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Homogenization of quasilinear elliptic-parabolic equations with respect to measures

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Abstract

We investigate the homogenization of quasilinear elliptic and degenerate elliptic-parabolic equations arising in nonlinear filtration and flow transport in saturated as well as unsaturated porous media. The main focus of the thesis is to study these equations on general multidimensional structures, which we characterize by a periodic positive measure μ on $\mathbb{R}^d$. Our approach contains the classical framework of homogenization on perforated domains and, more importantly, the investigation of networks of arbitrary, possibly nonconstant dimension.

To the aim of deriving effective macroscopic equations for nonlinear problems posed on these structures, we prove a new compactness result for bounded sequences $\{u_\varepsilon\}$ in the varying Sobolev spaces $H^{1,p}(\Omega, d\mu_\varepsilon)$, where the measures μ_ε are the nontrivial ε-rescalings of μ, namely $\mu_\varepsilon(B) := \varepsilon^d \mu(\varepsilon^{-1}B)$, and where ε is the typical microscopic length scale parameter.

The singular measure approach will also be justified by a fattening ansatz, where a measure μ^δ, absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, models a thin reinforced structure of thickness $\delta > 0$. We study in detail the two limit processes $\varepsilon \to 0$ and $\delta \to 0$ and show, at least for a large class of quasilinear problems, that the limits commute if the support of the singular measure μ, the weak limit of μ^δ as $\delta \to 0$, is sufficiently connected. On the other hand, by constructing explicit nontrivial counterexamples we will show that the limits do in general not commute on nonconnected structures, such that the homogenized equation will depend on the order we let the two parameters ε and δ tend to zero.

Zusammenfassung

Wir untersuchen die Homogenisierung von quasilinearen elliptischen bzw. degeneriert elliptisch-parabolischen Gleichungen, die ihre Anwendungen vor allem in der Modellierung von Strömungen durch gesättigte und ungesättigte poröse Medien finden. Im Zentrum der Betrachtung steht die Untersuchung dieser Gleichungen auf allgemeinen, multidimensionalen Strukturen, die durch ein periodisches, positives Maß μ auf $\mathbb{R}^d$ beschrieben werden. Unser Zugang beinhaltet den Standardfall der Homogenisierung auf perforierten Gebieten. Unser Hauptaugenmerk liegt jedoch vor allem auf Netzwerken beliebiger, möglicherweise nichtkonstanter Dimension.

Um effektive, makroskopische Gleichungen für nichtlineare Probleme, die auf diesen Strukturen gestellt sind, herzuleiten, beweisen wir ein neues Kompaktheitsresultat für beschränkte Folgen $\{u_\varepsilon\}$ in den variablen Sobolevräumen $H^{1,p}(\Omega, d\mu_\varepsilon)$, wobei die Maße μ_ε nichttriviale ε-Reskalierungen von μ sind, genauer $\mu_\varepsilon(B) := \varepsilon^d \mu(\varepsilon^{-1}B)$, und ε die mikroskopische Längenskala abbildet.

Unser Zugang mit singulären Maßen wird auch durch einen Andickungsansatz gerechtfertigt, bei dem ein Maß μ^δ, welches absolut stetig bezüglich des

Lebesgue Maßes auf $\mathbb{R}^d$ ist, die Dicke $\delta > 0$ einer dünnen, verstärkten Struktur beschreibt. Wir untersuchen detailliert die beiden Grenzübergänge $\varepsilon \to 0$ und $\delta \to 0$ und zeigen, jedenfalls für eine große Klasse von quasilinearen Problemen, dass die Limiten vertauschen wenn der Träger des singulären Maßes μ, dem schwachen Limes der Maße μ^δ für $\delta \to 0$, ausreichend zusammenhängend ist. Durch die Konstruktion expliziter nichttrivialer Gegenbeispiele weisen wir andererseits nach, dass die Limiten auf nicht zusammenhängenden Strukturen im Allgemeinen nicht vertauschen, so dass die homogenisierte Gleichung von der Reihenfolge abhängt, in der wir ε und δ gegen Null gehen lassen.

