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Thema

Iterative Connections and Abhyankar's Conjecture

Gutachter: Prof. Dr. Bernd Heinrich Matzat

PD Dr. Michael Dettweiler

Abstract

This thesis contains two major parts. In the first part, I introduce a new theory of *modules with iterative connection*. This theory unifies the theory of modules with connection in characteristic zero as given by N. Katz (see [Kat87]) and the theory of iterative differential modules in positive characteristic as given by B. H. Matzat und M. van der Put (see [Mat01] and [MvdP03]). The second part of this work is about the differential Abhyankar conjecture for iterative Picard-Vessiot extensions (IPV-extensions). This conjecture is concerned with the problem which linear algebraic groups occur as iterative differential Galois groups of IPV-extensions with restricted singular locus. In this thesis, I prove the differential Abhyankar conjecture for connected groups and give necessary and sufficient conditions for connected groups for being realisable with given singular points.

Zusammenfassung

Diese Doktorarbeit besteht im Großen aus zwei Teilen. Im ersten Teil entwickle ich eine neue Theorie von *Moduln mit iterativem Zusammenhang*. Diese Theorie vereinheitlicht die Theorie der Moduln mit Zusammenhang in Charakteristik Null, wie N. Katz sie in [Kat87] vorstellt, und die Theorie der iterativen Differential-Moduln von B. H. Matzat und M. van der Put (siehe [Mat01] und [MvdP03]). Im zweiten Teil der Arbeit geht es um die Differential-Abhyankar-Vermutung für iterative Picard-Vessiot-Erweiterungen (IPV-Erweiterungen). Diese Vermutung macht darüber Aussagen, welche lineare algebraische Gruppe als iterative Differential-Galoisgruppe von IPV-Erweiterungen mit eingeschränktem singulären Ort vorkommen. In dieser Arbeit beweise ich die Differential-Abhyankar-Vermutung für zusammenhängende Gruppen und gebe notwendige und hinreichende Kriterien für die Realisierbarkeit zusammenhängender Gruppen mit vorgegebenen Singularitäten an.

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Contents

Introduction

At the beginning of differential Galois theory, one was restricted to the case of characterictic zero. In this case N. Katz gave a general setting of modules with integrable connection to describe linear differential equations in several variables (see [Kat87]). These modules with integrable connection form a category $\mathbf{DE}(R/K)$ (here R denotes the differential ring and K the field of constants), which turns out to be a Tannakian category or even a neutral Tannakian category over K, if there is a fibre functor $\boldsymbol{\omega} : \mathbf{DE}(R/K) \to \mathbf{Vect}(K)$ (for example if R has a K-rational point). By the general properties of neutral Tannakian categories, this gives rise to a Galois theory for these linear differential equations (see for example [DM89]). But this approach of Katz only works in characteristic zero, mainly because in positive characteristic p, every p-th power is a constant with respect to any derivation on R. In particular, if there is a differential extension L of R, L would have additional constants, namely $L^p \setminus R$.

After a few attempts by K. Okugawa in 1963 and 1987 (see [Oku63] and [Oku87]), B. H. Matzat and M. van der Put started to set up a systematic approach to differential Galois theory in positive characteristic (see [MvdP03]). They used so called iterative derivations, which were first introduced by H. Hasse and F. K. Schmidt in [HS37]. In their notation, an iterative derivation on a ring R is a sequence of endomorphisms $\left(\partial^{(k)}\right)_{k\in\mathbb{N}}$ of the ring R satisfying some properties (cf. proposition 1.2 and the remarks following it), which imply that $\partial^{(1)}$ is a derivation and which would imply that $\partial^{(k)} = \frac{1}{k!}(\partial^{(1)})^k$, if the characteristic was zero. But this differential Galois theory developed by Matzat and van der Put only works for differential equations in one variable, and there has still been no systematic way for several variables.

In the first part of this thesis, I will introduce such a systematic description using so called higher differentials and iterative connections. This theory is completely independent of the characteristic. The characteristic will only take into account, when we look for special properties (which parameters determine an iterative derivation and so on). We will see that this theory of modules with iterative connection resp. integrable iterative connection is a generalisation of both the classical theory of modules with (integrable) connection in characteristic zero and the iterative differential theory of Matzat and van der Put over algebraic function fields. In section 4, it will be shown, that the category $\mathbf{ICon}(R/K)$ of modules with iterative connection over R and the category $\mathbf{ICon}_{int}(R/K)$ of modules with integrable iterative connection over R are both (neutral) Tannakian categories.

In getting the right setting, the main idea is to regard a higher derivation not as a sequence of maps $\left(\partial^{(k)} : R \to R\right)_{k\in\mathbb{N}}$ but as a homomorphism of algebras $\psi : R \to R[[T]]$ by summing up, in detail $\psi(r) := \sum_{k=0}^{\infty} \partial^{(k)}(r)T^k$. This leads to the notion of R-cgas (completions of graded R-algebras), which allows to generalise the definition of a higher derivation and to obtain a universal object $\hat{\Omega}_{R/K}$

with a universal higher derivation $\mathrm{d}_R : R \to \hat{\Omega}_{R/K}$, replacing the module of differentials $\Omega_{R/K}$ in classical theory. There have already been some attempts in this direction (see for example [Voj04], where P. Vojta defined an algebra of divided differentials), but they all didn't lead to an appropriate theory.

In the second part of the thesis, we will be concerned with the differential Abhyankar conjecture over algebraic function fields in positive characteristic.

So we will be in the case, for which Matzat and van der Put developed an iterative differential Galois theory. In more detail, for an iterative differential module (ID-module) M over the algebraic function field F, there is a minimal iterative differential extension field L/F (which is unique up to differential isomorphism), called iterative Picard-Vessiot extension (IPV-extension), such that $M \otimes_F L$ has a basis of differentially constant elements. The group of differential automorphisms of L over F is an algebraic subgroup of $\mathrm{GL}_n(K)$ ($n = \dim_F(M)$, K the field of constants), called the iterative differential Galois group $\mathrm{Gal}(L/F)$. It has already been shown by Matzat (see [Mat01], cor. 8.11) that for every reduced linear algebraic group $\mathcal{G}$ defined over K, there exists an IPV-extension with $\mathrm{Gal}(L/F) = \mathcal{G}(K)$. (We say that $\mathcal{G}$ can be realised as differential Galois group.) However, one wants to have realisations with few singular points. The differential Abhyankar conjecture states that a linear algebraic group $\mathcal{G}$ can be realised with singular locus inside a nonempty set S, if and only if $\mathcal{G}/p(\mathcal{G})$ can, where $p(\mathcal{G})$ denotes the subgroup of $\mathcal{G}$ generated by its unipotent elements. For finite groups $\mathcal{G}$, this conjecture becomes the classical Abhyankar conjecture, which has been proved by Raynaud and Harbater (see [Ray94], [Har94] and [Har95]).

In this work, we will give a realisation of connected groups which shows that the differential Abhyankar conjecture is also true for connected groups.

Nevertheless, the differential Abhyankar conjecture is not true in this form. In section 9.1, we will give an example of a non-connected group which is generated by unipotent elements but which is not realisable with one singularity. Since this example only works if the field of constants equals $\overline{\mathbb{F}}_p$, the differential Abhyankar conjecture might be true if the field of constants is not $\overline{\mathbb{F}}_p$.

Chapter 1 gives the definition of higher derivations in the general sense and in the special case which is equivalent to the higher derivations of Hasse and Schmidt. The R-cgas (completions of graded R-algebras) used here and throughout the whole thesis are defined in appendix A, together with some properties and notations related to R-cgas. In the first chapter, we also define higher derivations on modules and finally give an action of the field of constants K on the set of higher derivations, which turns out to be very useful later on to simplify a lot of calculations.

In chapter 2, the algebra of higher differentials $\hat{\Omega}_{R/K}$ is introduced together with the universal higher derivation d_R. We show that this universal higher derivation can be extended to an automorphism $\mathrm{d}_{\hat{\Omega}}$ of the K-algebra $\hat{\Omega}_{R/K}$ (see section 2.2). At last, we define higher connections on a module M as higher derivations over d_R, we define extensions of these higher connections to endomorphisms of $\hat{\Omega}_{R/K} \otimes M$ using the automorphism $\mathrm{d}_{\hat{\Omega}}$, and we show that over a regular local ring R, every finitely generated module with a higher connection is free.

In chapter 3, we focus on iterative derivations on the ring R (i. e. higher derivations with an additional composition law) and on iterative derivations on modules. The iterative derivations seem to be the appropriate replacement for the common derivations, because they are in one-to-one correspondence to those in characteristic zero, what will be shown later. We conclude the chapter with the definitions and some properties of iterative connections and integrable iterative connections, the central objects of the first part of this work.

The investigation of categorial properties is done in the forth chapter. There, we see that the category of modules with (arbitrary) higher connection is not a tensor category for lack of some morphisms regarding the dual object. But the category $\mathbf{ICon}(R/K)$ of modules with iterative connection and the category $\mathbf{ICon}_{int}(R/K)$ of modules with integrable iterative connection are tensor categories over K. Even more, together with the fibre functor $\boldsymbol{\omega} : \mathbf{ICon}(R/K) \to \mathbf{Mod}(R)$, that forgets the connection, the categories $\mathbf{ICon}(R/K)$ and $\mathbf{ICon}_{int}(R/K)$ are Tannakian categories and even neutral Tannakian categories over K, if R has a K-rational point. A short summary of the definitions of the categories used here is given in appendix B. In section 4.2, we sketch a generalisation of the previous to schemes.

The last two chapters of the first part concentrate on special properties related to the characteristic. In chapter 5, it is shown that for $\mathrm{char}(K) = 0$, (common) derivations, differentials and connections are in one-to-one correspondence to iterative derivations, higher differentials and iterative connections, what proves that the classical theory is obtained as a special case of the theory developed here.

In chapter 6, we show that modules with integrable iterative connection are in one-to-one correspondence to projective systems. This implies that the theory of iterative differential modules defined by Matzat in [Mat01] also is obtained as a special case.

In chapter 7, the first chapter of the second part, we start with some properties of iterative derivations in algebraic function fields in one variable, which will be necessary for later purposes. We recall the basic definitions and results of the iterative Picard-Vessiot theory, including methods for determining the iterative differential Galois group of an iterative Picard-Vessiot extension (IPV-extension).

In chapter 8, we then concentrate on questions regarding regularity both of iterative differential modules and of IPV-extensions. One point is that iterative differential modules (ID-modules) can be totally singular, i. e. they are singular in every place of the function field. This is a phenomenon that doesn't occur in characteristic zero. We also give criteria for deciding, whether an ID-module is totally singular or not, and for determining the points in which these modules are regular.

Finally in chapter 9, we discuss questions concerning the differential Abhyankar conjecture for IPV-extensions. We show that the conjecture is true for connected groups. Moreover, we show that every connected group can be realised with at most two singular points and in special cases with even less singular points. This has already been stated in [MvdP03] (see [MvdP03], thm. 7.1 (3) and cor. 7.7 (3)), but the proof sketched there has a gap (cf. the remark in section 9.5). The realisation of a connected group given here is splitted into several parts: The realisation of unipotently generated connected groups, the realisation of tori and the solution of embedding problems with unipotent kernel.

Part I

In this part of the thesis, the reader is introduced into the theory of iterative connections. The main result is given in section 4, namely that for a regular commutative ring R that is finitely generated as a K-algebra, the finitely generated modules with iterative connection form a Tannakian category and – if in addition $\mathrm{Spec}(R)$ has a K-rational point – even a neutral Tannakian category over K.

Furthermore in section 5, we show that in characteristic zero, the category $\mathbf{ICon}_{int}(R/K)$ of finitely generated modules with integrable iterative connection is equivalent to the category $\mathbf{DE}(R/K)$ of finitely generated modules with integrable (common) connection as introduced by Katz in [Kat87]. This shows that the theory of modules with iterative connection is a generalisation to all characteristics of the theory of modules with (common) connection.

At last in section 6, we consider the case that K has positive characteristic. Then if R is an algebraic function field, the category $\mathbf{ICon}(R/K)$ (and also $\mathbf{ICon}_{int}(R/K)$) is equivalent to the category $\mathbf{ID}_K(R)$ of modules with an iterative derivation as introduced by Matzat in [Mat01]. So the theory of modules with iterative connection is also a generalisation of this theory.

Notation Throughout this work, K denotes a field, R and $\tilde{R}$ denote integral domains, which are finitely generated K-algebras (or localisations of finitely generated K-algebras) and $f : R \to \tilde{R}$ denotes a homomorphism of K-algebras. Furthermore B denotes the completion of a graded algebra over $\tilde{R}$ (a $\tilde{R}$-cga for short), as defined in appendix A. M will always be a finitely generated R-module.

1 Higher Derivations

In this section we give the notion of higher derivations on rings and modules. The definition used here is different from that introduced by Hasse and Schmidt in [HS37]. In fact it is a generalisation which we will show later on. This more general definition is necessary to define the algebra of higher differentials as a universal object (see section 2.1).

1.1 Higher Derivations on Rings

Definition 1.1 *A* **higher derivation** *of R to B over K is a homomorphism of K-algebras $\psi : R \to B$ satisfying $\varepsilon \circ \psi = f : R \to B_0 = \tilde{R}$.*
The set of all higher derivations of R to B over K will be denoted by $\mathbf{HD}_{\boldsymbol{K}}\boldsymbol{(R, B)}$.
In the special case of $B = R[[T]]$ (and $\tilde{R} = R$) we set $\mathbf{HD}_{\boldsymbol{K}}\boldsymbol{(R)} := \mathrm{HD}_K(R, R[[T]])$.
For $\psi \in \mathrm{HD}_K(R)$ we define a sequence $\left(\psi^{(k)}\right)_{k\in\mathbb{N}}$ of maps $\psi^{(k)} : R \to R$ by the equation

$$\psi(r) = \sum_{k=0}^{\infty} \psi^{(k)}(r)T^k$$

for all $r \in R$.

Proposition 1.2 *For $\psi \in \mathrm{HD}_K(R)$ the maps $\psi^{(k)}$ are homomorphisms of K-modules and satisfy the following properties:*

$$\begin{aligned} \psi^{(0)} &= \mathrm{id}_R && (1)\\ \forall\, k \in \mathbb{N}, \forall\, r, s \in R : \psi^{(k)}(rs) &= \sum_{i+j=k} \psi^{(i)}(r)\psi^{(j)}(s) && (2) \end{aligned}$$

Furthermore a sequence $\left(\psi^{(k)}\right)_{k\in\mathbb{N}}$ of K-module-homomorphisms satisfying these two properties defines a higher derivation $\psi : R \to R[[T]]$ by the equation given above.

Proof $\psi = \sum_{k=0}^{\infty} \psi^{(k)}T^k$ is a homomorphism of K-modules, if and only if for all $r_1, r_2 \in R$ and $a_1, a_2 \in K$ we have

$$\begin{aligned} \sum_{k=0}^{\infty} \psi^{(k)}(a_1r_1 + a_2r_2)T^k &= \psi(a_1r_1 + a_2r_2) = a_1\psi(r_1) + a_2\psi(r_2) \\ &= a_1 \sum_{k=0}^{\infty} \psi^{(k)}(r_1)T^k + a_2 \sum_{k=0}^{\infty} \psi^{(k)}(r_2)T^k, \end{aligned}$$

i.e. if $\psi^{(k)}(a_1r_1 + a_2r_2) = a_1\psi^{(k)}(r_1) + a_2\psi^{(k)}(r_2)$ for all $k \in \mathbb{N}$, which means that $\psi^{(k)}$ is a homomorphism of K-modules for all $k \in \mathbb{N}$.
Since $\varepsilon \circ \psi = \psi^{(0)}$, we have $\mathrm{id}_R = \varepsilon \circ \psi$ if and only if $\psi^{(0)} = \mathrm{id}_R$.

At last, we have

$$\psi(rs) = \sum_{k=0}^{\infty} \psi^{(k)}(rs)T^k$$

and

$$\psi(r)\psi(s) = \left(\sum_{k=0}^{\infty} \psi^{(k)}(r)T^k\right)\left(\sum_{k=0}^{\infty} \psi^{(k)}(s)T^k\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} \psi^{(i)}(r)\psi^{(j)}(s)\right)T^k$$

for all $r, s \in R$. So ψ is a homomorphism of algebras if and only if the sequence $\left(\psi^{(k)}\right)_{k\in\mathbb{N}}$ satisfies the second property. $\square$

Remark More generally, for an arbitrary higher derivation $\psi \in \mathrm{HD}_K(R, B)$ we denote by $\psi^{(k)}$ the composition of ψ and the projection into the k-th homogeneous component of B. For every $r \in R$ we then have $\psi(r) = \sum_{k=0}^{\infty} \psi^{(k)}(r)$. (The right side is a series that converges in the topology of B.)
Note that this definition of $\psi^{(k)}$ slightly differs from that given for $\psi \in \mathrm{HD}_K(R)$ (as in the above definition $\psi^{(k)}(r)$ only is the coefficient of T^k), but it should always be clear from the context or not important, which of these two notations is used.
Remark As mentioned in the beginning, Hasse and Schmidt introduced another definiton of higher derivations, namely a sequence $\left(\psi^{(k)}\right)_{k\in\mathbb{N}}$ of homomorphisms of K-modules which satisfy the properties of proposition 1.2. So a higher derivation $\psi \in \mathrm{HD}_K(R)$ is exactly what Hasse and Schmidt called a higher derivation.

Proposition 1.3 *Let $S \subset R$ be a multiplicatively closed subset and let $\tilde{R} = R(y)$ be an integral extension of R such that the minimal polynomial of y, call it $g(X)$, has coefficients in R and $g'(y)$ is invertible in $\tilde{R}$, where $g'(X)$ denotes the formal derivative of $g(X)$. Then:*

1. *Every higher derivation $\psi \in \mathrm{HD}_K(R, B)$ to a $(S^{-1}R)$-cga B can be extended uniquely to a higher derivation $\psi_e \in \mathrm{HD}_K(S^{-1}R, B)$.*

2. *Every higher derivation $\psi \in \mathrm{HD}_K(R, B)$ to a $\tilde{R}$-cga B can be extended uniquely to a higher derivation $\psi_e \in \mathrm{HD}_K(\tilde{R}, B)$.*

Proof For the first part, by the universal property of localisation (cf. [Eis95], Ch.2) – applied to ψ and to $\varepsilon \circ \psi$ – we only have to show that for each $s \in S$ the image $\psi(s)$ is invertible in B.
Define $\sum_{i=0}^{\infty} b_i \in B$ inductively by

$$b_0 := s^{-1} \in S^{-1}R = B_0$$

and for all $k \geq 1$:

$$b_k := -s^{-1} \sum_{i=1}^{k} \psi^{(i)}(s) b_{k-i}.$$

Then we get:

$$\psi(s) \cdot \left(\sum_{i=0}^{\infty} b_i \right) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \psi^{(i)}(s) b_{k-i} = sb_0 + \sum_{k=1}^{\infty} \left(sb_k + \sum_{i=1}^{k} \psi^{(i)}(s) b_{k-i} \right) = 1.$$

The proof of the second part: Every extension of ψ is given by the image of y in B, i.e. by an element $\sum_{k=0}^{\infty} \eta_k \in B$ with $\eta_0 = y$ and $\sum\limits_{i=0}^{m} \psi(a_i) \left(\sum\limits_{k=0}^{\infty} \eta_k \right)^i = 0$, where $g(X) = \sum_{i=0}^{m} a_i X^i$. So we have to show that there exists a unique element with these properties.
Therefore let $\sum_{k=0}^{\infty} \eta_k \in B$ satisfy $\eta_0 = y$. The k-th homogenous component $(k > 0)$ of $\sum\limits_{i=0}^{m} \psi(a_i) \left(\sum\limits_{l=0}^{\infty} \eta_l \right)^i$ is then given by:

$$\begin{aligned}
\left(\sum_{i=0}^{m} \psi(a_i) \left(\sum_{l=0}^{\infty} \eta_l \right)^i \right)_k &= \sum_{i=0}^{m} \sum_{j+k_1+\dots+k_i=k} \psi^{(j)}(a_i) \eta_{k_1} \cdots \eta_{k_i} \\
&= \sum_{i=1}^{m} \psi^{(0)}(a_i) i \eta_0^{i-1} \eta_k + P(\psi^{(j)}(a_i), \eta_0, \eta_1, \dots, \eta_{k-1}) \\
&= \sum_{i=1}^{m} i a_i y^{i-1} \eta_k + P(\psi^{(j)}(a_i), \eta_0, \eta_1, \dots, \eta_{k-1}),
\end{aligned}$$

where $P(\psi^{(j)}(a_i), \eta_0, \eta_1, \dots, \eta_{k-1})$ denotes a polynomial expression in $\psi^{(j)}(a_i)$ ($j = 0, \dots, k$; $i = 0, \dots, m$) and $y = \eta_0, \eta_1, \dots, \eta_{k-1}$. (Here the second equality is obtained by sorting out all terms, in which η_k occurs.) Since $\sum_{i=1}^{m} i a_i y^{i-1} = g'(y)$ is invertible in $\tilde{R}$, the condition that the k-th homogenous component above equals 0 is equivalent to

$$\eta_k = -g'(y)^{-1} \cdot P(\psi^{(j)}(a_i), \eta_0, \eta_1, \dots, \eta_{k-1})$$

and therefore there is a unique $\sum\limits_{k=0}^{\infty} \eta_k \in B$, whose homogenous components can be calculated by the formula above. $\square$

Example 1.4 For a polynomial algebra $R = K[t_1, \dots, t_m]$, every higher derivation of R into some R-cga B is given by an m-tupel $(b_1, \dots, b_m)$ of elements of B

satisfying $\varepsilon(b_j) = t_j$ for all $j = 1, \ldots, m$.
The higher derivations $\phi_{t_j} \in \mathrm{HD}_K(K[t_1, \ldots, t_m])$ given by $\phi_{t_j}(t_i) = t_i$ for $i \neq j$ and $\phi_{t_j}(t_j) = t_j + T$ play an important role. In the classical context, $\phi_{t_j}^{(1)}$ is just the derivation with respect to t_j. We therefore call ϕ_{t_j} the **higher derivation with respect to** t_j. If $\tilde{R}$ is a localisation of $K[t_1, \ldots, t_m]$ or an integral extension as in the previous proposition, then the $\phi_{t_j} \in \mathrm{HD}_K(K[t_1, \ldots, t_m])$ uniquely extend to higher derivations on $\tilde{R}$. These derivations will also be referred to as **higher derivation with respect to** t_j and will also be denoted by ϕ_{t_j}.

Definition 1.5 *For* $\psi \in \mathrm{HD}_K(R)$ *we define* $\psi[[U]] \in \mathrm{HD}_{K[[U]]}(R[[U]])$ *by* $\psi[[U]](\sum_{i=0}^{\infty} a_i U^i) = \sum_{i=0}^{\infty} \psi(a_i) U^i$.
Using this we get a **composition of higher derivations** $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$ *by:*

$$\psi_1 \star \psi_2 : R \xrightarrow{\psi_2} R[[T]] \xrightarrow{T \mapsto U} R[[U]] \xrightarrow{\psi_1[[U]]} R[[U]][[T]] = R[[U, T]]$$

If $R[[U,T]]$ *is given the grading by total degree, we obviously have* $\psi_1 \star \psi_2 \in \mathrm{HD}_K(R, R[[U,T]])$.

Remark One easily calculates that for $r \in R$ we have

$$(\psi_1 \star \psi_2)(r) = \sum_{i,j \in \mathbb{N}} (\psi_1^{(i)} \circ \psi_2^{(j)})(r) U^j T^i. \tag{3}$$

Lemma 1.6 *Let* $\mu : R[[U,T]] \to R[[T]]$ *be the homomorphism of* R*-cgas defined by* $U \mapsto T$ *and* $T \mapsto T$*. Then the multiplication*

$$\psi_1 \cdot \psi_2 := \mu \circ (\psi_1 \star \psi_2) \tag{4}$$

for $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$ *defines a group structure on* $\mathrm{HD}_K(R)$.

Proof See [Mats89],§27. □

1.2 Higher Derivations on Modules

In this section we consider higher derivations on modules. Remind that M will denote a finitely generated R-module.

Definition 1.7 *Let $\psi : R \to B$ be a higher derivation of R in B over K. A* **(higher) ψ-derivation** *of M is a homomorphism of K-modules $\Psi : M \to B \otimes_R M$ with $(\varepsilon \otimes \mathrm{id}_M) \circ \Psi = f \otimes \mathrm{id}_M$ and $\Psi(rm) = \psi(r)\Psi(m)$ for all $r \in R, m \in M$. We denote by* $\mathbf{HD}_{\boldsymbol{K}}\boldsymbol{(M, \psi)}$ *the set of (higher) ψ-derivations of M. As in section 1.1, for $\psi \in \mathrm{HD}_K(R)$ and $\Psi \in \mathrm{HD}_K(M, \psi)$ we define a sequence of maps $\Psi^{(k)} : M \to M$ by writing $\Psi(m) = \sum_{k=0}^{\infty} \Psi^{(k)}(m)T^k$ for all $m \in M$.*

Proposition 1.8 *For $\psi \in \mathrm{HD}_K(R)$ and $\Psi \in \mathrm{HD}_K(M, \psi)$ the maps $\Psi^{(k)}$ are homomorphisms of K-modules and satisfy the following properties:*

$$\Psi^{(0)} = \mathrm{id}_M \tag{5}$$

$$\forall\, a \in R, m \in M : \Psi^{(k)}(am) = \sum_{i+j=k} \psi^{(i)}(a)\Psi^{(j)}(m) \tag{6}$$

Furthermore a sequence $\left(\Psi^{(k)}\right)_{k\in\mathbb{N}}$ of K-module-homomorphisms satisfying these two properties defines a ψ-derivation $\Psi : M \to M[[T]]$ by the equation given above.

Proof Analogous to the proof of proposition 1.2. □

Remark

1. For given $\psi \in \mathrm{HD}_K(R, B)$, every homomorphism of $\tilde{R}$-cgas $g : B \to \tilde{B}$ induces a map $g_* : \mathrm{HD}_K(M, \psi) \to \mathrm{HD}_K(M, g \circ \psi), \Psi \mapsto (g \otimes \mathrm{id}_M) \circ \Psi$.

2. Let $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$. Then as in definition 1.5, we can define the composition $\Psi_1 \star \Psi_2$ of two higher derivations $\Psi_i \in \mathrm{HD}_K(M, \psi_i)$ $(i = 1, 2)$, which is an element of $\mathrm{HD}_K(M, \psi_1 \star \psi_2)$, and the product $\Psi_1 \cdot \Psi_2 \in \mathrm{HD}_K(M, \psi_1\psi_2)$.

1.3 Action of K on Higher Derivations

We now regard the action of K on the set of higher derivations.[1] This action will be useful when giving a description of iterative derivations (see section 3), which is convenient for calculations.

Definition 1.9 *For $a \in K$ and $\psi \in \mathrm{HD}_K(R, B)$ we define a map $a.\psi : R \to B$ by $(a.\psi)^{(k)} := a^k \cdot \psi^{(k)}$ for all $k \in \mathbb{N}$, which is easily seen to be a higher derivation. (Here $a^0 := 1$ even if $a = 0$.)*
Also for a ψ-derivation $\Psi \in \mathrm{HD}_K(M, \psi)$ we define a map $a.\Psi : M \to B \otimes_R M$ by $(a.\Psi)^{(k)} := a^k \cdot \Psi^{(k)}$ for all $k \in \mathbb{N}$, which is an element of $\mathrm{HD}_K(M, a.\psi)$.

[1] The action given here actually is a special case of the action of K given in appendix A.

Proposition 1.10 *The definition above gives an action of the multiplicative monoid K on the set $\mathrm{HD}_K(R, B)$ for arbitrary $\tilde{R}$-cga B. Moreover this action commutes with the group structure on $\mathrm{HD}_K(R)$.*

Proof This is a special case of proposition A.5 in appendix A. $\square$

Corollary 1.11 *The set $\mathrm{Der}(R) := \{\psi^{(1)} \mid \psi \in \mathrm{HD}_K(R)\}$ is a vector space over K.*

Proof Let $\psi_1^{(1)}, \psi_2^{(1)} \in \mathrm{Der}(R)$ and $a_1, a_2 \in K$. Then by equation (3) and (4), for all $r \in R$ we have:

$$\begin{aligned}(a_1.\psi_1)(a_2.\psi_2)^{(1)}(r) &= \left((a_1.\psi_1)^{(1)} \circ (a_2.\psi_2)^{(0)}\right)(r) + \left((a_1.\psi_1)^{(0)} \circ (a_2.\psi_2)^{(1)}\right)(r) \\ &= a_1 \cdot \psi_1^{(1)}(r) + a_2 \cdot \psi_2^{(1)}(r).\end{aligned}$$

Therefore $a_1 \cdot \psi_1^{(1)} + a_2 \cdot \psi_2^{(1)} \in \mathrm{Der}(R)$. $\square$

Remark One could also define an action of R on $\mathrm{HD}_K(R, B)$ by the same rule. But we won't use this action, because it doesn't behave nicely. For example it doesn't commute with the multiplication in $\mathrm{HD}_K(R)$ and most properties that will be shown in section 3 and used later on are restricted to the action of K (and are not valid for arbitrary elements of R). However using the action of R one could see that the set $\mathrm{Der}(R)$ defined in the previous corollary is in fact an R-module, actually the R-module of (common) derivations of R.

2 Higher Differentials and Higher Connections

2.1 Higher Differentials

Theorem 2.1 *Up to isomorphism, there exists a unique R-cga $\hat{\Omega}_{R/K}$ together with a higher derivation $\mathrm{d}_R : R \to \hat{\Omega}_{R/K}$ satisfying the following universal property:*
For each $\tilde{R}$-cga B and higher derivation $\psi : R \to B$ there exists a unique homomorphism of $\tilde{R}$-cgas $\tilde{\psi} : \tilde{R} \otimes_R \hat{\Omega}_{R/K} \to B$ with $\tilde{\psi} \circ (1 \otimes \mathrm{d}_R) = \psi$. [2]

Proof We construct $\hat{\Omega}_{R/K}$. Uniqueness is given by the universal property.
Let $G = R[\mathrm{d}^{(k)}r \mid k \in \mathbb{N}_+, r \in R]$ be the polynomial algebra over R in the variables $\mathrm{d}^{(k)}r$ and let the degree of $\mathrm{d}^{(k)}r$ be k. Define $I \trianglelefteq G$ to be the ideal generated by the union of the sets

$$\begin{aligned}
&\{\mathrm{d}^{(k)}(r+s) - \mathrm{d}^{(k)}r - \mathrm{d}^{(k)}s \mid k \in \mathbb{N}_+; r, s \in R\}, \\
&\{\mathrm{d}^{(k)}a \mid k \in \mathbb{N}_+; a \in K\} \quad \text{ and} \\
&\{\mathrm{d}^{(k)}(rs) - \sum_{i=0}^{k} \mathrm{d}^{(i)}r \cdot \mathrm{d}^{(k-i)}s \mid k \in \mathbb{N}_+; r, s \in R\},
\end{aligned}$$

where we identify $\mathrm{d}^{(0)}r$ with r for all $r \in R$. Therefore I is a homogeneous ideal and we set $\hat{\Omega}_{R/K}$ as the completion of the graded algebra G/I. We also define the higher derivation $\mathrm{d}_R : R \to \hat{\Omega}_{R/K}$ by $\mathrm{d}_R(r) := \sum_{k=0}^{\infty} \mathrm{d}^{(k)}r$.[3]
The universal property is seen as follows: Let $\psi : R \to B$ be a higher derivation. Then we define an R-algebra-homomorphism $g : G \to B$ by $g(\mathrm{d}^{(k)}r) := \psi^{(k)}(r)$ for all $k > 0$ and $r \in R$. The properties of a higher derivation imply that I lies in the kernel of g, and therefore g factors through $\overline{g} : G/I \to B$ and we get a homomorphism of algebras $\hat{\Omega}_{R/K} \to B$ by extending $\overline{g}$ continuously and therefore a homomorphisms of $\tilde{R}$-cgas $\tilde{\psi} : \tilde{R} \otimes_R \hat{\Omega}_{R/K} \to B$.
On the other hand, the condition $\tilde{\psi} \circ (1 \otimes \mathrm{d}_R) = \psi$ forces this choice of g and so $\tilde{\psi}$ is unique. $\square$

Proposition 2.2 *(a) For every homomorphism of rings $f : R \to \tilde{R}$ there is a unique homomorphism of $\tilde{R}$-cgas $Df : \tilde{R} \otimes_R \hat{\Omega}_{R/K} \to \hat{\Omega}_{\tilde{R}/K}$ such that $\mathrm{d}_{\tilde{R}} \circ f = Df \circ (1 \otimes \mathrm{d}_R)$.*

(b) If $\tilde{R}$ is a localisation of R or $\tilde{R} = R(y)$, where y is integral over R, the minimal polynomial $g(X)$ of y has coefficients in R and $g'(y)$ is invertible in R, then Df is an isomorphism.

[2] In other words, $\hat{\Omega}_{R/K}$ is representing the functor $\mathrm{HD}_K(R, -)$.

[3] Here and in the following the residue class of $\mathrm{d}^{(k)}r \in G$ in $\hat{\Omega}_{R/K}$ will also be denoted by $\mathrm{d}^{(k)}r$.

Proof Since $\mathrm{d}_{\tilde R}\circ f$ is a higher derivation on R, part (a) follows from the universal property of $\hat\Omega_{R/K}$.
By proposition 1.3, in the two cases of part (b), every higher derivation on R to a $\tilde R$-cga extends uniquely to $\tilde R$. So there exists a (unique) higher derivation $\tilde{\mathrm{d}} : \tilde R \to \tilde R \otimes_R \hat\Omega_{R/K}$ extending $(1 \otimes \mathrm{d}_R)$. By the universal property of $\hat\Omega_{\tilde R/K}$, there exists a unique homomorphism of $\tilde R$-cgas $g : \hat\Omega_{\tilde R/K} \to \tilde R \otimes_R \hat\Omega_{R/K}$ such that $g \circ \mathrm{d}_{\tilde R} = \tilde{\mathrm{d}}$. Now we have

$$(g \circ Df) \circ (1 \otimes \mathrm{d}_R) = g \circ \mathrm{d}_{\tilde R} \circ f = \tilde{\mathrm{d}} \circ f = (1 \otimes \mathrm{d}_R)$$

and therefore by the universal property $(g \circ Df) = \mathrm{id}_{\tilde R \otimes_R \hat\Omega_{R/K}}$. Furthermore

$$(Df \circ g) \circ \mathrm{d}_{\tilde R} \circ f = Df \circ \tilde{\mathrm{d}} \circ f = Df \circ (1 \otimes \mathrm{d}_R) = \mathrm{d}_{\tilde R} \circ f.$$

By the unique extension of higher derivations this leads to $(Df \circ g) \circ \mathrm{d}_{\tilde R} = \mathrm{d}_{\tilde R}$ and finally we get $(Df \circ g) = \mathrm{id}_{\hat\Omega_{\tilde R/K}}$ by the universal property of $\hat\Omega_{\tilde R/K}$.
So Df and g are inverse to each other and so Df is an isomorphism. □

Theorem 2.3 *(a) Let $R = K[t_1, \dots, t_m]$ be the polynomial ring in m variables. Then $\hat\Omega_{R/K}$ is the completion of the polynomial algebra $R[\mathrm{d}^{(i)} t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]$.*

(b) Let $F/K(t_1, \dots, t_m)$ be a finite separable algebraic extension field. Then $\hat\Omega_{F/K}$ is the completion of the polynomial algebra $F[\mathrm{d}^{(i)} t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]$.

