THE INSTATIONARY MOTION OF A NAVIER-STOKES FLUID THROUGH A VESSEL WITH AN ELASTIC COVER

C. SURULESCU

ABSTRACT. We study here the time-dependent movement of a fluid through a vessel having an elastic cover and inflow and outflow sections, the rest of the boundary being rigid and fixed. The two media interact with each other. The fluid domain is moving in time. For the elastic structure we use plate equations and in order to describe the behavior of the fluid we consider Navier-Stokes equations with prescribed pressures at the inflow and at the outflow sides of the vessel. These are nonstandard boundary conditions. We prove the existence of a solution for the coupled problem.

1. INTRODUCTION

In this paper we consider a time-dependent 3D/2D fluid-elastic structure interaction problem. It can be formulated in the following way: a viscous, incompressible fluid flows through a vessel having an elastic plate as cover, one rigid, fixed bottom, two rigid, fixed opposite walls and the other two opposite walls are the inflow, respectively the outflow boundaries. The fluid domain is moving in time and is thus unknown: its shape depends on the displacement of the flexible plate, which in turn depends on the fluid stress acting on the elastic cover of the vessel. The mechanics of the two media are coupled through the position of the interface and the surface traction on the flexible part, which comes from the fluid. For the fluid we consider time-dependent Navier-Stokes equations with pressures prescribed on the inflow and outflow parts of the fluid boundary. These are nonstandard boundary conditions for a fluid flow. The elastic structure is viewed as a thin plate, which is clamped on its entire boundary. This problem can be viewed for instance as a rudimentary model for the study of blood flow

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in arteries. We show existence of a solution for the coupled system, by firstly solving a linearized approximate problem (with the method of Galerkin), then using Schauder's generalized fixed point theorem to show the existence of a solution for the approximate problem and eventually passing to the limit. The method follows the one in [CDEG02], where a similar problem has been studied, however in a different setting, since in that paper all boundaries of the fluid domain were rigid and fixed, excepting the cover. Our choice of boundary conditions for the fluid will lead to some difficulties in the proof; these will be overcome mainly by the right choice of the function spaces to be used and due to condition (17) below.

Flori & Orenga [FlOr98] studied the interaction of a compressible 3D fluid (Dirichlet boundary conditions on the rest of the boundary) with a thin plate. Other time-dependent fluid-structure interaction problems with time moving domains were considered for instance by Errate, Esteban and Maday [EEM94] (1D fluid, 1D structure), Litvinov [Litv96], Prouse [Prou71] and Beirao da Veiga [BdV01] (2D fluid, 1D structure) or by Desjardins, Esteban et al. [DEGL00] for the 3D case of a fluid interacting with an elastic structure having a finite number of elastic modes. Rigid bodies interacting with a fluid are studied for instance by Desjardins and Esteban [DeEs99a] and [DeEs99b] or Takahashi [Taka03]. For the time-dependent flow of a Stokes fluid through an elastic tube (small displacements, cylindrical domains) we refer e.g., to [ČaMi03] (effective 2D/1D problem) and to [Suru05] (3D/3D). For stationary 3D fluid/3D elastic structure interaction problems we refer to [Gran00], [Suru04].

2. Setting of the problem

Consider a viscous incompressible fluid which fills a vessel of length L, width l and high 1. The vessel has an inflow and an outflow boundary and as a cover an elastic plate of small thickness, fixed along its boundary. At the reference state the elastic plate occupies the domain $\Omega_s \times \{1\}$ and the fluid occupies at the initial state a domain

$$\Omega_{\eta_0} := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \ (x_1, x_2) \in \Omega_s, \ 0 < x_3 < 1 + \eta_0(x_1, x_2)\},$$

where $\Omega_s = (0, L) \times (0, l)^\dagger$ and η_0 is a given initial displacement of the elastic part.

†actually, Ω_s could be any Lipschitz domain of $\mathbf{R}^2$, but for the sake of clarity we took it here in this form

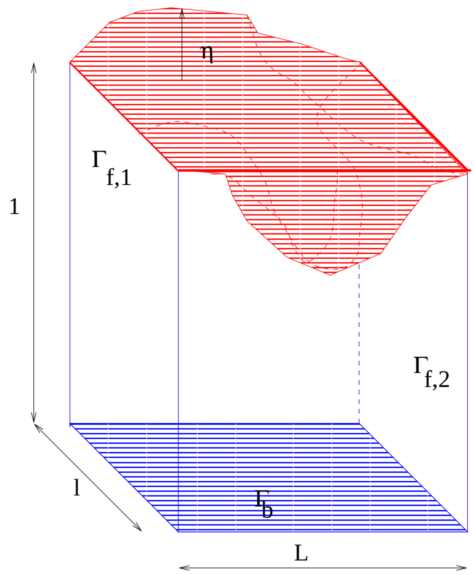

FIGURE 1. The box and the deformation of the elastic cover

We consider the longitudinal displacements of the elastic plate negligible. Thus we have for the elastic structure a 2D problem; the only relevant displacement is the transversal one and the corresponding equations for the motion of the elastic plate are then:

$$\partial_{tt}\eta + \Delta^2\eta + \gamma\Delta^2\partial_t\eta = g + G \text{ in } (0,T)\times\Omega_s \tag{1}$$

$$\eta = \partial_n\eta = 0 \text{ on } (0,T)\times\partial\Omega_s \tag{2}$$

$$\eta(0) = \eta_0,\ \partial_t\eta(0) = \eta_{01}, \tag{3}$$

where g is the exterior force applied to the elastic plate, η is the transversal displacement of the plate and $G := (F_f)_3$, where $\mathbf{F}_f$ is the surfacic force applied by the fluid on the structure.

As in [BdV01] or [CDEG02], a viscoelastic term $\gamma\Delta^2\partial_t\eta$ $(\gamma > 0)$ has been added to the usual equations of an elastic plate, in order to ensure the smoothness of the structure's velocity.

The domain occupied by the fluid at time t is:

$$\Omega_\eta(t) := \{(x_1,x_2,x_3)\in\mathbf{R}^3 \ :\ (x_1,x_2)\in\Omega_s,\ 0 < x_3 < 1+\eta(t,x_1,x_2)\}$$

and the equations for the fluid are:

$$\begin{aligned}
(4)\quad \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \text{ in } Q_{\eta,T} \\
(5)\quad \operatorname{div}\mathbf{u} &= 0 \text{ in } Q_{\eta,T} \\
(6)\quad \mathbf{u}\times\mathbf{n} &= 0 \text{ on } (0,T)\times\Gamma_f \\
(7)\quad \mathbf{u} &= 0 \text{ on } (0,T)\times(\Gamma_b\cup\Gamma_{sides}) \\
(8)\quad p &= p_{0i} \text{ on } (0,T)\times\Gamma_{f,i}\ (i=1,2) \\
(9)\quad \mathbf{u}(0) &= \mathbf{u}_0 \text{ in } \Omega_{\eta_0},
\end{aligned}$$

where $Q_{\eta,T}\subset\mathbf{R}^4$ is defined as

$$Q_{\eta,T} := \bigcup_{0<t<T}\{t\}\times\Omega_\eta(t),$$

$\mathbf{u}$ is the fluid velocity, ν is the kinematic viscosity of the fluid, p is the pressure, $\mathbf{f}$ the density of the external forces, p_{01} and p_{02} are the known pressures at the inflow and outflow ends and $\mathbf{u}_0$ is the initial velocity. $\Gamma_f = \Gamma_{f1}\cup\Gamma_{f2}$ represent the inflow, respectively the outflow boundaries (which we suppose to stay fixed), Γ_b stands for the rigid bottom of the box and Γ_{sides} represent the other two opposite boundaries of the fluid domain.

The adherence of the viscous fluid to the interface is expressed in the following equality of velocities:

$$(10)\quad \mathbf{u}(t,x_1,x_2,1+\eta(t,x_1,x_2)) = (0,0,\partial_t\eta(t,x_1,x_2)),\ (x_1,x_2)\in\Omega_s.$$

The fluid incompressibility (5) and the boundary conditions (6) and (7), toghether with (10) lead to the following compatibility condition:

$$(11)\qquad \int_{\Omega_s}\partial_t\eta - \int_{\Gamma_{f1}} u_1 + \int_{\Gamma_{f2}} u_1 = 0.$$

The surface force exerted by the fluid on the elastic wall is defined by:

$$(12)\quad \int_{\Omega_s}\mathbf{F}_f\cdot\tilde{\mathbf{v}} = \int_{\partial\Omega_\eta(t)-(\Gamma_f\cup\Gamma_b\cup\Gamma_{sides})}(-\nu(\nabla\times\mathbf{u})\times\mathbf{n}_t + p\cdot\mathbf{n}_t)\cdot\mathbf{v},\ \forall\mathbf{v},$$

where $\tilde{\mathbf{v}}(t,x_1,x_2) = \mathbf{v}(t,x_1,x_2,1+\eta(t,x_1,x_2))$, $\forall(x_1,x_2)\in\Omega_s$, $\mathbf{n}_t$ is the unit outer normal at $\Gamma_1(t) := \partial\Omega_\eta(t) - (\Gamma_f\cup\Gamma_b\cup\Gamma_{sides})$. Observe that (with an obvious notation) $d\Gamma_1(t) = \sqrt{1+(\partial_{x_1}\eta)^2+(\partial_{x_2}\eta)^2}dx_1dx_2$.

3. A priori estimates

In this section we make some a priori estimates, in order to motivate in a natural (and heuristical) way the introduction of the function spaces we will use in the definition of the weak solution, in the formulation of the main theorem of this section and, of course, in the proof of the result.

Testing the Navier-Stokes equations by $\mathbf{u}$, we obtain:

$$\int_{\Omega_\eta(t)} \partial_t \mathbf{u}\cdot\mathbf{u} + \int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} + \nu\int_{\Omega_\eta(t)} |\nabla\times\mathbf{u}|^2 = \int_{\Omega_\eta(t)} \mathbf{f}\cdot\mathbf{u}$$

$$+\int_{\partial\Omega_\eta(t)-(\Gamma_f\cup\Gamma_b\cup\Gamma_{sides})} [\nu(\nabla\times\mathbf{u})\times\mathbf{n}_t - p\cdot\mathbf{n}_t]\cdot\mathbf{u} - \int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\mathbf{u}.$$

Now using Reynold's transport formula, we observe that

$$\int_{\Omega_\eta(t)} [\partial_t\mathbf{u}\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}] = \frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)} |\mathbf{u}|^2.$$

Thus, for the fluid equations we get the following equality:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)} |\mathbf{u}|^2 + \nu\int_{\Omega_\eta(t)} |\nabla\times\mathbf{u}|^2 \tag{13}$$

$$+\int_{\partial\Omega_\eta(t)-(\Gamma_b\cup\Gamma_f\cup\Gamma_{sides})} [-\nu(\nabla\times\mathbf{u})\times\mathbf{n}_t + p\cdot\mathbf{n}_t]\cdot\mathbf{u} = \int_{\Omega_\eta(t)} \mathbf{f}\cdot\mathbf{u} - \int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\mathbf{u}.$$

Now test the equations of the elastic structure (with the supplementary term) with $\partial_t\eta$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_s} |\partial_t\eta|^2 + \frac{1}{2}\frac{d}{dt}\int_{\Omega_s} |\Delta\eta|^2 + \gamma\int_{\Omega_s} |\Delta\partial_t\eta|^2 \tag{14}$$

$$= \int_{\Omega_s} g\cdot\partial_t\eta + \int_{\Omega_s} G\cdot\partial_t\eta.$$

Adding (13) and (14) while keeping in mind (12), the definition of G and (10), we obtain the following energy equality:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)} |\mathbf{u}|^2 + \nu\int_{\Omega_\eta(t)} |\nabla\times\mathbf{u}|^2 + \frac{1}{2}\frac{d}{dt}\int_{\Omega_s} |\partial_t\eta|^2 + \frac{1}{2}\frac{d}{dt}\int_{\Omega_s} |\Delta\eta|^2$$

$$+\gamma\int_{\Omega_s} |\Delta\partial_t\eta|^2 = \int_{\Omega_\eta(t)} \mathbf{f}\cdot\mathbf{u} + \int_{\Omega_s} g\cdot\partial_t\eta - \int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\mathbf{u}. \tag{15}$$

The inequalities of Hölder and Young lead to:

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}|^2_{\mathbf{L}^2(\Omega_\eta(t))} + \nu|\nabla\times\mathbf{u}|^2_{\mathbf{L}^2(\Omega_\eta(t))} + \frac{1}{2}\frac{d}{dt}|\partial_t\eta|^2_{L^2(\Omega_s)} + \frac{1}{2}\frac{d}{dt}|\Delta\eta|^2_{L^2(\Omega_s)}$$

$$+\gamma||\partial_t\eta||^2_{H^2(\Omega_s)} \leq \frac{1}{2}|\mathbf{f}|^2_{\mathbf{L}^2(\Omega_\eta(t))} + \frac{1}{2}|\mathbf{u}|^2_{\mathbf{L}^2(\Omega_\eta(t))}$$

$$+\frac{1}{2}|g|^2_{L^2(\Omega_s)} + \frac{1}{2}|\partial_t\eta|^2_{L^2(\Omega_s)} + \frac{c_0}{2}|p_0|^2_{L^2(\Gamma_f)} + \frac{1}{2c_0}|\mathbf{u}|^2_{\mathbf{L}^2(\Gamma_f)}.$$

Now define

$$\tilde{\mathbf{u}} := \begin{cases} \mathbf{u} & \text{in } \Omega_\eta(t) \\ (0,0,\partial_t \eta) & \text{in } \mathcal{B}_K - \Omega_\eta(t) \end{cases},$$

where $\mathcal{B}_K = \Omega_s \times (0,K)$, $(K \geq 1+\eta(t,x_1,x_2)\ \forall (x_1,x_2) \in \Omega_s, \forall t \in [0,T])$ is a Lipschitz domain such that $\Omega_\eta(t) \subset \mathcal{B}_K$, $\forall t \in [0,T]$.

Then use condition (17) for the curl of the fluid's velocity in the next section to obtain the following inequality:

$$\frac{1}{2}\frac{d}{dt}|\mathbf{u}|^2_{\mathbf{L}^2(\Omega_\eta(t))} + \Big(\nu - \frac{c_\Gamma}{2c_0c_E}\Big)|\nabla \times \mathbf{u}|^2_{\mathbf{L}^2(\Omega_\eta(t))} + \frac{1}{2}\frac{d}{dt}|\partial_t\eta|^2_{L^2(\Omega_s)}$$

$$+\frac{1}{2}\frac{d}{dt}|\Delta\eta|^2_{L^2(\Omega_s)} + (\gamma - \frac{c_\Gamma c_1}{2c_0c_E})||\partial_t\eta||^2_{H^2(\Omega_s)} \leq \frac{1}{2}|\mathbf{f}|^2_{\mathbf{L}^2(\Omega_\eta(t))} + \frac{1}{2}|\mathbf{u}|^2_{\mathbf{L}^2(\Omega_\eta(t))}$$

$$+\frac{1}{2}|g|^2_{L^2(\Omega_s)} + \frac{1}{2}|\partial_t\eta|^2_{L^2(\Omega_s)} + \frac{c_0}{2}|p_0|^2_{L^2(\Gamma_f)},$$

where the constant c_Γ above is the constant in the inequality

$$|\mathbf{u}|^2_{\mathbf{L}^2(\Gamma_f)} \leq c_\Gamma|\tilde{\mathbf{u}}|^2_{\mathbf{H}^1(\mathcal{B}_K)}.$$

We assume $\nu \geq \frac{c_\Gamma}{2c_0c_E}$ and $\gamma \geq \frac{c_\Gamma c_1}{2c_0c_E}$. Then Gronwall's inequality leads to:

$$(16)\qquad \frac{1}{2}|\mathbf{u}(t,.)|^2_{\mathbf{L}^2(\Omega_\eta(t))} + \Big(\nu - \frac{c_\Gamma}{2c_0c_E}\Big)\int_0^t |\nabla \times \mathbf{u}|^2_{\mathbf{L}^2(\Omega_\eta(s))}ds$$

$$+\frac{1}{2}|\partial_t\eta(t,.)|^2_{L^2(\Omega_s)} + \frac{1}{2}|\Delta\eta|^2_{L^2(\Omega_s)} + (\gamma - \frac{c_\Gamma c_1}{2c_0c_E})\int_0^t ||\partial_t\eta(s,.)||^2_{H^2(\Omega_s)}ds$$

$$\leq \frac{e^t}{2}\left(|\mathbf{u}_0|^2_{\mathbf{L}^2(\Omega_{\eta_0})} + |\eta_{01}|^2_{L^2(\Omega_s)} + |\Delta\eta_0|^2_{L^2(\Omega_s)}\right)$$

$$+\frac{1}{2}\int_0^t e^{t-s}\left(|\mathbf{f}(s,.)|^2_{\mathbf{L}^2(\Omega_\eta(s))} + |g(s,.)|^2_{L^2(\Omega_s)} + c_0|p_0|^2_{L^2(\Gamma_f)}\right)ds.$$

Thus, we have proved the following proposition:

Proposition 3.1. *Assume that the data of the problem satisfy* $\mathbf{u}_0 \in \mathbf{L}^2(\Omega_{\eta_0})$, $\eta_{01} \in L^2(\Omega_s)$, $\eta_0 \in H^2_0(\Omega_s)$, $\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))$, $p_0 \in L^2(0,T;L^2(\Gamma_f))$ *and* $g \in L^2(0,T;L^2(\Omega_s))$. *Then the estimate (16) holds, so that one gets*

$$\mathbf{u} \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\eta(t))) \cap \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\eta(t)))$$

and

$$\eta \in W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H^2_0(\Omega_s)) \cap L^\infty(0,T;H^2_0(\Omega_s)).$$

Indeed, by the estimates (16) it follows that $\mathbf{u} \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\eta(t)))$ and $\nabla \times \mathbf{u} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_\eta(t)))$. We remark that we have $\nabla \times \tilde{\mathbf{u}} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_K))$, thus (via (17)) $\tilde{\mathbf{u}} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_K))$ and therefore $\mathbf{u} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\eta(t)))$.

So, function spaces of the form $\mathbf{L}^q(0,T;\mathbf{L}^r(\Omega_\delta(t)))$, $\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t)))$ have to be used, where $\Omega_\delta(t)$ has an analogous definition with $\Omega_\eta(t)$ above, with $\delta \in W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H_0^2(\Omega_s))$, so $\delta \in C^{0,1/2}(0,T;C^{0,q}(\bar{\Omega}_s))$, with $q \in [0,1)$. Observe that $\Omega_\delta(t)$ is an open set, but it is not necessarily Lipschitz, thus one has to take care when defining the functional spaces in $\Omega_\delta(t)$.