Contents

1 Introduction

The thesis is devoted to the homogenization of quasilinear elliptic and degenerate elliptic-parabolic equations posed on multidimensional structures. In the most general setting we consider the asymptotics $\varepsilon \to 0$ of the equation

$$\partial_t b(u_\varepsilon) - \operatorname{div} a_\varepsilon(\mu, x, t, b(u_\varepsilon), \nabla u_\varepsilon) \; = \; f_\varepsilon(\mu, x, t, b(u_\varepsilon)) \quad \text{in } \mathcal{D}^{'}(Q), \tag{1.1}$$

where $Q = \Omega \times (0, T)$ is the space-time cylinder, $\Omega \subset \mathbb{R}^d$ a bounded domain, $b : \mathbb{R} \to \mathbb{R}$ a continuous monotone function, μ a periodic Radon measure on $\mathbb{R}^d$ and $\varepsilon > 0$ a typical microscale parameter. The coefficients

$$a_\varepsilon(\mu, x, t, \cdot, \cdot) = a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, \cdot, \cdot)\mu_\varepsilon, \quad f_\varepsilon(\mu, x, t, \cdot) = f(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, \cdot)\mu_\varepsilon \tag{1.2}$$

are measure valued with respect to the ε-rescalings μ_ε of μ defined in (1.3) below and oscillate with period ε in the space and time variables. Stated more precisely, equation (1.1) tested with a smooth function $\varphi(x, t)$ compactly supported in Q, corresponds to the integral identity

$$\int_Q \left(-b(u_\varepsilon)\partial_t\varphi + a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, b(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla\varphi\right) d\mu_\varepsilon dt = \int_Q f(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, b(u_\varepsilon))\varphi \, d\mu_\varepsilon dt.$$

Our main application is the Richards equation, which models flow transport in unsaturated porous media. Here u is the matric potential, $b(u) = \Theta$ the water content and $a(\cdot, \Theta, \nabla u) = K(\cdot, \Theta)[\nabla u - \varrho\vec{g}]$ the hydraulic flux, where K is the conductivity and $\varrho\vec{g}$ a gravity term. We emphasize that the equation changes type from parabolic to elliptic if b has a vanishing derivative.
We characterize multidimensional structures by a positive Radon measure μ. By the choice $\mu = \mathcal{L}^d\lfloor A$, where A is the complement of a periodic system of holes, we recover the classical framework of homogenization on perforated domains. In general, we think of μ as a sum of Hausdorff measures $\mathcal{H}^{k_i}$ supported on k_i-dimensional periodic subsets of $\mathbb{R}^d$. Figure 1.1 illustrates that, possibly apart from the bulk, fluid flow can take place in highly permeable thin fissures (cf. the modeling in Section 3.2). Choosing an $\mathcal{H}^k$-component ($k < d$) in the support of μ may then model the network S of the fissures.

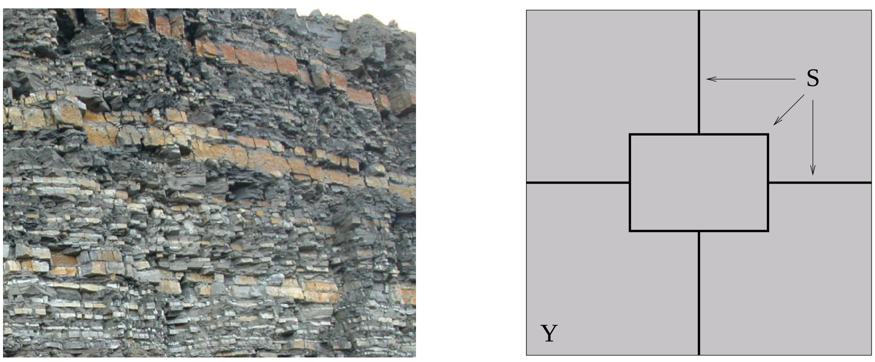


Figure 1.1: Stone pit and ansatz $\mu = \mathcal{L}^d\lfloor Y + \mathcal{H}^1\lfloor S$

Studying equation (1.1) on general multistructures will also be justified by a measure *fattening approach*, where we consider the asymptotics $\delta \to 0$ of δ-thickened structures (Chapter 5). As indicated above, the microscopic ε-periodic distribution of multidimensional structures gives rise to a family of rescaled measures

$$\mu_\varepsilon(B) := \varepsilon^d \mu(\tfrac{B}{\varepsilon}) \quad \text{for each Borel set } B \subset \mathbb{R}^d. \tag{1.3}$$