(c) Let $(R, \mathfrak{m})$ be a regular local ring of dimension m, let $t_1, \dots, t_m$ generate $\mathfrak{m}$ and assume that R is a localisation of a finitely generated K-algebra and that $R/\mathfrak{m}$ is a finite separable extension of K. Then $\hat\Omega_{R/K}$ is the completion of the polynomial algebra $R[\mathrm{d}^{(i)} t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]$.

Remark We will denote the completion of such a polynomial algebra by $R[[\mathrm{d}^{(i)} t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]]$, although it is not really a ring of power series, because it contains infinite sums of different variables.
Proof (a): Since for $P \in R$ the image $\mathrm{d}\big(P(t_1, \dots, t_m)\big) = P\big(\mathrm{d}(t_1), \dots, \mathrm{d}(t_m)\big)$ is a "power series" in $\mathrm{d}^{(i)} t_j$, $\hat\Omega_{R/K}$ is generated by the $\mathrm{d}^{(i)} t_j$ as a R-cga. On the other hand, since every choice of $\psi^{(i)}(t_j) \in R$ ($i \in \mathbb{N}_+, j = 1, \dots, m$) defines a higher derivation $\psi \in \mathrm{HD}_K(R)$, by the universal property of $\hat\Omega_{R/K}$ the $\mathrm{d}^{(i)} t_j$ are algebraically independent over R.
(b): This follows from part (a) and proposition 2.2(b), since $K(t_1, \dots, t_m)$ is a localisation of $K[t_1, \dots, t_m]$ and $F = K(t_1, \dots, t_m)[y]$ with an element $y \in F$ that is separable algebraic over $K(t_1, \dots, t_m)$. So the minimal polynomial

$g(X) \in K(t_1, \ldots, t_m)[X]$ of y satisfies $g'(y) \neq 0$, i.e. $g'(y)$ is invertible in F.
(c): We will show that $\hat{\Omega}_{R/K} \otimes_R (R/\mathfrak{m})$ is isomorphic to $(R/\mathfrak{m})[[\mathrm{d}^{(i)}t_j]]$. Then, since $\hat{\Omega}_{R/K} \otimes_R \mathrm{Quot}(R)$ is isomorphic to $\mathrm{Quot}(R)[[\mathrm{d}^{(i)}t_j]]$ (prop. 2.2 and part (b)), by [Hart77], Ch.II, lemma 8.9, it follows that $(\hat{\Omega}_{R/K})_k$ is a free R-module and that the residue classes of any basis of $(\hat{\Omega}_{R/K})_k$ is a basis of $(\hat{\Omega}_{R/K} \otimes_R R/\mathfrak{m})_k$. Hence we obtain $\hat{\Omega}_{R/K} = R[[\mathrm{d}^{(i)}t_j]]$.
First, let $\psi : R \to B$ be a higher derivation of R to a $R/\mathfrak{m}$-cga B. Then for all $k \in \mathbb{N}$ and $r_1, \ldots, r_{k+1} \in \mathfrak{m}$, we have

$$\psi^{(k)}(r_1 \cdots r_{k+1}) = \sum_{i_1 + \cdots + i_{k+1} = k} \psi^{(i_1)}(r_1) \cdots \psi^{(i_{k+1})}(r_{k+1}) = 0,$$

since in each summand at least one $i_j = 0$, and so $\psi^{(i_1)}(r_1) \cdots \psi^{(i_{k+1})}(r_{k+1}) \in \mathfrak{m}B = 0$. Therefore $\psi^{(k)}$ (and $\psi^{(i)}$ for $i < k$) factors through $R/\mathfrak{m}^{k+1}$.
Next, since $R/\mathfrak{m}$ is a finite separable extension of K, there is $\bar{y} \in R/\mathfrak{m}$ that generates the extension $K \subset R/\mathfrak{m}$. Let $g(X) \in K[X]$ be the minimal polynomial of $\bar{y}$, then starting with an arbitrary representative $y \in R$ for $\bar{y}$, using the Newton approximation $y_{n+1} = y_n - g(y_n)g'(y_n)^{-1}$, we obtain an element $\tilde{y}_k \in R$ such that $g(\tilde{y}_k) \equiv 0 \pmod{\mathfrak{m}^{k+1}}$ for given $k \in \mathbb{N}$. (Note that the Newton approximation is well defined and converges to a root of $g(X)$, since $\overline{g(y)} = g(\overline{y}) = 0 \in R/\mathfrak{m}$, so $g(y) \in \mathfrak{m}$, since $\overline{g'(y)} = g'(\overline{y}) \neq 0 \in R/\mathfrak{m}$, so $g(y) \in R^\times$ and so inductively for all $n \in \mathbb{N}$: $\bar{y}_{n+1} = \bar{y}_n = \bar{y} \in R/\mathfrak{m}$, $g(y_{n+1}) \in \mathfrak{m}$ and $g'(y_{n+1}) \in R^\times$.) This proves that for all $k \in \mathbb{N}$, the ring $R/\mathfrak{m}^{k+1}$ contains a subfield isomorphic to $R/\mathfrak{m}$.
Now by [Mats89], theorem 14.4, the associated graded ring $\mathrm{gr}(R)$ of R is isomorphic to the polynomial ring $(R/\mathfrak{m})[t_1, \ldots, t_m]$ and therefore we obtain $\mathrm{gr}(R/\mathfrak{m}^{k+1}) \cong (R/\mathfrak{m})[t_1, \ldots, t_m]/\mathfrak{n}^{k+1}$, where $\mathfrak{n}$ is the ideal generated by $\{t_1, \ldots, t_m\}$. Furthermore, since $R/\mathfrak{m}^{k+1}$ contains a subfield isomorphic to $R/\mathfrak{m}$, we see that the inclusion $\iota_k : (R/\mathfrak{m})[t_1, \ldots, t_m]/\mathfrak{n}^{k+1} \to R/\mathfrak{m}^{k+1}$ (given by the inclusion $K[t_1, \ldots, t_m]/\mathfrak{n}^{k+1} \subset R/\mathfrak{m}^{k+1}$ and $\bar{y} \mapsto \tilde{y}_k$) is an isomorphism.
Hence, every higher derivation $\psi_{\mathrm{gr}} : \mathrm{gr}(R) \to B$ into an $R/\mathfrak{m}$-cga B induces a higher derivation $\psi_R : R \to B$ on R by $\psi_R^{(k)} := \psi_{\mathrm{gr}}^{(k)} \circ \iota_k^{-1}$ $(K \in \mathbb{N})$ and vice versa. So $\hat{\Omega}_{R/K} \otimes_R R/\mathfrak{m} \cong \hat{\Omega}_{\mathrm{gr}(R)/K} \otimes_{\mathrm{gr}(R)} R/\mathfrak{m} = (R/\mathfrak{m})[[\mathrm{d}^{(i)}t_j]]$. □

Corollary 2.4 *If K is a perfect field and R is a regular ring, then the homogeneous components $(\hat{\Omega}_{R/K})_k$ $(k \in \mathbb{N})$ are projective R-modules.*

Proof For every maximal ideal $\mathfrak{m} \trianglelefteq R$, the localisation $R_\mathfrak{m}$ fulfills the conditions of theorem 2.3(c). And so by proposition 2.2, $R_\mathfrak{m} \otimes_R (\hat{\Omega}_{R/K})_k \cong (\hat{\Omega}_{R_\mathfrak{m}/K})_k$ is a free $R_\mathfrak{m}$-module. Hence by [Eis95], thm. A3.2, $(\hat{\Omega}_{R/K})_k$ is a projective R-module.
□

2.2 An Extension of the Universal Derivation

Notation We will sometimes omit indices when they are clear from the context. So for example in the following $\hat{\Omega}$ means $\hat{\Omega}_{R/K}$ and d means d_R, as there are no other rings mentioned.

Theorem 2.5 *For all $a \in K$ the mapping*

$$\mathrm{d}^{(i)}r \mapsto \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)}r,$$

where $i \in \mathbb{N}$ and $r \in R$, defines a continuous homomorphism of K-algebras $a.\mathrm{d}_{\hat{\Omega}} : \hat{\Omega} \to \hat{\Omega}$ satisfying the following three conditions:

1. *$a.\mathrm{d}_{\hat{\Omega}}$ extends the higher derivation $a.\mathrm{d} : R \to \hat{\Omega}$.*

2. *For all $a, b \in K$: $(a.\mathrm{d}_{\hat{\Omega}}) \circ (b.\mathrm{d}_{\hat{\Omega}}) = (a+b).\mathrm{d}_{\hat{\Omega}}$.*

3. *$0.\mathrm{d}_{\hat{\Omega}} = \mathrm{id}_{\hat{\Omega}}$.*

For short, we will write $\mathrm{d}_{\hat{\Omega}}$ instead of $1.\mathrm{d}_{\hat{\Omega}}$ and $-\mathrm{d}_{\hat{\Omega}}$ instead of $-1.\mathrm{d}_{\hat{\Omega}}$.

To prove the theorem we need a combinatorial lemma.

Lemma 2.6 *For all $i, k, l \in \mathbb{N}$ the following holds:*

1.
$$\sum_{i_1+i_2=i} \binom{k}{i_1}\binom{l}{i_2} = \binom{k+l}{i}$$

2.
$$\binom{i+l+k}{i+l}\binom{i+l}{i} = \binom{i+l+k}{i}\binom{l+k}{l}$$

Sketch of the proof Both identities are given by counting in two different ways: Given two disjoint sets M_1 and M_2 of order k resp. l, both sides of the first identity count the number of possibilities choosing i elements out of $M_1 \cup M_2$. The two sides of the second identity count the number of possibilities of partitioning a set of order $i+l+k$ into three subsets of order i,l and k.

Proof of theorem 2.5 We first prove that $a.\mathrm{d}_{\hat{\Omega}}$ is well defined: For $b \in K, i \in \mathbb{N}_+$ we have

$$a.\mathrm{d}_{\hat{\Omega}}\left(\mathrm{d}^{(i)}b\right) = \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)}(b) = 0.$$

For all $r, s \in R, i \in \mathbb{N}$ we have

$$\begin{aligned}
a.\mathrm{d}_{\hat{\Omega}}\left(\mathrm{d}^{(i)}(r+s)\right) &= \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)}(r+s) \\
&= \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)}(r) + \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)}(s) \\
&= a.\mathrm{d}_{\hat{\Omega}}\left(\mathrm{d}^{(i)}r + \mathrm{d}^{(i)}s\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
a.\mathrm{d}_{\hat{\Omega}}\left(\mathrm{d}^{(i)}(rs)\right) &= \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)}(rs) \\
&= \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \sum_{k+l=i+j} \mathrm{d}^{(k)}r \cdot \mathrm{d}^{(l)}s
\end{aligned}$$

and

$$\begin{aligned}
&a.\mathrm{d}_{\hat{\Omega}}\left(\sum_{i_1+i_2=i} \mathrm{d}^{(i_1)}r \cdot \mathrm{d}^{(i_2)}s\right) \\
&= \sum_{i_1+i_2=i} \left(\sum_{j_1=0}^{\infty} a^{j_1} \binom{i_1+j_1}{j_1} \mathrm{d}^{(i_1+j_1)}(r)\right) \left(\sum_{j_2=0}^{\infty} a^{j_2} \binom{i_2+j_2}{j_2} \mathrm{d}^{(i_2+j_2)}(s)\right) \\
&= \sum_{j=0}^{\infty} \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j}} a^{j_1+j_2} \binom{i_1+j_1}{i_1} \binom{i_2+j_2}{i_2} \mathrm{d}^{(i_1+j_1)}(r) \cdot \mathrm{d}^{(i_2+j_2)}(s) \\
&= \sum_{j=0}^{\infty} a^j \sum_{\substack{i_1+i_2=i \\ k+l=i+j}} \binom{k}{i_1} \binom{l}{i_2} \mathrm{d}^{(k)}(r) \cdot \mathrm{d}^{(l)}(s) \\
&= \sum_{j=0}^{\infty} a^j \sum_{k+l=i+j} \binom{k+l}{i} \mathrm{d}^{(k)}(r) \cdot \mathrm{d}^{(l)}(s).
\end{aligned}$$

So all relations are preserved and $a.\mathrm{d}_{\hat{\Omega}}$ is welldefined.

$a.\mathrm{d}_{\hat{\Omega}}$ extends $a.\mathrm{d}$ because for $r \in R$ we have

$$a.\mathrm{d}_{\hat{\Omega}}(r) = a.\mathrm{d}_{\hat{\Omega}}(\mathrm{d}^{(0)}r) = \sum_{j=0}^{\infty} a^j \binom{j}{j} \mathrm{d}^{(j)}(r) = a.\mathrm{d}(r).$$

At last, for $r \in R, i \in \mathbb{N}$ we have

$$\begin{aligned}
a.\mathrm{d}_{\hat{\Omega}}\left(b.\mathrm{d}_{\hat{\Omega}}(\mathrm{d}^{(i)}r)\right) &= a.\mathrm{d}_{\hat{\Omega}}\left(\sum_{j=0}^{\infty} b^j \binom{i+j}{j} \mathrm{d}^{(i+j)}r\right) \\
&= \sum_{j=0}^{\infty} b^j \binom{i+j}{j} \sum_{k=0}^{\infty} a^k \binom{i+j+k}{k} \mathrm{d}^{(i+j+k)}r \\
&= \sum_{j,k=0}^{\infty} b^j a^k \binom{i+j}{i} \binom{i+j+k}{i+j} \mathrm{d}^{(i+j+k)}r \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} b^j a^{n-j} \binom{i+j}{i} \binom{i+n}{i+j} \mathrm{d}^{(i+n)}r \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} b^j a^{n-j} \binom{n}{j}\right) \binom{i+n}{i} \mathrm{d}^{(i+n)}r \\
&= \sum_{n=0}^{\infty} (a+b)^n \binom{i+n}{i} \mathrm{d}^{(i+n)}r = (a+b).\mathrm{d}_{\hat{\Omega}}\left(\mathrm{d}^{(i)}r\right).
\end{aligned}$$

The identity $0.\mathrm{d}_{\hat{\Omega}} = \mathrm{id}_{\hat{\Omega}}$ is clear from the definition. □

Remark By the second and the third property we see, that $a.\mathrm{d}_{\hat{\Omega}}$ actually is an automorphism of $\hat{\Omega}$ for all $a \in K$. The endomorphisms $a.\mathrm{d}_{\hat{\Omega}}$ play an important role in the iterative theory, as will be seen in section 3.

Definition 2.7 *For $a \in K$ we decompose $a.\mathrm{d}_{\hat{\Omega}}$ into a sequence $\left((a.\mathrm{d}_{\hat{\Omega}})^{(k)}\right)_{k \in \mathbb{N}}$ of continuous endomorphisms of the K-module $\hat{\Omega}$ in the following way:*[4] *For a homogeneous element $\omega \in \hat{\Omega}_i$ of degree i we define*

$$(a.\mathrm{d}_{\hat{\Omega}})^{(k)}(\omega) := \mathrm{pr}_{i+k}(a.\mathrm{d}_{\hat{\Omega}}(\omega)) \in \hat{\Omega}_{i+k}.$$

It is clear that the series $\sum_{k=0}^{\infty}(a.\mathrm{d}_{\hat{\Omega}})^{(k)}$ converges against $a.\mathrm{d}_{\hat{\Omega}}$, at least pointwise, and that for all $k \in \mathbb{N}$ we have $(a.\mathrm{d}_{\hat{\Omega}})^{(k)} = a^k \cdot \mathrm{d}_{\hat{\Omega}}^{(k)}$.

Proposition 2.8 *For all $i, j \in \mathbb{N}$ we have:*

$$\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)} = \binom{i+j}{i} \mathrm{d}_{\hat{\Omega}}^{(i+j)}.$$

Proof For all $i, j \in \mathbb{N}$ and $\omega \in \hat{\Omega}$, the term $\left(\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)}\right)(\omega)$ is the coefficient of $a^i b^j$ in the expression $\left((a.\mathrm{d}_{\hat{\Omega}}) \circ (b.\mathrm{d}_{\hat{\Omega}})\right)(\omega)$. By theorem 2.5, we have $(a.\mathrm{d}_{\hat{\Omega}}) \circ (b.\mathrm{d}_{\hat{\Omega}}) = (a+b).\mathrm{d}_{\hat{\Omega}}$ and so $\left(\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)}\right)(\omega)$ is the coefficient of $a^i b^j$ in the expression $(a+b).\mathrm{d}_{\hat{\Omega}}(\omega) = \sum_{k=0}^{\infty}(a+b)^k \mathrm{d}_{\hat{\Omega}}^{(k)}(\omega)$, i. e. equals $\binom{i+j}{i}\mathrm{d}_{\hat{\Omega}}^{(i+j)}(\omega)$. □

[4] cf. appendix A

2.3 Higher Connections

Definition 2.9 *A* **higher connection** *on M is a* d*-derivation $\nabla \in \mathrm{HD}_K(M, \mathrm{d})$. If $\psi \in \mathrm{HD}_K(R, B)$ is a higher derivation, we define the higher ψ-derivation ∇_ψ on M by*

$$\nabla_\psi := (\tilde{\psi} \otimes \mathrm{id}_M) \circ \nabla : M \to \hat{\Omega}_{R/K} \otimes_R M \to B \otimes_R M \ .$$

For all $a \in K$ we define an endomorphism $a ._{\hat{\Omega}} \nabla : \hat{\Omega} \otimes_R M \to \hat{\Omega} \otimes_R M$ by

$$(a ._{\hat{\Omega}} \nabla)(\omega \otimes x) := a . \mathrm{d}_{\hat{\Omega}}(\omega) \cdot (a . \nabla)(x)$$

for all $\omega \in \hat{\Omega}$ and $x \in M$, i.e. $a ._{\hat{\Omega}} \nabla = (\mu_{\hat{\Omega}} \otimes \mathrm{id}_M) \circ (a . \mathrm{d}_{\hat{\Omega}} \otimes a . \nabla)$, where $\mu_{\hat{\Omega}}$ denotes the multiplication map in $\hat{\Omega}$.

Remark Be aware that in the previous definition the map $a . \mathrm{d}_{\hat{\Omega}} \otimes a . \nabla$ is a map from $\hat{\Omega} \otimes_R M$ to $\hat{\Omega} \otimes_{a.\mathrm{d}(R)} (\hat{\Omega} \otimes_R M) = (\hat{\Omega} \otimes_{a.\mathrm{d}(R)} \hat{\Omega}) \otimes_R M$ (the tensor product is taken over the image of R under $a . \mathrm{d}_{\hat{\Omega}}$!).

Lemma 2.10 *Let $(R, \mathfrak{m})$ be a regular local ring such that $R/\mathfrak{m}$ is a finite separable extension of K. By Noether normalization, R is a finite separable extension of $K[t_1, \dots, t_m]_{(t_1,\dots,t_m)}$, where $\{t_1, \dots, t_m\}$ is a minimal set of generators of $\mathfrak{m}$.*
Let $\phi_{t_j} \in \mathrm{HD}_K(R)$ ($j = 1, \dots, m$) denote the higher derivations with respect to t_j (cf. example 1.4).
Then for every $r \in R \setminus \{0\}$ there exist $k_1, \dots, k_m \in \mathbb{N}$ such that

$$\left(\phi_{t_m}^{(k_m)} \circ \dots \circ \phi_{t_1}^{(k_1)} \right)(r) \in R^\times$$

and for all $l_1, \dots, l_m \in \mathbb{N}$ with $l_j \le k_j$ ($j = 1, \dots, m$) and $l_i < k_i$ for some $i \in \{1, \dots, m\}$:

$$\left(\phi_{t_m}^{(l_m)} \circ \dots \circ \phi_{t_1}^{(l_1)} \right)(r) \notin R^\times$$

Proof Let $r \in R \setminus \{0\}$. Choose $E \in \mathbb{N}$ such that $r \in \mathfrak{m}^E$ and $r \notin \mathfrak{m}^{E+1}$. Then r can (uniquely) be written as

$$r = \sum_{\substack{\boldsymbol{e}=(e_1,\dots,e_m)\in\mathbb{N}^m \\ |\boldsymbol{e}|=E}} u_{\boldsymbol{e}} \boldsymbol{t}^{\boldsymbol{e}},$$

where $u_{\boldsymbol{e}} \in R$ and $u_{\boldsymbol{f}} \in R^\times$ for at least one $\boldsymbol{f} = (f_1, \dots, f_m)$.
(We use the usual notation of multiindices: $|\boldsymbol{e}| = e_1 + \dots + e_m$ and $\boldsymbol{t}^{\boldsymbol{e}} = t_1^{e_1} \cdots t_m^{e_m}$.)
For arbitrary $\boldsymbol{l} = (l_1, \dots, l_m) \in \mathbb{N}^m$ and $\boldsymbol{e} \in \mathbb{N}^m$ we have:

$$\left(\phi_{t_m}^{(l_m)} \circ \dots \circ \phi_{t_1}^{(l_1)} \right)(\boldsymbol{t}^{\boldsymbol{e}}) = \binom{e_1}{l_1} \cdots \binom{e_m}{l_m} \boldsymbol{t}^{\boldsymbol{e}-\boldsymbol{l}} = \begin{cases} 0 & \text{if } l_i > e_i \text{ for some } i \\ 1 & \text{if } l_j = e_j \text{ for all } j \\ \in \mathfrak{m} & \text{if } |\boldsymbol{l}| < |\boldsymbol{e}| \end{cases}$$

So if we choose $k_j = f_j$ $(j = 1, \ldots, m)$, we get

$$\begin{aligned}
\left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(r) &= \sum_{|\boldsymbol{e}|=E} \left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(u_{\boldsymbol{e}} \boldsymbol{t}^{\boldsymbol{e}}) \\
&= \sum_{|\boldsymbol{e}|=E} \sum_{\substack{0 \le l_j \le k_j \\ j=1,\ldots,m}} \left(\phi_{t_m}^{(k_m-l_m)} \circ \cdots \circ \phi_{t_1}^{(k_1-l_1)}\right)(u_{\boldsymbol{e}}) \left(\phi_{t_m}^{(l_m)} \circ \cdots \circ \phi_{t_1}^{(l_1)}\right)(\boldsymbol{t}^{\boldsymbol{e}}) \\
&\equiv u_{\boldsymbol{f}} \cdot 1 \pmod{\mathfrak{m}}.
\end{aligned}$$

So $\left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(r) \in u_{\boldsymbol{f}} + \mathfrak{m} \subset R^\times$, and for all $\boldsymbol{l} \in \mathbb{N}^m$ with $l_j \le k_j$ $(j = 1, \ldots, m)$ and $l_i < k_i$ for some i, we have $\left(\phi_{t_m}^{(l_m)} \circ \cdots \circ \phi_{t_1}^{(l_1)}\right)(r) \in \mathfrak{m} = R \setminus R^\times$, since $|\boldsymbol{l}| < E$. $\square$

Theorem 2.11 *Let $(R, \mathfrak{m})$ be a regular local ring such that $R/\mathfrak{m}$ is a finite separable extension of K and let M be a finitely generated R-module with a higher connection $\nabla \in \mathrm{HD}_K(M, \mathrm{d})$. Then M is a free R-module.*

Proof Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of M.
Assume that $x_1, \ldots, x_n$ are linearly dependent. Then there exists a nontrivial relation $\sum_{i=1}^n r_i x_i = 0$, with $r_i \in R$. Choose $E \in \mathbb{N}$ such that $r_j \in \mathfrak{m}^E$ for all $j = 1, \ldots n$ and $r_i \notin \mathfrak{m}^{E+1}$ for at least one i and without loss of generality let $r_1 \notin \mathfrak{m}^{E+1}$. Then choose $k_1, \ldots, k_m \in \mathbb{N}$ for r_1 as given in the previous lemma. Then

$$\begin{aligned}
0 &= \left(\nabla_{\phi_{t_m}}^{(k_m)} \circ \cdots \circ \nabla_{\phi_{t_1}}^{(k_1)}\right)\left(\sum_{i=1}^n r_i x_i\right) \\
&= \sum_{i=1}^n \sum_{\substack{0 \le l_j \le k_j \\ j=1,\ldots,m}} \left(\phi_{t_m}^{(l_m)} \circ \cdots \circ \phi_{t_1}^{(l_1)}\right)(r_i) \left(\nabla_{\phi_{t_m}}^{(k_m-l_m)} \circ \cdots \circ \nabla_{\phi_{t_1}}^{(k_1-l_1)}\right)(x_i) \\
&\equiv \sum_{i=1}^n \left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(r_i) \cdot x_i \pmod{\mathfrak{m}M}
\end{aligned}$$

Since $\left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(r_1) \in R^\times$, we get $x_1 \in \langle x_2, \ldots, x_n \rangle + \mathfrak{m}M$, so $M = \langle x_2, \ldots, x_n \rangle + \mathfrak{m}M$ and therefore by Nakayama's lemma $M = \langle x_2, \ldots, x_n \rangle$, in contradiction to the condition that $\{x_1, \ldots, x_n\}$ is minimal.
So $x_1, \ldots, x_n$ is a basis for M and in particular M is a free R-module. $\square$

3 Iterative Theory

3.1 Iterative Derivations

Definition 3.1 *A higher derivation $\phi \in \mathrm{HD}_K(R)$ is called an* **iterative derivation**, *if*

$$\phi \star \phi = \Delta \circ \phi,$$

where $\Delta : R[[T]] \to R[[U,T]]$ is the homomorphism of R-cgas defined by $T \mapsto U+T$.
In terms of the $\phi^{(k)}$, this identity is written as:

$$\forall\, i,j \in \mathbb{N} : \phi^{(i)} \circ \phi^{(j)} = \binom{i+j}{i} \phi^{(i+j)} \ .$$

We denote the set of iterative derivations on R by $\mathbf{ID}_{\boldsymbol{K}}(\boldsymbol{R})$.

Example 3.2 *If R is the polynomial ring $K[t_1,\ldots,t_m]$ or an extension of that ring as in proposition 1.3, the higher derivations ϕ_{t_j} with respect to t_j (cf. example 1.4) are iterative derivations. (For $K[t_1,\ldots,t_m]$ this is obvious and for the extensions, this follows from lemma 3.5.)*

Lemma 3.3 (characterisation of iterative derivations)
Let $\psi \in \mathrm{HD}_K(R)$ be a higher derivation. Then the following conditions are equivalent:

(i) *ψ is iterative,*

(ii) $\tilde{\psi} \circ \mathrm{d}_{\hat{\Omega}} = \psi[[T]] \circ \tilde{\psi}$,

(iii) *For all $a \in K$: $\tilde{\psi} \circ (a.\mathrm{d}_{\hat{\Omega}}) = (a.\psi[[T]]) \circ \tilde{\psi}$.*

If K is an infinite field, then this is also equivalent to

(iv) *For all $a,b \in K$: $(a.\psi)(b.\psi) = (a+b).\psi$,*

whereas for arbitrary K the conditions (i)-(iii) *only imply condition* (iv).

Proof For $a \in K$, $r \in R$ and $i \in \mathbb{N}$ we have:

$$\begin{aligned}
\tilde{\psi} \circ (a.\mathrm{d}_{\hat{\Omega}})(\mathrm{d}^{(i)} r) &= \tilde{\psi}\left(\sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)} r \right) \\
&= \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \psi^{(i+j)}(r) T^{i+j}
\end{aligned}$$

and

$$\begin{aligned}
(a.\psi[[T]]) \circ \tilde{\psi}(\mathrm{d}^{(i)}r) &= a.\psi[[T]]\left(\psi^{(i)}(r)T^i\right) \\
&= \sum_{j=0}^{\infty} a^j \psi^{(j)}(\psi^{(i)}(r))T^{i+j}.
\end{aligned}$$

So by comparing the coefficients of T^{i+j} one sees that condition (iii) is fulfilled if and only if $\tilde{\psi} \circ (a.\mathrm{d}_{\hat{\Omega}}) = (a.\psi[[T]]) \circ \tilde{\psi}$ is fulfilled for an arbitrary $a \in K \setminus \{0\}$ (e.g. $a = 1$, i.e. condition (ii)) and if and only if for all $i, j \in \mathbb{N}$ we have $\psi^{(j)} \circ \psi^{(i)} = \binom{i+j}{j}\psi^{(i+j)}$, i.e. ψ is iterative.

Furthermore we get for all $a, b \in K$:

$$\begin{aligned}
((a.\psi)(b.\psi))^{(k)} &= \sum_{i+j=k} (a.\psi)^{(i)} \circ (b.\psi)^{(j)} \\
&= \sum_{i+j=k} a^i b^j \psi^{(i)} \circ \psi^{(j)}, \text{ since } b \in K,
\end{aligned}$$

and

$$\begin{aligned}
((a+b).\psi)^{(k)} &= (a+b)^k \psi^{(k)} \\
&= \sum_{i+j=k} a^i b^j \binom{i+j}{i} \psi^{(i+j)}
\end{aligned}$$

So if ψ is iterative we obtain condition (iv) and if $\#K = \infty$ by comparing the coefficients of a^i we obtain from condition (iv) that ψ is iterative. □

Example 3.4 Condition (iv) is in fact weaker if K is finite. If for example $K = \mathbb{F}_q$ and $R = \mathbb{F}_q[t]$, then $\psi \in \mathrm{HD}_K(R)$ defined by $\psi(t) = t + 1 \cdot T^{2q-1}$ is not iterative, since

$$(2q-1)\psi^{(2q-1)}(t) = 2q - 1 \neq 0 = \psi^{(2q-2)}\left(\psi^{(1)}(t)\right).$$

On the other hand, for all $a \in \mathbb{F}_q$ we have $a^{2q-1} = a$ and so

$$\begin{aligned}
\big((a.\psi)(b.\psi)\big)^{(k)}(t) &= \sum_{i+j=k} a^i b^j \psi^{(i)}(\psi^{(j)}(t)) = a^k\psi^{(k)}(t) + a^{k-2q+1}b^{2q-1}\psi^{(k-2q+1)}(1) \\
&= \left\{ \begin{array}{cc} t & k = 0 \\ a^{2q-1} + b^{2q-1} = (a+b)^{2q-1} & k = 2q-1 \\ 0 & \text{otherwise} \end{array} \right\} = \big((a+b).\psi\big)^{(k)}(t)
\end{aligned}$$

for all $a, b \in K = \mathbb{F}_q$.

Remark Condition (iv) is very useful for calculations – even if K is finite. If one has to show that some higher derivation $\psi \in \mathrm{HD}_K(R)$ is iterative, one can often use the following trick:
Let $\tilde{R} := K^{\mathrm{sep}} \otimes_{\bar{K}\cap R} R$ be the maximal separable extension of R by constants. Then by proposition 1.3 the higher derivation ψ uniquely extends to a higher derivation $\psi_e \in \mathrm{HD}_K(\tilde{R}) = \mathrm{HD}_{K^{\mathrm{sep}}}(\tilde{R})$. Since $\#K^{sep} = \infty$, we can use condition (iv) to show that ψ_e is iterative and therefore ψ is iterative.
Whenever it will be shown that for all $a, b \in K^{\mathrm{sep}}$, $(a.\psi)(b.\psi) = (a+b).\psi$, this trick will be used, although we won't mention it explicitely.

The next lemma states some structural properties of $\mathrm{ID}_K(R)$.

Lemma 3.5 *1. If two iterative derivations $\phi_1, \phi_2 \in \mathrm{ID}_K(R)$ satisfy $\phi_1^{(i)} \circ \phi_2^{(j)} = \phi_2^{(j)} \circ \phi_1^{(i)}$ for all $i, j \in \mathbb{N}$, then $\phi_1\phi_2$ is again an iterative derivation.*

2. $\mathrm{ID}_K(R)$ is invariant under the action of K.

3. If $\tilde{R} \supset R$ is a ring extension such that every higher derivation on R uniquely extends to a higher derivation on $\tilde{R}$ (see proposition 1.3 for examples), then the extension $\phi_e \in \mathrm{HD}_K(\tilde{R})$ of an iterative derivation $\phi \in \mathrm{ID}_K(R)$ is again iterative.

Proof

1. By the given condition, for all $a, b \in K^{\mathrm{sep}}$, we have $(a.\phi_2)(b.\phi_1) = (b.\phi_1)(a.\phi_2)$ and so

$$\begin{aligned}(a.(\phi_1\phi_2))(b.(\phi_1\phi_2)) &= (a.\phi_1)(a.\phi_2)(b.\phi_1)(b.\phi_2) = (a.\phi_1)(b.\phi_1)(a.\phi_2)(b.\phi_2) \\ &= ((a+b).\phi_1)((a+b).\phi_2) = (a+b).(\phi_1\phi_2)\end{aligned}$$

   for all $a, b \in K^{\mathrm{sep}}$. Therefore by lemma 3.3 $\phi_1\phi_2$ is iterative.

2. Let $a \in K$ and $\phi \in \mathrm{ID}_K(R)$. Then for all $b, c \in K^{\mathrm{sep}}$ we have

$$(b.(a.\phi))(c.(a.\phi)) = (ba.\phi)(ca.\phi) = (ba+ca).\phi = (b+c).(a.\phi).$$

   So $a.\phi$ is iterative by lemma 3.3.

3. Let $\phi \in \mathrm{ID}_K(R)$ and $\phi_e \in \mathrm{HD}_K(\tilde{R})$ the unique extension. Then for all $a, b \in K^{\mathrm{sep}}$, $(a+b).\phi_e$ and $(a.\phi_e)(b.\phi_e)$ are both extensions of $(a+b).\phi \in \mathrm{HD}_K(R)$, hence equal. So ϕ_e is iterative.

