of D in the first argument of q_ε (as in Proposition 0.6.3). So there is no obvious way of verifying the contraction property in (E,q_ε).

Before specifying an adequate form of sequential compactness (so–called *transitional compactness*), we formulate the counterpart of Peano's existence theorem for ordinary differential equations. It results from Proposition 2.3.5 later.

Theorem 0.6.7 (Existence of right–hand forward solutions)
Assume that the tuple $(E,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}))$ *is transitionally compact (in the sense of Definition 0.6.9).*
Furthermore let $f : E \times [0,T] \longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *fulfill for every* $\varepsilon \in \mathcal{J}$

1. $M_\varepsilon \;:=\; \sup_{t,z}\ \alpha_\varepsilon^{\mapsto}(f(z,t)) \;<\; \infty,$
2. $c_\varepsilon(h) \;:=\; \sup_{t,z}\ \beta_\varepsilon(f(z,t))(h) \;<\; \infty, \quad c_\varepsilon(h) \longrightarrow 0$ *for* $h \downarrow 0,$
3. $\exists\ \widehat{\omega}_\varepsilon(\cdot):\quad Q_\varepsilon^{\mapsto}\Big(f(z_1,t_1),\ f(z_2,t_2)\Big) \;\le\; \widehat{\omega}_\varepsilon\Big(q_\varepsilon(z_1,z_2) + t_2 - t_1\Big)$
for all $0 \le t_1 \le t_2 \le T$ *and* $z_1, z_2 \in E$,
$\widehat{\omega}_\varepsilon(\cdot) \ge 0$ *nondecreasing,* $\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) \;=\; 0.$

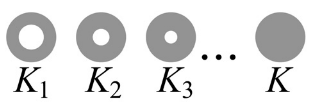
Then for every $x_0 \in E$, *there is a right–hand forward solution* $x(\cdot) : [0,T[\longrightarrow E$ *of the generalized mutational equation* $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ *in* $[0,T[$ *satisfying* $x(0) = x_0$ *(in the sense of Definition 0.6.5).*

So an adequate form of sequential compactness is needed. In Aubin's mutational analysis on metric spaces, Theorem 0.3.1 supposes the bounded closed balls to be compact, i.e. for every bounded sequence $(x_n)_{n\in I\!N}$ in (M,d), there exist a subsequence $(x_{n_j})_{j\in I\!N}$ and an element $x \in M$ with $d(x_{n_j},\ x) \longrightarrow 0$ (for $j \longrightarrow \infty$). Dispensing now with the symmetry of the distance, sequential compactness is to consist of two conditions.

Definition 0.6.8 $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *is called* two–sided sequentially compact (uniformly with respect to ε) *if for every* $z \in E$, $r_\varepsilon > 0\ (\varepsilon \in \mathcal{J})$ *and any sequence* $(x_n)_{n\in I\!N}$ *in* E *with* $\quad q_\varepsilon(z, x_n) \;\le\; r_\varepsilon \quad \forall\ n \in I\!N \quad \forall\ \varepsilon \in \mathcal{J}$
there exist a subsequence $(x_{n_j})_{j\in I\!N}$ *and an element* $x \in E$ *such that*

$$\left.\begin{array}{l} q_\varepsilon(x_{n_j},\ x) \longrightarrow 0 \\ q_\varepsilon(x,\ x_{n_j}) \longrightarrow 0 \end{array}\right. \qquad \textit{for } j \longrightarrow \infty \qquad \forall\ \varepsilon \in \mathcal{J}.$$

Some ostensible metric spaces have this compactness property in common like $(\mathcal{K}(I\!R^N),\ d\!l)$ or $\mathcal{K}(I\!R^N)$ supplied with the Pompeiu–Hausdorff excess $e^{\supset}$ in § 4.1.1, but in general, it is too restrictive. Indeed, $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ is not two–sided sequentially



compact since, for example, $K_n := \{\frac{1}{n+1} \le |x| \le 1\}$ and $K := I\!B_1$ satisfy $d\!l(K_n, K) = q_{\mathcal{K},N}(K_n, K) \longrightarrow 0\ \ (n \to \infty)$, but $q_{\mathcal{K},N}(K, K_n) \ge \frac{1}{2}$.

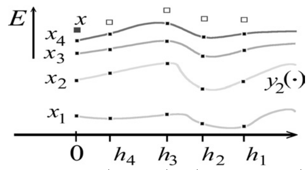
For this reason, we coin a more general term of sequential compactness that is particularly adapted for a sequence of Euler approximations at a fixed point of time. The following definition is again based on the key notion that the first argument of q_ε represents the earlier state whereas the second argument refers to the later element.

Definition 0.6.9 *Let Θ denote a nonempty set of maps $[0,1] \times E \longrightarrow E$. The tuple $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}, \Theta)$ is called* transitionally compact *if it has the property :*

Let $(x_n)_{n\in I\!N}$, $(h_j)_{j\in I\!N}$ be any sequences in E, $]0,1[$, respectively and $z \in E$ with $\sup_n q_\varepsilon(z, x_n) < \infty$ for each $\varepsilon \in \mathcal{J}$, $h_j \longrightarrow 0$. Moreover suppose $\vartheta_n : [0,1] \longrightarrow \Theta$ to be piecewise constant $(n \in I\!N)$ such that all curves $\vartheta_n(t)(\cdot, x) : [0,1] \longrightarrow E$ have a common modulus of continuity $(n \in I\!N,\ t \in [0,1],\ x \in E)$.
Each ϑ_n induces a function $y_n(\cdot) : [0,1] \longrightarrow E$ with $y_n(0) = x_n$ in the same piecewise way as forward transitions induce their own primitives mentioned after Definition 0.6.4 (i.e. using $\vartheta_n(t_m)\,(\cdot,\ y_n(t_m))$ in each interval $]t_m, t_{m+1}]$ in which $\vartheta_n(\cdot)$ is constant).
Then there exist a sequence $n_k \nearrow \infty$ of indices and $x \in E$ satisfying for each $\varepsilon \in \mathcal{J}$,

$$\limsup_{k \longrightarrow \infty} \; q_\varepsilon(x_{n_k}, x) = 0,$$

$$\limsup_{j \longrightarrow \infty} \; \sup_{k \geq j} \; q_\varepsilon(x, \; y_{n_k}(h_j)) = 0.$$



A nonempty subset $F \subset E$ is called transitionally compact *in $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}, \Theta)$ if the same property holds for any sequence $(x_n)_{n\in I\!N}$ in F (but $x \in F$ is* not *required).*

If $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is two–sided sequentially compact (uniformly with respect to ε) then the tuple $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}, \Theta)$ is transitionally compact for every nonempty set Θ of maps $[0,1] \times E \longrightarrow E$. Indeed, supposing $(x_n)_{n\in I\!N}$, $(h_j)_{j\in I\!N}$, $(\vartheta_n(\cdot))_{n\in I\!N}$, $(y_n(\cdot))_{n\in I\!N}$ as in Definition 0.6.9, there is a subsequence $(x_{n_j})_{j\in I\!N}$ and an element $x \in E$ such that

$$q_\varepsilon(x_{n_j},\ x) \longrightarrow 0, \qquad q_\varepsilon(x,\ x_{n_j}) \longrightarrow 0 \qquad \text{for } j \longrightarrow \infty \text{ and all } \varepsilon \in \mathcal{J}.$$

All curves $\vartheta_n(t)(\cdot, z) : [0,1] \longrightarrow E$ $(n \in I\!N,\ t \in [0,1],\ z \in E)$ are assumed to have a common modulus $\omega(\cdot)$ of continuity and thus,

$$\sup_{k\geq j}\ q_\varepsilon(x,\ y_{n_k}(h_j)) \;\leq\; \sup_{k\geq j}\ q_\varepsilon(x,\ x_{n_k}) + \omega(h_j) \;\longrightarrow\; 0 \qquad \text{for } j \longrightarrow 0.$$

Proving the existence of solutions in Theorem 0.6.7 starts with a sequence of Euler approximations. Transitional compactness and Cantor diagonal construction provide a subsequence that is converging to a limit function in an adequate way for each $\varepsilon \in \mathcal{J}$ and at every time of a countable dense subset of $[0,T]$. Due to Convergence Theorem 0.6.6, this limit is a right–hand forward solution. Details are presented in Proposition 2.3.5. Strictly speaking, it is sufficient to suppose that the values of all transitions $f(z,t)$ are in a transitionally compact subset of $(E,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}))$.

The third aspect of well–posed problems is uniqueness of solutions.
Since the ostensible metrics $(q_\varepsilon)_{\varepsilon\in\mathcal{J}}$ are not supposed to be positive definite in general, we are rather interested in estimates of the distance between solutions. As already mentioned in this section, estimating the distance between points of forward transitions is available only for elements of D in the first argument of q_ε (as in Proposition 0.6.3). So essentially, we have two possibilities : Either restricting ourselves to the comparison with elements of D (as in Proposition 0.6.10) or using an auxiliary function instead of the distance (as in Proposition 0.6.11).

Proposition 0.6.10 *Assume for the function* $f : E \times [0,T] \longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon))$ *and the curves* $x, y : [0,T[\longrightarrow E$

1. a) $\overset{\circ}{y}(\cdot) \ni f(y(\cdot), \cdot)$ *in* $[0,T[$ *(according to Definition 0.6.5),*
   b) $x(t) \in D$ *for all* $t \in [0,T[$,
   $$\limsup_{h\downarrow 0} \tfrac{1}{h}\; q_\varepsilon\Big(x(t+h),\; f(x(t),t)\,(h, x(t))\Big) \;=\; 0,$$
   $x(\cdot)$ *is uniformly continuous in time direction with respect to each* q_ε,
   c) $q_\varepsilon\Big(x(t), y(t)\Big) \;\le\; \limsup_{h\downarrow 0} q_\varepsilon\Big(x(t-h),\; y(t-h)\Big)$,
2. $M_\varepsilon \;:=\; \sup_{t,z}\; \alpha^{\mapsto}_\varepsilon(f(z,t)) \;<\; \infty$,
3. $\exists\; \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :\quad Q^{\mapsto}_\varepsilon\Big(f(z_1,t_1),\; f(z_2,t_2)\Big) \;\le\; L_\varepsilon \cdot q_\varepsilon(z_1,z_2) \;+\; \widehat{\omega}_\varepsilon(t_2 - t_1)$
   for all $0 \le t_1 \le t_2 \le T$ *and* $z_1, z_2 \in E$, *and* $\widehat{\omega}_\varepsilon(s) \searrow 0$ *for* $s \downarrow 0$.

Then, $q_\varepsilon\Big(x(t),\; y(t)\Big) \;\le\; q_\varepsilon\Big(x(0),\; y(0)\Big) \cdot e^{(L_\varepsilon + M_\varepsilon)\cdot t}$ *for all* $t \in [0,T[$ *and* $\varepsilon \in \mathcal{J}$.

Proposition 0.6.11 *Assume for* $f : E \times [0,T] \longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon))$, $x, y : [0,T[\longrightarrow E$

1. $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$, $\overset{\circ}{y}(\cdot) \ni f(y(\cdot), \cdot)$ *in* $[0,T[$,
2. $M_\varepsilon \;:=\; \sup_{t,z}\; \alpha^{\mapsto}_\varepsilon(f(z,t)) \;<\; \infty$,
3. $\exists\; \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :\quad Q^{\mapsto}_\varepsilon\Big(f(z_1,t_1),\; f(z_2,t_2)\Big) \;\le\; L_\varepsilon \cdot q_\varepsilon(z_1,z_2) \;+\; \widehat{\omega}_\varepsilon(t_2 - t_1)$
   for all $0 \le t_1 \le t_2 \le T$ *and* $z_1, z_2 \in E$, *and* $\widehat{\omega}_\varepsilon(s) \searrow 0$ *for* $s \downarrow 0$.

Furthermore suppose for each $t \in [0,T[$ *that the infimum*
$$\varphi_\varepsilon(t) \;:=\; \inf_{z\in D}\; \Big(q_\varepsilon(z,\; x(t)) \;+\; q_\varepsilon(z,\; y(t))\Big) \;<\; \infty$$
can be approximated by a minimizing sequence $(z_j)_{j\in I\!N}$ *in* D *and* $h_j \downarrow 0$ *with*
$$\frac{\sup_{k\ge j}\quad q_\varepsilon(z_j, z_k)}{\mathcal{T}_\Theta(f(z_j,t),\; z_j)} \;\longrightarrow\; 0 \qquad\qquad (j \longrightarrow \infty).$$
Then, $\varphi_\varepsilon(t) \;\le\; \varphi_\varepsilon(0)\; e^{(L_\varepsilon + M_\varepsilon)\cdot t}$.

Proving the last proposition, the basic idea consists in estimating both

$$h \longmapsto q_\varepsilon\Big(f(z_m,t)\,(h,z_m),\;\; x(t+h)\Big) \quad \text{and} \quad h \longmapsto q_\varepsilon\Big(f(z_m,t)\,(h,z_m),\;\; y(t+h)\Big)$$

(for small $h > 0$) with such a minimizing sequence $(z_m)_{m\in\mathbb{N}}$. Here assumptions about the time parameter $\mathcal{T}_\Theta(\cdot,\cdot) > 0$ are required for the first time. Roughly speaking, we need lower bounds of $\mathcal{T}_\Theta(f(z_m,t), z_m)$ for "preserving" the information while $m \longrightarrow \infty$. If $\mathcal{T}_\Theta(f(z_m,t), z_m)$ vanishes too quickly, then the comparison with $f(z_m,t)\,(\cdot, z_m)$ cannot be put into practice long enough for proving estimates that (might) imply uniqueness of solutions. Details and proofs are presented in § 2.3.3.

Mutational analysis of Jean–Pierre Aubin ([2, Aubin 99]) on a metric space (M,d) was sketched in § 0.3 and proves to be a special case of this concept on $(E,\, D,\, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$.

Indeed, define $E := M$, $D := M$, $\mathcal{J} := \{0\}$ and $q_0 := d$. Then, every transition ϑ on the metric space (M,d) (in the sense of Aubin) is a forward transition on (M,M,d) according to Definition 0.6.1 and, we set $\alpha_0^{\mapsto}(\vartheta) := \alpha(\vartheta)$, $\mathcal{T}_\Theta(\vartheta,x) := 1$ for all $x \in M$. Furthermore the key estimate

$$d\Big(\vartheta(h,\,x),\;\; \tau(h,\,y)\Big) \;\le\; d(x,y)\,\cdot\, e^{\alpha(\vartheta)\,h} \;+\; h\;D(\vartheta,\tau)\,\cdot\,\frac{e^{\alpha(\vartheta)\,h}-1}{\alpha(\vartheta)\;h} \qquad (*)$$

for arbitrary points $x,y \in M$ and time $h \in [0,1[$ has already been mentioned in § 0.3 and implies now several properties : Firstly, $Q_0^{\mapsto}(\vartheta,\tau) \le D(\vartheta,\tau)$ for all transitions ϑ,τ, Secondly, a transition ϑ belongs to the mutation of $x(\cdot) : [0,T[\longrightarrow M$ at time $t \in [0,T[$ (in the sense of Aubin), i.e.

$$\limsup_{h\downarrow 0}\;\; \tfrac{1}{h}\cdot d\Big(\vartheta(h,\,x(t)),\;\; x(t+h)\Big) \;=\; 0,$$

if and only if it satisfies

$$\limsup_{h\downarrow 0}\;\; \tfrac{1}{h}\,\Big(d\Big(\vartheta(h,\,y),\qquad x(t+h)\Big) \;-\; d(y,\,x(t))\,\cdot\, e^{\alpha(\vartheta)\cdot h}\Big) \;\le\; 0,$$

for all $y \in M$. Thus for a given function f from M to a set of transitions on (M,d), a Lipschitz continuous curve $x(\cdot) : [0,T[\longrightarrow M$ is solution of the mutational equation $\overset{\circ}{x}(t) \ni f(x(t))$ in the sense of Aubin if and only if it is a right–hand forward solution of the generalized mutational equation $\overset{\circ}{x}(t) \ni f(x(t))$ according to Definition 0.6.5. So Aubin's existence theorem 0.3.1 results directly from Theorem 0.6.7 in combination with Proposition 0.6.10.

0.7 Example : First–order geometric evolutions.

The concept of forward right–hand solutions extends mutational analysis of Jean–Pierre Aubin to ostensible metric spaces. As an immediate consequence, we can apply it to some geometric evolutions mentioned in § 0.3. An example consists of $(\mathcal{K}(I\!R^N), d\!\!l)$ (i.e. the nonempty compact subsets of $I\!R^N$ supplied with Pompeiu–Hausdorff distance $d\!\!l$) and the reachable sets $\vartheta_F : [0,1] \times \mathcal{K}(I\!R^N) \longrightarrow \mathcal{K}(I\!R^N)$ of differential inclusions $\dot{x}(\cdot) \in F(x(\cdot))$ a.e.

The so–called *Pompeiu–Hausdorff excess* is a first example of an ostensible metric on $\mathcal{K}(I\!R^N)$ that is very similar to the Pompeiu–Hausdorff distance $d\!\!l$, but not symmetric :

$$\begin{aligned} e^{\subset}(K_1, K_2) &:= \sup_{x \in K_1} \operatorname{dist}(x, K_2) \\ e^{\supset}(K_1, K_2) &:= \sup_{y \in K_2} \operatorname{dist}(y, K_1). \end{aligned}$$

for $K_1, K_2 \in \mathcal{K}(I\!R^N)$. Obviously, the link to the Pompeiu–Hausdorff distance is

$$d\!\!l(K_1, K_2) = \max\{ e^{\subset}(K_1, K_2), \quad e^{\supset}(K_1, K_2)\}$$

(see [2, Aubin 99], § 3.2 and [55, Rockafellar, Wets 98], § 4.C, for example).
$(\mathcal{K}(I\!R^N), \ e^{\supset})$ is two–sided sequentially compact since the metric space $(\mathcal{K}(I\!R^N), d\!\!l)$ is known to be compact (due to Corollary 4.1.2). Furthermore Filippov's Theorem A.1.2 about trajectories of differential inclusions implies

$$e^{\supset}\Big(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2)\Big) \ \le\ \Big(e^{\supset}(K_1, K_2) \ +\ t \cdot \sup_{I\!R^N} e^{\supset}\Big(F(\cdot), G(\cdot)\Big)\Big) \cdot e^{\lambda \cdot t}$$

for initial sets $K_1, K_2 \in \mathcal{K}(I\!R^N)$ and λ–Lipschitz maps $F, G : I\!R^N \rightsquigarrow I\!R^N$ with nonempty convex compact values and $\sup_{I\!R^N} e^{\supset}\Big(F(\cdot), G(\cdot)\Big) < \infty$ (according to Proposition 4.4.1). So these reachable sets induce forward transitions on the tuple $(\mathcal{K}(I\!R^N),\ \mathcal{K}(I\!R^N),\ e^{\supset})$ (as stated in Corollary 4.4.3) and, we conclude from Theorem 0.6.7

Proposition 0.7.1 *Let f be a function from $\mathcal{K}(I\!R^N) \times [0,T]$ to λ–Lipschitz maps $I\!R^N \rightsquigarrow I\!R^N$ with uniformly bounded, convex values in $\mathcal{K}(I\!R^N)$. Suppose*

$$\sup_{I\!R^N} e^{\supset}\left(f(K_1, t_1)(\cdot),\ \ f(K_2, t_2)(\cdot)\right) \ \le\ \omega\left(e^{\supset}(K_1, K_2) \ +\ t_2 - t_1\right)$$

for all $K_1, K_2 \in \mathcal{K}(I\!R^N)$ and $0 \le t_1 \le t_2 \le T$ with the modulus $\omega(\cdot)$ of continuity.

Then for every initial set $K_0 \in \mathcal{K}(I\!R^N)$, there exists a right–hand forward solution $K : [0, T[\longrightarrow (\mathcal{K}(I\!R^N),\ e^{\supset})$ of the generalized mutational equation $\overset{\circ}{K}(\cdot) \ni f(K(\cdot), \cdot)$ in $[0, T[$ with $K(0) = K_0$.

Assuming Lipschitz continuity of f with respect to the first arguments, we obtain even a relation between the Euler solution $K(\cdot)$ (that is constructed when proving Theorem 0.6.7) and any other right–hand forward solution with the same initial value. Roughly speaking, $K(t)$ is the largest subset of $I\!\!R^N$ among these solutions at time t. The proof is based on a further approach to estimates that is differing slightly from Proposition 0.6.11. We postpone the details to Proposition 2.3.10 and just state the immediate consequence of Corollary 4.4.4 :

Corollary 0.7.2 *In addition to the assumptions of Proposition 0.7.1, suppose that there exist* $L \geq 0$ *and a modulus* $\omega(\cdot)$ *of continuity with*

$$\sup_{I\!\!R^N} \; e^{\supset}\left(f(K_1,t_1)(\cdot),\; f(K_2,t_2)(\cdot)\right) \;\leq\; L \cdot e^{\supset}(K_1,K_2) \;+\; \omega(t_2-t_1)$$

for all $K_1, K_2 \in \mathcal{K}(I\!\!R^N)$ *and* $0 \leq t_1 \leq t_2 \leq T$.
Let $K(\cdot) : [0,T[\longrightarrow (\mathcal{K}(I\!\!R^N),\, e^{\supset})$ *be an Euler solution (i.e. it is constructed by Euler method according to the detailed proof of Proposition 0.6.7 given in Proposition 2.3.5).*

Then every other solution $M(\cdot) : [0,T[\longrightarrow (\mathcal{K}(I\!\!R^N),\, e^{\supset})$ *with* $M(0) = K(0)$ *fulfills*

$$\limsup_{h\downarrow 0} \; e^{\supset}(K(t),\; M(t+h)) \;=\; 0 \qquad \text{for all } t \in [0,T[.$$

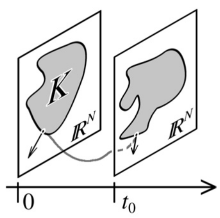
If $M(\cdot)$ *is continuous even with respect to* $d\!\!l$, *then* $M(t) \subset K(t)$ *for all* $t \in [0,T[$.

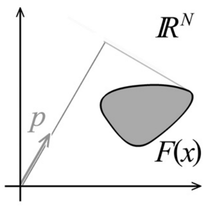
In regard to first–order geometric evolutions, we now use the ostensible metric $q_{\mathcal{K},N}$ that has already been suggested in § 0.5 :

$$\begin{aligned} q_{\mathcal{K},N}: \quad \mathcal{K}(I\!\!R^N) \times \mathcal{K}(I\!\!R^N) \;&\longrightarrow\; [0,\infty[\\ (K_1,\; K_2) \;&\longmapsto\; d\!\!l(K_1,K_2) \;+\; e^{\supset}\Big(\text{Graph } {}^{\flat}N_{K_1},\;\; \text{Graph } {}^{\flat}N_{K_2}\Big) \end{aligned}$$

with $N_K(x)$ denoting the limiting normal cone of $K \subset I\!\!R^N$ at $x \in \partial K$ (Def. 4.1.4),
${}^{\flat}N_K(x) \;:=\; N_K(x) \cap I\!\!B_1$.
Moreover, reachable sets of differential inclusions $\dot{x}(\cdot) \in F(x(\cdot))$ again form the basis for constructing forward transitions on $(\mathcal{K}(I\!\!R^N),\; \mathcal{K}_{C^{1,1}}(I\!\!R^N),\; q_{\mathcal{K},N})$.

First we focus on the evolution of limiting normal cones at the topological boundary. In connection with the obstacles due to normal cones (in § 0.5), the *Hamilton condition* has already been mentioned as a key tool.
It implies that roughly speaking, every boundary point x_0 of $\vartheta_F(t_0,K)$ and normal vector $\nu \in N_{\vartheta_F(t_0,K)}(x_0)$ have a trajectory and an adjoint arc linking x_0 to some $z \in \partial K$ and ν to $N_K(z)$, respectively.





Furthermore the trajectory and its adjoint arc fulfill a system of partial differential equations with the so–called *Hamiltonian function* of $F : I\!R^N \rightsquigarrow I\!R^N$,

$$\mathcal{H}_F : I\!R^N \times I\!R^N \longrightarrow I\!R^N, \quad (x,p) \longmapsto \sup_{y \in F(x)} p \cdot y$$

Although Proposition A.3.2 (2.) states this result in a very general form, we use only the following "smooth" version — due to later regularity conditions on F :

Proposition 0.7.3 *Suppose for the set–valued map* $F : I\!R^N \rightsquigarrow I\!R^N$

1. $F(\cdot)$ *has nonempty convex compact values,*
2. $\mathcal{H}_F(\cdot,\cdot)$ *is continuously differentiable on* $I\!R^N \times (I\!R^N \setminus \{0\})$,
3. *the derivative of* $\mathcal{H}_F$ *has linear growth on* $I\!R^N \times (I\!R^N \setminus I\!B_1)$, *i.e.* $\|D\mathcal{H}_F(x,p)\| \le const \cdot (1 + |x| + |p|)$ *for all* $x, p \in I\!R^N$, $|p| > 1$.

Let $K \in \mathcal{K}(I\!R^N)$ *be any initial set and* $t_0 > 0$.

For every boundary point $x_0 \in \partial\,\vartheta_F(t_0, K)$ *and normal vector* $\nu \in N_{\vartheta_F(t_0,K)}(x_0)$, *there exist a trajectory* $x(\cdot) \in C^1([0,t_0], I\!R^N)$ *and its adjoint* $p(\cdot) \in C^1([0,t_0], I\!R^N)$ *with*

$$\begin{cases} \dot{x}(t) = \frac{\partial}{\partial p}\,\mathcal{H}_F(x(t),\, p(t)) \in F(x(t)), & x(t_0) = x_0, \quad x(0) \in \partial K, \\ \dot{p}(t) = -\frac{\partial}{\partial x}\,\mathcal{H}_F(x(t),\, p(t)), & p(t_0) = \nu, \quad p(0) \in N_K(x(0)). \end{cases}$$

In short, the graph of normal cones at time t, i.e. Graph $N_{\vartheta_F(t,K)}(\cdot)|_{\partial\,\vartheta_F(t,K)}$, can be traced back to the beginning by means of the Hamiltonian system with $\mathcal{H}_F$.

These assumptions give a first hint about adequate conditions on $F : I\!R^N \rightsquigarrow I\!R^N$ for inducing forward transitions on $(\mathcal{K}(I\!R^N),\, \mathcal{K}_{C^{1,1}}(I\!R^N),\, q_{\mathcal{K},N})$. So the next question is whether these features are already sufficient.
An essential demand is that smooth compact subsets of $I\!R^N$ stay smooth for short times, i.e. strictly speaking, $\vartheta_F(t,K) \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ for each $K \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ and all small t. Considering the graph of unit normal vectors, i.e. Graph $\left(N_{\vartheta_F(t,K)}(\cdot) \cap \partial I\!B_1\right)\Big|_{\partial\,\vartheta_F(t,K)}$, it is equivalent to the request that the graph of a Lipschitz continuous function is preserving this property for a short time (at least). In [21, Caroff, Frankowska 96] and [37, Frankowska 2002], sufficient conditions are presented for a Hamiltonian system with given end points. Adapting its indirect proof for initial value problems, we obtain the autonomous version of Proposition A.4.6 :

Lemma 0.7.4 *Suppose for* $H : I\!\!R^N \times I\!\!R^N \longrightarrow I\!\!R$, $T > 0$ *and the Hamiltonian system*

$$\wedge \begin{cases} \dot{y}(t) &= \quad \frac{\partial}{\partial q} H(y(t),\, q(t)) \\ \dot{q}(t) &= -\frac{\partial}{\partial y} H(y(t),\, q(t)) \end{cases}$$

(*i*) $H(\cdot,\cdot)$ *is differentiable with locally Lipschitz continuous derivative,*

(*ii*) *every solution* $(y(\cdot), q(\cdot))$ *of the Hamiltonian system can be extended to* $[0, T]$ *and depends continuously on the initial data.*

Let M_0 *denote the graph of a Lipschitz continuous function* $K \longrightarrow I\!\!R$ *with* $K \in \mathcal{K}(I\!\!R^N)$ *and,* $M_t \subset I\!\!R^N \times I\!\!R$ *abbreviates the evolution of the initial set* M_0 *along the Hamiltonian system at time* $t \in [0, T]$.

Then there exist $\delta > 0$ *and* $\Lambda > 0$ *such that* M_t *is the graph of a* Λ*–Lipschitz continuous function for every* $t \in [0, \delta]$.

So applying this lemma to unit normals to reachable sets $\vartheta_F(t, K)$ of $K \in \mathcal{K}_{C^{1,1}}(I\!\!R^N)$ requires stronger conditions on $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ than the assumptions of Proposition 0.7.3. Indeed, the Hamiltonian $\mathcal{H}_F$ has to be in $C^{1,1}(I\!\!R^N \times (I\!\!R^N \setminus \{0\}))$ instead of C^1.
In fact, this lemma is a reason for choosing $\mathcal{K}_{C^{1,1}}(I\!\!R^N)$ as "test subset" of $\mathcal{K}(I\!\!R^N)$ — instead of compact sets with C^1 boundary, for example.
Motivated by recent results, we now introduce an abbreviation for these set–valued maps.

Definition 0.7.5 *For* $\lambda > 0$, $\mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!\!R^N, I\!\!R^N)$ *contains all* $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ *with*

1. $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ *has compact convex values,*
2. $\mathcal{H}_F(\cdot,\cdot) \in C^{1,1}(I\!\!B_R \times (I\!\!B_R \setminus \overset{\circ}{I\!\!B}_{\frac{1}{R}}))$ *for every radius* $R > 1$,
3. $\|\mathcal{H}_F\|_{C^{1,1}(I\!\!R^N \times \partial I\!\!B_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_F\|_{C^1(I\!\!R^N \times \partial I\!\!B_1)} + \mathrm{Lip}\; D\mathcal{H}_F|_{I\!\!R^N \times \partial I\!\!B_1} < \lambda\,.$

The key conclusion of Lemma 0.7.4 is that for all $F \in \mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!\!R^N, I\!\!R^N)$, $K \in \mathcal{K}_{C^{1,1}}(I\!\!R^N)$, there exist a time $\tau = \tau(F, K) > 0$ and a radius $\rho = \rho(F, K) > 0$ such that $\vartheta_F(t, K)$ is in $\mathcal{K}_{C^{1,1}}(I\!\!R^N)$ for each $t \in [0, \tau]$ and its radius of curvature has the lower bound ρ (see Proposition A.4.4). Moreover Proposition A.4.10 provides a lower estimate of $\tau(F, K) > 0$ if $\mathcal{H}_F$ is even *twice* continuously differentiable on $I\!\!R^N \times (I\!\!R^N \setminus \{0\})$.

The advantages of $\mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!\!R^N, I\!\!R^N)$ go beyond preserving regularity of $K \in \mathcal{K}_{C^{1,1}}(I\!\!R^N)$. Considering $\vartheta_F(t, K)$ $(0 \le t \le \tau)$, the lower bound $\rho(F, K) > 0$ of the radii of curvature implies that roughly speaking, "holes" of K cannot disappear up to time τ (due to Proposition A.2.10). Thus, the evolution of K is reversible in time, i.e. the initial set K can be reconstructed from each $\vartheta_F(t, K)$. To be more precise, Corollary A.5.2 ensures :

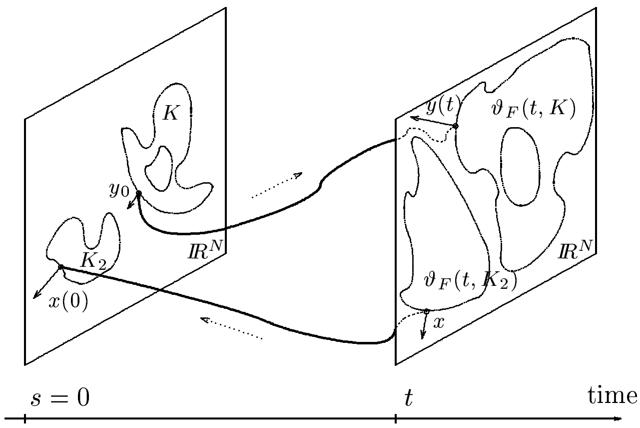
Proposition 0.7.6 *Let* $F : I\!R^N \rightsquigarrow I\!R^N$ *be a map of* $\mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$.

For every compact N–dimensional $C^{1,1}$ *submanifold* K *of* $I\!R^N$ *with boundary, there exist a time* $\tau > 0$ *and a radius* $\rho > 0$ *such that for all* $t \in [0, \tau[$,

1. $\vartheta_F(t, K) \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ *with radius of curvature* $\geq \rho$,
2. $K = I\!R^N \Big\backslash \vartheta_{-F}(t,\ I\!R^N \setminus \vartheta_F(t, K))$.

Statement (2.) provides a connection between the boundaries ∂K and $\partial\,\vartheta_F(t, K)$ — both in forward and backward time direction.

So applying Proposition 0.7.3 about adjoint arcs to $-F$ (instead of F), every boundary point $y_0 \in \partial K$ and each normal vector $\nu \in N_K(y_0)$ have a trajectory $y(\cdot)$ in the boundary and an adjoint arc up to time t.



This additional feature removes the second obstacle mentioned in § 0.5, i.e. we can estimate the distance

$$q_{\mathcal{K},N}\Big(\vartheta_F(t, K),\ \vartheta_F(t, K_2)\Big)$$

by means of $q_{\mathcal{K},N}(K, K_2)$ for all sets $K \in \mathcal{K}_{C^{1,1}}(I\!R^N)$, $K_2 \in \mathcal{K}(I\!R^N)$ and every time $t > 0$ sufficiently small.

Now we have achieved the goal of finding (nontrivial) forward transition on the tuple $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$. Indeed, the reachable sets ϑ_F of $F \in \mathrm{LIP}^{(\mathcal{H})}(I\!R^N, I\!R^N)$ prove to satisfy the conditions (of Definition 0.6.1) — as a consequence of the preceding results in detail. In particular, the time $\tau = \tau(F, K) > 0$ mentioned in Proposition 0.7.6 plays the role of $\mathcal{T}_\Theta(\vartheta_F, K)$ for each $K \in \mathcal{K}_{C^{1,1}}(I\!R^N)$. Lemma 4.4.23 and 4.4.25 state

Lemma 0.7.7 *Assume* $F, G \in \mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$, $K_1 \in \mathcal{K}_{C^{1,1}}(I\!R^N)$, $K_2, K \in \mathcal{K}(I\!R^N)$ *and* $0 < T < \mathcal{T}_\Theta(\vartheta_F, K_1)$. *Then, for every* $0 \leq s \leq t \leq T$,

$$q_{\mathcal{K},N}\Big(\vartheta_F(s, K),\ \vartheta_F(t, K)\Big) \ \leq\ \lambda\,(e^{\lambda T} + 2)\,\cdot\,(t - s)$$

$$q_{\mathcal{K},N}\Big(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2)\Big) \ \leq\ e^{(\Lambda_F + \lambda)\ t}\ \cdot\ \Big(q_{\mathcal{K},N}(K_1,\ K_2)\ +\ 4\,N\,t\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(D)}\Big)$$

with
$$\Lambda_F := 9\,e^{2\,\lambda\,t}\ \|\mathcal{H}_F\|_{C^{1,1}(D)} \ \leq\ 9\,e^{2\,\lambda\,t}\ \lambda,$$
$$D := I\!R^N \times \partial I\!B_1 \ \subset\ I\!R^N \times I\!R^N.$$

Proposition 0.7.8 *For every* $\lambda > 0$, *the reachable sets of the set–valued maps in* $\mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$ *induce forward transitions on* $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$ *with*

$$\begin{aligned} \alpha^{\mapsto}(\vartheta_F) &\overset{\text{Def.}}{=} 10\ \lambda \\ \beta(\vartheta_F)\,(t) &\overset{\text{Def.}}{=} \lambda\ \ (e^\lambda + 2)\ \cdot\ t, \\ Q^{\mapsto}(\vartheta_F, \vartheta_G) &\leq 4\ N\ \ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N \times \partial I\!B_1)}\,. \end{aligned}$$

Proof is given in Proposition 4.4.26.

In regard to the existence of right–hand forward solutions due to Theorem 0.6.7, we now focus on the assumption of transitional compactness. To be more precise, adequate subsets $\mathcal{K}_\circ(I\!R^N) \subset \mathcal{K}(I\!R^N)$, $\Theta \subset \mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$ are wanted such that $\mathcal{K}_\circ(I\!R^N)$ is transitionally compact in $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ \Theta)$.
Definition 0.6.9 provides the following condition on $\mathcal{K}_\circ(I\!R^N),\ \Theta$:

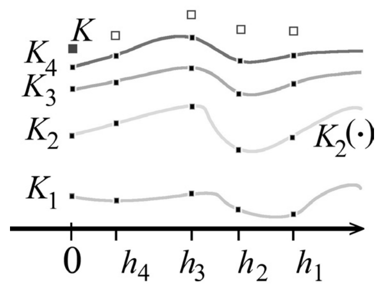
Let $(K_n)_{n \in I\!N}$, $(h_j)_{j \in I\!N}$ be sequences in $\mathcal{K}_\circ(I\!R^N)$, $]0,1[$, respectively with $h_j \downarrow 0$, $\sup_n\ q_{\mathcal{K},N}(I\!B_1, K_n) < \infty$. Suppose each $G_n : [0,1] \longrightarrow \Theta$ to be piecewise constant $(n \in I\!N)$ and set

$$\widetilde{G}_n\ :\ [0,1] \times I\!R^N \rightsquigarrow I\!R^N,\ \ (t,x) \longmapsto G_n(t)(x),$$

$$K_n(h)\ :=\ \vartheta_{\widetilde{G}_n}(h, K_n) \qquad \text{for } h \geq 0.$$

Then there exist a sequence $n_k \nearrow \infty$ of indices and $K \in \mathcal{K}(I\!R^N)$ satisfying

$$\begin{aligned} \limsup_{k \longrightarrow \infty}\ \ q_{\mathcal{K},N}(K_{n_k}(0),\ \ K) &= 0, \\ \limsup_{j \longrightarrow \infty}\ \sup_{k \geq j}\ \ q_{\mathcal{K},N}(K,\ \ K_{n_k}(h_j)) &= 0. \end{aligned}$$



The first condition, i.e. $q_{\mathcal{K},N}(K_{n_k},\ K) \longrightarrow 0\ \ (k \longrightarrow \infty)$, is not difficult to guarantee. Indeed, $(K_n)_{n \in I\!N}$ is bounded with respect to the Pompeiu–Hausdorff distance $d\!l$ and thus has a subsequence converging to a set K in $(\mathcal{K}(I\!R^N), d\!l)$. For the sake of simplicity, we abbreviate it as $(K_n)_{n \in I\!N}$ again. Moreover, it is well–known in nonsmooth analysis that every limiting normal vector of K can be approximated by normal vectors of K_n $(n \in I\!N)$ (see Proposition 4.1.6 and its Corollary 4.1.7). Thus, $q_{\mathcal{K},N}(K_n,\ K) \longrightarrow 0$.

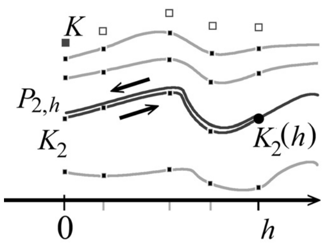
The properties of $q_{\mathcal{K},N}(K,\ K_n(h_j))$ for large $j, n \in I\!N$ are not so easy to handle. Investigating this question, we sketch some interesting results about the regularity of reachable sets. Probably the so–called *sets of positive erosion* are the the most regular subsets to be expected in general if topological changes (like "holes" disappearing) are not excluded a priori.

Roughly speaking, any sequence in $\mathcal{K}(I\!\!R^N)$ might have "topological holes" disappearing. In a figurative sense, information about the topological boundary is sometimes lost while the index is increasing.
In regard to transitional compactness, we are interested in which "information" on the initial boundaries ∂K_n $(n \in I\!\!N)$ is still relevant at a fixed time $h > 0$ (because only these features have to be taken into consideration when comparing with ∂K for $n \to \infty$).

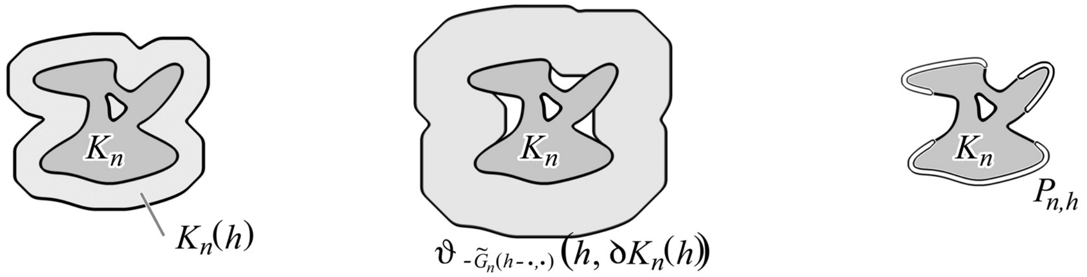
For $h > 0$ fixed and each $n \in I\!\!N$, some of the boundary points in ∂K_n can be connected with $\partial K_n(h)$ by a trajectory of $\widetilde{G}_n : [0,1] \times I\!\!R^N \rightsquigarrow I\!\!R^N$. They form the closed set

$$P_{n,h} \;:=\; K_n \,\cap\, \vartheta_{-\widetilde{G}_n(h-\,\cdot,\,\cdot)}(h,\, \partial\, K_n(h)) \;\subset\; \partial K_n.$$

Since G_{n_k} is assumed to be piecewise constant with values in $\mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!\!R^N, I\!\!R^N)$, Proposition 0.7.3 about adjoint arcs leads to



$$\mathbb{e}^{\supset}\Big(\mathrm{Graph}\ {}^\flat N_{K_n}\Big|_{P_{n,h}},\ \ \mathrm{Graph}\ {}^\flat N_{K_n(h)}\Big) \;\le\; \mathrm{const}(\lambda)\,\cdot\,h,$$
$$\mathbb{d}(K_n,\, K_n(h)) \;\le\; \mathrm{const}(\lambda)\,\cdot\,h.$$



So we achieve the goal of transitional compactness if we find sufficient conditions for

$$\mathbb{e}^{\supset}\Big(\mathrm{Graph}\ {}^\flat N_K,\ \ \mathrm{Graph}\ {}^\flat N_{K_n}\Big|_{P_{n,h}}\Big) \;\longrightarrow\; 0 \qquad\qquad (n \longrightarrow \infty)$$

with the fixed time $h > 0$ (sufficiently small).
Indeed, the triangle inequalities of $\mathbb{d}$ and $\mathbb{e}^{\supset}$ then imply

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\ \ \mathbb{d}\,(K,\, K_n(h)) \;&\le\; \limsup_{n\longrightarrow\infty}\ \Big(\mathbb{d}(K,\,K_n) \,+\, \mathbb{d}(K_n,\,K_n(h))\Big)\\
&\le\; 0 \,+\, \mathrm{const}(\lambda)\cdot h,\\
\limsup_{n\longrightarrow\infty}\ \ \mathbb{e}^{\supset}\Big(\mathrm{Graph}\ {}^\flat N_K,\ \mathrm{Graph}\ {}^\flat N_{K_n(h)}\Big) \;&\le\; 0 \,+\, \mathrm{const}(\lambda)\cdot h
\end{aligned}$$

and, we obtain a subsequence $(K_{n_k})_{k\,\in\, I\!\!N}$ such that

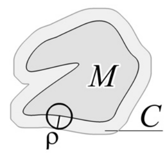
$$q_{\mathcal{K},N}\,(K,\; K_{n_k}(h)) \;\le\; \tfrac{1}{k} \,+\, \mathrm{const}(\lambda)\cdot h.$$

Choosing now a sequence $(h_j)_{j\,\in\,I\!\!N}$ with $h_j \searrow 0$ as time $h > 0$, the Cantor diagonal construction provides a subsequence again denoted by $(K_{n_k})_{k\,\in\,I\!\!N}$ satisfying

$$\limsup_{j\,\longrightarrow\,\infty}\ \sup_{k\,\ge\, j}\ q_{\mathcal{K},N}(K,\;K_{n_k}(h_j)) \;=\; 0.$$

An approach to these sufficient conditions is based on an interior sphere condition. Indeed, for $h > 0$ fixed and each $n \in I\!N$, the points of $P_{n,h}$ have in common that they belong to the boundary of $\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h, \partial K_n(h))$. Thus, we are interested in further regularity properties of such reachable sets.
On the one hand, we want to use geometric criteria without analytical restrictions and, on the other hand topological changes (like "holes" disappearing) must not be excluded a priori. The interior sphere condition proves to be an adequate synthesis and leads to the so–called *sets of positive erosion.*

Definition 0.7.9 *A closed subset* $C \subset I\!R^N$ *is said to have* positive erosion of radius $\rho > 0$ *if there exists a closed set* $M \subset I\!R^N$ *with* $C = \{x \in I\!R^N \mid dist(x, M) \leq \rho\}$
or equivalently, if it holds the interior sphere condition *of radius* ρ*, i.e. each* $x \in \partial C$ *has a ball* $B \subset I\!R^N$ *of radius* ρ *with* $x \in B \subset C$.
$\mathcal{K}^\rho_\circ(I\!R^N)$ *consists of all sets with positive erosion of radius* $\rho > 0$ *and, set* $\mathcal{K}_\circ(I\!R^N) := \bigcup_{\rho > 0} \mathcal{K}^\rho_\circ(I\!R^N)$.



The morphological term "erosion" is motivated by the fact that a set $C = \overline{C^\circ} \subset I\!R^N$ has positive erosion of radius $\rho > 0$ if and only if the closure $\overline{I\!R^N \setminus C}$ of its complement has *positive reach* in the sense of Federer ([35, Federer 59]) (proven in Corollary 4.3.3). This relation implies a collection of interesting regularity properties summarized in § 4.3. In fact, sets of positive erosion exemplify the most regular subsets of $I\!R^N$ that we can expect in this context.

The essential tool consists in sufficient conditions on a set–valued map ensuring that its reachable sets have positive erosion.
For every $\lambda, \rho > 0$, $\mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!R^N, I\!R^N)$ denotes the subset of $\mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$ with two additional conditions : Each $F \in \mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!R^N, I\!R^N)$ has compact convex values *with positive erosion of radius* ρ and, the Hamiltonian of F is even *twice* continuously differentiable.

Definition 0.7.10 *For any* $\lambda > 0$ *and* $\rho > 0$*, the set* $\mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!R^N, I\!R^N)$ *consists of all set–valued maps* $F : I\!R^N \leadsto I\!R^N$

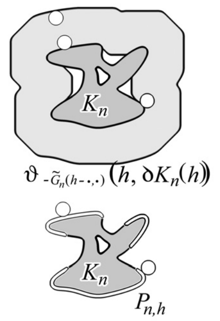
1. $F : I\!R^N \leadsto I\!R^N$ *has compact convex values in* $\mathcal{K}^\rho_\circ(I\!R^N)$.
2. $\mathcal{H}_F(\cdot,\cdot) \in C^2(I\!B_R \times (I\!B_R \setminus \overset{\circ}{I\!B}_{\frac{1}{R}}))$ *for every radius* $R > 1$,
3. $\|\mathcal{H}_F\|_{C^{1,1}(I\!R^N \times \partial I\!B_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_F\|_{C^1(I\!R^N \times \partial I\!B_1)} + \mathrm{Lip}\ D\mathcal{H}_F|_{I\!R^N \times \partial I\!B_1} < \lambda$.

Proposition 0.7.11 *For* $F_1 \ldots F_m \in \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(I\!\!R^N, I\!\!R^N)$, $\lambda,\ \rho > 0$ *and a partition* $0 \le \tau_0 < \tau_1 < \ldots < \tau_m = 1$ *of* $[0,1]$, *define the map* $\widetilde{G} : [0,1[\, \times I\!\!R^N \rightsquigarrow I\!\!R^N$ *as* $\widetilde{G}(t,x) := F_j(x)$ *for* $\tau_{j-1} \le t < \tau_j$. *Furthermore choose* $C \in \mathcal{K}(I\!\!R^N)$ *arbitrarily.*

Then there exist $\sigma > 0$ *and a time* $\widehat{\tau} \in\ ]0,1]$ *(depending only on* λ, ρ, C*) such that the reachable set* $\vartheta_{\widetilde{G}}(t, x_0)$ *has positive erosion of radius* $\sigma\, t$ *for any* $t \in\ ]0, \widehat{\tau}[$, $x_0 \in C$. *As an immediate consequence,* $\vartheta_{\widetilde{G}}(t, K_1)$ *has positive erosion of radius* $\sigma\, t$ *for all* $t \in]0, \widehat{\tau}[$ *and each initial subset* $K_1 \in \mathcal{K}(I\!\!R^N)$ *of* C.

Proof is also based on the Hamiltonian system for trajectories and their adjoint arcs (similar to Proposition 0.7.3). The key point now is to benefit from its symmetry and thus exchange the components (in the sense of Lemma A.7.3).
Roughly speaking, each normal vector is related with the same initial point $x_0 \in C$. So we choose the graph of a constant function as initial value and apply the tools of Hamiltonian systems obtaining the existence of smooth solutions up to time $\widehat{\tau}$ (at least). Details are presented in Proposition A.7.2.

In regard to transitional compactness, let us now return to the sequences $(K_n)_{n \in I\!\!N}$, $(\widetilde{G}_n)_{n \in I\!\!N}$ and the sets $P_{n,h} \stackrel{\text{Def.}}{=} K_n \cap \vartheta_{-\widetilde{G}_n(h - \cdot, \cdot)}(h,\ \partial K_n(h)) \subset \partial K_n \quad (n \in I\!\!N)$ for time $h \in]0,1]$ fixed. We have already found $K \in \mathcal{K}(I\!\!R^N)$ with $q_{\mathcal{K},N}(K_n, K) \longrightarrow 0$ $(n \longrightarrow \infty)$ and thus $d\!\!l(K_n, K) \longrightarrow 0$, in particular. So there exists $C \in \mathcal{K}(I\!\!R^N)$ large enough such that $K_n(s) \subset C$ for all $n \in I\!\!N$ and $s \in [0,1]$.
Suppose that $\widetilde{G}_n$ has all values in $\mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(I\!\!R^N, I\!\!R^N)$ with $\lambda, \rho > 0$. Proposition 0.7.11 provides $\sigma > 0$, $\widehat{\tau} \in]0,1]$ depending only on λ, ρ, C and, we assume $0 < h < \widehat{\tau}$ in addition. Then, $\vartheta_{-\widetilde{G}_n(h - \cdot, \cdot)}(h,\ \partial K_n(h))$ has positive erosion of radius $\sigma\, h$ and thus, K_n satisfies the exterior sphere condition (of radius $\sigma\, h$ not depending on n) at every point of $P_{n,h} \subset \partial K_n$.



If in addition, each K_n has positive erosion of arbitrary radius, we can show indirectly

$$e^{\supset}\Big(\text{Graph } {}^\flat N_K,\quad \text{Graph } {}^\flat N_{K_n}\Big|_{P_{n,h}}\Big) \longrightarrow 0 \qquad (n \longrightarrow \infty)$$

and finally, we obtain the result whose detailed proof is presented in Proposition 4.4.28.

Proposition 0.7.12 *For any* $\lambda, \rho > 0$, *the sets of positive erosion,* $\mathcal{K}_\circ(I\!\!R^N)$, *are transitionally compact in* $\Big(\mathcal{K}(I\!\!R^N),\ q_{\mathcal{K},N},\ \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(I\!\!R^N, I\!\!R^N)\Big)$.

So let us draw the main conclusion about first–order geometric evolution :
The set–valued maps of $\mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(I\!R^N, I\!R^N)$ induce forward transitions according to Proposition 0.7.8 and, together with the transitional compactness of Proposition 0.7.12, it implies the existence of right–hand forward solution due to Theorem 0.6.7 (and the remarks about its proof).

Theorem 0.7.13 *Let* $f : \mathcal{K}(I\!R^N) \times [0,T] \longrightarrow \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(I\!R^N, I\!R^N)$ *satisfy*

$$\left\| \mathcal{H}_{f(K_1,t_1)} - \mathcal{H}_{f(K_2,t_2)} \right\|_{C^1(I\!R^N \times \partial I\!B_1)} \le \omega(q_{\mathcal{K},N}(K_1,K_2) + t_2 - t_1)$$

for all $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $0 \le t_1 \le t_2 \le T$ *with a modulus* $\omega(\cdot)$ *of continuity and consider the reachable sets of maps in* $\mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(I\!R^N, I\!R^N)$ *as forward transitions on* $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$ *according to Proposition 0.7.8.*

Then for every initial set $K_0 \in \mathcal{K}(I\!R^N)$, *there exists a right–hand forward solution* $K : [0,T[\longrightarrow \mathcal{K}(I\!R^N)$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot),\cdot)$ *with* $K(0) = K_0$, *i.e.* $K : [0,T[\longrightarrow (\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ *is Lipschitz continuous in time direction (with Lipschitz constant depending only on* λ*) and, it satisfies*

$$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \left(q_{\mathcal{K},N}\Big(\vartheta_{f(K(t),t)}(h,M),\ K(t+h)\Big) - q_{\mathcal{K},N}(M,\ K(t)) \cdot e^{10\,\lambda\,h} \right) \le 0$$

for all $t \in [0,T[$, $M \in \mathcal{K}_{C^{1,1}}(I\!R^N)$.

It is not obvious how to prove uniqueness results by means of Proposition 0.6.11. Although a lower estimate of $\mathcal{T}_\Theta(f(M,t),M) > 0$ for $M \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ is presented in Proposition A.4.10, it might vanish too quickly for a minimizing sequence in $\mathcal{K}_{C^{1,1}}(I\!R^N)$ (in the sense of Proposition 0.6.11).

Similarly to Proposition 0.4.3, we also obtain existence of solutions for systems of semilinear evolution equations and first–order geometric evolutions (see Prop. 4.6.1).

Theorem 0.7.14 *Suppose :*

1. X *is a reflexive Banach space.*
2. *The linear operator* A *generates a* C^0 *semigroup* $(S(t))_{t \ge 0}$ *on* X.
3. *The dual operator* A' *of* A *has a countable family of eigenvectors* $\{v_j'\}_{j \in \mathcal{J}}$ $(\|v_j'\|_{X'} = 1)$ *spanning the dual space* X', *i.e.* $X' = \overline{\sum_{j \in \mathcal{J}} I\!R\, v_j'}$.
   λ_j *abbreviates the eigenvalue of* A' *belonging to* v_j' *and,* $q_j := q_{v_j'}$.
4. $p_\infty(x,y) := \sum_{k=1}^\infty 2^{-k} \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)}$, $\quad P_\infty(x,y) := \sum_{k=1}^\infty 2^{-k}\ q_{j_k}(x,y)$ *for* $x,y \in X$.

Furthermore assume for

$$\begin{array}{rcl} f: \; X \times \mathcal{K}(I\!R^N) \times [0,T] & \longrightarrow & X \\ g: \; X \times \mathcal{K}(I\!R^N) \times [0,T] & \longrightarrow & \mathrm{LIP}_\Lambda^{(\mathcal{H}^\rho_\circ)}(I\!R^N, I\!R^N) \; : \end{array}$$

5. $\|f\|_{L^\infty} < \infty, \quad 0 < \Lambda, \, \rho < \infty,$

6. $P_\infty\Big(f(x_1, K_1, t_1), \;\; f(x_2, K_2, t_2)\Big) \quad \le \; \omega\Big(p_\infty(x_1, x_2) + q_{\mathcal{K},N}(K_1, K_2) + t_2 - t_1\Big)$

7. $\big\|\mathcal{H}_{g(x_1,K_1,t_1)} - \mathcal{H}_{g(x_2,K_2,t_2)}\big\|_{C^1(I\!R^N \times \partial I\!B_1)} \; \le \; \omega\Big(p_\infty(x_1, x_2) + q_{\mathcal{K},N}(K_1, K_2) + t_2 - t_1\Big)$

for all $x_1, x_2 \in X, \; K_1, K_2 \in \mathcal{K}(I\!R^N), \; 0 \le t_1 \le t_2 \le T$ *with a modulus* $\omega(\cdot)$ *of continuity.*

Then for every $x_0 \in X$ *and* $K_0 \in \mathcal{K}(I\!R^N)$, *there exists a right–hand forward solution* $(x, K) \; : \; [0,T[\; \longrightarrow X \times \mathcal{K}(I\!R^N)$ *of the generalized mutational equations*

$$\wedge \left\{ \begin{array}{rcl} \overset{\circ}{x}(\cdot) & \ni & \Sigma_{f(x(\cdot), \, K(\cdot), \, \cdot)} \\ \overset{\circ}{K}(\cdot) & \ni & g(x(\cdot), \; K(\cdot), \; \cdot) \end{array} \right.$$

with $x(0) = x_0, \;\; K(0) = K_0$ *and, it fulfills*

a) $x : [0,T[\longrightarrow X$ *is a mild solution of the initial value problem*

$$\wedge \left\{ \begin{array}{rcl} \frac{d}{dt}\, x(t) & = & A\, x(t) \; + \; f(x(t), \; K(t), \; t) \\ x(0) & = & x_0 \end{array} \right.$$

i.e. $$x(t) \;=\; S(t)\, x_0 \;+\; \int_0^t S(t-s) \;\, f(x(s), \; K(s), \; s) \;\, ds.$$

b) $K : [0,T[\longrightarrow (\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ *is Lipschitz continuous in time direction,*
i.e. $q_{\mathcal{K},N}(K(s), \; K(t)) \; \le \; const(\Lambda, T) \cdot (t-s)$ *for all* $0 \le s < t < T$.

c) $\limsup_{h \downarrow 0} \; \frac{1}{h} \cdot \Big(q_{\mathcal{K},N}\Big(\vartheta_{g(x(t), \, K(t), \, t)}(h, \; M), \;\; K(t+h)\Big) \; - \; q_{\mathcal{K},N}(M, \; K(t)) \cdot e^{10\,\Lambda\, t}\Big) \; \le \; 0$
for every $M \in \mathcal{K}_{C^{1,1}}(I\!R^N), \;\; t \in [0,T[$.

0.8 Generalized mutational equations : Timed right–hand forward solutions.

The concept of right–hand forward solutions has two further aspects of generalization. They are sketched in this section although they might be less relevant to applications discussed in this thesis.

The first aspect is related to the triangle inequality. Obviously it plays the key rule whenever the distance between two points has to be estimated by means of a third element. Thus, it has been incorporated in Definition 0.5.1 of an "ostensible metric". In the preceding sections, the same feature of an ostensible metric space (E, q) occurred for several times : Considering $q(x, y)$, the first argument x refers to the state at an *earlier* point of time whereas the second argument y represents the *later* element.

In fact, this rule can be extended to the entire concept of right–hand forward solutions. We only need the possibility of distinguishing between the "earlier" and "later" element of E. For this reason, the product $\widetilde{E} := I\!R \times E$ with an additional time component is regarded instead of the nonempty set E.
The tilde usually symbolizes that a separate time component is taken into consideration.

So a function $\quad \widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[, \quad (\widetilde{x}_1, \widetilde{x}_2) \longmapsto \widetilde{q}(\widetilde{x}_1, \widetilde{x}_2)$
is to play the role of a distance. Then with respect to $\widetilde{x}_1 = (t_1, x_1)$, $\widetilde{x}_2 = (t_2, x_2) \in \widetilde{E}$, we usually consider the case $t_1 \le t_2$. In this time–directed situation, the principle of triangle inequality affects only points $\widetilde{z} = (s, z) \in \widetilde{E}$ whose time component s is between t_1 and t_2 $(t_1 \le s \le t_2)$ and, it is motivating the so–called *timed triangle inequality.* The term "timed" indicates that the (forward) time direction is taken into consideration by means of a separate time component.

Definition 0.8.1 *Set* $\widetilde{E} := I\!R \times E$. $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ *fulfills the so–called* timed triangle inequality *if for every* $(r, x), (s, y), (t, z) \in \widetilde{E}$ *with* $r \le s \le t$,

$$\widetilde{q}\Big((r, x),\ (t, z)\Big) \ \le\ \widetilde{q}\Big((r, x),\ (s, y)\Big) + \widetilde{q}\Big((s, y),\ (t, z)\Big).$$

$\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ *is called* timed ostensible metric *on* $\widetilde{E}$ *if it satisfies*

$$\widetilde{q}\Big((t, z),\ (t, z)\Big) = 0 \qquad \textit{(reflexive)}$$

$$\widetilde{q}\Big((r, x),\ (t, z)\Big) \le \widetilde{q}\Big((r, x),\ (s, y)\Big) + \widetilde{q}\Big((s, y),\ (t, z)\Big) \qquad \textit{(timed triangle inequality)}$$

for all $(r, x), (s, y), (t, z) \in \widetilde{E}$ *with* $r \le s \le t$.
$(\widetilde{E}, \widetilde{q})$ *is then called* timed ostensible metric space.

Every ostensible metric q on E induces a *timed* ostensible metric $\widetilde{q}$ on $\widetilde{E} \stackrel{\text{Def.}}{=} I\!\!R \times E$ according to $\widetilde{q}\Big((s,x),\ (t,y)\Big) := |t-s| + q(x,y)$. As a consequence, all statements about ostensible metric spaces result immediately from their more general counterparts about timed ostensible metric spaces.

Forward transitions are easy to adapt to timed ostensible metric spaces since there are only two new features to take into account (in comparison with Definition 0.6.1) : The arguments of each $\widetilde{q}_\varepsilon$ are sorted by time and, we have to specify the evolution of the time component. Condition (7.) of linear growth is mainly for the sake of simplicity. As a global abbreviation, define the projection $\pi_1 : \widetilde{E} \longrightarrow I\!\!R,\ \widetilde{x} = (t,x) \longmapsto t.$

Definition 0.8.2 *Let E be a nonempty set, $D \subset E$ and $\widetilde{E} \stackrel{\text{Def.}}{=} I\!\!R \times E$, $\widetilde{D} \stackrel{\text{Def.}}{=} I\!\!R \times D$. Moreover, $(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}$ denotes a countable family of timed ostensible metrics on $\widetilde{E}$. Assume for the map $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ and each $\varepsilon \in \mathcal{J}$*

1. $\widetilde{\vartheta}(0,\cdot) = \mathrm{Id}_{\widetilde{E}},$

2. $\limsup_{h \downarrow 0} \frac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\Big) = 0 \qquad \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[,$
   $\limsup_{h \downarrow 0} \frac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\Big) = 0 \qquad \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[,$

3. $\exists\ \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) < \infty : \sup_{\substack{\widetilde{x} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{x} \le \pi_1 \widetilde{y}}} \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{x}),\widetilde{\vartheta}(h,\widetilde{y})\big) - \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y})}{h \quad \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y})}\right)^+ \le \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$

4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) :\ ]0,1] \longrightarrow [0,\infty[:\ \ \beta_\varepsilon(\widetilde{\vartheta})(\cdot)$ *nondecreasing,* $\limsup_{h \downarrow 0} \beta_\varepsilon(\widetilde{\vartheta})(h) = 0,$
   $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\ \widetilde{\vartheta}(t,\widetilde{x})\Big) \le \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad \forall\ s < t \le 1,\ \widetilde{x} \in \widetilde{E},$

5. $\forall\ \widetilde{x} \in \widetilde{D} \quad \exists\ \mathcal{T}_\Theta = \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{x}) \in\ ]0,1] : \quad \widetilde{\vartheta}(t,\widetilde{x}) \in \widetilde{D} \qquad \forall\ t \in [0,\mathcal{T}_\Theta],$

6. $\limsup_{h \downarrow 0} \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\ \widetilde{x}),\ \widetilde{y}\Big) \ge \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}),\ \widetilde{y}\Big) \qquad \forall\ \widetilde{x} \in \widetilde{D},\ \widetilde{y} \in \widetilde{E},\ t \in\ ]0,\mathcal{T}_\Theta]$
   with $t + \pi_1 \widetilde{x} \le \pi_1 \widetilde{y},$

7. $\widetilde{\vartheta}\Big(h,\ (t,x)\Big) \in \{t+h\} \times E \qquad \forall\ (t,x) \in \widetilde{E},\ h \in [0,1].$

Then $\widetilde{\vartheta}$ is a so–called timed forward transition *on $(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$.*
$\widetilde{\Theta}^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ denotes a set of timed forward transitions on $(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$ assuming

$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau}) := \sup_{\substack{\widetilde{x} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{x} \le \pi_1 \widetilde{y}}} \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon(\widetilde{\vartheta}(h,\widetilde{x}),\ \widetilde{\tau}(h,\widetilde{y}))\ -\ \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y})\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\, h}}{h}\right)^+ < \infty$$

for all $\widetilde{\vartheta},\widetilde{\tau} \in \widetilde{\Theta}^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}),\ \ \varepsilon \in \mathcal{J}.$

As a next step, the definitions of primitives and solutions are extended in exactly the same way. It is just to point out that roughly speaking, the "test elements" of $\widetilde{D}$ always refer to earlier points of time.
Here we mention only "timed right–hand forward solutions" on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ explicitly before discussing the second aspect of generalization soon.

Definition 0.8.3 *For* $\widetilde{f} : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *given, a map* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a so–called* timed right–hand forward solution *of the generalized mutational equation*

$$\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$$

if $\widetilde{x}(\cdot)$ *is timed right–hand forward primitive of* $\widetilde{f}(\widetilde{x}(\cdot), \cdot) : [0,T[\longrightarrow \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, *i.e. for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \;\; t \in [0,T[\qquad \exists \;\widehat{\alpha}^{\mapsto}_\varepsilon(t) \;\geq\; \alpha^{\mapsto}_\varepsilon(\widetilde{f}(\widetilde{x}(t), t))$:
$$\limsup_{h \downarrow 0} \; \tfrac{1}{h} \left(\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t), t)\,(h, \widetilde{y}),\; \widetilde{x}(t+h)\Big) \;-\; \widetilde{q}_\varepsilon(\widetilde{y},\, \widetilde{x}(t)) \,\cdot\, e^{\widehat{\alpha}^{\mapsto}_\varepsilon(t)\cdot h} \right) \;\leq\; 0,$$
for all $\widetilde{y} \in \widetilde{D}$ *with* $\pi_1\, \widetilde{y} \leq \pi_1\, \widetilde{x}(t)$,

2. $\widetilde{x}(\cdot) : [0,T[\longrightarrow (\widetilde{E}, \widetilde{q}_\varepsilon)$ *is uniformly continuous in time direction,*
i.e. there is $\omega_\varepsilon(\widetilde{x}, \cdot) :]0,T[\longrightarrow [0,\infty[$ *such that* $\limsup_{h \downarrow 0} \; \omega_\varepsilon(\widetilde{x}, h) \;=\; 0$,
$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(s),\, \widetilde{x}(t)\Big) \;\leq\; \omega_\varepsilon(\widetilde{x},\, t-s) \qquad \text{for} \;\; 0 \leq s < t < T,$$

3. $\pi_1\, \widetilde{x}(t) \;=\; t \;+\; \pi_1\, \widetilde{x}(0) \qquad$ *for all* $t \in [0,T[$.

Indeed, all steps of section § 0.6 can be repeated with the timed triangle inequality (instead of the triangle inequality).
From the topological point of view, there is only one additional condition to suppose, i.e. the convergence with respect to the timed ostensible metric implies the convergence of the time components. It is the motivation for the following definition :

Definition 0.8.4 *Let* E *be a nonempty set,* $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R} \times E$, $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$.
$(\widetilde{E}, \widetilde{q})$ *is called* time continuous *if every sequence* $(\widetilde{x}_n = (t_n, x_n))_{n \in \mathbb{N}}$ *in* $\widetilde{E}$ *and element* $\widetilde{x} = (t,x) \in \widetilde{E}$ *with* $\widetilde{q}(\widetilde{x}_n, \widetilde{x}) \longrightarrow 0 \;\; (n \longrightarrow \infty)$ *fulfill* $\qquad t_n \longrightarrow t \qquad (n \longrightarrow \infty)$
(*i.e. the projection* $\pi_1(\cdot) : \widetilde{E} \longrightarrow \mathbb{R}$, $\widetilde{x} = (t,x) \longmapsto t$ *is right–sequentially continuous with respect to* $\widetilde{q}$).

In this chapter, all timed ostensible metric spaces are supposed to be time continuous.

The second aspect of generalization is related to the modified semigroup condition on transitions, i.e. condition (2.) of Definition 0.8.2. Using the Landau symbol $o(\cdot)$, it demands for every $\widetilde{x} \in \widetilde{E}$, $t \in [0,1[$ and $\varepsilon \in \mathcal{J}$

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\quad \widetilde{\vartheta}(t+h,\ \widetilde{x})\Big) \ = \ o(h) \\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\quad \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\Big) \ = \ o(h) \end{cases} \qquad \text{for } h \downarrow 0.$$

In short, the main idea now is to replace $o(h)$ with the other Landau symbol $O(h)$. Strictly speaking, each $\widetilde{\vartheta}$ has a parameter $\gamma_\varepsilon(\widetilde{\vartheta}) \in [0,\infty[$ (depending only on ε) with

$$\wedge \begin{cases} \limsup\limits_{h\downarrow 0} \ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\quad \widetilde{\vartheta}(t+h,\ \widetilde{x})\Big) \ \le \ \gamma_\varepsilon(\widetilde{\vartheta}) \\ \limsup\limits_{h\downarrow 0} \ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\quad \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\Big) \ \le \ \gamma_\varepsilon(\widetilde{\vartheta}) \end{cases}$$

for all $\widetilde{x} \in \widetilde{E}$, $t \in [0,1[$ and each $\varepsilon \in \mathcal{J}$. So the challenge is to incorporate this parameter in the concept of timed right–hand forward solutions.
The dependence of $\gamma_\varepsilon(\widetilde{\vartheta})$ on $\varepsilon \in \mathcal{J}$ exemplifies an additional feature for characterizing $\widetilde{\vartheta}$. Assuming $0 \in \overline{\mathcal{J}}$, we choose the asymptotic behavior of $\gamma_\varepsilon(\widetilde{\vartheta})$ (for $\varepsilon \longrightarrow 0$) as a further criterion and formulate now the most general definition of "timed forward transition" on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$:

Definition 0.8.5 *Let E be a nonempty set, $D \subset E$ and $\widetilde{E} \overset{\text{Def.}}{=} I\!R \times E$, $\widetilde{D} \overset{\text{Def.}}{=} I\!R \times D$. $(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}$ denotes a countable family of timed ostensible metrics on $\widetilde{E}$ and, $0 \in \overline{\mathcal{J}}$.
A map $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ is a so–called* timed forward transition of order $p \in I\!R$ *on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ if it fulfills the following conditions (for each $\varepsilon \in \mathcal{J}$)*

1. $\widetilde{\vartheta}(0,\cdot) \ = \ \mathrm{Id}_{\widetilde{E}}$,
2. $\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \ge 0 : \qquad \limsup\limits_{\varepsilon \longrightarrow 0} \ \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) \ = \ 0 \qquad$ *and*
$\limsup\limits_{h\downarrow 0} \ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\quad \widetilde{\vartheta}(t+h,\ \widetilde{x})\Big) \ \le \ \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[$,
$\limsup\limits_{h\downarrow 0} \ \frac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\quad \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\Big) \ \le \ \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[$,
3. $\exists\ \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) < \infty : \quad \sup\limits_{\substack{\widetilde{x}\,\in\,\widetilde{D},\ \widetilde{y}\,\in\,\widetilde{E} \\ \pi_1\,\widetilde{x}\,\le\,\pi_1\,\widetilde{y}}} \ \limsup\limits_{h\downarrow 0} \ \left(\dfrac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{x}),\, \widetilde{\vartheta}(h,\widetilde{y})\big) \ - \ \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y}) \ - \ \gamma_\varepsilon(\widetilde{\vartheta})\, h}{h \quad \big(\widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y}) + \gamma_\varepsilon(\widetilde{\vartheta})\, h\big)}\right)^{+} \ \le \ \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$
4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) : \ ]0,1] \longrightarrow [0,\infty[: \quad \beta_\varepsilon(\widetilde{\vartheta})(\cdot)$ *nondecreasing,* $\qquad \limsup\limits_{h\downarrow 0} \ \beta_\varepsilon(\widetilde{\vartheta})(h) \ = \ 0$,
$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\ \ \widetilde{\vartheta}(t,\widetilde{x})\Big) \ \le \ \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad \forall\ s < t \le 1,\ \ \widetilde{x} \in \widetilde{E}$,
5. $\forall\ \widetilde{x} \in \widetilde{D} \qquad \exists\ \mathcal{T}_\Theta = \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{x}) \in \,]0,1] : \quad \widetilde{\vartheta}(t,\widetilde{x}) \in \widetilde{D} \qquad \forall\ t \in [0, \mathcal{T}_\Theta]$,
6. $\limsup\limits_{h\downarrow 0} \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\ \widetilde{x}),\ \widetilde{y}\Big) \ \ge \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}),\ \widetilde{y}\Big) \qquad \forall\ \widetilde{x}\in\widetilde{D},\ \widetilde{y}\in\widetilde{E},\ t\in\,]0,\mathcal{T}_\Theta]$
with $\ t + \pi_1\,\widetilde{x} \le \pi_1\,\widetilde{y}$,
7. $\widetilde{\vartheta}\Big(h,\ (t,x)\Big) \ \in \ \{t+h\} \times E \qquad \forall\ (t,x) \in \widetilde{E},\ h \in [0,1]$.

$\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *denotes a set of timed forward transitions of order p on* $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *assuming*

$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau}) := \sup_{\substack{\widetilde{x} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{x} \le \pi_1 \widetilde{y}}} \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon(\widetilde{\vartheta}(h, \widetilde{x}),\ \widetilde{\tau}(h, \widetilde{y})) \ - \ \widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{y}) \cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\, h}}{h} \right)^+ < \infty$$

for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}),\ \ \varepsilon \in \mathcal{J}$.

In comparison with the preceding Definition 0.8.2, the new parameter $\gamma_\varepsilon(\widetilde{\vartheta})$ occurs in only two conditions so far, i.e. condition (2.), of course, and condition (3.) on $\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$,

$$\sup_{\substack{\widetilde{x} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{x} \le \pi_1 \widetilde{y}}} \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon(\widetilde{\vartheta}(h, \widetilde{x}),\ \widetilde{\vartheta}(h, \widetilde{y})) \ - \ \widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{y}) \ - \ \gamma_\varepsilon(\widetilde{\vartheta})\, h}{h \quad (\widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{y}) \ + \ \gamma_\varepsilon(\widetilde{\vartheta})\, h)} \right)^+ \le \ \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) \ < \ \infty\,.$$

As a consequence, we have to modify the substantial estimate that provided our starting point for generalizing Aubin's mutational analysis in § 0.6. So the more general counterpart of Proposition 0.6.3 is

Proposition 0.8.6 *Let* $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *be timed forward transitions,* $\varepsilon \in \mathcal{J}$, $\widetilde{x} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$ *with* $\pi_1 \widetilde{x} \le \pi_1 \widetilde{y}$ *and* $0 \le h < \mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{x})$. *Then,*

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h, \widetilde{x}),\ \widetilde{\tau}(h, \widetilde{y})\Big) \ \le \ \Big(\widetilde{q}_\varepsilon(\widetilde{x},\ \widetilde{y}) \ + \ h\ \Big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau})\Big)\Big) \ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\ h}.$$

Its proof is presented in Proposition 2.1.5.
Following basically the same track as in § 0.6 leads directly to the subsequent definitions of primitive and solution (formulated also in §§ 2.2, 2.3) :

Definition 0.8.7 *The function* $\widetilde{x} : [0, T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed right–hand forward primitive of *a map* $\widetilde{\vartheta} : [0, T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall\ \ t \in [0, T[\qquad \exists\ \ \widehat{\alpha}_\varepsilon^{\mapsto}(t) = \widehat{\alpha}_\varepsilon^{\mapsto}(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) \ge 0,\ \ \widehat{\gamma}_\varepsilon(t) = \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) \ge 0\ :$
   $$\widehat{\alpha}_\varepsilon^{\mapsto}(t) \ \ge\ \alpha_\varepsilon^{\mapsto}(\vartheta(t)), \qquad \widehat{\gamma}_\varepsilon(t) \ \ge\ \gamma_\varepsilon(\vartheta(t)), \qquad \limsup_{\varepsilon' \downarrow 0}\ \varepsilon'^{p} \cdot \widehat{\gamma}_{\varepsilon'}(t) \ = \ 0,$$
   $$\limsup_{h \downarrow 0}\ \tfrac{1}{h} \left(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(h, \widetilde{y}),\ \ \widetilde{x}(t{+}h)\Big) \ - \ \widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t)) \ \cdot\ e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t) \cdot h} \right) \ \le\ \widehat{\gamma}_\varepsilon(t),$$
   for all $\widetilde{y} \in \widetilde{D}$ *with* $\pi_1 \widetilde{y} \le \pi_1 \widetilde{x}(t)$,
2. $\widetilde{x}(\cdot) : [0, T[\longrightarrow (\widetilde{E}, \widetilde{q}_\varepsilon)$ *is uniformly continuous in time direction,*
   i.e. there is $\omega_\varepsilon(\widetilde{x}, \cdot) :]0, T[\longrightarrow [0, \infty[$ *such that* $\limsup_{h \downarrow 0}\ \omega_\varepsilon(\widetilde{x}, h) \ = \ 0,$
   $\widetilde{q}_\varepsilon\Big(\widetilde{x}(s),\ \widetilde{x}(t)\Big) \ \le\ \omega_\varepsilon(\widetilde{x},\ t - s) \qquad$ *for* $0 \le s < t < T$.
3. $\pi_1\, \widetilde{x}(t) \ = \ t \ + \ \pi_1\, \widetilde{x}(0) \qquad\qquad$ *for all* $t \in [0, T[$.

Definition 0.8.8 *For* $\widetilde{f} : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *given, a map* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand forward solution *of the generalized mutational equation*

$$\overset{\circ}{\widetilde{x}}(\cdot) \ \ni \ \widetilde{f}(\widetilde{x}(\cdot), \cdot)$$

if $\widetilde{x}(\cdot)$ *is timed right–hand forward primitive of* $\widetilde{f}(\widetilde{x}(\cdot), \cdot) : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, *i.e. for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \ t \in [0,T[\qquad \exists \ \widehat{\alpha}_\varepsilon^{\mapsto}(t) \ \geq \ \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{x}(t),t)), \quad \widehat{\gamma}_\varepsilon(t) \ \geq \ \gamma_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)) \ :$
$$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \left(\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\widetilde{y}), \ \widetilde{x}(t+h)\Big) \ - \ \widetilde{q}_\varepsilon(\widetilde{y}, \ \widetilde{x}(t)) \ \cdot \ e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t) \cdot h} \right) \ \leq \ \widehat{\gamma}_\varepsilon(t),$$
for all $\widetilde{y} \in \widetilde{D}$ *with* $\pi_1 \, \widetilde{y} \leq \pi_1 \, \widetilde{x}(t)$ *and* $\limsup_{\varepsilon' \downarrow 0} \ \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) \ = \ 0,$
2. $\widetilde{x}(\cdot) : [0,T[\longrightarrow (\widetilde{E}, \widetilde{q}_\varepsilon)$ *is uniformly continuous in time direction,*
3. $\pi_1 \, \widetilde{x}(t) \ = \ t \ + \ \pi_1 \, \widetilde{x}(0) \qquad$ *for all* $t \in [0,T[$.

In regard to existence and stability of solutions, the former results of § 0.6 indicate how to deal with the new parameter $\gamma_\varepsilon(\cdot)$. Uniform bounds are supposed in the same way as for the parameter $\alpha_\varepsilon^{\mapsto}(\cdot)$. Taking also the time direction into account, we obtain the following theorems about convergence and existence.Their detailed proofs are shown in Proposition 2.3.2 and 2.3.5, respectively.

Theorem 0.8.9 (of Convergence)
For each $\varepsilon \in \mathcal{J}$, *suppose the following properties of*

$$\begin{array}{lll} \widetilde{f}_m, \ \widetilde{f} : \ \widetilde{E} \times [0,T[& \longrightarrow & \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}) \qquad (m \in I\!N) \\ \widetilde{x}_m, \ \widetilde{x} : \qquad [0,T[& \longrightarrow & \widetilde{E} \ : \end{array}$$

1. $M_\varepsilon \ := \ \sup_{m,t,\widetilde{z}} \ \{ \, \alpha_\varepsilon^{\mapsto}(\widetilde{f}_m(\widetilde{z},t)) \, \} \ < \ \infty,$
$R_\varepsilon \ \geq \ \sup_{m,t,\widetilde{z}} \ \{ \, \widehat{\gamma}_\varepsilon(t, \, \widetilde{x}_m, \, \widetilde{f}_m(\widetilde{x}_m, \cdot)), \ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{z},t)), \ \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \, \}$
with $\limsup_{\varepsilon' \downarrow 0} \ \varepsilon'^p \cdot R_{\varepsilon'} \ = \ 0,$
2. $\limsup \ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_m(\widetilde{z}_1, t_1), \ \widetilde{f}_m(\widetilde{z}_2, t_2)\Big) \leq \ R_\varepsilon \quad$ *for* $m \to \infty, \quad t_2 - t_1 \downarrow 0,$
$\widetilde{q}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2) \to 0 \quad (\pi_1 \, \widetilde{z}_1 \leq \pi_1 \, \widetilde{z}_2),$
3. $\overset{\circ}{\widetilde{x}}_m(\cdot) \ \ni \ \widetilde{f}_m(\widetilde{x}_m(\cdot), \cdot) \qquad$ *in* $[0,T[$,
4. $\widehat{\omega}_\varepsilon(h) \ := \ \sup_m \ \omega_\varepsilon(\widetilde{x}_m, h) \ < \ \infty \quad$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*)* $\quad \forall \ h \in \,]0,T],$
$\limsup_{h \downarrow 0} \ \widehat{\omega}_\varepsilon(h) \ = \ 0,$

5. $\forall\ t_1, t_2 \in [0,T[, \quad t_3 \in]0,T[\quad \exists\ (m_j)_{j\in\mathbb{N}}$ *with* $m_j \nearrow \infty$ *and*

(i) $\limsup\ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t_1),t_1),\ \widetilde{f}_{m_j}(\widetilde{x}(t_1),t_1)\Big) \ \le\ R_\varepsilon \qquad (j \longrightarrow \infty)$

(ii) $\exists\ (\delta'_j)_{j\in\mathbb{N}}$ *in* $[0,1[\ :\quad \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_2),\ \widetilde{x}_{m_j}(t_2+\delta'_j)\Big) \longrightarrow 0, \qquad \delta'_j \longrightarrow 0,$
$\pi_1\,\widetilde{x}(t_2) \ \le\ \pi_1\,\widetilde{x}_{m_j}(t_2+\delta'_j).$

(iii) $\exists\ (\delta_j)_{j\in\mathbb{N}}$ *in* $[0,t_3[\ :\quad \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_3-\delta_j),\ \widetilde{x}(t_3)\Big) \longrightarrow 0, \qquad \delta_j \longrightarrow 0,$
$\pi_1\,\widetilde{x}_{m_j}(t_3-\delta_j) \ \le\ \pi_1\,\widetilde{x}(t_3),$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\,\cdot)$ *in* $[0,T[$.

Theorem 0.8.10 (Existence of timed right–hand forward solutions due to timed transitional compactness)

Assume that the tuple $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)))$ *is timed transitionally compact. Furthermore let* $\widetilde{f} : \widetilde{E}\times[0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *fulfill for every* $\varepsilon\in\mathcal{J}$

1. $M_\varepsilon \ :=\ \sup_{t,\widetilde{z}}\ \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{z},t)) \ <\ \infty,$

2. $c_\varepsilon(h) \ :=\ \sup_{t,\widetilde{z}}\ \beta_\varepsilon(\widetilde{f}(\widetilde{z},t))(h) \ <\ \infty, \qquad c_\varepsilon(h) \longrightarrow 0$ *for* $h\downarrow 0,$

3. $\exists\ R_\varepsilon\ :\ \sup_{t,\widetilde{z}}\ \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \ \le\ R_\varepsilon < \infty, \qquad \varepsilon'^p\, R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon'\downarrow 0,$

4. $\exists\ \widehat{\omega}_\varepsilon(\cdot) :\ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \ \le\ R_\varepsilon \ +\ \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2) + t_2 - t_1\Big)$
for all $0\le t_1\le t_2\le T$ *and* $\widetilde{z}_1,\widetilde{z}_2\in\widetilde{E}$ *with* $\pi_1\,\widetilde{z}_1 \le \pi_1\,\widetilde{z}_2,$
$\widehat{\omega}_\varepsilon(\cdot)\ge 0$ *nondecreasing,* $\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) \ =\ 0.$

Then for every $\widetilde{x}_0\in\widetilde{E}$, *there is a timed right–hand forward solution* $\widetilde{x} : [0,T[\longrightarrow\widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\,\cdot)$ *in* $[0,T[$ *with* $\widetilde{x}(0)=\widetilde{x}_0$.

Due to the general assumption of time continuity in this chapter, Definition 0.6.9 of transitional compactness can be applied literally to $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)))$ providing the term "*timed* transitionally compact" (see Definition 2.3.4 in detail).

Obviously these results can also be applied to systems with 2 components. Indeed, timed forward transitions $\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}_\varepsilon^{\,1})_{\varepsilon\in\mathcal{J}_1})$ and $\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon\in\mathcal{J}_1})$ induce timed forward transitions of order $\max\{p,p'\}$ on $\Big(\widetilde{E}_1\times\widetilde{E}_2,\ \widetilde{D}_1\times\widetilde{D}_2,\ (\widetilde{q}_\varepsilon^{\,1}+\widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon\in\mathcal{J}_1,\,\varepsilon'\in\mathcal{J}_2}\Big)$ (according to Lemma 2.4.2) and, the timed transitional compactness of each component

$$\Big(\widetilde{E}_1,\ (\widetilde{q}_\varepsilon^{\,1})_{\varepsilon\in\mathcal{J}_1},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}_\varepsilon^{\,1}))\Big) \quad\text{and}\quad \Big(\widetilde{E}_2,\ (\widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon'\in\mathcal{J}_2},\ \widetilde{\Theta}_{p'}^{\mapsto}(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}_{\varepsilon'}^{\,2}))\Big)$$

implies the same compactness property of the corresponding product (Lemma 2.4.3). For the sake of simplicity, we consider only elements $(\widetilde{x}_1,\widetilde{x}_2)\in\widetilde{E}_1\times\widetilde{E}_2$ with identical time component $\pi_1\,\widetilde{x}_1 = \pi_1\,\widetilde{x}_2$.

So under adequate assumptions about the right–hand side of the system, existence theorem 0.8.10 provides a timed right-hand forward solution

$$(\widetilde{x}_1,\ \widetilde{x}_2) :\ [0,T[\ \longrightarrow\ \widetilde{E}_1 \times \widetilde{E}_2$$

of the generalized mutational equations

$$(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot))^\circ\ \ni\ \Big(\widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot),\ \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot)\Big).$$

In this context, only *one* asymptotic demand (for $h \downarrow 0$) has to be fulfilled by both components $\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot)$ simultaneously, i.e. for each $\varepsilon \in \mathcal{J}_1,\ \varepsilon' \in \mathcal{J}_2$ and $t \in [0,T[$, there exist parameters $\widehat{\alpha}^{\mapsto},\ \widehat{\gamma}_{\varepsilon,\varepsilon'} \geq 0$ such that

$$\begin{aligned}\Delta_h \ :=\ & \widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{f}_1(\widetilde{x}_1(t),\ \widetilde{x}_2(t),\ t)\ (h,\widetilde{y}_1),\quad \widetilde{x}_1(t{+}h)\Big)\ -\ \widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\ \widetilde{x}_1(t))\ \cdot\ e^{\widehat{\alpha}^{\mapsto}\cdot h}\\ &+\ \widetilde{q}^{\,2}_{\varepsilon'}\Big(\widetilde{f}_2(\widetilde{x}_1(t),\ \widetilde{x}_2(t),\ t)\ (h,\widetilde{y}_2),\quad \widetilde{x}_2(t{+}h)\Big)\ -\ \widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{y}_2,\ \widetilde{x}_2(t))\ \cdot\ e^{\widehat{\alpha}^{\mapsto}\cdot h}\end{aligned}$$

satisfies $\displaystyle\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot\Delta_h\ \leq\ \widehat{\gamma}_{\varepsilon,\varepsilon'},\qquad \limsup_{\varepsilon,\varepsilon'\longrightarrow 0}\ (\varepsilon+\varepsilon')^{\max\{p,p'\}}\cdot\widehat{\gamma}_{\varepsilon,\varepsilon'}\ =\ 0$

for all $\widetilde{y}_1 \in \widetilde{D}_1,\ \widetilde{y}_2 \in \widetilde{D}_2$ with $\pi_1\,\widetilde{y}_1\ =\ \pi_1\,\widetilde{y}_2\ \leq\ \pi_1\,\widetilde{x}_1(t)\ =\ \pi_1\,\widetilde{x}_2(t)$.

It is not obvious that $(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot))$ is a timed right–hand forward solution of the system

$$\wedge\begin{cases}\overset{\circ}{\widetilde{x}}_1(\cdot)\ \ni\ \widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot)\\ \overset{\circ}{\widetilde{x}}_2(\cdot)\ \ni\ \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot)\end{cases}$$

(i.e. separately with respect to each component).
The main step is to adapt the convergence theorem to systems (as in Proposition 2.4.5). Then the tuples of Euler approximations in $\widetilde{E}_1,\ \widetilde{E}_2$, respectively, provide a curve $[0,T[\ \longrightarrow\ \widetilde{E}_1 \times \widetilde{E}_2$ whose components solve the generalized mutational equations separately. Details are given in Proposition 2.4.6.

Finally, we make some remarks about the role of $\gamma_\varepsilon(\cdot) \geq 0$.
Analytically speaking, this parameter gives the opportunity to introduce an additional limit process that follows the process of first–order approximation. This might be useful for multi–scale problems, for example, although they are not considered in this thesis.

However, $\gamma_\varepsilon(\cdot) \geq 0$ and its upper bounds (usually abbreviated as R_ε) are also of direct use for semilinear evolution equations here. Indeed, consider Proposition 0.4.2. Its continuity assumption about the right–hand side

$$\sum_{k=1}^{\infty}\ 2^{-k}\ q_{j_k}\Big(f(x,t_1),\ \ f(y,t_2)\Big)\ \leq\ \widehat{\omega}\Big(\sum_{k=1}^{\infty}\ 2^{-k}\ \tfrac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)}\ +\ |t_2-t_1|\Big)$$

(for all $x, y \in X$ and $t_1, t_2 \in [0,T]$ with a modulus $\widehat{\omega}(\cdot)$ of continuity) was to take more than one pseudo–metric $q_j \overset{\text{Def.}}{=} q_{v'_j}$ $(j \in \mathcal{J} = \{j_1, j_2, j_3 \ldots\})$ into account.

The corresponding parameters $\alpha^{\mapsto}(\cdot)$ are closely related with the eigenvalues of the infinitesimal generator A (as shown in Proposition 4.5.3 and Lemma 4.5.9). For this technical reason, we consider only a finite number of pseudo–metrics q_j simultaneously and define for all $x, y \in X$, $n \in I\!N$

$$p_n(x,y) := \sum_{k=1}^{n} 2^{-k} \frac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)}, \qquad P_n(x,y) := \sum_{k=1}^{n} 2^{-k} q_{j_k}(x,y).$$

Obviously, each p_n is a pseudo–metric on the reflexive Banach space X, but the preceding continuity assumption (of Proposition 0.4.2) implies merely

$$\begin{aligned} P_n\Big(f(x,t_1),\; f(y,t_2)\Big) &\le \widehat{\omega}\Big(p_n(x,y) + \sum_{k=n+1}^{\infty} 2^{-k} \frac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)} + |t_2-t_1|\Big) \\ &\le \widehat{\omega}\Big(p_n(x,y) + 2^{-n} + |t_2-t_1|\Big), \end{aligned}$$

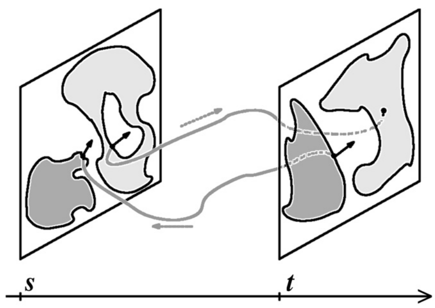
i.e. the continuity of the right–hand side (with respect to P_n, p_n) is not really guaranteed in the way we need *before* introducing the parameter $\gamma_n(\cdot)$.
Correspondingly, (timed) forward transitions with $\gamma_\varepsilon > 0$ are also used for proving Proposition 0.4.3 and 0.7.14 (dealing with semilinear evolution equations) in detail.

0.9 Generalized mutational equations : Timed right–hand backward solutions.

A second concept generalizing Aubin's mutational analysis is presented in chapter 3. It is motivated by the observation that roughly speaking, information about the evolving state might be lost in the course of time. Considering the geometric example of time–dependent sets, topological "holes" can disappear.

So the starting point is now to benefit from the information of the past as long as possible and basically, we take the states at time $t - h$ and t into consideration (for $h \downarrow 0$). This notion is indicated by the term "backward" and is symbolized by $\rightarrow\!\!\!\shortmid$ (representing the time axis).

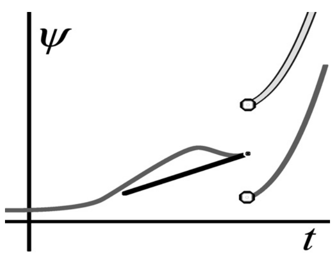
An immediate disadvantage of this approach is that we cannot overcome the second obstacle mentioned in § 0.5. Indeed, the figure on the right illustrates for any fixed time t that there might be a "hole" disappearing — no matter how small $t - s > 0$ is or how smooth the sets are at time t.



For this reason, we do not use the idea of distributions again (basically for defining the continuity parameter α in an adequate way). In particular, a "test set" $\widetilde{D} \subset \widetilde{E}$ is not required any longer.

Although this concept might be of less interest to (geometric) applications, we pursue this "backward" idea and present another analytical tool for dealing with the semicontinuity of (generalized) distance functions.

Local properties like subdifferentials can provide global information by means of Gronwall's Lemma (maybe in a modified form). This has already been essential in Aubin's mutational analysis and, we have extended this technique to (timed) ostensible metric spaces (in §§ 0.5, 0.6).



Now the states at time $t-h$ and t are considered (for $h \downarrow 0$). Correspondingly to Lemma 0.5.2, we again need a version of Gronwall's Lemma for semicontinuous functions — but in *backward* time direction. Indeed, Lemma 1.5.3 states :

Proposition 0.9.1

Let $\psi : [a,b] \longrightarrow I\!R,\ \ f, g \in C^0([a,b[, I\!R)$ *satisfy* $f(\cdot) \geq 0$ *and*

$$\psi(t) \;\geq\; \limsup_{h \downarrow 0} \;\psi(t+h), \qquad \forall\; t \in [a,\, b[,$$

$$\liminf_{h \downarrow 0} \;\frac{\psi(t) - \psi(t-h)}{h} \;\leq\; f(t) \;\cdot\; \limsup_{h \downarrow 0}\; \psi(t-h) \;+\; g(t) \;<\infty \qquad \forall\; t \in\,]a,\, b].$$

Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t) \;\leq\; \psi(a) \,\cdot\, e^{\mu(t)} \;+\; \int_a^t e^{\mu(t)-\mu(s)}\; g(s)\; ds$$

with $\mu(t) := \displaystyle\int_a^t f(s)\; ds.$

When applying this estimate to the distance between two points evolving along transitions, all these distances have to be finite of course. That is guaranteed by a general assumption (abbreviated as $(BUC^{\rightarrow})$) about any two curves that are uniformly continuous in time direction.

Furthermore we have to verify the first assumption of semicontinuity. In § 0.6, we overcame the corresponding difficulty by means of an additional condition on forward transitions (i.e. condition (6.) in Definition 0.6.1).
Now we prefer an alternative representing the second main difference in comparison with the "forward" concept : For two curves $\widetilde{x}, \widetilde{y} : [0,T] \longrightarrow (\widetilde{E}, \widetilde{q}_\varepsilon)$ in a timed ostensible metric space, we consider the upper limit

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\;\; \widetilde{y}(t^{++})\Big) \;\;:=\;\; \limsup_{\substack{k,l \,\downarrow\, 0\\ k\,<\,l}} \;\widetilde{q}_\varepsilon\Big(\widetilde{x}(t{+}k),\;\; \widetilde{y}(t{+}l)\Big)$$

instead of the distance $\widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{y}(t))$ because in regard to Proposition 0.9.1,

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\; \widetilde{y}(t^{++})\Big) \;\;\geq\;\; \limsup_{h \downarrow 0} \;\; \widetilde{q}_\varepsilon\Big(\widetilde{x}\,\big((t{+}h)^+\big)\,,\;\; \widetilde{y}\,\big((t{+}h)^{++}\big)\,\Big)$$

is obvious then. Moreover, the assumption $0 < k < l$ about the limit superior is to facilitate incorporating the time direction and applying the *timed* triangle inequality. General relations between $\widetilde{q}_\varepsilon(\widetilde{x}(t^+), \widetilde{y}(t^{++}))$ and $\widetilde{q}_\varepsilon(\widetilde{x}(t), \widetilde{y}(t))$ (in form of an inequality) are proven merely under additional conditions on $(\widetilde{E}, \widetilde{q}_\varepsilon)$ (as in Corollary 3.4.2).

These two modifications (i.e. the preceding limit superior and the "backward" notion without "test elements") lead, for example, to the following condition on a *timed backward transition* $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ of order p

$$\exists\;\; \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 \;:\quad \limsup_{\varepsilon \longrightarrow 0} \;\; \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) \;=\; 0 \quad \text{and} \quad \text{for all } t \in\,]0,1[,$$

$$\limsup_{h \downarrow 0} \;\; \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\;\; \widetilde{\vartheta}(t-h, \widetilde{x})),\;\;\; \widetilde{\vartheta}(t^{++}, \widetilde{x})\Big) \;\;\leq\;\; \gamma_\varepsilon(\widetilde{\vartheta})$$

$$\limsup_{h \downarrow 0} \;\; \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\;\;\; \widetilde{\vartheta}(h^{++},\;\; \widetilde{\vartheta}(t-h, \widetilde{x}))\Big) \;\;\leq\;\; \gamma_\varepsilon(\widetilde{\vartheta}).$$

In fact, considering the upper limit with respect to time (abbreviated as "+","++") requires an additional feature of "backward" transitions that provide a "forward" link with the initial values (roughly speaking), i.e.

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(0^+,\widetilde{\vartheta}(t,\widetilde{x})),\ \widetilde{\vartheta}(t^{++},\widetilde{x})\Big) \;=\; 0, \qquad \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+,\widetilde{x}),\ \widetilde{\vartheta}(0^{++},\widetilde{\vartheta}(t,\widetilde{x}))\Big) \;=\; 0.$$

From now on, $UC^{\rightarrow}(]0,1[,\ \widetilde{E},\ \widetilde{q}_\varepsilon)$ denotes the set of all functions $]0,1[\longrightarrow \widetilde{E}$ that are uniformly continuous in time direction with respect to $\widetilde{q}_\varepsilon$ and, we define

Definition 0.9.2 *Let E be a nonempty set, $\widetilde{E} \stackrel{\text{Def.}}{=} I\!R \times E$ and $(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}$ a countable family of timed ostensible metrics on $\widetilde{E}$ such that each $(\widetilde{E},\widetilde{q}_\varepsilon)$ is time continuous.*
$\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ is called timed backward transition of order $p \in I\!R$ *on $(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ if it satisfies for every $\varepsilon \in \mathcal{J}$,*

1. $\widetilde{\vartheta}(0,\cdot) \;=\; \mathrm{Id}_{\widetilde{E}},$

2. $\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 : \quad \limsup\limits_{\varepsilon \longrightarrow 0}\ \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) \;=\; 0$ *and*
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\ \widetilde{\vartheta}(t-h,\widetilde{x})),\ \ \widetilde{\vartheta}(t^{++},\widetilde{x})\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta}) \quad \forall\ t \in\ ]0,1[,$$
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+,\widetilde{x}),\ \ \widetilde{\vartheta}(h^{++},\ \widetilde{\vartheta}(t-h,\widetilde{x}))\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta}) \quad \forall\ t \in\ ]0,1[,$$

3. $$\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) \;:=\; \sup_{\substack{0<t<1\\ \widetilde{x},\,\widetilde{y}\,\in\\ UC^{\rightarrow}(]0,1[,\ \widetilde{E},\ \widetilde{q}_\varepsilon)}}\ \limsup_{h\downarrow 0}\ \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h^+,\widetilde{x}(t_h)),\,\widetilde{\vartheta}(h^{++},\widetilde{y}(t_h))\big) - \widetilde{q}_\varepsilon\big(\widetilde{x}(t_h^+),\,\widetilde{y}(t_h^{++})\big) - \gamma_\varepsilon(\widetilde{\vartheta})\,h}{h\quad \big(\widetilde{q}_\varepsilon\big(\widetilde{x}(t_h^+),\ \widetilde{y}(t_h^{++})\big) + \gamma_\varepsilon(\widetilde{\vartheta})\,h\big)}\right)^+$$
$$<\ \infty \qquad\qquad (\textit{with}\ \ t_h := t-h)$$

4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) :\ ]0,1] \longrightarrow [0,\infty[\ : \quad \beta_\varepsilon(\widetilde{\vartheta})(\cdot)$ *nondecreasing,* $\limsup\limits_{h\downarrow 0}\ \beta_\varepsilon(\widetilde{\vartheta})(h) \;=\; 0,$
$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\ \ \widetilde{\vartheta}(t,\widetilde{x})\Big) \;\leq\; \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad \forall\ 0 \leq s < t,\ \widetilde{x}\in\widetilde{E},$$

5. $$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(0^+,\widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t^{++},\widetilde{x})\Big) \;=\; 0 \qquad \forall\ t \in [0,1[,$$
$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+,\widetilde{x}),\ \ \widetilde{\vartheta}(0^{++},\widetilde{\vartheta}(t,\widetilde{x}))\Big) \;=\; 0 \qquad \forall\ t \in [0,1[,$$

6. $\widetilde{\vartheta}\Big(h,\ (t,x)\Big) \;\in\; \{t+h\} \times E \qquad \forall\ (t,x)\in\widetilde{E},\ h\in[0,1].$

Define for any timed backward transitions $\widetilde{\vartheta},\widetilde{\tau} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ and $\varepsilon \in \mathcal{J}$,

$$\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta},\widetilde{\tau}) \quad:=\quad \sup_{\substack{0<t<1\\ \widetilde{x}\in\widetilde{E}}}\ \limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\ \widetilde{\tau}(t-h,\widetilde{x})),\ \ \widetilde{\tau}(t^{++},\widetilde{x})\Big).$$

$\widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ denotes a set of timed backward transitions of order p on $(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ supposing for all $\widetilde{\vartheta},\widetilde{\tau} \in \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}),\ \widetilde{x}\in\widetilde{E},\ \varepsilon\in\mathcal{J}$ in addition

$$\wedge \begin{cases} \widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta},\widetilde{\tau}) \;<\; \infty\,, \\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(0^+,\widetilde{x}),\ \ \widetilde{\tau}(0^{++},\widetilde{x})\Big) \stackrel{\text{Def.}}{=} \limsup\limits_{k,l\downarrow 0\ (k<l)}\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\widetilde{x}),\ \ \widetilde{\tau}(l,\widetilde{x})\Big) \;=\; 0. \end{cases}$$

As mentioned before, we do not incorporate the general idea of distributions here and thus dispense with a "test set" $\widetilde{D} \subset \widetilde{E}$. So any points of $\widetilde{E}$ evolving along timed backward transitions can be compared directly. As a consequence of Gronwall's Lemma 0.9.1,

Proposition 0.9.3 *Every timed backward transitions* $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *and initial points* $\widetilde{x}, \widetilde{y} \in \widetilde{E}$ *(with* $\pi_1\, \widetilde{x} \leq \pi_1\, \widetilde{y}$*),* $t \in]0,1[$*,* $\varepsilon \in \mathcal{J}$ *satisfy*

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\ \widetilde{\tau}(t^{++}, \widetilde{y})\Big) \leq \widetilde{q}_\varepsilon(\widetilde{\vartheta}(0^+, \widetilde{x}), \widetilde{\tau}(0^{++}, \widetilde{y}))\quad e^{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})\, t} + \Big(\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) + 2\,\gamma_\varepsilon(\widetilde{\vartheta})\Big)\ \frac{e^{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})\, t} - 1}{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})}.$$

The detailed proof is given in Proposition 3.1.4. Here the parameter $\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})$ of the first argument is used — as in Aubin's mutational analysis in metric spaces (see § 0.3).

Now the terms "primitive" and "solution" are defined in a quite natural way — on the basis of first–order approximation.
The possibility of estimating the distance between any two evolving points motivates us to distinguish between "right–hand" and "left–hand backward primitive" and then state a corresponding estimate for primitives in Proposition 0.9.5 (proven in Proposition 3.2.3). This distinction, however, is only relevant for estimates, but not for existence results and so, we do not extend it to backward solutions explicitly.

Definition 0.9.4 $\widetilde{x} : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed right–hand backward primitive *of a map* $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$ *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall\ t \in]0,T[\qquad \exists\ \widehat{\gamma}_\varepsilon(t) = \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) : \qquad \gamma_\varepsilon(\widetilde{\vartheta}(t)) \leq \widehat{\gamma}_\varepsilon(t) < \infty,$
$\limsup_{h \downarrow 0} \frac{\widetilde{q}_\varepsilon\left(\widetilde{\vartheta}(t-h)\left(h^+, \widetilde{x}(t-h)\right),\ \widetilde{x}(t^{++})\right)}{h} \leq \widehat{\gamma}_\varepsilon(t), \qquad \limsup_{\varepsilon' \downarrow 0} \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) = 0,$

2. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$, *i.e. there is* $\omega_\varepsilon(\widetilde{x}, \cdot) :]0,T[\longrightarrow [0,\infty[$ *such that*
$\widetilde{q}_\varepsilon\Big(\widetilde{x}(s),\ \widetilde{x}(t)\Big) \leq \omega_\varepsilon(\widetilde{x},\ t-s)$ *for* $0 \leq s < t < T$, $\qquad \limsup_{h \downarrow 0} \omega_\varepsilon(\widetilde{x}, h) = 0,$

3. $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(0^+,\ \widetilde{x}(t)),\ \ \widetilde{x}(t^{++})\Big) = 0 \qquad \forall\ t \in [0,T[,$

4. $\pi_1\, \widetilde{x}(t) = t + \pi_1\, \widetilde{x}(0) \qquad \forall\ t \in [0,T[.$

$\widetilde{x} : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed left–hand backward primitive *of* $\widetilde{\vartheta}(\cdot)$
if it satisfies conditions (2.), (3.), (4.) and

1'. $\forall\ t \in]0,T[\qquad \exists\ \widehat{\gamma}_\varepsilon(t) = \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) : \qquad \gamma_\varepsilon(\widetilde{\vartheta}(t)) \leq \widehat{\gamma}_\varepsilon(t) < \infty,$
$\limsup_{h \downarrow 0} \frac{\widetilde{q}_\varepsilon\left(\widetilde{x}(t^+),\ \ \widetilde{\vartheta}(t-h)\left(h^{++}, \widetilde{x}(t-h)\right)\right)}{h} \leq \widehat{\gamma}_\varepsilon(t), \qquad \limsup_{\varepsilon' \downarrow 0} \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) = 0,$

Timed backward transitions induce their own right–hand and left–hand primitives due to condition (2.) of Definition 0.9.2. This result is easy to extend to piecewise constant functions $[0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon))$ (according to Lemma 3.2.2) and thus forms the basis of Euler approximations.

Proposition 0.9.5
Let $\widetilde{x}:[0,T[\longrightarrow \widetilde{E}$ *be a timed left–hand primitive of* $\widetilde{\vartheta}:[0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon))$ *and* $\widetilde{y}:[0,T[\longrightarrow \widetilde{E}$ *a timed right–hand primitive of* $\widetilde{\tau}:[0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon))$ *such that for each* $\varepsilon\in\mathcal{J}$,

$$\wedge\begin{cases} \alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(\cdot)) \;\le\; M_\varepsilon(\cdot) \;\in\; C^0([0,T[,\ ]0,\infty[),\\ \widehat{\gamma}_\varepsilon(\,\cdot\,,\widetilde{x},\widetilde{\vartheta}),\ \widehat{\gamma}_\varepsilon(\,\cdot\,,\widetilde{y},\widetilde{\tau}) \;\le\; R_\varepsilon(\cdot) \;\in\; C^0([0,T[,\ [0,\infty[),\\ \widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(\cdot),\ \widetilde{\tau}(\cdot)) \;\le\; c_\varepsilon(\cdot) \;\in\; C^0([0,T[,\ [0,\infty[),\\ \pi_1\,\widetilde{x}(0) \;=\; \pi_1\,\widetilde{y}(0). \end{cases}$$

Moreover, set $\mu_\varepsilon(t):=\displaystyle\int_0^t M_\varepsilon(s)\,ds.$

Then, for every $\varepsilon\in\mathcal{J}$ *and* $t\in]0,T[$, *these backward primitives fulfill the estimate*

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big) \;\le\; \widetilde{q}_\varepsilon\Big(\widetilde{x}(0^+),\ \widetilde{y}(0^{++})\Big)\ e^{\mu_\varepsilon(t)} \;+\; \int_0^t e^{\mu_\varepsilon(t)-\mu_\varepsilon(s)}\ \Big(c_\varepsilon(s)\;+\;5\,R_\varepsilon(s)\Big)\ ds.$$

Definition 0.9.6

For given $\widetilde{f}:\widetilde{E}\times[0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon))$, *a map* $\widetilde{x}:[0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand backward solution *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot)\ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand backward primitive of* $\widetilde{f}(\widetilde{x}(\cdot),\,\cdot\,):[0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, *i.e. for each* $\varepsilon\in\mathcal{J}$,

1. $\forall\ t\in]0,T[\quad \exists\,\widehat{\gamma}_\varepsilon(t):\quad \gamma_\varepsilon(\widetilde{f}(\widetilde{x}(t),t))\le\widehat{\gamma}_\varepsilon(t)<\infty,\quad \limsup_{\varepsilon'\downarrow 0}\ \varepsilon'^p\cdot\widehat{\gamma}_{\varepsilon'}(t)\;=\;0,$
   $\limsup_{h\downarrow 0}\ \frac{1}{h}\cdot\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t-h),\,t-h)\ (h^+,\ \widetilde{x}(t-h)),\quad \widetilde{x}(t^{++})\Big)\;\le\;\widehat{\gamma}_\varepsilon(t),$
2. $\widetilde{x}(\cdot)\in UC^{\rightarrow}([0,T[,\widetilde{E},\widetilde{q}_\varepsilon)$, *i.e. there is* $\omega_\varepsilon(\widetilde{x},\cdot):]0,T[\longrightarrow[0,\infty[$ *such that* $\widetilde{q}_\varepsilon\Big(\widetilde{x}(s),\ \widetilde{x}(t)\Big)\le\omega_\varepsilon(\widetilde{x},\,t-s)$ *for* $0\le s<t<T$, $\quad\limsup_{h\downarrow 0}\ \omega_\varepsilon(\widetilde{x},h)\;=\;0,$
3. $\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(0^+,\ \widetilde{x}(t)),\ \ \widetilde{x}(t^{++})\Big)\;=\;0 \qquad \forall\ t\in[0,T[,$
4. $\pi_1\,\widetilde{x}(t)\;=\;t\;+\;\pi_1\,\widetilde{x}(0) \qquad \forall\ t\in[0,T[.$

The first existence result is obtained in exactly the same way as for timed right–hand *forward* solutions in § 0.8. Indeed, Convergence Theorem 0.8.9 can be adapted to a sequence of timed right–hand backward solutions (as shown in Proposition 3.3.2) and then, Euler approximations in combination with timed transitional compactness lead to Proposition 3.3.3 stating

Theorem 0.9.7 (Existence of timed right–hand backward solutions due to timed transitional compactness)

Assume that the tuple $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\rightarrow\!\shortmid}(\widetilde{E},(\widetilde{q}_\varepsilon)))$ *is timed transitionally compact. Moreover let the function* $\widetilde{f}:\widetilde{E}\times[0,T]\longrightarrow\widetilde{\Theta}_p^{\rightarrow\!\shortmid}(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *satisfy for every* $\varepsilon\in\mathcal{J}$

1. $M_\varepsilon := \sup_{t,\widetilde{z}} \alpha_\varepsilon^{\rightarrow\!\shortmid}(\widetilde{f}(\widetilde{z},t)) < \infty,$
2. $c_\varepsilon(h) := \sup_{t,\widetilde{z}} \beta_\varepsilon(\widetilde{f}(\widetilde{z},t))(h) < \infty, \qquad c_\varepsilon(h) \longrightarrow 0$ *for* $h\downarrow 0,$
3. $\exists\ R_\varepsilon : \sup_{t,\widetilde{z}} \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \le R_\varepsilon < \infty, \qquad \varepsilon'^p\, R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon'\downarrow 0,$
4. $\exists\ \widehat{\omega}_\varepsilon(\cdot): \quad \widetilde{Q}_\varepsilon^{\rightarrow\!\shortmid}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \le R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2) + t_2 - t_1\Big)$
   for all $0\le t_1\le t_2\le T$ *and* $\widetilde{z}_1,\widetilde{z}_2\in\widetilde{E}$ *with* $\pi_1\,\widetilde{z}_1\le\pi_1\,\widetilde{z}_2,$
   $\widehat{\omega}_\varepsilon(\cdot)\ge 0$ *nondecreasing,* $\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) = 0.$

Then for every initial point $\widetilde{x}_0\in\widetilde{E}$, *there is a timed right–hand backward solution* $\widetilde{x}:[0,T[\,\longrightarrow\widetilde{E}$ *of* $\overset{\circ}{\widetilde{x}}(\cdot)\ni\widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *in* $[0,T[$ *with* $\widetilde{x}(0)=\widetilde{x}_0.$
If assumption (4.) is replaced by

4'. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon\ge 0: \quad \widetilde{Q}_\varepsilon^{\rightarrow\!\shortmid}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \le R_\varepsilon + L_\varepsilon\cdot\widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2) + \widehat{\omega}_\varepsilon(t_2-t_1)$
   for all $0\le t_1\le t_2\le T$ *and* $\widetilde{z}_1,\widetilde{z}_2\in\widetilde{E}$ *with* $\pi_1\,\widetilde{z}_1\le\pi_1\,\widetilde{z}_2,$
   $\widehat{\omega}_\varepsilon(\cdot)\ge 0$ *nondecreasing,* $\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) = 0.$

then any other timed right–hand backward solution $\widetilde{z}(\cdot)$ *(with* $\widetilde{z}(0)=\widetilde{x}_0$*) fulfills*

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})\Big) \le 6\,R_\varepsilon\,t\,e^{M_\varepsilon\,t}\,\Big(1+L_\varepsilon\,e^{(L_\varepsilon\,e^{M_\varepsilon\,T}+M_\varepsilon)\,T}\cdot t\Big) \qquad \forall\, t\in[0,T[,\ \varepsilon\in\mathcal{J}.$$

In particular, this theorem implies that a timed right–hand backward primitive with given initial value exists for every function $\widetilde{\vartheta}:[0,T[\,\longrightarrow\widetilde{\Theta}_p^{\rightarrow\!\shortmid}(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ satisfying

1. $M_\varepsilon := \sup_t \alpha_\varepsilon^{\rightarrow\!\shortmid}(\widetilde{\vartheta}(t)) < \infty$
2. $c_\varepsilon(h) := \sup_t \beta_\varepsilon(\widetilde{\vartheta}(t))(h) < \infty, \qquad c_\varepsilon(h)\longrightarrow 0$ for $h\downarrow 0$
3. $\exists\ R_\varepsilon : \sup_t \gamma_\varepsilon(\widetilde{\vartheta}(t)) \le R_\varepsilon, \quad \varepsilon'^p\ R_{\varepsilon'}\longrightarrow 0$ for $\varepsilon'\longrightarrow 0,$
   $\limsup_{0\le t_2-t_1\to 0}\ \widetilde{Q}_\varepsilon^{\rightarrow\!\shortmid}\Big(\widetilde{\vartheta}(t_1),\ \widetilde{\vartheta}(t_2)\Big) \le R_\varepsilon\,.$

So far all the existence results have been based on (timed) transitional compactness. Proposition 0.9.3 about the distance between *any* two evolving points opens up the possibility of supposing a weaker condition that is corresponding to completeness. Then, we need the preceding estimates for verifying even the Cauchy property of approximating sequences. So the assumption $R_\varepsilon(\cdot) = 0$ is obviously unavoidable. Moreover if a sequence fulfills the Cauchy condition with respect to every $\widetilde{q}_\varepsilon$ $(\varepsilon \in \mathcal{J})$, then its limit has to be the same for each ε.

For these reasons, we restrict ourselves to merely *one* timed ostensible metric $\widetilde{q}$ on $\widetilde{E} \stackrel{\text{Def.}}{=} I\!R \times E$ and consider timed backward transitions $\widetilde{\vartheta}$ on $(\widetilde{E}, \widetilde{q})$ with $\gamma(\widetilde{\vartheta}) = 0$. For the sake of simplicity, all transitions are also supposed to be *uniformly Lipschitz* continuous in time direction, i.e. there exists a constant $\beta^{\text{Lip}}(\widetilde{\vartheta}) \geq 0$ with

$$\widetilde{q}\Big(\widetilde{\vartheta}(s, \widetilde{x}),\ \widetilde{\vartheta}(t, \widetilde{x})\Big) \ \leq\ \beta^{\text{Lip}}(\widetilde{\vartheta}) \cdot (t - s)$$

for all $0 \leq s < t \leq 1$, $\widetilde{x} \in \widetilde{E}$.

Corollary 3.3.8 exemplifies the existence results due to completeness presented in § 3.3.3 :

Theorem 0.9.8 (Timed backward solutions in one–sided complete $(\widetilde{E}, \widetilde{q})$)

Suppose that the timed ostensible metric space $(\widetilde{E}, \widetilde{q})$ is one–sided complete, i.e. for any sequence $(\widetilde{z}_n)_n$ in $\widetilde{E}$ satisfying $\widetilde{q}(\widetilde{z}_m, \widetilde{z}_n) \longrightarrow 0$ for $m, n \longrightarrow \infty$ $(m < n)$, there is an element $\widetilde{z} \in E$ such that $\widetilde{q}(\widetilde{z},\ \widetilde{z}_n) \longrightarrow 0$, $\widetilde{q}(\widetilde{z}_n,\ \widetilde{z}) \longrightarrow 0$ $(n \longrightarrow \infty)$.

Furthermore suppose for any element $\widetilde{y} \in \widetilde{E}$ and all sequences $(\widetilde{y}_n)$, $(\widetilde{z}_n)$ in $\widetilde{E}$ that

$$\widetilde{q}(\widetilde{y}, \widetilde{y}_n) \longrightarrow 0, \qquad \widetilde{q}(\widetilde{y}_n, \widetilde{z}_n) \longrightarrow 0 \qquad (n \longrightarrow \infty), \qquad \pi_1\, \widetilde{y}_n \leq \pi_1\, \widetilde{z}_n$$

always imply $\widetilde{q}(\widetilde{y}, \widetilde{z}_n) \longrightarrow 0.$

Assume for $\widetilde{f} : \widetilde{E} \times [0, T] \longrightarrow \widetilde{\Theta}_0^{\rightarrow}(\widetilde{E}, \widetilde{q})$

1. *there exists $L > 0$ such that for any $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{E}$, $0 \leq t_1 \leq t_2 \leq 2\,T$ with $\pi_1\, \widetilde{x}_1 \leq \pi_1\, \widetilde{x}_2$:* $\widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}_1, t_1),\ \widetilde{f}(\widetilde{x}_2, t_2)\Big) \ \leq\ L \cdot \Big(\widetilde{q}(\widetilde{x}_1,\ \widetilde{x}_2) \ +\ t_2 - t_1\Big),$
2. $M \ := \ \sup_{\widetilde{x}, t}\ \alpha^{\rightarrow}(\widetilde{f}(\widetilde{x}, t)) \ <\ \infty,$
3. $c \ := \ \sup_{\widetilde{x}, t}\ \beta^{\text{Lip}}(\widetilde{f}(\widetilde{x}, t)) \ <\ \infty,$
4. $4\, L\, T\, e^{2\, M\, T} \ <\ 1.$

For every point $\widetilde{x}_0 \in \widetilde{E}$, there is a timed right–hand backward solution $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ in $[0, T[$ with $\widetilde{x}(0) = \widetilde{x}_0$ such that any other timed right–hand backward solution $\widetilde{z}(\cdot) \in \text{Lip}^{\rightarrow}([0, T[,\ \widetilde{E},\ \widetilde{q})$ with $\widetilde{z}(0) = \widetilde{x}_0$ fulfills $\widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})\Big) = 0$ *for all* $t \in [0, T[.$

There is no general inequality comparing $\widetilde{q}(\widetilde{x}(t),\ \widetilde{y}(t))$ with $\widetilde{q}(\widetilde{x}(t^+),\ \widetilde{y}(t^{++}))$ for two curves $\widetilde{x}, \widetilde{y} : [0,T[\longrightarrow \widetilde{E}$. In this context, the triangle inequality has an advantage over its timed counterpart because a general assumption about converging sequences implies equality in ostensible metric spaces.

Lemma 0.9.9 *Assume for the ostensible metric space (E,q) that left–convergence of a sequence $(z_n)_{n\in I\!N}$, i.e. $q(z, z_n) \longrightarrow 0$, always implies right–convergence of a subsequence $(z_{n_j})_{j\in I\!N}$, i.e. $q(z_{n_j}, z) \longrightarrow 0 \quad (j \longrightarrow \infty)$.*

Then, $q(x(t),\ y(t)) = q(x(t^+),\ y(t^{++}))$ for every $x(\cdot),\ y(\cdot) \in UC^{\to}([0,T],\ E,\ q)$ and $t \in]0,T[$.

The proof is a consequence of Lemma 1.4.7 (2.) and Corollary 3.4.2.
In short, the hypothesis of Lemma 0.9.9 for a (not timed) ostensible metric space (E,q) implies that upper limits (abbreviated as "+","++") are dispensable. As a consequence, we obtain the existence without time restrictions (like condition (4.) of Theorem 0.9.8). Indeed, Proposition 3.4.5 states

Theorem 0.9.10 (Long–time existence of backward solutions in ostensible metric spaces)
In addition to the assumptions of Lemma 0.9.9, let the ostensible metric space (E,q) be one–sided complete (as in Theorem 0.9.8)

Suppose for $f : E \times [0,T] \longrightarrow \Theta_0^{\to\!\!|}(E,q)$

1. *there exists $L > 0$ such that for any $x_1, x_2 \in E$, $0 \le t_1 \le t_2 \le T$,*
   $Q^{\to\!\!|}\Big(f(x_1,t_1),\ f(x_2,t_2)\Big) \le L \cdot \Big(q(x_1,\ x_2) + t_2 - t_1\Big),$
2. $M := \sup\limits_{x,t} \alpha^{\to\!\!|}(f(x,t)) < \infty,$
3. $c := \sup\limits_{x,t} \beta^{\mathrm{Lip}}(f(x,t)) < \infty,$

For every point $x_0 \in E$ there exists a right–hand backward solution $x : [0,T[\longrightarrow E$ of the generalized mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ in $[0,T[$ with $x(0) = x_0$ such that any other right–hand backward solution $z(\cdot)$ (with $z(0) = x_0$) fulfills

$$q\Big(x(t),\ z(t)\Big) = 0 \qquad \text{for all } t \in [0,T[.$$

Chapter 1

The triangle inequality in time direction and other preliminaries

The symmetry of a metric (on a nonempty set E) might prove to be an essential obstacle when describing evolutions that are not reversible in time. In this context, the interesting question arises which properties are relevant to a distance function on E.

The triangle inequality is necessary for estimating the distance between two elements by means of a third point. For this reason we cannot develop an approach completely without it. However there is an opportunity of weakening it by taking the time direction into consideration. Strictly speaking, the set $E \neq \emptyset$ is replaced by the product

$$\widetilde{E} \;:=\; I\!R \times E \;=\; \{\, \widetilde{x} = (t, x) \mid t \in I\!R,\; x \in E \,\}$$

with the first component t representing the time of the second component $x \in E$.
Now a function

$$\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[, \quad (\widetilde{x}_1, \widetilde{x}_2) \longmapsto \widetilde{q}(\widetilde{x}_1, \widetilde{x}_2)$$

is to play the role of a distance. Then in a figurative sense, the first argument $\widetilde{x}_1 = (t_1, x_1)$ refers to the *earlier* point whereas the second argument $\widetilde{x}_2 = (t_2, x_2)$ represents the *later* element of $\widetilde{E}$, i.e. we usually consider the case $t_1 \leq t_2$. In this time–directed situation, the principle of triangle inequality affects only points $\widetilde{z} = (s, z) \in \widetilde{E}$ whose time component s is between t_1 and t_2 $(t_1 \leq s \leq t_2)$ and, it is motivating the so–called *timed triangle inequality* in Definition 1.1.2.

This chapter provides the preliminaries for describing evolutions in $\widetilde{E} \overset{\text{Def.}}{=} I\!R \times E$, supplied with a countable family $(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}$ of distance functions $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$. It contains adjustments of topological terms, such as compactness and completeness, and it introduces the so–called *standard hypotheses* $(L^{\Rightarrow})$, $(R^{\Rightarrow})$, $(R^{\Leftarrow})$ on $\widetilde{E}$ that consist in sequential properties of closed spheres in $(\widetilde{E}, \widetilde{q})$. All of them are obvious in metric spaces, but they cannot be proven without the symmetry of $\widetilde{q}$ in general.

Dispensing with symmetry (of $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$) has the analytical consequence that the distance

$$[0, T[\longrightarrow [0, \infty[, \qquad t \longmapsto \widetilde{q}\,(\widetilde{x}(t),\ \widetilde{y}(t))$$

between two curves $\widetilde{x}(\cdot), \widetilde{y}(\cdot) : [0, T[\longrightarrow \widetilde{E}$ might not be continuous any longer — even if both $\widetilde{x}(\cdot)$ and $\widetilde{y}(\cdot)$ are quite "smooth" with respect to $\widetilde{q}$. For this reason, the last section provides generalized versions of Gronwall's Lemma for semicontinuous functions.

