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Kaluza-Klein Theories,
Branes and Cosmology

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Abstract

Kaluza-Klein theories are an elegant and highly predictive framework for unified theories. We relate the singularities appearing in the internal spaces of these models to braneworld scenarios, finding interesting connections between the two points of view. We discuss chiral fermion modes which are determined by the structure of the singularities.
In order to approach the Cosmological Constant Problem and the related dark energy discussion in this context, we study the cosmology of a toy model, six-dimensional Einstein-Maxwell theory. The time independence of the deficit angle branes is proven. Some special cases are solved, but for a solution of the most general case, many difficulties have still to be overcome. These difficulties are explained and a strategy for their possible solution is developed.

Zusammenfassung

Kaluza-Klein Theorien bilden einen eleganten und sehr vorhersagekräftigen Rahmen für vereinheitlichte Theorien. Wir bringen die Singularitäten, die in den internen Räumen dieser Modelle auftreten, mit Braneworld Szenarien in Verbindung und finden interessante Zusammenhänge zwischen den beiden Sichtweisen. Wir diskutieren chirale Fermionmoden, die durch die Struktur der Singularitäten bestimmt sind.
Um in diesem Zusammenhang das Problem der kosmologischen Konstante und die damit verknüpfte Diskussion um die dunkle Energie anzugehen, untersuchen wir die Kosmologie eines "Spielzeugmodells", sechsdimensionaler Einstein-Maxwell Theorie. Die Zeitunabhängigkeit der Defizitwinkel-Branes wird bewiesen. Einige Spezialfälle werden gelöst, aber für eine Lösung des allgemeinsten Falls müssen noch viele Schwierigkeiten überwunden werden. Diese Schwierigkeiten werden erklärt und eine Strategie zu ihrer möglichen Überwindung wird entwickelt.

Contents

1 Introduction

The present SU(3)× SU(2)× U(1) Standard Model of particle physics contains at least 19 parameters (3 gauge couplings, 10 parameters in the quark mass matrix, 3 lepton masses, 2 parameters in the Higgs sector and the Θ parameter in QCD), which is surely not a satisfying situation. The fermions of one generation belong to five different representations of the gauge group. The situation gets much better when one embeds the gauge group into a larger and simple unification group. Promising candidates are SU(5), SO(10) and E_6. Then there is only one gauge coupling, and some of the entries in the mass matrix get related. The fermions of one generation fit into one or two representations of the unification group. In SO(10) for example, they are contained in a 16-dimensional spinor representation. The 16th component is an additional right-handed neutrino which is a singlet with respect to SU(3)× SU(2)× U(1). It gets a large Majorana mass and explains in that way the smallness of the left-handed neutrino masses.

But still there are many open questions. What is the origin of the gauge group? Why does nature repeat itself in three generations with equivalent quantum numbers? What is the origin of the Higgs scalars? How are the Yukawa couplings, being responsible for the mass matrices, determined? Then there are questions of unexpected scales: Why is the scale of electroweak symmetry breaking so much smaller than the Planck mass and the unification scale? This is the gauge hierarchy problem. Why is the cosmological constant so much smaller than expected from the complicated vacuum structure of Quantum Field Theories? This is the cosmological constant problem. Finally there are questions raised by cosmological observations. What is the nature of the non-baryonic dark matter? What is the "dark energy" which leads to an accelerated expansion of the universe?

Theories with extra dimensions are a very attractive framework to study many of these questions. A particularly simple and economic ansatz is Kaluza-Klein theory. The higher dimensional Lagrangian may contain only the curvature scalar, the kinetic term of a fermion and a cosmological constant. Integrating out the extra dimensions, which are thought to be much too small to be resolved by measurement, one obtains an effective four-dimensional Lagrangian which may contain all the structures that are necessary for a realistic phenomenology: Gauge symmetries arise from isometries of the internal space, gauge fields and scalars are components of the higher dimensional metric and the observed fermions are components of one and the same higher dimensional spinor. Symmetry breaking is described by small deformations of the internal space. All effective four-dimensional couplings are related to the very few parameters of the original Lagrangian. This fact gives these theories a very high predictivity.

Kaluza-Klein theory is not intended as a fundamental theory. It does not tell us how to quantize gravity. The idea is rather that a "final theory" should be reached in two steps: At first one reaches unification a la Kaluza-Klein. Then one has to search for a quantum theory of gravity, of which the Kaluza-Klein theory is the classical limit.

From the mid 1980's on String theory absorbed most of the efforts in search for a "final theory". Recently the discovery of the D-brane solitons and other achievements within String theory lead to another development: The idea that we may live on a brane in a higher dimensional space. To be precise, this idea is not so new, but it found a new justification within the context of String theory. Anyway the new phenomenological brane models do not really make use of String theory. They are described purely in the language of General Relativity and are constructed with the intention to solve certain phenomenological problems, such as the gauge hierarchy problem. This development leads back to the idea of internal spaces with singularities (the branes) and a "warping" of the four-dimensional metric, a possibility which was earlier discussed in the Kaluza-Klein context:

If the internal space is compact, all the fermions in the effective four-dimensional theory are vector-like. Cusps or singularities in the internal space are therefore necessary to obtain chiral fermions. The recently discussed branes are such singularities, coming from a different theoretical background, concentrating on different aspects and using a slightly different language. The connection between these new models and the older Kaluza-Klein theories has not been described so far. This connection will be one of the main topics of this thesis.

A solution to the Cosmological Constant Problem and the Gauge Hierarchy Problem within the Kaluza-Klein framework remains to be found. But there are hints that these two problems may in fact be tightly related to each other. Furthermore, it was shown that with a "warping" of the four-dimensional metric, classical solutions exist with arbitrary Λ_4 (the effective 4D cosmological constant). It remains the question why a solution with Λ_4 so close to zero is selected. An answer may involve an understanding of the dynamics of the underlying quantum theory. Here we try instead to approach the problem within the classical theory. We imagine the possibility that, by some dynamical mechanism in the very early universe, the four-dimensional curvature is "driven away"; transferred into the warping for example. To investigate this possibility, we need to solve the classical field equations with relatively general initial conditions.

As a toy model for these studies, we choose six-dimensional Einstein-Maxwell theory [1]. This is not a pure Kaluza-Klein theory, because it contains already an abelian gauge field in the higher dimensional Lagrangian. It may be considered as an intermediate step of compactification, obtained from an even higher dimensional pure Kaluza-Klein theory which contains only gravity (and possibly a spinor), but this origin of the model in unimportant for our concerns. It has the advantage that it is relatively simple, but carries already all the features we need for our research: acceptable ground states, a curved internal space with appropriate singularities, and warped solutions with arbitrary Λ_4.

Altogether, we have several goals in this investigation:

- The features, history and status of pure Kaluza-Klein theories are reviewed, problems and perspectives are discussed.
- The link between these theories and the modern brane models is explained in detail. An equivalence between the two points of view is shown, leading to a kind of holographic principle and the notion of "holographic branes".

- The main goal of the research presented here is to find cosmological solutions of warped Kaluza-Klein theories in which the shape and size of the internal dimensions are time-dependent. Does internal space approach a stable shape which may lead to a realistic phenomenology? We study this question in our toy model, six-dimensional Einstein-Maxwell theory. The effective four-dimensional cosmological "constant" becomes a dynamical quantity. This may shed new light on the cosmological constant problem and the recent "dark energy" discussion.

- As a byproduct of the discussion of "warping" and branes, the general structure of maximally symmetric singularities in arbitrary dimensions is investigated and connected to the Kasner solutions.

The structure of the thesis is as follows: In chapter 2, the Kaluza-Klein ansatz for the unification of gravity and Yang-Mills theories is explained. The history of this ansatz is reviewed and some of its problems and successes are discussed. Furthermore the most popular brane models are introduced. The principal difference between codimension-one and higher codimension branes is worked out.

In chapter 3, our particular toy model is introduced: six-dimensional Einstein-Maxwell theory. The solutions are presented and the connection between geometrical quantities and the brane tensions is given. This links the old Kaluza-Klein or "bulk point of view" to the modern "brane point of view".

In chapter 4, fermions are discussed. The algebraic properties of spinor representaions in arbitrary dimensions and their dimensional reduction are reviewed. This raises the problem how to obtain chiral four-dimensional fermions, which finds a possible solution in the use of internal spaces with singularities, such as those discussed in our six-dimensional case. The wave functions of the chiral fermions are computed for this model and their number is related to the properties of the singularities. These fermions are shown to be confined to the "branes", and again the relation between bulk and brane point of view is given. Furthermore we introduce "holographic branes", for which both points of view are equivalent.

In chapter 5, the cosmology of six-dimensional Einstein-Maxwell theory is investigated. The most general metric consistent with the symmetries and the corresponding field equations are derived. Several choices of gauge are given, and the related difficulties are explained. Some special cases are solved, with and without the inclusion of fermions. In particular, late time cosmologies are discussed, in which the geometry of internal space is almost time-independent. We find that there are still many obstacles to overcome in order to solve for the early universe cosmology, which was our main motivation. In section 5.6, we summarize the open questions and develop a plan how they may be solved step by step in future research.

Chapter 6 is an outlook on the possibility to obtain a realistic phenomenology from 18-dimensional gravity with a Majorana-Weyl spinor. The structure of the mass matrices obtained from a slight deformation of internal space is outlined

In chapter 7, we summarize our results.

Finally an appendix discusses the general approximate behavior of the metric around highly symmetric singularities in arbitrary dimensions and links the result to the Kasner solutions known from anisotropic cosmologies.

1.1 Conventions

We use the metric signature $(-+++)$. The sign convention for the Riemann tensor is

$$R^{A}{}_{BCD} = -\Gamma^{A}{}_{BC,D} + \ \dots \tag{1}$$

The sign convention for the cosmological constant is such that a de Sitter spacetime has positive Λ. The Einstein equations are

$$G_{AB} \equiv R_{AB} - \frac{1}{2} R \, g_{AB} = -\Lambda \, g_{AB} + 8\pi G T_{AB}. \tag{2}$$

Since we are dealing with fermions, we have to distinguish between generally covariant and Lorentz indices.

Generally covariant indices:
μ, ν, λ are running over the four large dimensions,
i, j, k over the three large spatial dimensions,
α, β, γ over the internal dimensions,
A, B, C over all dimensions.

As Lorentz indices we use latin letters like a, b, m, n.

Indices in usual brackets denote that there is no summation. For instance, $G^{(i)}_{(i)}$ means one spatial diagonal component, not a sum over all three. In all other cases we use the Einstein sum convention.
To prevent confusion with space indices, we denote by Γ the "γ^5" matrix of the higher dimensional space, which anticommutes with all γ's of the Clifford algebra.
A tilde always denotes a corresponding quantity in the effective four-dimensional theory. For example, $\tilde{\Gamma}$ is the four-dimensional "γ^5" matrix. Exceptions are the 4D cosmological constant and the 4D Newton constant, which are denoted Λ_4 and G_4, respectively.
We often use 2D, 3D, ... as abbreviation for two-dimensional, three-dimensional, ...

2 Extra Dimensions

2.1 5D Kaluza-Klein Theory

The idea that electromagnetism and gravity can be unified by introducing a fifth dimensions is even older than General Relativity: In 1914 Gunnar Nordstroem [2] found that the equations of his scalar gravity theory were just an extension of Maxwell's equations when a fifth dimension was added to the four usual ones. Of course his theory was not generally covariant.

In 1919 Theodor Kaluza [3] discovered his famous five-dimensional theory, which is a pure Einstein gravity theory in 5D, but reduces to Einstein plus Maxwell in the effective four-dimensional world. This theory was re-invented by Oskar Klein in 1926 [4].

In this model, the fifth dimension is thought to be a small circle of radius r. We label the four usual coordinates x^μ and the fifth one y (ranging from 0 to $2\pi r$), and start with a line element

$$ds_5^2 = ds_4^2 + (dy + \beta A_\mu(x) dx^\mu)^2, \tag{3}$$

with

$$ds_4^2 = \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu \tag{4}$$

and β is a constant. The line element (3) is invariant under the transformations

$$\begin{aligned} y &\rightarrow y + \beta\chi(x), &(5)\\ A_\mu &\rightarrow A_\mu - \partial_\mu \chi(x). &(6) \end{aligned}$$

Note that all functions depend only on the x-coordinates, not on y. This is Kaluza's "cylindricity" condition. The action is the Einstein-Hilbert action,

$$S_5 = \frac{1}{16\pi G_5} \int d^4x\, dy\, \sqrt{-g}\, R, \tag{7}$$

where g is the determinant of the metric g_{AB}. Inserting the metric (3) and integrating over y, we get a four-dimensional action which is invariant under both four-dimensional general coordinate transformations and abelian gauge transformations,

$$S_4 = \int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{16\pi G_4} \tilde{R} + \frac{\beta^2}{64\pi G_4} \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \tag{8}$$

with

$$G_4 = \frac{G_5}{2\pi r}, \quad \tilde{g} = det(\tilde{g}_{\mu\nu}), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{9}$$

and $\tilde{R}$ the curvature scalar calculated from $\tilde{g}_{\mu\nu}$. The abelian gauge symmetry in four dimensions originates in the isometries of the fifth dimension. The standard Maxwell term is given for $\beta^2 = 16\pi G_4$. We see that β is essentially the Planck length.

How does the smallness of the fifth dimension enter the picture? It turns out that the size can be determined from the electric charge [5]. The reason is that charge is linked to the "winding number" of fields with respect to the fifth dimension. Consider for example a complex scalar field Φ. The kinetic term of Φ is

$$L_{kin} = g^{AB}\partial_A\Phi(\partial_B\Phi)^*. \tag{10}$$

The inverse of our metric is

$$\begin{aligned} g^{\mu\nu} &= \tilde{g}^{\mu\nu}, && (11)\\ g^{\mu 5} &= g^{5\mu} = -\beta\tilde{g}^{\mu\nu}A_\nu, && (12)\\ g^{55} &= 1 + \beta^2\tilde{g}^{\mu\nu}A_\mu A_\nu. && (13) \end{aligned}$$

If the field has the form $\Phi = \phi(x)\exp(\frac{in}{r}y)$ the kinetic term reduces to

$$L_{kin} = \tilde{g}^{\mu\nu}(\partial_\mu - \frac{in\beta}{r}A_\mu)\phi\left((\partial_\nu - \frac{in\beta}{r}A_\nu)\phi\right)^* + \frac{n^2}{r^2}\phi^*\phi. \tag{14}$$

So Φ corresponds to a four-dimensional field ϕ that couples to A_μ with coupling $n\beta/r$. Charge is quantized, and the elementary charge is $e = \beta/r$. As we saw, β is just the Planck length, so for a realistic e, r has to be only one order of magnitude smaller than the Planck length. The mass of ϕ can be inferred from the kinetic term of Φ. It is $m = n/r$ and has to be therefore only slightly smaller than the Planck mass (if $n \neq 0$). This is a general feature of Kaluza-Klein theories: Masses are either zero by some symmetry requirement, or of almost the order of the Planck mass and do therefore not appear in particle experiments.

How general was the metric ansatz we started with? The most general metric in five dimensions can be written in the form

$$\begin{aligned} g_{\mu\nu} &= \tilde{g}_{\mu\nu}(x,y) + \beta^2\phi(x,y)A_\mu(x,y)A_\nu(x,y), && (15)\\ g_{\mu 5} &= g_{5\mu} = \beta\phi(x,y)A_\mu(x,y), && (16)\\ g_{55} &= \phi(x,y), && (17) \end{aligned}$$

where ϕ is essentially the size of the fifth dimension. In the Kaluza-Klein ansatz we had $\phi = 1$ and everything depended only on x. Since every field quantity $F(x,y)$ is a periodic function of y, it admits a Fourier expansion

$$F(x,y) = \sum F^{(n)}(x)e^{iny/r}. \tag{18}$$

Kaluza's "cylindricity" condition means that only the zero modes ($n = 0$) appear. But this is automatically justified at energies well below the Planck scale, since all modes with $n \neq 0$ have effective four-dimensional masses of Planck order. At very high energies close to the Planck scale, the higher modes would have to be included in the action, of course.

But the Kaluza-Klein pioneers cheated in another way: They ignored the scalar field coming from g_{55}. It was always set constant in the early years, and the action was not varied with respect to it. If we take the fifth dimension serious [6], we have to include the field ϕ [7]. The determinant of the five-dimensional metric is then $det(g_{AB}) = det(\tilde{g}_{\mu\nu})\phi$.

When we plug this into the action, we get a factor of $\phi^{1/2}$ multiplying the 4D Ricci scalar, leading to a Brans-Dicke type tensor-scalar gravity theory plus gauge field in four dimensions. The $\tilde{R}$-term can be brought into standard form by a Weyl transformation of the metric: $g_{AB} \to \phi^{-1/3} g_{AB}$. Our zero mode metric is now

$$\bar{g}_{AB} = \phi^{-1/3}(x) \begin{pmatrix} \tilde{g}_{\mu\nu}(x) + \beta^2 A_\mu(x) A_\nu(x) \phi(x) & \beta A_\nu(x)\phi(x) \\ \beta A_\mu(x)\phi(x) & \phi(x) \end{pmatrix}. \tag{19}$$

When we insert this into the action (7) and integrate over y, we end up with the four-dimensional action

$$S_4 = \frac{1}{16\pi G_4} \int d^4x \, \sqrt{-\tilde{g}} \left(\tilde{R} + \frac{1}{4}\beta^2 F_{\mu\nu}F^{\mu\nu} + \frac{1}{6}\frac{\partial_\mu \phi \partial^\mu \phi}{\phi^2} \right). \tag{20}$$

By redefining the scalar field by its logarithm, its kinetic term can also be brought into a standard form. The corresponding field equations admit the vacuum solution

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad A_\mu = 0, \quad \phi = 1. \tag{21}$$

The scalar field is massless in this case, since the one-dimensional internal space is not curved. In higher dimensions this will change, and the scalar fields will generally have masses of the order of the compactification scale, like the non-zero modes of the other fields.

2.2 Kaluza-Klein Theories in more than five Dimensions

There was no need to extend the Kaluza-Klein idea beyond five dimensions until the importance of non-Abelian gauge theories was discovered. In 1963 B. DeWitt [8] suggested that a unification of Yang-Mills theories and gravitation could be achieved in a higher-dimensional Kaluza-Klein framework. A detailed discussion of this idea in the language of fibre bundles appears in the work of Ryszard Kerner [9]. The first complete derivation of the four-dimensional gravitational plus Yang-Mills plus scalar theory from a $(4+D)$-dimensional Einstein-Hilbert action was finally given by Cho and Freund in 1975 [10].

The vacuum spacetime is assumed to be a direct product $M^4 \times K$ of four-dimensional Minkowski space and a compact internal space K. In order to get the Yang-Mills term with gauge group G from dimensional reduction of a $(4+D)$ dimensional Einstein-Hilbert term, it is necessary to have Killing vectors ζ_a^α on the internal space (in the ground state) which represent this symmetry. (The index a is running over the dimension of G, labeling the ζ-vectors, and α is the coordinate index in internal space). This means

$$[\zeta_a, \zeta_b]^\alpha \equiv \zeta_a^\beta \zeta_{b,\beta}^\alpha - \zeta_b^\beta \zeta_{a,\beta}^\alpha = f_{ab}^c \zeta_c^\alpha, \tag{22}$$

$$\zeta_{a\alpha;\beta} + \zeta_{a\beta;\alpha} = 0. \tag{23}$$

Here $[\zeta_a, \zeta_b]$ is the standard Lie bracket and f_{ab}^c are the structure constants of G. The isometries of the internal space correspond now to the gauge transformations. The metric can, in zero-mode approximation, be written as follows :

$$g_{AB}(x,y) = \begin{pmatrix} \tilde{g}_{\mu\nu}(x) + \phi_{\alpha\beta}(y)\zeta_a^\alpha(y)\zeta_b^\beta(y)A_\mu^a(x)A_\nu^b(x) & \phi_{\alpha\beta}(y)\zeta_a^\alpha(y)A_\mu^a(x) \\ \phi_{\alpha\beta}(y)\zeta_b^\beta(y)A_\nu^b(x) & \phi_{\alpha\beta}(y) \end{pmatrix}. \tag{24}$$

We insert this metric into the $4+D$ dimensional Einstein action

$$S_{4+D} = \frac{1}{16\pi G_{4+D}} \int d^4x d^D y \sqrt{-g}(R^{(4+D)} + \Lambda), \tag{25}$$

where Λ is a cosmological constant. The resulting four-dimensional Lagrangian is

$$\begin{aligned} L_4 &= \frac{1}{16\pi G_{4+D}} \int d^D y \sqrt{-\tilde{g}} (\phi^{(D)})^{1/2} (\tilde{R}(x) + R^{(D)}(y) + \Lambda \\ &+ \frac{1}{4} \phi_{\alpha\beta}(y) \zeta_a^\alpha(y) \zeta_b^\beta(y) F_{\mu\nu}^a(x) F_{\lambda\rho}^b(x) \tilde{g}^{\mu\lambda}(x) \tilde{g}^{\nu\rho}(x)), \end{aligned} \tag{26}$$

where $\phi^{(D)}$ is the determinant of the internal metric $\phi_{\alpha\beta}$, $R^{(D)}$ is the corresponding curvature scalar and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c. \tag{27}$$

The four-dimensional Newton's constant is then

$$G_4 = G_{4+D} / \int d^D y \, (\phi^{(D)}(y))^{1/2} = G_{4+D}/V_{int} \tag{28}$$

with V_{int} the volume of the internal space. With a Weyl transformation one can again achieve a standard $\tilde{R}$-term. The zero mode ansatz does not contain any scalar fields describing x-dependent fluctuations of the internal metric $\phi_{\alpha\beta}$. One can show that these scalars have in general, as already mentioned, masses of the order of the compactification scale, because their excitation would lead to a change of the internal curvature, which is seen as a large energy shift in four dimensions.

In the five-dimensional case it was easy to find a vacuum solution that satisfies the field equations. This is not the case in the higher-dimensional models. In vacuum, with gauge fields and scalar fluctuations set to zero, the Einstein equations are

$$R_{AB} - \frac{1}{2} g_{AB}(R + \Lambda) = 0. \tag{29}$$

If four-space is to be flat $R_{\mu\nu} = 0$, it follows that $R + \Lambda = 0$. But then $R_{\alpha\beta}$ must vanish as well to fulfill eq.(29). This is possible only for an abelian gauge group, where internal space is a torus. For any non-abelian group, internal space has to be curved, $R_{\alpha\beta} \neq 0$. And so it is proven that there is no appropriate vacuum solution in this framework.

In the late 1970's and the early 1980's several possibilities were explored to surround this difficulty. Cremmer and Scherk [11] showed how the inclusion of additional Yang-Mills and scalar matter fields in the higher-dimensional theory would allow for a desired ground state, and Luciani [12] generalized their work. This of course destroys somehow the beauty of the Kaluza-Klein idea, which was essentially that Yang-Mills fields are **explained** by dimensional reduction and are not present in the fundamental action. Wetterich [13] suggested a compactification due to higher derivative terms of form R^2 which may become important when one approaches the Planck scale and may be relevant already at the compactification scale (which is not much smaller than the Planck scale). Another possibility is the inclusion of a "warp factor" [29] (a scale factor $a^2(y)$ multiplying the 4D metric), which makes the 4D metric dependent on the internal coordinates.

2.3 Branes

In the last few years, the emphasis in the development of theories with extra dimensions has shifted towards the "brane world" picture, which assumes that the Standard Model matter is confined to a four-dimensional submanifold - our observable spacetime - embedded in a higher-dimensional space. The idea is not really new [14, 15], but it was given a new motivation from the D-brane solitons in String Theory (for a review see [16]), and in particular by the work of Horava and Witten [17, 18]. Nevertheless, the usual brane models are purely phenomenological and make no real use of String theory.

There are three basic models, called ADD [19, 20], RS1 [21] and RS2 [22] by the names of their inventors (Arkani-Hamed, Dimopoulos and Dvali in the first case, Randall and Sundrum in the other two cases). These three models correspond to "large", "small" and infinite extra dimensions, respectively. ADD and RS1 were invoked with the intention to solve the gauge hierarchy problem, while RS2 shows how gravity can be "localized" at a brane, contradicting the usual assumption that the 4D Newton constant goes to zero when the size of an extra dimension goes to infinity. In all cases the Standard Model matter and gauge interactions are confined to the brane, while gravity - being the dynamics of spacetime itself - propagates through the entire space. The strength of gravity itself is given by the overlap of the massless graviton wave function with the brane.

ADD: Large Extra Dimensions: In the ADD scenario, the fundamental scale in the $4+D$ dimensional spacetime is the TeV scale, and the $4+D$ dimensional Planck Mass M is of that size. The large 4-dimensional Planck mass $M_p^{(4)}$ is due to the fact that the extra dimensions are so large. The brane has no tension and does therefore not affect the geometry of the higher-dimensional space, which is assumed to be a direct product of four-dimensional and a compact "internal" space (which would better be called "external" in this case, because it is orthogonal to our brane). The four-dimensional Newton constant is, as in the Kaluza-Klein theories, given by equation (28). In terms of Planck masses, with $G_{4+D} = M^{-(2+D)}$, we get

$$M_p^{(4)} = M(MR)^{D/2}, \tag{30}$$

where R is the average size of the extra dimensions. Assuming that $M \sim 1$ TeV, one calculates the value of R,

$$R \sim 10^{32/D} \times 10^{-17}\ cm. \tag{31}$$

On distances below R, strong deviations from Newton's law of gravity are expected. This law has been tested down to distances of about 0.1 mm, so $d = 1$ is excluded and $d = 2$ very improbable.

The ADD scenario has several problems:

- The question of why the Planck mass is so much larger than the weak scale is just replaced by the question of why the extra dimensions are so much larger than the weak scale (10^{-17} cm).
- The large size of the extra dimensions leads to the existence of light non-zero graviton Kaluza-Klein modes (called KK gravitons). Interacting with brane matter, these

may carry away a large amount of energy from the brane. The interaction rates can be computed and lead, in combination with astrophysical and cosmological observations, to strong constraints on ADD models and to a "possible but not very appealing" [23] early universe cosmology.

- The brane is assumed to carry no energy-momentum. But this changes when cosmological matter is added. This should lead to a breakdown of the higher-dimensional geometry.