In general, the support of μ_ε varies with ε, which is considerably more delicate than the oscillation of coefficients in the equation. To the aim of studying the homogenization of arbitrary multistructures, an adequate notion of two-scale convergence was introduced by Neuss-Radu [49] in some special cases, and in a general systematic treatment by Zhikov and Bouchitté et al. [14, 62]:
The sequence $u_\varepsilon \in L^p(\Omega, d\mu_\varepsilon)$ two-scale converges with respect to μ to a function $u \in L^p(\Omega \times Y, dx \otimes d\mu)$ and we write $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u$ (in $L^p(\Omega, d\mu_\varepsilon)$), if

$$\int_\Omega u_\varepsilon(x)\psi(x, \tfrac{x}{\varepsilon})\, d\mu_\varepsilon \to \int_{\Omega\times Y} u(x,y)\psi(x,y)\, dx d\mu(y) \quad \forall \psi \in \mathcal{D}(\Omega; \mathcal{C}^\infty_{\text{per}}(Y)), \tag{1.4}$$

where Y is the period of μ, typically the unitary cube of $\mathbb{R}^d$. The natural weak formulation of problem (1.1) comprises the Sobolev spaces $H^{1,p}(\Omega, d\mu_\varepsilon)$ with respect to the measure μ_ε. It is important to note that these spaces vary with ε and are not commonly contained in an adequate function space independent of ε. Classical extension techniques are not feasible, since in general they strongly depend on the concrete geometry under consideration. It turns out that the notion of *connectedness* of a measure (Section 2.3) is sufficiently flexible and at the same time of fundamental importance for the homogenization of associated multistructures. A systematic treatment on connectedness applicable to homogenization was first given by Zhikov [60, 61, 62] and, in a different framework, by Bouchitté and Fragala [14], where the authors derived a structure result for all possible two-scale limits of a sequence $\{u_\varepsilon, \nabla u_\varepsilon\}$ endowed with an uniform bound

$$\sup_{\varepsilon>0} \int_\Omega (|u_\varepsilon|^p + |\nabla u_\varepsilon|^p)\, d\mu_\varepsilon < \infty, \quad u_\varepsilon \in H^{1,p}(\Omega, d\mu_\varepsilon). \tag{1.5}$$

For instance, the two-scale limit u in (1.4) will be independent of the fast variable y, provided μ is *weakly p-connected* on the torus $\mathbb{T}$ (see Definition 2.3.3). However, these structure results do *not* suffice to study the homogenization of Richards equation (1.1), not even the corresponding elliptic problem

$$-\text{div}\, a(\tfrac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) = f(\tfrac{x}{\varepsilon}, u_\varepsilon), \quad u_\varepsilon \in H^{1,p}_0(\Omega, d\mu_\varepsilon) \tag{1.6}$$

with homogeneous Dirichlet boundary condition. Indeed, in order to study the asymptotics of (1.6), the dependence of the data on u_ε requires the *strong* two-scale convergence $u_\varepsilon \twoheadrightarrow u$, i.e. that in addition to (1.4) there holds

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x)|^p\, d\mu_\varepsilon(x) = \int_{\Omega\times Y} |u(x,y)|^p\, dx d\mu(y). \tag{1.7}$$

In some cases, the convergence in (1.7) can be derived *a posteriori* using the homogenized equation (see e.g. [44, 62, 64, 53]), which however does not work for equations of type (1.6). We prove, at least in this generality, a seemingly new result we may call the *rescaled Rellich property* for μ, which states that an estimate of type (1.5) yields (1.7) (see Theorem 2.4.5). We emphasize that this result is harder to obtain than the classical Rellich embedding theorem. It turns out that the measure μ at least needs to enjoy certain Poincaré-type inequalities studied by Hajlash and Koskela [36]. Moreover, the moving geometry of the support of μ_ε causes additional difficulties, which we show can be dealt with under some connectedness assumptions on μ. The rescaled Rellich property opens the door to the homogenization of equations (1.6) and (1.1) (Chapters 3,4) on multidimensional structures, which to the best of our knowledge has not been studied yet. Although we content ourselves with the periodic setting, we see no major obstacle to study the homogenization of equation (1.6) on random singular structures, which were thoroughly investigated by Piatnitski and Zhikov in a recent paper [53].
Our framework of homogenization with periodic measures includes in a natural way the case of thin reinforced structures concentrated along bars of some small thickness δ. For instance, the case of a one-dimensional ε-periodic structure εS (cf. Figure 1.1 and (1.3)) corresponds to the choice $\mu = \mathcal{H}^1 \lfloor S$ and, possibly, to a sequence of measures μ^δ associated with the fattened structure:

$$\mu^\delta = |S_\delta \cap Y|^{-1} \mathcal{L}^d \lfloor S_\delta \,, \qquad S_\delta := \{x \in \mathbb{R}^d :\ \mathrm{dist}\,(x, S) < \delta\}. \tag{1.8}$$

