Generalizing evolution equations in ostensible metric spaces: Timed right–hand sleek solutions provide uniqueness of 1^{st}–order geometric examples.

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Abstract. The mutational equations of Aubin extend dynamic systems to metric spaces. In first–order geometric evolutions, however, the topological boundary need not be continuous. Thus, a distribution–like extension to (timed) ostensible metric spaces was introduced in [14, 15, 16].
Here the notion of Petrov–Galerkin is implemented additionally, i.e. "test elements" need not belong to the same set as the values of solutions. Such a further freedom (in the general setting of timed ostensible metric spaces) has an advantage for geometric evolutions depending on the boundary : A new choice of timed ostensible metrics and "test sets" (in comparison with [15]) ensures not just existence, but also uniqueness of solutions.

Contents

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1 Introduction

Whenever different types of evolutions meet, they usually do not have an obvious vector space structure in common providing a basis for differential calculus. In particular, "shapes and images are basically sets, not even smooth" as Aubin stated ([2]). So he regards this obstacle as a starting point for extending ordinary differential equations to metric spaces – the so–called *mutational equations* ([2, 3, 4]).

Considering the example of time–dependent compact sets in $\mathbb{R}^N$, Aubin uses reachable sets of differential inclusions for describing a first–order approximation with respect to the Pompeiu–Hausdorff distance $d\!\!l$. However this approach (also called *morphological equations*) can hardly be applied to geometric evolutions depending on the topological boundary explicitly. Indeed, roughly speaking, "holes" of sets might disappear while evolving along differential inclusions and thus, analytically speaking, the topological boundary need not be continuous with respect to time.

This obstacle has been the motivation in [14], [16] for extending mutational equations to a nonempty set $\widetilde{E} = \mathbb{R} \times E$ (supplied with a separate real component of time) and a countable family of *timed ostensible metrics*, i.e. distance functions $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$ satisfying just the timed triangle inequality and $\widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{x}) = 0$ for each $\widetilde{x} \in \widetilde{E}$. The definitions leading to so–called *timed right–hand forward solutions* are summarized in § 2.

Two examples have already demonstrated the general character of this concept. In [15, Lorenz 2005], its tools are applied to both semilinear evolution equations in reflexive Banach spaces and first–order geometric evolutions simultaneously. Considering this geometric example in particular, the notion of solution is specified as curves of compact subsets of $\mathbb{R}^N$ that fulfill a fixed (structural) estimate while comparing shortly with the evolution of every compact N–dimensional submanifolds of $\mathbb{R}^N$ with $C^{1,1}$ boundary.

However, this example also demonstrates a weakness of the concept. In regard to uniqueness, the results of [14] require the assumption that, roughly speaking, compact sets with $C^{1,1}$ boundary do not loose their regularity too quickly (see Propositions 40, 42 in [14] as counterparts of Propositions 3.19, 3.21 here in § 3). However these conditions are not obvious to verify for the geometric example of [15].

This lack of uniqueness has motivated to generalize mutational equations once more. If we cannot dispense with the condition how long "test elements" preserve this feature, then we "expand" the set of test elements. For the previous geometric example in particular, it might be helpful to take *all* compact subsets into consideration (and not just the compact N–dimensional submanifolds of $\mathbb{R}^N$ with $C^{1,1}$ boundary). Then the missing continuity of topological boundaries forms an obstacle to timed right–hand forward solutions, though.

For overcoming this obstacle, we return to the basic notions when extending mutational equations : Timed right–hand forward solutions (in [14], [16]) can be interpreted as a form of *distributional solution.* Indeed, they do not rely on partial integration, but preserve a fixed structural estimate while comparing with the evolutions of all test elements shortly.
So the first essential idea now is similar to Petrov–Galerkin methods : *The test elements need not belong to the same set as the values of solutions.* Just some continuity properties have to be preserved when comparing their evolutions (along timed transitions) with each other.

Furthermore, the "mode" of comparing is changed. Indeed, timed forward transitions $\widetilde{\vartheta}$ (introduced in [14], [16]) are characterized by a parameter $\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$ (among other things). It concerns the continuity with respect to the initial point and is chosen uniformly for all test elements. Now we make this parameter depend on the test element $\widetilde{z}$ and use it only for comparing with all elements $\widetilde{\vartheta}(t, \widetilde{z})$ for t in a small interval $[0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})[\subset [0, 1]$ (that might also depend on ε). Such a dependence on the test element has an immediate impact on other quantities (like the distance between two transitions) and on the definition of "solution".
So as a key point of this paper, we investigate how to take these two additional degrees of freedom into consideration correctly. The terms "timed *sleek* transition" and "timed right–hand *sleek* solution" are coined in § 3. Then we obtain existence and stability results in essentially the same way as in [14], [16]. In particular, the proofs again require us to focus on how indices and parameters depend on each other. "Timed right–hand forward solutions" (as defined in in [14], [16]) and solutions in the sense of Aubin ([2]) are special cases. So in particular, the examples of [15] and [2] fulfill the assumptions about "timed right–hand sleek solutions" presented here in § 3.

In § 4, the new concept is applied to first–order geometric evolutions, i.e. the evolution of compact subsets of $\mathbb{R}^N$ may depend on nonlocal properties of both the current subset and its limiting normal cones at the boundary. In contrast to [15], we dispense with $C^{1,1}$ regularity of the "test elements" and distinguish between the basic set $\widetilde{E}$ and the set $\widetilde{\mathcal{D}}$ of test elements by an additional component instead :

$$\begin{aligned} \widetilde{E} &:= \mathbb{R} \times \{1\} \times \mathcal{K}(\mathbb{R}^N), \\ \widetilde{\mathcal{D}} &:= \mathbb{R} \times \{0\} \times \mathcal{K}(\mathbb{R}^N). \end{aligned}$$

This auxiliary component is just to indicate how the first component (of time) evolves along a transition $\widetilde{\vartheta}$. Indeed, for a set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ and a compact set $K \subset \mathbb{R}^N$, let $\vartheta_F(h, K) \subset \mathbb{R}^N$ denote the reachable set of K and the differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ (a.e.) at time $h \geq 0$. Then we distinguish between

$$\widetilde{\vartheta}_F(h, \widetilde{K}) := (t + h, \ 1, \ \vartheta_F(h, K)) \qquad \text{for } \widetilde{K} = (t, 1, K) \in \widetilde{E}$$

and

$$\widetilde{\vartheta}_F(h, \widetilde{K}) := (t, \ 0, \ \vartheta_F(h, K)) \qquad \text{for } \widetilde{K} = (t, 0, K) \in \widetilde{\mathcal{D}}.$$

The real time component comes into play (only) for comparing proximal normal cones in the new ostensible metric on $\widetilde{\mathcal{D}} \cup \widetilde{E}$: For a closed subset $C \subset \mathbb{R}^N$, $x \in \partial C$ and any $\rho > 0$, let $N^P_{C,\rho}(x) \subset \mathbb{R}^N$ consist of all proximal normal vectors $\eta \in N^P_C(x) \setminus \{0\}$ with the proximal radius $\geq \rho$ (thus it might be empty) and set ${}^\flat N^P_{C,\rho}(x) := N^P_{C,\rho}(x) \cap \mathbb{B}_1(0)$ (see Definition 4.1). With $d\!\!\!l$ denoting the Pompeiu–Hausdorff distance between compact subsets of $\mathbb{R}^N$, we define for any $\varepsilon \in [0,1]$ and $(s,\mu,C),\ (t,\nu,D) \in \widetilde{\mathcal{D}} \cup \widetilde{E}$

$$\widetilde{q}_{\mathcal{K},\varepsilon}((s,\mu,C),\ (t,\nu,D)) \;:=\; d\!\!\!l(C,D) + \limsup_{\kappa\downarrow 0} \int_\varepsilon^\infty \psi(\rho+\kappa+200\,\Lambda\,|t-s|)\cdot \operatorname{dist}\Big(\operatorname{Graph}\ {}^\flat N^P_{D,\,(\rho+\kappa+200\,\Lambda\,|t-s|)},\ \operatorname{Graph}\ {}^\flat N^P_{C,\,\rho}\Big)\, d\rho$$

with a fixed nonincreasing weight function $\psi \in C^\infty_0([0,2[)$, $\psi \geq 0$, and a parameter $\Lambda > 0$ (related with the differential inclusions inducing transitions).
This ostensible metric on $\widetilde{\mathcal{D}} \cup \widetilde{E}$ is motivated by the features of reachable sets of differential inclusions : Roughly speaking, when considering an arbitrary compact set $K \subset \mathbb{R}^N$ while evolving along a differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ (a.e.), its exterior spheres do not change very much for short times if the Hamiltonian function of F is C^2. To be more precise, Appendix A provides a connection between the exterior spheres of $\vartheta_F(t,K)$ and K (and vice versa) for small times $t > 0$:

Lemma 1.1 *Assume for the set–valued map* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *that its values are nonempty, compact, convex and that its Hamiltonian function is* $C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$ *with* $\|\mathcal{H}_F\|_{C^2(\mathbb{R}^N\times\,\partial\mathbb{B}_1)} < \Lambda$.
Then for every radius $r_0 \in]0,2]$, *there exists some time* $\tau = \tau(r_0,\Lambda) > 0$ *such that for any* $K \in \mathcal{K}(\mathbb{R}^N)$, $r \in [r_0,2]$ *and* $t \in [0,\tau[$,

1. *each* $x_1 \in \partial\vartheta_F(t,K)$ *and* $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ *with proximal radius* r *are linked to some* $x_0 \in \partial K$ *and* $\nu_0 \in N^P_K(x_0)$ *with proximal radius* $\geq r - 81\,\Lambda\,t$ *by a trajectory of* $\dot{x}(\cdot) \in F(x(\cdot))$ *a.e. and its adjoint arc, respectively.*

2. *each* $x_0 \in \partial K$ *and* $\nu_0 \in N^P_K(x_0)$ *with proximal radius* r *are linked to some* $x_1 \in \partial\vartheta_F(t,K)$ *and* $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ *with proximal radius* $\geq r - 81\,\Lambda\,t$ *by a trajectory of* $\dot{x}(\cdot) \in F(x(\cdot))$ *a.e. and its adjoint arc, respectively.*

So the difference of more than $200\,\Lambda\,t$ (in respect to proximal radii) proves to have two advantages. Firstly, $\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(s,\ \widetilde{K}),\ \widetilde{\vartheta}_F(t,\ \widetilde{K})\Big) \;\leq\; \Lambda\,(1 + \|\psi\|_{L^1}\ (e^\Lambda + 1))\ \cdot\ |t-s|$ holds for every initial element $\widetilde{K} \in \widetilde{E}$ and any times $0 \leq s < t \leq 1$ (Lemma 4.7).
Secondly, we can compare the evolution of arbitrary elements $\widetilde{K}_1 = (t_1, 0, K_1) \in \widetilde{\mathcal{D}}$, $\widetilde{K}_2 = (t_2, 1, K_2) \in \widetilde{E}$ with $t_1 \leq t_2$ while evolving along two set–valued maps $F,\ G$ (that satisfy the conditions of Lemma 1.1). According to Lemma 4.8, the different features of their time components, in particular, lead to

$$\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(h,\widetilde{K}_1),\ \widetilde{\vartheta}_G(h,\widetilde{K}_2)\Big) \;\leq\; e^{C(\Lambda)\cdot h}\cdot\Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1,\ \widetilde{K}_2) \;+\; \mathrm{C}\ h\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\,\partial\mathbb{B}_1)}\Big).$$

Thus the required continuity properties of timed sleek transitions are fulfilled — without any regularity assumptions about the compact subsets.

In regard to $\widetilde{q}_\varepsilon$, the additional limit superior with respect to $\kappa \downarrow 0$ has a geometric motivation. Appendix B investigates the proximal normal subsets $N^P_{K_n,\,\rho}(\cdot)$ for a converging sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets. Indeed, Proposition B.1 states for $K = \mathrm{Lim}_{n\to\infty} K_n \in \mathcal{K}(\mathbb{R}^N)$

$$\begin{array}{lrcll} & \mathrm{Limsup}_{n\to\infty} \,\mathrm{Graph}\ {}^\flat N^P_{K_n,\rho} & \subset & \mathrm{Graph}\ {}^\flat N^P_{K,\rho} & \text{for any } \rho > 0, \\ \text{but} & \mathrm{Graph}\ {}^\flat N^P_{K,\rho} & \subset & \mathrm{Liminf}_{n\to\infty}\,\mathrm{Graph}\ {}^\flat N^P_{K_n,\,r} & \text{for any } 0 < r < \rho \end{array}$$

and in general, we cannot dispense with the restriction $r < \rho$. Thus, it does not seem advisable to compare proximal subsets of identical proximal radii with each other (when verifying a form of sequential compactness, see Proposition 4.11 for details). So the comparison of proximal normal subsets is rather "epigraphical" (than pointwise) with respect to the proximal radius.

Combining the results about first–order geometric evolutions with semilinear evolution equations in reflexive Banach spaces (see [15]), we draw the following conclusion from timed right–hand sleek solutions :

Proposition 1.2 (Systems of semilinear evolution equations in Banach space and timed first–order geometric evolutions in $\mathbb{R}^N$)
Let X be a reflexive Banach space and $(S(t))_{t\geq 0}$ a C^0 semigroup on X with the infinitesimal generator A. Suppose that the dual operator A' of A has a countable family of unit eigenvectors $\{v'_j\}_{j\in\mathcal{J}}$ spanning the dual space X' and define

$$\begin{array}{rcll} q_j(x,y) & := & |\langle x-y,\ v'_j\rangle| & \textit{for}\ \ x,y\in X,\ \ j\in\mathcal{J}=\{j_1,j_2,j_3\,\ldots\}, \\ p_n(x,y) & := & \displaystyle\sum_{k=1}^{n} 2^{-k}\ \frac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)} & \textit{for}\ \ x,y\in X,\ \ n\in\mathbb{N}\cup\{\infty\}, \\ P_n(x,y) & := & \displaystyle\sum_{k=1}^{n} 2^{-k}\ q_{j_k}(x,y). & \end{array}$$

Let $\mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N,\mathbb{R}^N)$ denote the set of all set–valued maps satisfying the hypotheses of Lemma 1.1. Using the abbreviations $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R}\times\{1\}\times\mathcal{K}(\mathbb{R}^N)$, $\widetilde{\mathcal{D}} \overset{\text{Def.}}{=} \mathbb{R}\times\{0\}\times\mathcal{K}(\mathbb{R}^N)$, assume for

$$\begin{array}{rl} f: & X\times(\widetilde{\mathcal{D}}\cup\widetilde{E})\times[0,T] \longrightarrow X \\ g: & X\times(\widetilde{\mathcal{D}}\cup\widetilde{E})\times[0,T] \longrightarrow \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N,\mathbb{R}^N)\ : \end{array}$$

1. $\|f\|_{L^\infty} < \infty$
2. $P_\infty(f(x_1,\widetilde{K}_1,t_1),\ f(x_2,\widetilde{K}_2,t_2)) \leq \omega(p_\infty(x_1,x_2)+\widetilde{q}_{\mathcal{K},0}(\widetilde{K}_1,\widetilde{K}_2)+t_2-t_1)$
3. $\left\|\mathcal{H}_{g(x_1,\widetilde{K}_1,t_1)} - \mathcal{H}_{g(x_2,\widetilde{K}_2,t_2)}\right\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)} \leq \omega(p_\infty(x_1,x_2)+\widetilde{q}_{\mathcal{K},0}(\widetilde{K}_1,\widetilde{K}_2)+t_2-t_1)$

for all $x_1,x_2\in X$, $\widetilde{K}_1,\widetilde{K}_2\in\widetilde{E}$ $(\pi_1\widetilde{K}_1\leq\pi_1\widetilde{K}_2)$, $t_1\leq t_2$ with a modulus $\omega(\cdot)$ of continuity.

I.) *Then for every initial data* $x_0 \in X$ *and* $K_0 \in \mathcal{K}(\mathbb{R}^N)$, *there exists a tuple* $(x, K) : [0, T[\longrightarrow X \times \mathcal{K}(\mathbb{R}^N)$ *such that* $\widetilde{K} : [0, T[\longrightarrow \widetilde{E},\ t \longmapsto (t, 1, K(t))$ *satisfies*

a) $x : [0, T[\longrightarrow X$ *is a mild solution of the initial value problem*

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &= A\, x(t) \;+\; f(x(t),\, \widetilde{K}(t),\, t) \\ x(0) &= x_0 \end{cases}$$

i.e. $$x(t) \;=\; S(t)\, x_0 \;+\; \int_0^t S(t-s)\;\, f(x(s),\, \widetilde{K}(s),\, s)\;\, ds.$$

b) $K(0) = K_0$ *and* $\widetilde{K}(\cdot)$ *is Lipschitz continuous in forward time direction w.r.t.* $\widetilde{q}_{\mathcal{K},0}$, *i.e.* $\widetilde{q}_{\mathcal{K},0}(\widetilde{K}(s),\, \widetilde{K}(t)) \;\le\; const(\Lambda, T) \cdot (t-s)$ *for all* $0 \le s < t < T$.

c) $\limsup_{h \downarrow 0} \frac{1}{h} \cdot \Big(\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_{g(x(t),\, \widetilde{K}(t),\, t)}\, (h,\, \widetilde{Z}),\;\; \widetilde{K}(t{+}h)\Big) \;-\; \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\, \widetilde{K}(t)) \cdot e^{10\,\Lambda\, e^{2\,\Lambda}\cdot h}\Big) \;\le\; c\,\varepsilon$ *for every* $\varepsilon \in \,]0,1] \cap \mathbb{Q}$, *time* $t \in [0, T[$ *and test set* $\widetilde{Z} \in \,]-\infty, t] \times \{0\} \times \mathcal{K}(\mathbb{R}^N)$ *(with a constant* $c > 0$ *depending only on* Λ, T, K_0*).*
In particular, $\limsup_{h \downarrow 0} \frac{1}{h} \cdot d\!l\Big(\vartheta_{g(x(t),\, \widetilde{K}(t),\, t)}\, (h,\, K(t)),\;\; K(t{+}h)\Big) \;=\; 0$ *for all* t.

II.) *If, additionally,* $g : X \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times [0, T] \longrightarrow \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N, \mathbb{R}^N)$ *satisfies*

$$\Big\| \mathcal{H}_{g(x_1, \widetilde{Z}, t_1)} \;-\; \mathcal{H}_{g(x_2, \widetilde{K}, t_2)} \Big\|_{C^1(\mathbb{R}^N \times \partial \mathbb{B}_1)} \;\;\le\;\; L \cdot \widetilde{q}_{\mathcal{K},0}(\widetilde{Z}, \widetilde{K}) \;+\; \widehat{\omega}(t_2 - t_1)$$

for all $0 \le t_1 \le t_2 \le T$ *and* $\widetilde{Z} \in \widetilde{\mathcal{D}}$, $\widetilde{K} \in \widetilde{E}$ $(\pi_1\, \widetilde{Z} \le \pi_1\, \widetilde{K})$ *with a modulus* $\widehat{\omega}(\cdot)$ *of continuity and a Lipschitz constant* $L \ge 0$, *then the function* $K(\cdot)$ *is unique.*

Here the uniqueness in statement (II.) is the main new feature in comparison with [15]. It results from Proposition 4.14.

Finally, let us give a brief overview of this paper and its structure. In § 2, the key definitions presented in [14], [16] are summarized. They lead to so–called *timed right–hand forward solutions* and serve as a motivation for pointing out the new features here. Then in § 3, we introduce timed sleek transitions and follow basically the same steps up to existence and uniqueness results about so–called *timed right–hand sleek solutions.* The subsequent paragraph 4 contains the example of first–order geometric evolutions using the timed ostensible metrics $(\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in\,]0,1] \cap \mathbb{Q}}$. In particular, we verify that reachable sets of maps in $\mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N, \mathbb{R}^N)$ induce timed sleek transitions and investigate some required properties of sequential compactness. Appendix A provides the key tools of reachable sets (of differential inclusions) quoted here in Lemma 1.1. In the end, Appendix B relates the proximal normal subsets $N^P_{K_n,\,\rho}(\cdot)$ of a convergent sequence $(K_n)_{n \in \mathbb{N}}$ in $\mathcal{K}(\mathbb{R}^N)$ with its limit $K = \mathrm{Lim}_{n \to \infty} K_n$.

2 Timed right–hand forward solutions of mutational equations : Previous definitions.

Generalizing the mutational equations of Aubin in metric spaces ([2, 3, 4]), the so–called *timed right–hand forward solutions* (*of order p*) were defined in [14] and sufficient conditions ensure their existence (see also [16], Chapter 2). In this section, we summarize the main points – in preparation for a new step of generalization in § 3. This modification is also to weaken the restriction of "uniform" continuity on transitions and leads to so–called *timed sleek transitions* (*of order p*) in Definition 3.1.

As a first step, we specify the mathematical environment of our considerations. Similarly to metric spaces, a nonempty set E is to be supplied with a distance function. However, an additional real component provides the opportunity of sorting elements by time and for the same reason, we dispense with the symmetry of distance functions.

Definition 2.1 *Let E be a nonempty set and $\widetilde{E} := \mathbb{R} \times E$.*
$\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ *is called* timed ostensible metric *on* $\widetilde{E}$ *if it satisfies :*

$$\begin{array}{rcll} \widetilde{q}((t,z),\ (t,z)) & = & 0 & (\textit{reflexive}) \\ \widetilde{q}((r,x),\ (t,z)) & \leq & \widetilde{q}((r,x),\ (s,y)) + \widetilde{q}((s,y),\ (t,z)) & (\textit{timed triangle inequality}) \end{array}$$

for all $(r,x), (s,y), (t,z) \in \widetilde{E}$ *with* $r \leq s \leq t$. $(\widetilde{E}, \widetilde{q})$ *is called* timed ostensible metric space.

General assumptions for § 2.

1. Let E denote a nonempty set, $D \subset E$ a fixed subset of "test elements" and $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R} \times E$, $\widetilde{D} \overset{\text{Def.}}{=} \mathbb{R} \times D$, $\pi_1 : \widetilde{E} \longrightarrow \mathbb{R},\ (t,x) \longmapsto t$.
2. $\mathcal{J} \subset [0,1]^K$ abbreviates a countable index set with $K \in \mathbb{N}$, $0 \in \overline{\mathcal{J}}$.
3. $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ is a timed ostensible metric on $\widetilde{E}$ (for each $\varepsilon \in \mathcal{J}$).
4. Each $\widetilde{q}_\varepsilon$ is "time continuous", i.e. every sequence $(\widetilde{x}_n = (t_n, x_n))_{n \in \mathbb{N}}$ in $\widetilde{E}$ and $\widetilde{x} = (t,x) \in \widetilde{E}$ with $\widetilde{q}_\varepsilon(\widetilde{x}_n, \widetilde{x}) \longrightarrow 0$ $(n \longrightarrow \infty)$ fulfill $t_n \longrightarrow t$ $(n \longrightarrow \infty)$.