$\square$

It is easy to see that an element $\omega \in \hat{\Omega}_{R/K}$ equals zero if and only if for all higher derivations $\psi \in \mathrm{HD}_K(R)$ the image $\tilde{\psi}(\omega) \in R[[T]]$ equals zero. But it is not clear – nor even true – if there is a nonzero higher differential $\omega \in \hat{\Omega}_{R/K}$ such that $\tilde{\phi}(\omega) = 0$ for all iterative derivations $\phi \in \mathrm{ID}_K(R)$. We therefore make the following

Definition 3.6 *We say that* R **has enough iterative derivations**, *if for every nonzero* $\omega \in \hat{\Omega}_{R/K}$ *there exists an iterative derivation* $\phi \in \mathrm{ID}_K(R)$ *such that* $\tilde{\phi}(\omega) \neq 0$.

Example 3.7 *If* R/K *is an algebraic function field and* $\#K = \infty$, *then* R *has enough iterative derivations. This will be shown in section 6.*

Definition 3.8 *Let* M *be an* R*-module and* $\phi \in \mathrm{ID}_K(R)$. *A higher* ϕ*-derivation* $\Phi \in \mathrm{HD}_K(M, \phi)$ *is called an* **iterative ϕ-derivation**, *if* $\Phi \star \Phi = \Delta_*(\Phi)$, *where* $\Delta_*(\Phi) = (\Delta \otimes \mathrm{id}_M) \circ \Phi$ *(cf. remark 1.2). The set of iterative* ϕ*-derivations will be denoted by* $\mathbf{ID}_{\boldsymbol{K}}(\boldsymbol{M}, \boldsymbol{\phi})$.

Remark Note that there is no sense in defining an iterative derivation $\Phi \in \mathrm{HD}_K(M, \psi)$ for a non-iterative higher derivation $\psi \in \mathrm{HD}_K(R)$, because $\Phi \star \Phi \in \mathrm{HD}_K(M, \psi \star \psi)$, whereas $\Delta_*(\Phi) \in \mathrm{HD}_K(M, \Delta \circ \psi))$.

Lemma 3.9 *Let* $\phi \in \mathrm{ID}_K(R)$ *be an iterative derivation and* $\Psi \in \mathrm{HD}_K(M, \phi)$ *be a* ϕ*-derivation. Then* Ψ *is iterative, if and only if for all* $a, b \in K^{\mathrm{sep}}$ *the identity* $(a.\Psi)(b.\Psi) = (a + b).\Psi$ *holds.*

Proof Analogous to the proof of lemma 3.3. □

3.2 Iterative Connections

In the previous, we have seen that $\mathrm{d}_{\hat{\Omega}}$ satisfies the condition $\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)} = \binom{i+j}{i} \mathrm{d}_{\hat{\Omega}}^{(i+j)}$ and that for iterative derivations $\phi \in \mathrm{ID}_K(R)$ we have the "same" condition $\phi^{(i)} \circ \phi^{(j)} = \binom{i+j}{i} \phi^{(i+j)}$. This motivates the following definition of an iterative connection.

Definition 3.10 *A higher connection* ∇ *on* M *is called an* **iterative connection** *if for all* $i, j \in \mathbb{N}$ *the identity*

$$
{}_{\hat{\Omega}}\nabla^{(i)} \circ {}_{\hat{\Omega}}\nabla^{(j)} = \binom{i+j}{i} {}_{\hat{\Omega}}\nabla^{(i+j)}
$$

holds.[5]

[5]As defined in appendix A for the general case, ${}_{\hat{\Omega}}\nabla^{(i)}$ denotes that part of ${}_{\hat{\Omega}}\nabla$, that "increases degrees by i".

An iterative connection ∇ on M is called an **integrable iterative connection** *if for all commuting iterative derivations $\phi_1, \phi_2 \in \mathrm{ID}_K(R)$ (i. e. $\phi_1\phi_2 = \phi_2\phi_1$) the iterative derivations ∇_{ϕ_1} and ∇_{ϕ_2} commute.*
A higher connection ∇ on M is called an **involutive higher connection** *if*

$${}_{\hat{\Omega}}\nabla \circ -\nabla = 1 \otimes \mathrm{id}_M$$

as maps from M to $\hat{\Omega} \otimes M$.

In section 4 we will see the role which is played by the modules with involutive higher connections in questions about categorial properties. The notion of an integrable iterative connection is motivated by the correspondence to the integrable (common) connections in characteristic 0 (cf. section 5).

Theorem 3.11 *Let ∇ be a higher connection on M. Then:*

1. *∇ is iterative if and only if for all $a, b \in K^{\mathrm{sep}}$: $a.{}_{\hat{\Omega}}\nabla \circ b.{}_{\hat{\Omega}}\nabla = (a+b).{}_{\hat{\Omega}}\nabla$ and if and only if for all $a, b \in K^{\mathrm{sep}}$: $a.{}_{\hat{\Omega}}\nabla \circ b.\nabla = (a+b).\nabla$.*

2. *If ∇ is iterative, then for all iterative derivations $\phi \in \mathrm{ID}_K(R)$ the ϕ-derivation ∇_ϕ is again iterative. If R has enough iterative derivations then the converse is also true.*

Proof The first equivalence in 1. is seen by a similar calculation as in lemma 3.3. The second one is obtained by

$$\begin{aligned}
\left(a.{}_{\hat{\Omega}}\nabla \circ b.{}_{\hat{\Omega}}\nabla\right)(\omega \otimes m) &= a.\mathrm{d}_{\hat{\Omega}}\left(b.\mathrm{d}_{\hat{\Omega}}(\omega)\right) \cdot \left(a.{}_{\hat{\Omega}}\nabla \circ b.{}_{\hat{\Omega}}\nabla\right)(1 \otimes m) \\
&= (a.\mathrm{d}_{\hat{\Omega}} \circ b.\mathrm{d}_{\hat{\Omega}})(\omega) \cdot \left(a.{}_{\hat{\Omega}}\nabla \circ b.\nabla\right)(m)
\end{aligned}$$

and

$$\begin{aligned}
(a+b).{}_{\hat{\Omega}}\nabla(\omega \otimes m) &= (a+b).\mathrm{d}_{\hat{\Omega}}(\omega) \cdot \left((a+b).{}_{\hat{\Omega}}\nabla\right)(1 \otimes m) \\
&= (a.\mathrm{d}_{\hat{\Omega}} \circ b.\mathrm{d}_{\hat{\Omega}})(\omega) \cdot (a+b).\nabla(m),
\end{aligned}$$

for all $\omega \in \hat{\Omega}$ and $m \in M$.
For proving the second part, let $\phi \in \mathrm{ID}_K(R)$ and regard the following diagram:

$$\begin{tikzcd}
M \arrow[r, "b.\nabla"] \arrow[d, "b.\nabla_\phi"'] & \hat{\Omega} \otimes M \arrow[r, "a.\mathrm{d} \otimes a.\nabla"] \arrow[rr, bend left, "a.{}_{\hat{\Omega}}\nabla"] \arrow[d, "\left((a.\phi)[[T]] \circ \tilde{\phi}\right) \otimes a.\nabla_\phi"'] \arrow[dr, "\left(\tilde{\phi} \circ a.\mathrm{d}_{\hat{\Omega}}\right) \otimes a.\nabla_\phi" description] & \hat{\Omega} \otimes_{a.\mathrm{d}R} (\hat{\Omega} \otimes M) \arrow[r, "\mu \otimes \mathrm{id}_M"] \arrow[d, "\tilde{\phi} \otimes \tilde{\phi} \otimes \mathrm{id}_M"] & \hat{\Omega} \otimes M \arrow[d, "\tilde{\phi} \otimes \mathrm{id}_M"] \\
R[[T]] \otimes M \arrow[r, "(a.\phi)[[T]] \otimes a.\nabla_\phi"] \arrow[rrr, bend right, "(a.\nabla_\phi)[[T]]"'] & R[[T]] \otimes_{a.\phi(R)} M[[T]] \arrow[r, equal] & R[[T]] \otimes_{a.\phi(R)} M[[T]] \arrow[r, "\mu \otimes \mathrm{id}_M"] & R[[T]] \otimes M
\end{tikzcd}$$

The square on the left commutes, since

$$b.\nabla_\phi = b.\left((\tilde{\phi} \otimes \mathrm{id}_M) \circ \nabla\right) = (\tilde{\phi} \otimes \mathrm{id}_M) \circ (b.\nabla).$$

The lower triangle commutes by lemma 3.3, since ϕ is iterative. The upper triangle commutes, since $a.\nabla_\phi = (\tilde{\phi} \otimes \mathrm{id}_M) \circ (a.\nabla)$ and the square on the right commutes, since $\tilde{\phi}$ is a homomorphism of algebras. Furthermore the top of the diagram commutes by definition of $a._{\hat{\Omega}}\nabla$ and the bottom commutes, since $a.\nabla_\phi$ is a $(a.\phi)$-derivation.
So the whole diagram commutes and we obtain

$$(\tilde{\phi} \otimes \mathrm{id}_M) \circ (a._{\hat{\Omega}}\nabla) \circ (b.\nabla) = (a.\nabla_\phi)[[T]] \circ (b.\nabla_\phi) = (a.\nabla_\phi)(b.\nabla_\phi)$$

for all iterative derivations $\phi \in \mathrm{ID}_K(R)$.
If ∇ is iterative, we get

$$(a+b).\nabla_\phi = (\tilde{\phi} \otimes \mathrm{id}_M) \circ (a+b).\nabla = (\tilde{\phi} \otimes \mathrm{id}_M) \circ (a._{\hat{\Omega}}\nabla) \circ (b.\nabla) = (a.\nabla_\phi)(b.\nabla_\phi)$$

by the first part of this theorem and so by lemma 3.9, ∇_ϕ is iterative.
In turn, from the commuting diagram we see that if ∇_ϕ is iterative for an iterative derivation $\phi \in \mathrm{ID}_K(R)$, we get

$$(\tilde{\phi} \otimes \mathrm{id}_M) \circ (a._{\hat{\Omega}}\nabla) \circ (b.\nabla) = (\tilde{\phi} \otimes \mathrm{id}_M) \circ (a+b).\nabla$$

for those ϕ. So if R has enough iterative derivations and ∇_ϕ is iterative for all $\phi \in \mathrm{ID}_K(R)$ we obtain $(a._{\hat{\Omega}}\nabla) \circ (b.\nabla) = (a+b).\nabla$, i. e. ∇ is iterative. $\square$

Corollary 3.12 *Every iterative connection on M is an involutive higher connection.*

Proof If ∇ is iterative, then by the previous theorem, we have

$${}_{\hat{\Omega}}\nabla \circ -\nabla = 1._{\hat{\Omega}}\nabla \circ -1.\nabla = (1-1).\nabla = 0.\nabla = 1 \otimes \mathrm{id}_M.$$

So ∇ is involutive. $\square$

4 Categorial Properties

In this section we show – assuming a slight restriction to the ring R – that the finitely generated projective modules (i.e. locally free of finite rank) with higher connection form an abelian category and that the modules with integrable resp. iterative resp. involutive higher connection form full subcategories. Furthermore these subcategories form tensor categories over K and even form Tannakian categories (see appendix B for the notions of these categories that we use).

Notation From now on let in addition K be a perfect field and R be a regular commutative ring over K, which is the localisation of a finitely generated K-algebra, such that K is algebraically closed in R.

4.1 The Category of Modules with Iterative Connections

Notation In the following a pair (M, ∇) will always denote a finitely generated R-module M together with a higher connection $\nabla : M \to \hat{\Omega} \otimes_R M$, even if "finitely generated" is not mentioned.

Theorem 4.1 *Every finitely generated R-module M with higher connection ∇ is a projective R-module.*

Proof Since R is a finitely generated algebra over a field, R is a Noetherian ring. So every finitely generated R-module M is finitely presented and so by [Eis95], theorem A3.2, M is projective if and only if every localisation $M_\mathfrak{m}$ at a maximal ideal $\mathfrak{m} \trianglelefteq R$ is a free $R_\mathfrak{m}$-module. For $\mathfrak{m} \trianglelefteq R$ maximal, the connection ∇ can be extended to $M_\mathfrak{m}$ by $\nabla(s^{-1}m) = \mathrm{d}_{R_\mathfrak{m}}(s^{-1})\nabla(m)$ for $s \in R \setminus \mathfrak{m}, m \in M$. So $M_\mathfrak{m}$ is a module with higher connection over the local ring $R_\mathfrak{m}$. Since R is regular, $R_\mathfrak{m}$ is a regular local ring. Since R is finitely generated over K, the field $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ is a finite extension of K and since K is perfect, this extension is separable. Therefore we can apply theorem 2.11, i.e. $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module. $\square$

Definition 4.2 *Let (M_1, ∇_1) and (M_2, ∇_2) be R-modules with higher connections. Then we call $f \in \mathrm{Hom}_R(M_1, M_2)$ a* **morphism of modules with higher connections***, or a morphism for short, if the diagram*

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\nabla_1"] & M_2 \arrow[d, "\nabla_2"] \\ \hat{\Omega} \otimes_R M_1 \arrow[r, "\mathrm{id}_{\hat{\Omega}} \otimes f"] & \hat{\Omega} \otimes_R M_2 \end{tikzcd}$$

commutes. The set of all morphisms $f \in \mathrm{Hom}_R(M_1, M_2)$ will be denoted by $\mathbf{Mor}\big((\boldsymbol{M_1}, \boldsymbol{\nabla_1}), (\boldsymbol{M_2}, \boldsymbol{\nabla_2})\big)$. *If the connections are clear from the context we will sometimes omit them.*

Remark It is clear that the set of modules with higher connection and the sets of morphisms defined above form a category. We will denote this category by $\mathbf{HCon}(\boldsymbol{R/K})$. Furthermore the full subcategories of modules with involutive higher connection resp. iterative connection resp. integrable iterative connection will be denoted by $\mathbf{HCon}_{\boldsymbol{inv}}(\boldsymbol{R/K})$ resp. $\mathbf{ICon}(\boldsymbol{R/K})$ resp. $\mathbf{ICon}_{\boldsymbol{int}}(\boldsymbol{R/K})$. By corollary 3.12, we have a chain of inclusions $\mathbf{HCon}(R/K) \supset \mathbf{HCon}_{inv}(R/K) \supset \mathbf{ICon}(R/K) \supset \mathbf{ICon}_{int}(R/K)$. As the objects of $\mathbf{HCon}(R/K)$ are modules with an extra structure and the morphisms are special homomorphisms, we have a faithful functor $\boldsymbol{\omega} : \mathbf{HCon}(R/K) \to \mathbf{Mod}(R)$, that forgets the extra structure.

Definition 4.3 *Let (M_1, ∇_1) and (M_2, ∇_2) be R-modules with higher connection. Then we define a higher connection $\nabla_\oplus$ on $(M_1 \oplus M_2)$ by*

$$\nabla_\oplus : M_1 \oplus M_2 \xrightarrow{\nabla_1 \oplus \nabla_2} \hat{\Omega} \otimes M_1 \oplus \hat{\Omega} \otimes M_2 \xrightarrow{\cong} \hat{\Omega} \otimes (M_1 \oplus M_2)$$

and a higher connection $\nabla_\otimes$ on $M_1 \otimes_R M_2$ by

$$\begin{aligned} \nabla_\otimes : M_1 \otimes_R M_2 \xrightarrow{\nabla_1 \otimes \nabla_2} & \ (\hat{\Omega} \otimes_R M_1) \otimes_{d(R)} (\hat{\Omega} \otimes_R M_2) \xrightarrow{\cong} \\ \xrightarrow{\cong} & \ (\hat{\Omega} \otimes_{d(R)} \hat{\Omega}) \otimes_R (M_1 \otimes_R M_2) \xrightarrow{\mu \otimes \mathrm{id}} \hat{\Omega} \otimes_R (M_1 \otimes_R M_2). \end{aligned}$$

Furthermore we define a higher connection ∇_H on $\mathrm{Hom}_R(M_1, M_2)$ by the following:
For $f \in \mathrm{Hom}_R(M_1, M_2)$ the composition

$$M_1 \xrightarrow{-\nabla_1} \hat{\Omega} \otimes_R M_1 \xrightarrow{\mathrm{id}_{\hat{\Omega}} \otimes f} \hat{\Omega} \otimes_R M_2 \xrightarrow{{}_{\hat{\Omega}}\nabla_2} \hat{\Omega} \otimes_R M_2$$

is an element of $\mathrm{Hom}_R(M_1, \hat{\Omega} \otimes_R M_2)$, which can be regarded as an element of $\hat{\Omega} \otimes_R \mathrm{Hom}_R(M_1, M_2)$ by the canonical isomorphisms $\mathrm{Hom}_R(M_1, \hat{\Omega}^{(k)} \otimes_R M_2) \cong \hat{\Omega}^{(k)} \otimes_R \mathrm{Hom}_R(M_1, M_2)$ in each degree k. In this sense we define

$$\nabla_H(f) := {}_{\hat{\Omega}}\nabla_2 \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ -\nabla_1.$$

Theorem 4.4 *The category $\mathbf{HCon}(R/K)$ is an abelian category and $\mathbf{HCon}_{inv}(R/K)$, $\mathbf{ICon}(R/K)$ and $\mathbf{ICon}_{int}(R/K)$ are abelian subcategories.*

Proof For all $(M_1, \nabla_1), (M_2, \nabla_2) \in \mathbf{HCon}(R/K)$ the set $\mathrm{Mor}(M_1, M_2)$ is a subgroup of $\mathrm{Hom}_R(M_1, M_2)$ and so is an abelian group. Since $\mathbf{Mod}(R)$ is an abelian category, it is sufficiant to show that the kernels, direct sums and so on in the category $\mathbf{Mod}(R)$ can be equipped with a higher connection (resp. iterative connection ...) and that all necessary homomorphisms (like the inclusion map of the kernel into the module) are morphisms.
The trivial module $\{0\}$ with the zero map $0 : \{0\} \to \hat{\Omega} \otimes \{0\} = \{0\}$ as higher connection obviously fullfills the properties of a null object. The direct sum of

M_1 and M_2 together with the higher connection $\nabla_\oplus$ defined above is a biproduct for M_1 and M_2, since the natural inclusions $in_j : M_j \to M_1 \oplus M_2$ and the natural projections $pr_j : M_1 \oplus M_2 \to M_j$ $(j = 1, 2)$ are morphisms, what can easily be verified. Furthermore if ∇_1 and ∇_2 are iterative, integrable iterative or involutive higher connections, then so is $\nabla_\oplus$.

Next we show that kernels and cokernels exist. Let $f \in \mathrm{Mor}(M_1, M_2)$ be a morphism then the image of f is an object of $\mathbf{HCon}(R/K)$, because for all $f(y) \in \mathrm{Im}(f)$, we have $\nabla_2(f(y)) = (\mathrm{id}_{\hat{\Omega}} \otimes f)(\nabla_1(y)) \in \mathrm{Im}(\mathrm{id}_{\hat{\Omega}} \otimes f) = \hat{\Omega} \otimes \mathrm{Im}(f)$, i. e. ∇_2 can be restricted to a higher connection $\nabla_2|_{\mathrm{Im}(f)} : \mathrm{Im}(f) \to \hat{\Omega} \otimes \mathrm{Im}(f)$. So we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] \arrow[d, dashed, "\nabla_1|_{\mathrm{Ker}(f)}"'] & M_1 \arrow[r, "f"] \arrow[d, "\nabla_1"'] & \mathrm{Im}(f) \arrow[r] \arrow[d, "\nabla_2|_{\mathrm{Im}(f)}"'] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\mathrm{id}_{\hat{\Omega}} \otimes f) \arrow[r] & \hat{\Omega} \otimes M_1 \arrow[r, "\mathrm{id}_{\hat{\Omega}} \otimes f"] & \hat{\Omega} \otimes \mathrm{Im}(f) \arrow[r] & 0
\end{tikzcd}$$

But $\mathrm{Im}(f)$ is a projective R-module and therefore flat, so the short exact sequence $0 \to \mathrm{Ker}(f) \to M_1 \to \mathrm{Im}(f) \to 0$ stays exact after tensoring with an arbitrary R-module and so $\mathrm{Ker}(\mathrm{id}_{\hat{\Omega}} \otimes f) = \hat{\Omega} \otimes \mathrm{Ker}(f)$, which shows that $(\mathrm{Ker}(f), \nabla_1|_{\mathrm{Ker}(f)}) \in \mathbf{HCon}(R/K)$ and that the inclusion $\mathrm{Ker}(f) \hookrightarrow M_1$ is a morphism.

Furthermore we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(f) \arrow[r] \arrow[d, "\nabla_2|_{\mathrm{Im}(f)}"'] & M_2 \arrow[r] \arrow[d, "\nabla_2"'] & \mathrm{Coker}(f) \arrow[r] \arrow[d, dashed, "\overline{\nabla_2}"] & 0 \\
0 \arrow[r] & \hat{\Omega} \otimes \mathrm{Im}(f) \arrow[r] & \hat{\Omega} \otimes M_2 \arrow[r] & \hat{\Omega} \otimes \mathrm{Coker}(f) \arrow[r] & 0
\end{tikzcd}$$

(Remind that tensoring is always right exact). So $(\mathrm{Coker}(f), \overline{\nabla_2}) \in \mathbf{HCon}(R/K)$ and the epimorphism $M_2 \to \mathrm{Coker}(f)$ is a morphism. It is clear that the connections $\nabla_1|_{\mathrm{Ker}(f)}$ and $\overline{\nabla_2}$ will be iterative, integrable iterative or involutive higher connections, if ∇_1 and ∇_2 are.

At last, in an abelian category every monomorphism has to be an inclusion map of a kernel and every epimorphism has to be a projection map to a cokernel. But this is fulfilled in $\mathbf{HCon}(R/K)$, because if $f : M_1 \to M_2$ is a monomorphism then M_1 is the kernel of the projection $M_2 \to \mathrm{Coker}(f)$, and if f is an epimorphism then M_2 is the cokernel of the inclusion $\mathrm{Ker}(f) \to M_1$.

Therefore $\mathbf{HCon}(R/K)$, $\mathbf{HCon}_{inv}(R/K)$, $\mathbf{ICon}(R/K)$ and $\mathbf{ICon}_{int}(R/K)$ are abelian categories. □

Now we check, whether these categories are tensor categories over K ($\mathbf{HCon}(R/K)$ won't, whilest the others will). By the last theorem, they are all abelian, and by theorem 4.1, all modules that arise are projective and the category $\mathbf{Proj\text{-}Mod}(R)$ of finitely generated projective R-modules is known to

satisfy all properties apart from being an abelian category.[6]
So we define

- the tensor product of (M_1, ∇_1) and (M_2, ∇_2) by

$$(M_1, \nabla_1) \otimes (M_2, \nabla_2) := (M_1 \otimes_R M_2, \nabla_\otimes)$$

  (this tensor product is obviously associative and commutative),

- the unital object $\mathbf{1} := (R, \mathrm{d}_R)$ $(R \otimes_R M \to M, r \otimes m \mapsto rm$ is easily seen to be a morphism for all $M \in \mathbf{HCon}(R/K))$,

- the dual object to (M, ∇) by

$$(M, \nabla)^* := (M^*, \nabla^*),$$

  where $\nabla^*(f) := \mathrm{d}_{\hat{\Omega}} \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ (-\nabla) \in \mathrm{Hom}_R(M, \hat{\Omega})$[7] for $f \in M^* = \mathrm{Hom}_R(M, R)$ and

- the internal hom object of (M_1, ∇_1) and (M_2, ∇_2) by

$$\underline{\mathrm{Hom}}\left((M_1, \nabla_1), (M_2, \nabla_2)\right) := \left(\mathrm{Hom}_R(M_1, M_2), \nabla_H\right).$$

Furthermore we recognize that every endomorphism in $\mathrm{End}(\mathbf{1})$ is given by the image of $1 \in R$, which has to be constant, as $1 \in K$ is a constant. Since all constants are algebraic over K and K is algebraically closed in R, $\mathrm{End}(\mathbf{1})$ is isomorphic to K.

Lemma 4.5 *For all $(M_1, \nabla_1), (M_2, \nabla_2) \in \mathbf{HCon}(R/K)$ the isomorphism*

$$\iota_{M_1,M_2} : M_1^* \otimes_R M_2 \to \mathrm{Hom}_R(M_1, M_2), f \otimes m \mapsto \{v \mapsto f(v) \cdot m\}$$

is a morphism (and therefore an isomorphism) in $\mathbf{HCon}(R/K)$.

Proof For all $f \otimes m \in M_1^* \otimes_R M_2$ and for all $v \in M_1$, we have

$$\begin{aligned}
\nabla_H(\iota_{M_1,M_2}(f \otimes m))(v) &= \Big({}_{\hat{\Omega}}\nabla_2 \circ \big(\mathrm{id}_{\hat{\Omega}} \otimes \iota_{M_1,M_2}(f \otimes m)\big) \circ (-\nabla_1)\Big)(v) \\
&= {}_{\hat{\Omega}}\nabla_2\big((\mathrm{id}_{\hat{\Omega}} \otimes f)(-\nabla_1(v)) \otimes m\big) \\
&= (\mu \otimes \mathrm{id}_{M_2})\Big(\mathrm{d}_{\hat{\Omega}}\big((\mathrm{id}_{\hat{\Omega}} \otimes f)(-\nabla_1(v)\big) \otimes \nabla_2(m)\Big)
\end{aligned}$$

[6] See appendix B.

[7] Here we used that $\hat{\Omega} \otimes_R R \cong \hat{\Omega}$ and $\mathrm{Hom}_R(M, \hat{\Omega}) \cong \mathrm{Hom}_R(M, R) \otimes_R \hat{\Omega}$ (cf. the definition of ∇_H in definition 4.3).

and

$$
\begin{aligned}
&(\mathrm{id}_{\hat\Omega} \otimes \iota_{M_1,M_2})(\nabla_\otimes(f \otimes m))(v) \\
&\qquad = (\mathrm{id}_{\hat\Omega} \otimes \iota_{M_1,M_2})\Big(\big(\mathrm{d}_{\hat\Omega} \circ (\mathrm{id}_{\hat\Omega} \otimes f) \circ (-\nabla_1)\big) \otimes \nabla_2(m)\Big)(v) \\
&\qquad = \big(\mathrm{d}_{\hat\Omega} \circ (\mathrm{id}_{\hat\Omega} \otimes f) \circ (-\nabla_1)\big)(v) \cdot \nabla_2(m).
\end{aligned}
$$

So $\nabla_H \circ \iota_{M_1,M_2} = (\mathrm{id}_{\hat\Omega} \otimes \iota_{M_1,M_2}) \circ \nabla_\otimes$, i. e. ι_{M_1,M_2} is a morphism. $\square$

Lemma 4.6 *Let $(M, \nabla) \in$ **HCon**(R/K), and let $\varepsilon_M : M \otimes M^* \to R$ and $\delta_M : R \to M^* \otimes M$ be the homomorphisms given in the definition of a tensor category, i. e. $\varepsilon_M(m \otimes f) = f(m)$ and $\delta_M(1) = \iota^{-1}_{M,M}(\mathrm{id}_M)$. Then the following are equivalent:*

(i) ε_M is a morphism.

(ii) ∇ is involutive.

(iii) δ_M is a morphism.

Proof For $m \otimes f \in M \otimes M^*$, we have

$$
\begin{aligned}
&(\mathrm{id}_{\hat\Omega} \otimes \varepsilon_M)(\nabla_\otimes(m \otimes f)) \\
&\quad = (\mathrm{id}_{\hat\Omega} \otimes \varepsilon_M)\Big((\mu \otimes \mathrm{id}) \circ \big(\nabla(m) \otimes (\mathrm{d}_{\hat\Omega} \circ (\mathrm{id}_{\hat\Omega} \otimes f) \circ -\nabla)\big)\Big) \\
&\quad = (\mu \otimes \mathrm{id})\Big(\big(\mathrm{id}_{\hat\Omega} \otimes \big(\mathrm{d}_{\hat\Omega} \circ (\mathrm{id}_{\hat\Omega} \otimes f) \circ -\nabla\big)\big)(\nabla(m))\Big) \\
&\quad = \Big((\mu \otimes \mathrm{id}) \circ (\mathrm{d}_{\hat\Omega} \otimes \mathrm{d}_{\hat\Omega}) \circ (\mathrm{id}_{\hat\Omega \otimes \hat\Omega} \otimes f) \circ (-\mathrm{d}_{\hat\Omega} \otimes -\nabla) \circ \nabla\Big)(m) \\
&\quad = \Big(\mathrm{d}_{\hat\Omega} \circ (\mathrm{id}_{\hat\Omega} \otimes f) \circ (\mu \otimes \mathrm{id}) \circ (-\mathrm{d}_{\hat\Omega} \otimes -\nabla) \circ \nabla\Big)(m) \\
&\quad = \Big(\mathrm{d}_{\hat\Omega} \circ (\mathrm{id}_{\hat\Omega} \otimes f) \circ -_{\hat\Omega}\nabla \circ \nabla\Big)(m)
\end{aligned}
$$

and

$$\mathrm{d}_R(\varepsilon_M(m \otimes f)) = \mathrm{d}_R(f(m)) = (\mathrm{d}_R \circ f)(m).$$

Applying $-\mathrm{d}_{\hat\Omega}$ on both terms shows that ε_M is a morphism if and only if for all $f \in M^*$, $(\mathrm{id}_{\hat\Omega} \otimes f) \circ -_{\hat\Omega}\nabla \circ \nabla = 1 \otimes f \in \mathrm{Hom}(M, \hat\Omega \otimes R)$, i. e. if and only if $-_{\hat\Omega}\nabla \circ \nabla = 1 \otimes \mathrm{id}_M$, i. e. ∇ is involutive.

Since $\iota_{M,M}$ is an isomorphism in **HCon**(R/K), δ_M is a morphism if and only if $\iota_{M,M} \circ \delta_M$ is a morphism. Now

$$\nabla_H((\iota_{M,M} \circ \delta_M)(1)) = \nabla_H(\mathrm{id}_M) = {}_{\hat\Omega}\nabla \circ (\mathrm{id}_{\hat\Omega} \otimes \mathrm{id}_M) \circ -\nabla = {}_{\hat\Omega}\nabla \circ -\nabla$$

and

$$(\mathrm{id}_{\hat\Omega} \otimes (\iota_{M,M} \circ \delta_M))(\mathrm{d}_R(1)) = (\mathrm{id}_{\hat\Omega} \otimes (\iota_{M,M} \circ \delta_M))(1 \otimes 1) = 1 \otimes \mathrm{id}_M,$$

so δ_M is a morphism if and only if ${}_{\hat\Omega}\nabla \circ -\nabla = 1 \otimes \mathrm{id}_M$, i. e. ∇ is involutive. $\square$

Theorem 4.7 $\mathbf{HCon}_{inv}(R/K)$*,* $\mathbf{ICon}(R/K)$ *and* $\mathbf{ICon}_{int}(R/K)$ *are tensor categories over* K*.*

Proof Since we have already shown, that these categories are abelian, that $\mathbf{HCon}(R/K)$ is equipped with an associative and commutative tensor product and that ε_M and δ_M are morphisms if $(M,\nabla)\in\mathbf{HCon}_{inv}(R/K)$, it only remains to show that the three categories are closed under the constructions of tensorising and dualising. The unital object of the tensor product $\mathbf{1}=(R,\mathrm{d}_R)$ is clearly an element of all three categories.

We first show that $\mathbf{ICon}(R/K)$ is closed under these constructions. The proof for $\mathbf{HCon}_{inv}(R/K)$ is then obtained by replacing a by 1 and b by -1, since $0.\nabla=1\otimes\mathrm{id}_M:M\to\hat{\Omega}\otimes M$.

Let $(M_1,\nabla_1),(M_2,\nabla_2)\in\mathbf{ICon}(R/K)$, then for all $a,b\in K^{\mathrm{sep}}$:

$$\begin{aligned}(a._{\hat{\Omega}}\nabla_\otimes)\circ(b.\nabla_\otimes) &= (a._{\hat{\Omega}}\nabla_\otimes)\circ(\mu\otimes\mathrm{id})\circ(b.\nabla_1\otimes b.\nabla_2)\\ &= (\mu\otimes\mathrm{id})\circ\big((a.\mathrm{d}_{\hat{\Omega}}\otimes a.\nabla_1)\otimes(a.\mathrm{d}_{\hat{\Omega}}\otimes a.\nabla_2)\big)\circ(b.\nabla_1\otimes b.\nabla_2)\\ &= (\mu\otimes\mathrm{id})\circ\big((a+b).\nabla_1\otimes(a+b).\nabla_2\big)\\ &= (a+b).\nabla_\otimes.\end{aligned}$$

So $\nabla_\otimes$ is again iterative and $(M_1\otimes M_2,\nabla_\otimes)\in\mathbf{ICon}(R/K)$.

If $(M,\nabla)\in\mathbf{ICon}(R/K)$, then also $(M^*,\nabla^*)\in\mathbf{ICon}(R/K)$, because for all $a,b\in K^{\mathrm{sep}}, f\in M^*$:

$$\begin{aligned}a._{\hat{\Omega}}\nabla^*\big(b.\nabla^*(f)\big) &= \mu\circ a.(\mathrm{d}_{\hat{\Omega}}\otimes\mathrm{d}_{\hat{\Omega}})\circ\Big(\mathrm{id}_{\hat{\Omega}}\otimes\big(b.\mathrm{d}_{\hat{\Omega}}\circ(\mathrm{id}_{\hat{\Omega}}\otimes f)\circ(-b.\nabla)\big)\Big)\circ(-a.\nabla)\\ &= \mu\circ a.(\mathrm{d}_{\hat{\Omega}}\otimes\mathrm{d}_{\hat{\Omega}})\circ b.(\mathrm{d}_{\hat{\Omega}}\otimes\mathrm{d}_{\hat{\Omega}})\circ(\mathrm{id}_{\hat{\Omega}\otimes\hat{\Omega}}\otimes f)\circ -b.(\mathrm{d}_{\hat{\Omega}}\otimes\nabla)\circ -a.\nabla\\ &= a.\mathrm{d}_{\hat{\Omega}}\circ b.\mathrm{d}_{\hat{\Omega}}\circ(\mathrm{id}_{\hat{\Omega}}\otimes f)\circ(\mu\otimes\mathrm{id}_M)\circ -b.(\mathrm{d}_{\hat{\Omega}}\otimes\nabla)\circ -a.\nabla\\ &= (a+b).\mathrm{d}_{\hat{\Omega}}\circ(\mathrm{id}_{\hat{\Omega}}\otimes f)\circ -(a+b).\nabla\\ &= (a+b).\nabla^*(f).\end{aligned}$$

Therefore $\mathbf{ICon}(R/K)$ is a tensor category over K.