We investigate the commutativity of limits as the two parameters ε and δ tend to zero (Chapter 5). The classical procedure is to homogenize with respect to each μ^δ (see [27] and references therein), and then let δ tend to zero. The nonstandard approach investigated in this thesis is to homogenize with respect to the singular measure μ_ε, that is obtained as the weak limit of the ε-periodization μ_ε^δ of μ^δ according to (1.3). As pointed out in [14], in most cases the effective coefficients can be computed easier with respect to the singular structure. Therefore it is worth investigating whether the two procedures are equivalent or not, as indicated by the following diagram:

$$\begin{tikzcd}
(P_\varepsilon^\delta, \mu_\varepsilon^\delta) \arrow[r, "\varepsilon \to 0"] \arrow[d, "\delta \to 0"'] \arrow[dr, phantom, "\times"] & (P_{hom}^\delta, \mu^\delta) \arrow[d, "\delta \to 0"] \\
(P_\varepsilon^{sing}, \mu_\varepsilon) \arrow[r, "\varepsilon \to 0"] & (P_{hom}^{sing}, \mu)
\end{tikzcd}$$

Figure 1.2: Homogenization diagram

Main results We give here in telegram style the central new results of the thesis. For the complete set of assumptions on the data and the boundary conditions we refer to the mathematical formulation below, as well as to Section 2.3 for the notion of connectedness of a measure.

- **Elliptic problems**

  For every $p > 1$, strongly p-connected measures μ on $\mathbb{R}^d$ and operators $a = a(y, s, \xi)$ that are strictly monotone with respect to the gradient variable ξ, the homogenized equation for (1.6) reads

  $$-\operatorname{div} a^\star(u, \nabla u) \;=\; \bar{f}(u) \;\text{ in } \Omega, \quad u = 0 \;\text{ on } \partial\Omega. \tag{1.9}$$

  The effective coefficient $a^\star : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ can be computed via the solution $v : y \mapsto v(y, s, \xi)$ of the cell problem (1.15) defined below

  $$a^\star : (s, \xi) \mapsto \int_Y a(y, s, \xi + v(y, s, \xi))\, d\mu(y), \tag{1.10}$$

  and $a^\star$ inherits regularity and monotonicity from the coefficient a.

- **Degenerate elliptic-parabolic problems/Richards equation**

  If b is monotonically nondecreasing and μ strongly 2-connected on $\mathbb{R}^d$, and if the flux a_ε is of the form $a_\varepsilon = a(\varepsilon^{-1}x, s, \xi)\mu_\varepsilon$ with no explicit time dependence, and such that the coefficient $a = a(y, s)\xi$ separates in the gradient variable ξ, the homogenized equation for (1.1) reads

  $$\partial_t b(u) - \operatorname{div} a^\star(b(u), \nabla u) \;=\; \bar{f}(b(u)) \quad \text{in } Q, \tag{1.11}$$

  where $a^\star$ is given by (1.10). If the data a and f are sufficiently smooth, we can show uniqueness for (1.11). Moreover, for *strictly* monotone b (and $\mu = \mathcal{L}^d$) we will prove the first order corrector result

  $$\nabla u_\varepsilon - \nabla u - \nabla_y u_1(x, t, \tfrac{x}{\varepsilon}) \;\to\; 0 \quad \text{in } L^2(Q), \tag{1.12}$$

  and can consider also time oscillating data $a(\tau, \cdot) \mapsto a(\varepsilon^{-1}t, \cdot)$, where in the definition of $a^\star$ the coefficient a has to be averaged over the period $Y \times (0, 1)$ with respect to the cell solutions $v : (y, \tau) \mapsto v(y, \tau, s, \xi)$.

- **Two-parameter analysis**

  The homogenization diagram (Figure 1.2) will in general *not* commute. Explicit counterexamples comprising nonconnected singular structures (S, μ) will be constructed, where for semilinear equations the two limit processes lead respectively to different effective coefficients.

  If the measure μ is sufficiently *connected* and regular, the commutativity of the diagram holds for quasilinear elliptic problems (P_ε^δ) of the form

  $$-\operatorname{div}(K_\varepsilon^\delta(\mu^\delta, x, u_\varepsilon^\delta)\nabla u_\varepsilon^\delta) + \lambda u_\varepsilon^\delta \mu_\varepsilon^\delta \;=\; f_\varepsilon^\delta(\mu^\delta, x, u_\varepsilon^\delta) \quad \text{in } \Omega, \tag{1.13}$$

  where $\lambda > 0$ and $K_\varepsilon^\delta(\mu^\delta, x, \cdot) = K_\delta(\frac{x}{\varepsilon}, \cdot)\mu_\varepsilon^\delta$ is a δ-dependent, measure valued coefficient that oscillates with period ε. In particular, it is justified to study equations of type (1.6) and (1.1) on singular structures.