Now we specify tools for describing deformations in the tuple $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$. A map $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ is to define which point $\widetilde{\vartheta}(t, \widetilde{x}) \in \widetilde{E}$ is reached from the initial point $\widetilde{x} \in \widetilde{E}$ after time t. Of course, $\widetilde{\vartheta}$ has to fulfill some regularity conditions so that it may form the basis for a calculus of differentiation. Following [14], we define

Definition 2.2 *A map* $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ *is a so–called* timed forward transition of order p *on* $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *if it fulfills for each* $\varepsilon \in \mathcal{J}$

1. $\widetilde{\vartheta}(0, \cdot) = \mathrm{Id}_{\widetilde{E}}$,
2. $\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 :$ $\quad \limsup\limits_{\varepsilon \longrightarrow 0}\ \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0$ *and*

$$\limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[,$$

$$\limsup_{h \downarrow 0}\ \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x}))) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{E},\ t \in [0,1[,$$

3. $\exists\ \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) : \sup_{\substack{\widetilde{z} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{z} \leq \pi_1 \widetilde{y}}} \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{z}),\, \widetilde{\vartheta}(h,\widetilde{y})\big) - \widetilde{q}_\varepsilon(\widetilde{z},\widetilde{y}) - \gamma_\varepsilon(\widetilde{\vartheta})\, h}{h \quad \big(\widetilde{q}_\varepsilon(\widetilde{z},\widetilde{y}) + \gamma_\varepsilon(\widetilde{\vartheta})\, h\big)} \right)^+ \leq \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) < \infty$
4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) :\ ]0,1] \longrightarrow [0,\infty[:$ *nondecreasing*, $\lim_{h \downarrow 0} \beta_\varepsilon(\widetilde{\vartheta})(h) = 0,$
   $\widetilde{q}_\varepsilon(\widetilde{\vartheta}(s,\widetilde{y}),\ \widetilde{\vartheta}(t,\widetilde{y})) \leq \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad \forall\ s < t \leq 1,\ \widetilde{y} \in \widetilde{E},$
5. $\forall\ \widetilde{z} \in \widetilde{D}\ \ \exists\ \mathcal{T}_\Theta = \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{z}) \in]0,1] :\ \widetilde{\vartheta}(t,\widetilde{z}) \in \widetilde{D} \qquad \forall\ t \in [0, \mathcal{T}_\Theta],$
6. $\limsup_{h \downarrow 0}\ \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}) \geq \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\ \widetilde{y}) \qquad \forall\ \widetilde{z} \in \widetilde{D},\ \widetilde{y} \in \widetilde{E},\ t \leq \mathcal{T}_\Theta$
   $(t + \pi_1 \widetilde{z} \leq \pi_1 \widetilde{y}),$
7. $\widetilde{\vartheta}(h,\ (t,y)) \in \{t+h\} \times E \qquad \forall\ (t,y) \in \widetilde{E},\ h \in [0,1].$

Remark 2.3 The term “forward” and the symbol $\mapsto$ (representing the time axis) indicate that states at time $t+h$ are usually compared with elements at time t for $h \downarrow 0$.

Condition (2.) can be regarded as a weakened form of the semigroup property. It consists of two demands as $\widetilde{q}_\varepsilon$ need not be symmetric. Condition (3.) specifies the continuity property of $\widetilde{\vartheta}$ with respect to the initial point. In particular, the first argument of $\widetilde{q}_\varepsilon$ is restricted to elements $\widetilde{z}$ of the “test set” $\widetilde{D}$ and, $\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$ may be chosen larger than necessary. Thus, it is easier to define $\alpha_\varepsilon^{\mapsto}(\cdot) < \infty$ uniformly in some applications like the first–order geometric example of [15]. In condition (4.), all $\widetilde{\vartheta}(\cdot,\widetilde{y}) : [0,1] \longrightarrow \widetilde{E}$ $(\widetilde{y} \in \widetilde{E})$ are supposed to be equi–continuous (in time direction).
Condition (5.) guarantees that every element $\widetilde{z} \in \widetilde{D}$ stays in the “test set” $\widetilde{D}$ for short times at least. This assumption is required because estimates using the parameter $\alpha_\varepsilon^{\mapsto}(\cdot)$ can be ensured only within this period. Further conditions on $\mathcal{T}_\Theta(\widetilde{\vartheta},\cdot) > 0$ are avoidable for proving existence of solutions, but they are used for uniqueness (in [14]).
Condition (6.) forms the basis for applying Gronwall's Lemma that has been extended to semicontinuous functions in [14] (see Lemma 3.5). Indeed, every function $\widetilde{y} : [0,1] \longrightarrow \widetilde{E}$ with $\widetilde{q}_\varepsilon(\widetilde{y}(t-h),\ \widetilde{y}(t)) \longrightarrow 0$ (for $h \downarrow 0$ and each t) satisfies

$$\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t,\widetilde{z}),\ \widetilde{y}(t)\Big) \leq \limsup_{h \downarrow 0}\ \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t-h,\widetilde{z}),\ \widetilde{y}(t-h)\Big).$$

for all elements $\widetilde{z} \in \widetilde{D}$ with $\pi_1\, \widetilde{\vartheta}(\cdot,\widetilde{z}) \leq \pi_1\, \widetilde{y}(\cdot)$ and times $t \in\ ]\,0,\ \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{x})]$.

Remark 2.4 1. A set $\widetilde{E} \neq \emptyset$ supplied with only one function $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[$ can be regarded as easy (but important) example by setting $\mathcal{J} := \{0\}$, $\widetilde{q}_0 := \widetilde{q}$. Considering a timed forward transitions $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ of order 0, the condition $\limsup_{\varepsilon \longrightarrow 0} \varepsilon^0 \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0$ means $0 = 0^0 \cdot \gamma_0(\widetilde{\vartheta}) = \gamma_0(\widetilde{\vartheta})$ — due to the definition $0^0 \overset{\text{Def.}}{=} 1$.
So for all $\widetilde{x} \in \widetilde{E},\ t \in [0,1[$,

$$\wedge \begin{cases} \limsup_{h \downarrow 0}\ \frac{1}{h}\ \widetilde{q}\Big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\Big) = 0 \\ \limsup_{h \downarrow 0}\ \frac{1}{h}\ \widetilde{q}\Big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\Big) = 0. \end{cases}$$

2. For a set $E \neq \emptyset$ and $p \in \mathbb{R}$ given, every ostensible metric $q_\varepsilon : E \times E \longrightarrow [0, \infty[$ induces a *timed* ostensible metric $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ according to

$$\widetilde{q}_\varepsilon\Big((s,x),\ (t,y)\Big) \ := \ f(\varepsilon)\ |s-t| + q_\varepsilon(x,y) \qquad \text{for all } (s,x),\ (t,y) \in \widetilde{E}.$$

with a function $f(\varepsilon) = o(\varepsilon^p) \geq 0$ for $\varepsilon \downarrow 0$,

Then every $\vartheta : [0,1] \times E \longrightarrow E$ satisfying the conditions (1.)–(6.) for $(E, D, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$ induces a *timed* forward transition $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ of order p on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ by

$$\widetilde{\vartheta}\Big(h,\ (t,x)\Big) \ := \ \Big(t+h,\ \vartheta(h,x)\Big) \qquad \text{for all } (t,x) \in \widetilde{E},\ \ h \in [0,1].$$

So the statements of this paragraph can be applied to their counterparts without separate time component very easily. In particular, transitions on a metric space (M, d) (introduced by Aubin in [2], [3]) prove to be a special case on (M, M, d).

Definition 2.5

$\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ denotes a set of timed forward transitions on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ assuming

$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau}) \ := \sup_{\substack{\widetilde{z} \in \widetilde{D},\, \widetilde{y} \in \widetilde{E} \\ \pi_1 \widetilde{z} \leq \pi_1 \widetilde{y}}} \ \limsup_{h \downarrow 0} \ \left(\frac{\widetilde{q}_\varepsilon(\widetilde{\vartheta}(h, \widetilde{z}),\ \widetilde{\tau}(h, \widetilde{y})) \ - \ \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{y}) \cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\, h}}{h} \right)^+$$

to be finite for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, $\varepsilon \in \mathcal{J}$.

These definitions enable us to compare any element $\widetilde{y} \in \widetilde{E}$ with an "earlier test element" $\widetilde{z} \in \widetilde{D}$ (i.e. $\pi_1 \widetilde{z} \leq \pi_1 \widetilde{y}$) while evolving along two forward transitions. The key idea of timed right–hand forward solutions is to preserve the structural estimate of the next proposition while extending mutational equations to timed ostensible metrics and "distributional" features (in regard to a test set $\widetilde{D}$).

Proposition 2.6 *Let $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ be timed forward transitions, $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$ and $0 \leq t_1 \leq t_2 \leq 1$, $h \geq 0$ (with $\pi_1 \widetilde{z} \leq \pi_1 \widetilde{y}$, $t_1 + h < \mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{z})$). Then the following estimate holds*

$$\begin{aligned} &\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1 + h, \widetilde{z}),\ \widetilde{\tau}(t_2 + h, \widetilde{y})) \\ \leq\ &\Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \widetilde{\tau}(t_2, \widetilde{y})) \ + \ h \cdot (\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau}))\Big) \cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\ h} \end{aligned}$$

The next step is to define the term "timed right–hand forward primitive" for a curve $\widetilde{\vartheta}(\cdot) : [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ of timed forward transitions. Roughly speaking, a curve $\widetilde{x}(\cdot) : [0, T[\longrightarrow \widetilde{E}$ represents a primitive of $\widetilde{\vartheta}(\cdot)$ if at each time $t \in [0, T[$, the timed forward transition $\widetilde{\vartheta}(t)$ can be interpreted as a first–order approximation of $\widetilde{x}(t + \cdot)$. Combining this notion with the key estimate of Proposition 2.6, a vague meaning of "first–oder approximation" is provided : Comparing $\widetilde{x}(t+\cdot)$ with $\widetilde{\vartheta}(t)(\cdot, \widetilde{z})$ (for any earlier test element $\widetilde{z} \in \widetilde{D}$, $\pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t)$), the same estimate ought to hold as if the factor $\widetilde{Q}_\varepsilon^{\mapsto}(\cdot, \cdot)$ was 0. It motivates the following definition with the expression "right–hand" indicating that $\widetilde{x}(\cdot)$ appears in the second argument of the distances $\widetilde{q}_\varepsilon$ ($\varepsilon \in \mathcal{J}$) in condition (1.).

Definition 2.7 *The curve* $\widetilde{x}(\cdot) : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *is called* timed right–hand forward primitive of *a map* $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \quad t \in [0,T[\qquad \exists \quad \widehat{\alpha}_\varepsilon^{\mapsto}(t) \geq \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}(t)), \quad \widehat{\gamma}_\varepsilon(t) \geq \gamma_\varepsilon(\widetilde{\vartheta}(t)) :$
   $$\limsup_{h\downarrow 0} \; \tfrac{1}{h} \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t)\,(h,\widetilde{z}),\; \widetilde{x}(t+h)) \;-\; \widetilde{q}_\varepsilon(\widetilde{z},\; \widetilde{x}(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \Big) \;\leq\; \widehat{\gamma}_\varepsilon(t)$$
   for every $\widetilde{z} \in \widetilde{D}$ *with* $\pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{x}(t)$, $\quad \limsup_{\varepsilon'\downarrow 0} \; \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) \;=\; 0,$
2. $\widetilde{x}(\cdot)$ *is uniformly continuous in time direction with respect to* $\widetilde{q}_\varepsilon$,
   i.e. there is $\omega_\varepsilon(\widetilde{x},\cdot) :]0,T[\longrightarrow [0,\infty[$ *such that* $\limsup_{h\downarrow 0} \; \omega_\varepsilon(\widetilde{x},h) \;=\; 0,$
   $\widetilde{q}_\varepsilon(\widetilde{x}(s),\; \widetilde{x}(t)) \;\leq\; \omega_\varepsilon(\widetilde{x},\; t-s)$ *for* $0 \leq s < t < T$.
3. $\pi_1\,\widetilde{x}(t) \;=\; t \;+\; \pi_1\,\widetilde{x}(0)$ *for all* $t \in [0,T[$.

Remark 2.8 Timed forward transitions induce their own primitives. To be more precise, every constant function $\widetilde{\vartheta}(\cdot) : [0,1[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ with $\widetilde{\vartheta}(\cdot) = \widetilde{\vartheta}_0$ has the timed right–hand forward primitives $[0,1[\longrightarrow \widetilde{E}, \;\; t \longmapsto \widetilde{\vartheta}_0(t,\widetilde{x})$ with any $\widetilde{x} \in \widetilde{E}$ — as an immediate consequence of Proposition 2.6. This property is easy to extend to piecewise constant functions $[0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ and so it forms the basis for Euler approximations.

Definition 2.9 *For* $\widetilde{f} : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *given,* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand forward solution *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand forward primitive of* $\widetilde{f}(\widetilde{x}(\cdot),\,\cdot\,) : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$.

In [14, Lorenz 2005] (and [16]), these definitions form a basis for extending evolution equations to ostensible metric spaces. A special kind of compactness (so–called *timed transitional compactness*) proves to be sufficient for the existence of these solutions if the right–hand side $\widetilde{f}(\cdot,\cdot)$ is continuous. So a common environment for completely different types of evolutions is provided as the examples of [15] show.

3 Weaker conditions on continuity and test elements: Timed right–hand sleek solutions.

Similarly to semigroups in Banach spaces however, the assumptions about (uniform) continuity might form severe obstacles in applications. With regard to timed forward transitions $\widetilde{\vartheta}$, a bound of the parameter $\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$ is often difficult to verify. Thus, we want to weaken the "uniform" character of continuity assumptions. In particular, the choice of $\alpha_\varepsilon^{\mapsto}$, $\mathcal{T}_\Theta$ ought to be more flexible without losing the track to the final aim of existence. As second key aspect, we dispense with the assumption $\widetilde{D} \subset \widetilde{E}$ (similarly to the notion of Petrov–Galerkin methods). Finally $\widetilde{q}_\varepsilon$ need not be time continuous.

General assumptions for § 3.

1. Let E and $\mathcal{D}$ denote nonempty sets (not necessarily $\mathcal{D} \subset E$), $\widetilde{E} \stackrel{\text{Def.}}{=} \mathbb{R} \times E$, $\quad \widetilde{\mathcal{D}} \stackrel{\text{Def.}}{=} \mathbb{R} \times \mathcal{D}$, $\quad \pi_1 : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow \mathbb{R},\ (t,x) \longmapsto t.$
2. $\mathcal{J} \subset [0,1]^K$ abbreviates a countable index set with $K \in \mathbb{N}$, $0 \in \overline{\mathcal{J}}$.
3. $\widetilde{q}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0,\infty[$ satisfies the timed triangle inequality (for each index $\varepsilon \in \mathcal{J}$).
4. $i_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{D}} \longrightarrow \widetilde{E}$ fulfills $\widetilde{q}_\varepsilon(\widetilde{z},\ i_{\widetilde{\mathcal{D}}}\,\widetilde{z}) = 0,\ \ \pi_1\,\widetilde{z} = \pi_1\, i_{\widetilde{\mathcal{D}}}\,\widetilde{z}$ for every $\widetilde{z} \in \widetilde{\mathcal{D}}, \varepsilon \in \mathcal{J}$.

Definition 3.1 *A map* $\widetilde{\vartheta} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$ *is called* timed sleek transition of order p *on* $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *if it fulfills for each* $\varepsilon \in \mathcal{J}$

1. $\widetilde{\vartheta}(0,\cdot) = \mathrm{Id}_{\widetilde{\mathcal{D}} \cup \widetilde{E}}$,
2. $\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 :\qquad \limsup\limits_{\varepsilon \longrightarrow 0}\ \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0 \qquad$ *and*
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[,$$
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[,$$

3′. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}} \qquad \exists\ \ \alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z}) \in [0,\infty[,\quad \mathbb{T}_\varepsilon = \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z}) \in\,]0,1]:$
$$\limsup_{h\downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})\big) - \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\widetilde{y}) - \gamma_\varepsilon(\widetilde{\vartheta})\,h}{h\quad \big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\widetilde{y}) + \gamma_\varepsilon(\widetilde{\vartheta})\,h\big)}\right)^{+} \leq \alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z}) \quad \forall\ 0 \leq t < \mathbb{T}_\varepsilon,\ \ \widetilde{y} \in \widetilde{E}\ \ (t + \pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{y}),$$

4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) :\ ]0,1] \longrightarrow [0,\infty[\quad$ *modulus of continuity* :
$$\widetilde{q}_\varepsilon(\widetilde{\vartheta}(s,\widetilde{y}),\ \widetilde{\vartheta}(t,\widetilde{y})) \leq \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad \forall\ s < t \leq 1,\ \ \widetilde{y} \in \widetilde{E},$$
5. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}} :\qquad \widetilde{\vartheta}(t,\widetilde{z}) \in \widetilde{\mathcal{D}} \qquad \forall\ t \in [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})]$,
6. $\limsup\limits_{h\downarrow 0}\ \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}) \geq \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t,\widetilde{z}),\ \widetilde{y}) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \widetilde{y} \in \widetilde{E},\ t \leq \mathbb{T}_\varepsilon\ \ (t + \pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{y})$,

7′. $\widetilde{\vartheta}(h,\ (t,y)) \in \{t+h\} \times E \subset \widetilde{E} \qquad \forall\ (t,y) \in \widetilde{E},\quad h \in [0,1],$
$\pi_1\,\widetilde{\vartheta}(h,\ (t,z)) \leq t+h$ *nondecreasing w.r.t.* $h \qquad \forall\ (t,z) \in \widetilde{\mathcal{D}},\quad h \in [0,1].$

8′. $\limsup\limits_{h\downarrow 0}\ \tfrac{1}{h}\cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, i_{\widetilde{\mathcal{D}}}\,\widetilde{z})),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{z}))\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \qquad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ t < \mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z}).$

So in comparison with Definition 2.2 of a *timed forward transition* (*of order* p), two features are changed :
Firstly, in condition (3'.), the parameter $\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})$ (with any $\widetilde{z} \in \widetilde{\mathcal{D}}$ fixed) is chosen "uniformly" for comparing the evolution of any $\widetilde{y} \in \widetilde{E}$ with the elements $\widetilde{\vartheta}(t,\widetilde{z}) \in \widetilde{\mathcal{D}}$ $(0 \leq t < \mathbb{T}_\varepsilon(\vartheta,\widetilde{z}))$ — whereas condition (3.) of Definition 2.2 takes all $\widetilde{y} \in \widetilde{E}$ and every "test element" $\widetilde{z} \in \widetilde{\mathcal{D}}$ $(\pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{y})$ into consideration for $\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) < \infty$.
Roughly speaking, key new properties of *sleek* transitions $\widetilde{\vartheta}$ are that $\alpha_\varepsilon(\widetilde{\vartheta},\widetilde{z})$ may depend on the test element $\widetilde{z} \in \widetilde{\mathcal{D}}$ and $\mathbb{T}_\varepsilon(\widetilde{\vartheta},\widetilde{z})$ can depend on $\varepsilon \in \mathcal{J}$ additionally.

Secondly, we take into account that the "test set" $\widetilde{\mathcal{D}}$ need not be a subset of $\widetilde{E}$. In § 2, each distance function $\widetilde{q}_\varepsilon$ was supposed to be a timed ostensible metric and thus, reflexive in particular. To be more precise, $\widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{z}) = 0$ for all $\widetilde{z} \in \widetilde{D} \subset \widetilde{E}$ formed the basis for

1.) the triangle inequality of $\widetilde{Q}_\varepsilon^{\mapsto}$ (see [14, Lorenz 2005], Remarks 11, 18 (iv)) and
2.) estimating the distance between a timed forward transition $\widetilde{\vartheta}(\cdot, \widetilde{z})$ and a timed right–hand forward solution (see [14], Proposition 27).

Although we might dispense with such a triangle inequality of transitions, the second point will be relevant for proving estimates between solutions such as Proposition 3.19 later. So we need a further relation between every test element $\widetilde{z} \in \widetilde{\mathcal{D}}$ and its counterpart $i_{\widetilde{\mathcal{D}}}\,\widetilde{z} \in \widetilde{E}$ — in addition to the general assumption $\widetilde{q}_\varepsilon(\widetilde{z},\ i_{\widetilde{\mathcal{D}}}\,\widetilde{z}) = 0$. Condition (8'.) bridges this gap for each timed sleek transition and, (only) here $i_{\widetilde{\mathcal{D}}}\,\widetilde{z} \in \widetilde{E}$ occurs in the first argument of $\widetilde{q}_\varepsilon$ whereas $\widetilde{z} \in \widetilde{\mathcal{D}}$ appears in the second one.
Finally, condition (7'.) is restricting the time component of $\widetilde{\vartheta}(\cdot, \widetilde{z})$ (for every test element $\widetilde{z} \in \widetilde{\mathcal{D}}$) just qualitatively. This additional "degree of freedom" will prove to be an important advantage for the geometric example in § 4.

The common aim of these different approaches is to preserve the structural estimate stated in Proposition 2.6. So first the counterpart of $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau})$ is introduced and then we obtain the corresponding estimate in exactly the same way as in [14].

Definition 3.2
$\widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ denotes a set of timed sleek transitions on $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$ assuming

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \ := \sup_{\substack{t \,\leq\, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}),\ \widetilde{y} \in \widetilde{E} \\ t + \pi_1\, \widetilde{z} \,\leq\, \pi_1\, \widetilde{y}}} \ \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\tau}(h, \widetilde{y})\big) - \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}\big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z}) \cdot h}}{h} \right)^+$$

to be finite for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $\varepsilon \in \mathcal{J}$.