Remark 3.1. *(concerning the fluid equations)*

Notice that for a time dependent domain the presence of the convection term in the fluid's equations is essential for the stability conditions in the norms above to hold. Indeed, otherwise we couldn't simplify the cubic term in the transport formula of Reynolds and we couldn't majorize it either, since it doesn't have a precised sign. The problem with Stokes equations describing the fluid's behavior instead of the Navier-Stokes ones risks thus to be badly posed in a time-dependent domain (see also [EEM94]*).*

This kind of fluid problem (with boundary conditions involving the pressure) studied here has also been studied for the stationary case in [CMP94] *and in* [Bern00] *and for the time dependent case (with time moving domains) in* [ABC99] *and* [Luka97]. *In the latter ones the boundary condition (8) contains supplementary the term* $\frac{1}{2}|\mathbf{u}|^2$, *but the estimates obtained were only for the elliptic regularization of the penalized problem, with supplementary assumptions. The pressure boundary conditions used here for the fluid problem have also been considered in* [JäMi98] *for cylindrical domains; however, there the focus is on another topic, namely to analyze the motion of a viscous, incompressible fluid through a filter with finite thickness, but the fluid equations therein are the same as in this section, modulo, of course, the coupling conditions.*

$\diamondsuit$

4. Function spaces, trace operators, properties and assumptions

Let $T > 0$ and $\delta \in H^1(0,T;H_0^2(\Omega_s))$ $(\hookrightarrow C(0,T;H_0^1(\Omega_s) \cap C(\bar{\Omega}_s)))$ and, for $\alpha > 0$, $K > 0$, $K \geq 1 + \delta(t,x_1,x_2) \geq \alpha > 0$, $\forall\ (t,x_1,x_2) \in [0,T] \times \bar{\Omega}_s$ and $\delta = 0$ on $\partial\Omega_s$.

For every $t \in [0,T]$, $\Omega_\delta(t) := \{(x_1,x_2,x_3) \in \mathbf{R}^3 \ : \ (x_1,x_2) \in \Omega_s, \ 0 < x_3 < 1+\delta(t,x_1,x_2)\}$ is an open subset of $\mathbf{R}^3$; it is included in $\mathcal{B}_K := \Omega_s \times (0,K)$.

We define $Q_{\delta,T} \subset \mathbf{R}^4$ by

$$Q_{\delta,T} := \bigcup_{0<t<T} \{t\} \times \Omega_\delta(t)$$

and $\mathcal{B}_{K,T} := (0,T) \times \mathcal{B}_K$. The corresponding function spaces $L^q(\Omega_\delta(t))$, $H^1(\Omega_\delta(t))$ (for every t), $L^q(Q_{\delta,T})$, $H^1(Q_{\delta,T})$, $L^q(\mathcal{B}_{K,T})$, $H^1(\mathcal{B}_{K,T})$ etc. can be defined as usual.

We also define (in a natural way):

$$\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t))) := \{\mathbf{v} \in \mathbf{L}^2(Q_{\delta,T}) \ : \ \nabla \mathbf{v} \in \mathbf{L}^2(Q_{\delta,T})\},$$

$$\mathbf{L}^2(0,T;\mathbf{H}^1_0(\Omega_\delta(t))) := \overline{\mathcal{D}(Q_{\delta,T})}^{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t)))},$$

$$\mathcal{V}_\delta := \{\mathbf{v} \in \mathbf{C}^1(\bar{Q}_{\delta,T}) \ : \ \text{div } \mathbf{v} = 0, \ \mathbf{v} = 0 \text{ on } (0,T) \times (\Gamma_b \cup \Gamma_{sides}), \\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } (0,T) \times \Gamma_f\},$$

$$\mathbf{V}_\delta := \overline{\mathcal{V}_\delta}^{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t)))},$$

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_K)) \ : \ \text{div } \mathbf{v} = 0, \ \mathbf{v} = 0 \text{ on } (0,T) \times (\Gamma_b \cup \Gamma_{sides}), \\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } (0,T) \times \Gamma_f\}$$

and

$$\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\delta(t))) := \{\mathbf{v} \in \mathbf{L}^2(Q_{\delta,T}) \ : \ \text{sup ess}_{0<t<T} ||\mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))} < \infty\}.$$

Observe that for $\mathbf{V}_\delta$ we can give the following characterization:

$$\mathbf{V}_\delta = \{\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t))) \ : \ \text{div } \mathbf{v} = 0, \ \mathbf{v} = 0 \text{ on } (0,T) \times (\Gamma_b \cup \Gamma_{sides}), \\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } (0,T) \times \Gamma_f\}.$$

The following ellipticity condition (also encountered in [ABC99] or [Luka97], as a hypothesis) can be justified like in [CMP94] (Lemma 1.4) or in [GiRa86] (Theorem 3.9), observing that the divergence of the velocity at the interface stays zero (and extending the velocity in $\mathcal{B}_K - \Omega_\delta(t)$ by its value on the interface between the two media):

There exists a constant $c_E > 0$ such that $\forall \mathbf{u} \in \mathbf{V}$

$$|\nabla \times \mathbf{u}(t)|^2_{\mathbf{L}^2(\mathcal{B}_K)} \geq c_E ||\mathbf{u}(t)||^2_{\mathbf{H}^1(\mathcal{B}_K)}. \tag{17}$$

Let us now see how does the trace on $\partial\Omega_\delta(t) - (\Gamma_b \cup \Gamma_{sides} \cup \Gamma_f)$ make sense. Consider the mapping $\mathbf{C}^0(\overline{\Omega_\delta}(t)) \ni \mathbf{v} \overset{\gamma_{\delta(t)}}{\mapsto} \mathbf{v}(t,x_1,x_2,1+\delta(t,x_1,x_2))$.

Lemma 4.1. *For every $t \in [0,T]$, the mapping $\gamma_{\delta(t)} : \mathbf{C}^1(\overline{\mathcal{B}}_K)$ (respectively $\mathbf{C}^1(\bar{\Omega}_\delta(t))$) $\to \mathbf{C}^0(\bar{\Omega}_s)$ can be extended by continuity to a mapping from $\mathbf{H}^1(\mathcal{B}_K)$ (respectively $\mathbf{H}^1(\Omega_\delta(t))$) into $\mathbf{L}^2(\Omega_s)$.*

Proof. The proof is analogous to the one for Lemma 1 in [CDEG02]. □

Remark 4.1. *By the above lemma it also follows that for* $\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t)))$ *one has* $\gamma_{\delta(t)}(\mathbf{v}) \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_s))$.

◊

From now on, until the end of this subsection, we will assume that $\delta \in H^1(0,T;H_0^2(\Omega_s))$.

Lemma 4.2. *For every* $\psi \in H_0^1(\Omega_s)$ *there exists* $\mathbf{w} \in \mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}\cup\Gamma_f}(\Omega_\delta(t))$ *such that* $\gamma_{\delta(t)}(\mathbf{w}) = \psi$ *and* $||\mathbf{w}||_{\mathbf{H}^1(\Omega_\delta(t))} \leq c_\alpha ||\psi||_{H^1(\Omega_s)}$.

Proof. The proof is analogous to the one in [CDEG02], while taking care at the in- and outflow boundaries when defining $\mathbf{w}$. Since the proof does not differ essentially from the one in [CDEG02], we don't write it here. □

The following lemma gives a weak sense to the tangential trace on the time moving boundary:

Lemma 4.3. *For every* $t \in [0,T]$, *there exists a linear continuous operator* $\gamma^{tg}_{\delta(t)} : \mathbf{H}(\mathbf{curl},\Omega_\delta(t)) \to \mathbf{H}^{-1}(\Omega_s)$ *such that*

$$\gamma^{tg}_{\delta(t)}(\mathbf{v}) = \mathbf{v}(t,x_1,x_2,1+\delta(t,x_1,x_2)) \times \mathbf{n}_t,\ \forall (x_1,x_2) \in \Omega_s,$$

for all $\mathbf{v} \in \mathbf{C}^\infty(\bar{\Omega}_\delta(t))$, *where*

$$\mathbf{H}(\mathbf{curl},\Omega_\delta(t)) := \{\mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t)) : \mathbf{curl}\ \mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t))\}.$$

Proof. For the proof we use Lemma 4.2 and the following Green formula (see [GiRa86]):

$$< \mathbf{v} \times \mathbf{n}_t, \gamma_{\delta(t)}(\mathbf{w}) >_{-1,1;\Omega_s} = \int_{\Omega_\delta(t)} (\nabla \times \mathbf{v}) \cdot \mathbf{w} - \int_{\Omega_\delta(t)} \mathbf{v} \cdot (\nabla \times \mathbf{w});$$

$$\begin{aligned} &\left|\int_{\Omega_\delta(t)} (\nabla \times \mathbf{v}) \cdot \mathbf{w} - \int_{\Omega_\delta(t)} \mathbf{v} \cdot (\nabla \times \mathbf{w})\right| \leq \\ &\qquad \leq (||\mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))} + ||\nabla \times \mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))})||\mathbf{w}||_{\mathbf{H}^1(\Omega_\delta(t))}. \end{aligned}$$

□

A similar result can be stated for the normal trace on the time moving boundary:

Lemma 4.4. *For every* $t \in [0,T]$, *there exists a linear continuous operator* $\gamma^n_{\delta(t)} : \mathbf{H}(div,\Omega_\delta(t)) \to H^{-1}(\Omega_s)$ *such that*

$$\gamma^n_{\delta(t)}(\mathbf{v}) = \mathbf{v}(t,x_1,x_2,1+\delta(t,x_1,x_2)) \cdot \mathbf{n}_t,\ \forall (x_1,x_2) \in \Omega_s,$$

for all $\mathbf{v} \in \mathbf{C}^\infty(\bar{\Omega}_\delta(t))$, *where*

$$\mathbf{H}(div,\Omega_\delta(t)) := \{\mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t)) :\ div\ \mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t))\}.$$

Proof. For the proof we use Lemma 4.2 and the following Green formula (see again e.g., [GiRa86]):

$$< \mathbf{v}\cdot\mathbf{n}_t, \gamma_{\delta(t)}(\mathbf{w}) >_{-1,1;\Omega_s} = \int_{\Omega_\delta(t)} (\text{div } \mathbf{v})\cdot\mathbf{w} + \int_{\Omega_\delta(t)} \mathbf{v}\cdot(\nabla\mathbf{w});$$

$$|\int_{\Omega_\delta(t)} (\text{div } \mathbf{v})\cdot\mathbf{w} + \int_{\Omega_\delta(t)} \mathbf{v}\cdot(\nabla\mathbf{w})| \leq$$

$$\leq (||\mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))} + ||\text{div } \mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))})||\mathbf{w}||_{\mathbf{H}^1(\Omega_\delta(t))}.$$

□

The proof of the main result of this Section (Theorem 5.1) makes in its last part use of the following lemma:

Lemma 4.5.

$$\{\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}\cup\Gamma_1(t)}(\Omega_\delta(t)) \ : \ \mathbf{v}\times\mathbf{n} = 0 \ on \ \Gamma_f\}$$

$$= \{\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}}(\Omega_\delta(t)) \ : \ \gamma_{\delta(t)}(\mathbf{v}) = 0, \ \mathbf{v}\times\mathbf{n} = 0 \ on \ \Gamma_f\}.$$

Proof. "$\subseteq$": Let $\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}\cup\Gamma_1(t)}(\Omega_\delta(t))$ with $\mathbf{v}\times\mathbf{n} = 0$ on Γ_f. There are functions $\mathbf{v}_\epsilon$ in $\mathbf{H}^1(\Omega_\delta(t))$ with supp $\mathbf{v}_\epsilon$ compact such that $\mathbf{v}_\epsilon\to\mathbf{v}$ for $\epsilon\to 0$ and such that $\gamma_{\delta(t)}(\mathbf{v}_\epsilon) = 0$. Since (by Lemma 4.1) the mapping $\gamma_{\delta(t)} : \mathbf{H}^1(\Omega_\delta(t))\to\mathbf{L}^2(\Omega_s)$ is continuous, it follows that $\gamma_{\delta(t)}(\mathbf{v}) = 0$. $\mathbf{v}\times\mathbf{n} = 0$ on Γ_f is obviously satisfied, too.

"$\supseteq$": Let $\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}}(\Omega_\delta(t))$ with $\gamma_{\delta(t)}(\mathbf{v}) = 0$ and $\mathbf{v}\times\mathbf{n} = 0$ on Γ_f. The following Green formula can be proved by density arguments for $\boldsymbol{\psi}\in\mathbf{C}^1(\overline{\Omega}_\delta(t))$:

$$\int_{\Omega_\delta(t)} (\nabla\times\mathbf{v})\cdot\boldsymbol{\psi} = \int_{\Omega_\delta(t)} \mathbf{v}\cdot(\nabla\times\boldsymbol{\psi}) + \int_{\Omega_s} \gamma_{\delta(t)}(\mathbf{v})\times\begin{pmatrix} -\partial_{x_1}\delta \\ -\partial_{x_2}\delta \\ 1 \end{pmatrix}\cdot\gamma_{\delta(t)}(\boldsymbol{\psi}).$$

Thus, if $\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}}(\Omega_\delta(t))$ is such that $\gamma_{\delta(t)}(\mathbf{v}) = 0$ and $\mathbf{v}\times\mathbf{n} = 0$ on Γ_f, then one gets $\tilde{\mathbf{v}}\in\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides,K}\cup((\partial\Omega_s\times(\alpha,K))-\Gamma_f)}(\mathcal{B}_K)$, where $\Gamma_{sides,K} = ((0,L)\times\{0\}\times(0,K))\cup((0,L)\times\{l\}\times(0,K))$.

$$\tilde{\mathbf{v}} := \begin{cases} \mathbf{v} & \text{in} \quad \Omega_\delta(t) \\ 0 & \text{in} \quad \mathcal{B}_K - \Omega_\delta(t) \end{cases}$$

Thus, $\mathbf{v}_\beta(x_1,x_2,x_3) := \tilde{\mathbf{v}}(x_1,x_2,\beta x_3)$, $\beta\geq 1$ is in $\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}\cup\Gamma_1(t)}(\Omega_\delta(t))$ with $\mathbf{v}_\beta\times\mathbf{n} = 0$ on Γ_f. Since $\mathbf{v}_\beta \overset{\beta\to 1}{\to} \mathbf{v}$ (in $\mathbf{H}^1(\Omega_\delta(t))$), we have $\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}\cup\Gamma_1(t)}(\Omega_\delta(t))$ and $\mathbf{v}\times\mathbf{n} = 0$ on Γ_f. □

Lemma 4.6. *Let* $\mathbf{v} \in \mathbf{V}_\delta$ *such that for a.e.* t, $\gamma_{\delta(t)}(\mathbf{v}) = (0,0,b)$, *where* $b \in L^2(0,T;H^1_0(\Omega_s))$. *Then define the function*

$$\tilde{\mathbf{v}} := \begin{cases} \mathbf{v} & in \quad Q_{\delta,T} \\ (0,0,b) & in \quad \mathcal{B}_{K,T} - Q_{\delta,T} \end{cases}.$$

$\tilde{\mathbf{v}}$ *belongs to* $\mathbf{V}$ *and*

$$||\tilde{\mathbf{v}}||_{\mathbf{V}} \leq C(||\mathbf{v}||_{\mathbf{V}_\delta} + ||b||_{L^2(0,T;H^1_0(\Omega_s))}).$$

Proof. The proof follows observing that $\tilde{\mathbf{v}} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_K))$ (like in the proof of the previous lemma) and

$$||\tilde{\mathbf{v}}||_{\mathbf{V}} \leq ||\mathbf{v}||_{\mathbf{V}_\delta} + ||(0,0,b)||_{L^2(0,T;H^1(\mathcal{B}_K - \Omega_\delta(t)))}.$$

Then

$$||(0,0,b)||^2_{L^2(0,T;H^1(\mathcal{B}_K - \Omega_\delta(t)))} \leq (K-\alpha)||b||^2_{L^2(0,T;H^1_0(\Omega_s))} \leq K||b||^2_{L^2(0,T;H^1_0(\Omega_s))}.$$

It follows thus that $\tilde{\mathbf{v}} \in \mathbf{V}$, the rest of the properties necessary for $\tilde{\mathbf{v}}$ to be in $\mathbf{V}$ being clearly verified. □

Observe that the following lemma holds, too:

Lemma 4.7. *(Inequality of Poincaré)*

$$||\mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))} \leq ||\nabla \mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))} \ for\ \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_b \cup \Gamma_{sides}}(\Omega_\delta(t)).$$

5. Weak formulation and main result

With the function spaces in the previous section, we are now able to give the weak formulation of the problem.

Let $\eta_0 \in H^2_0(\Omega_s)$, $\mathbf{u}_0 \in \mathbf{L}^2(\Omega_{\eta_0})$, $p_0 \in L^2(0,T;H^{1/2}(\Gamma_f))$, $\eta_{01} \in L^2(\Omega_s)$ such that $\min_{\bar{\Omega}_s}(1+\eta_0) > 0$, div $\mathbf{u}_0 = 0$, $\mathbf{u}_0 = 0$ on $\Gamma_b \cup \Gamma_{sides}$, $\mathbf{u}_0 \times \mathbf{n} = 0$ on Γ_f, $\gamma^n_{\eta_0}(\mathbf{u}_0) = (0,0,\eta_{01}) \cdot \mathbf{n}_0$ on Ω_s, $\int_{\Omega_s} \eta_{01} - \int_{\Gamma_{f2}} u_1(0) + \int_{\Gamma_{f1}} u_1(0) = 0$.

Definition 5.1. $(\mathbf{u},\eta)$ *is a weak solution of (1)-(3), (4)-(10), (12) on* $[0,T)$ *if:*

- $\mathbf{u} \in \mathbf{V}_\eta \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\eta(t)))$,
- $\eta \in W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H^2_0(\Omega_s))$
- $\gamma_{\eta(t)}(\mathbf{u}) = (0,0,\partial_t\eta)$ *for a.e.* t
- *for all* $(\boldsymbol{\psi}, b) \in \mathcal{V}_\eta \times C^1(0,T;H^2_0(\Omega_s))$ *such that* $\boldsymbol{\psi}(t,x_1,x_2,1+\eta(t,x_1,x_2)) = (0,0,b(t,x_1,x_2))$, $(t,x_1,x_2) \in [0,T] \times \Omega_s$ *we have for a.e.* t

$$\int_{\Omega_\eta(t)} \mathbf{u}(t) \cdot \boldsymbol{\psi}(t) - \int_0^t \int_{\Omega_\eta(s)} \mathbf{u} \cdot \partial_t \boldsymbol{\psi} + \nu \int_0^t \int_{\Omega_\eta(s)} (\nabla \times \mathbf{u}) \cdot (\nabla \times \boldsymbol{\psi})$$

$$+\int_0^t\int_{\Omega_\eta(s)}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\psi}-\int_0^t\int_{\Omega_s}\partial_t\eta\partial_t b+\int_{\Omega_s}\partial_t\eta(t)b(t)+\int_0^t\int_{\Omega_s}\Delta\eta\Delta b$$

$$+\gamma\int_0^t\int_{\Omega_s}\Delta\partial_t\eta\Delta b=\int_0^t\int_{\Omega_\eta(s)}\mathbf{f}\cdot\boldsymbol{\psi}+\int_0^t\int_{\Omega_s}g\cdot b-\int_0^t\int_{\Gamma_f}p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi}$$

$$+\int_{\Omega_{\eta_0}}\mathbf{u}_0\boldsymbol{\psi}(0)+\int_{\Omega_s}\eta_{01}b(0) \tag{18}$$

We may now state the main result:

Theorem 5.1. *Suppose the assumptions stated before Definition 5.1 are satisfied and that* $\mathbf{f}\in\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))$, $g\in L^2(0,T;L^2(\Omega_s))$, $p_0\in L^2(0,T;\ H^{1/2}(\Gamma_f))$. $R:=\Omega_s\times(0,1)$ *represents the reference configuration of our problem.*

Then there exists $0<T<\infty$ *and a weak solution of the problem on* $[0,T]$, *which satisfies the following estimates:*

$$||\mathbf{u}||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\eta(t)))}+||\mathbf{u}||_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\eta(t)))}+||\partial_t\eta||_{L^\infty(0,T;L^2(\Omega_s))}$$

$$+||\Delta\eta||_{H^1(0,T;L^2(\Omega_s))}\le const\ (T,||\mathbf{u}_0||_{\mathbf{L}^2(\Omega_{\eta_0})},\ ||\mathbf{f}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))},$$

$$||g||_{L^2(0,T;L^2(\Omega_s))},\ ||\eta_0||_{H_0^2(\Omega_s)},\ ||\eta_{01}||_{L^2(\Omega_s)},||p_0||_{L^2(0,T;L^2(\Gamma_f))}).$$

Remark 5.2. *Notice that the theorem also implies the boundedness of* η *in the norm of* $H^1(0,T;H_0^2(\Omega_s))$, *thus also* $\eta\in C(0,T;H_0^1(\Omega_s))$.