For higher connections ∇_1 and ∇_2 and $\phi\in\mathrm{ID}_K(R)$, we have

$$\begin{aligned}(\nabla_\otimes)_\phi &= (\tilde{\phi}\otimes\mathrm{id})\circ(\mu\otimes\mathrm{id})\circ(\nabla_1\otimes\nabla_2)\\ &= (\mu\otimes\mathrm{id})\circ\big((\tilde{\phi}\otimes\mathrm{id}_{M_1})\otimes(\tilde{\phi}\otimes\mathrm{id}_{M_2})\big)\circ(\nabla_1\otimes\nabla_2)\\ &= (\mu\otimes\mathrm{id})\circ\big((\nabla_1)_\phi\otimes(\nabla_2)_\phi\big),\end{aligned}$$

from which it follows immediately that for integrable iterative connections ∇_1 and ∇_2, the iterative connection $\nabla_\otimes$ is integrable, too.

Finally, let $(M, \nabla) \in \mathbf{ICon}_{int}(R/K)$. Then for $\phi \in \mathrm{ID}_K(R)$ and all $f \in M^*$, we obtain

$$\begin{aligned}
\nabla^*_\phi(f) &= \tilde{\phi} \circ \mu \circ (\mathrm{d}_{\hat{\Omega}} \otimes \mathrm{d}_R) \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ -\nabla \\
&= \mu \circ (\tilde{\phi} \otimes \tilde{\phi}) \circ (\mathrm{d}_{\hat{\Omega}} \otimes \mathrm{d}_R) \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ -\nabla \\
&= \mu \circ \big((\tilde{\phi} \circ \mathrm{d}_{\hat{\Omega}}) \otimes (\tilde{\phi} \circ \mathrm{d}_R)\big) \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ -\nabla \\
&= \mu \circ \big((\phi[[T]] \circ \tilde{\phi}) \otimes \phi\big) \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ -\nabla \\
&= \mu \circ (\phi[[T]] \otimes \phi) \circ (\mathrm{id}_{R[[T]]} \otimes f) \circ (\tilde{\phi} \otimes \mathrm{id}_M) \circ -\nabla \\
&= \phi[[T]] \circ \mu \circ (\mathrm{id}_{R[[T]]} \otimes f) \circ -\nabla_\phi \\
&= \phi[[T]] \circ f[[T]] \circ -\nabla_\phi \ .
\end{aligned}$$

And so for commuting $\phi_1, \phi_2 \in \mathrm{ID}_K(R)$ and all $f \in M^*$:

$$\begin{aligned}
(\nabla^*_{\phi_1} \nabla^*_{\phi_2})(f) &= \phi_1[[T]] \circ \big(\phi_2[[T]] \circ f[[T]] \circ -\nabla_{\phi_2}\big)[[T]] \circ -\nabla_{\phi_1} \\
&= (\phi_1\phi_2)[[T]] \circ f[[T]] \circ -(\nabla_{\phi_2}\nabla_{\phi_1}) \\
&= (\phi_2\phi_1)[[T]] \circ f[[T]] \circ -(\nabla_{\phi_1}\nabla_{\phi_2}) = (\nabla^*_{\phi_2} \nabla^*_{\phi_1})(f).
\end{aligned}$$

Hence ∇^* also is an integrable iterative connection and therefore $\mathbf{ICon}_{int}(R/K)$ is a tensor category over K. □

Theorem 4.8 *The categories $\mathbf{HCon}_{inv}(R/K)$, $\mathbf{ICon}(R/K)$ and $\mathbf{ICon}_{int}(R/K)$ are Tannakian categories with the forgetful functor $\boldsymbol{\omega} : \mathbf{HCon}(R/K) \to \mathbf{Mod}(R)$ (restricted to the respective category) as fibre functor. If moreover R has a K-rational point, i. e. there exists a maximal ideal $\mathfrak{m} \trianglelefteq R$ with $K \cong R/\mathfrak{m}$, then these categories are neutral Tannakian categories with fibre functor $\boldsymbol{\omega}_K$: $\mathbf{HCon}(R/K) \xrightarrow{\boldsymbol{\omega}} \mathbf{Mod}(R) \xrightarrow{\otimes_R R/\mathfrak{m}} \mathbf{Vect}(K)$.*

Proof By construction, the functor $\boldsymbol{\omega}$ is a fibre functor and so the tensor categories $\mathbf{HCon}_{inv}(R/K)$, $\mathbf{ICon}(R/K)$ and $\mathbf{ICon}_{int}(R/K)$ are Tannakian categories. If R has a K-rational point, by [Del90].2.8, $\boldsymbol{\omega}_K$ is a fibre functor. This proves the second part. □

Remark One might ask whether the inclusions in the chain of categories $\mathbf{HCon}(R/K) \supset \mathbf{HCon}_{inv}(R/K) \supset \mathbf{ICon}(R/K) \supset \mathbf{ICon}_{int}(R/K)$ are strict or not.
Clearly, $\mathbf{HCon}(R/K) \neq \mathbf{HCon}_{inv}(R/K)$, because if for example M is a free R-module of dimension 1 with basis $b_1 \in M$, every $\omega = \sum_{j=0}^\infty \omega_j \in \hat{\Omega}_{R/K}$ with $\omega_0 = 1$ defines a higher connection $\nabla : M \to \hat{\Omega}_{R/K} \otimes_R M, b_1 \mapsto \omega \otimes b_1$, but in general this higher connection is not involutive, because if ∇ is involutive, ω satisfies the condition

$$0 = \left(-_{\hat{\Omega}}\nabla \circ \nabla\right)^{(2)}(b_1) = (2\omega_2 - \omega_1^2 + \mathrm{d}^{(1)}_{\hat{\Omega}}(\omega_1)) \otimes b_1.$$

(The only exception is the case, when R is algebraic over K, because in this case $\hat{\Omega}_{R/K} = R$ and hence all these categories are equivalent to $\mathbf{Mod}(R)$).

The last inclusion $\mathbf{ICon}(R/K) \supset \mathbf{ICon}_{int}(R/K)$ is strict in general, because in the next chapter we will see that in characteristic zero, the category $\mathbf{ICon}(R/K)$ is equivalent to the category of modules with (common) connection over R and $\mathbf{ICon}_{int}(R/K)$ is equivalent to the category of modules with integrable connection over R, and it is known that those two categories are different if for example $R = K(t_1, t_2)$. However, it is also known that every (common) connection is integrable, if $\mathrm{char}(K) = 0$ and R is an algebraic function field in one variable over K. In chapter 6, we will see that also $\mathbf{ICon}(R/K) = \mathbf{ICon}_{int}(R/K)$, if R is an algebraic function field (in one variable) over K and $\mathrm{char}(K) = p$.

It is yet not clear, if there exists a module with an involutive higher connection that is not iterative. However, if one regards the condition for an involutive higher connection more explicitly, there seems to be more choice for getting an involutive higher connection than for an iterative connection. We therefore make the following conjecture.

Conjecture If R is not algebraic over K, then there exist R-modules with involutive higher connection that are not iterative, i. e.

$$\mathbf{ICon}(R/K) \subsetneq \mathbf{HCon}_{inv}(R/K).$$

4.2 Higher Connections on Schemes

Throughout this section, let K be a perfect field, let X be a nonsingular, geometrically integral K-scheme, which is separated and of finite type over K and let $\mathcal{O}_X$ denote the structure sheaf of X.

Definition 4.9 *We define the* **sheaf of higher differentials on** X*, denoted by* $\hat{\boldsymbol{\Omega}}_{\boldsymbol{X/K}}$*, to be the sheaf associated to the presheaf given by*

$$U \mapsto \hat{\Omega}_{\mathcal{O}_X(U)/K}$$

for each open subset $U \subseteq X$ *and by the restriction maps*

$$D(\rho_V^U) : \hat{\Omega}_{\mathcal{O}_X(U)/K} \to \hat{\Omega}_{\mathcal{O}_X(V)/K}$$

for all open subsets $V \subseteq U \subseteq X$*, as defined in proposition 2.2, where* $\rho_V^U : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ *is the restriction map of* $\mathcal{O}_X$*.*

Remark By proposition 2.2, for all open subsets $V \subseteq U \subseteq X$, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\mathrm{d}_{\mathcal{O}_X(U)}} & \hat{\Omega}_{\mathcal{O}_X(U)/K} \\ \Big\downarrow{\scriptstyle \rho_V^U} & & \Big\downarrow{\scriptstyle D(\rho_V^U)} \\ \mathcal{O}_X(V) & \xrightarrow{\mathrm{d}_{\mathcal{O}_X(V)}} & \hat{\Omega}_{\mathcal{O}_X(V)/K} \end{array}$$

commutes and so the collection of maps $\mathrm{d}_{\mathcal{O}_X(U)}$ induces a morphism of sheaves of K-algebras $\mathrm{d}_X : \mathcal{O}_X \to \hat{\Omega}_{X/K}$.

Proposition 4.10 *If* X *is an affine scheme, then the presheaf* $U \mapsto \hat{\Omega}_{\mathcal{O}_X(U)/K}$ *already is a sheaf.*

Proof The given presheaf is a sheaf if and only if for all open subsets $U \subseteq X$ and all open coverings $\bigcup_{i\in I} U_i = U$, the sequence

$$0 \to \hat{\Omega}_{\mathcal{O}_X(U)/K} \to \prod_{i\in I} \hat{\Omega}_{\mathcal{O}_X(U_i)/K} \to \prod_{i,j\in I} \hat{\Omega}_{\mathcal{O}_X(U_i \cap U_j)/K}$$

is exact. Since this is a sequence of cgas, it suffices to show that the sequence is exact in each homogeneous component.
For every open subset $V \subseteq U$, $\mathcal{O}_X(V)$ is a localisation of $\mathcal{O}_X(U)$ and so by proposition 2.2, $\hat{\Omega}_{\mathcal{O}_X(V)/K} \cong \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \hat{\Omega}_{\mathcal{O}_X(U)/K}$. By corollary 2.4, the homogeneous components $(\hat{\Omega}_{\mathcal{O}_X(U)/K})_k$ ($k \in \mathbb{N}$) are projective $\mathcal{O}_X(U)$-modules and therefore tensoring with $(\hat{\Omega}_{\mathcal{O}_X(U)/K})_k$ is exact. So the sequence above is exact in each homogeneous component, if the sequence

$$0 \to \mathcal{O}_X(U) \to \prod_{i\in I} \mathcal{O}_X(U_i) \to \prod_{i,j\in I} \mathcal{O}_X(U_i \cap U_j)$$

is exact. But this is just the condition on $\mathcal{O}_X$ for being a sheaf. $\square$

As an immediate consequence of this proposition, we have the following corollary:

Corollary 4.11 *For every affine open subset $U \subseteq X$, we have $\hat{\Omega}_{X/K}(U) = \hat{\Omega}_{\mathcal{O}_X(U)/K}$.*

Definition 4.12 *Let M be a coherent $\mathcal{O}_X$-module. A* **higher connection on** *M is a morphism of sheaves $\nabla : M \to \hat{\Omega}_{X/K} \otimes_{\mathcal{O}_X} M$, which locally (i. e. on affine open subsets) is a higher connection in the sense of section 2.3. The higher connection ∇ is called* **involutive resp. iterative resp. integrable iterative** *if ∇ locally is an involutive higher resp. iterative resp. integrable iterative connection.*

Remark By theorem 4.1, every coherent $\mathcal{O}_X$-module M, that admits a higher connection $\nabla : M \to \hat{\Omega}_{X/K} \otimes_{\mathcal{O}_X} M$, is locally free and of finite rank.

Remark Following the notion of modules with higher connections over rings, we denote by $\mathbf{HCon}(X/K)$, $\mathbf{HCon}_{inv}(X/K)$, $\mathbf{ICon}(X/K)$ and $\mathbf{ICon}_{int}(X/K)$ the categories of coherent $\mathcal{O}_X$-modules with higher connections, with involutive higher connections, with iterative connections and with integrable iterative connections. By standard methods of algebraic geometry, one obtains that again $\mathbf{HCon}_{inv}(X/K)$, $\mathbf{ICon}(X/K)$ and $\mathbf{ICon}_{int}(X/K)$ are tensor categories over K and that they are Tannakian categories. And if X has a K-rational point, they are in fact neutral Tannakian categories over K.

Remark In the second part of this work, coherent modules with higher connections will occur from another point of view:
Let F/K be a field of finite transcendence degree over K, and let X be a nonsingular irreducible projective scheme over K with function field $K(X) = F$. Since $\mathcal{O}_X$ is a subsheaf of the constant sheaf $K(X)$, for every coherent $\mathcal{O}_X$-module $\tilde{M}$ with higher connection ∇, $F \otimes_{\mathcal{O}_X} \tilde{M}$ is an F-vector space with higher connection

$$\mathrm{d}_F \otimes \nabla : F \otimes_{\mathcal{O}_X} \tilde{M} \to \hat{\Omega}_{F/K} \otimes_F (F \otimes_{\mathcal{O}_X} \tilde{M}).$$

On the other hand, let M be an F-vector space with higher connection $\nabla : M \to \hat{\Omega}_{F/K} \otimes_F M$ and let $U \subseteq X$ be an open subset. If there exists a generating set $\{b_1, \dots, b_r\}$ for M such that for all $i = 1, \dots, r$, we have $\nabla(b_i) = \sum_{j=1}^{r} \omega_{ji} \otimes b_j$ with $\omega_{ji} \in \hat{\Omega}_{X/K}(U) \subset \hat{\Omega}_{F/K}$, then $\tilde{M} := \mathcal{O}_U b_1 + \dots + \mathcal{O}_U b_r \subset M$ is a coherent $\mathcal{O}_U$-module with higher connection $\nabla|_{\tilde{M}} : \tilde{M} \to \hat{\Omega}_{U/K} \otimes_{\mathcal{O}_U} \tilde{M}$ and the pair (M, ∇) is recovered from $(\tilde{M}, \nabla|_{\tilde{M}})$ in the way given above. In this case, we will call (M, ∇) regular on U.

5 Correspondence to the Classical Theory in Characteristic Zero

For $\mathrm{char}(K) = 0$, in general one gets the usual constructions of derivations, differentials and connections by restricting to the terms of degree 1. On the other hand these constructions can be uniquely extended to iterative derivations and iterative connections. Moreover integral connections, i. e. connections which preserve commutators of derivations, are corresponding to integrable iterative connections. This will be proven in this chapter.
So throughout this chapter, let K be a field of characteristic zero, R a regular ring, that is finitely generated as a K-algebra, and M a finitely generated R-module.

Proposition 5.1 *The map*

$$\mathrm{Der}(R/K) \longrightarrow \mathrm{ID}_K(R), \partial \mapsto \phi_\partial,$$

given by

$$\phi_\partial(r) := \sum_{n=0}^{\infty} \frac{1}{n!}\partial^n(r)T^n$$

for all $r \in R$, is a bijection and the inverse map is given by $\phi \mapsto \phi^{(1)}$.
For a given derivation ∂ on R and a corresponding iterative derivation ϕ_∂ the map $\mathrm{I} : \mathrm{Der}_R(M) \to \mathrm{ID}_K(M, \phi_\partial), \partial_M \mapsto \Phi_{\partial_M}$ *given by*

$$\Phi_{\partial_M}(m) := \sum_{n=0}^{\infty} \frac{1}{n!}\partial_M^n(m)T^n,$$

for all $m \in M$, is a bijection and the inverse map is given by $\Phi \mapsto \Phi^{(1)}$.

Proof Let $\partial \in \mathrm{Der}(R/K)$ be a derivation. Then for all $i, j \in \mathbb{N}$: $\frac{1}{i!}\partial^i \circ \frac{1}{j!}\partial^j = \binom{i+j}{i}\frac{1}{(i+j)!}\partial^{i+j}$. So ϕ_∂ is an iterative derivation. On the other hand, for every iterative derivation ϕ, one obtains $\phi^{(k)} = \frac{1}{k!}(\phi^{(1)})^k$ for all $k \in \mathbb{N}$ by applying the formula $\phi^{(i)} = \frac{1}{i}\phi^{(1)} \circ \phi^{(i-1)}$ inductively. Finally by proposition 1.2, for all $r, s \in R$ we have $\phi^{(1)}(rs) = r\phi^{(1)}(s) + \phi^{(1)}(r)s$, i. e. $\phi^{(1)} \in \mathrm{Der}(R/K)$.
The bijection $\mathrm{I} : \mathrm{Der}_R(M) \to \mathrm{ID}_K(M, \phi_\partial)$ is shown analogously. □

Proposition 5.2 *The R-module $(\hat{\Omega}_{R/K})_1$ is canonically isomorphic to the module of (usual) differentials $\Omega_{R/K}$ and $\mathrm{d}^{(1)} : R \to (\hat{\Omega}_{R/K})_1 \cong \Omega_{R/K}$ is the universal derivation.*

Proof The contruction of $(\hat{\Omega}_{R/K})_1$ in the proof of theorem 2.1 is the same as the usual construction of $\Omega_{R/K}$ (e.g. in [Hart77],II.8). □

Proposition 5.3 *For every iterative connection ∇ on M, the map $\nabla^{(1)}: M \to (\hat{\Omega}_{R/K})_1 \otimes M \cong \Omega_{R/K} \otimes M$ is a connection on M and every connection $\nabla^{(1)}$ on M extends uniquely to an iterative connection on M. Furthermore, ∇ is an integrable iterative connection if and only if $\nabla^{(1)}$ is an integrable connection.*

Proof Let ∇ be an iterative connection on M. Then for all $r \in R$ and $m \in M$, we have $\nabla^{(1)}(rm) = \mathrm{d}^{(1)}(r) \otimes m + r\nabla^{(1)}(m)$. So $\nabla^{(1)}$ is a connection. On the other hand, for a given connection $\nabla^{(1)}$ by the formula $\nabla^{(k)} = \frac{1}{k}\left({}_{\hat{\Omega}}\nabla^{(1)} \circ \nabla^{(k-1)}\right)$, one can inductively calculate maps $\nabla^{(k)} : M \to \hat{\Omega}_k \otimes M$ for all $k \in \mathbb{N}$, which build up an iterative connection $\nabla = \sum_{k=0}^{\infty} \nabla^{(k)}$ (same calculation as in proposition 5.1).
For proving the equivalence of the integrability conditions, remind that $\mathrm{Der}(R/K)$ is a free R-module and has a basis of commuting derivations (see [Hart77]). So $\nabla^{(1)}$ is integrable if and only if for all commuting derivations $\partial_1, \partial_2 \in \mathrm{Der}(R/K)$, we have $[(\nabla^{(1)})_{\partial_1}, (\nabla^{(1)})_{\partial_2}] = (\nabla^{(1)})_{[\partial_1,\partial_2]} = 0$, i. e. if for all $\partial_1, \partial_2 \in \mathrm{Der}(R/K)$ with $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$, the identity $(\nabla^{(1)})_{\partial_1} \circ (\nabla^{(1)})_{\partial_2} = (\nabla^{(1)})_{\partial_2} \circ (\nabla^{(1)})_{\partial_1}$ holds. Using the bijection in proposition 5.1, this is equivalent to the condition that for all commuting iterative derivations $\phi_{\partial_1}, \phi_{\partial_2} \in \mathrm{ID}_K(R)$ the iterative derivations $\nabla_{\phi_{\partial_1}}$ and $\nabla_{\phi_{\partial_2}}$ commute, because $\left(\nabla_{\phi_{\partial_1}}\right)^{(1)} = \left(\nabla^{(1)}\right)_{\partial_1}$. $\square$

Theorem 5.4 *The category $\mathbf{ICon}_{int}(R/K)$ of finitely generated R-modules with integrable iterative connection and the category $\mathbf{DE}(R/K)$ of finitely generated R-modules with integrable connection are equivalent.*

Proof This follows directly from the previous propositions. $\square$

6 Positive Characteristic

In this section, we regard the case that K has positive characteristic p. Contrary to characteristic zero, iterative derivations and iterative connections are not longer determined by the term of degree 1. Moreover, not every derivation $\partial \in \mathrm{Der}(R/K)$ can be extended to an iterative derivation ϕ with $\phi^{(1)} = \partial$, because the conditions on an iterative derivation imply $(\phi^{(1)})^p = p! \cdot \phi^{(p)} = 0$, i. e. at least ∂ has to be nilpotent.

But there are some other structural properties: The main result is that every module with integrable iterative connection gives rise to a projective system and vice versa, similar to the correspondence of projective systems and iterative differential modules over function fields given in [Mat01], Ch.2.2. In fact, when R is an algebraic function field, the projective systems defined here are equal to those defined by Matzat and so this shows that in this case the categories $\mathbf{ICon}(R/K)$, $\mathbf{ICon}_{int}(R/K)$, $\mathbf{Proj}_R$ and $\mathbf{ID}_R$[8] are equivalent.

For convenience, we will restrict to the case of fields over K, although this correspondence is true more generally.

In positive characteristic p, every finitely generated K-algebra (or localisation of a finitely generated K-algebra) R has a natural sequence of K-subalgebras $(R_l)_{l\in\mathbb{N}}$ given by $R_l := R^{p^l}$.[9] The following proposition gives a characterisation of this sequence by the higher differential:

Proposition 6.1 (Frobenius Compatibility) *For all $l \in \mathbb{N}$:*

$$R_l = \bigcap_{0<j<p^l} \mathrm{Ker}(\mathrm{d}_R^{(j)}).$$

Proof Since d_R is a homomorphism of algebras, $\mathrm{d}_R(R_l) = \mathrm{d}_R(R^{p^l}) \subset (\hat{\Omega}_{R/K})^{p^l}$ and therefore $\mathrm{d}_R^{(j)}(r) = 0$ $(0 < j < p^l)$ for all $r \in R_l$. The other inclusion is shown inductively: The case $l = 0$ is trivial. Now let $r \in R$ satisfy $\mathrm{d}_R^{(j)}(r) = 0$ for $0 < j < p^l$. By induction hypothesis $r \in R_{l-1}$. So there exists $t \in R$ with $t^{p^{l-1}} = r$. If $t \notin R^p$, then t is a separable element of R and we can find separating variables $t = t_1, t_2, \dots, t_m$ for R, i. e. $R/K[t_1, \dots, t_m]$ is a finite separable extension (or R is the localisation of a finite separable extension of $K[t_1, \dots, t_m]$). By localising and applying theorem 2.3 and proposition 2.2(b), we see that $\mathrm{d}_R^{(1)}(t) \neq 0$. And so

$$0 \neq \left(\mathrm{d}_R^{(1)}(t)\right)^{p^{l-1}} = \mathrm{d}_R^{(p^{l-1})}\left(t^{p^{l-1}}\right) = \mathrm{d}_R^{(p^{l-1})}(r),$$

[8] $\mathbf{Proj}_R$ denotes the category of projective systems over R and $\mathbf{ID}_R$ denotes the category of ID-modules (cf. [Mat01])

[9] Remember that we assumed K to be perfect and therefore $K^p = K$.

which is a contradiction. So $t \in R^p$ and $r \in R_l$. $\square$

Remark Since we are in positive characteristic, we have a Frobenius map on every ring: $\mathbf{F}_R : R \to R, r \mapsto r^p$.
The components $\mathrm{d}_R^{(k)}$ of d_R fulfill some kind of compatibility with the Frobenius maps $\mathbf{F}_R$ resp. $\mathbf{F}_{\hat{\Omega}}$, namely for all $k \in \mathbb{N}$,

$$\mathrm{d}_R^{(pk)} \circ \mathbf{F}_R = \mathbf{F}_{\hat{\Omega}} \circ \mathrm{d}_R^{(k)}.$$

(This follows directly from the fact, that d_R is a homomorphism of rings and that $\mathbf{F}$ multiplies degrees by p.) The proposition above then implies that an element $r \in R$ lies in the image of $\mathbf{F}_R$ if and only if it lies in the kernel of $\mathrm{d}_R^{(1)}$.

In the case of R being an algebraic function field in one variable, it was shown by F. K. Schmidt (see [Mat01], ch. 1.5) that for an iterative derivation $\phi \in \mathrm{ID}_K(R)$ satisfying $\phi^{(1)} \neq 0$, we have $R^{p^l} = \bigcap_{0<j<p^l} \mathrm{Ker}(\phi^{(j)})$.
So in this case we obtain the same sequence of subalgebras, when "only" regarding an iterative derivation instead of the universal derivation. This will be important in part II.

From now on, let K be a perfect field of characteristic $p > 0$ and F/K be a finitely generated field extension of transcendence degree m. Furthermore denote by $t_1, \dots, t_m$ a separable transcendence basis for F, i. e. F is a separable algebraic extension of the rational function field $K(t_1, \dots, t_m)$.

Definition 6.2 *A* **projective system over** F *is a sequence* $(M_l, \varphi_l)_{l\in\mathbb{N}}$ *with the following properties*

1. *For all* $l \in \mathbb{N}$, M_l *is an* F_l*-vector space of finite dimension.*
2. $\varphi_l : M_{l+1} \hookrightarrow M_l$ *is a monomorphism of* F_{l+1}*-vector spaces that uniquely extends to an isomorphism* $\mathrm{id}_{F_l} \otimes \varphi_l : F_l \otimes_{F_{l+1}} M_{l+1} \to M_l$.

A **morphism** $\boldsymbol{\alpha : (M_l, \varphi_l) \to (M'_l, \varphi'_l)}$ **of projective systems over** F *is a sequence* $\alpha = (\alpha_l)_{l\in\mathbb{N}}$ *of homomorphisms of vector spaces* $\alpha_l : M_l \to M'_l$ *satisfying* $\varphi'_l \circ \alpha_{l+1} = \alpha_l \circ \varphi_l$.

Proposition 6.3 *Every projective system* $(M_l, \varphi_l)_{l\in\mathbb{N}}$ *over* F *defines an integrable iterative connection* ∇ *on* $M := M_0$ *satisfying*

$$\bigcap_{0<j<p^l} \mathrm{Ker}(\nabla^{(j)}) = (\varphi_0 \circ \cdots \circ \varphi_{l-1})\,(M_l).$$

For a morphism $(\alpha_l)_{l\in\mathbb{N}} : (M_l, \varphi_l) \to (M'_l, \varphi'_l)$ of projective systems over F, the homomorphism of F-vector spaces $\alpha_0 : M = M_0 \to M' = M'_0$ is a morphism of modules with higher connection.

Proof (cf. [Mat01],2.8) By identifying M_l with its image $\varphi_0 \circ \cdots \circ \varphi_{l-1}(M_l)$ in M, we may assume that $M_l \subset M$ for all $l \in \mathbb{N}$. In order to define $\nabla^{(k)}$, choose $l \in \mathbb{N}$ such that $p^l > k$ and let $\{b_1, \dots, b_n\}$ be an F_l-basis for M_l. Then by the second property of a projective system, $\{b_1, \dots, b_n\}$ is an F-basis for M, so for all $v \in M$ we can find coefficients $a_i \in F$ such that $v = \sum_{i=1}^n a_i b_i$. Then define

$$\nabla^{(k)}(v) := \sum_{i=1}^n \mathrm{d}_F^{(k)}(a_i) b_i.$$

This definition is independent of the chosen basis, because given another F_l-basis $\{b'_1, \dots, b'_n\}$ for M_l, the base change matrix $C = (c_{ij})$ has coefficients in F_l and therefore

$$\begin{aligned}
\nabla'^{(k)}(v) &= \nabla'^{(k)}\left(\sum_{j=1}^n a_j b_j\right) = \nabla'^{(k)}\left(\sum_{i=1}^n \sum_{j=1}^n c_{ij} a_j b'_i\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \mathrm{d}_F^{(k)}(c_{ij} a_j) b'_i = \sum_{i=1}^n \sum_{j=1}^n c_{ij} \mathrm{d}_F^{(k)}(a_j) b'_i \\
&= \sum_{j=1}^n \mathrm{d}_F^{(k)}(a_j) b_j = \nabla^{(k)}(v).
\end{aligned}$$

The definition is also independent of the chosen l, because for $j > l$ every F_j-basis of M_j is also an F_l-basis for M_l.

Furthermore, by choosing an F_l-basis $\{b_1, \dots, b_n\}$ of M_l, one sees that an element $v = \sum_{i=0}^n a_i b_i \in M$ is in $\bigcap_{0<j<p^l} \mathrm{Ker}(\nabla^{(j)})$ if and only if $a_i \in \bigcap_{0<j<p^l} \mathrm{Ker}(\mathrm{d}_F^{(j)})$ for all i, i.e. if and only if $v \in M_l$.

If remains to show that ∇ is an integrable iterative connection. But by choosing an F_l-basis $\{b_1, \dots, b_n\}$ of M_l, one sees that the necessary conditions are fulfilled modulo degrees $\geq p^l$, since d_F is an integrable iterative connection. As l can be chosen arbitrary large, ∇ fulfills all conditions for being an integrable iterative connection.

Finally, let $(\alpha_l)_{l\in\mathbb{N}} : (M_l, \varphi_l) \to (M'_l, \varphi'_l)$ be a morphism of projective systems over F. We have to show, that $\nabla' \circ \alpha_0 = (\mathrm{id}_{\hat{\Omega}} \otimes \alpha_0) \circ \nabla$ or equivalently that for all $k \in \mathbb{N}$

$$\nabla'^{(k)} \circ \alpha_0(b_i) = (\mathrm{id}_{\hat{\Omega}} \otimes \alpha_0) \circ \nabla^{(k)}(b_i) \quad (i = 1, \dots, n),$$

where $\{b_1, \dots, b_n\}$ denotes an F-basis of M. But the last condition is seen easily by choosing $\{b_1, \dots, b_n\}$ to be an F_l-basis of M_l ($p^l > k$) and by reminding that

$\alpha_0(M_l) = \alpha_l(M_l) \subset M'_l$. $\square$

In the following, we will show that the converse is also true, i.e. a module with integrable iterative connection gives rise to a projective system over F. For this, we consider a monomial ordering on $\hat{\Omega}_{F/K} = F[[\mathrm{d}^{(i)}t_j]]$, namely the lexicographical order, where the variables are ordered by $\mathrm{d}^{(i_1)}t_{j_1} > \mathrm{d}^{(i_2)}t_{j_2}$ if $i_1 > i_2$ or if $i_1 = i_2$ and $j_1 > j_2$. The leading term of $\omega \in \hat{\Omega}_{F/K}$ (if it exists) is then denoted by $\mathrm{LT}(\omega)$.

Lemma 6.4 *Let $\omega \in \hat{\Omega}$ be homogeneous of degree p^l and $\omega \notin F\hat{\Omega}^{p^l}$. Let $\mathrm{d}^{(i_0)}t_{j_0}$ be the greatest variable with the property that there exist $e_0 \in \mathbb{N}$, $p \nmid e_0$ and a monomial $\omega' \in \hat{\Omega}$ such that $(\mathrm{d}^{(i_0)}t_{j_0})^{e_0}\omega'$ is a monomial term of ω. Let e_0 and ω' be chosen such that $(\mathrm{d}^{(i_0)}t_{j_0})^{e_0}\omega'$ is maximal amoung those monomials. Then for every $k \leq p^l(p-1)$, we have:*

$$\mathrm{LT}(\mathrm{d}_{\hat{\Omega}}^{(k)}(\omega)) \leq e_0 \mathrm{d}^{(i_0+p^l(p-1))}t_{j_0} \cdot (\mathrm{d}^{(i_0)}t_{j_0})^{e_0-1}\omega',$$

with equality if and only if $k = p^l(p-1)$ and $i_0 < p^l$.

Proof For $i \in \mathbb{N}$, $j \in \{1, \dots, m\}$, $e \in \mathbb{N}_+$ and $k \in \mathbb{N}$, we have

$$\mathrm{d}_{\hat{\Omega}}^{(k)}\left((\mathrm{d}^{(i)}t_j)^e\right) = \sum_{k_1+\cdots+k_e=k} \binom{i+k_1}{i} \cdots \binom{i+k_e}{i} \mathrm{d}^{(i+k_1)}t_j \cdots \mathrm{d}^{(i+k_e)}t_j.$$

So

$$\begin{aligned}
\mathrm{LT}\left(\mathrm{d}_{\hat{\Omega}}^{(k)}\left((\mathrm{d}^{(i)}t_j)^e\right)\right) &= e \cdot \binom{i+k}{i} \mathrm{d}^{(i+k)}t_j(\mathrm{d}^{(i)}t_j)^{e-1} \quad \text{if } e\binom{i+k}{i} \neq 0 \\
\mathrm{d}_{\hat{\Omega}}^{(k)}\left((\mathrm{d}^{(i)}t_j)^e\right) &= 0 \qquad \text{if } p \mid e \text{ and } p \nmid k \qquad \text{and} \\
\mathrm{d}_{\hat{\Omega}}^{(k)}\left((\mathrm{d}^{(i)}t_j)^e\right) &= \left(\mathrm{d}_{\hat{\Omega}}^{(\frac{k}{p})}\left((\mathrm{d}^{(i)}t_j)^{\frac{e}{p}}\right)\right)^p \quad \text{if } p \mid e \text{ and } p \mid k.
\end{aligned}$$

So for $k \leq p^l(p-1)$, a variable $\mathrm{d}^{(i)}t_j \neq \mathrm{d}^{(i_0)}t_{j_0}$ occuring in ω gives a contribution to $\mathrm{d}_{\hat{\Omega}}^{(k)}(\omega)$ of variables

(i) less than $\mathrm{d}^{(i_0+k)}t_{j_0}$ if it occurs in a power not divided by p and

(ii) less than $\mathrm{d}^{(i+\frac{k}{p})}t_j$ otherwise.