Mathematical formulation We may list here the more detailed, mathematical formulation of our main results before we compare them with existing literature. For the definition of all relevant function spaces, including the space $V^p_{\mathrm{pot}}(\mathbb{T}, d\mu)$ of periodic potential vectors, we refer to Section 2.2.

Theorem 1.1. *Let $p > 1$ and μ be doubling and strongly p-connected on $\mathbb{R}^d$. Let $a(y, s, \xi)$ and $f(y, s)$ be μ-measurable, Y-periodic in y and locally Hölder continuous in s, ξ. Assume $a(y, s, 0) = 0$, and strict monotonicity in ξ:*

$$[a(y, s, \xi_1) - a(y, s, \xi_2)] \cdot (\xi_1 - \xi_2) \geq c\,(1 + |s| + |\xi_1| + |\xi_2|)^{p-\alpha} |\xi_1 - \xi_2|^{\alpha}, \tag{1.14}$$

where $\alpha := \max\{p, 2\}$. Assume $|f(\cdot, s)| \leq C(1 + |s|^{\beta})$ for some $\beta \in [0, p-1)$. Then there exists a weak solution u_ε of problem (1.6), and up to a subsequence there holds $u_\varepsilon \twoheadrightarrow u$ two-scale strongly in $L^p(\Omega, d\mu_\varepsilon)$, where $u \in W^{1,p}_0(\Omega)$ is a solution of equation (1.9). The unique solution $v(\cdot, s, \xi) \in V^p_{\mathrm{pot}}(\mathbb{T}, d\mu)$ of the cell problem

$$\int_Y a(y, s, \xi + v(y, s, \xi)) \cdot \varphi(y)\, d\mu(y) \;=\; 0 \qquad \forall \varphi \in V^p_{\mathrm{pot}}(\mathbb{T}, d\mu) \tag{1.15}$$

determines the effective flux $a^\star$ according to (1.10).

Sketch of proof. Based on an uniform a priori estimate of type (1.5) on the sequence of solutions, the two-scale structure result (Theorem 2.4.4) yields

$$u_\varepsilon \rightharpoonup\!\!\rightharpoonup u \in W^{1,p}_0(\Omega)\,, \quad \nabla u_\varepsilon \rightharpoonup\!\!\rightharpoonup \nabla u + \nabla_y u_1\,, \quad \nabla_y u_1 \in L^p(\Omega; V^p_{\mathrm{pot}}(\mathbb{T}, d\mu)),$$

and the rescaled Rellich property asserts the crucial strong two-scale convergence of $\{u_\varepsilon\}$. The Hölder continuity of a and f with respect to s then allows to substitute (asymptotically) u_ε by u when passing to the limit in the weak formulation of problem (1.6). Using standard approximation techniques and monotonicity tricks we derive (Theorem 3.1.8) a two-scale homogenized problem, from which the corrector u_1, the effective coefficient $a^\star$ and the homogenized equation (1.9) can be derived. The well-posedness of the cell problem (Lemma 3.1.10) and hence of $a^\star$ relies on the connectedness of μ and on the structure conditions imposed on a. □

Now we give our homogenization result for the doubly nonlinear degenerate parabolic equation

$$(P_\varepsilon)\quad \begin{cases} \partial_t b(u_\varepsilon) - \operatorname{div} a_\varepsilon(\mu, x, b(u_\varepsilon), \nabla u_\varepsilon) &= f_\varepsilon(\mu, x, b(u_\varepsilon)) \;\text{ in } Q, \\ b(u_\varepsilon) &= b(u^0_\varepsilon) \qquad\quad\;\;\, \text{in } \Omega \times \{0\} \end{cases}$$

with a_ε and f_ε as in (1.2), where we assume a homogeneous Dirichlet condition on the lateral boundary. For the notion of a weak solution u_ε in the class $L^2(0, T; \tilde{H}^{1,2}_0(\Omega, d\mu_\varepsilon))$ we refer to Definition 4.2.5 on page 97. Time oscillating data as indicated in (1.1) and corrector results will, for strictly monotone b, be investigated in Section 4.3.