1.1 Definition of ostensible metric

Due to the well–known definition, a metric $d : M \times M \longrightarrow [0, \infty[$ on a nonempty set M has to fulfill three properties : positive definite, symmetric, triangle inequality. Now we introduce weakened terms for distance functions. It leads to so–called *ostensible metrics* that are mainly based on the triangle inequality. In topology, these generalized forms of distances are called *quasi–pseudo–metrics* (see e.g. [42, Kelly 63], [43, Künzi 92]), but for merely linguistic reasons we prefer the adjective "ostensible".

Definition 1.1.1 *Let E be a nonempty set.*
$q : E \times E \longrightarrow [0, \infty[$ *fulfills the so–called* triangle inequality *if for all* $x, y, z \in E$,

$$q(x, z) \ \le\ q(x, y) \ +\ q(y, z).$$

$q : E \times E \longrightarrow [0, \infty[$ *is called* ostensible metric *on E if it satisfies the conditions :*

1. $\forall\ x \in E : \quad q(x, x) \ = \ 0 \qquad$ (*reflexive*)
2. $\forall\ x, y, z \in E : \quad q(x, z) \ \le\ q(x, y) + q(y, z) \qquad$ (*triangle inequality*).

Then (E, q) *is called* ostensible metric space. □

Definition 1.1.2 *Furthermore set* $\widetilde{E} := I\!\!R \times E$. $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ *fulfills the so–called* timed triangle inequality *if for every* $(r, x),\ (s, y),\ (t, z) \in \widetilde{E}$ *with* $r \le s \le t$,

$$\widetilde{q}\Big((r, x),\ (t, z)\Big) \ \le\ \widetilde{q}\Big((r, x),\ (s, y)\Big) \ +\ \widetilde{q}\Big((s, y),\ (t, z)\Big).$$

$\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ *is called* timed ostensible metric *on $\widetilde{E}$ if it satisfies*

$$\widetilde{q}\Big((t, z),\ (t, z)\Big) \ = \ 0 \qquad (\textit{reflexive})$$
$$\widetilde{q}\Big((r, x),\ (t, z)\Big) \ \le\ \widetilde{q}\Big((r, x),\ (s, y)\Big) \ +\ \widetilde{q}\Big((s, y),\ (t, z)\Big) \qquad (\textit{timed triangle inequality})$$

for all $(r, x),\ (s, y),\ (t, z) \in \widetilde{E}$ *with* $r \le s \le t$.
$(\widetilde{E}, \widetilde{q})$ *is then called* timed ostensible metric space.

Remark. 1. In the literature on topology (e.g. [67, Wilson 31], [42, Kelly 63], [62, Stoltenberg 69], [43, Künzi 92]) a so–called *quasi–metric* $p : E \times E \longrightarrow [0,\infty[$ on a set E satisfies the triangle inequality and is positive definite, i.e.

$$p(x,y) = 0 \iff x = y \qquad \text{for every } x, y \in E.$$

A so–called *pseudo–metric* $p : E \times E \longrightarrow [0,\infty[$ on a set $E \neq \emptyset$ is characterized by the properties : reflexive (i.e. $p(x,x) = 0$ for all x), symmetric (i.e. $p(x,y) = p(y,x)$ for all x, y) and the triangle inequality. □

2. Each $q : E \times E \longrightarrow [0,\infty[$ satisfying the triangle inequality induces a function $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ in the way

$$\widetilde{q}\Big((s,x),\ (t,y)\Big) := |s-t| + q(x,y) \qquad \text{for all } (s,x),\ (t,y) \in \widetilde{E}$$

such that $\widetilde{q}$ fulfills the (timed) triangle inequality.
So the results about *timed* ostensible metric spaces $(\widetilde{E}, \widetilde{q})$ can be applied to ostensible metric spaces (E, q).

Now some abbreviations for continuous functions of time are introduced. Here the symbol $\rightarrow$ is to remind us of considering the forward time direction :

Definition 1.1.3 *Let $J \subset I\!R$ be nonempty, $D \subset E \neq \emptyset$, $q : E \times E \longrightarrow [0,\infty[$.*

1. $\mathrm{Lip}^{\rightarrow}(J,E,q)$ denotes the set of $f : J \longrightarrow E$ for which there is $L > 0$ with

$$\forall\ s,t \in J : \qquad s < t \implies q(f(s), f(t)) \leq L\,(t-s).$$

2. $UC^{\rightarrow}(J,E,q)$ abbreviates the set of uniformly continuous maps $f : J \longrightarrow E$ in the sense that there is some nondecreasing $\omega(f,\cdot) :]0,\infty[\longrightarrow [0,\infty[$

with $$\limsup_{h \downarrow 0}\ \omega(f,h) = 0$$

and $$\forall\ s,t \in J : \qquad s < t \implies q(f(s), f(t)) \leq \omega(f, t-s).$$

Such function $\omega(f,\cdot)$ is called modulus of continuity *(of $f(\cdot)$).*

3. The pair (E,q) satisfies the condition $(BUC^{\rightarrow})$ if and only if all functions $x, y \in UC^{\rightarrow}([0,T],E,q)$ (with $T < \infty$) fulfill $\sup_{[0,T]} q(x(\cdot), y(\cdot)) < \infty$.

Remark. If $q : E \times E \longrightarrow [0,\infty[$ satisfies the triangle inequality then property $(BUC^{\rightarrow})$ is trivial for (E,q) since for any $x, y \in UC^{\rightarrow}([0,T],E,q)$ and $t \in [0,T] \subset [0,\infty[$,

$$\begin{aligned} q(x(t), y(t)) &\leq q(x(t), x(T)) + q(x(T), y(0)) + q(y(0), y(t)) \\ &\leq \omega(x(\cdot), T) + q(x(T), y(0)) + \omega(y(\cdot), T). \end{aligned}$$

However, this conclusion does not hold for the timed triangle inequality in general.

1.2 One–sided and two–sided compactness

As a first step, we need a relationship between the convergence of a sequence in $(\widetilde{E}, \widetilde{q})$ and the asymptotic behavior of the time components.

Definition 1.2.1 *Let E be a nonempty set, $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R} \times E$, $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$. $(\widetilde{E}, \widetilde{q})$ is called* time continuous *if every sequence $(\widetilde{x}_n = (t_n, x_n))_{n \in \mathbb{N}}$ in $\widetilde{E}$ and element $\widetilde{x} = (t, x) \in \widetilde{E}$ with $\widetilde{q}(\widetilde{x}_n, \widetilde{x}) \longrightarrow 0$ $(n \longrightarrow \infty)$ fulfill $t_n \longrightarrow t$ $(n \longrightarrow \infty)$ (i.e. the projection $\pi_1(\cdot) : \widetilde{E} \longrightarrow \mathbb{R}$, $\widetilde{x} = (t, x) \longmapsto t$ is right–sequentially continuous with respect to $\widetilde{q}$).*

Generally speaking, constructing solutions (of evolution systems) by approximation is usually based on compactness or completeness. In this section, we are adapting the term of sequential compactness to $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ and distinguish between the order of arguments $\widetilde{x}_{n_j}$, $\widetilde{x}$ in the vanishing distance $\widetilde{q}_\varepsilon$:
$\widetilde{q}_\varepsilon(\widetilde{x}_{n_j}, \widetilde{x}) \longrightarrow 0$ $(j \longrightarrow \infty)$ is regarded as *right – convergence* of $(\widetilde{x}_{n_j})_{j \in \mathbb{N}}$ to $\widetilde{x}$ and
$\widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{x}_{n_j}) \longrightarrow 0$ as *left – convergence.*
The following definitions can be extended to tuples $(E, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$ without time component in a canonical way. In the case of ostensible metrics q_ε $(\varepsilon \in \mathcal{J})$, it proves to be equivalent to Definition 0.6.8.

Definition 1.2.2 *Let $E \neq \emptyset$ be a set, $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R} \times E$, $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$.*

1. $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* one–sided sequentially compact (uniformly with respect to ε) *if for every $\widetilde{z} \in \widetilde{E}$, $r_\varepsilon > 0$ $(\varepsilon \in \mathcal{J})$ and any sequence $(\widetilde{x}_n)_{n \in \mathbb{N}}$, in $\widetilde{E}$ satisfying*

$$\widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{x}_n) \leq r_\varepsilon \qquad \forall\ n \in \mathbb{N} \qquad \forall\ \varepsilon \in \mathcal{J}$$

there exist a subsequence $(\widetilde{x}_{n_j})_{j \in \mathbb{N}}$ and an element $\widetilde{x} \in \widetilde{E}$ such that

$$\widetilde{q}_\varepsilon(\widetilde{x}_{n_j}, \widetilde{x}) \longrightarrow 0 \qquad \text{for } j \longrightarrow \infty \qquad \forall\ \varepsilon \in \mathcal{J}.$$

2. $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed two–sided sequentially compact (uniformly with respect to ε) *if for every $\widetilde{z} \in \widetilde{E}$, $r_\varepsilon > 0$ $(\varepsilon \in \mathcal{J})$ and any sequences $(\widetilde{x}_n)_{n \in \mathbb{N}}$, $(\widetilde{y}_n)_{n \in \mathbb{N}}$ in E satisfying*

$$\begin{array}{lll} q_\varepsilon(\widetilde{x}_n, \widetilde{y}_n) \longrightarrow 0 & \text{for } n \longrightarrow \infty & \forall\ \varepsilon \in \mathcal{J} \\ q_\varepsilon(\widetilde{z}, \widetilde{x}_n),\ q_\varepsilon(\widetilde{z}, \widetilde{y}_n) \leq r_\varepsilon & \forall\ n \in \mathbb{N} & \forall\ \varepsilon \in \mathcal{J} \\ \pi_1\, \widetilde{x}_n < \pi_1\, \widetilde{y}_n & \forall\ n \in \mathbb{N} & \end{array}$$

there exist subsequences $(\widetilde{x}_{n_j})_{j \in \mathbb{N}}$, $(\widetilde{y}_{n_j})_{j \in \mathbb{N}}$ and an element $\widetilde{x} \in \widetilde{E}$ such that

$$\left.\begin{array}{l} q_\varepsilon(\widetilde{x}_{n_j}, \widetilde{x}) \longrightarrow 0 \\ q_\varepsilon(\widetilde{x}, \widetilde{y}_{n_j}) \longrightarrow 0 \end{array}\right. \qquad \text{for } j \longrightarrow \infty \qquad \forall\ \varepsilon \in \mathcal{J}.$$

1.3 One–sided and two–sided completeness

Now the notion of completeness is specified for a tuple (E, q).
In contrast to the preceding definitions of sequential compactness (in § 1.2), both left– and right–convergence are always demanded, i.e. the limiting point is to appear at both arguments of the vanishing distance q. The difference between "one–sided complete" and "two–sided complete" refers to the number of approximating sequences :

Definition 1.3.1 *Let E be a nonempty set and $q : E \times E \longrightarrow [0, \infty[$.*
(E, q) is called one–sided complete *if for every sequence (x_n) in E with $q(x_m, x_n) \longrightarrow 0$ for $m, n \longrightarrow \infty$ $(m < n)$, there is some element $x \in E$ such that*

$$q(x,\ x_n) \ \longrightarrow\ 0 \qquad q(x_n,\ x) \ \longrightarrow\ 0 \qquad\qquad (n \longrightarrow \infty).$$

(E, q) is called two–sided complete *if for any $(x_n)_{n\in I\!N}$, $(y_n)_{n\in I\!N}$ in E satisfying*

$$\left.\begin{array}{rcl} q(x_m, x_n) & \longrightarrow & 0 \\ q(y_n,\ y_m) & \longrightarrow & 0 \end{array}\right. \qquad \textit{for}\ \ m, n \longrightarrow \infty \quad (m\ <\ n),$$

$$q(x_n,\ y_n) \ \longrightarrow\ 0 \qquad \textit{for}\ \ n \longrightarrow \infty$$

there exists $x \in E$ such that

$$\left.\begin{array}{rcl} q(x_n,\ \ x) & \longrightarrow & 0 \\ q(x,\ \ y_n) & \longrightarrow & 0 \end{array}\right. \qquad \textit{for}\ \ n \longrightarrow \infty.$$

Remark. 1. Every one–sided complete (E, q) with q satisfying the triangle inequality is also two–sided complete. Indeed, for every sequences $(x_n)_{n\in I\!N}$, $(y_n)_{n\in I\!N}$ in E fulfilling

$$q(x_m, x_n) \ \longrightarrow\ 0 \qquad \text{for}\ \ m, n \longrightarrow \infty \quad (m\ <\ n),$$

$$q(x_n,\ y_n) \ \longrightarrow\ 0 \qquad \text{for}\ \ n \longrightarrow \infty$$

the triangle inequality implies for the limit x of $(x_n)_{n\in I\!N}$

$$q(x, y_n)\ \le\ q(x, x_n)\ +\ q(x_n, y_n)\ \longrightarrow\ 0 \qquad (n \longrightarrow \infty).$$

2. If (E, q) is one–sided complete and q fulfills the triangle inequality, any of these limits x, y of $(x_n)_{n\in I\!N}$ are equivalent in terms of $q(x, y) = 0$.

1.4 Standard hypotheses $(L^{\Rightarrow})$, $(R^{\Rightarrow})$, $(R^{\Leftarrow})$

Roughly speaking, standard hypotheses $(L^{\Rightarrow})$, $(R^{\Rightarrow})$, $(R^{\Leftarrow})$ on (E,q) are to provide sequential continuity for the distance from any fixed point.
To be more precise, we consider a sequence $(x_n)_{n\in\mathbb{N}}$ in E and its limiting point x. Then for any element $z \in E$, the distance between x and z is supposed to be approximated by the distance between x_n and z. This leads to different conditions on (E,q) depending on the left– or right–convergence of $(x_n)_{n\in\mathbb{N}}$ and on the consideration of $q(z,x)$ or $q(x,z)$. For a metric, all this properties are obvious consequences of symmetry and triangle inequality. However they do not hold for ostensible metrics in general.

Definition 1.4.1 *Let E be a nonempty set and $q : E \times E \longrightarrow [0,\infty[$.*
The left–hand spheres of (E,q) *are said to be* right–sequentially closed *if (and only if) every sequence $(x_n)_{n\in\mathbb{N}}$ in E and $x \in E$ with $q(x_n,x) \longrightarrow 0$ $(n \longrightarrow \infty)$ also fulfill*

$$\lim_{n\longrightarrow\infty} q(z,x_n) \;=\; q(z,x) \qquad \text{for every } z \in E.$$

This property is abbreviated as standard hypothesis $(L^{\Rightarrow})$ *for* (E,q).

Remark. The name of this property is due to the following restatement : For any $z \in E$, the *left–hand sphere* at z with radius $r > 0$ is defined as

$$S_r^{\text{left}}(z) \;:=\; \{x \in E \mid q(z,x) = r\}.$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $S_r^{\text{left}}(z)$ right–converging to $x \in E$, i.e. $q(x_n,x) \longrightarrow 0$ $(n \longrightarrow \infty)$. Then x is contained in $S_r^{\text{left}}(z)$ as well.
(Roughly speaking the limit x cannot be a better approximation of z than the sequence.)

Lemma 1.4.2 *1. For any timed ostensible metric space $(\widetilde{E},\widetilde{q})$ with standard hypothesis $(L^{\Rightarrow})$, the right–convergence $\widetilde{q}(\widetilde{x}_n,\widetilde{x}) \longrightarrow 0$ always implies $\widetilde{q}(\widetilde{x},\widetilde{x}_n) \longrightarrow 0$.*

2. Let $q : E \times E \longrightarrow [0,\infty[$ satisfy the triangle inequality. If the right–convergence of any sequence $(x_n)_{n\in\mathbb{N}}$, i.e. $q(x_n,x) \longrightarrow 0$, always guarantees $q(x,x_{n_j}) \longrightarrow 0$ $(j \longrightarrow \infty)$ for a subsequence $(x_{n_j})_{j\in\mathbb{N}}$, then (E,q) fulfills standard hypothesis $(L^{\Rightarrow})$.

□

Definition 1.4.3 *Let E be a nonempty set and $q : E \times E \longrightarrow [0, \infty[$.*
The right–hand spheres of (E, q) *are said to be* right–sequentially closed *if (and only if) every sequence $(x_n)_{n \in I\!N}$ in E and $x \in E$ with $q(x_n, x) \longrightarrow 0 \quad (n \longrightarrow \infty)$ also fulfill*

$$\lim_{n \longrightarrow \infty} q(x_n, z) \;=\; q(x, z) \qquad \textit{for every } z \in E.$$

This feature is abbreviated as standard hypothesis $(R^{\Rightarrow})$ *for (E, q).*

Remark. Correspondingly to the preceding Def. 1.4.1, the name of this property results from the following notion : The *right–hand sphere* at $z \in E$ with radius $r > 0$ is defined as $\quad S_r^{\text{right}}(z) \;:=\; \{x \in E \mid q(x, z) = r\}.$
Now suppose the sequence $(x_n)_{n \in I\!N}$ in $S_r^{\text{right}}(z)$ to be right–converging to $x \in E$, i.e. $q(x_n, x) \longrightarrow 0 \; (n \longrightarrow \infty)$. Then x is contained in $S_r^{\text{right}}(z)$ as well.

Lemma 1.4.4 *Let $q : E \times E \longrightarrow [0, \infty[$ satisfy the triangle inequality.*
If the right–convergence of any sequence $(x_n)_{n \in I\!N}$, i.e. $q(x_n, x) \longrightarrow 0$, always implies $q(x, x_{n_j}) \longrightarrow 0 \quad (j \longrightarrow \infty)$ for a subsequence $(x_{n_j})_{j \in I\!N}$, then (E, q) fulfills standard hypothesis $(R^{\Rightarrow})$.
So in particular, every ostensible metric q on E satisfying standard hypothesis $(L^{\Rightarrow})$ also fulfills standard hypothesis $(R^{\Rightarrow})$. □

The substantial benefit of standard hypothesis $(R^{\Rightarrow})$ affects right–convergent sequences of functions $[0, T] \longrightarrow \widetilde{E}$: If they are equi–continuous (in positive time direction) then so is the limit function, i.e. strictly speaking,

Proposition 1.4.5 *Suppose standard hypothesis $(R^{\Rightarrow})$ for $(\widetilde{E}, \widetilde{q})$ and let the functions $\widetilde{x}_n : [0, T] \longrightarrow \widetilde{E} \; (n \in I\!N)$ and $\widetilde{x} : J \longrightarrow \widetilde{E} \quad (J \subset [0, T])$ satisfy for any $s \in]0, T[$, $t \in J$, $h \in]0, T - t[$*

1. $\widetilde{q}\Big(\widetilde{x}_n(s),\; \widetilde{x}_n(s + h)\Big) \;\le\; \omega(h^+) \;\stackrel{\text{Def.}}{=}\; \limsup_{k \downarrow h} \omega(k)$
2. $\exists \;\; (\delta_n)_{n \in I\!N} : \quad \widetilde{q}\Big(\widetilde{x}_n(t - \delta_n),\; \widetilde{x}(t)\Big) \;\longrightarrow\; 0, \qquad \delta_n \downarrow 0 \qquad (n \longrightarrow \infty),$
3. $\pi_1 \, \widetilde{x}_n(\cdot) \;\le\; \pi_1 \, \widetilde{x}(\cdot)$ *nondecreasing* *for every $n \in I\!N$*

with the modulus $\omega(\cdot)$ of continuity (i.e. $\omega(\cdot)$ is nondecreasing and $\lim_{h \downarrow 0} \omega(h) = 0$).
Then $\quad \widetilde{q}\Big(\widetilde{x}(t),\; \widetilde{x}(t + h)\Big) \;\le\; \omega(h^+) \qquad$ *for every $t \in J$, $h > 0$ with $t + h \in J$.*

Proof. Assume that there exist $\eta > 0,\ t \in J,\ \ h > 0$ such that $t+h \in J$ and

$$\widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{x}(t+h)\Big) \ > \ \omega(h^+) + 2\,\eta.$$

Choose the sequences $\delta_n \downarrow 0,\ \ \delta'_n \downarrow 0$ satisfying

$$\widetilde{q}\Big(\widetilde{x}_n(t-\delta'_n),\ \widetilde{x}(t)\Big) \longrightarrow 0, \qquad \widetilde{q}\Big(\widetilde{x}_n(t+h-\delta_n),\ \widetilde{x}(t+h)\Big) \longrightarrow 0 \qquad (m \longrightarrow \infty).$$

Since the right–hand spheres of $(\widetilde{E}, \widetilde{q})$ are right–sequentially closed, we conclude for every $n \in I\!N$ sufficiently large

$$\begin{array}{rcll} \omega(h+\delta'_n) + \eta & < & \omega(h^+) + 2\,\eta & \\ & < & \widetilde{q}\Big(\widetilde{x}_n(t-\delta'_n),\ \widetilde{x}(t+h)\Big) & \\ & \leq & \widetilde{q}\Big(\widetilde{x}_n(t-\delta'_n),\ \widetilde{x}_n(t+h-\delta_n)\Big) & + \ \ \widetilde{q}\Big(\widetilde{x}_n(t+h-\delta_n),\ \widetilde{x}(t+h)\Big) \\ & \leq & \omega((h-\delta_n+\delta'_n)^+) & + \ \ \frac{\eta}{2} \\ & \leq & \omega(h+\delta'_n) & + \ \ \frac{\eta}{2} \end{array}$$

— a contradiction. □

Definition 1.4.6 *Let E be a nonempty set and $q : E \times E \longrightarrow [0,\infty[$.*
The right–hand spheres of (E,q) *are said to be* left–sequentially closed *if (and only if) every sequence $(x_n)_{n\,\in\,I\!N}$ in E and $x \in E$ with $q(x, x_n) \longrightarrow 0 \ \ (n \longrightarrow \infty)$ also fulfill*

$$\lim_{n\,\longrightarrow\,\infty} \ q(x_n, z) \ = \ q(x,z) \qquad \textit{for every } z \in E.$$

This property is abbreviated as standard hypothesis $(R^{\Leftarrow})$ *for (E,q).*

Lemma 1.4.7 *1. For any timed ostensible metric space $(\widetilde{E}, \widetilde{q})$ with standard hypothesis $(R^{\Leftarrow})$, the left–convergence $\widetilde{q}(\widetilde{x}, \widetilde{x}_n) \longrightarrow 0$ always implies $\widetilde{q}(\widetilde{x}_n, \widetilde{x}) \longrightarrow 0$.*

2. Let $q : E \times E \longrightarrow [0,\infty[$ satisfy the triangle inequality. If the left–convergence of any sequence $(x_n)_{n\in I\!N}$, i.e. $q(x,x_n) \longrightarrow 0$, always guarantees $q(x_{n_j}, x) \longrightarrow 0 \ (j \longrightarrow \infty)$ for a subsequence $(x_{n_j})_{j\in I\!N}$, then (E,q) fulfills standard hypothesis $(R^{\Leftarrow})$. □

As a final remark for one–sided sequentially compact (E,q) given, we suggest a way of constructing a function $q_{(R^{\Leftarrow})} : E \times E \longrightarrow [0,\infty[$ that satisfies standard hypothesis $(R^{\Leftarrow})$.

Proposition 1.4.8 *Let $q : E \times E \longrightarrow [0,\infty[$ be an ostensible metric on the nonempty set E that is one–sided sequentially compact and set for $x_1, x_2 \in E$,*

$$q_{(R^{\Leftarrow})}(x_1,x_2) \ := \ \sup \ \Big\{ q(y_1,x_2) \ \Big| \ y_1 \in E,\ \ q(x_1,y_1) = 0 \Big\}$$

Then $(E, q_{(R^{\Leftarrow})})$ fulfills standard hypothesis $(R^{\Leftarrow})$.

Proof. For showing the triangle inequality of $q_{(R^{\Leftarrow})}$, consider any $x_1, x_2, x_3, y_1 \in E$ with $q(x_1, y_1) = 0$.
Since q is an ostensible metric on E, we obtain $q(\cdot,\cdot) \le q_{(R^{\Leftarrow})}(\cdot,\cdot)$ and thus,

$$\begin{aligned} q(y_1, x_3) &\le q(y_1, x_2) + q(x_2, x_3) \\ &\le q_{(R^{\Leftarrow})}(x_1, x_2) + q_{(R^{\Leftarrow})}(x_2, x_3), \end{aligned}$$

i.e.

$$q_{(R^{\Leftarrow})}(x_1, x_3) \le q_{(R^{\Leftarrow})}(x_1, x_2) + q_{(R^{\Leftarrow})}(x_2, x_3).$$

Let $(x_n)_{n \in I\!N}$ be a sequence in E and $x \in E$ with $q_{(R^{\Leftarrow})}(x, x_n) \longrightarrow 0$.
For any $z \in E$ chosen arbitrarily, there exists a sequence $(y_n)_{n \in I\!N}$ such that for all n,

$$\wedge \begin{cases} q(y_n, z) & \ge q_{(R^{\Leftarrow})}(x_n, z) - \frac{1}{n}, \\ q(x_n, y_n) & = 0. \end{cases}$$

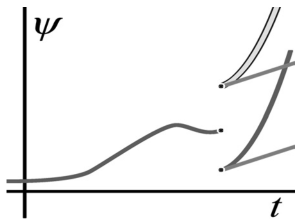
Due to $q(\cdot,\cdot) \le q_{(R^{\Leftarrow})}(\cdot,\cdot)$ on $E \times E$, $q(x, y_n) \le q(x, x_n) + q(x_n, y_n) \longrightarrow 0$ $(n \longrightarrow \infty)$.
As (E, q) is one–sided sequentially compact, there exist a sequence $n_j \nearrow \infty$ of indices and $y \in E$ satisfying $q(y_{n_j}, y) \longrightarrow 0$ $(j \longrightarrow \infty)$. In particular, $q(x, y) = 0$.
So, $\liminf\limits_{n \longrightarrow \infty} q_{(R^{\Leftarrow})}(x_n, z) \le \liminf\limits_{n \longrightarrow \infty} q(y_n, z) \le \limsup\limits_{n \longrightarrow \infty} q(y_{n_j}, y) + q(y, z) \le q_{(R^{\Leftarrow})}(x, z)$.
□

Remark. However, $q_{(R^{\Leftarrow})}$ does not satisfy $q_{(R^{\Leftarrow})}(x, x) = 0$ for all $x \in E$ in general.

1.5 Gronwall's Lemma for semicontinuous functions

Gronwall's Lemma provides a very useful tool for an upper estimate of a function $\psi : [a,b] \longrightarrow I\!R$ by means of its derivative. However the widespread versions suppose ψ to be continuous (at least).

In this section, the assumptions for ψ are weakened. Firstly, we demand an upper estimate only for a one–sided difference quotient (as its denominator tends to 0), but then ψ might still have a discontinuity in upward direction and of arbitrary height. So secondly, ψ has to be semicontinuous is a sense that is specified in the following three statements. Their indirect proofs follow all the same track.



Lemma 1.5.1 (Lemma of Gronwall : Subdifferential version I)
Let $\psi : [a,b] \longrightarrow I\!R,\ \ f, g \in C^0([a,b[, I\!R)$ *satisfy* $f(\cdot) \geq 0$ *and*

$$\begin{array}{rcll} \psi(t) & \leq & \limsup\limits_{h \downarrow 0}\ \psi(t-h), & \forall\ t \in\ ]a,\, b], \\ \psi(t) & \geq & \limsup\limits_{h \downarrow 0}\ \psi(t+h), & \forall\ t \in [a,\, b[, \\ \limsup\limits_{h \downarrow 0} \dfrac{\psi(t+h) - \psi(t)}{h} & \leq & f(t)\ \cdot\ \limsup\limits_{h \downarrow 0}\ \psi(t-h)\ +\ g(t) & \forall\ t \in\ ]a,\, b[. \end{array}$$

Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t)\ \leq\ \psi(a)\ \cdot\ e^{\mu(t)}\ +\ \int_a^t e^{\mu(t)-\mu(s)}\ g(s)\ ds$$

with $\mu(t) := \displaystyle\int_a^t f(s)\ ds.$

Proof. Let $\delta > 0$ be arbitrarily small. The proof is based on comparing ψ with the auxiliary function $\varphi_\delta : [a,b] \longrightarrow I\!R$ that uses $\psi(a) + \delta,\ g(\cdot) + \delta$ instead of $\psi(a),\ g(\cdot)$:

$$\varphi_\delta(t)\ :=\ \Big(\psi(a) + \delta\Big)\ e^{\mu(t)}\ +\ \int_a^t e^{\mu(t)-\mu(s)}\ (g(s) + \delta)\ ds.$$

Then, $\quad \varphi_\delta'(t)\ =\ f(t)\ \varphi_\delta(t) + g(t) + \delta \quad$ on $[a,b[$,
$\quad \varphi_\delta(t)\ >\ \psi(t) \quad$ for all $t \in [a,b[$ sufficiently close to a.

Assume now that there exists some $t_0 \in\ ]a,b]$ such that $\varphi_\delta(t_0) < \psi(t_0)$. Setting

$$t_1\ :=\ \inf\ \Big\{ t \in [a, t_0]\ \Big|\ \varphi_\delta(t) < \psi(t) \Big\},$$

we obtain $\quad \varphi_\delta(t_1)\ =\ \psi(t_1) \quad$ and $\quad a < t_1 < t_0 \quad$ because

$$\begin{aligned}
\varphi_\delta(t_1) &= \lim_{h\downarrow 0} \varphi_\delta(t_1-h) \geq \limsup_{h\downarrow 0} \psi(t_1-h) \geq \psi(t_1),\\
\varphi_\delta(t_1) &= \lim_{\substack{h\to 0\\ h\geq 0}} \varphi_\delta(t_1+h) \leq \limsup_{\substack{h\to 0\\ h\geq 0}} \psi(t_1+h) \leq \psi(t_1).
\end{aligned}$$

Thus, we conclude from the definition of t_1

$$\begin{aligned}
\liminf_{h\downarrow 0} \frac{\varphi_\delta(t_1+h)-\varphi_\delta(t_1)}{h} &\leq \limsup_{h\downarrow 0} \frac{\psi(t_1+h)-\psi(t_1)}{h}\\
\varphi_\delta'(t_1) &\leq f(t_1)\cdot \limsup_{h\downarrow 0} \psi(t_1-h) + g(t_1)\\
f(t_1)\,\varphi_\delta(t_1)+g(t_1)+\delta &\leq f(t_1)\cdot \limsup_{h\downarrow 0} \varphi_\delta(t_1-h) + g(t_1)\\
&\leq f(t_1)\cdot \varphi_\delta(t_1) + g(t_1)
\end{aligned}$$

— a contradiction. So $\varphi_\delta(\cdot)\geq\psi(\cdot)$ for any $\delta>0$. □

Remark. 1. If $\limsup_{h\downarrow 0} \psi(t-h) < \infty$ for all $t\in\,]a,b[$ then the second assumption in $]a,b[$ results from the third condition on ψ.

2. This and the following subdifferential versions of Gronwall's Lemma also hold if the functions $f,g:[a,b[\longrightarrow I\!R$ are only upper semicontinuous (instead of continuous). The proof is based on upper approximations of $f(\cdot)$, $g(\cdot)$ by continuous functions.

3. The condition $\limsup_{h\downarrow 0} \frac{\psi(t+h)-\psi(t)}{h} \leq f(t)\cdot\psi(t) + g(t)$ (supposed in the widespread forms of Gronwall's Lemma) is stronger than the third assumption of this lemma due to the semicontinuity condition $\psi(t) \leq \limsup_{h\downarrow 0} \psi(t-h)$.

Lemma 1.5.2 (Lemma of Gronwall : Subdifferential version II)
Let $\psi:[a,b]\longrightarrow I\!R$, $f,g\in C^0([a,b[,I\!R)$ *satisfy* $f(\cdot)\geq 0$ *and*

$$\begin{aligned}
\psi(t) &\leq \liminf_{h\downarrow 0} \psi(t-h), && \forall\ t\in\,]a,\,b],\\
\psi(t) &\geq \liminf_{h\downarrow 0} \psi(t+h), && \forall\ t\in[a,\,b[,\\
\liminf_{h\downarrow 0}\frac{\psi(t+h)-\psi(t)}{h} &\leq f(t)\cdot \liminf_{h\downarrow 0} \psi(t-h) + g(t) && \forall\ t\in\,]a,\,b[.
\end{aligned}$$

Then, for every $t\in[a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t) \leq \psi(a)\cdot e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)}\, g(s)\, ds$$

with $\mu(t) := \int_a^t f(s)\, ds$.

Proof differs from the preceding one in the point of time t_1 leading to a contradiction : For $\delta > 0$ arbitrarily small, set $\varphi_\delta : [a,b] \longrightarrow I\!R$,

$$\varphi_\delta(t) := \Big(\psi(a) + \delta\Big)\, e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)}\, (g(s) + \delta)\, ds.$$

Then,
$$\begin{array}{lll} \varphi'_\delta(t) = f(t)\,\varphi_\delta(t) + g(t) + \delta & \text{on } [a,b[, \\ \varphi_\delta(s_n) > \psi(s_n) & \text{for some sequence } s_n \downarrow a. \end{array}$$

Assume now that there exists some $t_0 \in\,]a,b]$ such that $\varphi_\delta(t_0) < \psi(t_0)$. Setting

$$t_1 := \inf \Big\{ t \in [a, t_0] \;\Big|\; \varphi_\delta(\cdot) < \psi(\cdot) \text{ in } [t, t_0] \Big\} > a,$$

we conclude $t_1 < t_0$ from the condition $\psi(t_0) \le \liminf_{h\downarrow 0} \psi(t_0 - h)$ and the continuity of $\varphi_\delta(\cdot)$. Moreover, $\varphi_\delta(t_1) = \psi(t_1)$ is a consequence of

$$\begin{aligned} \varphi_\delta(t_1) &= \lim_{h\downarrow 0} \varphi_\delta(t_1 - h) \ge \liminf_{h\downarrow 0} \psi(t_1 - h) \ge \psi(t_1), \\ \varphi_\delta(t_1) &= \lim_{h\downarrow 0} \varphi_\delta(t_1 + h) \le \liminf_{h\downarrow 0} \psi(t_1 + h) \le \psi(t_1). \end{aligned}$$

Thus, the definition of t_1 implies

$$\begin{aligned} \liminf_{h\downarrow 0} \frac{\varphi_\delta(t_1+h) - \varphi_\delta(t_1)}{h} &\le \liminf_{h\downarrow 0} \frac{\psi(t_1+h) - \psi(t_1)}{h} \\ \varphi'_\delta(t_1) &\le f(t_1) \cdot \liminf_{h\downarrow 0} \psi(t_1 - h) + g(t_1) \\ f(t_1)\,\varphi_\delta(t_1) + g(t_1) + \delta &\le f(t_1) \cdot \limsup_{h\downarrow 0} \varphi_\delta(t_1 - h) + g(t_1) \\ &\le f(t_1) \cdot \varphi_\delta(t_1) + g(t_1) \end{aligned}$$

— a contraduction. So $\varphi_\delta(\cdot) \ge \psi(\cdot)$ for any $\delta > 0$. □

Lemma 1.5.3 (Lemma of Gronwall : Subdifferential version III**)**
Let $\psi : [a,b] \longrightarrow I\!R$, $f, g \in C^0([a,b[, I\!R)$ *satisfy* $f(\cdot) \ge 0$ *and*

$$\begin{aligned} \psi(t) &\ge \limsup_{h\downarrow 0} \psi(t+h), && \forall\ t \in [a,\, b[, \\ \liminf_{h\downarrow 0} \frac{\psi(t) - \psi(t-h)}{h} &\le f(t) \cdot \limsup_{h\downarrow 0} \psi(t-h) + g(t) < \infty && \forall\ t \in\,]a,\, b]. \end{aligned}$$

Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t) \le \psi(a) \cdot e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)}\, g(s)\, ds$$

with $\mu(t) := \displaystyle\int_a^t f(s)\, ds.$

Proof. For any small $\delta > 0$, we consider again the auxiliary function $\varphi_\delta : [a, b[\longrightarrow I\!R$,

$$\varphi_\delta(t) := \Big(\psi(a) + \delta\Big)\, e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)}\, (g(s) + \delta)\; ds.$$

Then,

$$\begin{aligned} \varphi_\delta'(t) &= f(t)\,\varphi_\delta(t) + g(t) + \delta && \text{on } [a, b[, \\ \varphi_\delta(t) &> \psi(t) && \text{for all } t \in [a, b[\text{ sufficiently close to } a \end{aligned}$$

and

$$\psi(t) \le \limsup_{h \downarrow 0} \psi(t-h) \qquad \text{for all } t \in \,]a, b].$$

Assume now that there exists some $t_0 \in \,]a, b]$ such that $\varphi_\delta(t_0) < \psi(t_0)$. Setting

$$t_1 := \inf \Big\{ t \in [a, t_0] \;\Big|\; \varphi_\delta(t) \le \psi(t) \Big\},$$

we obtain $\varphi_\delta(t_1) = \psi(t_1)$ and $a < t_1 < t_0$ since

$$\begin{aligned} \varphi_\delta(t_1) &= \lim_{h \to 0^+} \varphi_\delta(t_1 - h) \;\ge\; \limsup_{h \to 0^+} \psi(t_1 - h) \;\ge\; \psi(t_1), \\ \varphi_\delta(t_1) &= \lim_{\substack{h \to 0 \\ h \ge 0}} \varphi_\delta(t_1 + h) \;\le\; \limsup_{\substack{h \to 0 \\ h \ge 0}} \psi(t_1 + h) \;\le\; \psi(t_1). \end{aligned}$$

This implies for all $h \in \,]0,\, t_1 - a[$

$$\begin{aligned} \frac{\varphi_\delta(t_1) - \varphi_\delta(t_1 - h)}{h} &< \frac{\psi(t_1) - \psi(t_1 - h)}{h} \\ \varphi_\delta'(t_1) &\le \liminf_{h \downarrow 0} \frac{\psi(t_1) - \psi(t_1 - h)}{h} \\ f(t_1)\,\varphi_\delta(t_1) + g(t_1) + \delta &\le f(t_1) \cdot \limsup_{h \downarrow 0} \psi\,(t_1 - h) + g(t_1) \\ &\le f(t_1) \cdot \limsup_{h \downarrow 0} \varphi_\delta(t_1 - h) + g(t_1) \\ &\le f(t_1) \cdot \varphi_\delta(t_1) + g(t_1) \end{aligned}$$

— a contradiction. So $\varphi_\delta(\cdot) \ge \psi(\cdot)$ for any $\delta > 0$. □

Remark. If the function $\psi : [a, b] \longrightarrow I\!R$ satisfies

$$\liminf_{h \downarrow 0} \frac{\psi(t) - \psi(t-h)}{h} \le f(t)\,\psi(t) + g(t) \qquad \forall\; t \in \,]a,\, b]$$

then it is lower semicontinuous in terms of $\psi(t) \le \limsup_{h \downarrow 0} \psi(t-h)$ and thus it fulfills the last assumption of Lemma 1.5.3.

Finally we present a modification with an integral assumption. Here the regularity condition on both $\psi(\cdot)$ and $g(\cdot)$ are weaker than in most of the widespread versions of Gronwall's Lemma since these functions need not be continuous (see e.g. [2, Aubin 99], Lemma 8.3.1). The proof is based on the same ideas that can be found in the literature, but it just takes advantage of them in a more detailed way.

Lemma 1.5.4 (Lemma of Gronwall : Integral version**)**

Let $\psi,\ g \in L^1([a,b], I\!R),\ f \in C^0([a,b])$ *satisfy* $\psi(\cdot),\ f(\cdot) \geq 0$ *and*

$$\psi(t) \ \leq\ g(t) \ +\ \int_a^t f(s)\,\psi(s)\,ds \qquad \text{for almost every } t \in [a,b].$$

Then, for almost every $t \in [a,b]$,

$$\psi(t) \ \leq\ g(t) \ +\ \int_a^t e^{\mu(t)-\mu(s)}\, f(s)\, g(s)\, ds$$

with $\mu(t) := \displaystyle\int_a^t f(s)\,ds.$

Assuming in addition that $g(\cdot)$ *is upper semicontinuous and that* $\psi(\cdot)$ *is lower semicontinuous or monotone, then this inequality holds for any* $t \in\,]a,b[$.

Proof. The function $\varphi : [a,b] \longrightarrow I\!R,\ \ t \longmapsto \displaystyle\int_a^t f(s)\,\psi(s)\ ds$ is absolutely continuous and satisfies for almost every $t \in [a,b]$ (since $f(\cdot) \geq 0$)

$$\varphi'(t) \ =\ f(t)\,\psi(t) \ \leq\ f(t)\,g(t) \ +\ f(t)\,\varphi(t).$$

Thus, $t \longmapsto e^{-\mu(t)}\,\varphi(t)$ is also absolutely continuous and has the weak derivative

$$\tfrac{d}{dt}\Big(e^{-\mu(t)}\,\varphi(t)\Big) \ =\ e^{-\mu(t)}\Big(\varphi'(t) \ -\ f(t)\,\varphi(t)\Big) \ \leq\ e^{-\mu(t)}\ f(t)\,g(t).$$

So we obtain for any $t \in [a,b]$

$$\begin{aligned} e^{-\mu(t)}\,\varphi(t) \ &\leq\ e^{-\mu(a)}\,\varphi(a) \ +\ \int_a^t e^{-\mu(s)}\ f(s)\,g(s)\ ds \\ \varphi(t) \ &\leq\ 0 \ +\ \int_a^t e^{\mu(t)-\mu(s)}\ f(s)\,g(s)\ ds \end{aligned}$$

and this estimate implies the assertion for almost every t.

Now suppose that $g(\cdot)$ is upper semicontinuous and that $\psi(\cdot)$ is lower semicontinuous or monotone. Then for every $t \in\,]a,b[$, there exists a sequence $(t_n)_{n\in I\!N}$ in $]a,b[$ such that $t_n \longrightarrow t \ \ (n \longrightarrow \infty)$,

$$\begin{aligned} \psi(t) \ &\leq\ \limsup_{n \longrightarrow \infty}\ \psi(t_n), \\ \psi(t_n) \ &\leq\ g(t_n) \ +\ \int_a^{t_n} e^{\mu(t_n)-\mu(s)} f(s)\,g(s)\ ds \qquad \forall\ n \in I\!N. \end{aligned}$$

As an easy consequence,

$$\begin{aligned} \psi(t) \ &\leq\ \limsup_{n \longrightarrow \infty}\ \Big(g(t_n) \ +\ \int_a^{t_n} e^{\mu(t_n)-\mu(s)}\ f(s)\,g(s)\ ds\Big) \\ &\leq\ g(t) \ +\ \int_a^t e^{\mu(t)-\mu(s)}\ f(s)\,g(s)\ ds. \end{aligned}$$

□

Chapter 2

Timed right–hand forward solutions of mutational equations

Extending evolution equations to geometric shapes motivated Jean–Pierre Aubin to develop the concept of mutational equations for metric spaces (M, d) in the nineties ([4, Aubin 93], [2, Aubin 99]).

For defining a derivative of a curve $x : [0, T] \longrightarrow (M, d)$, the idea of a velocity (or direction) has to be introduced for metric spaces (M, d). In a vector space X, every vector v induces a direction shifting each element $x \in X$ to the point $x + h\, v$ after some "time" $h > 0$. So it exemplifies a continuous map $[0, 1] \times X \longrightarrow X,\ (h, x) \longmapsto x + h\, v$. Extending this notion to a metric space (M, d) leads to a so–called *transition* $\vartheta : [0, 1] \times M \longrightarrow M$ that is demanded to fulfill 4 conditions (in [2, Aubin 99], Def. 1.1.2) :

1. $\vartheta(0, x) \ = \ x \qquad\qquad \forall\ x \in M,$
2. $\limsup\limits_{h \downarrow 0}\ \frac{1}{h} \cdot d\Big(\vartheta(h,\ \vartheta(t, x)),\ \ \vartheta(t + h,\ x)\Big) \ = \ 0 \qquad \forall\ x \in M,\ \ t \in [0, 1[,$
3. $\alpha(\vartheta) \ := \ \sup\limits_{x \neq y}\ \ \limsup\limits_{h \downarrow 0}\ \Big(\dfrac{d(\vartheta(h,\ x),\ \ \vartheta(h,\ y)) \ - \ d(x, y)}{h \ \ d(x, y)}\Big)^+ \ < \ \infty,$
4. $\beta(\vartheta) \ := \ \sup\limits_{x \in M}\ \ \limsup\limits_{h \downarrow 0}\ \dfrac{d(x,\ \ \vartheta(h,\ x))}{h} \qquad < \ \infty$

with the abbreviation $(r)^+ := \max(0, r)$ for every $r \in I\!\!R$.

In general, the derivative of a function plays the role of its first–order approximation. So considering now a curve $x : [0, T] \longrightarrow (M, d)$, a transition ϑ on (M, d) can be interpreted as a derivative of $x(\cdot)$ at time $t \in [0, T[$ if it satisfies

$$\limsup_{h \downarrow 0}\ \frac{1}{h} \cdot d\Big(x(t + h),\ \ \vartheta(h,\ x(t))\Big) \ = \ 0.$$

However this transition ϑ need not be unique. All transitions fulfilling this condition form the so–called *mutation* of $x(\cdot)$ at time t, abbreviated as $\overset{\circ}{x}(t)$.

The distance between transitions $\vartheta, \tau : [0,1] \times M \longrightarrow M$ affects the first–order approximation of each point of M evolving along ϑ, τ respectively :

$$D(\vartheta,\tau) \;:=\; \sup_{x \in M} \; \limsup_{h \downarrow 0} \; \frac{d(\vartheta(h,\, x),\;\; \tau(h,\, x))}{h}$$

These definitions form the basis of Jean–Pierre Aubin for an existence and uniqueness theorem of evolution equations in metric spaces (M,d) (Theorem 1.4.2 in [2, Aubin 99]). It generalizes the Cauchy–Lipschitz theorem for ordinary differential equations in $\mathbb{R}^N$:

Theorem *Assume that the closed bounded balls of the metric space (M,d) are compact. Let f be a function from M to a set of transitions on (M,d) satisfying*

1. $\exists\ \lambda > 0 : \qquad D(f(x),\;\, f(y)) \;\leq\; \lambda \cdot d(x,y) \qquad \forall\ x,y \in M$
2. $A := \sup_{x \in M} \; \alpha(f(x)) \;<\; \infty.$

Moreover suppose for $y(\cdot) : [0,T[\longrightarrow M$ that its mutation $\overset{\circ}{y}(t)$ is nonempty for each t. Then for every initial value $x_0 \in M$, there exists a unique solution $x(\cdot) : [0,T[\longrightarrow M$ of the mutational equation $\quad \overset{\circ}{x}(t) \ni f(x(t)) \quad$ for all $t \in [0,T[$, i.e.

$$\limsup_{h \downarrow 0} \; \tfrac{1}{h} \cdot d\Big(x(t+h),\;\; f(x(t))\,(h, x(t))\Big) \;=\; 0,$$

satisfying $x(0) = x_0$ and the inequality (for every $t \in [0,T[$)

$$d\Big(x(t),\;\; y(t)\Big) \;\leq\; d(x_0,\; y(0)) \cdot e^{(A+\lambda)\, t} \;+ \int_0^t e^{(A+\lambda)\,(t-s)} \cdot \inf_{\vartheta \in \overset{\circ}{y}(s)} D\Big(f(y(s)),\; \vartheta\Big) \;\; ds.$$

□

In fact, this last inequality implies the Lipschitz dependence of the solution $x(\cdot)$ on both the initial value and the right–hand side. The proof in [2, Aubin 99] is based on a combination of Euler method and fixed point argument. Essentially it uses an upper bound for the distance between two points $x, y \in M$ evolving along transitions ϑ, τ for some time $h \in [0,1[$

$$d\Big(\vartheta(h,\, x),\;\; \tau(h,\, y)\Big) \;\leq\; d(x,y) \cdot e^{\alpha(\vartheta)\, h} \;+\; h\;\, D(\vartheta,\tau) \cdot \tfrac{e^{\alpha(\vartheta)\, h}\, -\, 1}{\alpha(\vartheta)\;\, h} \qquad (*)$$

(see [2, Aubin 99], Lemma 1.1.3).
This key estimate $(*)$ results from Gronwall's Lemma applied to the Lipschitz continuous function $\qquad [0,1[\;\longrightarrow\; [0,\infty[\,, \qquad h \longmapsto d\Big(\vartheta(h,\, x),\;\; \tau(h,\, y)\Big).$

Now the concept of mutational equations (according to [2, Aubin 99]) is generalized in several respects :

- In contrast to a nonempty set E, the product $\widetilde{E} \stackrel{\text{Def.}}{=} I\!\!R \times E$ takes the time direction into consideration. Regarding these pairs with a separate time component is always abbreviated by a tilde.

- For quantifying distances between points, the metric d is replaced by a countable family of timed ostensible metrics $(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}$ on $\widetilde{E}$. Here the timed triangle inequality plays an important role and so, further assumptions about $\widetilde{q}_\varepsilon$ (like reflexivity) might be weakened even more.
  In fact, we prefer $\widetilde{E} \stackrel{\text{Def.}}{=} I\!\!R \times E$ to E only because the timed triangle inequality is a weaker condition than its counterpart without time component.
  Finally, supposing the index set $\mathcal{J} \subset [0,1]^\kappa$ $(\kappa \in I\!\!N)$ to be countable makes the Cantor diagonal construction available for proofs of existence.

- The basic notion of distributions is extended to timed ostensible metric spaces in a figurative sense : The first–order approximation of a curve $\widetilde{x} : [0,T] \longrightarrow \widetilde{E}$ at time t by a (generalized timed) transition $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ is replaced by the demand for preserving a key property when considering all elements of a given "test set" $\widetilde{D} \subset \widetilde{E}$.

  When proving the existence theorem in [2, Aubin 99], the substantial feature is the preceding estimate $(*)$. So we want it to be preserved principally for all points $\widetilde{y} \in \widetilde{E}$ (as before), but only for every "test element" $\widetilde{x} \in \widetilde{D}$. In particular for one and the same transition $\widetilde{\vartheta}$, this basic idea leads to the demand
  $$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\, \widetilde{x}),\;\; \widetilde{\vartheta}(h,\, \widetilde{y})\Big) \;\;\le\;\; \widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{y}) \;\cdot\; e^{\alpha\, h}$$
  for all elements $\widetilde{x} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$. It forms a basis for generalizing the mutation of $\widetilde{x}(\cdot) : [0,T] \longrightarrow \widetilde{E}$ at time t. Indeed, it motivates the condition
  $$\limsup_{h \downarrow 0} \; \tfrac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\, \widetilde{z}),\;\; \widetilde{x}(t+h)\Big) \;-\; \widetilde{q}_\varepsilon(\widetilde{z},\; \widetilde{x}(t)) \cdot e^{\widehat{\alpha}\, h}\Big) \;\le\; 0 \qquad \forall\; \widetilde{z} \in \widetilde{D},\; \varepsilon \in \mathcal{J}$$
  with a parameter $\widehat{\alpha} \;=\; \widehat{\alpha}(t, \widetilde{\vartheta}, \widetilde{x}(\cdot)) < \infty$.

  Applying this concept to examples, the adapted parameter $\alpha(\widetilde{\vartheta})$ of a generalized timed transition $\widetilde{\vartheta}$ has an important advantage. It will be much easier to find suitable upper bounds if only "test elements" $\widetilde{x}, \widetilde{\vartheta}(h, \widetilde{x}) \in \widetilde{D}$ are considered in the first arguments of $\widetilde{q}_\varepsilon$.
  For this reason, we demand in addition that every "test point" $\widetilde{x} \in \widetilde{D}$ is staying in $\widetilde{D}$ for some short time $\mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{x}) > 0$ while evolving along $\widetilde{\vartheta}$.

- In the concept of [2, Aubin 99], a transition ϑ on a metric space (M, d) has to fulfill the condition (comparable to a "first–order semigroup property") for each $x \in M$, t
$$d\Big(\vartheta(h,\ \vartheta(t,x)),\ \ \vartheta(t+h,\ x)\Big) \ = \ o(h).$$
Now we replace the Landau symbol $o(h)$ by $O(h)$, i.e. for characterizing a transition $\widetilde{\vartheta}$ on a tuple $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, there has to be a parameter $\gamma_\varepsilon(\widetilde{\vartheta}) \in [0, \infty[$ with
$$\wedge \begin{cases} \limsup\limits_{h \downarrow 0} \ \frac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\Big) \le \gamma_\varepsilon(\widetilde{\vartheta}) & \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[\\ \limsup\limits_{h \downarrow 0} \ \frac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\Big) \le \gamma_\varepsilon(\widetilde{\vartheta}) & \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[\end{cases}$$
because $\widetilde{q}_\varepsilon$ need not be symmetric. In general, $\gamma_\varepsilon(\widetilde{\vartheta}) \ge 0$ also depends on $\varepsilon \in \mathcal{J}$. So if $0 \in \overline{\mathcal{J}}$, a further characteristic of $\widetilde{\vartheta}$ is the asymptotic behavior of $\gamma_\varepsilon(\widetilde{\vartheta})$ as $\varepsilon \downarrow 0$. It is described by the *order* $p \in I\!R$ of $\widetilde{\vartheta}$ on the basis of $\limsup\limits_{\varepsilon \longrightarrow 0} \ \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) \ = \ 0.$

  Analytically speaking, this parameter $\gamma_\varepsilon(\widetilde{\vartheta}) \ge 0$ gives the opportunity to introduce an additional limit process that follows the process of first–order approximation. This might be useful for multi–scale problems, for example.
  Here we present a way how to take $\gamma_\varepsilon(\cdot)$ into consideration properly.

- For each (generalized) timed transition $\widetilde{\vartheta}$ on a tuple $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, the curves $\widetilde{\vartheta}(\cdot, \widetilde{x}) : [0,1] \longrightarrow \widetilde{E}$ $(\widetilde{x} \in \widetilde{E})$ have to be merely equi-continuous in positive time direction (and not uniformly Lipschitz) because this feature is used explicitly only in proofs.

In particular, each limit superior for first–order approximation still uses the information at the current time t and at a later point of time $t+h$ with $h > 0$ tending to 0. This aspect motivates the expression "forward" and is symbolized by $\mapsto$ (representing the time axis).
Furthermore the phrase "right–hand" comes from the following detail : When defining terms like "primitive" and "solution" for $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$, the function $\widetilde{x}(\cdot)$ always appears in the second argument of the vanishing distances $\widetilde{q}_\varepsilon$ $(\varepsilon \in \mathcal{J})$ (see Def. 2.2.1, 2.3.1).

Now so–called *timed forward transitions* $\widetilde{\vartheta}$ of order p on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ are defined precisely in § 2.1.
Then the key estimate $(*)$ (in a generalized form) is still fulfilled in $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$. Principally the proof is based on Gronwall's Lemma 1.5.1 for the semicontinuous function $h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(h, \widetilde{z}),\ \widetilde{\tau}(h, \widetilde{y}))$ with arbitrary $\widetilde{x} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$. Many of the following conclusions have this technique in common.

In § 2.2, the definition of *timed right–hand forward primitive* is formulated and, we present three ways for estimating the distance between a transition and a primitive. § 2.3 deals with *timed right–hand forward solutions* of generalized mutational equations : definition, stability, existence and estimates.

General assumptions for chapter 2. Let E be a nonempty set, $D \subset E$, $p \in I\!R$ and set $\widetilde{E} := I\!R \times E$, $\widetilde{D} := I\!R \times D$, $\pi_1 : \widetilde{E} \longrightarrow I\!R$, $(t, x) \longmapsto t$.
$\mathcal{J} \subset [0,1]^\kappa$ abbreviates a countable index set with $\kappa \in I\!N$, $0 \in \overline{\mathcal{J}}$.
Furthermore we assume for each function $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$

1. timed triangle inequality,
2. time continuity, i.e. every sequence $(\widetilde{x}_n = (t_n, x_n))_{n \in I\!N}$ in $\widetilde{E}$ and $\widetilde{x} = (t, x) \in \widetilde{E}$ with $\widetilde{q}(\widetilde{x}_n, \widetilde{x}) \longrightarrow 0$ $(n \longrightarrow \infty)$ fulfill $t_n \longrightarrow t$ $(n \longrightarrow \infty)$ (due to Def. 1.2.1).
3. reflexivity on $\widetilde{D}$, i.e. $\widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{z}) = 0$ for all $\widetilde{z} \in \widetilde{D}$.

2.1 Timed forward transitions

Definition 2.1.1 *A map* $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ *is a so–called* timed forward transition of order p *on* $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *if it fulfills the following timed conditions (for each* $\varepsilon \in \mathcal{J}$*)*

1. $\widetilde{\vartheta}(0,\cdot) \;=\; \mathrm{Id}_{\widetilde{E}},$

2. $\exists\; \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 : \qquad \limsup\limits_{\varepsilon \longrightarrow 0} \; \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) \;=\; 0 \qquad$ *and*
$$\limsup_{h \downarrow 0} \; \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\; \widetilde{\vartheta}(t,\widetilde{x})),\;\; \widetilde{\vartheta}(t+h,\; \widetilde{x})\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\; \widetilde{x} \in \widetilde{E},\; t \in [0,1[,$$
$$\limsup_{h \downarrow 0} \; \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+h,\; \widetilde{x}),\;\; \widetilde{\vartheta}(h,\; \widetilde{\vartheta}(t,\widetilde{x}))\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\; \widetilde{x} \in \widetilde{E},\; t \in [0,1[,$$

3. $$\exists\; \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) < \infty : \quad \sup_{\substack{\widetilde{x} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{x} \leq \pi_1 \widetilde{y}}} \; \limsup_{h \downarrow 0} \; \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{x}),\, \widetilde{\vartheta}(h,\widetilde{y})\big) \;-\; \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y}) \;-\; \gamma_\varepsilon(\widetilde{\vartheta})\, h}{h \quad \big(\, \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y}) + \gamma_\varepsilon(\widetilde{\vartheta})\, h\big)} \right)^{+} \;\leq\; \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$$

4. $\exists\; \beta_\varepsilon(\widetilde{\vartheta}) :\;]0,1] \;\longrightarrow [0,\infty[: \quad \beta_\varepsilon(\widetilde{\vartheta})(\cdot)$ *nondecreasing,* $\quad \limsup\limits_{h \downarrow 0} \; \beta_\varepsilon(\widetilde{\vartheta})(h) \;=\; 0,$
$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\;\; \widetilde{\vartheta}(t,\widetilde{x})\Big) \;\leq\; \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad\qquad \forall\; s < t \leq 1,\;\; \widetilde{x} \in \widetilde{E},$$

5. $\forall\; \widetilde{x} \in \widetilde{D} \qquad \exists\; \mathcal{T}_\Theta = \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{x}) \in \,]0,1] : \quad \widetilde{\vartheta}(t,\widetilde{x}) \in \widetilde{D} \qquad \forall\; t \in [0, \mathcal{T}_\Theta],$

6. $$\limsup_{h \downarrow 0} \; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\; \widetilde{x}),\; \widetilde{y}\Big) \;\geq\; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}),\; \widetilde{y}\Big) \qquad \forall\; \widetilde{x} \in \widetilde{D},\; \widetilde{y} \in \widetilde{E},\; t \in \,]0,\mathcal{T}_\Theta]$$
with $\;\; t + \pi_1 \widetilde{x} \leq \pi_1 \widetilde{y},$

7. $\widetilde{\vartheta}\Big(h,\; (t,x)\Big) \;\in\; \{t+h\} \times E \qquad\qquad \forall\; (t,x) \in \widetilde{E},\;\; h \in [0,1].$

$\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *denotes a set of timed forward transitions on* $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *assuming*
$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau}) \;:=\; \sup_{\substack{\widetilde{x} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{x} \leq \pi_1 \widetilde{y}}} \; \limsup_{h \downarrow 0} \; \left(\frac{\widetilde{q}_\varepsilon(\widetilde{\vartheta}(h,\widetilde{x}),\; \widetilde{\tau}(h,\widetilde{y})) \;-\; \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y}) \;\cdot\; e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\, h}}{h} \right)^{+} \;<\; \infty$$
for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}),\;\; \varepsilon \in \mathcal{J}.$

Remark. 1. A set $\widetilde{E} \neq \emptyset$ supplied with only one function $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ can be regarded as easy (but important) example by setting $\mathcal{J} := \{0\}$, $\widetilde{q}_0 := \widetilde{q}$. Considering a timed forward transitions $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ of order 0, the condition $\limsup\limits_{\varepsilon \longrightarrow 0} \varepsilon^0 \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0$ means $0 \;=\; 0^0 \cdot \gamma_0(\widetilde{\vartheta}) \;=\; \gamma_0(\widetilde{\vartheta})$ — due to the definition $0^0 \stackrel{\text{Def.}}{=} 1$. So it leads to the key property for all $\widetilde{x} \in \widetilde{E},\; t \in [0,1[$.
$$\wedge \begin{cases} \limsup\limits_{h \downarrow 0} \; \frac{1}{h} \;\; \widetilde{q}\Big(\widetilde{\vartheta}(h,\; \widetilde{\vartheta}(t,\widetilde{x})),\;\; \widetilde{\vartheta}(t+h,\; \widetilde{x})\Big) \;=\; 0 \\ \limsup\limits_{h \downarrow 0} \; \frac{1}{h} \;\; \widetilde{q}\Big(\widetilde{\vartheta}(t+h,\; \widetilde{x}),\;\; \widetilde{\vartheta}(h,\; \widetilde{\vartheta}(t,\widetilde{x}))\Big) \;=\; 0 \end{cases}$$

Then many of the following results do not depend on ε or $\gamma_\varepsilon(\cdot)$ (and its upper bounds) explicitly. So we do not mention the index ε there any longer and abbreviate the corresponding set of timed transitions (of order 0) as $\widetilde{\Theta}^{\mapsto}(\widetilde{E}, \widetilde{D}, \widetilde{q})$. In particular, the analogy to transitions in metric spaces (introduced by Aubin in [2], [4]) is more apparent.

2. For a set $E \neq \emptyset$, a family $q_\varepsilon : E \times E \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$ and $p \in I\!R$ given, let $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ be defined similarly to remark (2.) after Def. 1.1.2 :

$$\widetilde{q}_\varepsilon\Big((s,x),\ (t,y)\Big) \;:=\; f(\varepsilon)\,|s-t| + q_\varepsilon(x,y) \qquad \text{for all } (s,x),\ (t,y) \in \widetilde{E}.$$

with a function $f(\varepsilon) = o(\varepsilon^p) \geq 0$ for $\varepsilon \downarrow 0$,
Then every $\vartheta : [0,1] \times E \longrightarrow E$ satisfying the conditions (1.)–(6.) for $(E, D, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$ induces a *timed* forward transition $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ of order p on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ by

$$\widetilde{\vartheta}\Big(h,\ (t,x)\Big) \;:=\; \Big(t+h,\ \vartheta(h,x)\Big) \qquad \text{for all } (t,x) \in \widetilde{E},\ \ h \in [0,1].$$

As a consequence, the following statements about $\widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ can be applied to their counterparts without separate time component very easily. Correspondingly these functions $\vartheta : [0,1] \times E \longrightarrow E$ are called *forward transitions of order* p on $(E, D, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$ and abbreviated as $\Theta^{\mapsto}_p(E, D, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$.

3. Condition (4.) on a timed forward transition $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ states its uniform continuity (in positive time direction) with respect to $\widetilde{q}_\varepsilon$ for each $\varepsilon \in \mathcal{J}$, i.e.

$$\widetilde{\vartheta}(\cdot, \widetilde{x}) \in UC^{\rightarrow}([0,1], \widetilde{E}, \widetilde{q}_\varepsilon) \qquad \text{for any } \widetilde{x} \in \widetilde{E}.$$

Considering the inequality $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\ \widetilde{\vartheta}(t,\widetilde{x})\Big) \;\leq\; \beta_\varepsilon(\widetilde{\vartheta})((t-s)^+)$ for all $s < t$ instead has the merely technical advantage that the nondecreasing modulus of continuity $h \longmapsto \beta_\varepsilon(\widetilde{\vartheta})(h^+) \overset{\text{Def.}}{=} \limsup_{k \downarrow h} \beta_\varepsilon(\widetilde{\vartheta})(k)$ is upper semicontinuous in addition.

4. Condition (6.), the timed triangle inequality and the continuity of $\widetilde{\vartheta}(\cdot, \widetilde{x})$ imply

$$\limsup_{h \downarrow 0}\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h, \widetilde{x}),\ \widetilde{y}\Big) \;=\; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t, \widetilde{x}),\ \widetilde{y}\Big)$$

for all $\widetilde{\vartheta} \in \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, $\widetilde{x} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$, $0 < t < \mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{x})$, $\varepsilon \in \mathcal{J}$ with $t + \pi_1\,\widetilde{x} \leq \pi_1\,\widetilde{y}$. In particular, this property is weaker than standard hypothesis $(R^{\Rightarrow})$ because only elements $\widetilde{\vartheta}(t-h_n, \widetilde{x})$, $\widetilde{\vartheta}(t, \widetilde{x})$ of $\widetilde{D}$ appear in the first argument of $\widetilde{q}_\varepsilon$.

5. $\widetilde{Q}^{\mapsto}_\varepsilon \;:\; \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)) \times \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)) \longrightarrow [0, \infty[$ satisfies the triangle inequality. Indeed, for any $\widetilde{\vartheta}_1, \widetilde{\vartheta}_2, \widetilde{\vartheta}_3 \in \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ and $\widetilde{x} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$, $h \in [0,1]$, the general assumption $\widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{x}) = 0$ guarantees

$$\begin{aligned}
& \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_1(h,\widetilde{x}),\ \ \widetilde{\vartheta}_3(h,\widetilde{y})\Big) \;-\; \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y}) \cdot e^{\alpha^{\mapsto}_\varepsilon(\widetilde{\vartheta}_3)\,h} \\
\leq\quad & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_1(h,\widetilde{x}),\ \ \widetilde{\vartheta}_2(h,\widetilde{x})\Big) \;-\; \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{x}) \cdot e^{\alpha^{\mapsto}_\varepsilon(\widetilde{\vartheta}_2)\,h} \\
+\ & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_2(h,\widetilde{x}),\ \ \widetilde{\vartheta}_3(h,\widetilde{y})\Big) \;-\; \widetilde{q}_\varepsilon(\widetilde{x},\widetilde{y}) \cdot e^{\alpha^{\mapsto}_\varepsilon(\widetilde{\vartheta}_3)\,h}.
\end{aligned}$$

So after dividing by h, the limit superior for $h \downarrow 0$ leads to

$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}_1, \widetilde{\vartheta}_3) \;\le\; \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}_1, \widetilde{\vartheta}_2) \;+\; \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}_2, \widetilde{\vartheta}_3).$$

This triangle inequality motivates us to use the parameter $\alpha_\varepsilon^{\mapsto}(\vartheta_2)$ in the definition of $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}_1, \widetilde{\vartheta}_2)$ whereas the corresponding result for metric spaces (M, d) (according to [2]) takes the parameter $\alpha(\cdot)$ of the *first* argument ϑ_1 into consideration :

$$D(\vartheta_1, \vartheta_2) \;\le\; \sup_{x,y \in M} \; \limsup_{h \downarrow 0} \; \tfrac{1}{h} \cdot \Big(d\Big(\vartheta_1(h,\, x),\;\; \vartheta_2(h,\, y)\Big) \;-\; d(x,y) \cdot e^{\alpha(\vartheta_1)\, h}\Big)$$

When applying the right–hand forward generalization to examples like the nonempty compact subsets of $I\!R^N$, we suppose uniform bounds of $\alpha_\varepsilon^{\mapsto}(\cdot)$ for all transitions anyway.

The next lemma gives sufficient conditions for a timed forward transition $\widetilde{\vartheta}$ on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ that is even Lipschitz continuous in positive time direction. To be more precise, it implies that the uniform continuity in (4.) of Definition 2.1.1 can be replaced by Lipschitz continuity if for every $\varepsilon \in \mathcal{J}$,

$$\wedge \left\{ \begin{array}{rcll} \beta_\varepsilon^{\mathrm{Lip}}(\widetilde{\vartheta}) \;:=\; \sup\limits_{\widetilde{x} \in \widetilde{E}} \; \limsup\limits_{h \downarrow 0} \; \frac{\widetilde{q}_\varepsilon\left(\widetilde{x},\; \widetilde{\vartheta}(h,\widetilde{x})\right)}{h} & < & \infty, & \\ \liminf\limits_{h \downarrow 0} \;\; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{x}),\;\; \widetilde{\vartheta}(t,\widetilde{x})\Big) & = & 0 & \forall\; t \in [0,1[,\;\; \widetilde{x} \in \widetilde{E}. \end{array} \right.$$

Lemma 2.1.2 *For every timed forward transition $\widetilde{\vartheta}$ on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ with*

$$\wedge \left\{ \begin{array}{rcll} \beta_\varepsilon^{\mathrm{Lip}}(\widetilde{\vartheta}) \;:=\; \sup\limits_{\widetilde{x} \in \widetilde{E}} \; \limsup\limits_{h \downarrow 0} \; \frac{\widetilde{q}_\varepsilon\left(\widetilde{x},\; \widetilde{\vartheta}(h,\,\widetilde{x})\right)}{h} & < & \infty, & \forall\; \varepsilon \in \mathcal{J} \\ \liminf\limits_{h \downarrow 0} \;\; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{x}),\; \widetilde{\vartheta}(t,\widetilde{x})\Big) & = & 0 & \forall\; \varepsilon \in \mathcal{J},\; t \in [0,1[\end{array} \right.$$

and $\widetilde{x} \in \widetilde{E}$, the map $\widetilde{\vartheta}(\cdot, \widetilde{x})$ belongs to $\mathrm{Lip}^{\rightarrow}([0,1], \widetilde{E}, \widetilde{q}_\varepsilon)$, i.e. for any $0 \le s < t \le 1$,

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\; \widetilde{\vartheta}(t,\widetilde{x})\Big) \;\le\; (\beta_\varepsilon^{\mathrm{Lip}}(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\vartheta})) \cdot (t-s).$$

Proof. For $\varphi_\varepsilon :]s,1] \longrightarrow I\!R,\; t \longmapsto \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\; \widetilde{\vartheta}(t,\widetilde{x})\Big)$ (with $\varphi_\varepsilon(s) := 0$) and $t > s$, we conclude

$$\begin{array}{rcll} \varphi_\varepsilon(t) & \le & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}),\; \widetilde{\vartheta}(t-h,\widetilde{x})\Big) & + \qquad\qquad \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{x}),\; \widetilde{\vartheta}(t,\widetilde{x})\Big) \\ \varphi_\varepsilon(t) & \le & \limsup\limits_{h \downarrow 0} \;\; \varphi_\varepsilon(t-h) & + \;\; \liminf\limits_{h \downarrow 0} \; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{x}),\; \widetilde{\vartheta}(t,\widetilde{x})\Big) \\ & = & \limsup\limits_{h \downarrow 0} \;\; \varphi_\varepsilon(t-h) & \end{array}$$

and due to the timed triangle inequality,

$$\begin{array}{rcll} \varphi_\varepsilon(t+h) - \varphi_\varepsilon(t) & \le & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}),\; \widetilde{\vartheta}(t+h,\widetilde{x})\Big) & \\ & \le & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}),\; \widetilde{\vartheta}(h,\; \widetilde{\vartheta}(t,\widetilde{x}))\Big) & + \;\; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\; \widetilde{\vartheta}(t,\widetilde{x})),\; \widetilde{\vartheta}(t+h,\widetilde{x})\Big) \\ & \le & \beta_\varepsilon^{\mathrm{Lip}}(\widetilde{\vartheta}) \cdot h + o(h) & + \;\; \gamma_\varepsilon(\widetilde{\vartheta}) \cdot h + o(h) \end{array}$$

So the assertion is a consequence of Gronwall's Lemma 1.5.1. □

Here Gronwall's Lemma 1.5.1 for semicontinuous functions proves to be the main tool. For applying the same notion to other distances like e.g. $\psi_\varepsilon : t \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{x}), \widetilde{y}(t))$ (with a function $\widetilde{y}(\cdot) : [0,T] \longrightarrow \widetilde{E}$), we have to ensure the semicontinuity property $\psi_\varepsilon(t) \le \limsup_{h \downarrow 0} \psi_\varepsilon(t-h)$. It is the key point for using condition (6.) of Def. 2.1.1.

Lemma 2.1.3
Let $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, $\varepsilon \in \mathcal{J}$, $\widetilde{x} \in \widetilde{D}$, $0 < t < \mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{x})$, $\widetilde{y}(\cdot) : [0,t] \longrightarrow \widetilde{E}$
satisfy $\pi_1 \widetilde{\vartheta}(\cdot, \widetilde{x}) \le \pi_1 \widetilde{y}(\cdot)$ *increasing*
and $\widetilde{q}_\varepsilon\Big(\widetilde{y}(t-h),\ \widetilde{y}(t)\Big) \longrightarrow 0$ *for* $h \downarrow 0$.
Then,

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}),\ \widetilde{y}(t)\Big) \le \limsup_{h \downarrow 0} \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{x}),\ \widetilde{y}(t-h)\Big).$$

Proof. According to condition (6.) of Def. 2.1.1 and the timed triangle inequality,

$$\begin{aligned}
\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}),\ \widetilde{y}(t)\Big) &\le \limsup_{h \downarrow 0} \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\ \widetilde{x}),\ \widetilde{y}(t)\Big) \\
&\le \limsup_{h \downarrow 0} \ \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\ \widetilde{x}),\ \widetilde{y}(t-h)\Big) + \widetilde{q}_\varepsilon\Big(\widetilde{y}(t-h),\ \widetilde{y}(t)\Big)\Big) \\
&\le \limsup_{h \downarrow 0} \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\ \widetilde{x}),\ \widetilde{y}(t-h)\Big) + 0. \qquad \Box
\end{aligned}$$

As a first easy application of Gronwall's Lemma, we consider $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\vartheta})$ for any $\widetilde{\vartheta}$. Although $\widetilde{Q}_\varepsilon^{\mapsto} : \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)) \times \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)) \longrightarrow [0,\infty[$ satisfies the triangle inequality, it need not be reflexive, i.e. we cannot expect $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\vartheta}) = 0$ for every $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ in general. The parameter $\gamma_\varepsilon(\widetilde{\vartheta})$ provides an upper bound as stated by the following lemma :

Lemma 2.1.4 *Every timed transition* $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *fulfills*

$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\vartheta}) \le 3\,\gamma_\varepsilon(\widetilde{\vartheta}).$$

Proof is based on Gronwall's Lemma 1.5.1 applied to

$$\varphi_\varepsilon : [0,1] \longrightarrow [0,\infty[, \qquad h \longmapsto \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\widetilde{x}),\ \widetilde{\vartheta}(h,\widetilde{y})\Big)$$

with any $\widetilde{x} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$ $(\pi_1 \widetilde{x} \le \pi_1 \widetilde{y})$. The preceding Lemma 2.1.3 guarantees

$$\varphi_\varepsilon(h) \le \limsup_{k \downarrow 0} \varphi_\varepsilon(h-k).$$

Now choose $h \in [0,1[$, $\delta > 0$ arbitrarily and we obtain for any $k > 0$ small enough

$$\begin{aligned}
\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h+k,\widetilde{x}),\quad \widetilde{\vartheta}(k,\quad \widetilde{\vartheta}(h,\widetilde{x}))\Big) &\le (\gamma_\varepsilon(\widetilde{\vartheta}) + \delta)\,k, \\
\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\quad \widetilde{\vartheta}(h,\widetilde{y})),\ \widetilde{\vartheta}(h+k,\widetilde{y})\Big) &\le (\gamma_\varepsilon(\widetilde{\vartheta}) + \delta)\,k, \\
\frac{\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\widetilde{\vartheta}(h,\widetilde{x})),\ \widetilde{\vartheta}(k,\widetilde{\vartheta}(h,\widetilde{y}))\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\widetilde{x}),\ \widetilde{\vartheta}(h,\widetilde{y})\Big) - \gamma_\varepsilon(\widetilde{\vartheta})\,k}{k \cdot \Big\{\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h,\widetilde{x}),\ \widetilde{\vartheta}(h,\widetilde{y})\Big) + \gamma_\varepsilon(\widetilde{\vartheta})\,k\Big\}} &\le \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) + \delta.
\end{aligned}$$

So the timed triangle inequality leads to

$$\begin{aligned}
\varphi_\varepsilon(h+k) &= \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h+k,\widetilde{x}), \quad \widetilde{\vartheta}(h+k,\widetilde{y})\Big) \\
&\le \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h+k,\widetilde{x}), \quad \widetilde{\vartheta}(k, \quad \widetilde{\vartheta}(h,\widetilde{x}))\Big) \\
&\quad + \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k, \quad \widetilde{\vartheta}(h,\widetilde{x})), \ \widetilde{\vartheta}(k, \quad \widetilde{\vartheta}(h,\widetilde{y}))\Big) \\
&\quad + \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k, \quad \widetilde{\vartheta}(h,\widetilde{y})), \ \widetilde{\vartheta}(h+k,\widetilde{y})\Big) \\
&\le 2\ (\gamma_\varepsilon(\widetilde{\vartheta}) \ + \delta)\ k \\
&\quad + (\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) + \delta)\ k\ \Big(\varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\vartheta})\cdot k\Big) \ + \ \varphi_\varepsilon(h) \ + \ \gamma_\varepsilon(\widetilde{\vartheta})\cdot k,
\end{aligned}$$

i.e. $\quad \limsup_{k\downarrow 0} \frac{\varphi_\varepsilon(h+k)\,-\,\varphi_\varepsilon(h)}{k} \ \le\ (\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) + \delta)\ \cdot\ \varphi_\varepsilon(h) \ + \ 3\,(\gamma_\varepsilon(\widetilde{\vartheta}) + \delta).$

Since $\delta > 0$ is arbitrarily small, we conclude from Gronwall's Lemma 1.5.1

$$\begin{aligned}
\varphi_\varepsilon(h) &\le \varphi_\varepsilon(0)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})\cdot h} \ + \ 3\,\gamma_\varepsilon(\widetilde{\vartheta})\ \frac{e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})\cdot h}-1}{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})} \\
\limsup_{h\downarrow 0} \frac{\varphi_\varepsilon(h)\,-\,\varphi_\varepsilon(0)\cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})\cdot h}}{h} &\le 3\,\gamma_\varepsilon(\widetilde{\vartheta}).
\end{aligned}$$

□

The final result of this section is the upper estimate of the distance between two points while evolving along different timed transitions. In comparison with transitions on metric spaces (M,d) (according to [2, Aubin 99]), it generalizes the key estimate $(*)$ mentioned in the introduction of this chapter. So we continue this approach and use the inequality as a motivation for defining "primitives" and "solutions" in the next sections.

Proposition 2.1.5 *Let* $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *be timed forward transitions,* $\varepsilon \in \mathcal{J}$, $\widetilde{x} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$ *and* $0 \le t_1 \le t_2 \le 1$, $h \ge 0$ *(with* $\pi_1\widetilde{x} \le \pi_1\widetilde{y}$, $t_1+h < \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{x})$*). Then the following estimate holds*

$$\begin{aligned}
\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_1+h,\widetilde{x}),\ \widetilde{\tau}(t_2+h,\widetilde{y})\Big) \ \le\ & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_1,\widetilde{x}),\ \widetilde{\tau}(t_2,\widetilde{y})\Big) \quad\cdot\ e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\ h} \\
& + \ h\ \Big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau})\Big)\ \frac{e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\,h}-1}{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\,h}\,.
\end{aligned}$$

Proof. The auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1+h,\widetilde{x}),\ \widetilde{\tau}(t_2+h,\widetilde{y}))$ has the semicontinuity property $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0} \varphi_\varepsilon(h-k)$ due to the assumptions of $\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ and the preceding Lemma 2.1.3.

Moreover it fulfills for any $h \in [0,1[$ with $t_1 + h < \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{x})$

$$\limsup_{k\downarrow 0} \frac{\varphi_\varepsilon(h+k)-\varphi_\varepsilon(h)}{k} \ \le\ \alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\ \cdot\ \varphi_\varepsilon(h) \ + \ \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau}).$$

Indeed, for all $k > 0$ sufficiently small, the timed triangle inequality leads to

$$\begin{aligned}
\varphi_\varepsilon(h+k) \;\le\;& \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_1+h+k,\;\; \widetilde{x}),\quad \widetilde{\vartheta}(k,\; \widetilde{\vartheta}(t_1+h,\; \widetilde{x}))\Big)\\
&+\; \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k,\; \widetilde{\vartheta}(t_1+h, \widetilde{x})),\quad \widetilde{\tau}(k,\; \widetilde{\tau}(t_2+h,\; \widetilde{y}))\Big)\\
&+\; \widetilde{q}_\varepsilon\Big(\widetilde{\tau}(k,\; \widetilde{\tau}(t_2+h, \widetilde{y})),\quad \widetilde{\tau}(t_2+h+k,\quad \widetilde{y})\;\Big)\\
\le\;& \gamma_\varepsilon(\widetilde{\vartheta})\; k \;+\; \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau})\cdot k \;+\; \varphi_\varepsilon(h)\cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\,k} \;+\; \gamma_\varepsilon(\widetilde{\tau})\; k \;+\; o(k)
\end{aligned}$$

since $t_1+h+k < \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{x})$ implies $\widetilde{\vartheta}(t_1+h, \widetilde{x}),\; \widetilde{\vartheta}(t_1+h+k, \widetilde{x}) \in \widetilde{D}$.
Thus the claim results from Gronwall's Lemma 1.5.1. □

Remark. If $\alpha_\varepsilon^{\mapsto}(\widetilde{\tau}) = 0$, then the corresponding inequality is
$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_1+h, \widetilde{x}),\; \widetilde{\tau}(t_2+h, \widetilde{y})\Big) \;\le\; \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1, \widetilde{x}),\; \widetilde{\tau}(t_2, \widetilde{y})) \;+\; \Big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau})\Big)\cdot t.$

2.2 Timed right–hand forward primitives

Definition 2.2.1 *The function* $\widetilde{x} : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *is called* timed right–hand forward primitive of *a map* $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \quad t \in [0,T[\qquad \exists \ \widehat{\alpha}_\varepsilon^{\mapsto}(t) = \widehat{\alpha}_\varepsilon^{\mapsto}(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) \geq 0, \ \widehat{\gamma}_\varepsilon(t) = \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) \geq 0 \ :$
   $$\widehat{\alpha}_\varepsilon^{\mapsto}(t) \ \geq \ \alpha_\varepsilon^{\mapsto}(\vartheta(t)), \qquad \widehat{\gamma}_\varepsilon(t) \ \geq \ \gamma_\varepsilon(\vartheta(t)), \qquad \limsup_{\varepsilon' \downarrow 0} \ \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) \ = \ 0,$$
   $$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(h, \widetilde{y}), \ \widetilde{x}(t+h)\Big) \ - \ \widetilde{q}_\varepsilon(\widetilde{y}, \ \widetilde{x}(t)) \ \cdot \ e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \Big) \ \leq \ \widehat{\gamma}_\varepsilon(t),$$
   for all $\widetilde{y} \in \widetilde{D}$ *with* $\pi_1 \widetilde{y} \leq \pi_1 \widetilde{x}(t)$,
2. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$, *i.e. there is* $\omega_\varepsilon(\widetilde{x}, \cdot) :]0,T[\longrightarrow [0,\infty[$ *such that* $\widetilde{q}_\varepsilon\Big(\widetilde{x}(s), \ \widetilde{x}(t)\Big) \ \leq \ \omega_\varepsilon(\widetilde{x}, \ t-s)$ *for* $0 \leq s < t < T$, $\qquad \limsup_{h \downarrow 0} \ \omega_\varepsilon(\widetilde{x}, h) \ = \ 0$,
3. $\pi_1 \widetilde{x}(t) \ = \ t \ + \ \pi_1 \widetilde{x}(0)$ *for all* $t \in [0,T[$.

Remark. Let $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ be a timed right–hand forward primitive of $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$. For any $t \in]0,T[$, the map $\widetilde{x}(t+\cdot) : [0,T-t[\longrightarrow \widetilde{E}$ is a timed right–hand forward primitive of $\widetilde{\vartheta}(t+\cdot)$.
From now on we skip the attributes 'timed', 'right–hand', 'forward' of primitives in this chapter.

Timed transitions induce their own primitives – as an immediate consequence of Def. 2.1.1 and Prop. 2.1.5. This result is formulated in the following Lemma 2.2.2 so that we can use it explicitly later. Correspondingly, each piecewise constant function $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ has a primitive that is defined piecewise as well.

Lemma 2.2.2 *For every timed forward transition* $\widetilde{\vartheta}_0 \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *and* $\widetilde{z} \in \widetilde{E}$, *the function* $\widetilde{x} : [0,1[\longrightarrow \widetilde{E}, \ t \longmapsto \widetilde{\vartheta}_0(t, \widetilde{z})$ *is a primitive of* $\widetilde{\vartheta}(\cdot) := \widetilde{\vartheta}_0$. □

Now three ideas are presented how to estimate the distance between a primitive and a point evolving along a timed transition. An obstacle here is the common property of all preceding definitions that only points of $\widetilde{D}$ usually appear in the first argument of $\widetilde{q}_\varepsilon$. So essentially, we have two possibilities : Either restricting ourselves to the comparison with elements of $\widetilde{D}$ (as in Prop. 2.2.3) or using auxiliary functions for the distance (as in Propositions 2.2.4, 2.2.5).

Proposition 2.2.3 *Suppose* $\widetilde{\psi} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, $\widetilde{y} \in \widetilde{D}$, $t_1 \in [0, 1[$, $t_2 \in [0, T[$. *Let* $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ *be a timed primitive of* $\widetilde{\vartheta}(\cdot) : [0, T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *such that for each* $\varepsilon \in \mathcal{J}$,

$$\wedge \begin{cases} \widehat{\alpha}_\varepsilon^{\mapsto}(\cdot, \widetilde{x}, \widetilde{\vartheta}) & \leq & M_\varepsilon(\cdot), \\ \widehat{\gamma}_\varepsilon(\,\cdot\,, \widetilde{x}, \widetilde{\vartheta}) & \leq & R_\varepsilon(\cdot), \\ \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\psi},\ \widetilde{\vartheta}(\cdot)) & \leq & c_\varepsilon(\cdot), \\ t_1 + \pi_1\,\widetilde{y} & \leq & \pi_1\,\widetilde{x}(t_2) \end{cases}$$

with upper semicontinuous $M_\varepsilon, R_\varepsilon, c_\varepsilon : [0, T[\longrightarrow [0, \infty[$. *Set* $\mu_\varepsilon(h) := \displaystyle\int_{t_2}^{t_2+h} M_\varepsilon(s)\, ds$.

Then, for every $\varepsilon \in \mathcal{J}$ *and* $h \in]0, T[$ *with* $t_1 + h < \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{y})$,

$$\widetilde{q}_\varepsilon\Big(\widetilde{\psi}(t_1{+}h,\ \widetilde{y}),\ \ \widetilde{x}(t_2{+}h)\Big) \ \leq\ \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(t_1, \widetilde{y}),\ \widetilde{x}(t_2)\Big)\ \ e^{\mu_\varepsilon(h)}\ + \\ +\ \int_0^h e^{\mu_\varepsilon(h) - \mu_\varepsilon(s)}\ \Big(c_\varepsilon(t_2{+}s) + 2\, R_\varepsilon(t_2{+}s) + \gamma_\varepsilon(\widetilde{\psi})\Big)\ ds.$$

Proof. We follow the same track as in the proof of Prop. 2.1.5, consider the function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1 + h, \widetilde{y}),\ \widetilde{x}(t_2 + h))$. The semicontinuity property $\varphi_\varepsilon(h) \leq \limsup\limits_{k \downarrow 0}\ \varphi_\varepsilon(h - k)$ results from Lemma 2.1.3.
Furthermore we prove for any $h \in [0, T[$ with $t_1 + h < \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{y})$,

$$\limsup_{k \downarrow 0}\ \tfrac{\varphi_\varepsilon(h+k)\, -\, \varphi_\varepsilon(h)}{k}\ \leq\ M_\varepsilon(t_1{+}h)\,\cdot\,\varphi_\varepsilon(h)\ +\ c_\varepsilon(t_2{+}h) + 2\, R_\varepsilon(t_2{+}h) + \gamma_\varepsilon(\widetilde{\psi}).$$

In particular, this inequality implies $\varphi_\varepsilon(h) \geq \limsup\limits_{k \downarrow 0}\ \varphi_\varepsilon(h + k)$ since its right–hand side is finite. Thus, the claim results from Gronwall's Lemma 1.5.1 and its remark (2.).

For all $k > 0$ sufficiently small, the timed triangle inequality and Prop. 2.1.5 lead to

$$\begin{aligned} \varphi_\varepsilon(h + k)\ &=\ \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(t_1{+}h + k,\ \widetilde{y}),\ \ \widetilde{x}(t_2 + h + k)\Big) \\ &\leq\ \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(t_1{+}h + k,\ \widetilde{y}),\ \ \widetilde{\vartheta}(t_2{+}h)\Big(k,\ \widetilde{\psi}(t_1{+}h,\ \widetilde{y})\Big)\Big) \\ &\quad +\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_2{+}\,h)\Big(k, \widetilde{\psi}(t_1{+}h, \widetilde{y})\Big)\Big),\ \ \widetilde{x}(t_2 + h + k))\Big) \\ &\leq\ \Big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\psi},\ \widetilde{\vartheta}(t_2{+}h))\ +\ \gamma_\varepsilon(\widetilde{\psi})\ +\ \widehat{\gamma}_\varepsilon(t_2{+}h, \widetilde{x}, \widetilde{\vartheta})\Big)\ \tfrac{e^{M_\varepsilon(t_2+h)\,\cdot\,k} - 1}{M_\varepsilon(t_2+h)} \\ &\quad +\ \varphi_\varepsilon(h)\,\cdot\, e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t_2+h)\cdot k}\ +\ \widehat{\gamma}_\varepsilon(t_2{+}h, \widetilde{x}, \widetilde{\vartheta})\cdot k\ +\ o(k) \\ &\leq\ \varphi_\varepsilon(h)\,\cdot\, e^{M_\varepsilon(t_2+h)\cdot k}\ +\ \Big|c_\varepsilon(t) + \gamma_\varepsilon(\widetilde{\psi}) + 2\, R_\varepsilon(t)\Big|_{t = t_2 + h}\cdot\ k\ +\ o(k) \end{aligned}$$

since $t_1 + h + k < \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{y})$ implies $\widetilde{\psi}(t_1{+}h, \widetilde{y})$, $\widetilde{\psi}(t_1{+}h{+}k, \widetilde{y}) \in \widetilde{D}$.

□

The next proposition provides an upper bound of the auxiliary function

$$\varphi_\varepsilon(t) \;:=\; \inf_{\widetilde{z}\,\in\,\widetilde{D},\,\pi_1\,\widetilde{z}\,\leq\,t} \Big(\widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{\psi}(t,\widetilde{y})) \;+\; \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{x}(t))\Big)$$

for describing the distance between $\widetilde{\psi}(t,\widetilde{y})$ and a timed primitive $\widetilde{x}(t)$ without restricting to $\widetilde{\psi}(t,\widetilde{y}) \in \widetilde{D}$. The basic idea consists in estimating both

$$h \;\longmapsto\; \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_m),\; \widetilde{\psi}(t{+}h,\widetilde{y})\Big) \qquad \text{and} \qquad h \;\longmapsto\; \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_m),\; \widetilde{x}(t{+}h)\Big)$$

(for small $h > 0$) with a minimizing sequence $(\widetilde{z}_m)_{m\in I\!N}$ in $\widetilde{D}$. Here assumptions about the time parameter $\mathcal{T}_\Theta(\widetilde{\psi},\cdot) > 0$ are required for the first time. Roughly speaking, we need lower bounds of $\mathcal{T}_\Theta(\widetilde{\psi},\widetilde{z}_m)$ for "preserving" the information while $m \longrightarrow \infty$. If $\mathcal{T}_\Theta(\widetilde{\psi},\widetilde{z}_m)$ vanishes too quickly, then the comparison with $\widetilde{\psi}(\cdot,\widetilde{z}_m)$ cannot be put into practice long enough for proving estimates that (might) imply uniqueness of primitives.

Proposition 2.2.4 *Assume for a timed forward transition* $\widetilde{\psi} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, *a map* $\widetilde{\vartheta}(\cdot) : [0,1[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$, *a curve* $\widetilde{x} : [0,1[\longrightarrow \widetilde{E}$ *and* $\widetilde{y} \in \widetilde{E}$, $\lambda_\varepsilon > 0$

1. $\widetilde{x}(\cdot)$ *is a timed primitive of* $\widetilde{\vartheta}(\cdot)$ *with* $\pi_1\,\widetilde{x}(0) = \pi_1\,\widetilde{y} = 0$,
2. $$\begin{aligned} \alpha_\varepsilon^{\mapsto}(\widetilde{\psi}),\ \widehat{\alpha}_\varepsilon^{\mapsto}(\,\cdot\,,\widetilde{x},\widetilde{\vartheta}) \;&\leq\; M_\varepsilon \;<\; \infty \\ \widehat{\gamma}_\varepsilon(\,\cdot\,,\widetilde{x},\widetilde{\vartheta}) \;&\leq\; R_\varepsilon(\cdot) \\ \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\psi},\widetilde{\vartheta}(\cdot)) \;&\leq\; c_\varepsilon(\cdot), \qquad \textit{with upper semicontinuous } R_\varepsilon,\ c_\varepsilon : [0,1[\longrightarrow [0,\infty[. \end{aligned}$$
3. *for each* $t \in [0,1[$, $\quad \varphi_\varepsilon(t) \;:=\; \inf_{\widetilde{z}\,\in\,\widetilde{D},\,\pi_1\,\widetilde{z}\,\leq\,t} \Big(\widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{\psi}(t,\widetilde{y})) \;+\; \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{x}(t))\Big)$
   can be approximated by a minimizing sequence $(\widetilde{z}_n)_{n\,\in\,I\!N}$ *in* $\widetilde{D}$ *and* $h_n \downarrow 0$ *with*
   $\pi_1\widetilde{z}_m \;\leq\; \pi_1\,\widetilde{z}_n \;\leq\; t, \quad \widetilde{q}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n) \;\leq\; \lambda_\varepsilon\cdot h_m, \quad h_m \;<\; \mathcal{T}_\Theta(\widetilde{\psi},\widetilde{z}_m)$ *for all* $m < n$,

Then, $$\varphi_\varepsilon(t) \;\leq\; \varphi_\varepsilon(0)\,e^{M_\varepsilon\,t} \;+\; \int_0^t e^{M_\varepsilon\cdot(t-s)}\,\Big(c_\varepsilon(t) + 2\,R_\varepsilon(t) + 2\,\lambda_\varepsilon + 7\,\gamma_\varepsilon(\widetilde{\psi})\Big)\,ds.$$

Remark. If the above–mentioned minimizing sequence $(\widetilde{z}_n)$ in $\widetilde{D}$ fulfills

$$\frac{\sup_{n>m}\ \widetilde{q}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n)}{\mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j,t),\,\widetilde{z}_j)} \;\longrightarrow\; 0 \qquad\qquad (m \longrightarrow \infty)$$

then the estimate is fulfilled with $\lambda_\varepsilon = 0$. This provides a way to uniqueness results in the case of $R_\varepsilon(\cdot) = 0$, $\gamma_\varepsilon(\widetilde{\psi}) = 0$. The additional assumption (for $m \longrightarrow \infty$) is fulfilled particularly if $\widetilde{q}_\varepsilon$ is symmetric and $\widetilde{D}$ is dense $(\widetilde{E},\widetilde{q}_\varepsilon)$. □

Proof is based on the second subdifferential version of Gronwall's Lemma 1.5.2 : The timed triangle inequality implies for any $0 \leq t_1 \leq t_2 < 1$, $\widetilde{z} \in \widetilde{D}$ with $\pi_1 \, \widetilde{z} \leq t_1$

$$\begin{array}{lcl} \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{\psi}(t_2,\widetilde{y})) & \leq & \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{\psi}(t_1,\widetilde{y})) \;+\; \beta_\varepsilon(\widetilde{\psi})\,(t_2-t_1), \\ \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{x}(t_2)) & \leq & \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{x}(t_1)) \;+\; \omega_\varepsilon(\widetilde{x}(\cdot),\, t_2-t_1), \end{array}$$

As a consequence, $\varphi_\varepsilon(t) \;\leq\; \liminf_{h \downarrow 0} \; \varphi_\varepsilon(t-h)$ for every $t \in \,]0,1[$.

Now we prove for any $t \in [0,1[$

$$\liminf_{h \downarrow 0} \; \tfrac{\varphi_\varepsilon(t+h) \,-\, \varphi_\varepsilon(t)}{h} \;\leq\; M_\varepsilon \, \varphi_\varepsilon(t) \;+\; c_\varepsilon(t) + 2\, R_\varepsilon(t) + 2\,\lambda_\varepsilon + 7\,\gamma_\varepsilon(\widetilde{\psi}).$$

Let $(\widetilde{z}_n)_{n \in I\!N}$ denote a minimizing sequence in $\widetilde{D}$ and $h_n \downarrow 0$ according to cond. (3.), i.e.

$$\wedge \left\{ \begin{array}{ll} \pi_1 \widetilde{z}_m \;\leq\; \pi_1 \widetilde{z}_n \;\leq\; t, \quad \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \;\leq\; \lambda_\varepsilon \, h_m, \quad h_m \;<\; \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z}_m) & \text{for all } m < n, \\ \widetilde{q}_\varepsilon(\widetilde{z}_n,\, \widetilde{\psi}(t,\widetilde{y})) \;+\; \widetilde{q}_\varepsilon(\widetilde{z}_n,\, \widetilde{x}(t)) \;\longrightarrow\; \varphi_\varepsilon(t) & (n \longrightarrow \infty) \end{array} \right.$$

Due to Prop. 2.2.3 and Lemma 2.1.4, we obtain for every $0 \;<\; h \;\leq\; h_m \;<\; \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z}_m)$

$$\begin{array}{l} \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(h, \widetilde{z}_m),\;\; \widetilde{\psi}(t{+}h, \widetilde{y})\Big) \\ \leq\;\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_m,\; \widetilde{\psi}(t,\widetilde{y})\Big) \cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\psi})\,h} \;+\; \displaystyle\int_0^h e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\psi})\,\cdot\,(h-s)} \; \Big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\psi},\widetilde{\psi}) + 3\,\gamma_\varepsilon(\widetilde{\psi})\Big)\; ds \\ \leq\;\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_m,\; \widetilde{\psi}(t,\widetilde{y})\Big) \cdot e^{M_\varepsilon\,h} \;+\; \tfrac{e^{M_\varepsilon\,h}\,-\,1}{M_\varepsilon} \;\; 6\,\gamma_\varepsilon(\widetilde{\psi}) \end{array}$$

and

$$\begin{array}{l} \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(h, \widetilde{z}_m),\;\; \widetilde{x}(t{+}h)\Big) \\ \leq\;\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_m,\; \widetilde{x}(t)\Big) \cdot e^{M_\varepsilon\,h} \;+\; \displaystyle\int_0^h e^{M_\varepsilon\,\cdot\,(h-s)} \;\; \Big(c_\varepsilon(t{+}s) + 2\,R_\varepsilon(t{+}s) + \gamma_\varepsilon(\widetilde{\psi})\Big)\, ds. \end{array}$$

Firstly, $\varphi_\varepsilon(t{+}h) \;\leq\; \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_m),\;\widetilde{\psi}(t{+}h,\widetilde{y})\Big) \;+\; \widetilde{q}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_m),\;\widetilde{x}(t{+}h)\Big)$ results directly from its definition. Secondly, the timed triangle inequality implies for any $n > m$

$$\begin{array}{lclcl} \widetilde{q}_\varepsilon\Big(\widetilde{z}_m,\; \widetilde{\psi}(t,\widetilde{y})\Big) & \leq & \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \;+\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_n,\; \widetilde{\psi}(t,\widetilde{y})\Big) & \leq & \lambda_\varepsilon\, h_m \;+\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_n,\; \widetilde{\psi}(t,\widetilde{y})\Big), \\ \widetilde{q}_\varepsilon\Big(\widetilde{z}_m,\; \widetilde{x}(t)\Big) & \leq & \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \;+\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_n,\; \widetilde{x}(t)\Big) & \leq & \lambda_\varepsilon\, h_m \;+\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_n,\; \widetilde{x}(t)\Big) \end{array}$$

and $n \longrightarrow \infty$ leads to the estimate

$$\widetilde{q}_\varepsilon\Big(\widetilde{z}_m,\; \widetilde{\psi}(t,\widetilde{y})\Big) \;+\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_m,\; \widetilde{x}(t)\Big) \;\leq\; 2\,\lambda_\varepsilon\, h_m \;+\; \varphi_\varepsilon(t).$$

As a consequence,

$$\varphi_\varepsilon(t{+}h_m) \;\leq\; \Big(2\,\lambda_\varepsilon\, h_m {+} \varphi_\varepsilon(t)\Big)\; e^{M_\varepsilon\, h_m} \;+\; \int_0^{h_m} e^{M_\varepsilon\cdot(h_m-s)} \; \Big(c_\varepsilon(\cdot){+}2R_\varepsilon(\cdot){+}7\gamma_\varepsilon(\widetilde{\psi})\Big)_{t+s} \, ds.$$

So finally, $\liminf_{h \downarrow 0} \; \tfrac{\varphi_\varepsilon(t+h) \,-\, \varphi_\varepsilon(t)}{h} \;\leq\; M_\varepsilon\, \varphi_\varepsilon(t) \;+\; 2\,\lambda_\varepsilon + c_\varepsilon(t) + 2\,R_\varepsilon(t) + 7\,\gamma_\varepsilon(\widetilde{\psi}).$ □

Finally, the auxiliary function $\varphi_\varepsilon(\cdot)$ is modified with regard to the transition $\widetilde{\psi}(\cdot,\widetilde{y})$:

$$\varphi_\varepsilon(t) \;:=\; \inf_{\substack{\widetilde{z}\,\in\,\widetilde{D},\\ \pi_1\,\widetilde{z}\,\leq\,\pi_1\,\widetilde{x}(t)}} \Big(\widetilde{p}_\varepsilon(\widetilde{z},\, \widetilde{\psi}(t,\widetilde{y})) \;+\; \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{x}(t))\Big)$$

Here $\widetilde{p}_\varepsilon : \widetilde{E}\times\widetilde{E} \longrightarrow [0,\infty[$ represents a generalized distance function on $\widetilde{E}$ that has the additional advantage of symmetry (by assumption) and satisfies the triangle inequality (not just the *timed* one). Roughly speaking, $\widetilde{p}_\varepsilon$ might not take all the properties of elements $\widetilde{x},\widetilde{y}\in\widetilde{E}$ into consideration – compared with $\widetilde{q}_\varepsilon$. Anticipating the definitions of § 4.1 for a moment, the nonempty compact subsets of $I\!R^N$ give an example with $\widetilde{p}_\varepsilon := d\!l$ (Pompeiu–Hausdorff distance) and $\widetilde{q}_\varepsilon := q_{\mathcal{K},N}$.

In regard to timed transitions, the assumptions about $\widetilde{p}_\varepsilon$ have the advantage that they do not consider the comparison of two transitions. Instead we suppose continuity properties and that the distance $\widetilde{p}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2)$ between arbitrary points $\widetilde{z}_1,\widetilde{z}_2\in\widetilde{E}$ may grow exponentially at the most while evolving along $\widetilde{\psi}$.

Proposition 2.2.5 *Let* $\widetilde{p}_\varepsilon,\widetilde{q}_\varepsilon : \widetilde{E}\times\widetilde{E}\longrightarrow[0,\infty[$ $(\varepsilon\in\mathcal{J})$, $p\in I\!R$, $\widetilde{D}\overset{\text{Def.}}{=} I\!R\times D\subset\widetilde{E}$ *and* $\widetilde{\psi}\in\widetilde{\Theta}^{\mapsto}_p(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $\widetilde{\vartheta}(\cdot) : [0,1[\longrightarrow \widetilde{\Theta}^{\mapsto}_p(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$, $\widetilde{x} : [0,1[\longrightarrow\widetilde{E}$, $\widetilde{y}\in\widetilde{E}$, $\lambda_\varepsilon>0$ *satisfy the following conditions :*

1. *Each* $\widetilde{q}_\varepsilon$ *fulfills the timed triangle inequality and* $\widetilde{q}_\varepsilon(\widetilde{z},\widetilde{z})=0$ *for all* $\widetilde{z}\in\widetilde{D}$,
2. $\widetilde{p}_\varepsilon$ *is symmetric and satisfies the triangle inequality,*
3. $\widetilde{x}(\cdot)$ *is a timed primitive of* $\widetilde{\vartheta}(\cdot)$ *with* $\pi_1\,\widetilde{x}(0)\geq\pi_1\,\widetilde{y}$,
4. $\exists\; M_\varepsilon<\infty :$ $\alpha^{\mapsto}_\varepsilon(\widetilde{\psi}),\;\widehat{\alpha}^{\mapsto}_\varepsilon(\cdot,\widetilde{x},\widetilde{\vartheta}) \;\leq\; M_\varepsilon,$
   $\widetilde{p}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_1),\;\widetilde{\psi}(h,\widetilde{z}_2)\Big) \;\leq\; \widetilde{p}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2)\cdot e^{M_\varepsilon\,h}\quad\forall\;\widetilde{z}_1,\widetilde{z}_2,h$
   $\exists\; R_\varepsilon(\cdot)\geq 0 :$ $\gamma_\varepsilon(\widetilde{\psi}),\;\widehat{\gamma}_\varepsilon(\cdot,\widetilde{x},\widetilde{\vartheta}) \;\leq\; R_\varepsilon(\cdot),$
   $\limsup_{h\downarrow 0}\;\frac{\widetilde{p}_\varepsilon\big(\widetilde{\psi}(h,\widetilde{\psi}(t,\widetilde{y})),\,\widetilde{\psi}(t+h,\widetilde{y})\big)}{h} \;\leq\; R_\varepsilon(t),$
   $\widetilde{p}_\varepsilon\Big(\widetilde{\psi}(t-h,\widetilde{y}),\;\widetilde{\psi}(t,\widetilde{y})\Big)\longrightarrow 0$ *for* $h\downarrow 0$,
   $\widetilde{Q}^{\mapsto}_\varepsilon(\widetilde{\psi},\widetilde{\vartheta}(\cdot)) \;\leq\; c_\varepsilon(\cdot),$ *with upper semicontinuous* $R_\varepsilon,\,c_\varepsilon : [0,1[\longrightarrow[0,\infty[$,
5. *for each* $t\in[0,1[$, $\varphi_\varepsilon(t) := \inf_{\substack{\widetilde{z}\,\in\,\widetilde{D},\\ \pi_1\,\widetilde{z}\,\leq\,\pi_1\,\widetilde{x}(t)}} \Big(\widetilde{p}_\varepsilon(\widetilde{\psi}(t,\widetilde{y}),\;\widetilde{z}) \;+\; \widetilde{q}_\varepsilon(\widetilde{z},\,\widetilde{x}(t))\Big)$
   can be approximated by a minimizing sequence $(\widetilde{z}_n)_{n\in I\!N}$ *in* $\widetilde{D}$ *and* $h_n\downarrow 0$ *with*
   $\pi_1\widetilde{z}_m \;\leq\; \pi_1\,\widetilde{z}_n \;\leq\; \pi_1\,\widetilde{x}(t),\qquad \widetilde{p}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n)\;\leq\;\lambda_\varepsilon\cdot h_m,$
   $h_m \;<\; \mathcal{T}_\Theta(\widetilde{\psi},\widetilde{z}_m),\qquad \widetilde{q}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n)\;\leq\;\lambda_\varepsilon\cdot h_m$ *for all* $m<n$.

Then,
$$\varphi_\varepsilon(t) \;\leq\; \varphi_\varepsilon(0)\,e^{M_\varepsilon\,t} \;+\; \int_0^t e^{M_\varepsilon\cdot(t-s)}\,\Big(c_\varepsilon(t)+4\,R_\varepsilon(t)+2\,\lambda_\varepsilon\Big)\,ds.$$

Proof is based on the same version of Gronwall's Lemma 1.5.2 as the preceding Prop. 2.2.4 : $\varphi_\varepsilon(t) \le \liminf_{h \downarrow 0} \varphi_\varepsilon(t-h)$ results from cond. (1.), (2.) because for any $\widetilde{z} \in \widetilde{D}$ with $\pi_1 \widetilde{z} \le \pi_1 \widetilde{x}(t-h)$,

$$\begin{aligned} \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(t,\widetilde{y}),\ \widetilde{z}\Big) &\le \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(t,\widetilde{y}),\ \widetilde{\psi}(t-h,\widetilde{y})\Big) + \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(t-h,\widetilde{y}),\ \widetilde{z}\Big) \\ &\le \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(t-h,\widetilde{y}),\ \widetilde{\psi}(t,\widetilde{y})\Big) + \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(t-h,\widetilde{y}),\ \widetilde{z}\Big) \\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t)) &\le \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t-h)) + \omega_\varepsilon(\widetilde{x}(\cdot),h). \end{aligned}$$

For showing $\liminf_{h\downarrow 0} \frac{\varphi_\varepsilon(t+h)-\varphi_\varepsilon(t)}{h} \le M_\varepsilon\ \varphi_\varepsilon(t) + c_\varepsilon(t) + 4\,R_\varepsilon(t) + 2\,\lambda_\varepsilon$,

let $(\widetilde{z}_n)_{n\in I\!N}$ denote a minimizing sequence in $\widetilde{D}$ and $h_n \downarrow 0$ such that

$$\wedge \begin{cases} \pi_1\,\widetilde{z}_m \le \pi_1\,\widetilde{z}_n \le \pi_1\,\widetilde{x}(t), & \text{for all } m<n, \\ \widetilde{p}_\varepsilon(\widetilde{z}_m,\ \widetilde{z}_n),\ \widetilde{q}_\varepsilon(\widetilde{z}_m,\ \widetilde{z}_n) \le \lambda_\varepsilon \cdot h_m, \quad h_m < \mathcal{T}_\Theta(\widetilde{\psi},\ \widetilde{z}_m) & \\ \widetilde{p}_\varepsilon(\widetilde{\psi}(t,\widetilde{y}),\ \widetilde{z}_n) + \widetilde{q}_\varepsilon(\widetilde{z}_n,\ \widetilde{x}(t)) \longrightarrow \varphi_\varepsilon(t) & (n\longrightarrow\infty). \end{cases}$$

According to cond. (2.), (4.), we obtain for all $m<n$, $0<h\le h_m$

$$\begin{aligned} &\widetilde{p}_\varepsilon\Big(\widetilde{\psi}(t+h,\ \widetilde{y}),\ \widetilde{\psi}(h,\widetilde{z}_m)\Big) \\ =\ & \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_m),\ \widetilde{\psi}(t+h,\ \widetilde{y})\Big) \\ \le\ & \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_m),\ \widetilde{\psi}(h,\ \widetilde{\psi}(t,\widetilde{y}))\Big) + \widetilde{p}_\varepsilon\Big(\widetilde{\psi}(h,\ \widetilde{\psi}(t,\widetilde{y})),\ \widetilde{\psi}(t+h,\ \widetilde{y})\Big) \\ \le\ & \widetilde{p}_\varepsilon\Big(\widetilde{z}_m,\ \widetilde{\psi}(t,\widetilde{y})\Big) \cdot e^{M_\varepsilon h} + (R_\varepsilon(t)+o(1))\ h \\ \le\ & \Big(\lambda_\varepsilon\ h_m + \widetilde{p}_\varepsilon(\widetilde{z}_n,\ \widetilde{\psi}(t,\widetilde{y}))\Big) \cdot e^{M_\varepsilon h} + (R_\varepsilon(t)+o(1))\ h. \end{aligned}$$

Furthermore Prop. 2.2.3 implies for any $0 < h \le h_m < \mathcal{T}_\Theta(\widetilde{\psi},\ \widetilde{z}_m)$, $n>m$

$$\begin{aligned} &\widetilde{q}_\varepsilon\Big(\widetilde{\psi}(h,\widetilde{z}_m),\ \widetilde{x}(t+h)\Big) \\ \le\ & \widetilde{q}_\varepsilon(\widetilde{z}_m,\ \widetilde{x}(t)) \cdot e^{M_\varepsilon h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(c_\varepsilon(t+s) + 2\,R_\varepsilon(t+s) + \gamma_\varepsilon(\widetilde{\psi})\Big)\ ds \\ \le\ & \Big(\lambda_\varepsilon\ h_m + \widetilde{q}_\varepsilon(\widetilde{z}_n,\ \widetilde{x}(t))\Big) \cdot e^{M_\varepsilon h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(c_\varepsilon(t+s) + 3\,R_\varepsilon(t+s)\Big)\ ds \end{aligned}$$

and $n \longrightarrow \infty$ leads to

$$\begin{aligned} \varphi_\varepsilon(t+h_m) \le\ & \varphi_\varepsilon(t)\cdot e^{M_\varepsilon h_m} + 2\,\lambda_\varepsilon\ e^{M_\varepsilon h_m}\ h_m + (R_\varepsilon(t)+o(1))\ h_m \\ & + \int_0^{h_m} e^{M_\varepsilon\cdot(h_m-s)}\ \Big(c_\varepsilon(t+s) + 3\,R_\varepsilon(t+s)\Big)\ ds. \end{aligned}$$

So finally, $\liminf_{h\downarrow 0} \frac{\varphi_\varepsilon(t+h)-\varphi_\varepsilon(t)}{h} \le M_\varepsilon\ \varphi_\varepsilon(t) + c_\varepsilon(t) + 4\,R_\varepsilon(t) + 2\,\lambda_\varepsilon$. □

2.3 Timed right–hand forward solutions

2.3.1 Definition and convergence theorems

The term "primitive" (of a function $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)))$ is closely related to the expression "solution" $\widetilde{x}(\cdot)$ of a generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$.

Definition 2.3.1 *For* $\widetilde{f} : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *given, a map* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand forward solution *of the generalized mutational equation*

$$\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$$

if $\widetilde{x}(\cdot)$ *is timed right–hand forward primitive of* $\widetilde{f}(\widetilde{x}(\cdot), \cdot) : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, *i.e. for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \;\; t \in [0,T[\qquad \exists \;\; \widehat{\alpha}_\varepsilon^{\mapsto}(t) \geq \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{x}(t),t)), \quad \widehat{\gamma}_\varepsilon(t) \geq \gamma_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)) \;:$
$$\limsup_{h \downarrow 0} \; \tfrac{1}{h} \left(\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\widetilde{y}), \;\; \widetilde{x}(t+h)\Big) \; - \; \widetilde{q}_\varepsilon(\widetilde{y}, \; \widetilde{x}(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \right) \;\leq\; \widehat{\gamma}_\varepsilon(t),$$
for all $\widetilde{y} \in \widetilde{D}$ *with* $\pi_1 \widetilde{y} \leq \pi_1 \widetilde{x}(t)$ *and* $\limsup_{\varepsilon' \downarrow 0} \; \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) = 0$,
2. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$, *i.e. there is* $\omega_\varepsilon(\widetilde{x}, \cdot) :]0,T[\longrightarrow [0,\infty[$ *such that* $\widetilde{q}_\varepsilon\Big(\widetilde{x}(s), \; \widetilde{x}(t)\Big) \leq \omega_\varepsilon(\widetilde{x}, \; t-s)$ *for* $0 \leq s < t < T$, $\quad \limsup_{h \downarrow 0} \; \omega_\varepsilon(\widetilde{x}, h) = 0$,
3. $\pi_1 \, \widetilde{x}(t) \; = \; t \; + \; \pi_1 \, \widetilde{x}(0)$ *for all* $t \in [0,T[$.

Generally speaking, the existence of a solution can often be concluded from approximation. Seizing this well–tried notion here, we use Euler method in the next section. As a first step in this direction, the relevant kind of convergence has to be specified. It is to guarantee that the limit function of approximating solutions is a solution (in other words, it is to preserve the solution property).

Assumptions (5.ii), (5.iii) of the next proposition formulate a suitable form of convergence that might be subsumed under the generic term "two–sided graphically convergent". Obviously, it is weaker than pointwise convergence (with respect to time) and consists of two conditions with the limit function appearing in both arguments of $\widetilde{q}_\varepsilon$.
Admitting vanishing "time perturbations" $\delta_j, \delta'_j \geq 0$ exemplifies the basic idea that the first argument of $\widetilde{q}_\varepsilon$ usually refers to the earlier element whereas the second argument mostly represents the later point.

Proposition 2.3.2 (Convergence Theorem I)
Suppose the following properties of

$$\widetilde{f}_m, \ \widetilde{f}: \ \widetilde{E} \times [0,T[\ \longrightarrow \ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}) \qquad (m \in I\!N)$$
$$\widetilde{x}_m, \ \widetilde{x}: \ [0,T[\ \longrightarrow \ \widetilde{E} \ :$$

1. $M_\varepsilon \ := \ \sup_{m,t,\widetilde{z}} \{ \alpha_\varepsilon^{\mapsto}(\widetilde{f}_m(\widetilde{z},t)) \} \ < \ \infty,$

   $R_\varepsilon \ \geq \ \sup_{m,t,\widetilde{z}} \{ \widehat{\gamma}_\varepsilon(t, \widetilde{x}_m, \widetilde{f}_m(\widetilde{x}_m, \cdot)), \ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{z},t)), \ \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \}$

   with $\limsup_{\varepsilon' \downarrow 0} \ \varepsilon'^p \cdot R_{\varepsilon'} \ = \ 0,$
2. $\limsup \ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_m(\widetilde{z}_1, t_1), \ \widetilde{f}_m(\widetilde{z}_2, t_2)\Big) \ \leq \ R_\varepsilon$ *for* $m \to \infty, \quad t_2 - t_1 \downarrow 0,$

   $\widetilde{q}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2) \to 0 \quad (\pi_1 \widetilde{z}_1 \leq \pi_1 \widetilde{z}_2),$
3. $\overset{\circ}{\widetilde{x}}_m(\cdot) \ \ni \ \widetilde{f}_m(\widetilde{x}_m(\cdot), \cdot)$ *in* $[0,T[\,,$
4. $\widehat{\omega}_\varepsilon(h) \ := \ \sup_m \ \omega_\varepsilon(\widetilde{x}_m, h) \ < \ \infty$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*)* $\forall \ h \in \,]0,T],$

   $\limsup_{h \downarrow 0} \ \widehat{\omega}_\varepsilon(h) \ = \ 0,$
5. $\forall \ t_1, t_2 \in [0,T[, \quad t_3 \in \,]0,T[\quad \exists \ (m_j)_{j\in I\!N}$ *with* $m_j \nearrow \infty$ *and*

   (i) $\limsup \ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t_1), t_1), \ \widetilde{f}_{m_j}(\widetilde{x}(t_1), t_1)\Big) \ \leq \ R_\varepsilon \qquad (j \longrightarrow \infty)$

   (ii) $\exists \ (\delta_j')_{j\in I\!N}$ *in* $[0,1[\ :$ $\quad \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_2), \ \widetilde{x}_{m_j}(t_2+\delta_j')\Big) \longrightarrow 0, \qquad \delta_j' \longrightarrow 0,$

   $\pi_1 \widetilde{x}(t_2) \ \leq \ \pi_1 \widetilde{x}_{m_j}(t_2+\delta_j').$

   (iii) $\exists \ (\delta_j)_{j\in I\!N}$ *in* $[0,t_3[\ :$ $\quad \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_3-\delta_j), \ \widetilde{x}(t_3)\Big) \longrightarrow 0, \qquad \delta_j \longrightarrow 0,$

   $\pi_1 \widetilde{x}_{m_j}(t_3-\delta_j) \ \leq \ \pi_1 \widetilde{x}(t_3),$

for each $\varepsilon \in \mathcal{J}$*.*

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ \ni \ \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$*.*

Proof. The claimed uniform continuity of $\widetilde{x}(\cdot)$ results from assumption (4.) : Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon\Big(\widetilde{x}_m(t_1), \ \widetilde{x}_m(t_2)\Big) \ \leq \ \widehat{\omega}_\varepsilon(t_2 - t_1)$ for $0 \leq t_1 < t_2 < T$. Let $\varepsilon \in \mathcal{J}, \ 0 \leq t_1 < t_2 < T$ be arbitrary and choose $(\delta_j')_{j\in I\!N}, \ (\delta_j)_{j\in I\!N}$, for t_1, t_2 (according to cond. (5.ii), (5.iii)). For all $j \in I\!N$ large enough, $t_1 + \delta_j' < t_2 - \delta_j$ and so,

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{x}(t_1), \ \widetilde{x}(t_2)\Big) \\
&\leq \ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_1), \ \widetilde{x}_{m_j}(t_1+\delta_j')\Big) + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_1+\delta_j'), \ \widetilde{x}_{m_j}(t_2-\delta_j)\Big) + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_2-\delta_j), \ \widetilde{x}(t_2)\Big) \\
&\leq \qquad\qquad\qquad\qquad\qquad \widehat{\omega}_\varepsilon(t_2 - t_1) \qquad\qquad\qquad\qquad + o(1).
\end{aligned}$$

for $j \longrightarrow \infty$.

Now let $\varepsilon \in \mathcal{J}$, $\widetilde{y} \in \widetilde{D}$ and $t \in [0,T[$, $0 < h < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{y})$ be chosen arbitrarily. Condition (6.) of Def. 2.1.1 ensures for all $k \in]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\ (h,\widetilde{y}),\ \ \widetilde{x}(t+h)\Big) \ \le\ \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\ (h-k,\widetilde{y}),\ \ \widetilde{x}(t+h)\Big)\ +\ h^2.$$

According to cond. (5.i) – (5.iii), there exist sequences $(m_j)_{j\in I\!N}$, $(\delta_j)_{j\in I\!N}$, $(\delta'_j)_{j\in I\!N}$ satisfying $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta'_j \downarrow 0$, $\delta_j + \delta'_j < k$ and

$$\wedge \begin{cases} \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t),t), & \widetilde{f}_{m_j}(\widetilde{x}(t),t)\Big) \quad \le \quad R_\varepsilon + h^2, \\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+h-\delta_j), & \widetilde{x}(t+h)\Big) \quad \longrightarrow \quad 0, \\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), & \widetilde{x}_{m_j}(t+\delta'_j)\Big) \quad \longrightarrow \quad 0. \end{cases}$$

Thus, Proposition 2.2.3 implies for all $j \in I\!N$ large enough (depending on $\varepsilon, \widetilde{y}, t, h, k$),

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\ (h,\ \widetilde{y}),\ \ \widetilde{x}(t+h)\Big) \\
\le\ & \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\ (h-k,\ \widetilde{y}),\ \ \widetilde{x}_{m_j}(t+\delta'_j+h-k)\Big) \\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+\delta'_j+h-k),\ \ \widetilde{x}_{m_j}(t+h-\delta_j)\Big) \\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+h-\delta_j),\ \ \widetilde{x}(t+h)\Big) \qquad +\ h^2 \\
\le\ & \widetilde{q}_\varepsilon\Big(\widetilde{y},\ \widetilde{x}_{m_j}(t+\delta'_j)\Big)\ \cdot\ e^{M_\varepsilon\cdot(h-k)} \qquad + \\
&+\ \int_0^{h-k} e^{M_\varepsilon\cdot(h-k-s)}\ \Big(\widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big)\ +\ 3\,R_\varepsilon\Big)\ ds \\
&+\ \widehat{\omega}_\varepsilon(k-\delta_j-\delta'_j) \\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+h-\delta_j),\ \ \widetilde{x}(t+h)\Big) \qquad +\ h^2 \\
\le\ & \Big(\widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t))\ +\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t),\ \widetilde{x}_{m_j}(t+\delta'_j)\Big)\Big)\ \cdot\ e^{M_\varepsilon\cdot(h-k)}\ + \\
&+\ \int_0^{h} e^{M_\varepsilon\cdot(h-s)}\ \Big(\widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big)\ +\ 3\,R_\varepsilon\Big)\ ds \\
&+\ \widehat{\omega}_\varepsilon(k) \qquad +\ 2\,h^2 \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t))\ \cdot\ e^{M_\varepsilon\,h} \qquad +\ 3\,h^2\ +\ \widehat{\omega}_\varepsilon(k) \\
&+\ \int_0^{h} e^{M_\varepsilon\cdot(h-s)}\ \Big(R_\varepsilon + h^2 + \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big) + 3\,R_\varepsilon\Big)\ ds
\end{aligned}$$

i.e. for every $j \in I\!N$ sufficiently large

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\ (h,\ \widetilde{y}),\ \ \widetilde{x}(t+h)\Big) \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t)) \cdot e^{M_\varepsilon\,h}\ +\ \text{const}\cdot\ h\ (R_\varepsilon + h)\ +\ \widehat{\omega}_\varepsilon(k) \\
&+\ \int_0^{h} e^{M_\varepsilon\cdot(h-s)}\ \ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big)\ ds
\end{aligned}$$

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\;(h,\;\widetilde{y}),\;\;\widetilde{x}(t+h)\Big)\\
\le\quad &\widetilde{q}_\varepsilon(\widetilde{y},\;\widetilde{x}(t))\cdot e^{M_\varepsilon\,h} \;+\; \text{const}\cdot\; h\;(R_\varepsilon+h)\;+\;\widehat{\omega}_\varepsilon(k)\\
&+\;h\quad e^{M_\varepsilon\,h}\qquad \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\qquad \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta_j'}\Big)\\
&+\int_0^h e^{M_\varepsilon\cdot(h-s)}\;\;\widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta_j'},\;\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta_j'+s}\Big)\;ds.
\end{aligned}$$

Now $j\longrightarrow\infty$ and then $k\longrightarrow 0$ provide the estimate

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\;(h,\;\widetilde{y}),\;\;\widetilde{x}(t+h)\Big)\\
\le\quad &\widetilde{q}_\varepsilon(\widetilde{y},\;\widetilde{x}(t))\cdot e^{M_\varepsilon\,h}\;+\;\text{const}\cdot\;h\;(R_\varepsilon+h)\;+\;0\;+\;0\\
&+\;h\;e^{M_\varepsilon\,h}\;\limsup_{j\longrightarrow\infty}\;\sup_{0\le s\le h}\;\;\widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta_j'},\;\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t+\delta_j'+s}\Big).
\end{aligned}$$

So finally convergence assumption (2.) together with the equi-continuity of $(\widetilde{x}_m)$ ensures

$$\begin{aligned}
&\limsup_{h\downarrow 0}\;\tfrac{1}{h}\cdot\Big(\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\widetilde{y}),\;\widetilde{x}(t{+}h)\Big)\;-\;\widetilde{q}_\varepsilon(\widetilde{y},\;\widetilde{x}(t))\;\cdot\;e^{M_\varepsilon\,h}\Big)\\
\le\;&\limsup_{h\downarrow 0}\;\Big(\text{const}\cdot(R_\varepsilon{+}h)\;+\limsup_{j\longrightarrow\infty}\;\sup_{0\le s\le h}\;\widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\Big|_{t+\delta_j'},\;\widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\Big|_{t+\delta_j'+s}\Big)\Big)\\
=\;&\qquad\text{const}\cdot\;R_\varepsilon\qquad+\;R_\varepsilon\,.
\end{aligned}$$

□

The supposed form of convergence can be weakened slightly – but then stronger assumptions about $(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ are usually required. The next proposition gives an example : On the one hand we dispense with the left–convergence of a subsequence $(\widetilde{x}_{m_j})$ in assumption (5.iii), but on the other hand we use the standard hypotheses $(L^{\Rightarrow}), (R^{\Rightarrow})$ and modify condition (2.) on the equi–continuity of $(\widetilde{f}_m)$ (in a generalized sense). Here the two standard hypotheses do not imply the general equivalence of right– and left–convergence because $\widetilde{q}_\varepsilon$ need not be reflexive on $\widetilde{E}$ (but merely on $\widetilde{D}$).
Moreover the approximating solutions $\widetilde{x}_m(\cdot)$ are now defined on $[a_m,T[$ with $a_m\longrightarrow 0$.