RS1: "A Large Mass Hierarchy from a Small Extra Dimension": The Randall-Sundrum model is much more elegant and surrounds all the problems of the ADD scenario. There is only one extra dimension, and the five-dimensional space is a slice of an Anti de Sitter spacetime with negative cosmological constant Λ. The ground state line element is

$$ds^2 = e^{-2kz}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2, \tag{32}$$

with $k^2 = -\Lambda/6$. The exponential factor in front of the η is called "warp factor". The z-coordinate is restricted to the interval $[0, R]$. At these positions, $z = 0$ and $z = R$, two branes are located which act as "mirrors", so that the point $(x^\mu, -z)$ can be identified with $(x^\mu, +z)$ and $(x^\mu, R-z)$ with $(x^\mu, R+z)$. This means that we can make the topology of that "orbifold" space visible by continuing the z-coordinate beyond 0 and R and get a warp factor of e^{2kz} in the interval $[-R, 0]$ and $e^{2k(z-2R)}$ in the interval $[R, 2R]$. There is a jump of the first z-derivative of the warp factor at the brane positions, corresponding to delta functions in the second derivatives of the warp factor. In the Einstein equations, these delta functions must be matched by delta functions in the energy momentum tensor, the so-called "brane tensions" τ. These are given by action terms of the form

$$S_{brane} = \int d^4x dz\, \sqrt{-g}\, \tau_{brane}\delta(z - z_{brane}) = \int d^4x\, (-g(x^\mu, z_{brane}))^{1/2}\, \tau_{brane} \tag{33}$$

These matching conditions are a very simple special case of the Israel junction conditions, which determine the jump of the metric derivatives for arbitrary codimension-one hypersurfaces. In the Randall-Sundrum case one finds $\tau_1 = (-6\Lambda)^{1/2}/(8\pi G_5)$ for the brane at $z = 0$ and $\tau_2 = -(-6\Lambda)^{1/2}/(8\pi G_5)$ for the brane at $z = R$.

The massless gravitational fluctuations are of the form

$$ds^2 = e^{-2kz}(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dz^2. \tag{34}$$

(We ignore the so-called radion in this short discussion). Here $h_{\mu\nu}$ represents tensor fluctuations about Minkowski space and is the physical graviton of the four-dimensional effective theory. Massless vector zero modes like the A_μ in Kaluza-Klein theories do not exist here. Integrating the curvature term

$$S_{grav} = \int d^4x dz \frac{1}{16\pi G_5} e^{-2kz}\sqrt{-\tilde{g}}\tilde{R}, \tag{35}$$

one finds that the four-dimensional Newton constant is given by

$$G_4 = 2G_5k/(1 - e^{-2kR}), \tag{36}$$

which depends only slightly on R for $kR > 1$.

RS1 offers an interesting possibility to solve the hierarchy problem. Assume that we live on the negative tension brane at $z = R$. The physical masses can be determined by properly normalizing the fields. Consider for example a Higgs field confined to "our" brane, with mass parameter v_0:

$$\begin{aligned} S_H &= \int d^4x \, (-g(x^\mu, R))^{-1/2} \{ g^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - v_0^2)^2 \} && (37) \\ &= \int d^4x \, (-\tilde{g})^{1/2} e^{-4kR} \{ \tilde{g}^{\mu\nu} e^{2kR} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - v_0^2)^2 \}. && (38) \end{aligned}$$

After wave-function renormalization, $H \to e^{kR} H$, we obtain

$$S_{H,eff} = \int d^4x \, (-\tilde{g})^{1/2} \{ \tilde{g}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - e^{-2kR} v_0^2)^2 \}. \quad (39)$$

We see that the physical mass scale, set by the symmetry-breaking scale, is

$$v = e^{-kR} v_0. \quad (40)$$

This feature generalizes to arbitrary mass parameters on our brane. The physical mass will always be smaller by a factor e^{-kR}. If fundamental parameters like $(G_5)^{-1/3}$, k and v_0 are of Planck scale order, the TeV scale is produced on the brane if $e^{kR} \approx 10^{16}$, i.e. $kR \approx 50$. So, due to the exponential factor, even a small extra dimension can produce a large hierarchy.

In this treatment it looked like the fundamental scale is the Planck scale, and the TeV scale is a derived scale. But the opposite point of view is also possible. This can be seen by an appropriate rescaling of the metric, such that the warp factor is 1 at $z = R$ and e^{2kR} at $z = 0$.

RS2: Finite Gravity from an Infinite Extra Dimension: In the second Randall-Sundrum model, the setup is as before, but now there is only one brane, namely the one at $z = 0$, and the coordinate z goes from 0 to infinity. Thus we have an infinite extra dimension. The graviton zero mode, which is as before, decays exponentially in the z-direction, hence it is "localized" at the brane. The four-dimensional Newton constant is given by $G_4 = 2G_5 k$, cf. eq. (36). There is no longer a mass gap for the KK gravitons. Instead we now have a continuous spectrum, starting at $m = 0$. Randall and Sundrum argue that these KK gravitons couple only weakly to the brane matter, and hence produce only a very small correction to the Newton potential. So it was shown that one can have the usual four-dimensional gravity even in the presence of an infinite extra dimension. Note that RS2 does not offer a possibility to solve the hierarchy problem.

2.4 Higher Codimension Branes

The two Randall-Sundrum models contain codimension-one branes. These have the property that they cannot be seen by an "observer" in the bulk. The postion z_b of the brane cannot be determined by the bulk geometry. In other words, the bulk solution does not "feel" the closeness of a brane. From the point of view of an "observer" in the bulk, the brane could be located anywhere, at arbitrary z_b. Its only effect is a jump in the first derivative of the warp factor which can only be "seen" when z_b is reached. For that reason, codimension-one branes can be put in "by hand". One can arbitrarily choose the position and tension in order to fulfill certain phenomenological requirements, e.g. gauge hierarchy, orbifold symmetry [21], without affecting the bulk.

The situation is similar for cosmological solutions [24, 25, 26]. It is possible to put "by hand" arbitrary cosmological matter on a codimension-one brane. The only effect of the brane is a local jump of the first metric derivatives, determined by the Israel junction conditions. In fact, the cosmology of codimension-one branes can be seen in two ways, depending on the coordinate system one uses. First, one can regard the position of the brane as fixed. In this case (the brane-based point of view), the bulk cosmology seems to depend on the brane properties (its tension, energy and pressure) such that the time dependence of the bulk metric is generated by the brane. Alternatively, one can use coordinates in which the bulk geometry depends only on bulk quantities (the bulk-based point of view). Then the bulk is static if there are no source terms, or the bulk cosmology is driven by a bulk scalar field or something else. In these coordinates, the brane cosmology is an effect of the brane traveling through the bulk, showing that brane and bulk solutions are independent of each other (see [27] and references therein). So in the codimension-one case, we need two theories: one for the brane and one for the bulk.

An analogy for the difference between codimension one and two can be found in common physics: A charged particle, located between the plates of a capacitor, does not "feel" how close the plates are, since the electric field is constant, independent of the distance. A codimension one singularity (plate) is not detected in the bulk. This is different from a particle traveling through the field of a charged wire (codimension two) or of another point particle (codimension three). Here it feels the closeness of the source through the $1/r$- or $1/r^2$-behavior of the field. Similar statements are true for branes in higher dimensions.

In contrast to codimension one we find that for codimension two or larger the properties of the brane are determined by the bulk properties. If a similar situation holds for the excitations, the brane point of view becomes an option - one could equally well describe the physics by the properties of the bulk and its excitations. This situation has a familiar analogon in our usual four dimensional world, namely the black hole with metric given by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{41}$$

The parameter M can be seen as the mass of an object sitting at $r = 0$ which is intuitively correct if one considers a black hole created by a collapsed star. This corresponds to the brane point of view. However it could equally well be taken as simply a free parameter of the isotropic vacuum solution of the Einstein equations, without giving it a physical meaning. We may call this the bulk point of view. Without a way of probing the singularity directly the two points of view cannot be distinguished by observation.

Singular objects of codimension two or larger are much more restricted than those of codimension one. There is not much freedom for ad hoc adjustments of brane properties and localization of arbitrary fields on the brane, independently of the properties of the bulk. In that sense, models of codimension two or larger have more predictive power than codimension one brane models.

Recently codimension-two branes were discussed by Cline et al [28]. They claimed that the restrictions are so strong that it is not possible to have anything else than tension on an infinitely thin codim-2 brane. This point will be discussed later in this thesis. For the moment we notice that codim-2 is the largest codimension in which one can have an infinitely thin brane at all. The brane causes or is described by (depending on the point of view) a conical singularity with finite internal metric and without inducing any curvature in the bulk. But it can be observed from outside due to the deficit angle of the cone. For codimensions larger than two, the brane induces curvature in the bulk. If the brane were infinitely thin, this curvature would diverge at the brane, and the internal metric $g_{\mu\nu}$ would become infinite (see appendix A). Hence the brane has to be "regularized" in one or another way, giving it some finite size and some internal physics. Nevertheless it is possible that all the relevant physics of these regularized branes can be determined from the bulk point of view, as will be shown in chapter 4.

3 6D Einstein-Maxwell Theory and Deficit Angle Branes

Six-dimensional Einstein-Maxwell Theory has often been used as a toy model for "semi-realistic" Kaluza-Klein theories. It is the simplest theory admitting a ground state which leads to a non-Abelian gauge theory in four dimensions [1]. It also has less symmetric ground states which may be a hint to a possible solution of the Cosmological Constant Problem [29] and the Gauge Hierarchy Problem [30]. It is a standard framework in which to study codimension-two branes [28], and it may lead to chiral fermions [1, 32].
In this chapter, we will study geometric aspects of 6D Einstein-Maxwell theory. Fermions will be discussed in chapter 4. At first, pure gravity on a "warped" six-dimensional manifold with certain symmetries is analyzed and the connection to the Cosmological Constant Problem is explained. Then we turn to the effect of a Maxwell field on the same type of manifold. We show that it may lead to codimension-two branes with a magnetic monopole configuration. The connection between the older bulk point of view and the modern brane point of view is discussed in detail. Finally, we describe the maximally symmetric, unwarped solution found by Randjbar-Daemi, Salam and Strathdee.

3.1 6D Warped Geometry and the Cosmological Constant

We consider a six-dimensional manifold with line element

$$ds^2 = a^2(\rho)\tilde{g}_{\mu\nu}dx^\mu dx^\nu + b^2(\rho)d\theta^2 + d\rho^2. \tag{42}$$

Here $\tilde{g}_{\mu\nu}$ is the metric of a four-dimensional spacetime with constant curvature. Internal space is labeled by the radial coordinate ρ, running from 0 to ∞ or to a finite value $\bar{\rho}$, and by the angular coordinate θ, running from 0 to 2π. The nonzero Christoffel symbols, components of the Ricci tensor and the Ricci scalar are

$$\begin{aligned}
\Gamma^\mu{}_{\nu\lambda} &= \tilde{\Gamma}^\mu_{\nu\lambda}, & \Gamma^\rho{}_{\mu\nu} &= -a'a\tilde{g}_{\mu\nu}, & (43)\\
\Gamma^\mu{}_{\nu\rho} &= \tfrac{a'}{a}\,\delta^\mu_\nu, & \Gamma^\rho{}_{\theta\theta} &= -b'b, & (44)\\
\Gamma^\theta{}_{\theta\rho} &= \tfrac{b'}{b}, & & & (45)
\end{aligned}$$

$$\begin{aligned}
R_{\mu\nu} &= \tilde{R}_{\mu\nu} - \tilde{g}_{\mu\nu}(3a'^2 + a'a\frac{b'}{b} + a''a), & (46)\\
R_{\theta\theta} &= -4\frac{a'b'b}{a} - b''b, & (47)\\
R_{\rho\rho} &= -4\frac{a''}{a} - \frac{b''}{b}, & (48)
\end{aligned}$$

$$R = \frac{\tilde{R}}{a^2} - 12\frac{a'^2}{a^2} - 8\frac{a'b'}{ab} - 8\frac{a''}{a} - 2\frac{b''}{b}. \tag{49}$$

A prime denotes a derivative with respect to ρ, a tilde denotes a four-dimensional quantity derived from the metric $\tilde{g}_{\mu\nu}$. Let Λ be the six-dimensional cosmological constant and Λ_4 correspond to the four-dimensional spacetime,

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = -\Lambda_4\tilde{g}_{\mu\nu}. \tag{50}$$

The vacuum Einstein equations are then

$$
\begin{aligned}
G^\mu_\nu = R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu &= \left(-\frac{\Lambda_4}{a^2} + 3\frac{a''}{a} + 3\frac{a'^2}{a^2} + \frac{b''}{b} + 3\frac{a'b'}{ab}\right)\delta^\mu_\nu = -\Lambda\delta^\mu_\nu, &(51)\\
G^\theta_\theta = R^\theta_\theta - \frac{1}{2}R &= -\frac{2\Lambda_4}{a^2} + 4\frac{a''}{a} + 6\frac{a'^2}{a^2} = -\Lambda, &(52)\\
G^\rho_\rho = R^\rho_\rho - \frac{1}{2}R &= -\frac{2\Lambda_4}{a^2} + 4\frac{a'b'}{ab} + 6\frac{a'^2}{a^2} = -\Lambda. &(53)
\end{aligned}
$$

These are three equations for two functions, but due to the Bianchi identities, only two of them are independent. From the difference between the second and the third equation we see that

$$\frac{b'}{b} = \frac{a''}{a'} \tag{54}$$

and hence $b = Aa'$ with an arbitrary integration constant A. Plugging this into the first equation we see that it is just a combination of the second and its derivative. Defining a new variable z via

$$a = z^{2/5} \tag{55}$$

we may rewrite eq (52) as

$$z'' = -\frac{5}{8}\Lambda + \frac{5}{4}\Lambda_4 z^{1/5}. \tag{56}$$

This is the equation of motion of a "particle" in a "potential" V,

$$z'' = -\frac{\partial V}{\partial z}, \qquad V(z) = \frac{5}{16}\Lambda\, z^2 - \frac{25}{24}\Lambda_4\, z^{6/5} \tag{57}$$

where ρ plays the role of time. The relation between z and b is then

$$b = \frac{2}{5}A\, z' z^{-3/5}. \tag{58}$$

The "particle" should start "at rest" at $\rho = 0$, which means that we impose the boundary conditions $a' = 0$ and $b \propto \rho$ in the limit $\rho \to 0$. If the manifold should be smooth at $\rho = 0$ we must have $b \to \rho$ there which fixes the integration constant A. Otherwise there would be a conical singularity which may be identified as a brane. Define $z_0 = z(\rho = 0)$. Now there are four cases, depending on the signs of Λ, Λ_4 and $V(z_0)$:

1. If $\Lambda > 0$, $\Lambda_4 \leq 0$ arbitrary and $V(z_0) > 0$, the "particle" reaches $z = 0$ at a finite $\rho = \bar{\rho}$, and spacetime terminates in a singularity there. From eqs. (55,57,58) one can see that $a \to (\bar{\rho} - \rho)^{2/5}$ and $b \to (\bar{\rho} - \rho)^{-3/5}$ as $\rho \to \bar{\rho}$. In the simplest case $\Lambda_4 = 0$ and $z_0 = 1$ the solution is

$$a(\rho) = \cos^{2/5}(\omega\rho), \qquad b(\rho) = -\frac{2}{5}A\omega\sin(\omega\rho)\cos^{-3/5}(\omega\rho), \qquad \omega^2 = \frac{5}{8}\Lambda \tag{59}$$

and $\bar{\rho} = \pi/(2\omega)$.

The singularity at $\bar{\rho}$ corresponds to a type of higher dimensional black hole, with time replaced by a spacelike coordinate. Indeed, the properties of the singularity at $\bar{\rho}$ can best be understood in another coordinate system. By an appropriate rescaling of four-dimensional spacetime and introducing the variable $r = D(\bar{\rho} - \rho)^{2/5}$ with an appropriate constant D, the metric around $\bar{\rho}$, i.e. around $r = 0$, can be brought into the form

$$ds^2 \to \frac{M}{r^3} d\theta^2 + \frac{r^3}{M} dr^2 + r^2 g_{\mu\nu} dx^\mu dx^\nu. \tag{60}$$

Up to the signature, this is just the $r \to 0$-limit of the six-dimensional analogue of the Schwarzschild solution with mass parameter M, θ playing the role of time. Hence the singularity corresponds to a singular point in the five-dimensional space generated by the coordinates x^μ and ρ. We emphasize that in this case θ is the internal coordinate of the singularity and x^μ are external, complementary to the brane situation.

This type of solution was generalized to an arbitrary codimension by Randjbar-Daemi and Wetterich [31]. There appear singularities with similar properties as in the six-dimensional case. General properties of such singularities are discussed in the appendix A of this thesis.

This first type of solution exists also if $\Lambda < 0$, $\Lambda_4 < 0$. In this case the potential V has a maximum at some $z = z_{max}$. If $z_0 < z_{max}$, we have a solution of type 1, otherwise a solution of type 3.

2. If $\Lambda > 0$ and $\Lambda_4 > 0$, the potential V has a minimum at some $z = z_{min}$ with $V(z_{min}) < 0$. Now if also $V(z_0) < 0$, the "particle" will oscillate in the potential well. This means that it comes to rest at some finite $\rho = \bar{\rho}$, with behavior of a and b at $\bar{\rho}$ similar to $\rho = 0$: $a' \to 0$ and $b \propto (\bar{\rho} - \rho)$. Like at $\rho = 0$, there may be a conical defect which can be viewed as a brane. We will study this type of solution extensively in section 3.3.

3. If $\Lambda < 0$ and $\Lambda_4 \geq 0$, the slope of the potential is always negative. From eqs. (55,57,58) we see that both a and b diverge exponentially as $\rho \to \infty$, and spacetime does not terminate at finite ρ (except when infinity is shielded by a codimension-one brane, a possibility which was chosen in ref. [28], but which we do not consider).

4. An interesting borderline case between type 1 and type 2 appears when $\Lambda > 0$, $\Lambda_4 > 0$ and $V(z_0) = 0$. This is given for $z_0 = (10\Lambda_4/3\Lambda)^{5/4}$. The solution for this situation is

$$a(\rho) = a_0 \cos(\omega\rho), \qquad b(\rho) = A a_0 \omega \sin(\omega\rho), \qquad \omega^2 = \frac{\Lambda}{10}, \quad a_0 = z_0^{2/5}. \tag{61}$$

As in the first type of solutions, spacetime terminates at finite $\bar{\rho}$ and the warp factor a goes to zero there. But this time there is no singularity, all curvature scalars (including the square of the Riemann tensor) remain finite as $\rho \to \bar{\rho}$. In some sense, the roles of θ and x^μ are interchanged at $\bar{\rho}$ as compared to $\rho = 0$, since ρ becomes a kind of radial coordinate for the x^μ there, $a \propto (\bar{\rho} - \rho)$. One might imagine that the factor $a_0 k$ appearing in this proportionality leads to a deficit angle brane, but this time with codimension five. A closer look shows that this is not the case. Deficit angle branes appear only in codimension two (see appendix A).

There are also "unwarped" solutions in which the "particle" rests in a minimum or on a maximum of the potential (depending on the signs of Λ and Λ_4). These solutions, where only b depends on ρ, cannot be described with the potential method, and we will consider them later.

Which of these solutions can serve as a basis for a realistic model? There are several questions to study. At first we have to check that integrating out the extra dimensions gives a finite result, i.e. that the effective four-dimensional Newton's constant is finite,

$$\int d\rho \, d\theta \; a^2 \, b < \infty. \tag{62}$$

This inequality holds in all cases except the third one which will be discarded from now on. Another question is that of stability against small classical fluctuations. Lavrelashvili and Tinyakov [33] have explicitly shown that solutions of the first type are unstable. We conjecture that the situation is not much better in the other cases, and this is one of the reasons why the inclusion of a Yang-Mills field is helpful. Furthermore the number of chiral fermions is an important issue. Wetterich [32] showed that the addition of a Weyl spinor to the six-dimensional action leads to an infinite number of four-dimensional chiral fermions in the first case, which is due to the weird structure of the singularity at $\bar{\rho}$. In type 2 solutions the number of chiral fermions may be either zero or finite, as will be shown in chapter 4. This case is particularly interesting to us. The "borderline case" (type 4 solution) is not discussed here. But we already see that, after normalizing z_0 to 1 by a rescaling of the 4D metric, Λ_4 is of the same size as Λ, which is certainly not very promising.

Originally [29] the warped six-dimensional model was introduced in order to surround the cosmological constant problem. It was shown that solutions with $\Lambda_4 = 0$ exist for an arbitrary "true vacuum energy" Λ. The question remained why a solution with $\Lambda_4 = 0$ or very close to zero should be favored compared to those with larger λ_4. The authors of ref. [29] expressed their hope that quantum corrections or additional interactions would single out the solution with $\Lambda_4 = 0$. Our ansatz will be different. In chapter 5 we will consider cosmological solutions of the Einstein-Yang-Mills system. The arbitrariness of the four-dimensional cosmological constant could be due to the "absorption" of curvature by the warping along the internal space, such as the time evolution of the scale factor "absorbs" the curvature in ordinary cosmology. Now if the warping becomes time-dependent, the effective four-dimensional cosmological "constant" becomes a dynamical variable, i.e. effectively some type of quintessence. Our hope is that it is this dynamics which singles out a very small Λ_4.

3.2 6D Einstein-Maxwell Theory and Solutions

Now we turn to six-dimensional Einstein-Maxwell theory. The action is

$$S = \int d^6x \, \sqrt{-g} \left(\frac{-R + 2\Lambda}{16\pi G_6} + \frac{1}{4} F_{AB} F^{AB} \right), \tag{63}$$

where G_6 is the six-dimensional gravitational constant. The field equations are

$$G_A^B = R_A^B - \frac{1}{2} R \delta_A^B = -\Lambda \delta_A^B + 8\pi G_6 T_A^B, \tag{64}$$

$$T_A^B = (F_{AC}F^{BC} - \frac{1}{4} F_{CD}F^{CD}\delta_A^B), \tag{65}$$

$$\partial_A(\sqrt{g}F^{AB}) = 0. \tag{66}$$

Here T_A^B is the energy momentum tensor generated by the abelian gauge field strength F_{AB}. The spacetime symmetries require that $F_{\rho\theta}$ is the only non-vanishing component of the field strength tensor, since $F_{BC} = \partial_B A_C - \partial_C A_B$, $A_\mu = 0$ (by symmetry), $A_\rho = 0$ (by a suitable gauge transformation) and $A_\theta = \alpha(\rho)$. The Maxwell equations then imply

$$F_{\rho\theta} = C a^{-4} b, \tag{67}$$

where C is a constant of integration. Plugging the field strength (67) into our expression (65) for the bulk energy momentum tensor T_A^B one gets the non-vanishing components

$$T_\mu^\nu = -\frac{1}{2} C^2 a^{-8} \delta_\mu^\nu, \tag{68}$$

$$T_\theta^\theta = T_\rho^\rho = \frac{1}{2} C^2 a^{-8}. \tag{69}$$

When we insert this into the Einstein equations, we see that the relation between a and b remains the same as before and that the potential V gets an additional term,

$$V(z) = \frac{5}{16}\Lambda z^2 - \frac{25}{24}\Lambda_4 z^{6/5} + \frac{25}{12}\pi G_6 C^2 z^{-6/5}. \tag{70}$$

Now, for $\Lambda > 0$ and $C \neq 0$, the potential goes to infinity for $\rho \to 0$ and $\rho \to \infty$, so the solution is of type 2, whatever $V(z_0)$ or the sign of Λ_4 is. In addition to the attractive features of type 2 solutions already mentioned, we expect these solutions to be stable against classical perturbations, due to the presence of the Maxwell field.

By solving the system, we obtained the constants of integration Λ_4, C, A and z_0. The Einstein equations are obviously invariant under a rescaling with constant factor l:

$$a \to la, \qquad \Lambda_4 \to l^2 \Lambda_4, \qquad C \to l^4 C, \tag{71}$$

which corresponds to a change of scale for the four-dimensional coordinates

$$x^\mu \to l^{-1} x^\mu. \tag{72}$$

This freedom can be used to set $z_0 = 1$.

In the presence of charged fields, the gauge field A_θ has to fulfill certain requirements. Since the θ coordinate becomes singular at the two poles $\rho = 0$ and $\rho = \bar{\rho}$, consistent local coordinate systems must have $A_\theta = 0$ at these points (for a more detailed argument, see ref. [30]). Unless $C = 0$, at least two patches with different gauges are therefore needed to cover the whole internal space. If the gauge with $A_\theta(\rho = 0) = 0$ has $\lim_{\rho\to\bar{\rho}} A_\theta = \bar{m}$, the other gauge is obtained by subtraction of the constant $\bar{m}$. The gauge transformation

$$A_\theta \to A_\theta + \frac{1}{e}\partial_\theta \eta. \tag{73}$$

must be well-defined for the charged fields, which requires $\bar{m} = m/e$, with e the gauge coupling and m an integer "monopole number". The parameter C can be expressed in terms of m,

$$C = \frac{m}{e\int_0^{\bar{\rho}} d\rho\, a^4 b}. \tag{74}$$

In this case our general type 2 solution can be expressed in terms of the integration constants A and Λ_4 and the monopole number m.