Remark 3.3 The triangle inequality for $\widetilde{Q}_\varepsilon(\cdot, \cdot; \widetilde{z})$ cannot be expected to hold in general. Indeed for any sleek transitions $\widetilde{\vartheta}_1, \widetilde{\vartheta}_2, \widetilde{\vartheta}_3 \in \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ and $\widetilde{z} \in \widetilde{\mathcal{D}}$, $\widetilde{y} \in \widetilde{E}$, $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}_1, \widetilde{z})]$ with $t + \pi_1\, \widetilde{z} \leq \pi_1 \widetilde{y}$, the timed triangle inequality leads to

$$\begin{array}{rlll} & \frac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t+h, \widetilde{z}), & \widetilde{\vartheta}_3(h, \widetilde{y})) & - \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ \widetilde{y}) \ \cdot e^{\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z})\, h}\Big) \\ \leq & \frac{1}{h} \quad \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t+h, \widetilde{z}), & \widetilde{\vartheta}_2(h,\ i_{\widetilde{\mathcal{D}}}\, \widetilde{\vartheta}_1(t, \widetilde{z}))) & \\ + & \frac{1}{h} \quad \widetilde{q}_\varepsilon(\widetilde{\vartheta}_2(h,\ i_{\widetilde{\mathcal{D}}}\, \widetilde{\vartheta}_1(t, \widetilde{z})) & \widetilde{\vartheta}_2(h,\ \widetilde{\vartheta}_1(t, \widetilde{z}))) & \\ + & \frac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}_2(h,\ \widetilde{\vartheta}_1(t, \widetilde{z})), & \widetilde{\vartheta}_3(h, \widetilde{y})) & - \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ \widetilde{y}) \ \cdot e^{\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z})\, h}\Big). \end{array}$$

Supposing now $\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z}) \geq \alpha_\varepsilon(\widetilde{\vartheta}_3,\ \widetilde{\vartheta}_1(t, \widetilde{z}))$ in addition, we conclude from condition (8'.) on sleek transitions (Definition 3.1) and $\widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ i_{\widetilde{\mathcal{D}}}\, \widetilde{\vartheta}_1(t, \widetilde{z})) = 0$

$$\begin{array}{l} \limsup_{h \downarrow 0} \ \frac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t+h, \widetilde{z}),\ \widetilde{\vartheta}_3(h, \widetilde{y})) \ - \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ \widetilde{y}) \ \cdot e^{\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z})\, h}\Big) \\ \leq \ \ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}_1, \widetilde{\vartheta}_2; \widetilde{z}) \ + \ \gamma_\varepsilon(\widetilde{\vartheta}_2) \ + \ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}_2, \widetilde{\vartheta}_3;\ \widetilde{\vartheta}_1(t, \widetilde{z})). \end{array}$$

Proposition 3.4 *Let* $\widetilde{\vartheta}, \widetilde{\tau} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ *be timed sleek transitions of order* $p \in \mathbb{R}$ *on* $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$*. Furthermore suppose* $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $\widetilde{y} \in \widetilde{E}$ *and* $0 \le t_1 \le t_2 \le 1$, $h \ge 0$ *with* $\pi_1\, \widetilde{z} \le \pi_1\, \widetilde{y}$, $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$.
Then,
$$\begin{aligned} &\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1{+}h, \widetilde{z}),\ \widetilde{\tau}(t_2{+}h, \widetilde{y})) \\ &\le \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \widetilde{\tau}(t_2, \widetilde{y})) \ + \ h \cdot \Big(\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) + \gamma_\varepsilon(\widetilde{\tau})\Big)\Big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h}. \end{aligned}$$

Proof is based on the subsequent version of Gronwall's Lemma for semicontinuous functions. The auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1 + h, \widetilde{z}),\ \widetilde{\tau}(t_2 + h, \widetilde{y}))$ satisfies $\varphi_\varepsilon(h) \le \limsup_{k \downarrow 0} \varphi_\varepsilon(h - k)$ due to property (6.) of Definition 3.1.
Moreover it fulfills for any $h \in [0,1[$ with $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$
$$\limsup_{k \downarrow 0} \frac{\varphi_\varepsilon(h{+}k) - \varphi_\varepsilon(h)}{k} \ \le\ \alpha_\varepsilon(\widetilde{\tau}, \widetilde{z}) \cdot \varphi_\varepsilon(h) \ + \ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) + \gamma_\varepsilon(\widetilde{\tau}).$$
Indeed, for all $k > 0$ sufficiently small, the timed triangle inequality leads to
$$\begin{aligned} \varphi_\varepsilon(h+k) \ &\le\ \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1{+}h + k,\, \widetilde{z}),\ \ \widetilde{\tau}(k,\ \widetilde{\tau}(t_2{+}h,\ \widetilde{y}))) \\ &\quad + \ \widetilde{q}_\varepsilon(\widetilde{\tau}(k, \widetilde{\tau}(t_2{+}h, \widetilde{y})),\ \ \widetilde{\tau}(t_2{+}h + k,\ \ \widetilde{y})\ ) \\ &\le\ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \cdot k + \varphi_\varepsilon(h)\ e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ k} \ + \ \gamma_\varepsilon(\widetilde{\tau})\ k + o(k). \end{aligned}$$
□

Lemma 3.5 (Lemma of Gronwall for semicontinuous functions [14]**)**
Let $\psi : [a,b] \longrightarrow \mathbb{R}$, $f, g \in C^0([a,b[, \mathbb{R})$ *satisfy* $f(\cdot) \ge 0$ *and*
$$\begin{aligned} \psi(t) \ &\le\ \limsup_{h \downarrow 0}\ \psi(t-h), && \forall\ t \in\ ]a,\ b], \\ \psi(t) \ &\ge\ \limsup_{h \downarrow 0}\ \psi(t+h), && \forall\ t \in [a,\ b[, \\ \limsup_{h \downarrow 0} \frac{\psi(t+h) - \psi(t)}{h} \ &\le\ f(t) \cdot \limsup_{h \downarrow 0}\ \psi(t-h) \ + \ g(t) && \forall\ t \in\ ]a,\ b[. \end{aligned}$$
Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*
$$\psi(t) \ \le\ \psi(a) \cdot e^{\mu(t)} \ + \ \int_a^t e^{\mu(t) - \mu(s)}\ g(s)\ ds \qquad \text{with}\ \ \mu(t) := \int_a^t f(s)\ ds.$$

The structural estimate of Proposition 3.4 provides the key tool for applying the steps of [14] — after adapting the definitions of primitive and solution to timed sleek transitions. This modification is again based on comparing the evolutions with the initial points $\widetilde{\vartheta}_0(s, \widetilde{z}) \in \widetilde{\mathcal{D}}$, $0 \le s < \mathbb{T}_\varepsilon(\widetilde{\vartheta}_0, \widetilde{z})$, for any $\widetilde{z} \in \widetilde{\mathcal{D}}$ fixed (and the current transition $\widetilde{\vartheta}_0$).

Definition 3.6 *The curve* $\widetilde{x}(\cdot) : [0, T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed right–hand sleek primitive of *a map* $\widetilde{\vartheta}(\cdot) : [0, T[\longrightarrow \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \;\; t \in [0,T[\qquad\qquad \forall \;\; \widetilde{z} \in \widetilde{\mathcal{D}} \qquad$ *with* $\pi_1\, \widetilde{z} \le \pi_1\, \widetilde{x}(t)$:
$\exists \;\; \widehat{\alpha}_\varepsilon(t,\widetilde{z}) \ge \alpha_\varepsilon(\widetilde{\vartheta}(t),\widetilde{z}) \quad \exists \;\; \widehat{\gamma}_\varepsilon(t,\widetilde{z}) \ge \gamma_\varepsilon(\widetilde{\vartheta}(t)) : \quad \limsup_{\varepsilon' \downarrow 0} \; \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t,\widetilde{z}) \;=\; 0,$
$$\limsup_{h \downarrow 0} \tfrac{1}{h} \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s+h,\widetilde{z}),\;\; \widetilde{x}(t+h)\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s,\widetilde{z}),\;\; \widetilde{x}(t)\Big) \cdot e^{\widehat{\alpha}_\varepsilon(t,\widetilde{z})\cdot h} \Big) \le \widehat{\gamma}_\varepsilon(t,\widetilde{z})$$
for every $s \in [0,\; \mathbb{T}_\varepsilon(\widetilde{\vartheta}(t),\widetilde{z})[$ *with* $s + \pi_1\, \widetilde{z} \le \pi_1\, \widetilde{x}(t)$,

2. $\widetilde{x}(\cdot)$ *is uniformly continuous in time direction with respect to* $\widetilde{q}_\varepsilon$,
i.e. there is $\omega_\varepsilon(\widetilde{x},\cdot) :]0,T[\longrightarrow [0,\infty[$ *such that* $\limsup_{h\downarrow 0} \; \omega_\varepsilon(\widetilde{x},h) \;=\; 0,$
$\widetilde{q}_\varepsilon(\widetilde{x}(s),\; \widetilde{x}(t)) \;\le\; \omega_\varepsilon(\widetilde{x},\; t-s)$ *for* $0 \le s < t < T$.

3. $\pi_1\, \widetilde{x}(t) \;=\; t \;+\; \pi_1\, \widetilde{x}(0)$ *for all* $t \in [0,T[$.

Remark 3.7 Timed sleek transitions induce their own sleek primitives — as a direct consequence of Definition 3.1 and Proposition 3.4 (in the same way as in Remark 2.8 about timed *forward* transitions). Correspondingly, each piecewise constant function $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$ has a timed right–hand sleek primitive that is defined piecewise as well.

Definition 3.8 *For* $\widetilde{f} : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}_p(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$ *given, a map* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand sleek solution *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand forward primitive of* $\widetilde{f}(\widetilde{x}(\cdot),\,\cdot\,) : [0,T[\longrightarrow \widetilde{\Theta}_p(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$.

Considering timed right–hand *forward* solutions in [14], § 5 (and [16], Chapter 2), the main steps can now be applied to the new *sleek* versions. Two features have to be taken into account appropriately : firstly, $\widetilde{\mathcal{D}} \not\subset \widetilde{E}$ in general (but $i_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{D}} \longrightarrow \widetilde{E}$ "links" counterparts) and secondly, the dependence of parameters $\alpha_\varepsilon(\cdot,\widetilde{z})$, $\mathbb{T}_\varepsilon(\cdot,\widetilde{z})$ and $\widetilde{Q}_\varepsilon(\cdot,\cdot;\widetilde{z})$ on the test element $\widetilde{z}$ and $\varepsilon \in \mathcal{J}$.

Proposition 3.9 *Suppose* $\widetilde{\psi} \in \widetilde{\Theta}_p(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $t_1 \in [0,1[$, $t_2 \in [0,T[$, $\widetilde{z} \in \widetilde{\mathcal{D}}$. *Let* $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ *be a timed sleek primitive of* $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}_p(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$ *such that for each* $\varepsilon \in \mathcal{J}$, $t \in [0,T[$, *their parameters fulfill*
$$\wedge \begin{cases} \sup\limits_{0 \le s \le \min\{t,\mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})\}} & \widehat{\alpha}_\varepsilon(t,\widetilde{\psi}(s,\widetilde{z})) \;\le\; M_\varepsilon(t), \\ \sup\limits_{0 \le s \le \min\{t,\mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})\}} & \widehat{\gamma}_\varepsilon(t,\; \widetilde{\psi}(s,\widetilde{z})) \;\le\; R_\varepsilon(t), \\ & \widetilde{Q}_\varepsilon(\widetilde{\psi},\; \widetilde{\vartheta}(t);\; \widetilde{z}) \;\le\; c_\varepsilon(t) \end{cases}$$
with upper semicontinuous $M_\varepsilon, R_\varepsilon, c_\varepsilon : [0,T[\longrightarrow [0,\infty[$. *Set* $\mu_\varepsilon(h) := \displaystyle\int_{t_2}^{t_2+h} M_\varepsilon(s)\, ds$.
Then, for every $\varepsilon \in \mathcal{J}$ *and* $h \in]0,T[$ *with* $t_1 + h \;<\; \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})$, $t_1 + \pi_1\, \widetilde{z} \le \pi_1\, \widetilde{x}(t_2)$,
$$\begin{aligned} & \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1{+}h,\; \widetilde{z}),\;\; \widetilde{x}(t_2{+}h)) \\ \le \;\; & \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1,\widetilde{z}),\, \widetilde{x}(t_2)) \;\cdot\; e^{\mu_\varepsilon(h)} \;+\; \int_0^h e^{\mu_\varepsilon(h)-\mu_\varepsilon(s)} \;\; (c_\varepsilon(t_2{+}s) + 3\, R_\varepsilon(t_2{+}s))\;\, ds. \end{aligned}$$

Proof. We follow the same track as in the proof of Proposition 3.4 and consider the function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1+h, \widetilde{z}),\ \widetilde{x}(t_2+h))$. Firstly, $\varphi_\varepsilon(h) \leq \limsup_{k\downarrow 0} \varphi_\varepsilon(h-k)$ results from condition (6.) on sleek transitions (Def. 3.1) and the continuity of $\widetilde{x}(\cdot)$.

Furthermore we prove for any $h \in [0, T[$ with $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\psi}, \widetilde{z})$,

$$\limsup_{k\downarrow 0} \frac{\varphi_\varepsilon(h+k) - \varphi_\varepsilon(h)}{k} \leq M_\varepsilon(t_2+h) \cdot \varphi_\varepsilon(h) + c_\varepsilon(t_2+h) + 3\, R_\varepsilon(t_2+h).$$

In particular, this inequality implies $\varphi_\varepsilon(h) \geq \limsup_{k\downarrow 0} \varphi_\varepsilon(h+k)$ since its right–hand side is finite. Thus, the claim results from Gronwall's Lemma 3.5 – after approximating $M_\varepsilon(\cdot),\ R_\varepsilon(\cdot), c_\varepsilon(\cdot)$ by continuous functions from above.

For all small $k > 0$, the timed triangle inequality and Proposition 3.4 lead to

$$\begin{array}{rll}
\varphi_\varepsilon(h+k) & \leq & \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1+h+k,\ \widetilde{z}), \quad \widetilde{\vartheta}(t_2+h)\,(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1+h,\ \widetilde{z}))) \\
& + & \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_2+h)\,(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1+h, \widetilde{z})), \quad \widetilde{\vartheta}(t_2+h)\,(k,\ \widetilde{\psi}(t_1+h,\ \widetilde{z}))) \\
& + & \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_2+h)\,(k,\ \widetilde{\psi}(t_1+h, \widetilde{z})), \quad \widetilde{x}(t_2+h+k)) \\
& \leq & (\widetilde{Q}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(t_2+h);\ \widetilde{z}) + \widehat{\gamma}_\varepsilon(t_2+h,\ \widetilde{z}))\ e^{M_\varepsilon(t_2+h)\cdot k} \cdot k \\
& + & \gamma_\varepsilon(\widetilde{\vartheta}(t_2+h)) \cdot k + o(k) \\
& + & \varphi_\varepsilon(h)\, e^{\widehat{\alpha}_\varepsilon(t_2+h,\, \widetilde{\psi}(t_1+h,\, \widetilde{z}))\cdot k} + \widehat{\gamma}_\varepsilon(t_2+h,\ \widetilde{\psi}(t_1+h,\ \widetilde{z})) \cdot k + o(k) \\
& \leq & \varphi_\varepsilon(h)\, e^{M_\varepsilon(t_2+h)\cdot k} + \left|c_\varepsilon(t) + 3\, R_\varepsilon(t)\right|_{t=t_2+h} \cdot k + o(k)
\end{array}$$

since $t_1 + h + k < \mathbb{T}_\varepsilon(\widetilde{\psi}, \widetilde{z})$ implies $\widetilde{\psi}(t_1+h, \widetilde{z}),\ \widetilde{\psi}(t_1+h+k, \widetilde{z}) \in \widetilde{\mathcal{D}}$. □

With the objective of using Euler method for the existence of sleek solutions, we first have to specify an adequate type of convergence preserving the solution property. Assumptions (5.ii), (5.iii) of the next proposition might be subsumed under the term "two–sided graphically convergent". Obviously, it is weaker than pointwise convergence (with respect to time) and consists of two conditions with the limit function appearing in both arguments of $\widetilde{q}_\varepsilon$. Admitting vanishing "time perturbations" $\delta_j, \delta'_j \geq 0$ exemplifies the basic idea that the first argument of $\widetilde{q}_\varepsilon$ usually refers to the earlier element whereas the second argument mostly represents the later point.

Proposition 3.10 (Convergence Theorem)

Suppose the following properties of

$$\begin{array}{rcl}
\widetilde{f}_m,\ \ \widetilde{f}: \ \widetilde{E} \times [0, T[& \longrightarrow & \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}) \qquad (m \in \mathbb{N}) \\
\widetilde{x}_m,\ \ \widetilde{x}: \ [0, T[& \longrightarrow & \widetilde{E}\ :
\end{array}$$

1. $M_\varepsilon(\widetilde{z}) := \sup\limits_{m,t,\widetilde{y}} \{\alpha_\varepsilon(\widetilde{f}_m(\widetilde{y}, t),\ \widetilde{f}(\widetilde{x}(t), t)(h, \widetilde{z})) \mid 0 \leq h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t), t), \widetilde{z})\} < \infty,$

$R_\varepsilon(\widetilde{z}) \geq \sup\limits_{m,t,\widetilde{y},h} \{\widehat{\gamma}_\varepsilon(t,\ \widetilde{f}_m(\widetilde{x}_m, \cdot),\ \widetilde{f}(\widetilde{x}(t), t)(h, \widetilde{z})),\ \gamma_\varepsilon(\widetilde{f}_m(\widetilde{y}, t)),\ \gamma_\varepsilon(\widetilde{f}(\widetilde{y}, t))\}$

with $\limsup\limits_{\varepsilon'\downarrow 0} \varepsilon'^p \cdot R_{\varepsilon'}(\widetilde{z}) = 0,$

2. $\limsup \ \sup_{\substack{\widetilde{v}=\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z})\\ \tau<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}} \widetilde{Q}_\varepsilon\left(\widetilde{f}_m(\widetilde{y}_1,t_1),\ \widetilde{f}_m(\widetilde{y}_2,t_2);\ \widetilde{v}\right) \ \le\ R_\varepsilon(\widetilde{z})$

for $m \longrightarrow \infty,\quad t_2-t_1 \downarrow 0,\quad \widetilde{q}_\varepsilon(\widetilde{y}_1,\widetilde{y}_2) \longrightarrow 0\ \ (\pi_1\,\widetilde{y}_1 \le \pi_1\,\widetilde{y}_2),$

3. $\overset{\circ}{\widetilde{x}}_m(\cdot)\ \ni\ \widetilde{f}_m(\widetilde{x}_m(\cdot),\,\cdot)$ *in* $[0,T[\,,$ (*in the sense of Definition 3.8*)

4. $\widehat{\omega}_\varepsilon(h)\ :=\ \sup_m\ \omega_\varepsilon(\widetilde{x}_m,h)\ <\ \infty$ (*moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$)

$\limsup_{h\downarrow 0}\ \widehat{\omega}_\varepsilon(h)\ =\ 0,$

5. $\forall\ t_1,t_2 \in [0,T[,\quad t_3 \in\,]0,T[\quad \exists\ (m_j)_{j\in\mathbb{N}}$ *with* $m_j \nearrow \infty$ *and*

(*i*) $\limsup_{j\longrightarrow\infty}\ \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_1),t_1),\ \widetilde{f}_{m_j}(\widetilde{x}(t_1),t_1);\ \widetilde{z}\Big)\ \le\ R_\varepsilon(\widetilde{z}),$

(*ii*) $\exists\ (\delta'_j)_{j\in\mathbb{N}}:\ \delta'_j \searrow 0,\quad \widetilde{q}_\varepsilon(\widetilde{x}(t_2),\ \widetilde{x}_{m_j}(t_2+\delta'_j)) \longrightarrow 0,$
$\pi_1\,\widetilde{x}_{m_j}(t_2+\delta'_j)\ \searrow\ \pi_1\,\widetilde{x}(t_2).$

(*iii*) $\exists\ (\delta_j)_{j\in\mathbb{N}}:\ \delta_j \searrow 0,\quad \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t_3-\delta_j),\ \widetilde{x}(t_3)) \longrightarrow 0,$
$\pi_1\,\widetilde{x}_{m_j}(t_3-\delta_j)\ \nearrow\ \pi_1\,\widetilde{x}(t_3),$

for each $\varepsilon \in \mathcal{J}$ *and* $\widetilde{z} \in \widetilde{\mathcal{D}}$.

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{x}(\cdot),\,\cdot)$ *in* $[0,T[$.

Proof. The uniform continuity of $\widetilde{x}(\cdot)$ results from assumption (4.) :
Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon(\widetilde{x}_m(t_1),\ \widetilde{x}_m(t_2))\ \le\ \widehat{\omega}_\varepsilon(t_2-t_1)$ for $t_1<t_2<T$.
Let $\varepsilon \in \mathcal{J},\ 0 \le t_1 < t_2 < T$ be arbitrary and choose $(\delta'_j)_{j\in\mathbb{N}},\ (\delta_j)_{j\in\mathbb{N}},$ for $t_1,\ t_2$ (according to condition (5.ii), (5.iii)). For all $j \in \mathbb{N}$ large enough, we obtain $t_1+\delta'_j < t_2-\delta_j$ and so,

$$\begin{aligned}\widetilde{q}_\varepsilon(\widetilde{x}(t_1),\ \widetilde{x}(t_2))\ &\le\ \widetilde{q}_\varepsilon(\widetilde{x}(t_1),\ \widetilde{x}_{m_j}(t_1+\delta'_j))\ +\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t_1+\delta'_j),\ \widetilde{x}_{m_j}(t_2-\delta_j))\\ &\quad +\widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t_2-\delta_j),\ \widetilde{x}(t_2))\\ &\le\ o(1)\qquad\qquad +\ \widehat{\omega}_\varepsilon(t_2-t_1)\qquad \text{for } j\longrightarrow\infty.\end{aligned}$$

Now let $\varepsilon \in \mathcal{J},\ \widetilde{z} \in \widetilde{D}$ and $t \in [0,T[,\ \ 0 \le s < s+h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{z})$ be chosen arbitrarily with $s+\pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t)$. Condition (6.) of Definition 3.1 guarantees for all $k \in\,]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\widetilde{z}),\ \ \widetilde{x}(t+h))\ \le\ \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h-k,\widetilde{z}),\ \ \widetilde{x}(t+h))\ +\ h^2.$$

According to cond. (5.i) – (5.iii), there exist sequences $(m_j)_{j\in\mathbb{N}},\ (\delta_j)_{j\in\mathbb{N}},\ (\delta'_j)_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty,\ \ \delta_j \downarrow 0,\ \ \delta'_j \downarrow 0,\ \ \delta_j+\delta'_j < k$ and

$$\left\{\begin{array}{llcl}\widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t), & \widetilde{f}_{m_j}(\widetilde{x}(t),t);\ \widetilde{z}) & \le & R_\varepsilon+h^2,\\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t+h-\delta_j), & \widetilde{x}(t+h)) & \longrightarrow & 0,\\ \widetilde{q}_\varepsilon(\widetilde{x}(t), & \widetilde{x}_{m_j}(t+\delta'_j)) & \longrightarrow & 0,\\ & \pi_1\,\widetilde{x}_{m_j}(t+h-\delta_j) & \nearrow & \pi_1\,\widetilde{x}(t+h),\\ & \pi_1\,\widetilde{x}_{m_j}(t+\delta'_j) & \searrow & \pi_1\,\widetilde{x}(t).\end{array}\right.$$

Thus, Proposition 3.9 and Remark 3.3 imply for all large $j \in \mathbb{N}$ (depending on $\varepsilon, \widetilde{z}, t, h, k$),

$$\begin{aligned}
&\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\ \widetilde{z}), \quad \widetilde{x}(t+h)) \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h{-}k,\ \widetilde{z}), \quad \widetilde{x}_{m_j}(t{+}\delta'_j + h{-}k)) \\
&+\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t{+}\delta'_j + h{-}k), \quad \widetilde{x}_{m_j}(t{+}h - \delta_j)) \\
&+\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t{+}h - \delta_j), \quad \widetilde{x}(t{+}h)) \quad +\ h^2 \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}_{m_j}(t{+}\delta'_j)) \cdot e^{M_\varepsilon(\widetilde{z})\cdot(h-k)} \quad + \\
&+\ \int_0^{h-k} e^{M_\varepsilon(\widetilde{z})\cdot(h-k-\sigma)}\ (\widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta'_j+\sigma}\,;\ \widetilde{z})\ +\ 3\,R_\varepsilon(\widetilde{z}))\ d\sigma \\
&+\ \widehat{\omega}_\varepsilon(k - \delta_j - \delta'_j) \\
&+\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t{+}h - \delta_j), \quad \widetilde{x}(t{+}h)) \quad +\ h^2 \\
\le\ & \Big(\ \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t))\ +\ \widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{x}_{m_j}(t{+}\delta'_j))\Big) \cdot e^{M_\varepsilon(\widetilde{z})\cdot(h-k)}\ + \\
&+\ \int_0^{h} e^{M_\varepsilon(\widetilde{z})\cdot(h-\sigma)}\ \ \widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta'_j+\sigma}\,;\ \widetilde{z})\ d\sigma \\
&+\ \widehat{\omega}_\varepsilon(k) \quad +\ 2\,h^2\ +\ \mathrm{const}\cdot h\ R_\varepsilon(\widetilde{z}) \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)) \cdot e^{M_\varepsilon(\widetilde{z})\,h} \quad +\ \widehat{\omega}_\varepsilon(k)\ +\ 3\,h^2\ +\ \mathrm{const}\cdot h\ R_\varepsilon(\widetilde{z}) \\
&+\ \int_0^{h} e^{M_\varepsilon(\widetilde{z})\cdot(h-\sigma)} \Big(2\,R_\varepsilon(\widetilde{z}) + h^2 + \sup_{\substack{\widetilde{v}\,=\,\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z}) \\ \tau\,<\,\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}} \widetilde{Q}_\varepsilon(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta'_j+\sigma}\,;\widetilde{v})\Big) d\sigma
\end{aligned}$$