Proposition 2.3.3 (Convergence Theorem II)
For each $\varepsilon\in\mathcal{J}$, let $\widetilde{q}_\varepsilon:\widetilde{E}\times\widetilde{E}\longrightarrow[0,\infty[$ satisfy standard hypotheses $(L^{\Rightarrow})$, $(R^{\Rightarrow})$ (i.e. left–hand and right–hand spheres are right–sequentially closed) and $\widetilde{q}_\varepsilon(\widetilde{z},\widetilde{z})=0$ for all $\widetilde{z}\in\widetilde{D}$. Furthermore suppose the following properties of

$$\begin{aligned}
\widetilde{f}_m,\;\widetilde{f}:\;\;\widetilde{E}\times[0,T[\;&\longrightarrow\;\widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})\qquad(m\in I\!N)\\
\widetilde{x}_m:\;\;[a_m,T[\;&\longrightarrow\;\widetilde{E}\qquad\qquad(m\in I\!N)\\
\widetilde{x}:\;\;[\,0,\;T[\;&\longrightarrow\;\widetilde{E}\;:
\end{aligned}$$

$$\begin{array}{llll}
\textit{1.} & M_\varepsilon & := & \sup_{m,t,\widetilde{z}} \{ \alpha_\varepsilon^{\mapsto}(\widetilde{f}_m(\widetilde{z},t)) \} < \infty, \\
 & R_\varepsilon & \geq & \sup_{m,t,\widetilde{z}} \{ \widehat{\gamma}_\varepsilon(t, \widetilde{x}_m, \widetilde{f}_m(\widetilde{x}_m,\cdot)), \ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{z},t)), \ \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \} \\
 & \textit{with} & & \limsup_{\varepsilon' \downarrow 0} \ \varepsilon'^p \cdot R_{\varepsilon'} = 0,
\end{array}$$

2'. $\limsup \ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_m(\widetilde{z}_1,t_1), \ \widetilde{f}_m(\widetilde{z}_2,t_2)\Big) \leq R_\varepsilon$ *for* $m \longrightarrow \infty, \quad |t_2-t_1| \longrightarrow 0,$
$\min\{ \widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2), \ \widetilde{q}_\varepsilon(\widetilde{z}_2,\widetilde{z}_1) \} \longrightarrow 0,$

3. $\overset{\circ}{\widetilde{x}}_m(\cdot) \ni \widetilde{f}_m(\widetilde{x}_m(\cdot), \cdot)$ *in* $[a_m, T[$,

4. $\widehat{\omega}_\varepsilon(h) := \sup_m \omega_\varepsilon(\widetilde{x}_m, h) < \infty$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*)* $\forall \ h \in]0,T]$,
$\limsup_{h \downarrow 0} \ \widehat{\omega}_\varepsilon(h) = 0,$

5'. $\forall \ t_1 \in [0,T[, \ t_2, t_3 \in]0,T[\ \ \exists \ (m_j)_{j \in I\!N}$ *with* $m_j \nearrow \infty$ *and*

(*i*) $\limsup \ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t_1),t_1), \ \widetilde{f}_{m_j}(\widetilde{x}(t_1),t_1)\Big) \leq R_\varepsilon \qquad (j \longrightarrow \infty)$

(*ii*) $\exists \ (\delta_j)_{j\in I\!N}$ *in* $[0,t_2[$: $\widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_2-\delta_j), \ \widetilde{x}(t_2)\Big) \longrightarrow 0, \qquad \delta_j \longrightarrow 0,$
$\pi_1 \widetilde{x}_{m_j}(t_2-\delta_j) \leq \pi_1 \widetilde{x}(t_2),$

(*iii*) $\exists \ (\delta'_j)_{j\in I\!N}$ *in* $[0,t_3[$: $\widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_3-\delta'_j), \ \widetilde{x}(t_3)\Big) \longrightarrow 0, \qquad \delta'_j \longrightarrow 0,$
$\pi_1 \widetilde{x}_{m_j}(t_3-\delta'_j) \leq \pi_1 \widetilde{x}(t_3),$

(*iv*) $\exists \ (\delta''_j)_{j\in I\!N}$ *in* $]0, 1[$: $\widetilde{q}_\varepsilon\Big(\widetilde{x}(0), \ \widetilde{x}_{m_j}(\delta''_j)\Big) \longrightarrow 0, \qquad \delta''_j \longrightarrow 0,$
$\pi_1 \widetilde{x}_{m_j}(\delta''_j) \geq \pi_1 \widetilde{x}(0), \qquad \delta''_j \geq a_{m_j}$

for each $\varepsilon \in \mathcal{J}$.
Then, $\widetilde{x}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$.

Proof. Here the main benefit of standard hypothesis ($R^{\Rightarrow}$) is the uniform continuity of $\widetilde{x}(\cdot)$. Indeed assumption (4.) provides a uniform modulus of continuity for all $\widetilde{x}_m(\cdot)$ that is upper semicontinuous in addition :

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}_m(t_1), \ \widetilde{x}_m(t_2)\Big) \leq \widehat{\omega}_\varepsilon(t_2-t_1) \qquad \text{for any } a_m \leq t_1 < t_2 < T.$$

So considering adequate subsequences according to cond. (5'.), Prop. 1.4.5 guarantees

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t_1), \ \widetilde{x}(t_2)\Big) \leq \widehat{\omega}_\varepsilon(t_2-t_1) \qquad \text{for any } 0 < t_1 < t_2 < T.$$

The corresponding result for $t_1 = 0$ and any $0 < t_2 < T$ results from cond. (5'.iv) and the timed triangle inequality immediately.

Now let $\varepsilon \in \mathcal{J}$, $\widetilde{y} \in \widetilde{D}$ and $t \in]0,T[$, $0 < h < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{x}(t),t), \widetilde{y})$ be chosen arbitrarily with $\pi_1\widetilde{y} \leq \pi_1\widetilde{x}(t)$. Condition (6.) of Def. 2.1.1 ensures for all $k \in]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\widetilde{y}), \ \widetilde{x}(t+h)\Big) \leq \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\widetilde{y}), \ \widetilde{x}(t+h)\Big) + h^2.$$

According to cond. (5'.i) – (5'.iii) and standard hypothesis $(L^{\Rightarrow})$, there exist sequences $(m_j)_{j\in I\!N}$, $(\delta_j)_{j\in I\!N}$, $(\delta'_j)_{j\in I\!N}$ satisfying $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta'_j \downarrow 0$, $\delta_j + \delta'_j < k$ and

$$\wedge \begin{cases} \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t),t), & \widetilde{f}_{m_j}(\widetilde{x}(t),t)\Big) \;\le\; R_\varepsilon + h^2 & \forall\; j \in I\!N \\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+h-\delta_j), & \widetilde{x}(t+h)\Big) \;\longrightarrow\; 0 & (j \longrightarrow \infty) \\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t-\delta'_j), & \widetilde{x}(t)\Big) \;\longrightarrow\; 0 & (j \longrightarrow \infty) \\ \widetilde{q}_\varepsilon\Big(\widetilde{y}, & \widetilde{x}_{m_j}(t-\delta'_j)\Big) \;\le\; \widetilde{q}_\varepsilon\Big(\widetilde{y},\ \widetilde{x}(t)\Big) + h^2 & \forall\; j \in I\!N. \end{cases}$$

So the timed triangle inequality and Prop. 2.2.3 imply for all $j \in I\!N$ sufficiently large,

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{y}),\quad \widetilde{x}(t+h)\Big) \\
\le\;& \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k+\delta'_j,\ \widetilde{y}),\ \ \widetilde{x}_{m_j}(t+h-k)\Big) \\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+h-k),\quad \widetilde{x}_{m_j}(t+h-\delta_j)\Big) \\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+h-\delta_j),\quad \widetilde{x}(t+h)\Big) \;+\; h^2 \\
\le\;& \widetilde{q}_\varepsilon\Big(\widetilde{y},\ \widetilde{x}_{m_j}(t-\delta'_j)\Big) \cdot e^{M_\varepsilon\cdot(h-k+\delta'_j)} \;+ \\
&+ \int_0^{h-k+\delta'_j} e^{M_\varepsilon\cdot(h-k+\delta'_j-s)} \left(\widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t),t),\ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j+s}\Big) + 3\,R_\varepsilon\right) ds \\
&+\ \widehat{\omega}_\varepsilon(k-\delta_j) \\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+h-\delta_j),\quad \widetilde{x}(t+h)\Big) \;+\; h^2 \\
\le\;& \Big(\widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t)) + h^2\Big) \cdot e^{M_\varepsilon\cdot(h-k+\delta'_j)} \;+ \\
&+ \int_0^{h} e^{M_\varepsilon\cdot(h-s)} \left(\widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t),t),\ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j+s}\Big) + 3\,R_\varepsilon\right) ds \\
&+\ \widehat{\omega}_\varepsilon(k) \;+\; 2\,h^2 \\
\le\;& \widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t)) \cdot e^{M_\varepsilon\, h} \;+\; 3\,h^2\ e^{M_\varepsilon\, h} \;+\; \widehat{\omega}_\varepsilon(k) \\
&+ \int_0^{h} e^{M_\varepsilon\cdot(h-s)} \left(R_\varepsilon + h^2 + \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j+s}\Big) + 3\,R_\varepsilon\right) ds \\
\le\;& \widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t)) \cdot e^{M_\varepsilon\, h} \;+\; \text{const}\cdot\ h\,(R_\varepsilon + h) \;+\; \widehat{\omega}_\varepsilon(k) \\
&+\ h\cdot e^{M_\varepsilon\, h} \Big(\ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\quad \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j}\Big) \;+ \\
&+\ \sup_{0\le s\le h} \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j},\ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j+s}\Big)\Big)
\end{aligned}$$

Due to assumption (2'.), $j \longrightarrow \infty$ and then $k \longrightarrow 0$ lead to

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{y}),\quad \widetilde{x}(t+h)\Big) \\
\le\;& \widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t)) \cdot e^{M_\varepsilon\, h} \;+\; \text{const}\cdot\ h\,(R_\varepsilon + h) \;+\; 0 \;+\; 0 \\
&+\ h\cdot e^{M_\varepsilon\, h} \ \limsup_{j\longrightarrow\infty}\ \sup_{0\le s\le h} \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j},\ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\Big|_{t-\delta'_j+s}\Big).
\end{aligned}$$

So in combination with the equi-continuity of $(\widetilde{x}_m)$ due to assumption (4.), the uniform convergence of assumption (2'.) guarantees for any $t \in \,]0, T[$

$$\begin{aligned}
& \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t), t)\,(h, \widetilde{y}),\ \widetilde{x}(t+h)\Big) \ - \ \widetilde{q}_\varepsilon(\widetilde{y},\ \widetilde{x}(t)) \cdot e^{M_\varepsilon\, h} \Big) \\
\le\ & \limsup_{h \downarrow 0} \Big(\text{const} \cdot (R_\varepsilon + h) \ + \limsup_{j \longrightarrow \infty} \ \sup_{0 \le s \le h} \ \widetilde{Q}^{\mapsto}_\varepsilon \Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}, \cdot)\Big|_{t-\delta'_j},\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}, \cdot)\Big|_{t-\delta'_j+s} \Big)\Big) \\
=\ & \qquad \text{const} \cdot \ R_\varepsilon \qquad + \ R_\varepsilon\,.
\end{aligned}$$

Correspondingly for $t = 0$, condition (5'.iv) implies for all $j \in I\!N$ large enough

$$\begin{aligned}
& \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(0), 0)\,(h,\ \widetilde{y}),\quad \widetilde{x}(h)\Big) \\
\le \quad & \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(0), 0)\,(h - k - \delta''_j,\ \widetilde{y}),\quad \widetilde{x}_{m_j}(h-k)\Big) \\
& + \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(h-k), \qquad\qquad \widetilde{x}_{m_j}(h - \delta_j)\Big) \\
& + \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(h - \delta_j), \qquad\qquad \widetilde{x}(h)\Big) \ + \ h^2 \\
\le \quad & \widetilde{q}_\varepsilon\Big(\widetilde{y},\ \widetilde{x}_{m_j}(\delta''_j)\Big) \cdot e^{M_\varepsilon \cdot (h-k-\delta''_j)} \qquad + \\
& + \ \int_0^{h-k-\delta''_j} e^{M_\varepsilon \cdot (h-k-\delta''_j - s)} \Big(\widetilde{Q}^{\mapsto}_\varepsilon \Big(\widetilde{f}(\widetilde{x}(0), 0),\ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot), \cdot)\Big|_{\delta''_j + s} \Big) \ + \ 3\, R_\varepsilon \Big) \ ds \\
& + \ \widehat{\omega}_\varepsilon(k - \delta_j) \\
& + \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(h - \delta_j),\quad \widetilde{x}(h)\Big) \qquad\qquad + \ h^2
\end{aligned}$$

and we can repeat the same steps as for $t > 0$. □

2.3.2 Existence due to compactness

Roughly speaking, Convergence Theorem I (Prop. 2.3.2) provides sufficient conditions on a sequence $(\widetilde{x}_m(\cdot))_{m \in I\!N}$ of approximating solutions for converging to a timed right–hand forward solution $\widetilde{x}(\cdot)$ of $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$. The relevant form of convergence is given in its assumption (5.) :

5. $\forall\ t_1, t_3 \in [0, T[, \quad t_2 \in]0, T[\quad \exists\ (m_j)_{j \in I\!N}$ with $m_j \nearrow \infty$ and

(i) $\limsup\ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t_1), t_1),\ \widetilde{f}_{m_j}(\widetilde{x}(t_1), t_1)\Big) \ \le\ R_\varepsilon \qquad (j \longrightarrow \infty)$

(ii) $\exists\ (\delta_j)_{j \in I\!N}$ in $[0, t_2[$: $\quad \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_2 - \delta_j),\ \widetilde{x}(t_2)\Big) \longrightarrow 0, \quad \delta_j \longrightarrow 0,$
$\pi_1\, \widetilde{x}_{m_j}(t_2 - \delta_j) \ \le\ \pi_1\, \widetilde{x}(t_2),$

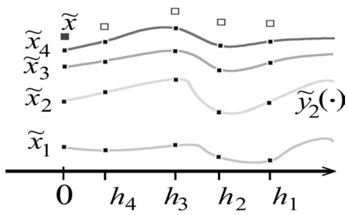
(iii) $\exists\ (\delta'_j)_{j \in I\!N}$ in $[0, 1[$: $\quad \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_3),\ \widetilde{x}_{m_j}(t_3 + \delta'_j)\Big) \longrightarrow 0, \quad \delta'_j \longrightarrow 0,$
$\pi_1\, \widetilde{x}(t_3) \ \le\ \pi_1\, \widetilde{x}_{m_j}(t_3 + \delta'_j).$

Our intention is to construct a timed right–hand forward solution of a generalized mutational equation by means of Euler method. For considering the family of $(\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}$, we prefer some form of compactness to a version of completeness. Thus in view of Convergence Theorem I (Prop. 2.3.2), we coin the following term :

Definition 2.3.4 *Let* $\widetilde{\Theta}$ *denote a nonempty set of maps* $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$.

The tuple $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\ \widetilde{\Theta})$ *is called* timed transitionally compact *if it has the following property :*
Let $(\widetilde{x}_n)_{n \in I\!N}$, $(h_j)_{j \in I\!N}$ *be any sequences in* $\widetilde{E}$, $]0, 1[$, *respectively and* $\widetilde{z} \in \widetilde{E}$ *with* $\sup_n\ \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{x}_n) < \infty$ *for each* $\varepsilon \in \mathcal{J}$, $h_j \longrightarrow 0$. *Moreover suppose* $\widetilde{\vartheta}_n : [0,1] \longrightarrow \widetilde{\Theta}$ *to be piecewise constant* $(n \in I\!N)$ *such that all curves* $\widetilde{\vartheta}_n(t)(\cdot, \widetilde{x}) : [0,1] \longrightarrow \widetilde{E}$ *have a common modulus of continuity* $(n \in I\!N,\ t \in [0,1],\ \widetilde{x} \in \widetilde{E})$.
Each $\widetilde{\vartheta}_n$ *induces a function* $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ *with* $\widetilde{y}_n(0) = \widetilde{x}_n$ *in the same (piecewise) way as timed forward transitions induce their own primitives according to Lemma 2.2.2 (i.e. using* $\widetilde{\vartheta}_n(t_m)\,(\cdot,\ \widetilde{y}_n(t_m))$ *in each interval* $]t_m, t_{m+1}]$ *in which* $\widetilde{\vartheta}_n(\cdot)$ *is constant).*
Then there exist a sequence $n_k \nearrow \infty$ *of indices and* $\widetilde{x} \in \widetilde{E}$ *satisfying for each* $\varepsilon \in \mathcal{J}$,

$$\begin{aligned} \lim_{k \to \infty}\ \ & \pi_1\, \widetilde{x}_{n_k} &&= \pi_1\, \widetilde{x}, \\ \limsup_{k \longrightarrow \infty}\ \ & \widetilde{q}_\varepsilon(\widetilde{x}_{n_k}, \widetilde{x}) &&= 0, \\ \limsup_{j \longrightarrow \infty}\ \sup_{k \ge j}\ \ & \widetilde{q}_\varepsilon(\widetilde{x},\ \ \widetilde{y}_{n_k}(h_j)) &&= 0. \end{aligned}$$



A nonempty subset $\widetilde{F} \subset \widetilde{E}$ *is called* timed transitionally compact in $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, \widetilde{\Theta})$ *if the same property holds for any sequence* $(\widetilde{x}_n)_{n \in\, ,I\!N}$ *in* $\widetilde{F}$ *(but* $\widetilde{x} \in \widetilde{F}$ *is* not *required).*

Remark. 1. The timed transitional compactness of $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)))$ implies that $(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is one–sided sequentially compact (uniformly with respect to ε) if $\widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)) \neq \emptyset$ (see Def. 1.2.2).
Indeed, for any sequence $(\widetilde{x}_n)_{n\in I\!N}$ in $\widetilde{E}$ and $\widetilde{z}\in\widetilde{E}$ with $\sup_n \widetilde{q}_\varepsilon(\widetilde{z},\widetilde{x}_n) < \infty$ $(\varepsilon\in\mathcal{J})$, we choose some $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$ and set $\widetilde{y}_n(\cdot) := \widetilde{\vartheta}(\cdot,\widetilde{x}_n)$. Then timed transitional compactness provides a sequence $n_k \nearrow \infty$ of indices and some $\widetilde{x}\in\widetilde{E}$ satisfying $\widetilde{q}_\varepsilon(\widetilde{x}_{n_k},\ \widetilde{x}) \longrightarrow 0$ $(k\longrightarrow\infty)$ for each $\varepsilon\in\mathcal{J}$.

2. Suppose that $(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is timed two–sided sequentially compact (uniformly with respect to ε). Then $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)))$ is timed transitionally compact since any sequences $(\widetilde{x}_n)$, (h_j), $(\vartheta_n(\cdot))$, $(\widetilde{y}_n)$ as in the preceding Def. 2.3.4 fulfill

$$\widetilde{q}_\varepsilon(\widetilde{x}_n,\ \widetilde{y}_n(h_n)) \ \leq\ c_\varepsilon(h_n) \longrightarrow 0 \qquad \text{for } n\longrightarrow\infty \text{ and every } \varepsilon\in\mathcal{J}.$$

So there exist a sequence $n_k \nearrow \infty$ of indices and $\widetilde{x}\in\widetilde{E}$ with

$$\widetilde{q}_\varepsilon(\widetilde{x}_{n_k},\ \widetilde{x}) \longrightarrow 0, \qquad \widetilde{q}_\varepsilon(\widetilde{x},\ \widetilde{y}_{n_k}(h_{n_k})) \longrightarrow 0 \qquad \text{for } k\longrightarrow\infty$$

and finally,

$$\widetilde{q}_\varepsilon(\widetilde{x},\ \widetilde{y}_{n_k}(h_j)) \ \leq\ \widetilde{q}_\varepsilon(\widetilde{x},\ \widetilde{y}_{n_k}(h_{n_k})) \ +\ c_\varepsilon(h_j) \qquad \text{for } h_{n_k} < h_j.$$

Proposition 2.3.5 (Existence of timed right–hand forward solutions due to timed transitional compactness)

Assume that the tuple $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)))$ *is timed transitionally compact. Furthermore let* $\widetilde{f}:\widetilde{E}\times[0,T]\longrightarrow\widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *fulfill for every* $\varepsilon\in\mathcal{J}$

1. $M_\varepsilon \ :=\ \sup_{t,\widetilde{z}}\ \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{z},t)) \ <\ \infty,$
2. $c_\varepsilon(h) \ :=\ \sup_{t,\widetilde{z}}\ \beta_\varepsilon(\widetilde{f}(\widetilde{z},t))(h) \ <\ \infty,$ $\qquad c_\varepsilon(h) \longrightarrow 0$ *for* $h\downarrow 0,$
3. $\exists\ R_\varepsilon\ :\ \sup_{t,\widetilde{z}}\ \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \ \leq\ R_\varepsilon<\infty,$ $\qquad \varepsilon'^p\ R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon'\downarrow 0,$
4. $\exists\ \widehat{\omega}_\varepsilon(\cdot):\ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \ \leq\ R_\varepsilon\ +\ \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2)+t_2-t_1\Big)$
   for all $0\leq t_1\leq t_2\leq T$ *and* $\widetilde{z}_1,\widetilde{z}_2\in\widetilde{E}$ *with* $\pi_1\,\widetilde{z}_1\leq\pi_1\,\widetilde{z}_2,$
   $\widehat{\omega}_\varepsilon(\cdot)\geq 0$ *nondecreasing,* $\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) \ =\ 0.$

Then for every $\widetilde{x}_0\in\widetilde{E}$, *there is a timed right–hand forward solution* $\widetilde{x}:[0,T[\longrightarrow\widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$ *in* $[0,T[$ *with* $\widetilde{x}(0)=\widetilde{x}_0$.

Proof is based on Euler method for an approximating sequence $(\widetilde{x}_n(\cdot))_{n\in I\!N}$ and Cantor diagonal construction for its limit $\widetilde{x}(\cdot)$.

For $n \in I\!N$ (with $2^n > T$) set

$$\begin{array}{lllll} h_n & := \frac{T}{2^n}, & t_n^j & := j\,h_n & \text{for } j = 0 \ldots 2^n, \\ \widetilde{x}_n(0) & := \widetilde{x}_0, & \widetilde{x}_0(\cdot) & := \widetilde{x}_0, & \\ \widetilde{x}_n(t) & := \widetilde{f}(\widetilde{x}_n(t_n^j),\, t_n^j)\,\Big(t - t_n^j,\ \widetilde{x}_n(t_n^j)\Big) & & & \text{for } t \in\,]t_n^j,\, t_n^{j+1}],\ j \le 2^n. \end{array}$$

The uniform modulus of continuity $c_\varepsilon(\cdot)$ can be replaced by a nondecreasing convex function $[0, T+1] \longrightarrow [0, \infty[$ such that all $\widetilde{x}_n(\cdot)$ are equi–continuous in the sense of

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}_n(s),\ \widetilde{x}_n(t)\Big) \ \le\ c_\varepsilon(t-s) \qquad \text{for any } 0 \le s < t < T + h_n \text{ and } \varepsilon \in \mathcal{J}.$$

Since $\mathcal{J}$ is countable there is a sequence $(j_k)_{k \in I\!N}$ with $\{j_1, j_2 \ldots\} = \mathcal{J} \subset [0,1]^\kappa$. Now for every $t \in]0,T[$, choose a decreasing sequence $(\delta_k(t))_{k \in I\!N}$ in $Q \cdot T$ satisfying

$$\begin{array}{lll} 0 \,<\, \delta_k(t) \,<\, \frac{h_k}{2}, & t + \delta_k(t) \,<\, T, & \\ c_{\varepsilon_j}(\delta_k(t)) \,<\, h_k & & \text{for any } j \in \{j_1 \ldots j_k\}. \end{array}$$

Then, $\widetilde{q}_{\varepsilon_j}\Big(\widetilde{x}_n(t),\ \widetilde{x}_n(t + \delta_k(t))\Big) \ \le\ h_k$ for any $j \in \{j_1 \ldots j_k\}$, $k, n \in I\!N$

and so $\widetilde{q}_\varepsilon\Big(\widetilde{x}_n(t),\ \widetilde{x}_n(t + \delta_k(t))\Big) \ \longrightarrow\ 0$ $(k \longrightarrow \infty)$ for every $\varepsilon \in \mathcal{J}$, uniformly in n. Thus for each $t \in]0,T[$ and any fixed $\varepsilon \in \mathcal{J}$, the timed transitional compactness of $(\widetilde{E}, (\widetilde{q}_\varepsilon), \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)))$ provides sequences $m_k \nearrow \infty$, $n_k \nearrow \infty$ $(m_k \le n_k)$ of indices and an element $\widetilde{x}(t) \in \widetilde{E}$ satisfying for every $k \in I\!N$

$$\wedge \begin{cases} \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_l}(t),\ \ \widetilde{x}(t)\Big) & \le\ \frac{1}{k}, \\ \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\Big(\widetilde{x}(t),\ \ \widetilde{x}_{n_l}(t + \delta_{m_k}(t))\Big) & \le\ \frac{1}{k}. \end{cases}$$

(In particular, each m_k, n_k may be replaced by larger indices preserving the properties.) For arbitrary $K \in I\!N$, these sequences $m_k,\, n_k \nearrow \infty$ can even be chosen in such a way that the estimates are fulfilled for the finite set of parameters $t \in Q_K :=]0,T[\, \cap\, I\!N \cdot h_K$ and $\varepsilon \in \mathcal{J}_K := \{\varepsilon_{j_1},\, \varepsilon_{j_2} \ldots \varepsilon_{j_K}\} \subset \mathcal{J}$ simultaneously.

Now the Cantor diagonal construction (with respect to the index K) provides subsequences (again denoted by) $m_k, n_k \nearrow \infty$ such that $m_k \le n_k$ and

$$\wedge \begin{cases} \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(t),\ \ \widetilde{x}(t)\Big) & \le\ \frac{1}{k} \\ \sup\limits_{l \ge k} & \widetilde{q}_\varepsilon\Big(\widetilde{x}(s),\ \ \widetilde{x}_{n_l}(s + \delta_{m_k}(s))\Big) & \le\ \frac{1}{k} \end{cases}$$

for every $K \in I\!N$ and all $\varepsilon \in \mathcal{J}_K$, $s, t \in Q_K$, $k \ge K$.

In particular, $\widetilde{q}_\varepsilon(\widetilde{x}(s),\ \widetilde{x}(t)) \ \le\ c_\varepsilon(t - s)$ for any $s, t \in Q_{I\!N} := \bigcup_K Q_K$ with $s < t$ and every $\varepsilon \in \mathcal{J}$. Moreover, the sequence $(\widetilde{x}_{n_k}(\cdot))_{k \in I\!N}$ fulfills for all $\varepsilon \in \mathcal{J}$, $K \in I\!N$, $t \in Q_K$ and sufficiently large $k, l \in I\!N$ (depending merely on ε, K)

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(t),\ \widetilde{x}_{n_l}(t + \delta_{m_l}(t))\Big) \ \le\ \frac{1}{k} + \frac{1}{l}\,.$$

For extending $\widetilde{x}(\cdot)$ to $t \in]0, T[\setminus Q_{I\!N}$, we apply the timed transitional compactness to $((\widetilde{x}_{n_k}(t))_{k \in I\!N}$ and obtain a subsequences $n_{l_j} \nearrow \infty$ of indices (depending on t) and an element $\widetilde{x}(t) \in \widetilde{E}$ satisfying for every $\varepsilon \in \mathcal{J}$,

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_{l_j}}(t), \ \ \widetilde{x}(t)\Big) & \longrightarrow \ 0, \\ \sup\limits_{i \geq j} \ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \ \ \widetilde{x}_{n_{l_i}}(t + \delta_{m_j}(t))\Big) & \longrightarrow \ 0 \end{cases} \qquad \text{for } \ j \longrightarrow \infty.$$

This implies the following convergence even uniformly in t (but not necessarily in ε)

$$\wedge \begin{cases} \limsup\limits_{K \longrightarrow \infty} \ \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(t - 2\,h_K), \ \ \widetilde{x}(t)\Big) & = \ 0, \\ \limsup\limits_{K \longrightarrow \infty} \ \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \ \ \widetilde{x}_{n_k}(t + 2\,h_K)\Big) & = \ 0. \end{cases}$$

Indeed, for $K \in I\!N$ fixed arbitrarily, there are $s = s(t, K) \in Q_K$ and $K' = K'(\varepsilon, K) \in I\!N$ with

$$t - 2\,h_K \ < \ s \ \leq \ t - h_K, \qquad K' \geq K,$$

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(s), \ \widetilde{x}_{n_l}(s + \delta_{m_l}(s))\Big) \ \leq \ \tfrac{1}{k} + \tfrac{1}{l} \qquad \text{for all} \quad k, l \geq K'.$$

So for any $k, l_j \geq K'$, we conclude from $0 \ < \ \delta_{m_{l_j}}(\cdot) \ < \ \frac{1}{2}\,h_{m_{l_j}} \ < \ \frac{1}{2}\,h_{l_j} \ \leq \ \frac{1}{2}\,h_K$

$$\begin{aligned} \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(t - 2\,h_K), \ \widetilde{x}(t)\Big) \ \leq \ & \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(t - 2\,h_K), \ \ \widetilde{x}_{n_k}(s)\Big) \\ & + \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(s), \ \ \widetilde{x}_{n_{l_j}}(s + \delta_{m_{l_j}}(s))\Big) \\ & + \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_{l_j}}(s + \delta_{m_{l_j}}(s)), \ \ \widetilde{x}_{n_{l_j}}(t)\Big) \\ & + \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_{l_j}}(t), \ \ \widetilde{x}(t)\Big) \\ \leq \ & \ c_\varepsilon(h_K) \ + \ \tfrac{1}{k} + \tfrac{1}{l_j} \ + \ c_\varepsilon(2\,h_K) \ + \ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_{l_j}}(t), \ \widetilde{x}(t)\Big) \end{aligned}$$

and $j \longrightarrow \infty$ leads to the uniform estimate $\widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(t - 2\,h_K), \ \widetilde{x}(t)\Big) \ \leq \ 2\ c_\varepsilon(2\,h_K) + \frac{2}{K}$.

The proof of $\limsup\limits_{K \longrightarrow \infty} \ \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \ \widetilde{x}_{n_k}(t + 2\,h_K)\Big) \ = \ 0$ is analogous (with $s' = s'(t, K) \in Q_K$ satisfying $t + h_K \leq s' < t + 2\,h_K$)

$$\begin{aligned} \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \ \widetilde{x}_{n_k}(t + 2\,h_K)\Big) \ \leq \ & \ \widetilde{q}_\varepsilon\Big(\ \widetilde{x}(t), \ \ \widetilde{x}_{n_{l_j}}(t + \delta_{m_j}(t))\Big) \\ & + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_{l_j}}(t + \delta_{m_j}(t)), \ \ \widetilde{x}_{n_{l_j}}(s')\Big) \\ & + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_{l_j}}(s'), \ \ \widetilde{x}_{n_k}(s' + \delta_{m_k}(s'))\Big) \\ & + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(s' + \delta_{m_k}(s')), \ \ \widetilde{x}_{n_k}(t + 2\,h_K)\Big) \\ \leq \ & \ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \ \widetilde{x}_{n_{l_j}}(t + \delta_{m_j}(t))\Big) + \ c_\varepsilon(2\,h_K) \ + \ \tfrac{1}{l_j} + \tfrac{1}{k} \ + \ c_\varepsilon(h_K) \end{aligned}$$

and $j \longrightarrow \infty$ provides the uniform estimate $\widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \ \widetilde{x}_{n_k}(t + 2\,h_K)\Big) \ \leq \ 2\ c_\varepsilon(2\,h_K) + \frac{2}{K}$.

Now we summarize the construction of $\widetilde{x}(\cdot)$ in the following notation : For each $\varepsilon \in \mathcal{J}$ and $j \in I\!N$, there exist $K_j \in I\!N$ (depending on ε, j) and $N_j \in I\!N$ (depending on ε, j, K_j) such that $N_j > K_j > N_{j-1}$ and

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\Big(\widetilde{x}_{N_j}(s - 2\,h_{K_j}), \quad \widetilde{x}(s)\Big) & \leq \frac{1}{j} \\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \quad \widetilde{x}_{N_j}(t + 2\,h_{K_j})\Big) & \leq \frac{1}{j} \end{cases}$$

for every $s, t \in [0, T[$.

Convergence Theorem I (Prop. 2.3.2) states that $\widetilde{x}(\cdot)$ is a timed right–hand forward solution of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ in $[0, T[$.
Indeed, set $\widetilde{g}_j : (\widetilde{z}, t) \longmapsto \widetilde{f}\Big(\widetilde{x}_{N_j}(t_{N_j}^{a+2} + 2\,h_{K_j}),\ t_{N_j}^{a+2} + 2\,h_{K_j}\Big)$ for $t_{N_j}^a \leq t < t_{N_j}^{a+1}$ and regard the sequence $t \longmapsto \widetilde{x}_{N_j}(t + 2\,h_{N_j} + 2\,h_{K_j})$ of solutions.
Obviously conditions (1.), (3.), (4.) of Prop. 2.3.2 result from the assumptions here. Furthermore, we obtain for any $0 \leq t < t' < T$ (with $t_{N_j}^a \leq t < t_{N_j}^{a+1}$, $t_{N_j}^b \leq t' < t_{N_j}^{b+1}$) and $j \in I\!N$, $\varepsilon \in \mathcal{J}$

$$\begin{aligned} & \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{g}_j(\widetilde{z}, t),\ \ \widetilde{g}_j(\widetilde{z}', t')\Big) \\ = \;& \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}\,\Big(\widetilde{x}_{N_j}(t_{N_j}^{a+2} + 2\,h_{K_j}),\ \ t_{N_j}^{a+2} + 2\,h_{K_j}\Big),\ \widetilde{f}\,\Big(\widetilde{x}_{N_j}(t_{N_j}^{b+2} + 2\,h_{K_j}),\ \ t_{N_j}^{b+2} + 2\,h_{K_j}\Big)\Big) \\ \leq \;& R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon\Big(\widetilde{x}_{N_j}(t_{N_j}^{a+2} + 2\,h_{K_j}),\ \widetilde{x}_{N_j}(t_{N_j}^{b+2} + 2\,h_{K_j})\Big) + (b-a)\ h_{N_j}\Big) \\ \leq \;& R_\varepsilon + \widehat{\omega}_\varepsilon\Big(c_\varepsilon(t' - t + 2\,h_{N_j}) + t' - t + 2\,h_{N_j}\Big) \\ \longrightarrow \;& R_\varepsilon \qquad \text{for } j \longrightarrow \infty,\ \ t' - t \downarrow 0 \text{ and all } \widetilde{z}, \widetilde{z}', \end{aligned}$$

i.e. condition (2.) of Prop. 2.3.2 is also satisfied by $(\widetilde{g}_j)_{j \in I\!N}$.
Finally for proving assumption (5.) of the Convergence Theorem I, we benefit from the convergence properties of the subsequence $(\widetilde{x}_{N_j})_{j \in I\!N}$ mentioned before. It ensures that for every $t \in [0, T[$ (with $t_{N_j}^a \leq t < t_{N_j}^{a+1}$),

$$\begin{aligned} & \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{x}(t),\ t),\ \ \widetilde{g}_j(\widetilde{x}(t),\ t)\Big) \\ = \;& \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}\,(\widetilde{x}(t),\ t)\,,\quad \widetilde{f}\Big(\widetilde{x}_{N_j}(t_{N_j}^{a+2} + 2\,h_{K_j}),\ \ t_{N_j}^{a+2} + 2\,h_{K_j}\Big)\Big) \\ \leq \;& R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t),\ \ \widetilde{x}_{N_j}(t_{N_j}^{a+2} + 2\,h_{K_j}))\Big) + 2\,h_{K_j} + t_{N_j}^{a+2} - t\Big) \\ \leq \;& R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t),\ \ \widetilde{x}_{N_j}(t + 2\,h_{K_j}))\Big) + c_\varepsilon(2\,h_{N_j}) + 2\,h_{K_j} + 2\,h_{N_j}\Big) \\ \longrightarrow \;& R_\varepsilon \qquad \text{for } j \longrightarrow \infty. \quad \square \end{aligned}$$

Remark. 1. Assumption (2.) is only to guarantee the uniform continuity of the Euler approximations $\widetilde{x}_n(\cdot)$. If this property results from other arguments, then we can dispense with this assumption and even with condition (4.) of Def. 2.1.1.

2. The proof shows that the compactness assumption can be weakened slightly. We only need that all $\widetilde{x}_n(t)$ $(0 < t < T,\ n \in \mathbb{N})$ are contained in a subset $\widetilde{F} \subset \widetilde{E}$ that is transitionally compact in $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)))$.
This modification is very useful if each transition $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ has all values in $\widetilde{F}$ after any positive time, i.e. $\widetilde{\vartheta}(t, \widetilde{x}) \in \widetilde{F}$ for all $0 < t \le 1$, $\widetilde{x} \in \widetilde{E}$. In particular, it does not require any additional assumptions about the initial value $\widetilde{x}_0 \in \widetilde{E}$ (see e.g. Prop. 4.4.28 and Corollary 4.4.29 for nonempty compact subsets of $\mathbb{R}^N$).

Corollary 2.3.6 (Existence of timed right–hand forward solutions due to timed two–sided sequential compactness)

Suppose that $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is timed two–sided sequentially compact (uniformly with respect to ε), Moreover let $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ satisfy for all $\varepsilon \in \mathcal{J}$

1. $M_\varepsilon := \sup_{t,\widetilde{z}} \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{z}, t)) < \infty,$
2. $c_\varepsilon(h) := \sup_{t,\widetilde{z}} \beta_\varepsilon(\widetilde{f}(\widetilde{z}, t))(h) < \infty, \qquad c_\varepsilon(h) \longrightarrow 0$ *for* $h \downarrow 0,$
3. $\exists\ R_\varepsilon : \sup_{t,\widetilde{z}} \gamma_\varepsilon(\widetilde{f}(\widetilde{z}, t)) \le R_\varepsilon < \infty, \qquad \varepsilon'^p\, R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon' \downarrow 0,$
4. $\exists\ \widehat{\omega}_\varepsilon(\cdot) : \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{z}_1, t_1),\ \widetilde{f}(\widetilde{z}_2, t_2)\Big) \le R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2) + t_2 - t_1\Big)$
   for all $0 \le t_1 \le t_2 \le T$ *and* $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$ *with* $\pi_1\, \widetilde{z}_1 \le \pi_1\, \widetilde{z}_2,$
   $\widehat{\omega}_\varepsilon(\cdot) \ge 0$ *nondecreasing,* $\limsup_{s\downarrow 0} \widehat{\omega}_\varepsilon(s) = 0.$

Then for every $\widetilde{x}_0 \in \widetilde{E}$, there is a timed right–hand forward solution $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ in $[0,T[$ with $\widetilde{x}(0) = \widetilde{x}_0$.

Proof results from Prop. 2.3.5 and remark (2.) after Def. 2.3.4 immediately. □

Proposition 2.3.7 (Existence of timed right–hand forward solutions due to one–sided sequential compactness and $(L^{\Rightarrow}), (R^{\Rightarrow})$)
Assume standard hypotheses $(L^{\Rightarrow}), (R^{\Rightarrow})$ for each $(\widetilde{E}, \widetilde{q}_\varepsilon)$ (i.e. due to Def. 1.4.1, 1.4.3, left–hand and right–hand spheres are right–sequentially closed) and suppose $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ to be one–sided sequentially compact (uniformly with respect to ε) (see Def. 1.2.2).

Moreover let $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *satisfy for every* $\varepsilon \in \mathcal{J}$

$$
\begin{array}{llll}
\text{1.} & M_\varepsilon := \sup_{t,\widetilde{z}} \; \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{z},t)) < \infty, & & \\
\text{2.} & c_\varepsilon(h) := \sup_{t,\widetilde{z}} \; \beta_\varepsilon(\widetilde{f}(\widetilde{z},t))(h) < \infty, & c_\varepsilon(h) \longrightarrow 0 & \text{for } h \downarrow 0, \\
\text{3.} & \exists\, R_\varepsilon : \sup_{t,\widetilde{z}} \; \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \le R_\varepsilon < \infty, & \varepsilon'^p\, R_{\varepsilon'} \longrightarrow 0 & \text{for } \varepsilon' \downarrow 0,
\end{array}
$$

$$
\text{4'.} \quad \exists\, \widehat{\omega}_\varepsilon(\cdot) : \quad \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \le R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\min\{\, \widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2),\ \widetilde{q}_\varepsilon(\widetilde{z}_2,\widetilde{z}_1)\,\} + |t_2 - t_1|\Big)
$$

for all $t_1,\ t_2 \in [0,T]$ *and* $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$,
$\widehat{\omega}_\varepsilon(\cdot) \ge 0$ *nondecreasing,* $\widehat{\omega}_\varepsilon(s) \longrightarrow 0$ *for* $s \downarrow 0$.

Then for every $\widetilde{x}_0 \in \widetilde{E}$, *there is a timed right–hand forward solution* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$ *with* $\widetilde{x}(0) = \widetilde{x}_0$.

Proof is based on Convergence Theorem II (Prop. 2.3.3) : Correspondingly to the preceding Proposition 2.3.5, we consider the Euler approximation for each $n \in I\!N$ (with $2^n > T$) setting

$$
\begin{array}{llll}
h_n := \frac{T}{2^n}, & t_n^j := j\, h_n & \text{for } j = 0 \ldots 2^n, & \\
\widetilde{x}_n(0) := \widetilde{x}_0, & \widetilde{x}_0(\cdot) := \widetilde{x}_0, & & \\
\widetilde{x}_n(t) := \widetilde{f}(\widetilde{x}_n(t_n^j),\ t_n^j)\,\Big(t - t_n^j,\ \widetilde{x}_n(t_n^j)\Big) & & \text{for } t \in\,]t_n^j,\ t_n^{j+1}], & j \le 2^n.
\end{array}
$$

The uniform modulus of continuity $c_\varepsilon(\cdot)$ is again supposed to be a nondecreasing convex function $[0,T{+}1] \longrightarrow [0,\infty[$ and so all $\widetilde{x}_n(\cdot)$ are equi–continuous in the sense of

$$
\widetilde{q}_\varepsilon\Big(\widetilde{x}_n(s),\ \widetilde{x}_n(t)\Big) \le c_\varepsilon(t-s) \qquad \text{for any } 0 \le s < t < T + h_n \text{ and } \varepsilon \in \mathcal{J}.
$$

Due to the one–sided compactness of $(\widetilde{E}, \widetilde{q}_\varepsilon)$ uniformly with respect to ε, the Cantor diagonal construction (with respect to the subsequent index l) leads to a sequence $n_k \nearrow \infty$ such that for every $l \in I\!N$ and $t \in Q_l :=]0,T[\cap I\!N \cdot h_l$, there is $\widetilde{x}(t) \in \widetilde{E}$ satisfying

$$
\sup_{\substack{s \in Q_l \\ k \ge l}} \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_k}(s),\ \widetilde{x}(s)\Big) \longrightarrow 0 \qquad (l \longrightarrow \infty) \qquad \text{for all } \varepsilon \in \mathcal{J}.
$$

According to Prop. 1.4.5, standard hypothesis $(R^{\Rightarrow})$ ensures $\widetilde{q}_\varepsilon(\widetilde{x}(s), \widetilde{x}(t)) \le c_\varepsilon(t-s)$ for every $s,t \in \bigcup_l Q_l$ $(s < t)$. In contrast to Prop. 2.3.5, we cannot use any form of left–convergence here (because $\widetilde{q}_\varepsilon$ need not be an ostensible metric on $\widetilde{E}$ and thus, standard hypotheses $(L^{\Rightarrow})$, $(R^{\Rightarrow})$ do not imply the equivalence of right– and left–convergence).

For extending $\widetilde{x}(\cdot)$ to $[0,T]$, we approximate each $t \in [0,T] \setminus \bigcup_l Q_l$ by $s_l := [\frac{t}{h_l}] \cdot h_l$. As another consequence of assuming one–sided compactness, there exist a subsequence $(s_{l_j})_{j \in I\!N}$ and some $\widetilde{x}(t) \in \widetilde{E}$ such that for all $\varepsilon \in \mathcal{J}$, $\widetilde{q}_\varepsilon\Big(\widetilde{x}(s_{l_j}),\ \widetilde{x}(t)\Big) \longrightarrow 0$ $(j \longrightarrow \infty)$. So we obtain an estimate (that is uniform in t, but not necessarily in ε) for every $l \in I\!N$

$$\begin{aligned} \widetilde{q}_\varepsilon\Big(\widetilde{x}(s_l),\ \widetilde{x}(t)\Big) &\le \limsup_{j \longrightarrow \infty} \Big(\widetilde{q}_\varepsilon\Big(\widetilde{x}(s_l),\ \widetilde{x}(s_{l_j})\Big) + \widetilde{q}_\varepsilon\Big(\widetilde{x}(s_{l_j}),\ \widetilde{x}(t)\Big)\Big) \\ &\le \limsup_{j \longrightarrow \infty} \Big(c_\varepsilon(h_l) + \widetilde{q}_\varepsilon\Big(\widetilde{x}(s_{l_j}),\ \widetilde{x}(t)\Big)\Big) \\ &= c_\varepsilon(h_l) \end{aligned}$$

and thus, $\quad \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_l}(s_l),\ \widetilde{x}(t)\Big) \longrightarrow 0 \quad (l \longrightarrow \infty) \quad$ uniformly in t for each $\varepsilon \in \mathcal{J}$.

Convergence Theorem II (Prop. 2.3.3) implies that $\widetilde{x}(\cdot)$ is a timed right–hand forward solution of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ in $[0,T[$.
Indeed, set $\widetilde{g}_l : (\widetilde{z}, t) \longmapsto \widetilde{f}\Big(\widetilde{x}_{n_l}(t^j_{n_l} - h_l),\ t^j_{n_l} - h_l\Big)$ for $t^j_{n_l} \le t < t^{j+1}_{n_l}$ $(t - h_l \ge 0)$ and regard the sequence $t \longmapsto \widetilde{x}_{n_l}(t - h_l)$ of solutions (on $[h_l, T]$).
Obviously conditions (1.), (3.), (4.) of Prop. 2.3.3 result from the hypotheses here. Moreover, we obtain for any $t, t' \in [h_l, T[$ (with $t^j_{n_l} \le t < t^{j+1}_{n_l}$, $t^k_{n_l} \le t' < t^{k+1}_{n_l}$) and $l \in I\!N$, $\varepsilon \in \mathcal{J}$, $\widetilde{z}, \widetilde{z}' \in \widetilde{E}$

$$\begin{aligned} \widetilde{Q}^{\mapsto}_\varepsilon\Big(\widetilde{g}_l(\widetilde{z}, t),\ \widetilde{g}_l(\widetilde{z}', t')\Big) &= \widetilde{Q}^{\mapsto}_\varepsilon\Big(\widetilde{f}(\widetilde{x}_{n_l}(t^j_{n_l} - h_l),\ t^j_{n_l} - h_l),\quad \widetilde{f}(\widetilde{x}_{n_l}(t^k_{n_l} - h_l),\ t^k_{n_l} - h_l)\Big) \\ &\le R_\varepsilon + \widehat{\omega}_\varepsilon\Big(c_\varepsilon(|t' - t| + 2\,h_{n_l}) + |t' - t| + 2\,h_{n_l}\Big) \\ &\longrightarrow R_\varepsilon \qquad \text{for } n \longrightarrow \infty,\ |t' - t| \longrightarrow 0, \end{aligned}$$

i.e. condition (2'.) of Prop. 2.3.3 is also satisfied by $(\widetilde{g}_{n_l})_{n \in I\!N}$.
Finally condition (5'.) of Convergence Theorem II results from the construction of $\widetilde{x}(\cdot)$ and continuity assumption (4'.) for $\widetilde{f}$. In particular, cond. (5'.i) is a consequence of

$$\begin{aligned} &\widetilde{Q}^{\mapsto}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),\ t),\ \widetilde{g}_l(\widetilde{x}(t),\ t)\Big) \\ &= \widetilde{Q}^{\mapsto}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),\ t),\quad \widetilde{f}(\widetilde{x}_{n_l}([\tfrac{t}{h_{n_l}}]\, h_{n_l} - h_l),\quad [\tfrac{t}{h_{n_l}}]\, h_{n_l} - h_l)\Big) \\ &\le R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_l}([\tfrac{t}{h_{n_l}}]\, h_{n_l} - h_l),\ \widetilde{x}(t)\Big) + t - [\tfrac{t}{h_{n_l}}]\, h_{n_l} + h_l\Big) \\ &\le R_\varepsilon + \widehat{\omega}_\varepsilon\Big(c_\varepsilon(h_l) + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_l}([\tfrac{t}{h_l}]\, h_l),\ \widetilde{x}(t)\Big) + h_{n_l} + h_l\Big) \\ &\le R_\varepsilon + \widehat{\omega}_\varepsilon\Big(c_\varepsilon(h_l) + \sup_s \widetilde{q}_\varepsilon\Big(\widetilde{x}_{n_l}([\tfrac{s}{h_l}]\, h_l),\ \widetilde{x}(s)\Big) + h_{n_l} + h_l\Big) \\ &\longrightarrow R_\varepsilon \end{aligned}$$

for $l \longrightarrow \infty$ and any $t \in]h_l, T[$. □

2.3.3 Estimates

Finally we extend the estimates of § 2.2 to timed right–hand forward solutions. To be more precise, Propositions 2.2.3, 2.2.4 and 2.2.5 find their counterparts here and the proofs are based on the same notions.
So the same obstacles as before keep us from estimates that are easy to apply : Due to the definitions, only elements of $\widetilde{D}$ usually appear in the first argument of $\widetilde{q}_\varepsilon$. Furthermore a solution $\widetilde{x}(\cdot)$ of $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ is required to fulfill the condition

$$\limsup_{h \downarrow 0} \; \tfrac{1}{h} \left(\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t), t)\,(h, \widetilde{y}),\; \widetilde{x}(t{+}h)\Big) \;-\; \widetilde{q}_\varepsilon(\widetilde{y},\, \widetilde{x}(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \right) \;\leq\; \widehat{\gamma}_\varepsilon(t)$$

with $\widetilde{x}(t+h)$ and $\widetilde{x}(t)$ merely in the second arguments of $\widetilde{q}_\varepsilon$. So we cannot expect an explicit estimate of $\widetilde{q}_\varepsilon(\widetilde{x}(t),\, \widetilde{y}(t))$ for timed right–hand forward solutions $\widetilde{x}(\cdot), \widetilde{y}(\cdot)$ in general.

Proposition 2.3.8 *Assume for the function* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *and curves* $\widetilde{x}, \widetilde{y} \in UC^{\to}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$

1. a) $\overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$,

b) $\widetilde{x}(t) \in \widetilde{D}$ *for all* $t \in [0,T[$,
$$\limsup_{h \downarrow 0} \tfrac{1}{h}\; \widetilde{q}_\varepsilon\Big(\widetilde{x}(t+h),\; \widetilde{f}(\widetilde{x}(t), t)\,(h, \widetilde{x}(t))\Big) \;\leq\; \gamma_\varepsilon(\widetilde{f}(\widetilde{x}(t), t)),$$

c) $\widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \widetilde{y}(t)\Big) \;\leq\; \limsup_{h \downarrow 0}\; \widetilde{q}_\varepsilon\Big(\widetilde{x}(t{-}h),\; \widetilde{y}(t{-}h)\Big), \quad \pi_1\, \widetilde{x}(0) \;=\; \pi_1\, \widetilde{y}(0) \;=\; 0,$

2. $M_\varepsilon \;:=\; \sup_{t, \widetilde{z}}\; \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{z}, t)) \;<\; \infty,$

3. $\exists\; R_\varepsilon < \infty : \; \sup_{t,\widetilde{z}}\; \gamma_\varepsilon(\widetilde{f}(\widetilde{z}, t)) \;\leq\; R_\varepsilon, \quad \varepsilon'^p\, R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon' \downarrow 0,$

4". $\exists\; \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon : \; \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{z}_1, t_1),\; \widetilde{f}(\widetilde{z}_2, t_2)\Big) \;\leq\; R_\varepsilon \;+\; L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2) \;+\; \widehat{\omega}_\varepsilon(t_2 - t_1)$
for all $0 \leq t_1 \leq t_2 \leq T$ *and* $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$ *with* $\pi_1\, \widetilde{z}_1 \leq \pi_1\, \widetilde{z}_2$,
$\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup_{s \downarrow 0}\; \widehat{\omega}_\varepsilon(s) \;=\; 0.$

Then, $\widetilde{q}_\varepsilon\Big(\widetilde{x}(t),\, \widetilde{y}(t)\Big) \;\leq\; \widetilde{q}_\varepsilon\Big(\widetilde{x}(0),\, \widetilde{y}(0)\Big) \cdot e^{(L_\varepsilon + M_\varepsilon)\cdot t} \;+\; 5\, R_\varepsilon\; \frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t} - 1}{L_\varepsilon + M_\varepsilon}$ *for all* t.

Proof is a consequence of Gronwall's Lemma 1.5.1 :
The auxiliary function $\varphi_\varepsilon(t) \;:=\; \widetilde{q}_\varepsilon\Big(\widetilde{x}(t), \widetilde{y}(t)\Big)$ satisfies the semicontinuity property $\varphi_\varepsilon(t) \;\leq\; \limsup_{h \downarrow 0}\; \varphi_\varepsilon(t{-}h)$ according to assumption (1.c).

Moreover, $\limsup_{h \downarrow 0}\; \frac{\varphi(t+h) \,-\, \varphi(t)}{h} \;\leq\; (L_\varepsilon {+} M_\varepsilon)\; \varphi_\varepsilon(t) \;+\; 5\, R_\varepsilon$ for all $t \in [0, T[$.

Indeed, the timed triangle inequality and Prop. 2.2.3 guarantee for all small $h > 0$

$$\begin{aligned}
&\varphi(t+h) \\
\leq\ & \widetilde{q}_\varepsilon\Big(\widetilde{x}(t{+}h),\qquad \widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{x}(t))\Big) \\
+\ & \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{x}(t)),\ \ \widetilde{y}(t{+}h)\Big) \\
\leq\ & \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{x}(t)),\ \ \widetilde{y}(t{+}h)\Big) \qquad + R_\varepsilon\, h + o(h) \\
\leq\ & \varphi_\varepsilon(t)\ e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\Big(5\,R_\varepsilon + L_\varepsilon\cdot\widetilde{q}_\varepsilon\Big(\widetilde{x}(t),\ \widetilde{y}(t{+}s)\Big) + \widehat{\omega}_\varepsilon(s)\Big)\,ds + o(h) \\
\leq\ & \varphi_\varepsilon(t)\ e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\Big(5\,R_\varepsilon + L_\varepsilon\cdot\Big(\varphi_\varepsilon(t) + \omega_\varepsilon(\widetilde{y},s)\Big) + \widehat{\omega}_\varepsilon(s)\Big)\,ds + o(h) \\
\leq\ & \varphi_\varepsilon(t)\ e^{M_\varepsilon\, h} + \tfrac{e^{M_\varepsilon\, h}-1}{M_\varepsilon}\Big(5\,R_\varepsilon + L_\varepsilon\cdot\ \varphi_\varepsilon(t)\Big) \qquad + o(h)
\end{aligned}$$

□

Proposition 2.3.9

Assume for $\widetilde{f}: \widetilde{E}\times[0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$ *and* $\widetilde{x},\widetilde{y}: [0,T[\ \longrightarrow \widetilde{E}$

1. $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot),\quad \overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot),\cdot)$ *in* $[0,T[,\quad \pi_1\,\widetilde{x}(0) = \pi_1\,\widetilde{y}(0) = 0,$
2. $M_\varepsilon := \sup_{t,\widetilde{z}} \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{z},t)) < \infty,$
3. $\exists\ R_\varepsilon < \infty:\ \sup_{t,\widetilde{z}} \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \leq R_\varepsilon,\qquad \varepsilon'^p\, R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon' \downarrow 0,$

4". $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon:\ \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \leq R_\varepsilon + L_\varepsilon\cdot\widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2) + \widehat{\omega}_\varepsilon(t_2-t_1)$
for all $0\leq t_1\leq t_2\leq T$ *and* $\widetilde{z}_1,\widetilde{z}_2\in\widetilde{E}$ *with* $\pi_1\,\widetilde{z}_1 \leq \pi_1\,\widetilde{z}_2,$
$\widehat{\omega}_\varepsilon(\cdot)\geq 0$ *nondecreasing,* $\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) = 0.$

Furthermore suppose the existence of $\lambda_\varepsilon > 0$ *such that for each* $t\in[0,T[,$ *the infimum*

$$\varphi_\varepsilon(t) := \inf_{\widetilde{z}\,\in\,\widetilde{D},\ \pi_1\,\widetilde{z}\,\leq t}\Big(\widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t)) + \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{y}(t))\Big) < \infty$$

can be approximated by a minimizing sequence $(\widetilde{z}_j)_{j\,\in\,I\!N}$ *in* $\widetilde{D}$ *and* $h_j\downarrow 0$ *with*

$$\pi_1\widetilde{z}_j \leq \pi_1\,\widetilde{z}_k \leq t,\quad \widetilde{q}_\varepsilon(\widetilde{z}_j,\widetilde{z}_k) \leq \lambda_\varepsilon\cdot h_j,\quad h_j < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j,t),\ \widetilde{z}_j)\quad \textit{for all}\ j<k.$$

Then, $\qquad \varphi_\varepsilon(t) \leq \varphi_\varepsilon(0)\ e^{(L_\varepsilon+M_\varepsilon)\cdot t} + 2\,\Big((L_\varepsilon{+}1)\,\lambda_\varepsilon + 4\,R_\varepsilon\Big)\cdot\frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t}-1}{L_\varepsilon+M_\varepsilon}.$

Remark. If the above–mentioned minimizing sequence $(\widetilde{z}_j)$ in $\widetilde{D}$ satisfies

$$\frac{\sup_{k>j}\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\widetilde{z}_k)}{\mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j,t),\ \widetilde{z}_j)} \longrightarrow 0 \qquad (j\longrightarrow\infty)$$

then, $\qquad \varphi_\varepsilon(t) \leq \varphi_\varepsilon(0)\ e^{(L_\varepsilon+M_\varepsilon)\cdot t} + 8\,R_\varepsilon\cdot\frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t}-1}{L_\varepsilon+M_\varepsilon}.$

So in the case of symmetric $\widetilde{q}_\varepsilon$ and $\widetilde{D}$ dense in $(\widetilde{E},\widetilde{q}_\varepsilon)$, we obtain that $R_\varepsilon = 0,\ \varphi_\varepsilon(0) = 0$ imply $\varphi_\varepsilon(\cdot) = 0$.

Proof follows the same track as for Prop. 2.2.4 and is based on the second subdifferential version of Gronwall's Lemma 1.5.2 :

$\varphi_\varepsilon(\cdot)$ satisfies $\varphi_\varepsilon(t) \le \liminf_{h\downarrow 0} \varphi_\varepsilon(t-h)$ for every $t \in]0,T[$ due to the timed triangle inequality and $\widetilde{x}, \widetilde{y} \in UC^{\rightarrow}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$. For showing

$$\liminf_{h\downarrow 0} \tfrac{\varphi_\varepsilon(t+h) - \varphi_\varepsilon(t)}{h} \le (L_\varepsilon + M_\varepsilon)\ \varphi_\varepsilon(t) + 2\,(L_\varepsilon+1)\,\lambda_\varepsilon + 8\,R_\varepsilon,$$

let $(\widetilde{z}_j)_{j\in I\!N}$ denote a minimizing sequence in $\widetilde{D}$ and $h_j \downarrow 0$ such that

$$\wedge \begin{cases} \pi_1 \widetilde{z}_j \le \pi_1 \widetilde{z}_k \le t, & \text{for all } j<k,\\ \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k) \quad \le \lambda_\varepsilon\, h_j, \quad h_j < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j,t),\ \widetilde{z}_j) & \\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\ \widetilde{x}(t)) + \widetilde{q}_\varepsilon(\widetilde{z}_j,\ \widetilde{y}(t)) \longrightarrow \varphi_\varepsilon(t) & (j \longrightarrow \infty) \end{cases}$$

Due to Prop. 2.2.3, we obtain for every $0 < h \le h_j < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j,t),\ \widetilde{z}_j),\ \ j<k$

$$\begin{aligned} &\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j,t)\,(h,\widetilde{z}_j),\ \ \widetilde{x}(t+h)\Big)\\ \le\ & \widetilde{q}_\varepsilon\Big(\widetilde{z}_j,\ \widetilde{x}(t)\Big)\cdot e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(R_\varepsilon + L_\varepsilon\cdot \widetilde{q}_\varepsilon\Big(\widetilde{z}_j,\ \widetilde{x}(t+s)\Big) + \widehat{\omega}_\varepsilon(s) + 3\,R_\varepsilon\Big)\ ds\\ \le\ & \widetilde{q}_\varepsilon\Big(\widetilde{z}_j,\ \widetilde{x}(t)\Big)\cdot e^{M_\varepsilon\, h} + \tfrac{e^{M_\varepsilon\, h}-1}{M_\varepsilon}\ \ \Big(L_\varepsilon\cdot \widetilde{q}_\varepsilon(\widetilde{z}_j,\ \widetilde{x}(t)) + 4\,R_\varepsilon\Big)\\ &\qquad + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(L_\varepsilon\cdot\omega_\varepsilon(\widetilde{x},s) \qquad + \widehat{\omega}_\varepsilon(s)\Big)\ ds\\ \le\ & \widetilde{q}_\varepsilon\Big(\widetilde{z}_k,\ \widetilde{x}(t)\Big)\cdot e^{M_\varepsilon\, h} + \tfrac{e^{M_\varepsilon\, h}-1}{M_\varepsilon}\ \ \Big(L_\varepsilon\cdot \widetilde{q}_\varepsilon(\widetilde{z}_k,\ \widetilde{x}(t)) + L_\varepsilon\,\lambda_\varepsilon\,h_j + 4\,R_\varepsilon\Big)\\ &+ \lambda_\varepsilon\,h_j \quad\cdot e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(L_\varepsilon\cdot\omega_\varepsilon(\widetilde{x},s) \qquad + \widehat{\omega}_\varepsilon(s)\Big)\ ds. \end{aligned}$$

The corresponding estimate for $\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j,t)\,(h,\widetilde{z}_j),\ \ \widetilde{y}(t+h)\Big)$ and $k \longrightarrow \infty$ provide for each $0 < h \le h_j$

$$\begin{aligned} \varphi_\varepsilon(t+h) \ \le\ & \quad \varphi_\varepsilon(t)\ \cdot\ e^{M_\varepsilon\, h} + 2\,\lambda_\varepsilon\,h_j\ e^{M_\varepsilon\, h}\\ &+ \quad \tfrac{e^{M_\varepsilon\, h}-1}{M_\varepsilon}\ \ \Big(L_\varepsilon\cdot\varphi_\varepsilon(t) + 2\,L_\varepsilon\,\lambda_\varepsilon\,h_j + 8\,R_\varepsilon\Big)\\ &+ \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \ \Big(L_\varepsilon\cdot\Big(\omega_\varepsilon(\widetilde{x},s) + \omega_\varepsilon(\widetilde{y},s)\Big) + 2\,\widehat{\omega}_\varepsilon(s)\Big)\ ds, \end{aligned}$$

So finally $h := h_j$ and $j \longrightarrow \infty$ lead to

$$\liminf_{h\downarrow 0} \tfrac{\varphi_\varepsilon(t+h) - \varphi_\varepsilon(t)}{h} \le (L_\varepsilon + M_\varepsilon)\ \varphi_\varepsilon(t) + 2\,(L_\varepsilon+1)\,\lambda_\varepsilon + 8\,R_\varepsilon. \qquad \Box$$

In the following counterpart of Prop. 2.2.5, it is a relevant point that the assumptions about $\widetilde{p}_\varepsilon$ do not consist in the comparison of two transitions of $\widetilde{f}$, i.e. regularity condition (9.) on $\widetilde{f}(\widetilde{v}_1,t_1),\ \widetilde{f}(\widetilde{v}_2,t_2)$ is used only with $\widetilde{Q}_\varepsilon^{\mapsto}$ (induced by $\widetilde{q}_\varepsilon$).

Proposition 2.3.10 *Suppose for* $\widetilde{p}_\varepsilon, \widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J})$, $p, \lambda_\varepsilon \geq 0$ *and* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, $\widetilde{x}, \widetilde{y} : [0,T[\longrightarrow \widetilde{E}$ *the following properties :*

1. $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)))$ *is timed transitionally compact,*
2. *each* $\widetilde{p}_\varepsilon$ *is symmetric and satisfies the triangle inequality,*
3. $\widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) := \inf\limits_{\substack{\widetilde{z}\,\in\,\widetilde{D},\\ \pi_1\,\widetilde{z}\,\leq\,\pi_1\,\widetilde{v}_2}} \Big(\widetilde{p}_\varepsilon(\widetilde{v}_1,\ \widetilde{z}) + \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{v}_2)\Big) < \infty$ *for any* $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$,
4. $\widetilde{x}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$ *constructed by Euler approximations according to the proof of Prop. 2.3.5,*
5. $\widetilde{y}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$ *with* $\pi_1\, \widetilde{x}(0) = \pi_1\, \widetilde{y}(0) = 0$,
6. $\exists\ M_\varepsilon < \infty :$
$$\begin{aligned} &\widehat{\alpha}_\varepsilon^{\mapsto}(\,\cdot\,, \widetilde{x},\ \widetilde{f}(\widetilde{x},\cdot)),\ \widehat{\alpha}_\varepsilon^{\mapsto}(\,\cdot\,, \widetilde{y},\ \widetilde{f}(\widetilde{y},\cdot)) &&\leq M_\varepsilon,\\ &\widetilde{p}_\varepsilon\Big(\widetilde{\psi}(h, \widetilde{z}_1),\ \ \widetilde{\psi}(h, \widetilde{z}_2)\Big) &&\leq \widetilde{p}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2)\cdot e^{M_\varepsilon\, h} \end{aligned}$$
for all $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E},\ \ h \in\,]0,T[,\ \ \widetilde{\psi} \in \Big\{\widetilde{f}(\widetilde{z}, s)\ \Big|\ \widetilde{z} \in \widetilde{E},\ 0 \leq s < T\Big\}$,
7. $\exists\ R_\varepsilon < \infty :$
$$\begin{aligned} &\widehat{\gamma}_\varepsilon(\,\cdot\,, \widetilde{x},\ \widetilde{f}(\widetilde{x},\cdot)),\ \widehat{\gamma}_\varepsilon(\,\cdot\,, \widetilde{y},\ \widetilde{f}(\widetilde{y},\cdot)) &&\leq R_\varepsilon,\\ &\limsup_{h\downarrow 0}\ \frac{\widetilde{p}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{\psi}(t,\widetilde{z})),\ \ \widetilde{\psi}(t{+}h, \widetilde{z})\big)}{h} &&\leq R_\varepsilon \end{aligned}$$
for all $\widetilde{z} \in \widetilde{E},\ \ t \in [0,T[,\ \ \widetilde{\psi} \in \Big\{\widetilde{f}(\widetilde{z}, s)\ \Big|\ \widetilde{z} \in \widetilde{E},\ 0 \leq s < T\Big\}$,
8. $\exists\ c_\varepsilon(\cdot) :$ $\quad \widetilde{p}_\varepsilon(\widetilde{\psi}(t,\widetilde{z}),\ \widetilde{\psi}(t{+}h,\ \widetilde{z})) + \beta_\varepsilon(\widetilde{\psi})\,(h) \leq c_\varepsilon(h)$
for all $\widetilde{z} \in \widetilde{E},\ \ t \in [0,T[,\ \ \widetilde{\psi} \in \Big\{\widetilde{f}(\widetilde{z}, s)\ \Big|\ \widetilde{z} \in \widetilde{E},\ 0 \leq s < T\Big\}$,
$c_\varepsilon(h) \longrightarrow 0$ *for* $h \downarrow 0$,
9. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :$ $\quad \widetilde{Q}_\varepsilon^{\mapsto}\Big(\widetilde{f}(\widetilde{v}_1, t_1),\ \widetilde{f}(\widetilde{v}_2, t_2)\Big) \leq R_\varepsilon + L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) + \widehat{\omega}_\varepsilon(t_2 - t_1)$
for all $0 \leq t_1 \leq t_2 \leq T$ *and* $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$ *with* $\pi_1\, \widetilde{v}_1 \leq \pi_1\, \widetilde{v}_2$,
$\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\quad \limsup\limits_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s) = 0$,
10. *for each* $\widetilde{v} \in \widetilde{E}$, $\ \delta > 0$, $\ 0 \leq s \leq t < T$, $\ 0 < h < 1$ *with* $t + h + \delta < T$, *the infimum* $\widetilde{\Delta}_\varepsilon\Big(\widetilde{f}(\widetilde{v}, s)\,(h, \widetilde{v}),\ \ \widetilde{y}(t{+}h{+}\delta)\Big)$ *can be approximated by a minimizing sequence* $(\widetilde{z}_n)_{n\,\in\,\mathbb{N}}$ *in* $\widetilde{D}$ *and* $h_n \downarrow 0$ *such that for all* $m < n$,
$$\begin{aligned} \pi_1 \widetilde{z}_m \ &\leq\ \pi_1\, \widetilde{z}_n \ \leq\ \pi_1\, \widetilde{y}(t{+}h{+}\delta), \qquad && \widetilde{p}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \ \leq\ \lambda_\varepsilon \cdot h_m,\\ h_m \ &<\ \mathcal{T}_\Theta(\widetilde{f}(\widetilde{v}, s),\ \widetilde{z}_m), && \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \ \leq\ \lambda_\varepsilon \cdot h_m. \end{aligned}$$

Then, $\quad \varphi_\varepsilon(t) := \limsup\limits_{\delta\,\downarrow\,0}\ \widetilde{\Delta}_\varepsilon\Big(\widetilde{x}(t),\ \widetilde{y}(t + \delta)\Big)$ *fulfills*
$$\varphi_\varepsilon(t) \ \leq\ \Big(\varphi_\varepsilon(0) \ + \ (5\, R_\varepsilon + 2\,\lambda_\varepsilon)\ t\Big)\ \ (1 + L_\varepsilon\, t)\ \ e^{2\, M_\varepsilon\, t}.$$

Proof. Let $(\widetilde{x}_n(\cdot))_{n \in I\!N}$ denote the sequence of Euler approximations according to the proof of Prop. 2.3.5, i.e. for each $n \in I\!N$ (with $2^n > T$) set

$$\begin{array}{lllll} b_n & := \frac{T}{2^n}, & t_n^j & := j\, b_n & \text{for } j = 0 \ldots 2^n, \\ \widetilde{x}_n(0) & := \widetilde{x}_0, & \widetilde{x}_0(\cdot) & := \widetilde{x}_0, & \\ \widetilde{x}_n(t) & := \widetilde{f}(\widetilde{x}_n(t_n^j),\, t_n^j) & \Big(t - t_n^j,\ \widetilde{x}_n(t_n^j)\Big) & & \text{for } t \in\,]t_n^j,\, t_n^{j+1}],\ \ j \le 2^n. \end{array}$$

Then the Cantor diagonal construction provided a subsequence $(\widetilde{x}_{n_k}(\cdot))_{k \in I\!N}$ with the additional property $\widetilde{q}_\varepsilon\Big(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t + 2\, b_k)\Big) \longrightarrow 0 \quad (k \longrightarrow \infty)$ for every $t \in [0, T[$,

Proposition 2.2.5 and condition (9.) imply for any $\delta > 0$ and $k \in I\!N$ (with $2\, b_k < \delta$)

$$\begin{aligned} &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(t + 2\, b_k),\ \widetilde{y}(t+\delta)\Big) \\ \le\ &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\, b_k),\ \widetilde{y}(\delta)\Big) \cdot e^{M_\varepsilon\, t} \\ &+ \int_0^t e^{M_\varepsilon \cdot (t-s)} \ \Big(R_\varepsilon + L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}([\tfrac{s+2\, b_k}{b_{n_k}}]\, b_{n_k}),\ \widetilde{y}(s+\delta)\Big) + \widehat{\omega}_\varepsilon(\delta) \\ &\qquad + 4\, R_\varepsilon + 2\, \lambda_\varepsilon\Big)\ ds. \end{aligned}$$

The triangle inequality of $\widetilde{p}_\varepsilon$ ensures $\widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_3) \ \le\ \widetilde{p}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) + \widetilde{\Delta}_\varepsilon(\widetilde{v}_2, \widetilde{v}_3)$ for any $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3 \in \widetilde{E}$ and so $\widetilde{\Delta}_\varepsilon$ fulfills the triangle inequality. As a consequence,

$$\begin{aligned} &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(t + 2\, b_k),\ \widetilde{y}(t+\delta)\Big) \\ \le\ &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\, b_k),\ \widetilde{y}(\delta)\Big) \cdot e^{M_\varepsilon\, t} \\ &+ \int_0^t e^{M_\varepsilon \cdot (t-s)} \ \Big(L_\varepsilon\ c_\varepsilon(b_{n_k}) + L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(s+2\, b_k),\ \widetilde{y}(s+\delta)\Big) \\ &\qquad + 5\, R_\varepsilon + 2\, \lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta)\Big)\ ds \\ \le\ &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\, b_k),\ \widetilde{y}(\delta)\Big) \cdot e^{M_\varepsilon\, t} + \Big(5\, R_\varepsilon + 2\, \lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta) + L_\varepsilon\ c_\varepsilon(b_{n_k})\Big)\ e^{M_\varepsilon\, t}\ t \\ &+ e^{M_\varepsilon\, t} \int_0^t e^{-M_\varepsilon\, s}\ \ L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(s+2\, b_k),\ \widetilde{y}(s+\delta)\Big)\ ds. \end{aligned}$$

Now the integral version of Gronwall's Lemma 1.5.4 provides an explicit upper bound

$$\begin{aligned} &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(t + 2\, b_k),\ \widetilde{y}(t+\delta)\Big) \cdot e^{-M_\varepsilon\, t} \\ \le\ &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\, b_k),\ \widetilde{y}(\delta)\Big) + \Big(5\, R_\varepsilon + 2\, \lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta) + L_\varepsilon\ c_\varepsilon(b_{n_k})\Big)\ t \\ &+ \int_0^t e^{-M_\varepsilon \cdot (t-s)}\ L_\varepsilon\ \Big(\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\, b_k),\ \widetilde{y}(\delta)\Big) + \Big(5\, R_\varepsilon + 2\, \lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta) + L_\varepsilon\ c_\varepsilon(b_{n_k})\Big)\ s\Big)\ ds \\ \le\ &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\, b_k),\ \widetilde{y}(\delta)\Big) \ \Big(1 + L_\varepsilon\ \tfrac{e^{M_\varepsilon\, t} - 1}{M_\varepsilon}\Big) \\ &+ \Big(5\, R_\varepsilon + 2\, \lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta) + L_\varepsilon\ c_\varepsilon(b_{n_k})\Big) \ \Big(t + L_\varepsilon\ \tfrac{e^{M_\varepsilon\, t} - 1 - M_\varepsilon\, t}{M_\varepsilon^2}\Big) \\ \le\ &\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\, b_k),\ \widetilde{y}(\delta)\Big) \ \Big(1 + L_\varepsilon\ t\ e^{M_\varepsilon\, t}\Big) \\ &+ \Big(5\, R_\varepsilon + 2\, \lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta) + L_\varepsilon\ c_\varepsilon(b_{n_k})\Big) \ \Big(t + L_\varepsilon\ \tfrac{t^2}{2}\ e^{M_\varepsilon\, t}\Big). \end{aligned}$$

So finally we obtain

$$
\begin{aligned}
&\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}(t),\ \widetilde{y}(t+\delta)\Big) \\
\le\ & \limsup_{k\longrightarrow\infty} \ \Big(\widetilde{p}_\varepsilon\Big(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t+2\,b_k)\Big) \ +\ \widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(t+2\,b_k),\ \ \widetilde{y}(t+\delta)\Big)\Big) \\
\le\ & \qquad\qquad 0 \qquad\qquad +\ \limsup_{k\longrightarrow\infty}\ \widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{y}(\delta)\Big)\quad (1+L_\varepsilon\ t)\quad e^{2\,M_\varepsilon\,t} \\
& \qquad\qquad\qquad\qquad +\Big(5\,R_\varepsilon + 2\,\lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta))\Big)\quad t \quad (1+L_\varepsilon\ t)\quad e^{2\,M_\varepsilon\,t} \\
\le\ & \qquad \Big(\ \widetilde{\Delta}_\varepsilon(\widetilde{x}(0),\ \widetilde{y}(\delta))\ +(5\,R_\varepsilon + 2\,\lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta)) \qquad t\Big)\ \cdot\ (1+L_\varepsilon\ t)\quad e^{2\,M_\varepsilon\,t}
\end{aligned}
$$

because $\widetilde{\Delta}_\varepsilon\Big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{y}(\delta)\Big) \ \le\ \widetilde{p}_\varepsilon\Big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{x}(0)\Big) \ +\ \widetilde{\Delta}_\varepsilon\Big(\widetilde{x}(0),\ \widetilde{y}(\delta)\Big).$ □

2.4 Systems of generalized mutational equations

Generalizing mutational equations in the presented way has the useful advantage that components of a system can come from different applications. In § 4.6 later, we give an example consisting of two components, namely a first–order geometric evolution and a C^0 semigroup on a reflexive Banach space.

To be more precise now, let $(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1})$ and $(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2})$ satisfy the general assumptions of this chapter. Furthermore $\widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1})$ abbreviates timed right–hand forward transitions of order p and, $\widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2})$ denotes timed right–hand forward transitions of order p'.

Convention in § 2.4. For the sake of simplicity, we always restrict ourselves to tuples $(\widetilde{x}_1, \widetilde{x}_2) \in \widetilde{E}_1 \times \widetilde{E}_2$ with $\pi_1\,\widetilde{x}_1 = \pi_1\,\widetilde{x}_2$, i.e. the components $\widetilde{x}_1 \in \widetilde{E}_1$, $\widetilde{x}_2 \in \widetilde{E}_2$ refer to the same point of time.

Strictly speaking, we consider elements $(t, x_1, x_2) \in I\!R \times E_1 \times E_2$ with sets $E_1, E_2 \neq \emptyset$ and prefer the notation $(\widetilde{x}_1,\ \widetilde{x}_2) \stackrel{\text{Def.}}{=} ((t, x_1),\ (t, x_2))$ in the style of preceding sections.

Definition 2.4.1 *For* $\widetilde{\vartheta}_1 \in \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1})$ *and* $\widetilde{\vartheta}_2 \in \widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2})$, *define*

$$\begin{aligned} \widetilde{\vartheta}_1 \times \widetilde{\vartheta}_2 :\ [0,1] \times \widetilde{E}_1 \times \widetilde{E}_2 &\longrightarrow \widetilde{E}_1 \times \widetilde{E}_2, \\ (h,\ \widetilde{x}_1,\ \widetilde{x}_2) &\longmapsto \Big(\widetilde{\vartheta}_1(h, \widetilde{x}_1),\ \widetilde{\vartheta}_2(h, \widetilde{x}_2)\Big). \end{aligned}$$

These maps $\widetilde{\vartheta}_1 \times \widetilde{\vartheta}_2$ induce timed forward transitions of order $\max\{p, p'\}$ on

$$\Big(\widetilde{E}_1 \times \widetilde{E}_2,\ \ \widetilde{D}_1 \times \widetilde{D}_2,\ \ (\widetilde{q}^{\,1}_\varepsilon + \widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2}\Big)$$

(according to the following Lemma 2.4.2).

So assuming transitional compactness of both components and suitable conditions on

$$(\widetilde{f}_1,\ \widetilde{f}_2)\ :\ [0,T] \times \widetilde{E}_1 \times \widetilde{E}_2 \ \longrightarrow\ \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1}) \times \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2})$$

the results of § 2.3.2 guarantee the existence of a timed right-hand forward solution $(\widetilde{x}_1,\ \widetilde{x}_2) :\ [0, T[\ \longrightarrow\ \widetilde{E}_1 \times \widetilde{E}_2$ of the generalized mutational equations

$$(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot))^\circ \ \ni\ \Big(\widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot),\ \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot)\Big)$$

in the sense of Proposition 2.4.4. In this context, only one asymptotic demand (for $h \downarrow 0$) has to be fulfilled by both components $\widetilde{x}_1(\cdot), \widetilde{x}_2(\cdot)$ simultaneously. For this reason, it is not obvious that $(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot))$ is a timed right–hand forward solution of the system

$$\wedge \begin{cases} \overset{\circ}{\widetilde{x}}_1(\cdot) \ \ni\ \widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot) \\ \overset{\circ}{\widetilde{x}}_2(\cdot) \ \ni\ \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot) \end{cases}$$

(i.e. separately with respect to each component).

Lemma 2.4.2 *The tuples of* $\widetilde{\Theta}^{\mapsto}_{p}(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}^{\,1}_{\varepsilon})_{\varepsilon\in\mathcal{J}_1}) \times \widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2})$ *induce timed right–hand forward transitions on* $\Big(\widetilde{E}_1\times\widetilde{E}_2,\ \ \widetilde{D}_1\times\widetilde{D}_2,\ \ (\widetilde{q}^{\,1}_{\varepsilon}+\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2}\Big)$ *with*

$$\begin{aligned}
\alpha^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2) &\overset{\text{Def.}}{=} \alpha^{\mapsto}_{\varepsilon}(\widetilde{\vartheta}_1) + \alpha^{\mapsto}_{\varepsilon'}(\widetilde{\vartheta}_2)\\
\beta_{\varepsilon,\varepsilon'}(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2) &\overset{\text{Def.}}{=} \beta_{\varepsilon}(\widetilde{\vartheta}_1) + \beta_{\varepsilon'}(\widetilde{\vartheta}_2)\\
\gamma_{\varepsilon,\varepsilon'}(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2) &\overset{\text{Def.}}{=} \gamma_{\varepsilon}(\widetilde{\vartheta}_1) + \gamma_{\varepsilon'}(\widetilde{\vartheta}_2)\\
\mathcal{T}_{\Theta}(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2,\ (\widetilde{x}_1,\widetilde{x}_2)) &\overset{\text{Def.}}{=} \min\{\mathcal{T}_{\Theta}(\widetilde{\vartheta}_1,\ \widetilde{x}_1),\ \mathcal{T}_{\Theta}(\widetilde{\vartheta}_2,\ \widetilde{x}_2)\} \quad \text{for } \widetilde{x}_1\in\widetilde{D}_1,\ \ \widetilde{x}_2\in\widetilde{D}_2,\\
\widetilde{Q}^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2,\ \ \widetilde{\tau}_1\times\widetilde{\tau}_2) &\le \widetilde{Q}^{1\,\mapsto}_{\varepsilon}(\widetilde{\vartheta}_1,\ \widetilde{\tau}_1) + \widetilde{Q}^{2\,\mapsto}_{\varepsilon'}(\widetilde{\vartheta}_2,\ \widetilde{\tau}_2)
\end{aligned}$$

for each $\varepsilon\in\mathcal{J}_1, \varepsilon'\in\mathcal{J}_2$. *In particular, these transitions* $\vartheta_1\times\vartheta_2$ *are of order* $\max\{p,p'\}$.

Proof. $(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2)\,(0,\cdot\,) = \mathrm{Id}_{\widetilde{E}_1\times\widetilde{E}_2}$ is obvious. So are the statements about $\beta_{\varepsilon,\varepsilon'}(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2)$, $\mathcal{T}_{\Theta}(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2,\ \cdot\,)$ and $\pi_1\,(\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2)\,(h,\widetilde{z}) = h+\pi_1\,\widetilde{z}$ for all $\widetilde{z}\in\widetilde{E}_1\times\widetilde{E}_2$. Thus, $\widetilde{\vartheta} := \widetilde{\vartheta}_1\times\widetilde{\vartheta}_2$ fulfills conditions (1.), (4.), (5.), (7.) on timed forward transitions stated in Definition 2.1.1. Condition (2.) with $\gamma_{\varepsilon,\varepsilon'}(\widetilde{\vartheta}) \overset{\text{Def.}}{=} \gamma_{\varepsilon}(\widetilde{\vartheta}_1)+\gamma_{\varepsilon'}(\widetilde{\vartheta}_2)$ results from

$$\begin{aligned}
&\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot(\widetilde{q}^{\,1}_{\varepsilon}+\widetilde{q}^{\,2}_{\varepsilon'})\Big(\widetilde{\vartheta}\Big(h,\ \widetilde{\vartheta}(t,\ (\widetilde{x}_1,\widetilde{x}_2))\Big),\quad \widetilde{\vartheta}\,(t+h,\ (\widetilde{x}_1,\widetilde{x}_2))\Big)\\
=\ &\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot(\widetilde{q}^{\,1}_{\varepsilon}+\widetilde{q}^{\,2}_{\varepsilon'})\Big(\widetilde{\vartheta}\Big(h,\ (\widetilde{\vartheta}_1(t,\widetilde{x}_1),\ \widetilde{\vartheta}_2(t,\widetilde{x}_2))\Big),\ \Big(\widetilde{\vartheta}_1(t+h,\widetilde{x}_1),\ \ \widetilde{\vartheta}_2(t+h,\widetilde{x}_2)\Big)\Big)\\
=\ &\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot\Big(\quad \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{\vartheta}_1(h,\ \widetilde{\vartheta}_1(t,\widetilde{x}_1)),\qquad \widetilde{\vartheta}_1(t+h,\widetilde{x}_1)\Big)\\
&\qquad\qquad +\ \widetilde{q}^{\,2}_{\varepsilon'}\Big(\widetilde{\vartheta}_2(h,\ \widetilde{\vartheta}_2(t,\widetilde{x}_2)),\qquad \widetilde{\vartheta}_2(t+h,\widetilde{x}_2)\Big)\Big)\\
\le\ &\qquad \gamma_{\varepsilon}(\widetilde{\vartheta}_1)\ +\ \gamma_{\varepsilon'}(\widetilde{\vartheta}_2)
\end{aligned}$$

for every $\widetilde{x}_1\in\widetilde{E}_1,\ \ \widetilde{x}_2\in\widetilde{E}_2,\ \ t\in[0,1[,\ \ \ \varepsilon\in\mathcal{J}_1,\ \ \varepsilon'\in\mathcal{J}_2$.

Furthermore we obtain for all elements $\widetilde{x}_1\in\widetilde{D}_1,\ \widetilde{y}_1\in\widetilde{E}_1$ and $\widetilde{x}_2\in\widetilde{D}_2,\ \widetilde{y}_2\in\widetilde{E}_2$ (with $\pi_1\,\widetilde{x}_1 = \pi_1\,\widetilde{x}_2 \le \pi_1\,\widetilde{y}_1 = \pi_1\,\widetilde{y}_2$)

$$\begin{aligned}
&\left(\frac{(\widetilde{q}^{\,1}_{\varepsilon}+\widetilde{q}^{\,2}_{\varepsilon'})\big(\widetilde{\vartheta}(h,(\widetilde{x}_1,\widetilde{x}_2)),\ \widetilde{\vartheta}(h,(\widetilde{y}_1,\widetilde{y}_2))\big)\ -\ (\widetilde{q}^{\,1}_{\varepsilon}+\widetilde{q}^{\,2}_{\varepsilon'})((\widetilde{x}_1,\widetilde{x}_2),\ (\widetilde{y}_1,\widetilde{y}_2))\ -\ \gamma_{\varepsilon,\varepsilon'}(\widetilde{\vartheta})\,h}{h\ \ \big((\widetilde{q}^{\,1}_{\varepsilon}+\widetilde{q}^{\,2}_{\varepsilon'})((\widetilde{x}_1,\widetilde{x}_2),\ (\widetilde{y}_1,\widetilde{y}_2))\ +\ \gamma_{\varepsilon,\varepsilon'}(\widetilde{\vartheta})\,h\big)}\right)^{+}\\
=\ &\frac{\Big(\widetilde{q}^{\,1}_{\varepsilon}\big(\widetilde{\vartheta}_1(h,\widetilde{x}_1),\ \widetilde{\vartheta}_1(h,\widetilde{y}_1)\big)\ +\ \widetilde{q}^{\,2}_{\varepsilon'}\big(\widetilde{\vartheta}_2(h,\widetilde{x}_2),\ \widetilde{\vartheta}_2(h,\widetilde{y}_2)\big)\ -\ \widetilde{q}^{\,1}_{\varepsilon}(\widetilde{x}_1,\widetilde{y}_1)\ -\ \widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{x}_2,\widetilde{y}_2)\ -\ \big(\gamma_{\varepsilon}(\widetilde{\vartheta}_1)+\gamma_{\varepsilon'}(\widetilde{\vartheta}_2)\big)\,h\Big)^{+}}{h\ \ \big(\widetilde{q}^{\,1}_{\varepsilon}(\widetilde{x}_1,\widetilde{y}_1)\ +\ \widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{x}_2,\widetilde{y}_2)\ \ +\ (\gamma_{\varepsilon}(\widetilde{\vartheta}_1)+\gamma_{\varepsilon'}(\widetilde{\vartheta}_2))\,h\big)}\\
\le\ &\left(\frac{\widetilde{q}^{\,1}_{\varepsilon}\big(\widetilde{\vartheta}_1(h,\widetilde{x}_1),\ \widetilde{\vartheta}_1(h,\widetilde{y}_1)\big)\ -\ \widetilde{q}^{\,1}_{\varepsilon}(\widetilde{x}_1,\widetilde{y}_1)\ -\ \gamma_{\varepsilon}(\widetilde{\vartheta}_1)\,h}{h\ \ \big(\widetilde{q}^{\,1}_{\varepsilon}(\widetilde{x}_1,\widetilde{y}_1)\ +\ \gamma_{\varepsilon}(\widetilde{\vartheta}_1)\,h\big)}\right)^{+}\ +\\
&\left(\frac{\widetilde{q}^{\,2}_{\varepsilon'}\big(\widetilde{\vartheta}_2(h,\widetilde{x}_2),\ \widetilde{\vartheta}_2(h,\widetilde{y}_2)\big)\ -\ \widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{x}_2,\widetilde{y}_2)\ -\ \gamma_{\varepsilon'}(\widetilde{\vartheta}_2)\,h}{h\ \ \big(\widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{x}_2,\widetilde{y}_2)\ +\ \gamma_{\varepsilon'}(\widetilde{\vartheta}_2)\,h\big)}\right)^{+}.
\end{aligned}$$

So, $\alpha^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{\vartheta}) \overset{\text{Def.}}{=} \alpha^{\mapsto}_{\varepsilon}(\widetilde{\vartheta}_1) + \alpha^{\mapsto}_{\varepsilon'}(\widetilde{\vartheta}_2)$ satisfies condition (3.) of Definition 2.1.1.

Similarly, the semicontinuity property

$$\limsup_{h\downarrow 0}\ (\widetilde{q}_\varepsilon^1+\widetilde{q}_{\varepsilon'}^2)\Big(\widetilde{\vartheta}(t-h,\ (\widetilde{x}_1,\widetilde{x}_2)),\ (\widetilde{y}_1,\widetilde{y}_2)\Big)\ \geq\ (\widetilde{q}_\varepsilon^1+\widetilde{q}_{\varepsilon'}^2)\Big(\widetilde{\vartheta}(t,\ (\widetilde{x}_1,\widetilde{x}_2)),\ (\widetilde{y}_1,\widetilde{y}_2)\Big)$$

of condition (6.) there (for all adequate $\widetilde{x}_1\in\widetilde{D}_1,\ \widetilde{y}_1\in\widetilde{E}_1,\ \ \widetilde{x}_2\in\widetilde{D}_2,\ \widetilde{y}_2\in\widetilde{E}_2$ and $t\in[0,\ \mathcal{T}_\Theta(\widetilde{\vartheta},\ (\widetilde{x}_1,\widetilde{x}_2))[\,)$ is easy to conclude from the corresponding feature of $\widetilde{\vartheta}_1,\widetilde{\vartheta}_2$.

Finally we have to prove the estimate $\widetilde{Q}^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{\vartheta},\ \widetilde{\tau})\ \leq\ \widetilde{Q}^{1\,\mapsto}_{\varepsilon}(\widetilde{\vartheta}_1,\ \widetilde{\tau}_1)\ +\ \widetilde{Q}^{2\,\mapsto}_{\varepsilon'}(\widetilde{\vartheta}_2,\ \widetilde{\tau}_2)$ for every $\widetilde{\vartheta}:=(\widetilde{\vartheta}_1,\widetilde{\vartheta}_2),\ \ \widetilde{\tau}:=(\widetilde{\tau}_1,\widetilde{\tau}_2)\ \in\ \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}_\varepsilon^1)_{\varepsilon\in\mathcal{J}_1})\times\widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}_{\varepsilon'}^2)_{\varepsilon'\in\mathcal{J}_2})$.
All $\widetilde{x}_1\in\widetilde{D}_1,\ \widetilde{y}_1\in\widetilde{E}_1$ and $\widetilde{x}_2\in\widetilde{D}_2,\ \widetilde{y}_2\in\widetilde{E}_2$ (with $\pi_1\,\widetilde{x}_1\ =\ \pi_1\,\widetilde{x}_2\ \leq\ \pi_1\,\widetilde{y}_1\ =\ \pi_1\,\widetilde{y}_2$) fulfill

$$\begin{aligned}
&(\widetilde{q}_\varepsilon^1+\widetilde{q}_{\varepsilon'}^2)\Big(\widetilde{\vartheta}(h,\ (\widetilde{x}_1,\widetilde{x}_2)),\ \ \widetilde{\tau}(h,\ (\widetilde{y}_1,\widetilde{y}_2))\Big)\ -\ (\widetilde{q}_\varepsilon^1+\widetilde{q}_{\varepsilon'}^2)\Big((\widetilde{x}_1,\widetilde{x}_2),\ (\widetilde{y}_1,\widetilde{y}_2)\Big)\ e^{\alpha^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{\vartheta})\cdot h}\\
=\ &\ \ \widetilde{q}_\varepsilon^1\Big(\widetilde{\vartheta}_1(h,\widetilde{x}_1),\ \ \widetilde{\tau}_1(h,\widetilde{y}_1)\Big)\ -\ \widetilde{q}_\varepsilon^1\,(\widetilde{x}_1,\ \widetilde{y}_1)\ e^{\alpha^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{\vartheta})\cdot h}\\
&+\ \widetilde{q}_{\varepsilon'}^2\Big(\widetilde{\vartheta}_2(h,\widetilde{x}_2),\ \ \widetilde{\tau}_2(h,\widetilde{y}_2)\Big)\ -\ \widetilde{q}_{\varepsilon'}^2\,(\widetilde{x}_2,\ \widetilde{y}_2)\ e^{\alpha^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{\vartheta})\cdot h}\\
\leq\ &\ \ \widetilde{q}_\varepsilon^1\Big(\widetilde{\vartheta}_1(h,\widetilde{x}_1),\ \ \widetilde{\tau}_1(h,\widetilde{y}_1)\Big)\ -\ \widetilde{q}_\varepsilon^1\,(\widetilde{x}_1,\ \widetilde{y}_1)\ e^{\alpha^{\mapsto}_{\varepsilon}(\widetilde{\vartheta}_1)\cdot h}\\
&+\ \widetilde{q}_{\varepsilon'}^2\Big(\widetilde{\vartheta}_2(h,\widetilde{x}_2),\ \ \widetilde{\tau}_2(h,\widetilde{y}_2)\Big)\ -\ \widetilde{q}_{\varepsilon'}^2\,(\widetilde{x}_2,\ \widetilde{y}_2)\ e^{\alpha^{\mapsto}_{\varepsilon'}(\widetilde{\vartheta}_2)\cdot h}
\end{aligned}$$

So the definition of $\widetilde{Q}^{1\,\mapsto}_{\varepsilon}(\widetilde{\vartheta}_1,\widetilde{\tau}_1),\ \ \widetilde{Q}^{2\,\mapsto}_{\varepsilon'}(\widetilde{\vartheta}_2,\widetilde{\tau}_2)$ guarantee the last claim immediately. □

Lemma 2.4.3
If both $\Big(\widetilde{E}_1,\ (\widetilde{q}_\varepsilon^1)_{\varepsilon\in\mathcal{J}_1},\ \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}_\varepsilon^1))\Big)$ *and* $\Big(\widetilde{E}_2,\ (\widetilde{q}_{\varepsilon'}^2)_{\varepsilon'\in\mathcal{J}_2},\ \widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}_{\varepsilon'}^2))\Big)$ *are timed transitionally compact, then so is*

$$\Big(\widetilde{E}_1\times\widetilde{E}_2,\ \ (\widetilde{q}_\varepsilon^1+\widetilde{q}_{\varepsilon'}^2)_{\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2},\ \ \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}_\varepsilon^1)_{\varepsilon\in\mathcal{J}_1})\times\widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}_{\varepsilon'}^2)_{\varepsilon'\in\mathcal{J}_2})\Big).$$

Proof results directly from Definition 2.3.4 (of timed transitional compactness) because in short, the convergence in $(\widetilde{E}_1\times\widetilde{E}_2,\ \ \widetilde{q}_\varepsilon^1+\widetilde{q}_{\varepsilon'}^2)$ is always equivalent to the convergence in both components $(\widetilde{E}_1,\ \widetilde{q}_\varepsilon^1),\ \ (\widetilde{E}_2,\ \widetilde{q}_{\varepsilon'}^2)$. □

So Proposition 2.3.5 guarantees the existence of a timed right–hand forward solution $\widetilde{x}(\cdot)\ =\ (\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot))\ :\ [0,T[\ \longrightarrow\ \widetilde{E}_1\times\widetilde{E}_2$ of the generalized mutational equation (with two components)

$$\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \Big(\widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot),\ \ \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot)\Big).$$

in the following sense :

Proposition 2.4.4 (Existence of timed right–hand forward solutions with two components)

Assume that $\Big(\widetilde{E}_1,\ (\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1},\ \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}^{\,1}_\varepsilon))\Big),\quad \Big(\widetilde{E}_2,\ (\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2},\ \widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}^{\,2}_{\varepsilon'}))\Big)$ *are timed transitionally compact.*

Furthermore let
$$\widetilde{f}_1 :\ \widetilde{E}_1\times\widetilde{E}_2\times[0,T] \longrightarrow \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\ \widetilde{D}_1,\ (\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1})$$
$$\widetilde{f}_2 :\ \widetilde{E}_1\times\widetilde{E}_2\times[0,T] \longrightarrow \widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2,\ \widetilde{D}_2,\ (\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon\in\mathcal{J}_1})$$
fulfill for every $\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2$

1. $M_{\varepsilon,\varepsilon'} := \sup\limits_{t,\widetilde{z}_1,\widetilde{z}_2} \alpha^{\mapsto}_{\varepsilon,\varepsilon'}(\widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t)\times\widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t)) < \infty,$
2. $c_{\varepsilon,\varepsilon'}(h) := \sup\limits_{t,\widetilde{z}_1,\widetilde{z}_2} \beta_{\varepsilon,\varepsilon'}(\widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t)\times\widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t))(h) \longrightarrow 0$ *for* $h\downarrow 0,$
3. $\exists\ R_{\varepsilon,\varepsilon'} : \sup\limits_{t,\widetilde{z}_1,\widetilde{z}_2} \gamma_{\varepsilon,\varepsilon'}(\widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t)\times\widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t)) \le R_{\varepsilon,\varepsilon'} < \infty,$
   with $(\varepsilon+\varepsilon')^{\max\{p,p'\}}\cdot R_{\varepsilon,\varepsilon'} \longrightarrow 0$ *for* $\varepsilon,\varepsilon'\downarrow 0,$
4. $\exists\ \widehat{\omega}_{\varepsilon,\varepsilon'}(\cdot) :\ \widetilde{Q}^{1\,\mapsto}_{\varepsilon}\Big(\widetilde{f}_1(\widetilde{y}_1,\widetilde{y}_2,t_1),\ \widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t_2)\Big) + \widetilde{Q}^{2\,\mapsto}_{\varepsilon'}\Big(\widetilde{f}_2(\widetilde{y}_1,\widetilde{y}_2,t_1),\ \widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t_2)\Big)$
   $\le\ R_{\varepsilon,\varepsilon'} + \widehat{\omega}_{\varepsilon,\varepsilon'}\Big(\widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\widetilde{z}_1) + \widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{y}_2,\widetilde{z}_2) + t_2 - t_1\Big)$
   for all $0\le t_1\le t_2\le T$ *and* $\widetilde{y}_1,\widetilde{z}_1\in\widetilde{E}_1,\ \ \widetilde{y}_2,\widetilde{z}_2\in\widetilde{E}_2$
   with $\pi_1\,\widetilde{y}_1 = \pi_1\,\widetilde{y}_2 \le \pi_1\,\widetilde{z}_1 = \pi_1\,\widetilde{z}_2,$
   $\widehat{\omega}_{\varepsilon,\varepsilon'}(\cdot)\ge 0$ *nondecreasing,* $\limsup\limits_{s\downarrow 0}\ \widehat{\omega}_{\varepsilon,\varepsilon'}(s) = 0.$

Then for every $\widetilde{x}^0 = (\widetilde{x}^0_1,\ \widetilde{x}^0_2)\in\widetilde{E}_1\times\widetilde{E}_2$ *(with* $\pi_1\,\widetilde{x}^0_1 = \pi_1\,\widetilde{x}^0_2$*), there exists a timed right–hand forward solution* $\widetilde{x}(\cdot) = (\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot)) :\ [0,T[\ \longrightarrow \widetilde{E}_1\times\widetilde{E}_2$ *of the generalized mutational equation (with two components)*
$$\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \Big(\widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot),\ \ \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot)\Big) \qquad \text{in } [0,T[$$
with $\widetilde{x}(0)=\widetilde{x}^0$ *in the following sense :*

Firstly, $(\widetilde{x}_1,\widetilde{x}_2)\in UC^{\to}([0,T[,\ \widetilde{E}_1\times\widetilde{E}_2,\ \widetilde{q}^{\,1}_\varepsilon+\widetilde{q}^{\,2}_{\varepsilon'})$ *for every* $\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2.$
Secondly, for each $\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2$ *and* $t\in[0,T[,$ *there exist parameters* $\widehat{\alpha}^{\mapsto},\ \widehat{\gamma}_{\varepsilon,\varepsilon'}\ge 0$ *such that*
$$\begin{aligned}\Delta_h\ :=\ & \widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{f}_1(\widetilde{x}_1(t),\ \widetilde{x}_2(t),\ t)\,(h,\widetilde{y}_1),\ \ \widetilde{x}_1(t+h)\Big) - \widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\ \widetilde{x}_1(t))\cdot e^{\widehat{\alpha}^{\mapsto}\cdot h}\\ &+\ \widetilde{q}^{\,2}_{\varepsilon'}\Big(\widetilde{f}_2(\widetilde{x}_1(t),\ \widetilde{x}_2(t),\ t)\,(h,\widetilde{y}_2),\ \ \widetilde{x}_2(t+h)\Big) - \widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{y}_2,\ \widetilde{x}_2(t))\cdot e^{\widehat{\alpha}^{\mapsto}\cdot h}\end{aligned}$$
satisfies $\limsup\limits_{h\downarrow 0}\ \frac{1}{h}\cdot\Delta_h\ \le\ \widehat{\gamma}_{\varepsilon,\varepsilon'},\qquad \limsup\limits_{\varepsilon,\varepsilon'\longrightarrow 0}\ (\varepsilon+\varepsilon')^{\max\{p,p'\}}\cdot\widehat{\gamma}_{\varepsilon,\varepsilon'} = 0$
for all $\widetilde{y}_1\in\widetilde{D}_1,\ \ \widetilde{y}_2\in\widetilde{D}_2$ *with* $\pi_1\,\widetilde{y}_1 = \pi_1\,\widetilde{y}_2 \le \pi_1\,\widetilde{x}_1(t) = \pi_1\,\widetilde{x}_2(t).$ □

So far it is not obvious that $\widetilde{x}_1(\cdot)$ and $\widetilde{x}_2(\cdot)$ are always timed right–hand forward solutions of the generalized mutational equations

$$\begin{array}{rcl} \overset{\circ}{\widetilde{x}}_1(\cdot) & \ni & \widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot\,) \\ \overset{\circ}{\widetilde{x}}_2(\cdot) & \ni & \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot\,) \end{array}$$

respectively. Proposition 2.3.5 states the existence of timed right–hand forward solutions due to timed transitional compactness and its proof is based on Euler approximations and the Cantor diagonal construction for selecting an adequate subsequence. This idea can be applied to

$$\Big(\widetilde{E}_1 \times \widetilde{E}_2,\quad (\widetilde{q}^{\,1}_\varepsilon + \widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2},\quad \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1}) \times \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2})\Big).$$

immediately. In addition, each component of the Euler approximations solves its own 'approximated' mutational equation in $\widetilde{E}_1$ and $\widetilde{E}_2$ respectively. So the key point now is to adapt Convergence Theorem 2.3.2 to the components of limit function $[0,T[\ \longrightarrow \widetilde{E}_1 \times \widetilde{E}_2$.

Proposition 2.4.5 (Convergence Theorem for Systems)
Suppose the following properties of

$$\begin{array}{llcl} \widetilde{f}^{(m)}_1,\ \ \widetilde{f}_1: & \widetilde{E}_1\times\widetilde{E}_2\times[0,T[& \longrightarrow & \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1}) \qquad (m\in I\!N) \\ \widetilde{f}^{(m)}_2,\ \ \widetilde{f}_2: & \widetilde{E}_1\times\widetilde{E}_2\times[0,T[& \longrightarrow & \widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2}) \qquad (m\in I\!N) \\ \widetilde{x}^{(m)}_1,\ \ \widetilde{x}_1: & [0,T[& \longrightarrow & \widetilde{E}_1 \\ \widetilde{x}^{(m)}_2,\ \ \widetilde{x}_2: & [0,T[& \longrightarrow & \widetilde{E}_2 \end{array}$$

for each $\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2$ *:*

1. a) $\overset{\circ}{\widetilde{x}}{}^{(m)}_1(\cdot) \ni \widetilde{f}^{(m)}_1(\widetilde{x}^{(m)}_1(\cdot),\ \widetilde{x}^{(m)}_2(\cdot),\ \cdot)$ *in* $[0,T[$,

b) $\overset{\circ}{\widetilde{x}}{}^{(m)}_2(\cdot) \ni \widetilde{f}^{(m)}_2(\widetilde{x}^{(m)}_1(\cdot),\ \widetilde{x}^{(m)}_2(\cdot),\ \cdot)$ *in* $[0,T[$,

2. a) $M_\varepsilon := \sup\limits_{m,t,\widetilde{z}_1,\widetilde{z}_2} \{\,\alpha^{\mapsto}_\varepsilon(\widetilde{f}^{(m)}_1(\widetilde{z}_1,\widetilde{z}_2,t))\,\} < \infty,$

$R_\varepsilon \geq \sup\limits_{m,t} \{\,\widehat{\gamma}_\varepsilon(t,\widetilde{x}^{(m)}_1,\ \widetilde{f}^{(m)}_1(\widetilde{x}^{(m)}_1,\widetilde{x}^{(m)}_2,\cdot)),\ \ \gamma_\varepsilon(\widetilde{f}^{(m)}_1(\cdot,\cdot,\cdot)),\ \ \gamma_\varepsilon(\widetilde{f}_1(\cdot,\cdot,\cdot))\,\}$

with $\limsup\limits_{\varepsilon\downarrow 0}\ \varepsilon^p \cdot R_\varepsilon = 0,$

b) $M_{\varepsilon'} := \sup\limits_{m,t,\widetilde{z}_1,\widetilde{z}_2} \{\,\alpha^{\mapsto}_{\varepsilon'}(\widetilde{f}^{(m)}_2(\widetilde{z}_1,\widetilde{z}_2,t))\,\} < \infty,$

$R_{\varepsilon'} \geq \sup\limits_{m,t} \{\,\widehat{\gamma}_{\varepsilon'}(t,\widetilde{x}^{(m)}_2,\ \widetilde{f}^{(m)}_2(\widetilde{x}^{(m)}_1,\widetilde{x}^{(m)}_2,\cdot)),\ \ \gamma_{\varepsilon'}(\widetilde{f}^{(m)}_2(\cdot,\cdot,\cdot)),\ \ \gamma_{\varepsilon'}(\widetilde{f}_2(\cdot,\cdot,\cdot))\,\}$

with $\limsup\limits_{\varepsilon'\downarrow 0}\ \varepsilon'^{(p')} \cdot R_{\varepsilon'} = 0,$

3. a) $\limsup\ \widetilde{Q}^{1\,\mapsto}_\varepsilon\Big(\widetilde{f}^{(m)}_1(\widetilde{y}_1,\widetilde{y}_2,t_1),\ \ \widetilde{f}^{(m)}_1(\widetilde{z}_1,\widetilde{z}_2,t_2)\Big) \leq R_\varepsilon$

b) $\limsup\ \widetilde{Q}^{2\,\mapsto}_{\varepsilon'}\Big(\widetilde{f}^{(m)}_2(\widetilde{y}_1,\widetilde{y}_2,t_1),\ \ \widetilde{f}^{(m)}_2(\widetilde{z}_1,\widetilde{z}_2,t_2)\Big) \leq R_{\varepsilon'}$ *for* $m\to\infty,\quad t_2-t_1\downarrow 0,$

and $\widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\widetilde{z}_1) + \widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{y}_2,\widetilde{z}_2) \longrightarrow 0$ *with* $\pi_1\,\widetilde{y}_1 = \pi_1\,\widetilde{y}_2 \leq \pi_1\,\widetilde{z}_1 = \pi_1\,\widetilde{z}_2$

4. a) $\widehat{\omega}_\varepsilon(h) \;:=\; \sup_m \; \omega_\varepsilon\,(\widetilde{x}_1^{(m)}, h) \;<\; \infty$ *(modulus of continuity w.r.t.* $\widetilde{q}^{\,1}_\varepsilon$*)*

b) $\widehat{\omega}_{\varepsilon'}(h) \;:=\; \sup_m \; \omega_{\varepsilon'}(\widetilde{x}_2^{(m)}, h) \;<\; \infty$ *(modulus of continuity w.r.t.* $\widetilde{q}^{\,2}_{\varepsilon'}$*)*

5. $\forall \; t_1, t_2 \in [0,T[, \quad t_3 \in \,]0,T[\quad \exists \; (m_j)_{j \in I\!N}$ *with* $m_j \nearrow \infty$ *and*

$$(i) \quad \limsup_{j \longrightarrow \infty} \; \widetilde{Q}^{1\,\mapsto}_\varepsilon\Big(\widetilde{f}_1(\widetilde{x}_1(t_1),\, \widetilde{x}_2(t_1),\, t_1), \quad \widetilde{f}_1^{(m_j)}(\widetilde{x}_1(t_1),\, \widetilde{x}_2(t_1),\, t_1)\Big) \;\leq\; R_\varepsilon$$

$$\limsup_{j \longrightarrow \infty} \; \widetilde{Q}^{2\,\mapsto}_{\varepsilon'}\Big(\widetilde{f}_2(\widetilde{x}_1(t_1),\, \widetilde{x}_2(t_1),\, t_1), \quad \widetilde{f}_2^{(m_j)}(\widetilde{x}_1(t_1),\, \widetilde{x}_2(t_1),\, t_1)\Big) \;\leq\; R_{\varepsilon'}$$

$$(ii) \quad \exists \; \delta'_j \searrow 0: \quad \widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{x}_1(t_2), \quad \widetilde{x}_1^{(m_j)}(t_2 + \delta'_j)\Big) \longrightarrow 0,$$

$$\widetilde{q}^{\,2}_{\varepsilon'}\Big(\widetilde{x}_2(t_2), \quad \widetilde{x}_2^{(m_j)}(t_2 + \delta'_j)\Big) \longrightarrow 0,$$

$$\pi_1\, \widetilde{x}_1(t_2) \;=\; \pi_1\, \widetilde{x}_2(t_2) \;\leq\; \pi_1\, \widetilde{x}_1^{(m_j)}(t_2 + \delta'_j) \;=\; \pi_1\, \widetilde{x}_2^{(m_j)}(t_2 + \delta'_j)$$

$$(iii) \quad \exists \; \delta_j \searrow 0: \quad \widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{x}_1^{(m_j)}(t_3 - \delta_j), \quad \widetilde{x}_1(t_3)\Big) \longrightarrow 0,$$

$$\widetilde{q}^{\,2}_{\varepsilon'}\Big(\widetilde{x}_2^{(m_j)}(t_3 - \delta_j), \quad \widetilde{x}_2(t_3)\Big) \longrightarrow 0,$$

$$\pi_1\, \widetilde{x}_1^{(m_j)}(t_3 - \delta_j) \;=\; \pi_1\, \widetilde{x}_2^{(m_j)}(t_3 - \delta_j) \;\leq\; \pi_1\, \widetilde{x}_1(t_3) \;=\; \pi_1\, \widetilde{x}_2(t_3).$$

Then, $\widetilde{x}_1(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}_1(\cdot) \ni \widetilde{f}_1(\widetilde{x}_1(\cdot),\, \widetilde{x}_2(\cdot),\, \cdot)$
and $\widetilde{x}_2(\cdot)$ *a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}_2(\cdot) \ni \widetilde{f}_2(\widetilde{x}_1(\cdot),\, \widetilde{x}_2(\cdot),\, \cdot)$.

Proof. We follow the same track as in Convergence Theorem 2.3.2. Here only $\widetilde{x}_1(\cdot)$ is considered since all conclusions can be drawn for $\widetilde{x}_2(\cdot)$ in the same way (due to the symmetry of assumptions).

As an abbreviation, set $\widetilde{x}(t) \;:=\; (\widetilde{x}_1(t),\, \widetilde{x}_2(t))$ and $\widetilde{x}^{(m)}(t) \;:=\; (\widetilde{x}_1^{(m)}(t),\, \widetilde{x}_2^{(m)}(t))$.

The uniform continuity of $\widetilde{x}_1(\cdot)$ with respect to each $\widetilde{q}^{\,1}_\varepsilon$ results from assumption (4.a): Each $\widetilde{x}_1^{(m)}(\cdot)$ satisfies $\widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{x}_1^{(m)}(s),\; \widetilde{x}_1^{(m)}(t)\Big) \;\leq\; \widehat{\omega}_\varepsilon(t-s)$ for $0 \leq s < t < T$.

Let $\varepsilon \in \mathcal{J}_1$, $0 \leq s < t < T$ be arbitrary and choose $(\delta'_j)_{j \in I\!N}$, $(\delta_j)_{j \in I\!N}$ for s, t (according to cond. (5.ii), (5.iii)). For all $j \in I\!N$ large enough, $s + \delta'_j < t - \delta_j$ and so,

$$\begin{aligned} &\widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{x}_1(s),\; \widetilde{x}_1(t)\Big) \\ \leq\; &\widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{x}_1(s),\, \widetilde{x}_1^{(m_j)}(s + \delta'_j)\Big) + \widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{x}_1^{(m_j)}(s + \delta'_j),\, \widetilde{x}_1^{(m_j)}(t - \delta_j)\Big) + \widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{x}_1^{(m_j)}(t - \delta_j),\, \widetilde{x}_1(t)\Big) \\ \leq\; &\qquad\qquad\qquad\qquad \widehat{\omega}_\varepsilon(t-s) \qquad\qquad\qquad\qquad + o(1). \end{aligned}$$

for $j \longrightarrow \infty$.

Now let $\varepsilon \in \mathcal{J}_1$, $\widetilde{y}_1 \in \widetilde{D}_1$ and $t \in [0,T[$, $0 < h < \mathcal{T}_\Theta\Big(\widetilde{f}_1(\widetilde{x}_1(t),\, \widetilde{x}_2(t),\, t),\; \widetilde{y}_1\Big)$ be chosen arbitrarily with $\pi_1\, \widetilde{y}_1 \;\leq\; \pi_1\, \widetilde{x}_1(t) \;=\; \pi_1\, \widetilde{x}_2(t)$. Condition (6.) of Def. 2.1.1 implies for all $k \in \,]0,h[$ sufficiently small

$$\widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{f}_1(\widetilde{x}(t),\, t)\,(h, \widetilde{y}_1),\;\; \widetilde{x}_1(t+h)\Big) \;\leq\; \widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{f}_1(\widetilde{x}(t),\, t)\,(h-k, \widetilde{y}_1),\;\; \widetilde{x}_1(t+h)\Big) \;+\; h^2.$$

According to cond. (5.i) – (5.iii), there exist sequences $(m_j)_{j\in\mathbb{N}}$, $(\delta'_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty$, $\delta'_j \downarrow 0$, $\delta_j \downarrow 0$, $\delta_j + \delta'_j < k$ and

$$\wedge \left\{ \begin{array}{llll} \widetilde{Q}^{1\,\mapsto}_{\varepsilon}\Big(\widetilde{f}_1(\widetilde{x}(t),t), & \widetilde{f}_1^{(m_j)}(\widetilde{x}(t),t)\Big) & \le & R_\varepsilon + h^2, \\ \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{x}_1^{(m_j)}(t+h-\delta_j), & \widetilde{x}_1(t+h)\Big) & \longrightarrow & 0, \\ \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{x}_1(t), & \widetilde{x}_1^{(m_j)}(t+\delta'_j)\Big) & \longrightarrow & 0, \\ \widetilde{q}^{\,2}_{\varepsilon'}\Big(\widetilde{x}_2(t), & \widetilde{x}_2^{(m_j)}(t+\delta'_j)\Big) & \longrightarrow & 0. \end{array} \right.$$

Thus, Proposition 2.2.3 implies for all $j \in \mathbb{N}$ large enough (depending on $\varepsilon, \widetilde{y}_1, t, h, k$),

$$\begin{array}{ll}
\multicolumn{2}{l}{\widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{f}_1(\widetilde{x}(t),t)\,(h,\ \widetilde{y}_1),\quad \widetilde{x}_1(t+h)\Big)} \\
\le & \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{f}_1(\widetilde{x}(t),t)\,(h-k,\ \widetilde{y}_1),\quad \widetilde{x}_1^{(m_j)}(t+\delta'_j+h-k)\Big) \\
+ & \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{x}_1^{(m_j)}(t+\delta'_j+h-k),\quad \widetilde{x}_1^{(m_j)}(t+h-\delta_j)\Big) \\
+ & \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{x}_1^{(m_j)}(t+h-\delta_j),\quad \widetilde{x}_1(t+h)\Big) \quad + \ h^2 \\
\le & \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{y}_1,\ \widetilde{x}_1^{(m_j)}(t+\delta'_j)\Big)\ \cdot\ e^{M_\varepsilon\cdot(h-k)} \quad + \\
+ & \displaystyle\int_0^{h-k} e^{M_\varepsilon\cdot(h-k-s)}\ \Big(\widetilde{Q}^{1\,\mapsto}_{\varepsilon}\Big(\widetilde{f}_1(\widetilde{x}(t),t),\ \widetilde{f}_1^{(m_j)}(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big)\ +\ 3\,R_\varepsilon\Big)\ ds \\
+ & \widehat{\omega}_\varepsilon(k-\delta_j-\delta'_j) \\
+ & \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{x}_1^{(m_j)}(t+h-\delta_j),\quad \widetilde{x}_1(t+h)\Big) \quad + \ h^2 \\
\le & \Big(\widetilde{q}^{\,1}_{\varepsilon}(\widetilde{y}_1,\ \widetilde{x}_1(t))\ +\ \widetilde{q}^{\,1}_{\varepsilon}\Big(\widetilde{x}_1(t),\ \widetilde{x}_1^{(m_j)}(t+\delta'_j)\Big)\Big)\ \cdot\ e^{M_\varepsilon\cdot(h-k)} \quad + \\
+ & \displaystyle\int_0^{h} e^{M_\varepsilon\cdot(h-s)}\ \Big(\widetilde{Q}^{1\,\mapsto}_{\varepsilon}\Big(\widetilde{f}_1(\widetilde{x}(t),t),\ \widetilde{f}_1^{(m_j)}(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big)\ +\ 3\,R_\varepsilon\Big)\ ds \\
+ & \widehat{\omega}_\varepsilon(k) \quad + \ 2\,h^2 \\
\le & \widetilde{q}^{\,1}_{\varepsilon}(\widetilde{y}_1,\ \widetilde{x}_1(t))\ \cdot\ e^{M_\varepsilon\,h} \quad +\ 3\,h^2\ +\ \widehat{\omega}_\varepsilon(k)\ +\ h\cdot e^{M_\varepsilon\,h}\ 3\,R_\varepsilon \\
+ & \displaystyle\int_0^{h} e^{M_\varepsilon\cdot(h-s)}\ \Big(R_\varepsilon + h^2 + \widetilde{Q}^{1\,\mapsto}_{\varepsilon}\Big(\widetilde{f}_1^{(m_j)}(\widetilde{x}(t),t),\ \widetilde{f}_1^{(m_j)}(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big)\Big)\ ds \\
\le & \widetilde{q}^{\,1}_{\varepsilon}(\widetilde{y}_1,\ \widetilde{x}_1(t))\cdot e^{M_\varepsilon\,h}\ +\ \mathrm{const}\cdot\ h\,(R_\varepsilon + h)\ +\ \widehat{\omega}_\varepsilon(k) \\
+ & h\quad e^{M_\varepsilon\,h}\quad \widetilde{Q}^{1\,\mapsto}_{\varepsilon}\Big(\widetilde{f}_1^{(m_j)}(\widetilde{x}(t),t),\quad \widetilde{f}_1^{(m_j)}(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j}\Big) \\
+ & \displaystyle\int_0^{h} e^{M_\varepsilon\cdot(h-s)}\quad \widetilde{Q}^{1\,\mapsto}_{\varepsilon}\Big(\widetilde{f}_1^{(m_j)}(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j},\ \widetilde{f}_1^{(m_j)}(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big)\ ds.
\end{array}$$

Now due to assumption (3.) and the choice of (δ'_j), the upper limits for $j \longrightarrow \infty$ and then $k \longrightarrow 0$ provide the estimate

$$\begin{aligned}
&\widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{f}_1(\widetilde{x}(t),t)\;(h,\,\widetilde{y}),\;\;\widetilde{x}_1(t+h)\Big)\\
\le\quad & \widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\;\widetilde{x}_1(t))\cdot e^{M_\varepsilon\,h}\quad + \;\text{const}\cdot\; h\;(R_\varepsilon+h)\;+\;0\;+\;0\\
&+\;\; h\;\, e^{M_\varepsilon\,h}\cdot\;\limsup_{j\longrightarrow\infty}\;\;\sup_{0\,\le\,s\,\le\,h}\;\;\widetilde{Q}^{1\,\mapsto}_\varepsilon\Big(\widetilde{f}^{(m_j)}_1(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j},\;\widetilde{f}^{(m_j)}_1(\widetilde{x}^{(m_j)}(\cdot),\cdot)\Big|_{t+\delta'_j+s}\Big).
\end{aligned}$$

So finally convergence assumption (3.) and the equi–continuity of $(\widetilde{x}^{(m)}_1)$, $(\widetilde{x}^{(m)}_2)$ implies

$$\begin{aligned}
&\limsup_{h\,\downarrow\,0}\;\;\tfrac{1}{h}\;\cdot\;\Big(\;\widetilde{q}^{\,1}_\varepsilon\Big(\widetilde{f}_1(\widetilde{x}(t),t)\,(h,\widetilde{y}),\;\;\widetilde{x}_1(t{+}h)\Big)\;\;-\;\;\widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\;\widetilde{x}_1(t))\;\cdot\;e^{M_\varepsilon\,h}\;\Big)\\
\le\;&\limsup_{h\,\downarrow\,0}\;\Big(\mathrm{c}\cdot(R_\varepsilon{+}h)+\limsup_{j\longrightarrow\infty}\;\sup_{0\,\le\,s\,\le\,h}\;\widetilde{Q}^{\mapsto}_\varepsilon\Big(\widetilde{f}^{(m_j)}_1(\widetilde{x}^{(m_j)},\cdot)\Big|_{t+\delta'_j},\;\widetilde{f}^{(m_j)}_1(\widetilde{x}^{(m_j)},\cdot)\Big|_{t+\delta'_j+s}\Big)\Big)\\
=\;&\qquad\quad \mathrm{c}\cdot\;R_\varepsilon\qquad+\;R_\varepsilon\,.
\end{aligned}$$

□

For applying this modified Convergence Theorem, we now make the assumptions about the components separately.