In the second case $i \leq p^{l-1}$, since $\omega \in \hat{\Omega}_{p^l}$, and so $i+\frac{k}{p} \leq p^{l-1}+p^{l-1}(p-1) = p^l$. So $\mathrm{d}^{(i+\frac{k}{p})}t_j < \mathrm{d}^{(i_0+p^l)}t_{j_0}$. Therefore the greatest variable that may occur is $\mathrm{d}^{(i_0+k)}t_{j_0}$ (or $\mathrm{d}^{(i_0+p^l)}t_{j_0}$ if $k < p^l$) and $\mathrm{d}^{(i_0+p^l(p-1))}t_{j_0}$ occurs if and only if $k = p^l(p-1)$ and

$\binom{i_0+p^l(p-1)}{i_0} \neq 0$, i. e. $i_0 \neq p^l$.
The greatest corresponding monomial then is

$$e_0 \mathrm{d}^{(i_0+p^l(p-1))} t_{j_0} \cdot (\mathrm{d}^{(i_0)} t_{j_0})^{e_0-1} \omega'.$$

$\square$

Proposition 6.5 *Every F-module M with integrable iterative connection ∇ defines a projective system (M_l, φ_l) over F, where $M_l := \bigcap_{0<i<p^l} \mathrm{Ker}(\nabla^{(i)})$ and $\varphi_l : M_{l+1} \to M_l$ is the inclusion map, and a morphism $f : (M, \nabla) \to (M', \nabla')$ of modules with higher connection defines a morphism $\alpha : (M_l, \varphi_l) \to (M'_l, \varphi'_l)$ of projective systems over F by $\alpha_l := f|_{M_l}$.*

Proof Since $\mathrm{d}^{(1)}t_1, \ldots, \mathrm{d}^{(1)}t_m$ is an F-basis of $\hat{\Omega}_1$, the kernel $\mathrm{Ker}(\nabla^{(1)})$ is equal to $\bigcap_{j=1}^m \mathrm{Ker}\left(\nabla^{(1)}_{\phi_{t_j}}\right)$ and since ∇ is integrable iterative, the endomorphisms $\nabla^{(1)}_{\phi_{t_j}} : M \to M$ commute and $\left(\nabla^{(1)}_{\phi_{t_j}}\right)^p = 0$ for all j.
Now let $M_1 := \mathrm{Ker}\left(\nabla^{(1)}\right)$ ($= \bigcap_{0<i<p^1} \mathrm{Ker}(\nabla^{(i)})$, since ∇ is iterative). Then M_1 is an F_1-vector space and

$$\dim_{F_1}(M_1) = \dim_{F_1}\left(\bigcap_{j=1}^m \mathrm{Ker}\left(\nabla^{(1)}_{\phi_{t_j}}\right)\right) \geq \frac{1}{p^m} \dim_{F_1}(M) = \dim_F(M).$$

On the other hand, an F_1-basis of M_1 is F-linearly independent in M and so $\dim_{F_1}(M_1) \leq \dim_F(M)$. So $\dim_{F_1}(M_1) = \dim_F(M)$ and the inclusion $\varphi_0 : M_1 \to M_0 = M$ extends to an isomorphism $\mathrm{id}_{F_0} \otimes \varphi_0 : F_0 \otimes_{F_1} M_1 \to M$.
Next, we will show that $\nabla(M_1) \subset (\hat{\Omega}_{F/K})^p \otimes_{F_1} M_1$. Since $(\hat{\Omega}_{F/K})^p = F_1[[(\mathrm{d}^{(i)}t_j)^p]] = F_1[[\mathrm{d}^{(pi)}(t_j^p)]]$ is isomorphic to $\hat{\Omega}_{F_1/K} = F_1[[\mathrm{d}^{(i)}(t_j^p)]]$ as an algebra by the map $\mathrm{d}^{(pi)}(t_j^p) \mapsto \mathrm{d}^{(i)}(t_j^p)$, this means that essentially $\nabla|_{M_1}$ is an integrable iterative connection on the F_1-module M_1. And then it follows inductively that $\dim_{F_{l+1}}(M_{l+1}) = \dim_{F_l}(M_l)$ and that, essentially, $\nabla|_{M_{l+1}}$ is an integrable iterative connection on the F_{l+1}-module M_{l+1}.
Since ∇ is iterative, it suffices to show that $\nabla^{(p^l)}(M_1) \subset (\hat{\Omega}_{F/K})^p \otimes_{F_1} M_1$ for all $l \geq 1$. So fix an F_1-basis $\boldsymbol{b} = (b_1, \ldots, b_n)$ of M_1 (written as a row) and let $A_l \in \mathrm{Mat}_n(\hat{\Omega}_{p^l})$ with $\nabla^{(p^l)}(\boldsymbol{b}) = \boldsymbol{b}A_l$.[10] From $0 = {}_{\hat{\Omega}}\nabla^{(p^l)}(\nabla^{(1)}(\boldsymbol{b})) = {}_{\hat{\Omega}}\nabla^{(1)}(\nabla^{(p^l)}(\boldsymbol{b})) = \boldsymbol{b}\mathrm{d}^{(1)}_{\hat{\Omega}}(A_l)$ we conclude $\mathrm{d}^{(1)}_{\hat{\Omega}}(A_l) = 0$. Assume there is an entry $\omega \in \hat{\Omega}_{p^l} \subset F[\mathrm{d}^{(i)}t_j \mid i = 1, \ldots, p^l, j = 1, \ldots, m]$ of A_l with $LT(\omega) = r\mathrm{d}^{(p^l)}t_j$ (for some $r \in F$ and $j \in \{1, \ldots, m\}$). Since $\mathrm{d}^{(1)}_{\hat{\Omega}}(r\mathrm{d}^{(p^l)}t_j) = \mathrm{d}^{(1)}(r)\mathrm{d}^{(p^l)}t_j + r\mathrm{d}^{(p^l+1)}t_j$ and for all other monomials of ω, the image under $\mathrm{d}^{(1)}_{\hat{\Omega}}$ doesn't contain the variable $\mathrm{d}^{(p^l+1)}t_j$, we obtain $\mathrm{d}^{(1)}_{\hat{\Omega}}(\omega) \neq 0$, a contradiction.

[10] For simplicitiy we use vector notations: $\boldsymbol{b}A_l$ denotes the row vector with j-th component $\sum_{i=1}^n (A_l)_{ij} b_i$, and ∇ and $\mathrm{d}_{\hat{\Omega}}$ are always applied to the components of a vector or a matrix.

So $\omega \in F[\mathrm{d}^{(i)}t_j \mid i = 1, \ldots, p^l - 1, j = 1, \ldots, m]$. Furthermore, since ∇ is iterative, ${}_{\hat{\Omega}}\nabla^{(p^l(p-1))} \circ \nabla^{(p^l)} = \binom{p^{l+1}}{p^l}\nabla^{(p^{l+1})} = 0$ and therefore

$$0 = {}_{\hat{\Omega}}\nabla^{(p^l(p-1))}(\boldsymbol{b}A_l) = \boldsymbol{b} \cdot \mathrm{d}_{\hat{\Omega}}^{(p^l(p-1))}(A_l) + \sum_{k=0}^{p^l(p-1)-1} \nabla^{(p^l(p-1)-k)}(\boldsymbol{b}) \cdot \mathrm{d}_{\hat{\Omega}}^{(k)}(A_l).$$

If $A_l \notin \mathrm{Mat}_n(F \cdot \hat{\Omega}^p)$, then by the previous lemma, $\mathrm{d}^{(p^l(p-1))}(A_l)$ has an entry with leading term $e_0 \mathrm{d}^{(i_0 + p^l(p-1))}t_{j_0} \left(\mathrm{d}^{(i_0)}t_{j_0}\right)^{e_0 - 1} \cdot \omega'$ for some $\omega' \in \hat{\Omega}$, $i_0 \leq p^l$ and $j_0 \in \{1, \ldots, m\}$ and the variables occuring in $\mathrm{d}_{\hat{\Omega}}^{(k)}(A_l)$ ($k < p^l(p-1) - 1$) are less than $\mathrm{d}^{(i_0 + p^l(p-1))}t_{j_0}$ and those occuring in $\nabla^{(p^l(p-1)-k)}(\boldsymbol{b})$ are even less than or equal to $\mathrm{d}^{(p^l(p-1))}t_m$. So we would have ${}_{\hat{\Omega}}\nabla^{(p^l(p-1))}(\boldsymbol{b}A_l) \neq 0$. Therefore $A_l \in \mathrm{Mat}_n(F\hat{\Omega}^p)$.
At last, since $\mathrm{d}_{\hat{\Omega}}^{(1)}(A_l) = 0$, in fact $A_l \in \mathrm{Mat}_n(\hat{\Omega}^p)$, which completes the proof. □

Theorem 6.6 *The category* $\mathbf{Proj}_F$ *of projective systems over* F *and the category* $\mathbf{ICon}_{int}(F/K)$ *are equivalent. Furthermore, if* F *is an algebraic function field in one variable over* K *and* $\phi \in \mathrm{ID}_K(F)$ *with* $\phi^{(1)} \neq 0$, *then they are also equivalent to the category* $\mathbf{ID}_F$ *of iterative differential modules over* (F, ϕ) *(cf. [Mat01]) and to the category* $\mathbf{ICon}(F/K)$.

Proof The first statement follows immediately from the previous two propositions, since the given maps are functors that are inverse to each other.
The proof of proposition 6.5 shows that the integrability condition is not necessary, when F is an algebraic function field in one variable. So $\mathbf{ICon}(F/K)$ is equivalent to $\mathbf{Proj}_F$, in this case. Furthermore, Matzat showed in [Mat01] that $\mathbf{ID}_F$ is equivalent to $\mathbf{Proj}_F$, too. □

Remark Let (M, ∇) be an F-module M with an integrable iterative connection ∇ and corresponding projective system $(M_l)_{l \in \mathbb{N}}$, and let $\boldsymbol{b} = (b_1, \ldots, b_n)$ be an F-basis of M. By the properties of a projective system, we could choose matrices $D_l \in \mathrm{GL}_n(F_l)$ ($l \in \mathbb{N}$) such that $\boldsymbol{b}D_0 \cdots D_{j-1}$ is an F_j-basis of M_j ($j = 0, 1, \ldots$). Then the image of an arbitrary element $\boldsymbol{ba} := \sum_{i=1}^n b_i a_i \in M$ (where $\boldsymbol{a} = (a_1, \ldots, a_n)^t$ with $a_i \in F$) by $\nabla^{(k)}$ can be calculated by

$$\nabla^{(k)}(\boldsymbol{ba}) = \boldsymbol{b}D_0 \cdots D_{l-1} \mathrm{d}_F^{(k)} \left(D_{l-1}^{-1} \cdots D_0^{-1} \boldsymbol{a}\right),$$

where $k < p^l$.

Part II

In this part of the thesis, we restrict to the case of an algebraic function field[11] F over an algebraically closed field of positive characteristic. In this case the modules with iterative connection are the same as iterative differential modules defined in [Mat01], which was shown in section 6.

So the iterative Picard-Vessiot theory (IPV-theory) developed by Matzat can be used: For an iterative Picard-Vessiot extension (IPV-extension) E/F, there is a Galois correspondence between the intermediate iterative differential fields L (i. e. $F \leq L \leq E$) and the (Zariski-)closed subgroups of the linear algebraic group $\mathrm{Gal}(L/F) = \mathrm{Aut}_{ID}(L/F)$. In what follows, we will investigate when a linear algebraic group can be realised as an iterative differential Galois group by an IPV-extension, which is regular outside a given nonempty set of places $S \subset \mathbb{P}_F$.

Notation Throughout this part of the thesis, K denotes an algebraically closed field of characteristic $p > 0$, F an algebraic function field over K, $\mathbb{P}_F$ the set of places of F and $\mathcal{C}_F$ a nonsingular projective model for F, i. e. a nonsingular projective curve over K with function field $K(\mathcal{C}_F) = F$. By [Hart77], ch. I, theorem 6.9, this curve is unique up to isomorphism and there is a one-to-one correspondence between the (closed) points of $\mathcal{C}_F$ and the places of F. For a point $x \in \mathcal{C}_F$, we denote by $\mathcal{O}_x \subset F$ the set of functions that are regular in x. It is a discrete valuation ring and therefore induces a valuation on F corresponding to the place $x \in \mathbb{P}_F$. Given a point $x \in \mathcal{C}_F$, we denote by $\mathrm{ord}_x(t)$ the image of an element $t \in F$ under this valuation. An element $s \in F$ with $\mathrm{ord}_x(s) = 1$ is called a local parameter for x.

For an open subset $U \subset \mathcal{C}_F$, we denote by $\mathcal{O}(U)$ the set of functions that are regular on U, i. e. $\mathcal{O}(U) = \bigcap_{x \in U} \mathcal{O}_x$.

For $l \in \mathbb{N}$, we denote by F_l, $(\mathcal{O}_x)_l$ resp. $(\mathcal{O}(U))_l$ the elements t in F, $\mathcal{O}_x$ resp. $\mathcal{O}(U)$ with $\mathrm{d}_F^{(k)}(t) = 0$ for all $0 < k < p^l$.[12]

[11] The term *algebraic function field* will always mean algebraic function field in one variable.

[12] Remind that d_F denotes the universal derivation $\mathrm{d}_F : F \to \hat{\Omega}_{F/K}$ and that F_l, $(\mathcal{O}_x)_l$ and $(\mathcal{O}(U))_l$ are subrings, as shown in proposition 6.1.

7 IPV-Extensions and ID-Galois Groups

7.1 Iterative Derivations in Algebraic Function Fields

Proposition 7.1 *Let $t \in F$ be a separating element (i. e. $F/K(t)$ is a finite separable extension). Then for every F-cga B and every $b \in B$ with $\varepsilon(b) = t$ there exists a unique higher derivation $\psi : F \to B$ satisfying $\psi(t) = b$.*
And the higher derivation ϕ_t given by $\phi_t(t) = t + T \in F[[T]]$ is an iterative derivation, the iterative derivation with respect to t.

Proof This is a special case of example 1.4 and example 3.2. □

Remark In example 1.4, we needed a transcendence basis to define the iterative derivations with respect to one of the basis elements. Since an algebraic function field has transcendence degree 1, every separating $t \in F$ itself is a transcendence basis for F.

Proposition 7.2 (chain rule) *Let $t \in F$ be separating, ϕ_t the iterative derivation with respect to t and let $\psi \in \mathrm{HD}_K(F)$. Then for all $r \in F$*

$$\psi(r) = \sum_{k=0}^{\infty} \phi_t^{(k)}(r) \left(\sum_{j=1}^{\infty} \psi^{(j)}(t) T^j \right)^k .$$

Proof Define a homomorphism of F-algebras $\lambda : F[[T]] \to F[[T]]$ by $\lambda(T) := \psi(t) - t = \sum_{j=1}^{\infty} \psi^{(j)}(t) T^j \in T \cdot F[[T]]$. Then $\lambda \circ \phi_t$ is a homomorphism of K-algebras and $\varepsilon \circ \lambda \circ \phi_t = \varepsilon \circ \phi_t = \mathrm{id}_F$, and therefore $\lambda \circ \phi_t$ is a higher derivation. Furthermore $(\lambda \circ \phi_t)(t) = \lambda(t + T) = t + \psi(t) - t = \psi(t)$ and so, by the previous proposition, $\lambda \circ \phi_t = \psi$, hence the formula above. □

Proposition 7.3 (chain rule for modules) *Let $t \in F$ be separating, ϕ_t the iterative derivation with respect to t and let $\psi \in \mathrm{HD}_K(F)$. Moreover let (M, ∇) be a module with iterative connection. Then for all $m \in M$*

$$\nabla_\psi(m) = \sum_{k=0}^{\infty} \nabla_{\phi_t}^{(k)}(m) \left(\sum_{j=1}^{\infty} \psi^{(j)}(t) T^j \right)^k .$$

Proof By theorem 6.6, an iterative connection leads to a projective system $(M_l)_{l \in \mathbb{N}}$ over F. So for an arbitrary $l \in \mathbb{N}$, choose an F_l-basis $\{b_1, \dots, b_n\}$ of M_l. Then for every $m \in M$, there are $a_1, \dots, a_n \in F$ with $m = \sum_{i=1}^{n} a_i b_i$ and therefore we get by the chain rule:

$$\begin{aligned}
\nabla_\psi(m) &= \nabla_\psi\left(\sum_{i=1}^n a_i b_i\right) \equiv \sum_{i=1}^n \psi(a_i) b_i \pmod{T^{p^l}} \\
&= \sum_{i=1}^n \sum_{k=0}^\infty \phi_t^{(k)}(a_i)\left(\sum_{j=1}^\infty \psi^{(j)}(t)T^j\right)^k b_i \\
&\equiv \sum_{k=0}^\infty \nabla_{\phi_t}^{(k)}(m)\left(\sum_{j=1}^\infty \psi^{(j)}(t)T^j\right)^k \pmod{T^{p^l}}.
\end{aligned}$$

Since l can be chosen arbitrary large, we get

$$\nabla_\psi(m) = \sum_{k=0}^\infty \nabla_{\phi_t}^{(k)}(m)\left(\sum_{j=1}^\infty \psi^{(j)}(t)T^j\right)^k.$$

$\square$

Lemma 7.4 *For each $n \in \mathbb{N}_+$ there exist $\tau_n \in K[X_1, \ldots, X_n, X_1^{-1}]$, such that for all separating variables $s, t \in F$ we have:*

$$\phi_t^{(n)}(s) = \tau_n\left(\phi_s^{(1)}(t), \ldots, \phi_s^{(n)}(t)\right).$$

Especially, $\phi_s^{(1)}(t) \neq 0$ for all separating $s, t \in F$.

Proof By the chain rule, for separating $s, t \in F$ we have:

$$s + T = \phi_s(s) = \sum_{k=0}^\infty \phi_t^{(k)}(s)\left(\sum_{j=1}^\infty \phi_s^{(j)}(t)T^j\right)^k.$$

And so, by comparing the coefficients, we obtain,

$$\begin{aligned}
1 &= \phi_t^{(1)}(s)\phi_s^{(1)}(t) \quad (\text{so } \phi_s^{(1)}(t) \neq 0) \text{ and} \\
0 &= \phi_t^{(n)}(s)\left(\phi_s^{(1)}(t)\right)^n + \sum_{k=1}^{n-1} \sum_{\substack{j_1+\cdots+j_k=n \\ j_i \geq 1}} \phi_t^{(k)}(s)\phi_s^{(j_1)}(t)\cdots\phi_s^{(j_k)}(t)
\end{aligned}$$

for $n > 1$. From this, one inductively obtains a formula for calculating $\phi_t^{(n)}(s)$ as a polynomial of $\phi_s^{(1)}(t), \ldots, \phi_s^{(n)}(t)$ and $\phi_s^{(1)}(t)^{-1}$. Replacing $\phi_s^{(j)}(t)$ by X_j gives the desired "polynomial" τ_n. $\square$

Theorem 7.5 *If $\#K = \infty$, then for every nonzero $\omega \in \hat{\Omega}_{F/K}$ there exists an iterative derivation $\phi \in \mathrm{ID}_K(F)$ such that $\tilde{\phi}(\omega) \neq 0$, i. e. F has enough iterative derivations.*

Proof Let $t \in F$ be a separating element. At first, we show that an element $\sigma \in K[X_1, \dots, X_n, X_1^{-1}]$ has to be zero, if for all separating elements $s \in F$, $\sigma(\phi_t^{(1)}(s), \dots, \phi_t^{(n)}(s)) = 0$:
Assume this is false and choose $j \in \{1, \dots, n\}$ maximal such that there is $0 \neq \sigma \in K[X_j, \dots, X_n] \subset K[X_1, \dots, X_n, X_1^{-1}]$ with $\sigma(\phi_t^{(j)}(s), \dots, \phi_t^{(n)}(s)) = 0$ for all separating $s \in F$. As j is maximal, there is a separating $s \in F$ such that

$$0 \neq \sigma(X_j, \phi_t^{(j+1)}(s), \dots, \phi_t^{(n)}(s)) \in F[X_j].$$

Then for almost all $a \in K$, $s+at^j$ also is separating[13], and $\phi_t^{(j)}(s+at^j) = \phi_t^{(j)}(s)+a$ and $\phi_t^{(k)}(s+at^j) = \phi_t^{(k)}(s)$ for all $k > j$. So for almost all $a \in K$ (i. e. in special infinitely many $a \in K$), $\sigma(\phi_t^{(j)}(s) + a, \phi_t^{(j+1)}(s), \dots, \phi_t^{(n)}(s)) = 0$ and therefore $\sigma(X_j, \phi_t^{(j+1)}(s), \dots, \phi_t^{(n)}(s)) = 0 \in F[X_j]$ in contradiction to the choice of s.
Next, we define a homomorphism of K-algebras $\chi_n : K[X_1, \dots, X_n, X_1^{-1}] \to K[X_1, \dots, X_n, X_1^{-1}]$ by $X_j \mapsto \tau_j$ (the τ_j given by the previous lemma). χ_n is an involution because for all separating $s \in F$ and $\sigma \in K[X_1, \dots, X_n, X_1^{-1}]$:

$$\begin{aligned} \left((\chi_n \circ \chi_n)(\sigma)\right) \left(\phi_t^{(1)}(s), \dots, \phi_t^{(n)}(s)\right) &= \chi_n(\sigma) \left(\phi_s^{(1)}(t), \dots, \phi_s^{(n)}(t)\right) \\ &= \sigma \left(\phi_t^{(1)}(s), \dots, \phi_t^{(n)}(s)\right) \end{aligned}$$

and so $(\chi_n \circ \chi_n)(\sigma) = \sigma$.
Now assume there is $0 \neq \omega \in \hat{\Omega}_{F/K}$ such that $\tilde{\phi}(\omega) = 0$ for all $\phi \in \mathrm{ID}_K(F)$. Then without loss of generality ω is homogeneous of degree n and so $\omega \in F[\mathrm{d}^{(1)}t, \dots, \mathrm{d}^{(n)}t] \cong F[X_1, \dots, X_n]$. Hence for all separating $s \in F$:

$$0 = \tilde{\phi}_s(\omega) = \omega \left(\phi_s^{(1)}(t), \dots, \phi_s^{(n)}(t)\right) = \chi_n(\omega) \left(\phi_t^{(1)}(s), \dots, \phi_t^{(n)}(s)\right).$$

So, by the previous, we get $\chi_n(\omega) = 0$ and therefore $\omega = \chi_n(\chi_n(\omega)) = 0$, a contradiction. $\square$

Remark Not all iterative derivations of F are given as the iterative derivation with respect to some separating $t \in F$. But since we won't use this fact, we won't proof it. See [Mat01], Ch. 1.5, for a description of all iterative derivations of F.

7.2 IPV-Extensions

In [Mat01], Ch. 3, Matzat has developed an iterative Picard-Vessiot theory in positive characteristic. In this section we give a summary of the main definitions and results.
In the following when we speak of the ID-field F, we mean a pair (F, ϕ), where $\phi \in \mathrm{ID}_K(F)$ is an iterative derivation satisfying $\phi^{(1)} \neq 0$. An ID-module over

[13] $r \in F$ is separating if and only if $\phi_t^{(1)}(r) \neq 0$.

(F, ϕ) is an F-vector space M with an iterative ϕ-derivation $\Phi \in \mathrm{ID}_K(M, \phi)$. Since F is an algebraic function field over K, Matzat showed that such an ID-module M determines a projective system over F. By the last section, this determines an integrable iterative connection ∇ on M. It is easy to see that this connection fulfills $\nabla_\phi = \Phi$. So there is no difference whether we consider ID-modules or modules with integrable iterative connections. We will therefore also call a pair (M, ∇) an ID-module.

Definition 7.6 *An* **iterative differential ring** *(ID-ring) over F is a ring $R \geq F$ with an iterative connection $d_R : R \to R \otimes_F \hat{\Omega}_{F/K}$[14] that is a higher derivation on R over K. An* **iterative differential field** *(ID-field) over F is an ID-ring L, that is a field. An ID-module over L is an L-vector space M equipped with a d_L-derivation $\nabla : M \to (\hat{\Omega}_{F/K} \otimes_F L) \otimes_L M$. For an ID-ring $R \geq L$, a matrix $Y \in \mathrm{GL}_n(R)$ is called a* **fundamental solution matrix** *for an ID-module (M, ∇) over L (with respect to a basis $\boldsymbol{b} = (b_1, \ldots, b_n)$ of M), if*

$$\nabla_{R \otimes_L M}(\boldsymbol{b}Y) := \nabla(\boldsymbol{b}) \cdot d_R(Y) = \boldsymbol{b}Y.^{15}$$

The ring R is called an **iterative Picard-Vessiot ring** *(IPV-ring) for M, if it satisfies the following conditions:*

1. *R is a simple ID-ring (i. e. has no non-trivial d_R-stable ideal).*
2. *There exists a fundamental solution matrix $Y \in \mathrm{GL}_n(R)$ for M.*
3. *R is generated over L by the coefficients of Y and $\det(Y)^{-1}$.*

Such a ring is an integral domain ([Mat01], prop. 3.2) and we call its quotient field E an **iterative Picard-Vessiot field** *(IPV-field). E/L is then called an* **IPV-extension** *and the group of iterative differential automorphisms of E over L (i. e. automorphisms that commute with the iterative connection) is called the* **iterative differential Galois group** *(ID-Galois group) $\mathrm{Gal}(E/L)$.*

Remark

1. d_R is an extension of d_F, because for all $t \in F$,

$$d_R(t) = d_R(t \cdot 1) = \mathrm{d}_F(t) \cdot d_R(1) = \mathrm{d}_F(t).$$

[14] Take care, that d_R is not the universal derivation. The similar notation is due to the fact, that for every iterative derivation $\phi \in \mathrm{ID}_K(F)$, d_R determines an iterative derivation on R extending ϕ (see the next remark).

[15] Like in section 6, we use vector notations and the higher connections are meant to be applied to the coefficients separately.

2. If we choose an iterative derivation $\phi \in \mathrm{ID}_K(F)$ with $\phi^{(1)} \neq 0$ and set $\phi_L := (\mathrm{id}_L \otimes \tilde{\phi}) \circ d_L \in \mathrm{ID}_K(L)$ and $\phi_R := (\mathrm{id}_R \otimes \tilde{\phi}) \circ d_R \in \mathrm{ID}_K(R)$, the pair (L, ϕ_L) is an ID-field and the pair (R, ϕ_R) is an ID-ring resp. IPV-ring for M in the sense of [Mat01]. On the other hand (R, d_R) and (R, ϕ_R) determine the same projective system over F and so (R, d_R) is determined by (R, ϕ_R). So the definition of ID-ring and IPV-ring given here is equivalent to the other. Furthermore for any other iterative derivation $\phi' \in \mathrm{ID}_K(F)$ with $\phi'^{(1)} \neq 0$, the pair (R, ϕ_R) determines a unique extension $\phi'_R \in \mathrm{ID}_K(R)$ of ϕ'.

In the following, we state some results that are all given in [Mat01] and we refer thereto for proofs.

L will denote an ID-field over F with field of constants K, (M, ∇) an ID-module over L with a basis $\boldsymbol{b} = (b_1, \dots, b_n)$ and $\nabla(\boldsymbol{b}) = \boldsymbol{b}A$ for a matrix $A \in \mathrm{GL}_n(L \otimes_F \hat{\Omega}_{F/K})$. Furthermore R denotes an IPV-ring for M and $E = \mathrm{Quot}(R)$ an IPV-field. $D_l \in \mathrm{GL}_n(L_l)$ are chosen such that $\boldsymbol{b}D_0 \cdot D_l$ is a basis of M_{l+1} for all $l \geq 0$.[16]

Proposition 7.7 *1. The IPV-ring R and the IPV-field E are unique up to iterative differential isomorphisms. (Thm. 3.4)*

2. The IPV-field E is a minimal field extension of L that contains a fundamental solution matrix for M. (Cor. 3.5)

3. An IPV-ring for M can be constructed in the following way: Let $U := L[X_{ij}(i,j = 1, \dots, n), \det(X)^{-1}]$ be a localisation of the polynomial ring in n^2 variables equipped with the integrable iterative connection defined by $d_U(X) := A^{-1}X \in \mathrm{GL}_n(\hat{\Omega}_{F/K} \otimes U)$. Next choose a maximal ID-ideal $P \trianglelefteq U$. Then U/P is an IPV-ring for M with fundamental solution matrix $Y := \overline{X}$, the image of X under the projection to U/P. (Thm. 3.4)

4. E contains no new constants. (Prop. 3.2)

5. Given two fundamental solution matrices $Y, \tilde{Y} \in \mathrm{GL}_n(E)$, there exist $C \in \mathrm{GL}_n(K)$ such that $\tilde{Y} = Y \cdot C$. (Prop. 3.3)

6. $\mathrm{Gal}(E/L)$ is a subgroup of $\mathrm{GL}_n(K)$. (Prop. 3.8)

7. There exists a reduced linear algebraic group $\mathcal{G} \leq \mathrm{GL}_n$ defined over K, such that $\mathrm{Gal}(E/L) = \mathcal{G}(K)$ under the inclusion above. (Thm. 3.10)

Proof of 6. Although this is proved in [Mat01], too, we give the proof here, to show how this inclusion is given:

[16] As usual, we set $L_l := \bigcap_{0<k<p^l} \mathrm{Ker}\left(d_L^{(k)}\right)$ and $M_l := \bigcap_{0<k<p^l} \mathrm{Ker}\left(\nabla^{(k)}\right)$.

If Y is a fundamental solution matrix and $\gamma \in \mathrm{Gal}(E/F)$, then since γ commutes with the higher derivation, $\gamma(Y)$ is again a fundamental solution matrix. By 5., there exists an element $C_\gamma \in \mathrm{GL}_n(K)$ such that $\gamma(Y) = YC_\gamma$. This defines a homomorphism $\varphi : \mathrm{Gal}(E/F) \to \mathrm{GL}_n(K), \gamma \mapsto C_\gamma$. If $C_\gamma = \mathbf{1}_n \in \mathrm{GL}_n(K)$, then $\gamma(Y) = Y$, but since E is generated by the coefficients of Y, this implies that $\gamma = \mathrm{id}_E$, showing that φ is injective. □

Theorem 7.8 (Galois correspondence) *Let L be an ID-field over F, let E/L be an IPV-extension for an ID-module M over L and let $\mathcal{G}$ be a reduced linear algebraic group such that $\mathcal{G}(K) = \mathrm{Gal}(E/L)$. Suppose*

$$\mathfrak{H} = \{\mathcal{H} \mid \mathcal{H} \leq \mathcal{G} \text{ is a Zariski closed reduced linear algebraic subgroup}\},$$

and

$$\mathfrak{E} = \{\tilde{E} \mid \tilde{E} \text{ is an intermediate ID-field } L \leq \tilde{E} \leq E\}.$$

Then the map $\chi : \mathfrak{H} \to \mathfrak{E}$ defined by $\chi(\mathcal{H}) := E^{\mathcal{H}(K)}$ is an anti-isomorphism of lattices with inverse given by $\chi^{-1}(\tilde{E}) = \mathcal{H}$, where $\mathcal{H}(K) = \mathrm{Gal}(E/\tilde{E})$. Further if $\mathcal{H} \in \mathfrak{H}$ is a normal subgroup of $\mathcal{G}$, then $\tilde{E} := E^{\mathcal{H}(K)}$ is an IPV-extension of L with Galois group $(\mathcal{G}/\mathcal{H})(K)$.

Proof See [Mat01], thm. 4.7. □

7.3 Determining the Galois Group

In the following, every linear algebraic group is supposed to be reduced and defined over K.

Theorem 7.9 *Using the notations above, if there exists a linear algebraic group $\mathcal{G} \leq \mathrm{GL}_n$, such that $D_l \in \mathcal{G}(L_l)$ for all $l \in \mathbb{N}$, then $\mathrm{Gal}(E/L) \leq \mathcal{G}(K)$.*

Proof See [Mat01], thm. 5.1. □

Theorem 7.10 *Let $\mathcal{G} \leq \mathrm{GL}_m$ and $\mathcal{H} \leq \mathrm{GL}_n$ be two linear algebraic groups and let $\Theta : \mathcal{G} \to \mathcal{H}$ be an epimorphism with reduced kernel. Let M and N be ID-modules over L with projective systems given by matrices $D_l \in \mathcal{G}(L_l)$ resp. $\tilde{D}_l \in \mathcal{H}(L_l)$ for all $l \in \mathbb{N}$ and let the IPV-fields for M resp. N be denoted by E_M resp. E_N.*
If $\tilde{D}_l = \Theta(D_l)$ for all l, then up to ID-isomorphism $E_M \geq E_N$ and $\mathrm{Gal}(E_M/E_N) \leq \mathrm{Ker}(\Theta)(K)$.

Proof See [Mat01], thm. 5.12. □

Remark If N is an ID-module over L with $\tilde{D}_l \in \mathcal{H}(L_l)$ and $\mathrm{Gal}(E_N/L) = \mathcal{H}(K)$. Then by choosing preimages $D_l \in \Theta^{-1}(\tilde{D}_l) \leq \mathcal{G}(L_l)$ (if possible) one obtains an ID-module M defined by the D_l and an IPV-field E_M for M such that $\mathrm{Gal}(E_M/L) \leq \mathcal{G}(K)$ and $\Theta(\mathrm{Gal}(E_M/L)) = \mathcal{H}(K)$.

Proposition 7.11 *Let R_1, R_2 be two IPV-rings over L with Galois groups* $\mathrm{Gal}(R_j/L) =: \mathcal{G}_j(K)$ *($j = 1, 2$). And assume that $R := R_1 \otimes_L R_2$ is a simple ID-ring. Then R is an IPV-extension over L with Galois group* $\mathrm{Gal}(R/L) = (\mathcal{G}_1 \times \mathcal{G}_2)(K)$.

Proof See [Mat01], proposition 7.9. □

Proposition 7.12 *Let $F = K(s, t)$ with some algebraic relation $f(s, t) = 0$. Let $\mathcal{C}'$ be the corresponding affine model and assume without loss of generality that $(0, 0) \in \mathcal{C}$ is a regular point. Then $F_l = K(s^{p^l}, t^{p^l})$ with some relation $f_l(s^{p^l}, t^{p^l}) = 0$ and model $\mathcal{C}'_l$. Let $\mathcal{G}$ be a linear algebraic group and let M be an ID-module over F with projective system defined by $D_l \in \mathcal{G}(F_l)$ ($l \in \mathbb{N}$). Assume that D_l ($l = 0, 1, \dots$) satisfy the following conditions:*

1. *For all $l \in \mathbb{N}$ there exists a rational map $\gamma_l : \mathcal{C}'_l \to \mathcal{G}$ such that $D_l = \gamma_l(s^{p^l}, t^{p^l}) \in \mathcal{G}(F_l)$ and $\gamma_l(0, 0) = 1_{\mathcal{G}(K)}$.*
2. *For all $m \in \mathbb{N}$ the set $\{\gamma_l(\mathcal{C}'_l(K)) \mid l \geq m\}$ generates $\mathcal{G}(K)$ as an algebraic group.*
3. *There exists a number $d \in \mathbb{N}$ such that* $\deg(\gamma_l) \leq dp^l$ *for all $l \in \mathbb{N}$, where* deg *denotes the maximum divisor degree of the matrix entries with respect to F.*
4. *If $l_0 < l_1 < \dots$ is a sequence of natural numbers l_i for which $\gamma_{l_i} \neq 1$, then* $\lim_{i \to \infty}(l_{i+1} - l_i) = \infty$.

Then M defines an IPV-extension E/F with Galois group isomorphic to $\mathcal{G}(K)$.