$\diamondsuit$

For the proof we will state firstly an approximate problem, whose solutions are built by regularizing the nonlinear convection terms and prove with the aid of Schauder's Second Fixed Point Theorem the existence of such an approximate solution (via linearization and Galerkin method). Then we prove some compactness properties, which will allow us further to pass to the limit in the approximate problem, in order to get the existence of the solution of our problem.

Before proceeding with the proof of this result we give the following remark, which is crucial for constructing approximate weak solutions of the problem:

Remark 5.3. *For the convective term* $\int_{\Omega_\eta(t)}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\psi}$ *the following expression is valid:*

$$\int_{\Omega_\eta(t)}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\psi}=\int_{\Omega_\eta(t)}(\nabla\times\mathbf{u})\times\mathbf{u}\cdot\boldsymbol{\psi}+\frac{1}{2}\int_{\Omega_s}(\partial_t\eta)^2b-$$

$$-\frac{1}{2}\int_{\Gamma_{f1}} u_1^2\psi_1 + \frac{1}{2}\int_{\Gamma_{f2}} u_1^2\psi_1.$$

◊

6. The approximate problem

6.1. **Constructing the approximate solution.** Let us now construct a sequence $(\mathbf{u}_\epsilon, \eta_\epsilon)_{\epsilon>0}$ of approximate weak solutions. Let $\mathbf{u}_0^\epsilon$, η_0^ϵ, η_{01}^ϵ be regularizations of the initial data such that div $\mathbf{u}_0^\epsilon = 0$, $\mathbf{u}_0^\epsilon(x_1, x_2, 1 + \eta_0^\epsilon(x_1, x_2)) = (0, 0, \eta_{01}^\epsilon(x_1, x_2))$, $\mathbf{u}_0^\epsilon = 0$ on $\Gamma_b \cup \Gamma_{sides}$, $\mathbf{u}_0^\epsilon \times \mathbf{n} = 0$ on Γ_f, $\int_{\Omega_s} \eta_{01}^\epsilon - \int_{\Gamma_{f1}} u_{0,1}^\epsilon + \int_{\Gamma_{f2}} u_{0,1}^\epsilon = 0$ and

$$\chi_{\Omega_{\eta_0 \epsilon}} \mathbf{u}_0^\epsilon \overset{\epsilon \to 0}{\to} \chi_{\Omega_{\eta_0}} \mathbf{u}_0 \text{ in } \mathbf{L}^2(\mathcal{B}_K),$$

where $\chi_{\Omega_{\eta_0 \epsilon}}$ is the characteristic function of $\Omega_{\eta_0 \epsilon}$ and $\chi_{\Omega_{\eta_0}}$ the characteristic function of Ω_{η_0}. We also demand $\eta_{01}^\epsilon \overset{\epsilon \to 0}{\to} \eta_{01}$ in $L^2(\Omega_s)$ and $\eta_0^\epsilon \overset{\epsilon \to 0}{\to} \eta_0$ in $H_0^2(\Omega_s)$.

The construction of such sequences can be done (like in [CDEG02]) in the following way: take $\eta_0^\epsilon \in \mathcal{D}(\Omega_s)$ with $\eta_0^\epsilon \overset{\epsilon \to 0}{\to} \eta_0$ in $H_0^2(\Omega_s)$. For ϵ small enough, if $\min_{\bar{\Omega}_s}(1 + \eta_0) \geq 2\alpha$, we have $\min_{\bar{\Omega}_s}(1 + \eta_0^\epsilon) \geq \frac{3\alpha}{2}$.

Define

$$\tilde{\mathbf{u}}_0 := \begin{cases} \mathbf{u}_0 \text{ in } \Omega_{\eta_0} \\ (0, 0, \eta_{01}) \text{ in } \mathcal{B}_{K+1} - \Omega_{\eta_0} \end{cases}.$$

Observe that div $\tilde{\mathbf{u}}_0 = 0$.

Construct

$$\tilde{\mathbf{u}}_0^\epsilon := \begin{cases} \tilde{\mathbf{u}}_0 \text{ in } \Omega_{\eta_0^\epsilon} \\ (0, 0, \eta_{01}^\epsilon) \text{ in } \mathcal{B}_{K+1} - \Omega_{\eta_0^\epsilon} \end{cases}.$$

Observe again that div $\tilde{\mathbf{u}}_0^\epsilon = 0$.

Now, consider

$$\mathbf{u}_0^{\epsilon,\beta}(x_1, x_2, x_3) = (\beta \tilde{\mathbf{u}}_{0,1}^\epsilon(x_1, x_2, \beta x_3), \beta \tilde{\mathbf{u}}_{0,2}^\epsilon(x_1, x_2, \beta x_3), \tilde{\mathbf{u}}_{0,3}^\epsilon(x_1, x_2, \beta x_3)),$$

$\beta \geq 1$, which is also solenoidal. For $\beta > 1$, $\mathbf{u}_0^{\epsilon,\beta} = (0, 0, \eta_{01}^\epsilon)$ in a neighbourhood of $\{(x_1, x_2, 1 + \eta_0^\epsilon(x_1, x_2)) : (x_1, x_2) \in \Omega_s\}$. If we regularize $\mathbf{u}_0^{\epsilon,\beta}$ in the standard way, it gives the required approximations on $\mathbf{u}_0$ and η_{01}.

The proof of the next proposition is done in the subsequent paragraphs of this subsection.

Proposition 6.1. *Let $\tilde{\mathbf{u}}_\epsilon^\sharp$ and $\eta_\epsilon^\sharp$ be regularizations of $\tilde{\mathbf{u}}_\epsilon$, respectively η_ϵ, $\tilde{\mathbf{u}}_\epsilon$ being the extension of $\mathbf{u}_\epsilon$, defined as in Lemma 4.6. (For more details on these regularizations we refer to the first step of the proof below).*

Then there exists a (weak) solution $(\mathbf{u}_\epsilon, \eta_\epsilon)$ *in the space*

$$(\mathbf{V}_{\eta_\epsilon^\sharp} \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))) \times (W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H_0^2(\Omega_s)))$$

with

- $\mathbf{u}_\epsilon(t,x_1,x_2,1+\eta_\epsilon^\sharp(t,x_1,x_2)) = (0,0,\partial_t\eta_\epsilon(t,x_1,x_2))$ *on* Ω_s,
- $\partial_t\mathbf{u}_\epsilon \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))$,
- $\partial_{tt}\eta_\epsilon \in L^2(0,T;L^2(\Omega_s))$,
- $\mathbf{u}_\epsilon(0) = \mathbf{u}_0^\epsilon$, $\eta_\epsilon(0) = \eta_0^\epsilon$, $\partial_t\eta_\epsilon(0) = \eta_{01}^\epsilon$
- *and*

$$\int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} \partial_t\mathbf{u}_\epsilon \cdot \boldsymbol{\psi}_\epsilon + \nu \int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} (\nabla \times \mathbf{u}_\epsilon)\cdot(\nabla \times \boldsymbol{\psi}_\epsilon) \tag{19}$$

$$+\int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} (\nabla \times \tilde{\mathbf{u}}_\epsilon^\sharp) \times \mathbf{u}_\epsilon \cdot \boldsymbol{\psi}_\epsilon + \frac{1}{2}\int_0^t \int_{\Omega_s} \partial_t\eta_\epsilon \partial_t\eta_\epsilon^\sharp b - \frac{1}{2}\int_0^t \int_{\Gamma_{f1}} u_{\epsilon,1}\tilde{u}_{\epsilon,1}^\sharp \psi_{\epsilon,1}$$

$$+\frac{1}{2}\int_0^t \int_{\Gamma_{f2}} u_{\epsilon,1}\tilde{u}_{\epsilon,1}^\sharp \psi_{\epsilon,1} + \int_0^t \int_{\Omega_s} \partial_{tt}\eta_\epsilon b + \int_0^t \int_{\Omega_s} \Delta\eta_\epsilon \Delta b + \gamma \int_0^t \int_{\Omega_s} \Delta(\partial_t\eta_\epsilon)\Delta b$$

$$= \int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} \mathbf{f}\cdot\boldsymbol{\psi}_\epsilon + \int_0^t \int_{\Omega_s} g \cdot b - \int_0^t \int_{\Gamma_f} p_0 \cdot \mathbf{n} \cdot \boldsymbol{\psi}_\epsilon,$$

$\forall \boldsymbol{\psi}_\epsilon \in \mathbf{V}_{\eta_\epsilon^\sharp}$, $b \in L^2(0,T;H_0^2(\Omega_s))$ *such that*

$$\boldsymbol{\psi}_\epsilon(t,x_1,x_2,1+\eta_\epsilon^\sharp(t,x_1,x_2)) = (0,0,b(t,x_1,x_2)) \text{ on } \Omega_s.$$

Remark 6.1. *We have used here Remark 5.3 to rewrite the convective term. Observe that the approximate solution satisfies the same estimates as those of the actual solution.*

Indeed, take $(\mathbf{u}_\epsilon, \partial_t\eta_\epsilon)$ *as a test function and notice that in virtue of Reynolds' transport formula and by the fact that the velocity of the moving boundary of the fluid domain* $\Omega_{\eta_\epsilon^\sharp}(t)$ *is* $(0,0,\partial_t\eta_\epsilon^\sharp)$ *and since we have* $\mathbf{u}_\epsilon(t,x_1,x_2,1+\eta_\epsilon^\sharp(t,x_1,x_2)) = (0,0,\partial_t\eta_\epsilon(t,x_1,x_2))$ *on* Ω_s, *it follows that*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\mathbf{u}_\epsilon|^2 = \int_{\Omega_{\eta_\epsilon^\sharp}(t)} \partial_t\mathbf{u}_\epsilon \cdot \mathbf{u}_\epsilon + \frac{1}{2}\int_{\Omega_s} (\partial_t\eta_\epsilon)^2 \partial_t\eta_\epsilon^\sharp$$

$$-\frac{1}{2}\int_0^t \int_{\Gamma_{f1}} (u_{\epsilon,1})^2 \tilde{u}_{\epsilon,1}^\sharp + \frac{1}{2}\int_0^t \int_{\Gamma_{f2}} (u_{\epsilon,1})^2 \tilde{u}_{\epsilon,1}^\sharp.$$

With calculations similar to those in Section 3 and choosing the constant c_3 in the involved Young's inequality such that:

$$\nu c_E \geq \frac{c_\Gamma}{2c_3} \text{ and } \gamma c_E \geq \frac{c_\Gamma c_1}{2c_3}$$

and with

$$c_4 = e^T \frac{1}{2}\Big[|\mathbf{u}_0^\epsilon|^2_{\mathbf{L}^2(\Omega^\sharp_{\eta_\epsilon}(0))} + |\eta^\epsilon_{01}|^2_{L^2(\Omega_s)} + |\Delta\eta^\epsilon_0|^2_{L^2(\Omega_s)} + ||\mathbf{f}||^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega^\sharp_{\eta_\epsilon}(t))}$$
$$+||g||^2_{L^2(0,T;L^2(\Omega_s))} + c_1||p_0||^2_{L^2(0,T;L^2(\Gamma_f))}\Big],$$

we get:

$$||\mathbf{u}_\epsilon||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_{\eta^\sharp_\epsilon}(t)))} + ||\nabla\times\mathbf{u}_\epsilon||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\eta^\sharp_\epsilon}(t)))}$$

$$+||\partial_t\eta_\epsilon||_{L^\infty(0,T;L^2(\Omega_s))} + ||\Delta\eta_\epsilon||_{H^1(0,T;L^2(\Omega_s))} \leq C, \tag{20}$$

where $C>0$ is a constant depending only on the data and not on ϵ.

Now taking into account condition (17), it follows (like in Subsection 3) that

$$||\mathbf{u}_\epsilon||_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_{\eta^\sharp_\epsilon}(t)))} \leq C, \tag{21}$$

with C depending on the data and on $\max_{[0,T]\times\bar{\Omega}_s}(1+\eta^\sharp_\epsilon)$*. Now use (20) and the fact that* $H^1(0,T;H^2_0(\Omega_s)) \hookrightarrow C^{0,1/2}(0,T;C^{0,q}(\bar{\Omega}_s))$ $(0 \leq q < 1)$ *to deduce that* $\min_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon)$ *doesn't depend on ϵ and so neither does* $\min_{[0,T]\times\bar{\Omega}_s}(1+\eta^\sharp_\epsilon)$*.*

$\diamondsuit$

Proof. (of Proposition 6.1) The proof is done in two main steps: in the first one we linearize the weak formulation for this approximate problem and show the existence of a unique solution of the linearized regularized approximate problem, with the aid of the Galerkin method. Under some supplementary regularity properties, we can pass to the limit in the Galerkin approximations and apply in a second step a version of the Generalized Schauder Fixed Point Theorem to prove the existence of the approximate weak solution $(\mathbf{u}_\epsilon, \eta_\epsilon)$.

6.2. **Step 1: The linearized approximate problem.** We linearize (19). Let $\delta \in H^1(0,T;H^2_0(\Omega_s))$, $\delta(0) = \eta^\epsilon_0$ and $K \geq 1 + \delta(t,x_1,x_2) \geq \alpha > 0$, $\forall (t,x_1,x_2) \in [0,T]\times\bar{\Omega}_s$ (α is such that $\min_{\bar{\Omega}_s}(1+\eta_0) \geq 2\alpha > 0$). Take $\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$.

Consider $\delta^\sharp_\epsilon = R^s_\epsilon(\delta)$ and $\mathbf{v}^\sharp_\epsilon = R^f_\epsilon(\mathbf{v})$ space-time regularizations of δ, respectively $\mathbf{v}$, such that: $R^s_\epsilon(\delta_\epsilon) \to \delta$ in $C([0,T]\times\bar{\Omega}_s)$ when $\delta_\epsilon \to \delta$ in $C([0,T]\times\bar{\Omega}_s)$, $\partial_t R^s_\epsilon(\delta_\epsilon) \to \partial_t\delta$ in $L^2(0,T;L^2(\Omega_s))$ when $\partial_t\delta_\epsilon \to$

$\partial_t \delta$ in $L^2(0,T;L^2(\Omega_s))$ and such that, if $\mathbf{v}$ is the limit of $\mathbf{v}_\epsilon$ in $\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$, then it is also the limit of $R^f_\epsilon(\mathbf{v}_\epsilon)$ in $\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$.

As in [CDEG02], we build $R^s_\epsilon(\delta)$ in the following way:

$$R^s_\epsilon(\delta) = S_\epsilon(\delta - \delta(t=0)) + \eta^\epsilon_0,$$

where S_ϵ is a space-time regularization such that $S_\epsilon(b)|_{t=0} = 0$ when $b(0) = 0$. In particular, observe that $R^s_\epsilon(\delta)|_{t=0} = \eta^\epsilon_0$. We may also suppose that $2K \geq 1 + \delta^\sharp_\epsilon(t,x_1,x_2) \geq \frac{\alpha}{2}$, $\forall (t,x_1,x_2) \in [0,T]\times\bar{\Omega}_s$.

The problem we want to solve is to find $(\mathbf{u}_\epsilon, \eta_\epsilon)$ such that:

- $\mathbf{u}_\epsilon \in \mathbf{V}_{\delta^\sharp_\epsilon} \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_{\delta^\sharp_\epsilon}(t)))$,
- $\eta_\epsilon \in W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H^2_0(\Omega_s))$,
- $\partial_t \mathbf{u}_\epsilon \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\delta^\sharp_\epsilon}(t)))$,
- $\partial_{tt}\eta_\epsilon \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_s))$,
- $\mathbf{u}_\epsilon(t,x_1,x_2,1+\delta^\sharp_\epsilon(t,x_1,x_2)) = (0,0,\partial_t\eta_\epsilon(t,x_1,x_2))$ on Ω_s,
- $\mathbf{u}_\epsilon(0) = \mathbf{u}^\epsilon_0$, $\eta_\epsilon(0) = \eta^\epsilon_0$, $\partial_t\eta_\epsilon(0) = \eta^\epsilon_{01}$
- and

$$\int_0^t \int_{\Omega_{\delta^\sharp_\epsilon}(s)} \partial_t \mathbf{u}_\epsilon \cdot \boldsymbol{\psi}_\epsilon + \nu \int_0^t \int_{\Omega_{\delta^\sharp_\epsilon}(s)} (\nabla\times\mathbf{u}_\epsilon)\cdot(\nabla\times\boldsymbol{\psi}_\epsilon) \tag{22}$$

$$+\int_0^t \int_{\Omega_{\delta^\sharp_\epsilon}(s)} (\nabla\times\mathbf{v}^\sharp_\epsilon)\times\mathbf{u}_\epsilon\cdot\boldsymbol{\psi}_\epsilon - \frac{1}{2}\int_0^t\int_{\Gamma_{f1}} u_{\epsilon,1}v^\sharp_{\epsilon,1}\cdot\psi_{\epsilon,1} + \frac{1}{2}\int_0^t\int_{\Gamma_{f2}} u_{\epsilon,1}v^\sharp_{\epsilon,1}\cdot\psi_{\epsilon,1}$$

$$+\frac{1}{2}\int_0^t\int_{\Omega_s}\partial_t\eta_\epsilon\partial_t\delta^\sharp_\epsilon b + \int_0^t\int_{\Omega_s}\partial_{tt}\eta_\epsilon b + \int_0^t\int_{\Omega_s}\Delta\eta_\epsilon\Delta b + \gamma\int_0^t\int_{\Omega_s}\Delta(\partial_t\eta_\epsilon)\Delta b$$

$$= \int_0^t \int_{\Omega_{\delta^\sharp_\epsilon}(s)} \mathbf{f}\cdot\boldsymbol{\psi}_\epsilon + \int_0^t\int_{\Omega_s} g\cdot b - \int_0^t\int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi}_\epsilon,$$

$\forall \boldsymbol{\psi}_\epsilon \in \mathbf{V}_{\delta^\sharp_\epsilon}$, $b \in L^2(0,T;H^2_0(\Omega_s))$ such that

$$\boldsymbol{\psi}_\epsilon(t,x_1,x_2,1+\delta^\sharp_\epsilon(t,x_1,x_2)) = (0,0,b(t,x_1,x_2)) \text{ on } \Omega_s.$$

Remark 6.2. *Observe that in this problem the test functions do not depend on the solution. Moreover (as in Remark 6.1), any solution of the above problem satisfies energy estimates which are independent on* ϵ.