Now $j \longrightarrow \infty$ and then $k \longrightarrow 0$ provide the estimate

$$\begin{aligned}
&\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\ \widetilde{z}),\ \widetilde{x}(t+h)) \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t)) \cdot e^{M_\varepsilon(\widetilde{z})\,h}\ +\ 0\ +\ \mathrm{const}\cdot\ h\ (R_\varepsilon(\widetilde{z}) + h) \\
&+\ h\ e^{M_\varepsilon(\widetilde{z})\,h} \cdot \limsup_{j\longrightarrow\infty}\ \sup_{0\,\le\,\sigma\,\le\,h}\ \sup_{\substack{\widetilde{v}\,=\,\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z}) \\ \tau\,<\,\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}} \widetilde{Q}_\varepsilon(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta'_j+\sigma}\,;\ \widetilde{v}).
\end{aligned}$$

Finally, convergence assumptions (2.),(5.ii) and the equi–continuity of $(\widetilde{x}_m)$ ensure

$$\limsup_{h\,\downarrow\,0}\ \limsup_{j\longrightarrow\infty}\ \sup_{0\,\le\,\sigma\,\le\,h}\ \sup_{\substack{\widetilde{v}\,=\,\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z}) \\ \tau\,<\,\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}} \widetilde{Q}_\varepsilon(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta'_j+\sigma}\,;\ \widetilde{v}) \quad \le \quad R_\varepsilon(\widetilde{z})$$

and thus,

$$\limsup_{h\,\downarrow\,0} \tfrac{1}{h} \Big(\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s{+}h,\widetilde{z}),\ \widetilde{x}(t{+}h)) - \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t))\cdot e^{M_\varepsilon(\widetilde{z})\,h}\Big) \le \mathrm{c}\cdot R_\varepsilon(\widetilde{z}).$$

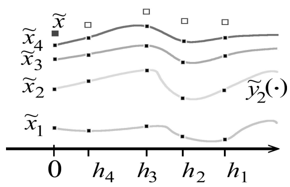
□

Similarly to ordinary differential equations, the convergence of approximations to a wanted solution usually results from assumptions about completeness or compactness. Here we prefer a suitable form of compactness since more than one distance function is involved. Still aiming to apply the Convergence Theorem 3.10 to Euler approximations, we introduce the following term (essentially as in [14], Definition 33) :

Definition 3.11 *Let $\widetilde{\Theta}$ denote a nonempty set of maps $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$. $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, \widetilde{\Theta})$ is called* timed transitionally compact *if it fulfills :*

Let $(\widetilde{x}_n)_{n \in \mathbb{N}}$, $(h_j)_{j \in \mathbb{N}}$ be any sequences in $\widetilde{E}$, $]0,1[$, respectively and $\widetilde{v} \in \widetilde{E}$ with $\sup_n |\pi_1 \widetilde{x}_n| < \infty$, $\sup_n \widetilde{q}_\varepsilon(\widetilde{v}, \widetilde{x}_n) < \infty$ for each $\varepsilon \in \mathcal{J}$, $h_j \longrightarrow 0$. Moreover suppose $\widetilde{\vartheta}_n : [0,1] \longrightarrow \widetilde{\Theta}$ to be piecewise constant (for each $n \in \mathbb{N}$) such that all curves $\widetilde{\vartheta}_n(t)(\cdot, \widetilde{x}) : [0,1] \longrightarrow \widetilde{E}$ have a common modulus of continuity $(n \in \mathbb{N}, t \in [0,1], \widetilde{x} \in \widetilde{E})$. Each $\widetilde{\vartheta}_n$ induces a function $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ with $\widetilde{y}_n(0) = \widetilde{x}_n$ in the same (piecewise) way as timed sleek transitions induce their own primitives according to Remark 3.7 (i.e. using $\widetilde{\vartheta}_n(t_m)(\cdot, \widetilde{y}_n(t_m))$ in each interval $]t_m, t_{m+1}]$ in which $\widetilde{\vartheta}_n(\cdot)$ is constant). Then there exist a sequence $n_k \nearrow \infty$ and $\widetilde{x} \in \widetilde{E}$ satisfying for each $\varepsilon \in \mathcal{J}$,

$$\begin{aligned}
&\lim_{k \to \infty} && \pi_1 \widetilde{x}_{n_k} &&= \pi_1 \widetilde{x}, \\
&\limsup_{k \longrightarrow \infty} && \widetilde{q}_\varepsilon(\widetilde{x}_{n_k}, \widetilde{x}) &&= 0, \\
\limsup_{j \longrightarrow \infty} & \sup_{k \geq j} && \widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{y}_{n_k}(h_j)) &&= 0.
\end{aligned}$$



A nonempty subset $\widetilde{F} \subset \widetilde{E}$ is called timed transitionally compact in *$(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, \widetilde{\Theta})$ if the same property holds for any sequence $(\widetilde{x}_n)_{n \in \mathbb{N}}$ in $\widetilde{F}$ (but $\widetilde{x} \in \widetilde{F}$ is* not *required).*

Proposition 3.12 (Existence of timed right–hand sleek solutions)

Assume that the tuple $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)))$ is timed transitionally compact. Furthermore let $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ fulfill for every $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$

1. $M_\varepsilon(\widetilde{z}) := \sup_{t_1, t_2, \widetilde{y}_1, \widetilde{y}_2} \{\alpha_\varepsilon(\widetilde{f}(\widetilde{y}_1, t_1), \widetilde{f}(\widetilde{y}_2, t_2)(h, \widetilde{z})) \mid 0 \leq h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{y}_2, t_2), \widetilde{z})\} < \infty,$
2. $c_\varepsilon(h) := \sup_{t, \widetilde{y}} \beta_\varepsilon(\widetilde{f}(\widetilde{y}, t))(h) < \infty, \qquad c_\varepsilon(h) \xrightarrow{h \downarrow 0} 0$
3. $\exists\, R_\varepsilon : \sup_{t, \widetilde{y}} \gamma_\varepsilon(\widetilde{f}(\widetilde{y}, t)) \leq R_\varepsilon < \infty, \qquad \varepsilon'^p\, R_{\varepsilon'} \xrightarrow{\varepsilon' \downarrow 0} 0$
4. $\exists\, \widehat{\omega}_\varepsilon(\cdot) : \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y}_1, t_1), \widetilde{f}(\widetilde{y}_2, t_2); \widetilde{z}\Big) \leq R_\varepsilon + \widehat{\omega}_\varepsilon\Big(\widetilde{q}_\varepsilon(\widetilde{y}_1, \widetilde{y}_2) + t_2 - t_1\Big)$
   for all $0 \leq t_1 \leq t_2 \leq T$ *and* $\widetilde{y}_1, \widetilde{y}_2 \in \widetilde{E}$ $(\pi_1 \widetilde{y}_1 \leq \pi_1 \widetilde{y}_2)$,
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup_{s \downarrow 0} \widehat{\omega}_\varepsilon(s) = 0.$

Then for every $\widetilde{x}_0 \in \widetilde{E}$, there is a timed right–hand sleek solution $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$.

Remark 3.13 The basic notion of its proof is easy to sketch. Indeed adapting the existence proof of *forward* solutions (in [14]) to *sleek* solutions here, we again start with Euler approximations $\widetilde{x}_n(\cdot) : [0, T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$,

$$\begin{aligned}
h_n &:= \tfrac{T}{2^n}, & t_n^j &:= j\, h_n & &\text{for } j = 0 \ldots 2^n, \\
\widetilde{x}_n(0) &:= \widetilde{x}_0, & \widetilde{x}_0(\cdot) &:= \widetilde{x}_0, \\
\widetilde{x}_n(t) &:= \widetilde{f}(\widetilde{x}_n(t_n^j), t_n^j)\,(t - t_n^j, \widetilde{x}_n(t_n^j)) & & & &\text{for } t \in\,]t_n^j, t_n^{j+1}], \ j \leq 2^n,
\end{aligned}$$

and then use Cantor diagonal construction (as $\mathcal{J}$ is assumed to be countable) in combination with timed transitional compactness. This leads to a function $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ with the property : For each $\varepsilon \in \mathcal{J}$ and $j \in \mathbb{N}$, there exist $K_j \in \mathbb{N}$ (depending on ε, j) and $N_j \in \mathbb{N}$ (depending on ε, j, K_j) such that $N_j > K_j > N_{j-1}$ and

$$\wedge \begin{cases} \widetilde{q}_\varepsilon(\widetilde{x}_{N_j}(s - 2\,h_{K_j}), \quad \widetilde{x}(s)\,) & \leq \frac{1}{j} \\ \widetilde{q}_\varepsilon(\widetilde{x}(t), \quad \widetilde{x}_{N_j}(t + 2\,h_{K_j})\,) & \leq \frac{1}{j} \end{cases}$$

for every $s, t \in [0,T[$. Due to Convergence Theorem 3.10 for $\widetilde{x}_{N_j}(\,\cdot\, + 2\,h_{N_j} + 2\,h_{K_j})$, the limit function $\widetilde{x}(\cdot)$ is a timed right–hand sleek solution. (For further details, see the proof of Proposition 36 in [14].)

Remark 3.14 Due to the fixed initial point $\widetilde{x}_0$, the compactness hypothesis can be weakened slightly. We only need that all $\widetilde{x}_n(t)$ $(0 < t < T,\, n \in \mathbb{N})$ are contained in a set $\widetilde{F} \subset \widetilde{E}$ that is transitionally compact in $(\widetilde{E},\, (\widetilde{q}_\varepsilon),\, \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)))$. This modification is useful if each transition $\widetilde{\vartheta} \in \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$ has all values in $\widetilde{F}$ after any positive time, i.e. $\widetilde{\vartheta}(t, \widetilde{x}) \in \widetilde{F}$ for all $0 < t \leq 1,\ \widetilde{x} \in \widetilde{E}$. In particular, it does not require additional assumptions about the initial value $\widetilde{x}_0 \in \widetilde{E}$.

Considering the geometric example of § 4, however, timed transitional compactness might be a very restrictive hypothesis. So we suggest a weaker condition of compactness — for the particular case that each $\widetilde{q}_\varepsilon$ is induced as supremum with respect to an additional parameter $\kappa \in \mathcal{I}$: $\quad \widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$.
Here $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ is a countable family of functions that need not satisfy the timed triangle inequality separately – in contrast to each $\widetilde{q}_\varepsilon$ $(\varepsilon \in \mathcal{J})$. We assume instead that every $\kappa \in \mathcal{I}$ has a counterpart $\kappa' \in \mathcal{I}$ with

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\leq\; \widetilde{q}_\varepsilon(\widetilde{y}_1, \widetilde{y}_2) \;+\; \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_2, \widetilde{y}_3)$$

for all $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $\pi_1\,\widetilde{y}_1 \leq \pi_1\,\widetilde{y}_2 \leq \pi_1\,\widetilde{y}_3$.
The key point now is : Supposing right–convergence with respect to each $\widetilde{q}_\varepsilon$ can be replaced by the hypothesis of right–convergence with respect to each $\widetilde{q}_{\varepsilon,\kappa}$ (and the latter might be easier to verify as in § 4, for example). In particular, assumption (5.iii) of the preceding Convergence Theorem 3.10 is modified.

Proposition 3.15 (Convergence Theorem II)
Assume $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ *such that each* $\kappa \in \mathcal{I}$ *has a counterpart* $\kappa' \in \mathcal{I}$ *fulfilling*

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\leq\; \widetilde{q}_\varepsilon(\widetilde{y}_1, \widetilde{y}_2) \;+\; \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_2, \widetilde{y}_3)$$

for all $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ *with* $\pi_1\,\widetilde{y}_1 \leq \pi_1\,\widetilde{y}_2 \leq \pi_1\,\widetilde{y}_3$.
In addition to hypotheses (1.)–(4.) of Proposition 3.10, suppose for all $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$ *and*

$$\begin{array}{rrcll} \widetilde{f}_m,\ \widetilde{f}: & \widetilde{E} \times [0,T[& \longrightarrow & \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}) & \quad (m \in \mathbb{N}) \\ \widetilde{x}_m,\ \widetilde{x}: & [0,T[& \longrightarrow & \widetilde{E}\ : & \end{array}$$

5'. $\forall\ t_1, t_2 \in [0,T[,\quad t_3 \in]0,T[\quad \exists\ (m_j)_{j\in\mathbb{N}}$ *with* $m_j \nearrow \infty$ *and*

$$\text{(i)}\quad \limsup_{j\longrightarrow\infty}\ \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_1),t_1),\ \widetilde{f}_{m_j}(\widetilde{x}(t_1),t_1);\ \widetilde{z}\Big)\ \leq\ R_\varepsilon(\widetilde{z})\quad \forall\ \widetilde{z}\in\widetilde{\mathcal{D}},$$

$$\text{(ii)}\quad \exists\ (\delta'_j)_{j\in\mathbb{N}}:\ \delta'_j \searrow 0,\quad \begin{array}{l}\widetilde{q}_\varepsilon(\widetilde{x}(t_2),\ \widetilde{x}_{m_j}(t_2+\delta'_j)) \longrightarrow 0\\ \pi_1\,\widetilde{x}_{m_j}(t_2+\delta'_j)\ \searrow\ \pi_1\,\widetilde{x}(t_2).\end{array}$$

$$\text{(iii')}\quad \exists\ (\delta_j)_{j\in\mathbb{N}}:\ \delta_j \searrow 0,\quad \begin{array}{l}\widetilde{q}_{\varepsilon,\kappa'}(\widetilde{x}_{m_j}(t_3-\delta_j),\ \widetilde{x}(t_3)) \longrightarrow 0\\ \pi_1\,\widetilde{x}_{m_j}(t_3-\delta_j)\ \nearrow\ \pi_1\,\widetilde{x}(t_3),\end{array}$$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$ *in* $[0,T[$.

Proof differs from the proof of Proposition 3.10 only in the additional supremum with respect to $\kappa \in \mathcal{I}$. Indeed, following the same track, the sufficiently large index $j \in \mathbb{N}$ (of the approximating sequences) now depends on $\kappa \in \mathcal{I}$ and its counterpart $\kappa' \in \mathcal{I}$ in addition. □

Now timed transitional compactness is now adapted for this modified condition on right–convergence and, we obtain the corresponding result about existence :

Definition 3.16 *Let* $\widetilde{\Theta}$ *denote a nonempty set of maps* $[0,1]\times\widetilde{E} \longrightarrow \widetilde{E}$. *Suppose* $\widetilde{q}_\varepsilon = \sup_{\kappa\in\mathcal{I}}\ \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}}\cup\widetilde{E})\times(\widetilde{\mathcal{D}}\cup\widetilde{E}) \longrightarrow [0,\infty[$ $(\varepsilon\in\mathcal{J}, \kappa\in\mathcal{I})$.
The tuple $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}}, \widetilde{\Theta})$ *is called* suitably transitionally compact *if it fulfills :*

Let $(\widetilde{x}_n)_{n\in\mathbb{N}},\ (h_j)_{j\in\mathbb{N}}$ *and* $\vartheta_n : [0,1] \longrightarrow \widetilde{\Theta},\ \ \widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ *(for each* $n\in\mathbb{N}$*) satisfy the assumptions of Definition 3.11. Then there exist a sequence* $n_k \nearrow \infty$ *and* $\widetilde{x}\in\widetilde{E}$ *satisfying for each* $\varepsilon\in\mathcal{J},\ \kappa\in\mathcal{I}$

$$\begin{array}{rcl}\lim\limits_{k\to\infty}\ \pi_1\,\widetilde{x}_{n_k} &=& \pi_1\,\widetilde{x},\\ \limsup\limits_{k\longrightarrow\infty}\ \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k},\ \widetilde{x}) &=& 0,\\ \limsup\limits_{j\longrightarrow\infty}\ \sup\limits_{k\geq j}\ \widetilde{q}_\varepsilon(\widetilde{x},\ \ \widetilde{y}_{n_k}(h_j)) &=& 0.\end{array}$$

Proposition 3.17 (Existence of timed right–hand sleek solutions II)
Assume $\widetilde{q}_\varepsilon = \sup_{\kappa\in\mathcal{I}}\ \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}}\cup\widetilde{E})^2 \longrightarrow [0,\infty[$ $(\varepsilon\in\mathcal{J},\kappa\in\mathcal{I})$ *such that each* $\kappa\in\mathcal{I}$ *has counterparts* $\kappa',\kappa''\in\mathcal{I}$ *fulfilling*

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1,\widetilde{y}_3)\ \leq\ \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_1,\widetilde{y}_2)\ +\ \widetilde{q}_{\varepsilon,\kappa''}(\widetilde{y}_2,\widetilde{y}_3)$$

for all $\widetilde{y}_1,\widetilde{y}_2,\widetilde{y}_3\in\widetilde{\mathcal{D}}\cup\widetilde{E}$ *with* $\pi_1\,\widetilde{y}_1\ \leq\ \pi_1\,\widetilde{y}_2\ \leq\ \pi_1\,\widetilde{y}_3$.
Furthermore let $\Big(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}}, \widetilde{\Theta}_p(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))\Big)$ *be suitably transitionally compact and* $\widetilde{f} : \widetilde{E}\times[0,T] \longrightarrow \widetilde{\Theta}_p(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *fulfill hypotheses (1.)–(4.) of Proposition 3.12.*

Then for every $\widetilde{x}_0\in\widetilde{E}$, *there is a timed right–hand sleek solution* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\ \cdot)$ *with* $\widetilde{x}(0) = \widetilde{x}_0$.

Remark 3.18 The proof is based on the same idea as for Proposition 3.12. Indeed, starting with the Euler approximations $\widetilde{x}_n(\cdot) : [0,T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$ of Remark 3.13, we conclude from the compactness hypothesis in combination with Cantor diagonal construction ($\mathcal{J}=\{\varepsilon_{j_1}, \varepsilon_{j_2} \dots\}$, $\mathcal{I}=\{\kappa_{i_1}, \kappa_{i_2} \dots\}$ are assumed to be countable at the most): With Q_K denoting the finite set $]0,T[\cap \mathbb{N} \cdot h_K$ of time steps for each $K \in \mathbb{N}$, there are sequences $m_k, n_k \nearrow \infty$ of indices and a function $\widetilde{x} : \bigcup_{K \in \mathbb{N}} Q_K \longrightarrow \widetilde{E}$ such that $m_k \leq n_k$,

$$\wedge \begin{cases} \sup\limits_{l \geq k} & \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_l}(t), \ \widetilde{x}(t)) & \leq \frac{1}{k} \\ \sup\limits_{l \geq k} & \widetilde{q}_{\varepsilon}(\widetilde{x}(s), \ \widetilde{x}_{n_l}(s + \frac{h_{m_k}}{2})) & \leq \frac{1}{k} \end{cases}$$

for every $K \in \mathbb{N}$ and all $\varepsilon \in \{\varepsilon_{j_1} \dots \varepsilon_{j_K}\} \subset \mathcal{J}$, $\kappa \in \{\kappa_{i_1} \dots \kappa_{i_K}\} \subset \mathcal{I}$, $s, t \in Q_K$, $k \geq K$. In particular, $\widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}(s), \ \widetilde{x}(t)) \ \leq \ c_\varepsilon(t-s)$ for any $s, t \in \bigcup_K Q_K$ with $s < t$ and all $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$. The supremum with respect to $\kappa \in \mathcal{I}$ implies $\widetilde{q}_\varepsilon(\widetilde{x}(s), \ \widetilde{x}(t)) \ \leq \ c_\varepsilon(t-s)$. Moreover, the sequence $(\widetilde{x}_{n_k}(\cdot))_{k \in \mathbb{N}}$ fulfills for all $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$, $K \in \mathbb{N}$, $t \in Q_K$ and sufficiently large $k, l \in \mathbb{N}$ (depending merely on ε, κ, K)

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k}(t), \ \widetilde{x}_{n_l}(t + \tfrac{h_{m_k}}{2})) \ \leq \ \tfrac{1}{k} + \tfrac{1}{l}\,.$$

For extending $\widetilde{x}(\cdot)$ to $t \in]0,T[\ \setminus \bigcup_K Q_K$, we apply the compactness hypothesis to $((\widetilde{x}_{n_k}(t))_{k \in \mathbb{N}}$ and obtain a subsequence $n_{l_j} \nearrow \infty$ of indices (depending on t) and an element $\widetilde{x}(t) \in \widetilde{E}$ satisfying for every $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$

$$\wedge \begin{cases} & \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_{l_j}}(t), \ \widetilde{x}(t)) & \longrightarrow 0, \\ \sup\limits_{i \geq j} & \widetilde{q}_\varepsilon(\widetilde{x}(t), \ \widetilde{x}_{n_{l_i}}(t + \frac{h_{m_j}}{2})) & \longrightarrow 0 \end{cases} \qquad \text{for } j \longrightarrow \infty.$$

It implies the following convergence even uniformly in t (but not necessarily in ε, κ)

$$\wedge \begin{cases} \limsup\limits_{K \longrightarrow \infty} \ \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k}(t - 2\,h_K), \ \widetilde{x}(t)\,) = 0, \\ \limsup\limits_{K \longrightarrow \infty} \ \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_\varepsilon(\widetilde{x}(t), \ \widetilde{x}_{n_k}(t + 2\,h_K)\,) = 0. \end{cases}$$

Indeed, the first property can be verified in exactly the same way as in the proof of [14], Proposition 36. For proving the second feature, we use $\widetilde{q}_\varepsilon(\widetilde{x}(t), \ \widetilde{x}(t')) \ \leq \ c_\varepsilon(t'-t)$ for every $t' \in \bigcup_K Q_K$ larger than t in combination with the corresponding convergence for all times in $\bigcup_K Q_K$.