3.3 Codimension-Two Branes

From now on we will almost only discuss type 2 solutions. But as long as only the local properties of a codimension-two brane are concerned, what we say is also true for the possible singularity at $\rho = 0$ in the other types of solutions.
In the modern language the conical singularities that may appear in the solutions would be called branes. In this section, the relation between the "old" and "modern" language is discussed, and the tension of the branes is calculated. We show that, at least at a geometrical level, the two points of view are equivalent.
At $\rho \to 0$, we saw that b vanishes linearly while a approaches a finite constant a_0 which can be rescaled to 1. The deficit angle λ can be defined via $b \to (1 - \lambda/2\pi)\rho$. Here, $\lambda = 0$ corresponds to $\rho = 0$ being a regular point in the internal space, whereas $\lambda \neq 0$ corresponds to a "defect" situated at $\rho = 0$ with deficit angle λ. This is what we call a deficit angle brane (DAB). The circumference of a circle in internal space at radius ρ is then $(2\pi - \lambda)\rho$ instead of $2\pi\rho$. A bulk test particle can measure the singularity by surrounding it, although the brane does not induce any curvature in the bulk. For $\lambda > 0$ the singularity is a familiar cone, whereas a negative deficit angle $\lambda < 0$ may be called an "anticone". We will denote by "cusps" all singular structures with $\lambda \neq 0$. The conical defect ($\lambda > 0$) is a straightforward generalization of a straight infinitely extended string in four dimensions, where the z-coordinate is now replaced by the cordinates $\vec{x}$ on the three-brane. If the space terminates at some finite $\bar{\rho}$, another DAB may be located at $\rho = \bar{\rho}$. Depending on the appearance of deficit angles we have two, one or zero "true" singularities, associated to a corresponding number of branes. The most generic solution has two branes at $\rho = 0$ and $\rho = \bar{\rho}$.
The original paper [30] has taken the point of view that the point $\rho = 0$ or $\bar{\rho}$ is not included into the manifold if a nonzero deficit angle occurs. The singularity was seen as a property of the bulk geometry, completely determined by the integration constants of the bulk solution. The modern "brane point of view" [28] asserts that an object called brane sits at $\rho = 0$ or $\bar{\rho}$ and determines the geometry due to its tension via the Einstein equations. These two descriptions describe exactly the same solution and are therefore equivalent. Different implications for physics for the two points of view could only arise

if objects would be located on the brane which cannot be described from a bulk point of view, as it is certainly possible for codimension-one branes. Then a brane point of view would be necessary in order to describe these objects. But, as we will discuss below, it seems unlikely to us that anything δ-function-like except pure tension can consistently be put on an infinitely thin deficit angle brane. If this conjecture turns out to be true, it would be unnecessary to speak of a brane, while the brane point of view can still be considered as being quite useful for intuition.
We first adopt the brane point of view where one or two cusps are included into the manifold as branes. We want to relate the properties of the branes to the free integration constants appearing in the "bulk point of view". The branes correspond to a δ-function singularity of the curvature tensor which may be seen as generated by a δ-function-like energy momentum tensor at that position, the brane tension. In order to calculate the brane tension, we follow the lines of ref. [34]. We first assume the brane to have a finite thickness ϵ and then take the limit $\epsilon \to 0$. The energy momentum tensor generated by the gauge field remains finite at $\rho = 0$ and $\bar{\rho}$, see eqs.(68), (69), so it cannot account for the singularity. The branes need to have some additional internal energy momentum tensor $\tilde{T}_A^B$. The Einstein equations inside the brane, $0 \leq \rho < \epsilon$, are then

$$G^\mu_\nu = \left(-\frac{\Lambda_4}{a^2} + 3\frac{a''}{a} + 3\frac{a'^2}{a^2} + \frac{b''}{b} + 3\frac{a'b'}{ab}\right)\delta^\mu_\nu = -\Lambda\delta^\mu_\nu + 8\pi G_6(T^\mu_\nu + \tilde{T}^\mu_\nu), \tag{75}$$

$$G^\theta_\theta = -\frac{2\Lambda_4}{a^2} + 4\frac{a''}{a} + 6\frac{a'^2}{a^2} = -\Lambda + 8\pi G_6(T^\theta_\theta + \tilde{T}^\theta_\theta), \tag{76}$$

$$G^\rho_\rho = -\frac{2\Lambda_4}{a^2} + 4\frac{a'b'}{ab} + 6\frac{a'^2}{a^2} = -\Lambda + 8\pi G_6(T^\rho_\rho + \tilde{T}^\rho_\rho). \tag{77}$$

The brane tension components can be defined as the integral over the components of the energy momentum tensor

$$\mu_i^{(\epsilon)} = -\int_0^\epsilon d\rho\, a^4 b\, \tilde{T}^{(i)}_{(i)}(\rho), \tag{78}$$

where $i = \nu, \theta, \rho$ and the brackets mean that there is no summation. Using eqs.(75)-(77) we can express the ρ-integrals over $\tilde{T}^{(i)}_{(i)}$ in terms of integrals over geometric quantities. Since we wish to consider the limit $\epsilon \to 0$, in which $\tilde{T}_A^B$ will diverge in order to give a finite tension, the contribution from the Λ-, Λ_4- and T_A^B-terms may be neglected in these integrals. As an example one obtains

$$\mu_\theta^{(\epsilon)} = -\frac{1}{8\pi G_6}\int_0^\epsilon d\rho\, a^4 b\, \left(4\frac{a''}{a} + 6\frac{a'^2}{a^2}\right). \tag{79}$$

For two particular combinations of brane tensions the ρ-integral can be performed explicitly:

$$\left(a^3 a' b\right)|_0^\epsilon = -2\pi G_6(\mu_\theta + \mu_\rho) \tag{80}$$

and

$$\left(a^4 b'\right)|_0^\epsilon = -8\pi G_6\left(\mu_\nu - \frac{3}{4}\mu_\theta + \frac{1}{4}\mu_\rho\right), \tag{81}$$

Here $|_0^\epsilon$ means the difference between the expression evaluated at $\rho = \epsilon$ and $\rho = 0$.
Up to this point we have only used the general form of the metric (42) and the higher dimensional Einstein equation. We implicitly assume that our model and solution is valid for $\rho \geq \epsilon$, whereas in the inner region $\rho < \epsilon$ more complicated physics may play a role, modifying the field equations but not the symmetries of the metric. (In this sense we define the energy momentum tensor in the inner region to include all parts in the field equations except the Einstein tensor.)
In order to proceed we need some additional information about the inner region. Within the brane point of view one assumes that there is no real singularity at $\rho = 0$. Sufficient resolution and understanding of the physics at extremely short distances should rather turn the brane into an extended object with finite thickness ϵ. In consequence, a manifold that is regular at $\rho = 0$ obeys

$$a'|_{\rho=0} = 0, \quad b'|_{\rho=0} = 1, \quad b|_{\rho=0} = 0, \tag{82}$$

and we choose a scaling of the four dimensional coordinates x^μ such that $a|_{\rho=0} = 1$. We next turn to our solution for small ρ. Since the "particle" starts at rest at $z = 1$ for $\rho = 0$ one finds by linearization

$$z(\rho) = 1 - \frac{\alpha}{2}\rho^2, \quad \alpha = \frac{\partial V}{\partial z}|_{z=1} \tag{83}$$

and therefore

$$a(\rho) = 1 - \frac{1}{5}\alpha\rho^2, \quad a'(\rho) = -\frac{2}{5}\alpha\rho, \quad b(\rho) = A\alpha\rho, \quad b'(\rho) = A\alpha. \tag{84}$$

Here α is related to the deficit angle λ by

$$b = (1 - \frac{\lambda}{2\pi})\rho = A\alpha\rho \tag{85}$$

or

$$b'(z \to 1) = 1 - \frac{\lambda}{2\pi} = A\,\frac{dV}{dz} = A\left(\frac{5}{8}\Lambda - \frac{5}{4}\Lambda_4 - \frac{5}{2}\pi G_6 C^2\right). \tag{86}$$

Up to corrections of the order $O(\epsilon)$ we infer

$$\left(a^3 a' b\right)|_0^\epsilon = 0, \qquad \left(a^4 b'\right)|_0^\epsilon = -\frac{\lambda}{2\pi}. \tag{87}$$

In the same approximation we note that the integrand in eq.(79) is of the order ϵ. This will not be changed by "regularizing" the brane in the inner region and we conclude $\mu_5^{(\epsilon)} = O(\epsilon^2)$. Combining this with eq.(87) and taking the limit $\epsilon \to 0$ we arrive at the final relation between the brane tensions and the deficit angle

$$\mu_\nu = \frac{\lambda}{16\pi^2 G_6}, \quad \mu_\theta = \mu_\rho = 0. \tag{88}$$

This equation constitutes the link between brane and bulk points of view. Within the brane point of view an object with tension $\mu_\nu \neq 0$, $\mu_\theta = \mu_\rho = 0$ produces a deficit angle in the geometry according to eq.(88). This in turn limits the allowed solutions of the Einstein equations. From the bulk point of view the general solution has free integration constants which are related to the deficit angle by virtue of eq.(86). One may consider λ as one of the independent integration constants. The discussion of the deficit angle at $\bar{\rho}$ proceeds in complete analogy. The general solution can therefore be characterized by two continuous deficit angles λ_0 and $\lambda_{\bar{\rho}}$ (at $\rho = 0$ and $\rho = \bar{\rho}$, respectively) and an integer monopole number m.

We observe that a positive brane tension μ_ν corresponds to a positive deficit angle. (In our conventions $\mu_\nu > 0$ means positive energy density and negative pressure.) We do not restrict our discussion to $\mu_\nu \geq 0$ and we will see in chapter 4 that a negative brane tension with negative deficit angle is particularly interesting.

3.4 The Spherically Symmetric Solution

Finally the "unwarped" solutions are to be discussed, which correspond to the situation where the "particle" is situated on a minimum or maximum of the potential V, and six-dimensional spacetime is a direct product of four-dimensional spacetime and internal space. In this case we have $a(\rho) = 1$ and the Einstein equations reduce to

$$-\Lambda_4 + \frac{b''}{b} = -\Lambda - 4\pi G_6 C^2 \tag{89}$$

$$-2\Lambda_4 = -\Lambda + 4\pi G_6 C^2. \tag{90}$$

The second equation gives a relation between Λ_4, Λ and C which is just the condition for the minimum/maximum of the potential to be located at $z = 1$. The first equation can then be integrated to

$$b(\rho) = A\sin(k\rho), \qquad k^2 = \frac{\Lambda}{2} + 6\pi G_6 C^2. \tag{91}$$

For $A = 1/k$ internal space is a sphere S^2 with radius $L = 1/k$, and for different A it has two equal deficit angles at the endpoints. Four-dimensional Minkowski space, i.e. $\Lambda_4 = 0$ is obtained when

$$\Lambda = 4\pi G_6 C^2, \qquad k^2 = 2\Lambda. \tag{92}$$

When we discuss cosmological solutions, C will become a function of time. So it will be better to classify a solution in terms of monopole numbers, which remain really constant. One obtains

$$F_{\rho\theta} = CA\sin k\rho \tag{93}$$

and we choose A_θ to be

$$A_\theta = \int_0^\rho d\rho' F_{\rho\theta} = -\frac{CA}{k}(\cos k\rho - 1) \tag{94}$$

In the presence of fields that couple to the gauge field, the difference between A_θ at $\rho = 0$ and $\bar{\rho}$ must again be an integer times $1/e$, so we infer

$$-\frac{CA}{k} = \frac{m}{2e} \tag{95}$$

or in the spherical case where $A = 1/k$

$$C = -\frac{m}{2e}k^2. \tag{96}$$

Plugging this into the expression for k^2 in equation (91), one obtains

$$k^2 = \frac{1}{3\pi G_6}\frac{e^2}{m^2}\left(1 \pm \sqrt{1 - 3\pi G_6 \Lambda \frac{m^2}{e^2}}\right), \tag{97}$$

and from this one gets

$$\Lambda_4 = \frac{2}{3}\Lambda - \frac{1}{9\pi G_6}\frac{e^2}{m^2}\left(1 \pm \sqrt{1 - 3\pi G_6 \Lambda \frac{m^2}{e^2}}\right). \tag{98}$$

So if $\Lambda < 0$, for each value of m there is one positive solution for k^2. If $\Lambda > 0$, there are two such solutions, provided Λ and m are not too large. The Minkowski solution $\Lambda_4 = 0$ is obtained for

$$\Lambda = \frac{1}{4\pi G_6}\frac{e^2}{m^2} \tag{99}$$

and the radius of the internal sphere takes the value

$$L^2 = 2\pi G_6 \frac{m^2}{e^2}. \tag{100}$$

If, more generally, $A = \tilde{A}/k$, all the previous relations hold with e substituted by $\tilde{A}e$. Finally we remark that even if no charged fields are present, this parametrization is still useful for cosmological purposes, since "m/e", which is now an arbitrary parameter, will still be constant, whereas C becomes time-dependent.

4 Chiral Fermions from Extra Dimensions

Obtaining chiral four-dimensional fermions from the dimensional reduction of a higher-dimensional theory is a nontrivial task. Witten [35] showed that it is in general impossible to get them from a pure Einstein theory if spacetime is a direct product of four-dimensional spacetime and a compact internal space. Wetterich [36] showed how the problem may be surrounded either by considering a generalized theory of gravity or by imposing a non-compact internal space. The second possibility is particularly interesting for us, since the internal spaces with singularities which were discussed in the previous section are non-compact. It turns out that some of these indeed admit chiral four-dimensional fermions [32, 37]. Another possibility to get chiral fermions is to couple them to a gauge field present in the higher dimensional theory [1].
In order to discuss these matters, we first work out the algebraic structure of spinors in more than four dimensions and show for which dimensionalities Weyl and Majorana constraints may be imposed [38]. Then the vielbein formalism and the spin connection are introduced in order to describe spinors on curved manifolds. The procedure of dimensional reduction is worked out [39]. The chirality index [40, 35] is introduced and the corresponding No-go-theorems are sketched. We investigate the appearance of chiral fermions in six-dimensional Einstein-Maxwell theory and show that their number and properties depend on the deficit angles in the internal space which in turn are related to integration constants of the "bulk" solution. The fermions are shown to be attached to the branes. Again we discuss in detail the connection between the older bulk point of view and the modern brane point of view and formulate a kind of holographic principle, leading us to the notion of "holographic branes".

4.1 Spinors in Arbitrary Dimensions

In this section we construct the spinor representations of the Lorentz group in arbitrarily many dimensions. The introduction presented here is based on a combination of the treatments given in ref.[38] and appendices of refs [41, 42].

We have to find gamma matrices obeying the Clifford algebra

$$\{\gamma^m, \gamma^m\} = 2\eta^{mn}. \tag{101}$$

Even Dimensions: $d = 2k$. Group the γ^n into k sets of anticommuting raising and lowering operators,

$$\gamma^{0\pm} = \frac{1}{2}(\pm\gamma^0 + \gamma^1), \tag{102}$$

$$\gamma^{a\pm} = \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1}), \quad a = 1, ..., k-1. \tag{103}$$

These satisfy

$$\left\{\gamma^{a+}, \gamma^{b-}\right\} = \delta^{ab}, \tag{104}$$

$$\left\{\gamma^{a+}, \gamma^{b+}\right\} = \left\{\gamma^{a-}, \gamma^{b-}\right\} = 0. \tag{105}$$

In particular, $(\gamma^{a+})^2 = (\gamma^{a-})^2 = 0$. It follows that by acting with the γ^{a-} we can find a spinor that obeys

$$\gamma^{a-}\zeta = 0 \tag{106}$$

for all a. Starting from ζ one can derive a representation of dimension 2^k by acting in all possible ways with the γ^{a+}. The states obtained can be labeled by $\mathbf{s} \equiv (\mathbf{s_0}, \mathbf{s_1}, ..., \mathbf{s_{k-1}})$, where each s_a is $\pm 1/2$:

$$\psi^{(\mathbf{s})} \equiv (\gamma^{(k-1)+})^{s_{k-1}+1/2} ... (\gamma^{0+})^{s_0+1/2}\zeta. \tag{107}$$

Taking the $\psi^{(\mathbf{s})}$ as a basis, the matrix elements of γ^n can be derived from the definitions and anticommutation relations. One finds an iterative expression starting in $d = 2$, where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{108}$$

Going from k to $k+1$,

$$\gamma^n = \tilde{\gamma}^n \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n = 0, ..., d-3 \quad , \tag{109}$$

$$\gamma^{d-2} = I \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{d-1} = I \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{110}$$

with $\tilde{\gamma}^n$ the $2^k \times 2^k$ Dirac matrices in $d-2$ dimensions and I the $2^k \times 2^k$ identity. The 2×2 matrices act on the index s_k, which is added in going from $2k$ to $2k+2$ dimensions. This representation gives the γs simple reality and symmetry properties: $\gamma^{n*} = \gamma^n$ for n even or $n = 1$, and $\gamma^{n*} = -\gamma^n$ otherwise; $\gamma^{nT} = -\gamma^{n*}$ for $n = 0$, and $\gamma^{nT} = \gamma^{n*}$ otherwise. The matrices

$$\Sigma^{mn} = \frac{1}{4i}[\gamma^m, \gamma^n] \tag{111}$$

satisfy the Lorentz algebra

$$i\,[\Sigma^{mn}, \Sigma^{pq}] = \eta^{np}\Sigma^{mq} + \eta^{mq}\Sigma^{np} - \eta^{nq}\Sigma^{mp} - \eta^{mp}\Sigma^{nq}. \tag{112}$$

The generators $\Sigma^{2a,2a+1}$ commute and can be simultaneously diagonalized. In terms of raising and lowering operators,

$$S_a = i^{\delta_{a,0}}\Sigma^{2a,2a+1} = \gamma^{a+}\gamma^{a-} - \frac{1}{2} \tag{113}$$

so $\psi^{(\mathbf{s})}$ is a simultaneous eigenstate of the S_a with eigenvalues s_a. The half-integer values show that this is a spinor representation, called the 2^k-dimensional Dirac representation. The elements of the Lorentz group in the neighborhood of the identity are represented by matrices of the form $1 + \frac{1}{2}i\epsilon_{mn}\Sigma^{mn}$.

The Dirac representation is in even dimensions reducible as a representation of the Lorentz algebra. Because Σ^{mn} is quadratic in the γs, the $\psi^{(\mathbf{s})}$ with even and odd numbers of $+1/2$ eigenvalues do not mix. Define

$$\Gamma = i^{-k}\gamma^0\gamma^1 ... \gamma^{d-1}, \tag{114}$$

which fulfills

$$\Gamma^2 = 1, \quad \{\Gamma, \gamma^n\} = 0, \quad \{\Gamma, \Sigma^{mn}\} = 0. \tag{115}$$

Noting that

$$\Gamma = 2^k S_0 S_1 ... S_{k-1}, \tag{116}$$

we see that Γ is diagonal in our basis, taking the value $+1$ when s_a include an even number of $-1/2$ and -1 for an odd number of $-1/2$ values. The 2^{k-1} states with Γ eigenvalue $+1$ form a Weyl representation of the Lorentz algebra, and those with eigenvalue -1 form a second.

The matrices γ^{n*} and $-\gamma^{n*}$ satisfy the same Clifford algebra as γ^n and so they must be related to them by a basis transformation (because of uniqueness). Indeed, we find in our representation from the reality properties, that the products

$$B_1 = \gamma^3 \gamma^5 ... \gamma^{d-1}, \quad B_2 = \Gamma B_1 \tag{117}$$

obey

$$B_1 \gamma^n B_1^{-1} = (-1)^{k-1} \gamma^{n*}, \quad B_2 \gamma^n B_2^{-1} = (-1)^k \gamma^{n*}. \tag{118}$$

For both of these matrices (and only for these or a linear combination of them) we have

$$B \Sigma^{mn} B^{-1} = -\Sigma^{mn*}. \tag{119}$$

It follows that the spinors ψ and $B^{-1}\psi^*$ transform in the same way under the Lorentz group, so the Dirac representation is its own conjugate. We can define charge conjugation by

$$\psi^c = B^{-1} \psi^* = \hat{C} \psi. \tag{120}$$

A Majorana condition demands $\psi^c = \psi$. Acting twice with the charge conjugation operator, it follows $\psi = B^* B \psi$ for those ψs that fulfill the condition. From the reality and anticommutation properties one finds

$$B_1^* B_1 = (-1)^{k(k-1)/2}, \quad B_2^* B_2 = (-1)^{(k-1)(k-2)/2}. \tag{121}$$

A Majorana condition is therefore only self-consistent if $k = 0 \bmod 4$ (with $B = B_1$), $k = 1 \bmod 4$ (with $B = B_1$ or $B = B_2$) or $k = 2 \bmod 4$ (with $B = B_2$).

A Majorana-Weyl spinor is possible if $B^* B = 1$ and charge conjugation commutes with chirality. Acting on Γ, one finds

$$B_1 \Gamma B_1^{-1} = B_2 \Gamma B_2^{-1} = (-1)^{k-1} \Gamma^*, \tag{122}$$

so for k odd each Weyl representation is its own conjugate, and for k even the Weyl representations are conjugate to each other. A Majorana-Weyl condition is therefore only possible if $k = 1 \bmod 4$.

For k even, $\hat{C}$ anticommutes with Γ. In a basis in which Γ has the form

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{123}$$

B^{-1} has the form

$$B^{-1} = \begin{pmatrix} 0 & \tilde{E} \\ E & 0 \end{pmatrix}, \tag{124}$$

where $\hat{C}^2 = 1$ requires $\tilde{E} = (E^{-1})^*$. In this basis, a Majorana spinor has the form

$$\psi_M = \begin{pmatrix} \chi \\ E\chi^* \end{pmatrix}. \tag{125}$$

It is completely described by the complex $2^{d/2-1}$ component spinor χ, which shows the equivalence of Weyl and Majorana spinors in these dimensions.

Odd dimensions: d=2k+1. One can just add $\gamma^{2k} = \Gamma$ to the γ-matrices from $d = 2k$ in order to satisfy the Clifford algebra in $d = 2k + 1$ dimensions. The Σs derived from these γs now form an irreducible representation of the Lorentz algebra. The conjugation of γ^{2k} (eq. (122)) is compatible with the conjugation of the other γs only for $B = B_1$ (eq. (118)). A Majorana condition is therefore possible for $k = 0$ or 1 mod 4 by virtue of eq.(121).

To summarize, we found for the several dimensions:

- If $d = 0$ or 4 mod 8, the Weyl representations are complex conjugate to each other. Majorana spinors exist and there is a one-to-one mapping between Majorana spinors and the spinors of each Weyl representation.
- If $d = 1$ or 3 mod 8, Majorana spinors exist, but there is no Weyl condition.
- If $d = 2$ mod 8, Majorana, Weyl and Majorana-Weyl spinors exist. The Weyl representations are self-conjugate ("real").
- If $d = 5$ or 7 mod 8, neither Weyl nor Majorana spinors are possible.
- If $d = 6$ mod 8, Weyl representations exist and are self-conjugate, but only "pseudo-real". Therefore Majorana spinors do not exist.

Gamma Products: From the anticommutation relations it can be seen that only anti-symmetric products of γ-matrices can be linearly independent. In fact, in even dimensions, none of the products

$$\gamma^{n_1 n_2 \dots n_p} = \gamma^{[n_1}\gamma^{n_2}\dots\gamma^{n_p]} \tag{126}$$

vanishes (the square is always proportional to the identity), and they are all linearly independent, since they all have different Lorentz and/or parity transformation rules. From their number (sum of binomial coefficients) one finds that they span the complete space of $2^k \times 2^k$ matrices. In odd dimensions $d = 2k + 1$ some of the products are related via

$$\gamma^{n_1 \dots n_p}\gamma^{2k} \sim \epsilon^{n_1 \dots n_d}\gamma_{n_{p+1}\dots n_d}. \tag{127}$$

So only products with $p \leq k$ are independent. These again span the complete space of $2^k \times 2^k$ matrices.

Fermion bilinears and mass terms: The matrices γ^{nT} and $-\gamma^{nT}$ also satisfy the Clifford algebra. Indeed, for even $d = 2k$ we find for

$$C_1 = B_1\gamma^0, \quad C_2 = B_2\gamma^0, \tag{128}$$

using the hermiticity property

$$\gamma^{n\dagger} = -\gamma^0\gamma^n(\gamma^0)^{-1}, \tag{129}$$

that

$$C_1\gamma^n C_1^{-1} = (-1)^k\gamma^{nT}, \quad C_2\gamma^n C_2^{-1} = (-1)^{k+1}\gamma^{nT}. \tag{130}$$

In odd dimensions, only C_1 acts uniformly on all γs. In all cases,

$$C\Sigma^{mn}C^{-1} = -\Sigma^{mnT}. \tag{131}$$

Now there are two ways to construct fermion bilinears. The first is the standard one known from four dimensions: $\bar{\psi} = \psi^\dagger\gamma^0$, and from the hermiticity properties of the Σ^{mn} one finds that $\bar{\psi}\psi$ is a Lorentz scalar. The other possibility is $\tilde{\psi} = \psi^T C$, and again $\tilde{\psi}\psi$ is a Lorentz scalar. Tensors can be constructed as

$$\bar{\psi}\gamma^{n_1 n_2 \dots n_p}\psi \quad or \quad \tilde{\psi}\gamma^{n_1 n_2 \dots n_p}\psi. \tag{132}$$

In even dimensions, one finds for a Weyl spinor $\psi_+ = \frac{1}{2}(1+\Gamma)\psi$:

$$\bar{\psi}_+ = \bar{\psi}\frac{1}{2}(1-\Gamma), \quad \tilde{\psi}_+ = \tilde{\psi}\frac{1}{2}(1+(-1)^{d/2}\Gamma). \tag{133}$$

So the tensors constructed from $\bar{\psi}_+$ vanish if the rank is even. For $d = 0$ mod 4 the tensors constructed from $\tilde{\psi}$ vanish if the rank is odd, for $d = 2$ mod 4 if the rank is even. From this follows that a mass term is forbidden for Weyl spinors if the dimension is $d = 2$ mod 4.

Another constraint comes from the Pauli principle [38]. It forbids a mass term for a Weyl or Majorana spinor in $d = 0$ mod 8 dimensions and for a Majorana spinor if $d = 1$ mod 8.

Spinors of SO(N) : For SO(N) the analysis is similar. The only difference lies essentially in ignoring the pair γ^0, γ^1, so that SO(N) is analogous to SO(N+1,1). The decompositions into Weyl and Majorana representations that were possible for $d+2$ dimensions in the Minkowski case are now possible in d dimensions. For the construction of bilinears one has $C = B$, hence $\tilde{\psi} = \psi^T B$, and $\bar{\psi} = \psi^\dagger$. Now for Weyl spinors

$$\bar{\psi}_+ = \bar{\psi}\frac{1}{2}(1+\Gamma), \tag{134}$$

and a mass term is always possible for them.

4.2 Spinors on Curved Spacetimes

The familiar formulation of gravity in terms of a metric tensor $g_{\mu\nu}$ is adequate for theories with matter fields restricted to scalars, vectors and tensors, but spinors need a different treatment. Unlike vectors and tensors, spinors have a Lorentz transformation rule that has no natural generalization to arbitrary coordinate systems. In order to deal with spinors, we have to introduce coordinate systems $\xi^a_X(x)$ that are locally inertial at any given point X. The transformation which leads from general coordinates into this locally inertial frame is described by the vielbein

$$e^a{}_\mu(X) \equiv \frac{\partial \xi^a_X(x)}{\partial x^\mu}|_{x=X}. \tag{135}$$

An action will be invariant under general coordinate transformations $x^\mu \to x'^\mu$ and local Lorentz transformations $\xi^a \to \xi'^a = \Lambda^a{}_b \xi^b$. Under a general coordinate transformation, the vielbein transforms as

$$e^a{}_\mu(x) \to e'^a{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a{}_\nu(x), \tag{136}$$

and under a local Lorentz transformation as

$$e^a{}_\mu(x) \to \Lambda^a{}_b(x) e^b{}_\mu(x). \tag{137}$$

Vectors may be regarded either as quantities V^a that transform as vectors under local Lorentz transformations, but as scalars under general coordinate transformations, or as quantities v^μ that transform as scalars under local Lorentz transformations but as vectors under general coordinate transformations, the two being related by

$$V^a = e^a{}_\mu v^\mu. \tag{138}$$

Similar relations hold for tensors. In particular one has

$$\eta^{ab} = e^a{}_\mu e^b{}_\nu g^{\mu\nu}. \tag{139}$$

Latin indices are raised and lowered with η_{ab}, greek ones with $g_{\mu\nu}$.
Now we turn to spinors. These transform under local Lorentz transformations according to

$$\psi(x) \to D(\Lambda(x))\psi(x), \tag{140}$$

where $D(\Lambda)$ is the spinor representaion of the Lorentz group. The derivative transforms under $\Lambda(x)$ as

$$\partial_\mu \psi \to D(\Lambda) \left\{ \partial_\mu \psi + D^{-1}(\Lambda) \left(\partial_\mu D(\Lambda) \right) \psi \right\}. \tag{141}$$

In order to get a covariant derivative, the second term in the brackets has to be cancelled by introducing a connection matrix Ω_μ with transformation property

$$\Omega_\mu \to D(\Lambda) \Omega_\mu D^{-1}(\Lambda) - \left(\partial_\mu D(\Lambda) \right) D^{-1}(\Lambda). \tag{142}$$

The covariant derivative can then be defined as

$$D_\mu \psi \equiv \partial_\mu \psi + \Omega_\mu \psi, \tag{143}$$

which transforms under local Lorentz transformations like ψ itself. We can write Ω_μ in the form

$$\Omega_\mu(x) = \frac{1}{2} i \Sigma_{ab} \omega_\mu^{ab}(x), \tag{144}$$

where ω_μ^{ab} is a representation-independent field known as the **spin connection**, which can be taken to be

$$\omega_\mu^{ab} = g^{\nu\lambda} e^a{}_\nu e^b{}_{\lambda;\mu}. \tag{145}$$

Here the semi-colon denotes an ordinary covariant derivative, constructed using the affine connection $\Gamma^\lambda_{\mu\nu}$. At first we observe that this is antisymmetric in a and b because $g^{\nu\lambda} e^a{}_\nu e^b{}_\lambda = \eta^{ab}$ has vanishing covariant derivative. In order to prove that the so constructed Ω_μ has the correct transformation properties, it is sufficient to show it for infinitesimal Lorentz transformations,

$$\Lambda^a{}_b(x) = \delta^a{}_b + \epsilon^a{}_b(x), \qquad \epsilon_{ab} = -\epsilon_{ba}, \tag{146}$$

$$D(\Lambda) = 1 + \frac{1}{2} i \epsilon^{ab} \Sigma_{ab}. \tag{147}$$

Now the transformation rule (142) becomes

$$\Omega_\mu \to \Omega_\mu + \frac{1}{2} i \epsilon^{ab} \left[\Sigma_{ab}, \Omega_\mu \right] - \frac{1}{2} i \Sigma_{ab} \partial_\mu \epsilon^{ab}. \tag{148}$$

Using the commutaion relations of the Σ_{ab} and the transformation property

$$e^{b\nu} \partial_\mu e^a{}_\nu \to e^{b\nu} \partial_\mu e^a{}_\nu + \epsilon^b{}_c e^{c\nu} \partial_\mu e^a{}_\nu + \epsilon^a{}_c e^{b\nu} \partial_\mu e^c{}_\nu + \partial_\mu \epsilon^{ab}, \tag{149}$$

one finds after some algebra that this rule indeed holds.