$\diamondsuit$

6.3. **Existence of a solution by the method of Galerkin.** So far we have dealt with equations set in a time-dependent domain. We now make a transformation, in order to rewrite our equations in the reference configuration $R := \Omega_s \times (0,1)$, which is a subset of $\mathbf{R}^3$, too, but which does not depend on time (it is fixed). We define the transformation by $\boldsymbol{\phi}_\epsilon : (0,T) \times R \to \Omega_{\delta_\epsilon^\sharp}(t)$,

$$\boldsymbol{\phi}_\epsilon(t, x_1, x_2, x_3) := (x_1, x_2, x_3(1 + \delta_\epsilon^\sharp(t, x_1, x_2))), \tag{23}$$

$\forall (x_1, x_2, x_3) \in R,\ t \in (0,T)$. Observe that $\boldsymbol{\phi}_\epsilon$ is smooth in space and time and that $\boldsymbol{\phi}_\epsilon(t,.)$ is a C^l-diffeomorphism. The time derivative of $\boldsymbol{\phi}_\epsilon$ is $\partial_t \boldsymbol{\phi}_\epsilon = (0, 0, x_3 \partial_t \delta_\epsilon^\sharp)$.

With this transformation and with the following notations:

$$\mathbf{u}_\epsilon^{\phi_\epsilon} := \mathbf{u}_\epsilon \circ \boldsymbol{\phi}_\epsilon,\ \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} := \boldsymbol{\psi}_\epsilon \circ \boldsymbol{\phi}_\epsilon,\ \mathbf{f}^{\phi_\epsilon} := \mathbf{f} \circ \boldsymbol{\phi}_\epsilon,\ p_0^{\phi_\epsilon} := p_0 \circ \boldsymbol{\phi}_\epsilon$$

$$J_\epsilon := \det\ \nabla \boldsymbol{\phi}_\epsilon,\ \mathbf{M}_\epsilon := \mathrm{cof}\ \nabla \boldsymbol{\phi}_\epsilon,\ \mathbf{n}^{\phi_\epsilon} := \frac{\mathbf{M}_\epsilon \cdot \mathbf{n}}{||\mathbf{M}_\epsilon \cdot \mathbf{n}||},\ d\sigma^{\phi_\epsilon} = ||\mathbf{M}_\epsilon \cdot \mathbf{n}|| d\sigma$$

the equations (22) become:

$$\int_0^t \int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon} \cdot \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} J_\epsilon + \nu \int_0^t \int_R ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}} \nabla) \times \mathbf{u}_\epsilon^{\phi_\epsilon}) \cdot ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}} \nabla) \times \boldsymbol{\psi}_\epsilon^{\phi_\epsilon}) \tag{24}$$

$$+ \int_0^t \int_R ((\mathbf{M}_\epsilon \nabla) \times \mathbf{v}_\epsilon^{\phi_\epsilon, \sharp}) \times \mathbf{u}_\epsilon^{\phi_\epsilon} \cdot \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} - \int_0^t \int_R (\partial_t \boldsymbol{\phi}_\epsilon \cdot (\mathbf{M}_\epsilon \nabla)) \mathbf{u}_\epsilon^{\phi_\epsilon} \cdot \boldsymbol{\psi}_\epsilon^{\phi_\epsilon}$$

$$- \frac{1}{2} \int_0^t \int_{\Gamma_{f1}} u_{\epsilon,1}^{\phi_\epsilon} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \cdot \psi_{\epsilon,1}^{\phi_\epsilon} J_\epsilon + \frac{1}{2} \int_0^t \int_{\Gamma_{f2}} u_{\epsilon,1}^{\phi_\epsilon} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \cdot \psi_{\epsilon,1}^{\phi_\epsilon} J_\epsilon$$

$$+ \frac{1}{2} \int_0^t \int_{\Omega_s} \partial_t \eta_\epsilon \partial_t \delta_\epsilon^\sharp b + \int_0^t \int_{\Omega_s} \partial_{tt} \eta_\epsilon b + \int_0^t \int_{\Omega_s} \Delta \eta_\epsilon \Delta b + \gamma \int_0^t \int_{\Omega_s} \Delta(\partial_t \eta_\epsilon) \Delta b$$

$$= \int_0^t \int_R \mathbf{f}^{\phi_\epsilon} \cdot \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} J_\epsilon + \int_0^t \int_{\Omega_s} g \cdot b - \int_0^t \int_{\Gamma_f} p_0^{\phi_\epsilon} \cdot \mathbf{M}_\epsilon \cdot \mathbf{n} \cdot \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} J_\epsilon,$$

$\forall \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} \in \mathbf{L}^2(0,T; \mathbf{H}^1_{0,\Gamma_b \cup \Gamma_{sides}}(R)),\ b \in L^2(0,T; H_0^2(\Omega_s))$ such that

$$\boldsymbol{\psi}_\epsilon^{\phi_\epsilon}(t, x_1, x_2, 1) = (0, 0, b(t, x_1, x_2)) \text{ on } \Omega_s,$$

$$\text{div } (\mathbf{M}_\epsilon^t \boldsymbol{\psi}_\epsilon^{\phi_\epsilon}) = 0 \text{ in } R,\ \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} \times \mathbf{n} = 0 \text{ on } \Gamma_f.$$

(observe that $\partial_t \mathbf{u}_\epsilon^{\phi_\epsilon}(t, \mathbf{x}) = \partial_t \mathbf{u}_\epsilon(t, \boldsymbol{\phi}_\epsilon(t, \mathbf{x})) + (\partial_t \boldsymbol{\phi}_\epsilon(t, \mathbf{x}) \cdot \nabla) \mathbf{u}_\epsilon(t, \boldsymbol{\phi}_\epsilon(t, \mathbf{x}))$, thus $\partial_t \mathbf{u}_\epsilon = \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon} - (\partial_t \boldsymbol{\phi}_\epsilon \cdot \frac{1}{J_\epsilon}(\mathbf{M}_\epsilon \nabla)) \mathbf{u}_\epsilon^{\phi_\epsilon})$.

The equality of the velocities at the interface becomes:

$$\mathbf{u}_\epsilon^{\phi_\epsilon}(t,x_1,x_2,1)=(0,0,\partial_t\eta_\epsilon(t,x_1,x_2)),\ (x_1,x_2)\in\Omega_s.$$

We now come to the construction of the Galerkin basis. We build a basis $\{\boldsymbol{\xi}_j^0\}_{j\in\mathbf{N}}$ of $\{\mathbf{v}\in\mathbf{H}^1(R)\ :\ \text{div}\ \mathbf{v}=0 \text{ in } R,\ \mathbf{v}\times\mathbf{n}=0 \text{ on } \Gamma_f,\ \mathbf{v}=0 \text{ on } \partial R-\Gamma_f\}$ by taking into account all the eigenfunctions of the Stokes problem (like in e.g., [JäMi98])

$$\begin{aligned}-\Delta\boldsymbol{\xi}_j^0+\nabla p_j^0 &= \lambda_j^\epsilon\boldsymbol{\xi}_j^0 \text{ in } R\\ \text{div}\ \boldsymbol{\xi}_j^0 &= 0 \text{ in } R\\ \boldsymbol{\xi}_j^0\times\mathbf{n} &= 0 \text{ on } \Gamma_f\\ p_j^0 &= 0 \text{ on } \Gamma_f\\ \boldsymbol{\xi}_j^0 &= 0 \text{ on } \partial R-\Gamma_f.\end{aligned}$$

Denote $\boldsymbol{\psi}_j^0:=\mathbf{M}_\epsilon^{-t}\boldsymbol{\xi}_j^0$. The family $\{\boldsymbol{\psi}_j^0\}_{j\in\mathbf{N}}$ is a basis of the space $\{\mathbf{v}\in\mathbf{H}^1(R)\ :\ \text{div}\ (\mathbf{M}_\epsilon^t\mathbf{v})=0 \text{ in } R,\ \mathbf{M}_\epsilon^t\mathbf{v}\times\mathbf{n}=0 \text{ on } \Gamma_f,\ \mathbf{v}=0 \text{ on } \partial R-\Gamma_f\}$ and the functions $\boldsymbol{\psi}_j^0$ are smooth in time, because $\mathbf{M}_\epsilon$ it is.

We also consider a basis $\{\rho_j\}_{j\in\mathbf{N}}$ of $\{b\in H_0^2(\Omega_s)\ :\ \int_{\Omega_s}b-\int_{\Gamma_{f1}}\psi_{\epsilon,1}^{\phi_\epsilon}+\int_{\Gamma_{f2}}\psi_{\epsilon,1}^{\phi_\epsilon}=0 \text{ with } \psi_{\epsilon,1}^{\phi_\epsilon}\in H^1_{0,\Gamma_b\cup\Gamma_{sides}}(R)\}$ and we build functions $\{\boldsymbol{\psi}_j^{*,\epsilon}\}_{j\in\mathbf{N}}$ such that div $(\mathbf{M}_\epsilon^t\boldsymbol{\psi}_j^{*,\epsilon})=0$ and $\boldsymbol{\psi}_j^{*,\epsilon}(t,x_1,x_2,1)=(0,0,\rho_j(x_1,x_2))$ on Ω_s. This can be done by solving a Stokes-like problem (similarly to e.g., [CMP94], while making the corresponding hypotheses):

$$\begin{aligned}-\Delta\boldsymbol{\psi}_j^{*,\epsilon}+(\mathbf{M}_\epsilon\nabla)p_j^{*,\epsilon} &= 0 \text{ in } R\\ \text{div}\ (\mathbf{M}_\epsilon^t\boldsymbol{\psi}_j^{*,\epsilon}) &= 0 \text{ in } R\\ \boldsymbol{\psi}_j^{*,\epsilon} &= 0 \text{ on } \Gamma_b\cup\Gamma_{sides}\\ \mathbf{M}_\epsilon^t\boldsymbol{\psi}_j^{*,\epsilon}\times\mathbf{n} &= 0 \text{ on } \Gamma_f\\ p_j^{*,\epsilon} &= q_j \text{ (known) on } \Gamma_f\\ \boldsymbol{\psi}_j^{*,\epsilon} &= (0,0,\rho_j) \text{ on } \partial R-(\Gamma_b\cup\Gamma_{sides}\cup\Gamma_f).\end{aligned}$$

Again, the functions $\boldsymbol{\psi}_j^{*,\epsilon}$ are smooth in time, because $\mathbf{M}_\epsilon$ is smooth in time.

We are now looking for $\eta_\epsilon^n:=\sum_{j=1}^n\beta_j(t)\rho_j+\eta_0^\epsilon$ and $\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}:=\sum_{j=1}^m\alpha_j(t)\boldsymbol{\psi}_j^{0,\epsilon}$ $+\sum_{l=1}^n\dot\beta_l(t)\boldsymbol{\psi}_l^{*,\epsilon}$ such that for all $1\le j\le m$,

$$\int_R\partial_t\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}\cdot\boldsymbol{\psi}_j^{0,\epsilon}J_\epsilon+\nu\int_R((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla)\times\mathbf{u}_\epsilon^{\phi_\epsilon,m,n})\cdot((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla)\times\boldsymbol{\psi}_j^{0,\epsilon})$$

$$+\int_R ((\mathbf{M}_\epsilon \nabla) \times \mathbf{v}_\epsilon^{\phi_\epsilon,\sharp}) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_j^{0,\epsilon} - \int_R (\partial_t \boldsymbol{\phi}_\epsilon \cdot (\mathbf{M}_\epsilon \nabla))\mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_j^{0,\epsilon}$$
$$-\frac{1}{2}\int_{\Gamma_{f1}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{j,1}^{0,\epsilon} J_\epsilon + \frac{1}{2}\int_{\Gamma_{f2}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{j,1}^{0,\epsilon} J_\epsilon$$
$$= \int_R \mathbf{f}^{\phi_\epsilon} \cdot \boldsymbol{\psi}_j^{0,\epsilon} J_\epsilon - \int_{\Gamma_f} p_0^{\phi_\epsilon} \mathbf{M}_\epsilon \cdot \mathbf{n} \cdot \boldsymbol{\psi}_j^{0,\epsilon} J_\epsilon \tag{25}$$

and for all $1 \leq l \leq n$

$$\int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon + \nu \int_R ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}) \cdot ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \boldsymbol{\psi}_l^{*,\epsilon})$$
$$+\int_R ((\mathbf{M}_\epsilon \nabla) \times \mathbf{v}_\epsilon^{\phi_\epsilon,\sharp}) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_l^{*,\epsilon} - \int_R (\partial_t \boldsymbol{\phi}_\epsilon \cdot (\mathbf{M}_\epsilon \nabla))\mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_l^{*,\epsilon}$$
$$+\frac{1}{2}\int_{\Omega_s} \partial_t \eta_\epsilon^n \partial_t \delta_\epsilon^\sharp \rho_l - \frac{1}{2}\int_{\Gamma_{f1}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{l,1}^{*,\epsilon} J_\epsilon + \frac{1}{2}\int_{\Gamma_{f2}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{l,1}^{*,\epsilon} J_\epsilon$$
$$+\int_{\Omega_s} \partial_{tt} \eta_\epsilon^n \rho_l + \int_{\Omega_s} \Delta \eta_\epsilon^n \Delta \rho_l + \gamma \int_{\Omega_s} \Delta(\partial_t \eta_\epsilon^n) \Delta \rho_l$$
$$= \int_R \mathbf{f}^{\phi_\epsilon} \cdot \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon + \int_{\Omega_s} g \cdot \rho_l - \int_{\Gamma_f} p_0^{\phi_\epsilon} \mathbf{M}_\epsilon \cdot \mathbf{n} \cdot \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon. \tag{26}$$

These equations are provided with the following initial conditions: $\beta_j(0) = 0$, $\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}(0) = \mathbf{u}_{\epsilon,0}^{\phi_\epsilon,m,n}$, where $\mathbf{u}_{\epsilon,0}^{\phi_\epsilon,m,n}$ is the projection of $\mathbf{u}_0^{\epsilon,\phi_\epsilon}$ on the finite dimensional space span $(\boldsymbol{\psi}_j^{0,\epsilon}, \boldsymbol{\psi}_l^{*,\epsilon})_{1\leq j\leq m, 1\leq l\leq n}$ and $\partial_t \eta_\epsilon^n(0) = \eta_{01}^{\epsilon,n}$, where $\eta_{01}^{\epsilon,n}$ is the projection of η_{01}^ϵ on the finite dimensional space span $(\rho_l)_{1\leq l\leq n}$.

We thus have a second order system of ordinary differential equations, whose coefficients are functions smooth in time (they are sums of terms of the form $\int_R \boldsymbol{\psi}_j^{0,\epsilon} \boldsymbol{\psi}_i^{0,\epsilon} J_\epsilon$, $\int_R \partial_t \boldsymbol{\psi}_j^{0,\epsilon} \boldsymbol{\psi}_i^{0,\epsilon} J_\epsilon$, $\nu \int_R ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \boldsymbol{\psi}_j^{0,\epsilon}) \cdot ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \boldsymbol{\psi}_i^{0,\epsilon})$, ..., $\frac{1}{2}\int_{\Omega_s} \rho_l \cdot \partial_t \rho_\epsilon^\sharp \rho_k$, $\int_{\Omega_s} \Delta \rho_l \Delta \rho_k$ etc.).

By a transformation of variables, the second order system can be reduced to a first order system of ODEs, having the mass matrix

$$\begin{pmatrix} \int_R \boldsymbol{\psi}_j^{0,\epsilon} \boldsymbol{\psi}_l^{0,\epsilon} J_\epsilon & \int_R \boldsymbol{\psi}_j^{0,\epsilon} \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon \\ \int_R \boldsymbol{\psi}_j^{0,\epsilon} \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon & \int_R \boldsymbol{\psi}_j^{*,\epsilon} \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon + \int_{\Omega_s} \rho_j \rho_l \end{pmatrix}.$$

This matrix can be written as a sum of two matrices, where for all $t \in [0, T]$ the first one is smooth and symmetric nonnegative and the second one is symmetric positive.

Thus, by the usual theory for the ODE systems, the system (25), (26) has a unique solution on $[0, T_{m,n}]$, for a $T_{m,n} > 0$.

Now multiply (25) with α_j and sum up for $j = 1, ..., m$ and multiply (26) with $\dot{\beta}_l$ and do the sumation for $l = 1, ..., n$. We add the two equations that are obtained in this way and get

$$\int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} J_\epsilon + \nu \int_R ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}) \cdot ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n})$$

$$-\frac{1}{2}\int_{\Gamma_{f1}} (u_{\epsilon,1}^{\phi_\epsilon,m,n})^2 v_{\epsilon,1}^{\phi_\epsilon,\sharp} J_\epsilon + \frac{1}{2}\int_{\Gamma_{f2}} (u_{\epsilon,1}^{\phi_\epsilon,m,n})^2 v_{\epsilon,1}^{\phi_\epsilon,\sharp} J_\epsilon + \frac{1}{2}\int_{\Omega_s} \partial_t \eta_\epsilon^n \partial_t \delta_\epsilon^\sharp \partial_t \eta_\epsilon^n$$

$$+\int_{\Omega_s} \Delta\eta_\epsilon^n \Delta\partial_t\eta_\epsilon^n + \gamma\int_{\Omega_s} \Delta(\partial_t\eta_\epsilon^n)\Delta\partial_t\eta_\epsilon^n - \int_R (\partial_t \boldsymbol{\phi}_\epsilon \cdot (\mathbf{M}_\epsilon\nabla))\mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}$$

$$+\int_{\Omega_s} \partial_{tt}\eta_\epsilon^n \partial_t \eta_\epsilon^n = \int_R \mathbf{f}^{\phi_\epsilon} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} J_\epsilon + \int_{\Omega_s} g\partial_t\eta_\epsilon^n - \int_{\Gamma_f} p_0^{\phi_\epsilon} \mathbf{M}_\epsilon \cdot \mathbf{n} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} J_\epsilon.$$

Now, clearly

$$\frac{1}{2}\frac{d}{dt}\int_R |\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}|^2 J_\epsilon = \int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} J_\epsilon + \frac{1}{2}\int_R |\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}|^2 \frac{d}{dt} J_\epsilon$$

(remember that R doesn't depend on t) and since with the aid of the Piola identity

$$\frac{d}{dt} J_\epsilon = \text{div } (\mathbf{M}_\epsilon^t \cdot \partial_t \boldsymbol{\phi}_\epsilon) J_\epsilon$$

and $\partial_t \boldsymbol{\phi}_\epsilon(t, x_1, x_2, 1) = (0, 0, \partial_t \delta_\epsilon^\sharp(t, x_1, x_2))$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_R |\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}|^2 J_\epsilon = \int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} J_\epsilon + \frac{1}{2}\int_R |\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}|^2 \text{div } (\mathbf{M}_\epsilon^t \partial_t \boldsymbol{\phi}_\epsilon) J_\epsilon$$

$$= \int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} J_\epsilon - \int_R (\partial_t \boldsymbol{\phi}_\epsilon \cdot (\mathbf{M}_\epsilon \nabla))\mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}$$

$$+\frac{1}{2}\int_{\Omega_s} \partial_t \eta_\epsilon^n \partial_t \delta_\epsilon^\sharp \partial_t \eta_\epsilon^n.$$

Now, analogously as in Section 3, we get the following energy estimates:

$$||\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(R))} + ||\nabla \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(R))}$$

$$+||\partial_t \eta_\epsilon^n||_{L^\infty(0,T;H_0^2(\Omega_s))} + ||\Delta\eta_\epsilon^n||_{L^2(0,T;L^2(\Omega_s))}$$

$$\leq \text{const } (T, ||\mathbf{u}_0||_{\mathbf{L}^2(\Omega_{\eta_0})},\ \epsilon,\ \alpha,\ ||g||_{L^2(0,T;L^2(\Omega_s))},\ ||\mathbf{f}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))},$$

$$||p_0||_{L^2(0,T;L^2(\Gamma_f))},\ ||\eta_0||_{H_0^2(\Omega_s)},\ ||\eta_{01}||_{L^2(\Omega_s)}). \tag{27}$$

Since these estimates do not depend on m and n, it follows that $T_{m,n} = T$.