Proposition 2.4.6 (Existence of timed right–hand forward solutions for systems of two generalized mutational equations)
Assume that $\Big(\widetilde{E}_1,\;(\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1},\;\widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\widetilde{D}_1,(\widetilde{q}^{\,1}_\varepsilon))\Big)$, $\Big(\widetilde{E}_2,\;(\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2},\;\widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2,\widetilde{D}_2,(\widetilde{q}^{\,2}_{\varepsilon'}))\Big)$ *are timed transitionally compact. Moreover for each* $\varepsilon\in\mathcal{J}_1,\;\varepsilon'\in\mathcal{J}_2$, *let*

$$\begin{aligned}
\widetilde{f}_1:\;\widetilde{E}_1\times\widetilde{E}_2\times[0,T]&\longrightarrow\widetilde{\Theta}^{\mapsto}_p(\widetilde{E}_1,\;\widetilde{D}_1,\;(\widetilde{q}^{\,1}_\varepsilon)_{\varepsilon\in\mathcal{J}_1})\\
\widetilde{f}_2:\;\widetilde{E}_1\times\widetilde{E}_2\times[0,T]&\longrightarrow\widetilde{\Theta}^{\mapsto}_{p'}(\widetilde{E}_2,\;\widetilde{D}_2,\;(\widetilde{q}^{\,2}_{\varepsilon'})_{\varepsilon\in\mathcal{J}_1})
\end{aligned}$$

fulfill

1. a) $M_\varepsilon \;:=\; \sup_{t,\widetilde{z}_1,\widetilde{z}_2}\;\alpha^{\mapsto}_\varepsilon(\widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t)) \;<\;\infty,$

b) $M_{\varepsilon'} \;:=\; \sup_{t,\widetilde{z}_1,\widetilde{z}_2}\;\alpha^{\mapsto}_{\varepsilon'}(\widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t)) \;<\;\infty,$

2. a) $c_\varepsilon(h) \;:=\; \sup_{t,\widetilde{z}_1,\widetilde{z}_2}\;\beta_\varepsilon\,(\widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t))\,(h) \;\longrightarrow 0$ *for* $h\downarrow 0$,

b) $c_{\varepsilon'}(h) \;:=\; \sup_{t,\widetilde{z}_1,\widetilde{z}_2}\;\beta_{\varepsilon'}\,(\widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t))\,(h) \;\longrightarrow 0$ *for* $h\downarrow 0$,

3. a) $\exists\;R_\varepsilon\;:\;\sup_{t,\widetilde{z}_1,\widetilde{z}_2}\;\gamma_\varepsilon\,(\widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t)) \;\le\; R_\varepsilon,\quad \varepsilon^p\;\cdot R_\varepsilon\;\longrightarrow 0$ *for* $\varepsilon\to 0$,

b) $\exists\;R_{\varepsilon'}\;:\;\sup_{t,\widetilde{z}_1,\widetilde{z}_2}\;\gamma_{\varepsilon'}\,(\widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t)) \;\le\; R_{\varepsilon'},\quad \varepsilon'^{(p')}\cdot R_{\varepsilon'}\longrightarrow 0$ *for* $\varepsilon'\to 0$,

4. $\exists$ *moduli* $\widehat{\omega}_\varepsilon(\cdot),\;\widehat{\omega}_{\varepsilon'}(\cdot)$ *of continuity :*

$$\begin{aligned}
\widetilde{Q}^{1\,\mapsto}_\varepsilon\Big(\widetilde{f}_1(\widetilde{y}_1,\widetilde{y}_2,t_1),\;\;\widetilde{f}_1(\widetilde{z}_1,\widetilde{z}_2,t_2)\Big)\;&\le\;\;R_\varepsilon\;+\widehat{\omega}_\varepsilon\,\Big(\widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\widetilde{z}_1)\;+\;\widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{y}_2,\widetilde{z}_2)\;+\;t_2{-}t_1\Big)\\
\widetilde{Q}^{2\,\mapsto}_{\varepsilon'}\Big(\widetilde{f}_2(\widetilde{y}_1,\widetilde{y}_2,t_1),\;\;\widetilde{f}_2(\widetilde{z}_1,\widetilde{z}_2,t_2)\Big)\;&\le\;\;R_{\varepsilon'}\;+\widehat{\omega}_{\varepsilon'}\Big(\widetilde{q}^{\,1}_\varepsilon(\widetilde{y}_1,\widetilde{z}_1)\;+\;\widetilde{q}^{\,2}_{\varepsilon'}(\widetilde{y}_2,\widetilde{z}_2)\;+\;t_2{-}t_1\Big)
\end{aligned}$$

for all $0\le t_1\le t_2\le T,\quad \widetilde{y}_1,\widetilde{z}_1\in\widetilde{E}_1,\quad \widetilde{y}_2,\widetilde{z}_2\in\widetilde{E}_2$ *and* $\varepsilon'\in\mathcal{J}_2$

with $\pi_1\,\widetilde{y}_1\;=\;\pi_1\,\widetilde{y}_2\;\le\;\pi_1\,\widetilde{z}_1\;=\;\pi_1\,\widetilde{z}_2,$

Then for every $\widetilde{x}_1^0 \in \widetilde{E}_1$ *and* $\widetilde{x}_2^0 \in \widetilde{E}_2$ *(with* $\pi_1\, \widetilde{x}_1^0 = \pi_1\, \widetilde{x}_2^0$*), there exist timed right–hand forward solutions* $\widetilde{x}_1(\cdot) : [0,T[\longrightarrow \widetilde{E}_1$, $\widetilde{x}_2(\cdot) : [0,T[\longrightarrow \widetilde{E}_2$ *of the generalized mutational equations*

$$\begin{aligned} \overset{\circ}{\widetilde{x}}_1(\cdot) &\ni \widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot\,) \\ \overset{\circ}{\widetilde{x}}_2(\cdot) &\ni \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot\,) \end{aligned}$$

with $\widetilde{x}_1(0) = \widetilde{x}_1^0$, $\widetilde{x}_2(0) = \widetilde{x}_2^0$.

Proof is based on the same combination of Euler method and Cantor diagonal construction as for Proposition 2.3.5. Applying it to

$$\Big(\widetilde{E}_1 \times \widetilde{E}_2,\quad (\widetilde{q}_\varepsilon^{\,1} + \widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon \in \mathcal{J}_1,\ \varepsilon' \in \mathcal{J}_2},\quad \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}_\varepsilon^{\,1})_{\varepsilon \in \mathcal{J}_1}) \times \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon' \in \mathcal{J}_2})\Big).$$

we obtain a sequence $\Big((\widetilde{x}_1^{(j)}(\cdot),\ \widetilde{x}_2^{(j)}(\cdot))\Big)_{n \in I\!N}$ of Euler approximations $[0,T[\longrightarrow \widetilde{E}_1 \times \widetilde{E}_2$, limit functions $(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot)) : [0,T[\longrightarrow \widetilde{E}_1 \times \widetilde{E}_2$ and a sequence $h_j \downarrow 0$ $(j \to \infty)$ with

$$\begin{cases} \sup\limits_{0 \le t < T} \ \widetilde{q}_\varepsilon^{\,1}\Big(\widetilde{x}_1^{(j)}(t - h_j),\ \widetilde{x}_1(t)\Big) \longrightarrow 0, & \sup\limits_{0 \le t < T} \ \widetilde{q}_\varepsilon^{\,1}\Big(\widetilde{x}_1(t),\ \widetilde{x}_1^{(j)}(t + h_j)\Big) \longrightarrow 0, \\ \sup\limits_{0 \le t < T} \ \widetilde{q}_{\varepsilon'}^{\,2}\Big(\widetilde{x}_2^{(j)}(t - h_j),\ \widetilde{x}_2(t)\Big) \longrightarrow 0, & \sup\limits_{0 \le t < T} \ \widetilde{q}_{\varepsilon'}^{\,2}\Big(\widetilde{x}_2(t),\ \widetilde{x}_2^{(j)}(t + h_j)\Big) \longrightarrow 0 \end{cases}$$

for $j \longrightarrow \infty$ and each $\varepsilon \in \mathcal{J}_1,\ \varepsilon' \in \mathcal{J}_2$.

So Convergence Theorem 2.4.5 for systems of mutational equations implies that $\widetilde{x}_1(\cdot)$ and $\widetilde{x}_2(\cdot)$ are timed right–hand forward solutions of

$$\begin{aligned} \overset{\circ}{\widetilde{x}}_1(\cdot) &\ni \widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot\,) \\ \overset{\circ}{\widetilde{x}}_2(\cdot) &\ni \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot\,) \end{aligned}$$

respectively. □

Chapter 3

Timed right–hand backward solutions of mutational equations

Now a second generalization of mutational equations is presented. In comparison with the concept of chapter 2, the main differences can be subsumed into two aspects :

Firstly, we consider the points of time $t-h,\ t$ for first–order approximation (instead of t and $t+h$). This idea motivates the term "backward" and is symbolized by $\rightarrow\!\!\!\shortmid$. Roughly speaking, detailed information might be lost while time is passing, e.g. in shape evolution, components of the boundary can disappear. So now the aim is to use the information of the past as long as possible. Then of course, another form of Gronwall's Lemma takes on a key role, namely Lemma 1.5.3 for ψ assuming

$$\begin{array}{rcl} \psi(t) & \geq & \limsup\limits_{h\downarrow 0}\ \psi(t+h), \\ \liminf\limits_{h\downarrow 0}\ \frac{\psi(t)-\psi(t-h)}{h} & \leq & f(t)\cdot \limsup\limits_{h\downarrow 0}\ \psi(t-h)\ +\ g(t)\ < \infty. \end{array}$$

So when applying it to the generalized distance between two curves $\widetilde{x}, \widetilde{y} : [0,T] \longrightarrow \widetilde{E}$, we have to guarantee the upper semicontinuity mentioned in the first condition. Here this goal is not achieved by a global assumption (like condition (6.) of Def. 2.1.1), but we consider the upper limit

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big)\ :=\ \limsup_{\substack{k,l\downarrow 0\\ k<l}}\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t+k),\ \widetilde{y}(t+l)\Big)$$

instead of the distance $\widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{y}(t))$. On the one hand, the upper semicontinuity is guaranteed for Gronwall's Lemma 1.5.3, but on the other hand these two versions of a distance are not equal in general.

Combining these modifications leads to the following condition on a timed backward transition $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ of order p (as a more general form of the semigroup property)

$$\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 : \quad \limsup_{\varepsilon \longrightarrow 0}\ \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) \;=\; 0 \quad \text{and} \quad \text{for all } t \in\,]0,1[,$$
$$\limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\ \widetilde{\vartheta}(t-h, \widetilde{x})),\quad \widetilde{\vartheta}(t^{++}, \widetilde{x})\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta})$$
$$\limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\quad \widetilde{\vartheta}(h^{++},\ \widetilde{\vartheta}(t-h, \widetilde{x}))\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta}).$$

As another consequence of the upper limits with respect to time, we need additional conditions for the "forward" link with initial elements (in a figurative sense), i.e.

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(0^+, \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t^{++}, \widetilde{x})\Big) \;=\; 0, \qquad \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\ \ \widetilde{\vartheta}(0^{++}, \widetilde{\vartheta}(t,\widetilde{x}))\Big) \;=\; 0.$$

The second essential difference to the forward generalization of chapter 2 is that we dispense with the basic idea of distributions, i.e. we prefer the direct (first–order) comparison between transitions to their effect on a "test set" $\widetilde{D} \subset \widetilde{E}$.
So in this concept, a timed backward transition $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ (of order p) represents the right–hand mutation of a curve $\widetilde{x} : [0,T] \longrightarrow \widetilde{E}$ at time $t \in\,]0,T[$ if

$$\limsup_{h \downarrow 0}\ \tfrac{1}{h}\ \cdot\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\ \widetilde{x}(t-h)),\ \ \widetilde{x}(t^{++})\Big) \;\leq\; \widehat{\gamma}_\varepsilon$$

with parameters $\widehat{\gamma}_\varepsilon \geq 0$ $(\varepsilon \in \mathcal{J})$ satisfying $\limsup_{\varepsilon \longrightarrow 0}\ \varepsilon^p \cdot \widehat{\gamma}_\varepsilon \;=\; 0$ and, the expression "right–hand" again refers to the fact that $\widetilde{x}(t)$ appears in the second argument of $\widetilde{q}_\varepsilon$.
The corresponding "left–hand" condition is

$$\limsup_{h \downarrow 0}\ \tfrac{1}{h}\ \cdot\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \ \widetilde{\vartheta}(h^{++},\ \widetilde{x}(t-h))\Big) \;\leq\; \widehat{\gamma}_\varepsilon.$$

In comparison with the distributive notion of chapter 2, the direct comparison here has two key advantages. Firstly, we obtain estimates for distance between "left–hand" and "right–hand" primitives (in Prop. 3.2.3). Secondly, two–sided completeness provides an alternative to transitional compactness for constructing solutions (in § 3.3.3).

Principally the order of this chapter provides many similarities to chapter 2.
In § 3.1, timed backward transitions of order p on $(\widetilde{E} \stackrel{\text{Def.}}{=} I\!R \times E,\ (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ are defined. The generalized distance functions $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J})$ are usually assumed to fulfill the timed triangle inequality and to have the so–called property $(BUC^{\rightarrow})$, i.e. $\sup\ \widetilde{q}_\varepsilon(\widetilde{x}(\cdot), \widetilde{y}(\cdot)) \;<\; \infty$ for all curves $\widetilde{x}, \widetilde{y} \in UC^{\rightarrow}([0,T], \widetilde{E}, \widetilde{q})$ $(T < \infty)$, $\varepsilon \in \mathcal{J}$.
This last condition is only introduced to make Gronwall's Lemma 1.5.3 applicable. In particular, the reflexivity of $\widetilde{q}_\varepsilon$ is usually not required since $\widetilde{q}_\varepsilon(\widetilde{x}(t^+), \widetilde{y}(t^{++}))$ does not compare curves $\widetilde{x}(\cdot),\ \widetilde{y}(\cdot)$ at the same point of time anyway.

In § 3.2, timed right–hand backward primitives are specified – consisting in their definition, the piecewise construction and the corresponding estimate. These results form the basis for timed right–hand backward solutions of generalized mutational equations in § 3.3. After the definition, we consider stability properties and prove existence in two different ways, i.e. transitional compactness and two–sided completeness in combination with standard hypotheses $(R^{\Rightarrow}), (R^{\Leftarrow})$.
Finally § 3.4 consists in an essential advantage of the triangle inequality in comparison with its *timed* counterpart. We consider an ostensible metric $q : E \times E \longrightarrow [0, \infty[$ on a nonempty set E. Then standard hypothesis $(R^{\Leftarrow})$ implies

$$q\Big(x(t^+),\ y(t^{++})\Big) \;=\; q\Big(x(t),\ y(t)\Big)$$

for all curves $x,\ y \in UC^{\rightarrow}([0,T[,\ E,\ q)$. Thus in short, the upper limits for $k, l \longrightarrow 0$ $(0 < k < l,$ denoted by "+", "++") can be omitted. As a consequence, Euler method leads to right–hand backward solution of generalized mutational equations — without restrictions on the time interval as in § 3.3.

General assumptions for chapter 3. Let E be a nonempty set, $\widetilde{E} := I\!R \times E$, $\pi_1 : \widetilde{E} \longrightarrow I\!R,\ (t,x) \longmapsto t$ and $p \in I\!R$. $\mathcal{J} \subset [0,1]^\kappa$ abbreviates a countable index set with $\kappa \in I\!N,\ 0 \in \overline{\mathcal{J}}$.
For each $\varepsilon \in \mathcal{J}$, the function $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ is always supposed to satisfy

1. timed triangle inequality,
2. time continuity, i.e. every sequence $(\widetilde{z}_n = (t_n, z_n))_{n \in I\!N}$ in $\widetilde{E}$ and $\widetilde{z} = (t, z) \in \widetilde{E}$ with $\widetilde{q}(\widetilde{z}_n, \widetilde{z}) \longrightarrow 0$ $(n \longrightarrow \infty)$ fulfill $t_n \longrightarrow t$ $(n \longrightarrow \infty)$ (due to Def. 1.2.1).
3. property $(BUC^{\rightarrow})$, i.e. $\sup\limits_{t \in [0,T]} \widetilde{q}_\varepsilon(\widetilde{x}(t), \widetilde{y}(t)) < \infty$ for all $\widetilde{x}, \widetilde{y} \in UC^{\rightarrow}([0,T], \widetilde{E}, \widetilde{q}_\varepsilon)$.

Notation.

$$\begin{aligned}
\widetilde{q}\Big(\widetilde{x}(s),\ \widetilde{y}(t^+)\Big) &:= \limsup_{k \downarrow 0}\ \widetilde{q}\Big(\widetilde{x}(s),\ \widetilde{y}(t+k)\Big),\\
\widetilde{q}\Big(\widetilde{x}(s^+),\ \widetilde{y}(t^+)\Big) &:= \limsup_{k \downarrow 0}\ \widetilde{q}\Big(\widetilde{x}(s+k),\ \widetilde{y}(t+k)\Big),\\
\widetilde{q}\Big(\widetilde{x}(s^+),\ \widetilde{y}(t^{++})\Big) &:= \limsup_{\substack{k,l \downarrow 0\\ k<l}}\ \widetilde{q}\Big(\widetilde{x}(s+k),\ \widetilde{y}(t+l)\Big)
\end{aligned}$$

for any $\widetilde{x}, \widetilde{y} : [0,T[\longrightarrow \widetilde{E}$ and $s, t \in [0,T[$.

3.1 Timed backward transitions

Definition 3.1.1 $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ *is called* timed backward transition of order p *on* $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *if for all* $\varepsilon \in \mathcal{J}$,

1. $\widetilde{\vartheta}(0, \cdot) \;=\; \mathrm{Id}_{\widetilde{E}}$,

2. $\exists\;\gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 : \qquad \limsup\limits_{\varepsilon \longrightarrow 0}\; \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) \;=\; 0$ *and*
$$\limsup_{h \downarrow 0}\; \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\; \widetilde{\vartheta}(t-h, \widetilde{x})),\quad \widetilde{\vartheta}(t^{++}, \widetilde{x})\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\; t \in\,]0,1[,$$
$$\limsup_{h \downarrow 0}\; \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\quad \widetilde{\vartheta}(h^{++},\; \widetilde{\vartheta}(t-h, \widetilde{x}))\Big) \;\leq\; \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\; t \in\,]0,1[,$$

3. $$\alpha_\varepsilon^{\rightarrow\!\shortmid}(\widetilde{\vartheta}) \;:=\; \sup_{\substack{0 \leq t < 1 \\ \widetilde{x},\, \widetilde{y} \,\in \\ UC^{\rightarrow}(]0,1[,\, \widetilde{E},\, \widetilde{q}_\varepsilon)}} \;\limsup_{h \downarrow 0}\; \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h^+, \widetilde{x}(t_h)),\, \widetilde{\vartheta}(h^{++}, \widetilde{y}(t_h))\big) - \widetilde{q}_\varepsilon\big(\widetilde{x}(t_h^+),\, \widetilde{y}(t_h^{++})\big) - \gamma_\varepsilon(\widetilde{\vartheta})\, h}{h \quad \big(\widetilde{q}_\varepsilon\big(\widetilde{x}(t_h^+),\; \widetilde{y}(t_h^{++})\big) + \gamma_\varepsilon(\widetilde{\vartheta})\, h\big)} \right)^+$$
$$<\; \infty \qquad\qquad (\textit{with}\;\; t_h := t - h)$$

4. $\exists\;\beta_\varepsilon(\widetilde{\vartheta}) :\;]0,1] \longrightarrow [0,\infty[\, : \quad \beta_\varepsilon(\widetilde{\vartheta})(\cdot)$ *nondecreasing,* $\quad \limsup\limits_{h \downarrow 0}\; \beta_\varepsilon(\widetilde{\vartheta})(h) \;=\; 0$,
$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s, \widetilde{x}),\;\; \widetilde{\vartheta}(t, \widetilde{x})\Big) \;\leq\; \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad \forall\; 0 \leq s < t,\; \widetilde{x} \in \widetilde{E},$$

5. $$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(0^+, \widetilde{\vartheta}(t, \widetilde{x})),\;\; \widetilde{\vartheta}(t^{++}, \widetilde{x})\Big) \;=\; 0 \qquad \forall\; t \in [0,1[,$$
$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\;\; \widetilde{\vartheta}(0^{++}, \widetilde{\vartheta}(t, \widetilde{x}))\Big) \;=\; 0 \qquad \forall\; t \in [0,1[,$$

6. $\pi_1\, \widetilde{\vartheta}(h,\; \widetilde{x}) \;=\; h \;+\; \pi_1\, \widetilde{x} \qquad \forall\; \widetilde{x} \in \widetilde{E},\; h \in [0,1].$

Define for any timed backward transitions $\widetilde{\vartheta}, \widetilde{\tau} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ *and* $\varepsilon \in \mathcal{J}$,
$$\widetilde{Q}_\varepsilon^{\rightarrow\!\shortmid}(\widetilde{\vartheta}, \widetilde{\tau}) \quad:=\quad \sup_{\substack{0 < t \leq 1 \\ \widetilde{x} \in \widetilde{E}}} \;\limsup_{h \downarrow 0}\; \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\; \widetilde{\tau}(t-h, \widetilde{x})),\;\; \widetilde{\tau}(t^{++}, \widetilde{x})\Big).$$
$\widetilde{\Theta}_p^{\rightarrow\!\shortmid}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *denotes a set of timed backward transitions of order* p *on* $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *supposing for all* $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\rightarrow\!\shortmid}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}),\; \widetilde{x} \in \widetilde{E},\; \varepsilon \in \mathcal{J}$ *in addition*
$$\wedge \begin{cases} \widetilde{Q}_\varepsilon^{\rightarrow\!\shortmid}(\widetilde{\vartheta}, \widetilde{\tau}) \;<\; \infty\,, \\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(0^+, \widetilde{x}),\;\; \widetilde{\tau}(0^{++}, \widetilde{x})\Big) \;\stackrel{\text{Def.}}{=}\; \limsup\limits_{k,l \downarrow 0\; (k<l)} \;\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(k, \widetilde{x}),\;\; \widetilde{\tau}(l, \widetilde{x})\Big) \;=\; 0. \end{cases}$$

Remark. 1. $\widetilde{E} \stackrel{\text{Def.}}{=} I\!\!R \times E$ supplied with only one function $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ provides an easy example by setting $\mathcal{J} := \{0\},\; \widetilde{q}_0 := \widetilde{q}$ as mentioned after Def. 2.1.1. For each timed backward transitions $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ of order 0, the condition $\limsup\limits_{\varepsilon \longrightarrow 0} \varepsilon^0 \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0$ means $0 = 0^0 \cdot \gamma_0(\widetilde{\vartheta}) = \gamma_0(\widetilde{\vartheta})$ — just by definition of $0^0 \stackrel{\text{Def.}}{=} 1$.

So $\widetilde{\vartheta}$ is demanded to fulfill for all $\widetilde{x} \in \widetilde{E},\ t \in]0,1[$

$$\wedge \begin{cases} \limsup_{h \downarrow 0} \ \frac{1}{h} \ \widetilde{q}\Big(\widetilde{\vartheta}(h^+, \ \widetilde{\vartheta}(t-h,\widetilde{x})), \quad \widetilde{\vartheta}(t^{++},\widetilde{x})\Big) \ = \ 0 \\ \limsup_{h \downarrow 0} \ \frac{1}{h} \ \widetilde{q}\Big(\widetilde{\vartheta}(t^+,\widetilde{x}), \quad \widetilde{\vartheta}(h^{++}, \ \widetilde{\vartheta}(t-h,\widetilde{x}))\Big) \ = \ 0 \end{cases}$$

Then the following results do not take the dependency on ε or $\gamma_\varepsilon(\cdot)$ into consideration (see e.g. the existence of solutions due to completeness in § 3.3.3). So we do not mention ε there and abbreviate the set of backward transitions of order 0 as $\widetilde{\Theta}_0^{\rightarrow}(\widetilde{E},\widetilde{q})$.

2. The definitions ensures only $\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta},\widetilde{\vartheta}) \ \le \ \gamma_\varepsilon(\widetilde{\vartheta})$ for all $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon))$. So the continuity assumption of a function $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is often replaced by a condition similar to

$$\limsup_{\substack{h \longrightarrow 0 \\ h \ge 0}} \ \sup_{t \in [0,T[} \ \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{\vartheta}(t), \ \widetilde{\vartheta}(t+h)\Big) \ \le \ \mathrm{const}(\widetilde{\vartheta}(\cdot),\varepsilon).$$

3. Condition (4.) on a timed backward transition $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ states its uniform continuity (in positive time direction) with respect to $\widetilde{q}_\varepsilon$ for every $\varepsilon \in \mathcal{J}$, i.e.

$$\widetilde{\vartheta}(\cdot,\widetilde{x}) \ \in \ UC^{\rightarrow}([0,1], \ \widetilde{E}, \ \widetilde{q}_\varepsilon) \qquad\qquad \text{for any } \widetilde{x} \in \widetilde{E}.$$

Sufficient conditions for Lipschitz continuity in positive time direction are not difficult to find. Gronwall's Lemma 1.5.3 leads to the answer that for every $\varepsilon \in \mathcal{J}$, we suppose

$$\wedge \begin{cases} \beta_\varepsilon^{\mathrm{Lip}}(\widetilde{\vartheta}) \ := \ \sup_{\substack{0<t\le 1 \\ \widetilde{x}\in\widetilde{E}}} \ \limsup_{h\downarrow 0} \ \frac{\widetilde{q}_\varepsilon\left(\widetilde{\vartheta}(t-h,\widetilde{x}), \ \widetilde{\vartheta}(t,\widetilde{x})\right)}{h} \ < \ \infty, \\ \qquad\qquad \limsup_{h\downarrow 0} \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}), \ \widetilde{\vartheta}(t+h,\widetilde{x})\Big) \ = \ 0 \qquad \forall \ t \in [0,1[, \ \ \widetilde{x} \in \widetilde{E} \end{cases}$$

as concluded from the next lemma.

Lemma 3.1.2 *For every timed backward transition* $\widetilde{\vartheta}$ *on* $(\widetilde{E},(\widetilde{q}_\varepsilon))$ *and* $\widetilde{x} \in \widetilde{E}$ *with*

$$\wedge \begin{cases} b_\varepsilon \ := \ \sup_{0<t\le 1} \ \limsup_{h\downarrow 0} \ \frac{\widetilde{q}_\varepsilon\left(\widetilde{\vartheta}(t-h,\widetilde{x}), \ \widetilde{\vartheta}(t,\widetilde{x})\right)}{h} \ < \ \infty, \qquad \forall \ \varepsilon \in \mathcal{J} \\ \qquad\quad \limsup_{h\downarrow 0} \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}), \ \widetilde{\vartheta}(t+h,\widetilde{x})\Big) \ = \ 0 \qquad \forall \ \varepsilon \in \mathcal{J}, \ t \in [0,1[\end{cases}$$

the map $\widetilde{\vartheta}(\cdot,\widetilde{x})$ *belongs to* $\mathrm{Lip}^{\rightarrow}([0,1],\widetilde{E},\widetilde{q}_\varepsilon)$, *i.e. for any* $0 \le s < t \le 1$ *and* $\varepsilon \in \mathcal{J}$,

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}), \ \widetilde{\vartheta}(t,\widetilde{x})\Big) \ \le \ b_\varepsilon \cdot (t-s).$$

Proof. For $\varphi_\varepsilon :]s,1] \longrightarrow I\!R, \ t \longmapsto \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}), \ \widetilde{\vartheta}(t,\widetilde{x})\Big)$ (with $\varphi_\varepsilon(s) := 0$) and $t > s$,

$$\varphi_\varepsilon(t+h) \ \le \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}), \ \widetilde{\vartheta}(t,\widetilde{x})\Big) + \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{x}), \ \widetilde{\vartheta}(t+h,\widetilde{x})\Big) \ \longrightarrow \ \varphi_\varepsilon(t) + 0 \quad (h \downarrow 0),$$

$$\varphi_\varepsilon(s+h) \ = \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(s,\widetilde{x}), \ \widetilde{\vartheta}(s+h,\widetilde{x})\Big) \qquad \longrightarrow \ 0 \stackrel{\text{Def.}}{=} \varphi_\varepsilon(s) \quad (h \downarrow 0)$$

and
$$\varphi_\varepsilon(t) - \varphi_\varepsilon(t-h) \ \le \ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{x}), \ \widetilde{\vartheta}(t,\widetilde{x})\Big) \ \le \ b_\varepsilon \cdot h + o(h).$$

So the assertion is a consequence of Gronwall's Lemma 1.5.3. □

As a next step, we consider now the semicontinuity properties of the generalized distance between uniformly continuous $\widetilde{x},\ \widetilde{y}\ :\ [0,T] \longrightarrow \widetilde{E}$. In particular, the upper limits $\widetilde{q}(\widetilde{x}(t),\ \widetilde{y}(t^+)),\quad \widetilde{q}(\widetilde{x}(t^+),\ \widetilde{y}(t^+)),\quad \widetilde{q}(\widetilde{x}(t^+),\ \widetilde{y}(t^{++}))$ are compared with each other and the last one proves to fulfill the assumptions of Gronwall's Lemma 1.5.3.

Proposition 3.1.3 *Suppose the timed triangle inequality for* $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ *and let the functions* $\widetilde{x}, \widetilde{y} : [0,T] \longrightarrow \widetilde{E}$ *satisfy for any* $t \in\,]0,T[,\quad h \in\,]0, T-t[$

$$\widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{x}(t+h)\Big) \ \le\ \omega(h), \qquad \widetilde{q}\Big(\widetilde{y}(t),\ \widetilde{y}(t+h)\Big) \ \le\ \omega(h),$$
$$\pi_1\,\widetilde{x}(\cdot),\ \pi_1\,\widetilde{y}(\cdot)\ \ \textit{nondecreasing}, \qquad \pi_1\,\widetilde{x}(\cdot)\ \le\ \pi_1\,\widetilde{y}(\cdot)$$

with the modulus $\omega(\cdot)$ *of continuity* ($\omega(\cdot)$ *is nondecreasing,* $\omega(h) \longrightarrow 0$ *for* $h \downarrow 0$).

Then $\varphi(t) := \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big) \ \stackrel{\text{Def.}}{=}\ \limsup_{k,l \downarrow 0\ (k<l)} \widetilde{q}\Big(\widetilde{x}(t+k),\ \widetilde{y}(t+l)\Big)$ *fulfills*

$$\begin{cases} \widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{y}(t+h)\Big) \ \le\ \liminf_{k\downarrow 0}\ \varphi(t+k)\ +\ 2\,\omega(h) \ \le\ \varphi(t)\ +\ 2\,\omega(h) \\ \varphi(t)\ \le\ \limsup_{k\downarrow 0}\ \widetilde{q}\Big(\widetilde{x}(t+k),\ \widetilde{y}(t+k)\Big) \ \stackrel{\text{Def.}}{=}\ \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^+)\Big), \\ \varphi(t)\ \ge\ \limsup_{k\downarrow 0}\ \varphi(t+k) \ \stackrel{\text{Def.}}{=}\ \varphi(t^+), \end{cases}$$

for every $t \in [0,T[,\ \ h \in\,]0,T-t[$. *In particular,*

$$\widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{y}(t^+)\Big) \ \le\ \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big) \ \le\ \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^+)\Big).$$

Proof. The timed triangle inequality guarantees for any $0 \le k_1 < k_2 \le l_2 < l_1 < T-t$

$$\begin{aligned} &\widetilde{q}\Big(\widetilde{x}(t+k_1),\ \widetilde{y}(t+l_1)\Big) \\ &\le\ \widetilde{q}\Big(\widetilde{x}(t+k_1),\ \widetilde{x}(t+k_2)\Big) \ +\ \widetilde{q}\Big(\widetilde{x}(t+k_2),\ \widetilde{y}(t+l_2)\Big) \ +\ \widetilde{q}\Big(\widetilde{y}(t+l_2),\ \widetilde{y}(t+l_1)\Big) \\ &\le\ \omega(k_2-k_1) \ +\ \widetilde{q}\Big(\widetilde{x}(t+k_2),\ \widetilde{y}(t+l_2)\Big) \ +\ \omega(l_1-l_2) \\ &\le\ \omega(k_2) \ +\ \widetilde{q}\Big(\widetilde{x}(t+k_2),\ \widetilde{y}(t+l_2)\Big) \ +\ \omega(l_1) \end{aligned}$$

and this implies the first two assertions.
Moreover for every $t \in [0,T[$ and $\eta > 0$, there exists $\rho = \rho(t,\eta) > 0$ such that

$$\widetilde{q}\Big(\widetilde{x}(t+k),\ \widetilde{y}(t+l)\Big) \ <\ \varphi(t)\ +\ \eta \qquad \text{for any } 0 < k < l < \rho.$$

For this reason, $\quad \varphi(t+k) \ \le\ \varphi(t)\ +\ \eta \qquad$ for any $0 < k < \rho$.

□

Finally we obtain an upper estimate of the generalized distance between two points $\widetilde{x}, \widetilde{y} \in \widetilde{E}$ evolving along timed backward transitions $\widetilde{\vartheta},\ \widetilde{\tau}$. Similarly to the forward generalization of chapter 2, it forms the basis for later results about existence and uniqueness of primitives and solutions.

Proposition 3.1.4 *Every timed backward transitions* $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *and initial points* $\widetilde{x}, \widetilde{y} \in \widetilde{E}$ *(with* $\pi_1\,\widetilde{x} \le \pi_1\,\widetilde{y}$*),* $t \in]0,1[$*,* $\varepsilon \in \mathcal{J}$ *satisfy*

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\ \widetilde{\tau}(t^{++}, \widetilde{y})\Big) \le \widetilde{q}_\varepsilon(\widetilde{\vartheta}(0^+, \widetilde{x}), \widetilde{\tau}(0^{++}, \widetilde{y}))\quad e^{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})\,t} + \Big(\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) + 2\,\gamma_\varepsilon(\widetilde{\vartheta})\Big)\ \frac{e^{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})\,t} - 1}{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})}.$$

Proof. The auxiliary function $\varphi_\varepsilon : [0,1] \longrightarrow [0,\infty[\,,\ t \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t^+, \widetilde{x}), \widetilde{\tau}(t^{++}, \widetilde{y}))$ is bounded due to assumption $(BUC^{\rightarrow})$ and Prop. 3.1.3.
Moreover it has the following property for any time $t \in\ ]0,1[$

$$\limsup_{h\downarrow 0}\ \frac{\varphi_\varepsilon(t) - \varphi_\varepsilon(t-h)}{h} \le \alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) \cdot \limsup_{h\downarrow 0} \varphi_\varepsilon(t_h) + \widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) + 2\,\gamma_\varepsilon(\widetilde{\vartheta}).$$

Indeed, for every $0 < k < k' < l' < l$ (and $t_h := t - h \ge 0$)

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+k, \widetilde{x}),\ \widetilde{\tau}(t+l, \widetilde{y})\Big) - \varphi_\varepsilon(t_h)\\
\le\ & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+k,\ \widetilde{x}),\ \widetilde{\vartheta}(h+k',\ \widetilde{\vartheta}(t_h, \widetilde{x}))\Big)\\
&+ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h+k',\ \widetilde{\vartheta}(t_h, \widetilde{x})),\ \widetilde{\vartheta}(h+l',\ \widetilde{\tau}(t_h, \widetilde{y}))\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h^+, \widetilde{x}),\ \widetilde{\tau}(t_h^{++}, \widetilde{y})\Big)\\
&+ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h+l',\ \widetilde{\tau}(t_h, \widetilde{y})),\ \widetilde{\tau}(t+l,\ \widetilde{y})\Big).
\end{aligned}$$

Let $\eta > 0$ be arbitrarily small. Then there exists $h_0 = h_0(\eta, t, \varepsilon) \in\ ]0,\ 1-t[$ such that

- $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\ \widetilde{\vartheta}(h^{++},\ \widetilde{\vartheta}(t_h, \widetilde{x}))\Big) \le (\gamma_\varepsilon(\widetilde{\vartheta}) + \eta) \cdot h$
- $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+,\ \widetilde{\tau}(t_h, \widetilde{y})),\ \widetilde{\tau}(t^{++}, \widetilde{y})\Big) \le (\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) + \eta) \cdot h$
- $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h^+, \widetilde{\vartheta}(t_h, \widetilde{x})),\ \widetilde{\vartheta}(h^{++}, \widetilde{\tau}(t_h, \widetilde{y}))\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h^+, \widetilde{x}),\ \widetilde{\tau}(t_h^{++}, \widetilde{y})\Big) - \gamma_\varepsilon(\widetilde{\vartheta})\,h$
  $\le \Big(\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) + \eta\Big) \cdot h \cdot \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h^+, \widetilde{x}),\ \widetilde{\tau}(t_h^{++}, \widetilde{y})\Big) + \gamma_\varepsilon(\widetilde{\vartheta})\,h\Big)$
  $= \Big(\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) + \eta\Big) \cdot h \cdot \Big(\varphi_\varepsilon(t_h) + \gamma_\varepsilon(\widetilde{\vartheta})\,h\Big)$

for all $h \in\ ]0, h_0]$.
Now for each $h \in]0, h_0]$, there is $\rho = \rho(h, \eta, t, \varepsilon) \in]0, \eta[$ satisfying for every $0 < k < l < \rho$,

- $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+k,\ \widetilde{x}),\ \widetilde{\vartheta}(h+l,\ \widetilde{\vartheta}(t_h,\ \widetilde{x}))\Big) \le (\gamma_\varepsilon(\widetilde{\vartheta}) + 2\,\eta) \cdot h,$
- $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h+k,\ \widetilde{\tau}(t_h, \widetilde{y})),\ \widetilde{\tau}(t+l, \widetilde{y})\Big) \le (\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) + 2\,\eta) \cdot h,$
- $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(h+k,\ \widetilde{\vartheta}(t_h, \widetilde{x})),\ \widetilde{\vartheta}(h+l,\ \widetilde{\tau}(t_h, \widetilde{y}))\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h^+, \widetilde{x}),\ \widetilde{\tau}(t_h^{++}, \widetilde{y})\Big)$
  $\le \Big(\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) + \eta\Big)\ h \cdot \Big(\varphi_\varepsilon(t_h) + \gamma_\varepsilon(\widetilde{\vartheta})\,h\Big) + (\gamma_\varepsilon(\widetilde{\vartheta}) + \eta)\ h.$

As a consequence, we get for all $h \in\ ]0, h_0]$ and $0 < k < l < \rho(h, \eta, t, \varepsilon)$

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t+k, \widetilde{x}),\ \widetilde{\tau}(t+l, \widetilde{y})\Big) - \varphi_\varepsilon(t_h)\\
\le\ & \Big(\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) + \eta\Big)\ h\ \Big(\varphi_\varepsilon(t_h) + \gamma_\varepsilon(\widetilde{\vartheta})\,h\Big) + \Big(\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) + 2\,\gamma_\varepsilon(\widetilde{\vartheta}) + 5\,\eta\Big)\ h.
\end{aligned}$$

So finally for all $h \in \,]0,\ h_0(\eta, t, \varepsilon)]$,

$$\frac{\varphi_\varepsilon(t) - \varphi_\varepsilon(t-h)}{h} \;\leq\; \Big(\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) + \eta\Big) \cdot \varphi_\varepsilon(t_h) \;+\; \widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta})\,(2 + \alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})\,h + \eta\,h) + 5\,\eta.$$

Furthermore, Prop. 3.1.3 guarantees $\varphi_\varepsilon(t) \;\geq\; \limsup_{h \downarrow 0} \varphi_\varepsilon(t+h)$.

Thus the claim results from Gronwall's Lemma 1.5.3.

□

Remark. 1. If $\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}) \;=\; 0$, then the corresponding inequality is

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\ \widetilde{\tau}(t^{++}, \widetilde{y})\Big) \;\leq\; \widetilde{q}_\varepsilon(\widetilde{\vartheta}(0^+, \widetilde{x}), \widetilde{\tau}(0^{++}, \widetilde{y})) \;+\; \Big(\widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) \;+\; 2\,\gamma_\varepsilon(\widetilde{\vartheta})\Big) \cdot t.$$

2. In particular for the same initial points $\widetilde{x} = \widetilde{y}$, we obtain

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t^+, \widetilde{x}),\ \widetilde{\tau}(t^{++}, \widetilde{x})\Big) \;\leq\; \Big(\widetilde{Q}^{\rightarrow}(\widetilde{\vartheta}, \widetilde{\tau}) \;+\; 2\,\gamma_\varepsilon(\widetilde{\vartheta})\Big)\; \frac{e^{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})\,t} - 1}{\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta})}\,.$$

This estimate is the essential conclusion of the additional condition on timed backward transitions in $\widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q})_{\varepsilon \in \mathcal{J}})$, i.e. $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(0^+, \widetilde{x}),\ \widetilde{\tau}(0^{++}, \widetilde{x})\Big) \;=\; 0$ for all $\widetilde{\vartheta}, \widetilde{\tau},\ \widetilde{x},\ \varepsilon$.

3.2 Timed right–hand backward primitives

Now the term of timed backward primitive is specified. For the first time we distinguish between "left–hand" and "right–hand" (with respect to the arguments of $\widetilde{q}_\varepsilon$). It is mainly relevant for estimating the distance between two primitives in Prop. 3.2.3 and it has already appeared in Prop. 2.3.8 of chapter 2 (but there we did not use again).

Backward transitions provide an important example inducing both of their own primitives. For constructing right–hand solutions by means of Euler method in § 3.3, we primarily need the comparison between a backward transition and a right–hand primitive given in Cor. 3.2.4. That is the only important application of left–hand primitives and we are going to restrict ourselves to right–hand solutions in § 3.3 because the corresponding results about left–hand solutions can be concluded from symmetrically adapted arguments. So the expression "left–hand" will not be relevant in later sections.

Definition 3.2.1 $\widetilde{x} : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *is called* timed right–hand backward primitive *of a map* $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$ *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall\ t \in]0,T[\qquad \exists\ \widehat{\gamma}_\varepsilon(t) = \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) : \qquad \gamma_\varepsilon(\widetilde{\vartheta}(t)) \;\le\; \widehat{\gamma}_\varepsilon(t) \;<\; \infty,$
$$\limsup_{h\downarrow 0} \frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h)\,\big(h^+, \widetilde{x}(t-h)\big),\ \widetilde{x}(t^{++})\big)}{h} \;\le\; \widehat{\gamma}_\varepsilon(t), \qquad \limsup_{\varepsilon'\downarrow 0}\ \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) \;=\; 0,$$
2. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$, *i.e. there is* $\omega_\varepsilon(\widetilde{x}, \cdot) :]0,T[\longrightarrow [0,\infty[$ *such that*
$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(s),\ \widetilde{x}(t)\Big) \;\le\; \omega_\varepsilon(\widetilde{x},\ t-s) \quad \text{for } 0 \le s < t < T, \qquad \limsup_{h\downarrow 0}\ \omega_\varepsilon(\widetilde{x}, h) \;=\; 0,$$
3. $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(0^+,\ \widetilde{x}(t)),\ \ \widetilde{x}(t^{++})\Big) \;=\; 0 \qquad \forall\ t \in [0,T[,$
4. $\pi_1\,\widetilde{x}(t) \;=\; t \;+\; \pi_1\,\widetilde{x}(0) \qquad \forall\ t \in [0,T[.$

$\widetilde{x} : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *is called* timed left–hand backward primitive *of* $\widetilde{\vartheta}(\cdot)$
if it satisfies conditions (*2.*), (*3.*), (*4.*) *and*

1'. $\forall\ t \in]0,T[\qquad \exists\ \widehat{\gamma}_\varepsilon(t) = \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{\vartheta}(\cdot)) : \qquad \gamma_\varepsilon(\widetilde{\vartheta}(t)) \;\le\; \widehat{\gamma}_\varepsilon(t) \;<\; \infty,$
$$\limsup_{h\downarrow 0} \frac{\widetilde{q}_\varepsilon\big(\widetilde{x}(t^+),\ \ \widetilde{\vartheta}(t-h)\,\big(h^{++}, \widetilde{x}(t-h)\big)\big)}{h} \;\le\; \widehat{\gamma}_\varepsilon(t), \qquad \limsup_{\varepsilon'\downarrow 0}\ \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) \;=\; 0,$$

Remark. Let $\widetilde{x}(\cdot)$ be a timed right–hand backward primitive of the function $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$. For any $t \in]0,T[$, the shifted map $\widetilde{x}(t+\cdot) : [0, T-t[\longrightarrow \widetilde{E}$ is a timed right–hand backward primitive of $\widetilde{\vartheta}(t+\cdot)$. (The corresponding statements are also obvious for timed left–hand backward primitives.)

Lemma 3.2.2 *For every* $\widetilde{\vartheta}_1, \widetilde{\vartheta}_2 \in \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *and* $t_0 \in]0,1[$, $\widetilde{z} \in \widetilde{E}$, *the function* $\widetilde{x} : [0,1[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$

$$\widetilde{x}(t) := \begin{cases} \widetilde{\vartheta}_1(t, \widetilde{z}) & \text{for } 0 \le t \le t_0 \\ \widetilde{\vartheta}_2(t - t_0, \ \widetilde{\vartheta}_1(t_0, \widetilde{z})) & \text{for } t_0 < t \le 1 \end{cases}$$

is both timed left–hand and timed right–hand backward primitive of $\widetilde{\vartheta} : [0,1[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$,

$$\widetilde{\vartheta}(t) := \begin{cases} \widetilde{\vartheta}_1 & \text{for } 0 \le t < t_0 \\ \widetilde{\vartheta}_2 & \text{for } t_0 \le t \le 1 \end{cases}.$$

Proof. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,1[, \widetilde{E}, \widetilde{q}_\varepsilon)$ is an immediate consequence of the timed triangle inequality. Moreover conditions (3.), (4.) results from properties (5.), (6.) of timed backward transitions (in Def. 3.1.1). Conditions (1.), (1'.) on backward primitives are trivial for all $t \neq t_0$.

At time $t = t_0$, we obtain for any $h \in]0, t_0[$ and $0 < k < k' < l' < l$ (with $t_h := t_0 - h$)

$$\begin{aligned} & \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_0 + k), \ \widetilde{\vartheta}(t_0 - h)\,\big(h + l, \ \widetilde{x}(t_0 - h)\big)\Big) \\ = \quad & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_2(k, \ \widetilde{\vartheta}_1(t_0, \widetilde{z})), \quad \widetilde{\vartheta}_1(h + l, \ \widetilde{\vartheta}_1(t_0 - h, \widetilde{z}))\Big) \\ \le \quad & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_2(k, \ \widetilde{\vartheta}_1(t_0, \widetilde{z})), \quad \widetilde{\vartheta}_1(k', \ \widetilde{\vartheta}_1(t_0, \widetilde{z}))\Big) \\ + \ & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_1(k', \widetilde{\vartheta}_1(t_0, \widetilde{z})), \quad \widetilde{\vartheta}_1(t_0 + l', \ \widetilde{z})\Big) \\ + \ & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_1(t_0 + l', \ \widetilde{z}), \quad \widetilde{\vartheta}_1(h + l, \ \widetilde{\vartheta}_1(t_0 - h, \widetilde{z}))\Big). \end{aligned}$$

The additional condition on timed backward transitions in $\widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ (Def. 3.1.1) states that the first term converges to 0 for $k, k' \longrightarrow 0$ $(0 < k < k')$.

Moreover, conditions (2.), (5.) on timed backward transitions (in Definition 3.1.1) state

$$\begin{aligned} \limsup_{h\downarrow 0} \ \tfrac{1}{h} \cdot \limsup_{l,l'\downarrow 0 \ (l'<l)} \ & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_1(t_0 + l', \ \widetilde{z}), \quad \widetilde{\vartheta}_1(h + l, \ \widetilde{\vartheta}_1(t_0 - h, \widetilde{z}))\Big) \le \gamma_\varepsilon(\widetilde{\vartheta}_1) \\ \limsup_{k',l'\downarrow 0 \ (k'<l')} \ & \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}_1(k', \ \widetilde{\vartheta}_1(t_0, \widetilde{z})), \ \widetilde{\vartheta}_1(t_0 + l', \ \widetilde{z})\Big) = 0. \end{aligned}$$

So $\widetilde{x}(\cdot)$ fulfills condition (1'.) on left–hand backward primitives at $t = t_0$, i.e.

$$\limsup_{h\downarrow 0} \ \tfrac{1}{h} \cdot \limsup_{l,l'\downarrow 0 \ (l'<l)} \ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_0 + k), \quad \widetilde{\vartheta}(t_0 - h)\,\big(h + l, \ \widetilde{x}(t_0 - h)\big)\Big) \le \gamma_\varepsilon(\vartheta_1).$$

The same reasons ensure also cond. (1.) on *right–hand* backward primitives at $t = t_0$. □

Remark. 1. As a consequence of the proof, the parameter $\widehat{\gamma}_\varepsilon(t) = \widehat{\gamma}_\varepsilon(t, \widetilde{x}, \widetilde{\vartheta})$ can be chosen as maximum of $\gamma_\varepsilon(\widetilde{\vartheta}_1)$ and $\gamma_\varepsilon(\widetilde{\vartheta}_2)$.

2. Supposing the moduli $\beta_\varepsilon(\vartheta_1)(\cdot)$, $\beta_\varepsilon(\vartheta_2)(\cdot)$ of continuity to be also convex, their pointwise maximum $h \longmapsto \max\{\beta_\varepsilon(\vartheta_1)(h), \ \beta_\varepsilon(\vartheta_2)(h)\}$ is also convex and provides a modulus of continuity for $\widetilde{x}(\cdot)$ due to the timed triangle inequality.

Condition (5.) on backward transitions (Def. 3.1.1) is used only in this proof explicitly.

Proposition 3.2.3
Let $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *be a timed left–hand primitive of* $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$ *and* $\widetilde{y} : [0,T[\longrightarrow \widetilde{E}$ *a timed right–hand primitive of* $\widetilde{\tau} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$ *such that for each* $\varepsilon \in \mathcal{J}$,

$$\wedge \begin{cases} \alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(\cdot)) & \leq & M_\varepsilon(\cdot) & \in & C^0([0,T[,\ ]0,\infty[), \\ \widehat{\gamma}_\varepsilon(\,\cdot\,,\widetilde{x},\widetilde{\vartheta}),\ \widehat{\gamma}_\varepsilon(\,\cdot\,,\widetilde{y},\widetilde{\tau}) & \leq & R_\varepsilon(\cdot) & \in & C^0([0,T[,\ [0,\infty[), \\ \widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(\cdot),\ \widetilde{\tau}(\cdot)) & \leq & c_\varepsilon(\cdot) & \in & C^0([0,T[,\ [0,\infty[), \\ \pi_1\,\widetilde{x}(0) & = & \pi_1\,\widetilde{y}(0). \end{cases}$$

Moreover, set $\mu_\varepsilon(t) := \displaystyle\int_0^t M_\varepsilon(s)\, ds.$
Then, for every $\varepsilon \in \mathcal{J}$ *and* $t \in\,]0,T[$, *these backward primitives fulfill the estimate*

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big) \;\leq\; \widetilde{q}_\varepsilon\Big(\widetilde{x}(0^+),\ \widetilde{y}(0^{++})\Big)\ e^{\mu_\varepsilon(t)} \ + \\ +\ \int_0^t e^{\mu_\varepsilon(t)-\mu_\varepsilon(s)}\ \Big(c_\varepsilon(s)\ +\ 5\,R_\varepsilon(s)\Big)\ ds.$$

Proof. Correspondingly to the proof of Prop. 3.1.4, we consider the bounded auxiliary function $\varphi_\varepsilon : [0,T[\longrightarrow I\!R,\quad t \longmapsto \widetilde{q}_\varepsilon(\widetilde{x}(t^+),\ \widetilde{y}(t^{++}))$ and prove for any $t \in\,]0,T[$

$$\limsup_{h\downarrow 0} \tfrac{\varphi_\varepsilon(t)\,-\,\varphi_\varepsilon(t-h)}{h} \;\leq\; M_\varepsilon(t_h)\,\cdot\,\limsup_{h\downarrow 0} \varphi_\varepsilon(t_h) \ +\ c_\varepsilon(t)\ +\ 5\,R_\varepsilon(t).$$

Since Proposition 3.1.3 guarantees $\varphi_\varepsilon(t) \,\geq\, \varphi_\varepsilon(t^+)$ for every $t \in [0,T[$, the claimed estimate results from Gronwall's Lemma 1.5.3.

For any $0 < k_1 < k_2 < k_3 < k_4 < k_5$ (and $t_h := t-h \geq 0$), we conclude from the timed triangle inequality

$$\begin{aligned} \widetilde{q}_\varepsilon\Big(\widetilde{x}(t+k_1),\ \widetilde{y}(t+k_5)\Big) \;\leq\;& \widetilde{q}_\varepsilon\Big(\,\widetilde{x}(t+k_1),\qquad \widetilde{\vartheta}(t_h)\,\Big(h+k_2,\ \widetilde{x}(t_h)\Big)\Big) \\ &+\ \widetilde{q}_\varepsilon\Big(\,\widetilde{\vartheta}(t_h)\,\Big(h+k_2,\ \widetilde{x}(t_h)\Big),\ \widetilde{\vartheta}(t_h)\,\Big(h+k_3,\ \widetilde{y}(t_h)\Big)\Big) \\ &+\ \widetilde{q}_\varepsilon\Big(\,\widetilde{\vartheta}(t_h)\,\Big(h+k_3,\ \widetilde{y}(t_h)\Big),\ \widetilde{\tau}(t_h)\,\Big(h+k_4,\ \widetilde{y}(t_h)\Big)\Big) \\ &+\ \widetilde{q}_\varepsilon\Big(\,\widetilde{\tau}(t_h)\,\Big(h+k_4,\ \widetilde{y}(t_h)\Big),\ \widetilde{y}(t+k_5)\Big). \end{aligned}$$

Let $\eta > 0$ be chosen arbitrarily. Then there exists $h_0 = h_0(\eta,t,\varepsilon) \in\,]0,T-t[$ such that

- $\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \ \widetilde{\vartheta}(t_h)\,\Big(h^{++},\ \widetilde{x}(t_h)\Big)\Big) \;\leq\; (\widehat{\gamma}_\varepsilon(t,\widetilde{x},\widetilde{\vartheta})\ +\eta)\ \cdot\ h$
- $\widetilde{q}_\varepsilon\Big(\widetilde{\tau}(t_h)\,\Big(h^+,\ \widetilde{y}(t_h)\Big),\ \ \widetilde{y}(t^{++})\Big) \;\leq\; (\widehat{\gamma}_\varepsilon(t,\widetilde{y},\widetilde{\tau})\ +\eta)\ \cdot\ h$
- $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)\,(h^+,\ \widetilde{x}(t_h)),\ \ \widetilde{\vartheta}(t_h)\,(h^{++},\widetilde{y}(t_h))\Big) \;-\; \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_h^+),\ \widetilde{y}(t_h^{++})\Big) \;-\; \gamma_\varepsilon(\widetilde{\vartheta}(t_h))\,h$
  $\leq\; \Big(\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(t_h))\ +\ \eta\Big)\ \cdot\ h\ \cdot\ \Big(\widetilde{q}_\varepsilon\Big(\widetilde{x}(t_h^+),\ \ \widetilde{y}(t_h^{++})\Big)\ +\ \gamma_\varepsilon(\widetilde{\vartheta}(t_h))\,h\Big)$
  $=\; \Big(\alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(t_h))\ +\ \eta\Big)\ \cdot\ h\ \cdot\ \Big(\varphi_\varepsilon(t_h)\qquad +\ \gamma_\varepsilon(\widetilde{\vartheta}(t_h))\,h\Big)$

for all $h \in\,]0,h_0]$.

Now for each $h \in]0, h_0]$, we get some $\rho = \rho(h, \eta, t, \varepsilon) \in]0, \eta[$ satisfying for all $0 < k < l < \rho$

$$\circ \quad \widetilde{q}_\varepsilon\Big(\widetilde{x}(t+k), \quad \widetilde{\vartheta}(t_h)\Big(h+l,\ \widetilde{x}(t_h)\Big)\Big) \leq (\widehat{\gamma}_\varepsilon(t, \widetilde{x}, \widetilde{\vartheta}) + 2\,\eta) \cdot h$$

$$\circ \quad \widetilde{q}_\varepsilon\Big(\widetilde{\tau}(t_h)\Big(h+k,\ \widetilde{y}(t_h)\Big), \quad \widetilde{y}(t+l)\Big) \leq (\widehat{\gamma}_\varepsilon(t, \widetilde{y}, \widetilde{\tau}) + 2\,\eta) \cdot h$$

$$\circ \quad \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)\Big(h+k,\ \widetilde{x}(t_h)\Big), \quad \widetilde{\vartheta}(t_h)\Big(h+l,\ \widetilde{y}(t_h)\Big)\Big)$$
$$\leq \Big(\alpha_\varepsilon^{\rightarrow\!\!\!\shortmid}(\widetilde{\vartheta}(t_h)) + \eta\Big)\ h \quad \Big(\varphi_\varepsilon(t_h) + \gamma_\varepsilon(\widetilde{\vartheta}(t_h))\, h\Big) + \varphi_\varepsilon(t_h) + (\gamma_\varepsilon(\widetilde{\vartheta}(t_h)) + \eta)\, h$$

$$\circ \quad \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)\Big(h+k,\ \widetilde{y}(t_h)\Big), \quad \widetilde{\tau}(t_h)\Big(h+l,\ \widetilde{y}(t_h)\Big)\Big)$$
$$\leq \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)\Big(h^+,\ \widetilde{y}(t_h)\Big), \quad \widetilde{\tau}(t_h)\Big(h^{++},\ \widetilde{y}(t_h)\Big)\Big) + \eta\ h.$$

As a consequence, all $h \in]0, h_0]$ and $0 < k_1 < k_2 < k_3 < k_4 < k_5 < \rho(h, \eta, t, \varepsilon)$ fulfill

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{x}(t+k_1),\ \widetilde{y}(t+k_5)\Big) \\
&\quad \leq (\widehat{\gamma}_\varepsilon(t, \widetilde{x}, \widetilde{\vartheta}) + 2\,\eta)\ h \\
&\quad\quad + \Big(\alpha_\varepsilon^{\rightarrow\!\!\!\shortmid}(\widetilde{\vartheta}(t_h)) + \eta\Big)\ h\ \Big(\varphi_\varepsilon(t_h) + \gamma_\varepsilon(\widetilde{\vartheta}(t_h))\, h\Big) + \varphi_\varepsilon(t_h) + (\gamma_\varepsilon(\widetilde{\vartheta}(t_h)) + \eta)\ h \\
&\quad\quad + \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)\Big(h^+,\ \widetilde{y}(t_h)\Big), \quad \widetilde{\tau}(t_h)\Big(h^{++},\ \widetilde{y}(t_h)\Big)\Big) + \eta\ h \\
&\quad\quad + (\widehat{\gamma}_\varepsilon(t, \widetilde{y}, \widetilde{\tau}) + 2\,\eta)\ h \\
&\leq \Big(2\,R_\varepsilon(t) + 6\,\eta\Big)\ h \\
&\quad\quad + \Big(M_\varepsilon(t_h)) + \eta\Big)\ h\ \ \varphi_\varepsilon(t_h) + \varphi_\varepsilon(t_h) + ((M_\varepsilon(t_h) + \eta)\, h + 1)\ R_\varepsilon(t_h)\, h \\
&\quad\quad + \Big(c_\varepsilon(t_h) + 2\,R_\varepsilon(t_h)\Big)\ \tfrac{1}{M_\varepsilon(t-h)}\ (e^{M_\varepsilon(t-h)\cdot h} - 1)
\end{aligned}$$

due to Proposition 3.1.4. So finally the continuity of $M_\varepsilon(\cdot)$, $R_\varepsilon(\cdot)$, $c_\varepsilon(\cdot)$ implies

$$\limsup_{h \downarrow 0}\ \tfrac{\varphi_\varepsilon(t) - \varphi_\varepsilon(t-h)}{h} \leq M_\varepsilon(t) \cdot \limsup_{h \downarrow 0} \varphi_\varepsilon(t_h) + c_\varepsilon(t) + 5\,R_\varepsilon(t). \qquad \square$$

Corollary 3.2.4 *Let* $\widetilde{y} : [0, T] \longrightarrow \widetilde{E}$ *be a timed right–hand backward primitive of* $\widetilde{\vartheta} : [0, T] \longrightarrow \widetilde{\Theta}_p^{\rightarrow\!\!\!\shortmid}(\widetilde{E}, (\widetilde{q}_\varepsilon))$ *and* $t, h, k > 0$ *such that* $t_h := t - h \geq 0$, $t + k < T$,

$$\widetilde{Q}_\varepsilon^{\rightarrow\!\!\!\shortmid}\Big(\widetilde{\vartheta}(t_h),\ \widetilde{\vartheta}(\cdot)\Big) \leq c_\varepsilon(\cdot) \in C^0([t_h, t+k]),$$
$$\widehat{\gamma}_\varepsilon(\,\cdot\,, \widetilde{y}, \widetilde{\vartheta}) \leq R_\varepsilon(\cdot) \in C^0([t_h, t+k]).$$

Then,

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)\Big(h + k^+,\ \widetilde{y}(t_h)\Big),\ \widetilde{y}(t + k^{++})\Big) \leq$$
$$\leq \int_{t-h}^{t+k} e^{\alpha_\varepsilon^{\rightarrow\!\!\!\shortmid}(\widetilde{\vartheta}(t_h))\cdot(t+k-s)}\ \Big(c_\varepsilon(s) + 5\,R_\varepsilon(s)\Big)\ ds.$$

Proof results from Prop. 3.2.3 and condition (3.) on right–hand primitives (Def. 3.2.1), i.e. $\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)(0^+,\ \widetilde{y}(t_h)),\ \widetilde{y}(t_h^{++})\Big) \overset{\text{Def.}}{=} \limsup_{\substack{k,l \downarrow 0 \\ k<l}}\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t_h)\,(k,\ \widetilde{y}(t_h)),\ \widetilde{y}(t_h + l)\Big) = 0 \quad \square$

3.3 Timed right–hand backward solutions

3.3.1 Definition and convergence theorem

Now we follow the same track as in chapter 2 about the timed forward generalization : The definition of "right–hand primitive" specifies the notion of a "right–hand solution". We do not extend this concept to "left–hand solutions" explicitly because the following conclusions can be repeated easily after permuting the arguments of $\widetilde{q}_\varepsilon$. So for the sake of simplicity, the expressions "right–hand", "timed" are sometimes left out in this section.

Definition 3.3.1

For given $\widetilde{f} : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon))$, *a map* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand backward solution *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand backward primitive of* $\widetilde{f}(\widetilde{x}(\cdot),\cdot) : [0,T[\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, *i.e. for each* $\varepsilon \in \mathcal{J}$,

1. $\forall\ t \in]0,T[\quad \exists\, \widehat{\gamma}_\varepsilon(t) : \quad \gamma_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)) \le \widehat{\gamma}_\varepsilon(t) < \infty, \quad \limsup_{\varepsilon'\downarrow 0} \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) = 0,$
   $\limsup_{h\downarrow 0} \ \frac{1}{h} \cdot \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t-h),\, t-h)\ (h^+,\ \widetilde{x}(t-h)),\quad \widetilde{x}(t^{++})\Big) \le \widehat{\gamma}_\varepsilon(t),$

2. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,T[,\widetilde{E},\widetilde{q}_\varepsilon)$, *i.e. there is* $\omega_\varepsilon(\widetilde{x},\cdot) :]0,T[\longrightarrow [0,\infty[$ *such that*
   $\widetilde{q}_\varepsilon\Big(\widetilde{x}(s),\ \widetilde{x}(t)\Big) \le \omega_\varepsilon(\widetilde{x},\ t-s)$ *for* $0 \le s < t < T$, $\quad \limsup_{h\downarrow 0} \ \omega_\varepsilon(\widetilde{x},h) = 0,$

3. $\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t),t)\,(0^+,\ \widetilde{x}(t)),\ \ \widetilde{x}(t^{++})\Big) = 0 \qquad \forall\ t \in [0,T[,$

4. $\pi_1\, \widetilde{x}(t) = t + \pi_1\, \widetilde{x}(0) \qquad \forall\ t \in [0,T[.$

For constructing solutions by Euler method later, a form of convergence is required that preserves the property of solving a mutational equation. The next proposition shows that we can use the same notion as for timed right–hand *forward* solutions (see Convergence Theorem 2.3.2 in § 2.3). Here it is described in assumptions (5.ii), (5.iii) and can be subsumed under the generic term "two–sided graphically convergent".
As an essential advantage of this similarity, the same type of compactness is useful for constructing backward solutions in § 3.3.2, namely timed transitional compactness of $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E},(\widetilde{q}_\varepsilon)))$.

Proposition 3.3.2 (Convergence Theorem)

For each $\varepsilon \in \mathcal{J}$, *suppose the following properties of*

$$\begin{array}{llcll} \widetilde{f}_m, \ \widetilde{f}: & \widetilde{E} \times [0,T[& \longrightarrow & \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}) & (m \in I\!N) \\ \widetilde{x}_m, \ \widetilde{x}: & [0,T[& \longrightarrow & \widetilde{E} \ : & \end{array}$$

1. $M_\varepsilon := \sup_{m,t,\widetilde{z}} \{ \alpha_\varepsilon^{\rightarrow}(\widetilde{f}_m(\widetilde{z},t)), \ \alpha_\varepsilon^{\rightarrow}(\widetilde{f}(\widetilde{z},t)) \} < \infty,$

   $c_\varepsilon(h) := \sup_{m,t,\widetilde{z}} \beta_\varepsilon(\widetilde{f}_m(\widetilde{z},t))(h) < \infty,$ *for every* $h \in]0,T[$,

   $R_\varepsilon \geq \sup_{m,t,\widetilde{z}} \{ \widehat{\gamma}_\varepsilon(t, \widetilde{x}_m, \widetilde{f}_m(\widetilde{x}_m, \cdot)), \ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{z},t)), \ \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \}$

   with $\limsup_{\varepsilon' \downarrow 0} \varepsilon'^p \cdot R_{\varepsilon'} = 0, \qquad \limsup_{h \downarrow 0} c_\varepsilon(h) = 0,$

2. $\limsup \ \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}_m(\widetilde{z}_1,t_1), \ \widetilde{f}_m(\widetilde{z}_2,t_2)\Big) \leq R_\varepsilon$ *for* $m \to \infty, \quad t_2 - t_1 \downarrow 0,$

   $\widetilde{q}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2) \to 0 \quad (\pi_1 \widetilde{z}_1 \leq \pi_1 \widetilde{z}_2),$

3. $\overset{\circ}{\widetilde{x}}_m(\cdot) \ni \widetilde{f}_m(\widetilde{x}_m(\cdot), \cdot)$ *in* $[0,T[$.

4. $\widehat{\omega}_\varepsilon(h) := \sup_m \omega_\varepsilon(\widetilde{x}_m, h) < \infty$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*)* $\forall \ h \in]0,T]$,

   $\limsup_{h \downarrow 0} \widehat{\omega}_\varepsilon(h) = 0,$

5. $\forall \ t_1, t_2, t_3 \in]0,T[\quad \exists \ (m_j)_{j \in I\!N}$ *with* $m_j \nearrow \infty$ *and*

   (i) $\limsup \ \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}(t_1),t_1), \ \widetilde{f}_{m_j}(\widetilde{x}(t_1),t_1)\Big) \leq R_\varepsilon \qquad (j \longrightarrow \infty)$

   (ii) $\exists \ (\delta_j)_{j \in I\!N}$ *in* $[0,t_2[$: $\widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_2 - \delta_j), \ \widetilde{x}(t_2)\Big) \longrightarrow 0, \quad \delta_j \longrightarrow 0,$

   $\pi_1 \widetilde{x}_{m_j}(t_2 - \delta_j) \leq \pi_1 \widetilde{x}(t_2),$

   (iii) $\exists \ (\delta'_j)_{j \in I\!N}$ *in* $[0,1[$: $\widetilde{q}_\varepsilon\Big(\widetilde{x}(t_3), \ \widetilde{x}_{m_j}(t_3 + \delta'_j)\Big) \longrightarrow 0, \quad \delta'_j \longrightarrow 0,$

   $\pi_1 \widetilde{x}(t_3) \leq \pi_1 \widetilde{x}_{m_j}(t_3 + \delta'_j).$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand backward solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$.

Proof. The claimed uniform continuity of $\widetilde{x}(\cdot)$ results from assumption (4.) : Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon\Big(\widetilde{x}_m(t_1), \ \widetilde{x}_m(t_2)\Big) \leq \widehat{\omega}_\varepsilon(t_2 - t_1)$ for $0 \leq t_1 < t_2 < T$. Let $\varepsilon \in \mathcal{J}$, $0 \leq t_1 < t_2 < T$ be arbitrary and choose $(\delta'_j)_{j \in I\!N}$, $(\delta_j)_{j \in I\!N}$ for t_1, t_2 according to cond. (5.ii), (5.iii). For all $j \in I\!N$ sufficiently large, $t_1 + \delta'_j < t_2 - \delta_j$ and as a consequence,

$$\begin{aligned} & \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_1), \ \widetilde{x}(t_2)\Big) \\ \leq \ & \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_1), \ \widetilde{x}_{m_j}(t_1 + \delta'_j)\Big) + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_1 + \delta'_j), \ \widetilde{x}_{m_j}(t_2 - \delta_j)\Big) + \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t_2 - \delta_j), \ \widetilde{x}(t_2)\Big) \\ \leq \ & \qquad\qquad \widehat{\omega}_\varepsilon(t_2 - t_1) \qquad\qquad + \ o(1) \qquad \text{for } j \longrightarrow \infty. \end{aligned}$$

Now for fixed $\varepsilon \in \mathcal{J}$ and any $\eta > 0$, we prove the existence of $h_0 = h_0(\varepsilon, \eta) \in]0, \eta[$ such that for every $t \in]0, T[$, $0 < k < l < h < h_0$ (with $t_h := t - h \geq 0$)

$$\begin{aligned} \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_h), t_h)\Big(h{+}k, \widetilde{x}(t_h)\Big),\ \widetilde{x}(t{+}l)\Big) &\leq 15\ e^{M_\varepsilon\, 2\eta}\,(\eta{+}R_\varepsilon)\ (h{+}l) + 2\,\widehat{\omega}_\varepsilon(l){+}3\,c_\varepsilon(l) \\ \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_h), t_h)\Big(h^+,\ \ \widetilde{x}(t_h)\Big),\ \widetilde{x}(t^{++})\Big) &\leq 15\ e^{M_\varepsilon\, 2\eta}\,(\eta{+}R_\varepsilon)\ \ h. \end{aligned}$$

Thus, $\widetilde{x}(\cdot)$ holds condition (1.) on backward solutions (in Def. 3.3.1), i.e.

$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_h), t_h)\Big(h^+,\ \widetilde{x}(t_h)\Big),\ \ \widetilde{x}(t^{++})\Big)\ \leq\ 15\, R_\varepsilon.$$

Furthermore we conclude condition (3.) from the preceding estimates, i.e.

$$\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(s), s)\,(0^+,\ \widetilde{x}(s)),\ \ \widetilde{x}(s^{++})\Big)\ =\ 0$$

Indeed, h_0 does not depend on t. So we obtain for any $s \in [0, T[$, $0 < h' < h'' < h_0$

$$\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(s),\ s)\,\Big(h',\ \widetilde{x}(s)\Big),\ \ \widetilde{x}(s+h'')\Big)\ \leq\ \mathrm{const}(M_\varepsilon, R_\varepsilon)\ \cdot\ (h'' + \widehat{\omega}_\varepsilon(h'') + c_\varepsilon(h'')).$$

For every $\eta \in]0, 1]$, assumption (2.) provides $h_0 = h_0(\varepsilon, \eta) \in]0, \eta[$, $N_0 = N_0(\varepsilon, \eta) \in I\!N$ such that all elements $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$, $t_1, t_2 \in [0, T[$, $m \in I\!N$ with

$$\widetilde{q}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2)\ \leq\ \widehat{\omega}_\varepsilon(2\,h_0), \qquad \pi_1\,\widetilde{z}_1 \leq \pi_1\,\widetilde{z}_2, \qquad 0\ <\ t_2 - t_1\ \leq\ 2\,h_0, \qquad m\ \geq\ N_0$$

satisfy
$$\widetilde{Q}_\varepsilon^{\rightarrow\!\!|}\Big(\widetilde{f}_m(\widetilde{z}_1, t_1),\ \widetilde{f}_m(\widetilde{z}_2, t_2)\Big)\ \leq\ R_\varepsilon + \eta.$$

Choose $t \in]0, T[$ and $0 < k < k' < l' < l < h < h_0$ arbitrarily with $t_h := t{-}h \geq 0$. Considering subsequences (that depend on $t{-}h$, t, $t+l$ and ε), assumptions (5.i)–(5.iii) guarantee the existence of sequences $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta'_j \downarrow 0$ such that

$$\wedge\left\{\begin{array}{rcll} \widetilde{q}_\varepsilon\Big(\ \widetilde{x}_{m_j}(t{+}l{-}\delta_j),\ \ \widetilde{x}(t{+}l)\Big) & \longrightarrow & 0, & \text{for } j \longrightarrow \infty, \\ \widetilde{q}_\varepsilon\Big(\ \widetilde{x}(t_h), \qquad\qquad \widetilde{x}_{m_j}(t_h{+}\delta'_j)\Big) & \longrightarrow & 0 & \text{for } j \longrightarrow \infty, \\ \limsup\limits_{j\longrightarrow\infty}\ \widetilde{Q}_\varepsilon^{\rightarrow\!\!|}\Big(\ \widetilde{f}(\widetilde{x}(t_h), t_h), \quad \widetilde{f}_{m_j}(\widetilde{x}(t_h), t_h)\Big) & \leq & R_\varepsilon. & \end{array}\right.$$

So there is an index $N_1 = N_1(\varepsilon, \eta, t, h, k, k', l, l')\ \geq\ N_0$ such that all $j \geq N_1$ satisfy

$$\wedge\left\{\begin{array}{rcl} \delta'_j + l'\ \ <\ \ l - \delta_j\ \ <\ \ l & < & h_0, \\ \widetilde{q}_\varepsilon\ \Big(\widetilde{x}_{m_j}(t{+}l{-}\delta_j),\ \ \widetilde{x}(t+l)\Big) & < & \min\{\eta\, h,\ \ \widehat{\omega}_\varepsilon(h)\}, \\ \widetilde{q}_\varepsilon\ \Big(\widetilde{x}(t_h), \qquad\quad \widetilde{x}_{m_j}(t_h + \delta'_j)\Big) & < & \min\{\eta\, h,\ \ \widehat{\omega}_\varepsilon(h)\}, \\ \widetilde{Q}_\varepsilon^{\rightarrow\!\!|}\Big(\widetilde{f}(\widetilde{x}(t_h), t_h), \quad \widetilde{f}_{m_j}(\widetilde{x}(t_h), t_h)\Big) & < & R_\varepsilon + \eta. \end{array}\right.$$

Now Proposition 3.1.3 and Corollary 3.2.4 imply for $m \geq N_1$ (with $t_{h,j} := t - h + \delta'_j$)

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_h+\delta'_j),\ t_h+\delta'_j)\ \Big(h+k',\ \widetilde{x}_{m_j}(t_h+\delta'_j)\Big),\quad \widetilde{x}(t+l)\Big)\\
\leq\ & \widetilde{q}_\varepsilon\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_{h,j}), t_{h,j})\ \Big(h+k',\ \widetilde{x}_{m_j}(t_{h,j})\Big),\quad \widetilde{x}_{m_j}(t+\delta'_j+l')\Big)\\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+\delta'_j+l'),\ \widetilde{x}_{m_j}(t+l-\delta_j)\Big)\ +\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+l-\delta_j),\ \widetilde{x}(t+l)\Big)\\
\leq\ & \widetilde{q}_\varepsilon\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_{h,j}), t_{h,j})\ \Big((h+k')^+,\ \widetilde{x}_{m_j}(t_{h,j})\Big), \widetilde{x}_{m_j}((t+\delta'_j+k')^{++})\Big) + \widehat{\omega}_\varepsilon(l'-k')\\
&+\ \widehat{\omega}_\varepsilon(l-\delta_j\ -\ l'-\delta'_j)\qquad\qquad +\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_{m_j}(t+l-\delta_j),\ \widetilde{x}(t+l)\Big)\\
\leq\ & \int_0^{h+k'} e^{M_\varepsilon\,(h+k'-s)}\ \Big(R_\varepsilon+\eta\ +\ 5\,R_\varepsilon\Big)\ ds\ +\ 2\,\widehat{\omega}_\varepsilon(l)\ +\ \eta\, h\\
\leq\ & \tfrac{e^{M_\varepsilon\,(h+l)}-1}{M_\varepsilon}\quad \Big(\eta\ +\ 6\,R_\varepsilon\Big)\ +\ 2\,\widehat{\omega}_\varepsilon(l)\ +\ \eta\, h\\
\leq\ & (h+l)\ e^{M_\varepsilon\,2\eta}\ \cdot\ 6\ \Big(\eta+\ R_\varepsilon\Big)\ +\ 2\,\widehat{\omega}_\varepsilon(l).
\end{aligned}$$

In this situation we also obtain for any $k_1 < k_2$ (with $0 < k < k_1 < k_2 < k'$)

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_h), t_h)\ \Big(h+k,\ \widetilde{x}(t_h)\Big),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_{h,j}), t_{h,j})\ \Big(h+k',\ \widetilde{x}_{m_j}(t_{h,j})\Big)\Big)\\
\leq\ & \widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_h), t_h)\Big(h+k,\ \widetilde{x}(t_h)\Big),\ \widetilde{f}_{m_j}(\widetilde{x}(t_h),\ t_h)\Big(h+k_1, \widetilde{x}(t_h)\Big)\Big)\\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{f}_{m_j}(\widetilde{x}(t_h), t_h)\Big(h+k_1,\ \widetilde{x}(t_h)\Big),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_{h,j}), t_{h,j})\Big(h+k_2,\ \widetilde{x}(t_h)\Big)\Big)\\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_{h,j}), t_{h,j})\Big(h+k_2,\ \widetilde{x}(t_h)\Big),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_{h,j}), t_{h,j})\Big(h+k',\ \widetilde{x}_{m_j}(t_{h,j})\Big)\Big)\\
\leq\ & \Big(\widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}(t_h), t_h),\ \widetilde{f}_{m_j}(\widetilde{x}(t_h), t_h)\Big)\ +2\,R_\varepsilon\Big)\ \tfrac{e^{M_\varepsilon\,(h+k)}-1}{M_\varepsilon}\ +\ c_\varepsilon(k_1-k)\\
&+\Big(\widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}_{m_j}(\widetilde{x}(t_h), t_h),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(t_{h,j}), t_{h,j})\Big)\ +2\,R_\varepsilon\Big)\ \tfrac{e^{M_\varepsilon\,(h+k_1)}-1}{M_\varepsilon}\ +\ c_\varepsilon(k_2-k_1)\\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_h),\ \widetilde{x}_{m_j}(t_h+\delta'_j)\Big)\ e^{M_\varepsilon\,(h+k_2)}\ +3\,R_\varepsilon\ \tfrac{e^{M_\varepsilon\,(h+k_2)}-1}{M_\varepsilon}\ +\ c_\varepsilon(k'-k_2)\\
\leq\ & \Big(R_\varepsilon+\eta\ +2\,R_\varepsilon\Big)\ \tfrac{e^{M_\varepsilon\,(h+k)}-1}{M_\varepsilon}\ +\ c_\varepsilon(k_1-k)\\
&+\ \Big(R_\varepsilon+\eta\ +2\,R_\varepsilon\Big)\ \tfrac{e^{M_\varepsilon\,(h+k_1)}-1}{M_\varepsilon}\ +\ c_\varepsilon(k_2-k_1)\\
&+\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(t_h),\ \widetilde{x}_{m_j}(t_h+\delta'_j)\Big)\ e^{M_\varepsilon\,(h+k_2)}\ +3\,R_\varepsilon\ \tfrac{e^{M_\varepsilon\,(h+k_2)}-1}{M_\varepsilon}\ +\ c_\varepsilon(k'-k_2)\\
\leq\ & \Big(2\,\eta+9\,R_\varepsilon\Big)\ \cdot\ (h+l)\ e^{M_\varepsilon\,2\eta}\ +\ \eta\, h\ \ e^{M_\varepsilon\,2\eta}\ +3\,c_\varepsilon(l)\\
\leq\ & 9\,\Big(\eta+\ R_\varepsilon\Big)\ \cdot\ (h+l)\ e^{M_\varepsilon\,2\eta}\ +3\,c_\varepsilon(l).
\end{aligned}$$

So the timed triangle inequality guarantees for any $t \in \,]0, T[$ and $0 < k < l < h < h_0$,

$$\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_h), t_h)\Big(h+k,\ \widetilde{x}(t_h)\Big),\ \widetilde{x}(t+l)\Big)\ \leq\ 15\ e^{M_\varepsilon\,2\eta}\ \Big(\eta+R_\varepsilon\Big)\ (h+l)\ +\ 2\,\widehat{\omega}_\varepsilon(l)+3\,c_\varepsilon(l).$$

□

3.3.2 Existence in $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ due to compactness

As a consequence of the preceding Convergence Theorem 3.3.2, we can (again) suppose timed transitional compactness of $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)))$ for proving the existence of a backward solution. Moreover, exactly the same steps of Cantor diagonal construction (as in Proposition 2.3.5 about *forward* solutions) lead from Euler approximations $\widetilde{x}_n(\cdot)$ to the limit function $\widetilde{x}(\cdot)$.

In contrast to forward solutions though, further assumptions about the right–hand side now enable us to estimate the distance between an Euler solution $\widetilde{x}(\cdot)$ and any other timed right–hand backward solution directly — by means of Proposition 3.2.3.

Proposition 3.3.3 (Existence of timed right–hand backward solutions due to timed transitional compactness)

Assume that the tuple $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)))$ *is timed transitionally compact. Moreover let the function* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *satisfy for every* $\varepsilon \in \mathcal{J}$

1. $M_\varepsilon := \sup_{t,\widetilde{z}} \alpha_\varepsilon^{\rightarrow}(\widetilde{f}(\widetilde{z},t)) < \infty,$
2. $c_\varepsilon(h) := \sup_{t,\widetilde{z}} \beta_\varepsilon(\widetilde{f}(\widetilde{z},t))(h) < \infty, \qquad c_\varepsilon(h) \longrightarrow 0$ *for* $h \downarrow 0,$
3. $\exists\ R_\varepsilon : \sup_{t,\widetilde{z}} \gamma_\varepsilon(\widetilde{f}(\widetilde{z},t)) \leq R_\varepsilon < \infty, \qquad \varepsilon'^p\, R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon' \downarrow 0,$
4. $\exists\ \widehat{\omega}_\varepsilon(\cdot) : \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \leq R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2) + t_2 - t_1\Big)$
   for all $0 \leq t_1 \leq t_2 \leq T$ *and* $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$ *with* $\pi_1\, \widetilde{z}_1 \leq \pi_1\, \widetilde{z}_2,$
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup_{s\downarrow 0} \widehat{\omega}_\varepsilon(s) = 0.$

Then for every initial point $\widetilde{x}_0 \in \widetilde{E}$, *there is a timed right–hand backward solution* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$ *with* $\widetilde{x}(0) = \widetilde{x}_0$.

If assumption (4.) is replaced by

4'. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon \geq 0 : \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}(\widetilde{z}_1,t_1),\ \widetilde{f}(\widetilde{z}_2,t_2)\Big) \leq R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2) + \widehat{\omega}_\varepsilon(t_2 - t_1)$
   for all $0 \leq t_1 \leq t_2 \leq T$ *and* $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$ *with* $\pi_1\, \widetilde{z}_1 \leq \pi_1\, \widetilde{z}_2,$
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup_{s\downarrow 0} \widehat{\omega}_\varepsilon(s) = 0.$

then any other timed right–hand backward solution $\widetilde{z}(\cdot)$ *(with* $\widetilde{z}(0) = \widetilde{x}_0$*) fulfills*

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})\Big) \leq 6\, R_\varepsilon\, t\, e^{M_\varepsilon t} \Big(1 + L_\varepsilon\, e^{(L_\varepsilon\, e^{M_\varepsilon T} + M_\varepsilon)\, T} \cdot t\Big) \qquad \forall\, t \in [0,T[,\ \ \varepsilon \in \mathcal{J}.$$

Proof. For each $n \in I\!N$ (with $2^n > T$) set

$$\begin{array}{lllll} h_n & := \frac{T}{2^n}, & t_n^j & := j\, h_n & \text{for } j = 0 \dots 2^n, \\ \widetilde{x}_n(0) & := \widetilde{x}_0, & \widetilde{x}_0(\cdot) & := \widetilde{x}_0, & \\ \widetilde{x}_n(t) & := \widetilde{f}(\widetilde{x}_n(t_n^j),\, t_n^j)\, \Big(t - t_n^j,\ \widetilde{x}_n(t_n^j)\Big) & & & \text{for } t \in\,]t_n^j,\, t_n^{j+1}],\ j \le 2^n. \end{array}$$

As a consequence of timed transitional compactness and Convergence Theorem 3.3.2, these Euler approximations $(\widetilde{x}_n(\cdot))_{n \in I\!N}$ provide a timed right–hand backward solution $\widetilde{x}(\cdot)$ in exactly the same way as for timed right–hand *forward* solutions (see Prop. 2.3.5). In particular, there are sequences $k_j, n_j \nearrow \infty$ of indices (depending on $\varepsilon \in \mathcal{J}$) with $\widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{x}_{n_j}(t + h_{k_j})) \longrightarrow 0 \quad (j \longrightarrow \infty)$ for all $t \in [0, T[$.

Now suppose that assumption (4.) is replaced by

4'. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon \ge 0:\ \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}(\widetilde{z}_1, t_1),\ \widetilde{f}(\widetilde{z}_2, t_2)\Big) \ \le\ R_\varepsilon \ +\ L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2) \ +\ \widehat{\omega}_\varepsilon(t_2 - t_1)$
for all $0 \le t_1 \le t_2 \le T$ and $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$ with $\pi_1\, \widetilde{z}_1 \le \pi_1\, \widetilde{z}_2$,
$\widehat{\omega}_\varepsilon(\cdot) \ge 0$ nondecreasing, $\limsup_{s \downarrow 0}\ \widehat{\omega}_\varepsilon(s) \ =\ 0.$

Let $\widetilde{z}(\cdot) : [0, T[\longrightarrow \widetilde{E}$ be any other backward solution of the generalized mutational equation $\overset{\circ}{\widetilde{z}}(\cdot) \ni \widetilde{f}(\widetilde{z}(\cdot), \cdot)$ with $\widetilde{z}(0) = \widetilde{x}_0$ and its modulus $\omega_\varepsilon(\widetilde{z}, \cdot)$ of continuity. Then for every $\varepsilon \in \mathcal{J}$ fixed and all $t \in [0, T[$, $n \in I\!N$, Prop. 3.2.3 leads to

$$\begin{aligned} & \widetilde{q}_\varepsilon\Big(\widetilde{x}_n(t^+),\ \widetilde{z}(t + h_n^{++})\Big) \ -\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_n(0^+),\ \widetilde{z}(h_n^{++})\Big)\ e^{M_\varepsilon t} \\ \le\ & \int_0^t e^{M_\varepsilon (t-s)} \Big(5\, R_\varepsilon + \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}_n([\tfrac{s}{h_n}]\, h_n),\ [\tfrac{s}{h_n}]\, h_n),\ \ \widetilde{f}(\widetilde{z}(s + h_n),\ s + h_n)\Big)\Big) ds \\ \le\ & \int_0^t e^{M_\varepsilon (t-s)} \Big(6\, R_\varepsilon + L_\varepsilon \quad \widetilde{q}_\varepsilon\Big(\widetilde{x}_n([\tfrac{s}{h_n}] h_n),\ \ \widetilde{z}(s + h_n)\Big) \ + \widehat{\omega}_\varepsilon(2\, h_n)\Big)\ ds \\ \le\ & \int_0^t e^{M_\varepsilon (t-s)} \Big(6\, R_\varepsilon + L_\varepsilon \cdot \sup_{0 \le r \le s}\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_n(r^+),\ \widetilde{z}(r + h_n^{++})\Big) \quad +\quad o_n\Big) \quad ds \end{aligned}$$

with a sequence $o_n \longrightarrow 0$ for $n \longrightarrow \infty$.
So $\varphi_{\varepsilon,n}(t) \ :=\ \sup_{0 \le s \le t}\ \widetilde{q}_\varepsilon\Big(\widetilde{x}_n(s^+),\ \widetilde{z}(s + h_n^{++})\Big)$ is nondecreasing and fulfills

$$\begin{aligned} \varphi_{\varepsilon,n}(t) \ &\le\ \omega_\varepsilon(\widetilde{z},\ h_n)\ e^{M_\varepsilon t} \ +\ \int_0^t e^{M_\varepsilon (t-s)}\ \Big(6\, R_\varepsilon \ +\ L_\varepsilon \cdot \varphi_{\varepsilon,n}(s) + o_n\Big)\ ds \\ &\le\ \omega_\varepsilon(\widetilde{z},\ h_n)\ e^{M_\varepsilon t} \ +\ \tfrac{e^{M_\varepsilon t} - 1}{M_\varepsilon}\ \Big(6\, R_\varepsilon \ + o_n\Big) \ +\ \int_0^t e^{M_\varepsilon (t-s)}\, L_\varepsilon\ \ \varphi_{\varepsilon,n}(s)\ ds \\ &\le\ o_n' \qquad\qquad\qquad +\ \tfrac{e^{M_\varepsilon t} - 1}{M_\varepsilon}\ \ 6\, R_\varepsilon \qquad\quad +\ \int_0^t e^{M_\varepsilon T}\quad L_\varepsilon\ \ \varphi_{\varepsilon,n}(s)\ ds \end{aligned}$$

with $o_n' \longrightarrow 0$ (for $n \longrightarrow \infty$). Due to the integral version of Gronwall's Lemma 1.5.4,

$$\begin{aligned} \varphi_{\varepsilon,n}(t) \ &\le\ \tfrac{e^{M_\varepsilon t} - 1}{M_\varepsilon}\, 6\, R_\varepsilon \ +\ o_n' \ +\ \int_0^t e^{L_\varepsilon\, e^{M_\varepsilon T}\, (t-s)} \ \cdot\ L_\varepsilon\, e^{M_\varepsilon T} \ \cdot\ \Big(\tfrac{e^{M_\varepsilon s} - 1}{M_\varepsilon}\, 6\, R_\varepsilon \ +\ o_n'\Big)\ ds \\ &\le\ 6\, R_\varepsilon\ \ \Big(\tfrac{e^{M_\varepsilon t} - 1}{M_\varepsilon} \ +\ L_\varepsilon\ e^{(L_\varepsilon\, e^{M_\varepsilon T} + M_\varepsilon)\, T}\ \tfrac{e^{M_\varepsilon t} - 1 - M_\varepsilon t}{M_\varepsilon^2}\Big) \ +\ \mathrm{const}(\varepsilon, T)\ \cdot\ o_n' \\ &\le\ 6\, R_\varepsilon\ t\, e^{M_\varepsilon t}\ \Big(1 \ +\ L_\varepsilon\ e^{(L_\varepsilon\, e^{M_\varepsilon T} + M_\varepsilon)\, T} \cdot t\Big) \qquad\qquad +\ \mathrm{const}(\varepsilon, T)\ \cdot\ o_n'. \end{aligned}$$

Then we obtain for any $0 < l < l'$

$$\begin{aligned}
&\widetilde{q}_\varepsilon\Big(\widetilde{x}(t+l),\ \widetilde{z}(t+l')\Big) \\
\le\ & \limsup_{j\longrightarrow\infty}\ \Big(\widetilde{q}\Big(\widetilde{x}(t+l),\ \widetilde{x}_{n_j}(t+l+2\,h_{k_j})\Big) + \widetilde{q}\Big(\widetilde{x}_{n_j}(t+l+2\,h_{k_j}),\ \widetilde{z}(t+l')\Big)\Big) \\
\le\ & \limsup_{j\longrightarrow\infty}\ \widetilde{q}\Big(\widetilde{x}_{n_j}(t+l+2\,h_{k_j}),\ \widetilde{z}(t+l')\Big) \\
\le\ & \limsup_{j\longrightarrow\infty}\ \Big(\widetilde{q}\Big(\widetilde{x}_{n_j}(s^+),\ \widetilde{z}((s+h_{n_j})^{++})\Big)\Big|_{s\,=\,t+l+2\,h_{k_j}} + \omega_\varepsilon(\widetilde{z},\ l' - l - 2\,h_{k_j} - h_{n_j})\Big) \\
\le\ & \limsup_{j\longrightarrow\infty}\ \varphi_{\varepsilon,n_j}(t+l+2\,h_{k_j}) + \omega_\varepsilon(\widetilde{z},\ l'-l) \\
\le\ & 6\,R_\varepsilon\cdot(t+l)\,e^{M_\varepsilon(t+l)}\ \Big(1+L_\varepsilon\ e^{(L_\varepsilon\,e^{M_\varepsilon T}+M_\varepsilon)\,T}\cdot(t+l)\Big) + \omega_\varepsilon(\widetilde{z},\ l'-l),
\end{aligned}$$

i.e. $$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})\Big)\ \le\ 6\,R_\varepsilon\ t\ e^{M_\varepsilon\,t}\ \Big(1+L_\varepsilon\ e^{(L_\varepsilon\ e^{M_\varepsilon\,T}+M_\varepsilon)\,T}\cdot t\Big) \qquad \text{for all } t,\ \varepsilon.$$

□

As an immediate consequence of the preceding proposition, we obtain the existence of timed right–hand backward primitives under adequate assumptions :

Corollary 3.3.4 (Existence of timed backward primitives in $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$)
Assume that the tuple $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)))$ *is timed transitionally compact. Furthermore suppose for* $\widetilde{\vartheta} : [0,T[\ \longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *and every* $\varepsilon \in \mathcal{J}$

1. $M_\varepsilon\ := \sup_t\ \alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(t)) < \infty$
2. $c_\varepsilon(h) := \sup_t\ \beta_\varepsilon(\widetilde{\vartheta}(t))(h) < \infty, \qquad c_\varepsilon(h) \longrightarrow 0$ *for* $h \downarrow 0$
3. $\exists\ R_\varepsilon \in [0,\infty[$:
   (i) $\sup_t\ \gamma_\varepsilon(\widetilde{\vartheta}(t)) \le R_\varepsilon$,
   (ii) $\limsup_{0\,\le\,t_2-t_1\,\to\,0}\ \widetilde{Q}_\varepsilon^{\rightarrow}\Big(\widetilde{\vartheta}(t_1),\ \widetilde{\vartheta}(t_2)\Big) \le R_\varepsilon$
   (iii) $\varepsilon'^p\ R_{\varepsilon'} \longrightarrow 0$ *for* $\varepsilon' \downarrow 0$.

Then for every initial point $\widetilde{x}_0 \in \widetilde{E}$, *there exists a timed right–hand backward primitive* $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ *of* $\widetilde{\vartheta}(\cdot)$ *with* $\widetilde{x}(0) = \widetilde{x}_0$ *such that any other timed right–hand backward primitive* $\widetilde{y}(\cdot)$ *(with* $\widetilde{y}(0) = \widetilde{x}_0$*) fulfills*

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big)\ \le\ 6\,R_\varepsilon\ \cdot\ e^{M_\varepsilon\,t}\ t \qquad \forall\ t\in[0,T],\ \varepsilon\in\mathcal{J}.$$

□

3.3.3 Existence in $(\widetilde{E}, \widetilde{q})$ due to completeness

The preceding result of existence for $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is based on assuming timed transitional compactness. Now it is replaced by weaker conditions corresponding to completeness. Then in comparison with the Euler notion of Prop. 3.3.3, the approximation has to provide even Cauchy sequences because uniform bounds are not enough any longer.

Proposition 3.2.3 represents the key tool for error estimates of approximations :

$$\widetilde{q}_\varepsilon\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big) \ \le\ \widetilde{q}_\varepsilon\Big(\widetilde{x}(0^+),\ \widetilde{y}(0^{++})\Big)\ \ e^{\mu_\varepsilon(t)}\ + \\ +\ \int_0^t e^{\mu_\varepsilon(t)-\mu_\varepsilon(s)}\ \Big(c_\varepsilon(s)\ +\ 5\,R_\varepsilon(s)\Big)\ ds$$

for a timed left–hand primitive $\widetilde{x} : [0,T[\,\longrightarrow \widetilde{E}$ of $\widetilde{\vartheta} : [0,T[\,\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$
and a timed right–hand primitive $\widetilde{y} : [0,T[\,\longrightarrow \widetilde{E}$ of $\widetilde{\tau} : [0,T[\,\longrightarrow \widetilde{\Theta}_p^{\rightarrow}(\widetilde{E}, (\widetilde{q}_\varepsilon))$
such that for each $\varepsilon \in \mathcal{J}$,

$$\wedge \left\{ \begin{array}{rcll} \alpha_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(\cdot)) & \le & M_\varepsilon(\cdot) & \in\ C^0([0,T[,\ ]0,\infty[), \\ \widehat{\gamma}_\varepsilon(\,\cdot\,,\widetilde{x},\widetilde{\vartheta}),\ \widehat{\gamma}_\varepsilon(\,\cdot\,,\widetilde{y},\widetilde{\tau}) & \le & R_\varepsilon(\cdot) & \in\ C^0([0,T[,\ [0,\infty[), \\ \widetilde{Q}_\varepsilon^{\rightarrow}(\widetilde{\vartheta}(\cdot),\ \widetilde{\tau}(\cdot)) & \le & c_\varepsilon(\cdot) & \in\ C^0([0,T[,\ [0,\infty[), \\ \pi_1\,\widetilde{x}(0) & = & \pi_1\,\widetilde{y}(0) & \end{array} \right.$$

and setting as an abbreviation, $\mu_\varepsilon(t) := \displaystyle\int_0^t M_\varepsilon(s)\ ds.$

For index $\varepsilon \in \mathcal{J}$ fixed, proving the Cauchy property requires the assumption $R_\varepsilon(\cdot) = 0$. Moreover if a sequence fulfills the Cauchy condition with respect to every $\widetilde{q}_\varepsilon$ $(\varepsilon \in \mathcal{J})$ then its limit has to be the same for each ε.
For this reason we consider now a nonempty set $\widetilde{E} \overset{\text{Def.}}{=} I\!R \times E$ with merely one function $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ and timed backward transitions of order 0, but the general assumptions stay essentially the same. (The parameter $\varepsilon \in \mathcal{J}$ is redundant as mentioned in the remark after Def. 3.1.1.)

General assumptions for § 3.3.3. Let E be a nonempty set, $\widetilde{E} := I\!R \times E$, $\pi_1 : \widetilde{E} \longrightarrow I\!R,\ (t,x) \longmapsto t$. The function $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ is supposed to satisfy

1. the timed triangle inequality,
2. time continuity, i.e. every sequence $(\widetilde{z}_n = (t_n, z_n))_{n\in I\!N}$ in $\widetilde{E}$ and $\widetilde{z} = (t,z) \in \widetilde{E}$ with $\widetilde{q}(\widetilde{z}_n, \widetilde{z}) \longrightarrow 0$ $(n \longrightarrow \infty)$ fulfill $t_n \longrightarrow t$ $(n \longrightarrow \infty)$ (due to Def. 1.2.1).
3. property $(BUC^{\rightarrow})$, i.e. $\sup\limits_{t\in[0,T]} \widetilde{q}(\widetilde{x}(t), \widetilde{y}(t)) < \infty$ for all $\widetilde{x}, \widetilde{y} \in UC^{\rightarrow}([0,T], \widetilde{E}, \widetilde{q})$.

Proposition 3.3.5 (Existence of timed backward primitives in $(\widetilde{E}, \widetilde{q})$)

Suppose that $(\widetilde{E}, \widetilde{q})$ is two–sided complete, i.e. according to Def. 1.3.1, for any sequences $(\widetilde{y}_n)_{n \in I\!N}$, $(\widetilde{z}_n)_{n \in I\!N}$ in $\widetilde{E}$ satisfying

$$\left.\begin{array}{lcl} \widetilde{q}(\widetilde{y}_m,\ \widetilde{y}_n) & \longrightarrow & 0 \\ \widetilde{q}(\widetilde{z}_n,\ \widetilde{z}_m) & \longrightarrow & 0 \end{array}\right\} \quad \text{for } m,n \longrightarrow \infty \quad (m < n),$$
$$q(\widetilde{y}_n,\ \widetilde{z}_n) \longrightarrow 0 \qquad \text{for } n \longrightarrow \infty$$

there exists $\widetilde{y} \in \widetilde{E}$ such that

$$\left.\begin{array}{lcl} \widetilde{q}(\widetilde{y}_n,\ \widetilde{y}) & \longrightarrow & 0 \\ \widetilde{q}(\widetilde{y},\ \widetilde{z}_n) & \longrightarrow & 0 \end{array}\right\} \quad \text{for } n \longrightarrow \infty.$$

Moreover assume for the function $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_0^{\rightarrow\!\shortmid}(\widetilde{E}, \widetilde{q})$

1. $\widetilde{Q}^{\rightarrow\!\shortmid}\Big(\widetilde{\vartheta}(t_1),\ \widetilde{\vartheta}(t_2)\Big) \longrightarrow 0 \qquad$ *for* $0 \le t_2 - t_1 \longrightarrow 0,$
2. $M := \sup\ \alpha^{\rightarrow\!\shortmid}(\widetilde{\vartheta}(\cdot)) < \infty,$
3. $c(h) := \sup\ \beta(\widetilde{\vartheta}(\cdot))(h) < \infty, \qquad c(h) \longrightarrow 0 \quad$ *for* $h \downarrow 0.$

Then for every initial point $\widetilde{x}_0 \in \widetilde{E}$, there exists a timed right–hand backward primitive $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ of $\widetilde{\vartheta}(\cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$ such that any other timed right–hand backward primitive $\widetilde{y}(\cdot)$ (with $\widetilde{y}(0) = \widetilde{x}_0$) fulfills $\qquad \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big) = 0 \qquad \forall\, t \in [0,T[.$

Proof. Using the abbreviations $h_n := \frac{T}{2^n}$, $t_n^j := j\, h_n$ $(j = 0 \ldots 2^n,\ n \in I\!N)$ the piecewise constant maps $\widetilde{\vartheta}_n : [0,T[\longrightarrow \widetilde{\Theta}_0^{\rightarrow\!\shortmid}(\widetilde{E}, (\widetilde{q}_\varepsilon))$ $(n \in I\!N)$

$$\widetilde{\vartheta}_n(\cdot) := \widetilde{\vartheta}(t_n^j) \qquad \text{on } [t_n^j,\ t_n^{j+1}[\quad (j = 0 \ldots 2^n - 1)$$

have the properties

$$\begin{array}{lll}
\text{a.)} & \sup\limits_{0 \le t \le T - h_n} \widetilde{Q}^{\rightarrow\!\shortmid}\Big(\widetilde{\vartheta}(t),\ \widetilde{\vartheta}_n(t + h_n)\Big) \longrightarrow 0 & \text{for } n \longrightarrow \infty, \\
\text{b.)} & \sup\limits_{0 \le t \le T} \widetilde{Q}^{\rightarrow\!\shortmid}\Big(\widetilde{\vartheta}_m(t),\ \widetilde{\vartheta}_n(t)\Big) \longrightarrow 0 & \text{for } m,n \longrightarrow \infty\ (m < n), \\
\text{c.)} & \sup\limits_{0 \le t \le T - h_m} \widetilde{Q}^{\rightarrow\!\shortmid}\Big(\widetilde{\vartheta}_n(t + h_n),\ \widetilde{\vartheta}_m(t + h_m)\Big) \longrightarrow 0 & \text{for } m,n \longrightarrow \infty\ (m < n), \\
\text{d.)} & \widetilde{Q}^{\rightarrow\!\shortmid}\Big(\widetilde{\vartheta}_n(s_1),\ \widetilde{\vartheta}_n(s_2)\Big) \longrightarrow 0 & \text{for } n \longrightarrow \infty,\ s_2 - s_1 \downarrow 0, \\
\text{e.)} & \sup\limits_{t,n}\ \alpha^{\rightarrow\!\shortmid}(\widetilde{\vartheta}_n(t)) \le M, & \\
\text{f.)} & \sup\limits_{t,n}\ \beta(\widetilde{\vartheta}_n(t))(h) \le c(h) & \text{for every } h \in\,]0,1].
\end{array}$$

Obviously each of them has a timed left–hand and timed right–hand backward primitive $\widetilde{x}_n(\cdot)$ defined as $\qquad \widetilde{x}_n(0) := \widetilde{x}_0, \qquad \widetilde{x}_n(t) := \widetilde{\vartheta}_n^j(t - t_n^j,\ \widetilde{x}_n(t_j)) \qquad$ for $t \in\,]t_n^j, t_n^{j+1}]$.

Now property (b.) and the estimate of Prop. 3.1.4 guarantee

$$\sup_{0 \le t \le T} \; \widetilde{q}\Big(\widetilde{x}_m(t^+),\ \widetilde{x}_n(t^{++})\Big) \longrightarrow 0 \quad \text{for } m,n \longrightarrow \infty \ (m<n)$$

and due to Prop. 3.1.3, for every $t \in \,]0,T[$,

$$\begin{aligned} \widetilde{q}\Big(\widetilde{x}_m(t-h_m),\ \widetilde{x}_n(t-h_n)\Big) \;&\le\; \widetilde{q}\Big(\widetilde{x}_m((t-h_m)^+),\ \widetilde{x}_n((t-h_m)^{++})\Big) \;+\; c(h_m - h_n) \\ &\longrightarrow\; 0 \qquad\qquad \text{for } m,n \longrightarrow \infty \ (m<n). \end{aligned}$$

Correspondingly, property (c.) has the consequence

$$\sup_{0 \le t \le T-h_m} \; \widetilde{q}\Big(\widetilde{x}_n((t+h_n)^+),\ \widetilde{x}_m((t+h_m)^{++})\Big) \longrightarrow 0 \quad \text{for } m,n \longrightarrow \infty \ (m<n)$$

and so we obtain for every $t \in [0,T[$,

$$\begin{aligned} \widetilde{q}\Big(\widetilde{x}_n(t+h_{n-1}),\ \widetilde{x}_m(t+h_{m-1})\Big) \;&\le\; \widetilde{q}\Big(\widetilde{x}_n((t+h_n\,+\,h_n)^+),\ \widetilde{x}_m((t+h_n\,+\,h_m)^{++})\Big) \\ &\quad + \; c(h_{m-1} \;-\; (h_n\,+\,h_m)) \\ &\longrightarrow\; 0 \qquad\qquad \text{for } m,n \longrightarrow \infty \ (m<n). \end{aligned}$$

As $(\widetilde{E},\widetilde{q})$ is assumed to be two–sided complete, there exists some $\widetilde{x}(t) \in \widetilde{E}$ such that

$$\wedge \begin{cases} \widetilde{q}\Big(\ \widetilde{x}(t), \qquad \widetilde{x}_n(t+h_{n-1})\Big) \longrightarrow 0 \\ \widetilde{q}\Big(\widetilde{x}_n(t-h_n),\ \ \widetilde{x}(t)\Big) \qquad \longrightarrow 0 \end{cases} \qquad \text{for } n \longrightarrow \infty.$$

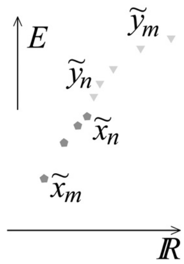
Convergence Theorem 3.3.2 implies that $\widetilde{x}(\cdot)$ is a timed right–hand backward primitive of $\widetilde{\vartheta}(\cdot)$. Its additional relation to other timed backward primitives is shown in exactly the same way as in Prop. 3.3.3 (with $L_\varepsilon = 0,\ R_\varepsilon = 0$ in condition (4'.) there). □

Now Euler method and two–sided completeness of $(\widetilde{E},\widetilde{q})$ lead to the existence of a timed–backward solution on a bounded time interval. For checking the Cauchy property of approximating sequences, more detailed estimates are required. So we restrict ourselves to timed backward transitions $\widetilde{\vartheta} \in \widetilde{\Theta}_0^{\rightarrow}(\widetilde{E},\widetilde{q})$ that are even Lipschitz continuous with respect to their time parameter, i.e.

$$\beta^{\text{Lip}}(\widetilde{\vartheta}) \;\stackrel{\text{Def.}}{=}\; \sup_{\substack{0<t\le 1\\ \widetilde{x}\in\widetilde{E}}} \; \limsup_{h\downarrow 0} \; \tfrac{1}{h} \cdot \widetilde{q}\Big(\widetilde{\vartheta}(t-h,\widetilde{x}),\ \ \widetilde{\vartheta}(t,\widetilde{x})\Big) \;<\; \infty$$

(see Lemma 3.1.2). This assumption facilitates the comparison with geometric series.

In comparison with preceding results about existence, there is a new essential aspect. Two–sided completeness considers two approximating sequences $(\widetilde{x}_n)_n,\ (\widetilde{y}_n)_n$ in $\widetilde{E}$ whose elements hold the Cauchy condition in opposite order, i.e.



$$\widetilde{q}(\widetilde{x}_m,\ \widetilde{x}_n) \longrightarrow 0, \qquad \widetilde{q}(\widetilde{y}_n,\ \widetilde{y}_m) \longrightarrow 0 \qquad \text{for } m,n \longrightarrow \infty \ (m<n).$$

Roughly speaking, $(\widetilde{x}_n)$ represents the approximation by *earlier* elements of $\widetilde{E} \stackrel{\text{Def.}}{=} I\!R \times E$ whereas $(\widetilde{y}_n)_{n\in I\!N}$ denotes *later* approximating points. Corresponding to this notion, the Euler method has to take the time direction into account.

Proposition 3.3.6 (Short–time existence of timed backward solutions in two–sided complete $(\widetilde{E}, \widetilde{q})$)

Let $(\widetilde{E}, \widetilde{q})$ *be two–sided complete and fulfill standard hypotheses* $(R^{\Rightarrow})$, $(R^{\Leftarrow})$, *i.e. right–hand spheres are both right– and left–sequentially closed (due to Def. 1.4.3, 1.4.6). Moreover,* $\widetilde{f} : \widetilde{E} \times [0, 2\,T] \longrightarrow \widetilde{\Theta}_0^{\rightarrow}(\widetilde{E}, \widetilde{q})$ *satisfies*

1. *there exists* $L > 0$ *such that for any* $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{E}$, $0 \le t_1 \le t_2 \le 2\,T$ *with* $\pi_1\,\widetilde{x}_1 \le \pi_1\,\widetilde{x}_2$: $\quad \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}_1, t_1),\ \widetilde{f}(\widetilde{x}_2, t_2)\Big) \ \le\ L \cdot \Big(\widetilde{q}(\widetilde{x}_1,\ \widetilde{x}_2) \ +\ t_2 - t_1\Big)$,
2. $M \ :=\ \sup_{\widetilde{x},t}\ \alpha^{\rightarrow}(\widetilde{f}(\widetilde{x}, t)) \ <\ \infty$,
3. $c \ \ :=\ \sup_{\widetilde{x},t}\ \beta^{\text{Lip}}(\widetilde{f}(\widetilde{x}, t)) \ <\ \infty$,
4. $4\,L\,T\,e^{2\,M\,T} \ <\ 1.$

For every point $\widetilde{x}_0 \in \widetilde{E}$, *there is a timed right–hand backward solution* $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ \ni\ \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0, T[$ *with* $\widetilde{x}(0) = \widetilde{x}_0$ *such that any other timed right–hand backward solution* $\widetilde{z}(\cdot) \in \text{Lip}^{\rightarrow}([0, T[,\ \widetilde{E}, \widetilde{q})$ *(with* $\widetilde{z}(0) = \widetilde{x}_0$*) fulfills* $\quad \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})\Big) = 0 \quad$ *for all* $t \in [0, T[$.

Proof is based on constructing approximations $\widetilde{x}_n,\ \widetilde{y}_n : [0, T_n] \longrightarrow \widetilde{E}$ with time shift :
For each $n \in \{2, 3, 4 \ldots\}$, $j = 0 \ldots 2^n$ set $\widehat{T} := 2\,T$,

$$\begin{aligned} h_n \ &:=\ \tfrac{\widehat{T}}{2^n}, & t_n^j \ &:=\ j\,h_n, \\ T_n \ &:=\ \widehat{T} - \sum_{j=2}^{n-1} h_j \ =\ T\ (1 + 2^{-n+2}), \\ \widetilde{y}_0(\cdot) \ &:=\ \widetilde{x}_0, & \widetilde{y}_n(0) \ &:=\ \widetilde{x}_0 \end{aligned}$$

and
$$\begin{aligned} \widetilde{x}_n(t) \ &:=\ \widetilde{f}(\widetilde{x}_0,\ 0)\ \Big(t,\ \ \widetilde{x}_0\Big) && \text{for } t \in [0,\ t_n^2], \\ \widetilde{x}_n(t) \ &:=\ \widetilde{f}(\widetilde{x}_{n-1}(t_n^{j-2}),\ t_n^{j-2})\ \Big(t - t_n^j,\ \ \widetilde{x}_n(t_n^j)\Big) && \text{for } t \in\ ]t_n^j,\ t_n^{j+1}] \cap\ ]t_n^2,\ \widehat{T}], \\ \widetilde{y}_n(t) \ &:=\ \widetilde{f}(\widetilde{y}_{n-1}(t_n^{j+3}),\ t_n^{j+3})\ \Big(t - t_n^j,\ \ \widetilde{y}_n(t_n^j)\Big) && \text{for } t \in\ ]t_n^j,\ t_n^{j+1}] \cap [\ 0,\ \widehat{T} - 3\,h_n], \\ \widetilde{y}_n(t) \ &:=\ \widetilde{f}(\widetilde{y}_{n-1}(\widehat{T}),\ \widehat{T})\ \Big(t - t_n^j,\ \ \widetilde{y}_n(t_n^j)\Big) && \text{for } t \in\ ]t_n^j,\ t_n^{j+1}] \cap\ ]\widehat{T} - 3\,h_n,\ \widehat{T}]. \end{aligned}$$

Then,

$$\begin{aligned} \widetilde{f}(\widetilde{x}_{n-1}(t_n^{j-2}),\ t_n^{j-2}) \ &\in\ \overset{\circ}{\widetilde{x}}_n(t) && \text{for } t \in\ ]t_n^j,\ t_n^{j+1}], \\ \widetilde{f}(\widetilde{y}_{n-1}(t_n^{j+3}),\ t_n^{j+3}) \ &\in\ \overset{\circ}{\widetilde{y}}_n(t) && \text{for } t \in\ ]t_n^j,\ t_n^{j+1}], \\ \widetilde{q}\Big(\widetilde{x}_n(s_1),\ \widetilde{x}_n(s_2)\Big) \ &\le\ c \cdot (s_2 - s_1) && \text{for } 0 \le s_1 < s_2 < \widehat{T}, \\ \widetilde{q}\Big(\widetilde{y}_n(s_1),\ \widetilde{y}_n(s_2)\Big) \ &\le\ c \cdot (s_2 - s_1) && \text{for } 0 \le s_1 < s_2 < \widehat{T}, \\ \pi_1\,\widetilde{x}_n(t) \ =\ \pi_1\,\widetilde{y}_n(t) \ &=\ \pi_1\,\widetilde{x}_0 \ + t && \text{for } 0 \le\ t\ \le \widehat{T}, \\ t_{n+1}^{2\,j} \ =\ \tfrac{2j}{2^{n+1}}\,\widehat{T} \ &=\ t_n^j. \end{aligned}$$

According to the following Lemma 3.3.7,

$$\sup_{0 \le t \le T_n} \widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1})^+, \;\; \widetilde{y}_n((t+h_n)^{++}))\Big) \le \mathrm{const}(c,L,M,T) \cdot h_n,$$
$$\sup_{h_n \le t \le \widehat{T}} \widetilde{q}\Big(\widetilde{x}_n((t-h_n)^+, \;\; \widetilde{x}_{n+1}((t-h_{n+1})^{++}))\Big) \le \mathrm{const}(c,L,M,T) \cdot h_n,$$
$$\sup_{0 \le t \le T_n} \widetilde{q}\Big(\widetilde{x}_n(t^+), \;\; \widetilde{y}_n(t^{++})\Big) \le \mathrm{const}(c,L,M,T) \cdot h_n$$

for every $n \in I\!N$ sufficiently large and thus,

$$\begin{aligned}
&\widetilde{q}\Big(\widetilde{x}_n(t-h_{n-1}), \;\; \widetilde{x}_{n+1}(t-h_n)\Big) \\
\le\;& \widetilde{q}\Big(\widetilde{x}_n((t-h_n-h_n)^+), \;\; \widetilde{x}_{n+1}((t-h_n-h_{n+1})^{++})\Big) + c \cdot (t-h_n-s)\Big|_{s=t-h_n-h_{n+1}} \\
\le\;& \mathrm{const}(c,L,M,T) \cdot h_n + c \cdot h_{n+1} \\
\le\;& \mathrm{const}(c,L,M,T) \cdot \tfrac{1}{2^n},
\end{aligned}$$

i.e. $\displaystyle\sup_{h_{m-1} < t \le T} \widetilde{q}\Big(\widetilde{x}_m(t-h_{m-1}), \;\; \widetilde{x}_n(t-h_{n-1})\Big) \longrightarrow 0$ for $m, n \longrightarrow \infty \;\; (m < n)$.

Correspondingly, we conclude for every $t \in [0, T[$

$$\begin{aligned}
&\widetilde{q}\Big(\widetilde{y}_{n+1}(t+h_n), \;\; \widetilde{y}_n(t+h_{n-1})\Big) \\
\le\;& \widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1}+h_{n+1})^+), \;\; \widetilde{y}_n((t+h_{n+1}+h_n)^{++})\Big) + c \cdot (h_{n-1}-h_{n+1} - h_n) \\
\le\;& \mathrm{const}(c,L,M,T) \cdot h_n + c \cdot h_{n+1} \\
\le\;& \mathrm{const}(c,L,M,T) \cdot \tfrac{1}{2^n}
\end{aligned}$$

i.e. $\displaystyle\sup_{0 \le t < T} \widetilde{q}\Big(\widetilde{y}_n(t+h_{n-1}), \;\; \widetilde{y}_m(t+h_{m-1})\Big) \longrightarrow 0$ for $m, n \longrightarrow \infty \;\; (m < n)$.

Furthermore,

$$\begin{aligned}
&\widetilde{q}\Big(\widetilde{x}_n(t-h_{n-1}), \;\; \widetilde{y}_n(t+h_{n-1})\Big) \\
\le\;& \widetilde{q}\Big(\widetilde{x}_n((t-h_{n-1})^+), \;\; \widetilde{y}_n((t-h_{n-1})^{++})\Big) + c \cdot 2\, h_{n-1} \le \mathrm{const}(c,L,M,T) \cdot \tfrac{1}{2^n}
\end{aligned}$$

i.e. $\displaystyle\sup_{h_{n-1} \le t < T-h_{n-1}} \widetilde{q}\Big(\widetilde{x}_n(t-h_{n-1}), \;\; \widetilde{y}_n(t+h_{n-1})\Big) \longrightarrow 0$ for $n \longrightarrow \infty$.

As $(\widetilde{E}, \widetilde{q})$ is two–sided complete, there exists $\widetilde{x}(t) \in \widetilde{E}$ for each $t \in]0, T[$ such that

$$\wedge \begin{cases} \widetilde{q}\Big(\widetilde{x}_n(t-h_{n-1}), \;\; \widetilde{x}(t)\Big) \longrightarrow 0 \\ \widetilde{q}\Big(\widetilde{x}(t), \;\; \widetilde{y}_n(t+h_{n-1})\Big) \longrightarrow 0 \end{cases} \quad \text{for } n \longrightarrow \infty.$$

The timed triangle inequality guarantees even the locally uniform convergence

$$\sup_{h_{m-1} < t \le T} \widetilde{q}\Big(\widetilde{x}_m(t-h_{m-1}), \;\; \widetilde{x}(t)\Big) \longrightarrow 0 \quad \text{for } m \longrightarrow \infty,$$
$$\sup_{0 \le t < T} \widetilde{q}\Big(\widetilde{x}(t), \;\; \widetilde{y}_m(t+h_{m-1})\Big) \longrightarrow 0 \quad \text{for } m \longrightarrow \infty.$$

The essential benefit of standard hypothesis $(R^{\Rightarrow})$ for $(\widetilde{E}, \widetilde{q})$ (i.e. right–hand spheres are right–sequentially closed) is put in a nutshell in the preceding Proposition 1.4.5 guaranteeing the Lipschitz continuity of $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$, i.e.

$$\widetilde{q}\Big(\widetilde{x}(s),\ \widetilde{x}(t)\Big) \ \leq\ c \cdot (t-s) \qquad \text{for } 0 \leq s < t < T.$$

Furthermore for every $t \in \,]0,T[$ and $k \in I\!N$ there exists an index $m = m(k) > k$ such that $\widetilde{q}\Big(\widetilde{y}_m(t + h_{m-1} - \frac{1}{k}),\ \widetilde{x}(t)\Big) \ <\ \frac{c+1}{k}, \qquad h_{m-1} < \frac{1}{k}.$
Indeed, the right–hand spheres of $(\widetilde{E}, \widetilde{q})$ are left–sequentially closed (due to standard hypothesis $(R^{\Leftarrow})$) and thus, $\lim\limits_{m \to \infty} \widetilde{q}\Big(\widetilde{y}_m(t+h_{m-1}-\frac{1}{k}),\ \widetilde{x}(t)\Big) = \widetilde{q}\Big(\widetilde{x}(t-\frac{1}{k}),\ \widetilde{x}(t)\Big) \leq \frac{c}{k}.$
So assumptions (1.), (4.), (5.ii), (5.iii) of the Convergence Theorem (Prop. 3.3.2) are satisfied by the translated functions $\widetilde{y}_n(\,\cdot\, + h_{n-1}) : [0, T - h_{n-1}[\longrightarrow \widetilde{E}$ $(n \in I\!N)$.

Now we prove that the functions $\widetilde{f}_n : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}_0^{\rightarrow}(\widetilde{E}, \widetilde{q})$ $(n \geq 2)$,

$$\begin{aligned} \widetilde{f}_n(\widetilde{z}, t) \ &:=\ \widetilde{f}\big(\widetilde{y}_{n-1}(t_n^{j+3} + h_{n-1}),\ \ t_n^{j+3} + h_{n-1}\big) \qquad \text{for } t \in [t_n^j,\ t_n^{j+1}[\ \cap\ ]0,T] \\ &:=\ \widetilde{f}\big(\widetilde{y}_{n-1}(t_n^{j+5}),\ \ t_n^{j+5}\big) \end{aligned}$$

fulfill the other conditions there.

For every $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}$, $s_1 \in [t_n^j,\ t_n^{j+1}[$, $s_2 \in [t_n^l,\ t_n^{l+1}[$ (with $s_1 \leq s_2$),

$$\begin{aligned} \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}_n(\widetilde{y}_1, s_1),\ \widetilde{f}_n(\widetilde{y}_2, s_2)\Big) \ &=\ \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{y}_{n-1}(t_n^{j+5}),\ t_n^{j+5}),\quad \widetilde{f}(\widetilde{y}_{n-1}(t_n^{l+5}),\ t_n^{l+5})\Big) \\ &\leq\ L\ \Big(\widetilde{q}(\widetilde{y}_{n-1}(t_n^{j+5}), \widetilde{y}_{n-1}(t_n^{l+5})) \ +\ t_n^{l+5} - t_n^{j+5}\Big) \\ &\leq\ L\ \Big(\qquad c \qquad +\ 1\Big)\ (t_n^l - t_n^j) \\ &\leq\ L\ \ (c+1) \qquad (s_2 - s_1 + 2\,h_n). \end{aligned}$$

Thus, $\quad \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}_n(\widetilde{z}_1, s_1),\ \widetilde{f}_n(\widetilde{z}_2, s_2)\Big) \longrightarrow 0 \quad$ for $\widetilde{q}(\widetilde{z}_1, \widetilde{z}_2) \to 0$, $\ s_2 - s_1 \downarrow 0$, $\ n \to \infty$.

Moreover we have for $t \in [t_n^j, t_n^{j+1}[$

$$\begin{aligned} &\widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}(t),\ t),\ \widetilde{f}_n(\widetilde{x}(t),\ t)\Big) \ =\ \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}(t),\ t),\quad \widetilde{f}(\widetilde{y}_{n-1}(t_n^{j+5}),\ t_n^{j+5})\Big) \\ &\leq\ L\ \Big(\widetilde{q}(\widetilde{x}(t),\ \widetilde{y}_{n-1}(t_n^{j+5})) \qquad\qquad +\ t_n^{j+5} - t\Big) \\ &\leq\ L\ \Big(\widetilde{q}(\widetilde{x}(t),\ \widetilde{y}_{n-1}(t + 4\,h_n)) \ +\ \widetilde{q}(\widetilde{y}_{n-1}(t + 4\,h_n),\ \widetilde{y}_{n-1}(t_n^{j+5})) \ +\ 5\,h_n\Big) \\ &\leq\ L\ \Big(\widetilde{q}(\widetilde{x}(t),\ \widetilde{y}_{n-1}(t + h_{n-2})) \ +\ c \cdot (t_n^{j+1} - t) \qquad +\ 5\,h_n\Big) \\ &\leq\ L\ \Big(\widetilde{q}(\widetilde{x}(t),\ \widetilde{y}_{n-1}(t + h_{n-2})) \ +\ (c+5)\ \ h_n\Big) \\ &\longrightarrow\ 0 \qquad\qquad (n \longrightarrow \infty). \end{aligned}$$

Each translated function $\widetilde{y}_n(\,\cdot\, + h_{n-1}) : [0, T - h_{n-1}[\longrightarrow \widetilde{E}$ is a timed right–hand backward primitive of $\widetilde{f}_n(\widetilde{y}_n(\cdot),\ \cdot\,)$. So the Convergence Theorem (Prop. 3.3.2) implies that $\widetilde{x}(\cdot)$ is a timed right–hand backward solution of $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$ in $[0,T[$.