Action of a symmetry group: How does an isometry group act on a spinor? Let the group be geometrically generated by the Killing vectors K_z. Then the definition of the Lie derivative can be extended to include spinors. The action of the group should leave the vielbein invariant, so the general coordinate change induced by K_z has to be combined with some local Lorentz transformation Λ_z accounting for that invariance. One obtains that the generators S_z acting on a spinor can be expressed as

$$S_z \psi = K_z{}^\mu D_\mu \psi + \frac{1}{2} i (D_\mu K_z{}^\nu) e^m{}_\mu e_{n\nu} \Sigma^{mn} \psi. \tag{150}$$

4.3 Dimensional Reduction of Fermions

In nature one observes that right-handed and left-handed fermions transform differently under the gauge group $SU(3) \times SU(2) \times U(1)$. (The quantum numbers are not "vector-like"). By gauge symmetry, a bare mass term is forbidden for such fermions. Masses occur only in the context of spontaneous symmetry breaking (SSB) through a Yukawa coupling to a scalar field which gets a nonzero vacuum expectation value (vev). A successful Kaluza-Klein theory would have to explain all the fermion quantum numbers from dimensional reduction of a fundamental spinor. Such attempts are described in chapter 6.

For the moment we are less ambitious and ask how we can get chiral four-dimensional fermions from dimensional reduction at all. We will see that there are strong constraints on such chiral models, but that the situation changes when we give up the restriction to compact internal spaces.

In the following we ignore SSB and regard chiral fermions as massless. On the other hand we assume that no small masses occur unless this is required by symmetry considerations. The reason for this is as follows: Without SSB, all natural masses that occur in Kaluza-Klein theories are almost of Planck mass order. (As we will show, the fermion masses are linked to the eigenvalues of the "internal" Dirac operator. The size of internal space is not far from the Planck scale, and this is the order at which eigenvalues are expected.) If a fermion is light just by chance and not by symmetry requirements, we expect that this will change dramatically when the parameters of the model are slightly shifted. Therefore a fine-tuning of parameters would be needed for such a non-required lightness, and we assume that this does not occur. To summarize this: We assume that the massless fermions are precisely the chiral ones.

We start with the the original Kaluza-Klein idea and assume at first that d-dimensional spacetime is a direct product of four-dimensional Minkowski space and a D-dimensional compact space K with isometry group G which appears as the gauge group in the effective four-dimensional world. We restrict ourselves to the case $D = 2 \bmod 4$. It was shown by Wetterich [39] and Witten [35] that chiral fermions cannot be obtained in any other case (this is still true for noncompact internal spaces). The space of d-dimensional spinors is the tensor product of the space of D-dimensional spinors and the space of 4-dimensional spinors. The Gamma matrices can be written

$$\gamma^{\mu} = \tilde{\gamma}^{\mu} \otimes I^{(D)}, \quad \gamma^{\alpha} = \tilde{\Gamma} \otimes \gamma^{\alpha(D)}, \tag{151}$$

where $\tilde{\gamma^{\mu}}$ are the 4D gamma matrices and $I^{(D)}$ is the $2^{D/2}$ dimensional unit matrix. An arbitrary d-dimensional spinor $\Psi(y,x)$ can always be "harmonically" expanded into representations of the group G:

$$\Psi(y,x) = \sum_{nHk} \psi_{nHk}(y)\phi_{nHk}(x). \tag{152}$$

Here the index H labels all irreducible representations of G that are contained in the infinite dimensional space of $2^{D/2}$-component spinor fields $\psi(y)$ corresponding to the internal space K. The index n runs over the components of H, and k counts how many times H is contained in $\psi(y)$. One can always normalize the ψ_{nHk} according to

$$\int d^D y \, \sqrt{-g} \psi_{n'H'k'}(y)^{\dagger} \psi_{nHk}(y) = \delta_{n'n}\delta_{H'H}\delta_{k'k}. \tag{153}$$

The Dirac operator is assumed to be hermitian. The fermion Lagrangian is then

$$L_\Psi = i\bar{\Psi}(\gamma^\mu D_\mu + \gamma^\alpha D_\alpha)\Psi. \tag{154}$$

For direct product spaces, the spin connections have no components mixing Minkowski space with internal space (this will change when a warping is introduced), and we have $D_\mu = \tilde{D}_\mu$ and $D_\alpha = D_\alpha^{(D)}$. Since $\gamma^\alpha D_\alpha$ is a G singlet operator and hence commutes with the generators of G, one has

$$\gamma^\alpha D_\alpha \psi_{nHk}(y) = \psi_{nHk'}(y) M^{(H)}_{k'k} \tag{155}$$

with a hermitian constant matrix $M^{(H)}_{k'k}$ for every representation H contained in $\psi(y)$. Inserting the harmonic expansion into the Lagrangian (154) gives

$$L_\Psi = i{\psi_{nHk}}^\dagger \psi_{n'H'k''}\ \bar{\phi}_{nHk}\tilde{\Gamma} M^{(H)}_{k''k'}\phi_{n'H'k'} + i{\psi_{nHk}}^\dagger\psi_{n'H'k'}\ \bar{\phi}_{nHk}\tilde{\gamma}^\mu\tilde{D}_\mu\phi_{n'H'k'}. \tag{156}$$

After carrying out the integration over the y coordinates, using the normalization (153) one obtains the effective four-dimensional Lagrangian

$$L^{(4)}_\Psi = i\bar{\phi}_{nHk}\tilde{\Gamma} M^{(H)}_{kk'}\phi_{nHk'} + i\bar{\phi}_{nHk}\tilde{\gamma}^\mu\tilde{D}_\mu\phi_{nHk}. \tag{157}$$

The second term is the usual kinetic term, and the first one is a mass term, obtained from the "mass operator" $\gamma^\alpha D_\alpha$. In fact, the matrices $M^{(H)}$ can be diagonalized by using unitary $k_H \times k_H$ matrices (where k_H is the number of $H's$ appearing in $\psi(y)$), without affecting the normalization (153).

We now turn to chirality. One has

$$\Gamma = \tilde{\Gamma} \otimes \Gamma^{(D)}. \tag{158}$$

If we start with a d-dimensional Weyl spinor, i.e. with a fixed Γ eigenvalue, then the eigenvalues of a $\Gamma^{(D)}$ eigenstate $\psi(y)$ in the harmonic expansion and the $\tilde{\Gamma}$ eigenvalue of the corresponding $\phi(x)$ are correlated. A $\Gamma^{(D)} = +1$ state $\psi^+(y)$ would belong to a left-handed four-dimensional fermion $\phi(x)$ and a $\Gamma^{(D)} = -1$ state $\psi^-(y)$ would belong to a right-handed four-dimensional fermion, or vice versa. Chiral four-dimensional fermions are therefore obtained if the ψ^+ and ψ^- states belong to different G-representations in the harmonic expansion. For then also the corresponding left- and right-handed four-dimensional fermions would transform differently. This can be seen as follows: The generators S_z of the symmetry group act on ψ_{nHk} via

$$S_z\psi_{nHk}(y) = \psi_{n'Hk}(y)(T_z^{(H)})_{n'n} \tag{159}$$

with a constant complex matrix $T_z^{(H)}$ for every representation H. The change of a d-dimensional spinor under an infinitesimal symmetry transformation is then

$$\begin{aligned} \delta_G\Psi(y,x) &= -\theta^z S_z\psi_{nHk}(y)\phi_{nHk}(x) && (160)\\ &= \psi_{n'Hk}(y)(-\theta^z(T_z^{(H)})_{n'n}\phi_{nHk}(x)) && (161)\\ &= \psi_{nHk}(y)\delta_G\phi_{nHk}(x). && (162) \end{aligned}$$

In order to get chiral four-dimensional fermions it is crucial to start with a d-dimensional Weyl (or Majorana-Weyl) spinor. A fixed Γ eigenvalue is necessary to get a correlation between D- and four-dimensional chirality. With a d-dimensional Dirac spinor one would get opposite correlations for the $\Gamma = 1$ and $\Gamma = -1$ eigenstates, and there would be a left-handed and a right-handed four-dimensional fermion for every ψ_{nHk}.

Distinguishing between complex, real and pseudoreal representations and using algebraic properties of the charge conjugation and helicity operators, one finds [39] that if there are chiral fermions at all, then they occur in pairs (starting from a d-dimensional Weyl spinor). There are either zero or at least two different massless fermions for any representation H. This unwanted degeneracy disappears only for a Majorana-Weyl spinor, which exists only for $d = 2$ mod 8. Thus a realistic Kaluza-Klein theory (with pure gravity and a massless spinor in the d-dimensional action) should have $8k + 2$ dimensions.

It remains the question whether chiral fermions can be obtained at all. This question is linked to the so-called **chirality index** N. Let n_C^+ be the number which denotes how many times a complex representaion C of the symmetry group appears in the expansion of $\psi^+(y)$, with corresponding numbers n_C^-, $n_{\bar{C}}^+$, $n_{\bar{C}}^-$, where $\bar{C}$ is the complex conjugate to C. The number $|N_C|$,

$$N_C = f_d(n_C^+ - n_C^- - n_{\bar{C}}^+ + n_{\bar{C}}^-) \tag{163}$$

denotes the total number of unpaired chiral fermions transforming as a representation C under the gauge group G. All other fermions can be paired to vector-like representations. These fermions will in general be massive, as explained in the beginning of this section. For $d = 2$ mod 4, D-dimensional charge conjugation implies

$$n_C^+ = n_{\bar{C}}^-, \quad n_C^- = n_{\bar{C}}^+. \tag{164}$$

The factor f_d is 1 for a Weyl spinor and $\frac{1}{2}$ for a Majorana-Weyl spinor, where ψ_C^+ and $\psi_{\bar{C}}^-$ are identified. The total chirality index is

$$N = \sum_C d_C |N_C|, \tag{165}$$

where d_C is the dimension of the representation C.

It is easy to show [40] that N does not change when the internal space K is deformed in accordance with the symmetry group G. If G is broken by the deformation to some smaller group $\tilde{G}$, the index N can only get smaller. So if $\bar{G}$ is the maximal possible symmetry group compatible with the topology and differentiable structure of K, and K can be obtained by deformation of such a maximally symmetric space $\bar{K}$, the index $\bar{N}$ corresponding to $\bar{K}$ is an upper limit for the index N corresponding to K. The presence and number of chiral fermions therefore depends essentially on the topology of the internal space K.

Unfortunately, a theorem by Atiyah and Hirzebruch (see ref. [35]) states that N is always zero for a compact space K. Possibilities to surround this no-go theorem are the inclusion of elementary gauge fields [35, 1] to which the fermions couple, or non-compact internal spaces [36]. Both of these possibilities are realized in our six-dimensional toy model.

4.4 Chiral Fermions from Deficit Angle Branes

We are now going to apply the techniques developed in the previous subsections to our six-dimensional case. The vielbein corresponding to the metric (42) can be chosen to be

$$e^m{}_\mu = a(\rho)\tilde{e}^m_\mu, \qquad e^m{}_\alpha = e^a{}_\mu = 0, \tag{166}$$
$$e^4{}_\theta = b(\rho)\cos\theta, \qquad e^5{}_\theta = b(\rho)\sin\theta, \tag{167}$$
$$e^4{}_\rho = \sin\theta, \qquad e^5{}_\rho = -\cos\theta. \tag{168}$$

Here we use the following conventions: Greek indices denote general coordinates, Latin indices correspond to the local inertial system. Indices μ, ν, m, n are running from 0 to 3, α, β, a, b from 4 to 5. For θ and ρ we use the coordinates themselves as indices to distinguish them from the Lorentz indices 4 and 5. A tilde again denotes the corresponding quantity in four dimensions. The non-vanishing components of the spin connection derived from this vielbein and the Christoffel symbols (43)-(45) are

$$\omega^{mn}_\mu = \tilde{\omega}^{mn}_\mu, \tag{169}$$
$$\omega^{m4}_\mu = -\omega^{4m}_\mu = a'\sin\theta\tilde{e}^m_\mu, \tag{170}$$
$$\omega^{m5}_\mu = -\omega^{5m}_\mu = -a'\cos\theta\tilde{e}^m_\mu, \tag{171}$$
$$\omega^{45}_\theta = -\omega^{54}_\theta = 1 - b' \tag{172}$$

The gamma matrices can be chosen to be $\gamma^m = \tilde{\gamma}^m$ (hence $\gamma^\mu = a^{-1}\tilde{\gamma}^\mu$), $\gamma^{4(5)} = \tilde{\Gamma} \otimes \tau_{1(2)}$, where $\tau_{1(2)}$ are the first and second Pauli matrix. In particular, the Lorentz generator Σ^{45} is $\frac{1}{2}\,\tilde{I} \otimes \tau_3$, where $\tilde{I}$ is the 4D unit matrix. The covariant derivatives for a spinor are

$$D_\mu\psi = \tilde{D}_\mu\psi + i(\Sigma_{m4}\sin\theta - \Sigma_{m5}\cos\theta)a'\tilde{e}^m_\mu\psi, \tag{173}$$
$$D_\theta\psi = \partial_\theta\psi + \frac{1}{2}i\tau_3(1-b')\psi, \tag{174}$$
$$D_\rho\psi = \partial_\rho\psi. \tag{175}$$

From this follows the Dirac operator

$$\begin{aligned}
\gamma^\mu D_\mu + \gamma^\alpha D_\alpha &= a^{-1}\tilde{\gamma}^\mu\tilde{D}_\mu + \gamma^\alpha D_\alpha + i\gamma^\mu(\Sigma_{m4}\sin\theta - \Sigma_{m5}\cos\theta)a'\tilde{e}^m &(176)\\
&= a^{-1}\tilde{\gamma}^\mu\tilde{D}_\mu + \gamma^\alpha D_\alpha + \gamma^m\frac{1}{2}(\gamma_m\gamma_4\sin\theta - \gamma_m\gamma_5\cos\theta)\frac{a'}{a} &(177)\\
&= a^{-1}\tilde{\gamma}^\mu\tilde{D}_\mu + \gamma^\alpha D_\alpha + 2(\gamma^4\sin\theta - \gamma^5\cos\theta)\frac{a'}{a} &(178)\\
&= a^{-1}\tilde{\gamma}^\mu\tilde{D}_\mu + \gamma^\alpha D_\alpha + 2\gamma^\rho\frac{a'}{a}. &(179)
\end{aligned}$$

The mass operator is therefore

$$M = a\Gamma^\alpha D_\alpha + a'\Gamma^\rho, \tag{180}$$

with Γ^α corresponding to the 2×2 gamma matrices of internal space. The charge operator Q is given by the action of the U(1) isometry group. Obviously a rotation $\theta \to \theta + \delta\theta$

has to be combined with a rotation of the "legs" e^4 and e^5 so that the vielbein remains invariant. This happens via the Lorentz generator Σ_{45}, and Q becomes

$$Q = -i\partial_\theta + \frac{1}{2}\tau_3 \tag{181}$$

with eigenvalues $n \pm 1/2$.
Consider now a six-dimensional Weyl spinor (uncharged with respect to the elementary gauge field A) and perform a harmonic expansion in eigenstates of Q:

$$\Psi(\rho,\theta,x) = \psi_{kn}(\rho,\theta)\phi_{kn}(x), \tag{182}$$

$$\psi_{kn}(\rho,\theta) = \left(\begin{array}{c} \psi^+_{kn} \\ \psi^-_{kn} \end{array}\right) = \left(\begin{array}{c} \chi^+_{kn}(\rho)\exp(in\theta) \\ \chi^-_{kn}(\rho)\exp(in\theta) \end{array}\right). \tag{183}$$

Summation over k and n is implied and k labels the modes with given n. Since M commutes with Q, k can be chosen to label the eigenstates of the mass operator. Here χ^+_{kn} and χ^-_{kn} are eigenstates of the internal Γ matrix τ_3 with opposite eigenvalues. Due to the six-dimensional Weyl constraint the positive eigenvalues of τ_3 are associated to left-handed four-dimensional Weyl spinors whereas the negative eigenvalues correspond to right-handed Weyl spinors,

$$\Psi(\rho,\theta,x) = \chi^+_{kn}(\rho)\exp(in\theta)\phi_{Lkn}(x) + \chi^-_{kn}(\rho)\exp(in\theta)\phi_{Rkn}(x). \tag{184}$$

Let $N^\pm(Q)$ be the number of zero mass modes of $\psi^\pm$ with charge $Q = n \pm 1/2$. In our case internal charge conjugation implies $N^-(Q) = N^+(-Q)$ (since $d = 2$ mod 4) and we can therefore restrict the analysis to the zero mass eigenmodes in ψ^+. Chiral fermions are obtained only if $N^+(Q) \neq N^+(-Q)$.
The zero mass modes are the solutions of

$$\Gamma^\alpha D_\alpha(a^2\psi_{0n}) = 0 \tag{185}$$

and we compute

$$\Gamma^\alpha D_\alpha = \left(\begin{array}{cc} 0 & D_- \\ D_+ & 0 \end{array}\right), \tag{186}$$

$$\begin{aligned} D_+ &= -i\,\exp(i\theta)\left(\partial_\rho + ib^{-1}\partial_\theta - \frac{1}{2}b^{-1}(1-b')\right), && (187)\\ D_- &= i\,\exp(-i\theta)\left(\partial_\rho - ib^{-1}\partial_\theta - \frac{1}{2}b^{-1}(1-b')\right). && (188)\end{aligned}$$

The solutions for the zero modes χ^+_{0n} were found to be

$$\chi^+_{0n}(\rho) = Ga^{-2}(\rho)b^{-1/2}(\rho)\exp((n+\frac{1}{2})I(\rho)), \tag{189}$$

where

$$I(\rho) = \int_{\rho_0}^{\rho} d\rho\, b^{-1}(\rho), \tag{190}$$

with ρ_0 an arbitrary point in the interval $(0, \bar{\rho})$ and G a normalization constant. Not all of these solutions make physically sense: The action for a spinor reads

$$
\begin{aligned}
S &= \int d^6x \sqrt{-g}\, i\bar{\Psi}(\gamma^\mu D_\mu + \gamma^\alpha D_\alpha)\Psi & (191)\\
&= \int d^4\, x \bar{\phi}_{kn} i\tilde{\gamma}^\mu D_\mu \phi_{kn} \times \int d\rho\, d\theta\, a^3 b\, \psi^\dagger_{kn}\psi_{kn} & (192)\\
&\quad + \int d^4x\, \bar{\phi}_{kn} i\tilde{\Gamma}\phi_{kn} \times \int d\rho\, d\theta\, a^3 b\, \psi^\dagger_{kn} M_k \psi_{kn}.
\end{aligned}
$$

The first term is the kinetic term and the second is the mass term. The spinors $\phi_{kn}(x)$ correspond to propagating fermions only if their kinetic term is finite after dimensional reduction. (This condition is equivalent to the condition that the action remains finite for an excitation $\phi_{kn}(x)$ which is local in four-dimensional space.) We therefore require the integral

$$\int d\rho\, a^3 b |\chi^+_{0n}|^2 \propto \int d\rho\, a^{-1} \exp((2n+1)I) \tag{193}$$

to be finite. Our task is therefore the determination of the values of Q (or n), for which the normalizability condition (193) is fulfilled.

Consider the type two solutions with deficit angles λ_0 and λ_1. Possible problems with normalizability can only come from the "cusps" at $\rho = 0$, $\rho = \bar{\rho}$ where b vanishes linearly, $b = (1 - \frac{\lambda_0}{2\pi})\rho$ and $b = (1 - \frac{\lambda_1}{2\pi})(\bar{\rho} - \rho)$, respectively. Therefore the function $I(\rho)$ diverges logarithmically at the cusps,

$$I(\rho) \to (1 - \frac{\lambda_0}{2\pi})^{-1} \ln \rho \quad for \quad \rho \to 0, \quad I(\rho) \to -(1 - \frac{\lambda_1}{2\pi})^{-1} \ln(\bar{\rho} - \rho) \quad for \quad \rho \to \bar{\rho}. \tag{194}$$

Thus the normalizability condition gives a constraint on the charge from each brane. Finiteness around $\rho = 0$ holds for

$$Q > -\frac{1}{2}(1 - \frac{\lambda_0}{2\pi}) \tag{195}$$

and finiteness around $\rho = \bar{\rho}$ requires

$$Q < \frac{1}{2}(1 - \frac{\lambda_1}{2\pi}). \tag{196}$$

For vanishing deficit angles $\lambda_0 = \lambda_1 = 0$ no massless spinors exist since Q is half integer and $-\frac{1}{2} < Q < \frac{1}{2}$ therefore has no solution. This situation also holds for positive deficite angles $\lambda_0 \geq 0$, $\lambda_1 \geq 0$. For $\lambda_0 = 0$ the massless spinors must have positive Q (cf. eq. 195) and exist if a cusp is present at $\bar{\rho}$ with negative deficit angle $\lambda_1 < 0$. In this case the chirality index depends on the deficit angle λ_1. Thus the massless spinors with positive Q are connected with the brane at $\bar{\rho}$ ($\lambda_1 < 0$). Inversely, the massless spinors with negative Q are associated with a brane at $\rho = 0$ ($\lambda_0 < 0$). In case of branes at $\rho = 0$ and $\rho = \bar{\rho}$ we find massless spinors both with positive and negative Q. For equal deficit angles $\lambda_0 = \lambda_1$ their number is equal, $N^+(Q) = N^-(Q)$. One concludes that a chiral imbalance (non-vanishing chirality index) is only realized if the two branes are associated with different deficit angles.

Holographic Branes [37]: In the remainder of this section the correspondence between the bulk and brane picture is discussed and "holographic branes" are introduced.

Starting from the bulk picture we have learned that the possible massless fermions are associated with the singularities or branes. For a given charge Q the left-handed fermions are linked to one brane and the right-handed ones to the other. A difference in the deficit angles of the two branes can therefore lead to chirality. This finds its correspondence within the brane point of view: in a certain sense the left-handed particles with $Q > 0$ "live" on the brane at $\bar{\rho}$ and those with $Q < 0$ on the other brane at $\rho = 0$. Indeed, for $Q > 0$ the probability density diverges for $\rho \to \bar{\rho}$,

$$\sigma^2\gamma^{1/2}|\chi_{0n}^+|^2 \; (\bar{\rho}-\rho)^{-\frac{2Q}{1-\lambda_1/2\pi}}, \quad |\chi_{0n}^+|^2 \; (\bar{\rho}-\rho)^{-(\frac{2Q}{1-\lambda_1/2\pi}+1)}, \tag{197}$$

with a corresponding behavior for $Q < 0$ and $\rho \to 0$.
In contrast to the behavior of the tension this concentration is, however, not of the δ-function type. It rather obeys an inverse power law singularity with a tail in the bulk. This type of brane fermions can be classified from the bulk geometry which must obey the corresponding field equations. More precisely, the number and charges of the chiral fermions on the brane are not arbitrary any more but can be computed as functions of the integration constants of the bulk geometry. This is a type of "holographic principle" which renders the model much more predictive - the arbitrariness of "putting matter on the brane" has disappeared. This predictive power extends to the more detailed properties of these fermions, like Yukawa couplings to the scalar modes of the model. These couplings can be computed without any knowledge of the details of the brane. The insensitivity with respect to the details of the brane is related to the dual nature of the wave function χ_0^+. Even though $\chi_0^+(\rho)$ diverges for $\rho \to \bar{\rho}$, the relevant integrals for the computation of the properties of the four-dimensional fermions converge for $\rho \to \bar{\rho}$. They are therefore dominated by the "tail" of the wave function in the bulk.
In analogy to the previous discussion we may imagine a "regularized brane" without singularity at $\bar{\rho}$. The existence of normalizable massless fermions then requires that also the mass operator and therefore the functional form (189) of the zero modes gets modified by the additional physics on the brane. (Otherwise the regular behavior of the metric $b \to (\bar{\rho}-\rho)$ would render the continuation of the zero mode for $\epsilon > 0$ into the inner region unnormalizable.) We can then imagine that the regularized wave function χ_0^+ reaches a constant, $\chi_0^+(\rho \to \bar{\rho}) = c_{\bar{\rho}}$, where the proper definition of eq.(184) everywhere on the manifold requires $c_\rho = 0$ for $n \neq 0$.
This "regularized picture" also sets the stage for the question if additional massless fermions could live on the brane without being detectable from the bulk. In the most general setting without further assumption the answer is positive. We still expect that the wave functions of such "pure brane fermions" have a tail in the region $\bar{\rho} - \rho > \epsilon$. In this bulk region the tail of such a wave function has to obey eq.(189). Nevertheless, we can now consider a value Q which violates the condition (196). Such a mode would look unnormalizable if continued to $\rho \to \bar{\rho}$ but may be rendered normalizable by the physics on the brane. In contrast to the modes obeying the condition (196) the physical properties of the corresponding four-dimensional fermion would be completely dominated by the physics on the brane, with negligible influence of the bulk geometry. Indeed, for

regularized branes the usual dimensional reduction by integration over the internal coordinates can be performed without distinction between 'pure brane fermions' and fermions obeying the conditions (195),(196). For the pure brane fermions the relevant integrals will be dominated by the brane region $\bar{\rho} - \rho < \epsilon$.
Unfortunately, without further knowledge of the physics on the brane the assumption of such pure brane fermions remains completely ad hoc, without any predictive power except that the charge Q should be larger than the bound (196). (Pure brane fermions would be needed for chirality in case of a positive deficit angle.) Postulating the existence of pure brane fermions without knowledge of the detailed physics on the brane amounts more or less to postulating that the physics of the fermions is as it is observed - this is not very helpful for an explanation of the properties of realistic quarks and leptons. This situation is very different for the chiral fermions obeying the bounds (195),(196) for which all observable properties are connected to the bulk geometry and therefore severely constrained for a given model.
As an interesting candidate for the computation of charges and couplings of quarks and leptons we therefore propose the notion of "holographic branes". For holographic branes all relevant excitations that are connected to observable particles in the effective four-dimensional world at low energies are of the type of the massless fermions obeying the constraints (195),(196). In other words, all relevant properties of the brane, including the excitations on the brane, are reflected by properties of the bulk geometry and bulk excitations. The holographical principle states that the observable properties can in principle be understood both from the brane and bulk point of view, with a one to one correspondence. In practice, the detailed properties of the brane are often not known such that actual computations of observable quantities can be performed in the bulk picture of a noncompact internal space with singularities.