Remark 6.3. *The dependence on ϵ and α of the above estimates relies on the transformation of domains we considered. However, if we transform back to the deformed configuration, the energy estimates we get (with calculations analogous to those in Section 3) no more depend on ϵ and on α:*

$$||\mathbf{u}_\epsilon^{m,n}||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_{\delta_\epsilon^\sharp}(t)))}+||\nabla\times\mathbf{u}_\epsilon^{m,n}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\delta_\epsilon^\sharp}(t)))}+||\partial_t\eta_\epsilon^n||_{L^\infty(0,T;H_0^2(\Omega_s))}$$
$$+||\Delta\eta_\epsilon^n||_{L^2(0,T;L^2(\Omega_s))}\le const\ (T,||\mathbf{u}_0||_{\mathbf{L}^2(\Omega_{\eta_0})},\ ||g||_{L^2(0,T;L^2(\Omega_s))},$$
$$||p_0||_{L^2(0,T;L^2(\Gamma_f))},\ ||\eta_0||_{H_0^2(\Omega_s)},\ ||\eta_{01}||_{L^2(\Omega_s)},\ ||\mathbf{f}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))}). \tag{28}$$

This will be a useful observation later, when passing to the limit with $\epsilon\to 0$.

$\diamondsuit$

6.4. **Some more estimates.** We now give some supplementary estimates for $\partial_t\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}$ and $\partial_{tt}\eta_\epsilon^n$ which are independent of m and n (though they depend on ϵ) and which will be needed in Subsection 7.

The following lemma holds:

Lemma 6.1. *There exists a constant $C(\epsilon,\alpha)>0$ such that C does not depend on m, n and*

$$||\partial_t\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(R))}+||\partial_{tt}\eta_\epsilon^n||_{L^2(0,T;L^2(\Omega_s))}\le C. \tag{29}$$

For the proof we refer to the Appendix.

Remark 6.4. *This lemma also provides us with an estimate for $\partial_t\mathbf{u}_\epsilon^{m,n}$ in $\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\delta_\epsilon^\sharp}(t)))$, since*

$$\partial_t\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}(t,x_1,x_2,x_3)$$
$$=\partial_t\mathbf{u}_\epsilon^{m,n}(t,\boldsymbol{\phi}_\epsilon(t,x_1,x_2,x_3))+(\partial_t\boldsymbol{\phi}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon^{m,n}(t,\boldsymbol{\phi}_\epsilon(t,x_1,x_2,x_3)).$$

$\diamondsuit$

Now thanks to the above estimates we are able to pass to limits in the discrete system for $m,n\to\infty$. We thus obtain a unique solution of (22), which satisfies the estimates (27), (28) and (29).

6.5. **Step 2: The fixed point theorem.** We have shown so far that for any $(\mathbf{v},\delta)$ with $\mathbf{v}\in\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$ and $\delta\in H^1(0,T;H_0^2(\Omega_s))$, $K\ge 1+\delta(t,x_1,x_2)\ge\alpha>0$, $\forall\ (t,x_1,x_2)\in[0,T]\times\bar{\Omega}_s$ there exists a unique solution $(\mathbf{u}_\epsilon,\eta_\epsilon)$ of (22) with the above mentioned properties.

Observe that by the previous estimates $\eta_\epsilon\in H^1(0,T;H_0^2(\Omega_s))$ and (by construction) that $\eta_\epsilon(0)=\eta_0^\epsilon$. The function

$$\tilde{\mathbf{u}}_\epsilon:=\begin{cases}\mathbf{u}_\epsilon \text{ in } \Omega_{\delta_\epsilon^\sharp}(t)\\ (0,0,\partial_t\eta_\epsilon) \text{ in } \mathcal{B}_{2K}-\Omega_{\delta_\epsilon^\sharp}(t)\end{cases}$$

belongs to $\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$.

We define the spaces

$$\mathbf{S} := \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K})) \times H^1(0,T;H_0^2(\Omega_s))$$

$$\mathbf{X} := \{(\mathbf{v},\delta) \in \mathbf{S} \ : \ ||(\mathbf{v},\delta)||_{\mathbf{S}} \leq C_X, \ \alpha \leq 1+\delta(t,x_1,x_2) \leq K \ , \\ \forall (t,x_1,x_2) \in [0,T] \times \bar{\Omega}_s, \ \delta(t=0) = \eta_0^\epsilon\}$$

and the mapping FP_ϵ:

$$\mathbf{X} \ni (\mathbf{v},\delta) \overset{FP_\epsilon}{\mapsto} (\tilde{\mathbf{u}}_\epsilon,\eta_\epsilon) \in \mathbf{S}.$$

The existence of a weak solution to the approximate problem (19) will be proved if we show that for every ϵ the mapping FP_ϵ has a fixed point. In order to do this, we intend to apply the Second Schauder Fixed Point Theorem (see [Zeid86], Section 9.3), whose hypotheses we verify in the following:

$\mathbf{S}$ is a reflexive, separable Banach space: this is clear.

$FP_\epsilon(\mathbf{X}) \subset \mathbf{X}$: by the estimates (28) it follows that $\sup_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon) \leq K$ (we take the bounds of the data sufficiently small). We also have

$$||\tilde{\mathbf{u}}_\epsilon||_{\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))} \leq ||\mathbf{u}_\epsilon||_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_{\delta_\epsilon^\sharp}(t)))} + 2K||\partial_t\eta_\epsilon||_{L^2(0,T;L^2(\Omega_s))}.$$

Thus we can choose C_X large enough with respect to the data and to K such that $||(\tilde{\mathbf{u}}_\epsilon,\eta_\epsilon)||_{\mathbf{S}} \leq C_X$.

We still have to verify that there exists some $T>0$ such that $0<\alpha \leq \min_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon)$. But T might depend on ϵ, which we don't want to, since further we intend to pass to the limit for $\epsilon \to 0$, so we show that T may be chosen independently of ϵ. Indeed, by (28) (after passing to limits for $m,n\to\infty$) we know that $\eta_\epsilon \in H^1(0,T;H_0^2(\Omega_s))$ and the bound doesn't depend on ϵ. Now using the imbedding $H^1(0,T;H_0^2(\Omega_s)) \hookrightarrow C^{0,1/2}(0,T;C^{0,q}(\bar{\Omega}_s))$ $(0 \leq q < 1)$ and the hypothesis $\min_{[0,T]\times\bar{\Omega}_s}(1+\eta_0^\epsilon) \geq \frac{3\alpha}{2} > 0$ required for the solution of the approximate problem in Subsection 6.1, it follows that there exists some $T>0$ independent of ϵ such that $\min_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon) \geq \alpha > 0$. Indeed, let us define the mapping $\Psi : [0,\infty) \to \mathbf{R}$, $\Psi(t) := \min_{[0,T]\times\bar{\Omega}_s,\ \epsilon>0}(1+\eta_\epsilon)$. Then $\Psi(0) = \min_{\bar{\Omega}_s,\ \epsilon>0}(1+\eta_0^\epsilon) \overset{hyp}{>} \frac{3\alpha}{2} > \alpha > 0$. It follows then that $\mathcal{U} = (\alpha,\infty)$ is a neighbourhood of $\Psi(0)$. Since Ψ is continuous with respect to time (for, η_ϵ it is), we have that $\lim_{t\searrow 0}\Psi(t) = \Psi(0)$ and by the definition of continuity $\exists\ \mathcal{W} = (-\epsilon,T_1)$ a neighbourhood of 0 $(T_1>0)$ such that $t\in\mathcal{W}$ implies $\Psi(t)\in\mathcal{U}$. We then take $0<T<T_1$ (such a T clearly exists and does not depend on ϵ).

<u>$FP_\epsilon : \mathbf{X} \subseteq \mathbf{S} \to \mathbf{X}$ is weakly sequentially continuous</u>: let $(\mathbf{v}_n, \delta_n)_{n\in\mathbf{N}} \subset \mathbf{X}$ such that $(\mathbf{v}_n, \delta_n) \overset{n\to\infty}{\rightharpoonup} (\mathbf{v}, \delta)$. We want to prove that

$$FP_\epsilon(\mathbf{v}_n, \delta_n) = (\tilde{\mathbf{u}}_\epsilon^n, \eta_\epsilon^n) \overset{n\to\infty}{\rightharpoonup} (\tilde{\mathbf{u}}_\epsilon, \eta_\epsilon) = FP_\epsilon(\mathbf{v}, \delta) \text{ in } \mathbf{S}.$$

We know (use (28)) that $(\tilde{\mathbf{u}}_\epsilon^n, \eta_\epsilon^n)_{n\in\mathbf{N}}$ is bounded in $\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K})) \times H^1(0,T;H_0^2(\Omega_s))$ (i.e. in $\mathbf{S}$) and (by (29) and especially Remark 6.4) that a subsequence of $(\chi_\epsilon^n \partial_t \mathbf{u}_\epsilon^n, \partial_{tt}\eta_\epsilon^n)$ (denoted in the same way) converges weakly in $\mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K})) \times L^2(0,T;L^2(\Omega_s))$, where we have denoted by χ_ϵ^n the characteristic function of $\Omega_{\delta_{n,\epsilon}^\sharp}(t)$. Let $(\widehat{\tilde{\mathbf{u}}}_\epsilon, \widehat{\eta}_\epsilon)$ be the weak limit of $(\tilde{\mathbf{u}}_\epsilon^n, \eta_\epsilon^n)$. We are done if we show that $(\widehat{\tilde{\mathbf{u}}}_\epsilon, \widehat{\eta}_\epsilon) = FP_\epsilon(\mathbf{v}, \delta)$.

Now, clearly $\widehat{\tilde{\mathbf{u}}}_\epsilon = (0,0,\partial_t\widehat{\eta}_\epsilon)$ in $\mathcal{B}_{2K} - \Omega_{\delta_\epsilon^\sharp}(t)$ and $\widehat{\tilde{\mathbf{u}}}_\epsilon(t,x_1,x_2,1+\delta_\epsilon^\sharp(t,x_1,x_2)) = (0,0,\partial_t\widehat{\eta}_\epsilon(t,x_1,x_2))$ for any $(x_1,x_2)\in\Omega_s$. Notice that $\operatorname{div}\widehat{\tilde{\mathbf{u}}}_\epsilon = 0$. We next intend to pass to the limit for $n\to\infty$ in the weak formulation satisfied by $(\mathbf{u}_\epsilon^n, \eta_\epsilon^n)$:

$$\int_0^t \int_{\Omega_{\delta_{n,\epsilon}^\sharp}(s)} \partial_t \mathbf{u}_\epsilon^n \cdot \boldsymbol{\psi}_\epsilon^n + \nu \int_0^t \int_{\Omega_{\delta_{n,\epsilon}^\sharp}(s)} (\nabla\times\mathbf{u}_\epsilon^n)\cdot(\nabla\times\boldsymbol{\psi}_\epsilon^n)$$

$$-\frac{1}{2}\int_0^t \int_{\Gamma_{f1}} u_{\epsilon,1}^n v_{n,\epsilon,1}^\sharp \cdot \psi_{\epsilon,1}^n + \frac{1}{2}\int_0^t \int_{\Gamma_{f2}} u_{\epsilon,1}^n v_{n,\epsilon,1}^\sharp \cdot \psi_{\epsilon,1}^n$$

$$+\frac{1}{2}\int_0^t \int_{\Omega_s} \partial_t\eta_\epsilon^n \partial_t\delta_{n,\epsilon}^\sharp b + \int_0^t\int_{\Omega_s} \partial_{tt}\eta_\epsilon^n b + \int_0^t\int_{\Omega_s}\Delta\eta_\epsilon^n\Delta b + \gamma\int_0^t\int_{\Omega_s}\Delta(\partial_t\eta_\epsilon^n)\Delta b$$

$$+\int_0^t \int_{\Omega_{\delta_{n,\epsilon}^\sharp}(s)} (\nabla\times\mathbf{v}_{n,\epsilon}^\sharp)\times\mathbf{u}_\epsilon^n\cdot\boldsymbol{\psi}_\epsilon^n = \int_0^t\int_{\Omega_{\delta_{n,\epsilon}^\sharp}(s)} \mathbf{f}\cdot\boldsymbol{\psi}_\epsilon^n + \int_0^t\int_{\Omega_s} g\cdot b - \int_0^t\int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi}_\epsilon^n,$$

$\forall \boldsymbol{\psi}_\epsilon^n \in \mathbf{V}_{\delta_{n,\epsilon}^\sharp}$, $b\in L^2(0,T;H_0^2(\Omega_s))$ such that

$$\boldsymbol{\psi}_\epsilon^n(t,x_1,x_2,1+\delta_{n,\epsilon}^\sharp(t,x_1,x_2)) = (0,0,b(t,x_1,x_2)) \text{ on } \Omega_s.$$

Observe that here the test functions depend on n, which could create problems when passing to the limit with n (actually, the main problem comes from the term $\nu\int_0^t\int_{\Omega_{\delta_{n,\epsilon}^\sharp}(s)}(\nabla\times\mathbf{u}_\epsilon^n)\cdot(\nabla\times\boldsymbol{\psi}_\epsilon^n)$, for which we do not have enough regularity). However, we can restrict ourselves to taking test functions which no more depend on n, but which are still admissible for sufficiently large n's. The idea is based on the one in [CDEG02], but because of our problem formulation we have to nontrivially modify that proof. Thus, take $\boldsymbol{\psi}_\epsilon^0 \in \mathcal{D}(Q_{\delta_\epsilon^\sharp,T})^\dagger$ such that div $\boldsymbol{\psi}_\epsilon^0 = 0$. Then $(\boldsymbol{\psi}_\epsilon^0, 0)$ is admissible for n large enough, since $\delta^n \to \delta$ uniformly for $n \to \infty$, thus for large n's the difference between δ^n and δ is very small. For $b \in L^2(0,T;H_0^2(\Omega_s))$ define

$$\boldsymbol{\psi}_\epsilon^1(b) := \begin{cases} (0,0,b) \text{ in } \mathcal{B}_{2K} - \Omega_{\delta_\epsilon^\sharp}(t) \\ \mathbf{B}(b) \text{ in } \Omega_{\delta_\epsilon^\sharp}(t) \end{cases},$$

where $\mathbf{B}(b)$ is such that div $\mathbf{B}(b) = 0$, $\mathbf{B}(b) \times \mathbf{n} = 0$ on Γ_f, $\mathbf{B}(b) = 0$ on $\Gamma_b \cup \Gamma_{sides}$ and, of course, $\mathbf{B}(b) = (0,0,b)$ on $\partial\Omega_{\delta_\epsilon^\sharp}(t) - (\Gamma_f \cup \Gamma_b \cup \Gamma_{sides})$ (solve for instance a Stokes problem with these boundary conditions and with prescribed pressures on Γ_f, like (for instance) in [Bern00] or in [CMP94]). Then $(\boldsymbol{\psi}_\epsilon^1(b), b)$ is admissible for all n.

With these test functions we can pass to the limit, since $(\mathbf{u}_\epsilon^n, \eta_\epsilon^n)$ is weakly convergent. Any test function $\boldsymbol{\psi}_\epsilon \in \mathbf{V}_{\delta_\epsilon^\sharp}$ with $\boldsymbol{\psi}_\epsilon(t, x_1, x_2, 1 + \delta_\epsilon^\sharp(t, x_1, x_2)) = (0, 0, b(t, x_1, x_2))$ on Ω_s can be written as $\boldsymbol{\psi}_\epsilon - \boldsymbol{\psi}_\epsilon^1(b) + \boldsymbol{\psi}_\epsilon^1(b)$ and $\boldsymbol{\psi}_\epsilon - \boldsymbol{\psi}_\epsilon^1(b)$ can be approximated by solenoidal functions of $\mathcal{D}(Q_{\delta_\epsilon^\sharp,T})$. It follows that $(\chi_\epsilon \widehat{\tilde{\mathbf{u}}}_\epsilon, \widehat{\eta}_\epsilon)$ is the unique solution of the linearized approximate problem associated to $(\mathbf{v}, \delta)$, thus $(\widehat{\tilde{\mathbf{u}}}_\epsilon, \widehat{\eta}_\epsilon) = FP_\epsilon(\mathbf{v}, \delta)$ and the whole sequence $(\tilde{\mathbf{u}}_\epsilon^n, \eta_\epsilon^n)_{n\in\mathbf{N}}$ converges (weakly) to $FP_\epsilon(\mathbf{v}, \delta)$.

$\mathbf{X}$ is nonempty, closed, bounded and convex: this is also clear.

We are able now to apply the Second Schauder Fixed Point Theorem to deduce that FP_ϵ has a fixed point (at least). But this means that there exists a weak solution of our approximate problem. □

In order to show the existence of a weak solution of our initial problem, we have to pass to the limit with $\epsilon \to 0$. For this we need compactness for $\tilde{\mathbf{u}}_\epsilon$ in $\mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))$ and for $\partial_t \eta_\epsilon$ in $L^2(0,T;L^2(\Omega_s))$. We therefore have to deduce some compactness results for the solution of the approximate problem, i.e. some bounds which should not depend on ϵ.

7. Compactness results

In the previous section we have seen that the solution we found for the approximate problem satisfies the estimates (20) and (21) and that $\tilde{\mathbf{u}}_\epsilon$ has the same regularity in the function spaces for $\mathcal{B}_{2K}$ as $\mathbf{u}_\epsilon$ in the function spaces for $\Omega_{\eta_\epsilon^\sharp}(t)$. But this is still not enough for us to pass to the limit with $\epsilon \to 0$, thus we need some stronger results, namely

† as usual, $\mathcal{D}(Q_{\delta_\epsilon^\sharp,T})$ denotes the space of $\mathcal{C}^\infty$ functions on $Q_{\delta_\epsilon^\sharp,T}$, with compact support

some compactness results. The method to use is the one involving time difference quotients and certain test functions, like for instance in [ABC99], [Salv85] or [Luka97].

Thus, let $h > 0$ be small enough and let us make the following notation: for any function $\mathbf{k}$ depending on time and space we denote $\mathbf{k}^-(t,.) := \mathbf{k}(t-h,.)$ and $\mathbf{k}^+(t,.) := \mathbf{k}(t+h,.)$.

Lemma 7.1. *Let $T > 0$ such that $\min_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon) \geq \alpha > 0$. Then for all $h > 0$ as above there holds:*

$$\int_0^T \int_{\mathcal{B}_{2K}} \chi_\epsilon |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2 + \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon - \partial_t \eta_\epsilon^-)^2 \leq Ch^{1/3} \tag{30}$$

and

$$\int_0^T \int_{\mathcal{B}_{2K}} |\chi_\epsilon \tilde{\mathbf{u}}_\epsilon - \chi_\epsilon^- \tilde{\mathbf{u}}_\epsilon^-|^2 \leq Ch^{1/3}, \tag{31}$$

where for $t < 0$ we extend η_ϵ by η_0^ϵ (thus $\partial_t\eta_\epsilon$ by 0) and $\tilde{\mathbf{u}}_\epsilon$ by 0. The constant C does not depend on ϵ and χ_ϵ is the characteristic function of $\Omega_{\eta_\epsilon^\sharp}(t)$.