Proof See [Mat01], lemma 8.6. □

Definition 7.13 *Let L/F be an IPV-extension with* $\mathrm{Gal}(L/F) \cong \mathcal{H}(K), \eta \mapsto C_\eta$, *for a linear algebraic group $\mathcal{H} \leq \mathrm{GL}_n$. And let $\mathcal{N} \leq \mathrm{GL}_n$ be a linear algebraic group on which $\mathcal{H}$ acts by conjugation in GL_n. Then there is an action of $\mathcal{H}(K)$ on the L-rational points of $\mathcal{N}$ given by*

$$C_\eta \star D := C_\eta \eta(D) C_\eta^{-1},$$

for all $C_\eta \in \mathcal{H}(K)$ and $D \in \mathcal{N}(L)$, where $\eta \in \mathrm{Gal}(L/F)$ acts on the matrix entries of $D \in \mathcal{N}(L) \leq \mathrm{GL}_n(L)$.
The subgroup of elements that are invariant under this action will be denoted by $\boldsymbol{\mathcal{N}(L)^{\mathcal{H}}}$ *and for all subrings $\mathcal{O} \subset L$ we define*

$$\mathcal{N}(\mathcal{O})^{\mathcal{H}} := \{D \in \mathcal{N}(\mathcal{O}) \leq \mathcal{N}(L) \mid \forall C_\eta \in \mathcal{H}(K) : C_\eta \star D = D\}.$$

Proposition 7.14 *With notations of the previous definition, assume there is a fundamental solution matrix* $Y \in \mathcal{H}(L)$ *with* $\eta(Y) = YC_\eta$ *for all* $\eta \in \mathrm{Gal}(L/F) \cong \mathcal{H}(K)$.[17] *Then*

$$\mathcal{N}(L)^{\mathcal{H}} = Y^{-1}\mathcal{N}(F)Y.$$

Proof For all $D \in \mathcal{N}(L)$ and $\eta \in \mathrm{Gal}(L/F)$:

$$\eta(YDY^{-1}) = \eta(Y)\eta(D)\eta(Y)^{-1} = YC_\eta\eta(D)C_\eta^{-1}Y^{-1} = Y(C_\eta \star D)Y^{-1}$$

and so $YDY^{-1} \in \mathcal{N}(F)$ if and only if $D \in \mathcal{N}(L)^{\mathcal{H}}$. □

Theorem 7.15 *Let* $\mathcal{G} \leq \mathrm{GL}_n$ *be a connected linear algebraic group and assume that* $\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}$ *is a semidirect product of two subgroups* $\mathcal{N}$ *and* $\mathcal{H}$ *of* $\mathcal{G}$. *Let* L/F *be an IPV-extension with fundamental solution matrix* $Y \in \mathcal{H}(L)$ *and* $\mathrm{Gal}(L/F) = \mathcal{H}(K)$. *Further, let* M *denote an ID-module over* L *with projective system defined by* $D_l \in \mathcal{N}(L_l)$ *and let* E/L *be an IPV-extension for* M *with fundamental solution matrix* $Z \in \mathcal{N}(E)$.
If for all $l \in \mathbb{N}$, D_l *is* $\mathcal{H}$*-invariant, i. e.* $D_l \in \mathcal{N}(L_l)^{\mathcal{H}}$, *then* E/F *is an IPV-extension with Galois group* $\mathrm{Gal}(E/F) = \mathrm{Gal}(E/L) \rtimes \mathcal{H}(K) \leq (\mathcal{N} \rtimes \mathcal{H})(K) = \mathcal{G}(K)$ *and* E *is generated over* F *by the coefficients of* YZ *(i. e.* YZ *is a fundamental solution matrix for* M*).*

Proof At first, let $C_l \in \mathcal{H}(F_l)$ $(l = 0, 1, \dots)$ be chosen such that $Y_l := C_{l-1}^{-1} \cdots C_0^{-1} Y \in \mathcal{H}(L_l)$. Then by proposition 7.14, $Y_l D_l Y_l^{-1} \in \mathcal{N}(F) \cap \mathcal{N}(L_l) = \mathcal{N}(F_l)$ and therefore

$$\tilde{D}_l := Y_l D_l Y_{l+1}^{-1} = Y_l D_l Y_l^{-1} C_l \in (\mathcal{N} \rtimes \mathcal{H})(F_l) \quad (l \in \mathbb{N}).$$

So the sequence $(\tilde{D}_l)_{l\in\mathbb{N}}$ defines an ID-module N over F and for all $k < p^{l+1}$:

$$\begin{aligned}
& \mathrm{d}_F^{(k)}\left(\tilde{D}_0 \cdots \tilde{D}_l\right) \cdot \left(\tilde{D}_0 \cdots \tilde{D}_l\right)^{-1} \\
= \; & d_L^{(k)}\left(YD_0Y_1^{-1}Y_1D_1Y_2^{-1} \cdots Y_lD_lY_{l+1}^{-1}\right) \cdot \left(YD_0Y_1^{-1} \cdots Y_lD_lY_{l+1}^{-1}\right)^{-1} \\
= \; & d_L^{(k)}\left(YD_0 \cdots D_l\right) Y_{l+1}^{-1} \cdot Y_{l+1} D_l^{-1} \cdots D_0^{-1} Y^{-1} \\
= \; & d_L^{(k)}\left(YD_0 \cdots D_l\right) \cdot D_l^{-1} \cdots D_0^{-1} Y^{-1} = d_L^{(k)}(YZ) \cdot Z^{-1}Y^{-1}.
\end{aligned}$$

Hence, $YZ \in \mathrm{GL}_n(E)$ is a fundamental solution matrix for N and $\tilde{E} := F(YZ) \leq E$ is an IPV-extension for N. Next, the projection $\Theta : \mathcal{N} \rtimes \mathcal{H} \to \mathcal{H}$ maps $\tilde{D}_l$ to C_l (since $Y_l D_l Y_l^{-1} \in \mathcal{N}(F_l)$) and therefore by theorem 7.10, L is a subfield of $\tilde{E}$ and so $Y \in \mathrm{GL}_n(\tilde{E})$ and $Z = Y^{-1}(YZ) \in \mathrm{GL}_n(\tilde{E})$, i. e. $\tilde{E} = E$.
So E/F is an IPV-extension, $\mathrm{Gal}(E/F) \leq (\mathcal{N} \rtimes \mathcal{H})(K)$ (since $\tilde{D}_l \in (\mathcal{N} \rtimes \mathcal{H})(F_l)$) and $\mathrm{Gal}(E/L) \leq \mathrm{Ker}(\Theta)(K) = \mathcal{N}(K)$.
So $\mathrm{Gal}(E/F) = \mathrm{Gal}(E/L) \rtimes \mathcal{H}(K) \leq (\mathcal{N} \rtimes \mathcal{H})(K)$. □

[17] By [Mat01], thm. 5.9, this is possible whenever $\mathcal{H}$ is connected.

8 Regularity

8.1 Differentially Stable Regular Rings

Definition 8.1 *Let $t \in F$ be a separating element. Then we denote by $U_t \subset \mathcal{C}_F$ the subset consisting of all points $y \in \mathcal{C}_F$ such that $t \in \mathcal{O}_y$ and $\mathcal{O}_y$ is ϕ_t-stable, i. e. $\phi_t^{(j)}(\mathcal{O}_y) \subseteq \mathcal{O}_y$ for all $j \in \mathbb{N}$.*

Proposition 8.2 *Let $t \in F$ be a separating element. Then*

$$\begin{aligned} U_t &= \{y \in \mathcal{C}_F \mid \operatorname{ord}_y(t) \geq 0 \textit{ and } \operatorname{ord}_y((dt)) = 0\} \\ &= \{y \in \mathcal{C}_F \mid \exists\, a \in K \textit{ such that } t - a \textit{ is a local parameter for } y\}, \end{aligned}$$

where (dt) denotes the divisor of the differential dt (as in [Sti93]). Especially, U_t is a (Zariski) open subset of $\mathcal{C}_F$.

Proof Let $U_t' := \{y \in \mathcal{C}_F \mid \operatorname{ord}_y(t) \geq 0 \text{ and } \operatorname{ord}_y((dt)) = 0\}$ and $U_t'' := \{y \in \mathcal{C}_F \mid \exists\, a \in K \text{ such that } t - a \text{ is a local parameter for } y\}$.
We will show $U_t' \subset U_t'' \subset U_t \subset U_t'$.

So let s be a local parameter for a given place $y \in U_t'$, then $0 = \operatorname{ord}_y((dt)) = \operatorname{ord}_y(\phi_s^{(1)}(t))$ and therefore $\operatorname{ord}_y(t) \leq 1$. Moreover we have $\operatorname{ord}_y(t-a) \leq 1$ for all $a \in K$, since $d(t-a) = dt$. As $\operatorname{ord}_y(t) \geq 0$, there exists an element $a \in K$ satisfying $\operatorname{ord}_y(t-a) > 0$, i.e. $\operatorname{ord}_y(t-a) = 1$ and so $t-a$ is a local parameter for y.

Now let $y \in U_t''$ and $t-a$ be a local parameter for y. Then $t-a$ is an element of $\mathcal{O}_y$ and $\mathcal{O}_y$ is ϕ_{t-a}-stable, since $\mathcal{O}_y$ is a finite separable extension of $K[t-a]_{(t-a)}$. So $t \in \mathcal{O}_y$ and, since $\phi_t = \phi_{t-a}$, the ring $\mathcal{O}_y$ is also ϕ_t-stable. This proves $y \in U_t$.

At last, let $y \in U_t$. Then $t \in \mathcal{O}_y$ and therefore $\operatorname{ord}_y(t) \geq 0$.
Let s be a local parameter for y. Then $\phi_s^{(1)}(t) \in \mathcal{O}_y$, since $t \in \mathcal{O}_y$ and $\mathcal{O}_y$ is ϕ_s-stable. Analogously, we get $\phi_t^{(1)}(s) \in \mathcal{O}_y$. But since $\phi_t^{(1)}(s) = (\phi_s^{(1)}(t))^{-1}$, the element $\phi_s^{(1)}(t)$ is invertible in $\mathcal{O}_y$ and so $\operatorname{ord}_y(\phi_s^{(1)}(t)) = 0$. This means $\operatorname{ord}_y((dt)) = 0$.

U_t' is open, since the conditions $\operatorname{ord}_y(t) \geq 0$ and $\operatorname{ord}_y((dt)) = 0$ are fulfilled for all but finitely many $y \in \mathcal{C}_F$. $\square$

Remark U_t is an affine variety. U_t is the maximal subset of $\mathcal{C}_F$ such that $t \in \mathcal{O}(U_t)$ and such that for all subsets $U \subset U_t$, the ring $\mathcal{O}(U)$ is ϕ_t-stable.

Proposition 8.3 *Let $t \in F$ be a separating element and $x \in U_t$. Then $\mathcal{O}_x$ has no nontrivial ϕ_t-stable ideal.*

Proof Choose $a \in K$ such that $t-a$ is a local parameter for x (cf. proposition 8.2). Then $\mathcal{O}_x$ is a regular local ring with maximal ideal generated by $t-a$ and $\mathcal{O}_x/(t-a) \cong K$, since K is algebraically closed. Since $\phi_t = \phi_{t-a}$, the proposition follows directly from lemma 2.10. $\square$

8.2 Differentially Stable Lattices

Recall that M is a vector space over F of finite dimension equipped with an iterative connection ∇.

Definition 8.4 *Let $\mathcal{O} \subset F$ be a subring. An* **$\mathcal{O}$-lattice** *in M is a free $\mathcal{O}$-submodule Λ of M, which contains an F-basis of M. An* **$\mathcal{O}$-pseudo-lattice** *in M is a finitely generated $\mathcal{O}$-submodule Λ of M, which contains an F-basis of M, i. e. Λ satisfies $M = F \otimes_{\mathcal{O}} \Lambda$.*
If $t \in F$ is separating and U is an open subset of U_t, then an $\mathcal{O}(U)$(-pseudo)-lattice Λ is called **ϕ_t-stable**, *if $\nabla_{\phi_t}(\Lambda) \subset \Lambda[[T]]$.*

Lemma 8.5 *Let $s, t \in F$ be separating elements, let $U \subset U_s \cap U_t$ and let Λ_U be a ϕ_s-stable $\mathcal{O}(U)$-pseudo-lattice in M. Then Λ_U is also ϕ_t-stable.*

Proof By the chain rule for modules, for all $m \in M$:

$$\sum_{k=0}^{\infty} \nabla_{\phi_t}^{(k)}(m) T^k = \sum_{k=0}^{\infty} \nabla_{\phi_s}^{(k)}(m) \left(\sum_{j=1}^{\infty} \phi_t^{(j)}(s) T^j \right)^k .$$

Now Λ_U is ϕ_s-stable, and so $\nabla_{\phi_s}^{(k)}(m) \in \Lambda_U$ for all $m \in \Lambda_U$. Moreover $\phi_t^{(j)}(s) \in \mathcal{O}(U)$, since $s \in \mathcal{O}(U)$ and $\mathcal{O}(U)$ is ϕ_t-stable.
So $\sum_{k=0}^{\infty} \nabla_{\phi_t}^{(k)}(m) T^k \in \Lambda_U[[T]]$ for $m \in \Lambda_U$, i. e. Λ_U is ϕ_t-stable. $\square$

Lemma 8.6 *Let $t \in F$ be separating and $U \subset U_t$ be an open subset. Then there exists at most one $\mathcal{O}(U)$-pseudo-lattice Λ in M, that is ϕ_t-stable.*

Proof Let Λ and Λ' be two ϕ_t-stable pseudo-lattices. Clearly, the intersection $\Lambda \cap \Lambda'$ also is ϕ_t-stable, and since for every $m \in M$, there are $\alpha, \alpha' \in \mathcal{O}(U)$ with $\alpha m \in \Lambda$ and $\alpha' m \in \Lambda'$ and hence with $\alpha\alpha' m \in \Lambda \cap \Lambda'$, the $\mathcal{O}(U)$-module $\Lambda \cap \Lambda'$ is an $\mathcal{O}(U)$-pseudo-lattice in M. So let without loss of generality be $\Lambda' \subset \Lambda$.
Now let $y \in U$ and define $\Lambda_y := \mathcal{O}_y \otimes_{\mathcal{O}(U)} \Lambda$ and $\Lambda'_y := \mathcal{O}_y \otimes_{\mathcal{O}(U)} \Lambda'$. Then $\Lambda'_y \subset \Lambda_y$ are two ϕ_t-stable $\mathcal{O}_y$-pseudo-lattices in M. Since $\mathcal{O}_y$ is a principal ideal domain, Λ'_y and Λ_y are in fact lattices in M and furthermore there exists an $\mathcal{O}_y$-basis $\{b_1, \dots b_n\}$ of Λ_y and $\alpha_1, \dots, \alpha_n \in \mathcal{O}_y$ such that $\{\alpha_1 b_1, \dots, \alpha_n b_n\}$ is an $\mathcal{O}_y$-basis of Λ'_y.
Now we show $\phi_t^{(k)}(\alpha_i) \in \mathcal{O}_y \cdot \alpha_i$ for all $i \in \{1, \dots, n\}$ and all $k \in \mathbb{N}$ by induction on k:

For $k = 0$ the claim is trivial. So let $k > 0$ and $\phi_t^{(j)}(\alpha_i) \in \mathcal{O}_y \cdot \alpha_i$ for all $i \in \{1, \ldots, n\}$ and all $0 \leq j < k$. Then

$$\begin{aligned} \phi_t^{(k)}(\alpha_i) b_i &= \nabla_{\phi_t}^{(k)}(\alpha_i b_i) - \sum_{j=0}^{k-1} \phi_t^{(j)}(\alpha_i) \nabla_{\phi_t}^{(k-j)}(b_i) \\ &\in \Lambda'_y + \alpha_i \cdot \Lambda_y. \end{aligned}$$

So $\phi_t^{(k)}(\alpha_i) b_i \in (\Lambda'_y + \alpha_i \cdot \Lambda_y) \cap \mathcal{O}_y \cdot b_i = \mathcal{O}_y \cdot \alpha_i b_i$, i. e. $\phi_t^{(k)}(\alpha_i) \in \mathcal{O}_y \cdot \alpha_i$.
Therefore $\mathcal{O}_y \cdot \alpha_i \neq 0$ is a ϕ_t-stable ideal of $\mathcal{O}_y$ and, by proposition 8.3, $\mathcal{O}_y \cdot \alpha_i = \mathcal{O}_y$.
So $\Lambda_y = \Lambda'_y$.
Since this holds for every $y \in U$, we get $\Lambda = \Lambda'$. □

In the following, we show that we can easily calculate differentially stable lattices, if the ID-module is 1-dimensional:
So let M be a 1-dimensional ID-module with basis b and projective system given by $(D_l)_{l \in \mathbb{N}}$, where $D_l \in \mathrm{GL}_1(F_l) = F_l^{\times}$.

For every $x \in \mathcal{C}_F$, we have $\mathrm{ord}_x(D_l) \in p^l\mathbb{Z}$, because $D_l \in F_l$. So $\sum\limits_{l=0}^{\infty} \mathrm{ord}_x(D_l) \in \mathbb{Z}_p$ is a welldefined p-adic integer. Since the product $D_0 \cdots D_l$ is uniquely determined by M and b up to $C \in F_{l+1}^{\times}$, the sum $\sum_{j=0}^{l} \mathrm{ord}_x(D_j) \pmod{p^{l+1}} = \mathrm{ord}_x(D_0 \cdots D_l) \pmod{p^{l+1}}$ is independent of the chosen sequence $(D_l)_{l \in \mathbb{N}}$ for the projective system, and hence $\sum_{l=0}^{\infty} \mathrm{ord}_x(D_l) \in \mathbb{Z}_p$ is independent of the chosen sequence $(D_l)_{l \in \mathbb{N}}$ for the projective system.

Proposition 8.7 *Let $x \in \mathcal{C}_F$, t a local parameter for x and $m \in \mathbb{Z}$. Then the $\mathcal{O}_x$-lattice $bt^m\mathcal{O}_x$ in M is ϕ_t-stable if and only if $m = \sum\limits_{j=0}^{\infty} \mathrm{ord}_x(D_j)$.*

Proof If $bt^m\mathcal{O}_x$ is ϕ_t-stable, then as it will be shown in corollary 8.14, we could choose $D'_j \in \mathrm{GL}_1((\mathcal{O}_x)_j) = (\mathcal{O}_x)_j^{\times}$, such that $bt^m D'_0 \cdots D'_l$ is an $(\mathcal{O}_x)_{l+1}$-basis of $bt^m\mathcal{O}_x \cap M_{l+1}$. So

$$\sum_{j=0}^{\infty} \mathrm{ord}_x(D_j) = \mathrm{ord}_x(t^m D'_0) + \sum_{j=1}^{\infty} \mathrm{ord}_x(D'_j) = m,$$

since $\mathrm{ord}_x(D'_j) = 0$ for all $j \in \mathbb{N}$.
On the other hand, if $\sum_{j=0}^{\infty} \mathrm{ord}_x(D_j) = m$, then for all $l \in \mathbb{N}$, there exists $\beta_l \in \mathcal{O}_x^{\times}$ such that $(D_0 \cdots D_l) = t^{m-m_l}\beta_l^{-1}$, where $m_l := m - \mathrm{ord}_x(D_0 \cdots D_l) \in p^{l+1}\mathbb{Z}$.
Since $m_l \in p^{l+1}\mathbb{Z}$, we have $t^{m_l} \in F_{l+1}$ for all $l \in \mathbb{N}$, and so for all $k < p^{l+1}$,

$$\begin{aligned} \nabla_{\phi_t}^{(k)}(bt^m) &= b \cdot (D_0 \cdots D_l) \phi_t^{(k)}\left((D_0 \cdots D_l)^{-1} t^m\right) \\ &= bt^m \cdot \beta_l^{-1} t^{-m_l} \phi_t^{(k)}\left(t^{m_l}\beta_l\right) \\ &= bt^m \cdot \beta_l^{-1} \phi_t^{(k)}\left(\beta_l\right) \in bt^m\mathcal{O}_x. \end{aligned}$$

Hence, $bt^m\mathcal{O}_x$ is ϕ_t-stable. □

8.3 Regular Points

Definition 8.8 *An ID-module M over F is called* **regular in $\boldsymbol{x} \in \boldsymbol{\mathcal{C}_F}$**, *if there exists a local parameter t for x, an open subset U of U_t and a ϕ_t-stable $\mathcal{O}(U)$-lattice in M. M is called* **regular on $\boldsymbol{V} \subset \boldsymbol{\mathcal{C}_F}$**, *if M is regular in every $x \in V$. We call M* **singular in $\boldsymbol{x} \in \boldsymbol{\mathcal{C}_F}$**, *if M is not regular in x. The set of points in which M is singular, is referred to as the* **singular locus** *of M. If M is singular in all points $x \in \mathcal{C}_F$, then M is called* **totally singular**.

Remark

1. By lemma 8.5, the local parameter t can be chosen arbitrarily.
2. If Λ_U and $\Lambda_{U'}$ are ϕ_t-stable $\mathcal{O}(U)$- resp. $\mathcal{O}(U')$-lattices ($U, U' \subset U_t$), then by lemma 8.6, their localisations to $\mathcal{O}(U \cap U')$ are equal.
3. Contrary to characteristic 0, totally singular ID-modules really exist, what will be shown later.
4. We will also show, that the singular locus always is a closed subset of $\mathcal{C}_F$, and that for every closed subset S of $\mathcal{C}_F$, there exists an ID-module with singular locus S.

Proposition 8.9 *The singular locus S of M is a closed subset of $\mathcal{C}_F$.*

Proof If M is totally singular, then $S = \mathcal{C}_F$ is a closed subset. Assume that M is not totally singular. So there exists $x \in \mathcal{C}_F$, a local parameter t for x, an open subset U of U_t and a ϕ_t-stable $\mathcal{O}(U)$-lattice Λ in M. Now for arbitrary $y \in U$, let s be a local parameter for y and $U' := U_s \cap U$. Then by lemma 8.5, the $\mathcal{O}(U')$-lattice $\mathcal{O}(U') \otimes_{\mathcal{O}(U)} \Lambda$ is ϕ_s-stable. Since $U' \subset \mathcal{C}_F$ is an open subset containing y, we obtain that M is regular in y. So the singular locus S is contained in $\mathcal{C}_F \setminus U$. Hence S is a finite set and therefore a closed subset of $\mathcal{C}_F$. $\square$

Proposition 8.10 *Let M be regular on an open subset $V \subseteq \mathcal{C}_F$, $t \in F$ separating and $\tilde{U}_t := U_t \cap V$. Then there exists a ϕ_t-stable $\mathcal{O}(\tilde{U}_t)$-pseudo-lattice in M.*

Proof For an arbitrary point $x \in \tilde{U}_t$ by definition, there exists a local parameter s for x, an open neighbourhood $U(x) \subset \mathcal{C}_F$ of x (without loss of generality $U(x) \subset \tilde{U}_t$) and a ϕ_s-stable $\mathcal{O}(U(x))$-lattice $\Lambda_{U(x)}$ in M. By lemma 8.5, $\Lambda_{U(x)}$ is also ϕ_t-stable. Now the sets $U(x)$ cover $\tilde{U}_t$ and the localisations of $\Lambda_{U(x)}$ and $\Lambda_{U(y)}$ to $\mathcal{O}(U(x) \cap U(y))$ coincide. So the lattices $\Lambda_{U(x)}$ can be glued together to a sheaf of $\mathcal{O}_{\tilde{U}_t}$-modules, which is induced by a finitely generated $\mathcal{O}(\tilde{U}_t)$-module $\Lambda_{\tilde{U}_t}$ (see [Hart77], Ch. II, Cor. 5.5). This module is ϕ_t-stable, since all localisations are ϕ_t-stable and we have

$$F \otimes_{\mathcal{O}(\tilde{U}_t)} \Lambda_{\tilde{U}_t} = F \otimes_{\mathcal{O}(U(x))} \mathcal{O}(U(x)) \otimes_{\mathcal{O}(\tilde{U}_t)} \Lambda_{\tilde{U}_t} = F \otimes_{\mathcal{O}(U(x))} \Lambda_{U(x)} = M.$$

So $\Lambda_{\tilde{U}_t}$ is a ϕ_t-stable $\mathcal{O}(\tilde{U}_t)$-pseudo-lattice. $\square$

Theorem 8.11 *Let $\boldsymbol{b} = (b_1, \dots, b_n)$ be a basis of M. Then M is not totally singular, if and only if for almost all $y \in \mathcal{C}_F$ the $\mathcal{O}_y$-lattice with basis $\boldsymbol{b}$ is ϕ_t-stable, where t denotes a local parameter for y.*

Proof If there exists a point $x \in \mathcal{C}_F$, in which M is regular, then by definition, there is a local parameter s for x and an open set $U \subset U_s$ such that there is a ϕ_s-stable $\mathcal{O}(U)$-lattice Λ. Let $A \in \mathrm{GL}_n(F)$ be chosen so that $\boldsymbol{b}A$ is a basis for Λ. But for almost all $y \in U$ (and hence for almost all $y \in \mathcal{C}_F$), we have $A \in \mathrm{GL}_n(\mathcal{O}_y)$, and therefore $\mathcal{O}_y \otimes_{\mathcal{O}(U)} \Lambda = \boldsymbol{b} \cdot \mathcal{O}_y^n$ for those y. So by lemma 8.5, for almost all $y \in \mathcal{C}_F$ the $\mathcal{O}_y$-lattice with basis $\boldsymbol{b}$ is ϕ_t-stable, where t denotes a local parameter for y.
On the other hand, let the $\mathcal{O}_y$-lattice with basis $\boldsymbol{b}$ be ϕ_t-stable for all y in a cofinite set $U \subset \mathcal{C}_F$, where t denotes a local parameter for y. Choose an $x \in U$, choose a local parameter s for x and let $\tilde{U} := U \cap U_s$. Then by lemma 8.5, for all $y \in \tilde{U}$, $\boldsymbol{b}\mathcal{O}_y^n$ is a ϕ_s-stable $\mathcal{O}_y$-lattice and so

$$\bigcap_{y \in \tilde{U}} \boldsymbol{b}\mathcal{O}_y^n = \boldsymbol{b}\mathcal{O}(\tilde{U})^n$$

is a ϕ_s-stable $\mathcal{O}(\tilde{U})$-lattice. Hence M is regular in x. $\square$

Theorem 8.12 *Let M be an ID-module which is not totally singular. Then M is regular in $x \in \mathcal{C}_F$ if and only if there exists a ϕ_s-stable $\mathcal{O}_x$-lattice in M, where s denotes a local parameter for x.*

Proof If M is regular in $x \in \mathcal{C}_F$, then we get a ϕ_s-stable $\mathcal{O}_x$-lattice in M by localising the ϕ_s-stable $\mathcal{O}(U)$-lattice in the definition of a regular point.
Now assume there exists a ϕ_s-stable $\mathcal{O}_x$-lattice in M and let $\boldsymbol{b}$ be an $\mathcal{O}_x$-basis of this lattice. Since M is not totally singular, for almost all $y \in \mathcal{C}_F$, the $\mathcal{O}_y$-lattice $\boldsymbol{b}\mathcal{O}_y^n$ is ϕ_t-stable (t a local parameter for y), by theorem 8.11. Furthermore, the proof of theorem 8.11 shows that M is regular in all these points, in particular M is regular in x. $\square$

Theorem 8.13 *Let $x \in \mathcal{C}_F$ be a point in which M is regular, t a local parameter for x, $U \subset U_t$ and Λ a ϕ_t-stable $\mathcal{O}(U)$-pseudo-lattice in M. Then for arbitrary $l \in \mathbb{N}$, there exists a generating set for Λ (as $\mathcal{O}(U)$-module) consisting of elements of M_l.*

Proof Denote by $\{b_1, \dots, b_n\}$ an F_l-basis of M_l. Since Λ is a pseudo-lattice in M, there exist $0 \neq a_i \in \mathcal{O}(U)$ such that $b_i a_i \in \Lambda$ ($i = 1, \dots, n$) and therefore $b_i a_i^{p^l} \in \Lambda \cap M_l =: \Lambda_l$ ($i = 1, \dots, n$). So Λ_l contains a basis of M_l and is a finitely generated $\mathcal{O}(U)_l$-module (finitely generated, because it is a submodule of the finitely generated $\mathcal{O}(U)_l$-module Λ), i. e. Λ_l is an $\mathcal{O}(U)_l$-pseudo-lattice in M_l. Hence $\mathcal{O}(U)\cdot\Lambda_l$ is an $\mathcal{O}(U)$-pseudo-lattice in M.

By assumption, Λ is ϕ_t-stable and so $\Lambda\cap M_l=\Lambda_l$ is ϕ_t-stable, too. Furthermore $\mathcal{O}(U)$ is ϕ_t-stable and so $\mathcal{O}(U)\cdot\Lambda_l$ is also ϕ_t-stable. Therefore by lemma 8.6, $\Lambda=\mathcal{O}(U)\cdot\Lambda_l$, which completes the proof. □

Corollary 8.14 *Let $x\in\mathcal{C}_F$ be a point in which M is regular, t a local parameter for x and Λ a ϕ_t-stable $\mathcal{O}_x$-lattice in M. Denote by $\boldsymbol{b}=(b_1,\dots,b_n)$ an $\mathcal{O}_x$-basis for Λ. Then there exist matrices $D_l\in\mathrm{GL}_n((\mathcal{O}_x)_l)$ $(l=0,1,\dots)$, such that $\boldsymbol{b}D_0\cdots D_j$ is an $(\mathcal{O}_x)_{j+1}$-basis of $\Lambda_{j+1}:=\Lambda\cap M_{j+1}$.*

Proof Since $(\mathcal{O}_x)_{l+1}$ is a local ring, every $(\mathcal{O}_x)_{l+1}$-pseudo-lattice in M_{l+1} is in fact an $(\mathcal{O}_x)_{l+1}$-lattice. By the previous theorem, an $(\mathcal{O}_x)_{l+1}$-basis of Λ_{l+1} also is an $(\mathcal{O}_x)_l$-basis of Λ_l. So there exists a base change matrix in $\mathrm{GL}_n((\mathcal{O}_x)_l)$. Starting from the $\mathcal{O}_x$-basis $(b_1,\dots,b_n)$ for Λ, we obtain all D_l $(l=0,1,\dots)$ step by step. □

In section 4.2, we defined higher connections on $\mathcal{O}_X$-modules, where X is a scheme. We also mentioned, when a higher connection on a $K(X)$-vector space should be called regular on an open subset $U\subseteq X$. We will now show, that this coincides with the definition of regularity in this section.

Proposition 8.15 *M is regular on an open subset $U\subseteq\mathcal{C}_F$, if and only if there is a coherent $\mathcal{O}_U$-module $\tilde{\Lambda}$ with a higher connection $\nabla':\tilde{\Lambda}\to\hat{\Omega}_{U/K}\otimes_{\mathcal{O}_U}\tilde{\Lambda}$ such that $F\otimes_{\mathcal{O}_U}\tilde{\Lambda}\cong M$ and ∇ equals $\mathrm{d}_F\otimes\nabla'$ as higher connections on $M=F\otimes_{\mathcal{O}_U}\tilde{\Lambda}$.*

Proof Let M be regular on $U\subseteq\mathcal{C}_F$. Then by proposition 8.10, for every separating $t\in F$ and $\tilde{U}_t:=U_t\cap U$, there is a ϕ_t-stable $\mathcal{O}(\tilde{U}_t)$-pseudo-lattice $\Lambda_{\tilde{U}_t}$ in M. We show that for $m\in\Lambda_{\tilde{U}_t}$, we have $\nabla(m)\in\hat{\Omega}_{\mathcal{O}(\tilde{U}_t)}\otimes_{\mathcal{O}(\tilde{U}_t)}\Lambda_{\tilde{U}_t}$:
For given $k\in\mathbb{N}$, choose $l\in\mathbb{N}$ with $p^l>k$ and choose a generating set $\{b_1,\dots,b_r\}$ for Λ with $b_i\in M_l$ (cf. theorem 8.13). Then there are $a_i\in\mathcal{O}(\tilde{U}_t)$ such that $m=\sum_{i=1}^r a_ib_i$ and therefore

$$\nabla^{(k)}(m)=\sum_{i=1}^r\nabla^{(k)}(a_ib_i)=\sum_{i=1}^r\mathrm{d}_F(a_i)\otimes b_i\in\left(\hat{\Omega}_{\mathcal{O}(\tilde{U}_t)/K}\right)_k\otimes_{\mathcal{O}(\tilde{U}_t)}\Lambda_{\tilde{U}_t}.$$

So $\nabla(m)\in\hat{\Omega}_{\mathcal{O}(\tilde{U}_t)}\otimes_{\mathcal{O}(\tilde{U}_t)}\Lambda_{\tilde{U}_t}$. Since the open sets $(U_t)_{t\in F\text{ sep.}}$ cover $\mathcal{C}_F$, we have $\bigcup_{t\in F\text{ sep.}}\tilde{U}_t=U$, and as in proposition 8.10, the pseudo-lattices $\Lambda_{\tilde{U}_t}$ can be glued together to a sheaf of $\mathcal{O}_U$-modules $\tilde{\Lambda}$. Since on the open covering $\{\tilde{U}_t\mid t\in F\text{ sep.}\}$, the higher connection ∇ restricts to a higher connection $\nabla_{\tilde{U}_t}:\Lambda_{\tilde{U}_t}\to\hat{\Omega}_{\tilde{U}_t/K}\otimes\Lambda_{\tilde{U}_t}$, these higher connections can be glued together to a higher connection $\nabla':\tilde{\Lambda}\to\hat{\Omega}_{U/K}\otimes_{\mathcal{O}_U}\tilde{\Lambda}$, that clearly fulfills $\nabla=\mathrm{d}_F\otimes\nabla'$.

For the converse, let $\tilde{\Lambda}$ be a coherent $\mathcal{O}_U$-module with higher connection ∇' satisfying the properties above and let $x \in U$ and t be a local parameter for x. Then $\tilde{\Lambda}(U \cap U_t)$ is a projective $\mathcal{O}_U(U \cap U_t)$-module (all localisations at maximal ideals are torsionfree, hence free) and so by [Eis95], thm. A3.2, there is an open neighbourhood $\tilde{U}_t \subset U \cap U_t$ of x, such that $\Lambda := \tilde{\Lambda}(\tilde{U}_t)$ is a free $\mathcal{O}(\tilde{U}_t)$-module. Finally, since $\nabla(\Lambda) = \nabla'(\Lambda) \subset \hat{\Omega}_{\mathcal{O}(\tilde{U}_t)/K} \otimes \Lambda$, we have

$$\nabla_{\phi_t}(\Lambda) = (\tilde{\phi}_t \otimes \mathrm{id}_\Lambda)(\nabla(\Lambda)) \subset \mathcal{O}(\tilde{U}_t)[[T]] \otimes \Lambda = \Lambda[[T]].$$

So M is regular in x. □

We now turn our attention to 1-dimensional ID-modules:

So let M be a 1-dimensional ID-module with basis b and projective system given by $(D_l)_{l\in\mathbb{N}}$, where $D_l \in \mathrm{GL}_1(F_l) = F_l^\times$.