Theorem 1.2. *Let b be monotonically nondecreasing and Lipschitz continuous with $b(0) = 0$. Let a, f and μ satisfy the assumptions of Theorem 1.1 for $p = 2$ with separation $a = a(y, s)\xi$, and assume $u_\varepsilon^0 \in L^2(\Omega, d\mu_\varepsilon)$. Then there exists a weak solution u_ε of problem (P_ε). If $u_\varepsilon^0 \twoheadrightarrow u^0 \in L^2(\Omega)$ two-scale strongly with respect to μ, then up to subsequences there holds*

$$u_\varepsilon \rightharpoonup\!\!\rightharpoonup u, \quad b(u_\varepsilon) \twoheadrightarrow b(u) \quad \text{in } L^2(Q, d\mu_\varepsilon \otimes dt) \tag{1.16}$$

two-scale weakly and strongly respectively, where $u \in L^2(0, T; H_0^1(\Omega))$ is a solution of the homogenized equation

$$\partial_t b(u) - \operatorname{div} a^\star(b(u), \nabla u) = \bar{f}(b(u)) \ \text{ in } Q, \ \ b(u) = b(u^0) \ \text{ in } \Omega \times \{0\}, \tag{1.17}$$

with $a^\star$ as in (1.10). If in addition a and f are Lipschitz continuous in s, and b Hölder continuous with exponent $1/2$, then the solution of (1.17) is unique.

Sketch of proof. Existence can be proven by the Rothe method of time discretization (Theorem 4.2.7), including an uniform a priori estimate

$$\|\partial_t b(u_\varepsilon)\|_{L^2 H_\varepsilon'} + \|b(u_\varepsilon)\|_{L^\infty L^2_{\mu_\varepsilon}} + \|u_\varepsilon\|_{L^2 H_\varepsilon} \le C, \tag{1.18}$$

where $H_\varepsilon := H_0^{1,2}(\Omega, d\mu_\varepsilon)$. Then (1.16) follows from estimate (1.18) and the monotonicity and regularity assumptions on b (Lemma 4.2.8). Equation (1.17) can be extracted from a two-scale homogenized problem, which can be derived as in the stationary case using similar monotonicity arguments (Theorem 4.2.10). The additional regularity of the data and, as a consequence, of the flux $a^\star$ (Lemma 3.2.11), enables us to apply a uniqueness result (Theorem 6.10) for equations of type (1.17) using the L^1-contraction principle. □

For the commutativity of the two-parameter diagram we will consider connected periodic networks (S, μ) on $\mathbb{R}^d$ and their approximations (S_δ, μ^δ). For their definition and the notions of weak and strong convergence in the variable $L^p(Y, d\mu^\delta)$-spaces we refer to Chapter 5.

Theorem 1.3. *Let (S, μ) be a connected periodic network in $\mathbb{R}^d$ and $\mu^\delta \rightharpoonup \mu$ according to (1.8). Let $(K_\delta, f_\delta)(y, s)$ be μ^δ-measurable, Y-periodic in y, locally Hölder continuous in s (uniformly in δ) and satisfy*

$$0 < c \le K_\delta(y, s) \le C, \quad |f_\delta(y, s)| \le C(1 + |s|^\beta) \quad \forall \delta > 0, \ \beta \in [0, 1). \tag{1.19}$$

Assume there exist functions $(K, f)(y, s)$ μ-measurable, Y-periodic in y, locally Hölder continuous in s, such that for any fixed s:

$$K_\delta(\cdot, s) \to K(\cdot, s) \ \text{ strongly}, \quad f_\delta(\cdot, s) \rightharpoonup f(\cdot, s) \ \text{ weakly in } L^2(Y, d\mu^\delta). \tag{1.20}$$

Then the two-parameter diagram starting from equation (1.13) (with $u_\varepsilon^\delta = 0$ on $\partial\Omega$) commutes, that means the functions $u, \tilde{u} \in H_0^1(\Omega)$ obtained respectively from the two limit processes are a solution of one and the same problem

$$(P) \qquad -\operatorname{div}(K^\star(u)\nabla u) + \lambda u = \bar{f}(u) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $K^\star$ is the effective tensor corresponding to K and the structure μ.