4.5 Fermions that Couple to the Gauge Field

The previous analysis can be easily generalized to fermions that couple to the gauge field A, say with charge e. The only difference is that D_θ gets an extra term $+ieA_\theta$. With the harmonic expansion

$$\Psi(\rho,\theta,x) = \psi_{kn}(\rho,\theta)\phi_{kn}(x), \tag{198}$$

$$\psi_{kn}(\rho,\theta) = \begin{pmatrix} \psi^+_{kn} \\ \psi^-_{kn} \end{pmatrix} = \begin{pmatrix} \chi^+_{kn}(\rho)\exp(in\theta) \\ \chi^-_{kn}(\rho)\exp(in\theta) \end{pmatrix}, \tag{199}$$

the zero modes are

$$\chi^+_{0n}(\rho) = Na^{-2}(\rho)b^{-1/2}(\rho)\exp(I^+(\rho)), \tag{200}$$

$$\chi^+_{0n}(\rho) = Na^{-2}(\rho)b^{-1/2}(\rho)\exp(I^-(\rho)), \tag{201}$$

where N is a normalization constant

$$I^+(\rho) = \int_{\rho_0}^{\rho} d\rho\, b^{-1}(n + \frac{1}{2} + eA_\theta), \tag{202}$$

$$I^-(\rho) = \int_{\rho_0}^{\rho} d\rho\, b^{-1}(-n + \frac{1}{2} - eA_\theta). \tag{203}$$

To each "left-handed" mode with quantum number n and charge e corresponds a "right-handed" mode with quantum number $-n$ and charge $-e$. In order to generalize the conditions (195,196), we choose the gauge in which $A_\theta \to 0$ at $\rho \to 0$ and $A_\theta \to m/e$ at $\rho \to \bar{\rho}$. In this gauge, the normalizability conditions for a "left-handed" zero mode χ_{0n}^{+} with charge e are

$$n + \frac{1}{2} > -\frac{1}{2}(1 - \frac{\lambda_0}{2\pi}), \tag{204}$$

$$n + m + \frac{1}{2} < \frac{1}{2}(1 - \frac{\lambda_1}{2\pi}). \tag{205}$$

For "right-handed" zero modes χ_{0n}^{-} with charge e one has to change the signs of m and n. For fermions with charge Ze one has to take Zm instead of m.

An important difference to the uncharged fermions is that now chiral fermions exist also if the deficit angles are zero. This is particularly the case in the relatively simple unwarped toy model with spherical internal space. The chiral zero modes of this model can be easily computed. For simplicity we take the radius of the sphere to unity, so that we have

$$b = \sin\rho, \quad A_\theta = \frac{e}{2m}(1 - \cos\rho). \tag{206}$$

We want to calculate the zero modes χ_{0n}^{+} with charge e. Such zero modes exist only if m is negative. If this is the case, the conditions (204,205) become

$$n \geq 0, \quad n < |m|. \tag{207}$$

So for any negative m, there are $|m|$ chiral fermions. The integral $I(\rho)$ can be performed explicitly and one obtains

$$\begin{aligned} \chi_{0n}^{+}(\rho) &= N(\sin\rho)^{-1/2} \exp \int_{\rho_0}^{\rho} d\rho \, \frac{n + \frac{1}{2} + \frac{m}{2}(1 - \cos\rho)}{\sin\rho} && (208) \\ &= \tilde{N}\,(\sin\rho)^{|m|-1-n}(1 - \cos\rho)^{n-(|m|-1)/2} && (209) \end{aligned}$$

In contrast to the fermions in the presence of branes, these modes are everywhere finite on the internal sphere. With an appropriate normalization one can show that

$$\sum_{n=0}^{|m|-1} |\chi_{0n}^{+}(\rho)|^2 = const. \tag{210}$$

This result is not surprising and shows once more the symmetry of the situation. In fact, the zero modes form an m dimensional irreducible SU(2) representation, corresponding to the internal "angular momentum" $l = \frac{m-1}{2}$.

5 Cosmology of 6D Einstein-Maxwell Theory

The simplest Kaluza-Klein cosmologies are described by two scale factors which are functions of time only: one for the three large dimensions and one for a highly symmetric internal space. The cosmology is then determined by two ordinary differential equations, which were explored more than twenty years ago [44, 45]. Here we will consider this as a very special case of much more general systems. We allow for a time-dependent warping and for deformed internal spaces, possibly with singularities. All quantities are now functions of time **and** an internal coordinate ρ. What are the features of such a cosmology?

In the static solutions of 6D Einstein-Maxwell theory, the four-dimensional cosmological constant was a free integration constant. If we drop the requirement of maximal 4D symmetry and allow for more general cosmologies, only keeping 3D maximal symmetry (i.e. spatial homogeneity and isotropy), the effective "dark energy" will become a dynamical quantity. The investigation of these dynamics was one of the main motivations for this work. Unfortunately the problem turns out to be very difficult in the most general situation. It is described by a complicated set of partial differential equations with many degrees of freedom. The general framework and some of its difficulties are described in section 5.1. At least we were able to find some qualitative features of the cosmology, such as the time independence of deficit angles, which is shown in section 5.2. We were also able to solve some special cases of higher symmetry. In section 5.3, the model in which internal space has the geometry of a sphere - from now on referred to as the "spherical model" - is discussed in the 6D picture and in the dimensionally reduced effective 4D picture. Some general features of Kaluza-Klein cosmology are visible from this simple example, in particular the appearance of a scalar field with an asymptotically exponential potential, related to the size of internal space. In section 5.5, we compute the 6D energy momentum tensor of the zero mode fermions and include them into the spherical model. General properties of possible late time cosmologies and the difficulties in their description are discussed in section 5.4. There is again a special case which is easily solved: If the internal space is exactly static, the possible equations of state and the ρ dependence of the energy momentum tensor can be strongly constrained. These solutions provide a new explanation of why the 4D cosmological constant is a free parameter. Nevertheless, a large number of questions remain unanswered. They are summarized in section 5.6, and the possible next steps towards a better understanding are outlined.

5.1 The Most General Metric

We have to find the most general metric consistent with the required symmetries: three-dimensional translation and rotation invariance, acting on the coordinates x^i, and a U(1) symmetry, acting on the coordinate $\theta \in [0, 2\pi]$. No real physical function should depend on x^i or θ (i.e. these coordinates should appear only in phases), and no direction in the three-dimensional space should be preferred. (For simplicity, we will take this space to be flat, so that the metric components g_{ij} are $a^2(t,\rho)\delta_{ij}$.) The latter condition forbids metric components g_{ti}, $g_{\rho i}$ and $g_{\theta i}$, since these would select preferred directions in three-space, e.g. by the three-vector (g_{t1}, g_{t2}, g_{t3}). The other off-diagonal metric components $g_{t\rho}$, $g_{t\theta}$ and $g_{\rho\theta}$ are allowed, as long as they are functions of t and ρ only.

This situation is unique to the six-dimensional case in the following sense: If there were D internal dimensions with $D > 2$, and $D-1$ of these dimensions, represented by coordinates θ_α, were symmetric under, say, SO(D), then the $g_{t\theta}$ and $g_{\rho\theta}$ components would be forbidden, because they would select preferred directions in the $D-1$-dimensional space. The difference is that a U(1) "rotation" is a translation rather than a rotation. In this sense a codimension-two spacetime is more complicated than a higher-dimensional one.

For the gauge field the situation is slightly different. For specific solutions (solitons) a component A_θ may be allowed even if the internal symmetry is larger than U(1). Think of the monopole solution on S^2. The θ-direction is then not preferred physically. A coordinate transformation may be accompanied by a gauge transformation, so that the transformed A-field lies in the new θ-direction. An analogous procedure does not work for the metric tensor, since the gauge transformations are the coordinate transformations themselves.

Up to now we have identified the most general metric consistent with the symmetries as

$$\begin{aligned} ds^2 = -c^2(t,\rho)dt^2 &+ a^2(t,\rho)(dx^i)^2 + b^2(t,\rho)d\theta^2 + n^2(t,\rho)d\rho^2 \\ &+ 2w(t,\rho)dtd\rho + 2u(t,\rho)dtd\theta + 2v(t,\rho)d\rho d\theta. \end{aligned} \tag{211}$$

The next step is to look how far this line element can be simplified by a coordinate transformation. Therefore one has to find the possible transformations consistent with the symmetries, which should still be represented by the new coordinates $x^{i'}$ and θ'. Global translations of θ and translations and rotations of x^i are of course allowed (these are just the isometries). Transformations can never depend on θ, since this would lead to functions depending on θ'; for example $t \to t' = t + \delta t(\theta)$, $\theta \to \theta' = \theta$ would imply $t = t' - \delta t(\theta')$, and so $f(t) \to f'(t', \theta')$ for any function f. Or if $\theta \to \theta'(\theta)$, we would get

$$g^{\theta\theta'} = \left(\frac{\partial\theta'}{\partial\theta}\right)^2 g^{\theta\theta}, \tag{212}$$

and so one must have $\partial\theta'/\partial\theta = const = 1$, since we would like to have θ' also in the interval $[0, 2\pi]$. Transformations of x^i cannot depend on t or ρ, since this would lead to forbidden components via

$$g^{ti'} = \frac{\partial t'}{\partial t}\frac{\partial x^{i'}}{\partial t} g^{tt}, \tag{213}$$

and similarly for $g^{\rho i}$. So we are left with the following possibilities:

$$\begin{aligned} x^i &\to x^{i'}(x^j), \qquad \theta \to \theta + \delta\theta(t,\rho), \\ t &\to t'(t,\rho), \qquad \rho \to \rho'(t,\rho). \end{aligned} \tag{214}$$

Obviously, the only effect of the x^i transformations could be a rescaling of three-dimensional space, so we can forget about them in this context. There are three off-diagonal metric components, $g_{t\rho}$, $g_{t\theta}$ and $g_{\rho\theta}$, and one might think that these can be removed by the three remaining coordinate transformations. It turns out that this is in general not true. The reason for that is essentially the U(1) symmetry. (In fact, the metric **can** always be diagonalized, but then in general the new coordinate θ' will not reflect the U(1) symmetry any more, and fields will depend on θ'.) To see this, consider the inverse of the metric. The components $g_{t\theta}$ and $g_{\rho\theta}$ will be zero if and only if $g^{t\theta}$ and $g^{\rho\theta}$ are zero. The condition that this happens after a coordinate transformation of the type (214) is

$$g^{t\theta'} = \frac{\partial t'}{\partial t}\left(g^{t\theta} + \frac{\partial\theta'}{\partial t}g^{tt} + \frac{\partial\theta'}{\partial\rho}g^{t\rho}\right) + \frac{\partial t'}{\partial\rho}\left(g^{\rho\theta} + \frac{\partial\theta'}{\partial t}g^{\rho t} + \frac{\partial\theta'}{\partial\rho}g^{\rho\rho}\right) = 0, \quad (215)$$

$$g^{\rho\theta'} = \frac{\partial\rho'}{\partial t}\left(g^{t\theta} + \frac{\partial\theta'}{\partial t}g^{tt} + \frac{\partial\theta'}{\partial\rho}g^{t\rho}\right) + \frac{\partial\rho'}{\partial\rho}\left(g^{\rho\theta} + \frac{\partial\theta'}{\partial t}g^{\rho t} + \frac{\partial\theta'}{\partial\rho}g^{\rho\rho}\right) = 0. \quad (216)$$

A solution of these differential equations implies either that the Jacobi determinant of the (ρ, t) transformation vanishes,

$$det\begin{pmatrix} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial\rho} \\ \frac{\partial\rho'}{\partial t} & \frac{\partial\rho'}{\partial\rho} \end{pmatrix} = 0, \quad (217)$$

which is not possible, or that the brackets vanish. But the second possibility consists of two conditions for the function θ', which can in general not be fulfilled simultaneously.

One concludes that generally only one of the two components $g^{t\theta}$ and $g^{\rho\theta}$ can be set to zero. A procedure to simplify the metric (211) could look as follows: Use the freedom for t' and ρ' to annihilate $g^{t\rho}$ and for one further simplification, e.g. to arrange that ${g_{tt}}' = -{g_{ii}}'$, i.e. to make time conformal with respect to space. Then use the freedom for θ' to annihilate either $g^{t\theta}$ or $g^{\rho\theta}$. The simplified line element is then

$$ds^2 = a^2(t,\rho)(-dt^2 + (dx^i)^2) + b^2(t,\rho)d\theta^2 + n^2(t,\rho)d\rho^2 + 2u(t,\rho)dtd\theta, \quad (218)$$

or similarly with $2v(t,\rho)d\rho d\theta$ instead of $2u(t,\rho)dtd\theta$. We will refer to these two possibilities as the "u-gauge" and the "v-gauge". In the effective four-dimensional picture u corresponds to the time component of an abelian gauge field (hence some kind of electric potential), since $g_{\theta\mu}$ integrated over internal space is the gauge field corresponding to the U(1) isometry. On the other hand v corresponds to a scalar field. The fact that a degree of freedom can be shifted between a scalar field and the component of a gauge field is a familiar fact in ordinary particle physics.

A very similar choice applies to the gauge field A. The three components A_t, A_ρ and A_θ are allowed by the symmetries. One can choose to set either A_t or A_ρ to zero by a gauge transformation.

Comparing this cosmological system to the static one from chapter 3, one finds that the ordinary differential equations are generalized to partial differential equations, containing t- and ρ-derivatives, and that the three functions a, b and A_θ are accompanied by three more functions: n, u or v, and A_t or A_ρ.

A full numerical analysis of this system would involve as initial conditions twelve functions of ρ (four metric and two gauge field components and their first time derivatives at some initial time t_0) which are subject to three constraint equations, namely the $(tt)-$, $(t\rho)-$ and $(t\theta)-$ components of Einstein's equations, which contain no second time derivatives. The time evolution is determined by the $(ii)-$, $(\theta\theta)-$, $(\rho\rho)-$ and $(\theta\rho)$- components of Einstein's equations and two equations for the gauge field. For completeness, the Einstein tensor and the energy momentum tensor of the gauge field are given in Appendix B.

We will not try to perform these numerics here. Instead we concentrate on two particular aspects of the subject:

1. Properties of the Codimension-two branes at the two endpoints of internal space;

2. Special cases of higher symmetry: the spherical model and late time cosmology.

Before we do so, some remarks on the difficulties in choosing a particular gauge are in order.

The Problem of finding a "natural" coordinate system: The coordinate systems with the properties defined above may be a bad choice under some circumstances. Problems may in particular arise from the relation $-g_{tt} = g_{ii} = a^2$. In usual 4D cosmology, time can be made conformal to space by a transformation $t \to \tau(t)$, which involves only a stretching of the time axis. On our 6D case, we need instead a more general transformation $t, \rho \to t', \rho'(t, \rho)$. Thereby time and ρ coordinate are mixed to some extent. This leads to the question: What is the "physical" time coordinate? The difficulty can be demonstrated in a four-dimensional example: the gravitational field of a wire.

Consider a wire in z-direction. A solution of Einstein's equations which describes its static gravitational field will in general have different functions $g_{zz}(\rho)$ and $g_{tt}(\rho)$ (in cylindrical coordinates). This corresponds to different functions $-g_{tt}$ and g_{ii} in the 6D case. One can now perform a coordinate transformation $t, \rho \to t', \rho'(t, \rho)$ to make time conformal, i.e. to give $-g_{tt}$ everywhere the correct value to be equal to g_{zz}. The transformed metric still describes the static gravitational field of a wire, but now the "physical" time independence is no longer visible in the metric functions, since now $g_{zz}(\rho) \to g_{zz}(\rho(\rho', t'))$, so that the metric depends on "time" in its new form. The solution still has the same timelike Killing vector, but its direction is no longer given by the new t-coordinate. The new time coordinate is "unphysical". An exception is the case where the equation of state of the wire is $w = -1$, i.e $T^t_t = T^z_z$. Then the symmetry between t and z would be "physical", and $-g_{tt}$ and g_{zz} would naturally have the same ρ-dependence. But in the other cases, with general equation of state, this choice of gauge would be unnatural.

The problem in six dimensions is similar. Indeed, there may be singularities at $\rho = 0$ different from the deficit angle branes, i.e. singularities not of the delta function type, but with divergent curvature and divergent metric components. These general singularities are probably not well described in our coordinates. It would be interesting to see how they evolve with time, but the question is: with **which** time? In the case of the wire there was a timelike Killing vector. Its direction identifies the natural time coordinate. But in our cosmological context, we are mainly interested in solutions with only **approximate** Killing vectors. It is necessary to find a coordinate system in which the deviations from a symmetry do not artificially blow up. We will come back to that point at the end of section 5.4.

5.2 Properties of the Branes

We will consider here a deficit angle brane at $\rho = 0$. (The discussion of a brane at $\bar{\rho}$ is completely analogous, of course.) To have a codimension-two brane at that place requires that the determinant of the metric vanishes there quadratically. (If det g would not vanish, the subspace with $\rho = 0$ would be of codimension one, and if it would vanish more than quadratically, it would be a different type of singularity.) In the u-gauge, this requirement reads

$$\det g = -a^6 n^2 (a^2 b^2 + u^2) \to 0 \quad as \quad \rho \to 0. \tag{219}$$

Since a and n should be finite (otherwise we would not have a brane but a different type of singularity) this means that both b and u must vanish at least linearly at $\rho = 0$. For b this was already clear, because $b(t, \rho)$ measures the radius of a circle in internal space at radius ρ.

To have an infinitely thin codimension-two brane means that the singularity at $\rho = 0$ is of the delta function type. Therefore curvature invariants and energy densities do not diverge as one approaches the brane (if they did, we would again have a different type of singularity), but have a delta function contribution precisely at $\rho = 0$. So the components of the Einstein tensor G^A_B should remain finite outside the brane. To have a well-defined time coordinate on the brane requires that the curvature contributions to the delta function are contained in the ρ-derivative terms, not in the time derivative terms.

These requirements imply strong constraints on the possible brane properties: The equation of state must be precisely $w = -1$ and the deficit angle must be time independent. To prove this, one shows at first that the off-diagonal metric component vanishes at least $\sim \rho^2$ at $\rho = 0$: In the u-gauge the $(\rho\rho)$ component of the Einstein tensor contains the term

$$\frac{u'^2}{4n^2(a^2b^2 + u^2)}, \tag{220}$$

where the prime again denotes derivation with respect to ρ. If u' were finite at the brane, this term would diverge as ρ^{-2}, which is not allowed by our requirements (there is no other term in G^ρ_ρ which could cancel this divergence). One concludes that u' vanishes at least $\sim \rho$ at $\rho = 0$, and so at least $u \sim \rho^2$ while $b \sim \rho$.

To prove that the equation of state is $w = -1$, consider the difference $G^t_t - G^{(i)}_{(i)}$. In the v-gauge, this difference does not contain any ρ-derivatives at all, only time derivatives which cannot contribute to the delta function singularity. Note that the ρ-derivative terms of refs. [28, 47] are absent here, since we have chosen $g_{tt} = -g_{ii}$ by choice of coordinates. In the u-gauge, the difference does contain ρ-derivatives, but these terms are damped with u^2/b^2 and are not sufficient to contribute to the singularity. Therefore the singular parts of G^t_t and $G^{(i)}_{(i)}$ are equal. The same is of course true for the corresponding components of the energy momentum tensor $T^t_t = -\varepsilon$ and $T^{(i)}_{(i)} = p$, so $w = \varepsilon^{(singular)}/p^{(singular)} = -1$.

To have a pure tension brane does not automatically imply that this tension (and hence the deficit angle) is a constant. One might imagine that there is some energy flow from the bulk onto the brane which would enlarge its tension and therefore the deficit angle. To show that this does not happen, consider the component G^t_ρ of the Einstein tensor in u-gauge. The "dangerous", i.e. possibly divergent part of this component is

$$\frac{b^2}{n(a^2b^2+u^2)}\left(\frac{\dot{b}'}{b}-\frac{b'}{b}\frac{\dot{n}}{n}\right), \tag{221}$$

where a dot denotes derivation with respect to time. With the approximation

$$b = f(t)\rho + O(\rho^2) \tag{222}$$

one sees that the "dangerous" term remains finite if and only if

$$\frac{\dot{f}}{f}=\frac{\dot{n}}{n} \tag{223}$$

in the vicinity of the brane. But this is just saying that if the extra dimensions are time varying at all, then the radius and circumference of a small circle around the brane grow by the same factor. Hence the deficit angle does not change with time.

This result is not surprising. Different to other singularities, there is no attractive force towards the brane which would compress any "cloud" of energy towards the center at $\rho = 0$. We saw that the curvature remains finite outside the brane, which is the geometric analogue of having no divergent gravitational forces. Therefore only an infinitesimal part of the cloud would reach the singularity, leading to no change of the tension.

Things will be different with other types of less symmetric codimension-two singularities, which can certainly occur in 6D cosmology. They probably induce attractive forces and will therefore be able to grow. But as was mentioned in the previous section, we do not yet know how to describe the cosmology of these other singularities in a meaningful way.

Time Independence of Monopole Numbers: From the energy momentum tensor given in Appendix B one can see that $\dot{A}_\theta$ always appears with a factor b^{-1}. At a deficit angle brane, this combination has to remain finite. Therefore $\dot{A}_\theta$ has to vanish at the branes. As long as no other types of singularities appear, this implies that the monopole number, which is essentially the difference between $A_\theta(\bar{\rho})$ and $A_\theta(0)$, is time-independent.

5.3 Cosmology of the Spherically Symmetric Model

The complicated system of equations described above is enormously simplified if internal space has the geometry of a sphere. We will use this simple system to present the effects and implications of dimensional reduction, and show the equivalence of the six-dimensional and the four-dimensional point of view. Furthermore we will find features of Kaluza-Klein cosmology which will surely generalize to some extent to more complicated systems.

In this section, we do not include any additional sources and describe a cosmology induced just by curvature. The gauge field serves only to compactify internal space and plays no role apart from that. The static solutions for that system were derived at the end of chapter 3, and we will make use of the relations derived there. The cosmology is given in terms of two scale functions: the 4D scale factor $a(t)$ (there is no warping in the presence of spherical symmetry) and the scale factor of internal space, $n(t)$. In contrast to the static case, n is no longer required to be equal to 1, so we may define the internal coordinates such that ρ goes from 0 to π, getting in this way rid of the quantity k used in chapter 3. The off-diagonal components of the metric have to vanish because of the higher symmetry, as was discussed in section 5.1, and the component $g_{\theta\theta}$ is given by

$$b(t,\rho) = n(t)\sin\rho. \tag{224}$$

It is straightforward to compute the Einstein tensor to this metric. All the off-diagonal components vanish. The vanishing of $G_{t\theta}$ implies that the corresponding component of the energy momentum tensor also vanishes:

$$T_{t\theta} = F_{t\rho}F_{\theta}{}^{\rho} = 0. \tag{225}$$

But $F_{\theta}{}^{\rho} \sim \partial_\rho A_\theta$ certainly does not vanish for a monopole solution, so on concludes $F_{t\rho} = 0$ and therefore

$$\partial_t A_\rho(t,\rho) = \partial_\rho A_t(t,\rho). \tag{226}$$

This means that the time and ρ-components of A are a pure gauge and can be removed by a gauge transformation. So we have, as in the static case, only to deal with the component A_θ. The vanishing of $G_{t\rho}$ implies

$$T_{t\rho} = F_{t\theta}F_{\rho}{}^{\theta} = 0, \tag{227}$$

and it follows that $\dot{A}_\theta = 0$. This means that the gauge field is static and from the field equation for F_{AB} one again infers that A_θ' is proportional to $\sin\rho$, where the proportionality factor can, for convenience, again be expressed in terms of a monopole number m:

$$A_5' = \frac{m}{2e}\sin\rho, \tag{228}$$

where e is a possible six-dimensional gauge coupling. The Einstein equations become

$$-G^t_t \equiv \frac{1}{n^2} + \frac{1}{a^2}\left(6\frac{\dot{a}\dot{n}}{an} + 3\frac{\dot{a}^2}{a^2} + \frac{\dot{n}^2}{n^2}\right) = \Lambda + 8\pi G_6\frac{m^2}{8e^2n^4} \tag{229}$$

$$-G^{(i)}_{(i)} \equiv \frac{1}{n^2} + \frac{1}{a^2}\left(2\frac{\ddot{a}}{a} + 2\frac{\ddot{n}}{n} + 2\frac{\dot{a}\dot{n}}{an} - \frac{\dot{a}^2}{a^2} + \frac{\dot{n}^2}{n^2}\right) = \Lambda + 8\pi G_6\frac{m^2}{8e^2n^4} \tag{230}$$

$$-G^\theta_\theta = -G^\rho_\rho \equiv \frac{1}{a^2}\left(3\frac{\ddot{a}}{a} + \frac{\ddot{n}}{n} + 2\frac{\dot{a}\dot{n}}{an}\right) = \Lambda - 8\pi G_6\frac{m^2}{8e^2n^4} \tag{231}$$

Only two of these three equations are independent due to the Bianchi identities. The (tt) component is a generalization of the Friedmann equation. The linear combination

$$-R^\theta_\theta - R^\rho_\rho \equiv \frac{1}{a^2}\left(\frac{\ddot{n}}{n} + 2\frac{\dot{a}\dot{n}}{an} + \frac{\dot{n}^2}{n^2}\right) = \frac{\Lambda}{2} - \frac{1}{n^2} + 8\pi G_6\frac{3m^2}{16e^2n^4} \tag{232}$$

will be useful when we compare the 6D and the effective 4D point of view. We will also need the Ricci scalar which is

$$R = \frac{\tilde{R}}{a^2} - 2\frac{1}{n^2} + \frac{1}{a^2}\left(4\frac{\ddot{n}}{n} + 8\frac{\dot{a}\dot{n}}{an} + 2\frac{\dot{n}^2}{n^2}\right). \tag{233}$$

In order to find a four-dimensional interpretation of the cosmological equations, one should perform a dimensional reduction the action.

$$\begin{aligned} S &= \int d^4x d\rho d\theta \sqrt{-g}\left(\frac{-R+2\Lambda}{16\pi G_6} + \frac{1}{4}F_{AB}F^{AB}\right) \qquad (234) \\ &= \int d^4x\, 4\pi\sqrt{-\tilde{g}}n^2\left[\frac{1}{16\pi G_6}\left(-\tilde{R} - \frac{2}{n^2} - \frac{1}{a^2}\left(4\frac{\ddot{n}}{n} + 8\frac{\dot{a}\dot{n}}{an} + 2\frac{\dot{n}^2}{n^2}\right) + 2\Lambda\right) + \frac{m^2}{8e^2n^4}\right] \\ &= \int d^4x\, 4\pi\sqrt{-\tilde{g}}\left[\frac{1}{16\pi G_6}\left(-n^2\tilde{R} - 2\partial_\mu n\partial^\mu n + 4(n^2)^{;\mu}{}_{;\mu}\right) + \tilde{V}(n)\right], \end{aligned}$$

where

$$\tilde{V}(n) = \frac{\Lambda}{8\pi G_6}n^2 - \frac{1}{8\pi G_6} + \frac{m^2}{8e^2}n^{-2}. \tag{235}$$

At this moment we are only interested in the effect of the time-dependent size of the internal dimensions on four-dimensional gravity. We therefore kept only the 4D curvature term and terms involving n, leaving out 4D perturbations of the gauge field and scalars which are non-singlets with respect to the isometry group of the sphere. One can see a typical feature of Kaluza-Klein theories: The factor n^2 multiplying the four-dimensional curvature scalar leads to a kind of Brans-Dicke theory. The strength of the gravitational coupling depends on the internal radius. This dependence may be absorbed by a Weyl scaling of the metric. Therefore we define the quantity $l = n/n_0$ and transform the metric via

$$\tilde{g}_{\mu\nu} \to l^{-2}\tilde{g}_{\mu\nu}. \tag{236}$$

Here n_0 is in principle arbitrary, but it is convenient to take it as the size of internal space in a "ground state", if such a ground state exists. By ground state we mean a stable solution with $\dot{n} = 0$. In chapter 3 we saw that for a large range of parameters two solutions with $\dot{n} =$ exist. In general we expect one of these to be stable and the other one to be unstable (This property will be shown below). For a special value of Λ (eq. 99), one of these two solutions has $\Lambda_4 = 0$, which is known to be classically stable. In this case we would take $n_0 = L$ with the L from eq.(100). Note that the second solution with the same Λ has internal radius and four-dimensional cosmological constant

$$\tilde{L} = \sqrt{3}L, \quad \Lambda_4 = \frac{1}{9\pi G_6}\frac{e^2}{m^2}, \tag{237}$$

see eqs (97) and (98).