Similarly to Remark 3.13 (and [14]), we summarize the construction of $\widetilde{x}(\cdot)$ in the following notation : For each $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$ and $j \in \mathbb{N}$, there exist $K_j \in \mathbb{N}$ (depending on ε, κ, j) and $N_j \in \mathbb{N}$ (depending on $\varepsilon, \kappa, j, K_j$) such that $N_j > K_j > N_{j-1}$ and

$$\wedge \begin{cases} \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{N_j}(s - 2\,h_{K_j}), \ \widetilde{x}(s)\,) & \leq \frac{1}{j} \\ \widetilde{q}_\varepsilon(\widetilde{x}(t), \ \widetilde{x}_{N_j}(t + 2\,h_{K_j})\,) & \leq \frac{1}{j} \end{cases}$$

for every $s, t \in [0,T[$. So Convergence Theorem II (Proposition 3.15) ensures that $\widetilde{x}(\cdot)$ is a timed right–hand sleek solution of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$. □

For concluding the existence from timed transitional compactness, we do not need any assumptions about the time parameter $\mathbb{T}_\varepsilon(\cdot, \widetilde{z}) > 0$ of sleek transitions.
The situation changes however for estimating the distance between solutions. Indeed, the definition of timed right–hand sleek solutions is based on comparisons with earlier elements (merely) of the form $\widetilde{\vartheta}(s, \widetilde{z}) \in \widetilde{\mathcal{D}}$ for $\widetilde{z} \in \widetilde{\mathcal{D}}$, $0 \leq s < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$. So two sleek solutions $\widetilde{x}(\cdot)$, $\widetilde{y}(\cdot)$ of the same initial value problem can hardly be compared with each other directly. Similarly to [14], auxiliary functions are used instead — like, for example,

$$\varphi_\varepsilon(t) \;:=\; \inf_{\substack{\widetilde{z} \in \widetilde{\mathcal{D}} \\ \pi_1 \widetilde{z} \leq \pi_1 \widetilde{x}(t)}} \Big(\widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{x}(t)) \;+\; \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{y}(t)) \Big).$$

Proposition 3.19 *Assume for the function* $\widetilde{f} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times [0,T] \longrightarrow \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$, *the curves* $\widetilde{x}, \widetilde{y} : [0,T[\longrightarrow \widetilde{E}$ *and some* $\varepsilon \in \mathcal{J}$

1. $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$, $\overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$ *(in the sense of Def. 3.8)*
   $\pi_1 \widetilde{x}(0) \;=\; \pi_1 \widetilde{y}(0) \;=\; 0,$
2. $M_\varepsilon \;\geq\; \sup\limits_{\widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\, t<T,\, \widetilde{z} \in \widetilde{\mathcal{D}}} \{\alpha_\varepsilon(\widetilde{f}(\widetilde{v},t),\, \widetilde{z}),\; \widehat{\alpha}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{z}),\; \widehat{\alpha}_\varepsilon(t, \widetilde{y}(\cdot), \widetilde{z})\},$
3. $R_\varepsilon \;\geq\; \sup\limits_{\widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\, t<T,\, \widetilde{z} \in \widetilde{\mathcal{D}}} \{\gamma_\varepsilon(\widetilde{f}(\widetilde{v},t)),\; \widehat{\gamma}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{z}),\; \widehat{\gamma}_\varepsilon(t, \widetilde{y}(\cdot), \widetilde{z})\}$
4. $\exists\; \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :\; \widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{z}, s),\; \widetilde{f}(\widetilde{v}, t);\; \widetilde{z}) \;\leq\; R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{v}) + \widehat{\omega}_\varepsilon(t-s)$
   for all $0 \leq s \leq t \leq T$ *and* $\widetilde{v} \in \widetilde{E}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$ *with* $\pi_1 \widetilde{z}_1 \leq \pi_1 \widetilde{v}$,
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup\limits_{s \downarrow 0} \; \widehat{\omega}_\varepsilon(s) \;=\; 0.$
5. $\forall\; t \in [0,T[:$ *the infimum* $\varphi_\varepsilon(t) \;:=\; \inf\limits_{\widetilde{z} \in \widetilde{\mathcal{D}},\, \pi_1 \widetilde{z} \leq t} \; (\widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{x}(t)) \;+\; \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{y}(t))) \;<\; \infty$
   can be approximated by a minimizing sequence $(\widetilde{z}_j)_{j \in \mathbb{N}}$ *in* $\widetilde{\mathcal{D}}$ *with*
   $$\pi_1 \widetilde{z}_j \;\leq\; \pi_1 \widetilde{z}_{j+1} \;\leq\; t, \qquad \frac{\sup_{k>j} \; \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k)}{\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\, \widetilde{z}_j)} \;\longrightarrow\; 0 \qquad (j \longrightarrow \infty).$$

Then, $\qquad \varphi_\varepsilon(t) \;\leq\; \varphi_\varepsilon(0) \,\cdot\, e^{(L_\varepsilon + M_\varepsilon) \cdot t} \;+\; 8\, R_\varepsilon\, t \,\cdot\, e^{(L_\varepsilon + M_\varepsilon) \cdot t}.$

Proof is based on a further subdifferential version of Gronwall's Lemma quoted in Lemma 3.20. $\varphi_\varepsilon(\cdot)$ satisfies $\varphi_\varepsilon(t) \;\leq\; \liminf\limits_{h \downarrow 0} \; \varphi_\varepsilon(t-h)$ for every $t \in]0,T[$ due to the timed triangle inequality and the continuity of $\widetilde{x}(\cdot)$, $\widetilde{y}(\cdot)$ (in time direction).
For showing $\qquad \liminf\limits_{h \downarrow 0} \; \frac{\varphi_\varepsilon(t+h) \,-\, \varphi_\varepsilon(t)}{h} \;\leq\; (L_\varepsilon + M_\varepsilon) \;\varphi_\varepsilon(t) \;+\; 8\, R_\varepsilon,$
let $(\widetilde{z}_j)_{j \in \mathbb{N}}$ denote a minimizing sequence in $\widetilde{\mathcal{D}}$ such that

$$\wedge \begin{cases} \pi_1 \widetilde{z}_j \;\leq\; \pi_1 \widetilde{z}_k \;\leq\; t, & \\ \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k) \qquad \leq\; \frac{1}{2\,j} \cdot \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\; \widetilde{z}_j) & \text{for all } j < k, \\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\; \widetilde{x}(t)) \;+\; \widetilde{q}_\varepsilon(\widetilde{z}_j,\; \widetilde{y}(t)) \;\longrightarrow\; \varphi_\varepsilon(t) & (j \longrightarrow \infty). \end{cases}$$

Now for every $h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\ \widetilde{z}_j),\ j < k,$ Proposition 3.9 and assumption (4.) imply

$$\begin{aligned}&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j, t)\ (h, \widetilde{z}_j),\ \ \widetilde{x}(t{+}h)\Big)\\ &\leq\ \widetilde{q}_\varepsilon\Big(\widetilde{z}_j,\ \widetilde{x}(t)\Big)\cdot e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon\Big(\widetilde{z}_j,\ \widetilde{x}(t{+}s)\Big) + \widehat{\omega}_\varepsilon(s)\ + 3\,R_\varepsilon\Big)\ ds.\end{aligned}$$

Setting the abbreviation $h_j := \frac{1}{2}\ \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\ \widetilde{z}_j) > 0,$ the approximating properties of $(\widetilde{z}_j)_{j\in\mathbb{N}}$ and the timed triangle inequality guarantee for any index $k > j$

$$\begin{aligned}&\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j, t)\ (h, \widetilde{z}_j),\ \ \widetilde{x}(t{+}h)\Big)\\ &\leq\ \widetilde{q}_\varepsilon\Big(\widetilde{z}_k,\ \widetilde{x}(t)\Big)\cdot e^{M_\varepsilon\, h} + \frac{e^{M_\varepsilon\, h} - 1}{M_\varepsilon}\ \Big(L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z}_k,\ \widetilde{x}(t)) + L_\varepsilon \cdot \tfrac{1}{j}\ h_j\ +\ 4\,R_\varepsilon\Big)\\ &\quad +\ \tfrac{1}{j}\ h_j\ \cdot e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(L_\varepsilon\cdot\omega_\varepsilon(\widetilde{x}, s) + \widehat{\omega}_\varepsilon(s)\Big)\ ds.\end{aligned}$$

The corresponding estimate for $\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j, t)\ (h, \widetilde{z}_j),\ \ \widetilde{y}(t{+}h)\Big)$ and $k \longrightarrow \infty,\ h := h_j,$ $j \longrightarrow \infty$ lead to $\displaystyle\liminf_{h\downarrow 0}\ \tfrac{\varphi_\varepsilon(t+h)\ -\ \varphi_\varepsilon(t)}{h}\ \leq\ (L_\varepsilon{+}M_\varepsilon)\ \varphi_\varepsilon(t)\ +\ 8\,R_\varepsilon.$

□

Lemma 3.20 (Lemma of Gronwall for semicontinuous functions II [14]**)**
Let $\psi : [a,b] \longrightarrow \mathbb{R},\ f, g \in C^0([a,b[, \mathbb{R})$ *satisfy* $f(\cdot) \geq 0$ *and*

$$\begin{aligned}\psi(t)\ &\leq\ \liminf_{h\downarrow 0}\ \psi(t-h), && \forall\ t \in\ ]a,\,b],\\ \psi(t)\ &\geq\ \liminf_{h\downarrow 0}\ \psi(t+h), && \forall\ t \in [a,\,b[,\\ \liminf_{h\downarrow 0}\ \tfrac{\psi(t+h)-\psi(t)}{h}\ &\leq\ f(t)\ \cdot\ \liminf_{h\downarrow 0}\ \psi(t-h)\ +\ g(t) && \forall\ t \in\ ]a,\,b[.\end{aligned}$$

Then, for every $t \in [a,b],$ *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t)\ \leq\ \psi(a)\ \cdot\ e^{\mu(t)}\ +\ \int_a^t e^{\mu(t)-\mu(s)}\ g(s)\ ds \qquad \textit{with } \mu(t) := \int_a^t f(s)\ ds.$$

Finally, the auxiliary function $\varphi_\varepsilon(\cdot)$ is modified with regard to $\widetilde{x}(\cdot)$:

$$\varphi_\varepsilon(t)\ :=\ \inf_{\widetilde{z}\in\widetilde{\mathcal{D}},\ \pi_1\widetilde{z}\leq\pi_1\widetilde{x}(t)}\ \big(\widetilde{p}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\ +\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{y}(t))\big)$$

Here $\widetilde{p}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ represents a generalized distance function on $\widetilde{\mathcal{D}} \cup \widetilde{E}$ that has the additional advantage of symmetry (by assumption) and satisfies the triangle inequality (not just the *timed* one). Roughly speaking, $\widetilde{p}_\varepsilon$ might not take all the properties of elements $\widetilde{x}, \widetilde{y} \in \widetilde{E}$ into consideration – compared with $\widetilde{q}_\varepsilon$.

In regard to timed transitions, the assumptions about $\widetilde{p}_\varepsilon$ do not consider the comparison of two different transitions. Instead we suppose continuity properties for each transition $\widetilde{\psi}$ only, e.g. the distance $\widetilde{p}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2)$ between arbitrary points $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$ may grow exponentially at the most while evolving along $\widetilde{\psi}$.

Proposition 3.21 *Suppose for* $\widetilde{p}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$, $p \in \mathbb{R}$, $\lambda_\varepsilon \geq 0$ *and* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$, $\widetilde{x}(\cdot), \widetilde{y}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ *the properties :*

1. $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)))$ *is timed transitionally compact,*
2. *each* $\widetilde{p}_\varepsilon$ *is symmetric and satisfies the triangle inequality,*
3. $\widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) := \inf\limits_{\substack{\widetilde{z} \in \widetilde{\mathcal{D}},\\ \pi_1 \widetilde{z} \leq \pi_1 \widetilde{v}_2}} (\widetilde{p}_\varepsilon(\widetilde{v}_1, \widetilde{z}) + \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{v}_2)) < \infty$ *for* $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$,
4. $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *constructed by Euler method according to Remark 3.13,*
5. $\widetilde{y}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$ *with* $\pi_1 \widetilde{x}(0) = \pi_1 \widetilde{y}(0) = 0$,
6. $\exists\ M_\varepsilon < \infty :$
$$\begin{aligned} &\widehat{\alpha}_\varepsilon(\cdot, \widetilde{x}(\cdot), \widetilde{z}),\ \widehat{\alpha}_\varepsilon(\cdot, \widetilde{y}(\cdot), \widetilde{z}) &&\leq M_\varepsilon, \\ &\widetilde{p}_\varepsilon(\widetilde{\psi}(h, \widetilde{v}_1),\ \widetilde{\psi}(h, \widetilde{v}_2)) &&\leq \widetilde{p}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) \cdot e^{M_\varepsilon h} \\ &\forall\ \widetilde{v}_1, \widetilde{v}_2 \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ \widetilde{z} \in \widetilde{\mathcal{D}},\ h \in]0,1],\ \widetilde{\psi} \in \{\widetilde{f}(\widetilde{w}, s) \mid \widetilde{w} \in \widetilde{E}, s < T\}, \end{aligned}$$
7. $\exists\ R_\varepsilon < \infty :$
$$\begin{aligned} &\widehat{\gamma}_\varepsilon(\,\cdot\,, \widetilde{x}(\cdot), \widetilde{z}),\ \widehat{\gamma}_\varepsilon(\,\cdot\,, \widetilde{y}(\cdot), \widetilde{z}) &&\leq R_\varepsilon, \\ &\limsup_{h \downarrow 0} \frac{\widetilde{p}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{\psi}(t, \widetilde{v})),\ \widetilde{\psi}(t+h, \widetilde{v})\big)}{h} &&\leq R_\varepsilon \\ &\forall\ \widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ \widetilde{z} \in \widetilde{\mathcal{D}},\ t \in [0,1[,\ \widetilde{\psi} \in \{\widetilde{f}(\widetilde{w}, s) \mid \widetilde{w} \in \widetilde{E}, s < T\}, \end{aligned}$$
8. $\exists\ c_\varepsilon(\cdot) :$
$$\begin{aligned} &\widetilde{p}_\varepsilon(\widetilde{\psi}(t, \widetilde{v}), \widetilde{\psi}(t+h, \widetilde{v})) + \beta_\varepsilon(\widetilde{\psi})(h) \leq c_\varepsilon(h) \\ &\forall\ \widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[,\ \widetilde{\psi} \in \{\widetilde{f}(\widetilde{w}, s) \mid \widetilde{w} \in \widetilde{E},\ s < T\}, \\ &c_\varepsilon(h) \longrightarrow 0 \quad \text{for } h \downarrow 0, \end{aligned}$$
9. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :$
$$\begin{aligned} &\widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{v}_1, t_1),\ \widetilde{f}(\widetilde{v}_2, t_2);\ \widetilde{z}) \leq R_\varepsilon + L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) + \widehat{\omega}_\varepsilon(t_2 - t_1) \\ &\text{for all } 0 \leq t_1 \leq t_2 \leq T,\quad \widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E},\ \widetilde{z} \in \widetilde{\mathcal{D}} \quad \text{with } \pi_1 \widetilde{v}_1 \leq \pi_1 \widetilde{v}_2, \\ &\widehat{\omega}_\varepsilon(\cdot) \geq 0 \text{ nondecreasing}, \qquad \limsup_{s \downarrow 0} \widehat{\omega}_\varepsilon(s) = 0, \end{aligned}$$
10. $\forall\ \widetilde{v} \in \widetilde{E},\ \delta > 0,\ 0 \leq s \leq t,\ 0 < h < 1$ *with* $t+h+\delta < T$, $h + \pi_1 \widetilde{v} \leq \pi_1 \widetilde{y}(t+h+\delta)$:
the infimum $\widetilde{\Delta}_\varepsilon(\widetilde{f}(\widetilde{v}, s)\,(h, \widetilde{v}),\ \widetilde{y}(t+h+\delta))$ *can be approximated by a minimizing sequence* $(\widetilde{z}_n)_{n \in \mathbb{N}}$ *in* $\widetilde{\mathcal{D}}$ *such that*
$$\pi_1 \widetilde{z}_m \leq \pi_1 \widetilde{z}_n \leq \pi_1 \widetilde{y}(t+h+\delta) \qquad \text{for all } m < n$$
$$\frac{\sup_{n > m}\ (\widetilde{p}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) + \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n))}{\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{v}, s),\ \widetilde{z}_m)} \longrightarrow 0 \qquad \text{for} \qquad m \longrightarrow \infty.$$

Then, $\varphi_\varepsilon(t) := \limsup\limits_{\delta \downarrow 0} \widetilde{\Delta}_\varepsilon(\widetilde{x}(t),\ \widetilde{y}(t + \delta))$ *fulfills the estimate*
$$\varphi_\varepsilon(t) \leq (\varphi_\varepsilon(0) + 5\, R_\varepsilon\ t)\ (1 + L_\varepsilon\, t)\ e^{2\, M_\varepsilon\, t}.$$

Remark 3.22 For verifying this estimate, we can follow exactly the same steps as in the proof of the (slightly more general) Proposition 42 in [14] because differences between *sleek* and *forward* timed transitions do not have any effect here (see [16], Proposition 2.3.10 alternatively). Roughly speaking, this analogy is due to the fact that

assumptions (6.), (7.), (9.) are uniform with respect to the "test element" $\widetilde{z} \in \widetilde{\mathcal{D}}$.
So let us just sketch the basic notion here :

Due to assumption (4.), there is a subsequence $(\widetilde{x}_{n_k}(\cdot))_{k \in \mathbb{N}}$ of Euler approximations (according to Remark 3.13) with the additional property $\widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t+2\,h_k)) \longrightarrow 0$ $(k \longrightarrow \infty)$ for every $t \in [0, T[$. So essentially, an estimate of $\widetilde{q}_\varepsilon(\widetilde{x}_{n_k}(t+2\,h_k),\ \widetilde{y}(t+\delta))$ is now needed for large k. Such a bound is stated in the subsequent lemma and results from Gronwall's Lemma 3.20 in the same way as Proposition 3.19.

Lemma 3.23 *Under the assumptions of preceding Proposition 3.21, choose* $t \in [0, T[$, $\delta > 0$, $\widetilde{\psi} \in \{\widetilde{f}(\widetilde{w}, s) \mid \widetilde{w} \in \widetilde{E}, s < T\} \subset \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$, $\widetilde{v} \in \widetilde{E}$ *with* $\pi_1 \widetilde{v} \le \pi_1 \widetilde{y}(0)$ *and, define* $\xi_\varepsilon(t) := \inf\limits_{\substack{\widetilde{z} \in \widetilde{\mathcal{D}},\\ \pi_1 \widetilde{z} \le \pi_1 \widetilde{y}(t+\delta)}} \Big(\widetilde{p}_\varepsilon(\widetilde{\psi}(t, \widetilde{v}),\ \widetilde{z}) + \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{y}(t+\delta))\Big).$

Then,

$$\xi_\varepsilon(t) \;\le\; \xi_\varepsilon(0)\, e^{M_\varepsilon\, t} + \int_0^t e^{M_\varepsilon \cdot (t-s)} \Big(\limsup_{s' \to s+\delta}\ \sup_{\widetilde{z} \in \widetilde{\mathcal{D}}} \widetilde{Q}_\varepsilon\Big(\widetilde{\psi},\ \widetilde{f}(\widetilde{y}(s'), s');\ \widetilde{z}\Big) + 4\, R_\varepsilon \Big)\, ds.$$

Remark 3.24 The estimate of Proposition 3.21 also holds in the situation of Proposition 3.17, i.e. particularly :

(i) Assume $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ with (at most) countably many $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ such that each $\kappa \in \mathcal{I}$ has counterparts $\kappa', \kappa'' \in \mathcal{I}$ fulfilling
$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\le\; \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_1, \widetilde{y}_2) + \widetilde{q}_{\varepsilon,\kappa''}(\widetilde{y}_2, \widetilde{y}_3)$$
for all $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $\pi_1 \widetilde{y}_1 \le \pi_1 \widetilde{y}_2 \le \pi_1 \widetilde{y}_3$.

(ii) Let $\Big(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}},\ (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon \in \mathcal{J}\\ \kappa \in \mathcal{I}}},\ \widetilde{\Theta}_p(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))\Big)$ be suitably transitionally compact in the sense of Definition 3.16 (instead of assumption (1.) of Proposition 3.21).

Indeed, Remark 3.18 guarantees that a subsequence $(\widetilde{x}_{n_k}(\cdot))_{k \in \mathbb{N}}$ of Euler approximations has again the additional property $\widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t+2\,h_k)) \longrightarrow 0$ $(k \longrightarrow \infty)$ for every $t \in [0, T[$. So the modification of its right–convergence is not relevant at all to the steps of adapting Proposition 3.21.

4 Example of first–order geometric evolutions

Now the concept of timed right–hand *sleek* solutions is applied to the evolution of compact subsets of $\mathbb{R}^N$. As key feature of *first–order* geometric evolutions, they may depend on nonlocal properties of the current compact set and its limiting normal cones at the boundary.

In [15, Lorenz 2005], such a geometric example is given for right–hand *forward* solutions. Indeed, the set $\mathcal{K}(\mathbb{R}^N)$ of all nonempty compact subsets of $\mathbb{R}^N$ is supplied with the ostensible metric

$$q_{\mathcal{K},N}(K_1, K_2) \;:=\; d\!\!l(K_1, K_2) \;+\; \text{dist}(\text{Graph } {}^\flat N_{K_2},\;\; \text{Graph } {}^\flat N_{K_1})$$

with $d\!\!l$ denoting the Pompeiu–Hausdorff distance on $\mathcal{K}(\mathbb{R}^N)$,
$N_K(x)$ the limiting normal cone of $K \in \mathcal{K}(\mathbb{R}^N)$ at $x \in \partial K$ (Def. 4.1),
${}^\flat N_K(x) \;:=\; N_K(x) \cap \mathbb{B} \;=\; \{v \in N_K(x) : |v| \le 1\}$.

$\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ consisting of all nonempty compact subsets with $C^{1,1}$ boundary is used for "test elements". Then for any parameter $\lambda > 0$ fixed, the set–valued maps $F : \mathbb{R}^N \leadsto \mathbb{R}^N$ satisfying

1. $F : \mathbb{R}^N \leadsto \mathbb{R}^N$ has nonempty compact convex values,
2. $\mathcal{H}_F(x,p) := \sup_{v \in F(x)} \; p \cdot v$ belongs to $C^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,
3. $\|\mathcal{H}_F\|_{C^{1,1}(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} \;\stackrel{\text{Def.}}{=}\; \|\mathcal{H}_F\|_{C^{1}(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} \;+\; \text{Lip } D\mathcal{H}_F|_{\mathbb{R}^N \times\, \partial\mathbb{B}_1} \;<\; \lambda$

induce forward transitions (of order 0) on $(\mathcal{K}(\mathbb{R}^N),\, \mathcal{K}_{C^{1,1}}(\mathbb{R}^N),\, q_{\mathcal{K},N})$ by means of their reachable sets $\vartheta_F(t, K) \;:=\; \{\, x(t) \,|\, x(\cdot) \in AC([0,t], \mathbb{R}^N),\; x(0) \in K,\; \dot{x}(\cdot) \in F(x(\cdot)) \text{ a.e.}\}$. Under stronger assumptions about the Hamiltonian $\mathcal{H}_F$, the required properties of transitional compactness are also verified in [15], § 4 and, so we obtain the existence of right–hand forward solutions (see [16], Section 4.4.4 alternatively).

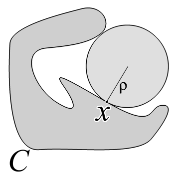
The estimates between forward solutions do not provide uniqueness though. Indeed, the smooth sets of $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ stay smooth for short times while evolving along such a differential inclusion, but there is no obvious lower bound of this period satisfying the approximating hypothesis such as condition (5.) of Proposition 3.19.

In this section, we introduce another timed ostensible metric for describing evolutions of compact subsets of $\mathbb{R}^N$ in Definition 4.2. When applying the new concept of timed *sleek* transitions, we benefit mainly from the facts that the time parameter $\mathbb{T}_\varepsilon(\cdot,\cdot)$ may depend on ε and that the "test set" $\widetilde{\mathcal{D}}$ need not be a subset of $\widetilde{E}$. Indeed, now we can use $\mathcal{K}(\mathbb{R}^N)$ also for "test elements" (i.e. restricting to $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ is dispensable). To be more precise, an additional index will enable us to distinguish whether a compact subset of $\mathbb{R}^N$ is regarded as "test element" or not :

$$\begin{aligned} E &:= \{1\} \times \mathcal{K}(\mathbb{R}^N), \\ \mathcal{D} &:= \{0\} \times \mathcal{K}(\mathbb{R}^N) \end{aligned} \qquad \text{and thus,} \qquad \begin{aligned} \widetilde{E} &:= \mathbb{R} \times \{1\} \times \mathcal{K}(\mathbb{R}^N), \\ \widetilde{\mathcal{D}} &:= \mathbb{R} \times \{0\} \times \mathcal{K}(\mathbb{R}^N). \end{aligned}$$

As a main advantage over the *forward* approach of [15], the estimate of Proposition 3.19 then implies uniqueness stated in Proposition 4.14.
From now on, fix a parameter $\Lambda > 0$ arbitrarily. It is used for both the timed ostensible metrics in Definition 4.2 and the set–valued maps (whose reachable sets are candidates for sleek transitions) in Definition 4.5.