Finally any other timed right–hand backward solution $\widetilde{z}(\cdot) \in \mathrm{Lip}^{\to}([0,T[, \widetilde{E}, \widetilde{q})$ with the initial value $\widetilde{z}(0) = \widetilde{x}_0$ fulfills $\widetilde{q}(\widetilde{x}(t^+), \widetilde{z}(t^{++})) = 0$ for any t. Indeed, for every n,

$$\begin{aligned}\widetilde{q}\Big(\widetilde{x}_n((s-h_{n-1})^+),\ \widetilde{z}(s^{++})\Big)\Big|_{s=t_n^2=h_{n-1}} &= \widetilde{q}\Big(\widetilde{f}(\widetilde{x}_0,0)\,\Big(0^+,\widetilde{x}_0\Big),\ \widetilde{z}(h_{n-1}{}^{++})\Big) \\ &\le \widetilde{q}\Big(\widetilde{f}(\widetilde{x}_0,0)\,\Big(0^+,\widetilde{x}_0\Big),\ \widetilde{z}(0^{++})\Big) + L_{\widetilde{z}}\cdot h_{n-1} \\ &= L_{\widetilde{z}}\cdot h_{n-1}\end{aligned}$$

with the Lipschitz constant $L_{\widetilde{z}} := \mathrm{Lip}^{\to}(\widetilde{z}(\cdot),\widetilde{q})$.

We conclude from the mutational equation of $\widetilde{x}_n(\cdot)$ and Prop. 3.2.3 that for $t \in]t_n^2, t_n^3]$

$$\begin{aligned}&\widetilde{q}\Big(\widetilde{x}_n((t-h_{n-1})^+),\ \widetilde{z}(t^{++})\Big)\\ \le\ &\widetilde{q}\Big(\widetilde{x}_n(0^+),\ \widetilde{z}(h_{n-1}{}^{++})\Big)\ e^{M\,(t-t_n^2)} + \int_{t_n^2}^t e^{M\,(t-s)}\ L\ \Big(\widetilde{q}\Big(\widetilde{x}_0,\ \widetilde{z}(s)\Big) + s\Big)\ ds\\ \le\ &L_{\widetilde{z}}\ h_{n-1}\ e^{M\,h_n} + h_n\ e^{M\,h_n}\ L\ (L_{\widetilde{z}}+2)\ 2\,h_n\\ \le\ &C_0\ h_n\end{aligned}$$

with a constant $C_0 = C_0(c, L, L_{\widetilde{z}}, M, T) > 0$ and correspondingly for $t \in]t_n^3, t_n^4]$,

$$\begin{aligned}&\widetilde{q}\Big(\widetilde{x}_n((t-h_{n-1})^+),\ \widetilde{z}(t^{++})\Big)\\ \le\ &\widetilde{q}\Big(\widetilde{x}_n(t_n^{1\,+}),\ \widetilde{z}(t_n^{3\,++})\Big)\ e^{M\,(t-t_n^3)} + \int_{t_n^3}^t e^{M\,(t-s)}\ L\ \Big(\widetilde{q}\Big(\widetilde{x}_0(t_j^1),\ \widetilde{z}(s)\Big) + s\Big)\ ds\\ \le\ &C_0\ h_n\ e^{M\,2\,h_n} + h_n\ e^{M\,h_n}\ L\ (C_0+2)\ 2\,h_n\\ \le\ &C_1\ h_n\end{aligned}$$

with $C_1 = C_1(c, L, L_{\widetilde{z}}, M, T) > 0$. This forms the basis for estimating at time $t \in]t_n^4, T[$

$$\begin{aligned}&\widetilde{q}\Big(\widetilde{x}_n((t-h_{n-1})^+),\ \widetilde{z}(t^{++})\Big) - \widetilde{q}\Big(\widetilde{x}_n((s-h_{n-1})^+),\ \widetilde{z}(s^{++})\Big)\Big|_{s=t_n^4}\ e^{M\,(t-t_n^4)}\\ \le\ &\int_{t_n^4}^t e^{M\,(t-s)}\ L\ \Big(\widetilde{q}\Big(\widetilde{x}_{n-1}([\tfrac{s}{h_n}]\,h_n - h_{n-1} - 2h_n),\ \widetilde{z}(s)\Big) + s - ([\tfrac{s}{h_n}]-4)\,h_n\Big)\ ds\\ \le\ &\int_{t_n^4}^t e^{M\,(t-s)}\ L\ \Big(\widetilde{q}\Big(\widetilde{x}_{n-1}((\sigma-h_{n-2})^+),\ \widetilde{z}(\sigma^{++})\Big)\Big|_{\sigma=[\frac{s}{h_n}]\,h_n} + L_{\widetilde{z}}\ h_n + 5\,h_n\Big)\ ds\\ \le\ &T\cdot e^{MT}\,L\,(L_{\widetilde{z}}+5)\cdot h_n + T\cdot L\,e^{MT}\cdot \sup_{h_{n-2}\le s\le t}\widetilde{q}\Big(\widetilde{x}_{n-1}((s-h_{n-2})^+),\ \widetilde{z}(s^{++})\Big),\end{aligned}$$

So $b_n := \sup_{h_{n-1}\le s<T} \widetilde{q}\Big(\widetilde{x}_n((s-h_{n-1})^+),\ \widetilde{z}(s^{++})\Big)$ satisfies the recursive inequality

$$\begin{aligned}b_n \le C_2\,h_n + L\,T\,e^{MT}\ b_{n-1} &\le (L\,T\,e^{MT})^n\ b_0 + \tfrac{C_2\,\widehat{T}}{2^n}\sum_{j=0}^{n-1}(2\,L\,T\,e^{MT})^j\\ &\le (L\,T\,e^{MT})^n\ b_0 + \tfrac{C_2\,\widehat{T}}{2^n}\ \tfrac{1}{1-2\,L\,T\,e^{MT}}\\ &\longrightarrow 0 \qquad (n\longrightarrow\infty)\end{aligned}$$

(with $C_2 = C_2(c, L, L_{\widetilde{z}}, M, T) > 0$) since $2\,L\,T\,e^{MT} < 1$.

Due to the construction of $\widetilde{x}(\cdot)$, the sequence $\widetilde{q}(\widetilde{x}_n(s-h_{n-1}),\ \widetilde{x}(s))$ also converges to 0 for $n \longrightarrow \infty$ and each s. So standard hypothesis $(R^{\Rightarrow})$ for $(\widetilde{E}, \widetilde{q})$ has the consequence $\widetilde{q}(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})) = 0$ for every $t \in]0,T[$ because for arbitrary $0 < k < l < T-t$,

$$\begin{aligned}
&\widetilde{q}\Big(\widetilde{x}(t+k),\ \widetilde{z}(t+l)\Big) \\
&= \lim_{n\longrightarrow\infty} \widetilde{q}\Big(\widetilde{x}_n(t+k-h_{n-1}),\ \widetilde{z}(t+l)\Big) \\
&\le \limsup_{n\longrightarrow\infty} \Big(\widetilde{q}\Big(\widetilde{x}_n(s-h_{n-1}),\ \widetilde{z}(s^+)\Big)\Big|_{s=t+k} + \widetilde{q}\Big(\widetilde{z}((t+k)^+,\ \widetilde{z}(t+l)\Big)\Big) \\
&\le \limsup_{n\longrightarrow\infty} \Big(b_n + L_{\widetilde{z}}\ (l-k)\Big) \\
&\le L_{\widetilde{z}}\ (l-k),
\end{aligned}$$

due to Proposition 3.1.3. □

Remark.

Currently it is not clear how to prove the second property (i.e. $\widetilde{q}(\widetilde{x}(t^+), \widetilde{z}(t^{++}) = 0)$ by means of the approximations $\widetilde{y}_n(\cdot + h_{n-1})$ $(n \in I\!N)$ since these functions use the translation of time.

Thus, for replacing standard hypothesis $(R^{\Rightarrow})$ here we need additional properties of the approximations $\widetilde{x}_n(\cdot)$. They can be guaranteed, for example, by the assumption that $(\widetilde{E}, \widetilde{q})$ is one–sided complete (see Corollary 3.3.8 later).

The essential step for the preceding proposition (about the existence of a timed backward solution) is the Cauchy property of the approximating sequences $(\widetilde{x}_n(\cdot - h_n))_{n\in I\!N}$, $(\widetilde{y}_n(\cdot + h_n))_{n\in I\!N}$ because then the solution $\widetilde{x}(\cdot)$ results from assuming that $(\widetilde{E}, \widetilde{q})$ is two–sided complete.

In the next lemma we collect the required statements about the Cauchy property. Proving the three parts is based on one and the same notion : First for the index n fixed, a recursive sequence provides an upper bound uniform with respect to time and resulting from Gronwall's Lemma (applied to the linear interpolation of the sequence). Then we obtain an upper estimate for every large n by a further recursive sequence.

Lemma 3.3.7 *Under the hypothesis of the preceding Prop. 3.3.6 (and its notation)*

$$\begin{aligned}
&a.)\quad \sup_{0\le t\le T_n} \widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1})^+,\ \ \widetilde{y}_n((t+h_n)^{++}))\Big) \le \mathrm{const}(c,L,M,T)\cdot h_n, \\
&b.)\quad \sup_{h_n\le t\le \widehat{T}} \widetilde{q}\Big(\widetilde{x}_n((t-h_n)^+,\ \ \widetilde{x}_{n+1}((t-h_{n+1})^{++}))\Big) \le \mathrm{const}(c,L,M,T)\cdot h_n, \\
&c.)\quad \sup_{0\le t\le T_n} \widetilde{q}\Big(\widetilde{x}_n(t^+),\ \ \widetilde{y}_n(t^{++})\Big) \le \mathrm{const}(c,L,M,T)\cdot h_n
\end{aligned}$$

for every $n \in I\!N$ sufficiently large.

Proof. a.) Prop. 3.2.3 and 3.1.3 lead to the uniform estimate for $t_n^j = t_{n+1}^{2j} < t \le t_n^{j+1}$

$$\widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1})^+),\ \widetilde{y}_n((t+h_n)^{++})\Big) - \widetilde{q}\Big(\widetilde{y}_{n+1}((s+h_{n+1})^+),\ \widetilde{y}_n((s+h_n)^{++})\Big)\Big|_{s=t_n^j}\ e^{M\,h_n}$$

$$\le \int_{t_n^j}^{t_{n+1}^{2j+1}} e^{M\,(t_{n+1}^{2j+1}-s)}\ \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{y}_n(t_n^j+h_{n+1}+3h_{n+1}),\ t_{n+1}^{2j+4}),\ \widetilde{f}(\widetilde{y}_{n-1}(t_n^j+h_n+3h_n),\ t_n^{j+4})\Big)\ ds$$

$$+ \int_{t_{n+1}^{2j+1}}^{t_n^{j+1}} e^{M\,(t_n^{j+1}-s)}\ \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{y}_n(t_{n+1}^{2j+1}+h_{n+1}+3h_{n+1}),\ t_{n+1}^{2j+5}),\ \widetilde{f}(\widetilde{y}_{n-1}(t_n^j+h_n+3h_n),\ t_n^{j+4})\Big)\ ds$$

$$\le h_{n+1}\,e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{y}_n(t_{n+1}^{2j+4}),\ \widetilde{y}_{n-1}(t_n^{j+4})\Big) + \tfrac{j+4}{2^n}\,\widehat{T} - \tfrac{2j+4}{2^{n+1}}\,\widehat{T}\Big)$$
$$+\ h_{n+1}\,e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{y}_n(t_{n+1}^{2j+5}),\ \widetilde{y}_{n-1}(t_n^{j+4})\Big) + \tfrac{j+4}{2^n}\,\widehat{T} - \tfrac{2j+5}{2^{n+1}}\,\widehat{T}\Big)$$

$$= h_{n+1}\,e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{y}_n(t_n^{j+1}+h_n),\ \widetilde{y}_{n-1}(t_n^{j+2}+h_{n-1})\Big) + 4\,h_{n+1}\Big)$$
$$+\ h_{n+1}\,e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{y}_n(t_{n+1}^{2j+3}+h_n),\ \widetilde{y}_{n-1}(t_n^{j+2}+h_{n-1})\Big) + 3\,h_{n+1}\Big)$$

$$\le h_{n+1}\,e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{y}_n((s+h_n)^+),\ \widetilde{y}_{n-1}((s+h_{n-1})^{++})\Big)\Big|_{s=t_n^{j+1}} + (2c+4)\,h_{n+1}\Big)$$
$$+\ h_{n+1}\,e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{y}_n((s+h_n)^+),\ \widetilde{y}_{n-1}((s+h_{n-1})^{++})\Big)\Big|_{s=t_{n+1}^{2j+3}} + (c+3)\,h_{n+1}\Big).$$

Using the abbreviations

$$a_0 := \widetilde{q}\Big(\widetilde{y}_{n+1}((t_n^0+h_{n+1})^+),\ \widetilde{y}_n((t_n^0+h_n)^{++})\Big) = \widetilde{q}\Big(\widetilde{y}_{n+1}(h_{n+1}^+),\ \widetilde{y}_n(h_n^{++})\Big),$$
$$b_n := \sup_{0\le t\le T_{n-1}} \widetilde{q}\Big(\widetilde{y}_n((t+h_n)^+),\ \widetilde{y}_{n-1}((t+h_{n-1})^{++})\Big)$$

for $n \ge 3$ fixed, a monotone recursive sequence $(a_j)_{j\in\mathbb{N}}$ provides an upper bound of

$$\widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1})^+),\ \widetilde{y}_n((t+h_n)^{++})\Big) \le a_{j+1}$$

for all $t \in\,]t_n^j, t_n^{j+1}]$ $(j = 0 \ldots 2^n)$ with $t_n^{j+1} \le t_{n+1}^{2j+3} \le T_{n-1}$, namely

$$a_{j+1} := e^{M\,h_n}\,a_j + L\,e^{M\,h_{n+1}}\,h_{n+1}\cdot 2\,b_n + L\,e^{M\,h_{n+1}}\,(3c+7)\,h_{n+1}^2\,.$$

In particular, this estimate holds for all $0 \le t \le T_{n-1} - 4\,h_{n+1} \overset{\text{Def.}}{=} T_n$.

The piecewise linear interpolation $a : [0,T_n] \longrightarrow \mathbb{R}$ with $a(t_n^j) = a_j$ is continuous, monotone increasing and fulfills for every $t \in\,]t_n^j, t_n^{j+1}] \cap [0,T_n]$

$$\limsup_{h\downarrow 0} \frac{a(t)-a(t-h)}{h} = \frac{a_{j+1}-a_j}{h_n} \le \frac{e^{M\,h_n}-1}{h_n}\ a_j + L\,e^{M\,h_{n+1}}\Big(b_n + (3c+7)\,h_{n+2}\Big)$$
$$\le M\,e^{M\,h_n}\ a(t) + L\,e^{M\,h_{n+1}}\Big(b_n + (3c+7)\,h_{n+2}\Big).$$

Now an upper bound of $\sup_j a_j = \sup_{[0,T_n]} a(\cdot)$ results from Gronwall's Lemma 1.5.3

$$\begin{aligned}\sup_j a_j &\le a_0\ e^{M\,(e^{M\,h_n})\,\widehat{T}} + \widehat{T}\ e^{M\,(e^{M\,h_n})\,\widehat{T}}\ L\,e^{M\,h_{n+1}} \Big(b_n + (3c+7)\,h_{n+2}\Big)\\ &\le a_0\ e^{M\,(e^{M\,h_n})\,\widehat{T}} + \widehat{T}\ e^{M\,(e^{M\,h_n})\,\widehat{T}}\ L\,e^{M\,h_{n+1}}\ b_n + C_0\ h_{n+2}.\end{aligned}$$

with a constant $C_0 = C_0(c, L, M, \widehat{T}) > 0$.

For a_0 in particular, the definitions of $\widetilde{y}_n(\cdot)$, $\widetilde{y}_{n+1}(\cdot)$ and Prop. 3.2.3 imply

$$\begin{aligned} a_0 &\le \widetilde{q}\Big(\widetilde{y}_{n+1}(h_{n+1}^{+}),\ \widetilde{y}_n(h_{n+1}^{++})\Big) + \widetilde{q}\Big(\widetilde{y}_n(h_{n+1}^{+}),\ \widetilde{y}_n(h_n^{++})\Big)\\ &\le h_{n+1}\ e^{M\,h_{n+1}}\ \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{y}_n(t_{n+1}^3),\ t_{n+1}^3),\ \ \widetilde{f}(\widetilde{y}_{n-1}(t_n^3),\ t_n^3)\Big) + c\ h_{n+1}\\ &\le h_{n+1}\ e^{M\,h_{n+1}}\ L\ \left(\widetilde{q}\Big(\widetilde{y}_n(t_{n+1}^3),\ \widetilde{y}_{n-1}(t_n^3)\Big) + \tfrac{3\,\widehat{T}}{2^n} - \tfrac{3\,\widehat{T}}{2^{n+1}}\right) + c\ h_{n+1}\\ &\le h_{n+1}\ e^{M\,h_{n+1}}\ L\ \left(\widetilde{q}\Big(\widetilde{y}_n(t_{n+1}^1 + h_n),\ \widetilde{y}_{n-1}(t_n^1 + h_{n-1})\Big) + 3\,h_{n+1}\right) + c\ h_{n+1}\\ &\le h_{n+1}\ e^{M\,h_{n+1}}\ L\ \Big(b_n + c\,(t_n^1 - t_{n+1}^1) + 3\,h_{n+1}\Big) + c\ h_{n+1}\\ &\le h_{n+1}\ e^{M\,h_{n+1}}\ L\ \ b_n + C_1(c, L, M, \widehat{T})\ h_{n+1}\end{aligned}$$

and as a consequence,

$$\begin{aligned}&\sup_t\ \widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1})^{+}),\ \widetilde{y}_n((t+h_n)^{++})\Big)\\ &\le a_0\ e^{M\,(e^{M\,h_n})\,\widehat{T}} + \widehat{T}\ L\ e^{M\,(e^{M\,h_n}\ \widehat{T}+h_{n+1})}\ b_n + C_0\ h_{n+2}\\ &\le \Big(h_{n+1}\,e^{M\,h_{n+1}} L\,b_n + C_1\ h_{n+1}\Big)\ e^{M\,(e^{M\,h_n})\,\widehat{T}} + \widehat{T}\ L\ e^{M\,(e^{M\,h_n}\ \widehat{T}+h_{n+1})}\ b_n + C_0\ h_{n+2}\\ &\le L\ e^{M\,(e^{M\,h_n}\ \widehat{T}+h_{n+1})}\ \Big(h_{n+1} + \widehat{T}\Big)\cdot b_n + C_2\cdot h_{n+1}\end{aligned}$$

with a positive constant $C_2 = C_2(c, L, M, \widehat{T})$.

So the sequence $(\widehat{b}_n)_{n\in I\!N}$ that is defined recursively by

$$\begin{aligned}\widehat{b}_1 &:= \widetilde{q}\Big(\widetilde{y}_2(h_2^{+}),\ \widetilde{y}_1(h_1^{++})\Big),\\ \widehat{b}_n &:= L\ e^{M\,(e^{M\,h_{n-1}})\,\widehat{T}+M\,h_n}\ \Big(h_n + \widehat{T}\Big)\cdot \widehat{b}_{n-1} + C_2\ h_n\end{aligned}$$

represents uniform upper estimates of $\widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1})^{+}),\ \widetilde{y}_n((t+h_n)^{++})\Big)$, i.e.

$$\sup_t\ \widetilde{q}\Big(\widetilde{y}_{n+1}((t+h_{n+1})^{+}),\ \widetilde{y}_n((t+h_n)^{++})\Big) \le \widehat{b}_n.$$

Due to assumption (4.) of Prop. 3.3.6 stating $2\,L\,\widehat{T}\,e^{M\,\widehat{T}} < 1$, there exist $\eta > 0$ and $N_0 = N_0(c, L, M, \widehat{T}) \in I\!N$ such that

$$L\ e^{M\,(e^{M\,h_{n-1}}\ \widehat{T}+h_n)}\ \Big(h_n + \widehat{T}\Big) < \eta < \tfrac{1}{2} \qquad \text{for all } n \ge N_0.$$

Thus,

$$\begin{aligned}\widehat{b}_{N_0+m} \le \eta\ \widehat{b}_{N_0+m-1} + C_2\ h_{N_0+m} &\le \eta^m\ \widehat{b}_{N_0} + \tfrac{C_2\,\widehat{T}}{2^{N_0+m}} \sum_{j=0}^{m-1} 2^j\ \eta^j\\ &\le \tfrac{1}{2^m}\ \widehat{b}_{N_0} + \tfrac{C_2\,\widehat{T}}{2^{N_0+m}}\ \tfrac{1}{1-2\,\eta}\\ &\le \operatorname{const}(c, L, M, \widehat{T})\ \ h_{N_0+m}.\end{aligned}$$

b.) We follow exactly the same track as in the first part of this proof :

Considering the subinterval for $j = 2$, Prop. 3.2.3 and 3.1.3 imply for every $t_n^2 < t \le t_n^3$

$$\begin{aligned}
&\widetilde{q}\Big(\widetilde{x}_n((t-h_n)^+),\ \widetilde{x}_{n+1}((t-h_{n+1})^{++})\Big) \\
&\quad - \ \widetilde{q}\Big(\widetilde{x}_n((s-h_n)^+),\ \widetilde{x}_{n+1}((s-h_{n+1})^{++})\Big)\Big|_{s=t_n^2} \ e^{M\,h_n} \\
&\le \int_{t_n^2}^{t_{n+1}^5} e^{M\,(t_{n+1}^5-s)}\ \widetilde{Q}^{\rightarrow\!\!|}\Big(\widetilde{f}(\widetilde{x}_0,\ 0),\quad \widetilde{f}(\widetilde{x}_n(t_n^2\ -h_{n+1}-2h_{n+1}),\ t_{n+1}^1)\Big)\ ds \\
&\ + \int_{t_{n+1}^5}^{t_n^3} e^{M\,(t_n^3-s)}\ \widetilde{Q}^{\rightarrow\!\!|}\Big(\widetilde{f}(\widetilde{x}_0,\ 0),\quad \widetilde{f}(\widetilde{x}_n(t_{n+1}^5-h_{n+1}-2h_{n+1}),\ t_{n+1}^2)\Big)\ ds \\
&\le \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,(c+1)\quad h_{n+1} \\
&\ + \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,(c+1)\ \ 2\,h_{n+1} \\
&\le \qquad e^{M\,h_{n+1}}\ L\,(c+1)\ \ 3\,h_{n+1}^2.
\end{aligned}$$

Correspondingly for $j = 3 \ldots 2^n - 1$, we obtain for all $t_n^j = t_{n+1}^{2j} < t \le t_n^{j+1}$

$$\begin{aligned}
&\widetilde{q}\Big(\widetilde{x}_n((t-h_n)^+),\ \widetilde{x}_{n+1}((t-h_{n+1})^{++})\Big) \\
&\quad - \ \widetilde{q}\Big(\widetilde{x}_n((s-h_n)^+),\ \widetilde{x}_{n+1}((s-h_{n+1})^{++})\Big)\Big|_{s=t_n^j} \ e^{M\,h_n} \\
&\le \int_{t_n^j}^{t_{n+1}^{2j+1}} e^{M\,(t_{n+1}^{2j+1}-s)}\ \widetilde{Q}^{\rightarrow\!\!|}\Big(\widetilde{f}(\widetilde{x}_{n-1}(t_n^j-h_n-2h_n),\qquad t_n^{j-3}), \\
&\qquad\qquad \widetilde{f}(\widetilde{x}_n(t_n^j-h_{n+1}-2h_{n+1}),\qquad t_{n+1}^{2j-3})\Big)\ ds \\
&\ + \int_{t_{n+1}^{2j+1}}^{t_n^{j+1}} e^{M\,(t_n^{j+1}-s)}\ \widetilde{Q}^{\rightarrow\!\!|}\Big(\widetilde{f}(\widetilde{x}_{n-1}(t_n^j-h_n-2h_n),\qquad t_n^{j-3}), \\
&\qquad\qquad \widetilde{f}(\widetilde{x}_n(t_{n+1}^{2j+1}-h_{n+1}-2h_{n+1}),\ t_{n+1}^{2j-2})\Big)\ ds \\
&\le \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{x}_{n-1}(t_n^{j-3}),\qquad \widetilde{x}_n(t_{n+1}^{2j-3})\Big)\qquad + \ \tfrac{2j-3}{2^{n+1}}\,\widehat{T} - \tfrac{j-3}{2^n}\,\widehat{T}\Big) \\
&\ + \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{x}_{n-1}(t_n^{j-3}),\qquad \widetilde{x}_n(t_n^{j-1})\Big)\qquad + \ \tfrac{2j-2}{2^{n+1}}\,\widehat{T} - \tfrac{j-3}{2^n}\,\widehat{T}\Big) \\
&= \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{x}_{n-1}(t_n^{j-1}-h_{n-1}),\ \ \widetilde{x}_n(t_{n+1}^{2j-1}-h_n)\Big)\qquad + \qquad 3\,h_{n+1}\Big) \\
&\ + \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{x}_{n-1}(t_n^{j-1}-h_{n-1}),\ \ \widetilde{x}_n(t_n^j\quad -h_n)\Big)\qquad + \qquad 4\,h_{n+1}\Big) \\
&\le \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{x}_{n-1}((s-h_{n-1})^+),\ \ \widetilde{x}_n((s-h_n)^{++})\Big)\Big|_{s=t_n^{j-1}}\ + \ (c+3)\,h_{n+1}\Big) \\
&\ + \ h_{n+1}\ e^{M\,h_{n+1}}\ L\,\Big(\widetilde{q}\Big(\widetilde{x}_{n-1}((s-h_{n-1})^+),\ \ \widetilde{x}_n((s-h_n)^{++})\Big)\Big|_{s=t_n^{j-1}}\ + (2c+4)\,h_{n+1}\Big).
\end{aligned}$$

Using now the abbreviations

$$a'_2 \;:=\; \widetilde{q}\Big(\widetilde{x}_{n-1}((t_n^2 - h_{n-1})^+),\; \widetilde{x}_n((t_n^2 - h_n)^{++})\Big) \;=\; \widetilde{q}\Big(\widetilde{x}_{n-1}(0^+),\; \widetilde{x}_n(h_n^{++})\Big) \;\le\; c \cdot h_n,$$

$$b' \;:=\; \sup_{h_{n-1} \le t \le \widehat{T}} \;\widetilde{q}\Big(\widetilde{x}_{n-1}((t - h_{n-1})^+),\; \widetilde{x}_n((t - h_n)^{++})\Big),$$

for n fixed, the recursive sequence $(a'_j)_{j \ge 2}$,

$$a'_{j+1} \;:=\; e^{M\,h_n}\, a'_j \;+\; L\, e^{M\,h_{n+1}}\, h_{n+1} \cdot 2\, b' \;+\; L\, e^{M\,h_{n+1}}\, (3c+7)\, h_{n+1}^2,$$

is monotone and satisfies for all $t \in\,]t_n^j, t_n^{j+1}]$ $\;(j = 2 \ldots 2^n - 1)$

$$\widetilde{q}\Big(\widetilde{x}_n((t - h_n)^+),\; \widetilde{x}_{n+1}((t - h_{n+1})^{++})\Big) \;\le\; a'_{j+1}.$$

Its piecewise linear interpolation $a : [t_n^2, \widehat{T}] \longrightarrow [0, \infty[$ with $a(t_n^j) = a'_j$ $(j = 2 \ldots 2^n)$ is continuous, monotone increasing and fulfills for every $t \in\,]t_n^j, t_n^{j+1}]$

$$\limsup_{h \downarrow 0} \frac{a(t)-a(t-h)}{h} \;=\; \frac{a'_{j+1}-a'_j}{h_n} \;\le\; \frac{e^{M\,h_n}-1}{h_n}\; a'_j \;+\; L\, e^{M\,h_{n+1}} \Big(b' \;+\; (3c+7)\, h_{n+2}\Big)$$

$$\le\; M\, e^{M\,h_n}\; a(t) \;+\; L\, e^{M\,h_{n+1}} \Big(b' \;+\; (3c+7)\, h_{n+2}\Big).$$

Gronwall's Lemma 1.5.3 guarantees (with a constant $C_3 = C_3(c, L, M, \widehat{T}) > 0$)

$$\sup_j\, a'_j \;\le\; c\, h_n \cdot e^{M\,(e^{M\,h_n})\,\widehat{T}} \;+\; \widehat{T}\; e^{M\,(e^{M\,h_n})\,\widehat{T}}\; L\, e^{M\,h_{n+1}}\, (b' + (3c+7)\, h_{n+2})$$

$$\le\; \widehat{T}\, L\, e^{M\,(e^{M\,h_n}\,\widehat{T} + h_{n+1})} \cdot b' \;+\; C_3\; h_n.$$

This inequality forms the basis for the rest of the proof exactly as in part (a.).

c.) Prop. 3.2.3 and 3.1.3 again guarantee for every $t_{n+1}^k < t \le t_{n+1}^{k+1}$, $2 \le k \le 2^n - 3$

$$\widetilde{q}\Big(\widetilde{x}_{n+1}(t^+),\; \widetilde{y}_{n+1}(t^{++})\Big) \;-\; \widetilde{q}\Big(\widetilde{x}_{n+1}(s^+),\; \widetilde{y}_{n+1}(s^{++})\Big)\Big|_{s = t_{n+1}^k}\; e^{M\,h_n}$$

$$\le\; \int_{t_{n+1}^k}^{t_{n+1}^{k+1}} e^{M\,(t_{n+1}^{k+1} - s)}\; \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}_n(t_{n+1}^k - 2h_{n+1}),\; t_{n+1}^{k-2}),\; \widetilde{f}(\widetilde{y}_n(t_{n+1}^k + 3h_{n+1}),\; t_{n+1}^{k+3})\Big)\; ds$$

$$\le\; h_{n+1}\; e^{M\,h_{n+1}}\; L\, \Big(\widetilde{q}\Big(\widetilde{x}_n(t_{n+1}^{k-2}),\; \widetilde{y}_n(t_{n+1}^{k+3})\Big) \;+\; \tfrac{k+3}{2^{n+1}}\, \widehat{T} - \tfrac{k-2}{2^{n+1}}\, \widehat{T}\Big)$$

$$\le\; h_{n+1}\; e^{M\,h_{n+1}}\; L\, \Big(\widetilde{q}\Big(\widetilde{x}_n(t_{n+1}^{k-2\,+}),\; \widetilde{y}_n(t_{n+1}^{k-2\,++})\Big) \;+\; c \cdot 5\, h_{n+1} \;+\; 5\, h_{n+1}\Big).$$

Defining the abbreviations (for fixed n)

$$a''_2 \;:=\; \widetilde{q}\Big(\widetilde{x}_{n+1}(t_{n+1}^{2\;+}),\; \widetilde{y}_{n+1}(t_{n+1}^{2\;++})\Big)$$

$$b'' \;:=\; \sup_{0 \le t \le T_n} \widetilde{q}\Big(\widetilde{x}_n(t^+),\; \widetilde{y}_n(t^{++})\Big),$$

the increasing recursive sequence $(a''_k)_{k \ge 2}$,

$$a''_{k+1} \;:=\; e^{M\,h_n}\, a''_k \;+\; L\, e^{M\,h_{n+1}}\, h_{n+1} \cdot b'' \;+\; 5\, L\, e^{M\,h_{n+1}}\, (c+1)\, h_{n+1}^2,$$

fulfills $\quad \widetilde{q}\Big(\widetilde{x}_{n+1}(t^+),\; \widetilde{y}_{n+1}(t^{++})\Big) \;\le\; a''_{k+1}. \quad$ for all $t \in\,]t_{n+1}^k, t_{n+1}^{k+1}] \cap\,]t_n^2, T_{n+1}]$.

Its piecewise linear interpolation $a : [t^2_{n+1}, T_{n+1}] \longrightarrow I\!R$ with $a(t^k_n) = a''_k$ is continuous, monotone increasing and satisfies for every $t \in \,]t^k_{n+1}, t^{k+1}_{n+1}]$

$$\limsup_{h \downarrow 0} \frac{a(t)-a(t-h)}{h} = \frac{a''_{j+1}-a''_j}{h_n} \leq \frac{e^{M\,h_n}-1}{h_n} \quad a''_j + L\,e^{M\,h_{n+1}} \Big(b'' + 5\,(c+1)\,h_{n+2}\Big)$$
$$\leq M\,e^{M\,h_n}\; a(t) + L\,e^{M\,h_{n+1}} \Big(b'' + 5\,(c+1)\,h_{n+2}\Big).$$

Gronwall's Lemma 1.5.3 provides the estimate

$$\begin{aligned} \sup_k\, a''_k &= \sup_{[t^2_n, T_{n+1}]} a(\cdot) \\ &\leq a''_2\; e^{M\,(e^{M\,h_n})\,\widehat{T}} + \int_0^{\widehat{T}} e^{M\,(e^{M\,h_n})\,(\widehat{T}-s)}\; L\,e^{M\,h_{n+1}}\,(b'' + 5\,(c+1)\,h_{n+2})\;ds \\ &\leq a''_2\; e^{M\,(e^{M\,h_n})\,\widehat{T}} + \widehat{T}\; e^{M\,(e^{M\,h_n})\,\widehat{T}}\; L\,e^{M\,h_{n+1}}\,(b'' + 5\,(c+1)\,h_{n+2}). \end{aligned}$$

So now we still need an adequate bound of a''_2 resulting again from Prop. 3.2.3

$$\begin{aligned} a''_2 &= \widetilde{q}\Big(\widetilde{f}(\widetilde{x}_0,0)(t^2_{n+1}{}^+, \widetilde{x}_0),\;\; \widetilde{y}_{n+1}(t^2_{n+1}{}^{++})\Big) \\ &\leq \int_0^{h_{n+1}} e^{M\,h_{n+1}}\;\; \widetilde{Q}^{\rightarrow\!\!|}\Big(\widetilde{f}(\widetilde{x}_0,0),\; \widetilde{f}(\widetilde{y}_n(t^3_{n+1}), t^3_{n+1})\Big)\;ds \\ &\quad + \int_{h_{n+1}}^{t^2_{n+1}} e^{M\,h_{n+1}}\;\; \widetilde{Q}^{\rightarrow\!\!|}\Big(\widetilde{f}(\widetilde{x}_0,0),\; \widetilde{f}(\widetilde{y}_n(t^4_{n+1}), t^4_{n+1})\Big)\;ds \\ &\leq h_{n+1}\; e^{M\,h_{n+1}}\;\; L\;\; \Big(\widetilde{q}\Big(\widetilde{x}_0,\; \widetilde{y}_n(t^3_{n+1})\Big) + 3\,h_{n+1} + \widetilde{q}\Big(\widetilde{x}_0,\; \widetilde{y}_n(t^4_{n+1})\Big) + 4\,h_{n+1}\Big) \\ &\leq h_{n+1}\; e^{M\,h_{n+1}}\;\; L\;\;\; 7\;(c+1)\;h_{n+1} \end{aligned}$$

and finally,

$$\sup_{0<t<T_{n+1}} \widetilde{q}\Big(\widetilde{x}_{n+1}(t^+),\; \widetilde{y}_{n+1}(t^{++})\Big) \leq \sup_k\, a''_k \leq L\,\widehat{T}\;e^{M\,(e^{M\,h_n}\;\widehat{T}+h_{n+1})}\;b'' + C_4\;h_{n+1}.$$

with a constant $C_4 = C_4(c, L, M, \widehat{T}) > 0$.

Now the same steps as in part (a.) result in assertion (c.). □

There is no obvious way of dispensing with standard hypotheses $(R^{\Rightarrow})$, $(R^{\Leftarrow})$ in the preceding proposition if $(\widetilde{E}, \widetilde{q})$ is assumed to be two–sided complete. Roughly speaking, these assumptions link the approximating sequences $(\widetilde{x}_n(\,\cdot\, - h_{n-1}))_{n\in I\!N}$ and $(\widetilde{y}_n(\,\cdot\, + h_{n-1})_{n\in I\!N}$ for constructing one and the same solution $\widetilde{x}(\cdot)$.

As an alternative to this approach, we now use only one approximating sequence. This requires the stronger assumption that $(\widetilde{E}, \widetilde{q})$ is one–sided complete, i.e. according to Def. 1.3.1, for any sequence $(\widetilde{z}_n)_n$ with $\widetilde{q}(\widetilde{z}_m, \widetilde{z}_n) \longrightarrow 0$ for $m, n \longrightarrow \infty$ $(m < n)$, there is an element $\widetilde{z} \in E$ such that $\widetilde{q}(\widetilde{z},\; \widetilde{z}_n) \longrightarrow 0$, $\widetilde{q}(\widetilde{z}_n,\; \widetilde{z}) \longrightarrow 0$ $(n \longrightarrow \infty)$.

Corollary 3.3.8 (Short–time existence of timed backward solutions in one–sided complete $(\widetilde{E}, \widetilde{q})$)

Suppose that $(\widetilde{E}, \widetilde{q})$ *is one–sided complete.*

Furthermore suppose for any element $\widetilde{y} \in \widetilde{E}$ *and all sequences* $(\widetilde{y}_n)$, $(\widetilde{z}_n)$ *in* $\widetilde{E}$ *that*

$$\widetilde{q}(\widetilde{y}, \widetilde{y}_n) \longrightarrow 0, \qquad \widetilde{q}(\widetilde{y}_n, \widetilde{z}_n) \longrightarrow 0 \qquad (n \longrightarrow \infty), \qquad \pi_1\, \widetilde{y}_n \le \pi_1\, \widetilde{z}_n$$

always imply $\qquad \widetilde{q}(\widetilde{y}, \widetilde{z}_n) \longrightarrow 0.$

Assume for $\widetilde{f} : \widetilde{E} \times [0, T] \longrightarrow \widetilde{\Theta}_0^{\rightarrow\!\!\shortmid}(\widetilde{E}, \widetilde{q})$

1. *there exists* $L > 0$ *such that for any* $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{E}$, $0 \le t_1 \le t_2 \le 2\,T$ *with* $\pi_1\, \widetilde{x}_1 \le \pi_1\, \widetilde{x}_2$: $\qquad \widetilde{Q}^{\rightarrow\!\!\shortmid}\Big(\widetilde{f}(\widetilde{x}_1, t_1),\ \widetilde{f}(\widetilde{x}_2, t_2)\Big) \ \le\ L \cdot \Big(\widetilde{q}(\widetilde{x}_1,\ \widetilde{x}_2)\ +\ t_2 - t_1\Big)$,
2. $M \ :=\ \sup\limits_{\widetilde{x}, t}\ \alpha^{\rightarrow\!\!\shortmid}(\widetilde{f}(\widetilde{x}, t))\ <\ \infty$,
3. $c \ \ :=\ \sup\limits_{\widetilde{x}, t}\ \beta^{\mathrm{Lip}}(\widetilde{f}(\widetilde{x}, t))\ <\ \infty$,
4. $4\,L\,T\,e^{2\,M\,T}\ <\ 1.$

For every point $\widetilde{x}_0 \in \widetilde{E}$, *there is a timed right–hand backward solution* $\widetilde{x} : [0, T[\ \longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0, T[$ *with* $\widetilde{x}(0) = \widetilde{x}_0$ *such that any other timed right–hand backward solution* $\widetilde{z}(\cdot) \in \mathrm{Lip}^{\rightarrow}([0, T[,\ \widetilde{E},\ \widetilde{q})$ *with* $\widetilde{z}(0) = \widetilde{x}_0$ *fulfills*

$$\widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})\Big) = 0 \qquad \textit{for all}\ \ t \in [0, T[.$$

Remark. This additional assumption about convergence in $(\widetilde{E}, \widetilde{q})$ does not result from the timed triangle inequality immediately because there are no restrictions on $\pi_1 \widetilde{y}$, $\pi_1\, \widetilde{y}_n$ (i.e. in particular, $\pi_1\, \widetilde{y}_n < \pi_1\, \widetilde{y}$ is also admitted).

Proof Following the same track of approximation as in the proof of Prop. 3.3.6, we obtain the sequence $(\widetilde{x}_n(\cdot))_{n \in I\!\!N}$ such that

$$\begin{array}{rcll}
\widetilde{f}(\widetilde{x}_0,\ 0) & \in & \overset{\circ}{\widetilde{x}}_n(t) & \text{for } t \in [0,\ t_n^2], \\
\widetilde{f}(\widetilde{x}_{n-1}(t_n^{j-2}),\ t_n^{j-2}) & \in & \overset{\circ}{\widetilde{x}}_n(t) & \text{for } t \in\ ]t_n^j,\ t_n^{j+1}]\ \ (j \ge 2), \\
\widetilde{q}\Big(\widetilde{x}_n(s_1),\ \ \widetilde{x}_n(s_2)\Big) & \le & c \cdot (s_2 - s_1) & \text{for } 0 \le s_1 < s_2 \le T, \\
\sup\limits_{h_{m-1} < t \le T} \widetilde{q}\Big(\widetilde{x}_m(t - h_{m-1}),\ \ \widetilde{x}_n(t - h_{n-1})\Big) & \longrightarrow & 0 & \text{for } m, n \longrightarrow \infty\ \ (m < n).
\end{array}$$

As $(\widetilde{E}, \widetilde{q})$ is one–sided complete, there exists $\widetilde{x}(t) \in \widetilde{E}$ with

$$\widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{x}_n(t - h_{n-1})\Big) \longrightarrow 0, \quad \widetilde{q}\Big(\widetilde{x}_n(t - h_{n-1}),\ \widetilde{x}(t)\Big) \longrightarrow 0 \qquad \text{for } n \longrightarrow \infty.$$

The assumption about convergence in $(\widetilde{E}, \widetilde{q})$ guarantees for $n \longrightarrow \infty$

$$\begin{array}{rcl}
\widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{x}_n(t)\Big) & \longrightarrow & 0, \\
\widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{x}_n(t + h_{n-2})\Big) & \longrightarrow & 0.
\end{array}$$

For applying the Convergence Theorem (Prop. 3.3.2) later, we prove now that the functions $\widetilde{f}_n : \widetilde{E} \times [0, T[\longrightarrow \widetilde{\Theta}_0^{\rightarrow}(\widetilde{E}, \widetilde{q}) \quad (n \in I\!N)$ fulfill the assumptions there,

$$\begin{aligned} \widetilde{f}_n(\widetilde{z}, t) &:= \widetilde{f}(\widetilde{x}_{n-1}(t_n^{j+2}), \qquad t_n^{j+2}) \qquad \text{for } t \in [t_n^j,\ t_n^{j+1}[\ \cap\ ]0, T] \\ &= \widetilde{f}(\widetilde{x}_{n-1}(t_n^{j-2} + h_{n-2}),\ t_n^{j-2} + h_{n-2}) \end{aligned}$$

For every $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E}, \quad s_1 \in [t_n^j,\ t_n^{j+1}[, \quad s_2 \in [t_n^l,\ t_n^{l+1}[\quad (\text{with } s_1 \le s_2)$,

$$\begin{aligned} \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}_n(\widetilde{y}_1, s_1),\ \widetilde{f}_n(\widetilde{y}_2, s_2)\Big) &= \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}_{n-1}(t_n^{j+2}),\ t_n^{j+2}), \quad \widetilde{f}(\widetilde{x}_{n-1}(t_n^{l+2}),\ t_n^{l+2})\Big) \\ &\le L\ \Big(\widetilde{q}(\widetilde{x}_{n-1}(t_n^{j+2}),\ \widetilde{x}_{n-1}(t_n^{l+2})) \ +\ t_n^{l+2} - t_n^{j+2}\Big) \\ &\le L\ \Big(\qquad c \qquad +\ 1\Big) \quad (t_n^l - t_n^j) \\ &\le L\ \ (c+1) \qquad (s_2 - s_1 + h_n). \end{aligned}$$

Thus, $\qquad \widetilde{Q}^{\rightarrow}\Big(\widetilde{f}_n(\widetilde{z}_1, s_1),\ \widetilde{f}_n(\widetilde{z}_2, s_2)\Big) \longrightarrow 0 \qquad \text{for } \widetilde{q}(\widetilde{z}_1, \widetilde{z}_2) \to 0, \quad s_2 - s_1 \downarrow 0, \quad n \to \infty.$

Moreover we have for $t \in [t_n^j, t_n^{j+1}[$

$$\begin{aligned} &\widetilde{Q}^{\rightarrow}\Big(\widetilde{f}(\widetilde{x}(t),\ t),\ \ \widetilde{f}_n(\widetilde{x}(t),\ t)\Big) \ = \ \widetilde{Q}^{\rightarrow}\Big(\ \widetilde{f}(\widetilde{x}(t),\ t),\ \ \widetilde{f}(\widetilde{x}_{n-1}(t_n^{j+2}),\ t_n^{j+2})\ \Big) \\ &\le\ L\ \Big(\ \widetilde{q}(\widetilde{x}(t),\ \widetilde{x}_{n-1}(t_n^{j+2})) \qquad\qquad +\ t_n^{j+2} - t\Big) \\ &\le\ L\ \Big(\ \widetilde{q}(\widetilde{x}(t),\ \widetilde{x}_{n-1}(t)) \ +\ \widetilde{q}(\widetilde{x}_{n-1}(t),\ \widetilde{x}_{n-1}(t_n^{j+2})) \ +\ 2\,h_n\Big) \\ &\le\ L\ \Big(\ \widetilde{q}(\widetilde{x}(t),\ \widetilde{x}_{n-1}(t)) \ +\ c \cdot (t_n^{j+2} - t) \qquad +\ 2\,h_n\Big) \\ &\le\ L\ \Big(\ \widetilde{q}(\widetilde{x}(t),\ \widetilde{x}_{n-1}(t)) \ +\ (2\,c+2)\ \ h_n\ \Big) \\ &\longrightarrow\ 0 \qquad\qquad (n \longrightarrow \infty). \end{aligned}$$

Each translated function $\widetilde{x}_n(\,\cdot\, + h_{n-2}) : [0, T - h_{n-2}[\longrightarrow \widetilde{E} \quad (n \in I\!N)$ is a timed right–hand backward primitive of $\widetilde{f}_n(\widetilde{x}_n(\,\cdot\, + h_{n-2}),\ \cdot\,)$. So the Convergence Theorem (Proposition 3.3.2) ensures that $\widetilde{x}(\cdot)$ is a timed right–hand backward solution of $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$ in $[0, T[$.

Finally any other timed right–hand backward solution $\widetilde{z}(\cdot) \in \mathrm{Lip}^{\rightarrow}([0, T[,\ \widetilde{E},\ \widetilde{q})$ (with $\widetilde{z}(0) = \widetilde{x}_0$) fulfills

$$\widetilde{q}(\widetilde{x}(t^+), \widetilde{z}(t^{++})) = 0 \qquad \text{for any } t.$$

Indeed, in the proof of Proposition 3.3.6 we concluded the convergence

$$\sup_{h_{n-1} \le s \le T} \ \widetilde{q}\Big(\widetilde{x}_n((s - h_{n-1})^+),\ \widetilde{z}(s^{++})\Big) \ \longrightarrow\ 0 \qquad (n \longrightarrow \infty)$$

from assumption (4.), i.e. $\ 4\,L\,T\,e^{2\,M\,T} \ <\ 1.$

For every $t \in [0, T[$ and $0 < k < k' < l < T - t$, the timed triangle inequality leads to

$$
\begin{aligned}
&\widetilde{q}\Big(\widetilde{x}(t+k),\ \widetilde{z}(t+l)\Big) \\
\le\ & \limsup_{n \longrightarrow \infty} \Big(\widetilde{q}\Big(\widetilde{x}(t+k), \widetilde{x}_n(t+k)) + \widetilde{q}\Big(\widetilde{x}_n(t+k),\ \ \widetilde{x}_n(t+k'-h_{n-1})\Big) \\
& \qquad\qquad + \widetilde{q}\Big(\widetilde{x}_n(t+k'-h_{n-1}), \quad \widetilde{z}(t+l)\Big)\Big) \\
\le\ & 0 \ + \ c\,(k'-k) \ + \ \limsup_{n \longrightarrow \infty}\ \widetilde{q}\Big(\widetilde{x}_n(t+k'-h_{n-1}), \quad \widetilde{z}(t+l)\Big) \\
\le\ & \qquad c\,(k'-k) \ + \ \limsup_{n \longrightarrow \infty}\ \widetilde{q}\Big(\widetilde{x}_n((s-h_{n-1})^+), \quad \widetilde{z}(s^{++})\ \Big)\Big|_{s=t+k'} + L_{\widetilde{z}}(l-k') \\
\le\ & \ \max\{c, L_{\widetilde{z}}\}\ \ (l-k)
\end{aligned}
$$

with $L_{\widetilde{z}} := \operatorname{Lip}^{\rightarrow} \widetilde{z}(\cdot)$. Thus, $\widetilde{q}(\widetilde{x}(t^+),\ \widetilde{z}(t^{++})) = 0.$ □

3.4 The advantage of (not timed) triangle inequality

In this section we dispense with the time component of $\widetilde{E} \stackrel{\text{Def.}}{=} I\!R \times E$ because the triangle inequality (instead of its *timed* counterpart) provides further simplifications. For the first time both in chapter 2 and 3, we prefer this stronger condition explicitly.

So now $q : E \times E \longrightarrow [0, \infty[$ for a set $E \neq \emptyset$ is to fulfill the triangle inequality. Then standard hypothesis $(R^{\Leftarrow})$ extends the relations between the upper limits

$$\widetilde{q}\Big(\widetilde{x}(t),\ \widetilde{y}(t^+)\Big) \ \leq\ \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^{++})\Big) \ \leq\ \widetilde{q}\Big(\widetilde{x}(t^+),\ \widetilde{y}(t^+)\Big)$$

of Proposition 3.1.3 and $\widetilde{q}(\widetilde{x}(t),\ \widetilde{y}(t))$ for any curves $x, y \in UC^{\rightarrow}([0,T],\ E,\ q)$. In fact, an ostensible metric q on E with standard hypothesis $(R^{\Leftarrow})$ even satisfies

$$q\Big(x(t),\ y(t^+)\Big) \ =\ q\Big(x(t^+),\ y(t^{++})\Big) \ =\ q\Big(x(t^+),\ y(t^+)\Big) \ =\ q\Big(x(t),\ y(t)\Big)$$

(Corollary 3.4.2) and so the upper limits abbreviated as "+", "++" can be omitted. This simplification serves as motivation for adapting the preceding Convergence Theorem in Corollary 3.4.4. Finally Euler method provides a timed right–hand backward solution of generalized mutational equations — without a restriction on the time interval as in Proposition 3.3.6 and Corollary 3.3.8 of the preceding section.

Lemma 3.4.1 *In addition to the triangle inequality and standard hypothesis* $(R^{\Leftarrow})$ *for* (E, q) *suppose that* $x, y : [0,T] \longrightarrow E$ *satisfy for any* $t \in [0, T[$

$$\limsup_{h \downarrow 0}\ q\Big(x(t),\ x(t+h)\Big) \ =\ 0, \qquad \limsup_{h \downarrow 0}\ q\Big(y(t),\ y(t+h)\Big) \ =\ 0.$$

Then, $\quad q\Big(x(t^+),\ y(t^+)\Big) \ \leq\ q\Big(x(t),\ y(t)\Big).$

Proof. Let $(h_n)_{n \in I\!N}$ be a sequence in $]0, T-t]$ such that $h_n \longrightarrow 0$ and

$$\lim_{n \to \infty}\ q\Big(x(t+h_n),\ y(t+h_n)\Big) \ =\ q\Big(x(t^+),\ y(t^+)\Big)$$

Then, $\quad q\Big(x(t+h_n),\ y(t+h_n)\Big) \ \leq\ q\Big(x(t+h_n),\ y(t)\Big) \ +\ q\Big(y(t),\ y(t+h_n)\Big)$

and $\quad q\Big(x(t),\ x(t+h_n)\Big) \ \longrightarrow\ 0 \quad$ for $n \longrightarrow \infty$

imply $\quad q\Big(x(t^+),\ y(t^+)\Big) \ \leq\ q\Big(x(t),\ y(t)\Big) \ +\ 0.$ □

Corollary 3.4.2 *Assume the triangle inequality and standard hypothesis* $(R^{\Leftarrow})$ *for* (E, q). *Then for every* $x, y \in UC^{\rightarrow}([0,T],\ E,\ q)$ *and* $t \in]0, T[$,

$$q\Big(x(t),\ y(t^+)\Big) \ \leq\ q\Big(x(t^+),\ y(t^{++})\Big) \ \leq\ q\Big(x(t^+),\ y(t^+)\Big) \ \leq\ q\Big(x(t),\ y(t)\Big).$$

If in addition, q *is an ostensible metric then these terms are equal.*

Proof. If (E, q) is an ostensible metric space then Lemma 1.4.7 (1.) implies

$$q\Big(x(t), y(t)\Big) \leq \limsup_{k \downarrow 0}\ \Big(q\Big(x(t),\ y(t+k)\Big) + q\Big(y(t+k),\ y(t)\Big)\Big) \ \leq\ q\Big(x(t),\ y(t^+)\Big) + 0.$$

□

Lemma 3.4.3 *Let q be an ostensible metric on E with standard hypothesis $(R^{\Leftarrow})$. Assume $c := \sup_{t \in [0,T[} \beta^{\mathrm{Lip}}(\vartheta(t)) < \infty$ for $\vartheta : [0,T[\longrightarrow \Theta_p^{\rightarrow\!\!|}(E,q)$ and let $x : [0,T[\longrightarrow E$ satisfy*

$$\wedge \begin{cases} \limsup_{h \downarrow 0} \frac{q\left(\vartheta(t-h)\left(h^+, x(t-h)\right),\ x(t^{++})\right)}{h} \;\leq\; R < \infty & \forall\ t \in]0,T[, \\ q\Big(x(t),\ x(t+h)\Big) \;\longrightarrow\; 0 \qquad \text{for } h \downarrow 0 & \forall\ t \in [0,T[, \\ q\Big(x(t-h),\ x(t)\Big) \;\longrightarrow\; 0 \qquad \text{for } h \downarrow 0 & \forall\ t \in]0,T[, \end{cases}$$

Then, $\quad q\Big(x(s),\ x(t)\Big) \;\leq\; (c+R)\cdot(t-s) \quad$ *for every* $0 \leq s < t < T$.
In particular, $x(\cdot)$ *is a right–hand backward primitive of* $\vartheta : [0,T[\longrightarrow \Theta_p^{\rightarrow\!\!|}(E,q)$.

Proof. is based on applying Gronwall's Lemma 1.5.3 to $\varphi(t) := q\Big(x(s),\ x(t^+)\Big)$
To be more precise, we are going to conclude the Lipschitz continuity of $\varphi(\cdot)$ from the first two assumptions about $x(\cdot)$. Finally the third condition of $x(\cdot)$ leads to the claimed Lipschitz continuity of $x(\cdot)$.
(We cannot apply Corollary 3.4.2 so far because $x \in UC^{\rightarrow}([0,T],\ E,\ q)$ is not supposed).
Indeed, as an immediate consequence of the definition, $\varphi(\cdot)$ is upper semicontinuous in terms of
$$\varphi(t) \;\geq\; \limsup_{h \downarrow 0}\ \varphi(t+h) \qquad \text{for all } t.$$
Besides, we obtain for every $0 < k < k' < l$ (with $t_h := t - h > s$ and $k < h$)

$$\begin{aligned} & q\Big(x(s),\ x(t+l)\Big) \;-\; q\Big(x(s),\ x(t_h+k)\Big) \\ \leq\;\; & q\Big(x(t_h+k),\ \ \vartheta(t_h)\,\big(h,\ x(t_h)\big)\Big) \\ & +\ c\cdot(h+k'\ -\ h) \\ & +\ q\Big(\vartheta(t_h)\,\big(h+k',\ x(t_h)\big),\ \ x(t+l)\Big) \end{aligned}$$

as a consequence of Lemma 3.1.2. Let $\eta > 0$ be chosen arbitrarily. There exists $h_0 = h_0(\eta,t)$ such that for every $h \in]0,h_0]$ and all positive $k' < l$ sufficiently small (depending on h,η,t)
$$\widetilde{q}\Big(\vartheta(t_h)\,\big(h+k',\ x(t_h)\big),\ \ x(t+l)\Big) \;<\; (R+\eta)\,h.$$
Since the right–hand spheres of (E,q) are left–sequentially closed, the assumption $q(x(t_h),\ x(t_h+k)) \longrightarrow 0\ \ (k \downarrow 0)$ guarantees $q(x(t_h+k),\ x(t_h)) \longrightarrow 0\ \ (k \downarrow 0)$ according to Lemma 1.4.7 (1.) and thus, for all $k > 0$ small enough
$$\begin{aligned} q\Big(x(t_h+k),\ \ \vartheta(t_h)\,\big(h,\ x(t_h)\big)\Big) \;&<\; q\Big(x(t_h),\ \ \vartheta(t_h)\,\big(h,\ x(t_h)\big)\Big) \;+\; \eta\,h \\ &\leq\; c\ h \;+\; \eta\,h. \end{aligned}$$
So finally, $\quad \varphi(t) \;\leq\; \varphi(t-h) \;+\; (c+R)\,h \;+\; 2\,\eta\,h \quad$ for every $h < h_0(\eta,t)$
and, Gronwall's Lemma 1.5.3 implies the Lipschitz continuity of $\varphi(\cdot)$.

It is still to prove that $x(\cdot)$ is a right–hand backward primitive of $\vartheta(\cdot)$. Strictly speaking, only condition (3.) on right–hand backward primitives (Def. 3.2.1) is missing so far, i.e. $$q\Big(\vartheta(t)\,(0^+,\,x(t)),\;\; x(t^{++})\Big) \;=\; 0 \qquad \text{for all } t \in [0,T[.$$
Obviously it results from Corollary 3.4.2. □

Now we adapt Convergence Theorem 3.3.2 to an ostensible metric space (E,q) with standard hypothesis $(R^{\Leftarrow})$ (i.e. right–hand spheres are left–sequentially closed) so that we can omit the upper limits abbreviated as "+", "++". In particular, the assumption about the convergence of $(x_n(\cdot))_{n\in I\!N}$ is simplified.
Considering just one ostensible metric q on E is merely for the sake of simplicity because this result is used for concluding long–time existence of a solution from one–sided completeness in Proposition 3.4.5 afterwards.

Corollary 3.4.4 (of Convergence Theorem 3.3.2)
In addition to standard hypothesis $(R^{\Leftarrow})$ for each ostensible metric space (E,q), suppose the following properties of

$$\begin{array}{llcl} f_m,\;\; f: & E\times[0,T[& \longrightarrow & \Theta_p^{\rightarrow}(E,q) \qquad (m\in I\!N) \\ x_m,\;\; x: & [0,T[& \longrightarrow & E\;: \end{array}$$

1. $$\begin{array}{lll} M & := & \sup_{m,t,z} \{\, \alpha^{\rightarrow}(f_m(z,t)),\; \alpha^{\rightarrow}(f(z,t)) \,\} \;<\; \infty, \\ c & := & \sup_{m,t,z} \beta^{\text{Lip}}(f_m(z,t)) \;<\; \infty, \\ R & \geq & \sup_{m,t,z} \{\, \widehat{\gamma}(t,\,x_m,\,f_m(x_m,\cdot)),\;\; \gamma(f_m(z,t)),\;\; \gamma(f(z,t)) \,\} \\ \textit{with} & & \limsup_{\varepsilon'\downarrow 0}\; \varepsilon'^p\cdot R_{\varepsilon'} \;=\; 0, \end{array}$$
2. $\limsup\; Q_\varepsilon^{\rightarrow}\Big(f_m(z_1,t_1),\;\; f_m(z_2,t_2)\Big) \;\leq\; R$ *for* $m\to\infty,\;\; t_2-t_1\downarrow 0,\;\; q(z_1,z_2)\to 0,$
3. $\limsup\; Q_\varepsilon^{\rightarrow}\Big(f(x(t),t),\; f_m(x(t),t)\Big) \;\leq\; R$ *for* $m\to\infty \quad \forall\; t\in[0,T[,$
4. $\overset{\circ}{x}_m(\cdot) \ni f_m(x_m(\cdot),\,\cdot)$ *in* $[0,T[\,.$

5.' $\forall\; t\in[0,T[\quad \exists\; (\delta'_m)_{m\in I\!N}$ *in* $[0,1[\,:\;\; q\Big(x(t),\, x_m(t+\delta'_m)\Big) \longrightarrow 0, \quad \delta'_m\longrightarrow 0.$

Then $x(\cdot)$ *is a right–hand backward solution of* $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\,\cdot)$ *in* $[0,T[$, *i.e. equivalently*

$$\wedge\left\{\begin{array}{llll} \exists\,\widehat{\gamma}(t)<\infty\;:\limsup_{h\downarrow 0}\; \frac{q\left(f(x(t-h),\,t-h)\,\left(h^+,\,x(t-h)\right),\;x(t^{++})\right)}{h} \;\leq\; \widehat{\gamma}(t) & & & \forall\; t\in]0,T[, \\ q\Big(x(t),\; x(t+h)\Big) \;\longrightarrow\; 0 & & \textit{for } h\downarrow 0 & \forall\; t\in[0,T[, \\ q\Big(x(t-h),\; x(t)\Big) \;\longrightarrow\; 0 & & \textit{for } h\downarrow 0 & \forall\; t\in]0,T[. \end{array}\right.$$

Proof. Due to standard hypothesis $(R^{\Leftarrow})$ for (E, q), Lemma 3.4.3 and assumption (1.) provide a uniform bound of the Lipschitz constants of $x_m(\cdot)$ with respect to q, i.e. $\mathrm{Lip}^{\rightarrow}(x_m(\cdot), q) \leq c$.
As q is an ostensible metric on E, condition (5.') guarantees

$$q\Big(x_m(t + \delta'_m),\ x(t)\Big) \longrightarrow 0 \qquad\qquad \text{for } m \longrightarrow \infty$$

due to Lemma 1.4.7 (1.) and thus, $q\Big(x_m(t),\ x(t)\Big) \longrightarrow 0.$
So Convergence Theorem 3.3.2 (for $\widetilde{E} = I\!R \times E$, $\widetilde{q}_\varepsilon((s, y), (t, z)) := |s - t| + q(y, z)$ and $\widetilde{x}_m(t) := (t, x_m(t))$) has the consequence that $x(\cdot)$ is a right–hand backward solution. Finally the equivalent description of the right–hand backward solution results from Lemma 3.4.3. □

Proposition 3.4.5 (Long–time existence of backward solutions in ostensible metric spaces)

Let the ostensible metric space (E, q) be one–sided complete and fulfill standard hypothesis $(R^{\Leftarrow})$ (i.e. its right–hand spheres are left–sequentially closed).

Assume for $f : E \times [0, T] \longrightarrow \Theta_0^{\rightarrow}(E, q)$

1. *there exists $L > 0$ such that for any $x_1, x_2 \in E$, $0 \leq t_1 \leq t_2 \leq T$,*
   $Q^{\rightarrow}\Big(f(x_1, t_1),\ f(x_2, t_2)\Big) \leq L \cdot \Big(q(x_1,\ x_2) + t_2 - t_1\Big),$
2. $M := \sup_{x,t} \alpha^{\rightarrow}(f(x, t)) < \infty,$
3. $c := \sup_{x,t} \beta^{\mathrm{Lip}}(f(x, t)) < \infty,$

For every point $x_0 \in E$ there exists a right–hand backward solution $x : [0, T[\longrightarrow E$ of the generalized mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ in $[0, T[$ with $x(0) = x_0$ such that any other right–hand backward solution $z(\cdot)$ (with $z(0) = x_0$) fulfills

$$q\Big(x(t),\ z(t)\Big) = 0 \qquad\qquad \textit{for all } t \in [0, T[.$$

Proof is based on the same ideas of Euler approximation as in Proposition 3.3.6. The preceding Corollary 3.4.4 then takes on the role of the Convergence Theorem 3.3.2. Furthermore Corollary 3.4.2 admits omitting upper limits denoted by "+" or "++". As a direct consequence, there are no time shifts required now for constructing Euler approximations. So we can avoid additional upper bounds of T (like assumption (4.) of Prop. 3.3.6) when proving the convergence or the comparison with other right–hand backward solutions.

For each $n \in I\!N$ and $j = 0 \ldots 2^n$ set $h_n := \frac{T}{2^n}, t_n^j := j\, h_n$, $x_n(0) := x_n^0 := x_0$ and

$$\begin{aligned} x_n(t) &:= f(x_n^j,\, t_n^j)\,\Big(t - t_n^j,\, x_n^j\Big) && \text{for } t \in]t_n^j,\, t_n^{j+1}], \\ x_n^{j+1} &:= x_n(t_n^{j+1}). \end{aligned}$$

We suppose n to be so large that $L\, h_n < 1$. Then,

$$\begin{aligned} f(x_n^j,\, t_n^j) &\in \overset{\circ}{x}_n(t) && \text{for } t \in]t_n^j,\, t_n^{j+1}], \\ q\Big(x_n(s_1),\, x_n(s_2)\Big) &\le c \cdot (s_2 - s_1) && \text{for } 0 \le s_1 < s_2 \le T, \\ t_{n+1}^{2\,j} &= \tfrac{2j}{2^{n+1}}\, T = t_n^j. \end{aligned}$$

The next aim is an upper estimate of $q(x_n(t), x_{n+1}(t))$ for $t_n^j \le t \le t_n^{j+1}$. In the first part of this interval (i.e. for $t_n^j = t_{2n}^{2j} \le t \le t_{n+1}^{2j+1}$) we conclude from Prop. 3.1.4

$$\begin{aligned} & q\Big(x_n(t),\, x_{n+1}(t)\Big) \\ \le\ & q\Big(x_n^j,\, x_{n+1}^{2j}\Big)\ e^{M\,(t-t_n^j)} + \int_0^{t-t_n^j} e^{M\,(t-t_n^j-s)}\ Q^{\rightarrow\!\!\mid}\Big(f(x_n^j, t_n^j),\ f(x_{n+1}^{2j}, t_n^j)\Big)\ ds \\ \le\ & q\Big(x_n^j,\, x_{n+1}^{2j}\Big)\ e^{M\,h_{n+1}} + \int_0^{t-t_n^j} e^{M\,(t-t_n^j-s)}\ L\ q\Big(x_n^j,\, x_{n+1}^{2j}\Big)\ ds \\ \le\ & q\Big(x_n^j,\, x_{n+1}^{2j}\Big)\ e^{M\,h_{n+1}}\ (1 + L\, h_{n+1}). \end{aligned}$$

Correspondingly, for $t_{2n}^{2j+1} \le t \le t_{n+1}^{2j+2} = t_n^{j+1}$,

$$\begin{aligned} & q\Big(x_n(t),\, x_{n+1}(t)\Big) \\ \le\ & q\Big(x_n(t_{2n}^{2j+1}),\, x_{n+1}^{2j+1}\Big)\ e^{M\,(t-t_{2n}^{2j+1})} \\ & + \int_0^{t-t_{2n}^{2j+1}} e^{M\,(t-t_{2n}^{2j+1}-s)}\ Q^{\rightarrow\!\!\mid}\Big(f(x_n^j, t_n^j),\ f(x_{n+1}^{2j+1}, t_{2n}^{2j+1})\Big)\ ds \end{aligned}$$

and due to assumption (1.),

$$\begin{aligned} & q\Big(x_n(t),\, x_{n+1}(t)\Big) \\ \le\ & q\Big(x_n(t_{2n}^{2j+1}),\, x_{n+1}^{2j+1}\Big)\ e^{M\,h_{n+1}} \\ & + h_{n+1}\ e^{M\,h_{n+1}}\ L\ \Big(q\Big(x_n^j,\, x_{n+1}^{2j+1}\Big) + t_{2n}^{2j+1} - t_n^j\Big) \\ \le\ & q\Big(x_n(t_{2n}^{2j+1}),\, x_{n+1}^{2j+1}\Big)\ e^{M\,h_{n+1}} \\ & + h_{n+1}\ e^{M\,h_{n+1}}\ L\ \Big(q\Big(x_n^j,\, x_n(t_{2n}^{2j+1})\Big) + q\Big(x_n(t_{2n}^{2j+1}),\, x_{n+1}^{2j+1}\Big) + h_{n+1}\Big) \\ \le\ & q\Big(x_n(t_{2n}^{2j+1}),\, x_{n+1}^{2j+1}\Big)\ e^{M\,h_{n+1}} \\ & + h_{n+1}\ e^{M\,h_{n+1}}\ L\ \Big(c\, h_{n+1} + q\Big(x_n(t_{2n}^{2j+1}),\, x_{n+1}^{2j+1}\Big) + h_{n+1}\Big) \\ \le\ & q\Big(x_n(t_{2n}^{2j+1}),\, x_{n+1}^{2j+1}\Big)\ e^{M\,h_{n+1}}\ (1 + L\, h_{n+1}) + L\,(c+1)\, e^{M\,h_{n+1}}\ h_{n+1}^2 \\ \le\ & q\Big(x_n^j,\, x_{n+1}^{2j}\Big)\ e^{2\,M\,h_{n+1}}\ (1 + L\, h_{n+1})^2 + L\,(c+1)\, e^{M\,h_{n+1}}\ h_{n+1}^2 \end{aligned}$$

Combining these results, we obtain for every $t_n^j \le t \le t_n^{j+1}$ (since $L\,h_{n+1} < 1$)

$$\begin{aligned} & q\Big(x_n(t),\ x_{n+1}(t)\Big) \\ \le\ & q\Big(x_n^j,\ x_{n+1}^{2j}\Big)\ e^{2\,M\,h_{n+1}}\ (1 + L\,h_{n+1})^2 + L\,(c+1)\ e^{M\,h_{n+1}}\ h_{n+1}^2 \\ \le\ & q\Big(x_n^j,\ x_{n+1}^{2j}\Big)\ e^{M\,h_n}\ (1 + 2L\,h_{n+1} + L^2\,h_{n+1}^2) + L\,(c+1)\ e^{M\,h_{n+1}}\ h_{n+1}^2 \\ \le\ & q\Big(x_n^j,\ x_{n+1}^{2j}\Big)\ e^{M\,h_n}\ (1 + 3\,L\,h_n) + L\,(c+1)\ e^{M\,h_n}\ h_n^2 \end{aligned}$$

and using the abbreviations

$$a \ :=\ e^{M\,h_n}\,(1 + 3\,L\,h_n), \qquad b\ :=\ L\,(c+1)\ e^{M\,h_n}\ h_n^2,$$

the induction principle provides the estimate for $n \ge n_0$ (with some $n_0 = n_0(L, M, T)$)

$$\begin{aligned} q\Big(x_n(t),\ x_{n+1}(t)\Big) \ &\le\ b \sum_{k=0}^{2^n-1} a^k \qquad\qquad \le\ \tfrac{b}{a-1}\ (a^{2^n} - 1) \\ &\le\ \tfrac{b}{a-1}\ \Big(e^{M\,h_n\,2^n}\ (1 + 3\,L\,h_n)^{2^n} - 1\Big) \ \le\ \tfrac{b}{a-1}\ \Big(e^{M\,T}\ e^{3\,L\,T} - 1\Big) \\ &\le\ h_n\ \ L\,(c+1)\ \frac{e^{M\,h_n}\ \ h_n}{e^{M\,h_n}\ (1 + 3\,L\,h_n)\ -\ 1}\ \ \Big(e^{M\,T}\ e^{3\,L\,T} - 1\Big) \\ &<\ \frac{T}{2^n}\ \ \frac{L\,(c+1)}{3L + M}\ e^{(3L+M)\,T}. \end{aligned}$$

Thus $(x_n)_{n\in I\!N}$ is a Cauchy sequence in terms of

$$\sup_t\ q\Big(x_m(t),\ x_n(t)\Big) \longrightarrow 0 \qquad\qquad \text{for } m, n \longrightarrow \infty \ \ (m \le n).$$

As (E, q) is one–sided complete, there is $x(t) \in E$ with $q\Big(x(t), x_n(t)\Big) \longrightarrow 0$ for $n \longrightarrow \infty$.

Now for applying Corollary 3.4.4 we prove that $f_n\ :\ E \times [0,T] \longrightarrow \Theta_0^{\rightarrow\!\shortmid}(E,q)$ $(n \in I\!N)$ satisfy the assumptions there

$$f_n(y,t)\ :=\ f(x_n^{j+1},\ t_n^{j+1}) \qquad\qquad \text{for } t \in [t_n^j,\ t_n^{j+1}[$$

For every $y_1, y_2 \in E$, $s_1 \in [t_n^j,\ t_n^{j+1}[$, $s_2 \in [t_n^l,\ t_n^{l+1}[$ (with $s_1 \le s_2$),

$$\begin{aligned} Q^{\rightarrow\!\shortmid}\Big(f_n(y_1,s_1),\ f_n(y_2,s_2)\Big) \ &=\ Q^{\rightarrow\!\shortmid}\Big(f(x_n(t_n^{j+1}),\ t_n^{j+1}),\ f(x_n(t_n^{l+1}),\ t_n^{l+1})\Big) \\ &\le\ L\ \Big(q(x_n(t_n^{j+1}), x_n(t_n^{l+1}))\ +\ t_n^{l+1} - t_n^{j+1}\Big) \\ &\le\ L\ \Big(\qquad c \qquad +\ 1\Big) \quad (s_2 - s_1 + h_n). \end{aligned}$$

Thus, $\quad Q^{\rightarrow\!\shortmid}\Big(f_n(y_1,s_1),\ f_n(y_2,s_2)\Big) \longrightarrow 0 \quad$ for $q(y_1,y_2) \to 0$, $s_2 - s_1 \downarrow 0$, $n \to \infty$.

Moreover we have for $t \in [t_n^j, t_n^{j+1}[$

$$\begin{aligned} Q^{\rightarrow\!\shortmid}\Big(f(x(t),\ t),\ f_n(x(t),\ t)\Big) \ &=\ Q^{\rightarrow\!\shortmid}\Big(f(x(t),\ t),\ f(x_n(t_n^{j+1}),\ t_n^{j+1})\Big) \\ &\le\ L\ \Big(q(x(t),\ x_n(t_n^{j+1})) \qquad\qquad + t_n^{j+1} - t\Big) \\ &\le\ L\ \Big(q(x(t),\ x_n(t))\ +\ q(x_n(t),\ x_n(t_n^{j+1}))\ +\ h_n\Big) \\ &\le\ L\ \Big(q(x(t),\ x_n(t))\ +\ c\,(t_n^{j+1} - t) \qquad +\ h_n\Big) \\ &\le\ L\ \Big(q(x(t),\ x_n(t))\ +\ (c+1)\ h_n\Big) \qquad \longrightarrow\ 0. \end{aligned}$$

Finally, each translated function $x_n(\cdot + h_n) : [0, T - h_n] \longrightarrow E \;\; (n \in \mathbb{N})$ is a right–hand backward primitive of $f_n(x_n(\cdot + h_n), \cdot)$, and

$$q\Big(x(t),\, x_n(t+h_n)\Big) \;\le\; q\Big(x(t),\, x_n(t)\Big) + c\,h_n \;\longrightarrow\; 0 \qquad (n \longrightarrow \infty).$$

So Corollary 3.4.4 guarantees that $x(\cdot)$ is a right–hand backward solution of the generalized mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ in $[0, T[$.

As a consequence of the integral version of Gronwall's Lemma 1.5.4, any other backward solution $z(\cdot)$ (with $z(0) = x_0$) fulfills $q(x(\cdot), z(\cdot)) = 0$ on $]0, T]$.
Indeed, for every $n \in \mathbb{N}$, $q(x_n(h), z(h)) \;=\; q\Big(f(x_0, 0)\,\big(h, x_0\big),\;\; z(h)\Big) \;\longrightarrow\; 0 \;\; (h \downarrow 0)$ and we conclude from the generalized mutational equation of $x_n(\cdot)$ and Prop. 3.2.3 that for $t \in\,]0, T]$

$$\begin{aligned}
& q\Big(x_n(t),\; z(t)\Big) \\
\le\; & \int_0^t e^{c\,(t-s)}\; L\; \Big(q\Big(x_n([\tfrac{s}{h_n}]\,h_n),\;\; z(s)\Big) \;+\; s - [\tfrac{s}{h_n}]\,h_n \Big)\; ds \\
\le\; & \int_0^t e^{c\,(t-s)}\; L\; \Big(q\Big(x_n([\tfrac{s}{h_n}]\,h_n),\;\; x_n(s)\Big) + q\Big(x_n(s),\; z(s)\Big) \;+\; h_n \Big)\; ds \\
\le\; & \int_0^t e^{c\,(t-s)}\; L\; \Big(c\,h_n + q\Big(x_n(s),\; z(s)\Big) \;+\; h_n \Big)\; ds \\
\le\; & L\,(c+1)\,h_n\; \tfrac{e^{c\,T}-1}{c} \;+\; L\,e^{c\,T} \cdot \int_0^t q\Big(x_n(s),\; z(s)\Big)\; ds.
\end{aligned}$$

So correspondingly to the proof of Prop. 3.3.3, the function $\varphi_n(t) := \sup_{0 < s \le t} q\Big(x_n(s),\; z(s)\Big)$ is nondecreasing and fulfills

$$\varphi_n(t) \;\le\; L\,(1 + \tfrac{1}{c})\,h_n\,e^{c\,T} \;+\; L\,e^{c\,T} \cdot \int_0^t \varphi_n(s)\; ds.$$

Due to the integral version 1.5.4 of Gronwall's Lemma,

$$\begin{aligned}
\varphi_n(t) \;&\le\; L\,(1 + \tfrac{1}{c})\,h_n\,e^{c\,T} \;+\; \int_0^t e^{L\,e^{c\,T}\,(t-s)} \cdot L\,e^{c\,T} \cdot L\,(1 + \tfrac{1}{c})\,h_n\,e^{c\,T}\; ds \\
&\le\; C_1(L, c, T)\;\; h_n \\
&\longrightarrow\; 0 \qquad\qquad (n \longrightarrow \infty)
\end{aligned}$$

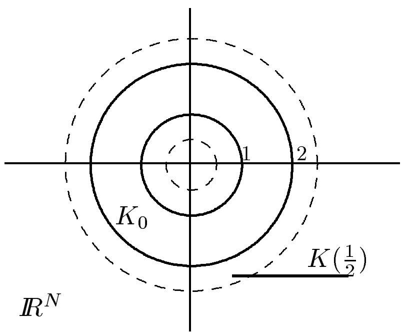
(with adequate choice of C_1).
According to the construction of $x(\cdot)$, the sequence $q(x(t),\, x_n(t)) \;\; (n \in \mathbb{N})$ also converges to 0. Thus, $q(x(t),\, z(t)) \;=\; 0$ for every $t \in\,]0, T]$. □

Chapter 4

Examples: Bounded subsets of $I\!R^N$ and C^0 semigroups on reflexive Banach spaces

Now the concepts of forward and backward solutions are applied to two main examples.

The first one belongs to the domain of shape evolution. Here we consider both the set $\mathcal{K}(I\!R^N)$ of nonempty compact subsets of $I\!R^N$ and the set $\Omega(I\!R^N)$ containing all nonempty bounded open subsets of $I\!R^N$ with several ostensible metrics. Some of them use the topological boundary and even the limiting normal cones explicitly.
A significant difference between considering $\mathcal{K}(I\!R^N)$ and $\Omega(I\!R^N)$ is shown by this very easy example :



Let $K_0 := I\!B_2 \setminus \overset{\circ}{I\!B}_1$ expand into all direction at a constant speed of 1. This leads to the compact sets

$$K(t) = \begin{cases} I\!B_{t+2} \setminus \overset{\circ}{I\!B}_{1-t} & \text{at time } t < 1, \\ I\!B_{t+2} & \text{at time } t \geq 1. \end{cases}$$

The same expansion makes the corresponding open set $O_0 := \overset{\circ}{K}_0 = \overset{\circ}{I\!B}_2 \setminus I\!B_1$ evolve into

$$O(t) = \begin{cases} \overset{\circ}{I\!B}_{t+2} \setminus I\!B_{1-t} & \text{at time } t \leq 1, \\ \overset{\circ}{I\!B}_{t+2} & \text{at time } t > 1. \end{cases}$$

So at the compact sets, the "hole" at 0 appears for all times $t < 1$ whereas the open sets show this "hole" up to time 1 inclusive. Roughly speaking, this particular feature of the boundary is preserved a moment longer. (We come back to this easy example in Appendix A.)

After defining the ostensible metrics $e^{\supset}$, $q_{\mathcal{K},N}$ on $\mathcal{K}(I\!R^N)$ and $q_{\Omega,\partial}$, q_{Ω,N_c} on $\Omega(I\!R^N)$ respectively, we consider some of their topological properties in § 4.1, 4.2 – particularly with respect to the terms introduced in chapter 1.
Concerning the regularity of the boundary, the so–called sets of positive erosion play an important role because roughly speaking, they represent the most regular sets to expect if topological changes (like "disappearing holes") are not excluded a priori. They are defined by means of an interior sphere condition and can be characterized by a collection of equivalent properties given in § 4.3.
For generalized mutational equations in $\mathcal{K}(I\!R^N)$ and $\Omega(I\!R^N)$, we need transitions. In § 4.4, their constructions are based on the deformation along smooth vector fields and differential inclusions respectively using some technical results of Appendix A.

The second main example deals with evolution equations in reflexive Banach spaces and is presented in § 4.5. Using the tools of C^0 semigroups, we consider semilinear equations whose mild solutions induce forward transitions. Demanding uniform continuity with respect to time on them is an obstacle to C^0 semigroups that we overcome by means of pseudo–metrics inducing the weak topology.

Finally § 4.6 deals with systems of these two main examples and thus provides existence results for a general type of free boundary problems.

4.1 Nonempty compact subsets of $I\!R^N$: $\mathcal{K}(I\!R^N)$

4.1.1 Pompeiu–Hausdorff excess $e^{\supset}$ and distance d

For any nonempty set $M \subset I\!R^N$, we define the *Pompeiu–Hausdorff distance* of $x \in I\!R^N$ as

$$\operatorname{dist}(x, M) \quad := \inf\{|x-y| \mid y \in M\}.$$

It induces the so–called *Pompeiu–Hausdorff excesses* between nonempty bounded subsets $M_1, M_2 \subset I\!R^N$

$$e^{\subset}(M_1, M_2) \;:=\; \sup_{x \in M_1} \operatorname{dist}(x, M_2)$$
$$e^{\supset}(M_1, M_2) \;:=\; \sup_{y \in M_2} \operatorname{dist}(y, M_1)$$

and furthermore the *Pompeiu–Hausdorff distance* between $M_1, M_2 \subset I\!R^N$

$$d(M_1, M_2) \;:=\; \max\{\, e^{\subset}(M_1, M_2),\;\; e^{\supset}(M_1, M_2)\}$$

(see [2, Aubin 99], § 3.2 and [55, Rockafellar, Wets 98], § 4.C, for example).
Moreover, set $I\!B_r(K) \;:=\; \{x \in I\!R^N \mid \operatorname{dist}(x, K) \le r\}$ for any $K \in \mathcal{K}(I\!R^N)$, $r \ge 0$ and as abbreviations, $I\!B_r := I\!B_r(0)$, $I\!B := I\!B_1(0) \subset I\!R^N$, $\|K\|_\infty := \sup_{z \in K} |z|$.

For the sake of completeness, the following lemma provides some well–known properties of $e^{\subset}$, $e^{\supset}, d\!\!l$. Most of them are obvious and their proofs are easy to find in the literature, see e.g. [2, Aubin 99], § 3.2, § 5.2 and [55, Rockafellar, Wets 98].

Proposition 4.1.1

1. $e^{\subset} : \mathcal{K}(I\!R^N) \times \mathcal{K}(I\!R^N) \longrightarrow [0,\infty[$ *satisfies the triangle inequality and* $e^{\subset}(K_1, K_2) \le \rho$ *is equivalent to* $K_1 \subset I\!B_\rho(K_2)$.
2. $e^{\supset} : \mathcal{K}(I\!R^N) \times \mathcal{K}(I\!R^N) \longrightarrow [0,\infty[$ *satisfies the triangle inequality and* $e^{\supset}(K_1, K_2) \le \rho$ *is equivalent to* $K_2 \subset I\!B_\rho(K_1)$.
3. $d\!\!l : \mathcal{K}(I\!R^N) \times \mathcal{K}(I\!R^N) \longrightarrow [0,\infty[$ *is a metric on* $\mathcal{K}(I\!R^N)$ *and* $$d\!\!l(K_1,K_2) = \sup_{x \in I\!R^N} \Big|\operatorname{dist}(x,K_1) - \operatorname{dist}(x,K_2)\Big| \qquad \text{for all } K_1, K_2 \in \mathcal{K}(I\!R^N).$$
4. *For any* $M \in \mathcal{K}(I\!R^N)$, *the nonempty compact subsets of* M *are sequentially compact with respect to* $d\!\!l$, *i.e. every sequence* $(K_n)_{n \in I\!N}$ *in* $\mathcal{K}(I\!R^N)$ *with* $K_n \subset M$ *has a subsequence* $(K_{n_j})_{j \in I\!N}$ *and a set* $K \in \mathcal{K}(I\!R^N)$ *such that* $$d\!\!l(K_{n_j}, K) \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

□

As an easy consequence, we obtain

Corollary 4.1.2

1. $(\mathcal{K}(I\!R^N), d\!\!l)$ *is a complete metric space.*
2. $(\mathcal{K}(I\!R^N), e^{\supset})$ *is two–sided sequentially compact,* *i.e. for every* $M \in \mathcal{K}(I\!R^N)$, $r > 0$ *and any sequence* (K_n) *in* $\mathcal{K}(I\!R^N)$ *with* $e^{\supset}(M, K_n) \le r$ *for all* n, *there exist a subsequence* $(K_{n_j})_{j \in I\!N}$ *and* $K \in \mathcal{K}(I\!R^N)$ *such that* $e^{\supset}(K_{n_j}, K) \longrightarrow 0$, $e^{\supset}(K, K_{n_j}) \longrightarrow 0$ $(j \longrightarrow \infty)$ *(see Def. 0.6.8).*
3. $(\mathcal{K}(I\!R^N), e^{\supset})$ *is one–sided complete, i.e. according to Def. 1.3.1, every* (K_n) *in* $\mathcal{K}(I\!R^N)$ *with* $e^{\supset}(K_m, K_n) \longrightarrow 0$ *for* $m, n \longrightarrow \infty$ $(m < n)$ *has a set* $K \in \mathcal{K}(I\!R^N)$ *such that* $e^{\supset}(K, K_n) \longrightarrow 0$, $e^{\supset}(K_n, K) \longrightarrow 0$.

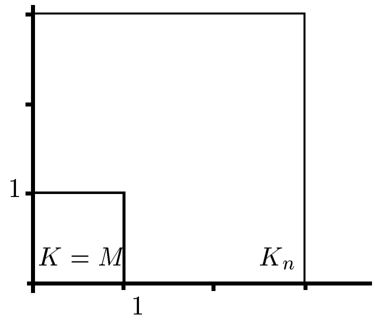
Proof results easily from the compactness of $(\mathcal{K}(I\!R^N), d\!\!l)$ and the general inequality $e^{\supset}(K_1, K_2) \le d\!\!l(K_1, K_2)$ for $K_1, K_2 \in \mathcal{K}(I\!R^N)$. □

Obviously the metric space $(\mathcal{K}(I\!R^N), d\!\!l)$ satisfies the standard hypotheses $(L^{\Rightarrow})$, $(R^{\Rightarrow})$, $(R^{\Leftarrow})$ (as an immediate consequence of symmetry and triangle inequality of $d\!\!l$).

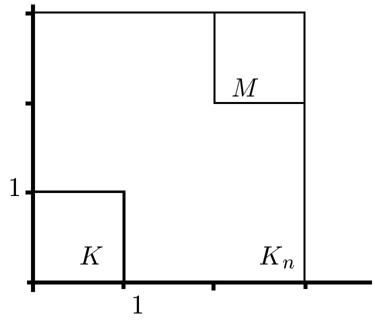
$(\mathcal{K}(\mathbb{R}^N),\, e^{\supset})$ does not fulfill any of the standard hypotheses $(L^{\Rightarrow}), (R^{\Rightarrow}), (R^{\Leftarrow})$ because $e^{\supset}$ is an ostensible metric, but right–convergence does not imply left–convergence and not vice versa. As an illustrative argument, we give the following simple counterexamples :

Example 4.1.3

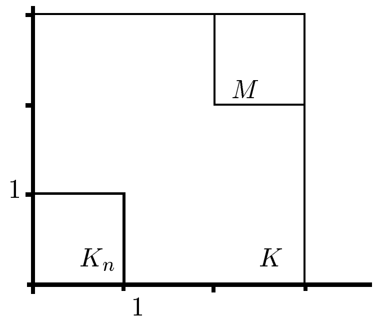
1. Define $K_n := [0,3]^N$ for each $n \in \mathbb{N}$ and $K := M := [0,1]^N$.
Then, on the one hand, $e^{\supset}(K_n, K) = 0 = e^{\supset}(M, K)$, but on the other hand we have $e^{\supset}(M, K_n) > 2$ for all n.
So $(\mathcal{K}(\mathbb{R}^N),\, e^{\supset})$ does not satisfy standard hypothesis $(L^{\Rightarrow})$.



2. Correspondingly the compact sets $K_n := [0,3]^N$, $K := [0,1]^N$, $M := [2,3]^N$ satisfy
$e^{\supset}(K_n, K) = 0$, $e^{\supset}(K_n, M) = 0$, but $e^{\supset}(K, M) > 2$ — contradicting standard hypothesis $(R^{\Rightarrow})$.



3. Finally setting $K := [0,3]^N$, $K_n := [0,1]^N$ $(n \in \mathbb{N})$ and $M := [2,3]^N$ implies
$e^{\supset}(K, K_n) = 0$, $e^{\supset}(K, M) = 0$, but $e^{\supset}(K_n, M) > 2$,
i.e. $(\mathcal{K}(\mathbb{R}^N),\, e^{\supset})$ does not fulfill standard hypothesis $(R^{\Leftarrow})$.



4.1.2 Ostensible metric $q_{\mathcal{K},N}$ considering normal cones

The Pompeiu–Hausdorff excess $e^{\supset}(K_1, K_2)$ does not distinguish between boundary points and interior points of the compact sets K_1, K_2. In this subsection an ostensible metric $q_{\mathcal{K},N}$ on $\mathcal{K}(\mathbb{R}^N)$ is defined that takes the boundaries into consideration explicitly. Strictly speaking, we even use the first–order approximation of the boundary represented by the limiting normal cones of a set. Following the well–known definitions like in [63, Vinter 2000], for example, these cones are specified :

Definition 4.1.4 *Let $C \subset I\!R^N$ be a nonempty closed set.*
A vector $\eta \in I\!R^N$ is said to be a proximal normal vector *to C at $x \in C$ if there exists $\alpha \geq 0$ with*

$$\eta \cdot (y - x) \;\leq\; \alpha \, |y - x|^2 \qquad \text{for all } y \in C.$$

The cone of all proximal normal vectors to C at x is called the proximal normal cone *to C at x and is abbreviated as $N^P_C(x)$.*
The so–called limiting normal cone $N_C(x)$ *to C at x consists of all vectors $\eta \in I\!R^N$ that can be approximated by sequences $(\eta_n)_{n \in I\!N}$, $(x_n)_{n \in I\!N}$ satisfying*

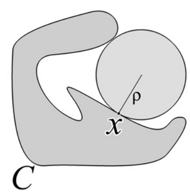
$$x_n \longrightarrow x, \qquad x_n \in C,$$
$$\eta_n \longrightarrow \eta, \qquad \eta_n \in N^P_C(x_n),$$

i.e. $\quad N_C(x) \overset{\text{Def.}}{=} \operatorname{Limsup}_{\substack{y \longrightarrow x \\ y \in C}} N^P_C(y).$
As a further abbreviation, we set ${}^\flat N_C(x) := N_C(x) \cap I\!B$. □

Convention. In the following we restrict ourselves to normal directions at boundary points, i.e. strictly speaking, Graph N_C and Graph ${}^\flat N_C$ are the abbreviations of Graph $N_C|_{\partial C}$, Graph ${}^\flat N_C|_{\partial C}$, respectively.

Remark. $\eta \in N^P_C(x)$, $|\eta| = 1$, is equivalent to the existence of $\rho > 0$ with $I\!B_\rho(x + \rho\,\eta) \cap C = \{x\}$ (see e.g. [63, Vinter 2000], Prop. 4.2.2). The supremum of all ρ with this property is called *proximal radius* of C at x in direction η and fulfills

$$\overset{\circ}{I\!B}_\rho \,(x + \rho\,\eta) \;\cap\; C \;=\; \emptyset.$$



Definition 4.1.5 *Set* $q_{\mathcal{K},N} : \mathcal{K}(I\!R^N) \times \mathcal{K}(I\!R^N) \longrightarrow [0, \infty[$,

$$q_{\mathcal{K},N}(K_1, K_2) \;:=\; d\!l(K_1, K_2) \;+\; e^{\supset}(\text{Graph } {}^\flat N_{K_1}, \;\text{Graph } {}^\flat N_{K_2}).$$

Obviously, the function $q_{\mathcal{K},N}$ is a quasi–metric on the set $\mathcal{K}(I\!R^N)$ of all nonempty compact subsets of $I\!R^N$, i.e. it is positive definite and satisfies the triangle inequality (see remark after Def. 1.1.2).
The properties of $q_{\mathcal{K},N}$ with respect to convergence depend on the relation between the normal cones of compact sets $K_n (n \in I\!N)$ and their limit $K = \operatorname{Lim}_{n \to \infty} K_n$ (if it exists). In general, they do not coincide of course, but each limiting normal vector of K can be approximated by limiting normal vectors of a subsequence $(K_{n_j})_{j \in I\!N}$.
This inclusion is regarded as well–known (see e.g. [6, Aubin 91], Theorem 8.4.6 or [26, Cornet, Czarnecki 99], Lemma 4.1) and it will be used quite often.

Proposition 4.1.6

Let $(M_k)_{k \in I\!N}$ be a sequence of closed subsets of $I\!R^N$ and set $M := \operatorname{Limsup}_{k \to \infty} M_k$.

Then, 1. Graph $N_M^P \subset \operatorname{Limsup}_{k \to \infty}$ Graph $N_{M_k}^P$,

2. Graph $N_M \subset \operatorname{Limsup}_{k \to \infty}$ Graph N_{M_k}. □

Corollary 4.1.7 *Let $(M_k)_{k \in I\!N}$ be a sequence of closed subsets of $I\!R^N$ whose limit $M := \operatorname{Lim}_{k\to\infty} M_k$ exists.*

Then Graph $N_M \subset \operatorname{Liminf}_{k \to \infty}$ Graph N_{M_k}.

In particular, $\partial M \subset \operatorname{Liminf}_{k \to \infty} \partial M_k$.

Proof. Otherwise there is some $(x, v) \in$ Graph $N_M \setminus \operatorname{Liminf}_{k \to \infty}$ Graph N_{M_k}. So there exist $\eta > 0$ and a sequence $(k_n)_{n \in I\!N}$ of indices such that $k_n \longrightarrow \infty \ (n \longrightarrow \infty)$,

$$\operatorname{dist}\Big((x, v), \ \text{Graph } N_{M_{k_n}} \Big) > \eta \qquad \text{for all } n.$$

However the preceding Proposition 4.1.6 concludes from $M = \operatorname{Limsup}_{n\to\infty} M_{k_n}$

$$\liminf_{n \longrightarrow \infty} \operatorname{dist}\Big((x, v), \ \text{Graph } N_{M_{k_n}} \Big) = 0$$

— a contradiction. □

Lemma 4.1.8 $(\mathcal{K}(I\!R^N), \, q_{\mathcal{K},N})$ *is one–sided sequentially compact and fulfills standard hypothesis* $(R^{\Leftarrow})$*, i.e. the right–hand spheres are left–sequentially closed.*

Proof results from Corollary 4.1.7 : For any sequence $(K_n)_{n \in I\!N}$ in $\mathcal{K}(I\!R^N)$ and $K \in \mathcal{K}(I\!R^N)$, the convergence $d\!\!l(K, K_n) \longrightarrow 0$ implies

$$\text{Graph } {}^\flat N_K^L \subset \operatorname{Liminf}_{n \to \infty} \text{Graph } {}^\flat N_{K_n}$$

and thus
$$\liminf_{n \longrightarrow \infty} \ e^{\supset}(\text{Graph } {}^\flat N_{K_n}, \ M') \le e^{\supset}(\text{Graph } {}^\flat N_K, \ M')$$

for all nonempty compact sets $M' \subset I\!R^N \times I\!R^N$. Then we obtain for every $M \in \mathcal{K}(I\!R^N)$

$$\begin{aligned} \liminf_{n \longrightarrow \infty} \ q_{\mathcal{K},N}\Big(K_n, M\Big) &= d\!\!l(K, M) + \liminf_{n \longrightarrow \infty} \ e^{\supset}\Big(\text{Graph } {}^\flat N_{K_n}, \ \text{Graph } {}^\flat N_M\Big) \\ &\le q_{\mathcal{K},N}\Big(K, M\Big). \end{aligned}$$

So $(\mathcal{K}(I\!R^N), \, q_{\mathcal{K},N})$ satisfies standard hypothesis $(R^{\Leftarrow})$, and the triangle inequality even guarantees
$$\lim_{n \longrightarrow \infty} q_{\mathcal{K},N}(K_n, M) = q_{\mathcal{K},N}(K, M) \qquad \text{if } q_{\mathcal{K},N}(K, K_n) \longrightarrow 0.$$

Let us now consider any sequence $(K_n)_{n \in I\!N}$ in $\mathcal{K}(I\!R^N)$ with $\sup_n \|K_n\|_\infty < \infty$. Due to Prop. 4.1.1 (4), there exist a subsequence $(K_{n_j})_{j \in I\!N}$ and a set $K \in \mathcal{K}(I\!R^N)$ such that $d\!\!l(K_{n_j}, K) \longrightarrow 0$. Cor. 4.1.7 ensures $e^{\supset}\Big(\text{Graph } {}^\flat N_{K_{n_j}}, \ \text{Graph } {}^\flat N_K^L\Big) \longrightarrow 0$ and so, $q_{\mathcal{K},N}(K_{n_j}, \, K) \longrightarrow 0 \quad (j \longrightarrow \infty)$. □

For stronger properties like two–sided sequential compactness, we usually have to restrict ourselves to a subset of $\mathcal{K}(I\!R^N)$ such that $d\!l(K_n, K) \longrightarrow 0$ always implies

$$\text{Graph}\ {}^\flat N_K \;=\; \text{Lim}_{n\to\infty}\ \text{Graph}\ {}^\flat N_{K_n}$$

(for a subsequence at least). The convex compact subsets of $I\!R^N$ exemplify this feature according to the following Lemma 4.1.9 and the well–known fact of Convex Analysis

$$N_K(x) \;=\; N_K^P(x) \;=\; \{\, y - x \mid \Pi_K(y) = x \,\}$$

for every *convex* subset $K \in \mathcal{K}(I\!R^N)$ and $x \in K$.

Lemma 4.1.9 *Let $(K_n)_{n\in I\!N}$ be a sequence in $\mathcal{K}(I\!R^N)$ such that $K := \text{Lim}_{n\to\infty} K_n$ exists in $\mathcal{K}(I\!R^N)$. Π_{K_n}, $\Pi_K : I\!R^N \leadsto I\!R^N$ denote the projections on K_n, K $(n \in I\!N)$ respectively. Then for every $x \in I\!R^N$,*

$$\text{Limsup}_{\substack{y\to x\\ n\to\infty}}\ \Pi_{K_n}(y) \;\subset\; \Pi_K(x),$$

$$\lim_{\substack{y\to x\\ n\to\infty}}\ \text{dist}(y, K_n) \;=\; \text{dist}(x, K).$$

Proof. The second claim is a simple consequence of the triangle inequality.

Let $r > 0$ and $n \in I\!N$ be arbitrary. For $y \in I\!B_r(x)$ given, choose any $z \in \Pi_{K_n}(y)$ and $\xi \in \Pi_K(z)$. Then, $|\xi - z| \;\le\; d\!l(K_n, K)$ and

$$\begin{aligned} |x-\xi| &\le |x-y| + |y-z| + |z-\xi| \\ &\le |x-y| + |y-x| + \text{dist}(x,K) + d\!l(K,K_n) + d\!l(K_n,K) \\ &\le 2\,r + \text{dist}(x,K) + 2\,d\!l(K_n,K). \end{aligned}$$

Thus, $\Pi_{K_n}(y) \;\subset\; I\!B_{d\!l(K_n,K)}\Big(K \,\cap\, I\!B_{2\,r\,+\,\text{dist}(x,K)\,+\,2\,d\!l(K_n,K)}(x)\Big)$ for any $y \in I\!B_r(x)$.

The set–valued map $[0,\infty[\ \leadsto\ I\!R^N,\quad r \longmapsto K \cap I\!B_r(x)$ is upper semicontinuous (due to [7, Aubin,Frankowska 90], Cor. 1.4.10) and in the closed interval $[\text{dist}(x,K), \infty[$, it is strict with compact values.

So for every $\eta > 0$ there exists $\rho = \rho(x,\eta)\ \in\ ]0,\eta[$ such that

$$K \,\cap\, I\!B_r(x) \;\subset\; I\!B_\eta\Big(\Pi_K(x)\Big) \qquad \text{for all } r \in \Big[\text{dist}(x,K),\ \text{dist}(x,K)+\rho\Big].$$

Due to $d\!l(K_n, K) \longrightarrow 0$ $(n \longrightarrow \infty)$, there is an index $m \in I\!N$ with $d\!l(K_n,K) \le \frac{\rho}{4}$ for all $n \ge m$. Thus we obtain for every $y \in I\!B_{\rho/4}(x)$ and $n \ge m$

$$\begin{aligned} \Pi_{K_n}(y) &\subset I\!B_{\frac{\rho}{4}}\Big(K \,\cap\, I\!B_{2\,\frac{\rho}{4}+\text{dist}(x,K)+2\,\frac{\rho}{4}}(x)\Big) \\ &= I\!B_{\frac{\rho}{4}}\Big(K \,\cap\, I\!B_{\text{dist}(x,K)+\rho}(x)\Big) \\ &\subset I\!B_{\frac{\rho}{4}}\Big(I\!B_\eta(\Pi_K(x))\Big) \\ &\subset I\!B_{2\,\eta}\Big(\Pi_K(x)\Big), \end{aligned}$$

i.e. $\text{Limsup}_{\substack{y\to x\\ n\to\infty}}\ \Pi_{K_n}(y) \;\subset\; \Pi_K(x).$ □

4.2 Nonempty bounded open subsets of $\mathbb{R}^N$: $\Omega(\mathbb{R}^N)$

4.2.1 Ostensible metric $q_{\Omega,\partial}$ considering topological boundary

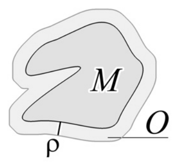
Definition 4.2.1 *The set of all nonempty bounded open subsets of* $\mathbb{R}^N$ *is abbreviated as* $\Omega(\mathbb{R}^N)$. *Moreover set* $q_{\Omega,\partial} : \Omega(\mathbb{R}^N) \times \Omega(\mathbb{R}^N) \longrightarrow [0, \infty[$,

$$q_{\Omega,\partial}\Big(O_1, O_2\Big) \;:=\; d\!\!l(\overline{O}_1,\, \overline{O}_2) \;+\; e\!\!e^{\supset}(\partial O_1,\, \partial O_2)$$

Obviously, $q_{\Omega,\partial}$ is an ostensible metric on $\Omega(\mathbb{R}^N)$ because $d\!\!l$, $e\!\!e^{\supset}$ are ostensible metrics on $\mathcal{K}(\mathbb{R}^N)$. Moreover $(\Omega(\mathbb{R}^N),\, q_{\Omega,\partial})$ is not one–sided sequentially compact as the simple example $O_n := \overset{\circ}{\mathbb{B}}_{1/n}$ $(n \in \mathbb{N})$ shows.

Roughly speaking, a significant difference between considering $O \in \Omega(\mathbb{R}^N)$ and its closure $\overline{O} \in \mathcal{K}(\mathbb{R}^N)$ is determined by the information on the topological boundaries of sequences. As mentioned in the section before, some features (like two–sided sequential compactness) usually require additional assumptions.
An example for appropriate conditions is given by the *sets of uniform positive erosion.* After introducing the criterion of definition now, we summarize their regularity properties in section 4.3. As a first essential advantage, they are "closed" in $\Omega(\mathbb{R}^N)$ in the sense of Prop 4.2.3 and fulfill $\partial O = \mathrm{Lim}_{j\to\infty}\, \partial O_{n_j}$ for a subsequence of $(O_n)_{n\in\mathbb{N}}$. A similar statement does not hold for sequences in $\mathcal{K}(\mathbb{R}^N)$ (even with uniform positive erosion according to Def. 4.3.1). Secondly sets of uniform positive erosion are two–sided sequentially compact with respect to $q_{\Omega,\partial}$ (as stated in Cor. 4.2.5).

Definition 4.2.2 *An nonempty open subset* $O \subset \mathbb{R}^N$ *is said to have* positive erosion of radius $\rho > 0$ *if there exists a closed set* $M \subset \mathbb{R}^N$ *with* $O = \overset{\circ}{\mathbb{B}}_\rho(M)$.
Moreover, a set $\mathcal{Q}$ *of open subsets of* $\mathbb{R}^N$ *has* uniform positive erosion *if each* $O \in \mathcal{Q}$ *has positive erosion of some radius* ρ *independent of* O.
$\Omega^\rho_\circ(\mathbb{R}^N)$ *denotes the set of bounded open subsets of* $\mathbb{R}^N$ *with positive erosion of radius* ρ.
$\Omega_\circ(\mathbb{R}^N)$ *abbreviates the set of all* $O \in \Omega(\mathbb{R}^N)$ *with positive erosion.*



Proposition 4.2.3 *Let* $(O_n)_{n\in\mathbb{N}}$ *be a sequence in* $\Omega^\rho_\circ(\mathbb{R}^N)$ *with* $\rho > 0$ *such that the limits* $C := \mathrm{Lim}_{n\to\infty}\, \overline{O_n}$ *and* $A := \mathrm{Lim}_{n\to\infty}\, \partial O_n$ *exist in* $\mathcal{K}(\mathbb{R}^N)$.
Then $O := C \setminus A \in \Omega(\mathbb{R}^N)$ *also has positive erosion of radius* ρ *and* $\partial O = A$.

Proof. O is bounded and open because A, C are compact and Corollary 4.1.7 leads to

$$\partial C \;\subset\; \mathrm{Liminf}_{n\longrightarrow\infty}\, \partial\, \overline{O_n} \;\subset\; \mathrm{Limsup}_{n\longrightarrow\infty}\, \partial\, O_n \;\subset\; A.$$

Due to Def. 4.2.2, there exists $M_n \in \mathcal{K}(I\!R^N)$ for each $n \in I\!N$ such that $O_n = \overset{\circ}{I\!B}_\rho(M_n)$. Since M_n is contained in the compact set $I\!B_{2\rho}(C)$ for all n sufficiently large, there are a subsequence of (M_n) (again denoted by $(M_n)_{n \in I\!N}$) and a compact set $M \subset C$ with $d\!\!l(M, M_n) \longrightarrow 0 \quad (n \longrightarrow \infty)$. The following Lemma 4.2.4 guarantees the convergence

$$d\!\!l\Big(I\!B_\rho(M),\, \overline{O_n}\Big) \;=\; d\!\!l(I\!B_\rho(M),\, I\!B_\rho(M_n)) \;\le\; d\!\!l(M,\, M_n) \;\longrightarrow\; 0 \qquad (n \longrightarrow \infty)$$

and thus $C = I\!B_\rho(M)$. Furthermore, the second statement of Lemma 4.1.9 implies

$$\begin{aligned} A &= \operatorname{Liminf}_{n\to\infty} \partial I\!B_\rho(M_n) &&= \Big\{ \lim_{n\to\infty} \operatorname{dist}(\,\cdot\,, M_n) = \rho \Big\} \\ &= \Big\{ \operatorname{dist}(\,\cdot\,,\, M) = \rho \Big\} &&= \partial\, I\!B_\rho(M) \end{aligned}$$

and so we obtain $O \;=\; C \setminus A \;= \overset{\circ}{I\!B}_\rho(M), \quad \partial O \;=\; A.$ □

Lemma 4.2.4 *For any compact subsets $K_1, K_2 \subset I\!R^N$ and $\rho > 0$,*

$$d\!\!l\Big(I\!B_\rho(K_1),\, I\!B_\rho(K_2)\Big) \;\le\; d\!\!l(K_1,\, K_2).$$

Proof. For any radius $\rho > 0$ and nonempty set $M \subset I\!R^N$,

$$\operatorname{dist}(\cdot,\, I\!B_\rho(M)) \;=\; \max\Big(0,\, \operatorname{dist}(\,\cdot\,, M) - \rho\Big) \;=\; \tfrac{1}{2}\Big(\operatorname{dist}(\,\cdot\,, M) - \rho \,+\, |\operatorname{dist}(\,\cdot\,, M) - \rho|\Big).$$

So the claim results from Prop. 4.1.1 (3.) and

$$\begin{aligned} &\operatorname{dist}(\,\cdot\,,\; I\!B_\rho(K_1)) \;-\; \operatorname{dist}(\,\cdot\,,\; I\!B_\rho(K_2)) \\ =\;& \tfrac{1}{2}\Big(\operatorname{dist}(\,\cdot\,, K_1) - \rho - \operatorname{dist}(\,\cdot\,, K_2) + \rho + \Big|\operatorname{dist}(\,\cdot\,, K_1) - \rho\Big| - \Big|\operatorname{dist}(\,\cdot\,, K_2) - \rho\Big|\Big) \\ \le\;& \tfrac{1}{2}\Big(\operatorname{dist}(\,\cdot\,, K_1) - \operatorname{dist}(\,\cdot\,, K_2) + \Big|\operatorname{dist}(\,\cdot\,, K_1) - \operatorname{dist}(\,\cdot\,, K_2)\Big|\Big) \\ \le\;& \Big|\operatorname{dist}(\,\cdot\,, K_1) - \operatorname{dist}(\,\cdot\,, K_2)\Big| \;\le\; d\!\!l(K_1, K_2). \end{aligned}$$

□

Corollary 4.2.5 *For arbitrary $\rho > 0$, $(\Omega^\rho_\circ(I\!R^N),\ q_{\Omega,\partial})$ is two–sided sequentially compact and one–sided complete.*

Proof. Consider any sequence $(O_n)_{n \in I\!N}$ in $\Omega(I\!R^N)$ with positive erosion of radius ρ and $R := \sup_n \|\overline{O}_n\|_\infty < \infty$. Now we prove the existence of $O \in \Omega^\rho_\circ(I\!R^N)$ satisfying $q_{\Omega,\partial}(O_{n_j}, O) \longrightarrow 0$, $q_{\Omega,\partial}(O, O_{n_j}) \longrightarrow 0$ for a subsequence $(O_{n_j})_{j \in I\!N}$. Indeed, due to the sequential compactness of $(\mathcal{K}(I\!B_R(0)), d\!\!l)$ (Prop. 4.1.1), there is a subsequence $(O_{n_j})_{j \in I\!N}$ such that $C := \operatorname{Lim}_{j\to\infty} \overline{O_{n_j}}$ and $A := \operatorname{Lim}_{j\to\infty} \partial O_{n_j}$ exist. According to Prop. 4.2.3, $O := C \setminus A$ has positive erosion of radius ρ with $\partial O = A$, i.e. $O \in \Omega^\rho_\circ(I\!R^N)$, $d\!\!l(\overline{O},\, \overline{O_{n_j}}) \longrightarrow 0$, $d\!\!l(\partial O,\, \partial O_{n_j}) \longrightarrow 0 \quad (j \longrightarrow \infty)$,
and thus, $q_{\Omega,\partial}(O, O_{n_j}) \longrightarrow 0$, $q_{\Omega,\partial}(O_{n_j}, O) \longrightarrow 0 \quad (j \longrightarrow \infty)$. □

4.2.2 Ostensible metric q_{Ω,N_c} considering interior normal cones

Definition 4.2.6 *Let* ${}^{\complement}O := I\!R^N \setminus O$ *denote the complement of any* $O \in \Omega(I\!R^N)$ *and set* $q_{\Omega,N_c} : \Omega(I\!R^N) \times \Omega(I\!R^N) \longrightarrow [0,\infty[$,

$$q_{\Omega,N_c}\Big(O_1, O_2\Big) := d\!\!l(\overline{O}_1, \overline{O}_2) + e^{\supset}\Big(\text{Graph } {}^{\flat}N_{{}^{\complement}O_1}, \text{ Graph } {}^{\flat}N_{{}^{\complement}O_2}\Big)$$

Obviously q_{Ω,N_c} is an ostensible metric on $\Omega(I\!R^N)$, but q_{Ω,N_c} is not a quasi–metric since it does not fulfill $q_{\Omega,N_c}(O_1,O_2) = 0 \implies O_1 = O_2$ for all $O_1, O_2 \in \Omega(I\!R^N)$. Moreover, $(\Omega(I\!R^N), q_{\Omega,N_c})$ is not one–sided sequentially compact as the easy example $O_n := \overset{\circ}{I\!B}_{1/n}$ $(n \in I\!N)$ shows once more.

For guaranteeing stronger properties like two–sided sequential compactness, we again use sets with uniform positive erosion, but now we have to take normal cones into consideration. Every set $O \in \Omega^{\rho}_{\circ}(I\!R^N)$ holds a relation between the interior normal cones $N_{{}^{\complement}O}(\cdot)$ and the projection $\Pi_{{}^{\complement}O} : O \longrightarrow I\!R^N$ on the complement ${}^{\complement}O$ quoted in § 4.3 :

$$\text{Graph } \Big(\Pi_{{}^{\complement}O}\Big|_{\overline{O} \cap \overset{\circ}{I\!B}_{\rho}({}^{\complement}O)}\Big)^{-1} = \text{Graph } \Big(\text{Id} + (N_{{}^{\complement}O} \cap \overset{\circ}{I\!B}_{\rho})\Big)\Big|_{\partial O} \qquad (*)$$

i.e. $$N_{{}^{\complement}O}(x) \cap \overset{\circ}{I\!B}_{\rho} = \Big\{ y - x \;\Big|\; y \in \overline{O},\ \text{dist}(y, \partial O) < \rho,\ \Pi_{{}^{\complement}O}(y) = x \Big\}$$

for every $x \in \partial O$.

Proposition 4.2.7 *Assume* $(O_n)_{n \in I\!N}$ *to be a sequence in* $\Omega^{\rho}_{\circ}(I\!R^N)$ $(\rho > 0)$ *such that* $C := \text{Lim}_{n\to\infty} \overline{O_n}$ *and* $A := \text{Lim}_{n\to\infty} \partial O_n$ *exist in* $\mathcal{K}(I\!R^N)$.
Then, $O := C \setminus A \in \Omega^{\rho}_{\circ}(I\!R^N)$ *fulfills* $\text{Lim}_{n\to\infty} \text{ Graph } {}^{\flat}N_{{}^{\complement}O_n} = \text{Graph } {}^{\flat}N_{{}^{\complement}O}$.

Proof. Prop. 4.2.3 states $O \in \Omega^{\rho}_{\circ}(I\!R^N)$ and $\partial O = A$. Now $\overline{O} = C = \text{Lim}_{n\to\infty} \overline{O_n}$ and the convergence of

$$\begin{aligned} \text{dist}(\,\cdot\,, {}^{\complement}O_n) &= \text{dist}(\,\cdot\,, \partial O_n) - \text{dist}(\,\cdot\,, \overline{O_n}) \\ &\longrightarrow \text{dist}(\,\cdot\,, A) - \text{dist}(\,\cdot\,, C) \qquad (n \longrightarrow \infty) \end{aligned}$$

imply ${}^{\complement}O = \text{Lim}_{n\to\infty} {}^{\complement}O_n$. So Corollary 4.1.7 leads to

$$\text{Graph } N_{{}^{\complement}O} \subset \text{Liminf}_{n\to\infty} \text{ Graph } N_{{}^{\complement}O_n}.$$

Moreover, for each $n \in I\!N$, the closed set $M_n := O_n \setminus \overset{\circ}{I\!B}_{\rho/2}(\partial O_n)$ satisfies $O_n = \overset{\circ}{I\!B}_{\rho/2}(M_n)$,

$$\text{Graph } \Big(\Pi_{{}^{\complement}O_n}\Big|_{\overline{O_n \setminus M_n}}\Big)^{-1} = \text{Graph } \Big(\text{Id} + \tfrac{\rho}{2}\,{}^{\flat}N_{{}^{\complement}O_n}\Big)\Big|_{\partial O_n}$$

due to the preceding remark $(*)$. The corresponding relations hold for the sets O and $M := O \setminus I\!B_{\rho/2}(\partial O)^{\circ} = \{x \in O \mid \text{dist}(x, {}^{\complement}O) \geq \frac{\rho}{2}\} \in \mathcal{K}(I\!R^N)$.

Furthermore, $\complement O = \text{Lim}_{n\to\infty} \complement O_n$ and Lemma 4.1.9 (applied to $\complement O_n \cap I\!B_R(0) \in \mathcal{K}(I\!R^N)$ with $R > 0$ sufficiently large) guarantee

$$\wedge \begin{cases} \text{Lim}_{n\to\infty} M_n &= M \\ \text{Limsup}_{n\to\infty} \text{ Graph } \Pi_{\complement O_n} &\subset \text{ Graph } \Pi_{\complement O} \end{cases}$$

and thus,

$$\begin{aligned} \text{Limsup}_{n\to\infty} \text{ Graph } \Pi_{\complement O_n}\Big|_{\overline{O_n\setminus M_n}} &\subset \text{ Graph } \Pi_{\complement O}\Big|_{\overline{O\setminus M}} \\ \text{Limsup}_{n\to\infty} \text{ Graph } \Big(\text{Id} + \tfrac{\rho}{2}\, {}^\flat N_{\complement O_n}\Big)\Big|_{\partial O_n} &\subset \text{ Graph } \Big(\text{Id} + \tfrac{\rho}{2}\, {}^\flat N_{\complement O}\Big)\Big|_{\partial O}. \end{aligned}$$

So finally we conclude from $\partial O = A = \text{Lim}_{n\to\infty} \partial O_n$

$$\text{Limsup}_{n\to\infty} \text{ Graph } {}^\flat N_{\complement O_n} \subset \text{ Graph } {}^\flat N_{\complement O}. \qquad \square$$

Corollary 4.2.8 *For every $\rho > 0$, $(\Omega^\rho_\circ(I\!R^N),\, q_{\Omega,N_c})$ is one–sided complete and two–sided sequentially compact.*

Proof results from Prop. 4.2.7 quite easily : For every sequence $(O_n)_{n\in I\!N}$ in $\Omega^\rho_\circ(I\!R^N)$ with $\sup_n \|\overline{O}_n\|_\infty < \infty$, we can consider a subsequence $(O_{n_j})_{j\in I\!N}$ such that $C := \text{Lim}_{j\to\infty} \overline{O_{n_j}}$ and $A := \text{Lim}_{j\to\infty} \partial O_{n_j}$ exist in $\mathcal{K}(I\!R^N)$ (due to Prop. 4.1.1 (4.)). Then $O := C \setminus A \in \Omega^\rho_\circ(I\!R^N)$ fulfills

$$d\!l(\overline{O}, \overline{O_{n_j}}) \longrightarrow 0, \quad d\!l\Big(\text{Graph } {}^\flat N_{\complement O},\ \text{Graph } {}^\flat N_{\complement O_{n_j}}\Big) \longrightarrow 0, \qquad (j \longrightarrow \infty),$$

i.e. $\quad q_{\Omega,N_c}(O,\, O_{n_j}) \longrightarrow 0, \quad q_{\Omega,N_c}(O_{n_j},\, O) \longrightarrow 0 \qquad (j \longrightarrow \infty).$

This two–sided convergence and the triangle inequality imply that $(\Omega^\rho_\circ(I\!R^N),\, q_{\Omega,N_c})$ is one–sided complete and two–sided sequentially compact. $\square$

Example 4.2.9 $(\Omega^\rho_\circ(I\!R^N),\, q_{\Omega,N_c})$ does not satisfy the standard hypotheses $(L^{\Rightarrow})$–$(R^{\Leftarrow})$ and thus, $(\Omega(I\!R^N), q_{\Omega,N_c})$ does not either. This results from easy counterexamples :

1. $O_n := \mathring{I\!B}_2(0) \setminus \{0\}$, $O := \mathring{I\!B}_2(0)$ have uniform positive erosion of radius 1 and fulfill $q_{\Omega,N_c}(O_n, O) = 0$, but $q_{\Omega,N_c}(O, O_n) \geq 1$. That is inconsistent with standard hypothesis $(L^{\Rightarrow})$.
2. $O_n := \mathring{I\!B}_2(0) \setminus \{0\}$, $O := \mathring{I\!B}_2(0)$, $M := \mathring{I\!B}_2(0) \setminus I\!B_{\frac{1}{2}}(0)$ belong to $\Omega^{\frac{1}{2}}_\circ(I\!R^N)$ as well and satisfy $q_{\Omega,N_c}(O_n, O) = 0$, $q_{\Omega,N_c}(O_n, M) = \frac{1}{2}$, but $q_{\Omega,N_c}(O, M) \geq 1$ — contradicting standard hypothesis $(R^{\Rightarrow})$.
3. $O_n := I\!B_1(0)^\circ$, $O := I\!B_1(0)^\circ \setminus \{0\}$ satisfy $q_{\Omega,N_c}(O, O_n) = 0$, $q_{\Omega,N_c}(O_n, O) = 1$. So standard hypothesis $(R^{\Leftarrow})$ does not hold, i.e. the right–hand spheres are not left–sequentially closed.

Seizing the suggestion of Prop. 1.4.8, we can bridge the third gap and "close" the right–hand spheres of $\Omega(I\!R^N)$ (with respect to a new function $q_{\Omega,N_c,(R^{\Leftarrow})}$) — although $(\Omega(I\!R^N), q_{\Omega,N_c})$ is not one–sided sequentially compact as we assumed in Prop. 1.4.8.

Proposition 4.2.10 *Setting*

$$q_{\Omega,N_c,(R^{\Leftarrow})}\Big(O_1, O_2\Big) := \sup\Big\{ q_{\Omega,N_c}(O, O_2) \;\Big|\; O \in \Omega(I\!R^N),\; q_{\Omega,N_c}(O_1, O) = 0 \Big\},$$

$(\Omega(I\!R^N),\ q_{\Omega,N_c,(R^{\Leftarrow})})$ *fulfills standard hypothesis* $(R^{\Leftarrow})$, *i.e. the right–hand spheres are left–sequentially closed.*

Remark. Here the condition $q_{\Omega,N_c}(O_1, O) = 0$ is equivalent to

$$\wedge \left\{ \begin{array}{rcl} \overline{O} & = & \overline{O_1} \\ \text{Graph } N_{^cO} & \subset & \text{Graph } N_{^cO_1}\,. \end{array} \right.$$

and thus has the consequence $\partial O \;=\; \partial\,^c O \;\subset\; \partial\,^c O_1 \;=\; \partial O_1$.
The set $O := \overset{\circ}{I\!B}_2(0) \setminus \partial\, I\!B_1(0) \;\in\; \Omega(I\!R^N)$ is an easy example for $q_{\Omega,N_c,(R^{\Leftarrow})}(O,O) > 0$ because $O' := \overset{\circ}{I\!B}_2(0)$ satisfies both $q_{\Omega,N_c}(O, O') = 0$ and $q_{\Omega,N_c}(O', O) \geq 1$.

Proof of Prop. 4.2.10. $q_{\Omega,N_c,(R^{\Leftarrow})}$ fulfills the triangle inequality. Indeed, let O_1, O_2, O_3 be arbitrary elements of $\Omega(I\!R^N)$ and $\eta > 0$. Then there is some $O_1' \in \Omega(I\!R^N)$ such that

$$q_{\Omega,N_c}(O_1, O_1') = 0, \qquad q_{\Omega,N_c,(R^{\Leftarrow})}(O_1, O_3) \leq q_{\Omega,N_c}(O_1', O_3) + \eta$$

and we conclude from the triangle inequality of q_{Ω,N_c}

$$\begin{array}{rcll} q_{\Omega,N_c}(O_1', O_3) & \leq & q_{\Omega,N_c}(O_1', O_2) & +\; q_{\Omega,N_c}(O_2, O_3) \\ & \leq & q_{\Omega,N_c,(R^{\Leftarrow})}(O_1, O_2) & +\; q_{\Omega,N_c,(R^{\Leftarrow})}(O_2, O_3). \end{array}$$

Now let (O_n) be a sequence in $\Omega(I\!R^N)$ and $M, O \in \Omega(I\!R^N)$ with $q_{\Omega,N_c,(R^{\Leftarrow})}(O, O_n) \longrightarrow 0$. There is a corresponding sequence $(O_n')_{n \in I\!N}$ in $\Omega(I\!R^N)$ such that

$$q_{\Omega,N_c}\;(O_n, O_n') = \;0, \qquad q_{\Omega,N_c,(R^{\Leftarrow})}(O_n, M) \;\leq\; q_{\Omega,N_c}(O_n', M) + \tfrac{1}{n}.$$

and consequently, $q_{\Omega,N_c,(R^{\Leftarrow})}(O, O_n') \;\leq\; q_{\Omega,N_c,(R^{\Leftarrow})}(O, O_n) + q_{\Omega,N_c}(O_n, O_n') \;\longrightarrow\; 0$ $(n \longrightarrow \infty)$.