Sketch of proof. The asymptotics $\varepsilon \to 0$ for fixed δ is covered by Theorem 1.1. The next step is to prove (Lemma 5.2.15) that for each $\varepsilon > 0$, the weak limits

$$u_\varepsilon^\delta \rightharpoonup u_\varepsilon\,, \quad \nabla u_\varepsilon^\delta \rightharpoonup \Phi_\varepsilon \quad \text{as } \delta \to 0 \text{ in } L^2(\Omega, d\mu_\varepsilon^\delta)$$

satisfy $u_\varepsilon \in H_0^{1,2}(\Omega, d\mu_\varepsilon)$ and $\Phi_\varepsilon = \nabla u_\varepsilon$. Combined with (1.20) and the regularity of the data this yields that u_ε is a solution of the singular problem ($P_\varepsilon^{\text{sing}}$) with coefficients K, f. For the asymptotics $\delta \to 0$ in (P_{hom}^δ) we need the *strong approximability property* of $\mu^\delta \rightharpoonup \mu$ (Lemma 5.2.14), which implies

$$V_{\text{pot}}^2(\mathbb{T}, d\mu^\delta) \ni w_\delta \rightharpoonup w \text{ weakly in } L^2(Y, d\mu^\delta) \;\Rightarrow\; w \in V_{\text{pot}}^2(\mathbb{T}, d\mu).$$

This stability result for potential vectors ensures that the cell solutions $\Lambda_\delta(\cdot, s)$ weakly converge to the cell solutions $\Lambda(\cdot, s)$ of the singular problem. Secondly, the uniform lower bound on K_δ yields (Lemma 5.2.14) that the quadratic form corresponding to $K_\delta^\star$ is strictly positive uniformly in δ and s. It follows that the sequence $\{u^\delta\}$ of solutions of problem (P_{hom}^δ) is actually bounded in $H_0^1(\Omega)$, and that its weak limit u is a solution of problem (P). □

Comparison with existing literature In the classical setting of the Lebesgue measure $\mu_\varepsilon \equiv \mu = \mathcal{L}^d$, the homogenization of equation (1.6) was first studied by Babuska [8] for $p = 2$, by Fusco and Moscariello [32] for arbitrary $p > 1$, by Pankov [52] and Allaire [1] with G-convergence and two-scale methods. For Radon measures μ the asymptotics of related energy functionals

$$J_\varepsilon(u_\varepsilon) \;=\; \int_\Omega j(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon)\, d\mu_\varepsilon\,, \quad z \mapsto j(y, z) \text{ convex}, \tag{1.21}$$

was studied by Zhikov [61] and by Bouchitté and Fragala [14]. In the special case when the coefficients do *not* depend on u_ε, the homogenization of equation (1.6) was investigated by Zhikov [60, 62] in the linear case and, most recently, by Lukkassen and Wall [44] for monotone operators $a = a(y, \xi)$. Hence Theorem 1.1 generalizes the results of [32] to the framework of Radon measures, respectively the results in [14, 44, 61] to the quasilinear equation (1.6) with additional dependence of the data on u_ε. As pointed out above, this is a nontrivial extension and at the same time fundamental for many applications. We also mention some recent studies on the homogenization of nonlinear elliptic operators on domains with asymptotically degenerating measure [3, 4] and on weighted Sobolev spaces [30], frameworks that either differ considerably from our singular measure approach or are merely special cases of our investigation. We also consider a singular nonlinear *double porosity* model (Section 3.3), where the coefficient a in equation (1.6) depends on ε in a more complicated way. This generalizes the analysis of Zhikov [62] and Bourgeat et al. [20] of the corresponding linear model.

The homogenization result for the quasilinear degenerate elliptic-parabolic equation (P_ε) subject to a general connected Radon measure μ is new. In the special case $\mu = \mathcal{L}^d$, the asymptotics of the problem

$$\partial_t b(u_\varepsilon) - \nabla \cdot a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) \;=\; f, \tag{1.22}$$

has been studied by Jian [41], Nandakumaran and Rajesh [48], Hou and Zhang [39], Chen, Deng and Ye [25] and Efendiev and Pankov [29], respectively for special classes of *strictly* monotone functions b. If the flux a depends more specifically on $b(u)$ and separates in the gradient variable as in Richards equation, we will allow b to be merely monotonically nondecreasing (cf. Theorem 1.2), which is also of physical relevance. Homogenization [39, 41, 48] and corrector results [48, Theorem 2.5] have been stated for equation (1.22), however under the (in general hard to be verified) presumption that the sequence of solutions $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty(Q)$. Moreover the proofs need an additional argument (see Section 4.3 for the details). In contrast, we will be able to derive the homogenized equation and a corrector result of type (1.12) for *any* strictly monotone b equipped with a mild growth condition, and without presuming any a priori bound on $\{u_\varepsilon\}$. Moreover we investigate uniqueness for the homogenized equation, which is not done in the papers mentioned.
Two-parameter homogenization, with periodicity ε and fracture thickness δ, has quite widely been studied (see e.g. [5, 11, 14, 18, 20, 21, 24, 27]). For linear equations, the commutativity of limits on some special networks was shown by Bourgeat, Chechkin, Lukkassen, Piatnitski and Zhikov [20, 24], and in a more general framework comprising two-parameter variational functionals