Definition 4.1 *Let $C \subset \mathbb{R}^N$ be a nonempty closed set.*
A vector $\eta \in \mathbb{R}^N$, $\eta \neq 0$, is said to be a proximal normal vector *to C at $x \in C$ if there exists $\rho > 0$ with* $\mathbb{B}_\rho(x + \rho \frac{\eta}{|\eta|}) \cap C = \{x\}$.
The supremum of all ρ with this property is called proximal radius *of C at x in direction η. The cone of all these proximal normal vectors is called the* proximal normal cone *to C at x and is abbreviated as $N^P_C(x)$.*



For any $\rho > 0$, the set $N^P_{C,\rho}(x) \subset \mathbb{R}^N$ consists of all vectors $\eta \in N^P_C(x) \setminus \{0\}$ with the proximal radius $\geq \rho$ (and thus might be empty). Furthermore ${}^\flat N^P_{C,\rho}(x) := N^P_{C,\rho}(x) \cap \mathbb{B}$.

The so–called limiting normal cone *$N_C(x)$ to C at x consists of all vectors $\eta \in \mathbb{R}^N$ that can be approximated by sequences $(\eta_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ satisfying*

$$\begin{aligned} x_n &\longrightarrow x, & x_n &\in C, \\ \eta_n &\longrightarrow \eta, & \eta_n &\in N^P_C(x_n), \end{aligned}$$

i.e. $\quad N_C(x) \overset{\text{Def.}}{=} \operatorname{Limsup}_{\substack{y \longrightarrow x \\ y \in C}} N^P_C(y).$

Definition 4.2 *Set* $\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) := \mathbb{R} \times \{1\} \times \mathcal{K}(\mathbb{R}^N)$, $\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) := \mathbb{R} \times \{0\} \times \mathcal{K}(\mathbb{R}^N)$.
For $\varepsilon, \kappa \in [0,1]$, define $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa} : (\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)) \times (\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)) \longrightarrow [0,\infty[$,

$$\begin{aligned} \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((s,\mu,C),\ (t,\nu,D)) := {} & d\!\!l(C,D) + \\ & \int_\varepsilon^\infty \psi(\rho+\kappa+200\,\Lambda\,|t-s|) \cdot \operatorname{dist}\Big(\operatorname{Graph}\ {}^\flat N^P_{D,\,(\rho+\kappa+200\,\Lambda\,|t-s|)}, \\ & \qquad \operatorname{Graph}\ {}^\flat N^P_{C,\,\rho}\Big)\ d\rho \end{aligned}$$

with a fixed nonincreasing weight function $\psi \in C^\infty_0([0,2[)$, $\psi \geq 0$, and set

$$\begin{aligned} \widetilde{q}_{\mathcal{K},\varepsilon}((s,\mu,C),\ (t,\nu,D)) &:= \sup_{\kappa \in \,]0,1]\, \cap\, \mathbb{Q}} \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((s,\mu,C),\ (t,\nu,D)) \\ &= \limsup_{\kappa \downarrow 0} \ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((s,\mu,C),\ (t,\nu,D)). \end{aligned}$$

In fact, the second component (being either 0 or 1) does not have any influence on $\widetilde{q}_{\mathcal{K},\varepsilon}$ and $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa}$. Its purpose will only be to determine the evolution of the time components for "test elements" and "normal" elements in a different way (as specified in Definition 4.6). Furthermore $\widetilde{q}_{\mathcal{K},\varepsilon}$ is not "time continuous" as it was assumed in § 2.

Lemma 4.3 *$\widetilde{q}_{\mathcal{K},\varepsilon}$ is a timed ostensible metric on $\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ for each $\varepsilon \in [0,1]$.*

Proof. Reflexivity (in the sense of Definition 2.1) is obvious. For verifying the timed triangle inequality, choose any $(t_1, \mu_1, K_1),\ (t_2, \mu_2, K_2),\ (t_3, \mu_3, K_3) \in \mathbb{R} \times \{0,1\} \times \mathcal{K}(\mathbb{R}^N)$ with $t_1 \leq t_2 \leq t_3$. Then, we obtain for every $\kappa, \kappa' > 0$

$$\begin{aligned}
& \operatorname{dist}\Big(\text{Graph } {}^{\flat}N^P_{K_3,\,(\rho+\kappa+\kappa'+200\,\Lambda\,(t_3-t_1))},\ \ \text{Graph } {}^{\flat}N^P_{K_1,\,\rho}\Big) \\
\leq \quad & \operatorname{dist}\Big(\text{Graph } {}^{\flat}N^P_{K_3,\,(\rho+\kappa+\kappa'+200\,\Lambda\,(t_3-t_1))},\ \ \text{Graph } {}^{\flat}N^P_{K_2,\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1))}\Big) \\
+ \quad & \operatorname{dist}\Big(\text{Graph } {}^{\flat}N^P_{K_2,\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1))},\ \ \text{Graph } {}^{\flat}N^P_{K_1,\,\rho}\Big).
\end{aligned}$$

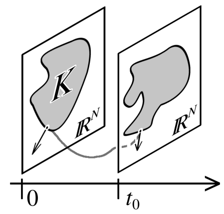
With regard to the weighted integral occurring in $\widetilde{q}_{\mathcal{K},\varepsilon,\,\kappa+\kappa'}((t_1,\mu_1,K_1),\ (t_3,\mu_3,K_3))$, a simple translation of coordinates (for the first distance term) and the monotonicity of ψ (related with the second distance term) imply

$$\begin{aligned}
& \widetilde{q}_{\mathcal{K},\varepsilon,\,\kappa+\kappa'}\,((t_1,\mu_1,K_1),\ (t_3,\mu_3,K_3)) \ \leq \\
& \leq\ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa'}((t_1,\mu_1,K_1),\ (t_2,\mu_2,K_2)) \ + \ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((t_2,\mu_2,K_2),\ (t_3,\mu_3,K_3)).
\end{aligned}$$

and thus the triangle inequality of $\widetilde{q}_\varepsilon$.

□

Now we focus on the evolution of limiting normal cones at the topological boundary and use the *Hamilton condition* as a key tool. It implies that roughly speaking, every boundary point x_0 of $\vartheta_F(t_0, K)$ and normal vector $\nu \in N_{\vartheta_F(t_0,K)}(x_0)$ have a trajectory and an adjoint arc linking x_0 to some $z \in \partial K$ and ν to $N_K(z)$, respectively.



Although the Hamilton condition is known in much more general forms (consider, for example, [22, Vinter 2000], Theorem 7.7.1 applied to proximal balls), we use only the well–known "smooth" version — due to later regularity conditions on F in the appendix.

Proposition 4.4 *Suppose for the set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$*

1. *$F(\cdot)$ has nonempty convex compact values,*
2. *$\mathcal{H}_F(x,p) := \sup_{v \in F(x)} p \cdot v$ is continuously differentiable in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$,*
3. *the derivative of $\mathcal{H}_F(\cdot,\cdot)$ has linear growth in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \mathbb{B}_1)$, i.e. $\|D\mathcal{H}_F(x,p)\| \leq const \cdot (1 + |x| + |p|)$ for all $x, p \in \mathbb{R}^N,\ |p| > 1$.*

Let $K \in \mathcal{K}(\mathbb{R}^N)$ be any initial set and $t_0 > 0$.

For every boundary point $x_0 \in \partial\,\vartheta_F(t_0,K)$ and normal vector $\nu \in N_{\vartheta_F(t_0,K)}(x_0)$, there are a solution $x(\cdot) \in C^1([0,t_0],\mathbb{R}^N)$ and its adjoint $p(\cdot) \in C^1([0,t_0],\mathbb{R}^N)$ satisfying

$$\left\{\begin{array}{llll}
\dot{x}(t) \ = & \frac{\partial}{\partial p}\ \mathcal{H}_F(x(t),\,p(t)) \ \in\ F(x(t)), & x(t_0) \ = \ x_0, & x(0) \ \in\ \partial K, \\
\dot{p}(t) \ = & -\,\frac{\partial}{\partial x}\ \mathcal{H}_F(x(t),\,p(t)), & p(t_0) \ = \ \nu, & p(0) \ \in\ N_K(x(0)).
\end{array}\right.$$

Definition 4.5 *For $\Lambda > 0$ fixed, the set $\mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$ consists of all set–valued maps $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ satisfying*

1. *$F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has nonempty compact convex values,*
2. *$\mathcal{H}_F(x,p) := \sup_{v \in F(x)} p \cdot v$ is twice continuously differentiable in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$,*
3. *$\|\mathcal{H}_F\|_{C^2(\mathbb{R}^N \times \partial \mathbb{B}_1)} < \Lambda$.*

These set–valued maps of $\mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$ induce the candidates for timed sleek transitions on $(\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in]0,1]})$ in the following sense :

Definition 4.6 *For any set–valued map $F \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$, element $(t, \mu, K) \in \mathbb{R} \times \{0,1\} \times \mathcal{K}(\mathbb{R}^N) = \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ and time $h > 0$, set*

$$\widetilde{\vartheta}_F(h,\ (t, \mu, K)) \ := \ (t + \mu\, h,\ \ \mu,\ \ \vartheta_F(h, K))$$

with the reachable set $\vartheta_F(h, K) \subset \mathbb{R}^N$ of the differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ a.e.

Lemma 4.7 *For every set–valued map $F \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$, initial element $\widetilde{K} = (b, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ and any times $0 \le s < t \le 1$,*

$$\widetilde{q}_{\mathcal{K},\varepsilon}\left(\widetilde{\vartheta}_F(s,\ \widetilde{K}),\ \ \widetilde{\vartheta}_F(t,\ \widetilde{K})\right) \ \le \ \Lambda\,(1 + \|\psi\|_{L^1}\ (e^\Lambda + 1)) \ \cdot\ |t - s|.$$

Proof. Obviously, the Pompeiu–Hausdorff distance satisfies for every $s, t \ge 0$

$$d\!\!l\left(\vartheta_F(s, K),\ \ \vartheta_F(t, K)\right) \ \le \ \sup_{\mathbb{R}^N} \ \|F(\cdot)\|_\infty \ \cdot\ (t - s) \ \le \ \Lambda\ (t - s).$$

Let $\tau(\varepsilon, \Lambda) > 0$ denote the time period mentioned in Corollary A.2. Without loss of generality, we can now assume $0 < t - s < \frac{1}{200\,\Lambda}\ \tau(\varepsilon, \Lambda)$ as a consequence of the timed triangle inequality.

For any $(x,p) \in \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t,K),\,(\rho + 200\,\Lambda\,(t-s))}$, $\rho \ge \varepsilon$ with $\rho + 200\,\Lambda\,(t-s) \le 2$, Corollary A.2 and Proposition 4.4 provide a solution $x(\cdot) \in C^1([s,t], \mathbb{R}^N)$ and its adjoint arc $p(\cdot) \in C^1([s,t], \mathbb{R}^N)$ satisfying

$$\begin{cases} \dot{x}(\sigma) \ = \ \ \ \frac{\partial}{\partial p}\ \mathcal{H}_F(x(\sigma),\, p(\sigma)) \ \in \ F(x(\sigma)), & x(t) \ = \ x, \quad x(s) \ \in \ \partial \vartheta_F(s, K), \\ \dot{p}(\sigma) \ = \ -\frac{\partial}{\partial x}\ \mathcal{H}_F(x(\sigma),\, p(\sigma)), & p(t) \ = \ p, \quad p(s) \ \in \ N^P_{\vartheta_F(s,K)}(x(s)) \end{cases}$$

and, $p(s)$ has proximal radius $\ge \ \rho + 200\,\Lambda\,(t-s) \ - \ 81\,\Lambda\,(t-s) \ > \ \rho$.

Obviously, $\mathcal{H}_F$ is (positively) homogeneous with respect to its second argument and thus, its definition implies $|\dot{p}(\sigma)| \le \Lambda\ |p(\sigma)|$ for all σ. Moreover $|p| \le 1$ implies that the projection of p on any cone is also contained in $\mathbb{B}_1$. So finally, we obtain

$$\begin{aligned} \mathrm{dist}\left((x,p),\ \ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(s,K),\,\rho}\right) \ &\le \ |x - x(s)| \ + \ |p - p(s)| \\ &\le \ \sup_{s \le \sigma \le t} \ \left(|\tfrac{\partial}{\partial x}\,\mathcal{H}_F| \ + \ |\tfrac{\partial}{\partial p}\,\mathcal{H}_F|\right)\Big|_{(x(\sigma),p(\sigma))} \cdot (t - s) \\ &\le \ \left(\Lambda\ e^{\Lambda\,t} \ \ + \ \Lambda\right) \cdot (t - s). \end{aligned}$$

□

Lemma 4.8 *For any $\varepsilon \in]0,1]$, let $\tau(\varepsilon,\Lambda) > 0$ denote the time period mentioned in Corollary A.2. Choose any set–valued maps $F, G \in \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N,\mathbb{R}^N)$, initial elements $\widetilde{K}_1 = (t_1, 0, K_1) \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K}_2 = (t_2, 1, K_2) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ with $t_1 \le t_2$.*
Then for all $h \in [0, \tau(\varepsilon,\Lambda)[$,

$$\begin{aligned}&\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(h,\widetilde{K}_1),\ \widetilde{\vartheta}_G(h,\widetilde{K}_2)\Big) \;\le\\ &\le\; e^{(\lambda_{\mathcal{H}}+\Lambda)\ h} \cdot \Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1,\ \widetilde{K}_2) \;+\; (1+4\,N\,\|\psi\|_{L^1})\ h\ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}\Big).\end{aligned}$$

with the abbreviation $\lambda_{\mathcal{H}} := 9\,\Lambda\,e^{2\,\Lambda\cdot\tau(\varepsilon,\Lambda)}$.

Proof. As presented in [15], Proposition 4.3, the well–known Theorem of Filippov provides the estimate of the Pompeiu–Hausdorff distance

$$\begin{aligned}d\!\!l\Big(\vartheta_F(h,K_1),\ \vartheta_G(h,K_2)\Big) &\le\; d\!\!l(K_1,K_2)\cdot e^{\Lambda\,h} \;+\; \sup_{\mathbb{R}^N} d\!\!l\Big(F(\cdot),G(\cdot)\Big)\cdot\frac{e^{\Lambda\,h}-1}{\Lambda}\\ &\le\; d\!\!l(K_1,K_2)\cdot e^{\Lambda\,h} \;+\; \sup_{\mathbb{R}^N\times\partial\mathbb{B}_1} |\mathcal{H}_F-\mathcal{H}_G|\cdot h\ e^{\Lambda\,h}.\end{aligned}$$

According to Definition 3.6, $\widetilde{\vartheta}_F(h,\widetilde{K}_1) \in \{t_1\}\times\{0\}\times\mathcal{K}(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$ and $\widetilde{\vartheta}_G(h,\widetilde{K}_2) \in \{t_2+h\}\times\{1\}\times\mathcal{K}(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$.
So for any $\kappa \in]0,1]\cap\mathbb{Q}$ and $\rho \ge \varepsilon$ with $\rho+\kappa+200\,\Lambda\,(t_2-t_1+h) \le 2$, we need an upper bound of $\operatorname{dist}\Big(\mathrm{Graph}\ {}^{\flat}N^P_{\vartheta_G(h,K_2),\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1+h))},\ \mathrm{Graph}\ {}^{\flat}N^P_{\vartheta_F(h,K_1),\,\rho}\Big)$.
Choose $\delta > 0$, $x \in \partial\,\vartheta_G(h,K_2)$ and $p \in N^P_{\vartheta_G(h,K_2)}(x)\cap\partial\mathbb{B}_1$ with proximal radius $\ge \rho+\kappa+200\,\Lambda\,(t_2-t_1+h)$ arbitrarily. According to Corollary A.2 and Proposition 4.4, there are a solution $x(\cdot) \in C^1([0,h],\mathbb{R}^N)$ and its adjoint arc $p(\cdot) \in C^1([0,h],\mathbb{R}^N)$ fulfilling

$$\begin{array}{rclrcl}\dot{x}(\cdot) &=& \frac{\partial}{\partial p}\,\mathcal{H}_G(x(\cdot),\ p(\cdot)) \in G(x(\cdot)), & \dot{p}(\cdot) &=& -\frac{\partial}{\partial x}\,\mathcal{H}_G(x(\cdot),\ p(\cdot)) \in \Lambda\,|p(\cdot)|\cdot\mathbb{B}\\ x(0) &\in& \partial K_2, & p(0) &\in& N^P_{K_2}(x(0)),\\ x(h) &=& x, & p(h) &=& p,\end{array}$$

and the proximal radius at $x(0)$ in direction $p(0)$ is $\ge \rho+\kappa+200\,\Lambda\,(t_2-t_1+h) - 81\,\Lambda\,h > \rho+\kappa+100\,\Lambda\,h+200\,\Lambda\,(t_2-t_1)$. Gronwall's Lemma guarantees $e^{-\Lambda\,h} \le |p(\cdot)| \le e^{\Lambda\,h}$ and so, $p(0)\,e^{-\Lambda\,h} \in {}^{\flat}N^P_{K_2}(x(0))\setminus\{0\}$.
Now let $(y_0,\widehat{q}_0)$ denote an element of $\mathrm{Graph}\ {}^{\flat}N^P_{K_1,\,(\rho+100\,\Lambda\,h)}$ with $\widehat{q}_0 \ne 0$ and

$$\begin{aligned}&\Big|(y_0,\widehat{q}_0) \;-\; \big(x(0),\ p(0)\,e^{-\Lambda\,h}\big)\Big| \;\le\\ &\le\; \operatorname{dist}\Big(\mathrm{Graph}\ {}^{\flat}N^P_{K_2,\,(\rho+\kappa+100\,\Lambda\,h+200\,\Lambda\,(t_2-t_1))},\ \mathrm{Graph}\ {}^{\flat}N^P_{K_1,\,(\rho+100\,\Lambda\,h)}\Big) \;+\; \delta.\end{aligned}$$

As a further consequence of Corollary A.2, we obtain a solution $y(\cdot) \in C^1([0,h],\mathbb{R}^N)$ and its adjoint arc $q(\cdot)$ satisfying

$$\begin{array}{rclrcl}\dot{y}(\cdot) &=& \frac{\partial}{\partial p}\,\mathcal{H}_F(y(\cdot),\ q(\cdot)), & \dot{q}(\cdot) &=& -\frac{\partial}{\partial y}\,\mathcal{H}_F(y(\cdot),\ q(\cdot)) \in \Lambda\,|q(\cdot)|\cdot\mathbb{B}\\ y(0) &=& y_0, & q(0) &=& \widehat{q}_0\ e^{\Lambda\,h} \ne 0,\\ y(h) &\in& \partial\,\vartheta_F(h,K_1), & q(h) &\in& N^P_{\vartheta_F(h,K_1)}(y(h))\end{array}$$

and the proximal radius at $y(h)$ in direction $q(h)$ is $\ge \rho+100\,\Lambda\,h \,-\, 81\,\Lambda\,h > \rho$.

According to [15], Lemma 4.22, the derivative of $\mathcal{H}_F$ is $\lambda_{\mathcal{H}}$–Lipschitz continuous on $\mathbb{R}^N \times (\mathbb{B}_{e^{\Lambda \cdot \tau(\varepsilon,\Lambda)}} \setminus \overset{\circ}{\mathbb{B}}_{e^{-\Lambda \cdot \tau(\varepsilon,\Lambda)}})$ with the abbreviation $\lambda_{\mathcal{H}} := 9\,\Lambda\, e^{2\,\Lambda \cdot \tau(\varepsilon,\Lambda)}$. Thus. the Theorem of Cauchy–Lipschitz leads to

$$\begin{aligned} &\operatorname{dist}\Big((x,p),\ \text{Graph}\ {}^\flat N^P_{\vartheta_F(h,K_1),\,\rho}\Big) \qquad \le \Big|(x,\,p)\ -\ (y(h),\,q(h))\Big| \\ &\le\ e^{\lambda_{\mathcal{H}} \cdot h} \cdot \Big|(x(0),\,p(0))\ -\ (y_0,\,\widehat{q}_0\, e^{\Lambda\, h})\Big|\ +\ \tfrac{e^{\lambda_{\mathcal{H}} \cdot h} - 1}{\lambda_{\mathcal{H}}} \cdot \sup_{0 \le s \le h} \big|D\,\mathcal{H}_F - D\,\mathcal{H}_G\big|\Big|_{(x(s),\,p(s))}. \end{aligned}$$

$\mathcal{H}_F$ and $\mathcal{H}_G$ are positively homogenous with respect to the second argument and thus,

$$\begin{aligned} \Big|\tfrac{\partial}{\partial x_j}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\,p(s))}\Big| &\le\ e^{\Lambda\, h}\ \ \|D\mathcal{H}_F - D\mathcal{H}_G\|_{C^0(\mathbb{R}^N \times \partial\mathbb{B}_1)}, \\ \Big|\tfrac{\partial}{\partial p_j}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\,p(s))}\Big| &\le\ \ 2 \cdot\ \ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}. \end{aligned}$$

So we obtain

$$\begin{aligned} &\operatorname{dist}\Big((x,p),\ \text{Graph}\ {}^\flat N^P_{\vartheta_F(h,K_1),\,\rho}\Big) \\ &\le\ e^{(\lambda_{\mathcal{H}}+\Lambda)\ h}\ \ \Big|(x(0),\,p(0)\,e^{-\Lambda\, h})\ -\ (y_0,\widehat{q}_0)\Big|\ +\ e^{\lambda_{\mathcal{H}}\, h}\, h \cdot 4\,N\, e^{\Lambda\, h}\ \ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} \end{aligned}$$

and, since $\delta > 0$ is arbitrarily small and $|p| = 1$,

$$\begin{aligned} &\operatorname{dist}\Big(\text{Graph}\ {}^\flat N^P_{\vartheta_G(h,K_2),\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1+h))},\ \ \text{Graph}\ {}^\flat N^P_{\vartheta_F(h,K_1),\,\rho}\Big) \\ &\le\ e^{(\lambda_{\mathcal{H}}+\Lambda)\ h}\ \cdot\ \Big\{\operatorname{dist}\Big(\text{Graph}\ {}^\flat N^P_{K_2,\,(\rho+\kappa+100\,\Lambda\, h+200\,\Lambda\,(t_2-t_1))},\ \ \text{Graph}\ {}^\flat N^P_{K_1,\,(\rho+100\,\Lambda\, h)}\Big)\ + \\ &\qquad\qquad +\ 4\,N\,h\ \cdot\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}\Big\}. \end{aligned}$$

With regard to $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa}\Big(\widetilde{\vartheta}_F(h,\widetilde{K}_1),\ \widetilde{\vartheta}_G(h,\widetilde{K}_2)\Big)$, integrating over ρ and the monotonicity of the weight function ψ (supposed in Definition 3.2) leads to the claimed estimate for all $h \in [0, \tau(\varepsilon,\Lambda)[$. □

Corollary 4.9 *Under the assumptions of Lemma 4.8,*

$$\begin{aligned} &\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(t{+}h,\widetilde{K}_1),\ \widetilde{\vartheta}_G(h,\widetilde{K}_2)\Big)\ \le \\ &\le\ e^{(\lambda_{\mathcal{H}}+\Lambda)\ h}\ \cdot\ \Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{K}_1),\ \widetilde{K}_2)\ +\ (1 + 4\,N\,\|\psi\|_{L^1})\ h\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times\,\partial\mathbb{B}_1)}\Big). \end{aligned}$$

for all $h,\ t \ge 0$ *with* $t + h < \tau(\varepsilon,\Lambda)$ *and*
$\widetilde{K}_1 = (t_1, 0, K_1) \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K}_2 = (t_2, 1, K_2) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *with* $t_1 \le t_2$.