In terms of the rescaled metric the action reads (and we omit the total divergence term)

$$S = \int d^4x\, \sqrt{-\tilde{g}}\left[\frac{n_0^2}{4G_6}\left(-\tilde{R} + 4l^{-2}\partial_\mu l\partial^\mu l\right) + V(l)\right], \tag{238}$$

where

$$V(l) = \frac{\Lambda n_0^2}{2G_6} l^{-2} - \frac{1}{2G_6} l^{-4} + \frac{\pi m^2}{2e^2 n_0^2} l^{-6}. \tag{239}$$

One infers that the four-dimensional Newton constant is

$$G_4 = \frac{G_6}{4\pi n_0^2}. \tag{240}$$

In the case where the stable ground state solution has $\Lambda_4 = 0$, the potential is

$$V_{\Lambda_4=0}(l) = \frac{l^{-2}}{4G_6}(1 - l^{-2})^2 \tag{241}$$

and the Newton constant is independent of G_6,

$$G_{4,\Lambda_4=0} = \frac{e^2}{8\pi^2 m^2}. \tag{242}$$

Note that all dimensions come out correct: The coordinates ρ and θ are angles and hence are dimensionless, in contrast to x^μ. The lengths are therefore absorbed by the metric components $g_{\theta\theta}$ and $g_{\rho\rho}$, so that n has dimension $(length)$ and still $\int d^4x d\rho d\theta (-g)^{1/2} = (length)^6$. The 6D gauge coupling e has dimension $(mass)^{-1}$. Now both A_θ (by eq.(228)) and $F_{\theta\rho}$ have dimension $(mass)$, although the second quantity is a derivative of the first, and anyway $F_{\theta\rho}F^{\theta\rho}$ has dimension $(mass)^6$, because the indices are raised with $g^{\theta\theta}$ and $g^{\rho\rho}$. The 4D gravitational constant has dimension $(mass)^{-2}$, whereas G_6 has $(mass)^{-4}$. So everything fits to make the action dimensionless.

It is interesting that the strength of four-dimensional gravitation is determined by the six-dimensional gauge coupling and not by the six-dimensional Newton constant.

There is one final step to perform with the action in order to bring it to a standard form. We have to redefine the scalar field so that it has a standard kinetic term:

$$\phi = \frac{\ln l}{\sqrt{2\pi G_4}}. \tag{243}$$

We finally end up with the action

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{-\tilde{R}}{16\pi G_4} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(l(\phi)) \right]. \tag{244}$$

So one finds that a four-dimensional observer sees a scalar field which is given by the logarithm of the size of internal space. The Friedmann equation and the scalar field equation of motion for the above system are:

$$3\frac{\dot{\tilde{a}}^2}{\tilde{a}^2} = 8\pi G_4 \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right), \tag{245}$$

$$\ddot{\phi} + 3\frac{\dot{\tilde{a}}}{\tilde{a}}\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{246}$$

Here a dot denotes a derivative with respect to "usual" time $\tilde{t}$. From these 4D effective field equations one can easily get back to the 6D equations: Go back to conformal time replacing $\partial/\partial\tilde{t}$ by $\partial/(\tilde{a}\partial t)$. Then one has to get back from the Weyl scaled to the original metric. This happens via the substitution $\tilde{a} \to la$. After these two modifications, eqs (245, 246) become exactly eqs (229,233). In this way we have checked that the six-dimensional and the four-dimensional point of view are equivalent.

What are the qualitative features of such a cosmology? At first look at the case in which the ground state has $\Lambda_4 = 0$. The potential (241) reads in terms of ϕ:

$$V_{\Lambda_4=0}(\phi) = \frac{1}{4G_6} \exp(-\sqrt{8\pi G_4}\ \phi)(1 - \exp(-\sqrt{8\pi G_4}\ \phi))^2. \tag{247}$$

This is the result of ref. [45]. The potential has a minimum at $\phi = 0$ and a maximum at $\phi_{max} = (8\pi G)^{-1/2} \ln 3$, corresponding to the stable solution with $n = n_0$ and the unstable solution with $n = \sqrt{3}n_0$, respectively. If the system starts with $\phi < \phi_{max}$, ϕ and hence n will perform damped oscillations around the ground state. This oscillation is accompanied by bouncing epochs of four-dimensional expansion. The expansion comes to rest asymptotically, and four-dimensional spacetime approaches Minkowski space. If the system starts instead with $\phi > \phi_{max}$, internal space will always grow. The scalar field potential decreases exponentially for large ϕ.

In a finite range of the parameter space around the combination leading to $\Lambda_4 = 0$, the potential will have a very similar shape. The ground state has $\Lambda_4 \neq 0$, but apart from that, the system will behave qualitatively as described above.

Again (like at the end of chapter 3) one can generalize the above solutions by substituting $b \to \tilde{A}b$. This corresponds to two equal deficit angle branes at the poles of the former sphere. When at the same time e is substituted by $\tilde{A}e$, the cosmological solutions are exactly the same.

The appearance of scalar fields with exponentially decreasing potential is a quite general feature in Kaluza-Klein theories, since the steps in performing the dimensional reduction (integration, Weyl scaling, rescaling of the scalar field) always involve similar structures. In particular, the Weyl scaling brings the internal radius l into the denominator of its kinetic term, so that the rescaling to a standard kinetic term requires taking the logarithm of l. The potential will then naturally involve exponential functions of $\phi \sim \ln l$. If the internal volume is large enough, corresponding to the region $\phi > \phi_{max}$, the tendency to expand will in general dominate over the forces which tend to shrink the internal space down to the ground state. The possible importance of quintessence fields with exponential potential for cosmology has been thoroughly discussed [46]. Note however that the exponential potential region of Kaluza-Klein scalar fields does not lead to a realistic cosmology. The permanently growing size of internal space would lead to a strong time dependence of coupling constants, far beyond the observational bounds. For a realistic model it is necessary to start with $\phi < \phi_{max}$, so that the ground state is approached.

5.4 Late Time Cosmology with a Perfect Fluid

Now we drop the restriction to spherical symmetry and ask instead how a realistic late time cosmology with matter and radiation can be realized. The effective four-dimensional couplings and masses depend on the size and shape of the extra dimensions. Cosmological observations imply that these masses and couplings have varied only very slightly since Big Bang Nucleosynthesis. At least we can say that the effective four-dimensional scale factor changes much faster than the parameters of particle physics, such as the fine structure constant α. This leads to strong restrictions on the time dependence of the internal metric.

A zero change of particle parameters is certainly realized if the shape and size of internal space, and also the warping, do not change at all. In terms of the metric this means

$$a = a_0(t)a_1(\rho), \quad b = b(\rho), \quad n = 1 \tag{248}$$

in an appropriate coordinate system. This can be seen as follows: Go to the v-gauge, i.e. $g_{t\theta} = 0$, and assume that the internal metric components $g_{\rho\rho}$, $g_{\rho\theta}$ and $g_{\theta\theta}$ all depend only on ρ. The component $g_{\rho\theta}$ can then be removed by a θ transformation $\theta \to \theta' = \theta + \delta\theta(\rho)$. In this special case the transformation does not induce a component $g_{t\theta}$, since $\delta\theta$ does not dependent on t. So the off-diagonal components of the metric have been removed. The condition $n = 1$ can then be obtained from $n = n(\rho)$ by a coordinate transformation $\rho \to \rho'(\rho)$. Finally the product structure of a follows from the time independence of the warping.

From this ansatz one can again compute the Einstein tensor. The off-diagonal components are again identically zero, from which one again deduces that A_t and A_ρ are a pure gauge and that $\dot{A}_\theta = 0$. The other components are

$$G^t_t = 3\frac{a_1''(\rho)}{a_1(\rho)} + 3\frac{{a_1'}^2(\rho)}{a_1^2(\rho)} + \frac{b''(\rho)}{b(\rho)} + 3\frac{a_1'(\rho)b'(\rho)}{a_1(\rho)b(\rho)} - \frac{1}{a_1^2(\rho)}3\frac{\dot{a}_0^2(t)}{a_0^4(t)} \tag{249}$$

$$G^{(i)}_{(i)} = 3\frac{a_1''(\rho)}{a_1(\rho)} + 3\frac{{a_1'}^2(\rho)}{a_1^2(\rho)} + \frac{b''(\rho)}{b(\rho)} + 3\frac{a_1'(\rho)b'(\rho)}{a_1(\rho)b(\rho)} - \frac{1}{a_1^2(\rho)}\left(2\frac{\ddot{a}_0(t)}{a_0^3(t)} - \frac{\dot{a}_0^2(t)}{a_0^4(t)}\right) \tag{250}$$

$$G^\theta_\theta = 4\frac{a_1''(\rho)}{a_1(\rho)} + 6\frac{{a_1'}^2(\rho)}{a_1^2(\rho)} - \frac{1}{a_1^2(\rho)}3\frac{\ddot{a}_0(t)}{a_0^3(t)} \tag{251}$$

$$G^\rho_\rho = 6\frac{{a_1'}^2(\rho)}{a_1^2(\rho)} + 4\frac{a_1'(\rho)b'(\rho)}{a_1(\rho)b(\rho)} - \frac{1}{a_1^2(\rho)}3\frac{\ddot{a}_0(t)}{a_0^3(t)} \tag{252}$$

They contain two types of terms: First there are terms containing ρ-derivatives. These terms depend only on ρ. And then there are terms $a_1^{-2}\dot{a}_0^2/a_0^4$ and $a_1^{-2}\ddot{a}_0/a_0^3$. The ρ-dependence of a and b, which is described by the first type of terms, is therefore determined by time-independent terms on the right hand side of Einstein's equations, such as the six-dimensional cosmological constant Λ and the gauge field source term ${A_\theta'}^2/b^2$. On the other hand, the time dependence of the scale factor a_0 is determined by - possibly time-dependent - source terms T^A_B which have to fulfill several conditions: The ρ-dependence of T^A_B is fixed by the warping,

$$T^A_B(\rho, t) = \tilde{T}^A_B(t)a_1^{-2}(\rho). \tag{253}$$

The difference $G^\rho_\rho - G^\theta_\theta$ is time-independent, from which follows that $T^\rho_\rho = T^\theta_\theta$. ($T^A_B$ is defined to be only the part of the energy momentum tensor which governs the time evolution). We define

$$\tilde{T}^t_t = -\varepsilon, \quad \tilde{T}^{(i)}_{(i)} = p_1, \quad \tilde{T}^\theta_\theta = \tilde{T}^\rho_\rho = p_2. \tag{254}$$

The combination $G^\mu_\mu - G^\theta_\theta - G^\rho_\rho$ is also time-independent, implying

$$-\varepsilon + 3p_1 - 2p_2 = 0. \tag{255}$$

This relates the two equations of state $w_1 = p_1/\varepsilon$ and $w_2 = p_2/\varepsilon$. Finally, if all these conditions are fulfilled, the equations are consistent and energy-momentum is conserved provided that

$$\varepsilon(t) \sim a_0(t)^{-3(1+w_1)}. \tag{256}$$

This is the same relation as in usual four-dimensional cosmology. The solution for a_0 is then

$$a_0 \sim t^{2/(3w_1+1)}. \tag{257}$$

For a relativistic fluid, $p_1 = \varepsilon/3$, one would get

$$p_2^{(r)} = 0, \quad \varepsilon^{(r)} \sim a_0^{-4}, \quad a_0^{(r)} \sim t. \tag{258}$$

For a non-relativistic fluid, $p_1 = 0$, one would get

$$p_2^{(nr)} = -\varepsilon/2, \quad \varepsilon^{(nr)} \sim a_0^{-3}, \quad a_0^{(nr)} \sim t^2. \tag{259}$$

How does a four-dimensional cosmological constant Λ_4 appear in this picture? A Λ_4-term has $p_1 = -\varepsilon$ and eq. (255) therefore requires $p_2 = -2\varepsilon$, which is a rather unusual equation of state. One can add to the right hand side of each $(\mu\mu)$-component of Einstein's equations a zero in the form of

$$0 = \frac{\Lambda_4}{a_1^2} - \frac{\Lambda_4}{a_1^2}, \tag{260}$$

and to the $(\theta\theta)$ and $(\rho\rho)$ components twice these terms. The $+\Lambda_4$ term is used to change the ρ-dependence of a_1, b and A_θ. The $-\Lambda_4$ term generates a time-dependence of a_0. This is exactly what was described in a slightly different way in chapter 3, when 6D Einstein-Maxwell theory was introduced: A part of the 6D curvature (the amount is arbitrary, this is why Λ_4 is an integration constant) is taken from the 2D internal curvature and the warping and is transferred into 4D curvature. The reason we can do this is that the Λ_4 term (and only this term) fits to both types of terms on the left hand side of Einstein's equations: the time-independent ρ-derivative terms and the time-derivative terms with ρ-dependence a_1^{-2}.

Note that the time-dependence of the scale factor in de Sitter space is in conformal time not exponential as with "usual" time, but of the form

$$a_0^{(\Lambda_4)} \sim (t_0 - t)^{-1}, \tag{261}$$

i.e. a_0 diverges at a finite time t_0.

To summarize, the late time cosmology derived from 6D Einstein-Maxwell theory with a perfect fluid turned out to be very restricted, when one requires that internal space is absolutely static. The ρ-dependence of the metric is the same as in the static case, derived in chapter 3. The time evolution of the scale factor is described by the usual 4D Friedmann-Robertson-Walker cosmology, depending on w_1. The requirement of a static internal space fixes w_2 as a function of w_1. The 4D cosmological constant is still a free parameter.

We will refer to the cosmologies derived above as "perfect" late time cosmologies. This is certainly not the most general realistic case. In general, the ρ dependence of the energy momentum tensor will be different from $a_1^{-2}(\rho)$ and will therefore lead to a time dependent perturbation of the shape of internal space. In the dimensionally reduced effective theory this would appear as an interaction of the fluid with supermassive scalar fields. In the six-dimensional picture, the perturbed shape will be expressed in time-dependent perturbations of the ρ-dependent metric functions, for example

$$b(\rho, t) = b_0(\rho) + \delta b(\rho, t), \tag{262}$$

with $\delta b/b$ of the order $(\varepsilon/M_c^4)^\alpha$, where ε is the 4D energy density of the fluid, M_c is the compactification scale and α is a positive and model-dependent parameter. Time derivative terms such as $\dot{b}^2/b^2$ are suppressed with $(\delta b/b)^2$ compared to $\dot{a}^2/a^2$. The time dependence of 4D coupling constants will therefore still be very small, as required by observational bounds. Nevertheless there will be additional ρ-derivative terms like $\delta b''/b$ which may be of the same order as the cosmological term $\dot{a}^2/a^2$. These additional terms might cancel the "wrong" ρ-dependence of the energy momentum tensor, restoring a realistic Friedmann cosmology.

Unfortunately it seems that such a cancellation does not take place in the coordinate system we have chosen. The difference between the (tt) and the (ii) component of the Einstein tensor contains only time derivatives (in the v-gauge; in the u-gauge there would be additional ρ-derivative terms, but these are too strongly suppressed). This seems to lead to a "wrong" cosmological behavior. We suspect that this is only a consequence of choosing the "wrong" coordinate system, in which the "true" cosmological situation is not visible. If we had allowed for $g_{tt} \neq -g_{ii}$, $G^t_t - G^{(i)}_{(i)}$ would contain terms like $\frac{g''_{tt}}{g_{tt}} - \frac{g''_{ii}}{g_{ii}}$ which might induce the required cancellation. The effective 4D picture strongly suggests that a slight excitation of supermassive scalar fields does not prevent a realistic Friedmann cosmology. Nevertheless, the situation has to be clarified. This will be subject to future research.

5.5 Cosmology with Relativistic Fermions

Now we want to include fermions into the model. Since we are interested in possible late time cosmologies, we will assume the density of these fermions to be small compared to the other variables which determine the structure of internal space: the 6D cosmological constant and the magnetic flux. The fermions can therefore be considered as a perturbation, and their wave functions will be to a good approximation given by the static solutions derived in chapter 4. Only the massless modes are excited in a late time cosmology. We have to compute the energy momentum tensor of these modes. Since they are massless, we expect a relativistic equation of state, $w_1 = 1/3$. Consistency with the conditions derived in the previous section requires then $w_2 = 0$ for a "perfect" cosmology. (We expect that the constraints on the equations of state still hold, at least approximately, even when the cosmology is not "perfect", i.e. when scalar fields are excited. But this remains to be proven). We will show that both relations are indeed fulfilled.

The energy momentum tensor of the fermions is given by the expectation value

$$T^A{}_B = \left\langle \frac{1}{2} i\bar{\Psi}\gamma^A D_B \Psi \ + h.c. \right\rangle, \tag{263}$$

where $\gamma^A = \gamma^a e_a{}^A$. Usually an energy momentum tensor contains also a piece $L\delta^A_B$, where L is the Lagrangian density, but this vanishes here, since $L = 0$ for solutions of the Dirac equation. The covariant derivative D_B contains the spin connection and a possible coupling to the gauge field. We assume that the distribution of the fermions is homogeneous and isotropic in three-dimensional space. This forbids any components containing 3D spatial indices except for diagonal ones, $T^{(i)}_{(i)}$. The effective 4D Dirac equation implies

$$\bar{\psi}(x)\tilde{\gamma}^\mu \tilde{D}_\mu \psi(x) = 0, \tag{264}$$

where a tilde again denotes a four-dimensional operator. From isotropy then follows

$$\bar{\psi}(x)\tilde{\gamma}^{(i)}\tilde{D}_{(i)}\psi(x) = -\frac{1}{3}\bar{\psi}(x)\tilde{\gamma}^t \tilde{D}_t \psi(x). \tag{265}$$

This does not automatically imply that $w_1 = 1/3$. It remains to be proven that the effective 4D energy momentum tensor is really given by

$$\tilde{T}^{(\mu)}_{(\mu)} = \left\langle \bar{\psi}\tilde{\gamma}^{(\mu)}\tilde{D}_{(\mu)}\psi \right\rangle. \tag{266}$$

This is not yet clear, since the covariant derivative D_μ contains terms additional to $\tilde{D}_\mu$, compare eq. (173). Hence a more detailed analysis is necessary, which is needed anyway to compute the internal components of the energy momentum tensor.

We assume that the different fermion modes do not mix and have arbitrary phases with respect to each other, so that all expectation values of the type $\left\langle \psi_i^\dagger \psi_j \right\rangle$, $\left\langle \bar{\psi}_i \psi_j \right\rangle$, $\left\langle \psi_i^\dagger \partial_\mu \psi_j \right\rangle$ and $\left\langle \bar{\psi}_i \partial_\mu \psi_j \right\rangle$ vanish for different modes, $i \neq j$. Then we can compute the energy momentum tensor for each mode separately. We therefore take $\Psi(x, \theta, \rho) = \psi(x)\zeta(\theta, \rho)$, where ζ is one particular zero mode and splits up into a θ-dependent and a ρ-dependent part, $\zeta = e^{in\theta}\chi(\rho)$. Furthermore we assume that with each mode also the corresponding antiparticle (with opposite handedness, opposite "winding number" n and opposite charge q with respect to gauge field) is excited with the same density, so that no net charges appear. Every operator $\gamma^A D_A$ is a tensor product (or a sum of tensor products) of an operator acting on the 4D part $\psi(x)$ and an operator acting on the internal part $\zeta(\rho, \theta)$ of the fermions. We will refer to these as the 4D and the 2D part of the operator. Recall some of the relations obtained in chapter 4:

$$\gamma^m = \tilde{\gamma}^m \otimes 1, \quad \gamma^\mu = a^{-1}\tilde{\gamma}^\mu \otimes 1, \quad \gamma^{4(5)} = \tilde{\Gamma} \otimes \tau_{1(2)}, \tag{267}$$

$$
\begin{aligned}
D_\mu &= \tilde{D}_\mu + i(\Sigma_{m4}\sin\theta - \Sigma_{m5}\cos\theta)a'\tilde{e}^m_\mu && (268)\\
&= \tilde{D}_\mu + \frac{1}{2}\frac{a_1'}{a_1}\gamma_\mu\gamma_\rho, && (269)\\
D_\theta &= \partial_\theta + \frac{1}{2}i\tau_3(1-b') + iqeA_\theta, && (270)\\
D_\rho &= \partial_\rho. && (271)
\end{aligned}
$$

The covariant derivatives D_θ and D_ρ have only 2D parts. The first term of D_μ is purely 4D, whereas the second has both a 4D and a 2D part.

We will now show that all components of the energy momentum tensor vanish or cancel each other with exception of the part obtained from $\gamma^{(\mu)}\tilde{D}_{(\mu)}$. Most terms vanish because $\bar{\psi}(x)\tilde{\Gamma}\psi(x)$ is zero for a chiral fermion. This term occurs in $T^\theta{}_\theta$, $T^\theta{}_\rho$, $T^\rho{}_\theta$ and $T^\rho{}_\rho$, since the 4D part of $\gamma^\alpha D_\beta$ is just $\tilde{\Gamma}$. Therefore these components all vanish. The second part in D_μ contains $\gamma_\mu\gamma_\rho$. In $T^{(\mu)}{}_{(\mu)}$, the γ_μ is multiplied with γ^μ to give 1, and the 4D part of γ_ρ is again $\tilde{\Gamma}$, and so $\bar{\psi}\tilde{\Gamma}\psi$ occurs again. The remaining part of $T^{(\mu)}{}_{(\mu)}$ is

$$
T^{(\mu)}{}_{(\mu)} = \left\langle \bar{\psi}(x)\tilde{\gamma}^{(\mu)}\tilde{D}_{(\mu)}\psi(x)\right\rangle a_1^{-1}(\rho)|\chi(\rho)|^2. \tag{272}
$$

It has obviously $w_1 = 1/3$, as derived from the 4D Dirac equation, and we also showed that $w_2 = 0$. It remains to be proven that the remaining off-diagonal components are also zero. We have

$$
\begin{aligned}
T^t{}_\rho &= \left\langle \frac{1}{2}i\bar{\Psi}\gamma^t D_\rho\Psi \ + h.c.\right\rangle = \left\langle \frac{1}{2}i(\bar{\psi}\tilde{\gamma}^t\psi)\otimes(a_1^{-1}\chi^\dagger\partial_\rho\chi) \ + h.c.\right\rangle && (273)\\
&= \left\langle \frac{1}{2}i(\psi^\dagger\psi)\otimes(a_1^{-1}\chi^\dagger\partial_\rho\chi) \ + h.c.\right\rangle. && (274)
\end{aligned}
$$

This is purely imaginary and is therefore cancelled by the hermitian conjugate.

$$
T^\rho{}_t = \left\langle \frac{1}{2}i\bar{\Psi}\gamma^\rho\tilde{D}_t\Psi + \frac{1}{4}i\bar{\Psi}\gamma^\rho\gamma_t\gamma_\rho\frac{a_1'}{a_1}\Psi + h.c.\right\rangle \tag{275}
$$

The first term contains as 4D part $\bar{\psi}\tilde{\Gamma}\tilde{D}_t\psi$ which is zero because of chirality. The second term simplifies to

$$
T^\rho{}_t = \left\langle \frac{1}{4}i(\psi^\dagger\psi)\otimes(a_1'\chi^\dagger\chi) + h.c.\right\rangle \tag{276}
$$

This is again purely imaginary and cancels with the hermitian conjugate.

$$
\begin{aligned}
T^t{}_\theta &= \left\langle \frac{1}{2}i\bar{\Psi}\gamma^t D_\theta\Psi \ + h.c.\right\rangle && (277)\\
&= \left\langle \frac{1}{2}i(\psi^\dagger\psi)\otimes a_1^{-1}\zeta^\dagger(\partial_\theta + \frac{1}{2}i\tau_3(1-b') + iqeA_\theta)\zeta + h.c.\right\rangle && (278)\\
&= \left\langle -(\psi^\dagger\psi)\otimes a_1^{-1}\chi^\dagger\chi(n \pm \frac{1}{2}(1-b') + qeA_\theta)\right\rangle && (279)
\end{aligned}
$$

This is real and does not vanish. A net charge would indeed lead to a nonvanishing $T^t{}_\theta$. Such a component would certainly, through Einstein's equations, force one of the off-diagonal metric components to become nonzero, since $G^t{}_\theta$ would identically vanish for a diagonal metric. Hereby is demonstrated that the inclusion of the off-diagonal metric components in the general discussion was really necessary. But fortunately we assumed here that the for each particle the corresponding antiparticle is also present. This has opposite n, opposite τ_3 eigenvalue and opposite q. Its $T^t{}_\theta$ component has therefore the same value but the opposite sign, and the terms cancel in the total energy momentum tensor. Finally

$$T^\theta{}_t = \left\langle \frac{1}{2} i\bar{\Psi}\gamma^\theta \tilde{D}_t \Psi + \frac{1}{4} i\bar{\Psi}\gamma^\theta\gamma_t\gamma_\rho \frac{a_1'}{a_1}\Psi + h.c. \right\rangle \tag{280}$$

The first term contains again $\bar{\psi}\tilde{\Gamma}\tilde{D}_t\psi$ as 4D part and therefore vanishes. The second term simplifies to

$$T^\theta{}_t = \left\langle (\psi^\dagger\psi) \otimes (a_1'\zeta^\dagger\tau_3\zeta) \right\rangle, \tag{281}$$

which is again in general nonzero and needs the antiparticle for cancellation (due to the opposite τ_3 eigenvalue). This completes our calculation of the energy momentum tensor for relativistic fermions.