Proof. We show first that (30) implies (31). Indeed,

$$\begin{aligned} |\chi_\epsilon \tilde{\mathbf{u}}_\epsilon - \chi_\epsilon^- \tilde{\mathbf{u}}_\epsilon^-|^2 &= |\chi_\epsilon \tilde{\mathbf{u}}_\epsilon - \chi_\epsilon \tilde{\mathbf{u}}_\epsilon^- + \chi_\epsilon \tilde{\mathbf{u}}_\epsilon^- - \chi_\epsilon^- \tilde{\mathbf{u}}_\epsilon^-|^2 \\ &\leq C(|\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2 + |\chi_\epsilon - \chi_\epsilon^-|^2 \cdot |\tilde{\mathbf{u}}_\epsilon^-|^2). \end{aligned}$$

Now we integrate between 0 and T and on $\mathcal{B}_{2K}$ and use (30) for the first term in the right hand side to obtain

$$\begin{aligned} \int_0^T \int_{\mathcal{B}_{2K}} |\chi_\epsilon \tilde{\mathbf{u}}_\epsilon - \chi_\epsilon^- \tilde{\mathbf{u}}_\epsilon^-|^2 &\leq Ch^{1/3} + C\int_0^T \int_{\mathcal{B}_{2K}} |\chi_\epsilon - \chi_\epsilon^-| \cdot |\tilde{\mathbf{u}}_\epsilon^-|^2 \\ &\leq Ch^{1/3} + C\int_0^T |\chi_\epsilon - \chi_\epsilon^-|_{\mathbf{L}^3(\mathcal{B}_{2K})} |\tilde{\mathbf{u}}_\epsilon^-|^2_{\mathbf{L}^3(\mathcal{B}_{2K})} \\ &\leq Ch^{\frac{1}{3}} + C||\tilde{\mathbf{u}}_\epsilon||^2_{\mathbf{L}^4(0,T;\mathbf{L}^3(\mathcal{B}_{2K}))} \Big(\int_0^T |\chi_\epsilon - \chi_\epsilon^-|^2_{\mathbf{L}^3(\mathcal{B}_{2K})}\Big)^{\frac{1}{2}}. \end{aligned}$$

Now, $\tilde{\mathbf{u}}_\epsilon$ is bounded in $\mathbf{L}^\infty(0,T;\mathbf{L}^2(\mathcal{B}_{2K})) \cap \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$ (since the solution of the approximate problem satisfies (20) and (21), see also the begin of this section), thus $\tilde{\mathbf{u}}_\epsilon$ belongs to $\mathbf{L}^2(0,T;\mathbf{L}^6(\mathcal{B}_{2K})) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))$ and consequently $\tilde{\mathbf{u}}_\epsilon$ is bounded in $\mathbf{L}^4(0,T;\mathbf{L}^3(\mathcal{B}_{2K}))$

(see, for instance, [Lion69], p.73). It remains to show the boundedness of the other factor in the right hand side of the last inequality above. For this, remember that $\partial_t \eta_\epsilon^\sharp$ is bounded in $L^\infty(0,T;L^2(\Omega_s))$ independently of ϵ (see (20)), therefore we can calculate

$$\int_{\mathcal{B}_{2K}} |\chi_\epsilon - \chi_\epsilon^-|^3 = \int_{\mathcal{B}_{2K}} |\chi_\epsilon - \chi_\epsilon^-| = \int_{\Omega_{\eta_\epsilon^\sharp}(t)} d\mathbf{x} - \int_{\Omega_{\eta_\epsilon^\sharp}(t-h)} d\mathbf{x}$$

$$= \int_{\Omega_s} |\eta_\epsilon^\sharp(t) - \eta_\epsilon^\sharp(t-h)| = \int_{\Omega_s} |\int_{t-h}^{t} \partial_t \eta_\epsilon^\sharp(s) ds| \le Ch$$

Thus, $|\chi_\epsilon - \chi_\epsilon^-|^2_{\mathbf{L}^3} \le Ch^{2/3}$ and it follows that

$$\int_0^T \int_{\mathcal{B}_{2K}} |\chi_\epsilon \tilde{\mathbf{u}}_\epsilon - \chi_\epsilon^- \tilde{\mathbf{u}}_\epsilon^-|^2 \le Ch^{1/3}.$$

For the proof of (30), remember that the solution $(\mathbf{u}_\epsilon, \eta_\epsilon)$ of the approximate problem satisfies

$$(32) \qquad \int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} \partial_t \mathbf{u}_\epsilon \cdot \boldsymbol{\psi}_\epsilon + \nu \int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} (\nabla \times \mathbf{u}_\epsilon) \cdot (\nabla \times \boldsymbol{\psi}_\epsilon)$$

$$+ \int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} (\nabla \times \tilde{\mathbf{u}}_\epsilon^\sharp) \times \mathbf{u}_\epsilon \cdot \boldsymbol{\psi}_\epsilon + \frac{1}{2} \int_0^T \int_{\Omega_s} \partial_t \eta_\epsilon \partial_t \eta_\epsilon^\sharp b - \frac{1}{2} \int_0^T \int_{\Gamma_{f1}} u_{\epsilon,1} \tilde{u}_{\epsilon,1}^\sharp \psi_{\epsilon,1}$$

$$+ \frac{1}{2} \int_0^T \int_{\Gamma_{f2}} u_{\epsilon,1} \tilde{u}_{\epsilon,1}^\sharp \psi_{\epsilon,1} + \int_0^T \int_{\Omega_s} \partial_{tt} \eta_\epsilon b + \int_0^T \int_{\Omega_s} \Delta \eta_\epsilon \Delta b + \gamma \int_0^T \int_{\Omega_s} \Delta(\partial_t \eta_\epsilon) \Delta b$$

$$= \int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(s)} \mathbf{f} \cdot \boldsymbol{\psi}_\epsilon + \int_0^T \int_{\Omega_s} g \cdot b - \int_0^T \int_{\Gamma_f} p_0 \cdot \mathbf{n} \cdot \boldsymbol{\psi}_\epsilon,$$

$\forall \boldsymbol{\psi}_\epsilon \in \mathbf{V}_{\eta_\epsilon^\sharp}$, $b \in L^2(0,T;H_0^2(\Omega_s))$ such that

$$\boldsymbol{\psi}_\epsilon(t, x_1, x_2, 1 + \eta_\epsilon^\sharp(t, x_1, x_2)) = (0, 0, b(t, x_1, x_2)) \text{ on } \Omega_s.$$

As for the test functions, we make the following choice: similarly as in [CDEG02], we take for $\beta > 1$

$$\mathbf{v}_\beta(x_1, x_2, x_3) := (\beta v_1(x_1, x_2, \beta x_3), \beta v_2(x_1, x_2, \beta x_3), v_3(x_1, x_2, \beta x_3))$$

and then (like in [ABC99], [Luka97] or [Salv85]) put

$$\boldsymbol{\psi}_\epsilon := \int_{t-h}^{t} (\tilde{\mathbf{u}}_\epsilon)_\beta(s)ds, \quad b = \int_{t-h}^{t} \partial_t\eta_\epsilon(s)ds.$$

Observe that $\boldsymbol{\psi}_\epsilon \in \mathbf{H}^1(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$ and $b \in H^1(0,T;H_0^2(\Omega_s))$ and that the functions chosen as above are admissible test functions.

Indeed, we also notice that

$$||\eta_\epsilon - \eta_\epsilon^-||_{L^\infty((0,T)\times\bar{\Omega}_s)} \leq Ch^{1/3}; \ ||\eta_\epsilon^\sharp - (\eta_\epsilon^\sharp)^-||_{L^\infty((0,T)\times\bar{\Omega}_s)} \leq Ch^{1/3},$$

since η_ϵ is bounded in $H^1(0,T;H_0^2(\Omega_s))$ and this space is imbedded with continuous injection into $C^{0,1/2}(0,T;C^{0,q}(\Omega_s))$ $(0 \leq q < 1)$.

Thus for an adequate β we have that

$$\boldsymbol{\psi}_\epsilon(t,x_1,x_2,1+\eta_\epsilon^\sharp(t,x_1,x_2)) = (0,0,\int_{t-h}^{t} \partial_t\eta_\epsilon(s,x_1,x_2)ds) \text{ on } \Omega_s.$$

Next we shall analyze the boundedness of each term in (32) (with the previously considered test functions): the sum of the first, the fourth, the fifth and the sixth terms in (32) writes (after anew using the transport formula):

$$\int_{\Omega_{\eta_\epsilon^\sharp}(T)} \mathbf{u}_\epsilon(T)\int_{T-h}^{T}(\tilde{\mathbf{u}}_\epsilon)_\beta - \frac{1}{2}\int_0^T\int_{\Omega_s}(\int_{t-h}^{t}\partial_t\eta_\epsilon)\partial_t\eta_\epsilon\partial_t\eta_\epsilon^\sharp - \int_0^T\int_{\Omega_{\eta_\epsilon^\sharp}(t)}\mathbf{u}_\epsilon\cdot[(\tilde{\mathbf{u}}_\epsilon)_\beta - (\tilde{\mathbf{u}}_\epsilon)_\beta^-]$$

(we used here that $\tilde{\mathbf{u}}_\epsilon = 0$ for $t < 0$).

For the last integral above we have:

$$-\int_0^T\int_{\Omega_{\eta_\epsilon^\sharp}(t)}\mathbf{u}_\epsilon\cdot[(\tilde{\mathbf{u}}_\epsilon)_\beta - (\tilde{\mathbf{u}}_\epsilon)_\beta^-]$$

$$= -\int_0^T\int_{\Omega_{\eta_\epsilon^\sharp}(t)}\mathbf{u}_\epsilon\cdot(\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-) - \int_0^T\int_{\Omega_{\eta_\epsilon^\sharp}(t)}\tilde{\mathbf{u}}_\epsilon[(\tilde{\mathbf{u}}_\epsilon)_\beta - \tilde{\mathbf{u}}_\epsilon - ((\tilde{\mathbf{u}}_\epsilon)_\beta^- - \tilde{\mathbf{u}}_\epsilon^-)]$$

and

$$|\int_0^T\int_{\Omega_{\eta_\epsilon^\sharp}(t)}\tilde{\mathbf{u}}_\epsilon[(\tilde{\mathbf{u}}_\epsilon)_\beta - \tilde{\mathbf{u}}_\epsilon - ((\tilde{\mathbf{u}}_\epsilon)_\beta^- - \tilde{\mathbf{u}}_\epsilon^-)]|$$

$$\leq 2||\mathbf{u}_\epsilon||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))}||(\tilde{\mathbf{u}}_\epsilon)_\beta - \tilde{\mathbf{u}}_\epsilon||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))} \leq C(\beta-1) \leq Ch^{1/3}$$

(β chosen adequately). Now, for $-\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} \mathbf{u}_\epsilon \cdot (\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-)$ we make the following calculations:

$$\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} \mathbf{u}_\epsilon \cdot (\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-) = \int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\mathbf{u}_\epsilon|^2 - \int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} \mathbf{u}_\epsilon \cdot \tilde{\mathbf{u}}_\epsilon^- \tag{33}$$

$$= -\frac{1}{2}\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\mathbf{u}_\epsilon|^2 + \frac{1}{2}\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\tilde{\mathbf{u}}_\epsilon^-|^2 - \frac{1}{2}\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2$$

$$-\frac{1}{2}\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\mathbf{u}_\epsilon|^2 + \frac{1}{2}\int\limits_0^{T-h} \int_{\Omega_{\eta_\epsilon^\sharp}(t+h)} |\tilde{\mathbf{u}}_\epsilon^-|^2 - \frac{1}{2}\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2$$

$$= \frac{1}{2}\int\limits_0^{T-h} \int_{\mathcal{B}_{2K}} (\chi_\epsilon^+ - \chi_\epsilon)|\tilde{\mathbf{u}}_\epsilon|^2 - \frac{1}{2}\int\limits_{T-h}^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\mathbf{u}_\epsilon|^2 - \frac{1}{2}\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2.$$

It is clear that the second term in the right hand side of this last identity is negative and that analogously as in the proof of (30)$\Rightarrow$(31) it can be shown that $\int\limits_0^T \int_{\mathcal{B}_{2K}} (\chi_\epsilon^+ - \chi_\epsilon)|\tilde{\mathbf{u}}_\epsilon|^2 \le Ch^{1/3}$. This implies that

$$\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} \mathbf{u}_\epsilon \cdot (\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-) \le Ch^{1/3} - \frac{1}{2}\int\limits_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2.$$

Thus the sum of the first, 4th, 5th and 6th terms in (32) is majorized by:

$$\int_{\Omega_{\eta_\epsilon^\sharp}(T)} \mathbf{u}_\epsilon(T) \int\limits_{T-h}^T (\tilde{\mathbf{u}}_\epsilon)_\beta - \frac{1}{2}\int\limits_0^T \int\limits_{\Omega_s} (\int\limits_{t-h}^t \partial_t\eta_\epsilon)\partial_t\eta_\epsilon\partial_t\eta_\epsilon^\sharp - \frac{1}{2}\int\limits_0^T \int\limits_{\Omega_{\eta_\epsilon^\sharp}(t)} |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2 + Ch^{1/3}.$$

Now, the second term above is majorized as follows:

$$|-\frac{1}{2}\int\limits_0^T \int_{\Omega_s} (\int\limits_{t-h}^t \partial_t\eta_\epsilon)\partial_t\eta_\epsilon^\sharp\partial_t\eta_\epsilon| \le \frac{1}{2}\int\limits_0^T ||\partial_t\eta_\epsilon||_{L^\infty(\Omega_s)}||\partial_t\eta_\epsilon^\sharp||_{L^2(\Omega_s)}||\eta_\epsilon - \eta_\epsilon^-||_{L^2(\Omega_s)}$$

and

$$||\eta_\epsilon - \eta_\epsilon^-||_{L^2(\Omega_s)} \leq \int_{t-h}^{t} ||\partial_t \eta_\epsilon||_{L^2(\Omega_s)} \leq C\sqrt{h}\Big(\int_{t-h}^{t} ||\partial_t \eta_\epsilon||^2_{L^2(\Omega_s)}\Big)^{1/2} \leq C\sqrt{h} \leq Ch^{1/3},$$

thus with the help of the energy estimates we have that this term is bounded, too, by $Ch^{1/3}$.

The second term in (32) is

$$\nu \int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} (\nabla \times \mathbf{u}_\epsilon) \cdot (\nabla \times \int_{t-h}^{t} (\tilde{\mathbf{u}}_\epsilon)_\beta)$$

$$\leq \nu \int_0^T ||\nabla \times \mathbf{u}_\epsilon||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))} \int_{t-h}^{t} ||\nabla \times (\tilde{\mathbf{u}}_\epsilon)_\beta||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(s))}$$

$$\leq \nu\sqrt{h} \int_0^T ||\nabla \times \mathbf{u}_\epsilon||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))} \Big(\int_{t-h}^{t} ||\nabla \times (\tilde{\mathbf{u}}_\epsilon)_\beta||^2_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))}\Big)^{1/2}$$

$$\leq C\sqrt{h} \leq Ch^{1/3}$$

(by the Cauchy-Schwarz inequality and the fact that we have $\int_{t-h}^{t} (\tilde{\mathbf{u}}_\epsilon)_\beta ds \in \mathbf{H}^1(0,T;\mathbf{H}^1(\mathcal{B})))$.

We now come to the convective term, which is majorized as follows:

$$|\int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} (\nabla \times \tilde{\mathbf{u}}_\epsilon^\sharp) \times \mathbf{u}_\epsilon \cdot (\int_{t-h}^{t} (\tilde{\mathbf{u}}_\epsilon)_\beta)| \leq$$

$$\leq \int_0^T ||\nabla \times \tilde{\mathbf{u}}_\epsilon^\sharp||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))} ||\mathbf{u}_\epsilon||_{\mathbf{L}^4(\Omega_{\eta_\epsilon^\sharp}(t))} \int_{t-h}^{t} ||(\tilde{\mathbf{u}}_\epsilon)_\beta||_{\mathbf{L}^4(\Omega_{\eta_\epsilon^\sharp}(t))}$$

$$\leq \sqrt{h} \int_0^T ||\nabla \times \tilde{\mathbf{u}}_\epsilon^\sharp||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))} ||\mathbf{u}_\epsilon||_{\mathbf{L}^4(\Omega_{\eta_\epsilon^\sharp}(t))} \Big(\int_{t-h}^{t} ||(\tilde{\mathbf{u}}_\epsilon)_\beta||^2_{\mathbf{L}^4(\Omega_{\eta_\epsilon^\sharp}(t))}\Big)^{1/2}$$

$$\leq \sqrt{h} \int_0^T ||\nabla \times \tilde{\mathbf{u}}_\epsilon^\sharp||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))} ||\mathbf{u}_\epsilon||_{\mathbf{L}^4(\Omega_{\eta_\epsilon^\sharp}(t))} ||(\tilde{\mathbf{u}}_\epsilon)_\beta||^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))}$$

$$\leq C\sqrt{h} \leq Ch^{1/3}.$$

For the integral term containing the second time derivative of the displacement we have:

$$\int_0^T \int_{\Omega_s} \partial_{tt}\eta_\epsilon (\int_{t-h}^t \partial_t \eta_\epsilon ds) = \int_0^T \int_{\Omega_s} \left\{ \partial_t \Big[\partial_t \eta_\epsilon (\int_{t-h}^t \partial_t \eta_\epsilon ds) \Big] - \partial_t \eta_\epsilon \partial_t (\int_{t-h}^t \partial_t \eta_\epsilon ds) \right\}$$

$$= \int_{\Omega_s} \partial_t \eta_\epsilon (T) \int_{T-h}^T \partial_t \eta_\epsilon - \int_0^T \int_{\Omega_s} \partial_t \eta_\epsilon \partial_t (\eta_\epsilon - \eta_\epsilon^-)$$

(remember that $\partial_t \eta_\epsilon = 0$ for $t < 0$).