Lemma 8.16 *M is totally singular if and only if for infinitely many $x \in \mathcal{C}_F$, $\sum_{l=0}^{\infty} \mathrm{ord}_x(D_l) \neq 0$. If M is not totally singular, then M is regular in all $x \in \mathcal{C}_F$ for which $\sum_{l=0}^{\infty} \mathrm{ord}_x(D_l) \in \mathbb{Z}$.*

Proof This follows immediately from proposition 8.7, theorem 8.11 and theorem 8.12. □

Corollary 8.17 *If M is regular in all points different from a point $x \in \mathcal{C}_F$. Then M is also regular in x.*

Proof Let $(D_l)_{l\in\mathbb{N}}$ be a sequence giving the projective system associated to M. If M is regular in all points $y \neq x$. Then $\sum_{l=0}^{\infty} \mathrm{ord}_y(D_l) \in \mathbb{Z}$ for all $y \neq x$. But for all $l \in \mathbb{N}$, we have $\mathrm{ord}_x(D_l) = -\sum_{y\in\mathcal{C}_F\setminus\{x\}} \mathrm{ord}_x(D_l)$ and therefore

$$\sum_{l=0}^{\infty} \mathrm{ord}_x(D_l) = -\sum_{y\in\mathcal{C}_F\setminus\{x\}} \sum_{l=0}^{\infty} \mathrm{ord}_y(D_l) \in \mathbb{Z} \subset \mathbb{Z}_p.$$

Hence by lemma 8.16, M is regular in x, too. □

Example 8.18 Let $F = K(t)$ be the rational function field in one variable and choose a sequence $(a_n)_{n\in\mathbb{N}}$ of distinct elements of K. Define a 1-dimensional ID-module M with projective system given by $D_l := (t-a_l)^{p^l(p-1)} \in F_l^\times$, $l \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \mathrm{ord}_{(t-a)}(D_j) = p^l(p-1)$ for $a = a_l$ and so $\sum_{j=0}^{\infty} \mathrm{ord}_{(t-a)}(D_j) \neq 0$ for infinitely many $(t-a) \in \mathcal{C}_F$. Hence by lemma 8.16, M is totally singular. Further, for all $a \notin \{a_l \mid l \in \mathbb{N}\}$, we have $\sum_{j=0}^{\infty} \mathrm{ord}_{(t-a)}(D_j) = 0 \in \mathbb{Z}$ and $\sum_{j=0}^{\infty} \mathrm{ord}_{\infty}(D_j) = \sum_{j=0}^{\infty}(p^j - p^{j+1}) = 1 \in \mathbb{Z}$ and for all $l \in \mathbb{N}$, $\sum_{j=0}^{\infty} \mathrm{ord}_{(t-a_l)}(D_j) = p^l(p-1) \in \mathbb{Z}$. So by proposition 8.7, for all $x \in \mathcal{C}_F$ and local parameter s for x, there exists a ϕ_s-stable $\mathcal{O}_x$-lattice.

This example shows, that totally singular ID-modules exist. It also shows, that an ID-module can be singular in some points even if for all points $x \in \mathcal{C}_F$, there exist differentially stable $\mathcal{O}_x$-lattices. However, in theorem 8.12, we have seen that this doesn't happen, if M is not totally singular.

Notation In the following, we denote by $\mathcal{J}$ the Jacobian variety of $\mathcal{C}_F$ and by $T_p(\mathcal{J})$ the p-adic Tate-module of $\mathcal{J}$, i. e. $T_p(\mathcal{J}) = \varprojlim \mathcal{J}[p^n]$, where $\mathcal{J}[p^n]$ denote the points of p^n-torsion of $\mathcal{J}$.

The set of isomorphism classes of 1-dimensional ID-modules over F will be denoted by $\mathrm{Isom}_{\mathcal{C}_F,1}$. With multiplication given by the tensor product, $\mathrm{Isom}_{\mathcal{C}_F,1}$ is an abelian group. Further, let

$$\mathrm{Div}^0(\mathcal{C}_F, \mathbb{Z}_p) := \left\{ f : \mathcal{C}_F \to \mathbb{Z}_p \,\middle|\, \forall\, l \in \mathbb{N} : |supp(f \pmod{p^l})| < \infty \text{ and } \sum_{x \in \mathcal{C}_F} f(x) = 0 \right\}$$

and let $\mathrm{H}(\mathcal{C}_F)$ be the group of principal divisors on $\mathcal{C}_F$, which can be regarded as a subgroup of $\mathrm{Div}^0(\mathcal{C}_F, \mathbb{Z}_p)$.

Theorem 8.19 *(cf. [MvdP03], prop. 4.2) There is a short exact sequence of abelian groups*

$$0 \to T_p(\mathcal{J}) \to \mathrm{Isom}_{\mathcal{C}_F,1} \xrightarrow{\chi} \frac{\mathrm{Div}^0(\mathcal{C}_F, \mathbb{Z}_p)}{\mathrm{H}(\mathcal{C}_F)} \to 0$$

where the homomorphism χ is given in the following way: For an ID-module M with basis b, calculate a sequence $(D_l)_{l \in \mathbb{N}}$ of elements in $\mathrm{GL}_1(F_l)$ such that $bD_0 \cdots D_l$ is an F_l-basis for M_l. Then $\chi([M])$ is represented by the map $x \mapsto \sum_{l=0}^{\infty} \mathrm{ord}_x(D_l) \in \mathbb{Z}_p$.

Proposition 8.20 *For every closed subset $S \subseteq \mathcal{C}_F$, there exists an ID-module with singular locus S.*

Proof We have already seen (cf. example 8.18), that there exist totally singular ID-modules, i. e. ID-modules with singular locus equal to $\mathcal{C}_F$. If $S \neq \mathcal{C}_F$, i. e. S is finite, and if $\#S \geq 2$ or $S = \emptyset$, then for all $x \in S$ choose $\alpha_x \in \mathbb{Z}_p \setminus \mathbb{Z}$ such that $\sum_{x \in S} \alpha_x = 0$. The map $f : \mathcal{C}_F \to \mathbb{Z}_p$ defined by $f(x) := \alpha_x$ for $x \in S$ and $f(x) := 0$ for $x \notin S$ is an element of $\mathrm{Div}^0(\mathcal{C}_F, \mathbb{Z}_p)$. So by theorem 8.19, there is a 1-dimensional ID-module M such that $\chi([M])$ is represented by f. By theorem 8.16, M is not totally singular and M is singular exactly in the points in S, i. e. S is the singular locus of M.

If remains to show, that S occurs as singular locus, when $\#S = 1$. For this choose an element $t \in F$, whose pole divisor $(t)_\infty$ has support equal to S, and

define a projective system for a 2-dimensional ID-module M with basis (b_1, b_2) by choosing matrices $D_l := \left(\begin{smallmatrix} 1 & a_l \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(F_l)$, $l \in \mathbb{N}$, where $a_l = t^{p^l}$. Then for all $y \notin S$, $D_l \in \mathrm{GL}_2((\mathcal{O}_y)_l)$ and hence by theorem 8.11 and theorem 8.12, M is regular outside S. On the other hand, if M is also regular in the point $x \in S$. Then there exists a differentially stable $\mathcal{O}_x$-lattice Λ in M. Since $b_1\mathcal{O}_x \subset M \cap b_1 F$ is the unique differentially stable $\mathcal{O}_x$-lattice in $M \cap b_1 F$ and since $\boldsymbol{b}\mathcal{O}_x^2/b_1\mathcal{O}_x \subset M/(M \cap b_1 F)$ is the unique differentially stable $\mathcal{O}_x$-lattice in $M/(M \cap b_1 F)$, we have $\Lambda = \boldsymbol{b}\mathcal{O}_x^2$.
Let s be a local parameter for x. Then for all $l \in \mathbb{N}$:

$$\nabla_{\phi_s}^{(p^l)}(b_2) = -b_1 \sum_{i=0}^{l} \phi_s^{(p^l)}(t^{p^i}) = -b_1 \sum_{i=0}^{l} \left(\phi_s^{(p^{l-i})}(t)\right)^{p^i}.$$

Since $t \notin \mathcal{O}_x$, there exists a minimal $j \geq 0$ such that $\phi_s^{(p^j)}(t) \notin \mathcal{O}_x$ and so

$$\nabla_{\phi_s}^{(p^j)}(b_2) = -b_1 \left(\phi_s^{(p^j)}(t) + \sum_{i=1}^{j} \left(\phi_s^{(p^{j-i})}(t)\right)^{p^i}\right) \notin \Lambda,$$

a contradiction. Therefore M is singular in $x \in S$. □

We now regard regular points of IPV-extensions:

Definition 8.21 *Let L/F be an IPV-extension for M. Then L/F is called* **regular in** $x \in \mathcal{C}_F$*, if M is regular in x. Otherwise we call L/F* **singular in** $x \in \mathcal{C}_F$*. The set of points in which L/F is singular, is referred to as the* **singular locus** *of L/F.*

Proposition 8.22 *If L/F is regular in $x \in \mathcal{C}_F$ and t is a local parameter for x. Then there exists a monomorphism of iterative differential fields $(L, d_L) \hookrightarrow \big(K((t)), d_{K((t))}\big)$, where $K((t))$ is regarded as the completion of F with respect to the valuation corresponding to x and $d_{K((t))}$ is the continuous extension of d_F.*

Proof Let $A \in \mathrm{Mat}_n(\hat{\Omega}_{F/K})$ satisfy $\nabla(\boldsymbol{b}) = \boldsymbol{b}A$ and let $A_x \in \mathrm{GL}_n(F)$ be such that $\boldsymbol{b}A_x^{-1}$ is a basis for the ϕ_t-stable $\mathcal{O}_x$-lattice Λ. Then by corollary 8.14, there are $D_j \in \mathrm{GL}_n((\mathcal{O}_x)_j)$ $(j = 0, 1, \dots)$ such that $\boldsymbol{b}A_x^{-1}D_0 \cdots D_l$ is an $(\mathcal{O}_x)_{l+1}$-basis of $\Lambda_{l+1} := \Lambda \cap M_{l+1}$. Moreover, we can choose D_j in such a way, that $D_j|_{t=0} = \mathbf{1}_n$. Hence $D_0 \cdots D_l \equiv D_0 \cdots D_{l+1} \pmod{t^{p^{l+1}}}$ and therefore the matrix entries of the sequence $(D_0 \cdots D_l)_{l \in \mathbb{N}}$ converge in the completion $\widehat{\mathcal{O}}_x \cong K[[t]]$. Let $D \in \mathrm{Mat}_n(K[[t]])$ denote the limit, then $D \in \mathrm{GL}_n(K[[t]])$, since $D|_{t=0} = \mathbf{1}_n$, and for all $k \in \mathbb{N}$ we have $\mathrm{pr}_k(A) = A_x^{-1} D d_{K((t))}^{(k)}(D^{-1}A_x)$ and therefore $A = A_x^{-1} D d_{K((t))}(D^{-1}A_x) \in \mathrm{Mat}_n\left(K((t)) \otimes \hat{\Omega}_{F/K}\right)$.
So the ID-field $K((t))$ has a fundamental solution matrix $A_x^{-1}D$ and therefore the IPV-field L is iterative differentially isomorphic to a subfield of $K((t))$. □

8.4 Iterative Differential Closure

Let L/F be an IPV-extension for an ID-module M with Galois group $\mathcal{H}(K) \leq \mathrm{GL}_n(K)$ and singular locus inside a finite set $S \subset \mathcal{C}_F$. Denote by $Y \in \mathrm{GL}_n(L)$ a fundamental solution matrix for M with respect to a basis $\boldsymbol{b} = (b_1, \dots, b_n)$ of M. And denote by $\mathcal{O} \subset F$ the ring of regular functions on $\mathcal{C}_F \setminus S$.

Definition 8.23 *For a point $x \in \mathcal{C}_F \setminus S$, the* **iterative differential closure of** *$\mathcal{O}_x$ is by definition the largest subring $\mathcal{O}_{L,x} \subset L$ of L such that for all local parameters t for x and all iterative differential embeddings $\iota : (L, d_L) \hookrightarrow (K((t)), d_{K((t))})$, the image $\iota(\mathcal{O}_{L,x})$ of $\mathcal{O}_{L,x}$ is contained in $K[[t]]$. The* **iterative differential closure of** $\mathcal{O}$ *is by definition the subring*

$$\mathcal{O}_L := \bigcap_{x \in \mathcal{C}_F \setminus S} \mathcal{O}_{L,x}.$$

Proposition 8.24 *For $x \in \mathcal{C}_F \setminus S$ let $A_x \in \mathrm{GL}_n(F)$ be a matrix such that $\boldsymbol{b}A_x^{-1}$ is a basis for the ϕ_t-stable $\mathcal{O}_x$-lattice in M. Then*

$$\mathcal{O}_x[A_x Y] \subseteq \mathcal{O}_{L,x}.$$

Proof Let $x \in \mathcal{C}_F \setminus S$ and t be a local parameter for x. Since Y is a fundamental solution matrix with respect to $\boldsymbol{b}$, $A_x Y$ is a fundamental solution matrix with respect to $\boldsymbol{b}A_x^{-1}$. So by propostion 8.22, there is an ID-monomorphism $\iota : L \to K((t))$ such that $\iota(A_x Y) \in \mathrm{Mat}_n(K[[t]])$. Since two fundamental solution matrices differ by a matrix $C \in \mathrm{GL}_n(K)$, for every ID-monomorphism $\iota : L \to K((t))$, we have $\iota(A_x Y) \in C \cdot \mathrm{Mat}_n(K[[t]]) = \mathrm{Mat}_n(K[[t]])$. So $\iota(\mathcal{O}_x[A_x Y]) \subseteq K[[t]]$ for all ID-monomorphisms $\iota : L \to K((t))$. So $\mathcal{O}_x[A_x Y] \subseteq \mathcal{O}_{L,x}$. □

Remark Obviously, for all $x \in \mathcal{C}_F \setminus S$, we have

$$\mathcal{O}_{L,x} \cap F = \mathcal{O}_x.$$

The next theorem is a refinement of proposition 7.14.

Theorem 8.25 *Let $\mathcal{N} \rtimes \mathcal{H} \leq \mathrm{GL}_n$ be a semidirect product of connected linear algebraic groups. Let L/F be an IPV-extension with $\mathrm{Gal}(L/F) \cong \mathcal{H}(K)$ for an ID-module M over F and assume there is a fundamental solution matrix $Y \in \mathcal{H}(\mathcal{O}_L)$ for M with respect to an appropriate basis $\boldsymbol{b} = (b_1, \dots, b_n)$. Assume that for any $x \in \mathcal{C}_F \setminus S$ there are $C_l \in \mathcal{H}((\mathcal{O}_x)_l)$ ($l = 0, 1, \dots$) such that $C_{l-1}^{-1} \cdots C_0^{-1} Y =: Y_l \in \mathcal{H}((\mathcal{O}_{L,x})_l)$. Then*

$$\mathcal{N}((\mathcal{O}_{L,x})_l)^{\mathcal{H}} = Y_l^{-1} \mathcal{N}((\mathcal{O}_x)_l) Y_l.$$

Proof By proposition 7.14, we have

$$\mathcal{N}(L)^{\mathcal{H}} = Y^{-1}\mathcal{N}(F)Y = Y^{-1}C_0 \cdots C_{l-1}\mathcal{N}(F)C_{l-1}^{-1} \cdots C_0^{-1}Y = Y_l^{-1}\mathcal{N}(F)Y_l.$$

Furthermore, since $Y_l \in \mathcal{H}\left((\mathcal{O}_{L,x})_l\right)$, we have $D \in \mathcal{N}\left((\mathcal{O}_{L,x})_l\right)$ if and only if $Y_l D Y_l^{-1} \in \mathcal{N}\left((\mathcal{O}_{L,x})_l\right)$. And therefore

$$\begin{aligned} \mathcal{N}\left((\mathcal{O}_{L,x})_l\right)^{\mathcal{H}} &= \mathcal{N}\left((\mathcal{O}_{L,x})_l\right) \cap \mathcal{N}(L)^{\mathcal{H}} = Y_l^{-1}\mathcal{N}\left((\mathcal{O}_{L,x})_l\right)Y_l \cap Y_l^{-1}\mathcal{N}(F)Y_l \\ &= Y_l^{-1}\mathcal{N}\left((\mathcal{O}_x)_l\right)Y_l. \end{aligned}$$

□

9 Realisation with Restricted Singular Locus

In this chapter, we consider the problem, for which linear algebraic group $\mathcal{G}$ the group $\mathcal{G}(K)$ of K-rational points occurs as the Galois group of an IPV-extension E/F with singular locus inside a given finite set $S \subset \mathcal{C}_F$. (We say that $\mathcal{G}$ is realisable over F regularly outside S.)

9.1 On the Abhyankar Conjecture

Conjecture (Differential Abhyankar Conjecture) Assume F is an algebraic function field (in one variable) over K with nonsingular projective model $\mathcal{C}_F$, and let $\emptyset \neq S \subset \mathcal{C}_F$ be a finite subset. Suppose $\mathcal{G}$ is a linear algebraic group over K and let $p(\mathcal{G})$ denote the (normal) subgroup of $\mathcal{G}$ generated by all unipotent elements. Then $\mathcal{G}$ is realisable as an iterative differential Galois group over F regularly outside S if and only if $\mathcal{G}/p(\mathcal{G})$ is.

Raynaud and Harbater have proved this conjecture for finite groups $\mathcal{G}$ (see [Ray94], [Har94] and [Har95]). In the next sections, I will prove this conjecture for connected groups.
However, this conjecture is not true in this generality as the following example shows:
Let $K = \overline{\mathbb{F}}_2$ be the algebraic closure of the field of two elements and let $\mathbb{D}_\infty := \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ be the infinite dihedral group, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{G}_m$ be inverting the elements. So since $\mathrm{char}(K) = 2$ and all elements of $\mathbb{D}_\infty \setminus \mathbb{G}_m$ have order 2, $\mathbb{D}_\infty$ is unipotently generated. Therefore by the Abhyankar conjecture, $\mathbb{D}_\infty$ should be realisable with at most one singular point over $K(t)$.

Theorem 9.1 *Let $K = \overline{\mathbb{F}}_2$ and let $F = K(t)$ be the rational function field over K. Then the infinite dihedral group $\mathbb{D}_\infty := \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ is not realisable over F with only one singular point.*

Proof Assume E/F is an IPV-extension with Galois group $\mathrm{Gal}(E/F) = \mathbb{D}_\infty(K)$ and singular locus inside $S \subset \mathcal{C}_F$ with $\#S = 1$, and without loss of generality let $S = \{\infty\}$. Then the fixed field $L := E^{\mathbb{G}_m(K)}$ is a finite extension of F with Galois group $\mathrm{Gal}(L/F) = \mathbb{Z}/2\mathbb{Z}$ and L/F is regular outside S. But such an extension is given as $L = K(t, s)$ with $s^2 + s = f(t)$, where $f(t) \in K[t] \subset F$. Now E/L is an IPV-extension with Galois group $\mathbb{G}_m(K)$ and singular locus inside $S_L := \{\infty_L\}$, the place of L lying over $\infty \in \mathcal{C}_F$. (Since ∞ is ramified in L/F, there is only one place ∞_L over ∞.)
Let M be a 1-dimensional ID-module over L with IPV-field E, then $[M] \in \mathrm{Isom}_{\mathcal{C}_L,1}$ has infinite order, since the Galois group is infinite. But a short calculation shows, that $\mathcal{J}_L$ has no 2-torsion and so $T_2(\mathcal{J}_L) = 0$. Therefore the homomorphism χ in theorem 8.19 is an isomorphism. By corollary 8.17, since

M is singular in at most one point, M is regular in all points and $\chi([M]) \in \mathcal{J}_L$. Finally, by the general theory on Jacobian varieties, since $K = \overline{\mathbb{F}}_2$, the Jacobian $\mathcal{J}_L$ has no element of infinite order. So $[M]$ cannot have infinite order. $\square$

Remark This proof only works if $K = \overline{\mathbb{F}}_2$, because this is the only case (in characteristic 2) for which $\mathcal{J}_L$ has no element of infinite order. Furthermore if K is an algebraically closed field of characteristic 2 and $K \neq \overline{\mathbb{F}}_2$, then in fact one can find the desired ID-module M, and $\mathbb{D}_\infty$ can be realised with exactly one singular point.

One might wonder if this example only occurs in characteristic 2, because there is no other p-group acting nontrivially on $\mathbb{G}_m$. But this example can be generalised to arbitrary characteristic p by regarding the group

$$\mathcal{G} := \{(a_1, \ldots, a_p) \in \mathbb{G}_m^p \mid a_1 \cdots a_p = 1\}$$

on which $\mathbb{Z}/p\mathbb{Z}$ acts by cyclic permutation of the components.

In the next sections, we restrict to connected linear algebraic groups. We will show that every reduced connected linear algebraic group $\mathcal{G}$ is realisable regularly outside a set $S \subset \mathcal{C}_F$ of order $\#S = 2$ for any algebraic function field F/K.

The proof will show that the Abhyankar conjecture is true for connected groups.

9.2 Dividing the Problem of Realisation

We first give a splitting of a connected linear algebraic group $\mathcal{G}$ into parts that are easier to handle with, regarding the problem of realisation.

Notation So let $\mathcal{G}$ be a reduced connected linear algebraic group, $R(\mathcal{G})$ its radical and $\mathcal{R}_u := R_u(\mathcal{G})$ its unipotent radical. Furthermore let $\mathcal{T}_0$ be a maximal torus of $R(\mathcal{G})$ and $\mathcal{Z} := C_{\mathcal{G}}(\mathcal{T}_0)$ the centraliser of $\mathcal{T}_0$ in $\mathcal{G}$ and let $[\mathcal{Z}, \mathcal{Z}]$ denote the commutator subgroup of $\mathcal{Z}$.

Theorem 9.2 *The inclusions of the subgroups $\mathcal{R}_u$, $\mathcal{T}_0$ and $[\mathcal{Z}, \mathcal{Z}]$ into $\mathcal{G}$ induce an epimorphism of algebraic groups*

$$\mathcal{R}_u \rtimes \big(\mathcal{T}_0 \times [\mathcal{Z}, \mathcal{Z}]\big) \longrightarrow \mathcal{G},$$

where the action of $\mathcal{T}_0 \times [\mathcal{Z}, \mathcal{Z}]$ on $\mathcal{R}_u$ is given by conjugation in $\mathcal{G}$.

For the proof, we need some lemmata:

Lemma 9.3 *$\mathcal{G}$ is generated by $\mathcal{R}_u$ and $\mathcal{Z}$.*

Proof Let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$. Then $\mathcal{T} \cap R(\mathcal{G})$ is a maximal torus of $R(\mathcal{G})$ and therefore conjugate to $\mathcal{T}_0$. Since $R(\mathcal{G}) = \mathcal{R}_u \cdot \mathcal{T}_0$, there exists $a \in \mathcal{R}_u$

such that $\mathcal{T} \cap R(\mathcal{G}) = a\mathcal{T}_0 a^{-1}$. And so, we get

$$C_{\mathcal{G}}(\mathcal{T}) \subset C_{\mathcal{G}}\big(\mathcal{T} \cap R(\mathcal{G})\big) = C_{\mathcal{G}}(a\mathcal{T}_0 a^{-1}) = aC_{\mathcal{G}}(\mathcal{T}_0)a^{-1} = a\mathcal{Z}a^{-1}.$$

So the union of all conjugates $a\mathcal{Z}a^{-1}$ contains all Cartan subgroups of $\mathcal{G}$. By [Spr98], thm. 6.4.5(iii), and [Spr98], lemma 2.2.3, these conjugates generate $\mathcal{G}$ and therefore $\mathcal{R}_u$ and $\mathcal{Z}$ generate $\mathcal{G}$. □

Lemma 9.4 *A connected linear algebraic group $\mathcal{H}$ is generated by its radical $R(\mathcal{H})$ and its commutator subgroup $[\mathcal{H}, \mathcal{H}]$.*

Proof By [Spr98], cor. 8.1.6(i), the factor group $\mathcal{H}/R_u(\mathcal{H})$ is generated by its radical and its commutator subgroup. But since $R(\mathcal{H}/R_u(\mathcal{H})) = R(\mathcal{H})/R_u(\mathcal{H})$ and $[\mathcal{H}/R_u(\mathcal{H}), \mathcal{H}/R_u(\mathcal{H})] = ([\mathcal{H},\mathcal{H}]R_u(\mathcal{H}))/R_u(\mathcal{H})$, $\mathcal{H}$ is generated by $R(\mathcal{H})$ and $[\mathcal{H},\mathcal{H}]$. □

Lemma 9.5 *$R(\mathcal{Z})$ equals the identity component of $R(\mathcal{G}) \cap \mathcal{Z}$ (denoted by $(R(\mathcal{G}) \cap \mathcal{Z})^\circ$) and $R_u(\mathcal{Z}) = (\mathcal{R}_u \cap \mathcal{Z})^\circ$. Furthermore $\mathcal{T}_0$ is a maximal torus of $R(\mathcal{Z})$.*

Proof By [Spr98], thm. 6.4.7, a Borel subgroup of $\mathcal{Z} = Z_{\mathcal{G}}(\mathcal{T}_0)$ is the intersection of $\mathcal{Z}$ with a Borel subgroup of $\mathcal{G}$ containing $\mathcal{T}_0$. Since $\mathcal{T}_0$ lies in the radical of $\mathcal{G}$, $\mathcal{T}_0$ is contained in every Borel subgroup of $\mathcal{G}$. And therefore:

$$\begin{aligned} R(\mathcal{Z}) &= \left(\bigcap_{\substack{\tilde{B}\subset\mathcal{Z}\\ \text{Borel}}} \tilde{B}\right)^\circ = \left(\bigcap_{\substack{B\subset\mathcal{G}\\ \text{Borel}}} B \cap \mathcal{Z}\right)^\circ \\ &= (R(\mathcal{G}) \cap \mathcal{Z})^\circ \\ &= ((\mathcal{R}_u \rtimes \mathcal{T}_0) \cap \mathcal{Z})^\circ = ((\mathcal{R}_u \cap \mathcal{Z}) \rtimes \mathcal{T}_0)^\circ \\ &= (\mathcal{R}_u \cap \mathcal{Z})^\circ \times \mathcal{T}_0, \end{aligned}$$

since $\mathcal{T}_0$ is central in $\mathcal{Z}$. Since all elements of $(\mathcal{R}_u \cap \mathcal{Z})^\circ$ are unipotent, we obtain $R_u(\mathcal{Z}) = (\mathcal{R}_u \cap \mathcal{Z})^\circ$. If follows immediately from $R(\mathcal{Z}) = (\mathcal{R}_u \cap \mathcal{Z})^\circ \times \mathcal{T}_0$, that $\mathcal{T}_0$ is a maximal torus of $R(\mathcal{Z})$. □

Proof of theorem 9.2 Since by definition of $\mathcal{Z}$, the elements of $\mathcal{T}_0 \subset \mathcal{G}$ and of $\mathcal{Z} \subset \mathcal{G}$ commute, the map $\mathcal{T}_0 \times [\mathcal{Z},\mathcal{Z}] \to \mathcal{G}$ induced by the inclusions is a homomorphism of algebraic groups, and therefore the map $\mathcal{R}_u \rtimes \big(\mathcal{T}_0 \times [\mathcal{Z},\mathcal{Z}]\big) \longrightarrow \mathcal{G}$ also is a homomorphism of algebraic groups. So it is sufficient to show, that these subgroups generate $\mathcal{G}$.
By lemma 9.4, $\mathcal{Z}$ is generated by $R(\mathcal{Z})$ and $[\mathcal{Z},\mathcal{Z}]$ and therefore by lemma 9.5, it is generated by $(\mathcal{R}_u \cap \mathcal{Z})^\circ$, $\mathcal{T}_0$ and $[\mathcal{Z},\mathcal{Z}]$. So by lemma 9.3, $\mathcal{G}$ is generated by

$\mathcal{R}_u$, $(\mathcal{R}_u \cap \mathcal{Z})^\circ$, $\mathcal{T}_0$ and $[\mathcal{Z},\mathcal{Z}]$, i. e. by $\mathcal{R}_u$, $\mathcal{T}_0$ and $[\mathcal{Z},\mathcal{Z}]$. $\square$

At last, we give a structural property of commutator subgroups in positive characteristic:

Theorem 9.6 *Let $\mathcal{H}$ be a connected linear algebraic group (over an algebraically closed field of positive characteristic p). Then the commutator subgroup $[\mathcal{H},\mathcal{H}]$ is unipotently generated.*

Proof The group $\mathcal{H}/R_u(\mathcal{H})$ is reductive and so by [Spr98], cor. 8.1.6(ii), the group $[\mathcal{H}/R_u(\mathcal{H}), \mathcal{H}/R_u(\mathcal{H})]$ is semisimple, and then by [Spr98], thm. 8.1.5(i), unipotently generated. Let $a \in [\mathcal{H},\mathcal{H}]$ be a representative of a unipotent element of $[\mathcal{H}/R_u(\mathcal{H}), \mathcal{H}/R_u(\mathcal{H})]$. So there is $n \in \mathbb{N}$ such that $a^{p^n} \in R_u(\mathcal{H})$. But since all elements of $R_u(\mathcal{H})$ are unipotent, there is $m \in \mathbb{N}$ such that $1 = \left(a^{p^n}\right)^{p^m} = a^{p^{n+m}}$. So a itself is unipotent.
Since $[\mathcal{H},\mathcal{H}]$ is generated by $R_u(\mathcal{H})$ and by representatives of a generating set of $[\mathcal{H}/R_u(\mathcal{H}), \mathcal{H}/R_u(\mathcal{H})]$, it is unipotently generated. $\square$

Corollary 9.7 *$[\mathcal{Z},\mathcal{Z}]$ is unipotently generated.*

Proof By [Spr98], thm. 6.4.7(i), centralisers of tori are always connected. So the statement follows from the previous theorem. $\square$

Remark By the previous results, the realisation of connected groups as Galois groups can be reduced to realising tori and unipotently generated groups and to realising unipotent groups equivariantly.

9.3 Realisation of Tori and Unipotently Generated Groups

In [Mat01], Matzat has already proved that unipotently generated groups can be realised with one singular point, and in [MvdP03], Matzat and van der Put have proved that tori and connected unipotently generated groups can be realised with two singular points. But for the realisation of the direct product of those two, we need to be able to realise the torus in such a way, that the corresponding IPV-extension is linearly disjoint to that of the unipotently generated group (cf. prop. 7.11).
Notation Let $S \subset \mathcal{C}_F$ be a set with $\#S = 2$, let $\mathcal{O} = \mathcal{O}(\mathcal{C}_F \setminus S)$ be the ring of regular functions on $\mathcal{C}_F \setminus S$ and $\mathcal{O}_l = \mathcal{O} \cap F_l$.

Proposition 9.8 *A connected unipotently generated group $\mathcal{G}$ can be realised as a Galois group over F regularly outside S.*

Sketch of the proof Choose unipotent subgroups $\mathcal{U}_i \leq \mathcal{G}$ $(i = 1, \dots, k)$ which generate $\mathcal{G}$. Then for all $l \in \mathbb{N}$ and $i = 1, \dots k$, $\mathcal{U}_i(\mathcal{O}_l)$ is a free $\mathcal{O}_l$-module of dimension $\dim(\mathcal{U}_i)$, since $\mathcal{U}_i$ is unipotent. So we can choose a sequence $(D_l)_{l\in\mathbb{N}}$ of matrices with $D_l \in \mathcal{U}_i(\mathcal{O}_l)$ for some $i \in \{1, \dots, k\}$, that satisfies the conditions of proposition 7.12. Therefore the D_l define an ID-module whose IPV-extension has Galois group $\mathcal{G}(K)$, and since for all $l \in \mathbb{N}$, $D_l \in \mathcal{U}_i(\mathcal{O}_l) \leq \mathcal{G}(\mathcal{O}_l)$, this IPV-extension is regular outside S.

Next, we consider 1-dimensional ID-modules M, because an IPV-extension E for M has Galois group $\mathrm{Gal}(E/F) \leq \mathrm{GL}_1(K) = \mathbb{G}_m(K)$ and every torus defined over K is isomorphic to $\mathbb{G}_m^k$ for some $k \in \mathbb{N}$.
So let M be an ID-module with basis b and projective system given by $(D_l)_{l\in\mathbb{N}}$, where $D_l \in \mathrm{GL}_1(F_l) = F_l^\times$.

Theorem 9.9 *Let $M_1, \dots, M_r$ be 1-dimensional ID-modules over F and let L/F be an IPV-extension for $M_1 \oplus \dots \oplus M_r$. If $[M_1], \dots, [M_r] \in \mathrm{Isom}_{\mathcal{C}_F,1}$ generate a free abelian group of rank r, then $\mathrm{Gal}(L/F)$ is isomorphic to $\mathbb{G}_m^r(K)$.*

Proof For $i = 1, \dots, r$, let b_i be a basis element for M_i and $(D_{i,l})_{l\in\mathbb{N}}$ be a sequence corresponding to the ID-module-structure of M_i. Further let $U = F[X_1, \dots, X_r, X_1^{-1}, \dots, X_r^{-1}]$ be an ID-ring via $d_U^{(k)}(X_i) = \mathrm{d}_F^{(k)}(D_{i,0} \cdots D_{i,l}) \cdot (D_{i,0} \cdots D_{i,l})^{-1} X_i$ for all $k < p^l$ and $i = 1, \dots, r$. If $I \trianglelefteq U$ is a maximal ID-ideal, then obviously U/I is an IPV-ring for $M_1 \oplus \dots \oplus M_r$, i. e. $L \cong U/I$ and $\mathrm{Gal}(L/F)$ is isomorphic to $\mathbb{G}_m^r(K)$ if and only if $I = (0)$.
Assume that I is not trivial. Since d_U stabilizes monomials, I is generated by elements of the form $X_1^{e_1} \cdots X_r^{e_r} - a$, where $e_1, \dots, e_r \in \mathbb{Z}$ and $a \in F$. Choose such an element and define a 1-dimensional ID-module M (resp. its projective system over F) by the sequence $(D_l)_{l\in\mathbb{N}}$, where $D_l := D_{1,l}^{e_1} \cdots D_{r,l}^{e_r} \in F_l^\times$, with respect to a basis b. Then define the ID-ring $U' := F[Y, Y^{-1}]$ with $d_{U'}^{(k)}(Y) = \mathrm{d}_F^{(k)}(D_0 \cdots D_l) \cdot (D_0 \cdots D_l)^{-1} Y$ for all $k < p^l$. So the ideal $(Y - a) \trianglelefteq U'$ is an ID-ideal, hence M is a trivial ID-module and so $[M] = 0 \in \mathrm{Isom}_{\mathcal{C}_F,1}$. But by construction, $[M] = e_1[M_1] + \dots e_r[M_r] \in \mathrm{Isom}_{\mathcal{C}_F,1}$, which contradicts the assumption that $[M_1], \dots, [M_r]$ are $\mathbb{Z}$-linearly independent. $\square$

Corollary 9.10 *Every torus $\mathcal{T}$ can be realised regularly outside S (where $\#S = 2$) by an IPV-extension L/F. Furthermore this IPV-extension can be chosen linearly disjoint to any other given IPV-extension L'/F.*
$\mathcal{T}$ can be realised even without any singular point, if and only if the Jacobian variety $\mathcal{J}$ of $\mathcal{C}_F$ has an element of infinite order or if $\mathcal{J}$ has p-torsion.