Since $\partial O_n' \;=\; \partial\,^c O_n'$ are compact subsets of $I\!B_1(\overline{O})$ (for all n large enough) we can take a subsequence into consideration (again denoted by O_n') such that the limit $A := \text{Lim}_{n\to\infty}\, \partial\, O_n'$ exists.
Then, $A \subset \partial O$ as $q_{\Omega,N_c}(O, O_n') \longrightarrow 0$ implies $\text{Limsup}_{n\to\infty} \text{Graph } N_{^cO_n'} \;\subset\; \text{Graph } N_{^cO}$. Furthermore, $\partial \overline{O} \;\subset\; A$ because $\overline{O} = \text{Lim}_{n\to\infty}\, \overline{O_n'}$ and Corollary 4.1.7 result in

$$\partial\, \overline{O} \;\subset\; \text{Liminf}_{n\to\infty}\, \partial\, \overline{O_n'} \;\subset\; A.$$

So $O' := \overline{O} \setminus A$ is open, bounded and fulfills

1. $O \subset O' \subset \overline{O'} = \overline{O}$,
2. $\complement O' = I\!R^N \Big\backslash (\overline{O} \setminus A) = (I\!R^N \setminus \overline{O}) \cup A = (\complement O)^\circ \cup A$,
3. $\partial O' = \overline{O'} \cap \overline{\complement O'} = \overline{O} \cap \overline{((\complement O)^\circ \cup A)} = \overline{O} \cap A = A$,
4. $\overline{\complement O'} = \mathrm{Lim}_{n\to\infty} \overline{\complement O'_n}$
   because $\mathrm{dist}(\,\cdot\,, \complement M') = \mathrm{dist}(\,\cdot\,, \partial M') - \mathrm{dist}(\,\cdot\,, \overline{M'})$ for all $M' \in \Omega(I\!R^N)$ and
   $d\!\!l(\overline{O'}, \overline{O'_n}) = d\!\!l(\overline{O}, \overline{O'_n}) \longrightarrow 0$,
   $d\!\!l(\partial O', \partial O'_n) = d\!\!l(A, \partial O'_n) \longrightarrow 0$
   lead to $d\!\!l(\overline{\complement O'}, \overline{\complement O'_n}) \longrightarrow 0 \qquad (n \longrightarrow \infty)$.

Corollary 4.1.7 and $q_{\Omega,N_c,(R^{\Leftarrow})}(O, O'_n) \longrightarrow 0$, respectively, guarantee

$$\mathrm{Graph}\ N_{\complement O'} \subset \mathrm{Liminf}_{n\to\infty}\ \mathrm{Graph}\ N_{\complement O'_n} \subset \mathrm{Limsup}_{n\to\infty}\ \mathrm{Graph}\ N_{\complement O'_n} \subset \mathrm{Graph}\ N_{\complement O},$$

i.e. $\quad q_{\Omega,N_c}(O, O') = 0$

and $\quad \liminf_{n\longrightarrow\infty}\ \mathrm{dist}\Big(\mathrm{Graph}\ {}^\flat\! N_{\complement M},\ \mathrm{Graph}\ {}^\flat\! N_{\complement O'_n}\Big) \le \mathrm{dist}\Big(\mathrm{Graph}\ {}^\flat\! N_{\complement M},\ \mathrm{Graph}\ {}^\flat\! N_{\complement O'}\Big).$

So finally we obtain

$$\liminf_{n\longrightarrow\infty}\ q_{\Omega,N_c,(R^{\Leftarrow})}(O_n, M) \le \liminf_{n\longrightarrow\infty}\ q_{\Omega,N_c}(O'_n, M) \le q_{\Omega,N_c}(O', M) \le q_{\Omega,N_c,(R^{\Leftarrow})}(O, M)$$

□

4.3 Sets of positive erosion in $I\!R^N$

A set $C \subset I\!R^N$ of positive erosion provides a counterpart of sets of positive reach that were introduced by Federer ([35, Federer 59]). In terms of partial differential equation, its topological boundary is characterized by a uniform interior sphere condition whereas sets of positive reach are based on a uniform exterior sphere condition. For underlining this analogy, we prefer the term "erosion" that is borrowed from morphology.
The definition of open sets with positive erosion has already been given in Def. 4.2.2. Now we extend this concept to closed subsets of $I\!R^N$ correspondingly :

Definition 4.3.1 *A closed subset $C \subset I\!R^N$ is said to have* positive erosion of radius *$\rho > 0$ if there exists a closed set $M \subset I\!R^N$ such that $C = I\!B_\rho(M)$.*
Moreover, a set $\mathcal{F}$ of closed subsets of $I\!R^N$ has uniform positive erosion *if (and only if) each $C \in \mathcal{F}$ has positive erosion of some radius ρ independent of C.*
$\mathcal{K}^\rho_\circ(I\!R^N)$ contains all nonempty compact subsets of $I\!R^N$ with positive erosion of radius ρ.
$\mathcal{K}_\circ(I\!R^N)$ abbreviates the set of all $K \in \mathcal{K}(I\!R^N)$ with positive erosion, i.e.

$$\mathcal{K}_\circ(I\!R^N) := \bigcup_{\rho > 0} \mathcal{K}^\rho_\circ(I\!R^N).$$

Meanwhile sets of positive reach in $\mathbb{R}^N$ have been investigated extensively. In particular, they were generalized to Hilbert spaces leading to so–called *proximally smooth* sets. Now we summarize some of their characterizing properties :
Theorem 4.1 in [25, Clarke,Stern,Wolenski 95] states the following equivalence for a closed set X of a Hilbert space H (Π_X abbreviates the projection on X) :

1. X is proximally smooth of radius $\rho > 0$, i.e.
   $d_X := \text{dist}(\cdot, X)$ is continuously differentiable on $U_X(\rho) := \{0 < d_X(\cdot) < \rho\}$.
2. $\Pi_X(u) \neq \emptyset$ for all $u \in U_X(\rho)$ and the Gâteaux derivative $d'_X(u)$ exists.
3. $\Pi_X(u) \neq \emptyset$ for all $u \in U_X(\rho)$ and for every $r \in]0, \rho[$, one has
   $$d_X(\cdot) + d_{Y(r)}(\cdot) = r \qquad \text{in } U_X(r)$$
   with $Y(r) := \{u \in H \mid d_X(u) \geq r\}$, $d_{Y(r)} := \text{dist}(\,\cdot\,,\ Y(r))$.
4. $\Pi_X(u) \neq \emptyset$ for all $u \in U_X(\rho)$ and for any $x \in X$, every proximal normal $v \in N_X^P(x)$, $|v| = 1$, can be realized by a ρ–ball, i.e. $\mathring{\mathbb{B}}_\rho(x + \rho\, v) \cap X = \emptyset$.
5. For every $r \in]0, \rho[$ and $u \in H$ such that $d_X(u) = r$, one has $N^P_{\mathbb{B}_r(X)}(u) \neq \{0\}$.
6. The proximal subdifferential of d_X, $\partial_P d_X(u)$, is nonempty for all $u \in U_X(\rho)$.

More equivalent properties are proven in [53, Poliquin,Rockafellar,Thibault 2000] :

7. d_X is Fréchet differentiable on $U_X(\rho)$.
8. $d_X^2|_{U_X(\rho)} \in C^{1+}$, i.e. differentiable with locally Lipschitz continuous derivative.
9. Whenever $x_1, x_2 \in X$ and $v_i \in N_X(x_i)$, $|v_i| < \rho$ $(i = 1, 2)$, one has
   $$\langle v_1 - v_2,\ x_1 - x_2\rangle \ \geq\ -\,|x_1 - x_2|^2.$$
   If $u \in U_X(\rho)$ and $x = \Pi_X(u)$, then $\{x\} = \Pi_X\Big(x + [0, 1[\ \frac{u-x}{|u-x|}\Big)$.
10. Π_X is single-valued and (strongly–weakly) continuous on $U_X(\rho)$.
11. *Global Shapiro property :* $\text{dist}(x' - x,\ T_X^B(x)) \ \leq\ \frac{1}{2\,\rho}\ |x' - x|^2 \quad \forall\ x, x' \in X.$

Then Π_X is (single-valued) monotone on $U_X(\rho)$ and Lipschitz continuous on $U_X(r)$ for every radius $r \in]0, \rho[$, with
$$\text{Graph}\left(\Pi_X\Big|_{\overline{U_r(X)}}\right)^{-1} \ =\ \text{Graph}\left(\text{Id} + (N_X \cap \mathring{\mathbb{B}}_r)\right)\Big|_{\partial X}.$$
If $X \subset H$ is weakly closed (which is always the case if H is finite–dimensional) then the preceding conditions are equivalent to

12. Π_X is single-valued in $U_X(\rho)$.

In case of $H = \mathbb{R}^N$, a proximally smooth set X of radius ρ has positive reach ρ (in the sense of Federer) and another equivalent property is ([24, Clarke,Ledyaev,Stern 97])

13. $X = (\mathbb{B}_r(X))_r$ for all $r \in]0, \rho[$
    (with $Z_r := \{\, d_{\mathbb{R}^N \setminus Z}(\cdot) \geq r\,\} \ =\ Z \setminus \mathring{\mathbb{B}}_r\ (\partial Z)\ \subset \mathring{Z}$ for any set $Z \subset \mathbb{R}^N$).

Moreover, Corollary 4.15 of [25, Clarke,Stern,Wolenski 95] guarantees the additional consequences :

- For every $x \in X \subset I\!R^N$, one has $N_X^P(x) \; = \; N_X(x) \; = \; N_X^C(x)$.
- For each $r \in \,]0, \rho[$ and $u \in I\!R^N$ such that $d_X(u) = r$, one has
$$N^P_{I\!B_r(X)}(u) \;=\; N_{I\!B_r(X)}(u) \;=\; N^C_{I\!B_r(X)}(u) \;=\; [0,\infty[\,\cdot\,(u-x)$$
where x is the unique closest point to u in X.
- For each $r \in \,]0, \rho[$, the boundary of $I\!B_r(X)$ is a C^1–manifold.

□

The close relationship between sets of positive erosion and positive reach is stated in

Proposition 4.3.2 *A nonempty open set $O \subset I\!R^N$ has positive erosion of radius $\rho > 0$ if and only if $X := I\!R^N \setminus O$ has positive reach of radius ρ.*

Proof results from the properties (3.) and (13.) respectively :

Assume first that the closed set $X \overset{\text{Def.}}{=} I\!R^N \setminus O$ has positive reach of radius $\rho > 0$. Then we obtain (for any $r \in \,]0, \rho[\,$)
$$u \in O \quad\Longleftrightarrow\quad d_X(u) > 0 \quad\overset{(3.)}{\Longleftrightarrow}\quad d_{Y(r)}(u) < r \quad\Longleftrightarrow\quad u \in \overset{\circ}{I\!B}_r\,(Y(r)).$$
Now suppose that O has positive erosion of radius ρ, i.e. $O = \overset{\circ}{I\!B}_\rho\,(M)$ for some closed set M. The triangle inequality implies $O = \overset{\circ}{I\!B}_r\,(I\!B_{\rho-r}(M))$ for any $r \in \,]0, \rho]$. Morphologically speaking,
$$Y(r) \overset{\text{Def.}}{=} \{u \in I\!R^N \mid d_X(u) \geq r\} \;=\; \{u \in O \mid d_{\partial O}(u) \geq r\} \;=\; O \setminus \overset{\circ}{I\!B}_r\,(\partial O)$$
is the closing of $I\!B_{\rho-r}(M)$ with respect to the structuring element $\overset{\circ}{I\!B}_r$ and this implies $I\!B_{\rho-r}(M) \subset Y(r)$. So we obtain $O = \overset{\circ}{I\!B}_r(I\!B_{\rho-r}(M)) \subset \overset{\circ}{I\!B}_r(Y(r)) \subset O$ for all $r \in]0, \rho]$, i.e. $\overset{\circ}{I\!B}_r\,(Y(r)) = O$. Considering the complements leads to property (13.) for X. □

Corollary 4.3.3 *A nonempty closed set $C \subset I\!R^N$ has positive erosion of radius $\rho > 0$ if and only if $X := \overline{I\!R^N \setminus C}$ has positive reach of radius ρ and $C = \overline{C^\circ}$.*

Proof results from Prop. 4.3.2 since $C = \overline{C^\circ}$ implies $\overline{I\!R^N \setminus C} \;=\; I\!R^N \setminus C^\circ$. □

4.4 Transitions on $\mathcal{K}(I\!R^N)$ and $\Omega(I\!R^N)$

4.4.1 Differential inclusions for $(\mathcal{K}(I\!R^N),\ e^{\supset})$ and $(\mathcal{K}(I\!R^N), d\!l)$

Filippov's Theorem A.1.2 provides the key tool for estimating the Pompeiu–Hausdorff excess between reachable sets of differential inclusions. For the sake of completeness, we mention immediate consequences that have already been shown in [6, Aubin 91] and [2, Aubin 99], § 3.7, for example.

Proposition 4.4.1 *Let $F,\ G : I\!R^N \rightsquigarrow I\!R^N$ be Lipschitz continuous maps with nonempty compact convex values.*
Then for every compact sets $K_1, K_2 \in \mathcal{K}(I\!R^N)$ and time $t > 0$, the reachable sets fulfill

$$e^{\supset}\Big(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2)\Big) \ \le\ e^{\supset}(K_1, K_2)\ \cdot\ e^{\lambda_F \cdot t}\ +\ \sup_{R(t)\, I\!B}\ e^{\supset}\Big(F(\cdot), G(\cdot)\Big) \cdot \tfrac{e^{\lambda_F \cdot t} - 1}{\lambda_F}$$

$$R(t)\ :=\ \|K_2\|_\infty + \sup_{K_2}\ \|G(\cdot)\|_\infty \cdot \tfrac{e^{\mathrm{Lip}\, G \,\cdot\, t} - 1}{\mathrm{Lip}\, G}, \qquad \lambda_F\ :=\ \mathrm{Lip}\ F.$$

Supposing $\lambda \ge \max\{\mathrm{Lip}\, F,\ \mathrm{Lip}\, G\}$ *and* $\sup_{I\!R^N} d\!l(F(\cdot), G(\cdot)) < \infty$ *in addition, the Pompeiu–Hausdorff distance between the reachable sets satisfies*

$$d\!l\Big(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2)\Big) \ \le\ d\!l(K_1, K_2)\ \cdot\ e^{\lambda\, t}\ +\ \sup_{I\!R^N}\ d\!l\Big(F(\cdot), G(\cdot)\Big) \cdot \tfrac{e^{\lambda\, t} - 1}{\lambda}.$$

Proof. For every point $x_2 \in \vartheta_G(t, K_2)$, there is a trajectory $x_2(\cdot) \in AC([0,t], I\!R^N)$ of $\dot{x}_2(\cdot) \in G(x_2(\cdot))$ (almost everywhere) with $x_2(0) \in K_2$, $x_2(t) = x_2$.
Now let $z_1 \in K_1$ satisfy the condition $|z_1 - x_2(0)| \le e^{\supset}(K_1, K_2)$. Then Filippov's Theorem A.1.2 provides a solution $x_1(\cdot) \in AC([0,t], I\!R^N)$ of $\dot{x}_1(\cdot) \in F(x_1(\cdot))$ a.e. with the properties $x_1(0) = z_1$ and

$$\begin{aligned}
&\mathrm{dist}(x_2,\ \vartheta_F(t, K_1)) \ \le\ |x_1(t) - x_2(t)| \\
&\le\ e^{\supset}(K_1, K_2)\ \cdot\ e^{\lambda_F \cdot t}\ +\ \int_0^t e^{\lambda_F \cdot (t-s)}\ \mathrm{dist}\Big(\dot{x}_2(s),\ F(x_2(s))\Big) \quad ds \\
&\le\ e^{\supset}(K_1, K_2)\ \cdot\ e^{\lambda_F \cdot t}\ +\ \int_0^t e^{\lambda_F \cdot (t-s)}\ \ e^{\supset}\Big(F(x_2(s)),\ G(x_2(s))\Big)\ ds.
\end{aligned}$$

Furthermore,
$$\begin{aligned}
|x_2(t) - x_2(0)| \ &\le\ \int_0^t \|G(x_2(s))\|_\infty\ ds \\
&\le\ \int_0^t \Big(\sup_{K_2}\ \|G(\cdot)\|_\infty\ +\ \mathrm{Lip}\ G \cdot |x_2(s) - x_2(0)|\Big)\ ds
\end{aligned}$$

and the integral version of Gronwall's Lemma 1.5.4 implies $\sup_{[0,t]} |x_2(\cdot)| \ \le\ R(t)$.
The consequence for the Pompeiu–Hausdorff distance is obvious. □

These estimates provide sufficient conditions of both forward and backward transitions (of order 0) on $(\mathcal{K}(I\!R^N),\ e^{\supset})$ and $(\mathcal{K}(I\!R^N), d\!l)$ in the sense of Def. 2.1.1, 3.1.1, respectively. Moreover $(\mathcal{K}(I\!R^N),\ e^{\supset})$ and $(\mathcal{K}(I\!R^N), d\!l)$ are two–sided sequentially compact according to Prop. 4.1.1 and Cor. 4.1.2. So we can apply the results about right–hand forward and backward solutions of § 2.3 and § 3.3.

Definition 4.4.2 *For any $\lambda > 0$, the set of λ–Lipschitz continuous maps $F : I\!R^N \rightsquigarrow I\!R^N$ with nonempty compact convex values and $\sup_{x \in I\!R^N} \|F(x)\|_\infty < \infty$ is denoted by* $\mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$.

Corollary 4.4.3 *For every $\lambda \geq 0$, the reachable sets of $\mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$ induce forward transitions (of order 0) on $(\mathcal{K}(I\!R^N), \mathcal{K}(I\!R^N),\ e^{\supset})$ and on $(\mathcal{K}(I\!R^N), \mathcal{K}(I\!R^N), d\!l)$.*

Proof. Def. A.1.1 of reachable sets implies for all $F : I\!R^N \rightsquigarrow I\!R^N,\ M \subset I\!R^N,\ s,t \geq 0$

$$\vartheta_F(t+s, M) \;=\; \vartheta_F(t,\ \vartheta(s,M)).$$

So, $d\!l\Big(\vartheta_F(t+s,K),\ \vartheta_F(t,\ \vartheta_F(s,K))\Big) \;=\; 0$ for every map $F \in \mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$ and initial set $K \in \mathcal{K}(I\!R^N),\ \ s,t \geq 0$.

Furthermore Prop. 4.4.1 guarantees for each $F, G \in \mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$

$$\sup_{K_1,K_2 \in \mathcal{K}(I\!R^N)} \ \limsup_{h \downarrow 0} \ \frac{e^{\supset}\big(\vartheta_F(h,K_1),\ \vartheta_F(h,K_2)\big) - e^{\supset}\big(K_1,\ K_2\big)}{h \ \cdot \ e^{\supset}\big(K_1,\ K_2\big)} \;\leq\; \lim_{h\downarrow 0} \frac{e^{\lambda h}-1}{h} \;=\; \lambda \;=:\; \alpha^{\mapsto}(\vartheta_F),$$

$$Q^{\mapsto}(\vartheta_F, \vartheta_G) \stackrel{\text{Def.}}{=} \sup_{K_1,K_2 \in \mathcal{K}(I\!R^N)} \ \limsup_{h\downarrow 0} \ \left(\frac{e^{\supset}\big(\vartheta_F(h,K_1),\ \vartheta_G(h,K_2)\big) \ - \ e^{\supset}\big(K_1,\ K_2\big)\cdot e^{\alpha^{\mapsto}(\vartheta_G)\cdot h}}{h}\right)^+$$

$$= \sup_{K_1,K_2 \in \mathcal{K}(I\!R^N)} \ \limsup_{h\downarrow 0} \ \left(\frac{e^{\supset}\big(\vartheta_F(h,K_1),\ \vartheta_G(h,K_2)\big) \ - \ e^{\supset}\big(K_1,\ K_2\big)\cdot e^{\lambda h}}{h}\right)^+$$

$$\leq \sup_{I\!R^N} e^{\supset}(F(\cdot),\ G(\cdot)) \quad \leq \quad \sup_{I\!R^N} \|F(\cdot)\|_\infty \ + \ \sup_{I\!R^N} \|G(\cdot)\|_\infty,$$

$$\sup_{K \in \mathcal{K}(I\!R^N)} \ e^{\supset}\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big) \;\leq\; \sup_{I\!R^N} \|F(\cdot)\|_\infty \cdot \ (t-s) \quad \text{for all } 0 \leq s \leq t$$

and obviously, $\mathcal{T}_\Theta(\vartheta_F, K) = 1$. The same estimates hold for $d\!l$ correspondingly.

In particular, $d\!l\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big) \;\leq\; \sup_{I\!R^N} \|F(\cdot)\|_\infty \cdot \ |t-s|$ for all $s,t \geq 0,\ \ K$ and the triangle inequality bridge the last gap for $(\mathcal{K}(I\!R^N),\ \mathcal{K}(I\!R^N),\ e^{\supset})$:

$$\limsup_{h\downarrow 0} \ e^{\supset}\Big(\vartheta_F(t-h, K_1),\ \ K_2\Big) \;=\; e^{\supset}\Big(\vartheta_F(t,K_1),\ \ K_2\Big)$$

for every $K_1, K_2 \in \mathcal{K}(I\!R^N),\ \ t \in\,]0,1]$. □

Remark. The estimate of $Q^{\mapsto}(\vartheta_F, \vartheta_G)$ provides the motivation for assuming the Lipschitz constant λ uniformly : In the general Definition 2.1.1 of $Q^{\mapsto}(\vartheta_F, \vartheta_G)$, we take the parameter $\alpha^{\mapsto}(\vartheta_G)$ (related with the *second* transition) into consideration. It serves the particular purpose that the triangle inequality of $Q^{\mapsto}$ is a simple consequence (see remark (5.) after Definition 2.1.1).
On the other hand. the estimate of $e^{\supset}(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2))$ in Proposition 4.4.1 uses the Lipschitz constant of F (instead of G). Thus, we restrict ourselves to the uniform upper bound λ.

Corollary 4.4.4 *Consider the reachable sets of* $\mathrm{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$ *as forward transitions (of order* 0) *on* $(\mathcal{K}(\mathbb{R}^N), \mathcal{K}(\mathbb{R}^N), e^{\supset})$ *and on* $(\mathcal{K}(\mathbb{R}^N), \mathcal{K}(\mathbb{R}^N), d\!\!l)$, *respectively. Let* $f : \mathcal{K}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$ *satisfy* $\sup_{K,\,t,\,x} \|f(K,t)(x)\|_\infty < \infty$ *and*
$$\sup_{\mathbb{R}^N}\ e^{\supset}\left(f(K_1,t_1)(\cdot),\ f(K_2,t_2)(\cdot)\right)\ \le\ \omega\left(e^{\supset}(K_1,K_2)\ +\ t_2 - t_1\right)$$
for all $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$ *and* $0 \le t_1 \le t_2 \le T$ *with the modulus* $\omega(\cdot)$ *of continuity.*

Then for every initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$, *there exists a right–hand forward solution* $K : [0,T[\longrightarrow (\mathcal{K}(\mathbb{R}^N),\ e^{\supset})$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot)\ \ni\ f(K(\cdot),\cdot)$ *in* $[0,T[$ *with* $K(0) = K_0$.

Suppose in addition that there exist $L \ge 0$ *and a modulus* $\omega(\cdot)$ *of continuity with*
$$\sup_{\mathbb{R}^N}\ e^{\supset}\left(f(K_1,t_1)(\cdot),\ f(K_2,t_2)(\cdot)\right)\ \le\ L\cdot e^{\supset}(K_1,K_2)\ +\ \omega(t_2 - t_1)$$
for all $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$ *and* $0 \le t_1 \le t_2 \le T$. *Let* $K(\cdot) : [0,T[\longrightarrow (\mathcal{K}(\mathbb{R}^N), e^{\supset})$ *be an Euler solution (i.e. constructed by Euler method according to the proof of Prop. 2.3.5). Then every other solution* $M(\cdot)$ *with* $M(0) = K(0)$ *satisfies* $e^{\supset}(K(t),\ M(t^+))\ =\ 0$.
The corresponding statements hold for $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ *instead of* $(\mathcal{K}(\mathbb{R}^N),\ e^{\supset})$.

Proof. The existence results from Cor. 2.3.6 in both cases since $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and $(\mathcal{K}(\mathbb{R}^N),\ e^{\supset})$ are two–sided sequentially compact (due to Prop. 4.1.1 and Cor. 4.1.2). The comparison with an Euler solution is a consequence of Prop. 2.3.10 and $\mathcal{T}_\Theta(\cdot,\cdot) \equiv 1$. Indeed setting $p := d\!\!l$, $q := e^{\supset}$, the triangle inequality implies for all $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$
$$\Delta(K_1,K_2)\ \overset{\text{Def.}}{=}\ \inf_{M\,\in\,\mathcal{K}(\mathbb{R}^N)}\ \Big(p(K_1,M)\ +\ q(M,K_2)\Big)\ =\ e^{\supset}(K_1,K_2)$$
because on the one hand, $\Delta(K_1,K_2) \le e^{\supset}(K_1,K_2)$ is obvious and on the other hand, $e^{\supset}(K_1,K_2)\ \le\ e^{\supset}(K_1,M)\ +\ e^{\supset}(M,K_2)\ \le\ d\!\!l(K_1,M)\ +\ e^{\supset}(M,K_2)$ for all M.
So Proposition 2.3.10 guarantees $\limsup_{\delta\downarrow 0}\ e^{\supset}(K(t),\ M(t+\delta))\ =\ 0.$ □

Similarly, the set–valued maps of $\text{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$ induce backward transitions of order 0.

Corollary 4.4.5 *For each parameter $\lambda \geq 0$, the reachable sets of* $\text{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$ *also provide backward transitions (of order 0) on $(\mathcal{K}(\mathbb{R}^N), e^{\supset})$ and $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$.*

Proof. Like every metric space, $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ fulfills standard hypothesis $(R^{\Leftarrow})$. So Corollary 3.4.2 states for all continuous $K_1, K_2 : [0,T] \longrightarrow (\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and $t \in [0,T[$

$$d\!\!l(K_1(t),\ K_2(t)) \ = \ d\!\!l(K_1(t^+),\ K_2(t^+)) \ = \ d\!\!l(K_1(t^+),\ K_2(t^{++})).$$

For this reason we can omit the limit superior that is denoted by "+", "++" when checking the conditions of Definition 3.1.1 for $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$.

The corresponding simplification is valid for $(\mathcal{K}(\mathbb{R}^N), e^{\supset})$. Indeed, for every map $F \in \text{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$ and $K \in \mathcal{K}(\mathbb{R}^N)$, the reachable set $\vartheta_F(\cdot, K) : [0, \infty[\rightsquigarrow \mathbb{R}^N$ is Lipschitz continuous with respect to $d\!\!l$ and thus,

$$\begin{aligned}
e^{\supset}\Big(\vartheta_F(s^+, K_1),\ \vartheta_G(t^{++}, K_2)\Big) &\stackrel{\text{Def.}}{=} \limsup_{k,l \downarrow 0\ (k<l)} e^{\supset}\Big(\vartheta_F(s+k,\ K_1),\ \vartheta_G(t+l,\ K_2)\Big)\\
&\leq \limsup_{k,l \downarrow 0\ (k<l)} \Big(d\!\!l\ \Big(\vartheta_F(s+k,\ K_1),\ \vartheta_F(s,\ K_2)\Big) +\\
&\qquad e^{\supset}\Big(\vartheta_F(s,\ K_1),\ \vartheta_G(t,\ K_2)\Big) +\\
&\qquad d\!\!l\ \Big(\vartheta_G(t,\ K_1),\ \vartheta_G(t+l,\ K_2)\Big)\Big)\\
&= e^{\supset}\Big(\vartheta_F(s,\ K_1),\ \vartheta_G(t,\ K_2)\Big)
\end{aligned}$$

for any $F, G \in \text{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$, $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$, $s, t \geq 0$.
In the same way, we get $e^{\supset}\Big(\vartheta_F(s, K_1),\ \vartheta_G(t, K_2)\Big) \leq e^{\supset}\Big(\vartheta_F(s^+, K_1),\ \vartheta_G(t^{++}, K_2)\Big)$.

$\vartheta_F(0, K) = K$ and $d\!\!l\Big(\vartheta_F(t, K),\ \vartheta_F(h,\ \vartheta_F(t-h, K))\Big) = 0$ is trivial for all $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $K \in \mathcal{K}(\mathbb{R}^N)$, $0 \leq h \leq t$. Considering $(\mathcal{K}(\mathbb{R}^N), e^{\supset})$ first, Prop. 4.4.1 implies for all $F, G \in \text{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$

$$\begin{aligned}
\alpha^{\rightarrow}(\vartheta_F) &\stackrel{\text{Def.}}{=} \sup_{\substack{0<t<1\\ K_1, K_2 \in\\ UC^{\rightarrow}(]0,1[, \mathcal{K}(\mathbb{R}^N), e^{\supset})}} \limsup_{h \downarrow 0} \left(\frac{e^{\supset}\big(\vartheta_F(h, K_1(t_h)), \vartheta_F(h, K_2(t_h))\big) - e^{\supset}\big(K_1(t_h), K_2(t_h)\big)}{h \quad e^{\supset}\big(K_1(t_h),\ K_2(t_h)\big)}\right)^+\\
&\leq \limsup_{h \downarrow 0} \frac{e^{\lambda h} - 1}{h} \ = \ \lambda
\end{aligned}$$

(with the abbreviation $t_h := t - h$) and

$$\begin{aligned}
Q^{\rightarrow}(\vartheta_F, \vartheta_G) &\stackrel{\text{Def.}}{=} \sup_{\substack{0<t<1\\ K \in \mathcal{K}(\mathbb{R}^N)}} \limsup_{h \downarrow 0} \frac{1}{h} \cdot e^{\supset}\Big(\vartheta_F(h,\ \vartheta_G(t_h, K)),\ \vartheta_G(t, K)\Big)\\
&= \sup_{\substack{0<t<1\\ K \in \mathcal{K}(\mathbb{R}^N)}} \limsup_{h \downarrow 0} \frac{1}{h} \cdot e^{\supset}\Big(\vartheta_F(h,\ \vartheta_G(t_h, K)),\ \vartheta_G(h,\ \vartheta_G(t_h, K))\Big)\\
&\leq \sup_{\mathbb{R}^N} e^{\supset}\Big(F(\cdot),\ G(\cdot)\Big) \ < \ \infty\,.
\end{aligned}$$

The same estimates hold for $d\!\!l$ instead of $e^{\supset}$. □

Corollary 4.4.6 *Consider the reachable sets of* $\mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$ *as backward transitions (of order* 0*) on* $(\mathcal{K}(I\!R^N), e^{\supset})$ *and on* $(\mathcal{K}(I\!R^N), d\!l)$*, respectively.*
Assume for $f : \mathcal{K}(I\!R^N) \times [0,T] \longrightarrow \mathrm{LIP}_\lambda(I\!R^N, I\!R^N)$ *both* $\sup_{K,\,t,\,x} \|f(K,t)(x)\|_\infty < \infty$
and $\sup_{I\!R^N} e^{\supset}(f(K_1,t_1),\ f(K_2,t_2)) \;\le\; \omega\,(e^{\supset}(K_1,K_2) \,+\, t_2 - t_1)$
for all $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $0 \le t_1 \le t_2 \le T$ *with the modulus* $\omega(\cdot)$ *of continuity.*

Then for each initial set $K_0 \in \mathcal{K}(I\!R^N)$*, there is a right–hand backward solution* $K : [0,T[\, \longrightarrow (\mathcal{K}(I\!R^N), e^{\supset})$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot),\cdot)$ *in* $[0,T[$ *with* $K(0) = K_0$.

If there are a constant $L \ge 0$ *and a modulus* $\omega(\cdot) \ge 0$ *of continuity in addition with*

$$\sup_{I\!R^N} e^{\supset}(f(K_1,t_1),\ f(K_2,t_2)) \;\le\; L \cdot e^{\supset}(K_1,K_2) \,+\, \omega(t_2 - t_1)$$

for all $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $0 \le t_1 \le t_2 \le T$*, then* $K(\cdot)$ *can be chosen in such a way that any other right–hand backward solution* $M(\cdot)$ *with* $M(0) = K_0$ *satisfies*

$$e^{\supset}(K(\cdot), M(\cdot)) = 0.$$

The corresponding statements hold for $(\mathcal{K}(I\!R^N), d\!l)$ *instead of* $(\mathcal{K}(I\!R^N), e^{\supset})$.

Proof results from Proposition 3.3.3 because $(\mathcal{K}(I\!R^N), d\!l)$ and $(\mathcal{K}(I\!R^N), e^{\supset})$ are two–sided sequentially compact (according to Prop. 4.1.1 and Cor. 4.1.2, respectively). □

4.4.2 Smooth vector fields for $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$

In Definition 4.1.5 we introduced the quasi–metric $q_{\mathcal{K},N} : \mathcal{K}(I\!R^N) \times \mathcal{K}(I\!R^N) \longrightarrow [0, \infty[$,

$$q_{\mathcal{K},N}(K_1, K_2) \;:=\; d\!l(K_1, K_2) \;+\; e^{\supset}(\text{Graph }{}^{\flat}N_{K_1},\;\; \text{Graph }{}^{\flat}N_{K_2}).$$

Roughly speaking, the Pompeiu–Hausdorff distance $d\!l(K_1, K_2)$ takes all points of the sets K_1, K_2 symmetrically into account. The second term is nonsymmetric and can be regarded as the *graphical* distance from the unit normal vectors in ${}^{\flat}N_{K_2}$ to ${}^{\flat}N_{K_1}$. According to Lemma 4.1.8, $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ is one–sided sequentially compact and fulfills standard hypothesis $(R^{\Leftarrow})$, i.e. the right–hand spheres are left–sequentially closed.

Now autonomous vector fields of class $C^{1,1}$ induce both forward transitions on $(\mathcal{K}(I\!R^N),\ \mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$ and backward transitions on $(\mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$ (of order 0) by means of the ordinary differential equation : The reachable set of $f \in C^{1,1}(I\!R^N, I\!R^N)$ and the initial set $K \in \mathcal{K}(I\!R^N)$ at time $t \geq 0$ is

$$\vartheta_f(t, K) \;\stackrel{\text{Def.}}{=}\; \bigcup_{x\,\in\,K} \; y(t; x)$$

with the unique solution $y(\,\cdot\,; x) \in C^1([0,\ t], I\!R^N)$ of the initial value problem

$$\wedge \left\{ \begin{array}{rcl} \frac{d}{d\,s}\, y(s; x) &=& f(y(s; x)) \\ y(0; x) &=& x. \end{array} \right.$$

Obviously ϑ_f has again the semigroup property $\quad \vartheta_f(t + h,\ K) \;=\; \vartheta_f(h,\ \vartheta_f(t, K))$. Furthermore the evolution is always reversible in time because the solutions of these initial value problems are unique.

The two following lemmas provide the key estimates for transitions : Considering one vector field and one initial set first, we prove the Lipschitz continuity of the reachable set with respect to time. Afterwards the distance between reachable sets is estimated after a given time, but for two vector fields and different initial sets.

Lemma 4.4.7 *For every vector field* $f \in C^1(I\!R^N, I\!R^N)$ *with* $\|Df\|_\infty \leq \Lambda$ *and* $K \in \mathcal{K}(I\!R^N)$, $T > 0$, *the map* $\vartheta_f(\cdot, K)$ *is in* $\text{Lip}^{\to}([0, T],\ \mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$:

$$q_{\mathcal{K},N}\Big(\vartheta_f(s, K),\ \vartheta_f(t, K)\Big) \;\leq\; C \,\cdot\, (t - s) \qquad \textit{for all } \; 0 \leq s < t \leq T$$

with $C \;:=\; 1 \;+\; \Big(\Lambda \;+\; \sup_K |f|\Big)\; e^{2\,\Lambda\,T}$.

Proof. Due to $\|Df\|_\infty \le \Lambda$, Gronwall's Lemma 1.5.1 provides the estimate for $t \ge 0$

$$d\!\!l\Big(K,\ \vartheta_f(t,K)\Big) \;\le\; \sup_K |f| \cdot \tfrac{e^{\Lambda t}-1}{\Lambda}.$$

For any $0 \le s < t \le T$, the semigroup property $\vartheta_f(t,K) = \vartheta_f(t-s,\ \vartheta_f(s,K))$ leads to

$$\begin{aligned} d\!\!l\Big(\vartheta_f(s,K),\ \vartheta_f(t,K)\Big) &\le \sup_{\vartheta_f(s,K)} |f| \cdot \tfrac{e^{\Lambda(t-s)}-1}{\Lambda} \\ &\le \sup_{I\!B_R(K)} |f| \cdot e^{\Lambda T}\,(t-s) \\ &\le \Big(\sup_K |f| + \Lambda R\Big) \cdot e^{\Lambda T}\,(t-s) \\ &\le \sup_K |f|\ e^{\Lambda T} \cdot e^{\Lambda T}\,(t-s) \end{aligned}$$

with $R = R(K,T,\Lambda) := \frac{1}{\Lambda} \cdot \sup_K |f| \cdot (e^{\Lambda T} - 1)$.

Moreover, according to Corollary A.3.3 (1.), every boundary point $x \in \partial\,\vartheta_f(t,K)$ and $p \in N_{\vartheta_f(t,K)}(x)$ is reached by the (unique) trajectory $x(\cdot) \in C^1([s,t], I\!R^N)$ and the adjoint $p(\cdot) \in C^1([s,t], I\!R^N)$ satisfying

$$\begin{array}{lllllll} x(s) & \in & \vartheta_f(s,K), & x(t) & = & x, & \\ p(s) & \in & N_{\vartheta_f(s,K)}(x(s)), & p(t) & = & p, & \\ \dot{x}(\cdot) & = & f(x(\cdot)), & \dot{p}(\cdot) & = & -p(\cdot)\cdot Df(x(\cdot)) & \text{in } [s,t]. \end{array}$$

If $0 < |p| \le 1$ in addition, Gronwall's Lemma guarantees $p(s) \neq 0$ and

$$|p(s) - p| \;\le\; e^{\Lambda(t-s)} - 1$$

because for $\tau \in [s,t]$,

$$\begin{aligned} \Big|\tfrac{d}{d\tau}\Big(p(t-\tau) - p\Big)\Big| &= |\dot{p}(t-\tau)| && \le\ \Lambda\,|p(t-\tau)| \\ &\le \Lambda\,\Big(|p(t-\tau) - p| + |p|\Big) && \le\ \Lambda\,|p(t-\tau) - p| + \Lambda. \end{aligned}$$

Furthermore, $|p| \le 1$ implies that the projection of p on the ray $[0,\infty[\,\cdot\,p(s)$ is also contained in the unit ball $I\!B_1(0)$. So,

$$\text{dist}\Big(p,\ {}^{\flat}N_{\vartheta_f(s,K)}(x(s))\Big) \;\le\; \text{dist}\Big(p,\ [0,\infty[\,\cdot\,p(s)\Big) \;\le\; |p - p(s)| \;\le\; e^{\Lambda(t-s)} - 1$$

and finally we obtain

$$\text{dist}\Big(\text{Graph}\ {}^{\flat}N_{\vartheta_f(t,K)},\ \text{Graph}\ {}^{\flat}N_{\vartheta_f(s,K)}\Big) \;\le\; e^{\Lambda(t-s)} - 1 \;\le\; \Lambda\, e^{\Lambda T}\,(t-s). \qquad \square$$

The deformation along vector fields is reversible in time, i.e. $K = \vartheta_{-f}(t,\ \vartheta_f(t,K))$ for every $f \in C^1(I\!R^N, I\!R^N)$ and initial set $K \in \mathcal{K}(I\!R^N)$. Thus, we obtain Lipschitz continuity in negative time direction as well.

Corollary 4.4.8 *For every vector field* $f \in C^1(I\!R^N, I\!R^N)$ *with* $\|Df\|_\infty \le \Lambda$ *and* $K \in \mathcal{K}(I\!R^N)$, $T > 0$, *the map* $\vartheta_f(\cdot,K) : [0,T] \longrightarrow (\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ *is Lipschitz continuous, i.e.* $q_{\mathcal{K},N}\Big(\vartheta_f(s,K),\ \vartheta_f(t,K)\Big) \;\le\; C \cdot |t-s|$ *for all* $s,t \in [0,T]$ $\square$

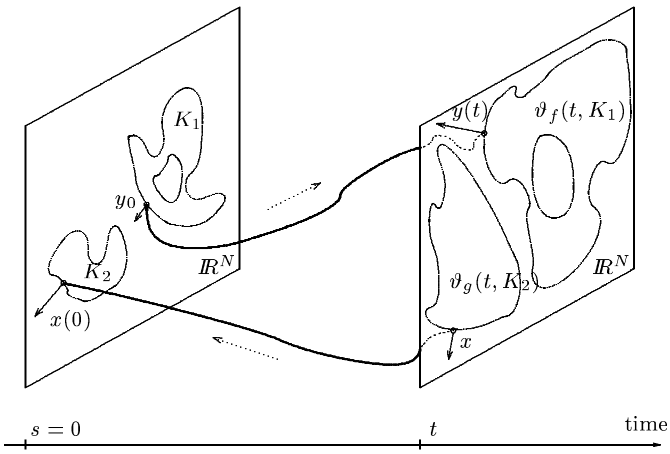
Lemma 4.4.9 *Assume for $f, g \in C^{1,1}(I\!\!R^N, I\!\!R^N)$ that $\|f-g\|_\infty < \infty$ and $\|Df\|_{C^{0,1}},\ \|Dg\|_\infty \le \Lambda$.*
Then for any $K_1,\ K_2 \in \mathcal{K}(I\!\!R^N)$ and $t \ge 0$, the following estimate holds

$$q_{\mathcal{K},N}\Big(\vartheta_f(t,K_1),\ \vartheta_g(t,K_2)\Big) \;\le\; q_{\mathcal{K},N}(K_1,K_2)\ e^{(L+\Lambda)\,t} \;+\; 2\ e^{(L+2\,\Lambda)\,t}\ \ t\ \|f-g\|_{C^1}$$

with $L = L(\Lambda,t) := \Lambda\ (2+e^{2\,\Lambda\,t})$.

Proof. Applying Prop. 4.4.1 to ordinary differential equations leads to

$$d\!\!\!l\Big(\vartheta_f(t,K_1),\ \vartheta_g(t,K_2)\Big) \;\le\; d\!\!\!l(K_1,K_2)\cdot e^{\Lambda\,t} + \|f-g\|_\infty \cdot \tfrac{e^{\Lambda\,t}-1}{\Lambda}.$$



Now choose $x \in \partial\,\vartheta_g(t,K_2)$ and $p \in {}^\flat N_{\vartheta_g(t,K_2)}(x) \setminus \{0\}$ arbitrarily. Correspondingly to the proof of Lemma 4.4.7, Corollary A.3.3 (1.) provides the trajectory

$$x(\cdot) \in C^1([0,t], I\!\!R^N)$$

and its adjoint arc

$$p(\cdot) \in C^1([0,t], I\!\!R^N)$$

satisfying

$$\begin{array}{rclrcl} x(0) & \in & K_2, & x(t) & = & x,\\ p(0) & \in & N_{K_2}(x(0)), & p(t) & = & p,\\ \dot x(\cdot) & = & g(x(\cdot)), & \dot p(\cdot) & = & -\,p(\cdot)\cdot Dg(x(\cdot)). \end{array}$$

Then Gronwall's Lemma ensures $\|\,p(\cdot)\,\|_\infty \le |p|\ e^{\Lambda\,t} \le e^{\Lambda\,t}$.

Now let $(y_0, \widehat{q}_0) \in \text{Graph}\ N_{K_1}$ be in the projection of $\Big(x(0), p(0)\,e^{-\Lambda\,t}\Big) \in \text{Graph}\ {}^\flat N_{K_2}$ on $\text{Graph}\ {}^\flat N_{K_1}$: $\Big|(y_0,\widehat{q}_0) - \Big(x(0),\ p(0)\,e^{-\Lambda\,t}\Big)\Big| \le e^{\supset}\Big(\text{Graph}\ {}^\flat N_{K_1},\ \text{Graph}\ {}^\flat N_{K_2}\Big)$.

Due to $f, g \in C^{1,1}$ and Corollary A.3.3 (2.), the initial value problem

$$\wedge \left\{ \begin{array}{rclrcl} \dot y(\cdot) & = & f(y(\cdot)), & \dot q(\cdot) & = & -\,q(\cdot)\cdot Df(y(\cdot))\\ y(0) & = & y_0, & q(0) & = & \widehat{q}_0\ e^{\Lambda\,t} \end{array}\right.$$

has the unique solution $(y(\cdot), q(\cdot))$ satisfying both

$$\begin{array}{rcl} y(t) & \in & \partial\,\vartheta_f(t,K_1),\\ q(t) & \in & N_{\vartheta_f(t,K_1)}(y(t)),\\ \|\,q(\cdot)\,\|_\infty & \le & |q(0)|\ e^{\Lambda\,t} \;\le\; e^{2\,\Lambda\,t} \end{array}$$

and the estimate

$$\begin{aligned}
&\Big|\Big(x(t), p(t)\Big) - \Big(y(t), q(t)\Big)\Big| \\
\le\ & e^{\widehat{L}\,t} \quad \Big|\Big(x(0), p(0)\Big) - (y_0, \widehat{q}_0\, e^{\Lambda\,t})\Big| \\
&+ \tfrac{e^{\widehat{L}\,t}-1}{\widehat{L}} \sup_{\tau} \Big|\Big(f(x(\tau)),\ -p(\tau)\cdot Df(x(\tau))\Big) - \Big(g(x(\tau)),\ -p(\tau)\cdot Dg(x(\tau))\Big)\Big| \\
\le\ & e^{\widehat{L}\,t} \quad \Big|\Big(x(0), p(0)\Big) - (y_0, \widehat{q}_0\, e^{\Lambda\,t})\Big| + \tfrac{e^{\widehat{L}\,t}-1}{\widehat{L}} \Big(\|f-g\|_\infty + e^{2\,\Lambda\,t}\,\|Df - Dg\|_\infty\Big)
\end{aligned}$$

with the (local) Lipschitz constant $\widehat{L}$ of the Hamiltonian function

$$\mathbb{R}^N \times \mathbb{B}_{e^{2\Lambda t}} \longrightarrow \mathbb{R}^N \times \mathbb{R}^N, \qquad (z, \rho) \longmapsto (f(z),\ -\rho\cdot Df(z)).$$

This estimate results from the Theorem of Cauchy-Lipschitz (see e.g. [2, Aubin 99], Theorem 1.4.1). As an upper bound of $\widehat{L}$, we obtain

$$\begin{aligned}
\widehat{L}\ &\le\ \text{Lip } f\ +\ \Big(\text{Lip } \text{Id}_{\mathbb{R}^N}\Big)\,\|Df\|_\infty\ +\ e^{2\,\Lambda\,t}\cdot \text{Lip } Df \\
&\le\ \Lambda\ +\ \|Df\|_\infty\ +\ e^{2\,\Lambda\,t}\cdot \text{Lip } Df \\
&\le\ \Lambda\ (2 + e^{2\,\Lambda\,t})\ \stackrel{\text{Def.}}{=}\ L.
\end{aligned}$$

Moreover, the initial assumption $|p| \le 1$ implies that the projection of p on any (nonempty) cone is contained in the closed unit ball $\mathbb{B}_1(0)$ and thus,

$$\begin{aligned}
&\text{dist}\Big(\ (x,p),\ \ \text{Graph}\ {}^\flat N_{\vartheta_f(t,K_1)}\Big) \\
\le\ & \text{dist}\Big(\ (x,p),\ \ \{y(t)\} \times ([0,\infty[\,\cdot\, q(t)\ \cap\ \mathbb{B}_1)\Big) \\
\le\ & \text{dist}\Big(\ (x,p),\ \ \{y(t)\} \times\ [0,\infty[\,\cdot\, q(t)\ \Big) \\
\le\ & \Big|\Big(x(t), p(t)\Big) - \Big(y(t), q(t)\Big)\Big| \\
\le\ & e^{L\,t} \quad \Big|\Big(x(0), p(0)\Big) - (y_0, \widehat{q}_0\, e^{\Lambda\,t})\Big| + \tfrac{e^{L\,t}-1}{L} \Big(\|f-g\|_\infty + e^{2\,\Lambda\,t}\,\|Df - Dg\|_\infty\Big) \\
\le\ & e^{(L+\Lambda)\,t}\ \Big|\Big(x(0), p(0)\,e^{-\Lambda\,t}\Big) - (y_0, \widehat{q}_0)\Big| + e^{L\,t}\cdot t\ \Big(\|f-g\|_\infty + e^{2\,\Lambda\,t}\,\|Df - Dg\|_\infty\Big) \\
\le\ & e^{(L+\Lambda)\,t}\ \ e^{\supset}\Big(\text{Graph}\ {}^\flat N_{K_1},\ \text{Graph}\ {}^\flat N_{K_2}\Big) + e^{L\,t}\cdot t\ \Big(\|f-g\|_\infty + e^{2\,\Lambda\,t}\,\|Df - Dg\|_\infty\Big).
\end{aligned}$$

So finally we get with $L = L(\Lambda, t) \stackrel{\text{Def.}}{=} \Lambda\ (2 + e^{2\,\Lambda\,t})$

$$\begin{aligned}
&q_{\mathcal{K},N}\Big(\vartheta_f(t, K_1),\ \vartheta_g(t, K_2)\Big) \\
=\ & d\!\!l(K_1, K_2) \qquad + \quad e^{\supset}\Big(\text{Graph}\ {}^\flat N_{\vartheta_f(t,K_1)},\ \ \text{Graph}\ {}^\flat N_{\vartheta_g(t,K_2)}\Big) \\
\le\ & d\!\!l(K_1, K_2)\ e^{\Lambda\,t} + e^{\Lambda\,t}\cdot t\ \|f-g\|_\infty\ +\ e^{(L+\Lambda)\,t}\quad e^{\supset}\Big(\text{Graph}\ {}^\flat N_{K_1},\ \text{Graph}\ {}^\flat N_{K_2}\Big) \\
&\qquad +\ e^{L\,t}\cdot t\ \ \Big(\|f-g\|_\infty + e^{2\,\Lambda\,t}\,\|Df - Dg\|_\infty\Big) \\
\le\ & q_{\mathcal{K},N}(K_1, K_2)\ e^{(L+\Lambda)\,t}\quad +\quad (e^{\Lambda\,t} + e^{(L+2\,\Lambda)\,t})\ \ t\ \ \|f-g\|_{C^1} \\
\le\ & q_{\mathcal{K},N}(K_1, K_2)\ e^{(L+\Lambda)\,t}\quad +\qquad 2\ e^{(L+2\,\Lambda)\,t}\ \ t\ \ \|f-g\|_{C^1}\,. \qquad \square
\end{aligned}$$

Proposition 4.4.10 *For every $\Lambda \geq 0$, the reachable sets of the vector fields $f \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ with $\|f\|_\infty < \infty$, $\|Df\|_{C^{0,1}} \leq \Lambda$ induce forward transitions on $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}(\mathbb{R}^N),\ q_{\mathcal{K},N})$ of order 0 with*

$$\begin{aligned} \alpha^{\mapsto}(\vartheta_f) &\overset{\text{Def.}}{=} 4\,\Lambda, \\ \beta(\vartheta_f)\,(t) &\overset{\text{Def.}}{=} \Big(1 + \|f\|_{C^1} \cdot e^{2\,\|Df\|_\infty}\Big) \cdot t, \\ Q^{\mapsto}(\vartheta_f, \vartheta_g) &\leq 2\ \|f-g\|_{C^1}\,. \end{aligned}$$

Proof. The semigroup property $\vartheta_f(t+h, K) = \vartheta_f(h,\ \vartheta_f(t,K))$ implies for the quasi–metric $q_{\mathcal{K},N}$ and all $K \in \mathcal{K}(\mathbb{R}^N),\ t,h \geq 0$

$$\begin{aligned} q_{\mathcal{K},N}\Big(\vartheta_f(h,\ \vartheta_f(t,K)),\quad \vartheta_f(t+h,K)\Big) &= 0, \\ q_{\mathcal{K},N}\Big(\vartheta_f(t+h,K),\quad \vartheta_f(h,\ \vartheta_f(t,K))\Big) &= 0. \end{aligned}$$

Furthermore Lemma 4.4.9 has the consequences

$$\begin{aligned} &\sup_{K_1,K_2 \in \mathcal{K}(\mathbb{R}^N)} \ \limsup_{h \downarrow 0} \ \left(\frac{q_{\mathcal{K},N}\big(\vartheta_f(h,K_1), \vartheta_f(h,K_2)\big) - q_{\mathcal{K},N}\big(K_1,K_2\big)}{h \qquad q_{\mathcal{K},N}\big(K_1,K_2\big)}\right)^+ \\ \leq\ & \limsup_{h \downarrow 0} \ \frac{\exp(\Lambda \cdot (3+e^{2\,\Lambda\,h}) \cdot h) - 1}{h} \ = \ 4\,\Lambda \ \overset{\text{Def.}}{=} \ \alpha^{\mapsto}(\vartheta_f) \end{aligned}$$

and

$$\begin{aligned} Q^{\mapsto}(\vartheta_f, \vartheta_g) &\overset{\text{Def.}}{=} \sup_{\substack{K_1,K_2 \in \\ \mathcal{K}(\mathbb{R}^N)}} \ \limsup_{h \downarrow 0} \ \left(\frac{q_{\mathcal{K},N}\big(\vartheta_f(h,K_1), \vartheta_g(h,K_2)\big) - q_{\mathcal{K},N}(K_1,K_2) \cdot e^{\alpha^{\mapsto}(\vartheta_g)\cdot h}}{h}\right)^+ \\ &= \sup_{K_1,K_2} \ \limsup_{h \downarrow 0} \ \left(\frac{q_{\mathcal{K},N}\big(\vartheta_f(h,K_1), \vartheta_g(h,K_2)\big) - q_{\mathcal{K},N}(K_1,K_2) \cdot e^{4\,\Lambda \cdot h}}{h}\right)^+ \\ &\leq \sup_{K_1,K_2} \ \limsup_{h \downarrow 0} \ \Big(q_{\mathcal{K},N}(K_1,\ K_2)\ \frac{\exp(\Lambda\,h\,(3+e^{2\,\Lambda\,h})) - \exp(4\,\Lambda\,h)}{h} \\ &\qquad\qquad + \ 2 \cdot \|f-g\|_{C^1} \cdot e^{\Lambda\,h\,(4+e^{2\,\Lambda\,h})}\Big) \\ &= \qquad 2 \cdot \|f-g\|_{C^1}. \end{aligned}$$

Lemma 4.4.7 states $q_{\mathcal{K},N}\Big(\vartheta_f(s,K),\ \vartheta_f(t,K)\Big) \leq \Big(1 + \|f\|_{C^1} \cdot e^{2\,\|Df\|_\infty}\Big) \cdot (t-s)$ for all $K \in \mathcal{K}(\mathbb{R}^N)$, $0 \leq s \leq t \leq 1$, i.e. $\vartheta_f(\,\cdot\,,K) \in \text{Lip}^{\to}([0,1],\ \mathcal{K}(\mathbb{R}^N),\ q_{\mathcal{K},N})$. Furthermore the reversibility in time leads to

$$\begin{aligned} q_{\mathcal{K},N}\Big(\vartheta_f(t,K),\ \vartheta_f(s,K)\Big) &= q_{\mathcal{K},N}\Big(\vartheta_f(t,K),\ \vartheta_{-f}(t-s,\ \vartheta_f(t,K)\Big) \\ &\leq \text{const}(\|f\|_{C^1}) \cdot |t-s| \end{aligned}$$

for every $0 \leq s \leq t \leq 1$, and so the triangle inequality ensures

$$\limsup_{h \downarrow 0} \ q_{\mathcal{K},N}\Big(\vartheta_f(t-h,K_1),\ K_2\Big) = q_{\mathcal{K},N}\Big(\vartheta_f(t,K_1),\ K_2\Big)$$

for all $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$, $t \in\,]0,1]$. □

In respect of forward solutions of generalized mutational equations, we need some compactness property for applying the results of § 2.3.2. For $(\mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$ however, two sided–sequential compactness does not hold in general as mentioned in the remark after Lemma 4.1.8. Instead of restricting ourselves to *convex* compact subsets of $I\!R^N$, we prove now the following modification of transitional compactness :

Lemma 4.4.11 *For any $\Lambda \geq 0$ and each $n \in I\!N$, let $f_n : [0,T] \longrightarrow C^{1,1}(I\!R^N, I\!R^N)$ be piecewise constant with $\sup_{0 \leq t \leq T} \| f_n(t)(\cdot) \|_{C^{1,1}} \leq \Lambda$. Choosing $K \in \mathcal{K}(I\!R^N)$, set*

$$\widetilde{f}_n \ : \ [0,T] \times I\!R^N \rightsquigarrow I\!R^N, \quad (t,x) \longmapsto f_n(t)(x),$$
$$K_n(h) \ := \ \vartheta_{\widetilde{f}_n}(h, K) \qquad \textit{for } h \in [0,T].$$

Then there is a sequence $n_k \nearrow \infty$ of indices such that for each $h \in [0,T]$, a compact set $K(h)$ fulfills $\quad q_{\mathcal{K},N}\Big(K_{n_k}(h),\ K(h)\Big) \longrightarrow 0, \quad q_{\mathcal{K},N}\Big(K(h),\ K_{n_k}(h)\Big) \longrightarrow 0 \quad (k \to \infty).$

Proof For each $n \in I\!N$, the Hamiltonian system

$$\wedge \begin{cases} \dot{x}(t) \ = \ \widetilde{f}_n(t,\ x(t)) & x(0) \ = \ x_0 \\ \dot{p}(t) \ = \ -p(t) \cdot \frac{\partial}{\partial x} \widetilde{f}_n(t,\ x(t)) & p(0) \ = \ p_0 \end{cases} \qquad (*)$$

is piecewise autonomous and its right–hand side is locally Lipschitz continuous in each subinterval of autonomy because $f_n : [0,T] \longrightarrow C^{1,1}(I\!R^N, I\!R^N)$ is piecewise constant. So its flow $\Phi_n : [0,T] \times I\!R^N \times I\!R^N \longrightarrow I\!R^N \times I\!R^N,\ (t,\ x_0,\ p_0) \longmapsto (x(t),\ p(t))$ has the following properties for every $t \in [0,T]$:

1. $\Phi_n(t,\cdot,\cdot) : I\!R^N \times I\!R^N \longrightarrow I\!R^N \times I\!R^N$ is locally bi–Lipschitz homeomorphism
2. $\Phi_n(t,\cdot,\cdot)\ (\text{Graph } N_K) \ = \ \text{Graph } N_{K_n(t)}$ (due to Cor. A.3.3)
3. $\Phi_n(t,\cdot,\cdot)\ \big(\text{Graph } {}^\flat N_K\big) \ \subset \ \text{Graph } (N_{K_n(t)}(\cdot) \cap e^{\Lambda t}\, I\!B)$
4. $\Phi_n(t,\cdot,\cdot)^{-1}\big(\text{Graph } {}^\flat N_{K_n(t)}\big) \ \subset \ \text{Graph } (N_K(\cdot) \cap e^{\Lambda t}\, I\!B)$.

Due to $\sup_{n,t} \| f_n(t)(\cdot) \|_{C^{1,1}} \leq \Lambda$, every reachable set $K_n(h)$ $(n \in I\!N,\ h \in [0,T])$ is contained in $I\!B_{\Lambda T}(K)$ and, in the compact product $[0,T] \times I\!B_{2(\Lambda T+1)}(K) \times I\!B_{e^{2\Lambda T}}$, the sequence $(\Phi_n)_n$ is equi–continuous and uniformly bounded.

So the Theorem of Arzela–Ascoli provides a sequence $n_k \nearrow \infty$ of indices and a function $\Phi \in C^0([0,T] \times I\!R^N \times I\!R^N)$ such that for $k \longrightarrow \infty$,

$$\Phi_{n_k} \longrightarrow \Phi \quad \text{uniformly in } [0,T] \times I\!B_{2(\Lambda T+1)}(K) \times I\!B_{e^{2\Lambda T}}.$$

In particular, considering the flows Φ_{n_k} $(k \in I\!N)$ has the technical advantage that the sequence (n_k) of indices does not depend on $t \in [0,T]$.

Now for every $t \in [0,T]$, the limit

$$\begin{aligned} K(t) \;:=\; \mathrm{Lim}_{k\to\infty}\, \vartheta_{\widetilde{f}_{n_k}}(t,K) \;&=\; \mathrm{Lim}_{k\to\infty} \;\; \pi_1\, \Phi_{n_k}(t,\cdot,\cdot)\,(K \times \mathbb{B}_1) \\ &=\; \pi_1 \;\; \Phi\,(t,\cdot,\cdot)\,(K \times \mathbb{B}_1) \end{aligned}$$

exists and Cor. 4.1.7 guarantees $\quad \mathrm{Graph}\, N_{K(t)} \;\subset\; \mathrm{Liminf}_{k\to\infty}\, \mathrm{Graph}\, N_{K_{n_k}(t)}$
i.e. $\qquad q_{\mathcal{K},N}\,(K_{n_k}(t),\, K(t)) \longrightarrow 0 \qquad (k \longrightarrow \infty).$

We still have to prove the inclusion $\;\mathrm{Limsup}_{k\to\infty}\; \mathrm{Graph}\; N_{K_{n_k}(t)} \;\subset\; \mathrm{Graph}\, N_{K(t)},$
or (equivalently) $\qquad \mathrm{Limsup}_{k\to\infty}\;\; \Phi_{n_k}(t,\cdot,\cdot)\,\big(\mathrm{Graph}\; {}^\flat N_K\big) \quad \subset\; \mathrm{Graph}\; N_{K(t)}.$
For arbitrary $x \in \partial K$ and $p \in N_K^P(x) \cap \partial\mathbb{B}_1$, there is some radius $\rho \in\,]0, \frac14[$ satisfying $K \,\cap\, \mathbb{B}_\rho(x+\rho\, p) \;=\; \{x\}$. $\quad$ So setting as an abbreviation

$$\begin{aligned} O_k \;&:=\; \vartheta_{\widetilde{f}_{n_k}}(t,\; \mathbb{B}_\rho(x{+}\rho\, p))^\circ \;&&=\; \vartheta_{\widetilde{f}_{n_k}}\Big(t,\; \overset{\circ}{\mathbb{B}}_\rho\,(x{+}\rho\, p)\Big), \\ O \;&:=\; \pi_1\, \Phi(t,\; \mathbb{B}_\rho(x{+}\rho\, p),\; \mathbb{B}_1)^\circ \;&&=\; \pi_1\, \Phi\Big(t,\; \overset{\circ}{\mathbb{B}}_\rho\,(x{+}\rho\, p),\; \mathbb{B}_1\Big), \end{aligned}$$

$q_k := \pi_2 \Phi_{n_k}(t,x,p) \neq 0$ is a limiting normal vector of both ${}^{\complement}O_k = \overline{\mathbb{R} \setminus \vartheta_{\widetilde{f}_{n_k}}(t,\; \mathbb{B}_\rho(x{+}\rho\, p))}$
and $\;K_{n_k}(t)\;$ at $\;y_k \;:=\; \pi_1\, \Phi_{n_k}(t,x,p)\;$ (due to Cor. A.3.3).
Furthermore Cor. A.6.4 provides a lower bound $\;r = r(\rho,\Lambda,T) \in\;]0,\rho[\;$ such that each O_k has positive erosion of radius r. Due to $\;\overline{O} \;=\; \mathrm{Lim}_{k\to\infty}\, \overline{O_k},\;\; \partial O \;=\; \mathrm{Lim}_{k\to\infty}\, \partial O_k,$
Prop. 4.2.7 and § 4.3 imply $\;O \in \Omega^r_\circ(\mathbb{R}^N)\;$ and $\;\mathrm{Lim}_{k\to\infty}\, \mathrm{Graph}\; {}^\flat N_{{}^\complement O_k} \;=\; \mathrm{Graph}\; {}^\flat N_{{}^\complement O}$
$=\; \mathrm{Graph}\; {}^\flat N^P_{{}^\complement O}$. $\quad$ So now we conclude $\quad \Phi(t,x,p) \;\in\; \mathrm{Graph}\;\; N^P_{K(t)}\;$ from

$$\Phi(t,x,p) \;=\; \lim_{k\to\infty}\; (y_k, q_k) \;\subset\; \mathrm{Limsup}_{k\to\infty}\; \{y_k\} \times N_{{}^\complement O_k}(y_k) \;\subset\; \mathrm{Graph}\;\; N^P_{{}^\complement O}$$

and from the fact that the first component $\pi_1\, \Phi(t,\cdot,\mathbb{B}_1)$ maps $K \cap \mathbb{B}_\rho(x+\rho\, p) \;=\; \{x\}$ to $\quad K(t) \,\cap\, \overline{O} \;=\; \{\pi_1\, \Phi(t,x,0)\}, \quad$ i.e. $\quad K(t) \;\subset\; {}^\complement O.$

As a consequence of continuity, $\qquad \Phi(t,\cdot,\cdot)\,(\mathrm{Graph}\; {}^\flat N_K) \;\subset\; \mathrm{Graph}\; N_{K(t)}$
and finally, $\mathrm{Lim}_{k\to\infty}\, \Phi_{n_k}(t,\cdot,\cdot)\,(\mathrm{Graph}\; {}^\flat N_K) \;\subset\; \Phi(t,\cdot,\cdot)\,(\mathrm{Graph}\; {}^\flat N_K) \;\subset\; \mathrm{Graph}\; N_{K(t)}.$

□

Corollary 4.4.12 *Consider the reachable sets of vector fields* $g \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$, $\|g\|_{C^{1,1}} \leq \Lambda$, *as forward transitions (of order* 0*) on* $(\mathcal{K}(\mathbb{R}^N), \mathcal{K}(\mathbb{R}^N), q_{\mathcal{K},N})$.
Let $f : \mathcal{K}(\mathbb{R}^N) \times [0,T] \;\longrightarrow\; C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ *satisfy* $\quad \sup_{K,\,t}\; \|\, f(K,t)(\cdot)\, \|_{C^{1,1}} \leq \Lambda$

and $\qquad \sup_{\mathbb{R}^N}\; \Big\| f(K_1,t_1) \;-\; f(K_2,t_2) \Big\|_{C^1} \;\leq\; \omega\,(q_{\mathcal{K},N}(K_1,K_2) \;+\; t_2 - t_1)$

for all $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$ *and* $0 \leq t_1 \leq t_2 \leq T$ *with the modulus* $\omega(\cdot)$ *of continuity.*

Then for every initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$, *there exists a right–hand forward solution* $K : [0,T[\, \longrightarrow (\mathcal{K}(\mathbb{R}^N), q_{\mathcal{K},N})$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot) \;\ni\; f(K(\cdot),\cdot)$ *in* $[0,T[$ *with* $K(0) = K_0$.

Moreover, the solution is unique if there are $L \geq 0$ *and a modulus* $\omega(\cdot)$ *of continuity with* $\sup_{I\!R^N} \left\| f(K_1,t_1) - f(K_2,t_2) \right\|_{C^1} \leq L \cdot q_{\mathcal{K},N}(K_1,K_2) + \omega(t_2-t_1)$ *for all* $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $0 \leq t_1 \leq t_2 \leq T$ *in addition.*

Proof. The existence results from Proposition 2.3.5 and remark (2.) after its proof. Indeed, according to Lemma 4.4.11, the Euler approximations are contained in a subset that is transitionally compact in $\Big(\mathcal{K}(I\!R^N),\ q_{\mathcal{K},N},\ \big\{g \in C^{1,1}(I\!R^N, I\!R^N) \mid \|g\|_{C^{1,1}} \leq \Lambda\big\}\Big)$.

Under the additional Lipschitz assumption about f, any other solution $M(\cdot)$ satisfies $q_{\mathcal{K},N}(K(t), M(t^+)) = 0$ for all t — as a consequence of Proposition 2.3.10 (applied to $p(K_1,K_2) := q_{\mathcal{K},N}(K_1,K_2) + q_{\mathcal{K},N}(K_2,K_1)$, $q := q_{\mathcal{K},N}$) since $\mathcal{T}_\Theta(\cdot,\cdot) \equiv 1$ and

$$\Delta(K_1,K_2) \stackrel{\text{Def.}}{=} \inf_{K \in \mathcal{K}(I\!R^N)} \Big(p(K_1,\ K) + q(K,\ K_2)\Big) = p(K_1,K_2).$$

Finally Corollary 3.4.2 implies $q_{\mathcal{K},N}(K(t), M(t)) = q_{\mathcal{K},N}(K(t), M(t^+)) = 0$ because $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ fulfills standard hypothesis $(R^{\Leftarrow})$ (due to Lemma 4.1.8). □

Proposition 4.4.13 *For any* $\Lambda \geq 0$ *fixed, the reachable sets of all vector fields* $f \in C^{1,1}(I\!R^N, I\!R^N)$ *with* $\|f\|_\infty < \infty$, $\|Df\|_{C^{0,1}} \leq \Lambda$ *induce backward transitions on* $(\mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$ *of order* 0 *with*

$$\begin{aligned}
\alpha^{\rightarrow}(\vartheta_f) &\leq 4\,\Lambda,\\
\beta(\vartheta_f)\,(t) &\stackrel{\text{Def.}}{=} \Big(1 + \|f\|_{C^1} \cdot e^{2\,\|Df\|_\infty}\Big) \cdot t,\\
Q^{\rightarrow}(\vartheta_f, \vartheta_g) &\leq 2\ \|f-g\|_{C^1}.
\end{aligned}$$

Proof. $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ fulfills standard hypothesis $(R^{\Leftarrow})$ according to Lemma 4.1.8. Thus Corollary 3.4.2 guarantees $q_{\mathcal{K},N}(K_1(t^+),\ K_2(t^{++})) = q_{\mathcal{K},N}(K_1(t),\ K_2(t))$ for all $K_1,\ K_2 \in UC^{\rightarrow}([0,1[,\ \mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$ and $t \in [0,1[$.

Correspondingly to the preceding proof of forward transitions (Prop. 4.4.10), Lemma 4.4.9 implies for every $K_1,\ K_2 \in UC^{\rightarrow}(]0,1[,\ \mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$, $K \in \mathcal{K}(I\!R^N)$ and $t \in]0,1[$

$$\begin{aligned}
&\limsup_{h\downarrow 0} \left(\frac{q_{\mathcal{K},N}\big(\vartheta_f(h^+, K_1(t_h)),\, \vartheta_f(h^{++}, K_2(t_h))\big) - q_{\mathcal{K},N}\big(K_1(t_h^+),\, K_2(t_h^{++})\big)}{h \qquad q_{\mathcal{K},N}\big(K_1(t_h^+),\ K_2(t_h^{++})\big)}\right)^+\\
=\ &\limsup_{h\downarrow 0} \left(\frac{q_{\mathcal{K},N}\big(\vartheta_f(h, K_1(t_h)),\, \vartheta_f(h, K_2(t_h))\big) - q_{\mathcal{K},N}\big(K_1(t_h),\, K_2(t_h)\big)}{h \qquad q_{\mathcal{K},N}\big(K_1(t_h),\ K_2(t_h)\big)}\right)^+\\
\leq\ &\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot \Big(e^{\Lambda\,h\,(3+e^{2\,\Lambda\,h})} - 1\Big) \quad = \quad 4\,\Lambda \qquad \text{with } t_h := t-h \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot q_{\mathcal{K},N}\Big(\vartheta_f(h^+,\ \vartheta_g(t-h,K)),\quad \vartheta_g(t^{++},K)\Big)\\
=\ &\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot q_{\mathcal{K},N}\Big(\vartheta_f(h,\quad \vartheta_g(t-h,K)),\ \ \vartheta_g(h,\ \vartheta_g(t-h,K))\Big)\\
\leq\ &\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot 2\ h\ \|f-g\|_{C^1}\ \ e^{\Lambda\,h\,(4+e^{2\,\Lambda\,h})} \ =\ 2\ \|f-g\|_{C^1}.
\end{aligned}$$

Finally $\vartheta_f(\,\cdot\,,K) \in \text{Lip}^{\rightarrow}([0,1], \mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ results from Lemma 4.4.7. □

Corollary 4.4.14 *Consider the reachable sets of vector fields* $g \in C^{1,1}(I\!R^N, I\!R^N)$, $\|g\|_{C^{1,1}} \le \Lambda$, *as backward transitions (of order 0) on* $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$.
Let $f : \mathcal{K}(I\!R^N) \times [0,T] \longrightarrow C^{1,1}(I\!R^N, I\!R^N)$ *satisfy* $\sup_{K,\,t} \| f(K,t)(\cdot) \|_{C^{1,1}} \le \Lambda$
and $\sup_{I\!R^N} \Big\| f(K_1,t_1) - f(K_2,t_2) \Big\|_{C^1} \le \omega\,(q_{\mathcal{K},N}(K_1,K_2) + t_2 - t_1)$
for all $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $0 \le t_1 \le t_2 \le T$ *with the modulus* $\omega(\cdot)$ *of continuity.*

Then for each initial set $K_0 \in \mathcal{K}(I\!R^N)$, *there exists a right–hand backward solution* $K : [0,T[\longrightarrow (\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot),\cdot)$ *in* $[0,T[$ *with* $K(0) = K_0$.

Moreover, the solution is unique if there are $L \ge 0$ *and a modulus* $\omega(\cdot)$ *of continuity with* $\sup_{I\!R^N} \Big\| f(K_1,t_1) - f(K_2,t_2) \Big\|_{C^1} \le L \cdot q_{\mathcal{K},N}(K_1,K_2) + \omega(t_2 - t_1)$
for all $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $0 \le t_1 \le t_2 \le T$ *in addition.*

Proof of existence results from Prop. 3.3.3 because according to Lemma 4.4.11, the Euler approximations are contained in a transitionally compact subset of

$$\Big(\mathcal{K}(I\!R^N),\ q_{\mathcal{K},N},\ \{g \in C^{1,1}(I\!R^N, I\!R^N) \mid \|g\|_{C^{1,1}} \le \Lambda\}\Big).$$

Under the additional assumption of Lipschitz continuity, we also obtain that any other solution $M(\cdot) : [0,T[\longrightarrow \mathcal{K}(I\!R^N)$ with $M(0) = K_0$ satisfies $q_{\mathcal{K},N}(K(t^+),\ M(t^{++})) = 0$ for all $t \in [0,T[$. Now $(\mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$ fulfills standard hypothesis $(R^{\Leftarrow})$ (due to Lemma 4.1.8) and thus Corollary 3.4.2 implies the uniqueness of solutions. □

4.4.3 Differential inclusions for $(\Omega(I\!R^N), q_{\Omega,\partial})$

In the two preceding sections, we presented examples of forward transitions on $\mathcal{K}(I\!R^N)$ with the additional property that we can choose $\mathcal{K}(I\!R^N)$ also as "test set" (abbreviated as D in chapter 2).

Difficulties in choosing D adequately are usually related to the estimates of $\alpha^{\mapsto}(\vartheta_F)$ and $Q^{\mapsto}(\vartheta_F, \vartheta_G)$. Roughly speaking, they often use some version of reversibility in time : When proving Lemma 4.4.9 about $q_{\mathcal{K},N}\,(\vartheta_f(t,K_1),\ \vartheta_g(t,K_2))$, for example, we follow a trajectory $x(\cdot)$ (of the vector field g) in backward time direction and then look for a suitable trajectory $y(\cdot)$ of f in positive time direction. Here the reversibility in time is to guarantee that such a counterpart $y(\cdot)$ stays in the boundary of the reachable set until the given time (at which we started).

Now the next example is an ostensible metric that does not imply this property in general and, we start with $(\Omega(I\!R^N), q_{\Omega,\partial})$ because its subset $(\Omega^\rho_\circ(I\!R^N),\ q_{\Omega,\partial})$ (of sets with uniform positive erosion of radius ρ) is two–sided sequentially compact (due to Corollary 4.2.5).

As a consequence of Appendix A.4 and Appendix A.5, standard hypothesis $(\overset{\circ}{\mathcal{H}})$ leads to $\Omega_{C^{1,1}}(I\!R^N)$ as a candidate for the "test set" (abbreviated as D in chapter 2). Indeed, we prove in Corollary A.5.4 that smooth open initial sets stay in $\Omega_{C^{1,1}}(I\!R^N)$ for a short time (at least) and their evolution is reversible in time within this period.
Adapting the approach of the preceding § 4.4.2, we now obtain a new class of set–valued maps $I\!R^N \rightsquigarrow I\!R^N$ whose reachable sets induce forward transitions of *positive* order on $(\Omega(I\!R^N),\ \Omega_{C^{1,1}}(I\!R^N),\ q_{\Omega,\partial})$ as stated in Proposition 4.4.20.

Definition 4.4.15 *The* Hamiltonian $\mathcal{H}_F : I\!R^N \times I\!R^N \longrightarrow I\!R$ *of* $F : I\!R^N \rightsquigarrow I\!R^N$ *is* $\mathcal{H}_F(x,p) := \sup_{y\in F(x)} p\cdot y.$ $\quad \mathrm{LIP}^{(\overset{\circ}{\mathcal{H}})}_\lambda(I\!R^N, I\!R^N)$ *contains all maps* $F : I\!R^N \rightsquigarrow I\!R^N$ *with*

1. $F : I\!R^N \rightsquigarrow I\!R^N$ *has compact convex values with nonempty interior,*
2. $\mathcal{H}_F(\cdot,\cdot) \in C^{1,1}(I\!B_R \times (I\!B_R \setminus \overset{\circ}{I\!B}_{\frac{1}{R}}))$ *for every radius* $R > 1$,
3. $\|\mathcal{H}_F\|_{C^{1,1}(I\!R^N\times\,\partial I\!B_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_F\|_{C^1(I\!R^N\times\,\partial I\!B_1)} + \mathrm{Lip}\ D\mathcal{H}_F|_{I\!R^N\times\,\partial I\!B_1} < \lambda\,.$

Remark. 1. Condition (3.) and Lemma A.2.4 have the consequences that $F : I\!R^N \rightsquigarrow I\!R^N$ is λ–Lipschitz continuous and $\sup_{I\!R^N} \|F(\cdot)\| \le \lambda.$

2. The derivative of $\mathcal{H}_F$ has linear growth as in standard hypotheses $(\mathcal{H}), (\overset{\circ}{\mathcal{H}})$, i.e. there exists a constant $\gamma_F > 0$ with

$$\Big\| D\,\mathcal{H}_F(x,p)\Big\|_{\mathcal{L}(I\!R^N\times I\!R^N, I\!R)} \le \gamma_F \cdot (1 + |x| + |p|)$$

for all $x, p \in I\!R^N$ $(|p| \ge 1)$ according to Def. A.4.2.
Indeed, Lemma A.2.4 (1.) ensures for the partial derivative of $\mathcal{H}_F$ with respect to $p \neq 0$

$$\Big|\tfrac{\partial}{\partial p}\,\mathcal{H}_F(x,p)\Big| \le \|F(x)\|_\infty \le \sup \|F(\cdot)\|_\infty \le \lambda.$$

For the partial derivative of $\mathcal{H}_F(x,p)$ with respect to x, we follow the same track as in the proof of Prop. A.4.11 and apply Lemma A.4.15 of Ward to the marginal function

$$\varphi(x) := \inf\{-p\cdot y \mid y \in F(x)\} = -\mathcal{H}_F(x,p)$$

with $p \neq 0$ fixed. Setting $y_x := \frac{\partial}{\partial p}\mathcal{H}_F(x,p) = (N_{F(x)}|_{\partial F(x)})^{-1}(p)$, Lemma A.4.15 (1.) states for every direction $u \in I\!R^N$

$$\begin{aligned} D^c_\uparrow \varphi(x)\,(u) &\le \inf_v \ \{-p\cdot v \mid (u,v) \in T^c_{\mathrm{Graph}\ F}(x, y_x)\} \\ &= \inf_v \ \{-p\cdot v \mid \quad v \in D^c F(x,y_x)\,(u)\}. \end{aligned}$$

The λ–Lipschitz continuity of F implies the λ–Lipschitz continuity of each $D^c F(x, y_x)$ according to the remark after Def. A.4.13. So we conclude from $0 \in D^c F(x, y_x)\,(0)$

$$-\tfrac{\partial}{\partial x}\,\mathcal{H}_F(x,p)\cdot u \;=\; D^c_\uparrow\varphi(x)\,(u) \;\le\; |p|\;(0+\lambda\,|u|) \qquad \text{for any } u$$

and thus, $$\left|\tfrac{\partial}{\partial x}\,\mathcal{H}_F(x,p)\right| \;\le\; \lambda\;|p|.$$

Now for maps of $\mathrm{LIP}^{(\mathring{\mathcal{H}})}_\lambda(I\!R^N, I\!R^N)$, the properties of their reachable sets are checked with respect to their role as forward transitions on $(\Omega(I\!R^N),\ \Omega_{C^{1,1}}(I\!R^N),\ q_{\Omega,\partial})$. The Lipschitz continuity in positive time direction is quite obvious.

Lemma 4.4.16 *For every map* $F \in \mathrm{LIP}^{(\mathring{\mathcal{H}})}_\lambda(I\!R^N, I\!R^N)$, *initial set* $O \in \Omega(I\!R^N)$ *and time* $0 \le s \le t \le T$,

$$q_{\Omega,\partial}\Big(\vartheta_F(s,O),\;\vartheta_F(t,O)\Big) \;\le\; 2\,\cdot\,\sup_{I\!R^N}\,\|F(\cdot)\|_\infty\,\cdot\,(t-s) \;\le\; 2\,\lambda\,\cdot\,(t-s).$$

Proof. The Pompeiu–Hausdorff distance fulfills the estimate

$$d\!l\Big(\vartheta_F(s,K),\;\vartheta_F(t,K)\Big) \;\le\; \sup_{I\!R^N}\,\|F(\cdot)\|_\infty\,\cdot\,(t-s) \;\le\; \lambda\,(t-s).$$

In respect to $e^{\supset}\Big(\partial\,\vartheta_F(s,O),\;\partial\,\vartheta_F(t,O)\Big)$, each point $x \in \partial\,\vartheta_F(t,O) \subset \overline{\vartheta_F(t,O)} = \vartheta_F(t,\overline{O})$ is attained by a trajectory $x(\cdot) \in AC([0,t], I\!R^N)$ of F with $x(0) \in \overline{O}$. As an indirect consequence of Filippov's Theorem A.1.2, $x(s) \in \partial\,\vartheta_F(s,O)$ for any s and thus, $\mathrm{dist}\Big(x,\;\partial\,\vartheta_F(s,O)\Big) \;\le\; |x(t)-x(s)| \;\le\; \sup_{I\!R^N}\,\|F(\cdot)\|_\infty\,\cdot\,(t-s).$ □

The parameter $\alpha^{\mapsto}(\cdot)$ of forward transitions results from the comparison between the reachable sets of different maps in $\mathrm{LIP}^{(\mathring{\mathcal{H}})}_\lambda(I\!R^N, I\!R^N)$ and different initial sets. Here we need a new function for two bounded subsets M_1, M_2 of $I\!R^N$. $m\!m(M_1,M_2)$ abbreviates the maximum distance between elements of M_1 and points of M_2.
In particular, $m\!m(M,M) > 0$ for every bounded set $M \subset I\!R^N$ with more than 1 element. So these forward transitions on $(\Omega(I\!R^N),\ \Omega_{C^{1,1}}(I\!R^N),\ q_{\Omega,\partial})$ are not of order 0 as we conclude from Lemma 4.4.18 later.

Definition 4.4.17 *For nonempty bounded subsets* $M_1, M_2 \subset I\!R^N$ *define*

$$m\!m(M_1,M_2) \;:=\; \sup\;\Big\{\,|x-y|\;\Big|\;x \in M_1,\;\; y \in M_2\Big\}$$

Remark. Obviously, $m\!m$ is symmetric and satisfies the triangle inequality for all nonempty bounded subsets, but it is not an ostensible metric. Moreover it fulfills
$d\!l(K_1,K_3) \;\le\; m\!m(K_1,K_3) \;\le\; m\!m(K_1,K_2) \;+\; d\!l(K_2,K_3)$ for all $K_j \in \mathcal{K}(I\!R^N)$.

Lemma 4.4.18 *Suppose for* $F,\ G \in \mathrm{LIP}_\lambda^{(\overset{\circ}{\mathcal{H}})}(I\!R^N, I\!R^N)$, $O_1,\ O_2 \in \Omega(I\!R^N)$ *and* $\rho, T > 0$ *that the closure of each* $\vartheta_F(t, O_1) \in \Omega(I\!R^N)$ $(t \in [0,T])$ *has positive reach of radius* ρ.
Then, for every $t \in [0,T[$,

$$q_{\Omega,\partial}\Big(\vartheta_F(t,O_1),\ \vartheta_G(t,O_2)\Big) \ \le\ q_{\Omega,\partial}(O_1,O_2)\cdot e^{\lambda t} + 2\cdot \sup_{I\!R^N} \mathrm{m}(F(\cdot),\ G(\cdot))\cdot \tfrac{e^{\lambda t}-1}{\lambda}.$$

Proof. Prop. 4.4.1 is already providing an estimate for the Pompeiu–Hausdorff distance between the closures of reachable sets :

$$d\!l\Big(\vartheta_F(t,\overline{O}_1),\ \vartheta_G(t,\overline{O}_2)\Big) \ \le\ d\!l(\overline{O}_1,\overline{O}_2)\cdot e^{\lambda t} + \sup_{I\!R^N} d\!l\Big(F(\cdot),G(\cdot)\Big)\cdot \tfrac{e^{\lambda t}-1}{\lambda}.$$

For any boundary point x of $\vartheta_G(t,O_2) \in \Omega(I\!R^N)$, we need an upper bound of its distance from $\partial\,\vartheta_F(t,O_1)$.
As a frequently used consequence of Filippov's Theorem A.1.2, x is attained by a trajectory $x(\cdot) \in AC([0,t], I\!R^N)$ of G with $x(0) \in \partial O_2$. Now choose $y_0 \in \partial O_1$ satisfying $|y_0 - x(0)| \le e^{\supset}(\partial O_1,\ \partial O_2)$.
Due to Cor. A.5.3, the evolution of O_1 along F is reversible in time on $[0,T[$, i.e.

$$I\!R^N \setminus O_1 \ =\ \vartheta_{-F}\Big(t,\ I\!R^N \setminus \vartheta_F(t,O_1)\Big).$$

So there exists a trajectory $\widehat{y}(\cdot) \in AC([0,t], I\!R^N)$ of $-F$ satisfying $\widehat{y}(t) = y$ and $\widehat{y}(0) \in I\!R^N \setminus \vartheta_F(t,O_1)$. Since y belongs to $\partial O_1 \subset \overline{O}_1$, we obtain $\widehat{y}(0) \in \vartheta_F(t,\overline{O}_1) = \overline{\vartheta_F(t,O_1)}$ and thus, $\widehat{y}(0) \in \partial\,\vartheta_F(t,O_1)$. Finally, Gronwall's Lemma provides an upper bound of $|x - \widehat{y}(0)|$:

$$\begin{aligned}
|x(s) - \widehat{y}(t-s)| \ &\le\ |x(0) - \widehat{y}(t)| + \int_0^s \Big|\,\dot{x}(r) - \tfrac{d}{dr}\,\widehat{y}(t-r)\Big|\ dr \\
&\le\ |x(0) - \widehat{y}(t)| + \int_0^s \mathrm{m}\Big(G(x(r)),\ F(\widehat{y}(t-r))\Big)\ dr \\
&\le\ |x(0) - \widehat{y}(t)| + \int_0^s \Big(\mathrm{m}\Big(G(x(r)),\ F(x(r))\Big) \\
&\qquad\qquad + \mathrm{Lip}\ F\cdot |x(r) - \widehat{y}(t-r)|\Big)\ dr
\end{aligned}$$

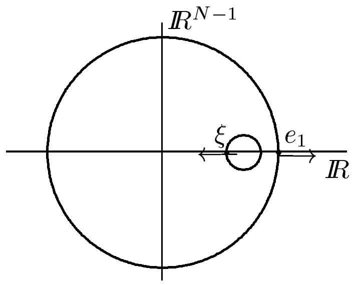
for all $0 \le s \le t$ implies

$$\begin{aligned}
\mathrm{dist}(x,\ \partial\,\vartheta_F(t,O_1)) \ &\le\ |x - \widehat{y}(0)| \\
&\le\ |x(0) - \widehat{y}(t)|\quad e^{\lambda t} + \sup_{I\!R^N} \mathrm{m}(F(\cdot),\ G(\cdot))\ \ \tfrac{e^{\lambda t}-1}{\lambda} \\
&\le\ e^{\supset}(\partial O_1,\ \partial O_2)\ \ e^{\lambda t} + \sup_{I\!R^N} \mathrm{m}(F(\cdot),\ G(\cdot))\ \ \tfrac{e^{\lambda t}-1}{\lambda}.
\end{aligned}$$

□

The estimate of the last lemma cannot be improved essentially with respect to F, G since the trajectory $x(\cdot)$ and its counterpart $\widehat{y}(t - \cdot)$ might move in opposite directions as the next example shows briefly.

Example 4.4.19 Roughly speaking, a small ball is contained in the unit ball close to the boundary : $I\!\!B_r((1-2\,r)\,e_1) \;\subset\; I\!\!B_1(0) \;\subset\; I\!\!R^N$ with $r \ll 1$, $e_1 := (1, 0 \ldots 0) \in I\!\!R^N$.
Set $F(\cdot) := I\!\!B_1$ and consider $\xi := x(0) = (1-3\,r)\,e_1$. Then e_1 is the unique projection of ξ on $\partial I\!\!B_1$ and the boundary trajectories $x(\cdot)$, $y(\cdot)$ of F starting in ξ and e_1 respectively are also unique : $x(t) = \xi - t$, $y(t) = e_1 + t$. Furthermore they keep moving in opposite directions and $|x(t) - y(t)| = |\xi - e_1| + 2\,t = |\xi - e_1| + 2\,\mathrm{m}(I\!\!B, I\!\!B)\,t$.



In particular, the example shows that $q_{\Omega,\partial}\,(\vartheta_F(t, O_1),\; \vartheta_F(t, O_2))$ depends on $\mathrm{m}(F(\cdot), F(\cdot))$ explicitly, i.e. it does not satisfy the inequality

$$q_{\Omega,\partial}\,(\vartheta_F(t, O_1),\; \vartheta_F(t, O_2)) \;\leq\; q_{\Omega,\partial}(O_1, O_2) \,\cdot\, e^{\mathrm{const}\,\cdot\, t}$$

in general. This dependence implies even $Q^{\mapsto}(\vartheta_F, \vartheta_F) > 0$ and then Lemma 2.1.4 excludes forward transitions of order 0. So we consider the countable family $q_\varepsilon := q_{\Omega,\partial}$ ($\varepsilon \in Q\!\!\!\!Q^+ := Q\!\!\!\!Q \cap \,]0, \infty[$) of identical ostensible metrics on $\Omega(I\!\!R^N)$ instead. The result consists in forward transitions of *positive* order. Here the order is not specified in detail because the scaling of ε does not provide any canonical criterion.

Proposition 4.4.20 *For every $\lambda \geq 0$, the reachable sets of maps in* $\mathrm{LIP}_\lambda^{(\mathring{\mathcal{H}})}(I\!\!R^N, I\!\!R^N)$ *induce forward transitions (of* positive *order) on* $(\Omega(I\!\!R^N),\; \Omega_{C^{1,1}}(I\!\!R^N),\; (q_{\Omega,\partial})_{\varepsilon \in Q\!\!\!\!Q^+})$ *with*

$$\begin{aligned}
\gamma_\varepsilon(\vartheta_F) &\stackrel{\text{Def.}}{=} 4\,\lambda, \\
\alpha_\varepsilon^{\mapsto}(\vartheta_F) &\stackrel{\text{Def.}}{=} 2\,\lambda, \\
\beta_\varepsilon(\vartheta_F)\,(t) &\stackrel{\text{Def.}}{=} 2\,\lambda \,\cdot\, t, \\
Q_\varepsilon^{\mapsto}(\vartheta_F, \vartheta_G) &\leq 2 \,\cdot\, \sup_{I\!\!R^N}\; \mathrm{m}(F(\cdot),\, G(\cdot)) \;\leq\; 4\,\lambda.
\end{aligned}$$

Proof. The semigroup property of reachable sets has the immediate consequence

$$\begin{aligned}
q_{\Omega,\partial}\Big(\vartheta_F(h,\; \vartheta_F(t, O)),\quad \vartheta_F(t+h,\; O)\Big) &= 0, \\
q_{\Omega,\partial}\Big(\vartheta_F(t+h,\; O),\quad \vartheta_F(h,\; \vartheta_F(t, O))\Big) &= 0
\end{aligned}$$

for all $F \in \mathrm{LIP}_\lambda^{(\mathring{\mathcal{H}})}(I\!\!R^N, I\!\!R^N)$, $O \in \Omega(I\!\!R^N)$, $h, t \geq 0$ since $q_{\Omega,\partial}$ is an ostensible metric.

Due to Cor. A.4.5, every map $F \in \mathrm{LIP}_\lambda^{(\mathring{\mathcal{H}})}(I\!R^N, I\!R^N)$ and initial set $O_1 \in \Omega_{C^{1,1}}(I\!R^N)$ lead to a time $\mathcal{T}_\Theta(\vartheta_F, O_1) > 0$ and a radius $\rho > 0$ such that $\vartheta_F(t, O_1) \in \Omega_{C^{1,1}}(I\!R^N)$ and its closure has radius of curvature $\geq \rho$ for any $t \in [0,\ \mathcal{T}_\Theta(\vartheta_F, O_1)]$.

Setting $\gamma_\varepsilon(\vartheta_F) := 4\,\lambda$, Lemma 4.4.18 implies for all $O_1 \in \Omega_{C^{1,1}}(I\!R^N)$, $O_2 \in \Omega(I\!R^N)$

$$\begin{array}{rl} & \limsup\limits_{h\downarrow 0} \left(\dfrac{q_{\Omega,\partial}\big(\vartheta_F(h,O_1),\vartheta_F(h,O_2)\big) - q_{\Omega,\partial}\big(O_1,O_2\big) - \gamma_\varepsilon(\vartheta_F)\ h}{h \quad \big(q_{\Omega,\partial}(O_1,O_2) + \gamma_\varepsilon(\vartheta_F)\ h\big)}\right)^+ \\ \leq & \limsup\limits_{h\downarrow 0} \dfrac{1}{h\ \big(q_{\Omega,\partial}(O_1,O_2) + \gamma_\varepsilon(\vartheta_F)\ h\big)} \Big(q_{\Omega,\partial}(O_1,\ O_2)\cdot\big(e^{\lambda h} - 1\big) \\ & \qquad\qquad + \ 2\cdot \sup\ m\!\!m(F(\cdot),F(\cdot)) \cdot \dfrac{e^{\lambda h}-1}{\lambda} - 4\,\lambda\, h\Big)^+ \\ \leq & \limsup\limits_{h\downarrow 0} \left(\dfrac{e^{\lambda h}-1}{h} + \dfrac{2\cdot 2\,\lambda\cdot e^{\lambda h}\ h - 4\,\lambda\, h}{h\ \big(0 + 4\,\lambda\, h\big)}\right)^+ = 2\,\lambda \stackrel{\text{Def.}}{=} \alpha^{\mapsto}(\vartheta_F) \end{array}$$

and for every $F, G \in \mathrm{LIP}_\lambda^{(\mathring{\mathcal{H}})}(I\!R^N, I\!R^N)$

$$\begin{array}{rl} Q_\varepsilon^{\mapsto}(\vartheta_F,\vartheta_G) = & \sup\limits_{\substack{O_1 \in \Omega_{C^{1,1}}(I\!R^N)\\ O_2 \in \Omega(I\!R^N)}} \limsup\limits_{h\downarrow 0} \left(\dfrac{q_{\Omega,\partial}\big(\vartheta_F(h,O_1),\vartheta_G(h,O_2)\big) - q_{\Omega,\partial}(O_1,O_2)\cdot e^{2\,\lambda\cdot h}}{h}\right)^+ \\ \leq & \sup\limits_{\substack{O_1 \in \Omega_{C^{1,1}}(I\!R^N)\\ O_2 \in \Omega(I\!R^N)}} \limsup\limits_{h\downarrow 0} \Big(q_{\Omega,\partial}(O_1,\ O_2)\ \frac{1}{h}\ \big(e^{\lambda h} - e^{2\,\lambda h}\big)^+ \\ & \qquad\qquad + \ 2\cdot\sup\ m\!\!m(F(\cdot),\ G(\cdot))\cdot\dfrac{e^{\lambda h}-1}{\lambda\, h}\Big) \\ = & \qquad 2\cdot\sup\ m\!\!m(F(\cdot),\ G(\cdot)). \end{array}$$

Moreover Lemma 4.4.16 states $q_{\Omega,\partial}\Big(\vartheta_F(s,O),\ \vartheta_F(t,O)\Big) \leq 2\,\lambda\cdot(t-s)$ for any $0 \leq s \leq t \leq 1$ and $O \in \Omega(I\!R^N)$.

The last part of the claim is to show

$$\limsup_{h\downarrow 0}\ q_{\Omega,\partial}\Big(\vartheta_F(t-h,\ O_1),\ O_2\Big) \geq q_{\Omega,\partial}\Big(\vartheta_F(t,O_1),\ O_2\Big)$$

for all $F \in \mathrm{LIP}_\lambda^{(\mathring{\mathcal{H}})}(I\!R^N, I\!R^N)$, $O_1 \in \Omega_{C^{1,1}}(I\!R^N)$, $O_2 \in \Omega(I\!R^N)$ and $0 < t < \mathcal{T}_\Theta(\vartheta_F, O_1)$. For every initial set $O_1 \in \Omega_{C^{1,1}}(I\!R^N)$, Cor. A.5.3 ensures the reversibility in time in the interval $[0, \mathcal{T}_\Theta(\vartheta_F, O_1)[$, i.e. for all $0 < h < t < \mathcal{T}_\Theta(\vartheta_F, O_1)$

$$I\!R^N \Big\backslash\ \vartheta_F(t-h,\ O_1) = \vartheta_{-F}\Big(h,\ I\!R^N \setminus \vartheta_F(t,O_1)\Big) \in \Omega_{C^{1,1}}(I\!R^N).$$

Thus, $\quad \partial\,\vartheta_F(t-h,\ O_1) \subset \vartheta_{-F}\Big(h,\ \partial\,\vartheta_F(t,O_1)\Big)$

and we get $\quad e^{\supset}\Big(\partial\,\vartheta_F(t,O_1),\ \partial\,\vartheta_F(t-h,\ O_1)\Big) \longrightarrow 0 \qquad$ for $h \downarrow 0$.

So the last claim results from the triangle inequality for $q_{\Omega,\partial}$.

□

On the one hand we now have forward transitions (of positive order) on the tuple $(\Omega(I\!R^N),\ \Omega_{C^{1,1}}(I\!R^N),\ (q_{\Omega,\partial})_{\varepsilon\in Q^+})$, but on the other hand, only subsets of $\Omega(I\!R^N)$ like $(\Omega^\rho_\circ(I\!R^N),\ q_{\Omega,\partial})$ (for each $\rho > 0$) are two–sided sequentially compact.
For combining these results, we use Cor. A.6.5 stating that open sets of positive erosion are preserving this property while evolving along differential inclusions (with maps in $\overset{\circ}{\text{LIP}}{}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N))$:

Corollary 4.4.21
Consider the forward transitions of positive order on $(\Omega_\circ(I\!R^N),\ \Omega_{C^{1,1}}(I\!R^N),\ (q_{\Omega,\partial})_{\varepsilon\in Q^+})$ *induced by the reachable sets of maps in* $\overset{\circ}{\text{LIP}}{}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$ *(according to Prop. 4.4.20). Moreover let* $f : \Omega_\circ(I\!R^N) \times [0,T] \longrightarrow \overset{\circ}{\text{LIP}}{}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$ *fulfill*

$$\sup_{I\!R^N} \ \mathrm{m}(F(\cdot),\ G(\cdot)) \ \le\ 4\,\lambda \ +\ \omega\,(q_{\Omega,\partial}(O_1, O_2)\ +\ t_2 - t_1)$$

for all $O_1, O_2 \in \Omega_\circ(I\!R^N)$ *and* $0 \le t_1 \le t_2 \le T$ *with a modulus* $\omega(\cdot)$ *of continuity.*

Then for every initial set $O_0 \in \Omega_\circ(I\!R^N)$, *there exists a right–hand forward solution* $O : [0,T[\longrightarrow \Omega_\circ(I\!R^N)$ *of the generalized mutational equation* $\overset{\circ}{O}(\cdot) \ni f(O(\cdot),\cdot)$ *in* $[0,T[$ *with* $O(0) = O_0$.

Proof. Let $O_0 \in \Omega(I\!R^N)$ have positive erosion of radius $\rho_0 > 0$. Then Cor. A.6.5 ensures that the values of all Euler approximations $[0,T[\longrightarrow \Omega(I\!R^N)$ have uniform positive erosion (with a lower bound ρ of the radius depending only on $\lambda,\ \rho_0,\ T$).
So we conclude the existence of a solution $O(\cdot) : [0,T[\longrightarrow \Omega^\rho_\circ(I\!R^N)$ from Corollary 2.3.6 since $(\Omega^\rho_\circ(I\!R^N), q_{\Omega,\partial})$ is two–sided sequentially compact (due to Cor. 4.2.5). □

4.4.4 Differential inclusions for $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$

The reachable sets of maps in $\overset{\circ}{\text{LIP}}{}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$ $(\lambda > 0)$ induce forward transitions on $(\Omega(I\!R^N),\ \Omega_{C^{1,1}}(I\!R^N),\ q_{\Omega,\partial})$ as just shown in § 4.4.3 and, we can benefit from the two–sided sequential compactness of $(\Omega^\rho_\circ(I\!R^N), q_{\Omega,\partial})$ for any $\rho > 0$ (due to Cor. 4.2.5). This topological advantage is asking a high price : In general, the transitions are not of order 0 any longer. Roughly speaking, this consequence is due to dispensing with the detailed information on normals in Lemma 4.4.18, i.e. we cannot know in which directions related boundary trajectories move (and the "worst case" of opposite directions leads to the dependence on $\mathrm{m}(F(\cdot),\ G(\cdot))$).
So we want to take more information on normal cones at the boundary into consideration.

Adjoint arcs of trajectories provide a powerful tool for describing the evolution of the limiting normal cones – in the sense of Prop. A.3.2. Since they refer to *exterior* normals, we consider $\mathcal{K}(I\!R^N)$ supplied with the quasi–metric $q_{\mathcal{K},N}$.
Moreover standard hypothesis $(\mathcal{H})$ implies that the corresponding Hamiltonian system (of each trajectory and its adjoint) is an ordinary differential equation with Lipschitz continuous right–hand side and thus, its solutions are unique. So we prove in Cor. A.5.2 that smooth compact initial sets stay in $\mathcal{K}_{C^{1,1}}(I\!R^N)$ for a short time (at least) and that their evolution is reversible in time within this period.

This leads to another class $\mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$ of set–valued maps $F: I\!R^N \leadsto I\!R^N$ as candidates for forward transitions on $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$. In comparison with $\mathrm{LIP}_\lambda^{(\mathring{\mathcal{H}})}(I\!R^N, I\!R^N)$, we dispense with the condition that every value has nonempty interior. So the remark after Def. 4.4.15 is also correct for $\mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N)$, i.e. every set–valued map $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N)$ is λ–Lipschitz continuous with $\sup_{I\!R^N} \|F(\cdot)\| \leq \lambda$ and fulfills standard hypothesis $(\mathcal{H})$.

Definition 4.4.22 $\mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$ *contains all maps* $F: I\!R^N \leadsto I\!R^N$ *satisfying*

1. $F: I\!R^N \leadsto I\!R^N$ *has compact convex values,*
2. $\mathcal{H}_F(\cdot,\cdot) \in C^{1,1}(I\!B_R \times (I\!B_R \setminus \mathring{I\!B}_{\frac{1}{R}}))$ *for every radius* $R > 1$,
3. $\|\mathcal{H}_F\|_{C^{1,1}(I\!R^N \times\, \partial I\!B_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_F\|_{C^1(I\!R^N \times\, \partial I\!B_1)} + \mathrm{Lip}\ D\mathcal{H}_F|_{I\!R^N \times\, \partial I\!B_1} < \lambda\,.$

Lemma 4.4.23 *For every* $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$ *and* $K \in \mathcal{K}(I\!R^N)$, $0 \leq s \leq t \leq T$,

$$q_{\mathcal{K},N}\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big) \ \leq\ \lambda\,(e^{\lambda T} + 2)\ \cdot\ (t-s).$$

Proof. Obviously, the Pompeiu–Hausdorff distance satisfies for every $s, t \geq 0$

$$d\!l\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big) \ \leq\ \sup_{I\!R^N} \|F(\cdot)\|_\infty\ \cdot\ (t-s) \ \leq\ \lambda\,(t-s).$$

Furthermore Prop. A.3.2 guarantees that for every $0 \leq s < t$, $x \in \partial\,\vartheta_F(t,K)$ and $p \in {}^\flat N_{\vartheta_F(t,K)}(x)$, there exist a trajectory $x(\cdot) \in AC([s,t], I\!R^N)$ and its adjoint arc $p(\cdot) \in AC([s,t], I\!R^N)$ satisfying

$$\begin{array}{rcll}
\dot p(\tau) & \in & \overline{co}\ \Big\{-q\ \Big|\ (q, \dot x(\tau)) \in \partial^L\, \mathcal{H}_F(x(\tau), p(\tau))\Big\} & \text{for almost every } \tau,\\
p(\tau)\cdot \dot x(\tau) & = & \max\ p(\tau)\cdot F(x(\tau)) & \text{for almost every } \tau,\\
|\dot p(\tau)| & \leq & \lambda\ |p(\tau)| & \text{for almost every } \tau,\\
p(s) & \in & N_{\vartheta_F(s,K)}(x(s)), \qquad p(t) = p, & \\
x(s) & \in & \partial\,\vartheta_F(s,K), \qquad x(t) = x. &
\end{array}$$

Due to standard hypothesis $(\mathcal{H})$, each $\partial^L \mathcal{H}_F(x(\tau), p(\tau))$ is single–valued and the Hamiltonian system $(-\dot{p}(\tau),\ \dot{x}(\tau)) = D\,\mathcal{H}_F(x(\tau), p(\tau))$ (for every $\tau \in [s,t]$) has Lipschitz continuous right–hand side. Correspondingly to earlier conclusions, $|p| \le 1$ implies that the projection of p on any cone is also contained in $I\!B_1$ and thus,

$$\begin{aligned}
\operatorname{dist}\Big((x,p),\ \operatorname{Graph}{}^\flat N_{\vartheta_F(s,K)}\Big) &\le |x - x(s)| + \operatorname{dist}\Big(p,\ {}^\flat N_{\vartheta_F(s,K)}(x(s))\Big)\\
&= |x - x(s)| + \operatorname{dist}\Big(p,\ N_{\vartheta_F(s,K)}(x(s))\Big)\\
&\le |x - x(s)| + |p - p(s)|\\
&\le \sup_{s \le \tau \le t} \Big(|\tfrac{\partial}{\partial x}\mathcal{H}_F| + |\tfrac{\partial}{\partial p}\mathcal{H}_F|\Big)\Big|_{(x(\tau),p(\tau))} \cdot (t-s)\\
&\le \sup_{s \le \tau \le t} \Big(\lambda\,|p(\tau)| + \|F(x(\tau))\|_\infty\Big) \cdot (t-s)\\
&\le \Big(\lambda\, e^{\lambda t} + \lambda\Big) \cdot (t-s). \quad \square
\end{aligned}$$

Lemma 4.4.24 *For every* $F \in \mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$ *and radius* $R > 1$, *the product* $9\,R^2\,\lambda$ *is a Lipschitz constant of the derivative* $D\mathcal{H}_F$ *restricted to* $I\!R^N \times (I\!B_R \setminus \overset{\circ}{I\!B}_{\frac{1}{R}})$.

Proof results from the fact that $\mathcal{H}_F(x,p)$ is positively homogenous with respect to p: For every $(x,p) \in I\!R^N \times (I\!B_R \setminus \overset{\circ}{I\!B}_{\frac{1}{R}})$, we conclude from $\mathcal{H}_F(x,p) = |p|\ \mathcal{H}_F(x, \frac{p}{|p|})$

$$\begin{aligned}
\frac{\partial\,\mathcal{H}_F(x,p)}{\partial p_j} &= \frac{\partial}{\partial p_j}|p| \cdot \mathcal{H}_F(x, \tfrac{p}{|p|}) + |p| \cdot \sum_{k=1}^N \frac{\partial}{\partial p_k}\mathcal{H}_F\big|_{(x,\frac{p}{|p|})} \cdot \frac{\partial}{\partial p_j}\frac{p_k}{|p|}\\
&= \frac{p_j}{|p|} \cdot \mathcal{H}_F(x, \tfrac{p}{|p|}) + |p| \cdot \sum_{k=1}^N \frac{\partial}{\partial p_k}\mathcal{H}_F\big|_{(x,\frac{p}{|p|})} \cdot \Big(-\frac{p_j\,p_k}{|p|^3} + \frac{\delta_{jk}}{|p|}\Big)\\
&= \frac{p_j}{|p|} \cdot \Big(\mathcal{H}_F(x, \tfrac{p}{|p|}) - \frac{p}{|p|} \cdot \frac{\partial}{\partial p}\mathcal{H}_F\big|_{(x,\frac{p}{|p|})}\Big) + \frac{\partial}{\partial p_j}\mathcal{H}_F\big|_{(x,\frac{p}{|p|})}.
\end{aligned}$$

So the Lipschitz constant of $p \longmapsto \frac{\partial}{\partial p_j}\mathcal{H}_F(x,p)$ has the upper bound

$$\begin{aligned}
&\mathrm{Lip}\ (p \mapsto \tfrac{p_j}{|p|}) \cdot \Big(\|\mathcal{H}_F\|_{C^0(I\!R^N \times \partial I\!B_1)} + 1 \cdot \|\tfrac{\partial}{\partial p}\mathcal{H}_F\|_{C^0(I\!R^N \times \partial I\!B_1)}\Big)\\
&+ 1 \cdot \mathrm{Lip}\ (p \mapsto \tfrac{p}{|p|}) \Big(\mathrm{Lip}\ \mathcal{H}_F|_{I\!R^N \times \partial I\!B_1} + \mathrm{Lip}\ (p \mapsto \tfrac{p}{|p|}) \cdot \|\tfrac{\partial}{\partial p}\mathcal{H}_F\|_{C^0(I\!R^N \times \partial I\!B_1)}\\
&\qquad + 1 \cdot \mathrm{Lip}\ \tfrac{\partial}{\partial p}\mathcal{H}_F|_{I\!R^N \times \partial I\!B_1}\Big)\\
&+ \mathrm{Lip}\ (p \mapsto \tfrac{p}{|p|})\ \ \mathrm{Lip}\ \tfrac{\partial}{\partial p}\mathcal{H}_F|_{I\!R^N \times \partial I\!B_1}\\
\le\ & R\ \|\mathcal{H}_F\|_{C^1(I\!R^N \times \partial I\!B_1)} + R\,(1+R)\ \|D\,\mathcal{H}_F\|_{C^0(I\!R^N \times \partial I\!B_1)} + 2\,R\ \mathrm{Lip}\ \tfrac{\partial}{\partial p}\mathcal{H}_F|_{I\!R^N \times \partial I\!B_1}\\
\overset{R>1}{\le}\ & 3\,R^2\ \|\mathcal{H}_F\|_{C^{1,1}(I\!R^N \times \partial I\!B_1)}\,.
\end{aligned}$$

and $x \longmapsto \frac{\partial}{\partial p_j}\mathcal{H}_F(x,p)$ has the Lipschitz constant $\le 3\,\|D\mathcal{H}_F\|_{C^{0,1}(I\!R^N \times \partial I\!B_1)} \le 3\,\lambda$. Furthermore, $\frac{\partial}{\partial x_j}\mathcal{H}_F(x,p) = |p| \cdot \frac{\partial}{\partial x_j}\mathcal{H}_F|_{(x,\,\frac{p}{|p|})}$ has the consequence

$$\mathrm{Lip}\ \Big(x \mapsto \frac{\partial \mathcal{H}_F(x,p)}{\partial x_j}\Big) \le R \cdot \lambda, \quad \mathrm{Lip}\ \Big(p \mapsto \frac{\partial \mathcal{H}_F(x,p)}{\partial x_j}\Big) \le R\,\lambda + R \cdot \lambda\,R \overset{R>1}{\le} 2\,R^2\,\lambda. \ \square$$

Lemma 4.4.25 *Assume for* $F, G \in \mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!R^N, I\!R^N)$, $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $T > 0$ *that all the sets* $\vartheta_F(t, K_1) \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ $(0 \le t \le T)$ *have uniform positive reach. Then, for every* $t \in [0, T[$,

$$q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\ \vartheta_G(t,K_2)\Big) \ \le$$
$$\le\ e^{(\Lambda_F+\lambda)\,t}\ \cdot\ \Big(q_{\mathcal{K},N}(K_1,\ K_2)\ +\ 4\,N\,t\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N\times\partial I\!B_1)}\Big)$$

with $\Lambda_F \ := \ 9\,e^{2\lambda t}\ \|\mathcal{H}_F\|_{C^{1,1}(I\!R^N\times\partial I\!B_1)} \ \le\ 9\,e^{2\lambda t}\,\lambda\ <\ \infty.$

Proof. Prop. 4.4.1 and Lemma A.2.4 provide the first estimate

$$\begin{aligned} d\!\!l\Big(\vartheta_F(t,K_1),\ \vartheta_G(t,K_2)\Big) \ &\le\ d\!\!l(K_1,K_2)\cdot e^{\lambda t} \ +\ \sup_{I\!R^N} d\!\!l\Big(F(\cdot),G(\cdot)\Big)\ \cdot\ \tfrac{e^{\lambda t}-1}{\lambda}\\ &\le\ d\!\!l(K_1,K_2)\cdot e^{\lambda t} \ +\ \sup_{I\!R^N\times\partial I\!B_1} |\mathcal{H}_F - \mathcal{H}_G|\ \cdot\ \tfrac{e^{\lambda t}-1}{\lambda}. \end{aligned}$$

So we need an upper bound of $e^{\supset}\Big(\mathrm{Graph}\ {}^\flat N_{\vartheta_F(t,K_1)},\ \mathrm{Graph}\ {}^\flat N_{\vartheta_G(t,K_2)}\Big)$ and follow the same track of adjoint arcs as in the proof of Lemma 4.4.9.

Choose $x \in \partial\,\vartheta_G(t,K_2)$, $p \in N_{\vartheta_G(t,K_2)}(x) \cap \partial I\!B_1$ and $\delta > 0$ arbitrarily. According to Prop. A.3.2 and standard hypothesis $(\mathcal{H})$, there exist a trajectory $x(\cdot) \in C^1([0,t], I\!R^N)$ of G and its adjoint arc $p(\cdot) \in C^1([0,t], I\!R^N)$ with

$$\begin{aligned} \dot x(\cdot) &= \tfrac{\partial}{\partial p}\,\mathcal{H}_G(x(\cdot),\,p(\cdot)) \in G(x(\cdot)), & \dot p(\cdot) &= -\tfrac{\partial}{\partial x}\,\mathcal{H}_G(x(\cdot),\,p(\cdot))\\ x(0) &\in \partial K_2, & p(0) &\in N_{K_2}(x(0)),\\ x(t) &= x, & p(t) &= p,\\ & & |\dot p(\cdot)| &\le \lambda\ |p(\cdot)|. \end{aligned}$$

Gronwall's Lemma 1.5.1 guarantees $0 < e^{-\lambda t} \le |p(\cdot)| \le e^{\lambda t}$ and thus,

$$p(0)\,e^{-\lambda t} \in {}^\flat N_{K_2}(x(0)) \setminus \{0\}.$$

Now let $(y_0, \widehat q_0)$ denote an element of $\mathrm{Graph}\ {}^\flat N_{K_1}$ with $\widehat q_0 \ne 0$ and

$$\Big|(y_0,\widehat q_0)\ -\ \Big(x(0),\ p(0)\,e^{-\lambda t}\Big)\Big|\ \le\ e^{\supset}\Big(\mathrm{Graph}\ {}^\flat N_{K_1},\ \mathrm{Graph}\ {}^\flat N_{K_2}\Big)\ +\ \delta.$$

Assuming that all the sets $\vartheta_F(s, K_1) \in \mathcal{K}(I\!R^N)$ $(s \in [0,t])$ have uniform positive reach implies the reversibility in time according to Prop. A.5.1 :

$$I\!R^N \setminus K_1 \ =\ \vartheta_{-F}(t,\ I\!R^N \setminus \vartheta_F(t,K_1)).$$

So in particular, y_0 is a boundary point of $I\!R^N \setminus \overset{\circ}{K}_1 = \vartheta_{-F}(t,\ \overline{I\!R^N \setminus \vartheta_F(t,K_1)})$ and $-\widehat q_0$ belongs to its limiting normal cone at y_0. As a consequence of Prop. A.3.2 again, there exist a trajectory $\widehat y(\cdot) \in AC([0,t], I\!R^N)$ of $-F$ and its adjoint arc $\widehat q(\cdot)$ satisfying

$$\begin{aligned} \tfrac{d}{dt}\,\widehat y(\cdot) &\in -F(\widehat y(\cdot)), & \tfrac{d}{dt}\,\widehat q(\tau) &\in \overline{co}\ \Big\{-q\ \Big|\ (q,\ \tfrac{d}{dt}\,\widehat y(\tau)) \in \partial^L\,\mathcal{H}_{-F}(\widehat y(\tau), \widehat q(\tau))\Big\}\\ \widehat y(0) &\in \partial\,\vartheta_F(t,K), & \widehat q(0) &\in N_{\overline{I\!R^N\setminus\vartheta_F(t,K_1)}}(\widehat y(0)),\\ \widehat y(t) &= y_0, & \widehat q(t) &= -\widehat q_0\ e^{\lambda t}\ \ne\ 0, \end{aligned}$$

Due to $\vartheta_F(t, K_1) \in \mathcal{K}_{C^{1,1}}(I\!R^N)$, $-\widehat{q}(0)$ is contained in the normal cone of $\vartheta_F(t, K_1)$ at $\widehat{y}(0)$ consisting of exactly one direction. Moreover, $\mathcal{H}_{-F}(z, v) = \mathcal{H}_F(z, -v)$ for all z, v and standard hypothesis $(\mathcal{H})$ ensure $\mathcal{H}_{-F} \in C^{1,1}(I\!R^N \times (I\!R^N \setminus \{0\}))$.
So $q(s) := -\widehat{q}(t-s) \neq 0$ is the adjoint arc of the trajectory $y(s) := \widehat{y}(t-s)$ of F and,

$$\begin{array}{rclrcl} \dot{y}(\cdot) &=& \frac{\partial}{\partial p}\,\mathcal{H}_F(y(\cdot),\, q(\cdot)), & \dot{q}(\cdot) &=& -\frac{\partial}{\partial y}\,\mathcal{H}_F(y(\cdot),\, q(\cdot)) \\ y(0) &=& y_0, & q(0) &=& \widehat{q}_0\; e^{\lambda\, t}, \\ y(t) &\in& \partial\,\vartheta_F(t, K_1), & q(t) &\in& N_{\vartheta_F(t,K_1)}(y(t)). \end{array}$$

According to Lemma 4.4.24, the derivative of $\mathcal{H}_F$ is Λ_F–Lipschitz continuous on $I\!R^N \times (I\!B_{e^{\lambda T}} \setminus \mathring{I\!B}_{e^{-\lambda T}})$. Thus. the Theorem of Cauchy–Lipschitz leads to

$$\begin{array}{rl} & \mathrm{dist}\Big((x,p),\;\; \mathrm{Graph}\;\; {}^\flat\! N_{\vartheta_F(t,K_1)}\Big) \\ = & \mathrm{dist}\Big((x,p),\;\; \mathrm{Graph}\;\; N_{\vartheta_F(t,K_1)}\Big) \\ \le & \Big|(x,\, p) \;-\; (y(t),\, q(t))\Big| \\ \le & e^{\Lambda_F \cdot t} \cdot \Big|(x(0),\, p(0)) \;-\; (y_0,\, \widehat{q}_0\; e^{\lambda\, t})\Big| \;+\; \frac{e^{\Lambda_F \cdot t} - 1}{\Lambda_F} \cdot \sup\limits_{0 \le s \le t} \;|D\,\mathcal{H}_F - D\,\mathcal{H}_G|\Big|_{(x(s),\, p(s))}. \end{array}$$

$\mathcal{H}_F$ and $\mathcal{H}_G$ are positively homogenous with respect to the second argument and thus,

$$\begin{array}{rcl} \Big|\frac{\partial}{\partial x_j}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\, p(s))}\Big| &\le& |p(s)|\;\; \Big|\frac{\partial}{\partial x_j}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\, \frac{p(s)}{|p(s)|})}\Big| \\ &\le& e^{\lambda\, t}\;\; \|D\mathcal{H}_F - D\mathcal{H}_G\|_{C^0(I\!R^N \times \partial I\!B_1)}, \\ \Big|\frac{\partial}{\partial p_j}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\, p(s))}\Big| &\le& \Big|(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\, \frac{p(s)}{|p(s)|})}\Big| \;+\; 2\,\Big|\frac{\partial}{\partial p}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\, \frac{p(s)}{|p(s)|})}\Big| \\ &\le& 2 \cdot\;\; \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N \times \partial I\!B_1)}. \end{array}$$

So we obtain

$$\begin{array}{rl} & \mathrm{dist}\Big((x,p),\;\; \mathrm{Graph}\;\; {}^\flat\! N_{\vartheta_F(t,K_1)}\Big) \\ \le & e^{(\Lambda_F + \lambda)\; t}\;\; \Big|(x(0),\, p(0)\, e^{-\lambda\, t}) \;-\; (y_0, \widehat{q}_0)\Big| \;+\; e^{\Lambda_F\, t}\; t\; \cdot 4\; N\; e^{\lambda t}\;\; \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N \times \partial I\!B_1)} \end{array}$$

and, since $\delta > 0$ is arbitrarily small and $|p| = 1$,

$$\begin{array}{rl} & e^{\supset}\Big(\mathrm{Graph}\;\; {}^\flat\! N_{\vartheta_F(t,K_1)},\;\; \mathrm{Graph}\;\; {}^\flat\! N_{\vartheta_G(t,K_2)}\Big) \\ \le & e^{(\Lambda_F + \lambda)\; t}\; \cdot\; \Big\{ e^{\supset}\Big(\mathrm{Graph}\; {}^\flat\! N_{K_1},\;\; \mathrm{Graph}\; {}^\flat\! N_{K_2}\Big) \;+\; 4\, N\, t\; \cdot\; \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N \times \partial I\!B_1)} \Big\}. \end{array}$$

□

Remark. The extended Hamilton condition (in Prop. A.3.1) is our (only) tool for identifying boundary trajectories in forward time direction and, the adjoint arc is an essential part of that proposition. For this reason we cannot adapt the preceding notions to the ostensible metric q_{Ω, N_c} on $\Omega(I\!R^N)$ directly since adjoint arcs describe the *exterior* normals.

Proposition 4.4.26 *For every $\lambda \geq 0$, the reachable sets of the set–valued maps in* $\mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$ *induce forward transitions (of order 0) on* $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$ *with*

$$\begin{aligned} \alpha^{\mapsto}(\vartheta_F) &\stackrel{\text{Def.}}{=} 10\ \lambda \\ \beta(\vartheta_F)\,(t) &\stackrel{\text{Def.}}{=} \lambda\ (e^\lambda + 2)\ \cdot\ t, \\ Q^{\mapsto}(\vartheta_F, \vartheta_G) &\leq 4\ N\ \ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N \times \partial I\!B_1)}\,. \end{aligned}$$

Proof. The semigroup property of reachable sets implies again

$$\begin{aligned} q_{\mathcal{K},N}\Big(\vartheta_F(h,\ \vartheta_F(t,K)),\quad \vartheta_F(t+h,\ K)\Big) &= 0, \\ q_{\mathcal{K},N}\Big(\vartheta_F(t+h,\ K),\quad \vartheta_F(h,\ \vartheta_F(t,K))\Big) &= 0 \end{aligned}$$

for all $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$, $K \in \mathcal{K}(I\!R^N)$, $h, t \geq 0$ since $q_{\mathcal{K},N}$ is a quasi–metric. According to Prop. A.4.4, every set–valued map $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$ and initial set $K_1 \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ lead to a time $\mathcal{T}_\Theta(\vartheta_F, K_1) > 0$ and a radius $\rho > 0$ such that $\vartheta_F(t, K_1) \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ has radius of curvature $\geq \rho$ for any $t \in [0,\ \mathcal{T}_\Theta(\vartheta_F, K_1)]$. So Lemma 4.4.25 guarantees for all $K_1 \in \mathcal{K}_{C^{1,1}}(I\!R^N)$, $K_2 \in \mathcal{K}(I\!R^N)$

$$\begin{aligned} &\limsup_{h \downarrow 0}\ \left(\frac{q_{\mathcal{K},N}\big(\vartheta_F(h,K_1),\,\vartheta_F(h,K_2)\big)\ -\ q_{\mathcal{K},N}\big(K_1,K_2\big)}{h \qquad q_{\mathcal{K},N}\big(K_1,K_2\big)}\right)^+ \\ \leq\ &\limsup_{h \downarrow 0}\ \tfrac{1}{h}\ \left(e^{\left(9\,e^{2\,\lambda\,h}\ \lambda + \lambda\right)\cdot h}\ -\ 1\right) \ =\ 10\ \lambda\ \stackrel{\text{Def.}}{=}\ \alpha^{\mapsto}(\vartheta_F) \end{aligned}$$

and for every $F, G \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$

$$\begin{aligned} Q^{\mapsto}(\vartheta_F, \vartheta_G) &= \sup_{\substack{K_1 \in \mathcal{K}_{C^{1,1}}(I\!R^N) \\ K_2 \in \mathcal{K}(I\!R^N)}}\ \limsup_{h \downarrow 0}\ \left(\frac{q_{\mathcal{K},N}\big(\vartheta_F(h,K_1),\vartheta_G(h,K_2)\big)\ -\ q_{\mathcal{K},N}(K_1,K_2)\cdot e^{10\,\lambda\cdot h}}{h}\right)^+ \\ &\leq \sup_{\substack{K_1 \in \mathcal{K}_{C^{1,1}}(I\!R^N) \\ K_2 \in \mathcal{K}(I\!R^N)}}\ \limsup_{h \downarrow 0}\ \Big(q_{\mathcal{K},N}(K_1,\ K_2)\ \tfrac{1}{h}\ \left(e^{\left(9\,e^{2\,\lambda\,h}\ \lambda + \lambda\right)\cdot h}\ -\ e^{10\,\lambda\,h}\right) \\ &\qquad\qquad +\ 4\ N\cdot\|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N\times\partial I\!B_1)}\ \cdot\ e^{\left(9\,e^{2\,\lambda\,h}\ \lambda + \lambda\right)\cdot h}\Big) \\ &= 4\ N\cdot\|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(I\!R^N\times\partial I\!B_1)}. \end{aligned}$$

Moreover Lemma 4.4.23 states $\quad q_{\mathcal{K},N}\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big)\ \leq\ \lambda\,(e^\lambda + 2)\ \cdot\ (t-s)$ for any $0 \leq s \leq t \leq 1$ and $K \in \mathcal{K}(I\!R^N)$.