Now, for the first term in the right hand side above we write:

$$| \int_{\Omega_s} \partial_t \eta_\epsilon (T) (\int_{T-h}^T \partial_t \eta_\epsilon) | \leq ||\partial_t \eta_\epsilon (T)||_{L^2(\Omega_s)} \int_{T-h}^T ||\partial_t \eta_\epsilon||_{L^2(\Omega_s)} \leq Ch$$

and for the second one we write (like in (33)):

$$\begin{aligned}
&- \int_0^T \int_{\Omega_s} \partial_t \eta_\epsilon \partial_t (\eta_\epsilon - \eta_\epsilon^-) \\
= \;& -\frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon)^2 + \frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon^-)^2 - \frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon - \partial_t \eta_\epsilon^-)^2 \\
= \;& -\frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon)^2 + \frac{1}{2} \int_{-h}^{T-h} \int_{\Omega_s} (\partial_t \eta_\epsilon)^2 - \frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon - \partial_t \eta_\epsilon^-)^2 \\
= \;& -\frac{1}{2} \int_{T-h}^T \int_{\Omega_s} (\partial_t \eta_\epsilon)^2 - \frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon - \partial_t \eta_\epsilon^-)^2 \\
\leq \;& -\frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon - \partial_t \eta_\epsilon^-)^2,
\end{aligned}$$

thus

$$\int_0^T \int_{\Omega_s} \partial_{tt} \eta_\epsilon (\int_{t-h}^t \partial_t \eta_\epsilon ds) \leq Ch - \frac{1}{2} \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon - \partial_t \eta_\epsilon^-)^2.$$

Next,

$$|\int_0^T \int_{\Omega_s} \Delta\eta_\epsilon \Delta(\int_{t-h}^t \partial_t\eta_\epsilon)| \leq \int_0^T ||\Delta\eta_\epsilon||_{L^2(\Omega_s)} \int_{t-h}^t ||\Delta(\partial_t\eta_\epsilon)||_{L^2(\Omega_s)}$$

$$\leq \sqrt{h}\int_0^T ||\Delta\eta_\epsilon||_{L^2(\Omega_s)}\Big(\int_{t-h}^t ||\Delta(\partial_t\eta_\epsilon)||^2_{L^2(\Omega_s)}\Big)^{1/2} \leq C\sqrt{h} \leq Ch^{1/3}$$

and

$$|\gamma\int_0^T \int_{\Omega_s} \Delta(\partial_t\eta_\epsilon)\Delta(\int_{t-h}^t \partial_t\eta_\epsilon)| \leq \gamma\int_0^T ||\Delta\partial_t\eta_\epsilon||_{L^2(\Omega_s)} \int_{t-h}^t ||\Delta(\partial_t\eta_\epsilon)||_{L^2(\Omega_s)}$$

$$\leq \gamma\sqrt{h}\int_0^T ||\Delta\partial_t\eta_\epsilon||_{\mathbf{L}^2(\Omega_s)}\Big(\int_{t-h}^t ||\Delta(\partial_t\eta_\epsilon)||^2_{\mathbf{L}^2(\Omega_s)}\Big)^{1/2} \leq C\sqrt{h} \leq Ch^{1/3}.$$

We have arrived at the terms of the right hand side. For the first one we have analogously as for the usual terms above:

$$\int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} \mathbf{f}(\int_{t-h}^t (\tilde{\mathbf{u}}_\epsilon)_\beta) \leq \int_0^T ||\mathbf{f}||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))} \int_{t-h}^t ||(\tilde{\mathbf{u}}_\epsilon)_\beta||_{\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t))} \leq C\sqrt{h}$$

and for the second one the bound is (analogously, again) Ch. The last of the right hand side terms is bounded by $C\sqrt{h}$, taking into account the energy estimates for the approximate solution and the hypothesis made on p_0.

All the above estimates imply that

$$\int_0^T \int_{\Omega_s} (\partial_t\eta_\epsilon - \partial_t\eta_\epsilon^-)^2 + \int_0^T \int_{\Omega_{\eta_\epsilon^\sharp}(t)} |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2 \leq Ch^{1/3}$$

and this completes the proof of this lemma. □

By the lemma above and the Frechet-Kolmogorov characterization of a relatively compact subset of $\mathbf{L}^2$ it follows that $\chi_\epsilon\tilde{\mathbf{u}}_\epsilon$ is relatively compact in $\mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))$ and $\partial_t\eta_\epsilon$ is relatively compact in $L^2(0,T;L^2(\Omega_s))$, thus $\tilde{\mathbf{u}}_\epsilon$ is relatively compact in $\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))$.

8. Passage to the limit

Let $T > 0$ such that $\inf_\epsilon \min_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon) \geq \alpha > 0$. By the energy estimates, the usual Sobolev imbeddings and the compactness results in the previous section we have the following convergences:

$$
\begin{aligned}
\eta_\epsilon &\overset{\epsilon\to 0}{\to} \eta \text{ in } C^{0,1/2}(0,T;C^{0,q}(\bar{\Omega}_s)) \ (0 \leq q < 1)\\
\eta_\epsilon &\overset{\epsilon\to 0}{\rightharpoonup} \eta \text{ in } H^1(0,T;H^2(\Omega_s))\\
\partial_t\eta_\epsilon &\overset{\epsilon\to 0}{\to} \partial_t\eta \text{ in } L^2(0,T;L^2(\Omega_s))\\
\tilde{\mathbf{u}}_\epsilon &\overset{\epsilon\to 0}{\to} \widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))\\
\tilde{\mathbf{u}}_\epsilon &\overset{\epsilon\to 0}{\rightharpoonup} \widehat{\mathbf{u}} \text{ in } L^2(0,T;H^1_{0,\Gamma_b\cup\Gamma_{sides}}(\mathcal{B}_{2K}))\\
\chi_\epsilon\tilde{\mathbf{u}}_\epsilon &\overset{\epsilon\to 0}{\to} \chi\widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))\\
\eta^\sharp_\epsilon &\overset{\epsilon\to 0}{\to} \eta \text{ in } C^{0,1/2}(0,T;C^{0,q}(\bar{\Omega}_s)) \ (0 \leq q < 1)\\
\partial_t\eta^\sharp_\epsilon &\overset{\epsilon\to 0}{\to} \partial_t\eta \text{ in } L^2(0,T;L^2(\Omega_s))\\
\tilde{\mathbf{u}}^\sharp_\epsilon &\overset{\epsilon\to 0}{\to} \widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K})),
\end{aligned}
\tag{34}
$$

where we have denoted by $(\widehat{\mathbf{u}}, \eta)$ the limit of any subsequence $(\tilde{\mathbf{u}}_\epsilon, \eta_\epsilon)_{\epsilon>0}$ of the sequence $(\tilde{\mathbf{u}}_\epsilon, \eta_\epsilon)_{\epsilon>0}$.

Furthermore,

$$
\chi_\epsilon\nabla\times\mathbf{u}_\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \chi\nabla\times\widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K})).
\tag{35}
$$

Indeed, observe that there exists some $\boldsymbol{\xi}$ such that $\chi_\epsilon\nabla\times\mathbf{u}_\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \boldsymbol{\xi}$ in $\mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))$; since $\eta^\sharp_\epsilon \overset{\epsilon\to 0}{\to} \eta$ in $C^{0,\frac{1}{2}}(0,T;C^{0,q}(\bar{\Omega}_s))$, $\boldsymbol{\xi} = 0$ in $\mathcal{B}_{2K,T} - Q_{\eta^\sharp_\epsilon,T}$ and $\boldsymbol{\xi}|_{Q_{\eta,T}} = \nabla\times\widehat{\mathbf{u}}$. Also notice that we have $\chi_\epsilon\nabla\times\tilde{\mathbf{u}}^\sharp_\epsilon \overset{\epsilon\to 0}{\rightharpoonup} \chi\nabla\times\widehat{\mathbf{u}}$ in $\mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))$.

Next, we want the equality of velocities on the interface to be preserved in the limit, thus we are now concerned with the limit for $\epsilon\to 0$ of

$$
\mathbf{u}_\epsilon(t,x_1,x_2,1+\eta^\sharp_\epsilon(t,x_1,x_2)) = (0,0,\partial_t\eta_\epsilon(t,x_1,x_2)) \text{ on } (0,T)\times\Omega_s.
$$

Now, $(0,0,\partial_t\eta_\epsilon(t,x_1,x_2)) \to (0,0,\partial_t\eta)$ in $L^2(0,T;L^2(\Omega_s))$ and for the left hand side of the above equality let us define

$$
\boldsymbol{\nu}_\epsilon := \begin{cases} (0,0,\partial_t\eta_\epsilon) \text{ in } \mathcal{B}_{2K} - C_{\alpha/2} \\ \mathcal{R}(0,0,\partial_t\eta_\epsilon) \text{ in } C_{\alpha/2} \end{cases},
$$

where $\mathcal{R}$ is a lifting from $\mathbf{H}^{1/2}(\Omega_s\times\{\alpha/2\})$ to $\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}}(C_{\alpha/2})$ such that we have div $\mathcal{R}(0,0,\partial_t\eta_\epsilon) = 0$ in $C_{\alpha/2}$, $\mathcal{R}(0,0,\partial_t\eta_\epsilon)|_{(\Gamma_b\cup\Gamma_{sides})\cap\partial C_{\alpha/2}} =$

0 and $\mathcal{R}(0,0,\partial_t\eta_\epsilon)\times\mathbf{n}=0$ on $\Gamma_f\cap\partial C_{\alpha/2}$ (again, one can solve for instance a Stokes problem in $C_{\alpha/2}$ with these boundary conditions and with prescribed pressures on $\Gamma_f\cap\partial C_{\alpha/2}$).

Then notice that $\tilde{\mathbf{u}}_\epsilon-\boldsymbol{\nu}_\epsilon\in\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$ and its bound in this space does not depend on ϵ. It follows that there exists a subsequence $(\tilde{\mathbf{u}}_\epsilon-\boldsymbol{\nu}_\epsilon)_{\epsilon>0}$ of $(\tilde{\mathbf{u}}_\epsilon-\boldsymbol{\nu}_\epsilon)_{\epsilon>0}$ and a function $\boldsymbol{\nu}_0:=\widehat{\mathbf{u}}-\boldsymbol{\nu}$ ($\boldsymbol{\nu}$ is the limit of $\boldsymbol{\nu}_\epsilon$) such that $\tilde{\mathbf{u}}_\epsilon-\boldsymbol{\nu}_\epsilon\overset{\epsilon\to0}{\rightharpoonup}\boldsymbol{\nu}_0$ in $\mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$. Notice that we have $\boldsymbol{\nu}_0=0$ on $\Gamma_b\cup\Gamma_{sides}$, $\boldsymbol{\nu}_0\times\mathbf{n}=0$ on Γ_f and $\boldsymbol{\nu}_0=0$ in $Q_{\eta+\delta,T}$ for all $\delta>0$ (since $\eta_\epsilon^\sharp\to\eta$ (uniformly)). We have $\boldsymbol{\nu}_0\in\mathbf{L}^2(0,T;\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}\cup\Gamma_1(t)}(\Omega_\eta(t)))$, thus by Lemma 4.5 it follows that $\gamma_{\eta(t)}(\boldsymbol{\nu}_0)=0$ and, since $\gamma_{\eta(t)}(\boldsymbol{\nu})=(0,0,\partial_t\eta)$, that $\gamma_{\eta(t)}(\widehat{\mathbf{u}})=(0,0,\partial_t\eta)$, thus the equality of velocities is preserved on the interface.

Now we can pass to the limit for $\epsilon\to0$ in the weak formulation (approximate problem):

$$-\int_0^t\int_{\Omega_{\eta_\epsilon^\sharp}(s)}\mathbf{u}_\epsilon\cdot\partial_t\boldsymbol{\psi}_\epsilon+\int_{\Omega_{\eta_\epsilon^\sharp}(t)}\mathbf{u}_\epsilon(t)\boldsymbol{\psi}_\epsilon(t)+\nu\int_0^t\int_{\Omega_{\eta_\epsilon^\sharp}(s)}(\nabla\times\mathbf{u}_\epsilon)\cdot(\nabla\times\boldsymbol{\psi}_\epsilon)$$

$$+\int_0^t\int_{\Omega_{\eta_\epsilon^\sharp}(s)}(\nabla\times\tilde{\mathbf{u}}_\epsilon^\sharp)\times\mathbf{u}_\epsilon\cdot\boldsymbol{\psi}_\epsilon+\frac{1}{2}\int_0^t\int_{\Omega_s}\partial_t\eta_\epsilon\partial_t\eta_\epsilon^\sharp b-\int_0^t\int_{\Omega_s}\partial_t\eta_\epsilon\partial_t b$$

$$+\int_{\Omega_s}\partial_t\eta_\epsilon(t)b(t)+\int_0^t\int_{\Omega_s}\Delta\eta_\epsilon\Delta b+\gamma\int_0^t\int_{\Omega_s}\Delta(\partial_t\eta_\epsilon)\Delta b-\frac{1}{2}\int_0^t\int_{\Gamma_{f1}}u_{\epsilon,1}\tilde{u}^\sharp_{\epsilon,1}\psi_{\epsilon,1}$$

$$+\frac{1}{2}\int_0^t\int_{\Gamma_{f2}}u_{\epsilon,1}\tilde{u}^\sharp_{\epsilon,1}\psi_{\epsilon,1}=\int_0^t\int_{\Omega_{\eta_\epsilon^\sharp}(s)}\mathbf{f}\cdot\boldsymbol{\psi}_\epsilon+\int_0^t\int_{\Omega_s}g\cdot b-\int_0^t\int_{\Gamma_f}p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi}_\epsilon$$

$$+\int_{\Omega_{\eta_0}}\mathbf{u}_0^\epsilon\cdot\boldsymbol{\psi}_\epsilon(0)+\int_{\Omega_s}\eta_{01}^\epsilon\cdot b(0),$$

for a.e. t and for all $(\boldsymbol{\psi}_\epsilon,b)\in\boldsymbol{\mathcal{V}}_{\eta_\epsilon^\sharp}\times C^1(0,T;H_0^2(\Omega_s))$ such that

$$\boldsymbol{\psi}_\epsilon(t,x_1,x_2,1+\eta_\epsilon^\sharp(t,x_1,x_2))=(0,0,b(t,x_1,x_2))\text{ on }\Omega_s.$$

Observe again that the fluid test functions depend a priori on ϵ, which might be bad for instance when passing to the limit in the third term of the above identity. We therefore consider test functions which do not depend on ϵ and which are admissible for ϵ small enough. The idea of constructing such test functions is analogous to the one in Subsection 6.5, where we have constructed test functions for the fluid which did not depend on n. The construction is very much alike with the one therein, the difference being that we now use the convergences for $\epsilon \to 0$ instead of those for $n \to \infty$ and relations (34).

Now with the test functions constructed as above we can pass to the limit in the corresponding weak formulation for $\epsilon \to 0$ and we get the existence of a weak solution of (18) on $(0, T)$ with the estimates in Section 3.

9. Some concluding remarks

The problem with both the cover and the bottom of the box being elastic can be treated in a similar way. One has to take care at the formulation of the corresponding function spaces, at the extensions to the Lipschitz domain including $\Omega_{\eta,\mu}(t)$ for any $t \in [0, T]$ ($\mu(t, x_1, x_2)$ denotes here the displacement of the bottom of the box) and also at the transformation leading to a problem on a fixed domain $R := (0, L) \times (0, l) \times (0, 1)$. The transformation corresponding to (23) in Subsection 6 would have the form $\boldsymbol{\phi}_\epsilon : (0, T) \times R \to \Omega_{\delta^\sharp_\epsilon, \pi^\sharp_\epsilon}(t)$, with

$$\boldsymbol{\phi}_\epsilon(t, x_1, x_2, x_3) := (x_1, x_2, x_3(1+\delta^\sharp_\epsilon(t, x_1, x_2)-\pi^\sharp_\epsilon(t, x_1, x_2))+\pi^\sharp_\epsilon(t, x_1, x_2)),$$

$\forall (x_1, x_2, x_3) \in R,\ t \in (0, T)$, where $\pi^\sharp_\epsilon$ has the same meaning for μ_ϵ as the meaning of $\delta^\sharp_\epsilon$ for η_ϵ.

Furthermore, we have neglected here the longitudinal displacements of the elastic cover, which is largely accepted, but not quite realistic. The present proof seems to fail when longitudinal displacements are considered, because one needs to controle the second derivatives in space for the velocity of the fluid, which does not follow from the a priori estimates. A different plate model and/or another proof might be needed in order to handle this case.

Thinking about modeling blood flow in arteries, one would like to have a more realistic model. In a forthcoming paper we shall treat the time-dependent problem of a Navier-Stokes fluid moving inside a cylinder whose wall is a thin elastic shell (again with a viscoelastic term), fixed on its boundary. We'll consider Navier-Stokes equations to model the fluid motion and assume that the velocities at both ends of the tube are zero (one could also take nonzero velocities). The boundary conditions for the fluid are thus less "realistic", but considering an elastic cylindrical shell instead of a box with elastic cover (and bottom) is closer to reality. We shall prove again the existence of a solution to the coupled problem.

10. Appendix

Proof. (of Lemma 6.1) The proof is similar to the one of Lemma 8 in [CDEG02], however with some non trivial changes, due to the form of the boundary conditions for the fluid. Multiply (25) by $\dot{\alpha}_j$ and add the corresponding equations for $j = 1, ..., m$. Also multiply (26) by $\ddot{\beta}_l$ an sum up for $l = 1, ..., n$. This is the same thing as to test our system with $(\sum_{j=1}^{m} \dot{\alpha}_j \boldsymbol{\psi}_j^{0,\epsilon} + \sum_{l=1}^{n} \ddot{\beta}_l \boldsymbol{\psi}_l^{*,\epsilon}, \sum_{l=1}^{n} \ddot{\beta}_l \rho_l)$. This is not $(\partial_t \mathbf{u}_\epsilon^{\phi_\epsilon, m, n}, \partial_{tt}\eta_\epsilon^n)$, because the Galerkin functions $\boldsymbol{\psi}_j^{0,\epsilon}$ and $\boldsymbol{\psi}_l^{*,\epsilon}$ depend on time.

Denote

$$\lambda_j := \begin{cases} \alpha_j, & 1 \le j \le m \\ \dot{\beta}_{j-m}, & m+1 \le j \le n+m \end{cases}$$

and

$$\boldsymbol{\psi}_j := \begin{cases} \boldsymbol{\psi}_j^0, & 1 \le j \le m \\ \boldsymbol{\psi}_{j-m}^*, & m+1 \le j \le n+m \end{cases} .$$

In the following we shall omit the indices of the functions implied in the calculations. We have:

$$\int_R |\sum_j \dot{\lambda}_j \boldsymbol{\psi}_j|^2 J + \int_R \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \cdot \sum_j \lambda_j \partial_t \boldsymbol{\psi}_j J$$

$$+\nu \int_R ((\frac{\mathbf{M}}{\sqrt{J}}\nabla)\times \mathbf{u}^{\phi_\epsilon})\cdot((\frac{\mathbf{M}}{\sqrt{J}}\nabla)\times \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j) - \int_R (\partial_t \boldsymbol{\phi}_\epsilon^{\phi_\epsilon} \cdot (\mathbf{M}\nabla))\mathbf{u}^{\phi_\epsilon} \cdot \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j$$

$$+\frac{1}{2}\int_R ((\mathbf{M}\nabla) \times \mathbf{v}^\sharp) \times \mathbf{u}^{\phi_\epsilon} \cdot \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j + \frac{1}{2}\int_{\Omega_s} \partial_t \eta \partial_t \delta^\sharp \partial_{tt}\eta$$

$$-\frac{1}{2}\int_{\Gamma_{f1}} u_1^{\phi_\epsilon} v_1^\sharp \cdot \sum_j \dot{\lambda}_j \boldsymbol{\psi}_{j,1} J + \frac{1}{2}\int_{\Gamma_{f2}} u_1^{\phi_\epsilon} v_1^\sharp \cdot \sum_j \dot{\lambda}_j \boldsymbol{\psi}_{j,1} J + \int_{\Omega_s} (\partial_{tt}\eta)^2$$

$$+\gamma \int_{\Omega_s} \Delta\partial_t \eta \Delta \partial_{tt}\eta + \int_{\Omega_s} \Delta\eta\Delta\partial_{tt}\eta$$

$$= \int_R \mathbf{f}^{\phi_\epsilon} \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j J - \int_{\Gamma_f} p_0^{\phi_\epsilon} \cdot \mathbf{n} \cdot \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j J + \int_{\Omega_s} g \partial_{tt}\eta.$$