Proof The subset $\mathrm{Div}^0(\mathcal{C}_F, S, \mathbb{Z}_p)$ of $\mathrm{Div}^0(\mathcal{C}_F, \mathbb{Z}_p)$ consisting of the maps $f : \mathcal{C}_F \to \mathbb{Z}_p$, with $f(x) = 0$ for $x \notin S$, is a free $\mathbb{Z}_p$-module of rank 1, i. e. a free

$\mathbb{Z}$-module of infinite rank. So $\mathrm{Div}^0(\mathcal{C}_F, S, \mathbb{Z}_p)/(\mathrm{H}(\mathcal{C}_F) \cap \mathrm{Div}^0(\mathcal{C}_F, S, \mathbb{Z}_p))$ contains a free $\mathbb{Z}$-module of infinite rank, and therefore its inverse image $\chi^{-1}\left(\mathrm{Div}^0(\mathcal{C}_F, S, \mathbb{Z}_p)/(\mathrm{H}(\mathcal{C}_F) \cap \mathrm{Div}^0(\mathcal{C}_F, S, \mathbb{Z}_p))\right) \subset \mathrm{Isom}_{\mathcal{C}_F,1}$ contains a free $\mathbb{Z}$-module of infinite rank. So we can find the desired IPV-extension L.
The Jacobian is equal to the subgroup of $\frac{\mathrm{Div}^0(\mathcal{C}_F, \mathbb{Z}_p)}{\mathrm{H}(\mathcal{C}_F)}$ whose elements are represented by maps $f : \mathcal{C}_F \to \mathbb{Z} \subset \mathbb{Z}_p$. Hence the preimages under χ are exactly the ID-modules which are regular on $\mathcal{C}_F$. If the Jacobian has p-torsion, then the p-adic Tate-module $T_p(\mathcal{J})$ is nonzero and therefore a free $\mathbb{Z}$-module of infinite rank. Hence the image of $T_p(\mathcal{J})$ in $\mathrm{Isom}_{\mathcal{C}_F,1}$ is a free $\mathbb{Z}$-module of infinite rank. Since for all $[M] \in \mathrm{Isom}_{\mathcal{C}_F,1}$ in this image, we have $\chi([M]) = 0$, these modules are all regular on $\mathcal{C}_F$. So the subgroup $\chi^{-1}(\mathcal{J})$ has an element of infinite order, if and only if $\mathcal{J}$ has an element of infinite order or $\mathcal{J}[p] \neq 0$. By [MvdP03], thm. 7.1 (4), if $\chi^{-1}(\mathcal{J})$ has an element of infinite order, then it contains a free $\mathbb{Z}$-submodule of infinite rank. Hence every torus can be realised without singularities and such that the IPV-extension is linearly disjoint to any other given IPV-extension. □

9.4 Equivariant Realisation of Unipotent Groups

Notation Let $\mathcal{H}$ be a reduced linear algebraic group, $\mathcal{U}$ a connected unipotent group and $\mathcal{G} = \mathcal{U} \rtimes \mathcal{H}$ a semidirect product. Furthermore let L/F be an IPV-extension with Galois group $\mathrm{Gal}(L/F) = \mathcal{H}(K)$ and singular locus inside a finite set $\emptyset \neq S \subset \mathcal{C}_F$, such that for all $x \in \mathcal{C}_F \setminus S$ and all $l \in \mathbb{N}$ there exists a fundamental solution matrix $Y_l \in \mathcal{H}\left((\mathcal{O}_{L,x})\right)$.

Theorem 9.11 *$\mathcal{U}$ can be realised $\mathcal{H}$-equivariantly over L regularly outside S, i. e. there is an IPV-extension E/L with Galois group $\mathcal{U}(K)$ such that E/F is an IPV-extension with Galois group $\mathcal{G}(K) = (\mathcal{U} \rtimes \mathcal{H})(K)$ with singular locus inside S.*

Proof Let $\mathcal{A} \trianglelefteq \mathcal{U}$ be a minimal nontrivial $\mathcal{H}$-invariant connected normal subgroup of $\mathcal{U}$, i. e. $\mathcal{A} \neq 1$ is a connected normal subgroup of $\mathcal{U}$ that is invariant under the action of $\mathcal{H}$ and is minimal amoungst those. Since the center $C(\mathcal{A})$ of $\mathcal{A}$ is a characteristic subgroup of $\mathcal{A}$, it is $\mathcal{H}$-invariant and a normal subgroup of $\mathcal{U}$. Since $\mathcal{A}$ is unipotent, $C(\mathcal{A})$ is nontrivial and so by minimality of $\mathcal{A}$, we get $C(\mathcal{A}) = \mathcal{A}$, i. e. $\mathcal{A}$ is abelian.
Further if $\mathcal{A}' \lneq \mathcal{A}$ is a non-connected $\mathcal{H}$-invariant normal subgroup of $\mathcal{U}$, then $\mathcal{A}'$ is finite, because its identity component $(\mathcal{A}')^\circ$ is also $\mathcal{H}$-invariant and normal in $\mathcal{U}$ and hence trivial by minimality of $\mathcal{A}$.
First case: $\mathcal{A} = \mathcal{U}$, i. e. there is no nontrivial normal subgroup of $\mathcal{U}$ – apart from 1 and $\mathcal{U}$ – that is invariant under the action of $\mathcal{H}$.
Now, every sequence $D_l \in \mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}}$ $(l = 0, 1, \dots)$ defines an IPV-extension E/L with $\mathrm{Gal}(E/L) \subset \mathcal{U}(K)$ and $\mathrm{Gal}(E/F) \subset (\mathcal{U} \rtimes \mathcal{H})(K)$, which is regular outside S (cf. section 8.4). Then $\mathrm{Gal}(E/L)$ is an $\mathcal{H}$-invariant subgroup of $\mathcal{U}(K)$. But since

$\mathcal{U}$ is abelian and $\mathcal{H}$-simple, we obtain that $\mathrm{Gal}(E/L)$ is finite or $\mathrm{Gal}(E/L) = \mathcal{U}(K)$.

We have to show that the D_l can be chosen such that E/L is not finite: For this, we consider the set of all IPV-extensions that are defined by sequences $D_l \in \mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}}$, $l \in \mathbb{N}$.

Let E' and E'' be extensions defined by $D'_l \in \mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}}$ resp. $D''_l \in \mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}}$ and let Y' and Y'' be the corresponding fundamental solution matrices. Then the map $\alpha : E' \to E''$, defined by $\alpha(Y') = Y''$ and $\alpha|_L = \mathrm{id}_L$, is a differential isomorphism if and only if for all $l \in \mathbb{N}$ we have: $(D'_0 \cdots D'_l)^{-1}(D''_0 \cdots D''_l) \in \mathcal{U}((\mathcal{O}_L)_{l+1})^{\mathcal{H}}$.

Therefore we have a one-to-one correspondence between differential isomorphism classes of those IPV-extensions $L(Y')$ and the infinite product

$$\prod_{l\geq 0} \mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}} / \mathcal{U}((\mathcal{O}_L)_{l+1})^{\mathcal{H}}.$$

But $\mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}}$ is an $(\mathcal{O}_F)_l$-module for all l and therefore a K-vector space. So $\mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}} / \mathcal{U}((\mathcal{O}_L)_{l+1})^{\mathcal{H}}$ is a K-vector space and its dimension is greater than 1, because it is a nontrivial torsionfree $(\mathcal{O}_F)_{l+1}$-module and $\dim_K((\mathcal{O}_F)_{l+1}) > 1$. Hence the dimension of the infinite product as K-vector space is uncountable ($\geq 2^{\mathbb{N}}$).

Those IPV-extensions whose Galois group is finite are given by maximal ideals in the ring $U := L[X_{ij}, \det(X)^{-1}]$. Since every maximal ideal is given by n^2 polynomials, the L-vector space of n^2-tuples of polynomials in U gives an upper bound to the number of those IPV-extensions with finite Galois group. But since this is an L-vector space of countable dimension and L is a K-vector space of countable dimension, the set of n^2-tuples of polynomials in U is a K-vector space of countable dimension.

Thus for dimensional reasons, there exists an IPV-extension $E' = L(Y')$ with $\mathrm{Gal}(E'/L) = \mathcal{U}(K)$ and $\mathrm{Gal}(E'/F) = (\mathcal{U} \rtimes \mathcal{H})(K)$.

Second case: $\mathcal{A} \neq \mathcal{U}$ and there exists an $\mathcal{H}$-equivariant isomorphism $\mathcal{A} \rtimes (\mathcal{U}/\mathcal{A}) \to \mathcal{U}$.

Then $\big(\mathcal{A} \rtimes (\mathcal{U}/\mathcal{A})\big) \rtimes \mathcal{H} \cong \mathcal{A} \rtimes \big((\mathcal{U}/\mathcal{A}) \rtimes \mathcal{H}\big)$ and by induction we can assume that $\mathcal{U}/\mathcal{A}$ is realised $\mathcal{H}$-equivariantly as Galois group $\mathrm{Gal}(\tilde{E}/L)$, such that $\tilde{E}/F$ has singular locus inside S. (The dimension of $\mathcal{U}/\mathcal{A}$ is less than that of $\mathcal{U}$.) So it remains to realise $\mathcal{A}$ over $\tilde{E}$ by $((\mathcal{U}/\mathcal{A}) \rtimes \mathcal{H})$-invariant matrices $D_l \in \mathcal{A}((\mathcal{O}_{\tilde{E}})_l)$. But an $((\mathcal{U}/\mathcal{A}) \rtimes \mathcal{H})$-invariant connected normal subgroup of $\mathcal{A}$ is a normal subgroup of $\mathcal{U}$ (since it is $\mathcal{U}/\mathcal{A}$-invariant) and is $\mathcal{H}$-invariant, so equals 1 or $\mathcal{A}$, by minimality of $\mathcal{A}$. Hence we are in the first case.

Third case: $\mathcal{A} \neq \mathcal{U}$ and there doesn't exist an $\mathcal{H}$-equivariant isomorphism $\mathcal{A} \rtimes (\mathcal{U}/\mathcal{A}) \to \mathcal{U}$.
We first show that the map $\alpha : \mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}} \to (\mathcal{U}/\mathcal{A})\,((\mathcal{O}_L)_l)^{\mathcal{H}}$, induced by the projection, is surjective:
Since $\mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}}$ and $\mathcal{U}((\mathcal{O}_L)_l)^{\mathcal{H}}$ are $(\mathcal{O})_l$-modules it suffices to show that for all $x \in \mathcal{C}_F \setminus S$ the localised map $\alpha_x : \mathcal{U}((\mathcal{O}_{L,x})_l)^{\mathcal{H}} \to (\mathcal{U}/\mathcal{A})\,((\mathcal{O}_{L,x})_l)^{\mathcal{H}}$ is surjective. But by choosing a fundamental solution matrix $Y_l \in \mathcal{H}\,((\mathcal{O}_{L,x})_l)$ (which exists by assumption), we get a commutative diagram

$$\begin{tikzcd}
\mathcal{U}((\mathcal{O}_x)_l) \arrow[r, "\alpha_x"] \arrow[d, "()^{Y_l}"'] & (\mathcal{U}/\mathcal{A})((\mathcal{O}_x)_l) \arrow[d, "()^{Y_l}"'] \\
\mathcal{U}((\mathcal{O}_{L,x})_l)^{\mathcal{H}} \arrow[r, "\alpha_x"] & (\mathcal{U}/\mathcal{A})\,((\mathcal{O}_{L,x})_l)^{\mathcal{H}},
\end{tikzcd}$$

where $()^{Y_l}$ denotes conjugation by Y_l. The vertical maps are isomorphisms and the upper horizontal map is an epimorphism and so the lower horizontal map also is an epimorphism.
Now let $(\mathcal{U}/\mathcal{A})$ be realised $\mathcal{H}$-equivariantly as $(\mathcal{U}/\mathcal{A})(K) = \mathrm{Gal}(\tilde{E}/L)$ by matrices $\tilde{D}_l \in (\mathcal{U}/\mathcal{A})\,((\mathcal{O}_L)_l)^{\mathcal{H}}$ $(l = 0, 1, \dots)$. Then choose preimages $D_l \in \alpha^{-1}(\tilde{D}_l)$. These define an IPV-extension E'/L with Galois group $\mathcal{U}'(K) = \mathrm{Gal}(E'/L) \subseteq \mathcal{U}$ and $\mathcal{U}' \to \mathcal{U}/\mathcal{A}$ is surjective (cf. [Mat01], thm. 5.12). Since the D_l are $\mathcal{H}$-invariant and $\mathcal{U}'$ is generated by the D_l as an algebraic group, $\mathcal{U}'$ is $\mathcal{H}$-invariant, and therefore $\mathcal{A} \cap \mathcal{U}'$ is $\mathcal{H}$-invariant. Furthermore $\mathcal{A} \cap \mathcal{U}'$ is normal in $\mathcal{U}'$ (since $\mathcal{A}$ is normal in $\mathcal{U}$) and normal in $\mathcal{A}$ (since $\mathcal{A}$ is abelian), so it is normal in $\mathcal{A}\mathcal{U}' = \mathcal{U}$. By minimality of $\mathcal{A}$, we get that $\mathcal{A} \cap \mathcal{U}'$ is finite or $\mathcal{A} \cap \mathcal{U}' = \mathcal{A}$. If $\mathcal{A} \cap \mathcal{U}'$ is finite, then we have $1 = (\mathcal{A} \cap \mathcal{U}')^\circ = \mathcal{A} \cap (\mathcal{U}')^\circ$, since $\mathcal{A}$ is connected and $\mathcal{A}$ and $\mathcal{U}'$ are unipotent. And so $(\mathcal{U}')^\circ \cong \mathcal{U}/\mathcal{A}$ and $\mathcal{U} \cong \mathcal{A} \rtimes (\mathcal{U}')^\circ \cong \mathcal{A} \rtimes (\mathcal{U}/\mathcal{A})$ as $\mathcal{H}$-groups. But by assumption, there doesn't exist such an isomorphism. So $\mathcal{A} \cap \mathcal{U}' = \mathcal{A}$ and therefore $\mathcal{U}' = \mathcal{U}$. Hence E'/L is the desired IPV-extension. □

9.5 Realisation of Connected Groups

As a summary of the previous sections, we have the following theorem:

Theorem 9.12 *Every connected linear algebraic group $\mathcal{G}$ can be realised as iterative differential Galois group of an IPV-extension E/F which has at most two singular points. If the Jacobian variety of $\mathcal{C}_F$ has p-torsion or if $\mathcal{G}$ is unipotently generated, then $\mathcal{G}$ can be realised even with at most one singular point.*

Proof Denote by $S \subset \mathcal{C}_F$ the set of two points in $\mathcal{C}_F$, which may be singular in the IPV-extension. Choose a maximal torus $\mathcal{T}_0$ of the radical $R(\mathcal{G})$, then with the notations of section 9.2, we have an epimorphism

$$\pi : \tilde{\mathcal{G}} := \mathcal{R}_u \rtimes (\mathcal{T}_0 \times [\mathcal{Z}, \mathcal{Z}]) \longrightarrow \mathcal{G}.$$

By corollary 9.7 and proposition 9.8, we can realise $[\mathcal{Z}, \mathcal{Z}]$ by an IPV-extension L'/F regularly outside S and by corollary 9.10, we can realise $\mathcal{T}_0$ by an IPV-extension L/F regularly outside S such that $L \otimes_F L'$ is an IPV-extension of F with Galois group $(\mathcal{T}_0 \times [\mathcal{Z}, \mathcal{Z}])(K)$. Since L and L' are regular outside S, this extension is also regular outside S. Then by theorem 9.11, there is an IPV-extension $\tilde{E}/L$ with Galois group $\mathcal{R}_u(K)$ such that $\tilde{E}/F$ is an IPV-extension with $\mathrm{Gal}(\tilde{E}/F) = \tilde{\mathcal{G}}(K)$ and $\tilde{E}/F$ is regular outside S. Hence, the fixed field under $\mathrm{Ker}(\pi)$, $E := \tilde{E}^{\mathrm{Ker}(\pi)}$ is an IPV-extension over F with $\mathrm{Gal}(E/F) = \left(\tilde{\mathcal{G}}/\mathrm{Ker}(\pi)\right)(K) = \mathcal{G}(K)$ and E/F has singular locus inside S.

The last statement is then clear, because if $\mathcal{J}[p] \neq 0$, then $\mathcal{T}_0$ can be realised without singularities, by corollary 9.10. If $\mathcal{G}$ is unipotently generated, $\mathcal{G}$ can be realised with only one singular point, by proposition 9.8. □

Remark In [MvdP03], cor. 7.7 (3), it has already been stated that $\mathcal{G}$ can be realised regularly outside a non empty set S, if the torus $\mathcal{T}_0$ can be realised regularly outside S, but the proof given there doesn't work in general:

Assume $\#S = 2$ and, for simplicity, let $\mathcal{T}_0 = \mathbb{G}_m$ be the 1-dimensional torus and let $\mathcal{T}_0$ be realised regularly outside S, i. e. we have an ID-module with projective system given by matrices $D_l \in \mathcal{T}_0(F_l) = F_l^{\times}$.

If F is not a rational function field, then for all $l \in \mathbb{N}$, where $D_l \notin K^{\times}$ (in particular for infinitely many l), there exists a point $x_l \in \mathcal{C}_F \setminus S$ with $\mathrm{ord}_{x_l}(D_l) \neq 0$, because the support of a nontrivial principal divisor has at least three elements. So if we choose an increasing sequence $(l_j)_{j \in \mathbb{N}}$ with $\lim_{j\to\infty}(l_{j+1} - l_j) = \infty$ and if we define a new ID-module N with projective system given by $D'_k := 1$, if $k \notin \{l_j\}$, and $D'_k := (D_j)^{p^{l_j - j}} \in F_{l_j}$, if $k = l_j$, then the IPV-extension for this ID-module also has differential Galois group $\mathcal{T}_0(K)$, by proposition 7.12. But in general for the x_l defined above, we get $\sum_{k=0}^{\infty} \mathrm{ord}_{x_l}(D'_k) \neq 0$. This means that in general N is totally singular (if there are infinitely many such x_l), or N has at least one additional singular point, namely a point $x \in \mathcal{C}_F \setminus S$, for which there are infinitely many $l \in \mathbb{N}$ such that $x = x_l$.

Hence, also in the “interlacing with gaps” of the matrices for the torus and the unipotently generated group, as given in the proof of [MvdP03], cor. 7.7 (3), there might occur new singularities.

Corollary 9.13 *The differential Abhyankar conjecture is true for connected groups.*

Proof The factor group $\mathcal{G}/p(\mathcal{G})$ is a torus. If $\mathcal{G}/p(\mathcal{G}) = 1$, then $\mathcal{G}$ is unipotently generated and therefore can be realised with one singularity. If $\mathcal{G}/p(\mathcal{G}) \neq 1$, then by the previous theorem, both $\mathcal{G}$ and $\mathcal{G}/p(\mathcal{G})$ can be realised with singular locus inside a nonempty set S, if and only if $\#S \geq 2$ or if $\#S = 1$ and the Jacobian $\mathcal{J}$ has an element of infinite order or an element of order p. $\square$

Appendices

A Completions of Graded Algebras

In this appendix we regard completions of graded algebras over a ring R (R-cgas for short).
Let R be a commutative ring.

Definition A.1 *A commutative R-algebra B is called an* R-**cga** *, if B is the completion of a connected graded R-algebra $\bigoplus_{i=0}^{\infty} B_i$, where the completion is taken with respect to the filtration given by the ideals $I_k := \bigoplus_{i=k}^{\infty} B_i$. We call B_i the i-th homogeneous component of B. The augmentation map will be denoted by $\varepsilon : B \to B_0 = R$. More general, the projection map to the i-th homogeneous component will be denoted by $\mathrm{pr}_i : B \to B_i$.*

Proposition A.2 *Let B be an R-cga. Then as an R-module B is isomorphic to the direct product $\prod_{k=0}^{\infty} B_k$.*

Proof By definition the completion is the inverse limit $\varprojlim_{n\in\mathbb{N}} (\bigoplus_{k=0}^{n} B_k)$ (see also [Eis95]). But this limit is obviously isomorphic to $\prod_{k=0}^{\infty} B_k$. $\square$

Example A.3

1. *The ring of formal power series $R[[T]]$ is an R-cga, with i-th homogeneous component $R \cdot T^i$.*
2. *The ring R itself is the trivial R-cga with $(R)_i = 0$ for $i > 0$.*

Remark According to the notation of a power series as an infinite sum, we will denote elements of an R-cga B by $\sum_{i=0}^{\infty} b_i$, where $b_i \in B_i$. This notation is also justified by the fact, that, indeed, $\sum_{i=0}^{\infty} b_i$ is the limit of the sequence of partial sums $(\sum_{i=0}^{n} b_i)_{n\in\mathbb{N}}$ in the given topology, or in other words that $\sum_{i=0}^{\infty} b_i$ is a convergent series.

Definition A.4 *Let B and $\tilde{B}$ be R-cgas. A homomorphism of R-algebras $f : B \to \tilde{B}$ is called a homomorphism of R-cgas, if f is a continuous extension of a homomorphism of graded R-algebras $g : \bigoplus_{k=0}^{\infty} B_k \to \bigoplus_{k=0}^{\infty} \tilde{B}_k$.*

Remark Since $\bigoplus_{k=0}^{\infty} B_k$ is dense in B, the continuous extension of a given homomorphism of graded R-algebras is unique. So the category of commutative connected graded R-algebras and the category of R-cgas are equivalent.

In this thesis, we sometimes have to consider more general homomorphisms between R-cgas, too. So let $K \subset R$ be a subring, B and $\tilde{B}$ be R-cgas and let

$f : B \to \tilde{B}$ be a continuous homomorphism of K-modules (or even K-algebras). Then we define "homogeneous components" $f^{(i)} : B \to \tilde{B}$ ($i \in \mathbb{Z}$) of f to be the continuous homomorphisms of K-modules given by

$$f^{(i)}|_{B_j} := \mathrm{pr}_{i+j} \circ f|_{B_j} : B_j \to \tilde{B}_{i+j}$$

for all $j \in \mathbb{N}$ (set $\tilde{B}_{i+j} := 0$ for $i+j < 0$). The $f^{(i)}$ uniquely determine f, because for all $b_j \in B_j$, $\sum_{i=-j}^{\infty} f^{(i)}(b_j)$ converges to $f(b_j)$.

Such a continuous homomorphism of K-modules $f : B \to \tilde{B}$ is called **positive**, if $f^{(i)} = 0$ for $i < 0$.

Proposition A.5 *The monoid $(K, \cdot)$ acts on the set $\mathrm{Hom}_K^+(B, \tilde{B})$ of positive continuous homomorphisms of K-modules by*

$$(a.f)^{(i)} := a^i \cdot f^{(i)} \quad (i \geq 0)$$

for all $a \in K$, $f \in \mathrm{Hom}_K^+(B, \tilde{B})$. If f is a homomorphism of algebras, then $a.f$ also is a homomorphism of algebras. Furthermore for $f \in \mathrm{Hom}_K^+(B, \tilde{B})$, $g \in \mathrm{Hom}_K^+(\tilde{B}, \tilde{\tilde{B}})$ and $a \in K$, we have

$$a.(g \circ f) = a.g \circ a.f,$$

i. e. the action of K commutes with compositions.

Proof It is clear, that for all $a \in K$ and $f \in \mathrm{Hom}_K^+(B, \tilde{B})$, $a.f$ is a positive continuous homomorphism. If f is a homomorphism of algebras, then for all $b, c \in B$:

$$\begin{aligned} a.f(bc) &= \sum_{k=0}^{\infty} a^k f^{(k)}(bc) = \sum_{k=0}^{\infty} a^k \sum_{i+j=k} f^{(i)}(b) f^{(j)}(c) \\ &= \left(\sum_{i=0}^{\infty} a^i f^{(i)}(b) \right) \cdot \left(\sum_{j=0}^{\infty} a^j f^{(j)}(c) \right) = a.f(b) \cdot a.f(c), \end{aligned}$$

i. e. $a.f$ is a homomorphism of algebras.
Next, it is clear from the definition that $1.f = f$ and $a_1.(a_2.f) = (a_1 a_2).f$ for all $a_1, a_2 \in K, f \in \mathrm{Hom}_K^+(B, \tilde{B})$, i. e. this defines an action of the monoid K. Now let $f \in \mathrm{Hom}_K^+(B, \tilde{B})$, $g \in \mathrm{Hom}_K^+(\tilde{B}, \tilde{\tilde{B}})$ and $a \in K$. Then for all $b \in B$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned} ((a.g) \circ (a.f))^{(k)}(b) &= \sum_{i+j=k} (a.g)^{(i)} \circ (a.f)^{(j)}(b) = \sum_{i+j=k} a^i g^{(i)} \left(a^j f^{(j)}(b) \right) \\ &= \sum_{i+j=k} a^i a^j g^{(i)} \left(f^{(j)}(b) \right) = a^k \sum_{i+j=k} \left(g^{(i)} \circ f^{(j)} \right)(b) \\ &= (a.(g \circ f))^{(k)}(b) \end{aligned}$$

So $(a.g) \circ (a.f) = a.(g \circ f)$. $\square$

Remark Some special maps, that are used in this thesis are the higher derivations on rings and modules (cf. sections 1.1 and 1.2) – maps in $\mathrm{Hom}_K^+(R, B)$ resp. $\mathrm{Hom}_K^+(M, B \otimes_R M)$ –, the extension $\mathrm{d}_{\hat{\Omega}}$ of the universal derivation to the algebra of higher differentials – a map in $\mathrm{Hom}_K^+(\hat{\Omega}, \hat{\Omega})$ (cf. section 2.1) – and at last the extensions of higher connections on M to maps in $\mathrm{Hom}_K^+(\hat{\Omega} \otimes_R M, \hat{\Omega} \otimes_R M)$ (cf. section 2.3).

B Definitions of Some Categories

In this appendix we give an overview of the definitions of some special categories, such as the notion of a Tannakian category. We don't give all the details but refer to other books, if for example one doesn't know the universal property of the kernel of a morphism.

Definition B.1 (Abelian Category)
A category $\mathcal{C}$ is called an **abelian category** *if the following conditions hold:*

1. *For all objects X, Y of $\mathcal{C}$, the set of morphisms $\mathrm{Mor}(X, Y)$ is an abelian group.*
2. *There exists a null object $\mathbf{0} \in \mathrm{Ob}(\mathcal{C})$.*[18]
3. *For all objects X, Y of $\mathcal{C}$, there exists a biproduct $X \oplus Y \in \mathrm{Ob}(\mathcal{C})$.*
4. *For all morphisms f of $\mathcal{C}$, the kernel $\mathrm{Ker}(f)$ and the cokernel $\mathrm{Coker}(f)$ of f exist.*
5. *For every monomorphism $f \in \mathrm{Mor}(X, Y)$, there exists a morphism $g : Y \to Z$ such that $X \cong \mathrm{Ker}(g)$ and for every epimorphism $f \in \mathrm{Mor}(Y, X)$, there exists a morphism $g : Z \to Y$ such that $X \cong \mathrm{Coker}(g)$.*

For the next definition, the definition of a tensor category over a field K, we follow the notion of P. Deligne in [Del90] and B. H. Matzat in [Mat01]. There also exist other notions of a tensor category. For example, what we call a tensor category is called a "rigid abelian K-linear ACU $\otimes$-category" by S. Saavedra in [Saa72] or a "K-linear rigid abelian tensor category with $K \cong \mathrm{End}(\mathbf{1})$" by P. Deligne and J. Milne in [DM89].

Definition B.2 (Tensor Category over K)
A category $\mathcal{C}$ is called a **tensor category over a field** *K if the following conditions hold:*

1. *$\mathcal{C}$ is an abelian category.*
2. *There exists a biadditive functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called* **tensor product**, *that is associative and commutative.*
3. *There exists a* **unital object** *$\mathbf{1}_{\mathcal{C}}$ for $\otimes$.*
4. *For all $X \in \mathcal{C}$, there exists an object $X^* \in \mathcal{C}$ (called* **dual of** *X) and morphisms $\varepsilon_X \in \mathrm{Mor}(X \otimes X^*, \mathbf{1}_{\mathcal{C}})$ (called* **evaluation***) and $\delta_X \in \mathrm{Mor}(\mathbf{1}_{\mathcal{C}}, X^* \otimes X)$ (called* **coevaluation***), such that:*

[18] All the object that are assumed to exists are defined by their universal properties.

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X \otimes \delta_X"] \arrow[dr, "\mathrm{id}_X"'] & X \otimes X^* \otimes X \arrow[d, "\varepsilon_X \otimes \mathrm{id}_X"] \\
& X
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
X^* \arrow[r, "\delta_X \otimes \mathrm{id}_{X^*}"] \arrow[dr, "\mathrm{id}_{X^*}"'] & X^* \otimes X \otimes X^* \arrow[d, "\mathrm{id}_{X^*} \otimes \varepsilon_X"] \\
& X^*
\end{tikzcd}
$$

commute.

5. $\mathrm{End}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}) \cong K$.

Remark Let $\mathcal{C}$ be a tensor category. Then for all objects $X, Y \in \mathcal{C}$, the functor $T \mapsto \mathrm{Mor}(T \otimes X, Y)$ is representable and the representing object, denoted by $\underline{\mathrm{Hom}}(X, Y)$, is called the **internel hom** of X and Y.

The following proposition is a collection of some useful results, that are all proved in [Del90]:

Proposition B.3 *Let $\mathcal{C}$ be a tensor category over a field K. Then:*

1. *For all $X, Y \in \mathcal{C}$, there is an isomorphism $\iota_{X,Y} : X^* \otimes Y \to \underline{\mathrm{Hom}}(X, Y)$.*
2. *For all $X \in \mathcal{C}$, the bidual $(X^*)^*$ is isomorphic to X.*
3. *For all $X, Y \in \mathcal{C}$, $X^* \otimes Y^*$ is isomorphic to $(X \otimes Y)^*$.*

Example B.4 *For a commutative ring R, the category* **Mod**(R) *of finitely generated R-modules is an abelian category.*
The category **Proj-Mod**(R) *of finitely generated projective R-modules is in general not abelian, but satisfies the properties 2.-4. of a tensor category, with the usual tensor product, $\mathbf{1} = R$, $X^* = \mathrm{Hom}_R(X, \mathbf{1})$, $\varepsilon_X : X \otimes X^* \to \mathbf{1}, x \otimes \alpha \mapsto \alpha(x)$ and $\mathbf{1} \xrightarrow{\delta_X} X^* \otimes X \xrightarrow{\iota_{X,X}} \mathrm{Hom}_R(X, X), r \mapsto r \cdot \mathrm{id}_X$.*

Definition B.5 ((Neutral) Tannakian Category)
A tensor category $\mathcal{T}$ over a field K is called a **Tannakian category** *if there exists a scheme $S \neq \emptyset$ over K and a functor $\boldsymbol{\omega} : \mathcal{T} \to$* **Mod**$(S)$ *(so called* **fibre functor***) which*

1. *respects the tensor product,* [19]
2. *is K-linear and*
3. *is exact.*

$\mathcal{T}$ is called a **neutral Tannakian category** *if it admits a fibre functor $\boldsymbol{\omega} : \mathcal{T} \to$* **Vect**$(K)$.

[19] see again [Del90] for more details

References

[Del70] Deligne, P.: *Equations Différentielle à Points Singuliers Réguliers.* Springer Lecture Notes 163 (1970)

[Del90] Deligne, P.: *Catégories Tannakiennes*, in Grothendieck Festschrift, Vol II. Birkhäuser Boston, pp. 111-195 (1990)

[DM89] Deligne, P.. Milne, J.: *Tannakian Categories*, in Hodge Cycles, Motives and Shimura Varieties. Springer Lecture Notes 900:101-228 (1989)

[Eis95] Eisenbud, D.: *Commutative Algebra with a View Toward Algebraic Geometry.* Graduate Texts in Mathematics 150, Springer (1995)

[Har94] Harbater, D.: *Abhyankar's conjecture on Galois groups over curves.* Invent. Math., 117:1-25 (1994)

[Har95] Harbater, D.: *Fundamental groups of curves in characteristic p.* Proc. of the Int. Congress of Math., Zürich 1994, Birkhäuser Basel, pp. 656-666 (1995)

[Hart77] Hartshorne, R.: *Algebraic Geometry.* Graduate Texts in Mathematics 52, Springer (1977)

[HS37] Hasse, H.. Schmidt, F.K.: *Noch eine Begründung der Theorie des höheren Differentialquotienten in einem algebraischen Funktionenkörper in einer Unbestimmten.* J. Reine Angew. Math., 177:215-237 (1937)

[Kat87] Katz, N.: *On the calculation of some differential Galois groups.* Invent. Math., 87:13-61 (1987)

[Mat01] Matzat, B.H.: *Differential Galois Theory in Positive Characteristic*, notes written by J. Hartmann. IWR-Preprint 2001-35 (2001)

[Mats80] Matsumura, H.: *Commutative Algebra, second edition.* Mathematics Lecture Notes Series 56 (1980)

[Mats89] Matsumura, H.: *Commutative ring theory.* Cambridge studies in adv. math. 8 (1989)

[MvdP03] Matzat, B.H.. van der Put, M.: *Iterative Differential Equations and the Abhyankar Conjecture.* J. Reine Angew. Math., 557:1-52 (2003)

[Oku63] Okugawa, K.: *Basic properties of differential fields of an arbitrary characteristic and the Picard-Vessiot theory.* J. Math. Kyoto Univ. 2, 2:295-322 (1963)

[Oku87] Okugawa, K.: *Differential Algebra of Nonzero Characteristic*. Kinokuniya Company Ltd., Tokyo (1987)

[Ray94] Raynaud, M.: *Revêtements de la droite affine en charactéristique p*, Invent. Math. 116:425-462 (1994)

[Saa72] Saavedra, S.: *Catégories Tannakiennes*. Springer Lecture Notes 265 (1972)

[Spr98] Springer, T. A.: *Linear Algebraic Groups, Second Edition*. Birkhäuser (1998)

[Sti93] Stichtenoth, H.: *Algebraic Function Fields and Codes*. Universitext, Springer (1993)

[Voj04] Vojta, P.: *Jets via Hasse-Schmidt Derivations*. eprint arXiv:math/0407113 (2004)