Finally we have to show for all $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(I\!R^N, I\!R^N)$, $K_1 \in \mathcal{K}_{C^{1,1}}(I\!R^N)$, $K_2 \in \mathcal{K}(I\!R^N)$ and $0 < t < \mathcal{T}_\Theta(\vartheta_F, K_1)$

$$\limsup_{h \downarrow 0}\ q_{\mathcal{K},N}\Big(\vartheta_F(t-h,\ K_1),\ K_2\Big)\ \geq\ q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\ K_2\Big).$$

Choosing $K_1 \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ arbitrarily, Prop. A.5.1 ensures the reversibility in time in the interval $[0, \mathcal{T}_\Theta(\vartheta_F, K_1)[$, i.e. for every $0 < h < t < \mathcal{T}_\Theta(\vartheta_F, K_1)$

$$I\!R^N \Big\backslash\ \vartheta_F(t-h, K_1)\ =\ \vartheta_{-F}\Big(h,\ \ I\!R^N\ \backslash\ \vartheta_F(t,K_1)\Big)\ \in\ \Omega_{C^{1,1}}(I\!R^N).$$

Due to standard hypothesis $(\mathcal{H})$, the flow of the Hamiltonian system even induces a Lipschitz homeomorphism between Graph $N_{\vartheta_F(t-h,K_1)}$ and Graph $N_{\vartheta_F(t,K_1)}$ since each limiting normal cone contains exactly one direction and $N_{\vartheta_F(t,K_1)}(\cdot) = -N_{\overline{I\!\!R^N \setminus \vartheta_F(t,K_1)}}(\cdot)$. Thus, Graph $N_{\vartheta_F(t,K_1)} = \mathrm{Lim}_{h \downarrow 0}$ Graph $N_{\vartheta_F(t-h,K_1)}$ and finally

$$q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\ \vartheta_F(t-h,\ K_1)\Big) \longrightarrow 0 \qquad \text{for } h \downarrow 0.$$

So the last claim results from the triangle inequality. □

Finally we use standard hypothesis $(\mathcal{H}^\rho_\circ)$ for guaranteeing the transitional compactness :

Definition 4.4.27 *For any $\lambda > 0$ and $\rho > 0$, the set $\mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!\!R^N, I\!\!R^N)$ consists of all set–valued maps $F : I\!\!R^N \rightsquigarrow I\!\!R^N$*

1. *$F : I\!\!R^N \rightsquigarrow I\!\!R^N$ has compact convex values in $\mathcal{K}^\rho_\circ(I\!\!R^N)$.*
2. *$\mathcal{H}_F(\cdot,\cdot) \in C^2(I\!\!B_R \times (I\!\!B_R \setminus \overset{\circ}{I\!\!B}_{\frac{1}{R}}))$ for every radius $R > 1$,*
3. *$\|\mathcal{H}_F\|_{C^{1,1}(I\!\!R^N \times\, \partial I\!\!B_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_F\|_{C^1(I\!\!R^N \times\, \partial I\!\!B_1)} + \mathrm{Lip}\ D\mathcal{H}_F|_{I\!\!R^N \times\, \partial I\!\!B_1} < \lambda\,.$*

Remark. $\mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!\!R^N, I\!\!R^N)$ is a subset of $\mathrm{LIP}^{(\mathcal{H})}_\lambda(I\!\!R^N, I\!\!R^N)$ and its maps fulfill standard hypothesis $(\mathcal{H}^\rho_\circ)$ (see Definition A.7.1). In particular, they make points evolve into sets of positive erosion according to Proposition A.7.2.

Proposition 4.4.28

For any $\lambda, \rho > 0$, consider the maps $F \in \mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!\!R^N, I\!\!R^N)$ (i.e. their reachable sets, strictly speaking) as forward transitions of order 0 on $(\mathcal{K}(I\!\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!\!R^N),\ q_{\mathcal{K},N})$.

Then $\mathcal{K}_\circ(I\!\!R^N)$ is transitionally compact in $\Big(\mathcal{K}(I\!\!R^N),\ q_{\mathcal{K},N},\ \mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!\!R^N, I\!\!R^N)\Big)$ in the following sense (see Definition 2.3.4) :

Let $(K_n)_{n \in I\!\!N}$, $(h_j)_{j \in I\!\!N}$ be sequences in $\mathcal{K}_\circ(I\!\!R^N)$ and $]0,1[$, respectively with $h_j \downarrow 0$, $\sup_n\ q_{\mathcal{K},N}(I\!\!B_1, K_n) < \infty$. Suppose each $G_n : [0,1] \longrightarrow \mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(I\!\!R^N, I\!\!R^N)$ to be piecewise constant $(n \in I\!\!N)$ and set

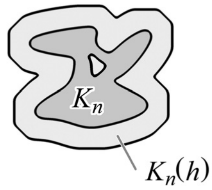
$$\widetilde{G}_n \ :\ [0,1] \times I\!\!R^N \rightsquigarrow I\!\!R^N,\quad (t,x) \longmapsto G_n(t)(x),$$
$$K_n(h) \ :=\ \vartheta_{\widetilde{G}_n}(h, K_n) \qquad \textit{for } h \geq 0.$$

Then there exist a sequence $n_k \nearrow \infty$ of indices and $K \in \mathcal{K}(I\!\!R^N)$ satisfying

$$\limsup_{k \longrightarrow \infty}\ q_{\mathcal{K},N}(K_{n_k}(0),\ \ K) \;=\; 0,$$
$$\limsup_{j \longrightarrow \infty}\ \sup_{k \geq j}\ q_{\mathcal{K},N}(K,\ \ K_{n_k}(h_j)) \;=\; 0.$$

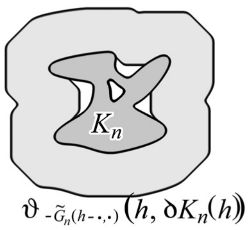
Proof. $(\mathcal{K}(I\!R^N),\ q_{\mathcal{K},N})$ is one–sided sequentially compact according to Lemma 4.1.8. So considering a subsequence (again denoted by) $(K_n)_{n \in I\!N}$ leads to the existence of a set $K \in \mathcal{K}(I\!R^N)$ with $q_{\mathcal{K},N}(K_n, K) \longrightarrow 0 \quad (n \longrightarrow \infty)$.
In particular, this convergence implies $d\!l(K_n, K) \longrightarrow 0$ and thus,

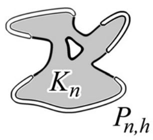
$$d\!l(K, K_n(h)) \ \le\ d\!l(K, K_n) + \lambda\, h \ \longrightarrow\ \lambda\, h \qquad \text{for } n \longrightarrow \infty.$$



Now we want to prove that K satisfies the claim by choosing subsequences of (K_n) for countably many times (and applying the Cantor diagonal construction).
An important tool here is Prop. A.7.2. It ensures the existence of $\sigma = \sigma(\lambda, \rho, K) > 0$ and $\widehat{h} = \widehat{h}(\lambda, \rho, K) \in\ ]0, 1]$ such that $\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ z)$ has positive erosion of radius $\sigma\, h$ for every $h \in\ ]0, \widehat{h}]$ and $z \in I\!B_1(K)$. In the following, we assume without loss of generality $0 < h_j < \widehat{h}$ and $K_n(h) \subset I\!B_1(K)$ for all $j, n \in I\!N$, $h \in [0, \widehat{h}]$.

So the asymptotic properties of $\quad e\!\!\!\!e^{\supset}\big(\text{Graph } {}^\flat\! N_K,\ \text{Graph } {}^\flat\! N_{K_n(h)}\big) \qquad (n \longrightarrow \infty)$ have to be investigated for each $h \in\ ]0, \widehat{h}]$.
Due to Def. 4.1.4, every limiting normal cone results from the neighboring proximal normal cones, i.e. $N_C(x) \overset{\text{Def.}}{=} \operatorname{Limsup}_{\substack{y \longrightarrow x \\ y \in C}} N_C^P(y)$ for all nonempty $C \subset I\!R^N$, $x \in \partial C$.
Thus, $\ \text{Graph } N_C\ =\ \overline{\text{Graph } N_C^P}\ $ and from now on, we confine our considerations to $e\!\!\!\!e^{\supset}\Big(\text{Graph } {}^\flat\! N_K,\ \text{Graph } {}^\flat\! N^P_{K_n(h)}\Big)$ for any $h \in\ ]0, \widehat{h}]$.





The intersection $\quad P_{n,h}\ :=\ K_n\ \cap\ \vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ \partial\, K_n(h))$ is a subset of ∂K_n.
More precisely, it consists of all points $x \in K_n$ such that a trajectory of $\widetilde{G}_n$ starts in x and reaches $\partial\, K_n(h)$ at time h.
In addition, every boundary point y of $K_n(h)$ is attained by such a trajectory.
Taking now adjoint arcs into account, the Hamiltonian system in Prop. A.3.2 (1.) provides the following estimate for every $n \in I\!N$ (similarly to Lemma 4.4.23)

$$e\!\!\!\!e^{\supset}\Big(\text{Graph } {}^\flat\! N_{K_n}\Big|_{P_{n,h}},\ \ \text{Graph } {}^\flat\! N^P_{K_n(h)}\Big)\ \le\ \text{const}(\lambda)\ \cdot\ h.$$

Furthermore, $N^P_{\overline{I\!R^N \setminus K_n}}(x) \neq \emptyset$ for all $x \in \partial K_n$, due to $K_n \in \mathcal{K}_\circ(I\!R^N)$ and § 4.3.
In particular, $\quad N^P_{K_n}(x) \quad \neq \emptyset$ for all $x \in P_{n,h}$ because $\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ \partial\, K_n(h))$ has positive erosion of radius $\sigma\, h$ (due to Prop. A.7.2) and

$$K_n\ \cap\ \Big(\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ \partial\, K_n(h))\Big)^{\circ}\ =\ \emptyset.$$

So, $N^P_{\overline{\mathbb{R}^N \setminus K_n}}(x) \;=\; -\,N^P_{K_n}(x)$ contain exactly one direction for every point $x \in P_{n,h}$ according to [24, Clarke,Ledyaev,Stern 97], Lemma 6.4.
The positive erosion of K_n implies that $\overline{\mathbb{R}^N \setminus K_n}$ has positive reach due to Cor. 4.3.3 and thus, $N^P_{\overline{\mathbb{R}^N \setminus K_n}}(x) = N_{\overline{\mathbb{R}^N \setminus K_n}}(x) = N^C_{\overline{\mathbb{R}^N \setminus K_n}}(x)$ are containing exactly one direction (with $N^C_M(x)$ denoting the Clarke normal cone of $M \subset \mathbb{R}^N$ at x). As a consequence of a well–known result in [23, Clarke 83], we obtain that $N^C_{K_n}(x) \;=\; -\,N^C_{\overline{\mathbb{R}^N \setminus K_n}}(x)$ consist of exactly one direction for all $x \in P_{n,h}$ and so, $N^C_{K_n}(x) \;=\; N_{K_n}(x) \;=\; N^P_{K_n}(x)$.
In addition, the proximal radius of K_n at each $x \in P_{n,h}$ (in its unique proximal direction) is $\geq \sigma\, h \;>\; 0$ because $\vartheta_{-\,\widetilde{G}_n(h\,-\,\cdot,\,\cdot)}(h,\; \partial\, K_n(h))$ has positive erosion of radius $\sigma\, h$.

Now we conclude from the last property that for every $h \in \,]0, \widehat{h}]$,

$$e^{\supset}\Big(\text{Graph } {}^\flat N_K, \quad \text{Graph } {}^\flat N^P_{K_n}\Big|_{P_{n,h}}\Big) \;\longrightarrow\; 0 \qquad\qquad (n \longrightarrow \infty).$$

Indeed, assume that there exist $\delta > 0$ and sequences $n_k \nearrow \infty$, $(x_k)_{k \,\in\, \mathbb{N}}$ satisfying

$$x_k \;\in\; P_{n_k,h} \;\subset\; \partial K_{n_k},$$
$$\text{dist}\Big(\{x_k\} \times {}^\flat N^P_{K_{n_k}}(x_k), \;\; \text{Graph } {}^\flat N_K\Big) \;\;\geq\;\; \delta.$$

$\eta_k \in N^P_{K_{n_k}}(x_k) \cap \partial \mathbb{B}_1$ is unique for each k and, $\qquad K_{n_k} \,\cap\, \mathbb{B}_{\sigma\, h}(x_k + \sigma\, h \cdot \eta_k) = \{x_k\}$.
Considering subsequences (again denoted by) $(n_k)_{k\,\in\,\mathbb{N}}$, $(x_k)_{k\,\in\,\mathbb{N}}$, $(\eta_k)_{k\,\in\,\mathbb{N}}$ leads to $x \in K$ and $\eta \in \mathbb{R}^N$ with $x_k \longrightarrow x$, $\quad \eta_k \longrightarrow \eta \quad (k \longrightarrow \infty)$.
Then $K_{n_k} \;\subset\; \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_{\sigma\, h}(x_k + \sigma\, h \cdot \eta_k)$ (for each k) and $K = \text{Lim}_{n \to \infty}\, K_n$ imply

$$K \;\;\subset\;\; \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_{\sigma\, h}(x + \sigma\, h \cdot \eta),$$

i.e. $\eta \in N^P_K(x)$ – contradicting the initial assumption with δ.

As a consequence, we obtain the estimate for every $h \in \,]0, \widehat{h}]$,

$$\limsup_{n \,\longrightarrow\, \infty} \;\; e^{\supset}\Big(\text{Graph } {}^\flat N_K, \quad \text{Graph } {}^\flat N^P_{K_n(h)}\Big) \;\;\leq\;\; \text{const}(\lambda) \,\cdot\, h.$$

For proving transitional compactness of $\mathcal{K}_\circ(\mathbb{R}^N)$ in $(\mathcal{K}(\mathbb{R}^N),\, q_{\mathcal{K},N},\, \text{LIP}^{(\mathcal{H}^\rho_\circ)}_\lambda(\mathbb{R}^N, \mathbb{R}^N))$, a monotone sequence $(h_j)_{j\,\in\,\mathbb{N}}$ in $]0, \widehat{h}]$ with $h_j \longrightarrow 0$ is given.
Applying the Cantor diagonal construction once more, we obtain a subsequence (again denoted by) $(K_{n_k})_{k\,\in\,\mathbb{N}}$ satisfying for every $j \in \mathbb{N}$, $k \geq j$

$$e^{\supset}\Big(\text{Graph } {}^\flat N_K, \quad \text{Graph } {}^\flat N^P_{K_{n_k}(h_j)}\Big) \;\;\leq\;\; \text{const}(\lambda) \,\cdot\, h_j \;+\; \tfrac{1}{k}\,,$$

and thus, $\qquad \limsup\limits_{j \,\longrightarrow\, \infty} \;\; \sup\limits_{k\,\geq\, j} \;\; q_{\mathcal{K},N}(K,\;\, K_{n_k}(h_j)) \qquad = \;\; 0.$

□

Corollary 4.4.29 *Let* $f : \mathcal{K}(I\!R^N) \times [0,T] \longrightarrow \text{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(I\!R^N, I\!R^N)$ *satisfy*

$$\big\|\mathcal{H}_{f(K_1,t_1)} - \mathcal{H}_{f(K_2,t_2)}\big\|_{C^1(I\!R^N \times \partial I\!B_1)} \;\le\; \omega(q_{\mathcal{K},N}(K_1,K_2) + t_2 - t_1)$$

for all $K_1, K_2 \in \mathcal{K}(I\!R^N)$ *and* $0 \le t_1 \le t_2 \le T$ *with a modulus* $\omega(\cdot)$ *of continuity and consider the reachable sets of maps in* $\text{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(I\!R^N, I\!R^N)$ *as forward transitions on* $(\mathcal{K}(I\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!R^N),\ q_{\mathcal{K},N})$ *according to Proposition 4.4.26.*

Then for every initial set $K_0 \in \mathcal{K}(I\!R^N)$, *there exists a right–hand forward solution* $K : [0,T[\, \longrightarrow \mathcal{K}(I\!R^N)$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot) \ni f(K(\cdot),\cdot)$ *with* $K(0) = K_0$.

Proof results from Prop. 4.4.28 along with Prop. 2.3.5 and the remark after its proof.

4.5 Mild solutions of semilinear equations in reflexive Banach spaces

In the field of evolution equations, C^0 semigroups play a central role. Properly speaking, a family $(S(t))_{t\ge 0}$ of bounded linear operators on a Banach space X is called a *strongly continuous (one–parameter) semigroup* or C^0 *semigroup* if it satisfies

$$\wedge \begin{cases} S(t_1+t_2) &= S(t_1) \circ S(t_2) \\ S(0) &= \text{Id}_X \end{cases}$$

for all $t_1, t_2 \ge 0$ and if $[0,\infty[\, \longrightarrow X,\ t \longmapsto S(t)\,x$ is continuous for each $x \in X$ (see e.g. [51, Pazy 83], [34, Engel,Nagel 2000]).

This condition of continuity prevents us from applying mutational equations to examples of C^0 semigroups directly. Indeed, they are not uniformly continuous with respect to $|\cdot|_X$ in general. So the weak topology is to overcome this obstacle, i.e. we consider the pseudo–metrics $(x,y) \longmapsto |\langle x-y, v'\rangle|$ $(v' \in X')$.

General assumptions for § 4.5.

1. X is a reflexive Banach space.
2. The linear operator A generates a C^0 semigroup $(S(t))_{t\ge 0}$ on X with $\|S(t)\|_{\mathcal{L}(X,X)} \le \widehat{\eta} \cdot e^{\eta\, t}$ for all $t \ge 0$.
3. The dual operator A' of A has a countable family of eigenvectors $\{v_j'\}_{j\in\mathcal{J}}$ $(|v_j'|_{X'} = 1)$ spanning the dual space X', i.e. $X' = \overline{\sum_{j\in\mathcal{J}} I\!R\, v_j'}$.

   λ_j abbreviates the eigenvalue of A' belonging to the eigenvector v_j'.

Example 4.5.1 1. Consider a normal compact operator $A : H \longrightarrow H$ on a separable Hilbert space H generating a C^0 semigroup $(S(t))_{t \geq 0}$.
Then there exists a countable orthonormal system $(e_i)_{i \in \mathcal{I}}$ of eigenvectors of A satisfying $H = \ker A \oplus \overline{\sum_{i \in \mathcal{I}} I\!R\, e_i}$ (see e.g. [66, Werner 2002], Th. VI.3.2). Since H is separable, $(e_i)_{i \in \mathcal{I}}$ induces a *countable* orthonormal basis $(e_i)_{i \in \widehat{\mathcal{I}}}$ of H with $A\, e_i = 0$ for all $i \in \widehat{\mathcal{I}} \backslash \mathcal{I}$. In fact, each e_i $(i \in \widehat{\mathcal{I}})$ is also eigenvector of A' corresponding to the complex conjugate eigenvalue because A is normal (see [66, Werner 2002], Lemma VI.3.1, for example). So the general assumptions of this section are satisfied.

Symmetric integral operators of Hilbert–Schmidt type provide typical examples of A:

$$\begin{array}{lll} H := L^2(O) & \text{with} & O \subset I\!R^N \text{ open, nonempty,} \\ k(\cdot,\cdot) \in L^2(O \times O) & \text{with} & k(x,y) = \overline{k(y,x)} \text{ for all } x, y, \\ A\, u := \int_O k(\cdot, y)\, u(y)\, dy & \text{for} & u \in L^2(O). \end{array}$$

Then $A : L^2(O) \longrightarrow L^2(O)$ is symmetric and compact (see [68, Yosida 78], chapter X, § 2, example 2). Furthermore it generates even a uniformly continuous semigroup because A is bounded : $\|A\| \leq \|k(\cdot,\cdot)\|_{L^2(O \times O)}$.

2. An example of more general interest is the generator $A : D_A \longrightarrow H$ $(D_A \subset H)$ of a C^0 semigroup $(S(t))_{t \geq 0}$ on a Hilbert space H — assuming that the resolvent $R(\lambda_0, A) := (\lambda_0 \cdot \mathrm{Id}_H - A)^{-1} : H \longrightarrow H$ is compact and normal for some λ_0.
For the same reasons as before, there exists a countable orthonormal system $(e_i)_{i \in \mathcal{I}}$ of eigenvectors of $R(\lambda_0, A)$ satisfying $H = \ker R(\lambda_0, A) \oplus \overline{\sum_{i \in \mathcal{I}} I\!R\, e_i}$. The resolvent $R(\lambda_0, A)$ is injective (by definition) and so, $H = \overline{\sum_{i \in \mathcal{I}} I\!R\, e_i}$.
$R(\lambda_0, A)\, e_i = \mu_i \cdot e_i$ implies $\mu_i \neq 0$ and that e_i is eigenvector of A corresponding to the eigenvalue $\lambda_0 - \frac{1}{\mu_i}$ since $(\lambda_0 - A)\, e_i = (\lambda_0 - A) \cdot \frac{1}{\mu_i} R(\lambda_0, A)\, e_i = \frac{1}{\mu_i} e_i$.

This case opens the door for considering strongly elliptic differential operators in divergence form with smooth (time–independent) coefficients, for example.

Definition 4.5.2

1. *For every* $j \in \mathcal{J}$, *define the pseudo–metric* $q_j(x,y) := |\langle x - y,\, v'_j \rangle|$ *on* X.
2. *For each* $v \in X$, *the function* $\tau_v : [0,1] \times X \longrightarrow X$ *is defined as*

$$\tau_v(h,x) := S(h)\, x + \int_0^h S(h-s)\, v\, ds.$$

Remark. In the literature about C^0 semigroups, $\tau_v(\cdot, x) : [0,1] \longrightarrow X$ is called *mild solution* of the initial value problem $\frac{d}{dt}\, u(t) = A\, u(t) + v, \quad u(0) = x \in X$ (see [51, Pazy 83], Definition 4.2.3 and [34, Engel,Nagel 2000], for example). It is even continuously differentiable since the inhomogeneous term v does not depend on t. □

Proposition 4.5.3 *For $v \in X$ fixed, the function $\tau_v : [0,1] \times X \longrightarrow X$ satisfies the following conditions on forward transitions of order 0 on $(X, X, (q_j)_{j \in \mathcal{J}})$ (see Def. 2.1.1) :*

1. $\tau_v(0, \cdot\,) = \mathrm{Id}_X$,
2. $q_j\Big(\tau_v(h,\ \tau_v(t,x)),\ \tau_v(t+h, x)\Big) = 0 = q_j\Big(\tau_v(t+h,x),\ \tau_v(h,\ \tau_v(t,x))\Big)$ *for all* $x \in X,\ t,h \in [0,1]$ *with* $t+h \leq 1$,
3. $\sup\limits_{\substack{x,y \in X \\ q_j(x,y) \neq 0}} \ \limsup\limits_{h \downarrow 0} \ \left(\frac{q_j\left(\tau_v(h,x),\, \tau_v(h,y)\right) \,-\, q_j(x,y)}{h \quad q_j(x,y)} \right)^+ \leq |\lambda_j|\,.$

Moreover for every radius $R > 0$ and index $j \in \mathcal{J}$, there is a modulus $\omega_j(\cdot)$ of continuity (depending only on A and v_j) such that for all $t_1, t_2 \in [0,1]$, $x \in X$ $(|x| \leq R)$,

$$q_j\Big(\tau_v(t_1,x),\ \tau_v(t_2,x)\Big) \leq R \cdot \omega_j(|t_2 - t_1|).$$

Finally, the functions $\tau_v,\ \tau_w : [0,1] \times X \longrightarrow X$ related to $v, w \in X$ respectively fulfill

$$Q_j^{\mapsto}(\tau_v, \tau_w) \overset{\text{Def.}}{=} \sup_{x,\, y \in X} \ \limsup_{h \downarrow 0} \ \left(\frac{q_j\left(\tau_v(h,x),\, \tau_w(h,y)\right) \,-\, q_j(x,y) \cdot e^{|\lambda_j|\, h}}{h} \right)^+ \leq q_j(v,w).$$

In preparation of the proof, we summarize the essential tools about C^0 semigroups. The first lemma bridges the gap between the semigroup operators and their dual counterparts. It is the first of two reasons for assuming the Banach space X to be reflexive. Afterwards Lemma 4.5.5 implies that each v_j' $(j \in \mathcal{J})$ is eigenvector of every dual operator $S(t)'$ $(t \geq 0)$ belonging to the eigenvalue $e^{\lambda_j\, t}$.

Lemma 4.5.4
Let $(S(t))_{t \geq 0}$ be a C^0 semigroup on a reflexive Banach space with generator A. Then the dual operators $S(t)'$ $(t \geq 0)$ provide a C^0 semigroup on the dual space and its generator is the dual operator A'.

Proof is given in [51, Pazy 83], Cor. 1.10.6 and [34, Engel,Nagel 2000], Prop. I.5.14. □

Lemma 4.5.5 *The eigenspaces of the generator A and of the C^0 semigroup operators $S(t)$ $(t \geq 0)$, respectively, fulfill for every complex μ*

$$\ker\,(\mu - A) = \bigcap_{t \geq 0} \ker\,\left(e^{\mu\, t} - S(t)\right).$$

Proof is presented in detail in [34, Engel,Nagel 2000], Corollary IV.3.8. □

Proof of Prop. 4.5.3. The first assertion results directly from the definition of τ_v and, the second claim is a consequence of the semigroup property $\tau_v(h,\ \tau_v(t,x)) = \tau_v(t+h,x)$.

Furthermore we obtain for every $x,\ y \in X$ and $h \in [0,1]$ with $q_j(x,y) \neq 0$

$$\begin{aligned} & q_j\Big(\tau_v(h,x),\ \ \tau_v(h,y)\Big) && - \ q_j(x,y) \\ \leq\ & \big|\,\langle S(h)\,x - S(h)\,y,\ v_j'\rangle\,\big| && - \ \big|\,\langle x-y,\ v_j'\rangle\,\big| \\ \leq\ & \big|\,\langle\ x-y, \qquad S(h)'\,v_j'\rangle\,\big| && - \ \big|\,\langle x-y,\ v_j'\rangle\,\big| \\ \leq\ & \big|\,\langle\ x-y, \qquad (S(h)' - \mathrm{Id}_{X'})\,v_j'\rangle\,\big| \end{aligned}$$

and thus, $\limsup\limits_{h\downarrow 0} \frac{q_j\left(\tau_v(h,x),\,\tau_w(h,y)\right)\ -\ q_j(x,y)}{h} \ \leq\ |\,\langle x-y,\ A'\,v_j'\rangle\,| \ \leq\ |\lambda_j|\cdot|\,\langle x-y,\ v_j'\rangle\,|$

since v_j' is assumed to be eigenvector of A'. So the third claim is satisfied, i.e.

$$\sup_{\substack{x,y\,\in\,X \\ q_j(x,y)\,\neq\,0}} \ \limsup_{h\downarrow 0} \ \left(\frac{q_j\left(\tau_v(h,x),\,\tau_v(h,y)\right)\ -\ q_j(x,y)}{h \quad q_j(x,y)}\right)^{+} \ \leq\ |\lambda_j|.$$

The claimed continuity of $\tau_v(\cdot,x) : [0,1] \longrightarrow X \ \ (x \in X, |x| \leq R)$ results from the strong continuity of $(S(t)')_{t\geq 0}$ (according to Lemma 4.5.4).
Indeed, for every $t_1,\ t_2 \in [0,1]$ and $x \in X$ with $|x| \leq R$,

$$\begin{aligned} q_j\Big(S(t_1)\,x,\ \ S(t_2)\,x\Big) \ &\leq\ \big|\,\langle S(t_2)\,x - S(t_1)\,x,\ v_j'\rangle\,\big| \\ &\leq\ \big|\,\langle x,\ (S(t_2)' - S(t_1)')\ v_j'\rangle\,\big| \\ &\leq\ R\ \ \big|(S(t_2)' - S(t_1)')\ v_j'\ \big|. \end{aligned}$$

Finally we prove $Q_j^{\mapsto}(\tau_v,\tau_w) \ \leq\ |v-w|$ for arbitrary $v,\ w \in X$.
Indeed, the definition of τ_v, τ_w and Lemma 4.5.5 provide for every $x, y \in X$ and $h \in]0,1]$

$$\begin{aligned} q_j\Big(\tau_v(h,x),\ \ \tau_w(h,w)\Big) \ &=\ \left|\Big\langle S(h)\,(x-y) \ + \int_0^h S(h-s)\,(v-w)\ ds,\ \ v_j'\Big\rangle\right| \\ &\leq\ \big|\,\langle x-y,\ S(h)'\,v_j'\rangle\big| \ + \int_0^h |\langle v-w,\ \ S(h-s)'\,v_j'\rangle|\ \ ds \\ &\leq\ \big|\,\langle x-y,\ v_j'\rangle\big|\cdot e^{|\lambda_j|\,h} \ +\ |\langle v-w,\ v_j'\rangle|\ \cdot \int_0^h e^{|\lambda_j|\,(h-s)}\ ds \\ &\leq\ q_j(x,y)\ \cdot e^{|\lambda_j|\,h} \ +\ q_j(v,w)\ \cdot \frac{e^{|\lambda_j|\,h}-1}{|\lambda_j|} \\ &\leq\ \Big(q_j(x,y) \ +\ q_j(v,w)\ h\Big)\ \cdot\ e^{|\lambda_j|\,h} \end{aligned}$$

and thus,

$$Q_j^{\mapsto}(\tau_v,\tau_w) \ \stackrel{\text{Def.}}{=}\ \sup_{x,\,y\,\in\,X}\ \limsup_{h\downarrow 0}\ \left(\frac{q_j\left(\tau_v(h,x),\,\tau_w(h,y)\right)\ -\ q_j(x,y)\cdot e^{|\lambda_j|\,h}}{h}\right)^{+} \ \leq\ q_j(v,w).$$

□

As a direct consequence of this proposition, we get $q_j(\tau_v(t-h,x),\, y) \longrightarrow q_j(\tau_v(t,x),\, y)$ for $h \downarrow 0$ and all x, y, t. So there is only one reason why τ_v is *not* a forward transition on $(X, X, (q_j)_{j\in\mathcal{J}})$ in the strict sense of Definition 2.1.1 :
Considering $\tau_v(\cdot, x) : [0,1] \longrightarrow X$, the modulus of continuity can be chosen uniformly only for all $x \in B_R \subset X$ (with any finite radius R), but not for all $x \in X$ in general. This gap does not really prevent us from applying the results of chapter 2. Indeed, for concluding the existence of right–hand forward solutions from two–sided sequential compactness, we only need the uniform continuity of Euler approximations in positive time direction. (It has already been mentioned in remark (1.) after proving Prop. 2.3.5.) This property results directly from the preceding Proposition 4.5.3 and the following a priori estimate.

Lemma 4.5.6 *For every $g \in L^\infty([0,T], X)$ and $x_0 \in X$ with $|x_0| \le R$, the function*

$$\xi : [0,T] \longrightarrow X, \quad t \longmapsto S(h)\, x_0 + \int_0^h S(h-s)\ g(s)\ ds$$

has the upper bound $\quad \|\xi\|_{L^\infty} \le R \cdot \widehat{\eta}\ e^{\eta T} + \|g\|_{L^\infty} \cdot \widehat{\eta}\ \frac{e^{\eta T}-1}{\eta}.$ □

Assuming X to be reflexive has the second advantage that the compactness properties of $(X, (q_j)_{j\in\mathcal{J}})$ are quite obvious.

Lemma 4.5.7 *$(X, (q_j)_{j\in\mathcal{J}})$ is one–sided sequentially compact (uniformly with respect to j), i.e. for every sequence $(x_n)_{n\in I\!N}$ in X with $\sup_n\, q_j(0, x_n) < \infty$ (for all j), there exist a subsequence $(x_{n_k})_{k\in I\!N}$ and $x \in X$ satisfying $q_j(x_{n_k}, x) \longrightarrow 0 \quad (k \to \infty)$.*

So $(X, (q_j)_{j\in\mathcal{J}})$ is also two–sided sequentially compact (uniformly with respect to j) since each q_j is symmetric.

Proof. The general assumption $X' = \overline{\sum_{j\in\mathcal{J}} I\!R\, v_j'}$ of this section guarantees that $(q_j)_{j\in\mathcal{J}}$ induces the weak topology of the Banach space X. Due to the well–known theorem of uniform boundedness, $\sup_n\, q_j(0, x_n) < \infty$ (for all j) implies that $(x_n)_{n\in I\!N}$ is bounded in X.
So there exist a subsequence $(x_{n_k})_{k\in I\!N}$ converging weakly to some $x \in X$ for $k \longrightarrow \infty$ because closed balls of reflexive Banach spaces are always weakly sequentially compact.
Thus, $\quad q_j(x_{n_k},\, x) = |\,\langle x_{n_k} - x,\, v_j'\rangle\,| \longrightarrow 0 \quad$ for $k \longrightarrow \infty$ and all $j \in \mathcal{J}$. □

Now we conclude the existence of right–hand forward solutions from the results of § 2.3.2. In the next proposition, the assumptions about f might be regarded as unfavorable. Indeed we suppose the continuity with respect to each linear form v'_j $(j \in \mathcal{J})$ separately in a Banach space X of possibly infinite dimension.

Proposition 4.5.8 *In addition to the general assumptions about* $X, A, S(\cdot)$ *of this section 4.5, let* $f : X \times [0,T] \longrightarrow X$ *satisfy* $\|f\|_{L^\infty} < \infty$ *and for each* $j \in \mathcal{J}$,

$$q_j\Big(f(x_1,t_1),\; f(x_2,t_2)\Big) \;\le\; \omega_j\Big(q_j(x_1,x_2) + |t_2 - t_1|\Big) \quad \text{for all } x_1, x_2,\ t_1, t_2$$

with a modulus $\omega_j(\cdot)$ *of continuity.*

Then for every initial vector $x_0 \in X$, *there exists a right–hand forward solution* $x(\cdot) : [0,T[\longrightarrow X$ *of the generalized mutational equation* $\overset{\circ}{x}(\cdot) \ni \tau_{f(x(\cdot),\cdot)}$ *in* $[0,T[$ *with* $x(0) = x_0$ *i.e. for each* $j \in \mathcal{J}$, $x(\cdot) \in UC^{\rightarrow}([0,T[,\ X,\ q_j)$ *and*

$$\limsup_{h \downarrow 0}\ \tfrac{1}{h}\ \Big(\, q_j\Big(\tau_{f(x(t),t)}\,(h,y),\ \ x(t{+}h)\Big) \;-\; q_j(y,\ x(t)) \,\cdot\, e^{|\lambda_j|\,h} \Big) \;\le\; 0,$$

holds for all $y \in X$, $t \in [0,T[$.
Supposing $q_j\Big(f(x_1,t_1),\ f(x_2,t_2)\Big) \le L_j \cdot q_j(x_1,x_2) + \widehat{\omega}_j(t_2 - t_1)$ *for all* x_1, x_2, t_1, t_2, j *with* $L_j \ge 0$ *and a modulus* $\widehat{\omega}_j(\cdot)$ *of continuity, this solution is unique.*

Proof results directly from Corollary 2.3.6 since $(X, (q_j)_{j \in \mathcal{J}})$ is two–sided sequentially compact (according to Lemma 4.5.7). If in addition, f is Lipschitz continuous with respect to the first argument, the uniqueness is a consequence of Prop. 2.3.8 because the "test set" is X and each q_j is symmetric. □

For overcoming this obstacle (of continuity with respect to each v'_j separately), several pseudo–metrics q_j $(j \in \mathcal{J})$ are considered simultaneously. To be more precise, we replace the family q_j $(j \in \mathcal{J} = \{j_1, j_2, j_3 \ldots\})$ with the pseudo–metrics p_n on X

$$p_n(x,y) \;:=\; \sum_{k=1}^{n} 2^{-k}\ \frac{q_{j_k}(x,y)}{1 \,+\, q_{j_k}(x,y)} \qquad (n \in I\!N).$$

Reflexivity and symmetry of p_n are obvious and, the triangle inequality results from the auxiliary function $[0,\infty[\longrightarrow [0,1]$, $r \longmapsto \frac{r}{1+r}$ being increasing and concave.
The key advantage of $(p_n)_{n \in I\!N}$ is that we can take finitely many q_j into consideration and estimate the rest uniformly. So in short, the existence results of § 2.3.2 hold with the parameter $R_\varepsilon > 0$ arbitrarily small.

Lemma 4.5.9 *For $v \in X$ fixed, the function $\tau_v : [0,1] \times X \longrightarrow X$ satisfies the following conditions on forward transitions of* positive *order on $(X, X, (p_n)_{n \in I\!N})$*

1. $\tau_v(0, \cdot) = \mathrm{Id}_X$,
2. $p_n\Big(\tau_v(h, \tau_v(t,x)), \; \tau_v(t+h, x)\Big) = 0 = p_n\Big(\tau_v(t+h,x), \; \tau_v(h, \tau_v(t,x))\Big)$ *for all* $x \in X$, $t, h \in [0,1]$ *with* $t + h \leq 1$,
3. $\displaystyle\sup_{\substack{x,y \in X \\ p_n(x,y) \neq 0}} \limsup_{h \downarrow 0} \left(\frac{p_n\big(\tau_v(h,x), \tau_v(h,y)\big) - p_n(x,y)}{h \quad p_n(x,y)} \right)^+ \leq \mu_n$.

with the abbreviation $\mu_n := \max\limits_{k=1 \dots n} |\lambda_{j_k}|$.
Moreover for every radius $R > 0$ and index $n \in I\!N$, there is a modulus $\omega_n(\cdot)$ of continuity (depending only on A and n) such that for all $t_1, t_2 \in [0,1]$, $x \in X$ $(|x| \leq R)$,

$$p_n\Big(\tau_v(t_1,x), \; \tau_v(t_2,x)\Big) \leq R \cdot \omega_n(|t_2 - t_1|).$$

$\tau_v, \tau_w : [0,1] \times X \longrightarrow X$ *related to $v, w \in X$ respectively satisfy*

$$\begin{aligned} P_n^{\mapsto}(\tau_v, \tau_w) &\stackrel{\text{Def.}}{=} \sup_{x,y \in X} \limsup_{h \downarrow 0} \left(\frac{p_n\big(\tau_v(h,x), \tau_w(h,y)\big) - p_n(x,y) \cdot e^{\mu_n h}}{h} \right)^+ \\ &\leq \sum_{k=1}^{n} 2^{-k} \; q_{j_k}(v,w) \quad \leq \quad |v - w| \,. \end{aligned}$$

Proof results from Proposition 4.5.3 about forward transitions on $(X, X, (q_j)_{j \in \mathcal{J}})$ because the auxiliary function $[0, \infty[\longrightarrow [0,1], \; r \longmapsto \frac{r}{1+r}$ is increasing and concave. Indeed, assertions (1.), (2.) are obvious. Moreover we obtain for all $x, y \in X$, $h \in [0,1]$

$$\begin{aligned} & p_n\Big(\tau_v(h,x), \; \tau_v(h,y)\Big) - p_n(x,y) \\ \leq & \sum_{k=1}^{n} 2^{-k} \left(\frac{q_{j_k}\big(\tau_v(h,x), \tau_v(h,y)\big)}{1 + q_{j_k}\big(\tau_v(h,x), \tau_v(h,y)\big)} - \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)} \right) \\ \leq & \sum_{k=1}^{n} 2^{-k} \left(\frac{q_{j_k}(x,y) \; e^{|\lambda_{j_k}| \, h}}{1 + q_{j_k}(x,y) \; e^{|\lambda_{j_k}| \, h}} - \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)} \right) \\ \leq & \sum_{k=1}^{n} 2^{-k} \left(\frac{q_{j_k}(x,y) \; e^{\mu_n h}}{1 + q_{j_k}(x,y) \; e^{\mu_n h}} - \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)} \right) \\ = & \sum_{k=1}^{n} 2^{-k} \; \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)} \; \frac{e^{\mu_n h} \; (1 + q_{j_k}(x,y)) - 1 - q_{j_k}(x,y) \; e^{\mu_n h}}{1 + q_{j_k}(x,y) \; e^{\mu_n h}} \\ \leq & \sum_{k=1}^{n} 2^{-k} \; \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)} \; \frac{e^{\mu_n h} - 1}{1 + q_{j_k}(x,y) \; e^{\mu_n h}} \\ \leq & \qquad p_n(x,y) \qquad h \quad \mu_n \; e^{\mu_n h}. \end{aligned}$$

The claimed continuity results directly from the corresponding property with respect to q_j stated in Proposition 4.5.3.

Finally, the triangle inequality of the concave function $[0,\infty[\longrightarrow [0,1], \;\; r \longmapsto \frac{r}{1+r}$ implies for every $v, w \in X, \; x, y \in X$ and $h \in [0,1]$,

$$\begin{aligned}
p_n\Big(\tau_v(h,x), \; \tau_w(h,y)\Big) &\overset{\text{Def.}}{=} \sum_{k=1}^{n} 2^{-k} \; \frac{q_{j_k}\big(\tau_v(h,x), \, \tau_w(h,y)\big)}{1 + q_{j_k}\big(\tau_v(h,x), \, \tau_w(h,y)\big)} \\
&\le \sum_{k=1}^{n} 2^{-k} \; \frac{\big(q_{j_k}(x,y) + q_{j_k}(v,w)\; h\big) \; e^{\mu_n \, h}}{1 \, + \, \big(q_{j_k}(x,y) + q_{j_k}(v,w)\; h\big) \; e^{\mu_n \, h}} \\
&\le \sum_{k=1}^{n} 2^{-k} \; \left(\frac{q_{j_k}(x,y) \;\, e^{\mu_n \, h}}{1 \, + \, q_{j_k}(x,y) \;\, e^{\mu_n \, h}} \; + \; \frac{q_{j_k}(v,w) \; h \;\, e^{\mu_n \, h}}{1 \, + \, q_{j_k}(v,w) \; h \;\, e^{\mu_n \, h}} \right) \\
&\le \sum_{k=1}^{n} 2^{-k} \; \left(\frac{q_{j_k}(x,y) \;\, e^{\mu_n \, h}}{1 \, + \, q_{j_k}(x,y)} \; + \; \frac{q_{j_k}(v,w) \; h \;\, e^{\mu_n \, h}}{1 \, + \, q_{j_k}(v,w) \; h \;\, e^{\mu_n \, h}} \right) \\
&\le p_n(x,y) \;\; e^{\mu_n \, h} \; + \; \sum_{k=1}^{n} 2^{-k} \; \frac{q_{j_k}(v,w)}{1 \, + \, q_{j_k}(v,w) \; h \;\, e^{\mu_n \, h}} \; h \;\, e^{\mu_n \, h}
\end{aligned}$$

and thus,

$$\begin{aligned}
P_n^{\mapsto}(\tau_v, \tau_w) &\overset{\text{Def.}}{=} \sup_{x,\, y \,\in X} \; \limsup_{h \downarrow 0} \; \left(\frac{p_n\big(\tau_v(h,x), \, \tau_w(h,y)\big) - p_n(x,y) \cdot e^{\mu_n \, h}}{h} \right)^{+} \\
&\le \limsup_{h \downarrow 0} \; \sum_{k=1}^{n} 2^{-k} \; \frac{q_{j_k}(v,w)}{1 \, + \, q_{j_k}(v,w) \; h \;\, e^{\mu_n \, h}} \;\; e^{\mu_n \, h} \\
&\le \sum_{k=1}^{n} 2^{-k} \; q_{j_k}(v,w) \\
&\le |v - w|.
\end{aligned}$$

□

Proposition 4.5.10 *In addition to the general assumptions about $X, A, S(\cdot)$ of § 4.5, let $f : X \times [0,T] \longrightarrow X$ fulfill $\|f\|_{L^\infty} < \infty$ and*

$$\sum_{k=1}^{\infty} 2^{-k} \; q_{j_k}\Big(f(x_1,t_1), \; f(x_2,t_2)\Big) \;\le\; \widehat{\omega}\Big(\limsup_{n \longrightarrow \infty} \; p_n(x_1,x_2) \, + \, |t_2 - t_1|\Big)$$

for all $x_1, x_2 \in X$ and $t_1, t_2 \in [0,T]$ with a modulus $\widehat{\omega}(\cdot)$ of continuity.
For each $x_0 \in X$, there exists a mild solution $x : [0,T[\longrightarrow X$ of the initial value problem

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &= \; A\, x(t) \; + \; f(x(t),t) \\ \quad x(0) &= \; x_0 \end{cases}$$

i.e.
$$x(t) \;=\; S(t)\, x_0 \; + \; \int_0^t S(t-s) \;\; f(x(s), \, s) \;\, ds.$$

Remark. Considering the continuity assumption about f, the series is finite due to $\|f\|_{L^\infty} < \infty$ and, it is an upper bound of $p_n(f(x_1,t_1), \; f(x_2,t_2))$ for every $n \in \mathbb{N}$.

The main steps for proving this proposition are summarized in the following lemmas. The existence results of chapter 2 provide a forward solution $x(\cdot) : [0,T[\longrightarrow (X, (p_n)_n)$ of the generalized mutational equation $\overset{\circ}{x}(\cdot) \ni \tau_{f(x(\cdot),\,\cdot)}$.
Restricting ourselves to the linear forms v'_j $(j \in \mathcal{J})$, $x(\cdot)$ can be regarded a weak solution of the initial value problem. Then Lemma 4.5.12 of John M. Ball ensures the existence of a unique weak solution (with respect to all linear forms of X') and it is even a mild solution.

Lemma 4.5.11 *Suppose the assumptions of Proposition 4.5.10.*
Then for every initial vector $x_0 \in X$, *there exists a right–hand forward solution* $x(\cdot) : [0,T[\longrightarrow (X, (p_n)_n)$ *of the generalized mutational equation* $\overset{\circ}{x}(\cdot) \ni \tau_{f(x(\cdot),\,\cdot)}$ *in* $[0,T[$ *with* $x(0) = x_0$ *in the sense that for each* $n \in I\!N$, $x(\cdot) \in UC^{\rightarrow}([0,T[,\ X,\ p_n)$ *and*

$$\limsup_{n \longrightarrow \infty} \ \limsup_{h \downarrow 0} \ \tfrac{1}{h} \left(p_n\Big(\tau_{f(x(t),t)}\,(h,y),\ x(t+h)\Big) \ - \ p_n(y,\ x(t)) \cdot e^{\mu_n h} \right) \ \le \ 0,$$

holds for all $y \in X$, $t \in [0,T[$.

In particular, $x(\cdot)$ *has the following properties :*

1. $\limsup_{h \downarrow 0} \ \frac{1}{h} \cdot p_n\Big(\tau_{f(x(t),t)}\,(h, x(t)),\ x(t+h)\Big) \ = \ 0$ *for every* $t \in [0,T[$, $n \in I\!N$.
2. $x(\cdot)$ *is bounded in* X.
3. $[0,T[\ \longrightarrow \ X,\ t \longmapsto \langle f(x(t),\,t),\ v'_j\rangle$ *is continuous for every* $j \in \mathcal{J}$.
4. $f(x(\cdot),\cdot) \ \in \ L^\infty([0,T[,\ X)$.
5. $]0,T[\ \longrightarrow \ I\!R,\ t \longmapsto \langle\, x(t),\ v'_j \,\rangle$ *is continuously differentiable for each* $j \in \mathcal{J}$,
$$\tfrac{d}{dt}\,\langle\, x(t),\ v'_j \,\rangle \ = \ \langle\, x(t),\ A'\, v'_j \,\rangle \ + \ \langle\, f(x(t),t),\ v'_j \,\rangle.$$

Proof is based on Corollary 2.3.6 (about existence due to sequential compactness). Indeed, the sequence $(p_n)_{n \in I\!N}$ of pseudo–metrics induces the weak topology on the reflexive Banach space X. So X is weakly sequentially compact and thus, $(X, (p_n)_{n \in I\!N})$ is two–sided sequentially compact (uniformly with respect to n).
Choosing $\delta > 0$ arbitrarily small, there is $M \in I\!N$ with $\sum_{k=1}^{n} 2^{-k} \ge 1 - \delta$ for all $n \ge M$.
So, $p_n(x_1, x_2) \ \le \ \limsup_{k \longrightarrow \infty} \ p_k(x_1, x_2) \ \le \ p_n(x_1,x_2) + \delta$ for every $n \ge M$, $x_1, x_2 \in X$ and in particular, $\quad p_n\Big(f(x_1,t_1),\ f(x_2,t_2)\Big) \ \le \ \widehat{\omega}\Big(\delta + p_n(x_1,x_2) \ + \ |t_2 - t_1|\Big).$

Now the steps of Corollary 2.3.6 provide a right–hand forward solution

$$x : [0,T[\ \longrightarrow \ (X, (p_n)_{n \ge M})$$

in the sense that $x(\cdot) \in UC^{\rightarrow}([0,T[,\ X,\ p_n)$ and for all $y \in X,\ t \in [0,T[,\ n \geq M$,

$$\limsup_{h\downarrow 0} \ \tfrac{1}{h} \left(p_n\Big(\tau_{f(x(t),t)}\,(h,y),\ x(t{+}h)\Big) \ - \ p_n(y,\ x(t)) \ \cdot\ e^{\mu_n\, h} \right) \ \leq \ \text{const} \cdot \widehat{\omega}(\delta).$$

Since $\delta > 0$ is arbitrarily small, we conclude for every vector $y \in X$, time $t \in [0,T[$

$$\limsup_{n\longrightarrow\infty} \ \limsup_{h\downarrow 0} \ \tfrac{1}{h} \left(p_n\Big(\tau_{f(x(t),t)}\,(h,y),\ x(t{+}h)\Big) \ - \ p_n(y,\ x(t)) \ \cdot\ e^{\mu_n\, h} \right) \ \longrightarrow \ 0$$

(In short, we have just applied Cor. 2.3.6 with its parameter $R_\varepsilon > 0$ arbitrarily small.)

1. Setting $y := x(t)$, we get $\lim_{n\longrightarrow\infty} \limsup_{h\downarrow 0} \ \tfrac{1}{h}\cdot p_n\Big(\tau_{f(x(t),t)}\,(h,x(t)),\ x(t{+}h)\Big) \ = \ 0.$
Obviously, the definition of p_n implies $p_{n-1} \leq p_n$ for all $n \in I\!N$ and so the claimed property (1.) holds true for every $t \in [0,T[$ and $n \in I\!N$.

2. $x(\cdot)$ is bounded in X, i.e. $\|x\|_{L^\infty} < \infty$.
Indeed, the proof of existence (in § 2.3.2) uses Euler approximations $x_n(\cdot)$ that are uniformly bounded according to Lemma 4.5.6. Moreover for each time $t \in\,]0,T[$, a subsequence of $(x_n(t))_{n\,\in\, I\!N}$ converges weakly to $x(t)$ and thus, $|x(t)| \ \leq\ \limsup_{n\longrightarrow\infty} |x_n(t)|$.

3. The function $[0,T[\longrightarrow X,\ t \longmapsto \langle f(x(t),t),\ v_j'\rangle$ is continuous for each $j \in \mathcal{J}$.
Indeed, for any $\delta > 0$, there exists an index M with $\displaystyle\sum_{k=1}^{n} 2^{-k} \geq 1-\delta$ for all $n \geq M$.
So, $\displaystyle\sum_{k\,=\,1}^{\infty} 2^{-k}\ q_{j_k}\Big(f(x(s),s),\ f(x(t),t)\Big) \ \leq\ \widehat{\omega}\Big(\delta + p_n(x(s),x(t)) \ +\ |t-s|\Big)$ for all s,t.
Fixing some $n \geq M$, $x(\cdot) \in UC^{\rightarrow}([0,T[,\ X,\ p_n)$ and the symmetry of p_n provide

$$2^{-k}\ q_{j_k}\Big(f(x(s),s),\ f(x(t),t)\Big) \ \leq\ \widehat{\omega}(2\,\delta) \qquad \text{for any small } |t-s| \text{ and all } k.$$

4. $\langle f(x(\cdot),\cdot),\ v'\rangle \ \in\ L^1([0,T[,\ I\!R)$ for every linear form $v' \in X'$ results from the general assumption that $(v_j')_{j\in\mathcal{J}}$ is spanning the dual space X'.
Indeed, for every $\delta \in]0,1]$, there is a finite linear combination w' of $(v_j')_{j\in\mathcal{J}}$ satisfying $\|\,v' - w'\,\|_{X'} < \delta$. As a consequence of the preceding property (3.), $[0,T[\longrightarrow I\!R$, $t \longmapsto \langle f(x(\cdot),\cdot),\ w'\rangle$ is continuous. Furthermore it is bounded by $(\|v'\|_{X'}+1)\ \|f\|_{L^\infty}$. Thus, the Convergence Theorem of Lebesgue guarantees $\langle f(x(\cdot),\cdot),\ v'\rangle \in L^1([0,T[,\ I\!R)$.

Now we conclude $f(x(\cdot),\cdot) \in L^\infty([0,T[,X)$ from the Measurability Theorem of Pettis and the assumption $\|f\|_{L^\infty} < \infty$. Indeed, $f(x(\cdot),\cdot)$ is weakly Lebesgue–measurable. Moreover, the Banach space X is separable since its dual space X' is supposed to be separable (see e.g. [68, Yosida 78], Appendix of chapter V, § 4).
So $f(x(\cdot),\cdot) : [0,T[\longrightarrow X$ is (strongly) Lebesgue–measurable due to the Theorem of Pettis (stated and proven in [68, Yosida 78], chapter V, § 4, for example).

5. Defining p_n by means of $(q_j)_{j\in\mathcal{J}}$ leads to $x(\cdot) \in UC^{\to}([0,T], X, q_j)$ and

$$\limsup_{h\downarrow 0} \frac{1}{h} \left| \left\langle \tau_{f(x(t),t)}(h, x(t)) - x(t+h), \; v'_j \right\rangle \right| = 0$$

for every time $t \in [0,T]$ and index $j \in \mathcal{J}$. Thus,

$$\limsup_{h\downarrow 0} \left| \left\langle \frac{\tau_{f(x(t),t)}\big(h, x(t)\big) - x(t)}{h} - \frac{x(t+h) - x(t)}{h}, \; v'_j \right\rangle \right| = 0.$$

Due to Def. 4.5.2 of $\tau_{f(x(t),t)}(h,\cdot)$, the limit of $\frac{\tau_{f(x(t),t)}\big(h, x(t)\big) - x(t)}{h}$ for $h \downarrow 0$ exists and,

$$\begin{aligned} \lim_{h\downarrow 0} \frac{\tau_{f(x(t),t)}\big(h, x(t)\big) - x(t)}{h} &= A\,x(t) + f(x(t),t) \end{aligned}$$

So we obtain

$$\begin{aligned} \lim_{h\downarrow 0} \left\langle \frac{x(t+h) - x(t)}{h}, \; v'_j \right\rangle &= \langle A\,x(t) + f(x(t),t), \; v'_j \rangle \\ &= \langle x(t), \; A'\, v'_j \rangle + \langle f(x(t),t), \; v'_j \rangle \\ &= \lambda_j \; \langle x(t), \; v'_j \rangle + \langle f(x(t),t), \; v'_j \rangle \end{aligned}$$

and, the right–hand side is continuous with respect to t — as a consequence of property (3.) and $x(\cdot) \in UC^{\to}([0,T], X, q_j)$. These two properties guarantee that $]0,T[\longrightarrow I\!R, \; t \longmapsto \langle x(t), \, v'_j \rangle$ is continuously differentiable for every $j \in \mathcal{J}$ (see e.g. [51, Pazy 83], Corollary 2.1.2). □

The following lemma provides existence and uniqueness of a weak solution $z(\cdot)$ of

$$\wedge \begin{cases} \frac{d}{dt}\, z(t) = A\, z(t) + f(x(t),t) \\ \quad z(0) = x_0 \end{cases}$$

because in this section, A has been supposed to be the infinitesimal generator of the C^0 semigroup $(S(t))_{t\geq 0}$. Furthermore this weak solution is even a mild solution. Then the proof of Proposition 4.5.10 is based on the uniqueness of $z(\cdot)$.

Lemma 4.5.12 ([8, Ball 77]**)** *Let A be a densely defined closed linear operator on a real or complex Banach space Y and $g \in L^1([0,T], Y)$.*
There exists for each $y \in Y$ a unique weak solution $u(\cdot)$ of

$$\wedge \begin{cases} \frac{d}{dt}\, u(t) = A\, u(t) + g(t) \quad \text{on }]0,T] \\ \quad u(0) = x \end{cases}$$

i.e. for every $v' \in D(A') \subset Y'$, $\langle u(\cdot), v' \rangle \in AC([0,T])$ and

$$\frac{d}{dt} \langle u(t), \, v' \rangle = \langle u(t), \, A'\, v' \rangle + \langle g(t), \, v' \rangle \qquad \text{for almost all } t,$$

if and only if A is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$, and in this case $u(t)$ is given by $\quad u(t) = S(t)\,x + \int_0^t S(t-s)\; g(s)\; ds.$ □

Proof of Proposition 4.5.10 Let $x(\cdot) : [0,T[\longrightarrow X$ abbreviate the right–hand forward solution considered in Lemma 4.5.11 and,

$$z(t) \;:=\; S(t)\,x_0 \;+\; \int_0^t S(t-s)\;\, f(x(s),s)\;\, ds$$

is the unique weak solution of the initial value problem

$$\wedge \begin{cases} \frac{d}{dt}\,z(t) &=\; A\,z(t) \;+\; f(x(t),t) \\ \quad z(0) &=\; x_0 \end{cases}$$

according to Lemma 4.5.12. Then for every index $j \in \mathcal{J}$, the function $[0,T[\longrightarrow I\!R$, $t \longmapsto \langle\, x(t) - z(t),\; v_j'\,\rangle$ is continuous, bounded and satisfies

$$\langle\, x(t) - z(t),\; v_j'\,\rangle \;=\; \int_0^t \langle\, x(s) - z(s),\;\; A'\,v_j'\,\rangle\;\, ds \;=\; \lambda_j \cdot \int_0^t \langle\, x(s) - z(s),\; v_j'\,\rangle\;\, ds.$$

So the integral version 1.5.4 of Gronwall's Lemma (applied to $|\langle\, x(\cdot) - z(\cdot),\; v_j'\,\rangle|$) leads to $\langle\, x(\cdot) - z(\cdot),\; v_j'\,\rangle \;\equiv\; 0$ for all $j \in \mathcal{J}$ and thus,

$$x(t) \;=\; z(t) \;\stackrel{\text{Def.}}{=}\; S(t)\,x_0 \;+\; \int_0^t S(t-s)\;\, f(x(s),s)\;\, ds. \qquad \square$$

4.6 Systems of evolution in $(\mathcal{K}(I\!R^N), q_{\mathcal{K},N})$ and a reflexive Banach space

The forward generalization of mutational equations can be applied to systems in a very easy way as presented in § 2.4. Now the last two sections § 4.4.4, § 4.5 provide the starting point for an example in $X \times \mathcal{K}(I\!R^N)$ (with a reflexive Banach space X). To cut a long story short, we consider a curve $[0,T[\longrightarrow X \times \mathcal{K}(I\!R^N),\ t \longmapsto (x(t), K(t))$ whose first component $x(t) \in X$ is a mild solution of a semilinear equation and whose second component $K(t)$ evolves along differential inclusions,

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &= A\, x(t) + f(x(t),\, K(t),\, t), \qquad x(0) = x_0, \\ \overset{\circ}{K}(\cdot) &\ni \qquad\qquad g(x(t),\, K(t),\, t), \qquad K(0) = K_0. \end{cases}$$

A main point here is that both deformations (i.e. the semilinear equation and the differential inclusions induced by g) depend on $x(t)$ and $K(t)$ (including its normal cones). So this type of evolution belongs to a generalized class of free boundary problems.

General assumptions for § 4.6.

1. X is a reflexive Banach space.
2. The linear operator A generates a C^0 semigroup $(S(t))_{t\geq 0}$ on X with $\|S(t)\|_{\mathcal{L}(X,X)} \leq \widehat{\eta} \cdot e^{\eta\, t}$ for all $t \geq 0$.
3. The dual operator A' of A has a countable family of eigenvectors $\{v'_j\}_{j\in\mathcal{J}}$ $(|v'_j|_{X'} = 1)$ spanning the dual space X', i.e. $X' = \overline{\sum_{j\in\mathcal{J}} I\!R\, v'_j}$.

   λ_j abbreviates the eigenvalue of A' belonging to the eigenvector v'_j and, $\mu_n := \max_{k=1\dots n} |\lambda_{j_k}|$ with $\mathcal{J} = \{j_1. j_2,\, j_3\, \dots\}$.
4. $q_j(x,y) := |\langle x-y,\ v'_j\rangle|$ for $x, y \in X,\ j \in \mathcal{J}$,

   $p_n(x,y) := \sum_{k=1}^{n} 2^{-k}\ \frac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)}$ for $x, y \in X,\ n \in I\!N \cup \{\infty\}$,

   $P_n(x,y) := \sum_{k=1}^{n} 2^{-k}\ q_{j_k}(x,y)$.
5. $\tau_v : [0,1]\times X \longrightarrow X, \quad (h,x) \longmapsto S(h)\, x + \int_0^h S(h-s)\ v\ ds \ \ (v \in X)$.
6. For $\Lambda, \rho > 0$ fixed, each $F \in \mathrm{LIP}_\Lambda^{(\mathcal{H}_\circ^\rho)}(I\!R^N, I\!R^N)$ (see Def. 4.4.27) induces a forward transition ϑ_F (of order 0) on $(\mathcal{K}(I\!R^N),\, \mathcal{K}_{C^{1,1}}(I\!R^N),\, q_{\mathcal{K},N})$ by means of reachable sets according to Proposition 4.4.26.

Proposition 4.6.1

In addition to the general assumptions about X, A, $S(\cdot)$ of § 4.6, suppose for

$$f: \quad X \times \mathcal{K}(I\!\!R^N) \times [0,T] \longrightarrow X$$
$$g: \quad X \times \mathcal{K}(I\!\!R^N) \times [0,T] \longrightarrow \mathrm{LIP}_\Lambda^{(\mathcal{H}^\varrho_\circ)}(I\!\!R^N, I\!\!R^N) \ :$$

1. $\|f\|_{L^\infty} < \infty$
2. $P_\infty\Big(f(x_1,K_1,t_1), \ \ f(x_2,K_2,t_2)\Big) \leq \omega\Big(p_\infty(x_1,x_2) + q_{\mathcal{K},N}(K_1,K_2) + t_2 - t_1\Big)$
3. $\Big\|\mathcal{H}_{g(x_1,K_1,t_1)} - \mathcal{H}_{g(x_2,K_2,t_2)}\Big\|_{C^1(I\!\!R^N \times \partial I\!\!B_1)} \leq \omega\Big(p_\infty(x_1,x_2) + q_{\mathcal{K},N}(K_1,K_2) + t_2 - t_1\Big)$

for all $x_1, x_2 \in X$, $K_1, K_2 \in \mathcal{K}(I\!\!R^N)$, $0 \leq t_1 \leq t_2 \leq T$ *with a modulus* $\omega(\cdot)$ *of continuity.*

Then for every $x_0 \in X$ *and* $K_0 \in \mathcal{K}(I\!\!R^N)$, *there exists a right–hand forward solution* $(x, K) : [0,T[\longrightarrow X \times \mathcal{K}(I\!\!R^N)$ *of the generalized mutational equations*

$$\wedge \begin{cases} \overset{\circ}{x}(\cdot) & \ni \ \tau_{f(x(\cdot),\, K(\cdot),\, \cdot)} \\ \overset{\circ}{K}(\cdot) & \ni \ \vartheta_{g(x(\cdot),\, K(\cdot),\, \cdot)} \end{cases}$$

with $x(0) = x_0$, $K(0) = K_0$ *and, it fulfills*

a) $x : [0,T[\longrightarrow X$ *is a mild solution of the initial value problem*

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &= A\, x(t) + f(x(t),\, K(t),\, t) \\ x(0) &= x_0 \end{cases}$$

i.e. $$x(t) = S(t)\, x_0 + \int_0^t S(t-s)\ f(x(s),\, K(s),\, s)\ ds.$$

b) $K(\cdot) \in \mathrm{Lip}^{\to}([0,T[,\ \mathcal{K}(I\!\!R^N),\ q_{\mathcal{K},N})$, *i.e.*
$q_{\mathcal{K},N}(K(s),\, K(t)) \leq$ *const*$(\Lambda, T) \cdot (t-s)$ *for all* $0 \leq s < t < T$.

c) $\limsup_{h \downarrow 0} \ \frac{1}{h} \cdot \Big(q_{\mathcal{K},N}\Big(\vartheta_{g(x(t),\, K(t),\, t)}\,(h,\ M),\ \ K(t+h)\Big) - q_{\mathcal{K},N}(M,\ K(t)) \cdot e^{10\,\Lambda\, t}\Big) \leq 0$
for every $M \in \mathcal{K}_{C^{1,1}}(I\!\!R^N)$, $t \in [0,T[$.

Proof results from Prop. 2.4.6 about timed right–hand forward solutions of systems.

To be more precise, Lemma 4.5.6 and the assumption $\|f\|_{L^\infty} < \infty$ here provide a uniform upper bound for all Euler approximations (with respect to the norm $|\cdot|_X$). So Lemma 4.5.9 ensures that all transitions (of order 0) on $(X, X, (p_n)_{n \in I\!\!N})$ induced by f fulfill the conditions of Proposition 2.4.6.
In regard to the forward transitions on $(\mathcal{K}(I\!\!R^N),\ \mathcal{K}_{C^{1,1}}(I\!\!R^N),\ q_{\mathcal{K},N})$ induced by g, the missing conditions (of Prop. 2.4.6) result directly from Propositions 4.4.26, 4.4.28.

So we obtain a right–hand forward solution $(x, K) : [0, T[\longrightarrow X \times \mathcal{K}(I\!R^N)$ of the two generalized mutational equations mentioned above with $x(0) = x_0$, $K(0) = K_0$. In particular, claim (b), i.e. $K(\cdot) \in \text{Lip}^{\rightarrow}([0, T[, \mathcal{K}(I\!R^N), q_{\mathcal{K},N})$, results from the uniform Lipschitz continuity of its Euler approximations (according to Lemma 4.4.23) and the details proving Convergence Theorem 2.4.5 for systems.

According to Lemma 4.5.12 (quoted from [8, Ball 77]), the initial value problem

$$\wedge \begin{cases} \frac{d}{dt}\, z(t) &= A\, z(t) \;+\; f(x(t),\ K(t),\ t) \\ z(0) &= x_0 \end{cases}$$

has a unique mild solution $z(\cdot)$ and, we conclude $x(\cdot) = z(\cdot)$ from Lemma 4.5.11 in exactly the same way as we did for proving Proposition 4.5.10. □

Appendix A

Tools of differential inclusions

This appendix provides a collection of properties for the reachable sets of differential inclusions giving a quite general example of shape evolution. In particular, we use adjoint arcs for describing the time–dependent limiting normal cones and find sufficient conditions for preserving smooth boundaries (for a short time at least).
These results mainly form the basis for constructing forward and backward transitions on $\mathcal{K}(I\!R^N)$, $\Omega(I\!R^N)$ in § 4.4.

A.1 Filippov's Theorem for differential inclusions

Following the well–known convention, we define the solutions of a differential inclusion in the sense of Carathéodory as it is described e.g. in [7, Aubin,Frankowska 90]. The Theorem of Filippov represents the counterpart for the Theorem of Cauchy–Lipschitz about ordinary differential equations.

Definition A.1.1 *Let* $\widetilde{F} : [0,T] \times I\!R^N \rightsquigarrow I\!R^N$ *be a set–valued map.*
A function $x : [0,T] \longrightarrow I\!R^N$ *is called* trajectory *or* solution of the differential inclusion $\dot{x}(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ *a.e. if* $x(\cdot)$ *is absolutely continuous and its (weak) derivative* $\dot{x}(\cdot)$ *satisfies* $\dot{x}(t) \in \widetilde{F}(t, x(t))$ *for almost every* $t \in [0,T]$.

The reachable set *of* $\widetilde{F}$ *and a nonempty initial set* $M \subset I\!R^N$ *at time* $t \in [0,T]$ *contains the points* $x(t)$ *of all trajectories* $x(\cdot)$ *starting in* M, *i.e.*

$$\vartheta_{\widetilde{F}}(t, M) \;:=\; \Big\{ x(t) \in I\!R^N \;\Big|\; \begin{array}{l} x(\cdot) \in AC([0,t],\, I\!R^N), \quad x(0) \in M, \\ \dot{x}(\cdot) \in \widetilde{F}(\cdot, x(\cdot)) \;\; \text{almost everywhere in } [0,t] \end{array} \Big\}. \qquad \square$$

Proposition A.1.2 (Generalized Theorem of Filippov)
Let $\mathcal{O}$ be a relatively open subset of $[0,T] \times I\!R^N$. Take a set–valued map $\widetilde{F} : \mathcal{O} \leadsto I\!R^N$, an arc $y(\cdot) \in AC([0,T], I\!R^N)$, a point $\eta \in I\!R^N$ and $\delta \in \,]0,\infty]$ such that

$$\mathcal{N}(y,\delta) \;:=\; \bigcup_{0 \le t \le T} \{t\} \times I\!B_\delta(y(t)) \;\subset\; \mathcal{O}.$$

Assume that

(i) $\widetilde{F}(t,z) \neq \emptyset$ *is closed for every* $(t,z) \in \mathcal{N}(y,\delta)$ *and*
Graph $\widetilde{F}$ *is* $\mathcal{L}^1 \times \mathcal{B}^N$ *measurable,*

(ii) *there exists* $k(\cdot) \in L^1([0,T])$ *such that* $\widetilde{F}(t,z_1) \subset \widetilde{F}(t,z_2) + k(t)\,|z_1 - z_2| \cdot I\!B_1$ *for all* $z_1, z_2 \in I\!B_\delta(y(t))$ *and almost every* $t \in [0,T]$.

Suppose further

$$e^{\|k\|_{L^1}} \cdot \Big(|\eta - y(0)| \;+\; \int_0^T \operatorname{dist}\Big(\dot{y}(t),\; \widetilde{F}(t,y(t))\Big)\, dt\Big) \;\le\; \delta.$$

Then there exists a trajectory $x(\cdot) \in AC([0,T], I\!R^N)$ *of* $\dot{x}(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ *a.e. satisfying* $x(0) = \eta$ *and*

$$\|x - y\|_{L^\infty} \;\le\; |\eta - y(0)|\; e^{\|k\|_{L^1}} \;+\; \int_0^T e^{\int_t^T k(s)\,ds}\; \operatorname{dist}\Big(\dot{y}(t),\; \widetilde{F}(t,y(t))\Big)\; dt$$

Now assume that (i) *and* (ii) *are replaced by the stronger hypotheses*

(i′) $\widetilde{F}(t,z) \neq \emptyset$ *is convex and compact for every* $(t,z) \in \mathcal{N}(y,\delta)$,

(ii′) *there exist* $\omega(\cdot) : [0,\infty[\longrightarrow [0,\infty[$ *and* $k_\infty \in \,]0,\infty[$ *such that* $\lim_{h \downarrow 0} \omega(h) = 0$,
$\widetilde{F}(t_1,z_1) \;\subset\; \widetilde{F}(t_2,z_2) \;+\; \Big(k_\infty\,|z_1 - z_2| \;+\; \omega(|t_1 - t_2|)\Big)\; I\!B_1$
for all $(t_1,z_1),\ (t_2,z_2) \in \mathcal{N}(y,\delta)$.

If $y(\cdot)$ *is continuously differentiable, then the trajectory* $x(\cdot)$ *can be chosen as a continuously differentiable function too.*

Proof is given in [63, Vinter 2000], Theorem 2.4.3, for example. □

A.2 The boundary of open reachable sets

Now we consider the relationship between the reachable sets of a bounded open set $O \subset I\!R^N$ and its closure $\overline{O} \in \mathcal{K}(I\!R^N)$ – with respect to their topological boundaries, in particular.

If $F : I\!R^N \rightsquigarrow I\!R^N$ is Lipschitz continuous and its compact convex values have nonempty interior, then roughly speaking, the differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ (a.e.) has a smoothing effect on its reachable sets. As a consequence, the boundary of $\vartheta_F(t, O)$ is the upper limit of $\partial\,\vartheta_F(t, \overline{O})$ in positive time direction, i.e.

$$\partial\,\vartheta_F(t, O) \;=\; \mathrm{Limsup}_{s \uparrow t}\;\partial\,\vartheta_F(t, \overline{O})$$

according to the following Prop. A.2.8. The proof is based on a smooth selection of $I\!R^N \rightsquigarrow I\!R^N,\;\; x \longmapsto F(x)^\circ$ whose existence is shown by means of the Steiner point in Prop. A.2.6. This relationship implies that in the product $]0, \infty[\times I\!R^N$, the boundary of $\mathrm{Graph}\;\vartheta_F(\cdot, \overline{O})$ is the graph of the boundary $\partial\,\vartheta_F(\cdot, O)$ (Cor. A.2.9).

Finally we mention a similar result for compact initial sets $K \subset I\!R^N$: Assuming that each $\vartheta_F(t, K)$ $(t \geq 0)$ has uniform positive reach of radius ρ, Prop. A.2.10 states

$$\partial\,\mathrm{Graph}\;\vartheta_F(t, K)\;\cap\;(]0, \infty[\,\times I\!R^N) \;=\; \mathrm{Graph}\;\;\partial\,\vartheta_F(t, K)|_{]0,\infty[}$$

As a useful consequence, we obtain $\partial\,\vartheta_F(t, O) = \partial\,\vartheta_F(t, \overline{O})$ if the boundaries are sufficiently smooth. This result provides a connection between smooth open and closed reachable sets and will be used in § A.4 and § A.5 for extending results about compact sets to their open counterparts.

As a well–known tool for selections of convex sets, we use the so–called *Steiner point* that is already introduced and investigated e.g. in [7, Aubin,Frankowska 90], § 9.4.1.

Definition A.2.1 *For a convex subset $C \in \mathcal{K}(I\!R^N)$, the point*

$$s_N(C) \;:=\; \tfrac{N}{\mathcal{L}^{N-1}(\partial I\!B_1)} \int_{\partial I\!B_1} p\;\;\sigma_C(p)\;\;d\omega_p$$

is called Steiner point *or* curvature centroid *or* Krümmungsschwerpunkt *of C.*
The support function *of C is defined as* $\quad \sigma_C : I\!R^N \longrightarrow I\!R,\;\; p \longmapsto \sup\limits_{y \in C}\; p \cdot y.$ □

Lemma A.2.2 *For all convex sets $C, C_1, C_2 \in \mathcal{K}(I\!R^N)$ and parameters $\lambda, \mu \in I\!R$,*

$$\begin{aligned} s_N(C) &\in C, \\ s_N(\lambda\, C_1 + \mu\, C_2) &= \lambda\, s_N(C_1) \;+\; \mu\, s_N(C_2) \\ s_N(C) &= \tfrac{1}{\mathcal{L}^N(I\!B_1)} \int_{I\!B_1} m(\partial^L \sigma_C(p))\;\;dp \\ |s_N(C_1) \,-\, s_N(C_2)| &\leq N \,\cdot\, d\!\!l(C_1, C_2). \end{aligned}$$

with $m(A)$ abbreviating the minimal selection of a nonempty closed set $A \subset I\!R^N$. □

Lemma A.2.3 *For every convex compact set $C \subset I\!R^N$ with nonempty interior, the Steiner point $s_N(C)$ is contained in the interior of C.*

Proof. Assume that $s := s_N(C)$ is contained in the topological boundary of C and choose a unit vector $p \in N_C(s)$. Then $(C - s) \cdot p \subset]-\infty, 0]$ implies $\sigma_C(p) = s \cdot p$. The assumption $C^\circ \neq \emptyset$ leads to a closed ball $B \subset I\!R^N$ of radius $r > 0$ with $B \subset C^\circ$. So $\sigma_C(-p) \geq s \cdot (-p) + 2r$ and due to the continuity of $\sigma_C(\cdot)$ in $I\!R^N \setminus \{0\}$, there is a neighborhood U of $-p$ with $\sigma_C(q) \geq s \cdot q + r$ for all $q \in U$.
Now Lemma A.2.4 (1.) implies $\partial^L \sigma_C(q) \cdot q \geq s \cdot q + r$ for any $q \in U$
and thus, all q close to $-p$ satisfy $\partial^L \sigma_C(q) \cdot (-p) \geq s \cdot (-p) + \frac{r}{2}$
Finally, we get $s \cdot p = \frac{1}{\mathcal{L}^N(I\!B_1)} \int_{I\!B_1} m(\partial^L \sigma_C(q)) \cdot p \; dq \leq s \cdot p - \frac{r}{2}$ — a contradiction. □

Lemma A.2.4
The support functions of any nonempty compact convex sets $C, D \subset I\!R^N$ satisfies

1. $z \in \partial^L \sigma_C(p) \iff p \in N_C(z) = N_C^P(z) \iff z \in C, \; p \cdot z = \sigma_C(p)$
2. $e^{\complement}(C, D) = \sup_{|p| \leq 1} \sigma_C(p) - \sigma_D(p) = \sup_{|p| = 1} \sigma_C(p) - \sigma_D(p).$

Proof in [55, Rockafellar, Wets 98], Example 11.4 and [2, Aubin 99], Prop. 3.2.8. □

Lemma A.2.5 *If $F : I\!R^M \rightsquigarrow I\!R^N$ is (locally) Lipschitz continuous with nonempty compact convex values, then the topological boundary $I\!R^M \rightsquigarrow I\!R^N$, $x \longmapsto \partial F(x)$ is also (locally) Lipschitz continuous.*
Moreover $F(\cdot)$ and $\partial F(\cdot)$ have the (local) Lipschitz constants in common.

Proof. Lemma A.2.4 (2.) implies the local Lipschitz continuity of $I\!R^M \longrightarrow I\!R$, $x \longmapsto \sigma_{F(x)}(p)$ (with the same Lipschitz constant as F) for every $p \in \partial I\!B_1 \subset I\!R^N$.
Choose $R > 1$, $x_1, x_2 \in I\!B_R \subset I\!R^M$ and a boundary point $y_1 \in \partial F(x_1)$ arbitrarily. Furthermore set $\lambda_R := \text{Lip}\, F|_{R I\!B}$ and let $p \in I\!R^N$ satisfy $p \cdot y_1 = \sigma_{F(x_1)}(p)$ and $|p| = 1$. Then $\sigma_{F(x_2)}(p) \leq \sigma_{F(x_1)}(p) + \lambda_R \, |x_1 - x_2|$ implies for every $\delta > 0$

$$y_1 + \Big(\lambda_R \, |x_1 - x_2| + \delta\Big) \cdot p \;\notin\; F(x_2).$$

Moreover the Lipschitz continuity of $F|_{R I\!B}$ leads to an element $y_2 \in F(x_2)$ satisfying $|y_2 - y_1| \leq \lambda_R \, |x_1 - x_2|$. So the convex hull of $y_1 + (\lambda_R \, |x_1 - x_2| + \delta) \cdot p$ and y_2 contains a boundary point $y_2' \in \partial F(x_2)$ with $|y_2' - y_1| \leq \lambda_R \, |x_1 - x_2| + \delta$. □

Proposition A.2.6 *Let $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ be a Lipschitz continuous map whose values are compact and convex with nonempty interior.*
Then there exists a selection $f \in C^\infty(I\!\!R^N, I\!\!R^N)$ of $I\!\!R^N \rightsquigarrow I\!\!R^N, \quad x \longmapsto F(x)^\circ$.

Proof. We show the existence only for the restriction to any set $K \in \mathcal{K}(I\!\!R^N)$. Then after covering $I\!\!R^N$ with compact sets, a locally finite smooth partition of unit provides the extension to $I\!\!R^N$ because each open set $F(x)^\circ,\ x \in I\!\!R^N$, is convex.

The Steiner point induces a Lipschitz selection of $F(\cdot)^\circ$ due to Lemmas A.2.2, A.2.3. So for every $K \in \mathcal{K}(I\!\!R^N)$, the infimum $\varepsilon = \varepsilon(F, K) := \frac{1}{2} \inf\limits_{I\!\!B_1(K)} \text{dist}(s_N(F(\cdot)),\ \partial\, F(\cdot))$ is positive because Lemma A.2.5 guarantees the continuity of this distance function, i.e.

$$I\!\!B_{2\,\varepsilon}\,(s_N(F(x))) \;\subset\; F(x) \qquad\qquad \text{for all } x \in I\!\!B_1(K).$$

Now convolving $s_N(F(\cdot))$ with a smooth auxiliary function $\varphi \geq 0$ of sufficiently small compact support (and $\|\varphi\|_{L^1} = 1$), we obtain a smooth selection of $F(\cdot)^\circ|_K$.

□

After these technical preliminaries about a smooth selection of $F(\cdot)^\circ$, the next lemma provides the key conclusion from assuming that values of F with nonempty interior. It states that after arbitrarily short time, each interior point of the closure $\vartheta_F(t, \overline{O}) = \overline{\vartheta_F(t, O)}$ evolves like the points of the open set $\vartheta_F(t, O)$.

Lemma A.2.7 *Suppose for the Lipschitz continuous map $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ that all its values are compact convex and have nonempty interior. Furthermore let y be an interior point of the reachable set $\vartheta_F(t, \overline{O}) = \overline{\vartheta_F(t, O)}$ for some $t > 0$ and $O \in \Omega(I\!\!R^N)$.*
Then, $\quad \vartheta_F(h, y) \subset \vartheta_F(t + h, O) \quad$ for all $h > 0$.

Proof. There is some $\rho > 0$ with $I\!\!B_{4\,\rho}(y) \subset \vartheta_F(t, \overline{O})$ and Prop. A.2.6 provides a selection $f \in C^\infty(I\!\!R^N, I\!\!R^N)$ of $F(\cdot)^\circ$. Now we can choose small $h_0,\ \varepsilon > 0$ such that

$$\begin{array}{rcll} \vartheta_F(h,\ y) & \subset & I\!\!B_\rho(y) & \text{for all } h \in\ ]0, h_0], \\ \vartheta_{-f}(h,\ I\!\!B_\rho(y)) & \subset & I\!\!B_{2\,\rho}(y) & \text{for all } h \in\ ]0, h_0], \\ \vartheta_F(h,\ I\!\!B_{2\,\rho}(y)) & \subset & I\!\!B_{3\,\rho}(y) & \text{for all } h \in\ ]0, h_0], \\ I\!\!B_\varepsilon(f(z)) & \subset & F(z) & \text{for all } z \in I\!\!B_{4\,\rho}(y). \end{array}$$

The last 2 inclusions imply $\vartheta_f(h, z) + \varepsilon\, h \cdot I\!\!B \subset \vartheta_F(h, z)$ for any $z \in I\!\!B_{2\,\rho}(y),\ h \in]0, h_0]$.

Due to $\vartheta_F(t,\overline{O}) = \overline{\vartheta_F(t,O)}$, the open set $\vartheta_F(t,O) \cap \overset{\circ}{I\!B}_{2\rho}(y)$ is dense in the ball $I\!B_{2\rho}(y) \subset \vartheta_F(t,\overline{O})$ and thus, we obtain

$$\begin{aligned}
\vartheta_F(h,y) &\subset \vartheta_f\Big(h,\ \vartheta_{-f}(h,\ I\!B_\rho(y))\Big) \\
&\subset \vartheta_f\Big(h,\ I\!B_{2\rho}(y)\Big) \\
&= \overline{\vartheta_f\Big(h,\ \vartheta_F(t,O) \cap \overset{\circ}{I\!B}_{2\rho}(y)\Big)} \\
&\subset \vartheta_f\Big(h,\ \vartheta_F(t,O) \cap \overset{\circ}{I\!B}_{2\rho}(y)\Big) + \varepsilon\, h \cdot I\!B \\
&\subset \vartheta_F\Big(h,\ \vartheta_F(t,O) \cap \overset{\circ}{I\!B}_{2\rho}(y)\Big) \\
&\subset \vartheta_F(t+h,\ O)
\end{aligned}$$

for all $h \in \,]0,h_0]$. For $h > h_0$, the semigroup property of reachable sets ensures

$$\vartheta_F(h,y) = \vartheta_F\Big(h-h_0,\ \vartheta_F(h_0,y)\Big) \subset \vartheta_F\Big(h-h_0,\ \vartheta_F(t+h_0,O)\Big) = \vartheta_F(t+h,\ O). \quad \square$$

So now we have the tools for proving a main result of this section :

Proposition A.2.8 *Assume for the Lipschitz continuous map $F : I\!R^N \leadsto I\!R^N$ that all its values are compact convex and have nonempty interior.*
For every initial set $O \in \Omega(I\!R^N)$ and time $t > 0$, the topological boundary of the reachable set fulfills

$$\partial\,\vartheta_F(t,O) = \mathrm{Limsup}_{s\uparrow t}\ \partial\,\vartheta_F(s,\overline{O}).$$

Proof. The inclusion $\mathrm{Limsup}_{s\uparrow t}\ \partial\,\vartheta_F(s,\overline{O}) \subset \partial\,\vartheta_F(t,O)$ results from Filippov's Theorem A.1.2 : For any time $t > 0$ and element $x \in \mathrm{Limsup}_{s\uparrow t}\ \partial\,\vartheta_F(s,\overline{O})$, consider sequences (s_n), (x_n) with $s_n \uparrow t$, $x_n \longrightarrow x$ and $x_n \in \partial\,\vartheta_F(s_n,\overline{O})$ for all $n \in I\!N$. Due to Filippov's Theorem, each $\vartheta_F(s_n,\overline{O})$ is closed and $\vartheta_F(t,\overline{O}) = \overline{\vartheta_F(t,O)}$. Thus, $x \in \mathrm{Limsup}_{n\to\infty}\,\vartheta_F(s_n,\overline{O}) \subset \vartheta_F(t,\overline{O}) \subset \overline{\vartheta_F(t,O)}$.
Moreover for each n, there is $z_n \in I\!B_{\frac{1}{n}}(x_n) \setminus \vartheta_F(s_n,\overline{O})$. In particular, $z_n \notin \vartheta_F(s_n,\overline{O})$ is equivalent to $\vartheta_{-F}(s_n,z_n) \subset I\!R^N \setminus \overline{O} \subset I\!R^N \setminus O$. As $I\!R^N \setminus O$ is closed, $n \longrightarrow \infty$ leads to $\vartheta_{-F}(t,x) \subset I\!R^N \setminus O$, i.e. $x \notin \vartheta_F(t,O)$.
So finally, $x \in \overline{\vartheta_F(t,O)} \setminus \vartheta_F(t,O) = \partial\,\vartheta_F(t,O)$ since $\vartheta_F(t,O)$ is open.

Now we prove the inclusion $\partial\,\vartheta_F(t,O) \subset \mathrm{Limsup}_{s\uparrow t}\ \partial\,\vartheta_F(s,\overline{O})$.
Each $x \in \partial\,\vartheta_F(t,O) \subset \vartheta_F(t,\overline{O})$ is attained by a trajectory $x(\cdot) \in AC([0,t], I\!R^N)$ of F with $x(0) \in \overline{O}$. If $x(s)$ was an interior point of $\vartheta_F(s,\overline{O})$ for some time $s \in [0,t[$, the preceding Lemma A.2.7 would imply $x = x(t) \in \vartheta_F(t-s,\ x(s)) \subset \vartheta_F(t,O)$ – a contradiction. So $x(s) \in \partial\,\vartheta_F(s,\overline{O})$ for each $s \in [0,t[$. $\square$

Corollary A.2.9 *Suppose* $F : I\!R^N \rightsquigarrow I\!R^N$ *to be a Lipschitz continuous map whose values are compact convex and have nonempty interior.*
Then for every $O \in \Omega(I\!R^N)$, *the graph of* $\vartheta_F(\cdot, \overline{O}) : [0, \infty[\rightsquigarrow I\!R^N$ *satisfies*

1. $$\big(\text{Graph}\, \vartheta_F(\,\cdot\,, \overline{O})\big)^\circ = \bigcup_{t>0} (\{t\} \times \vartheta_F(t, O))$$
2. $$\partial\ \text{Graph}\, \vartheta_F(\,\cdot\,, \overline{O}) = (\{0\} \times O) \cup \bigcup_{t\geq 0} (\{t\} \times \partial\, \vartheta_F(t, O))$$

Proof. 1. "$\supset$" Filippov's Theorem A.1.2 implies for every trajectory $x(\cdot) \in AC([0,t], I\!R^N)$ of F with $x(0) \in O$ that there is a radius $\rho > 0$ satisfying
$$\bigcup_{0\leq s\leq t} (\{s\} \times I\!B_\rho(x(s))) \subset \text{Graph}\, \vartheta_F(\,\cdot\,, O) \subset \text{Graph}\, \vartheta_F(\,\cdot\,, \overline{O}).$$
The Lipschitz continuity of F and Gronwall's Lemma guarantee that $x(\cdot)$ is Lipschitz continuous. So setting $\lambda_x := \text{Lip}\ x(\cdot)$, $r := \frac{\rho}{1+\lambda_x}$, we get for every $\tau \in]0, t[$
$$\begin{aligned} I\!B_r(\tau, x(\tau)) \cap ([0,t] \times I\!R^N) &\subset \bigcup_{\substack{s\in[0,t]\\ |s-\tau|\leq r}} \{s\} \times I\!B_r(x(\tau)) \\ &\subset \bigcup_{\substack{s\in[0,t]\\ |s-\tau|\leq r}} \{s\} \times I\!B_{r+\lambda_x\cdot|\tau-s|}(x(s)) \\ &\subset \bigcup_{s\in[0,t]} \{s\} \times I\!B_\rho(x(s)) \quad \subset \text{Graph}\, \vartheta_F(\,\cdot\,, \overline{O}), \end{aligned}$$
i.e. $(\tau,\ x(\tau))$ is an interior point of Graph $\vartheta_F(\,\cdot\,, \overline{O})$.

"$\subset$" Let (t, z) be any interior point of Graph $\vartheta_F(\,\cdot\,, \overline{O})$. Then, $t > 0$ and $z \notin \text{Limsup}_{s\uparrow t}\ \partial\, \vartheta_F(s, \overline{O})$ because each set $\{s\} \times \partial\, \vartheta_F(s, \overline{O})$ is contained in the boundary of Graph $\vartheta_F(\,\cdot\,, \overline{O})$. So we conclude from Prop. A.2.8
$$\begin{aligned} z &\in \vartheta_F(t, \overline{O}) \setminus \text{Limsup}_{s\uparrow t}\ \partial\, \vartheta_F(s, \overline{O}) \\ &= \overline{\vartheta_F(t, O)} \setminus \partial\, \vartheta_F(t, O) \\ &= \vartheta_F(t, O). \end{aligned}$$

2. results from statement (1.) :
$$\begin{aligned} \partial\ \text{Graph}\, \vartheta_F(\,\cdot\,, \overline{O}) &= \text{Graph}\, \vartheta_F(\,\cdot\,, \overline{O}) \setminus \big(\text{Graph}\, \vartheta_F(\,\cdot\,, \overline{O})\big)^\circ \\ &= \bigcup_{t\geq 0} (\{t\} \times \vartheta_F(t, \overline{O})) \setminus \bigcup_{t>0} (\{t\} \times \vartheta_F(t, O)) \\ &= (\{0\} \times \overline{O}) \cup \bigcup_{t>0} \Big(\{t\} \times (\vartheta_F(t, \overline{O}) \setminus \vartheta_F(t, O))\Big) \\ &= (\{0\} \times \overline{O}) \cup \bigcup_{t>0} \Big(\{t\} \times (\overline{\vartheta_F(t, O)} \setminus \vartheta_F(t, O))\Big) \\ &= (\{0\} \times \overline{O}) \cup \bigcup_{t>0} \Big(\{t\} \times \partial\, \vartheta_F(t, O)\Big) \end{aligned}$$
□

A relationship between boundaries similar to Cor. A.2.9 (2.) holds also for compact initial sets K. The key assumption here is that the reachable sets $\vartheta_F(t, K)$ (for every t) are proximally smooth of a uniform radius ρ. Roughly speaking, it guarantees that no converging sequence $(x_n)_{n \in I\!N}$ of boundary points (i.e. $x_n \in \partial\, \vartheta_F(t_n, K)$ with $t_n \uparrow t$) can lead to an interior point of $\vartheta_F(t, K)$. This statement is not correct in general without supposing the radius ρ to be uniform as Example A.2.11 shows easily.

Proposition A.2.10 *Suppose for* $F : I\!R^N \leadsto I\!R^N$, $K \in \mathcal{K}(I\!R^N)$ *and* $\rho > 0$ *that the map* $[0, T] \leadsto I\!R^N$, $t \longmapsto \vartheta_F(t, K)$ *is* λ*–Lipschitz continuous (with respect to* $d\!l$*) and each set* $\vartheta_F(t, K)$ $(0 \le t \le T)$ *has positive reach of radius* ρ*.*
Then the topological boundary of $\text{Graph } \vartheta_F(\cdot, K)|_{[0,T]}$ *in* $I\!R \times I\!R^N$ *is*

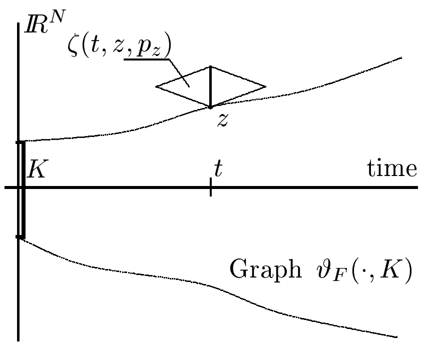
$$\{0\} \times K \;\cup\; \bigcup_{0<t<T} \{t\} \times \partial\vartheta_F(t, K) \;\cup\; \{T\} \times \vartheta_F(T, K).$$

Proof. The inclusion

$$\{0\} \times K \;\cup\; \bigcup_{0<t<T} \{t\} \times \partial\vartheta_F(t, K) \;\cup\; \{T\} \times \vartheta_F(T, K) \;\subset\; \partial\,\text{Graph } \vartheta_F(\cdot, K)|_{[0,T]}$$

is obvious. Due to the Lipschitz continuity of $\vartheta_F(\cdot, K)$, we only have to show

$$\partial\,\text{Graph } \vartheta_F(\cdot, K) \;\cap\; (]0, T[\times I\!R^N) \;\subset\; \bigcup_{0<t<T} \{t\} \times \partial\,\vartheta_F(t, K).$$



Every point $z \in \partial\,\vartheta_F(t, K)$ $(0 \le t \le T)$ and any unit vector $p_z \in N^P_{\vartheta_F(t,K)}(z) = N_{\vartheta_F(t,K)}(z)$ satisfy $\overset{\circ}{I\!B}_\rho(z + \rho\, p_z) \cap \vartheta_F(t, K) = \emptyset$ and thus, $\Big(\{t\} \times \overset{\circ}{I\!B}_\rho(z + \rho\, p_z)\Big) \cap \text{Graph } \vartheta_F(\cdot, K) = \emptyset.$ The λ–Lipschitz continuity of $\vartheta_F(\cdot, K)$ implies $\zeta(t, z, p_z) \cap \text{Graph } \vartheta_F(\cdot, K) = \emptyset$ for

$$\zeta(t, z, p_z) := \Big\{ (s, y) \in I\!R \times I\!R^N \;\Big|\; |z + \rho\, p_z - y| < \rho - \lambda\, |s - t| \Big\} \in \Omega(I\!R \times I\!R^N).$$

Now choose $(t, x) \in \partial\,\text{Graph } \vartheta_F(\cdot, K)$ with $0 < t < T$ arbitrarily. The continuity of $\vartheta_F(\cdot, K)$ guarantees that $\text{Graph } \vartheta_F(\cdot, K)$ is closed and thus, it contains (t, x). Moreover there are sequences $(t_n)_{n \in I\!N}$, $(x_n)_{n \in I\!N}$ in $]0, T[$, $I\!R^N$, respectively, satisfying $(t_n, x_n) \notin \text{Graph } \vartheta_F(\cdot, K)$ for every $n \in I\!N$ and $(t_n, x_n) \longrightarrow (t, x)$ $(n \longrightarrow \infty)$. For each $n \in I\!N$, let z_n be an element of the projection $\Pi_{\vartheta_F(t_n,K)}(x_n) \subset \partial\vartheta_F(t_n, K)$. Then, $0 < |x_n - z_n| = \text{dist}(x_n, \vartheta_F(t_n, K)) \le |x_n - x| + \text{dist}(x, \vartheta_F(t_n, K)) \longrightarrow 0$ and $p_n := \frac{x_n - z_n}{|x_n - z_n|} \in N^P_{\vartheta_F(t_n,K)}(z_n) \cap \partial I\!B_1.$

As mentioned before, we obtain for each $n \in I\!N$

$$\zeta(t_n, z_n, p_n) \cap \text{Graph}\, \vartheta_F(\cdot, K) = \emptyset.$$

Considering adequate subsequences (again denoted by) $(t_n)_{n \in I\!N}$, $(x_n)_{n \in I\!N}$, $(p_n)_{n \in I\!N}$ leads to the additional convergence $p_n \longrightarrow p \in \partial I\!B_1$ $(n \longrightarrow \infty)$. So finally

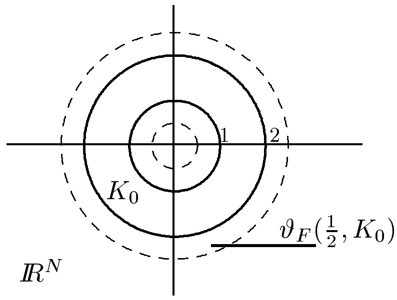
$$\zeta(t, x, p) \cap \text{Graph}\, \vartheta_F(\cdot, K) = \emptyset$$

because $\vartheta_F(\cdot, K)$ is continuous and $\overline{\zeta(t, x, p)} = \text{Lim}_{n \to \infty}\, \zeta(t_n, z_n, p_n)$.
In particular, $\overset{\circ}{I\!B}_\rho(x + \rho p) \cap \vartheta_F(t, K) = \emptyset$ implies $x \in \partial \vartheta_F(t, K)$. □

Example A.2.11 shows that Prop. A.2.10 is not correct if we do not assume a *uniform* radius ρ of proximal smoothness. This deformation has already been mentioned in the beginning of this chapter.
Set $F(x) := I\!B_1$ for all $x \in I\!R^N$ and the compact initial set $K_0 := I\!B_2 \setminus \overset{\circ}{I\!B}_1$. Then,

$$\vartheta_F(t, K_0) = I\!B_{t+2} \setminus \overset{\circ}{I\!B}_{1-t} \quad \text{for } 0 \le t < 1,$$
$$\vartheta_F(t, K_0) = I\!B_{t+2} \quad \text{for } t \ge 1.$$

So all reachable sets are proximally smooth. and $(1, 0)$ is contained in the boundary of Graph $\vartheta_F(\cdot, K)$, but $0 \notin \partial \vartheta_F(1, K)$.



A.3 Adjoint arcs for the evolution of limiting normal cones

The following extended Hamilton condition (Prop. A.3.1) is the key tool for describing the evolution of limiting normal cones along differential inclusions. Strictly speaking, it states a necessary condition for minimizers of an optimal control problem (with a differential inclusion $\dot{x}(\cdot) \in \widetilde{F}(\cdot, x(\cdot))$ a.e.). With respect to normal cones of reachable sets, it guarantees the existence of an adjoint arc $p(\cdot)$ that, roughly speaking, follows the normal direction along a trajectory $x(\cdot)$ (see Prop. A.3.2). The limiting subdifferential of the *Hamiltonian*

$$\mathcal{H}_{\widetilde{F}}(t, x, p) := \sigma_{\widetilde{F}(t,x)}(p) = \sup\, p \cdot \widetilde{F}(t, x)$$

provides a differential inclusion for $(x(\cdot), p(\cdot))$.
This result proves to be particularly useful if the Hamiltonian is continuously differentiable – in the case of a vector field $f \in C^1(I\!R^N, I\!R^N)$, for example.

Proposition A.3.1 (Extended Hamilton Condition)

Let $x(\cdot) \in AC([S,T], I\!R^N)$ *be a local minimizer (with respect to perturbations in* $AC([0,T], I\!R^N)$ *) of the problem*

$$\begin{aligned} & g(y(S), y(T)) \longrightarrow \min \\ & \textit{over} \;\; y(\cdot) \in AC([S,T], I\!R^N) \;\; \textit{satisfying} \\ & \qquad \dot y(t) \in \widetilde F(t, y(t)) \qquad \textit{for almost every}\, t \in [S,T], \\ & \qquad (y(S),\, y(T)) \in C \subset I\!R^N \times I\!R^N. \end{aligned}$$

Assume also that

(G1) *g is locally Lipschitz continuous;*

(G2)′ $\widetilde F(t,x) \neq \emptyset$ *is convex for each* (t,x), $\widetilde F$ *is* $\mathcal{L} \times \mathcal{B}$ *measurable, and* Graph $\widetilde F(t,\,\cdot\,)$ *is closed for each* $t \in [S,T]$.

Suppose, furthermore, that either of the following hypotheses is satisfied :

(a) *There exist* $k \in L^1([S,T])$ *and* $\varepsilon > 0$ *such that for almost every t*
$$\widetilde F(t,x_1) \cap \Big(\dot x(t) + \varepsilon\, k(t)\, I\!B\Big) \subset \widetilde F(t,x_2) + k(t)\, |x_1 - x_2|\, I\!B$$
for all $x_1, x_2 \in I\!B_\varepsilon(x(t))$.

(b) *There exist* $k \in L^1([S,T])$, $\overline K > 0$, *and* $\varepsilon > 0$ *such that the following two conditions are satisfied for almost every* $t \in [S,T]$ *and all* $x_1, x_2 \in I\!B_\varepsilon(x(t))$
$$\widetilde F(t,x_1) \cap \Big(\dot x(t) + \varepsilon\, I\!B\Big) \subset \widetilde F(t,x_2) + k(t)\, |x_1 - x_2|\, I\!B,$$
$$\inf \Big\{ |v - \dot x(t)| \;\Big|\; v \in \widetilde F(t,x_1) \Big\} \le \overline K\, |x_1 - x(t)|\,.$$

Then there exist an arc $p(\cdot) \in AC([S,T], I\!R^N)$ *and a constant* $\lambda \ge 0$ *such that*

(i) $(p(\cdot),\, \lambda) \neq (0,0)$,

(ii) $\dot p(t) \in co \Big\{ \eta \in I\!R^N \;\Big|\; (\eta, p(t)) \in N_{\text{Graph}\, \widetilde F(t,\,\cdot\,)}(x(t),\, \dot x(t)) \Big\}$ *for almost every t*

(iii) $\Big(p(S),\, -p(T)\Big) \in \lambda\, \partial^L g(x(S),\, x(T)) + N_C(x(S),\, x(T))$.

Condition (ii) implies

(iv) $p(t) \cdot \dot x(t) = \sup \Big(p(t) \cdot \widetilde F(t, x(t))\Big)$ *for almost every t*

(v) $\dot p(t) \in co \Big\{ -q \in I\!R^N \;\Big|\; (q, \dot x(t)) \in \partial^L\, \mathcal{H}_{\widetilde F}(t,\cdot,\cdot)|_{(x(t),p(t))} \Big\}$ *for almost every t.*

Proof is presented in [63, Vinter 2000], Theorem 7.7.1, for example. □

Remark. This adjoint $p(\cdot)$ also satisfies $|\dot p(t)| \le k(t)\, |p(t)|$ for almost every t as an immediate consequence of statement (ii) and the so–called *Mordukhovich criterion* (see e.g. [55, Rockafellar, Wets 98], Theorem 9.40). □

Now we use the adjoint arcs for the evolution of proximal and limiting normal cones :

Proposition A.3.2 *Let $\widetilde{F} : I\!R \times I\!R^N \rightsquigarrow I\!R^N$ be measurable and strict with compact convex values, $K \in \mathcal{K}(I\!R^N)$ and $t_0 > 0$. Moreover assume that $\widetilde{F}(t, \cdot) : I\!R^N \rightsquigarrow I\!R^N$ is $k(t)$–Lipschitz continuous for almost every t with some $k(\cdot) \in L^1(I\!R)$.*

1. Suppose for the trajectory $x(\cdot) \in AC([0, t_0], I\!R^N)$ of $\dot{x}(\cdot) \in \widetilde{F}(\cdot\,,\, x(\cdot))$ that $x(0) \in K$, $x_0 := x(t_0) \in \partial\, \vartheta_{\widetilde{F}}(t_0, K)$.
Then for any $\nu \in N^P_{\vartheta_{\widetilde{F}}(t_0,K)}(x_0)$, there is an adjoint $p(\cdot) \in AC([0, t_0], I\!R^N)$ satisfying

$$\begin{array}{rcll}
\dot{p}(t) & \in & \overline{co} \left\{ \; q \;\middle|\; (q, p(t)) \in N_{Graph\, \widetilde{F}(t,\cdot)}(x(t),\, \dot{x}(t)) \right\} & \text{for almost every } t,\\
\dot{p}(t) & \in & \overline{co} \left\{ -\, q \;\middle|\; (q, \dot{x}(t)) \in \partial^L\, \mathcal{H}_{\widetilde{F}}(t, \cdot, \cdot)\,|_{(x(t),p(t))} \right\} & \text{for almost every } t,\\
p(t) \cdot \dot{x}(t) & = & \max\ p(t) \cdot \widetilde{F}(t, x(t)) & \text{for almost every } t,\\
|\dot{p}(t)| & \le & k(t)\ |p(t)| & \text{for almost every } t,\\
p(0) & \in & N_K(x(0)), & \\
p(t_0) & = & \nu. &
\end{array}$$

2. For every $x_0 \in \partial\vartheta_{\widetilde{F}}(t_0, K)$ and $\nu \in N_{\vartheta_{\widetilde{F}}(t_0,K)}(x_0)$, there exist a trajectory $x(\cdot) \in AC([0, t_0], I\!R^N)$ of $\dot{x}(\cdot) \in \widetilde{F}(\cdot\,,\, x(\cdot))$ and an adjoint $p(\cdot) \in AC([0, t_0], I\!R^N)$ satisfying the preceding properties and $x(0) \in K$, $x(t_0) = x_0$.

Proof 1. For $\rho > 0$ small enough, we conclude $I\!B_\rho(x_0 + \rho\,\nu) \cap \vartheta_{\widetilde{F}}(t_0, K) = \{x_0\}$ from the assumption $\nu \in N^P_{\vartheta_{\widetilde{F}}(t_0,K)}(x_0)$. Then $x(\cdot)$ is a minimizer of the problem

$$\begin{array}{l}
g(y(0), y(t_0)) \;:=\; \frac{1}{2} \left| y(t_0) - \Big(x_0 + \rho\,\nu\Big) \right|^2 \;\longrightarrow\; \min\\
\text{over} \quad y(\cdot) \in AC([0, t_0], I\!R^N) \ \text{satisfying}\\
\qquad \dot{y}(t) \in \widetilde{F}(t, y(t)) \qquad \text{for almost every } t \in [0, t_0],\\
\qquad (y(0),\, y(t_0)) \,\in\, K \times I\!R^N.
\end{array}$$

So due to Prop. A.3.1, there are an adjoint $q(\cdot) \in AC([0, t_0], I\!R^N)$ and $\lambda \ge 0$ such that

$$\begin{array}{rl}
(i) & (q(\cdot),\, \lambda) \ne (0, 0),\\
(ii) & \dot{q}(t) \in \text{co} \left\{ \eta \in I\!R^N \;\middle|\; (\eta, q(t)) \in N_{\text{Graph}\, \widetilde{F}(t,\,\cdot\,)}(x(t),\, \dot{x}(t)) \right\} \quad \text{for almost every } t\\
(iii) & \Big(q(0),\, -q(t_0)\Big) \in \lambda\ \partial^L g(x(0),\, x(t_0)) \ + N_{K \times I\!R^N}(x(0),\, x(t_0)),\\
& \qquad = \lambda\ \Big(0, x(t_0) - x_0 - \rho\,\nu\Big) \ + N_K(x(0)) \times \{0\}\\
(iv) & q(t) \cdot \dot{x}(t) = \sup \Big(q(t) \cdot \widetilde{F}(t, x(t))\Big) \quad \text{for almost every } t\\
(v) & \dot{q}(t) \in \text{co} \left\{ -\,\xi \in I\!R^N \;\middle|\; (\xi, \dot{x}(t)) \in \partial^L\, \mathcal{H}_{\widetilde{F}}(t, \cdot, \cdot)|_{(x(t),q(t))} \right\} \quad \text{for almost every } t\\
(vi) & |\dot{q}(t)| \le k(t)\ |q(t)| \quad \text{for almost every } t.
\end{array}$$

Assuming $\lambda = 0$ and property (iii) imply $q(t_0) = 0$. So property (vi) and Gronwall's Lemma 1.5.4 lead to $q(\cdot) = 0$ — contradicting property (i). Thus, $\lambda > 0$.
Setting $p(t) := \frac{1}{\lambda\,\rho}\, q(t)$, the adjoint $p(\cdot)$ also fulfills the conditions (ii), (iv)–(vi) and $p(0) \in N_K(x(0))$, $p(t_0) = \nu$.

2. is a consequence of the first part by means of approximation.
For every $x_0 \in \partial\vartheta_{\widetilde{F}}(t_0, K)$ and $\nu \in N_{\vartheta_{\widetilde{F}}(t_0,K)}(x_0) \setminus \{0\}$, there exist sequences $(\nu_n)_{n\,\in\,I\!N}$, $(x_n(\cdot))_{n\,\in\,I\!N}$ satisfying

$$\begin{array}{llll} \dot{x}_n(\cdot) \in \widetilde{F}(\cdot, x_n(\cdot)) & \text{a.e. in } [0, t_0], & \nu_n \in N^P_{\vartheta_{\widetilde{F}}(t_0,K)}(x_n(t_0)) \setminus \{0\}, & \\ x_n(t_0) \longrightarrow x_0 & & \nu_n \longrightarrow \nu & (n \longrightarrow \infty), \\ x_n(0) \in K. & & & \end{array}$$

For each $n \in I\!N$, the first part provides an adjoint $p_n(\cdot) \in AC([0, t_0], I\!R^N)$ with $p_n(t_0) = \nu_n$. Due to Gronwall's Lemma 1.5.4 and the inequality $|\dot{p}_n(t)| \le k(t)\ |p_n(t)|$, the sequence $(p_n(\cdot))$ is equi-continuous. So the Theorems of Arzela–Ascoli and Dunford–Pettis ensure the convergence of subsequences (again denoted by) $(x_n(\cdot))$, $(p_n(\cdot))$ such that $x(\cdot) := \lim_{n\longrightarrow\infty} x_n(\cdot) \in AC([0, t_0], I\!R^N)$, $p(\cdot) := \lim_{n\longrightarrow\infty} p_n(\cdot) \in AC([0, t_0], I\!R^N)$ fulfill

$$\begin{array}{lll} x_n(\cdot) \longrightarrow x(\cdot), & p_n(\cdot) \longrightarrow p(\cdot) & \text{uniformly}, \\ \dot{x}_n(\cdot) \longrightarrow \dot{x}(\cdot), & \dot{p}_n(\cdot) \longrightarrow \dot{p}(\cdot) & \text{weakly in } L^1([0, t_0], I\!R^N). \end{array}$$

Finally, $n \longrightarrow \infty$ provides that $p(\cdot)$ satisfies the claimed properties of an adjoint arc (see e.g. Convergence Theorem 7.2.1 in [7, Aubin,Frankowska 90]). □

Restricting to autonomous differential equations (instead of differential inclusions), we benefit from the additional fact that time is reversible. So the limiting normal vectors of the initial set can be described by the normals of a later reachable set.

Corollary A.3.3 *Let $f \in C^1(I\!R^N, I\!R^N)$ have linear growth, $K \in \mathcal{K}(I\!R^N)$, $t > 0$.*

1. Then for every boundary point $x \in \partial\,\vartheta_f(t, K)$ and $p \in N_{\vartheta_f(t,K)}(x)$, there exist the unique trajectory $x(\cdot) \in C^1([0, t], I\!R^N)$ and its adjoint $p(\cdot) \in C^1([0, t], I\!R^N)$ with

$$\begin{array}{ll} x(0) \in K, & x(t) = x, \\ p(0) \in N_K(x(0)), & p(t) = p, \\ \dot{x}(s) = f(x(s)), & \dot{p}(s) = -p(s) \cdot Df(x(s)) \quad \text{for every } s \in \,]0, t[. \end{array}$$

2. Moreover for every initial point $x \in \partial\,K$ and $p \in N_K(x)$, there exist the unique trajectory $x(\cdot) \in C^1([0, t], I\!R^N)$ and its adjoint $p(\cdot) \in C^1([0, t], I\!R^N)$ satisfying

$$\begin{array}{ll} x(0) = x, & x(t) \in \partial\,\vartheta_f(t, K), \\ p(0) = p, & p(t) \in N_{\vartheta_f(t,K)}(x(t)), \\ \dot{x}(s) = f(x(s)), & \dot{p}(s) = -p(s) \cdot Df(x(s)) \quad \text{for every } s \in \,]0, t[. \end{array}$$

Proof. 1. The linear growth of f implies that $M := \bigcup_{0 \le s \le t} \vartheta_f(s, K)$ is compact. So f is Lipschitz continuous on M. Prop. A.3.2 guarantees the existence of a trajectory $x(\cdot) \in AC([0,t], I\!R^N)$ and an adjoint $p(\cdot) \in AC([0,t], I\!R^N)$ fulfilling

$$\begin{array}{rclrcl} x(0) & \in & K, & x(t) & = & x, \\ p(0) & \in & N_K(x(0)), & p(t) & = & p, \\ (-\dot{p}(s),\ \dot{x}(s)) & \in & \overline{co}\ \partial^L \mathcal{H}_f(x(s),\ p(s)) & & & \text{for almost every } s \in \,]0,t[. \end{array}$$

Due to $f \in C^1(I\!R^N, I\!R^N)$, the Hamiltonian $\mathcal{H}_f$ is in $C^1(I\!R^N \times I\!R^N)$ and so,

$$\partial^L \mathcal{H}_f(x,p) = \Big(\tfrac{\partial}{\partial x}\mathcal{H}_f(x,p),\ \tfrac{\partial}{\partial p}\mathcal{H}_f(x,p)\Big) = \Big(p \cdot Df(x),\ f(x)\Big).$$

Thus, $x(\cdot),\ p(\cdot)$ are continously differentiable satisfying

$$\dot{x}(s) = f(x(s)), \qquad \dot{p}(s) = -p(s) \cdot Df(x(s)) \qquad \text{for every } s \in \,]0,t[.$$

2. The solutions of initial value problems with $\dot{x}(\cdot) = f(x(\cdot))$ are always unique — as a well–known consequence of Gronwall's Lemma. Thus, the evolution along this ordinary differential equation is reversible in time, i.e. $\vartheta_f(s,K) = \vartheta_{-f}(t-s,\ \vartheta_f(t,K))$ for every $0 \le s \le t$. Now the claim results directly from the first part applied to $-f(\cdot)$, $x(t-\cdot)$, $p(t-\cdot)$. □

A.4 Differential inclusions preserving smooth sets shortly : Standard hypotheses $(\mathcal{H}), (\mathring{\mathcal{H}}), (\widetilde{\mathcal{H}})$

Now sufficient conditions on maps $\widetilde{F} : [0,T] \times I\!\!R^N \rightsquigarrow I\!\!R^N$, $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ are presented such that their reachable sets preserve the smoothness of initial sets for a short time (at least).
Here we consider compact N–dimensional $C^{1,1}$ submanifolds of $I\!\!R^N$ with boundary. In comparison with class C^1, it has the advantage that both the set K and its complement $I\!\!R^N \setminus K$ have positive reach. Due to Corollary 4.3.3, this is equivalent to the property that both K and $I\!\!R^N \setminus K$ have positive erosion. Then proximal and limiting normal cone coincide at every boundary point of K and contain exactly one direction.

After introducing the so–called standard hypotheses $(\widetilde{\mathcal{H}}), (\mathcal{H})$ in Def. A.4.2, their role as sufficient conditions for $C^{1,1}$ submanifolds with boundary is ensured by Prop. A.4.4. Due to § A.2, standard hypothesis $(\mathring{\mathcal{H}})$ forms the basis for applying the results to open sets (in Cor. A.4.5). Under slightly stricter assumptions of $\mathcal{H}_{\widetilde{F}}$, we even obtain a lower bound of the time how long the reachable sets stay smooth (in Prop. A.4.10).
Finally some sufficient conditions on a map $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ are given for satisfying standard hypothesis $(\mathring{\mathcal{H}})$ (in Prop. A.4.11), but these properties are not necessary as Example A.4.16 shows.

Definition A.4.1 $\mathcal{K}_{C^{1,1}}(I\!\!R^N)$ *abbreviates the set of all nonempty compact* N*–dimensional* $C^{1,1}$ *submanifolds of* $I\!\!R^N$ *with boundary.*
Furthermore set $\Omega_{C^{1,1}}(I\!\!R^N) := \{K^\circ \mid K \in \mathcal{K}_{C^{1,1}}(I\!\!R^N)\} \subset \Omega(I\!\!R^N)$.

Definition A.4.2 *A set–valued map* $\widetilde{F} : [0,T] \times I\!\!R^N \rightsquigarrow I\!\!R^N$ *fulfills the so–called* standard hypothesis $(\widetilde{\mathcal{H}})$ *if*

1. $\widetilde{F}$ *is measurable and has nonempty compact convex values,*
2. $\mathcal{H}_{\widetilde{F}}(t,\cdot,\cdot) : I\!\!R^N \times (I\!\!R^N \setminus \{0\}) \longrightarrow I\!\!R$ *is continuously differentiable for all* $t \in [0,T]$,
3. *for every* $R > 1$, *there exists* $k_R(\cdot) \in L^1([0,T])$ *such that the derivative of* $\mathcal{H}_{\widetilde{F}}(t,\cdot,\cdot)$ *restricted to* $I\!\!B_R \times (I\!\!B_R \setminus \mathring{I\!\!B}_{\frac{1}{R}})$ *is* $k_R(t)$ *–Lipschitz continuous for almost every* $t \in [0,T]$,
4. *there exists* $\gamma_{\widetilde{F}} \in L^1([0,T])$ *such that for all* $x, p \in I\!\!R^N$ $(|p| \geq 1)$,
$$\left\| \partial_{(x,p)} \mathcal{H}_{\widetilde{F}}(t,x,p) \right\|_{\mathcal{L}(I\!\!R^N \times I\!\!R^N, I\!\!R)} \leq \gamma_{\widetilde{F}}(t) \cdot (1 + |x| + |p|).$$

Correspondingly for a set–valued map $F : I\!R^N \rightsquigarrow I\!R^N$, *the* standard hypothesis $(\mathcal{H})$ *comprises the following conditions on* $\mathcal{H}_F(x,p) := \sup\ p \cdot F(x)$

1'. *F has nonempty compact convex values,*

2'. *for every* $R > 1$, $\mathcal{H}_F(\cdot,\cdot) \in C^{1,1}(I\!B_R \times (I\!B_R \setminus \overset{\circ}{I\!B}_{\frac{1}{R}}))$,

3'. *the derivative of* $\mathcal{H}_F$ *has linear growth, i.e. there is some* $\gamma_F > 0$ *with*
$\left\| D\,\mathcal{H}_F(x,p) \right\|_{\mathcal{L}(I\!R^N \times I\!R^N, I\!R)} \le \gamma_F \cdot (1 + |x| + |p|)$ *for all* $x, p \in I\!R^N$ $(|p| \ge 1)$.

$F : I\!R^N \rightsquigarrow I\!R^N$ *fulfills the so–called* standard hypothesis $(\overset{\circ}{\mathcal{H}})$ *if in addition to* $(\mathcal{H})$, *the values of* F *have nonempty interior.*

Obviously every vector field $f \in C^{1,1}(I\!R^N, I\!R^N)$ (regarded as a set–valued map) fulfills standard hypothesis $(\mathcal{H})$, but not $(\overset{\circ}{\mathcal{H}})$. Considering a map $F : I\!R^N \rightsquigarrow I\!R^N$ that is not single–valued, sufficient conditions for standard hypothesis $(\overset{\circ}{\mathcal{H}})$ are presented in Prop. A.4.11. For the moment we mention just two easy consequences of $(\widetilde{\mathcal{H}})$ for the set–valued map $\widetilde{F} : [0,T] \times I\!R^N \rightsquigarrow I\!R^N$:

Lemma A.4.3 *Standard hypothesis* $(\widetilde{\mathcal{H}})$ *for* $\widetilde{F} : [0,T] \times I\!R^N \rightsquigarrow I\!R^N$ *implies :*

1. $\widetilde{F}(t,\cdot) : I\!R^N \rightsquigarrow I\!R^N$ *is locally Lipschitz continuous for almost every* $t \in [0,T]$.
   In particular, $\text{Lip}\ \widetilde{F}(t,\cdot)|_{I\!B_R} \le (2+R)\ \ \gamma_{\widetilde{F}}(t)$.
2. $\widetilde{F}(t,x) \subset \gamma_{\widetilde{F}}(t)\ (2 + |x|) \cdot I\!B$ *for all* $x \in I\!R^N$ *and almost every* $t \in [0,T]$.
3. *The values of* $\widetilde{F}$ *are strictly convex.*

Proof. 1. According to the preceding Lemma A.2.4 (2.), the Hausdorff distance between values of $\widetilde{F}(t,\cdot)$ can be estimated by means of their support functions since $\widetilde{F}$ has nonempty convex compact values, i.e. in terms of the Hamiltonian $\mathcal{H}_{\widetilde{F}}$, we get for every $R > 0$ and $x, y \in I\!B_R \subset I\!R^N$

$$\begin{aligned} d\!l(F(t,x),\ F(t,y)) &\le \sup_{|p|=1} \left|\mathcal{H}_{\widetilde{F}}(t,x,p) - \mathcal{H}_{\widetilde{F}}(t,y,p)\right| \\ &\le \sup_{|p|=1}\ \sup_{I\!B_R}\ \left|\partial_p \mathcal{H}_{\widetilde{F}}(t,\cdot,p)\right| \cdot |x-y| \\ &\le \gamma_{\widetilde{F}}(t)\ \ (2+R) \cdot |x-y|. \end{aligned}$$

2. As a consequence of Lemma A.2.4 (1.), every $y \in \widetilde{F}(t,x) \setminus \{0\}$ satisfies

$$|y| = \tfrac{y}{|y|} \cdot y \le \mathcal{H}_{\widetilde{F}}(t,x,\tfrac{y}{|y|}) \le \left| \tfrac{y}{|y|} \cdot \partial_p\, \mathcal{H}_{\widetilde{F}}(t,x,\tfrac{y}{|y|}) \right| \le \gamma_{\widetilde{F}}(t) \cdot (2 + |x|)$$

3. results from Lemma A.2.4 as well because the subdifferential of

$$I\!R^N \setminus \{0\} \longrightarrow 0, \qquad p \longmapsto \mathcal{H}_{\widetilde{F}}(t,x,p) = \sigma_{\widetilde{F}(t,x)}(p)$$

is single–valued (by assumption). □

The main benefit of these hypotheses $(\mathcal{H})$, $(\widetilde{\mathcal{H}})$ is to preserve the smooth boundary of a compact initial set for a short time at least.

Proposition A.4.4 *Assume standard hypothesis* $(\widetilde{\mathcal{H}})$ *for* $\widetilde{F} : [0,T] \times I\!R^N \rightsquigarrow I\!R^N$. *For every initial set* $K \in \mathcal{K}_{C^{1,1}}(I\!R^N)$, *there exist* $\tau = \tau(\widetilde{F}, K) > 0$ *and* $\rho = \rho(\widetilde{F}, K) > 0$ *such that* $\vartheta_{\widetilde{F}}(t, K)$ *is also a* N*–dimensional* $C^{1,1}$ *submanifold of* $I\!R^N$ *with boundary for all* $t \in [0, \tau]$ *and its radius of curvature is* $\geq \rho$ *(i.e. both* $\vartheta_{\widetilde{F}}(t, K)$ *and its complement* $I\!R^N \setminus \vartheta_{\widetilde{F}}(t, K)$ *have positive erosion of radius* ρ*).*

Corollary A.4.5 *Under standard hypothesis* $(\mathring{\mathcal{H}})$ *for the map* $F : I\!R^N \rightsquigarrow I\!R^N$, *every initial set* $O \in \Omega_{C^{1,1}}(I\!R^N)$ *leads to* $\rho, \tau > 0$ *satisfying* $\vartheta_F(t, O) \in \Omega_{C^{1,1}}(I\!R^N)$ *for all* $t \in [0, \tau]$ *and the radius of curvature of each* $\overline{\vartheta_F(t, O)}$ *is* $\geq \rho$.