Proof results directly from Lemma 4.8 since

$$\begin{aligned} \widetilde{\vartheta}_F(t{+}h,\ \widetilde{K}_1)\ &=\ \{t_1\} \times \{0\} \times \vartheta_F(t{+}h, K_1)\ \ =\ \ \widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}_F(t,\widetilde{K}_1)), \\ \widetilde{\vartheta}_F(t,\ \widetilde{K}_1)\ &\in\ \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N). \end{aligned}$$

□

Proposition 4.10 *The maps* $\widetilde{\vartheta}_F$ *of all set–valued* $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ *introduced in Def. 4.6 induce timed sleek transitions of order* 0 *on* $(\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in]0,1] \cap \mathbb{Q}})$ *with*

$$\begin{array}{rcl} \alpha_\varepsilon(\widetilde{\vartheta}_F, \cdot) & \stackrel{\text{Def.}}{=} & 10\ \Lambda\ \ e^{2\Lambda \cdot \tau(\varepsilon,\Lambda)}, \\ \beta_\varepsilon(\widetilde{\vartheta}_F)\,(t) & \stackrel{\text{Def.}}{=} & \Lambda\ \ (1 + \|\psi\|_{L^1}\ \ (e^\Lambda + 1))\ \cdot\ t, \\ \mathbb{T}_\varepsilon(\widetilde{\vartheta}_F, \cdot) & \stackrel{\text{Def.}}{=} & \min\{\tau(\varepsilon,\Lambda),\ 1\} \quad (\textit{mentioned in Corollary A.2}), \\ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}_F, \widetilde{\vartheta}_G; \cdot) & \le & (1 + 4\,N\,\|\psi\|_{L^1})\ \cdot\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}\,. \end{array}$$

Proof. The semigroup property of reachable sets implies

$$\begin{array}{rcl} \widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}_F(t,\widetilde{K})),\quad \widetilde{\vartheta}_F(t+h,\ \widetilde{K})\Big) & = & 0, \\ \widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(t+h,\ \widetilde{K}),\quad \widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}_F(t,\widetilde{K}))\Big) & = & 0 \end{array}$$

for all $F \in \mathrm{LIP}^{(C^2)}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$, $\widetilde{K} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, $h, t \ge 0$, $\varepsilon \in]0,1]$ since $\widetilde{q}_{\mathcal{K},\varepsilon}$ is reflexive. Thus, condition (2.) on timed sleek transitions (in Definition 3.1) is satisfied.

As an obvious choice of $i_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \longrightarrow \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, define $i_{\widetilde{\mathcal{D}}}((t,0,K)) := (t,1,K)$. In particular, it fulfills $\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z}, i_{\widetilde{\mathcal{D}}}\,\widetilde{Z}) = 0$ and $\pi_1\,\widetilde{Z} = \pi_1\,i_{\widetilde{\mathcal{D}}}\,\widetilde{Z}$ for all $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$. Definition 4.6 has the immediate consequences

$$\begin{array}{rcll} \widetilde{\vartheta}_F(0,\widetilde{K}) & = & \widetilde{K} & \text{for all } \ \widetilde{K} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N), \\ \widetilde{\vartheta}_F(h,\widetilde{Z}) & \in & \{\pi_1\,\widetilde{Z}\}\times\{0\}\times\mathcal{K}(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) & \text{for all } \ \widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N), \quad h \in [0,1], \\ \widetilde{\vartheta}_F(h,\widetilde{K}) & \in & \{h + \pi_1\,\widetilde{K}\}\times\{1\}\times\mathcal{K}(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) & \text{for all } \ \widetilde{K} \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N), \quad h \in [0,1], \\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}(t, i_{\widetilde{\mathcal{D}}}\widetilde{Z})),\ \widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}(t,\widetilde{Z}))) & = & 0 & \text{for all } \ \widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N), t, h \in [0,1], \end{array}$$

i.e. conditions (1.), (5.), (7'.), (8'.) of Definition 3.1 hold.

Set $\mathbb{T}_\varepsilon(\widetilde{\vartheta}_F, \cdot) \stackrel{\text{Def.}}{=} \min\{\tau(\varepsilon,\Lambda), 1\}$ with the time parameter $\tau(\varepsilon,\Lambda) > 0$ mentioned in Corollary A.2. Then, Corollary 4.9 guarantees for all $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K} \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}_F, \widetilde{Z})[$ with $t + \pi_1\,\widetilde{Z}\ \le\ \pi_1\,\widetilde{K}$

$$\limsup_{h\downarrow 0}\ \left(\frac{\widetilde{q}_{\mathcal{K},\varepsilon}\big(\widetilde{\vartheta}_F(t+h,\widetilde{Z}),\, \widetilde{\vartheta}_F(h,\widetilde{K})\big) - \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{Z}),\widetilde{K})}{h\quad \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{Z}),\widetilde{K})}\right)^{+}\ \le\ \lambda_{\mathcal{H}} + \Lambda\ \le\ 10\,\Lambda\,e^{2\Lambda\cdot\tau(\varepsilon,\Lambda)}.$$

Furthermore Lemma 4.7 implies condition (4.) of Definition 3.1 with the modulus

$$\beta_\varepsilon(\widetilde{\vartheta}_F)\,(t)\ \stackrel{\text{Def.}}{=}\ \Lambda\,(1 + \|\psi\|_{L^1}\ \ (e^\Lambda + 1))\ \cdot\ t$$

and, we obtain for all $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $F, G \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ that

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}_F,\ \widetilde{\vartheta}_G;\ \widetilde{Z})\ \le\ (1 + 4\,N\,\|\psi\|_{L^1})\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}\,.$$

Finally condition (6.) of Definition 3.1 has to be verified, i.e.

$$\limsup_{h\downarrow 0}\ \ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t-h,\ \widetilde{Z}),\ \widetilde{K})\quad \ge\quad \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{Z}),\ \widetilde{K})$$

for all $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K} \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}_F, \widetilde{Z})]$ with $t + \pi_1\,\widetilde{Z} \le \pi_1\,\widetilde{K}$.
Indeed, Lemma 4.7 guarantees $d\!\!\!l\Big(\vartheta_F(t-h, Z),\ \vartheta_F(t,Z)\Big) \longrightarrow 0$ for $h \downarrow 0$ and any set $Z \in \mathcal{K}(\mathbb{R}^N)$. So according to the subsequent Proposition B.1 (1.),

$$\mathrm{Limsup}_{h\downarrow 0}\ \ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t-h,Z),\,\rho}\ \subset\ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t,Z),\,\rho}$$

and thus, we obtain for every $\widetilde{Z} = (a, 0, Z) \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K} = (b, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, $\rho > 0$, $\kappa \in]0, 1]$ and $t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}_F, \widetilde{Z})]$ with $a + t \le b$

$$\begin{array}{ll} \limsup\limits_{h \downarrow 0} & \operatorname{dist}\Big(\text{Graph } {}^\flat N^P_{K,\,(\rho+\kappa+200\,\Lambda\,|b-a|)},\ \ \text{Graph } {}^\flat N^P_{\vartheta_F(t-h,Z),\,\rho}\Big) \\ \ge & \operatorname{dist}\Big(\text{Graph } {}^\flat N^P_{K,\,(\rho+\kappa+200\,\Lambda\,|b-a|)},\ \ \text{Graph } {}^\flat N^P_{\vartheta_F(t,Z),\,\rho}\Big). \end{array}$$

Due to $\pi_1\, \widetilde{\vartheta}_F(t-h,\, \widetilde{Z}) = a = \pi_1\, \widetilde{\vartheta}_F(t, \widetilde{Z})$, this inequality implies the wanted condition (6.) of Definition 3.1 with respect to $\widetilde{q}_{\mathcal{K},\varepsilon}$. □

In § 3, the results about the existence of timed right–hand sleek solutions are based on appropriate forms of (transitional) compactness (see Definitions 3.11, 3.16). Considering a converging sequence of compact sets, some features of their proximal cones are summarized in Appendix B. In particular, Graph $N^P_{K,\rho} \subset \operatorname{Limsup}_{n\to\infty}$ Graph $N^P_{K_n,\rho}$ does not hold for every radius $\rho > 0$ in general. For this reason, we now prefer the second approach using "suitably transitionally compact" and Proposition 3.17.

Proposition 4.11 $(\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon,\kappa})_{\varepsilon,\kappa \in]0,1] \cap \mathbb{Q}},\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in]0,1] \cap \mathbb{Q}},\ \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N))$ *is suitably transitionally compact* (*in the sense of Definition 3.16*).

Proof. Applying Definition 3.16 to this tuple, the situation is the following : Let $(\widetilde{K}_n = (t_n, 1, K_n))_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ be sequences in $\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ and $]0, 1[$, respectively, with $h_j \downarrow 0$ and $\sup_n |t_n| < \infty$, $\sup_n\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1, \widetilde{K}_n) < \infty$. Furthermore suppose each $G_n : [0, 1] \longrightarrow \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ to be piecewise constant $(n \in \mathbb{N})$ and set

$$\begin{array}{l} \widetilde{G}_n\ :\ [0,1] \times \mathbb{R}^N \leadsto \mathbb{R}^N, \quad (t, x) \longmapsto G_n(t)(x), \\ \widetilde{K}_n(h)\ :=\ \{t_n + h\} \times \{1\} \times \vartheta_{\widetilde{G}_n}(h, K_n)\ \in\ \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \qquad \text{for } h \ge 0. \end{array}$$

We have to prove the existence of a sequence $n_k \nearrow \infty$ of indices and an element $\widetilde{K} = (t, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ satisfying $t_{n_k} \longrightarrow t$ $(k \longrightarrow \infty)$ and for every $\varepsilon, \kappa \in]0, 1] \cap \mathbb{Q}$

$$\begin{array}{rcl} \limsup\limits_{k\longrightarrow\infty}\ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}(\widetilde{K}_{n_k}(0),\ \ \widetilde{K}) &=& 0, \\ \limsup\limits_{j\longrightarrow\infty}\ \sup\limits_{k\ge j}\ \ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K},\ \ \widetilde{K}_{n_k}(h_j)) &=& 0. \end{array}$$

Closed bounded balls in $(\mathbb{R}, |\cdot|)$ and $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ are known to be compact. So there are a subsequence (again denoted by) $(\widetilde{K}_n = (t_n, 1, K_n))_{n\in\mathbb{N}}$ and $\widetilde{K} = (t, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ with $d\!\!l(K_n, K) \le \frac{1}{n}$ and $t_n \longrightarrow t$ $(n \longrightarrow \infty)$. Proposition B.1 (3.) ensures for all $\rho, \kappa > 0$

$$\operatorname{dist}\Big(\text{Graph } {}^\flat N^P_{K,\,\rho+\kappa},\ \ \text{Graph } {}^\flat N^P_{K_n,\,\rho}\Big) \longrightarrow 0 \qquad (n \longrightarrow \infty)$$

and thus, $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa}(\widetilde{K}_n,\ \widetilde{K}) \longrightarrow 0.$

Furthermore, $\operatorname{dist}\Big(\text{Graph } {}^\flat N^P_{K_n,\,\rho},\ \ \text{Graph } {}^\flat N^P_{K,\,\rho}\Big) \longrightarrow 0 \qquad (n \longrightarrow \infty)$

results from Proposition B.1 (1.) for every $\rho > 0$ and so, Lebesgue's Dominated Convergence Theorem guarantees

$$\int_0^2 \operatorname{dist}\Big(\text{Graph } {}^\flat N^P_{K_n,\,\rho},\ \ \text{Graph } {}^\flat N^P_{K,\,\rho}\Big)\, d\rho \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

In particular, we can choose a subsequence (again denoted by) $(\widetilde{K}_n = (t_n, 1, K_n))_{n \in \mathbb{N}}$ with the additional properties $|t - t_n| < \frac{h_j}{2}$ for all $n > j$ and

$$\int_0^2 \operatorname{dist}\Big(\operatorname{Graph}\ {}^\flat N^P_{K_n,\,\rho},\ \operatorname{Graph}\ {}^\flat N^P_{K,\,\rho}\Big)\ d\rho \ \le\ \tfrac{1}{n \cdot \|\psi\|_{L^\infty}} \quad \text{for all } n \in \mathbb{N}.$$

Similarly to the preceding Lemma 4.7, the Hamilton condition (of Proposition 4.4) provides the following upper bound of $\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K},\ \widetilde{K}_n(h_j))$ for every $j \in \mathbb{N}$ and all $n > j$

$$\begin{array}{rll}
& d\!\!l(K,\ \vartheta_{\widetilde{G}_n}(h_j, K_n)) & \\
+ & \sup\limits_{\kappa>0} \displaystyle\int_\varepsilon^\infty & \psi(\rho{+}\kappa{+}200\,\Lambda\,|t - t_n + h_j|)\ \cdot \\
& & \operatorname{dist}\Big(\operatorname{Graph}\ {}^\flat N^P_{K_n(h_j),\,(\rho+\kappa+200\,\Lambda\,|t-t_n+h_j|)},\ \ \operatorname{Graph}\ {}^\flat N^P_{K,\,\rho}\Big)\ \ d\rho \\
\le & d\!\!l(K,\ K_n) + \Lambda\, h_j & \\
+ & \sup\limits_{\kappa>0} \displaystyle\int_\varepsilon^\infty & \psi(\rho{+}\kappa{+}200\,\Lambda\,|t - t_n + h_j|)\ \cdot \\
& & \operatorname{dist}\Big(\operatorname{Graph}\ {}^\flat N^P_{K_n(h_j),\,(\rho+\kappa+100\,\Lambda\,h_j)},\qquad \operatorname{Graph}\ {}^\flat N^P_{K_n,\,\rho}\Big)\ \ d\rho\ +\ \tfrac{1}{n} \\
\le & d\!\!l(K,\ K_n) + \Lambda\, h_j \quad + \quad \sup\limits_{\kappa>0}\ \Lambda\ (e^\Lambda + 1)\ \|\psi\|_{L^1}\ \cdot\ h_j\ \ +\ \ \tfrac{1}{n}\,, & \\
\text{i.e.} & \sup\limits_{n>j}\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K},\ \widetilde{K}_n(h_j))\ \longrightarrow\ 0 \quad \text{for}\ \ j \longrightarrow \infty. &
\end{array}$$

□

Corollary 4.12 *For any $R > 0$, set $\widetilde{\mathcal{K}}^{\rightarrow}_R(\mathbb{R}^N) \stackrel{\text{Def.}}{=} \mathbb{R} \times \{1\} \times \mathcal{K}(\mathbb{B}_R(0)) \subset \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ and $\widetilde{\mathcal{K}}^{\curlyvee}_R(\mathbb{R}^N) \stackrel{\text{Def.}}{=} \mathbb{R} \times \{0\} \times \mathcal{K}(\mathbb{B}_R(0)) \subset \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$.*
Then, $\Big(\widetilde{\mathcal{K}}^{\rightarrow}_R(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon,\kappa})_{\varepsilon,\kappa \in\,]0,1]\,\cap\,\mathbb{Q}},\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in\,]0,1]\,\cap\,\mathbb{Q}},\ \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)\Big)$ is also suitably transitionally compact. □

Applying Proposition 3.17 to this tuple provides the existence of timed right–hand sleek forward solutions :

Proposition 4.13
Regard the maps $\widetilde{\vartheta}_F$ of all set–valued $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ (defined in Def. 4.5, 4.6) as timed sleek transitions of order 0 on $(\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in\,]0,1] \cap \mathbb{Q}})$ according to Proposition 4.10.
For $\widetilde{f} : \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \times [0, T] \longrightarrow \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$, suppose the existence of a modulus $\widehat{\omega}(\cdot)$ of continuity with $\|\mathcal{H}_{\widetilde{f}(\widetilde{K}_1, t_1)} - \mathcal{H}_{\widetilde{f}(\widetilde{K}_2, t_2)}\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} \ \le\ \widehat{\omega}\Big(\widetilde{q}_{\mathcal{K},0}(\widetilde{K}_1, \widetilde{K}_2) + t_2 - t_1\Big)$ for all $0 \le t_1 \le t_2 \le T$ and $\widetilde{K}_1, \widetilde{K}_2 \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ $(\pi_1\, \widetilde{K}_1 \le \pi_1\, \widetilde{K}_2)$.

Then for every initial element $\widetilde{K}_0 \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, there exists a timed right–hand sleek solution $\widetilde{K} : [0, T[\longrightarrow \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ of the generalized mutational equation $\overset{\circ}{\widetilde{K}}(\cdot) \ni \widetilde{f}(\widetilde{K}(\cdot), \cdot)$ with $\widetilde{K}(0) = \widetilde{K}_0$.

Proof results directly from Prop. 3.17. Indeed, $\widetilde{q}_{\mathcal{K},0}$ and $\widetilde{q}_{\mathcal{K},\varepsilon}$ (for any $\varepsilon > 0$) satisfy

$$\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1, \widetilde{K}_2) \;\leq\; \widetilde{q}_{\mathcal{K},0}(\widetilde{K}_1, \widetilde{K}_2) \;\leq\; \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1, \widetilde{K}_2) + \|\psi\|_{L^\infty} \left(\|K_1\|_\infty + \|K_2\|_\infty + 2\right) \cdot \varepsilon$$

for all $\widetilde{K}_j = (t_j, \mu_j, K_j) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$ (abbreviating $\|K_1\|_\infty \overset{\text{Def.}}{=} \sup_{x\in K_1} |x|$). In particular, for any element $\widetilde{K}_0 = (t_0, 1, K_0) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ fixed, all Euler approximations (mentioned in Remark 3.13) are uniformly bounded, i.e. $\|K_n(t)\|_\infty \leq \|K_0\|_\infty + \Lambda\, t$. So setting $R := \|K_0\|_\infty + \Lambda\, T + 1$, we combine Proposition 3.17 with Corollary 4.12 and Remark 3.14. □

In comparison with previous results in [15], an essential advantage of *sleek* solutions is that Proposition 3.19 guarantees uniqueness :

Proposition 4.14 *For* $\widetilde{f} : (\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)) \times [0,T] \longrightarrow \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N, \mathbb{R}^N)$, *suppose that there exist a modulus* $\widehat{\omega}(\cdot)$ *of continuity and a constant* $L \geq 0$ *satisfying*

$$\|\mathcal{H}_{\widetilde{f}(\widetilde{Z},s)} - \mathcal{H}_{\widetilde{f}(\widetilde{K},t)}\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} \;\leq\; L \cdot \widetilde{q}_{\mathcal{K},0}(\widetilde{Z}, \widetilde{K}) + \widehat{\omega}(t-s)$$

for all $0 \leq s \leq t \leq T$ *and* $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K} \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ $(\pi_1\, \widetilde{Z} \leq \pi_1\, \widetilde{K})$.

Then for every initial element $\widetilde{K}_0 \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, *the timed right–hand sleek solution* $\widetilde{K} : [0,T[\longrightarrow \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{K}}(\cdot) \ni \widetilde{f}(\widetilde{K}(\cdot), \cdot)$, $\widetilde{K}(0) = \widetilde{K}_0$ *is unique.*

Proof results from Proposition 3.19 for the same reasons as we have just obtained the preceding Proposition 4.13. For any $\widetilde{K}_0 = (t_0, 1, K_0) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ fixed and $j = 1, 2$, let

$$\widetilde{K}_j : [0,T] \longrightarrow \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N), \quad t \longmapsto (t_0 + t,\ 1,\ K_j(t))$$

denote timed right–hand sleek solutions of the same initial value problem.
The uniform continuity of $\widetilde{K}_j(\cdot)$ (according to Definition 3.8) ensures a common bound $R > 0$ such that $\|K_j(t)\|_\infty < R$ for all $t \in [0,T]$, $j = 1, 2$. So from now on, we restrict ourselves to $\widetilde{\mathcal{K}}^{\rightarrow}_R(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ and $\widetilde{\mathcal{K}}^{\curlyvee}_R(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$ respectively. Then,

$$\|\mathcal{H}_{\widetilde{f}(\widetilde{Z},s)} - \mathcal{H}_{\widetilde{f}(\widetilde{K},t)}\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} \;\leq\; L\, \|\psi\|_{L^\infty}\, (2\,R + 2) \cdot \varepsilon + L \cdot \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z}, \widetilde{K}) + \widehat{\omega}(t-s)$$

for all $0 \leq s \leq t \leq T$ and $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}_R(\mathbb{R}^N)$, $\widetilde{K} \in \widetilde{\mathcal{K}}^{\rightarrow}_R(\mathbb{R}^N)$ $(\pi_1\, \widetilde{Z} \leq \pi_1\, \widetilde{K})$.
In view of Prop. 3.19, the minimizing elements $\widetilde{Z}_j := \left(t_0 + t - \frac{3}{200\,\Lambda},\ 0,\ K_1(t)\right)$ $(j \in \mathbb{N})$ lead to

$$\inf_{\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}_R(\mathbb{R}^N),\, \pi_1 \widetilde{Z} \leq t_0 + t} \Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\ \widetilde{K}_1(t)) + \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\ \widetilde{K}_2(t))\Big) \;=\; d\!\!l(K_1(t),\ K_2(t))$$

and thus, we obtain for every $t \in [0,T]$ and $\varepsilon \in \,]0,1] \cap \mathbb{Q}$

$$d\!\!l(K_1(t),\ K_2(t)) \;\leq\; 8 \cdot L\, \|\psi\|_{L^\infty}\, 2\,(R+1)\ \varepsilon \cdot t\ e^{(L + 10\,\Lambda\, e^{2\,\Lambda}) \cdot t} \;\overset{\varepsilon \downarrow 0}{\longrightarrow}\; 0.$$

□

A Tools of reachable sets of differential inclusions

In this appendix, we investigate the proximal radius of boundary points while sets are evolving along differential inclusions. Compact balls and their complements exemplify the key features for short times (as stated in Proposition A.1). So they lead to the main results about proximal radii in both forward and backward time direction as a corollary.