Consider now the spherical model. One condition for a "perfect" late time cosmology was that the energy momentum tensor is "warped" with $a_1^{-2}(\rho)$. On the other hand we computed that the energy momentum tensor of a fermion mode is "warped" with $a_1^{-1}(\rho)|\chi(\rho)|^2$. In the spherical model there is no warping at all, $a_1 = 1$, and $\sum |\chi_i|^2$ is also constant (by symmetry) if all zero modes are excited with the same probability. In this case we would really get a "perfect" late time cosmology, provided the sphere starts with internal scale factor $n < n_{max}$ and therefore converges to the ground state $n = n_0$, as discussed in section 5.2. The cosmology would be a Friedmann cosmology with relativistic matter and with possible inclusion of a cosmological constant (the Λ_4 of the ground state).

If only some of the fermion zero modes on the sphere are excited, this would break the spherical symmetry, and the cosmology can no longer be "perfect". Non-singlet scalar fields would be excited. This is the case mentioned in the discussion at the end of section 5.4 and is not yet fully understood. But since the stability of the ground state of the spherical model has been proven [1], all scalars have huge masses and can certainly not disturb the effective 4D cosmology too much.

The situation is much worse if there are deficit angle branes. Now the fermion distribution $|\chi(\rho)|^2$ is singular at one of the branes. Even if the fermion density is very small, it is not clear how far they can be considered as a small perturbation in the vicinity of the brane. They may alter the structure of the singularity, transforming it into a thick brane. The effects on the effective 4D picture may be small, since only a small part of internal space is involved, but this conclusion is not so obvious. Furthermore, if the deviation from a sphere is large, we do not know very much about the stability of the "ground state". A perturbation of the warping or the internal metric (excitation of scalars in the 4D picture) may have drastic consequences. The cosmology of this general case is still completely unclear and requires a much better understanding of the ground states with non-compact internal space.

5.6 Open Questions

We are still far away from a complete understanding of the cosmological evolution of higher dimensional theories, even in this relatively simple six-dimensional example. Our goal was to explore if there is a chance to solve the Cosmological Constant Problem (and maybe similarly the Gauge Hierarchy Problem) in such a dynamical context. Would a universe, starting from arbitrary initial conditions, automatically evolve towards the universe we observe today, with its small 4D curvature? This goal is at the moment completely out of sight.

But we think that we achieved a good basic picture of the situation. The two most serious difficulties were identified and explored to some extent. They are

1. the complexity of the dynamics and
2. the problem to find the 4D interpretation of a 6D solution, which is mainly identical with the problem to find a physically meaningful coordinate system.

A way to deal with these problems is to look for special cases which can be solved and then to generalize step by step. The beginning of this long road has been taken, and we believe that we have reached a good starting position for future research. The next steps towards our final goal could be:

- a better understanding of the qualitative features of the general ansatz, especially of the off-diagonal metric components and the correspondingly appearing gauge field components A_t or A_ρ. Under which circumstances are these fields excited? Maybe there is a large class of solutions in which these components can be ignored. This would open new prospects of numerical analysis, since much fewer functions would then be involved. We have seen that the off-diagonal components are certainly involved if there are net charges (see section 5.5, discussion of T^t_θ). Is this maybe the only case in which these fields are forced out of their ground state?

- an investigation of codimension-two singularities different from the deficit angle branes, or their 4D analogues: the gravitational field of a wire with arbitrary equation of state. A step into this direction has recently been made by Vinet and Cline [47] in the context of thick branes. One could first look for static solutions with three independent functions, $g_{tt}(\rho)$, $g_{ii}(\rho)$ and $g_{\theta\theta}(\rho)$ and see what happens when a solution is forced into a coordinate system with conformal time $-g_{tt} = g_{ii}$ or a frame constrained in any other way. This would certainly help a bit for a better understanding of the coordinate problem.

  Proceeding further, it will be interesting to see how bulk energy like the magnetic flux and fermions react on such more general singularities. Does a part of them fall into the singularity like into a black hole? How would this look like in the effective four-dimensional theory? And would such an effect change the "equation of state" of the singularity?

- a better understanding of the late time cosmology with fermions. One could perform the following computation: Take the spherical ground state with monopole number

$m = 2$. The fermion zero modes will then form a doublet. Assume that for some reasons only one of the two modes is excited. The excitation will backreact on the geometry and lead to a slight deformation of the sphere. In the effective 4D picture the deformation would be expressed as the excitation of a component of a scalar SU(2) triplet. With some effort it should be possible to solve the cosmology of this particular example in the 6D and the effective 4D picture and check the consistency of the two pictures. This will certainly also shed new light on the coordinate problem.

- To surround the coordinate problem, it may be useful to look for more mathematical, coordinate free descriptions. Maybe "asymptotic Killing vectors" can be formulated and looked for in a gauge invariant formalism.

When all these steps have been successfully solved, one may face the most interesting part of the problem: the early universe cosmology. Starting with arbitrary initial conditions, are there dynamical reasons why three spatial dimensions become large, with almost zero effective cosmological constant, while the other dimensions remain small?

To summarize: The major achievement of this chapter is not a solution, but a detailed **description** of the Kaluza-Klein cosmology problem. The difficulties that have to be faced were discussed, and a **strategy** how to overcome these difficulties was outlined.

6 How to construct a Realistic Kaluza-Klein Theory

Our toy model, six-dimensional Einstein-Maxwell theory, was very useful for the investigation of many typical features of Kaluza-Klein theories. But it contains by far not enough symmetries to account for the SU(3) × SU(2) × U(1) Standard Model and its rather complicated fermion spectrum. Going beyond our simple model, how far can one get in constructing a realistic theory? Is it possible to get the correct low energy symmetries and the correct particle spectrum? And if this can be achieved, may we even hope to explain the structure of the mass matrices in this way (hierarchy, mixings)?

Edward Witten [48] showed that, if we want to get the gauge groups from pure higher dimensional gravity, at least seven extra dimensions are needed. He constructed explicitly seven-dimensional internal spaces containing the Standard Model gauge group. Wetterich [39] showed that, in order to get also a realistic fermion spectrum, the total dimensionality must be 2 mod 8. Combining the two requirements, one ends up with at least 18 dimensions. The corresponding 14-dimensional internal space must be non-compact, since otherwise the chirality index would vanish. A Majorana-Weyl spinor with 256 real components could then be sufficient to reduce to all the observed fermions in the effective four-dimensional theory. Indeed, if 11 of the extra dimensions form an S^{11} and another one is "radial" similar to the ρ coordinate in our toy model, a first step of compactification could lead to the six-dimensional SO(12) model [49, 50, 51, 52], which turns out to be almost completely successful in explaining the observed world and will be discussed in section 6.2. This gives us hope that our discussion of a specific six-dimensional gauge theory was a good choice and may be very useful for the investigation of more realistic theories.

It should be mentioned that the 18-dimensional model with a Majorana-Weyl fermion contains a gravitational anomaly [53]. On the other hand, the six-dimensional SO(12) theory is anomaly free.

6.1 Fermion Mass Matrices

How can one reproduce the quantum numbers, hierarchies of masses and mixing angles for quarks and leptons?

In contrast to four-dimensional unified gauge theories, higher dimensional theories have typically only a few free parameters. If such theories lead to a realistic four-dimensional model after dimensional reduction, the spectrum of fermion masses should be highly predictable. Four-dimensional gauge fields and scalars often correspond to different components of the same higher dimensional field. In this section we outline in a general framework the steps leading to predictions concerning the mass matrices. An understanding of these matrices is related to an understanding of the origin and couplings of the low-energy weak Higgs doublet. It is proposed that a fine structure of scales at the unification scale is responsible for the observed structures. In the next section the application of these methods to a particular example, the six-dimensional SO(12) model, is sketched.

In Kaluza-Klein theories, massless fermions have a similar status as massless gauge bosons: the zero mass is guaranteed by symmetry (plus topology) and any mass term involves the breaking of symmetry. A symmetry breaking pattern would look as follows: Let $\tilde{G}$ be the symmetry of the underlying $d = D + 4$-dimensional Lagrangian and define

$$\tilde{H} = SU(3)_C \times SU(2)_L \times U(1)_Y, \qquad H = SU(3)_C \times U(1)_{em}. \tag{282}$$

We start with a "compactified" first approximate ground state with symmetry $G \subset \tilde{G}$ and compactification scale M_c. The term "compactified" does not necessarily mean that the D-dimensional internal space has to be compact, but it should have finite volume. The solution may be unstable and is just an approximation to the true ground state. The symmetry group G contains higher dimensional gauge groups and the isometries of internal space. It should be of the form $G = F \times K$, where F is either $\tilde{H}$ itself or contains it as a subgroup, like $SU(5)$ or $SO(10)$. The group K serves as a generalized generation group and may contain discrete subgroups. (We have omitted here additional four-dimensional isometries like Poincare invariance or similar, which are of course also contained in G.) This highly symmetric ground state is, possibly in several steps, spontaneously broken to a second approximate ground state with symmetry $\tilde{H}$. The scales of this symmetry breaking are M_1, M_2 ... somewhat below M_c. This is what was already mentioned as a "fine structure" of scales. Ratios $M_1/M_c \approx \frac{1}{4}$ may sometimes be sufficient. Small ratios of quark and lepton masses are then induced by various powers of M_i/M_c, as we will discuss. The appearance of a fine structure may either be directly related to small quantities of D-dimensional internal space like $1/D$, the ratio of "radius" to volume L^D/V, inverse monopole numbers $1/N$ etc. or it may result from geometric properties of particular solutions. Finally, at the much lower scale M_L, $\tilde{H}$ is spontaneously broken to H, which is then the symmetry group of the true ground state. So we have the chain

$$\tilde{G} \longrightarrow G = F \times K \longrightarrow \tilde{H} \longrightarrow H. \tag{283}$$

The fermion mass problem can then be split into several pieces:

1. Identify the chiral quarks and leptons in the first approximate ground state and find all their quantum numbers with respect to G.

2. Identify the scalar fields with the appropriate quantum numbers in order to couple to some of the fermion bilinears.

3. Compute the Yukawa couplings of these scalars to the relevant fermions.

4. Analyze the mixing between the scalars and see if the lowest scalar mass eigenstate, identified as the Higgs field, may serve to generate the observed fermion mass ratios.

At each step one has to check if the result is still compatible with observational bounds. These bounds are very restrictive, and it will not be easy to find a realistic model.

Step 1: Fermion Quantum Numbers: At first one has to be aware of the possibility that different solutions, although having the same topology, may contain a different number of generations. This is because these solutions may belong to different "deformation classes" of $\tilde{H}$, leading to a different chirality index. In this way the number of generations may change in the course of symmetry breaking. We assume that such a problem does not occur here in the breaking of G. Otherwise an analysis of the G-symmetric state would be completely useless as an approximation.

For a realistic model, one has to identify three $\tilde{H}$ standard generations. The fermions should have a rich structure with respect to the generation group K, so that even fermions in the same generation have different K quantum numbers in order to give the necessary mass relations. All mass ratios of an order of magnitude or more should follow from symmetry considerations, since all allowed Yukawa couplings to the same scalar are typically of the same order.

Step 2: Scalar Doublets: The various fermion bilinears appearing as entries in mass matrices couple to color singlet electrically neutral $SU(2)_L$ doublet scalar fields with different K quantum numbers. Such scalar fields appear in the harmonic expansion of internal metric or gauge field components or of scalars which are already present in the d-dimensional theory. The latter do occur naturally if this theory was already obtained from an even more fundamental theory. Step 2 involves the identification of such doublets and of their possible couplings to the fermion bilinears allowed by their K quantum numbers.

The low-energy Higgs doublet must be a linear combination of these fields. It should mainly consist of a leading doublet which couples to the top quark but is forbidden by K symmetry to couple to other quarks or charged leptons. (We do not talk about neutrinos here, because we expect that some scalar VEVs induce large right-handed neutrino masses and assure in this way small left-handed neutrino masses.) There should be a small admixture of another doublet which couples only to bottom, tau and charm, and so on.

A realistic model requires that all entries in the mass matrices with different orders of magnitude are coupled to different doublets. This does not yet mean that these orders of magnitudes will turn out correct, since we do not yet know the structure of the low-energy Higgs doublet. But it assures that IF the Higgs turns out to have the required admixtures, then the observed ratios would be obtained. This requirement is already very restrictive, and a model which survives step 2 has passed an important test.

Step 3: Yukawa Couplings: One has to find the internal wave functions for the relevant components in the harmonic expansions on the first approximate ground state. (In the true ground state, symmetry breaking will induce small corrections to these wave functions and to the Yukawa couplings.) Such expansions exist for scalars similar as for spinors:

$$\Phi(x,y) = \sum_i \varphi_i(y)\phi_i(x). \tag{284}$$

The field Φ could for example be an internal gauge field component. The effective four-dimensional Yukawa coupling would then arise from the higher dimensional gauge coupling and would be determined by integrating the relevant wave functions over internal space. If a coupling between φ_i and the zero mode fermion bilinear $\bar{\psi}_{0j}\psi_{0k}$ is allowed, the coupling h_{jki} would be of the form

$$h_{jki} = \bar{g} \int d^D y \; g_D^{1/2} a^4 \; \bar{\psi}_{0j}(y)\varphi_i(y)\psi_{0k}(y), \tag{285}$$

where $\bar{g}$ is the higher dimensional gauge coupling and $a^2(y)$ is a possible warp factor. If Φ would instead be a higher dimensional scalar field, there would be a higher dimensional Yukawa coupling $\bar{f}$ instead of $\bar{g}$ in front of the integral. We see that the many effective Yukawa couplings are all related to the very few couplings of the underlying theory, times some integrals which are either zero for symmetry reasons or expected to be roughly of the same magnitude.

Step 4: Mixing of Scalars: How to identify the low-energy Higgs doublet? In the limit of unbroken G, doublets with different G quantum numbers cannot mix. A breaking of G involves scalar fields which are singlets with respect to $\tilde{H}$ but not with respect to G. These scalars acquire vacuum expectation values and induce mixings between the doublets via their couplings. These mixings are proportional to various powers of M_i/M_c, depending on the power of singlets needed to produce a G invariant by coupling to the doublets.

If the low-energy Higgs doublet ϕ_L has only a small admixture γ_i of a given doublet d_i, the vacuum expectation value of d_i,

$$\langle d_i \rangle = \gamma_i \langle \phi_L \rangle \tag{286}$$

will be small compared to $\langle \phi_L \rangle$ and this reflects itself in a small entry to the fermion mass matrices. To compute the γ_i's for a specific model, one has to identify the scalar singlets and their possible couplings to the doublets. From this one obtains the powers of M_i/M_c appearing in the scalar mass matrices. If these powers lead to a realistic hierarchy in the mixings to the "leading doublet" which couples to the top quark, step 4 has been successful. No model has passed this test so far, which shows the very high predictivity of Kaluza-Klein theories.

Even if a model passes this test, the problem is not solved completely. It remains the question if the "leading doublet" in the lowest mass eigenstate ϕ_L is really the one that couples to the top quark. To investigate this, a more detailed analysis of the ground state is needed. Furthermore, the question why a solution with such a small Higgs mass is selected (the gauge hierarchy problem) has not even been touched.

6.2 Six-Dimensional SO(12) Theory

The six-dimensional SO(12) gauge theory, which we discuss here to illustrate the methods described in the previous section, could be obtained from 18-dimensional pure gravity coupled to a Majorana-Weyl spinor, or from another more fundamental theory. We will not refer to such a possible origin here. It is encouraging to see how far one can get with this relatively simple and economic model. Solutions exist where almost the complete structure of mass matrices is reproduced. Only one detail turns out wrong in each of these possibilities while all the other ratios appear to be of the right size. Here we will roughly explain the several steps and state the results without giving proofs.

The six-dimensional action involves the Einstein term, an $SO(12)$ gauge field, and a Majorana-Weyl spinor in each of the two pseudoreal 32-dimensional spinor representations of $SO(12)$, Ψ_1 and Ψ_2. These spinors contain 16-dimensional representations of an $SO(10)$ subgroup, which are the standard fermion generations known from $SO(10)$ unification. Six-dimensional scalars should be also present and appear naturally if the model was obtained from a more fundamental theory. Here we need only one scalar in the fifth rank antisymmetric tensor representation of $SO(12)$ to generate some features of the required mass relations. There are only two couplings: The 6D gauge coupling $\bar{g}$ and one 6D Yukawa coupling $\bar{f}$.

In the first approximate ground state, two dimensions are compactified on a sphere, with the gauge field in a generalized monopole configuration. The geometry is exactly the same as in the Einstein-Maxwell solutions discussed throughout this thesis. The only difference is the more complicated structure of the gauge field. The gauge configuration can be brought into the form

$$A_\theta = \frac{1}{2\bar{g}} \hat{N}(\pm 1 - \cos\theta), \qquad A_\rho = A_\mu = 0, \tag{287}$$

$$\hat{N} = m(T_{12} + T_{34}) + p(T_{56} + T_{78} + T_{9,10}) + nT_{11,12}. \tag{288}$$

The T's are the generators of a Cartan subalgebra of $SO(12)$, and m, p and n are monopole numbers. The symmetry of this approximate ground state is at least

$$G = SU(3)_C \times SU(2)_L \times U(1)_R \times U(1)_{B-L} \times U(1)_G \times SU(2)_G. \tag{289}$$

Here $U(1)_G$ corresponds to the generator $T_{11,12}$, and $SU(2)_G$ corresponds to the isometries on the S^2, combined with gauge transformations to preserve the form of the gauge configuration. The product $U(1)_G \times SU(2)_G$ serves as generalized generation group K. There are additional discrete symmetries, such as parity reflections and the reflection of one of the two spinors:

$$\Gamma: \; \Psi_1 \to \Psi_1, \quad \Psi_2 \to -\Psi_2. \tag{290}$$

All these symmetries have to be taken into account in the quantum number analysis.

Deriving the fermion quantum numbers with respect to K, one finds that the number of massless generations is given by n. In the case of three generations, the possibility $n = 3$, $p = m = 1$ is the only one which survives step 2. In all the other cases the assignment of required scales to certain entries in the mass matrices would fail for some reasons based on symmetry. This is even before the mixing of the doublets has been discussed at all! Even at this early stage, all possibilities except one are excluded, which shows the high predictivity of the Kaluza-Klein framework.

For the case $n = 3$, $p = m = 1$, one finds six scalar weak doublets which can have Yukawa couplings to the zero mode fermions. Two of them, H_1 and H_2, are contained in the internal components of the gauge field, whereas the four others, d_1, d_2, d_3 and

d_4, appear in the harmonic expansion of the six-dimensional scalar field. The field H_1 can couple only to one charge 2/3 quark, which is thereby identified as the top quark. Furthermore, any scalar couples only to fermion bilinears which have entries of similar magnitude in the mass matrices. Therefore, IF the VEVs of the scalar doublets fulfill certain relations, a realistic mass hierarchy may be obtained.

The harmonic expansion of the six-dimensional scalar field contains also $\tilde{H}$ singlets which may break G down to $\tilde{H}$. Mixings between the doublets are induced by these fields. The ratios between these mixings can be estimated from group theoretical considerations. Assuming that H_1 is the "leading doublet" (since it couples to the top quark), one finds that the mixing pattern comes out almost as required, but not completely (the Cabibbo angle comes out wrong) [52]. It remains to be seen if these difficulties can be cured by some modifications of the model, or if another model will be more successful.

Similar to the 6D Einstein-Maxwell toy model, the 6D SO(12) model also shows how the Cosmological Constant Problem and the Gauge Hierarchy Problem may be connected. Again there exist solutions where the sphere is deformed in a way such that (at least) one pole becomes singular (a brane). In this large class of solutions, the 4D cosmological constant as well as the weak symmetry breaking scale become free integration constant. It becomes a dynamical problem why solutions are selected in which these two parameters are so small.

7 Conclusions

We have studied properties of theories with extra dimensions by exploring a particular example: six-dimensional Einstein-Maxwell theory with four "warped" large dimensions and a two-dimensional, possibly non-compact, internal space. This toy model has not enough symmetry to explain all the structures in the world (Standard Model gauge group etc.) but carries already many of the features which are believed to be important for a more realistic model.

The work we have carried out can be divided into four parts.

- The achievements made in Kaluza-Klein theories about twenty years ago were carried together and summarized.
- Kaluza-Klein Theories were compared to the more modern brane models which are motivated by results of String Theory, and the two different points of view were combined, leading to the notion of "holographic branes".
- The cosmology of our six-dimensional toy model was explored, which was the main motivation for this work.
- As a generalization of the singularities appearing in the six-dimensional model, properties of symmetric singularities in arbitrary dimensions were studied.

The last issue was rather a byproduct of our research. We have therefore put it into an appendix.

In the following, we summarize the results of all four issues separately.

Review of Kaluza-Klein Theories: Kaluza-Klein theories have the advantage that they usually contain only very few parameters. Several four-dimensional fields correspond to different components of the **same** higher dimensional field. This makes these theories very predictive. Many different Yukawa couplings, for example, are related to the same higher dimensional coupling. Relations in the fermion mass matrices are therefore much richer than in four-dimensional unification, leading to strong constraints on such models. The number of chiral fermion generations is determined by an index of mainly topological nature. Gauge groups arise naturally from the isometries of internal space. Gauge and scalar fields are (mostly) components of the higher dimensional metric. Spontaneous Symmetry Breaking is described as a slight deformation of the internal geometry.

The Gauge Hierarchy Problem and the Cosmological Constant Problem are not yet solved in this context, but there are hints that both problems may be linked [30], i.e. that both "small numbers" are two facets of one and the same underlying feature of the model.

Kaluza-Klein theories provide a very beautiful and promising framework of unification. But they are not intended as a fundamental theory. Starting from a very simple and economic higher dimensional Lagrangian, they are able to explain the matter content of the universe and to relate all effective four-dimensional forces to higher dimensional gravity. But they are classical theories and do not tell us how to quantize such a higher dimensional gravity. A self-consistent and predictive quantum theory of gravity does not exist at the moment, but there are interesting ideas towards that direction. Such a theory may arise from String Theory [42], Loop Quantum Gravity [54], non-perturbative methods [55], Spinor Gravity [56], or a combination of them.

Kaluza-Klein Theories versus Branes: Chiral fermions cannot be obtained from pure higher dimensional gravity if the internal space is compact. Internal spaces with cusps or singularities are therefore a general feature of successful Kaluza-Klein theories. Such singularities in a higher dimensional space are also present in a very modern class of phenomenological models motivated by String Theory: the so-called braneworlds. In principle, these brane models have a very different point of view compared to Kaluza-Klein theories. All particles and gauge interactions are located on the brane. In contrast, the observable particles in Kaluza-Klein theories are zero modes of the internal space (called "bulk" in the brane models). In a maximally symmetric internal space, the probability density of these zero modes would be constant over the entire "bulk", which is just the opposite of the delta-function-like distribution in the braneworlds. Our intention was to see how these two types of models are linked.

We considered a conical singularity, which appears naturally in the two-dimensional internal space of our six-dimensional toy model. In the modern point of view, this would be called a codimension-two brane. In the Kaluza-Klein point of view, it would be just a subspace which cannot be included into the manifold. We showed how the brane tension from the brane point of view can be translated into integration constants appearing in the solution of the field equations from the Kaluza-Klein point of view. Both descriptions are therefore equivalent. This was easily generalized to a case with two branes, one at each pole of internal space.

We also investigated the wave functions of the chiral fermions. They are peaked at the brane, and their probability density becomes singular at that position. This fits to the brane point of view: The fermions are located on the brane. But their "tail" into the bulk is such that all their physical properties (Yukawa couplings etc.) can be computed from bulk integrals, which is a basic feature of the Kaluza-Klein point of view. In this way we connected the two types of models. We called singular subspaces "holographic branes" if such a connection is possible. The term "holographic" is justified, since all properties of the brane and of the particles which are located there can be translated into geometric properties of the "bulk" and parameters of the underlying Kaluza-Klein theory.

We noticed that such a connection is not possible for codimension-one branes. The difference is that a conical singularity can be expressed as a property of the surrounding space, determined by the deficit angle, which can be "measured" by surrounding the singularity. In contrast, a codimension-one brane cannot be detected by a "bulk observer", and it is impossible to surround it.

Cosmology of six-dimensional Einstein-Maxwell Theory: A major question of all theories with extra dimensions is why the effective four-dimensional space is so much flatter than the internal dimensions, or equivalently: why the 4D cosmological constant Λ_4 is so small. We wanted to look for a dynamical solution of this problem in the context of our relatively simple toy model. Solutions exist with arbitrary Λ_4. Cosmological observations are consistent with $\Lambda_4/M_p \approx 10^{-121}$ (where M_p is the Planck mass). Why would a solution with such a small Λ_4 be selected? Starting from arbitrary initial conditions, what kind of solutions would be asymptotically approached? Is there some mechanism that drives late time cosmology towards a small Λ_4? For generic initial conditions, the shape of internal space will be time-dependent, so that the time evolution is described by a complicated set of partial differential equations.

Our intention was to study the cosmology of a six-dimensional model with the following isometries: translation and rotation invariance of the three large spatial dimensions, and a U(1) symmetry generated by translation invariance of the internal angle θ. All metric components are then functions of t and ρ (the second internal coordinate) only. We derived the most general metric consistent with the isometries and showed how far it can be simplified by coordinate transformations. Unfortunately, it is in general not possible to bring the metric into a diagonal form and simultaneously keep the functions θ-independent. The field equations for this metric turned out to be so complicated that it was impossible to attack the problem directly. Another difficulty, apart from the complexity, concerns the four-dimensional interpretation of six-dimensional solutions. In order to bring the metric into a specific form, we had to perform coordinate transformations $t, \rho \rightarrow t', \rho'(t, \rho)$. By this procedure, the time and ρ dimensions are mixed to some extent. It is not clear in which cases the time parametrized by t' corresponds to the "physical" time that we observe in our effective four-dimensional world. To overcome these difficulties, a much better understanding of the solutions will be necessary.

But we were able to solve some special cases and to find some generic properties of the system. It was shown that the codimension-two branes (conical singularities) cannot have an equation of state different from $w = -1$ and that the deficit angle associated with such a brane is always time-independent. (However there may be codimension-two singularities of a different type, behaving more like black holes.)