Proof of Cor. A.4.5 results from Prop. A.4.4 and Cor. A.2.9, Prop. A.2.10 :
As a consequence of Filippov's Theorem A.1.2, $\overline{\vartheta_F(t, O)} = \vartheta_F(t, \overline{O}) \in \mathcal{K}(I\!R^N)$ and according to Def. A.4.1, $\overline{O} \in \mathcal{K}_{C^{1,1}}(I\!R^N)$. So Prop. A.4.4 provides $\rho, \tau > 0$ such that $\vartheta_F(t, \overline{O}) \in \mathcal{K}_{C^{1,1}}(I\!R^N)$ for all $t \in [0, \tau]$ and each radius of curvature is $\geq \rho$.
Cor. A.2.9 (2.) and Prop. A.2.10 imply $\bigcup_{t>0} (\{t\} \times \partial\, \vartheta_F(t, O)) = \bigcup_{t>0} \left(\{t\} \times \partial\, \vartheta_F(t, \overline{O})\right)$.
Thus, $\vartheta_F(t, O) = \vartheta_F(t, \overline{O}) \setminus \partial\, \vartheta_F(t, \overline{O}) = \left(\vartheta_F(t, \overline{O})\right)^\circ \in \Omega_{C^{1,1}}(I\!R^N)$. □

Proof of Prop. A.4.4 is based on the following lemma :

Lemma A.4.6 *Suppose for* $H : [0,T] \times I\!R^N \times I\!R^N \longrightarrow I\!R$, $\psi : I\!R^N \longrightarrow I\!R^N$ *and the Hamiltonian system*

$$\wedge \begin{cases} \dot{y}(t) = \frac{\partial}{\partial q} H(t,\, y(t),\, q(t)), & y(0) = y_0 \\ \dot{q}(t) = -\frac{\partial}{\partial y} H(t,\, y(t),\, q(t)), & q(0) = \psi(y_0) \end{cases} \qquad (*)$$

the following properties :

1. $H(t, \cdot, \cdot)$ *is differentiable for every* $t \in [0, T]$,
2. *for every* $R > 0$, *there exists* $k_R \in L^1([0,T])$ *such that the derivative of* $H(t, \cdot, \cdot)$ *is* $k_R(t)$*–Lipschitz continuous on* $I\!B_R \times I\!B_R$ *for almost every* t,
3. ψ *is locally Lipschitz continuous,*
4. *every solution* $(y(\cdot), q(\cdot))$ *of the Hamiltonian system* $(*)$ *can be extended to* $[0, T]$ *and depends continuously on the initial data in the following sense :*
   Let each $(y_n(\cdot), q_n(\cdot))$ *be a solution satisfying* $y_n(t_n) \longrightarrow z_0$, $q_n(t_n) \longrightarrow q_0$ *for some* $t_n \longrightarrow t_0$, $z_0, q_0 \in I\!R^N$. *Then* $(y_n(\cdot), q_n(\cdot))_{n \in I\!N}$ *converges uniformly to a solution* $(y(\cdot), q(\cdot))$ *of the Hamiltonian system with* $y(t_0) = z_0$, $q(t_0) = q_0$.

For a compact set $K \subset I\!R^N$ *and* $t \in [0,T]$, *define*

$$M_t^{\mapsto}(K) := \Big\{ (y(t),\, q(t)) \;\Big|\; (y(\cdot),\, q(\cdot)) \text{ solves system } (*), \;\; y_0 \in K \Big\} \subset I\!R^N \times I\!R^N.$$

Then there exist $\delta > 0$ *and* $\lambda > 0$ *such that* $M_t^{\mapsto}(K)$ *is the graph of a* λ*–Lipschitz continuous function for every* $t \in [0,\delta]$.

Proof of Lemma A.4.6 (*indirect*) [37, Frankowska 2002], Lemma 5.5 states the corresponding result for the Hamiltonian system with $y(T) = y_T$, $q(T) = q_T$ given (without mentioning the uniform Lipschitz constant λ explicitly). Now for pointing out the indirect character of the proof there, we seize its notion and adapt it to the initial conditions at $t = 0$ explicitly :
Assumption (4.) and $K \in \mathcal{K}(I\!R^N)$ imply $\bigcup\limits_{0 \le t \le T} M_t^{\mapsto}(K) \subset I\!B_R \times I\!B_R$ for some $R > 0$.

Suppose that the claim is false. Then there exists a sequence $(t_n)_{n \in I\!N}$ in $]0,T]$ with $t_n \longrightarrow 0$ such that either $M_{t_n}^{\mapsto}(K)$ is not the graph of a Lipschitz function or the corresponding Lipschitz constants converge to ∞. In both cases, we can find distinct solutions $(y_n^1(\cdot),\, q_n^1(\cdot))$, $(y_n^2(\cdot),\, q_n^2(\cdot))$, $n \in I\!N$, of the Hamiltonian system $(*)$ with

$$\varepsilon_n := \frac{|\, y_n^1(t_n) - y_n^2(t_n)\,|}{|\, q_n^1(t_n) - q_n^2(t_n)\,|} \longrightarrow 0 \qquad \text{for } n \longrightarrow \infty.$$

Assumption (2.) provides the estimate

$$\begin{aligned} |y_n^1(t) - y_n^2(t)| \le\;& |y_n^1(t_n) - y_n^2(t_n)| + \int_t^{t_n} k_R(s) \,\Big(|y_n^1(s) - y_n^2(s)| + |q_n^1(s) - q_n^2(s)|\Big)\, ds \\ \le\;& \varepsilon_n\, |q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s) \,\Big(|y_n^1(s) - y_n^2(s)| + |q_n^1(s) - q_n^2(s)|\Big)\, ds \end{aligned}$$

for all $t \in [0, t_n]$, and the integral version of Gronwall's Lemma 1.5.4 leads to a constant $C_1 > 0$ (independent of n) with

$$|y_n^1(t) - y_n^2(t)| \le C_1 \Big(\varepsilon_n\, |q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s)\; |q_n^1(s) - q_n^2(s)|\; ds\Big).$$

Due to $\sup\limits_n \varepsilon_n < \infty$, we obtain a constant $C_2 > 0$ such that for all $n \in I\!N$, $t \in [0, t_n]$,

$$\begin{aligned} |q_n^1(t) - q_n^2(t)| \le\;& |q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s)\Big(|y_n^1(s) - y_n^2(s)| + |q_n^1(s) - q_n^2(s)|\Big) ds \\ \le\;& C_2 \,\Big(|q_n^1(t_n) - q_n^2(t_n)| + \int_t^{t_n} k_R(s)\; |q_n^1(s) - q_n^2(s)|\; ds\Big). \end{aligned}$$

As a consequence of Gronwall's Lemma 1.5.4 again, there is another constant $C_3 > 0$ (independent of n) with $|q_n^1(t) - q_n^2(t)| \le C_3\, |q_n^1(t_n) - q_n^2(t_n)|$ for all n, $t \in [0, t_n]$.
So in particular,

$$\varepsilon_n' := \sup_{0 \le t \le t_n} \frac{|\, y_n^1(t) - y_n^2(t)\,|}{|\, q_n^1(t_n) - q_n^2(t_n)|} \le C_1 \,\Big(\varepsilon_n + C_3 \int_0^{t_n} k_R(s)\; ds\Big) \longrightarrow 0 \qquad \text{for } n \longrightarrow 0.$$

Similarly we get a constant $C_4 = C_4(\|k_R\|_{L^1}) > 0$ fulfilling

$$|q_n^1(t_n) - q_n^2(t_n)| \le C_4\, |q_n^1(0) - q_n^2(0)| = C_4\, |\psi(y_n^1(0)) - \psi(y_n^2(0))|$$

for all $n \in I\!N$ sufficiently large. Indeed, for all $t \in [0, t_n]$, assumption (2.) ensures

$$\begin{aligned} |q_n^1(t) - q_n^2(t)| &\le |q_n^1(0) - q_n^2(0)| + \int_0^t k_R(s) \Big(|y_n^1(s) - y_n^2(s)| + |q_n^1(s) - q_n^2(s)|\Big)\, ds \\ &\le |q_n^1(0) - q_n^2(0)| + \int_0^t k_R(s) \Big(\varepsilon_n'\, |q_n^1(t_n) - q_n^2(t_n)| + |q_n^1(s) - q_n^2(s)|\Big)\, ds \end{aligned}$$

and Gronwall's Lemma 1.5.4 provides a constant $C_4 = C_4(\|k_R\|_{L^1}) > 0$ such that

$$|q_n^1(t_n) - q_n^2(t_n)| \le \tfrac{C_4}{2}\, |q_n^1(0) - q_n^2(0)| + \mathrm{const}(\|k_R\|_{L^1})\ \varepsilon_n'\ |q_n^1(t_n) - q_n^2(t_n)|$$

for every $n \in I\!N$. Due to $\varepsilon_n' \longrightarrow 0$, we obtain $|q_n^1(t_n) - q_n^2(t_n)| \le C_4\, |q_n^1(0) - q_n^2(0)|$ for all $n \in I\!N$ large enough.

So finally,

$$\begin{aligned} \frac{|\psi(y_n^1(0)) - \psi(y_n^2(0))|}{|y_n^1(0) - y_n^2(0)|} &= \frac{|q_n^1(0) - q_n^2(0)|}{|q_n^1(t_n) - q_n^2(t_n)|} \cdot \frac{|q_n^1(t_n) - q_n^2(t_n)|}{|y_n^1(0) - y_n^2(0)|} \\ &\ge \frac{1}{C_4} \cdot \frac{1}{\varepsilon_n'} \\ &\longrightarrow \infty \qquad \text{for } n \longrightarrow \infty \end{aligned}$$

— contradicting the local Lipschitz continuity of ψ at each cluster point of $(y_n^1(0))_n$. □

Proof of Prop. A.4.4. Standard hypothesis $(\widetilde{\mathcal{H}})$ for $\widetilde{F} : [0,T] \times I\!R^N \leadsto I\!R^N$ implies conditions (1.), (4.) of the preceding Lemma A.4.6 for the Hamiltonian $\mathcal{H}_{\widetilde{F}}$.
Assuming that $K \in \mathcal{K}(I\!R^N)$ is a N–dimensional $C^{1,1}$ submanifold of $I\!R^N$ with boundary, the unit *exterior* normal vectors of K (restricted to ∂K) can be extended to a Lipschitz continuous function $\psi : I\!R^N \longrightarrow I\!R^N$. Furthermore, choose $\varphi \in C^\infty(I\!R, I\!R)$ with

$$\varphi(s) = 0 \quad \text{for } s \le \tfrac{1}{4}, \qquad \varphi(s) = 1 \quad \text{for } s \ge \tfrac{1}{2}$$

and set $H(t,x,p) := \mathcal{H}_{\widetilde{F}}(t,x,p) \cdot \varphi(|p|)$ for $(t,x,p) \in [0,T] \times I\!R^N \times I\!R^N$.
Then H satisfies condition (2.) of Lemma A.4.6 in addition.

Consider now the differential equations

$$\wedge \begin{cases} \dot{x}(t) = \frac{\partial}{\partial p} H(t, x(t), p(t)), & x(0) = x_0 \\ \dot{p}(t) = -\frac{\partial}{\partial x} H(t, x(t), p(t)), & p(0) = \psi(x_0) \end{cases} \qquad (*)$$

for arbitrary $x_0 \in \partial K$. Due to $|\psi(\cdot)| = 1$ on ∂K and $H \in C^{1,1}$, there exists $\tau_1 > 0$ such that $|p(t)| > \frac{1}{2}$ for all solutions $(x(\cdot), p(\cdot))$ of $(*)$ with $x_0 \in \partial K$ and every $t \in [0, \tau_1]$.
In particular, $H(t,x(t),p(t)) = \mathcal{H}_{\widetilde{F}}(t,x(t),p(t))$, $DH(t,x(t),p(t)) = D\mathcal{H}_{\widetilde{F}}(t,x(t),p(t))$.

Setting

$$\widehat{M_t^{\mapsto}}(\partial K) := \Big\{ (x(t),\ \lambda\, p(t)) \;\Big|\; (x(\cdot),\ p(\cdot)) \text{ solves system } (*),\ \ x_0 \in \partial K,\ \ \lambda \geq 0 \Big\},$$

Prop. A.3.2 states Graph $N_{\vartheta_{\widetilde{F}}(t,K)}(\cdot) \;\subset\; \widehat{M_t^{\mapsto}}(\partial K)$ for all $t \in [0, \tau_1]$.
Furthermore Lemma A.4.6 yields $\tau \in \,]0, \tau_1[$ and $\lambda_M > 0$ such that

$$M_t^{\mapsto}(\partial K) := \Big\{ (x(t),\ p(t)) \;\Big|\; (x(\cdot),\ p(\cdot)) \text{ solves system } (*),\ \ x_0 \in \partial K \Big\} \subset I\!R^N \times I\!R^N$$

is the graph of a λ_M–Lipschitz continuous function for each $t \in [0, \tau]$.

Then for every point $z \in \partial \vartheta_{\widetilde{F}}(t, K)$, the limiting normal cone $N_{\vartheta_{\widetilde{F}}(t,K)}(z)$ contains exactly one direction and, its unit vector depends on z in a Lipschitz continuous way. (The Lipschitz constant is uniformly bounded by $2\,\lambda_M$ since the choice of τ_1 ensures $|p(\cdot)| > \frac{1}{2}$ on $[0, \tau_1]$ for each solution of $(*)$.)
So the compact set $\vartheta_{\widetilde{F}}(t, K)$ is N–dimensional $C^{1,1}$ submanifold of $I\!R^N$ with boundary for all $t \in [0, \tau]$ and its radius of curvature has a uniform lower bound.

□

However the indirect proof of Lemma A.4.6 does not enable us to find a lower bound of the time $\tau(\widetilde{F}, K)$. For this purpose, stricter regularity conditions on the Hamiltonian system permit taking the second variation into consideration. This notion leads to the following equivalence in [21, Caroff, Frankowska 96] and [37, Frankowska 2002].

Lemma A.4.7 ([37, Frankowska 2002], Theorem 5.3)
Assume for $H : [0, T] \times I\!R^N \times I\!R^N \longrightarrow I\!R$, $\psi : I\!R^N \longrightarrow I\!R^N$ and the Hamiltonian system

$$\wedge \begin{cases} \dot{y}(t) &= \frac{\partial}{\partial q} H(t,\ y(t),\ q(t)), \qquad y(T) = y_T \\ \dot{q}(t) &= -\frac{\partial}{\partial y} H(t,\ y(t),\ q(t)), \qquad q(T) = \psi(y_T) \end{cases} \tag{$**$}$$

the following properties :

1. *$H(t, \cdot, \cdot)$ is* twice *continuously differentiable for every $t \in [0, T]$,*
2. *for every $R > 0$, there exists $k_R \in L^1([0, T])$ such that the derivative of $H(t, \cdot, \cdot)$ is $k_R(t)$–Lipschitz continuous on $I\!B_R \times I\!B_R$ for almost every t,*
3. *ψ is locally Lipschitz continuous,*
4. *every solution $(y(\cdot), q(\cdot))$ of the Hamiltonian system $(**)$ can be extended to $[0, T]$ and depends continuously on the initial data in the following sense :*
   Let each $(y_n(\cdot), q_n(\cdot))$ be a solution satisfying $y_n(t_n) \longrightarrow z_0$, $q_n(t_n) \longrightarrow q_0$ for some $t_n \longrightarrow t_0$, $z_0, q_0 \in I\!R^N$. Then $(y_n(\cdot), q_n(\cdot))_{n \in I\!N}$ converges uniformly to a solution $(y(\cdot), q(\cdot))$ of the Hamiltonian system with $y(t_0) = z_0$, $q(t_0) = q_0$.

For a compact set $K \subset I\!R^N$ *and* $t \in [0,T]$, *define*

$$M_t^{\rightarrow\!|}(K) \;:=\; \Big\{ (y(t),\, q(t)) \;\Big|\; (y(\cdot),\, q(\cdot)) \text{ solves system } (**),\;\; y_T \in K \Big\} \;\subset\; I\!R^N \times I\!R^N.$$

Then the following statements are equivalent :

(i) *For all* $t \in [0,T]$, $M_t^{\rightarrow\!|}(K)$ *is the graph of a locally Lipschitz continuous function,*

(ii) *For every solution* $(y(\cdot), q(\cdot)) : [0,T] \longrightarrow I\!R^N \times I\!R^N$ *of system* $(**)$ *and each cluster point* $Q_T \in \operatorname{Limsup}_{z \to y_T} \{\nabla\, \psi(z)\}$, *the following matrix Riccati equation has a matrix–valued solution* $Q(\cdot)$ *on* $[0,T]$

$$\wedge \begin{cases} \partial_t\, Q \;+\; \frac{\partial^2 H}{\partial p\, \partial x}(t,\, y(t),\, q(t))\;\; Q \;+\; Q\;\; \frac{\partial^2 H}{\partial x\, \partial p}(t,\, y(t),\, q(t)) \\ \qquad +\; Q\;\; \frac{\partial^2 H}{\partial p^2}(t,\, y(t),\, q(t))\;\; Q \;+\; \frac{\partial^2 H}{\partial x^2}(t,\, y(t),\, q(t)) \;=\; 0, \\ \hfill Q(T) \;=\; Q_T. \end{cases}$$

If one of these equivalent properties is satisfied and if ψ *is (continuously) differentiable, then* $M_t^{\rightarrow\!|}(K)$ *is even the graph of a (continuously) differentiable function.* □

Corollary A.4.8 *Consider the Hamiltonian system* $(*)$ *of Lemma A.4.6 and assume cond. (1.)–(4.) of Lemma A.4.7 for* $H : [0,T]\times I\!R^N\times I\!R^N \longrightarrow I\!R$, $\psi : I\!R^N \longrightarrow I\!R^N$. *Then for every* $K \in \mathcal{K}(I\!R^N)$, *the following statements are equivalent :*

(i) *For all* $t \in [0,T]$,

$$M_t^{\mapsto}(K) \;:=\; \Big\{ (y(t),\, q(t)) \;\Big|\; (y(\cdot),\, q(\cdot)) \text{ solves system } (*),\;\; y_0 \in K \Big\}$$

is the graph of a locally Lipschitz continuous function,

(ii) *For any solution* $(y(\cdot), q(\cdot)) : [0,T] \longrightarrow I\!R^N \times I\!R^N$ *of the initial value problem* $(*)$ *and each cluster point* $Q_0 \in \operatorname{Limsup}_{z \to y_0} \{\nabla\, \psi(z)\}$, *the following matrix Riccati equation has a solution* $Q(\cdot)$ *on* $[0,T]$

$$\wedge \begin{cases} \partial_t\, Q \;+\; \frac{\partial^2 H}{\partial p\, \partial x}(t,\, y(t),\, q(t))\;\; Q \;+\; Q\;\; \frac{\partial^2 H}{\partial x\, \partial p}(t,\, y(t),\, q(t)) \\ \qquad +\; Q\;\; \frac{\partial^2 H}{\partial p^2}(t,\, y(t),\, q(t))\;\; Q \;+\; \frac{\partial^2 H}{\partial x^2}(t,\, y(t),\, q(t)) \;=\; 0, \\ \hfill Q(0) \;=\; Q_0. \end{cases}$$

If one of these equivalent properties is satisfied and if ψ *is (continuously) differentiable, then* $M_t^{\mapsto}(K)$ *is even the graph of a (continuously) differentiable function.*

Proof is an immediate consequence of Lemma A.4.7 applied to $-H(T - \cdot, \cdot, \cdot)$ because for any solution $(y(\cdot), q(\cdot))$ of the initial value problem $(*)$, $(y(T - \cdot), q(T - \cdot))$ solves the corresponding system $(**)$. □

For preventing singularities of $Q(\cdot)$, the following comparison principle provides a bridge to solutions of a *scalar* Riccati equation. So it is the basis for a lower estimate of time τ (i.e. how long the boundary of $\vartheta_{\widetilde{F}}(t, K)$ stays smooth at least).
Here our main aim is to present a short way to an explicit lower bound in Prop. A.4.10 and for this reason we accept the disadvantage that it might not be optimal.

Lemma A.4.9 (Comparison theorem for the matrix Riccati equation, [56, Royden 88], Theorem 2**)**

Let $A_j, B_j, C_j : [0,T[\longrightarrow \mathbb{R}^{N,N}$ $(j = 0,1,2)$ *be bounded matrix–valued functions such that each* $M_j(t) := \begin{pmatrix} A_j(t) & B_j(t) \\ B_j(t)^T & C_j(t) \end{pmatrix}$ *is symmetric.*
Assume that $U_0,\ U_2 : [0,T[\longrightarrow \mathbb{R}^{N,N}$ *are solutions of the matrix Riccati equation*

$$\tfrac{d}{dt}\, U_j \;=\; A_j \;+\; B_j\, U_j \;+\; U_j\, B_j^T \;+\; U_j\, C_j\, U_j$$

with $M_2(\cdot) \geq M_0(\cdot)$ *(i.e.* $M_2(t) - M_0(t)$ *is positive semi–definite for every* t*).*

Then, given symmetric $U_1(0) \in \mathbb{R}^{N,N}$ *with*

$$U_2(0) \;\geq\; U_1(0) \;\geq\; U_0(0), \qquad M_2(\cdot) \;\geq\; M_1(\cdot) \;\geq\; M_0(\cdot),$$

there exists a solution $U_1 : [0,T[\longrightarrow \mathbb{R}^{N,N}$ *of the corresponding Riccati equation with matrix* $M_1(\cdot)$*. Moreover,* $U_2(t) \;\geq\; U_1(t) \;\geq\; U_0(t)$ *for all* $t \in [0,T[$. □

Proposition A.4.10 *In addition to standard hypothesis* $(\widetilde{\mathcal{H}})$ *for* $\widetilde{F} : [0,T] \times \mathbb{R}^N \leadsto \mathbb{R}^N$ *suppose that* $\mathcal{H}_{\widetilde{F}}(t,\cdot,\cdot) : \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \longrightarrow \mathbb{R}$ *is twice continuously differentiable for each* $t \in [0,T]$*.*
Let $K \in \mathcal{K}(\mathbb{R}^N)$ *be a* N*–dimensional* $C^{1,1}$ *submanifold of* $\mathbb{R}^N$ *with boundary whose radius of curvature is larger than* $\rho = \rho(\partial K) > 0$ *and set*

$$\begin{aligned}
R(t) \;&:=\; 1 + \|K\|_\infty + (\|K\|_\infty + 2)\, \Big(1 + \|\gamma_{\widetilde{F}}\|_{L^1} \cdot e^{\|\gamma_{\widetilde{F}}\|_{L^1}}\Big) \cdot \int_0^t \gamma_{\widetilde{F}}(s)\, ds \;>\; 1,\\
\widehat{R}(t) \;&:=\; \max\; \Big\{1 + (R(t)+1)\, \Big(1 + \|\gamma_{\widetilde{F}}\|_{L^1} \cdot e^{\|\gamma_{\widetilde{F}}\|_{L^1}}\Big) \cdot \int_0^t \gamma_{\widetilde{F}}(s)\, ds,\;\; R(t)\Big\},\\
\widehat{\tau} \;&\in\;]0,T[\text{ small such that } (R(T)+1) \int_0^{\widehat{\tau}} \gamma_{\widetilde{F}}\, ds + \int_0^{\widehat{\tau}} k_{R(T)+1}\, ds \;\leq\; 1 - \tfrac{1}{R(T)},\\
\mu \;&:=\; \sup_{\substack{0 \,\leq\, t \,\leq\, \widehat{\tau} \\ |x| \,\leq\, R(T) \\ \frac{1}{\widehat{R}(T)} \,\leq\, |p| \,\leq\, \widehat{R}(T)}} \left\| \begin{pmatrix} \frac{\partial^2}{\partial x^2}\, \mathcal{H}_{\widetilde{F}}(t,x,p) & \frac{\partial^2}{\partial x\, \partial p}\, \mathcal{H}_{\widetilde{F}}(t,x,p) \\ \frac{\partial^2}{\partial p\, \partial x}\, \mathcal{H}_{\widetilde{F}}(t,x,p) & \frac{\partial^2}{\partial p^2}\, \mathcal{H}_{\widetilde{F}}(t,x,p) \end{pmatrix} \right\|_{\mathcal{L}(\mathbb{R}^{2N}, \mathbb{R}^{2N})} < \infty,\\
\tau \;&:=\; \min\; \Big\{\widehat{\tau},\;\; \tfrac{1}{\mu}\, \big(\tfrac{\pi}{2} - \arctan \tfrac{1}{\rho}\big)\Big\}.
\end{aligned}$$

Then the compact set $\vartheta_{\widetilde{F}}(t, K)$ *also belongs to* $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ *for every* $t \in [0, \tau[$.

Proof. Due to Lemma A.4.3, $\widetilde{F}(t,\cdot)$ is locally Lipschitz continuous and has linear growth, i.e. $\text{Lip } \widetilde{F}(t,\cdot)|_{\mathbb{B}_R} \le (2+R)\ \gamma_{\widetilde{F}}(t)$, $\widetilde{F}(t,x) \subset \gamma_{\widetilde{F}}(t)\ (2+|x|)\cdot \mathbb{B}$ for all $R>0$, $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. So the integral version of Gronwall's Lemma 1.5.4 leads to the inclusion

$$\vartheta_{\widetilde{F}}(t,z) \;\subset\; z \;+\; (2+|z|)\,(1+\|\gamma_{\widetilde{F}}\|_{L^1}\cdot e^{\|\gamma_{\widetilde{F}}\|_{L^1}}) \int_0^t \gamma_{\widetilde{F}}(s)\,ds \;\cdot\; \mathbb{B}$$

for all $t \in [0,T]$, $z \in \mathbb{R}^N$ and thus, $\vartheta_{\widetilde{F}}(t,K) \subset (R(t)-1)\cdot\mathbb{B}$.
Moreover, standard hypothesis $(\widetilde{\mathcal{H}})$ guarantees for all $r>1$, $x \in \mathbb{B}_r \subset \mathbb{R}^N$, $p \in \mathbb{R}^N$ and almost every $t \in [0,T]$

$$\begin{cases} \left\|\partial_{(x,p)}\,\mathcal{H}_{\widetilde{F}}(t,x,p)\right\|_{\mathcal{L}(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R})} \le \gamma_{\widetilde{F}}(t)\cdot(1+|x|+|p|) & \text{if} \quad |p| \ge 1,\\ \left\|\partial_{(x,p)}\,\mathcal{H}_{\widetilde{F}}(t,x,p)\right\|_{\mathcal{L}(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R})} \le \gamma_{\widetilde{F}}(t)\cdot(2+|x|) + k_r(t) & \text{if} \quad \frac{1}{r} \le |p| \le 1. \end{cases}$$

Firstly we consider the Hamiltonian system of $\mathcal{H}_{\widetilde{F}}$ for initial points $x_0 \in \partial K$

$$\wedge \begin{cases} \dot{x}(t) = \frac{\partial}{\partial p}\,\mathcal{H}_{\widetilde{F}}(t,\,x(t),\,p(t)), & x(0) = x_0,\\ \dot{p}(t) = -\frac{\partial}{\partial x}\,\mathcal{H}_{\widetilde{F}}(t,\,x(t),\,p(t)), & \{p(0)\} = N_K(x_0)\cap\partial\mathbb{B}_1. \end{cases} \qquad (***)$$

Gronwall's Lemma 1.5.4 yields the upper bound of every solution $(x(\cdot),p(\cdot))$

$$\begin{aligned} |p(t)| &\le 1 + \int_0^t \gamma_{\widetilde{F}}(s)\cdot(1+R(s))\,ds \\ &\quad + \int_0^t e^{\int_s^t \gamma_{\widetilde{F}}\,dr}\cdot\gamma_{\widetilde{F}}(s)\cdot\int_0^s \Big(\gamma_{\widetilde{F}}(r)\cdot(1+R(r))\Big)\,dr\;ds \\ &\le 1 + (R(t)+1)\,\Big(1+\|\gamma_{\widetilde{F}}\|_{L^1}\cdot e^{\|\gamma_{\widetilde{F}}\|_{L^1}}\Big)\cdot\int_0^t \gamma_{\widetilde{F}}(s)\,ds \\ &\le \widehat{R}(t) \end{aligned}$$

for all $t \in [0,T]$. Moreover for each $t \in [0,T]$ with $|p(\cdot)| \ge \frac{1}{R(T)+1}$ on $[0,t]$, it fulfills

$$\begin{aligned} |p(t)| &\ge 1 - \int_0^t \Big(\gamma_{\widetilde{F}}(s)\cdot(2+R(s)-1) + k_{R(T)+1}(s)\Big)\,ds \\ &\ge 1 - (R(T)+1)\cdot\int_0^t \gamma_{\widetilde{F}}(s)\,ds - \int_0^t k_{R(T)+1}(s)\,ds. \end{aligned}$$

So in particular, $|p(\cdot)| \ge \frac{1}{R(T)} \ge \frac{1}{\widehat{R}(T)}$ on $[0,\widehat{\tau}]$.

Secondly we construct $H : [0,\widehat{\tau}]\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow \mathbb{R}$ easily such that it satisfies the assumptions of Corollary A.4.8 and

$$H(t,\cdot,\cdot) = \mathcal{H}_{\widetilde{F}}(t,\cdot,\cdot), \qquad \partial^j_{(x,p)}H(t,\cdot,\cdot) = \partial^j_{(x,p)}\mathcal{H}_{\widetilde{F}}(t,\cdot,\cdot) \qquad \text{in } \mathbb{R}^N\times(\mathbb{R}^N\setminus\mathbb{B}_{\frac{1}{2\,\widehat{R}(T)}})$$

for every $t \in [0,\widehat{\tau}]$, $j=1,2$. Choose a smooth auxiliary function $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ with

$$\varphi(s) = 0 \quad \text{for } s \le \tfrac{1}{8\,\widehat{R}(T)}, \qquad \varphi(s) = 1 \quad \text{for } s \ge \tfrac{1}{2\,\widehat{R}(T)}$$

and set $\quad H(t,x,p) := \mathcal{H}_{\widetilde{F}}(t,x,p)\cdot\varphi(|p|) \quad$ for $t \in [0,\widehat{\tau}]$, $x,p \in \mathbb{R}^N$.

Furthermore the exterior unit normal vector of K can be extended to a Lipschitz continuous function $\psi : I\!R^N \longrightarrow I\!R^N$. In respect of the limiting subdifferential of ψ, the curvature assumption of ∂K guarantees

$$\partial^L \psi(x) \subset \tfrac{1}{\rho}\, I\!B \qquad \text{for all } x \text{ close to } \partial K$$

and thus,

$$\operatorname{Limsup}_{z \to x} \{\nabla \psi(z)\} \subset \tfrac{1}{\rho}\, I\!B \qquad \text{for all } x \in \partial K.$$

Thirdly, the scalar Riccati equation

$$\wedge \begin{cases} \frac{d}{dt} u &= a + a\, u^2 \\ u(0) &= u_0 \end{cases}$$

has the bounded solution $u(t) = \tan(a\,t + \arctan u_0) = \frac{u_0 + \tan(a\,t)}{1 - u_0 \cdot \tan(a\,t)}$ in the interval $[0, \tau[\; \subset \Big[0,\ \frac{1}{\mu}\,(\frac{\pi}{2} - \arctan \frac{1}{\rho})\Big[$ for $u_0 = \frac{1}{\rho}$, $a = \mu$ and for $u_0 = -\frac{1}{\rho}$, $a = -\mu$.

Let us now apply these preparations to the Hamiltonian system of Corollary A.4.8

$$\wedge \begin{cases} \dot{y}(t) &= \frac{\partial}{\partial q} H(t,\, y(t),\, q(t)), \qquad y(0) = y_0 \\ \dot{q}(t) &= -\frac{\partial}{\partial y} H(t,\, y(t),\, q(t)), \qquad q(0) = \psi(y_0) \end{cases} \qquad (*)$$

for $y_0 \in \partial K$. Each solution $(y(\cdot), q(\cdot))$ satisfies $|y(t)| \le R(T)$, $\frac{1}{\widehat{R}(T)} \le |q(t)| \le \widehat{R}(T)$ for all $t \in [0, \widehat{\tau}]$. As a consequence, $H(t,\cdot,\cdot) = \mathcal{H}_{\widetilde{F}}(t,\cdot,\cdot)$ close to $(y(t), q(t))$. According to the comparison principle in Lemma A.4.9, the matrix Riccati equation

$$\wedge \begin{cases} \partial_t Q + \frac{\partial^2 H}{\partial p\, \partial x}(t,\, y(t),\, q(t))\; Q + Q\; \frac{\partial^2 H}{\partial x\, \partial p}(t,\, y(t),\, q(t)) \\ \qquad + Q\; \frac{\partial^2 H}{\partial p^2}(t,\, y(t),\, q(t))\; Q + \frac{\partial^2 H}{\partial x^2}(t,\, y(t),\, q(t)) = 0, \\ \qquad Q(0) = Q_0 \end{cases}$$

has a solution $Q(\cdot)$ in the interval $[0, \tau[$ for every symmetric matrix $Q_0 \in I\!R^{N,N}$ with $-\frac{1}{\rho} \cdot \mathrm{Id}_{I\!R^N} \le Q_0 \le \frac{1}{\rho} \cdot \mathrm{Id}_{I\!R^N}$ because the definition of μ guarantees

$$-\mu \cdot \mathrm{Id}_{I\!R^{2N}} \le \begin{pmatrix} \frac{\partial^2}{\partial x^2}\, \mathcal{H}_{\widetilde{F}}(t,x,p) & \frac{\partial^2}{\partial x\, \partial p}\, \mathcal{H}_{\widetilde{F}}(t,x,p) \\ \frac{\partial^2}{\partial p\, \partial x}\, \mathcal{H}_{\widetilde{F}}(t,x,p) & \frac{\partial^2}{\partial p^2}\, \mathcal{H}_{\widetilde{F}}(t,x,p) \end{pmatrix} \le \mu \cdot \mathrm{Id}_{I\!R^{2N}}$$

for all $0 \le t \le \widehat{\tau}$, $|x| \le R(T)$ and $\frac{1}{\widehat{R}(T)} \le |p| \le \widehat{R}(T)$.
Finally Corollary A.4.8 implies that

$$M_t^{\mapsto}(\partial K) := \Big\{ (y(t),\, q(t)) \;\Big|\; (y(\cdot),\, q(\cdot)) \text{ solves system } (*),\ \ y_0 \in \partial K \Big\} \subset I\!R^N \times I\!R^N$$

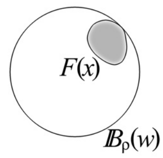
is the graph of a locally Lipschitz continuous function for all $t \in [0, \tau[$.

As in the proof of Prop. A.4.4, we obtain for every point $z \in \partial \vartheta_{\widetilde{F}}(t, K)$ that the limiting normal cone $N_{\vartheta_{\widetilde{F}}(t,K)}(z)$ contains exactly one direction and its unit vector depends on z in a Lipschitz continuous way. So $\vartheta_{\widetilde{F}}(t, K)$ is a compact N–dimensional $C^{1,1}$ submanifold of $I\!R^N$ with boundary for all $t \in [0, \tau[$. □

Finally we mention sufficient conditions of a set–valued map $F : I\!R^N \rightsquigarrow I\!R^N$ for satisfying standard hypothesis $(\overset{\circ}{\mathcal{H}})$. However these three properties are not necessary as Example A.4.16 shows.

Proposition A.4.11 *Let $F : I\!R^N \rightsquigarrow I\!R^N$ have following properties :*

1. *F has convex compact values with a* locally uniform enclosing sphere condition, *i.e. for any $R>1$, there exists a radius $\rho = \rho(R) > 0$ such that for all $x \in I\!B_\rho$, $z \in \partial F(x)$, a closed ball $I\!B_\rho(w) \subset I\!R^N$ satisfies*
$$F(x) \subset I\!B_\rho(w), \qquad F(x) \cap \partial I\!B_\rho(w) = \{z\}.$$



2. *F is locally Lipschitz continuous (with respect to $d\!l$),*
3. Graph F *is a $2N$–dimensional $C^{1,1}$ submanifold of $I\!R^{2N}$ with boundary.*

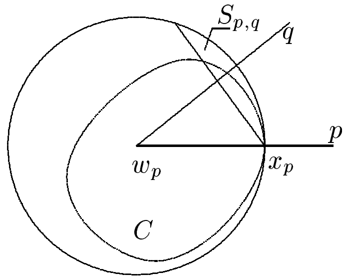
Then, $\mathcal{H}_F \in C^{1,1}(I\!R^N \times (I\!R^N \setminus \{0\}))$.
If F is (globally) Lipschitz continuous in addition, F fulfills standard hypothesis $(\overset{\circ}{\mathcal{H}})$.

Proving this proposition is based mainly on the relation between F and the derivatives of the Hamiltonian $\mathcal{H}_F$. So first we consider the partial derivative of $\mathcal{H}_F(x,p) = \sigma_{F(x)}(p)$ with respect to $p \neq 0$. Roughly speaking, the enclosing sphere condition implies that the inverse map of limiting normal cones (of a fixed convex set) is Lipschitz continuous.

Lemma A.4.12 *Suppose the* enclosing sphere condition *for the convex set $C \in \mathcal{K}(I\!R^N)$, i.e. there is a positive radius ρ such that for every boundary point $z \in \partial C$ there exists a closed ball $I\!B_\rho(w) \subset I\!R^N$ with $\quad C \subset I\!B_\rho(w), \quad C \cap \partial I\!B_\rho(w) = \{z\}$.*

Then the subdifferential $\partial^L \sigma_C(\cdot)|_{\partial I\!B_1}$ of the support function σ_C is single–valued and Lipschitz continuous (with Lipschitz constant $\leq 2\rho$).

Proof. The enclosing sphere condition implies that C is strictly convex. So $\partial^L \sigma_C(p)$ is single–valued for every $p \in \partial I\!B_1$ due to Lemma A.2.4 (2.).
For arbitrary $p, q \in I\!R^N$ with $|p| = |q| = 1$, set $x_p, x_q \in C$ such that
$$\partial^L \sigma_C(p) = \{x_p\}, \qquad \partial^L \sigma_C(q) = \{x_q\}.$$
According to the enclosing sphere condition, there is a closed ball $I\!B_\rho(w_p) \subset I\!R^N$ with
$$C \subset I\!B_\rho(w_p), \qquad C \cap \partial I\!B_\rho(w_p) = \{x_p\}.$$
$S_{p,q} := \{x \in I\!B_\rho(w_p) \mid q \cdot x \geq q \cdot x_p\}$ contains x_q (since $q \cdot x_q = \sigma_C(q) \geq q \cdot x_p$) and thus we obtain the estimate $\quad |x_q - x_p| \leq \text{diam}\, S_{p,q} \leq 2\rho$.
Now we prove $\text{diam}\, S_{p,q} \leq 2\rho\,|p-q|$ if $|p-q| < 1$ (i.e. in an equivalent way, $p \cdot q > \frac{1}{2}$).



For every $x \in S_{p,q}$, the definitions ensure

$$0 \;<\; \rho \;\; q \cdot p \;=\; q \cdot (x_p - w_p) \;\leq\; q \cdot (x - w_p) \;\leq\; \rho$$

and $S_{p,q} \subset \mathbb{B}_\rho(w_p)$ implies

$$\begin{aligned}\operatorname{dist}\Big(x,\;\; w_p + [0, \infty[\,\cdot\, q\Big)^2 \;&\leq\; \rho^2 \;-\; |q \cdot (x - w_p)|^2 \\ &\leq\; \rho^2 \;-\; \rho^2\;(q \cdot p)^2 \\ &=\; \rho^2 \;\cdot\; \Big(1 - (q \cdot p)^2\Big).\end{aligned}$$

For the difference $x - y$ of any points $x, y \in S_{p,q}$, we distinguish between the component in the direction of q and its normal component and thus,

$$\begin{aligned}(\operatorname{diam}\; S_{p,q})^2 \;&\leq\; \Big(\rho - \rho\,(q \cdot p)\Big)^2 + \Big(2\,\rho\,\sqrt{1 - (q \cdot p)^2}\Big)^2 \\ &=\; \rho^2\,\Big(1 \;-\; q \cdot p\Big)^2 \;+\; 4\,\rho^2\;\;\Big(1 - (q \cdot p)^2\Big) \\ &=\; \rho^2\,\Big(5 \;-\; 2\,q \cdot p \;-\; 3\,(q \cdot p)^2\Big) \\ &=\; \rho^2\,\Big(3 \qquad\qquad\; -\; 3\,(q \cdot p)^2 \;+\; |p - q|^2\Big) \\ &=\; \rho^2\,\Big(3\,(1 + q \cdot p)\;\;\;(1 - q \cdot p) \;\;+\; |p - q|^2\Big) \\ &\leq\; \rho^2\,\Big(3 \cdot \quad 2 \quad\cdot\; \tfrac{1}{2}\;|p - q|^2 \;+\; |p - q|^2\Big) \;\;=\;\; 4\;\rho^2\;\;|p - q|^2 \qquad \square\end{aligned}$$

As a next step, sufficient conditions are to guarantee the Lipschitz continuity of the partial derivative $\frac{\partial}{\partial p}\mathcal{H}_F(x, p)$ with respect to both x and p. Lemma A.2.4 (1.) has already given the relation to the inverse of normal cones : $\partial^L \sigma_{F(x)}(p) \;=\; \Big(N_{F(x)}\Big|_{\partial F(x)}\Big)^{-1}(p)$. So now we specify the link with the tangent cones of the graph of F according to [7, Aubin,Frankowska 90].

For considering the partial derivative of $\mathcal{H}_F(x, p)$ with respect to x afterwards, the so–called *graphical derivative* are also introduced – in connection with tangent cones.

Definition A.4.13

Let $S \subset \mathbb{R}^N$ *be a nonempty closed subset and* $x \in S$.

$$\begin{aligned}T_S^c(x) \;&:=\; \Big\{u \in \mathbb{R}^N \;\Big|\; \liminf_{h \downarrow 0}\; \tfrac{1}{h} \cdot \operatorname{dist}\Big(x + h\,u,\;\; S\Big) \;=\; 0\Big\}, \\ T_S^a(x) \;&:=\; \Big\{u \in \mathbb{R}^N \;\Big|\; \lim_{h \downarrow 0}\; \tfrac{1}{h} \cdot \operatorname{dist}\Big(x + h\,u,\;\; S\Big) \;=\; 0\Big\}\end{aligned}$$

are called the contingent cone *and the* adjacent cone *of* M *at* x, *respectively.*
For a set–valued map $F : \mathbb{R}^M \leadsto \mathbb{R}^N$, *the* contingent derivative $D^c F(x, y) : \mathbb{R}^M \leadsto \mathbb{R}^N$ *and the* adjacent derivative $D^a F(x, y) : \mathbb{R}^M \leadsto \mathbb{R}^N$ *at* $(x, y) \in$ Graph F *are defined by*

$$\text{Graph } D^c F(x, y) \;:=\; T^c_{\text{Graph } F}(x, y), \qquad \text{Graph } D^a F(x, y) \;:=\; T^a_{\text{Graph } F}(x, y). \qquad \square$$

Remark. 1. According to [7, Aubin,Frankowska 90], Prop. 5.1.4, the λ–Lipschitz continuity of a set–valued map $F : I\!R^M \leadsto I\!R^N$ ensures that the contingent derivative $D^c F(x,y)$ is λ–Lipschitz continuous with nonempty values for every $(x,y) \in \text{Graph } F$.

2. If $F : I\!R^M \leadsto I\!R^N$ has compact convex values in addition, then [7, Aubin, Frankowska 90], Prop. 5.2.6 states for every $(x,y) \in \text{Graph } F$ that the images of the adjacent derivative $D^a F(x,y)$ are convex and

$$\begin{aligned} D^a F(x,y)\,(0) &= T^c_{F(x)}(y) \\ D^a F(x,y)\,(u) + D^a F(x,y)\,(0) &= D^a F(x,y)\,(u) \qquad \text{for all } u. \;\square \end{aligned}$$

Lemma A.4.14 *Assume for the set–valued map* $F : I\!R^M \leadsto I\!R^N$:

1. *F has convex compact values with a* locally uniform enclosing sphere condition, *i.e. for any $R > 1$, there is $\rho = \rho(R) > 0$ such that for all $x \in I\!B_\rho$, $z \in \partial F(x)$, a closed ball $I\!B_\rho(w) \subset I\!R^N$ satisfies* $F(x) \subset I\!B_\rho(w), \quad F(x) \cap \partial I\!B_\rho(w) = \{z\}.$
2. *F is locally Lipschitz continuous (with respect to $d\!l$),*
3. $T^c_{\text{Graph } F}(\cdot) \cap I\!B_1 : \partial\,\text{Graph } F \leadsto I\!R^M \times I\!R^N$ *is locally Lipschitz continuous (with respect to $d\!l$).*

Set $\lambda_R := \max \left\{ \text{Lip } F|_{R I\!B},\ \text{Lip} \left(T^c_{\text{Graph } F}(\cdot) \cap I\!B_1 \right)\Big|_{\partial\,\text{Graph } F \cap (I\!B_R \times I\!R^N)} \right\}$ *for* $R > 1$. *Then,*

(i) $\partial\,\text{Graph } F \leadsto I\!R^N, \quad (x,y) \longmapsto T^c_{F(x)}(y) \cap I\!B_1$ *is locally Lipschitz continuous.*

(ii) *The limiting subdifferential of the support function* $I\!R^M \times \partial I\!B_1 \leadsto I\!R^N$, $(x,p) \longmapsto \partial^L \sigma_{F(x)}(p) = \left(N_{F(x)}\Big|_{\partial F(x)} \right)^{-1}(p)$ *is single–valued and locally Lipschitz continuous (with constant* $\leq (1 + 2\,\rho(R)) \cdot (1 + 3\,\lambda_R)^3$ *on* $I\!B_R \times \partial I\!B_1$).

Proof of Lemma A.4.14.

(i) The local Lipschitz continuity and the convex values of F guarantee

$$T^a_{\text{Graph } F}(x,y) \;\cap\; (\{0\} \times I\!R^N) \;=\; \{0\} \times T^c_{F(x)}(y)$$

for all $x \in I\!R^M$ according to the preceding remark (after Def. A.4.13).
Now choose $R > 1$, $x_1, x_2 \in I\!B_R \subset I\!R^M$, $y_1 \in \partial F(x_1)$, $y_2 \in \partial F(x_2)$ and $v_1 \in T^c_{F(x_1)}(y_1)$ with $|v_1| \leq 1$ arbitrarily. Due to assumption (3.), the adjacent cone and the contingent cone of Graph F coincide at every boundary point.

Furthermore there is a vector $(u_2, v_2) \in T^c_{\text{Graph } F}(x_2, y_2) \cap I\!B_1$ with

$$|(u_2, v_2) - (0, v_1)| \leq \lambda_R \cdot (|x_1 - x_2| + |y_1 - y_2|).$$

As another consequence of the preceding remark, there exists $v_2' \in D^c F(x_2, y_2)(0)$ (i.e. $(0, v_2') \in T^c_{\text{Graph } F}(x_2, y_2)$) satisfying

$$|v_2' - v_2| \leq \lambda_R \cdot |0 - u_2| \leq \lambda_R^2 \cdot (|x_1 - x_2| + |y_1 - y_2|).$$

So in particular, $v_2' \in T^c_{F(x_2)}(y_2)$ and

$$\begin{aligned} \text{dist}(v_1, T^c_{F(x_2)}(y_2)) &\leq |v_1 - v_2'| \leq |v_1 - v_2| + |v_2 - v_2'| \\ &\leq \lambda_R (\lambda_R + 1) \cdot (|x_1 - x_2| + |y_1 - y_2|). \end{aligned}$$

Finally $|v_1| \leq 1$ implies that the projection of v_1 on the cone $T^c_{F(x_2)}(y_2) = T^a_{F(x_2)}(y_2)$ is also contained in $I\!B_1$. Thus,

$$\text{dist}(v_1, T^c_{F(x_2)}(y_2) \cap I\!B_1) \leq \lambda_R (\lambda_R + 1) \cdot (|x_1 - x_2| + |y_1 - y_2|).$$

(ii) Lemma A.2.4 (1.) states $\partial^L \sigma_{F(x)}(p) = \left(N_{F(x)}\Big|_{\partial F(x)} \right)^{-1}(p)$ for all $x, p \neq 0$. We have already mentioned in Lemma A.4.12 that the limiting subdifferential $\partial^L \sigma_{F(x)}(\cdot)$ of the support function is single–valued on $\partial I\!B_1$ due to the enclosing sphere condition. Consider now any $R > 1$, $x_1, x_2 \in I\!B_R \subset I\!R^M$ and $p_1, p_2 \in I\!R^N$ with $|p_1| = |p_2| = 1$ and set $\{y_1\} = \partial^L \sigma_{F(x_1)}(p_1) \subset \partial F(x_1)$. Set $\mu_R := \lambda_R (\lambda_R + 1)$.

Then Lemma A.2.5 provides an element $y_2' \in \partial F(x_2)$ with $|y_2' - y_1| \leq \lambda_R \cdot |x_1 - x_2|$, and according to statement (i), the restriction

$$\partial \text{Graph } F \cap (I\!B_R \times I\!R^N) \leadsto I\!R^M, \quad (x, y) \longmapsto T^c_{F(x)}(y) \cap I\!B_1 = T^a_{F(x)}(y) \cap I\!B_1$$

is μ_R–Lipschitz continuous. The so–called Theorem of Walkup–Wets ([55, Rockafellar, Wets 98], Theorem 11.36) implies that the map of polar cones

$$\partial \text{Graph } F \cap (I\!B_R \times I\!R^N) \leadsto I\!R^M, \quad (x, y) \longmapsto N_{F(x)}(y) \cap I\!B_1 = T^a_{F(x)}(y)^* \cap I\!B_1$$

is μ_R–Lipschitz continuous as well. So there is a unit vector $q_2 \in N_{F(x_2)}(y_2')$ satisfying

$$|q_2 - p_1| \leq 3 \mu_R (1 + \lambda_R) \cdot |x_2 - x_1|$$

because for every $\delta > 0$, we get an element $q_2' \in N_{F(x_2)}(y_2') \cap I\!B_1 \setminus \{0\}$ with

$$|q_2' - p_1| \leq \mu_R \cdot (|x_2 - x_1| + |y_2' - y_1|) + \delta \leq \mu_R (1 + \lambda_R) \cdot |x_2 - x_1| + \delta$$

and then,

$$\vee \begin{cases} \left| \frac{q_2'}{|q_2'|} - p_1 \right|^2 = 2 - 2 \frac{q_2' \cdot p_1}{|q_2'|} \leq 2 - 8 q_2' \cdot p_1 \leq 4 |q_2' - p_1|^2 & \text{if } q_2' \cdot p_1 > \frac{1}{4}, \\ \left| \frac{q_2'}{|q_2'|} - p_1 \right|^2 = 2 - 2 \frac{q_2' \cdot p_1}{|q_2'|} \leq 4 \qquad\qquad \leq 8 |q_2' - p_1|^2 & \text{if } q_2' \cdot p_1 \leq \frac{1}{4}. \end{cases}$$

Finally the enclosing sphere condition (of assumption (1.)) and Lemma A.4.12 ensure for the unique element $y_2 \in \partial^L \sigma_{F(x_2)}(p_2)$

$$\begin{aligned}
|y_2 - y_1| &\le |y_2 - y_2'| + |y_2' - y_1| \\
&\le 2\,\rho(R) \cdot |p_2 - q_2| + \lambda_R \cdot |x_2 - x_1| \\
&\le 2\,\rho(R) \cdot \Big(|p_2 - p_1| + |p_1 - q_2|\Big) + \lambda_R \cdot |x_2 - x_1| \\
&\le \Big(2\,\rho(R) \cdot (1 + 3\,\mu_R \ (1+\lambda_R)\ ) + \lambda_R\Big) \cdot \Big(|x_2 - x_1| + |p_2 - p_1|\Big). \\
&\le \Big(2\,\rho(R) \cdot (1 + 3\,\lambda_R \ (1+\lambda_R)^2) + \lambda_R\Big) \cdot \Big(|x_2 - x_1| + |p_2 - p_1|\Big). \\
&\le \Big(2\,\rho(R) \cdot (1 + 3\,\lambda_R) \cdot (1+\lambda_R)^2 + \lambda_R\Big) \cdot \Big(|x_2 - x_1| + |p_2 - p_1|\Big). \\
&\le (2\,\rho(R)+1)\ (1 + 3\,\lambda_R)^3 \cdot \Big(|x_2 - x_1| + |p_2 - p_1|\Big).
\end{aligned}$$

□

The partial derivative of $\mathcal{H}_F(x,p) \overset{\text{Def.}}{=} \sup\ p \cdot F(x)$ with respect to x is closely related to the graphical derivatives of F. Quite general results about this type of functions with supremum or infimum (so–called *marginal functions*) are quoted in the next lemma.

Lemma A.4.15 ([65, Ward 94], Theorem 3.1 and Prop. 3.5)
Let $\varphi : I\!R^M \longrightarrow\ ]-\infty, \infty]$ *denote the* marginal function *of* $f : I\!R^M \times I\!R^N \longrightarrow\ ]-\infty,\infty]$ *and a set–valued map* $G : I\!R^M \leadsto I\!R^N$, *i.e.* $\varphi(x) := \inf\,\{f(x,y) \mid y \in G(x)\}$.
Moreover set $R(x) := \{y \in G(x) \mid \varphi(x) = f(x,y)\}$ *for every* $x \in I\!R^M$ *and*

$$\begin{aligned}
D^c_\uparrow \varphi(x)\,(u) &:= \liminf_{\substack{h \downarrow 0,\\ v \to u}} \frac{\varphi(x+h\,v) - \varphi(x)}{h}, \\
D^a_\uparrow \varphi(x)\,(u) &:= \limsup_{h \downarrow 0}\ \inf_{v \to u} \frac{\varphi(x+h\,v) - \varphi(x)}{h}
\end{aligned}$$

abbreviate the lower and upper epiderivative of φ, *respectively.*

1. *Suppose that* f *is lower semicontinuous,* Graph G *is closed and the asymptotic limiting subdifferential* $\partial^\infty f$ *satisfies* $-N_{\text{Graph } G}(x,y) \cap \partial^\infty f(x,y) = \{0\}$.
   Then, for every $x, u \in I\!R^M$, $y \in R(x)$,
   $$\begin{aligned}
   D^a_\uparrow \varphi(x)\,(u) &\le \inf_v\ \{D^a_\uparrow f(x,y)\,(u,v) \mid (u,v) \in T^a_{\text{Graph } G}(x,y)\} \\
   D^c_\uparrow \varphi(x)\,(u) &\le \inf_v\ \{D^c_\uparrow f(x,y)\,(u,v) \mid (u,v) \in T^a_{\text{Graph } G}(x,y)\} \\
   D^c_\uparrow \varphi(x)\,(u) &\le \inf_v\ \{D^a_\uparrow f(x,y)\,(u,v) \mid (u,v) \in T^c_{\text{Graph } G}(x,y)\}
   \end{aligned}$$

2. *Suppose the following condition : For every* $\varepsilon > 0$, $t_j \downarrow 0$ *and* $u_j \longrightarrow u$, *there is a bounded sequence* $(v_j)_{j \in I\!N}$ *such that for all* j *sufficiently large,* $y + t_j\,v_j \in G(x + t_j\,u_j)$, $f(x + t_j\,u_j,\ y + t_j\,v_j) \le \varphi(x + t_j\,u_j) + t_j\,\varepsilon$.
   Then, $D^c_\uparrow \varphi(x)\,(u) \ \ge\ \inf_v\ \{D^c_\uparrow f(x,y)\,(u,v) \mid (u,v) \in T^c_{\text{Graph } G}(x,y)\}$. □

Proof of Prop. A.4.11 According to assumption (3.) for the map $F : I\!\!R^N \leadsto I\!\!R^N$, Graph F is a $2N$–dimensional $C^{1,1}$ submanifold of $I\!\!R^{2N}$ with boundary. So the (exterior) unit normal vector of Graph F is unique and depends on the boundary point in a locally Lipschitz continuous way. Then for every $(x, y) \in \partial\,\text{Graph}\, F$, the contingent cone $T^c_{\text{Graph}\, F}(x, y)$ is a half-space and

$$\partial\,\text{Graph}\, F \leadsto I\!\!R^N \times I\!\!R^N, \qquad (x, y) \longmapsto T^c_{\text{Graph}\, F}(x, y) \,\cap\, I\!\!B_1$$

is locally Lipschitz continuous, i.e. $F : I\!\!R^N \leadsto I\!\!R^N$ satisfies the assumptions of the preceding Lemma A.4.14.

In particular, we conclude from Lemma A.4.14 (ii) that the partial derivative $\frac{\partial}{\partial p}\mathcal{H}_F$ of the Hamiltonian $\mathcal{H}_F : (x, p) \longmapsto \sigma_{F(x)}(p)$ is locally Lipschitz continuous on $I\!\!R^N \times \partial I\!\!B_1$ Thus, $\frac{\partial}{\partial p}\mathcal{H}_F \in C^{0,1}(I\!\!B_R \times (I\!\!B_R \setminus \overset{\circ}{I\!\!B}_{\frac{1}{R}}))$ for every radius $R > 1$ because the function $p \longmapsto \frac{\partial}{\partial p}\,\mathcal{H}_F(x, p) = \Big(N_{F(x)}\Big|_{\partial F(x)}\Big)^{-1}(p)$ depends only on $\frac{p}{|p|}$.

Now we have to prove that the partial derivative $\frac{\partial}{\partial x}\,\mathcal{H}_F(x, p)$ exists and is locally Lipschitz continuous on $I\!\!R^N \times (I\!\!R^N \setminus \{0\})$. Choosing any $R > 1$, $p \in I\!\!R^N$ with $|p| = 1$, we want to apply Lemma A.4.15 to the Lipschitz continuous function $\varphi : \overset{\circ}{I\!\!B}_R \longrightarrow I\!\!R$,

$$\varphi(x) \;:=\; \inf\{\, -p \cdot y \mid y \in F(x)\} \;=\; -\,\mathcal{H}_F(x, p).$$

For every $x \in \overset{\circ}{I\!\!B}_R \subset I\!\!R^N$, the element $y_x \in F(x)$ with $-p \cdot y_x = \varphi(x) = -\sigma_{F(x)}(p)$ is unique and fulfills $y_x \in \partial^L \sigma_{F(x)}(p) = \Big(N_{F(x)}\Big|_{\partial F(x)}\Big)^{-1}(p)$ according to Lemma A.2.4. So y_x depends on x, p in a locally Lipschitz continuous way due to Lemma A.4.14 (ii), i.e. particularly, the assumptions of Lemma A.4.15 (2.) are fulfilled. Thus,

$$\begin{aligned} D^c_\uparrow\,\varphi(x)\,(u) \;&=\; \inf_v \;\; \{-p \cdot v \;\mid\; (u, v) \in T^c_{\text{Graph}\, F}(x, y_x)\,\} \\ &\overset{\text{Def.}}{=} \inf_v \;\; \{-p \cdot v \;\mid\; \quad v \;\in D^c F(x, y_x)\,(u)\,\}. \end{aligned}$$

$p \in N_{F(x)}(y_x)$ is equivalent to $-p \cdot w \geq 0$ for all $w \in T^c_{F(x)}(y) = D^a F(x, y_x)\,(0)$ (using the remark after Def. A.4.13). Setting $\lambda_R \;:=\; \text{Lip}\; F|_{R\,I\!\!B}$, the λ_R–Lipschitz continuity of $D^c F(x, y_x) = D^a F(x, y_x)$ provides the following estimates for $D^c_\uparrow\,\varphi(x)\,(u)$

$$\begin{aligned} \inf\,\Big(-\,p \cdot D^c F(x, y_x)\,(u)\Big) \;&\geq\; |p| \cdot (0 \,-\, \lambda_R\,|u|) \;=\; -\,\lambda_R\,|u| \;>\; -\,\infty\,, \\ \inf\,\Big(-\,p \cdot D^c F(x, y_x)\,(u)\Big) \;&\leq\; |p| \cdot (0 \,+\, \lambda_R\,|u|) \;=\; \lambda_R\,|u| \end{aligned}$$

and thus, $\qquad D^c_\uparrow\,\varphi(x)\,(u) \;=\; \inf\limits_v \;\{-p \cdot v \;\mid\; (u, v) \in \partial\, T^c_{\text{Graph}\, F}(x, y_x),\; |v| \leq \lambda_R\,|u|\,\}.$

Since $T^c_{\text{Graph}\, F}(x, y_x)$ is even a half-space, its topological boundary $\partial\, T^c_{\text{Graph}\, F}(x, y_x)$ is a subspace of $I\!\!R^{2\,N}$ and $D^c_\uparrow\varphi(x) : I\!\!R^N \longrightarrow I\!\!R$ is linear, uniformly bounded for all $x \in \overset{\circ}{I\!\!B}_R$.

Moreover Lemma A.4.15 implies $D_\uparrow^c \varphi(x) = D_\uparrow^a \varphi(x) : \mathbb{R}^N \longrightarrow \mathbb{R}$ because of

$$\begin{array}{rcll} D_\uparrow^c \varphi(x)(\cdot) & \leq & D_\uparrow^a \varphi(x)(\cdot) & \text{(in general)}, \\ T^c_{\text{Graph } F}(x, y_x) & = & T^a_{\text{Graph } F}(x, y_x) & \text{(in particular)}. \end{array}$$

This means for the Lipschitz continuous function $\varphi : \mathring{\mathbb{B}}_R \longrightarrow \mathbb{R}$ that its lower and upper Dini derivatives coincide (according to [7, Aubin, Frankowska 90], Prop. 6.1.7, 6.2.3). So $\mathcal{H}_F(\cdot, p) = -\varphi : \mathring{\mathbb{B}}_R \longrightarrow \mathbb{R}$ has the (uniformly bounded) directional derivative

$$\begin{array}{rcl} D_x \mathcal{H}_F(x,p) : \mathbb{R}^N & \longrightarrow & \mathbb{R}, \\ u & \longmapsto & \sup_v \{ p \cdot v \mid (u,v) \in T^c_{\text{Graph } F}(x, y_x) \} = \\ & & \sup_v \{ p \cdot v \mid \quad v \in D^c F(x, y_x)(u) \cap \lambda_R \, |u| \cdot \mathbb{B} \}. \end{array}$$

Finally this directional derivative (restricted to $u \in \mathbb{B}_1$) depends on $x \in \mathring{\mathbb{B}}_R$ and $p \in \partial \mathbb{B}_1$ in a Lipschitz continuous way : As mentioned before, $(x,p) \longmapsto y_x$ is Lipschitz continuous on $\mathring{\mathbb{B}}_R \times \partial \mathbb{B}_1$ and the set–valued map

$$\partial \, \text{Graph } F \cap (\mathbb{B}_R \times \mathbb{R}^N) \leadsto \mathbb{R}^N \times \mathbb{R}^N, \qquad (x,y) \longmapsto T^c_{\text{Graph } F}(x,y) \cap \mathbb{B}_1$$

is Lipschitz continuous (with respect to $\mathbb{d}$). This implies the Lipschitz continuity of the following maps successively :

$$\begin{array}{llllcl} \mathring{\mathbb{B}}_R & \leadsto \mathbb{R}^N, & x & \longmapsto & & T^c_{\text{Graph } F}(x, y_x) \cap (\mathbb{B}_1 \times \mathbb{B}_{\lambda_R}), \\ \mathring{\mathbb{B}}_R \times \partial \mathbb{B}_1 & \leadsto \mathbb{R}, & (x,p) & \longmapsto & & (0,p) \cdot \Big(T^c_{\text{Graph } F}(x, y_x) \cap (\mathbb{B}_1 \times \mathbb{B}_{\lambda_R}) \Big), \\ \mathring{\mathbb{B}}_R \times \partial \mathbb{B}_1 & \leadsto \mathbb{R}, & (x,p) & \longmapsto & \sup & (0,p) \cdot \Big(T^c_{\text{Graph } F}(x, y_x) \cap (\mathbb{B}_1 \times \mathbb{B}_{\lambda_R}) \Big) = \\ & & & & = & \text{Graph } \; D_x \mathcal{H}_F(x,p) \quad \cap (\mathbb{B}_1 \times \mathbb{B}_{\lambda_R}) \quad = \\ & & & & = & \text{Graph } \Big(D_x \mathcal{H}_F(x,p) \Big) \Big|_{\mathbb{B}_1} \end{array}$$

So $\mathcal{H}_F \in C^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$.

For proving standard hypothesis $(\mathcal{H})$, we still need the linear growth of $D\mathcal{H}_F(\cdot,\cdot)$. So now we assume in addition that F is (globally) λ–Lipschitz continuous. Then $\frac{\partial}{\partial p} \mathcal{H}_F(x,p) \in F(x)$ implies $\left| \frac{\partial}{\partial p} \mathcal{H}_F(x,p) \right| \leq \|F(x)\|_\infty \leq \|F(0)\|_\infty + \lambda \, |x|$. Furthermore the directional derivative $D_x \mathcal{H}_F(x,p)$ satisfies (for every $p \neq 0$)

$$\|D_x \mathcal{H}_F(x,p)\|_{\mathcal{L}(\mathbb{R}^N, \mathbb{R})} \leq \lambda \, |p|. \qquad \square$$

Example A.4.16 shows that the conditions of Prop. A.4.11 are not necessary. Set $e_1 := (1, 0 \ldots 0) \in \mathbb{R}^N$, $C := \mathbb{B}_1(0) \cap \mathbb{B}_1(e_1)$ and $F : \mathbb{R}^N \leadsto \mathbb{R}^N$, $x \longmapsto C$. Obviously, $\mathcal{H}_F$ does not depend on x explicitly. C is compact, convex with nonempty interior and satisfies the enclosing sphere condition of Lemma A.4.12. So $\frac{\partial}{\partial p} \mathcal{H}_F(x,p)$ is Lipschitz continuous with respect to $p \in \partial \mathbb{B}_1$. Thus, $\mathcal{H}_F \in C^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$.

A.5 Proximally smooth reachable sets and standard hypotheses $(\mathcal{H}), (\overset{\circ}{\mathcal{H}})$ imply reversibility in time

The extended Hamilton condition leads to a necessary condition on boundary points $x \in \partial\,\vartheta_F(t,K)$ and their limiting normal cones in Prop. A.3.2. If each set $\vartheta_F(t,K)$ $(0 \le t \le T)$ has positive reach of radius ρ, then standard hypothesis $(\mathcal{H})$ turns adjoint arcs into sufficient conditions and, we conclude that the evolution of reachable sets is reversible with respect to time — in the sense of Proposition A.5.1.
This last property is well–known for autonomous differential equations (with a Lipschitz vector field), but obviously it is not correct for differential inclusions in general. The reversibility in time makes the preceding results about adjoint arcs for normal cones available in positive time direction. So it proves to be an essential step for estimating parameters like $\alpha^{\mapsto}(\vartheta_F)$, $Q^{\mapsto}(\vartheta_F, \vartheta_G)$ when applying the right–hand forward generalization to $\mathcal{K}(I\!R^N)$, $\Omega(I\!R^N)$ and autonomous differential inclusions in § 4.4.

Proposition A.5.1 *Suppose standard hypothesis* $(\mathcal{H})$ *for the map* $F : I\!R^N \rightsquigarrow I\!R^N$. *Assume for* $K_0 \in \mathcal{K}(I\!R^N)$ *and* $\rho > 0$ *that each compact set* $K_t := \vartheta_F(t, K_0)$ $(0 \le t \le T)$ *has positive reach of radius* ρ.
Then for every $0 \le s \le t < T$, $\quad K_s \;=\; I\!R^N \Big\backslash\, \vartheta_{-F}(t-s,\; I\!R^N \backslash K_t)$.

Before proving this proposition, we mention some consequences : Starting with a $C^{1,1}$ submanifold $K \in \mathcal{K}_{C^{1,1}}(I\!R^N)$, Prop. A.4.4 guarantees the existence of $\rho, \tau > 0$ such that every $\vartheta_F(t,K)$ $(0 \le t \le \tau)$ has positive reach of radius ρ. So we obtain directly

Corollary A.5.2 *Suppose standard hypothesis* $(\mathcal{H})$ *for the map* $F : I\!R^N \rightsquigarrow I\!R^N$.
For every compact N*–dimensional* $C^{1,1}$ *submanifold* K *of* $I\!R^N$ *with boundary, there exist a time* $\tau > 0$ *and a radius* $\rho > 0$ *such that for all* $t \in [0, \tau[$,

1. $\vartheta_{\widetilde{F}}(t,K) \;\in\; \mathcal{K}_{C^{1,1}}(I\!R^N)$ *with radius of curvature* $\ge \rho$,
2. $K \;=\; I\!R^N \Big\backslash\, \vartheta_{-F}(t,\; I\!R^N \setminus \vartheta_F(t,K))$. □

This form of reversibility in time can be extended to open sets by means of § A.2. According to Cor. A.2.9 and Prop. A.2.10 in particular, standard hypothesis $(\overset{\circ}{\mathcal{H}})$ for $F : I\!R^N \rightsquigarrow I\!R^N$ guarantees $\quad \partial\,\vartheta_F(t,O) \;=\; \partial\,\vartheta_F(t,\overline{O}), \quad \vartheta_F(t,O) \;=\; \big(\vartheta_F(t,\overline{O})\big)^{\circ}$ for $O \in \Omega(I\!R^N)$ and $t,\ \rho > 0$ if the closure $\vartheta_F(s,\overline{O})$ has positive reach of radius ρ for all $s \in [0,t]$. (This link has already been used for Cor. A.4.5.) Now we conclude

Corollary A.5.3 *In addition to standard hypothesis* $(\mathring{\mathcal{H}})$ *for* $F : I\!R^N \rightsquigarrow I\!R^N$, *assume for* $O \in \Omega(I\!R^N)$, $\rho > 0$ *that each set* $\vartheta_F(t, \overline{O})$ $(0 \le t \le T)$ *has positive reach of radius* ρ. *Then, for every* $t \in [0,T[$, $O = I\!R^N \Big\backslash \vartheta_{-F}\Big(t,\ I\!R^N \setminus \vartheta_F(t,O)\Big)$. □

Corollary A.5.4 *Suppose standard hypothesis* $(\mathring{\mathcal{H}})$ *for the map* $F : I\!R^N \rightsquigarrow I\!R^N$ *and let* $O \in \Omega_{C^{1,1}}(I\!R^N)$. *Then there exists* $\rho, \tau > 0$ *such that for all* $t \in [0, \tau[$,

1. $\vartheta_F(t, O) \in \Omega_{C^{1,1}}(I\!R^N)$ *with radius of curvature* $\ge \rho$,
2. $O = I\!R^N \Big\backslash \vartheta_{-F}(t,\ I\!R^N \setminus \vartheta_F(t,O))$. □

In Prop. A.5.1, we even suppose a uniform radius ρ of positive reach for $K_t \overset{\text{Def.}}{=} \vartheta_F(t, K_0)$. The essential advantage for the proof is the relation between the boundaries of $K_t \subset I\!R^N$ and Graph $(t \longmapsto K_t) \subset I\!R \times I\!R^N$ stated in Prop. A.2.10 :

$$\partial\,\text{Graph}\ \vartheta_F(\,\cdot\,, K_0)|_{[0,T]} \;=\; (\{0\} \times K_0) \,\cup \bigcup_{0<t<T} (\{t\} \times \partial\vartheta_F(t, K_0)) \,\cup\, (\{T\} \times \vartheta_F(T, K_0))\,.$$

Proof of Prop. A.5.1 $\vartheta_F(s, K_0) \subset I\!R^N \setminus \vartheta_{-F}(t-s,\ I\!R^N \setminus K_t)$ is an indirect consequence of definitions : It is equivalent to $\vartheta_F(s, K_0) \cap \vartheta_{-F}(t-s,\ I\!R^N \setminus K_t) = \emptyset$. For every point $y \in \vartheta_F(s, K_0) \cap \vartheta_{-F}(t-s,\ I\!R^N \setminus K_t)$, there exist two trajectories $x_1(\cdot) \in AC([0,s], I\!R^N)$, $x_2(\cdot) \in AC([0,t-s], I\!R^N)$ of $\dot{x}_1 \in F(x_1)$, $\dot{x}_2 \in -F(x_2)$, respectively, with $x_1(0) \in K_0$, $x_1(s) = y = x_2(t-s)$, $x_2(0) \notin K_t$.
Then the function $x(\cdot) : [0,t] \longrightarrow I\!R^N$,

$$x(\sigma) \;:=\; \begin{cases} x_1(\sigma) & \text{for } \ 0 \le \sigma \le s, \\ x_2(t-\sigma) & \text{for } \ s < \sigma \le t \end{cases}$$

is a solution of $\dot{x}(\cdot) \in F(x(\cdot))$ almost everywhere satisfying $x(0) \in K_0$, $x(t) \notin K_t$ — contradicting the definition of $K_t \overset{\text{Def.}}{=} \vartheta_F(t, K_0)$.

For proving the inverse inclusion indirectly at time $s = 0$, we assume the existence of a time $t \in [0,T[$ and a point $y_0 \in I\!R^N$ with $y_0 \notin K_0 \cup \vartheta_{-F}(t,\ I\!R^N \setminus K_t)$.
As an immediate consequence of $y_0 \notin \vartheta_{-F}(t,\ I\!R^N \setminus K_t)$, the reachable set $\vartheta_F(t, y_0)$ is contained in $K_t \overset{\text{Def.}}{=} \vartheta_F(t, K_0)$. Now set $\tau := \inf\{s \in [0,t] \mid \vartheta_F(s, y_0) \subset \vartheta_F(s, K_0)\}$.
In particular, $\tau > 0$ due to $y_0 \notin K_0$.
and $\vartheta_F(\tau, y_0) \subset \vartheta_F(\tau, K_0)$ due to the continuity of the reachable sets.
There are sequences $\tau_n \nearrow \tau$ and $(x_n(\cdot))_{n \in I\!N}$ in $AC([0,T], I\!R^N)$ satisfying

$$\dot{x}_n(\cdot) \in F(x_n(\cdot)) \quad \text{a.e.}, \qquad x_n(0) = y_0, \qquad x_n(\tau_n) \notin \vartheta_F(\tau_n, K_0).$$

Then for each $n \in I\!N$, we obtain

$$\begin{array}{rclr} x_n(s) & \notin & \vartheta_F(s, K_0) & \text{for every } s \in [0, \tau_n], \\ x_n(s) & \in & \vartheta_F(s, K_0) & \text{for every } s \in [\tau, T\,]. \end{array}$$

Furthermore Lemma A.4.3 provides uniform bounds and the equicontinuity of all $x_n(\cdot)$, $n \in I\!N$. So the Theorems of Arzela–Ascoli and Dunford–Pettis lead to subsequences (again denoted by) $(\tau_n)_{n \in I\!N}$, $(x_n(\cdot))_{n \in I\!N}$ and a function $x(\cdot) \in AC([0,T], I\!R^N)$ with

$$\begin{array}{rclr} x_n(\cdot) & \longrightarrow & x(\cdot) & \text{uniformly in } [0,T], \\ \dot{x}_n(\cdot) & \rightharpoonup & \dot{x}(\cdot) & \text{in } L^1([0,T], I\!R^N). \end{array}$$

$x(\cdot)$ is a solution of $\dot{x}(\cdot) \in F(x(\cdot))$ (almost everywhere) according to the compactness of trajectories (see e.g. [63, Vinter 2000], Theorem 2.5.3). In addition, it fulfills

$$\begin{array}{rclr} x(0) & = & y_0, & \\ x(s) & \notin & \vartheta_F(s, K_0) & \text{for every } s \in [0, \tau[, \\ x(s) & \in & \vartheta_F(s, K_0) & \text{for every } s \in [\tau, T], \end{array}$$

and thus, $(\tau, x(\tau))$ is a boundary point of Graph $\vartheta_F(\cdot, K_0)$.
Prop. A.2.10 and $0 < \tau \leq t < T$ ensure $x_\tau := x(\tau) \in \partial K_\tau \overset{\text{Def.}}{=} \partial\, \vartheta_F(\tau, K_0)$.

Moreover, $K_\tau \overset{\text{Def.}}{=} \vartheta_F(\tau, K_0)$ is supposed to have positive reach. So its limiting and proximal normal cone coincide at each boundary point and thus,

$$\emptyset \neq N_{\vartheta_F(\tau,K_0)}(x_\tau) = N^P_{\vartheta_F(\tau,K_0)}(x_\tau) \subset N^P_{\vartheta_F(\tau,\,y_0)}(x_\tau).$$

For every unit vector $\nu \in N_{\vartheta_F(\tau,K_0)}(x_\tau)$, Proposition A.3.2 leads to a trajectory $z(\cdot) \in AC([0,\tau], I\!R^N)$ of F and its adjoint arc $q(\cdot) \in AC([0,\tau], I\!R^N)$ satisfying

$$\wedge \left\{ \begin{array}{rclrcl} \dot{z}(t) & = & \frac{\partial}{\partial q}\, \mathcal{H}_F(z(t),\, q(t)), & z(\tau) & = & x_\tau, \\ \dot{q}(t) & = & -\frac{\partial}{\partial z}\, \mathcal{H}_F(z(t),\, q(t)), & q(\tau) & = & \nu \end{array} \right.$$

and $z(0) \in K_0$. Besides, the same Cauchy problem is solved by $x(\cdot)$ and its adjoint. $\mathcal{H}_F \in C^{1,1}$ implies the uniqueness of solutions and, its consequence $z(0) = x(0) \notin K_0$ leads to a contradiction. Thus, $I\!R^N \setminus \vartheta_{-F}(t,\, I\!R^N \setminus K_t) \subset K_0$.
Finally the corresponding inclusion for any $0 < s \leq t < T$ results from the semigroup property of reachable sets. □

Remark. 1. The map $\mathcal{K}(I\!R^N) \rightsquigarrow I\!R^N$, $K_0 \longmapsto I\!R^N \setminus \vartheta_{-F}(t,\, I\!R^N \setminus \vartheta_F(t, K_0))$ generalizes the morphological operation of closing (of sets in $\mathcal{K}(I\!R^N)$) that was introduced by Minkowski and is usually defined as

$$\mathcal{P}(X) \rightsquigarrow X, \qquad K \longmapsto (K - t\,B) \ominus (-t\,B) \overset{\text{Def.}}{=} \{\, y \in X \mid y - t\,B \subset K - t\,B \,\}$$

for a vector space X and fixed $B \subset X$, $t > 0$ (see e.g. [2, Aubin 99], Def. 3.3.1).

2. In [11, Barron, Cannarsa, Jensen, Sinestrari 99], the viscosity solutions of the Hamilton–Jacobi equation $\partial_t u + H(t,x,Du) = 0$ are investigated and roughly speaking, the continuous differentiability of u is concluded from the reversibility in time :
If $u : [0,T] \times I\!R^N \longmapsto I\!R$ is a continuous viscosity solution of $\partial_t u + H(t, \cdot, Du) = 0$ and $v(t,x) := u(T-t,x)$ is a viscosity solution of $\partial_t v - H(T-t,\cdot,Dv) = 0$ then adequate assumptions of H ensure $u \in C^1(]0,T[\,\times I\!R^N)$.
Referring to the relation between reachable sets and level sets of viscosity solutions, we draw an inverse conclusion as we assume smoothness and obtain the reversibility in time.

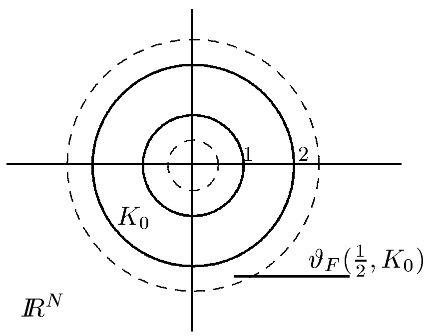
3. Furthermore it is shown for some optimal control problems in [11] that the continuous viscosity solution u of the Hamilton–Jacobi equation is even in $C^1([0,T]\times I\!R^N)$ if both $u(0,\cdot)$ and $u(T,\cdot)$ are of class C^1.
In the geometric context here, we cannot restrict ourselves to regularity assumptions about K_0 and $\vartheta_F(T,K_0)$ as the following Example A.5.5 shows.

4. The reversibility in time (in the sense of Prop. A.5.1) can also be regarded as recovering the initial data. Further results about this problem have already been published in [57, Rzeżuchowski 97] and [58, Rzeżuchowski 99], for example, but they usually assume other conditions. Either the initial set consists of only one point or the Hamiltonian function $\mathcal{H}_F$ is of class C^2.

Example A.5.5 is the same as Example A.2.11 and shows now two further aspects :
Firstly, we cannot restrict ourselves to regularity conditions at time $t = 0$ and $t = T$. Secondly, the reversibility in time does not result directly from the positive reach of every compact set $\vartheta_F(t,K_0)$ $(0 \le t \le T)$.
Set $F(x) := I\!B_1$ for all $x \in I\!R^N$ and the compact initial set $K_0 := I\!B_2 \setminus \mathring{I\!B}_1$. Then,

$$\vartheta_F(t,K_0) = I\!B_{t+2} \setminus \mathring{I\!B}_{1-t} \quad \text{for } 0 \le t < 1,$$
$$\vartheta_F(t,K_0) = I\!B_{t+2} \quad \text{for } t \ge 1.$$

So all these sets have positive reach.
Obviously this evolution is not reversible in time because roughly speaking, the hole at 0 disappears at time $t = 1$.



This example gives a hint how to 'realize' this singularity. Starting with the interior of K_0 leads to the open reachable set $\vartheta_F(1,K_0^\circ) = \mathring{I\!B}_3 \setminus \{0\}$ (as stated by Cor. A.2.9).

A.6 Standard hypothesis ($\mathcal{H}$) preserves sets of positive erosion

Sets of positive erosion have very useful features with regard to sequential compactness and regularity at the boundary (as mentioned in § 4.2.1, § 4.2.2 and § 4.3). For this reason we are interested in sufficient conditions on a map $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ guaranteeing that initial sets of positive erosion preserve this property.

Prop. A.6.1 gives an answer for some control systems and has already been proven in [44, Lorenz 2003]. Now we show (in Prop. A.6.2) that standard hypothesis ($\mathcal{H}$) for $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ implies its assumptions. Strictly speaking, each map $F : I\!\!R^N \rightsquigarrow I\!\!R^N$ satisfying standard hypothesis ($\mathcal{H}$) has a semiconcave parameterization as it is required for Prop. A.6.1. Then the differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ (a.e.) is regarded as a control system with sufficiently smooth right–hand side.
In particular, this property has the advantage that results of optimal control theory (e.g. in [37, Frankowska 2002]) can be applied to these differential inclusions.

Proposition A.6.1 ([44, Lorenz 2003], Theorem 2.1)
Assume for a complete separable metric space $\mathcal{Z}$, a function $f : [0,T] \times I\!\!R^N \times \mathcal{Z} \longrightarrow I\!\!R^N$ and a set–valued map $U : [0,T] \rightsquigarrow \mathcal{Z}$

1. $f(\cdot, x, u) : [0,T] \longrightarrow I\!\!R^N$ *is measurable* *for any* (x,u),
2. $f(t,\cdot,\cdot) : I\!\!R^N \times \mathcal{Z} \longrightarrow I\!\!R^N$ *is continuous* *for a.e.* $t \in [0,T]$,
3. $\exists\ k \in L^1([0,T]) :$ $f(t,\cdot,u)$ *is* $k(t)$*–Lipschitz* *for a.e.* $t \in [0,T]$ *and any* $u \in U(t)$,
4. $\exists\ \mu < \infty :$ $\frac{\partial}{\partial x} f(t,\cdot,u)$ *is* μ*–Lipschitz* *for any* $t \in [0,T]$ *and any* $u \in U(t)$,
5. $\exists\ \gamma \in L^1([0,T]) :$ $\sup\limits_{u \in U(t)} |f(t,0,u)| \le \gamma(t)$ *for a.e.* $t \in [0,T]$,
6. $U(\cdot)$ *is measurable and has closed nonempty images.*

Let $K \in \mathcal{K}(I\!\!R^N)$ have positive erosion of radius $\rho_K > 0$.
Then for every $t \in [0,T]$, the reachable set $\vartheta_{f(\cdot,\cdot,U)}(t,K)$ has positive erosion of radius
$$r(t) \;\ge\; \mathrm{const}(\mu,\gamma) \cdot \frac{\rho_K}{1+\rho_K\,T}\; e^{-4 \cdot \int_0^t |k|\,ds} \;>\; 0. \qquad \square$$

Proposition A.6.2 *Under standard hypothesis ($\mathcal{H}$) for $F : I\!\!R^N \rightsquigarrow I\!\!R^N$, there exists a function $\psi \in C^{1,1}(I\!\!R^N \times I\!\!R^N,\ I\!\!R^N)$ with $F(x) = \psi(x, I\!\!B_1)$ for every $x \in I\!\!R^N$ and* $\|\psi\|_{C^{1,1}(K \times I\!\!B_1)} \le \mathrm{const}(\|\mathcal{H}_F\|_{C^{1,1}(K \times \partial I\!\!B_1)},\ N)$ *for every $K \in \mathcal{K}(I\!\!R^N)$.*

Proof. Due to standard hypothesis $(\mathcal{H})$, the Hamiltonian $\mathcal{H}_F : (x,p) \longmapsto \sigma_{F(x)}(p)$ belongs to $C^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$. So the function of Steiner points of $F(\cdot)$

$$s_N(F(\cdot)) : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad x \longmapsto \tfrac{N}{\mathcal{L}^{N-1}(\partial \mathbb{B}_1)} \int_{\partial \mathbb{B}_1} p \;\; \mathcal{H}_F(x,p) \;\; d\omega_p$$

is differentiable with locally Lipschitz continuous derivative : $s_N(F(\cdot)) \in C^{1,1}(\mathbb{R}^N)$ and $\|s_N(F(\cdot))\|_{C^{1,1}(K)} \leq N \cdot \|\mathcal{H}_F\|_{C^{1,1}(K \times \partial \mathbb{B}_1)}$ for every $K \in \mathcal{K}(\mathbb{R}^N)$.

Now we define $\psi : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$,

$$\psi(x,p) := \begin{cases} s_N(F(x)) + \Big(\mathcal{H}_F(x, \frac{p}{|p|}) - s_N(F(x)) \cdot \frac{p}{|p|}\Big) \; |p|^8 \; p & \text{for } p \neq 0, \\ s_N(F(x)) & \text{for } p = 0. \end{cases}$$

Then $F(x) = \psi(x, \mathbb{B}_1)$ is an obvious consequence of $F(x) \in \mathcal{K}(\mathbb{R}^N)$ being convex. Moreover ψ is continuously differentiable in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$ and its derivative is locally Lipschitz continuous. Finally the factor $|p|^8$ ensures that ψ has these properties even in $\mathbb{R}^N \times \mathbb{R}^N$ and in particular, $\|\psi\|_{C^{1,1}(K \times \mathbb{B}_1)} \leq \text{const}(\|\mathcal{H}_F\|_{C^{1,1}(K \times \partial \mathbb{B}_1)}, N)$ for every $K \in \mathcal{K}(\mathbb{R}^N)$. □

So Proposition A.6.1 in combination with Proposition A.6.2 leads to

Corollary A.6.3 *Suppose standard hypothesis* $(\mathcal{H})$ *for the map* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *and let* $K \in \mathcal{K}(\mathbb{R}^N)$ *have positive erosion of radius* $\rho_K > 0$.
Then for every $T > 0$, *each reachable set* $\vartheta_F(t, K)$ $(t \in [0,T])$ *has positive erosion of radius* $\rho(t) \geq C_1 \cdot \frac{\rho_K}{1 + \rho_K\, T}\, e^{-C_2\, t}$ *with constants* $C_k = C_k(N, \|\mathcal{H}_F\|_{C^{1,1}(\mathbb{B}_R \times \partial \mathbb{B}_1)})$ *and* $R > 0$ *satisfying* $\bigcup_{0 \leq s \leq T} \vartheta_F(s, K) \subset \mathbb{B}_R$. □

As a more general consequence of Prop. A.6.1, sets of positive erosion are also preserved by some maps $\widetilde{G} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ that are piecewise constant with respect to time. In particular, the same estimate of the radius holds.

Corollary A.6.4 *Assume standard hypothesis* $(\mathcal{H})$ *for* $F_1, F_2 \dots F_m : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. *For a partition* $0 = \tau_0 < \tau_1 < \ldots < \tau_m = T$ *of* $[0,T]$, *define* $\widetilde{G} : [0,T[\, \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *as* $\widetilde{G}(t,x) := F_{j+1}(x)$ *for* $\tau_j \leq t < \tau_{j+1}$.
Moreover let $K \in \mathcal{K}(\mathbb{R}^N)$ *have positive erosion of radius* $\rho_K > 0$.

Then for every time $t \in [0,T[$, *the reachable set* $\vartheta_{\widetilde{G}}(t,K)$ *has positive erosion of radius* $\rho(t) \geq C_1 \cdot \frac{\rho_K}{1 + \rho_K\, t}\, e^{-C_2\, t}$ *with constants* $C_k = C_k(N, \max_j \|\mathcal{H}_{F_j}\|_{C^{1,1}(\mathbb{B}_R \times \partial \mathbb{B}_1)})$ *and* $R > 0$ *satisfying* $\bigcup_{0 \leq s < T} \vartheta_{\widetilde{G}}(s, K) \subset \mathbb{B}_R$. □

In § A.2 we investigated the relation between the reachable set $\vartheta_F(t, O)$ of $O \in \Omega(\mathbb{R}^N)$ and its closure $\overline{\vartheta_F(t, O)} = \vartheta_F(t, \overline{O})$ if the values of F have nonempty interior. Now these results bridge the gap between the preceding Corollary A.6.4 and open sets of positive erosion :

Corollary A.6.5 *Assume standard hypothesis $(\mathring{\mathcal{H}})$ for $F_1, F_2 \dots F_m : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. For a partition $0 = \tau_0 < \tau_1 < \dots < \tau_m = T$ of $[0, T]$, define $\widetilde{G} : [0, T[\, \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ as $\widetilde{G}(t, x) := F_{j+1}(x)$ for $\tau_j \le t < \tau_{j+1}$.*
Let $O \in \Omega(\mathbb{R}^N)$ have positive erosion of radius ρ_0.

Then for every time $t \in [0, T[$, the reachable set $\vartheta_{\widetilde{G}}(t, O)$ has positive erosion of radius $\rho(t) \ge C_1 \cdot \frac{\rho_0}{1 + \rho_0\, t}\, e^{-C_2\, t}$ with constants $C_k = C_k(N, \max_j \|\mathcal{H}_{F_j}\|_{C^{1,1}(\mathbb{B}_R \times \partial \mathbb{B}_1)})$ and $R > 0$ satisfying $$\bigcup_{0 \le s < T} \vartheta_{\widetilde{G}}(s, \overline{O}) \subset \mathbb{B}_R.$$

Proof. According to Cor. A.6.4, the closed reachable set $\overline{\vartheta_{\widetilde{G}}(t, O)} = \vartheta_{\widetilde{G}}(t, \overline{O})$ has positive erosion of radius $\ge C_1 \cdot \frac{\rho_0}{1 + \rho_0\, t}\, e^{-C_2\, t} =: \widehat{\rho}(t)$. As an abbreviation, define $M(t) := \vartheta_{\widetilde{G}}(t, \overline{O}) \setminus \mathring{\mathbb{B}}_{\widehat{\rho}(t)}\big(\partial\, \vartheta_{\widetilde{G}}(t, \overline{O})\big)$ so that $\mathbb{B}_{\widehat{\rho}(t)}(M(t)) = \vartheta_{\widetilde{G}}(t, \overline{O})$. Now we show $\vartheta_{\widetilde{G}}(t, O) = \mathring{\mathbb{B}}_{\widehat{\rho}(t)}\big(\mathrm{Limsup}_{s \uparrow t}\ M(t)\big)$ for every time $t \in \,]0, T[$.

Since $\widetilde{G}$ is piecewise constant with respect to time, Prop. A.2.8 guarantees
$$\partial\, \vartheta_{\widetilde{G}}(t, O) = \mathrm{Limsup}_{s \uparrow t}\ \partial\, \vartheta_{\widetilde{G}}(s, \overline{O}) \qquad \text{for every } t \in \,]0, T[$$
and Cor. A.2.9 implies $\big(\mathrm{Graph}\ \vartheta_F(\,\cdot\,, \overline{O})\big)^\circ = \bigcup_{t > 0} \left(\{t\} \times \vartheta_F(t, O)\right).$
So for each $x \in \vartheta_{\widetilde{G}}(t, O)$, there is $\varepsilon > 0$ with $]t-\varepsilon,\ t+\varepsilon[\, \times \mathbb{B}_{2\varepsilon}(x) \subset \mathrm{Graph}\ \vartheta_{\widetilde{G}}(\,\cdot\,, \overline{O})$, i.e. particularly $\mathbb{B}_{2\varepsilon}(x) \subset \vartheta_{\widetilde{G}}(s, \overline{O}) = \mathbb{B}_{\widehat{\rho}(s)}(M(s))$ for all $s \in \,]t-\varepsilon,\ t]$.
As a consequence, $\mathbb{B}_{\varepsilon}(x) \subset \mathbb{B}_{\widehat{\rho}(t)+\delta}\big(\mathrm{Limsup}_{s \uparrow t}\ M(s)\big)$ for arbitrary $\delta > 0$
and thus, $x \in \mathring{\mathbb{B}}_{\widehat{\rho}(t)}\big(\mathrm{Limsup}_{s \uparrow t}\ M(s)\big)$.

Furthermore every point $y \in \mathring{\mathbb{B}}_{\widehat{\rho}(t)}\big(\mathrm{Limsup}_{s \uparrow t}\ M(t)\big)$ is contained in the ball $\mathbb{B}_{\widehat{\rho}(t)-4\varepsilon}\big(\mathrm{Limsup}_{s \uparrow t}\ M(t)\big)$ with some small $\varepsilon > 0$. Due to the continuity of $\widehat{\rho}(\cdot)$, there exists a sequence $s_n \uparrow t$ with $y \in \mathbb{B}_{\widehat{\rho}(s_n)-3\varepsilon}(M(s_n))$, i.e. $\mathbb{B}_{2\varepsilon}(y) \subset \vartheta_{\widetilde{G}}(s_n, \overline{O})$. Now Lemma A.2.7 implies $\vartheta_{\widetilde{G}}(t - s_n,\ \mathring{\mathbb{B}}_{2\varepsilon}(y)) \subset \vartheta_{\widetilde{G}}(t, O)$ for all $n \in \mathbb{N}$ and finally we obtain $\mathbb{B}_{\varepsilon}(y) \subset \vartheta_{\widetilde{G}}(t - s_n,\ \mathring{\mathbb{B}}_{2\varepsilon}(y)) \subset \vartheta_{\widetilde{G}}(t, O)$ for all n large enough.

□

A.7 Standard hypothesis $(\mathcal{H}^\rho_\circ)$ makes points evolve into sets of positive erosion

Our aim still consists in sufficient conditions for the positive erosion of $\vartheta_F(t, K)$. Weakening the assumption about the initial set $K \in \mathcal{K}_\circ(I\!R^N)$ in Cor. A.6.3 usually requires stronger properties of the map $F : I\!R^N \rightsquigarrow I\!R^N$ than standard hypothesis $(\mathcal{H})$ (see Definition A.4.2).

Definition A.7.1 *For any* $\rho > 0$, *a set–valued map* $F : I\!R^N \rightsquigarrow I\!R^N$ *satisfies the so–called* standard hypothesis $(\mathcal{H}^\rho_\circ)$ *if it has the following properties :*

1. *F has convex values in* $\mathcal{K}^\rho_\circ(I\!R^N)$,
2. *for every* $R > 1$, $\mathcal{H}_F(\cdot,\cdot) \in C^2(I\!B_R \times (I\!B_R \setminus \overset{\circ}{I\!B}_{\frac{1}{R}}))$,
3. *the derivative of* $\mathcal{H}_F$ *has linear growth, i.e. there is some* $\gamma_F > 0$ *with*
$$\Big\| D\,\mathcal{H}_F(x,p) \Big\|_{\mathcal{L}(I\!R^N \times I\!R^N, I\!R)} \le \gamma_F \cdot (1 + |x| + |p|) \qquad \textit{for all } x, p \in I\!R^N \ (|p| \ge 1).$$

Remark. Standard hypothesis $(\mathcal{H}^\rho_\circ)$ differs from its counterpart $(\mathcal{H})$ in two respects : The values of F have uniform positive erosion (additionally) and its Hamiltonian is even twice continuously differentiable in $I\!R^N \times (I\!R^N \setminus \{0\})$. This second restriction has the advantage that we can apply the tools of matrix Riccati equation (mentioned in Lemma A.4.7 — Lemma A.4.9).

Proposition A.7.2 *Let* $F_1 \dots F_m : I\!R^N \rightsquigarrow I\!R^N$ *hold standard hypothesis* $(\mathcal{H}^\rho_\circ)$ *and*
$$\|\mathcal{H}_{F_j}\|_{C^{1,1}(I\!R^N \times\, \partial I\!B_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_{F_j}\|_{C^{1}(I\!R^N \times\, \partial I\!B_1)} + \text{Lip } D\mathcal{H}_{F_j}|_{I\!R^N \times\, \partial I\!B_1} < \lambda$$
for some $\lambda, \rho > 0$. *Moreover for a partition* $0 \le \tau_0 < \tau_1 < \dots < \tau_m = 1$ *of* $[0,1]$, *define the map* $\widetilde{G} : [0,1[\, \times I\!R^N \rightsquigarrow I\!R^N$ *as* $\widetilde{G}(t,x) := F_j(x)$ *for* $\tau_{j-1} \le t < \tau_j$. *Furthermore choose* $K \in \mathcal{K}(I\!R^N)$ *arbitrarily.*

Then there exist $\sigma > 0$ *and a time* $\widehat{\tau} \in \,]0,1]$ *(depending only on* λ, ρ, K*) such that the reachable set* $\vartheta_{\widetilde{G}}(t, x_0)$ *has positive erosion of radius* $\sigma\, t$ *for any* $t \in \,]0, \widehat{\tau}[$, $x_0 \in K$. *As an immediate consequence,* $\vartheta_{\widetilde{G}}(t, K_1)$ *has positive erosion of radius* $\sigma\, t$ *for all* $t \in]0, \widehat{\tau}[$ *and each initial subset* $K_1 \in \mathcal{K}(I\!R^N)$ *of* K.

The proof of this proposition uses matrix Riccati equations for Hamiltonian systems, but these tools of § A.4 consider initial values induced by a Lipschitz function ψ. So roughly speaking, we exchange the two components $(x(\cdot), p(\cdot))$ (of a trajectory and its adjoint) preserving the Hamiltonian structure of their differential equations :

Lemma A.7.3 *Assume the Hamiltonian system for* $x(\cdot),\ p(\cdot) \in AC([0,T], I\!R^N)$

$$\wedge \begin{cases} \dot{x}(t) &= \ \ \frac{\partial}{\partial p} H_1(t,\ x(t),\ p(t)) \\ \dot{p}(t) &= -\frac{\partial}{\partial x} H_1(t,\ x(t),\ p(t)) \end{cases} \qquad a.e.\ in\ [0,T]$$

with sufficiently smooth $H_1 : [0,T] \times I\!R^N \times I\!R^N \longrightarrow I\!R$. *Moreover set*

$$y(t) := -p(t), \qquad q(t) := x(t) \qquad H_2(t,\ \xi,\ \zeta) := H_1(t,\ \zeta,\ -\xi).$$

Then the absolutely continuous functions $(y(\cdot), q(\cdot))$ *satisfy the Hamiltonian system*

$$\wedge \begin{cases} \dot{y}(t) &= \ \ \frac{\partial}{\partial q} H_2(t,\ y(t),\ q(t)) \\ \dot{p}(t) &= -\frac{\partial}{\partial y} H_2(t,\ y(t),\ q(t)) \end{cases} \qquad a.e.\ in\ [0,T].$$

□

Proof of Prop. A.7.2. The uniform bound λ of $\|\mathcal{H}_{F_j}\|_{C^{1,1}(I\!R^N \times\, \partial I\!B_1)}$ $(j = 1 \ldots m)$ and Gronwall's Lemma lead to a radius $R = R(\lambda, K) > 1$ and a time $T = T(\lambda, K) \in]0,1[$ such that 1. $\vartheta_{\widetilde{G}}(t,K) \subset I\!B_R$ for all $t \in [0,1]$,

2. for every trajectory $x(\cdot)$ of $\widetilde{G}$ starting in K, each adjoint $p(\cdot)$ with $\frac{1}{2} \leq |p(0)| \leq 2$ fulfills $\frac{1}{R} < |p(\cdot)| < R$, $|p(\cdot) - p(0)| < \frac{1}{4\,R}$ on $[0,T]$

(as we have already shown for Prop. A.4.10). So a smooth cut–off function again provides a map $H_1 : [0,T] \times I\!R^N \times I\!R^N \longrightarrow I\!R$ that satisfies the assumptions of Corollary A.4.8 and $H_1(t,\cdot,\cdot) = \mathcal{H}_{\widetilde{G}}(t,\cdot,\cdot)$, $\partial^j_{(x,p)} H_1(t,\cdot,\cdot) = \partial^j_{(x,p)} \mathcal{H}_{\widetilde{G}}(t,\cdot,\cdot)$ in $I\!R^N \times (I\!R^N \setminus I\!B_{\frac{1}{2R}})$ for every $t \in [0,T]$, $j = 1, 2$.

Now we use the transformation of the preceding Lemma A.7.3 and define

$$H_2 :\ [0,T] \times I\!R^N \times I\!R^N \longrightarrow I\!R, \qquad (t,\xi,\zeta) \longmapsto H_1(t,\ \zeta,\ -\xi)$$

still holding the conditions of Corollary A.4.8. As a consequence, we obtain for any initial point $x_0 \in K$ and time $\tau \in\ ]0,T]$ that the following statements are equivalent :

(*i*) For all $t \in [0,\tau]$, the set M^1_t of all points $(p(t),\ x(t))$ with solutions $(x(\cdot), p(\cdot)) \in AC([0,t], I\!R^N \times I\!R^N)$ of

$$\wedge \begin{cases} \dot{x}(s) &= \ \ \frac{\partial}{\partial p} H_1(s,\ x(s),\ p(s)), & x(0) = x_0 \\ \dot{p}(s) &= -\frac{\partial}{\partial x} H_1(s,\ x(s),\ p(s)), & p(0) \in I\!B_2 \setminus \overset{\circ}{I\!B}_{\frac{1}{2}} \end{cases}$$

is the graph of a continuously differentiable function f_t.

(*ii*) For all $t \in [0,\tau]$, the set M^2_t of all points $(y(t),\ q(t))$ with solutions $(y(\cdot), q(\cdot)) \in AC([0,t], I\!R^N \times I\!R^N)$ of

$$\wedge \begin{cases} \dot{y}(s) &= \ \ \frac{\partial}{\partial q} H_2(s,\ y(s),\ q(s)), & y(0) \in I\!B_2 \setminus \overset{\circ}{I\!B}_{\frac{1}{2}} \\ \dot{q}(s) &= -\frac{\partial}{\partial y} H_2(s,\ y(s),\ q(s)), & q(0) = x_0 \end{cases}$$

is the graph of a continuously differentiable function g_t (and $g_t(\xi) = f_t(-\xi)$).

(iii) For any solution $(y,q) : [0,t] \longrightarrow I\!R^N \times I\!R^N$ of the initial value problem (ii) $(t \le \tau)$, there exists a solution $Q : [0,t] \longrightarrow I\!R^{N \times N}$ of the Riccati equation

$$\wedge \begin{cases} \dot{Q} \;+\; \frac{\partial^2 H_2}{\partial q\, \partial y}(s,\, y(s),\, q(s))\;\; Q \;+\; Q\;\; \frac{\partial^2 H_2}{\partial y\, \partial q}(s,\, y(s),\, q(s)) \\ \quad +\; Q\;\; \frac{\partial^2 H_2}{\partial q^2}(t,\, y(s),\, q(s))\;\; Q \;+\; \frac{\partial^2 H_2}{\partial y^2}(s,\, y(s),\, q(s)) \;=\; 0, \\ \qquad Q(0) \;=\; 0. \end{cases}$$

(iv) For any solution $(x,p) : [0,t] \longrightarrow I\!R^N \times I\!R^N$ of the initial value problem (i) $(t \le \tau)$, there exists a solution $Q : [0,t] \longrightarrow I\!R^{N \times N}$ of the Riccati equation

$$\wedge \begin{cases} \dot{Q} \;-\; \frac{\partial^2 H_1}{\partial x\, \partial p}(s,\, x(s),\, p(s))\;\; Q \;-\; Q\;\; \frac{\partial^2 H_1}{\partial p\, \partial x}(s,\, x(s),\, p(s)) \\ \quad +\; Q\;\; \frac{\partial^2 H_1}{\partial x^2}(s,\, x(s),\, p(s))\;\; Q \;+\; \frac{\partial^2 H_1}{\partial p^2}(s,\, x(s),\, p(s)) \;=\; 0, \\ \qquad Q(0) \;=\; 0. \end{cases}$$

Now we give a criterion for the choice of $\widehat{\tau}$: Setting

$$\mu \;=\; \mu(\lambda, K) \;:=\; \sup_{\substack{0 \le t \le T \\ |x| \le R \\ \frac{1}{R} \le |p| \le R}} \left\| \begin{pmatrix} \frac{\partial^2}{\partial p^2}\, \mathcal{H}_{\widetilde{G}}(t,x,p) & -\frac{\partial^2}{\partial x\, \partial p}\, \mathcal{H}_{\widetilde{G}}(t,x,p) \\ -\frac{\partial^2}{\partial p\, \partial x}\, \mathcal{H}_{\widetilde{G}}(t,x,p) & \frac{\partial^2}{\partial x^2}\, \mathcal{H}_{\widetilde{G}}(t,x,p) \end{pmatrix} \right\|_{\mathcal{L}(I\!R^{2N}, I\!R^{2N})}$$

the comparison theorem for the matrix Riccati equation (Lemma A.4.9) guarantees existence and uniqueness of such a solution $Q : [0,t] \longrightarrow I\!R^{N \times N}$ for any $t < \min\{T,\, \frac{\pi}{2\,\mu}\}$ because the scalar Riccati equation $\frac{d}{dt}\, u \;=\; a + a\, u^2$, $u(0) = 0$ has the solution $u(t) \;=\; \tan(a\, t)$ on $[0, \frac{\pi}{2\,|a|}[$ (for $a = \pm\mu$). Furthermore we obtain $\|Q(t)\| \;\le\; \tan(\mu\, t)$. Standard hypothesis $(\mathcal{H}^\rho_\circ)$ for $F_1 \ldots F_m$ leads to a constant $\sigma = \sigma(\lambda, \rho, K) > 0$ with

$$\xi \cdot \frac{\partial^2}{\partial p^2}\, \mathcal{H}_{\widetilde{G}}(t,x,p)\;\; \xi \;\;\ge\;\; 4\, \sigma \left| \xi \;-\; \frac{\xi \cdot p}{|p|^2}\; p \right|^2$$

for all $t \in [0,T]$, $|x| \le R$, $\frac{1}{R} \le |p| \le R$, ξ. Using the abbreviation $D(t,x,p)$ for $-\frac{\partial^2 \mathcal{H}_{\widetilde{G}}}{\partial x\, \partial p}(t,x,p)\;\; Q(t) \;-\; Q(t)\;\; \frac{\partial^2 \mathcal{H}_{\widetilde{G}}}{\partial p\, \partial x}(t,x,p) \;+\; Q(t)\;\; \frac{\partial^2 \mathcal{H}_{\widetilde{G}}}{\partial x^2}(t,x,p)\;\; Q(t) \;\in\; I\!R^{N \times N}$, choose $\widehat{\tau} = \widehat{\tau}(\lambda, \rho, K) > 0$ small enough s.t. $\widehat{\tau} < \min\{\, T,\, \frac{\pi}{2\,\mu},\, \frac{1}{\lambda}\,\}$, $\|D(t,x,p)\| \;\le\; \sigma$ for every $t \in [0, \widehat{\tau}]$, $|x| \le R$, $\frac{1}{R} \le |p| \le R$.

As a next step, we show that the solution $Q(t)$ of (iv) (restricted to $[0, \widehat{\tau}]$) has the upper bound $-\,\sigma\, t$ in a $(N-1)$–dimensional subspace of $I\!R^N$. Indeed, let $(x(\cdot),\, p(\cdot)) \in AC([0, \widehat{\tau}],\, I\!R^N \times I\!R^N)$ be a solution of the Hamiltonian system (i) and choose an arbitrary unit vector $\xi \in I\!R^N$ with $|\xi \cdot p(0)| < \frac{1}{4\,R}$.

Then the auxiliary function $\varphi : [0, \widehat{\tau}] \;\longrightarrow\; I\!R^N$, $\;t \;\longmapsto\; \xi \cdot Q(t)\, \xi + \sigma t \left| \xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right|^2$ satisfies $\varphi(0) = 0$ and is absolutely continuous with

$$\begin{aligned} \dot{\varphi}(t) \;&=\; \xi \cdot \dot{Q}(t)\; \xi \;+\; \sigma \left| \xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right|^2 \;+\; \sigma\, t\; \left(\xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right) \cdot\; \frac{d}{dt} \left(\frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right) \\ &=\; \xi \cdot \dot{Q}(t)\; \xi \;+\; \sigma \left| \xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right|^2 \;+\; \sigma\, t\; \left(\xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t) \right) \cdot\; \frac{\xi \cdot p(t)}{|p(t)|^2}\;\; \dot{p}(t) \end{aligned}$$

as $\xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\; p(t)$ is perpendicular to $p(t)$.

$$\begin{aligned}
\dot\varphi(t) &\le (-4+1+1)\ \sigma \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right|^2 + \sigma\, t \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right| \tfrac{|\xi|\ |p(t)|}{|p(t)|^2}\, |\dot p(t)| \\
&\le -2\ \sigma \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right|^2 + \sigma\, t \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right| \lambda \\
&\le \sigma \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right| \cdot \Big(-2 \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right| + \lambda\, t\Big) \\
&\le \sigma \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right| \cdot \Big(-2 \Big(1 - \tfrac{\xi\cdot p(t)}{|p(t)|}\Big) + \lambda\, t\Big) \\
&\le \sigma \left|\xi - \tfrac{\xi\cdot p(t)}{|p(t)|^2}\, p(t)\right| \cdot \Big(-2 \Big(1 - \tfrac{1}{2}\Big) + \lambda\, t\Big) \\
&\le 0
\end{aligned}$$

because $|p(t)-p(0)| < \frac{1}{4R}$, $\frac{1}{R} \le |p(t)| \le R$ and $|\xi\cdot p(0)| < \frac{1}{4R}$ imply $\frac{\xi\cdot p(t)}{|p(t)|} < \frac{1}{2}$. So we obtain $\varphi(t) \le 0$ for all $t \in [0,\widehat\tau]$ and as a consequence, $Q(t) \le -\sigma\, t\cdot \mathrm{Id}$ is fulfilled in the subspace of $I\!R^N$ perpendicular to $p(t)$.

Finally we need the geometric interpretation for concluding the positive erosion of $\vartheta_{\widetilde G}(t,x_0)$ (of radius $\sigma\, t$) for each $t \in\,]0,\widehat\tau[$ and $x_0 \in K$.
As mentioned before, the existence of the solution $Q(\cdot)$ on $[0,\widehat\tau[$ implies for all $t \in [0,\widehat\tau[$ that the sets M_t^1, M_t^2 are graphs of continuously differentiable functions f_t, g_t respectively. Moreover Prop. A.3.2 guarantees

$$\text{Graph } N_{\vartheta_{\widetilde G}(t,x_0)} \subset \Big\{(x(t),\ \lambda\, p(t)) \ \Big|\ (x(\cdot),p(\cdot)) \text{ solves } (i),\ \lambda\ge 0\Big\} \overset{\text{Def.}}{=} \bigcup_{\lambda\ge 0} \text{Graph } (\lambda\, f_t^{-1}).$$

So we obtain for every $t \in\,]0,\widehat\tau[$ that each $p \in I\!R^N\setminus\{0\}$ belongs to the limiting normal cone of a unique boundary point $z \in \partial\,\vartheta_{\widetilde G}(t,x_0)$ (and $z = z(p)$ is continuously diff.). In particular, the projection on $\vartheta_{\widetilde G}(t,x_0)$ is a single–valued function in $I\!R^N$ and thus, $\vartheta_{\widetilde G}(t,x_0)$ is convex for all $t \in\,]0,\widehat\tau[$ (see e.g. [25, Clarke,Stern,Wolenski 95], Cor. 4.12). So it is sufficient to consider the limiting normal cones of $\vartheta_{\widetilde G}(t,x_0)$ *locally* at every boundary point.

For each solution $(x(\cdot),p(\cdot))$ of the Hamiltonian system (i), set $(y(\cdot),q(\cdot)) := (-p(\cdot),\, x(\cdot))$ again and let $(U(\cdot),V(\cdot)) : [0,t] \longrightarrow I\!R^{N\times N} \times I\!R^{N\times N}$ denote the solution of the linearized system

$$\wedge \begin{cases}
\dot U(s) = \frac{\partial^2}{\partial y\,\partial q} H_2(s,\ y(s),\ q(s))\ \ U(s) + \frac{\partial^2}{\partial q^2} H_2(s,\ y(s),\ q(s))\ \ V(s),\\
\dot V(s) = -\ \frac{\partial^2}{\partial y^2} H_2(s,\ y(s),\ q(s))\ \ U(s) - \frac{\partial^2}{\partial q\,\partial y} H_2(s,\ y(s),\ q(s))\ \ V(s),\\
U(0) = \mathrm{Id}_{I\!R^{N\times N}}, \qquad V(0) = 0.
\end{cases}$$

Then for any $s \in]0,t]$ and initial direction $u_0 \in I\!R^N\setminus\{0\}$, the tuple $(U(s)\,u_0,\ V(s)\,u_0)$ belongs to the contingent cone of $M_s^2 \subset I\!R^N \times I\!R^N$ at $(y(s),q(s))$ (due to well–known properties of variational equations, see e.g. [37, Frankowska 2002]).

Since M_s^2 is the graph of a continuously differentiable function g_s, we conclude that firstly, this cone $T^c_{M_s^2}(y(s), q(s))$ is a N–dimensional subspace of $I\!R^N \times I\!R^N$ and secondly, $|V(s)\,u_0| \le \text{const}(\lambda) \cdot |U(s)\,u_0|$ (according to remark (1.) after Def. A.4.13). The latter property and the uniqueness of the linearized system ensure $U(s)\,u_0 \neq 0$ for all $u_0 \neq 0$ and thus, $U(s)$ is invertible. Comparing now the dimensions leads to

$$T^c_{M_s^2}(y(s), q(s)) \;=\; (U(s),\, V(s))\; I\!R^N$$

and $V(s)\,U(s)^{-1}$ is the derivative of the C^1 function g_s at $y(s)$.
So $-V(s)\,U(s)^{-1}$ is the derivative of the C^1 function $f_s = g_s(-\,\cdot\,)$ at $p(s) = -y(s)$. Moreover it is easy to check that $V(s)\,U(s)^{-1}$ satisfies the matrix Riccati equation (iii) and thus, its uniqueness implies $V(s)\,U(s)^{-1} \;=\; Q(s)$ for $0 < s \le t < \widehat{\tau}$.

Together with the preceding upper bound of $Q(t)$, we obtain for every time $t \in \,]0, \widehat{\tau}[$ that the derivative of f_t at $p(t)$ is bounded by $\sigma\, t$ from below in a $(N{-}1)$–dimensional subspace of $I\!R^N$. Since $\vartheta_{\widetilde{G}}(t, x_0)$ is convex, it implies that $\vartheta_{\widetilde{G}}(t, x_0)$ has positive erosion of radius $\sigma\, t$. □

Meanwhile, Hélène Frankowska and Piermarco Cannarsa have investigated the same question of regularity independently. Their direct geometric approach is differing completely from Proposition A.7.2 and implies that the Hamiltonian $\mathcal{H}_{F_j}$ need not be even *twice* continuously differentiable (see [17, Cannarsa, Frankowska 2004]).

Bibliography

[1] Ambrosio, L. (2000) : Geometric evolution problems, distance function and viscosity solutions, in : Buttazzo, G. (ed.) et al., *Calculus of variations and partial differential equations. Topics on geometrical evolution problems and degree theory*, Springer-Verlag

[2] Aubin, J.-P. (1999) : *Mutational and Morphological Analysis : Tools for Shape Evolution and Morphogenesis*, Birkhäuser-Verlag, Systems and Control: Foundations and Applications

[3] Aubin, J.-P. (1997) : *Dynamic Economic Theory. A Viability Approach*, Springer-Verlag, Studies in Economic Theory 5

[4] Aubin, J.-P. (1993): Mutational equations in metric spaces, *Set-Valued Analysis* 1, pp. 3-46

[5] Aubin, J.-P. (1992) : A note on differential calculus in metric spaces and its applications to the evolution of tubes, *Bull. Pol. Acad. Sci., Math.* 40, No.2, pp. 151-162

[6] Aubin, J.-P. (1991) : *Viability Theory*, Birkhäuser-Verlag, Systems and Control: Foundations and Applications

[7] Aubin, J.-P. & Frankowska, H. (1990): *Set-Valued Analysis*, Birkhäuser-Verlag, Systems and Control : Foundations and Applications

[8] Ball, J.M. (1977) : Strongly continuous semigroups, weak solutions, and the variation of constants formula, *Proc. Am. Math. Soc.* 63, pp. 370-373

[9] Barles, G., Soner, H.M. & Souganidis, P.E. (1993) : Front propagation and phase field theory, *SIAM J. Control Optimization* 31, No.2, pp. 439-469

[10] Barles, G. & Souganidis, P. (1998) : A new approach to front propagation problems: theory and applications, *Arch. Ration. Mech. Anal.* 141, No.3, pp. 237-296

[11] Barron, E.N., Cannarsa, P., Jensen, R. & C. Sinestrari (1999) : Regularity of Hamilton–Jacobi equations when forward is backward, *Indiana Univ. Math. J.* 48, No.2, pp. 385-409

[12] Beer, G. (1993) : *Topologies on Closed and Closed Convex Sets*, Dordrecht: Kluwer Academic Publishers, Mathematics and its Applications 268

[13] Bellettini, G. & Novaga, M. (1998) : Comparison results between minimal barriers and viscosity solutions for geometric evolutions, *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.* 26, No.1, pp. 97-131

[14] Bellettini, G. & Novaga, M. (1997) : Minimal barriers for geometric evolutions, *J. Differ. Equations* 139, No.1, pp. 76-103

[15] Brakke, K. (1978) : *The motion of a surface by its mean curvature*, Princeton University Press

[16] Calcaterra, C. & Bleecker, D.(2000) : Generating flows on metric spaces, *J. Math. Anal. Appl.* 248, No.2, pp. 645-677

[17] Cannarsa, P. & Frankowska, H. (2004) : Interior sphere property of attainable sets and time optimal control problems, Preprint 38 of EU Research Training Network "Evolution Equations for Deterministic and Stochastic Systems"

[18] Cardaliaguet, P. (2001) : Front propagation problems with nonlocal terms II, *J. Math. Anal. Appl.* 260, No.2, pp. 572-601

[19] Cardaliaguet, P. (2000) : On front propagation problems with nonlocal terms, *Adv. Differ. Equ.* 5, No.1-3, pp. 213-268

[20] Cardaliaguet, P. & Pasquignon, D. (2001) : On the approximation of front propagation problems with nonlocal terms, *M2AN, Math. Model. Numer. Anal.* 35, No.3, pp. 437-462

[21] Caroff, N. & Frankowska, H. (1996): Conjugate points and shocks in nonlinear optimal control, *Trans. Am. Math. Soc.* 348, No.8, pp. 3133-3153

[22] Chen, Yun-G., Giga, Y. & Goto, S. (1991) : Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Differ. Geom.* 33, No.3, pp. 749-786

[23] Clarke, F.H. (1983) : *Optimization and Nonsmooth Analysis*, Wiley-Interscience, Canadian Mathematical Society Series of Monographs and Advanced Texts

[24] Clarke, F.H., Ledyaev, Yu.S. & Stern R.J. (1997) : Complements, approximations, smoothings & invariance properties, *J. Convex Anal.* 4, No.2, pp. 189-219

[25] Clarke, F.H., Stern, R.J. & Wolenski, P.R. (1995) : Proximal smoothness and the lower-C^2 property, *J. Convex Anal.* 2, No.1/2, pp. 117-144

[26] Cornet, B. & Czarnecki, M.-O. (1999): Smooth normal approximations of epi-Lipschitzian subsets of $\mathbb{R}^n$, *SIAM J. Control Optim.* 37, No.3, pp. 710-730

[27] De Giorgi, E. (1994) : *Barriers, boundaries, motion of manifolds*, Lectures held in Pavia, Italy

[28] Delfour, M.C. & Zolésio, J.-P. (2001) : *Shapes and geometries. Analysis, differential calculus, and optimization*, SIAM, Advances in Design and Control, 4

[29] Demongeot, J., Kulesa, P. & Murray, J.D. (1997) : Compact set valued flows. II: Applications in biological modelling, *C. R. Acad. Sci., Paris, Sér.* II, *Fasc. b* 324, No.2, pp. 107-115

[30] Demongeot, J. & Leitner, F. (1996) : Compact set valued flows. I: Applications in medical imaging, *C. R. Acad. Sci., Paris, Sér.* II, *Fasc. b* 323, No.11, pp. 747-754

[31] Doyen, L. (1995) : Mutational equations for shapes and vision-based control, *J. Math. Imaging Vis.* 5, No.2, pp. 99-109

[32] Doyen, L. (1993) : Filippov and invariance theorems for mutational inclusions of tubes, *Set-Valued Anal.* 1, No.3, pp. 289-303

[33] Doyen, L., Najman, L. & Mattioli, J. (1995) : Mutational equations of the morphological dilation tubes, *J. Math. Imaging Vis.* 5, No.3, pp. 219-230

[34] Engel, K.-J. & Nagel, R. (2000) : *One–Parameter Semigroups of Linear Evolution Equations*, Springer-Verlag, Graduate Texts in Mathematics 194

[35] Federer, H. (1959) : Curvature measures, *Trans. Am. Math. Soc.* 93, pp. 418-491

[36] Federer, H. (1969) : *Geometric measure theory*, Springer-Verlag, Grundlehren der mathematischen Wissenschaften 153

[37] Frankowska, H. (2002) : Value function in optimal control, in : Agrachev, A. A. (ed.), *Mathematical control theory*, ICTP Lect. Notes. 8, pp. 515–653

[38] Frankowska, H. (1989) : Optimal trajectories associated with a solution of the contingent Hamilton-Jacobi equation, *Appl. Math. Optimization* 19, No 3, pp. 291-311

[39] Frankowska, H., Plaskacz, S. & Rzeżuchowski, T. (1995) : Measurable viability theorems and the Hamilton–Jacobi–Bellman equation, *J. Differ. Equations* 116, No 2, pp. 265-305

[40] Gautier, S. & Pichard, K. (2003) : Viability results for mutational equations with delay, *Numer. Funct. Anal. Optimization* 24, No.3-4, pp. 273-284

[41] Gorre, A. (1997) : Evolutions of tubes under operability constraints, *J. Math. Anal. Appl.* 216, No.1, pp. 1-22

[42] Kelly, J.C. (1963) : Bitopological spaces, *Proc. Lond. Math. Soc.*, III. Ser. 13, pp. 71-89

[43] Künzi, H.-P. (1992) : Complete quasi–pseudo–metric spaces, *Acta Math. Hung.* 59, No.1/2, pp. 121-146

[44] Lorenz, T. (2003) : Boundary regularity of reachable sets of control systems, Preprint 40 of EU Research Training Network "Evolution Equations for Deterministic and Stochastic Systems", submitted to *Systems & Control letters*

[45] Lorenz, T. (2001) : Set-valued maps for image segmentation, *Comput. Vis. Sci.* 4, No.1, pp. 41-57

[46] Najman, L. (1995) : Euler method for mutational equations, *J. Math. Anal. Appl.* 196, No.3, pp. 814-822

[47] Panasyuk, A.I. (1995) : Quasidifferential equations in a complete metric space under conditions of the Carathéodory type. I, *Differ. Equations* 31, No.6, pp. 901-910

[48] Panasyuk, A.I. (1995) : Quasidifferential equations in a complete metric space under Carathéodory-type conditions. II, *Differ. Equations* 31, No.8, pp. 1308-1317

[49] Panasyuk, A.I. (1992) : Properties of solutions of a quasidifferential approximation equation and the equation of an integral funnel, *Differ. Equations* 28, No.9, pp. 1259-1266

[50] Panasyuk, A.I (1985) : Quasidifferential equations in metric spaces, *Differ. Equations* 21, pp. 914-921

[51] Pazy, A. (1983) : *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Applied Mathematical Sciences 44

[52] Pichard, K. & Gautier, S. (2000) : Equations with delay in metric spaces: The mutational approach, *Numer. Funct. Anal. Optimization* 21, No.7-8, pp. 917-932

[53] Poliquin, R.A., Rockafellar, R.T & Thibault, L. (2000) : Local differentiability of distance functions, *Trans. Am. Math. Soc.* 352, No.11, pp. 5231-5249

[54] Ribeiro, H. (1943) : Sur les espaces à métrique faible, *Port. Math.* 4, pp. 21-40

[55] Rockafellar, R.T. & Wets, R. (1998) : *Variational Analysis*, Springer-Verlag, Grundlehren der mathematischen Wissenschaften 317

[56] Royden, H.L. (1988) : Comparison theorems for the matrix Riccati equation, *Commun. Pure Appl. Math.* 41, No.5, pp. 739-746

[57] Rzeżuchowski, T. (1997) : Boundary solutions of differential inclusions and recovering the initial data, *Set–Valued Analysis* 5, pp. 181-193

[58] Rzeżuchowski, T. (1999) : Continuous parameterization of attainable sets by solutions of differential inclusions, *Set–Valued Analysis* 7, pp. 347-355

[59] Sokolowski, J. & Zolésio, J.-P. (1992) : *Introduction to Shape Optimization: Shape Sensitivity Analysis*, Springer-,Verlag, Series in Computational Mathematics 16

[60] Soner, H.M. (1993) : Motion of a set by the curvature of its boundary, *J. Differ. Equations* 101, No.2, pp. 313-372

[61] Souganidis, P.E. (1997) : Front propagation: Theory and applications, in : Capuzzo Dolcetta, I. (ed.) et al., *Viscosity solutions and applications*. Springer-Verlag. Lect. Notes Math. 1660, pp. 186-242

[62] Stoltenberg, R.A. (1969) : On quasi-metric spaces, *Duke Math. J.* 36, pp. 65-71

[63] Vinter, R. (2000) : *Optimal Control*, Birkhäuser-Verlag, Systems and Control: Foundations and Applications

[64] Walter, W. (1993) : *Gewöhnliche Differentialgleichungen*, Springer-Verlag

[65] Ward, D.E. (1994) : Epiderivatives of the marginal function in nonsmooth parametric optimization, *Optimization* 31, No.1, pp. 47-61

[66] Werner, D. (2002) : *Funktionalanalysis* (4th edition), Springer–Verlag

[67] Wilson, W.A. (1931) : On quasi–metric spaces, *Amer. J. Math.* 53, pp. 675-684

[68] Yosida, K. (1978) : *Functional Analysis* (5th edition), Springer–Verlag, Grundlehren der mathematischen Wissenschaften 123

Index