Proposition A.1 *Let F be any set–valued map of $\mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ (according to Definition 4.5) and $B := \mathbb{B}_r(x_0) \subset \mathbb{R}^N$ a compact ball of positive radius r.*
Then there exists a time $\tau = \tau(r, \Lambda) > 0$ such that for all times $t \in [0, \tau(r,\Lambda)[$,

1. *$\vartheta_F(t, B)$ is convex and has radius of curvature $\geq r - 9\,\Lambda\,(1+r)^2\,t$,*
2. *$\vartheta_F(t, \mathbb{R}^N \setminus B)$ is concave and has radius of curvature $\geq r - 9\,\Lambda\,(1+r)^2\,t$,*

Restricting ourselves to $0 < r \leq 2$, the time $\tau(r, \Lambda) > 0$ can be chosen as an increasing function of r. The claim of Prop. A.1 does not include however that $r - 9\Lambda(1+r)^2\,t \geq 0$ for all $t \in [0, \tau(r,\Lambda)[$ (because then it is not immediately clear how to choose $\tau(r, \Lambda) > 0$ as increasing with respect to all $r \in]0, 2]$).
As an equivalent formulation of statement (1.), the convex set $\vartheta_F(t, B)$ has *positive erosion* of radius $\rho(t) \geq r - 9\Lambda(1+r)^2\,t$, i.e. there is some $K_t \subset \mathbb{R}^N$ with $\vartheta_F(t, B) = \mathbb{B}_{\rho(t)}(K_t)$ (as defined in [17, Lorenz 2003], [15], for example). The question of preserving positive erosion has already been investigated in [17] and in [7, Cannarsa, Frankowska 2004] under different assumptions. In fact, the results of [17] even imply that $\vartheta_F(t, B)$ has positive erosion of radius $\geq \mathrm{const}(\Lambda) \cdot \frac{r}{1+r\,t} \cdot e^{-\mathrm{const}(\Lambda)\cdot t}$ for *every* $t \geq 0$.
So strictly speaking, statement (2.) is of more interest here. It ensures that $\vartheta_F(t, \mathbb{R}^N \setminus B)$ has *positive reach* of radius $\rho(t) \geq r - 9\,\Lambda\,(1+r)^2\,t$ (in the sense of Federer [12]), i.e. for each point $y \in \partial\,\vartheta_F(t, \mathbb{R}^N \setminus B)$, there exists an "exterior" ball $\mathbb{B}_{\rho(t)}(y_0) \subset \mathbb{R}^N$ with $y \in \partial\,\mathbb{B}_{\rho(t)}(y_0)$ and $\vartheta_F(t, \mathbb{R}^N \setminus B) \cap \overset{\circ}{\mathbb{B}}_{\rho(t)}(y_0) = \emptyset$. Roughly speaking, the proofs of these two statements just differ in a sign and thus, both of them are mentioned here.

Applying Proposition A.1 to adequate proximal balls, the inclusion principle of reachable sets and Proposition 4.4 have the immediate consequence :

Corollary A.2 *For every set–valued map $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ and radius $r_0 \in]0, 2]$, there exists some $\tau = \tau(r_0, \Lambda) > 0$ such that for any $K \in \mathcal{K}(\mathbb{R}^N)$, $r \in [r_0, 2]$ and $t \in [0, \tau[$,*

1. *each $x_1 \in \partial\vartheta_F(t, K)$ and $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ with proximal radius r are linked to some $x_0 \in \partial K$ and $\nu_0 \in N^P_K(x_0)$ with proximal radius $\geq r - 81\,\Lambda\,t$ by a trajectory of $\dot{x}(\cdot) \in F(x(\cdot))$ and its adjoint arc, respectively.*

2. *each $x_0 \in \partial K$ and $\nu_0 \in N^P_K(x_0)$ with proximal radius r are linked to some $x_1 \in \partial\vartheta_F(t, K)$ and $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ with proximal radius $\geq r - 81\,\Lambda\,t$ by a trajectory of $\dot{x}(\cdot) \in F(x(\cdot))$ and its adjoint arc, respectively.*

For describing the time–dependent limiting normals, we use adjoint arcs and benefit from the Hamiltonian system they are satisfying together with the trajectories (as quoted in Prop. 4.4). In short, the graph of normal cones at time t, Graph $N_{\vartheta_F(t,K)}(\cdot)|_{\partial\,\vartheta_F(t,K)}$, can be traced back to the beginning by means of the Hamiltonian system with $\mathcal{H}_F$. Roughly speaking, we now take the next order into consideration and, the matrix Riccati equation provides an analytical access to geometric properties like curvature. The next lemma motivates the assumption $\mathcal{H}_F \in C^2$ for all maps $F \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$.

Lemma A.3

Suppose for $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *and the Hamiltonian system*

$$\wedge \begin{cases} \dot y(t) &= \frac{\partial}{\partial q} H(t,\, y(t),\, q(t)), \qquad y(0) = y_0 \\ \dot q(t) &= -\frac{\partial}{\partial y} H(t,\, y(t),\, q(t)), \qquad q(0) = \psi(y_0) \end{cases} \tag{$*$}$$

the following properties :

1. $H(t,\cdot,\cdot)$ *is* twice *continuously differentiable for every* $t \in [0,T]$.
2. *for every* $R > 0$, *there exists* $k_R \in L^1([0,T])$ *such that the derivative of* $H(t,\cdot,\cdot)$ *is* $k_R(t)$*–Lipschitz continuous on* $\mathbb{B}_R \times \mathbb{B}_R$ *for almost every* t,
3. ψ *is locally Lipschitz continuous,*
4. *every solution* $(y(\cdot), q(\cdot))$ *of the Hamiltonian system* $(*)$ *can be extended to* $[0,T]$ *and depends continuously on the initial data in the following sense :*
   Let each $(y_n(\cdot), q_n(\cdot))$ *be a solution satisfying* $y_n(t_n) \longrightarrow z_0$, $q_n(t_n) \longrightarrow q_0$ *for some* $t_n \longrightarrow t_0$, $z_0, q_0 \in \mathbb{R}^N$. *Then* $(y_n(\cdot), q_n(\cdot))_{n\in\mathbb{N}}$ *converges uniformly to a solution* $(y(\cdot), q(\cdot))$ *of the Hamiltonian system with* $y(t_0) = z_0$, $q(t_0) = q_0$.

Then for every initial set $K \in \mathcal{K}(\mathbb{R}^N)$, *the following statements are equivalent :*

(i) *For all* $t \in [0,T]$,
$$M_t^{\mapsto}(K) := \Big\{ (y(t),\, q(t)) \;\Big|\; (y(\cdot),\, q(\cdot)) \text{ solves system } (*),\; y_0 \in K \Big\}$$
is the graph of a locally Lipschitz continuous function,

(ii) *For any solution* $(y(\cdot), q(\cdot)) : [0,T] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ *of the initial value problem* $(*)$ *and each cluster point* $Q_0 \in \mathrm{Limsup}_{z\to y_0} \{\nabla\,\psi(z)\}$, *the following matrix Riccati equation has a solution* $Q(\cdot)$ *on* $[0,T]$

$$\wedge \begin{cases} \partial_t\, Q + \frac{\partial^2 H}{\partial p\, \partial x}(t,\, y(t),\, q(t))\; Q + Q\; \frac{\partial^2 H}{\partial x\, \partial p}(t,\, y(t),\, q(t)) \\ \qquad + Q\; \frac{\partial^2 H}{\partial p^2}(t,\, y(t),\, q(t))\; Q + \frac{\partial^2 H}{\partial x^2}(t,\, y(t),\, q(t)) = 0, \\ \qquad Q(0) = Q_0. \end{cases}$$

If one of these equivalent properties is satisfied and if ψ *is (continuously) differentiable, then* $M_t^{\mapsto}(K)$ *is even the graph of a (continuously) differentiable function.*

Proof is given in [13, Frankowska 2002], Theorem 5.3 for the same Hamiltonian system but with $y(T) = y_T$, $q(T) = q_T$ given. So this lemma is an immediate consequence considering $-H(T - \cdot\,, \cdot\,, \cdot\,)$ and $(y(T - \cdot\,),\, q(T - \cdot\,))$. □

Remark A.4 In addition to the final statement of Lemma A.3, well–known properties of variational equations (see e.g. [13, Frankowska 2002]) imply that $Q(t)$ is the derivative of the C^1 function with graph $M_t^{\mapsto}(K)$ at the point $y(t)$.

For preventing singularities of $Q(\cdot)$, the following comparison principle provides a bridge to solutions of a *scalar* Riccati equation.

Lemma A.5 (Comparison theorem for the matrix Riccati equation, [21, Royden 88], Theorem 2)

Let $A_j, B_j, C_j : [0,T[\longrightarrow \mathbb{R}^{N,N}$ $(j = 0,1,2)$ *be bounded continuous matrix–valued functions such that each* $M_j(t) := \begin{pmatrix} A_j(t) & B_j(t) \\ B_j(t)^T & C_j(t) \end{pmatrix}$ *is symmetric.*

Assume that $U_0,\ U_2 : [0,T[\longrightarrow \mathbb{R}^{N,N}$ *are solutions of the matrix Riccati equation*

$$\tfrac{d}{dt}\, U_j \;=\; A_j \;+\; B_j\, U_j \;+\; U_j\, B_j^T \;+\; U_j\, C_j\, U_j$$

with $M_2(\cdot) \geq M_0(\cdot)$ *(i.e.* $M_2(t) - M_0(t)$ *is positive semi–definite for every* t*).*

Then, given symmetric $U_1(0) \in \mathbb{R}^{N,N}$ *with*

$$U_2(0) \;\geq\; U_1(0) \;\geq\; U_0(0), \qquad M_2(\cdot) \;\geq\; M_1(\cdot) \;\geq\; M_0(\cdot),$$

there exists a solution $U_1 : [0,T[\longrightarrow \mathbb{R}^{N,N}$ *of the corresponding Riccati equation with matrix* $M_1(\cdot)$*. Moreover,* $U_2(t) \;\geq\; U_1(t) \;\geq\; U_0(t)$ *for all* $t \in [0,T[$. $\square$

Proof of Proposition A.1 (1) is based on applying Lemma A.3 to the boundary $K := \partial\, \mathbb{B}_r(0)$ and its exterior unit normals, i.e. $\psi(x) := \frac{x}{r}$, after assuming $B = \mathbb{B}_r(0)$ without loss of generality. Obviously, ψ can be extended to $\psi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$.
(Statement (2.) of Proposition A.1 is shown in the same way – just with inverse signs, i.e. $\widehat{\psi}(x) := -\frac{x}{r}$ instead. So we do not formulate this part in detail.)

For every point $y_0 \in \partial\, \mathbb{B}_r$, there exist a solution $y(\cdot) \in C^1([0,\infty[, \mathbb{R}^N)$ and its adjoint $q(\cdot) \in C^1([0,\infty[, \mathbb{R}^N)$ satisfying

$$\begin{cases} \dot y(t) \;=\; \frac{\partial}{\partial q}\, \mathcal{H}_F(y(t),\, q(t)) \;\in\; F(y(t)), & y(0) \;=\; y_0, \\ \dot q(t) \;=\; -\frac{\partial}{\partial y}\, \mathcal{H}_F(y(t),\, q(t)), & q(0) \;=\; \psi(y_0) \end{cases} \qquad (*)$$

and, $F \in \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N, \mathbb{R}^N)$ implies the a priori bounds $|y(t) - y_0| \leq \Lambda\, t,\ e^{-\Lambda\, t} \leq |q(t)| \leq e^{\Lambda\, t}$.
So after restricting to the finite time interval I_r (specified later), a simple cut-off function provides a twice continuously differentiable extension of $\mathcal{H}_F$ to $\mathbb{R}^N \times \mathbb{R}^N$ and finally, Lemma A.3 can be applied to $\partial \mathbb{B}_r$, ψ and $\mathcal{H}_F$.
Furthermore $\mathcal{H}_F(x,p) \overset{\text{Def.}}{=} \sup_{v \in F(x)}\, p \cdot v$ is positively homogenous with respect to p and thus, the second derivatives of $\mathcal{H}_F$ are bounded by $9\, \Lambda\, R^2$ on $\mathbb{R}^N \times (\mathbb{B}_R \setminus \overset{\circ}{\mathbb{B}}_{\frac{1}{R}})$ (according to [15], Lemma 4.22). Together with the preceding a priori bounds, we obtain

$$\|\, D^2\, \mathcal{H}_F(y(t), q(t))\, \|_{\mathrm{Lin}(\mathbb{R}^{2N}, \mathbb{R}^{2N})} \;\leq\; 9\, \Lambda\, e^{2\, \Lambda\, t}.$$

Let $Q(\cdot)$ denote the solution of the matrix Riccati equation

$$\wedge \begin{cases} \partial_t Q \;+\; \frac{\partial^2 \mathcal{H}_F}{\partial p\,\partial x}(y(t),\, q(t))\; Q \;+\; Q\; \frac{\partial^2 \mathcal{H}_F}{\partial x\,\partial p}(y(t),\, q(t)) \\ \quad +\; Q\; \frac{\partial^2 \mathcal{H}_F}{\partial p^2}(y(t),\, q(t))\; Q \;+\; \frac{\partial^2 \mathcal{H}_F}{\partial x^2}(y(t),\, q(t)) \;=\; 0, \\ \qquad Q(0) \;=\; \nabla\psi(y_0) \;=\; \frac{1}{r}\cdot \mathrm{Id}_{\mathbb{R}^N}. \end{cases}$$

Due to the comparison principle of Lemma A.5, $Q(\cdot)$ exists (at least) as long as the two scalar Riccati equations $\partial_t\, u_\pm \;=\; \pm\, 9\,\Lambda\, e^{2\,\Lambda\,t} \;\pm\; 9\,\Lambda\, e^{2\,\Lambda\,t}\, u_\pm^2, \qquad u_\pm(0) = \frac{1}{r}$ have finite solutions and within this period, they fulfill $u_-(t)\cdot \mathrm{Id}_{\mathbb{R}}^N \le Q(t) \le u_+(t)\cdot \mathrm{Id}_{\mathbb{R}}^N$. In fact, we get the explicit solutions on $I_r := [\,0,\; \frac{1}{2\,\Lambda}\cdot \log\,(1 + \frac{\pi}{9} \;-\; \frac{2}{9}\cdot \arctan\,\frac{1}{r})[$, namely $u_\pm(t) \;=\; \tan\,(\,\pm\,\frac{9}{2}\,(e^{2\,\Lambda\,t} - 1) \;+\; \arctan\,\frac{1}{r}),$
So $Q(t)$ is positive definite with eigenvalues $\ge\; u_-(t)$ for every time t of the (maybe smaller) interval $I'_r \;:=\; I_r \;\cap\; [0, \frac{1}{2\,\Lambda}\cdot \log\,(1 \;+\; \frac{2}{9}\cdot \arctan\,\frac{1}{r})[$.

Now we focus on the geometric interpretation of $Q(\cdot)$.
Due to Lemma A.3, $M_t^{\mapsto}(\partial\,\mathbb{B}_r) \;:=\; \{\,(y(t),\, q(t))\,|\,(y(\cdot),\, q(\cdot)) \text{ solves system } (*),\; |y_0| = r\,\}$ is the graph of a continuously differentiable function and, $Q(t)$ is its derivative at $y(t)$ (according to Remark A.4). Furthermore the Hamilton condition of Prop. 4.4 ensures

$$\text{Graph } N_{\vartheta_F(t,\mathbb{B}_r)}(\cdot) \;\subset\; \Big\{ (y(t),\;\; \lambda\, q(t)) \;\Big|\; (y(\cdot),\, q(\cdot)) \text{ solves system } (*),\;\; |y_0| = r,\;\; \lambda \ge 0 \Big\}$$

and thus, the graph property of $M_t^{\mapsto}(\partial\,\mathbb{B}_r)$ implies that each $q(t)$ is a normal vector to the smooth reachable set $\vartheta_F(t, \mathbb{B}_r)$ at $y(t)$.
As $q(t) \ne 0$ need not have norm 1, the eigenvalues of $Q(t)$ are not always identical to the principal curvatures $(\kappa_j)_{j=1\ldots N}$ of $\vartheta_F(t, \mathbb{B}_r)$ at $y(t)$, but they provide bounds :

$$e^{-\Lambda\,t}\;\cdot\; u_-(t) \;\le\; \kappa_j \;\le\; e^{\Lambda\,t}\;\cdot\; u_+(t) \qquad\qquad (\text{due to } \;\; e^{-\Lambda\,t} \;\le\; |q(t)| \;\le\; e^{\Lambda\,t}).$$

Thus, $\vartheta_F(t, \mathbb{B}_r)$ is convex for all times $t \in I'_r$ and, so the *local* properties of principal curvatures have the *nonlocal* consequence that $\vartheta_F(t, \mathbb{B}_r)$ has positive erosion of radius

$$\rho(t) \;\ge\; \frac{1}{e^{\Lambda\,t}\,\cdot\, u_+(t)} \;\ge\; r - 9\,\Lambda\,(1+r)^2\; t \qquad\qquad \text{for all } \; t \in I'_r$$

Indeed, the linear estimate at the end is shown by means of the auxiliary function $t \mapsto \frac{1}{e^{\Lambda\,t}\,\cdot\, u_+(t)} - r + 9\,\Lambda\,(1+r)^2\; t$ that is 0 at $t = 0$, has positive derivative at $t = 0$ and is convex (due to nonnegative second derivative in I'_r).
Finally, the time $\tau(r, \Lambda) > 0$ is chosen as minimum of $\frac{1}{2\,\Lambda}\cdot \log\,(1 + \frac{\pi}{9} \;-\; \frac{2}{9}\cdot \arctan\,\frac{1}{r})$, $\frac{1}{2\,\Lambda}\cdot \log\,(1 \,+\, \frac{2}{9}\cdot \arctan\,\frac{1}{r})$. The linear estimate need not be positive in $[0, \tau(r, \Lambda)[$ though.
□

B Tools of proximal normals

Comparing the proximals normals of a converging sequence $(K_n)_{n \in \mathbb{N}}$ in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ with the normals of its limit $K \in \mathcal{K}(\mathbb{R}^N)$, the following inclusion is well known

$$\text{Graph } N_K^P \quad \subset \quad \text{Limsup}_{n \to \infty} \text{ Graph } N_{K_n}^P$$

(see e.g. [5, Aubin 91], Theorem 8.4.6 or [11, Cornet, Czarnecki 99], Lemma 4.1). Of course, the equality here is not fulfilled in general. A key advantage of the subset $N_{K,\rho}^P$ (for $\rho > 0$) now is that an inverse inclusion is satisfied. This feature is very useful for the preceding Proposition 4.10 and for verifying the suitable transitional compactness in Proposition 4.11.

Proposition B.1 *Let $(K_n)_{n \in \mathbb{N}}$ be a converging sequence in $\mathcal{K}(\mathbb{R}^N)$ and K its limit. $\Pi_{K_n}, \Pi_K : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ denote the projections on K_n, K $(n \in \mathbb{N})$ respectively. Then,*

1. $\text{Limsup}_{n \to \infty} \text{ Graph } {}^\flat N_{K_n,\rho}^P \quad \subset \quad \text{Graph } {}^\flat N_{K,\rho}^P$ *for any* $\rho > 0$,
2. $\text{Limsup}_{\substack{y \to x \\ n \to \infty}} \Pi_{K_n}(y) \quad \subset \quad \Pi_K(x)$ *for any* $x \in \mathbb{R}^N$,
3. $\text{Graph } {}^\flat N_{K,\rho}^P \quad \subset \quad \text{Liminf}_{n \to \infty} \text{ Graph } {}^\flat N_{K_n,r}^P$ *for any* $0 < r < \rho$.

Proof. (1.) Choose any converging sequence $((x_{n_j}, p_{n_j}))_{j \in \mathbb{N}}$ with $p_{n_j} \in N_{K_{n_j},\rho}^P(x_{n_j}) \cap \partial\mathbb{B}$ and set $x := \lim_{j \to \infty} x_{n_j} \in K$, $p := \lim_{j \to \infty} p_{n_j} \in \partial\mathbb{B}$. According to Definition 4.1, each K_{n_j} is contained in the complement of the open ball with center $x_{n_j} + \rho\, p_{n_j}$ and radius ρ,

$$K_{n_j} \quad \subset \quad \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_\rho\big(x_{n_j} + \rho\, p_{n_j}\big)\,.$$

As an indirect consequence, $j \longrightarrow \infty$ leads to $K \subset \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_\rho(x + \rho\, p)$, i.e. $p \in N_{K,\rho}^P(x)$.

(2.) has already been shown in [16, Lorenz 2004], Lemma 4.1.9. For the sake of completeness, we give the full proof shortly :
Let $r > 0$ and $n \in \mathbb{N}$ be arbitrary. For $y \in \mathbb{B}_r(x)$ given, choose any $z \in \Pi_{K_n}(y)$ and $\xi \in \Pi_K(z)$. Then, $|\xi - z| \leq d\!\!l(K_n, K)$ and

$$\begin{aligned} |x - \xi| &\leq |x - y| + |y - z| + |z - \xi| \\ &\leq |x - y| + |y - x| + \text{dist}(x, K) + d\!\!l(K, K_n) + d\!\!l(K_n, K) \\ &\leq 2\,r + \text{dist}(x, K) + 2\, d\!\!l(K_n, K). \end{aligned}$$

Thus, $\Pi_{K_n}(y) \quad \subset \quad \mathbb{B}_{d\!\!l(K_n,K)}\Big(K \cap \mathbb{B}_{2\,r + \text{dist}(x,K) + 2\, d\!\!l(K_n,K)}(x)\Big)$ for any $y \in \mathbb{B}_r(x)$.

The set–valued map $[0, \infty[\; \rightsquigarrow \mathbb{R}^N$, $r \longmapsto K \cap \mathbb{B}_r(x)$ is upper semicontinuous (due to [6, Aubin,Frankowska 90], Cor. 1.4.10) and in the closed interval $[\text{dist}(x, K), \infty[$, it is strict with compact values.
So for every $\eta > 0$, there exists $\rho = \rho(x, \eta) \in \,]0, \eta[$ such that

$$K \cap \mathbb{B}_r(x) \quad \subset \quad \mathbb{B}_\eta\Big(\Pi_K(x)\Big) \qquad \text{for all } r \in \Big[\text{dist}(x, K),\ \text{dist}(x, K) + \rho\Big].$$

Due to $d\!\!l(K_n, K) \longrightarrow 0$ $(n \longrightarrow \infty)$, there is an index $m \in \mathbb{N}$ with $d\!\!l(K_n, K) \leq \frac{\rho}{4}$ for all $n \geq m$. Thus we obtain for every $y \in \mathbb{B}_{\rho/4}(x)$ and $n \geq m$

$$\begin{array}{rcl} \Pi_{K_n}(y) & \subset & \mathbb{B}_{\frac{\rho}{4}}\Big(K \,\cap\, \mathbb{B}_{2\,\frac{\rho}{4}+\mathrm{dist}(x,K)+2\,\frac{\rho}{4}}(x)\Big) \;\;=\;\; \mathbb{B}_{\frac{\rho}{4}}\Big(K \,\cap\, \mathbb{B}_{\mathrm{dist}(x,K)+\rho}(x)\Big) \\ & \subset & \mathbb{B}_{\frac{\rho}{4}}\Big(\mathbb{B}_\eta(\Pi_K(x))\Big) \qquad\qquad\qquad\quad \subset \;\; \mathbb{B}_{2\,\eta}\Big(\Pi_K(x)\Big), \end{array}$$

i.e. $\quad \mathrm{Limsup}_{\substack{y\to x\\ n\to\infty}} \; \Pi_{K_n}(y) \;\subset\; \Pi_K(x).$

(3.) Choose any $x \in \partial K$ and $p \in N^P_{K,\rho}(x) \neq \emptyset$ with $|p| = 1$. Then x is the unique projection of $x + \delta\, p$ on K for every $\delta \in]0, \rho[$. Considering now a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \Pi_{K_n}(x + \delta\, p) \subset K_n$, the preceding statement (2.) implies $x_n \longrightarrow x$ and, the definition of proximal normal guarantees $p_n := \frac{x+\delta\, p - x_n}{|x+\delta\, p - x_n|} \in {}^\flat N^P_{K_n}(x_n)$ converging to p. Finally the proximal radius of p_n is $\geq |x + \delta\, p - x_n| \geq \delta - |x - x_n|$, and thus, $(x, p) \in \mathrm{Liminf}_{n\to\infty}$ Graph ${}^\flat N^P_{K_n,\,r}$ for every positive $r < \delta < \rho$. □

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