A rather simple case is given when internal space is a sphere and has therefore an SU(2) isometry group, not only U(1). In this case the metric depends only on t, not on ρ, so that the field equations become ordinary differential equations, which can be easily solved. The result is not new [45], but we repeated the calculations to illustrate the procedure of dimensional reduction and the method of Weyl scaling, and to show how scalar fields with asymptotically exponential potentials arise in Kaluza-Klein cosmology. These solutions are easily generalized to the case with two equal branes, one at each pole of internal space. As another new extension of these solutions, we computed the energy momentum tensor of the chiral fermion zero modes and showed that they indeed behave like a relativistic fluid in the effective four-dimensional picture, and that they induce a standard Friedmann-Robertson-Walker cosmology (at least as long as no scalars are excited).

"Late time cosmologies" were also discussed in general. There are strong observational bounds on the time-dependence of coupling constants, which implies that the geometry of internal space has to be almost time-independent. The special case in which this time independence is total was called "perfect late time cosmology". Such a solution can be only obtained if strong constraints on the ρ dependence and the equations of state of the six-dimensional energy momentum tensor are fulfilled. We also discussed how the constraint on the ρ dependence is relaxed even if only a tiny time-dependent perturbation appears in the internal space.

Finally, we developed a plan for possible future research in order to overcome the many remaining difficulties and open questions. We are still far away from a full understanding of the subject, in particular as far as the early universe is concerned. But at least we have solved some special cases and understood the nature of many of the complications to be faced, which we believe to be a considerable progress. We have reached a good starting position for further investigations.

Symmetric Singularities in Arbitrary Dimensions The conical codimension-two singularity, which was discussed throughout this thesis, has the very special property that it is of a delta function type. The curvature is entirely concentrated a $\rho = 0$, without a "tail" in the bulk. It can be measured from outside only by surrounding it, recognizing the deficit angle. The metric remains finite on the singular subspace at $\rho = 0$. This is the reason why it can be easily included into the manifold.

These properties are specific to the codimension-two case. A singularity with D_1 internal and $D_2 + 1$ external dimensions is in the case of highest symmetry described by two scale functions, $a(\rho)$ and $b(\rho)$, where ρ is the distance from the singularity. We show that these functions in general exhibit a generalized Kasner behavior in the vicinity of $\rho = 0$, which means that a and b are proportional to powers p_1 and p_2 of ρ, obeying the relation

$$D_1 p_1 + D_2 p_2 = D_1 p_1^2 + D_2 p_2^2 = 1. \tag{291}$$

If $D_2 = 1$, one of the solutions is $p_1 = 0$, $p_2 = 1$. This corresponds to the conical singularity. In all other cases, one of the exponents is negative, implying divergent metric components and curvature. The codim-2 case is therefore very special.

The asymptotic Kasner behavior near singularities is a universal property and does not depend on the topology or signature of the involved subspaces.

To summarize: We found that Kaluza-Klein theories provide a unified and highly predictive framework into which most of the structure of this world can be embedded. The modern braneworld scenarios lead to a new point of view concerning the singularities appearing in these theories. An understanding of the Cosmological Constant Problem and dark energy in this higher dimensional context requires a better understanding of the corresponding early universe cosmology. We think that we have pushed the frontier a little bit forward into that direction.

Appendix A: Symmetric Vacuum Singularities in Arbitrary Dimensions

In order to understand the very special properties of codimension-two branes, it is necessary to consider it in context with other highly symmetric singularities in higher dimensions. That is the purpose of this Appendix. It is shown that all maximally symmetric vacuum solutions are generalizations of the Kasner metric in the vicinity of the singularity. The Kasner solutions [57] of Einstein's equations describe an anisotropic vacuum cosmology and the metric is given by

$$ds^2 = -dt^2 + \sum t^{2p_i}(dx^i)^2, \tag{292}$$

$$\sum p_i = \sum p_i^2 = 1. \tag{293}$$

This is valid in arbitrary dimensions, but it is not the most general anisotropic vacuum cosmology. There are also the so-called Mixmaster solutions [58] with chaotic behavior in the vicinity of the singularity. For these solutions, an expansion of the metric in powers of t is not possible. Mixmaster-type singularities exist only if the metric contains at least three independent functions of time. Here we consider metrics which contain only two independent functions of one variable.

Our general ansatz is

$$ds^2 = \pm d\rho^2 + a^2(\rho)\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + b^2(\rho)\tilde{g}_{\alpha\beta}(y)dy^\alpha dy^\beta. \tag{294}$$

Here ρ is the generalization of the time coordinate in the Kasner solutions. If $g_{\rho\rho}$ were also a function of ρ, it could be made equal to ± 1 by a transformation $\rho \to \rho'(\rho)$. Let s be the sign of $g_{\rho\rho}$, $s = -1$ for ρ timelike and $s = +1$ for ρ spacelike. The metrics $\tilde{g}_{\mu\nu}$ and $\tilde{g}_{\alpha\beta}$ describe maximally symmetric spaces (or spacetimes) with dimensions D_1 and D_2 and with Ricci tensors

$$\tilde{R}_{\mu\nu} = \Lambda_1 \tilde{g}_{\mu\nu}, \qquad \tilde{R}_{\alpha\beta} = \Lambda_2 \tilde{g}_{\alpha\beta}. \tag{295}$$

As an example, our codimension-two branes would correspond to $D_1 = 4$ and $D_2 = 1$. The Schwarzschild solution would have $D_1 = 1$ (time) and $D_2 = 2$ (a sphere).

The full $D_1 + D_2 + 1$-dimensional Ricci tensor derived from the metric (294) is

$$\begin{aligned}
R_{\mu\nu} &= \tilde{g}_{\mu\nu}\left[\Lambda_1 - a^2 s\left((D_1-1)\frac{a'^2}{a^2} + D_2\frac{a'b'}{ab} + \frac{a''}{a}\right)\right], && (296)\\
R_{\alpha\beta} &= \tilde{g}_{\alpha\beta}\left[\Lambda_2 - b^2 s\left((D_2-1)\frac{b'^2}{b^2} + D_1\frac{a'b'}{ab} + \frac{b''}{b}\right)\right], && (297)\\
R_{\rho\rho} &= -D_1\frac{a''}{a} - D_2\frac{b''}{b}. && (298)
\end{aligned}$$

In vacuum, the Ricci tensor vanishes. This gives three equations, of which only two are independent due to the Bianchi identities. We are interested in solutions which are singular at $\rho = 0$. Furthermore we assume that a and b can be expanded in powers of ρ in the vicinity of the singularity,

$$a = c_1 \rho^{p_1} + \; ... \quad , \qquad b = c_2 \rho^{p_2} + \; ... \tag{299}$$

The exponents p_1 and p_2 need not be integer.

At first we look for the case $\Lambda_1 = \Lambda_2 = 0$. Plugging the ansatz (299) into the Ricci tensor, one obtains from $R_{\rho\rho} = 0$ the necessary condition

$$D_1 p_1^2 + D_2 p_2^2 = D_1 p_1 + D_2 p_2. \tag{300}$$

From the equations $R_{\mu\nu} = 0$ and $R_{\alpha\beta} = 0$ one gets either $p_1 = p_2 = 0$, in which we are not interested since this would not be singular (except for a possible codimension-one brane at $\rho = 0$, which cannot be detected from outside), or

$$D_1 p_1 + D_2 p_2 = 1. \tag{301}$$

Together these two conditions for p_1 and p_2 are just the Kasner conditions. There are always two solutions to (300) and (301), namely

$$p_1^{(\pm)} = \frac{1}{D_1(D_1 + D_2)}(D_1 \pm \sqrt{D_1 D_2 (D_1 + D_2 - 1)}), \tag{302}$$

$$p_2^{(\pm)} = \frac{1}{D_2(D_1 + D_2)}(D_2 \mp \sqrt{D_1 D_2 (D_1 + D_2 - 1)}). \tag{303}$$

These are the exponents which were already derived by Randjbar-Daemi and Wetterich [31] who considered generalizations to the Rubakov-Shaposhnikov solutions [29] in arbitrary dimensions. They appear also in Ruth Gregory's p-brane solutions [59] when one expands the metric around the singularities.

One of the exponents is always positive, the other negative, and one has always $0 < |p_{1,2}| < 1$. There is a single exception: If one of the dimensions, say D_2, is equal to one, then one of the solutions is $p_1 = 0$, $p_2 = 1$. This is just our well-known deficit angle solution, where a approaches a constant and b vanishes linearly. The arbitrariness of the deficit angle appears here due to the arbitrariness of the constant c_2 in the ansatz (299). It is the only solution which has brane character in the sense that there are finite metric components on the singularity. All the other solutions have only vanishing and divergent metric components at $\rho = 0$.

As a consistency check, one finds that the other solution (apart from $p_1 = 0$, $p_2 = 1$) with $D_1 = 4$ and $D_2 = 1$ is $p_1 = 2/5$, $p_2 = -3/5$, which we recognize as the exponents of the Rubakov-Shaposhnikov solutions.

We may call ρ and the dimensions corresponding to positive p "external" with respect to the singularity. The latter have angular character in the sense that they shrink to zero size at $\rho = 0$. The other dimensions may be called "internal" to the singularity.

As a next step, we include the curvature terms Λ_1 and Λ_2. One easily shows that this does not modify the character of the singularities of the solutions we have found so far.

Consider Λ_1 (the same applies to Λ_2). That term has to be cancelled by a constant term in the derivative part of the $R_{\mu\nu} = 0$ equation. Each of the terms in the derivative part diverges like $\rho^{2(p_1-1)}$ (see eq.(296)). The Kasner conditions were necessary to induce a cancellation between these terms. The addition of a constant is negligible compared to the divergent parts in the above solutions. It modifies only non leading order terms in the expansions of a and b. In the special case $p_1 = 0$, $p_2 = 1$, Λ_1 modifies the term a_2 in the expansion of a,

$$a^{(DAB)}(\rho) = a_0 + a_2\rho^2 + \ ... \tag{304}$$

(exponents between 0 and 2 do not appear in this case, since a' must vanish at least linearly at $\rho = 0$ in order to prevent $\frac{a'b'}{ab}$ from diverging in (296)). The case $p_1 = 1$, $p_2 = 0$ implies $D_1 = 1$ and therefore $\Lambda_1 = 0$, because a one-dimensional space has no curvature.

Hence the Kasner-type solutions still exist in the presence of the curvatures Λ_1 and Λ_2. Nevertheless there may be new additional solutions. One convinces oneself that the only new possibility is $p_1 = p_2 = 1$. This follows from the structure of the Ricci tensor. The possibility requires $D_1, D_2 \geq 2$ and has

$$\frac{\Lambda_1}{c_1^2} = \frac{\Lambda_2}{c_2^2} = s(D_1 + D_2 - 1). \tag{305}$$

The D-dimensional unit sphere S^D has $\Lambda = D - 1$. The line element of the new solutions is therefore

$$ds^2 = \pm d\rho^2 + \frac{D_1 - 1}{D_1 + D_2 - 1}\rho^2 d\Sigma_1^2 + \frac{D_2 - 1}{D_1 + D_2 - 1}\rho^2 d\Sigma_2^2. \tag{306}$$

Here $d\Sigma_1^2$ and $d\Sigma_2^2$ are the line elements of a D_1- and a D_2- dimensional unit sphere (or de Sitter spacetime, or the corresponding hyperbolic space if ρ is timelike). How are these solutions to be understood? Take, for simplicity, the case $s = +1$. If there were just one $(D_1 + D_2)$-dimensional sphere, the line element for $\Lambda = D_1 + D_2 - 1$ would simply be

$$\tilde{ds^2} = d\rho^2 + \rho^2 d\Sigma^2. \tag{307}$$

This is the metric in spherical coordinates for a $(D_1 + D_2 + 1)$-dimensional Euclidean space, which is of course a vacuum solution. At $\rho = 0$, there is only a coordinate singularity, not a physical one. In the solutions (306) there is instead a product of two spheres with the same radial coordinate ρ, but with a "wrong" radius to surface ratio. The Ricci tensor does not "see" the difference between the two line elements (306) and (307). But the full curvature tensor does, and there is a true singularity at the center of (306), as we will show now.

In order to distinguish between coordinate singularities and true singularities one has to consider the square of the Riemann tensor. For the metric (294) it is

$$\begin{aligned} R_{ABCD}R^{ABCD} \quad &= 2\tfrac{D_1}{D_1-1}\tfrac{\Lambda_1^2}{a^4} + 2\tfrac{D_2}{D_2-1}\tfrac{\Lambda_2^2}{b^4} + 2D_1(D_1-1)\tfrac{a'^4}{a^4} + 2D_2(D_2-1)\tfrac{b'^4}{b^4} \\ &+4D_1D_2\tfrac{a'^2b'^2}{a^2b^2} + 4D_1\tfrac{a''^2}{a^2} + 4D_2\tfrac{b''^2}{b^2} - 4s\left(D_1\Lambda_1\tfrac{a'^2}{a^4} + D_2\Lambda_2\tfrac{b'^2}{b^4}\right). \end{aligned} \tag{308}$$

For the Kasner-type solutions the dominant terms in the vicinity of $\rho = 0$ are those which do not contain Λ_1 or Λ_2. They all diverge with ρ^{-4} and are all non-negative, so they cannot cancel each other. So there is always a true singularity at $\rho = 0$. The only exception is again the case $p_1 = 0$, $p_2 = 1$ (or vice versa), for which there are no divergent terms at all in the square of the Riemann tensor. In this case there may be a pure coordinate singularity at $\rho = 0$, or a singularity of the delta function type, which can be detected from outside by surrounding it, but not from the curvature. This is what we called deficit angle branes.

For the non Kasner type $p_1 = p_2 = 1$ solutions, all terms, including those with $\Lambda_{1,2}$, diverge as ρ^{-4}, and one computes

$$R_{ABCD}R^{ABCD} = \frac{2}{\rho^4}\frac{D_1 D_2}{(D_1-1)(D_2-1)}(D_1+D_2-1)(D_1+D_2-2). \tag{309}$$

This always implies a true singularity at $\rho = 0$.

Inclusion of sources: How does the presence of sources like matter, radiation, magnetic flux or a cosmological constant modify the structure of the Kasner-type singularities? A source which remains finite at $\rho = 0$, like a cosmological constant, can of course only be relevant at large ρ. For a given model it may determine the global structure of possible solutions. But it does not generate any new types of singularities, and if a singularity is given at $\rho = 0$, it obviously cannot change or modify the divergence of curvature. Such a modification can only occur if the energy momentum tensor on the right hand side of Einstein's equations diverges as fast as the derivative terms on the left hand side. These derivative terms, like $(a'/a)^2$, diverge as ρ^{-2} in the Kasner-type solutions. On the other hand, the volume measure $\sqrt{g}$ of the constant ρ hypersurfaces is proportional to ρ, due to the Kasner condition $D_1p_1 + D_2p_2 = 1$. So one needs a source which diverges at least like $1/(\text{volume})^2$. This happens for example in the Reissner-Nordstroem black hole where an electric field changes the structure of the singularity. A second example is the magnetic flux in our six-dimensional model, which forbids the Rubakov-Shaposhnikov singularity (the energy would diverge too strong if a would go to zero) and changes it to a deficit angle brane or a pure coordinate singularity.

In many cases, the divergence of energy momentum is not strong enough to destroy the Kasner behavior. In the cosmological Kasner model, matter and radiation are irrelevant for the geometry at early times. But they are important at later times where they make the anisotropy disappear and lead to the late-time universe we observe today, expanding with the same rate in all directions. In those cases the Kasner singularities still exist as solutions, but new, additional types of singularities are also possible, such as the Big Bang singularity in Friedmann cosmology.

The Schwarzschild Black Hole from a Kasner point of view: The universality of the Kasner exponents, independent of signature and topology, can be impressively demonstrated in comparing our six-dimensional solutions to a Schwarzschild Black Hole,

$$ds^2_{(BH)} = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{310}$$

In the 6D case, $D_1 = 4$ corresponds to a four-dimensional constant curvature spacetime, while the Black Hole has $D_1 = 2$, corresponding to a sphere, parametrized by coordinates θ and ϕ. Both spacetimes have $D_2 = 1$. In the first case, D_2 corresponds to the spacelike coordinate θ, with topology S^1. In the second case, D_2 corresponds to the timelike coordinate t, with topology R. One has to express the black hole metric in terms of the coordinate $\rho(r)$ which has $g_{\rho\rho} = \pm 1$ (+1 outside, -1 inside the Schwarzschild horizon), and expand it around the two critical points $r = 0$ and $r = 2M$.

At $r = 0$, the t-dimension becomes infinitely large with $g_{tt} \sim \rho^{-1/3}$, while the sphere becomes infinitely small with $g_{\theta\theta} \sim \rho^{2/3}$. Therefore t is the internal coordinate of the singularity and the other dimensions are external. This corresponds to the Rubakov-Shaposhnikov singularity in six dimensions, where θ is internal.

Now turn to $r = 2M$. Here the sphere has a finite size, $g_{\theta\theta} = (2M)^2$. The t-dimension becomes infinitely small, with $g_{tt} \sim (\rho - \rho(2M))^2$. We have therefore $p_1 = 0$ and $p_2 = 1$. The Schwarzschild horizon corresponds to the deficit angle brane! Notice that the Schwarzschild horizon is really a codimension-two (and not one!) object, since time becomes an external dimension due to $g_{tt} \to 0$, like the angle θ in the corresponding six-dimensional solution. Of course we cannot speak of a deficit angle here, because of the different topology of time. If there were a delta function singularity at $r = 2M$, it could not be detected from outside, since it is not possible to surround it along a closed timelike (!) curve, which would be the procedure equivalent to surrounding the brane in the 6D model along the θ-direction.

Appendix B: Einstein Equations in the u-gauge

In the following, the Einstein tensor derived from the metric (218) is given. We use the abbreviation $q^2 \equiv u^2/(a^2 b^2)$.

$$
\begin{aligned}
G^t_t &= -\frac{1}{a^2(1+q^2)}\left(3\frac{\dot{a}^2}{a^2}+3\frac{\dot{a}\dot{b}}{ab}+3\frac{\dot{a}\dot{n}}{an}+\frac{\dot{b}\dot{n}}{bn}\right)+\frac{1}{n^2}\left(3\frac{a'^2}{a^2}-3\frac{a'n'}{an}+3\frac{a''}{a}\right) \qquad (311)\\
&+ \frac{1}{n^2(1+q^2)}\left(3\frac{a'b'}{ab}-\frac{b'n'}{bn}-\frac{n'u'q^2}{2nu}+\frac{b''}{b}+\frac{u''q^2}{2u}\right)\\
&+ \frac{q^2}{n^2(1+q^2)^2}\left(\frac{a'b'}{ab}+\frac{b'^2}{b^2}+\frac{a'u'}{au}(1+\frac{3}{2}q^2)-\frac{3b'u'q^2}{2bu}+\frac{u'^2}{4u^2}(1-q^2)\right)\\
G^{(i)}_{(i)} &= -\frac{1}{a^2(1+q^2)^2}\{-\frac{\dot{a}^2}{a^2}(1-q^2)+\frac{\dot{a}\dot{b}}{ab}(1+4q^2)+(\frac{\dot{a}}{a}+\frac{\dot{b}}{b})\frac{\dot{n}}{n}(1+2q^2) \qquad (312)\\
&+ \frac{\dot{b}^2q^2}{b^2}-\left(2\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{n}}{n}\right)\frac{\dot{u}}{u}q^2+\left(2\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{n}}{n}\right)(1+q^2)\}\\
&+ \frac{1}{n^2(1+q^2)^2}\{\frac{a'^2}{a^2}(3+5q^2+q^4)+\frac{a'b'}{ab}(3+5q^2)-\frac{a'n'}{an}(3+5q^2+2q^4)\\
&- \frac{b'n'}{bn}(1+q^2)+\frac{b'^2}{b^2}q^2+2\frac{a'u'}{au}q^2-2\frac{b'u'}{bu}q^2-\frac{n'u'}{nu}(q^2+q^4)\\
&+ \frac{u'^2}{4u^2}(3q^2-q^4)+\frac{a''}{a}(3+5q^2+2q^4)+\frac{b''}{b}(1+q^2)+\frac{u''}{u}(q^2+q^4)\}\\
G^\theta_\theta &= -\frac{1}{a^2(1+q^2)^2}\{3\frac{\dot{a}^2q^2}{a^2}+3\frac{\dot{a}\dot{b}q^2}{ab}+\frac{\dot{a}\dot{n}}{an}(2+3q^2)+\frac{\dot{b}\dot{n}q^2}{bn} \qquad (313)\\
&- \frac{\dot{a}\dot{u}q^2}{au}-\frac{\dot{n}\dot{u}q^2}{nu}+3\frac{\ddot{a}}{a}(1+q^2)+\frac{\ddot{n}}{n}(1+q^2)\}\\
&+ \frac{1}{n^2(1+q^2)^2}\{\frac{a'^2}{a^2}(6+10q^2+3q^4)+\frac{a'b'q^2}{ab}-\frac{a'n'}{an}(4+7q^2+3q^4)\\
&+ \frac{3a'u'}{2au}q^2-\frac{b'u'}{2bu}q^2-\frac{n'u'}{2nu}(q^2+q^4)+\frac{u'^2}{4u^2}(q^2-q^4)\\
&+ \frac{a''}{a}(4+7q^2+3q^4)+\frac{u''}{2u}(q^2+q^4)\}\\
G^\rho_\rho &= -\frac{1}{a^2(1+q^2)^2}\{3\frac{\dot{a}^2}{a^2}q^2+\frac{\dot{a}\dot{b}}{ab}(2+6q^2)+\frac{\dot{b}^2}{b^2}q^2-3\frac{\dot{a}\dot{u}}{au}q^2-\frac{\dot{b}\dot{u}}{bu}q^2 \qquad (314)\\
&+ 3\frac{\ddot{a}}{a}(1+q^2)+\frac{\ddot{b}}{b}(1+q^2)\}+\frac{1}{n^2(1+q^2)^2}\frac{a'^2}{a^2}(6+9q^2+3q^4)\\
&+ \frac{1}{n^2(1+q^2)}\left(4\frac{a'b'}{ab}+3\frac{a'u'}{au}q^2+\frac{u'^2}{4u^2}q^2\right)\\
G^t_\rho &= \frac{1}{a^2(1+q^2)}\left(3\frac{\dot{a}'}{a}+\frac{\dot{b}'}{b}-3\frac{\dot{n}a'}{na}-\frac{\dot{n}b'}{nb}\right) \qquad (315)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{a^2(1+q^2)^2}\left(-3\frac{\dot a a'}{a^2} - \frac{\dot b a'}{ba} + 3\frac{\dot a b'}{ab}q^2 + \frac{\dot b b'}{b^2}q^2 - 3\frac{\dot a u'}{au}q^2 - \frac{\dot b u'}{bu}q^2\right)\\
G^t_\theta &= \frac{u}{a^2n^2(1+q^2)}\left(\frac{b'n'}{bn} - \frac{n'u'}{2nu} - \frac{b''}{b} + \frac{u''}{2u}\right) \qquad (316)\\
&+ \quad \frac{u}{a^2n^2(1+q^2)^2}\left(-\frac{a'b'}{ab}(2+3q^2) - \frac{b'^2}{b^2}q^2 + \frac{a'u'}{au}(1+\frac{3}{2}q^2) - \frac{b'u'}{bu}(\frac{1}{2}-q^2) - \frac{u'^2}{2u^2}\right)\\
G^\theta_\rho &= \frac{u}{a^2b^2(1+q^2)}\left(3\frac{\dot n a'}{na} + \frac{\dot n u'}{2nu} - 3\frac{\dot a'}{a} - \frac{\dot u'}{2u}\right) \qquad (317)\\
&= \frac{u}{a^2b^2(1+q^2)^2}\left(3\frac{\dot a a'}{a^2} + \frac{\dot b a'}{ba} + 2\frac{\dot a b'}{ab} + \frac{\dot u b'}{ub} + \frac{\dot a u'}{au}(1+\frac{3}{2}q^2) - \frac{\dot b u'}{2bu} + \frac{\dot u u'}{2u^2}\right).
\end{aligned}
$$

For the gauge field we choose the gauge $A_\rho = 0$. The corresponding energy momentum tensor

$$
T^A_B = F^{AC}F_{BC} - \frac{1}{4}F^{CD}F_{CD}\delta^A_B \qquad (318)
$$

is then

$$
\begin{aligned}
T^t_t &= \frac{1}{2(1+q^2)}\left(-\frac{\dot A_\theta^2}{a^2b^2} - \frac{A_t'^2}{a^2n^2} - \frac{A_\theta'^2}{n^2b^2}\right) && (319)\\
T^{(i)}_{(i)} &= \frac{1}{2(1+q^2)}\left(\frac{\dot A_\theta^2}{a^2b^2} + \frac{A_t'^2}{a^2n^2} - \frac{A_\theta'^2}{n^2b^2} - 2u\frac{A_t'A_\theta'}{a^2b^2n^2}\right) && (320)\\
T^\theta_\theta &= \frac{1}{2(1+q^2)}\left(-\frac{\dot A_\theta^2}{a^2b^2} + \frac{A_t'^2}{a^2n^2} + \frac{A_\theta'^2}{n^2b^2}\right) && (321)\\
T^\rho_\rho &= \frac{1}{2(1+q^2)}\left(\frac{\dot A_\theta^2}{a^2b^2} - \frac{A_t'^2}{a^2n^2} + \frac{A_\theta'^2}{n^2b^2} + 2u\frac{A_t'A_\theta'}{a^2b^2n^2}\right) && (322)\\
T^t_\rho &= -\frac{\dot A_\theta A_\theta'}{a^2b^2(1+q^2)} && (323)\\
T^t_\theta &= -\frac{A_t'A_\theta'}{a^2n^2(1+q^2)} + u\frac{A_\theta'^2}{a^2b^2n^2(1+q^2)} && (324)\\
T^\theta_\rho &= \frac{\dot A_\theta A_t'}{a^2b^2(1+q^2)}. && (325)
\end{aligned}
$$

Finally we give the field equations for the gauge field. They are

$$
\begin{aligned}
\partial_A(\sqrt{-g}F^{A\theta}) &\equiv \partial_\rho\left(\frac{a^4}{nb\sqrt{1+q^2}}A_\theta' + \frac{ua^2}{nb\sqrt{1+q^2}}A_t'\right) - \partial_t\left(\frac{a^2n}{b\sqrt{1+q^2}}\dot A_\theta\right) = 0 && (326)\\
\partial_A(\sqrt{-g}F^{At}) &\equiv \partial_\rho\left(-\frac{a^2b}{n\sqrt{1+q^2}}A_t' + \frac{ua^2}{nb\sqrt{1+q^2}}A_\theta'\right) = 0 && (327)\\
\partial_A(\sqrt{-g}F^{A\rho}) &\equiv \partial_t\left(\frac{a^2b}{n\sqrt{1+q^2}}A_t' - \frac{ua^2}{nb\sqrt{1+q^2}}A_\theta'\right) = 0. && (328)
\end{aligned}
$$

The first of the three equations determines the time evolution of A_θ. The other two equations relate A_t to A_θ. They imply that

$$\left(\frac{a^2 b}{n\sqrt{1+q^2}}A_t' - \frac{ua^2}{nb\sqrt{1+q^2}}A_\theta'\right) = const \tag{329}$$

in the entire six-dimensional spacetime.

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