After some calculations, we see that

$$||\sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \sqrt{J}||_{\mathbf{L}^2(R)} + \frac{\nu}{2}\frac{d}{dt}\int_R (\frac{\mathbf{M}}{\sqrt{J}}\nabla \times \mathbf{u}^{\phi_\epsilon})^2 + \frac{\gamma}{2}\frac{d}{dt}\int_{\Omega_s} (\Delta\partial_t\eta)^2$$

$$+||\partial_{tt}\eta||_{L^2(\Omega_s)}^2 = -\int_R \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \sum_j \lambda_j \partial_t \boldsymbol{\psi}_j J - \frac{1}{2}\int_{\Omega_s} \partial_t\eta\partial_t\delta^\sharp\partial_{tt}\eta$$

$$-\frac{1}{2}\int_R ((\mathbf{M}\nabla)\times\mathbf{v}^\sharp)\times\mathbf{u}^{\phi_\epsilon}\cdot\sum_j \dot\lambda_j\boldsymbol{\psi}_j+\nu\int_R(\partial_t(\frac{\mathbf{M}}{\sqrt{J}})\nabla\times\mathbf{u}^{\phi_\epsilon})(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\mathbf{u}^{\phi_\epsilon})$$

$$+\nu\int_R(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\mathbf{u}^{\phi_\epsilon})(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\sum_j\lambda_j\partial_t\boldsymbol{\psi}_j)+\int_R(\partial_t\boldsymbol{\phi}_\epsilon^{\phi_\epsilon}\cdot(\mathbf{M}\nabla))\mathbf{u}^{\phi_\epsilon}\cdot\sum_j\dot\lambda_j\boldsymbol{\psi}_j$$

$$-\frac{1}{2}\int_{\Gamma_{f1}} u_1^{\phi_\epsilon}v_1^\sharp\cdot\sum_j\dot\lambda_j\boldsymbol{\psi}_{j,1}J+\frac{1}{2}\int_{\Gamma_{f2}} u_1^{\phi_\epsilon}v_1^\sharp\cdot\sum_j\dot\lambda_j\boldsymbol{\psi}_{j,1}J+\int_{\Omega_s}(\Delta\partial_t\eta)^2$$

$$-\frac{d}{dt}\int_{\Omega_s}\Delta\eta\Delta\partial_t\eta+\int_R\mathbf{f}^{\phi_\epsilon}\sum_j\dot\lambda_j\boldsymbol{\psi}_jJ-\int_{\Gamma_f}p_0^{\phi_\epsilon}\cdot\mathbf{n}\cdot\sum_j\dot\lambda_j\boldsymbol{\psi}_jJ+\int_{\Omega_s}g\partial_{tt}\eta.$$

Now integrating from 0 to t we get:

$$\int_0^t||\sum_j\dot\lambda_j\boldsymbol{\psi}_j\sqrt{J}||_{\mathbf{L}^2(R)}+\frac{\nu}{2}\int_R(\frac{\mathbf{M}(t)}{\sqrt{J(t)}}\nabla\times\mathbf{u}^{\phi_\epsilon}(t))^2+\frac{\gamma}{2}||\Delta\partial_t\eta(t)||^2_{L^2(\Omega_s)}$$

$$+\int_0^t||\partial_{tt}\eta||^2_{L^2(\Omega_s)}=\frac{\nu}{2}\int_R(\frac{\mathbf{M}(0)}{\sqrt{J(0)}}\nabla\times\mathbf{u}^{\phi_\epsilon}(0))^2+\frac{\gamma}{2}||\Delta\partial_t\eta(0)||^2_{L^2(\Omega_s)}$$

$$-\int_0^t\int_R\sum_j\dot\lambda_j\boldsymbol{\psi}_j\sum_j\lambda_j\partial_t\boldsymbol{\psi}_jJ-\frac{1}{2}\int_0^t\int_{\Omega_s}\partial_t\eta\partial_t\delta^\sharp\partial_{tt}\eta$$

$$-\frac{1}{2}\int_0^t\int_R((\mathbf{M}\nabla)\times\mathbf{v}^\sharp)\times\mathbf{u}^{\phi_\epsilon}\cdot\sum_j\dot\lambda_j\boldsymbol{\psi}_j+\nu\int_0^t\int_R(\partial_t(\frac{\mathbf{M}}{\sqrt{J}})\nabla\times\mathbf{u}^{\phi_\epsilon})(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\mathbf{u}^{\phi_\epsilon})$$

$$+\nu\int_0^t\int_R(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\mathbf{u}^{\phi_\epsilon})(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\sum_j\lambda_j\partial_t\boldsymbol{\psi}_j)+\int_0^t\int_R(\partial_t\boldsymbol{\phi}_\epsilon^{\phi_\epsilon}\cdot(\mathbf{M}\nabla))\mathbf{u}^{\phi_\epsilon}\cdot\sum_j\dot\lambda_j\boldsymbol{\psi}_j$$

$$-\frac{1}{2}\int_0^t\int_{\Gamma_{f1}}u_1^{\phi_\epsilon}v_1^\sharp\cdot\sum_j\dot\lambda_j\psi_{j,1}J+\frac{1}{2}\int_0^t\int_{\Gamma_{f2}}u_1^{\phi_\epsilon}v_1^\sharp\cdot\sum_j\dot\lambda_j\psi_{j,1}J+\int_0^t\int_{\Omega_s}(\Delta\partial_t\eta)^2$$

$$-\int_{\Omega_s}\Delta\eta(t)\Delta\partial_t\eta(t)+\int_{\Omega_s}\Delta\eta(0)\Delta\partial_t\eta(0)+\int_0^t\int_R\mathbf{f}^{\phi_\epsilon}\sum_j\dot\lambda_j\boldsymbol{\psi}_jJ$$

$$-\int_0^t\int_{\Gamma_f}p_0^{\phi_\epsilon}\cdot\mathbf{n}\cdot\sum_j\dot\lambda_j\boldsymbol{\psi}_jJ+\int_0^t\int_{\Omega_s}g\partial_{tt}\eta.$$

Now we have to estimate the terms in the right hand side such that their bounds should not depend on m, n. Notice that there are no problems with the terms depending on the initial data, since these are smooth. For the rest of the terms we have ($C > 0$ is a constant depending on the data, on K, α and ϵ, but not on m and n):

$$|-\int_0^t \int_R \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \sum_j \lambda_j \partial_t \boldsymbol{\psi}_j J|$$

$$\leq C|| \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \sqrt{J}||_{\mathbf{L}^2(0,t;\mathbf{L}^2(R))} || \sum_j \lambda_j \partial_t \boldsymbol{\psi}_j||_{\mathbf{L}^2(0,t;\mathbf{L}^2(R))}$$

We have $\sum_j \lambda_j \partial_t \boldsymbol{\psi}_j = \sum_{j=1}^m \alpha_j \partial_t \mathbf{M}^{-t} \boldsymbol{\xi}_j^0 + \sum_{j=1}^n \dot{\beta}_j \partial_t \boldsymbol{\psi}_j^* = \sum_{j=1}^m \alpha_j \partial_t \mathbf{M}^{-t}\mathbf{M}^t \boldsymbol{\psi}_j^0 + \sum_{j=1}^n \dot{\beta}_j \partial_t \boldsymbol{\psi}_j^*$ (remember the definition of $\boldsymbol{\psi}_j^0$). Since $\partial_t \mathbf{M}^{-t}\mathbf{M}^t$ is bounded in $\mathbf{L}^\infty$ (with bounds depending on ϵ), it is

$$\begin{aligned} || \sum_{j=1}^m \alpha_j \partial_t \mathbf{M}^{-t}\mathbf{M}^t \boldsymbol{\psi}_j^0||_{\mathbf{L}^2(0,t;\mathbf{H}^1(R))} &\leq C|| \sum_{j=1}^m \alpha_j \boldsymbol{\psi}_j^0||_{\mathbf{L}^2(0,t;\mathbf{H}^1(R))} \\ &\leq C||\mathbf{u}^{\phi_\epsilon}||_{\mathbf{L}^2(0,T;\mathbf{H}^1(R))} \leq C \end{aligned}$$

In order to estimate the other term of the above sum, we observe that $\partial_t \boldsymbol{\psi}_j^*$ satisfies the following problem (remember the way $\boldsymbol{\psi}_j^*$ was defined):

$$\begin{aligned} -\Delta \partial_t \boldsymbol{\psi}_j^* + (\mathbf{M}\nabla)\partial_t p_j^* &= -(\partial_t \mathbf{M}\nabla) p_j^* \text{ in } R \\ \text{div } (\mathbf{M}^t \partial_t \boldsymbol{\psi}_j^*) &= -\text{div } (\partial_t \mathbf{M}^t \boldsymbol{\psi}_j^*) \text{ in } R \\ \partial_t \boldsymbol{\psi}_j^* &= 0 \text{ on } \partial R - \Gamma_f \\ \partial_t \boldsymbol{\psi}_j^* \times \mathbf{n} &= 0 \text{ on } \Gamma_f \\ \partial_t p_j^* &= 0 \text{ on } \Gamma_f \end{aligned}$$

By linearity, $\sum_{j=1}^n \dot{\beta}_j \partial_t \boldsymbol{\psi}_j^*$ is a solution, too, of the same type of problem, thus

$$|| \sum_{j=1}^n \dot{\beta}_j \partial_t \boldsymbol{\psi}_j^*||_{\mathbf{H}^1(R)} \leq C(|| \sum_{j=1}^n \dot{\beta}_j p_j^*||_{\mathbf{L}^2(R)} + || \sum_{j=1}^n \dot{\beta}_j \boldsymbol{\psi}_j^*||_{\mathbf{H}^1(R)}).$$

But from the definition of $\boldsymbol{\psi}_j^*$ it follows that

$$\begin{aligned} || \sum_{j=1}^n \dot{\beta}_j p_j^*||_{\mathbf{L}^2(R)} + || \sum_{j=1}^n \dot{\beta}_j \boldsymbol{\psi}_j^*||_{\mathbf{H}^1(R)} &\leq C|| \sum_{j=1}^n \dot{\beta}_j \rho_j||_{H^{1/2}(\Omega_s)} \\ &\leq C||\partial_t \eta||_{H^{1/2}(\Omega_s)}, \end{aligned}$$

thus

$$||\sum_{j=1}^{n}\dot{\beta}_j\partial_t\boldsymbol{\psi}_j^*||_{\mathbf{L}^2(0,t;\mathbf{H}^1(R))} \leq C||\partial_t\eta||_{L^2(0,t;H^{1/2}(\Omega_s))}. \tag{36}$$

Therefore, $\sum_{j=1}^{n}\dot{\beta}_j\partial_t\boldsymbol{\psi}_j^*$ is bounded independently of m, n in $\mathbf{L}^2(0,t;\mathbf{H}^1(R))$, because $\partial_t\eta$ can be bounded in $L^2(0,t;H_0^2(\Omega_s))$ by a constant depending only on the data and on ϵ, due to the viscous type term added to the plate equations (see also the a priori estimates).

Consequently,

$$|-\int_0^t\int_R\sum_j\dot{\lambda}_j\boldsymbol{\psi}_j\sum_j\lambda_j\partial_t\boldsymbol{\psi}_jJ| \leq C+\frac{1}{8}||\sum_j\dot{\lambda}_j\boldsymbol{\psi}_j||^2_{\mathbf{L}^2(0,t;\mathbf{L}^2(R))}.$$

Clearly,

$$|-\frac{1}{2}\int_0^t\int_{\Omega_s}\partial_t\eta\partial_t\delta^\sharp\partial_{tt}\eta| \leq C+\frac{1}{6}||\partial_{tt}\eta||_{L^2(0,t;L^2(\Omega_s))}$$

and

$$|-\frac{1}{2}\int_0^t\int_R((\mathbf{M}\nabla)\times\mathbf{v}^\sharp)\times\mathbf{u}^{\phi_\epsilon}\cdot\sum_j\dot{\lambda}_j\boldsymbol{\psi}_j| \leq C+\frac{1}{8}||\sum_j\dot{\lambda}_j\boldsymbol{\psi}_j\sqrt{J}||^2_{\mathbf{L}^2(0,t;\mathbf{L}^2(R))},$$

since $\mathbf{M}$, $\mathbf{v}^\sharp$ and $\mathbf{u}^{\phi_\epsilon}$ are all bounded independently of m, n.

The term $\nu\int_0^t\int_R(\partial_t(\frac{\mathbf{M}}{\sqrt{J}})\nabla\times\mathbf{u}^{\phi_\epsilon})(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\mathbf{u}^{\phi_\epsilon})$ is bounded independently of m, n, for, $\mathbf{u}^{\phi_\epsilon}$ is bounded in $\mathbf{L}^2(0,t;\mathbf{H}^1(R))$ and $\partial_t(\frac{\mathbf{M}}{\sqrt{J}})$ is bounded in $\mathbf{L}^\infty$, independently of m, n.

Similarly, the term $\nu\int_0^t\int_R(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\mathbf{u}^{\phi_\epsilon})(\frac{\mathbf{M}}{\sqrt{J}}\nabla\times\sum_j\lambda_j\partial_t\boldsymbol{\psi}_j)$ can be estimated by a constant which does not depend on m, n, due to the bound of $\frac{\mathbf{M}}{\sqrt{J}}$ in $\mathbf{L}^\infty$, to the fact that $\mathbf{u}^{\phi_\epsilon}\in\mathbf{L}^2(0,t;\mathbf{H}^1(R))$ and (see (36)) that $\sum_j\lambda_j\partial_t\boldsymbol{\psi}_j\in\mathbf{L}^2(0,t;\mathbf{H}^1(R))$.

Further,

$$|\int_0^t\int_R(\partial_t\boldsymbol{\phi}_\epsilon^{\phi_\epsilon}\cdot(\mathbf{M}\nabla))\mathbf{u}^{\phi_\epsilon}\cdot\sum_j\dot{\lambda}_j\boldsymbol{\psi}_j| \leq C+\frac{1}{8}||\sum_j\dot{\lambda}_j\boldsymbol{\psi}_j\sqrt{J}||^2_{\mathbf{L}^2(0,t;\mathbf{L}^2(R))},$$

since $\partial_t\boldsymbol{\phi}_\epsilon^{\phi_\epsilon}$ is bounded independently of m, n.

Next,

$$|-\frac{1}{2}\int_0^t\int_{\Gamma_{f1}} u_1^{\phi_\epsilon} v_1^\sharp \cdot \sum_j \dot{\lambda}_j \psi_{j,1} J| \leq \frac{1}{2}|\int_0^t\int_{\Gamma_f} \mathbf{u}^{\phi_\epsilon}\mathbf{v}^\sharp \cdot \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j J|$$

$$\overset{p.\ int\ w.r.t.\ time}{\leq} \frac{1}{4}|\int_{\Gamma_f}(\mathbf{u}^{\phi_\epsilon}(t))^2\mathbf{v}^\sharp(t)J(t)| + \frac{1}{4}|\int_{\Gamma_f}(\mathbf{u}^{\phi_\epsilon}(0))^2\mathbf{v}^\sharp(0)J(0)|$$

$$+\frac{1}{4}|\int_0^t\int_{\Gamma_f}(\mathbf{u}^{\phi_\epsilon})^2\partial_t(\mathbf{v}^\sharp J)| + \frac{1}{2}|\int_0^t\int_{\Gamma_f}\sum_j \lambda_j \partial_t \boldsymbol{\psi}_j \mathbf{v}^\sharp \sum_j \lambda_j \boldsymbol{\psi}_j J|.$$

Using the regularity of the solution in Subsection 6 and the boundedness of $\sum_j \lambda_j \partial_t \boldsymbol{\psi}_j$ in $\mathbf{L}^2(0,t;\mathbf{H}^1(R))$ (actually proved before for a.e. $t \in (0,T)$), each of the above terms is bounded by a constant C, independently on m, n, thus

$$|-\frac{1}{2}\int_0^t\int_{\Gamma_{f1}} u_1^{\phi_\epsilon} v_1^\sharp \cdot \sum_j \dot{\lambda}_j \psi_{j,1} J| \leq C.$$

The next term is to be treated in the same way.

The term $\int_0^t ||\Delta\partial_t\eta||^2_{L^2(\Omega_s)}$ is clearly bounded independently of m, n, thanks to the estimates (27).

Next,

$$\begin{aligned}|-\int_{\Omega_s}\Delta\eta(t)\Delta\partial_t\eta(t)| &\leq \frac{1}{4}||\Delta\partial_t\eta(t)||^2_{L^2(\Omega_s)} + C||\Delta\eta(t)||^2_{L^2(\Omega_s)}\\ &\leq C + \frac{1}{4}||\Delta\partial_t\eta(t)||^2_{L^2(\Omega_s)} \leq C\end{aligned}$$

$$|\int_0^t\int_R \mathbf{f}^{\phi_\epsilon}\sum_j \dot{\lambda}_j \boldsymbol{\psi}_j J| \leq C + \frac{1}{8}||\sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \sqrt{J}||^2_{\mathbf{L}^2(0,t;\mathbf{L}^2(R))},$$

$$|\int_0^t\int_{\Omega_s} g\partial_{tt}\eta| \leq C + \frac{1}{6}||\partial_{tt}\eta||^2_{L^2(0,t;L^2(\Omega_s))}$$

and we finally take care of the term $-\int_0^t\int_{\Gamma_f} p_0^{\phi_\epsilon}\cdot \mathbf{n}\cdot \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j J$.

By integration by parts with respect to time, this term becomes:

$$-\int_0^t \int_{\Gamma_f} \partial_t p_0 \cdot \mathbf{n} \cdot \sum_j \lambda_j \boldsymbol{\psi}_j J - \int_0^t \int_{\Gamma_f} p_0 \cdot \mathbf{n} \cdot \sum_j \lambda_j \partial_t \boldsymbol{\psi}_j J$$

$$-\int_0^t \int_{\Gamma_f} p_0 \cdot \mathbf{n} \sum_j \lambda_j \boldsymbol{\psi}_j \partial_t J + \int_{\Gamma_f} p_0(t) \cdot \mathbf{n} \sum_j \lambda_j(t) \boldsymbol{\psi}_j(t) J(t)$$

$$-\int_{\Gamma_f} p_0(0) \cdot \mathbf{n} \cdot \sum_j \lambda_j(0) \boldsymbol{\psi}_j(0) J(0)$$

Due to the estimates we made for $-\int_0^t \int_R \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \sum_j \lambda_j \partial_t \boldsymbol{\psi}_j J$ and to the estimates for η and $\mathbf{u}^{\phi_\epsilon}$, each of these terms is bounded by a constant independent of m, n, if we assume that $\partial_t p_0 \in L^2(0,t;L^2(\Gamma_f))$.

Therefore, from the above calculations it follows that

$$\frac{1}{2}||\sum_j \dot{\lambda}_j \boldsymbol{\psi}_j \sqrt{J}||^2_{\mathbf{L}^2(0,t;\mathbf{L}^2(R))} + \frac{1}{2}||\partial_{tt}\eta||^2_{L^2(0,t;L^2(\Omega_s)} + \frac{1}{4}||\Delta \partial_t \eta(t)||^2_{L^2(\Omega_s)} \leq C.$$

Since $\partial_t \mathbf{u}^{\phi_\epsilon} = \sum_j \dot{\lambda}_j \boldsymbol{\psi}_j + \sum_j \lambda_j \partial_t \boldsymbol{\psi}_j$ and $\sqrt{J} \geq \sqrt{\frac{\alpha}{2}}$, it follows that

$$||\partial_t \mathbf{u}^{\phi_\epsilon}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(R))} + ||\partial_{tt}\eta||_{L^2(0,T;L^2(\Omega_s))} \leq C,$$

with $C > 0$ independent on m, n (it only depends on the data, on K, α and ϵ). This is actually what we intended to prove, if we remember the convention of omitting the indices of the involved functions.

□

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Institut für Angewandte Mathematik, INF 294, 69120 Heidelberg, Germany

E-mail address: `christina.surulescu@iwr.uni-heidelberg.de`