$$J_\varepsilon^\delta(u) = \int_\Omega j(\nabla u)\, d\mu_\varepsilon^\delta\,, \quad z \mapsto j(z) \text{ convex}, \tag{1.23}$$

by Bouchitté and Fragala [14]. In the latter case, the authors showed that in the framework of letting δ depend on ε, the Γ-limit of $J_\varepsilon^{\delta(\varepsilon)}$ is the same whatever the choice of the sequence $\delta(\varepsilon)$, provided the underlying measure μ is strongly connected. In the nonconnected case, Bellieud and Bouchitté [11] have shown that the limit energy may in addition to $j(z)$ contain a nonlocal term which depends on the velocity of convergence to zero of $\delta(\varepsilon)$.
Theorem 1.3 generalizes the commutativity results to the class (1.13) of quasilinear equations, where we consider the framework of letting one parameter fixed, including the singular measure approach with $\delta = 0$ as $\varepsilon \to 0$. In this setting we also construct our counterexamples to noncommutativity for a suitable class of semilinear equations.

Outline of thesis In Chapter 2 we introduce the setting for the homogenization with respect to Radon measures, including two-scale structure and compactness results needed to treat nonlinear problems. In Chapter 3 and Chapter 4 we study the homogenization of quasilinear elliptic and degenerate elliptic-parabolic equations respectively, where an extra section is dedicated to Richards equation. In Chapter 5 we investigate the two-parameter case and study necessary and sufficient conditions for the (non-)commutativity of limits. In the appendix we collect the basic notation significant for the thesis, as well as some technical lemmas to which we refer in the text.

2 The measure setting

In this preparatory chapter we develop an adequate setting for the homogenization of multidimensional structures with respect to Radon measures. This includes a suitable notion of two-scale convergence and an intensive study of Lebesgue and Sobolev spaces related to a measure μ and its rescalings μ_ε defined in (1.3). Moreover we introduce the notion of connectedness of a measure, which is closely related to classical Poincaré-type inequalities. Connectedness plays a significant role for the homogenization theory in this field. On this basis we are able to prove several auxiliary lemmas, that will lead to the main results of this chapter: The structure theorem for all possible two-scale limits of bounded sequences $\{u_\varepsilon, \nabla u_\varepsilon\}$ as in (1.5), and the rescaled Rellich property, which is crucial for the homogenization of nonlinear problems. Let us briefly summarize the notation and basic assumptions relevant for this chapter which will also hold, unless otherwise stated, for the rest of the thesis.

Preliminaries and notation

Let Ω be an open, bounded and connected subset of $\mathbb{R}^d$ with smooth boundary and Y the unitary cube of $\mathbb{R}^d$. We always assume that μ is a positive, normalized, Y-periodic Radon measure on $\mathbb{R}^d$, which satisfies $\mu(\partial Y) = 0$ without loss of generality. If $\mathcal{L}^d$ is the Lebesgue measure in $\mathbb{R}^d$, we denote by $m := (\mathcal{L}^d \lfloor \Omega) \otimes (\mu \lfloor Y)$ the product measure. For any $q \in [1, \infty]$, we set

$$L^q_\mu := L^q(\mathbb{R}^d, d\mu),\ L^q_{\mu,\mathrm{loc}} := L^q_{\mathrm{loc}}(\mathbb{R}^d, d\mu),\ L^q_m := L^q(\Omega \times Y, dm).$$

We call $\mathbb{T}$ the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ and identify functions on $\mathbb{T}$ with Y-periodic functions on $\mathbb{R}^d$, that means $L^q_\mu(\mathbb{T}) := L^q(\mathbb{T}, d\mu)$ is the space of functions in $L^q(Y, d\mu)$ extended by Y-periodicity to the whole of $\mathbb{R}^d$. For $q \in [1, \infty)$, the norms in L^q_μ, $L^q_\mu(\mathbb{T})$ and L^q_m are respectively abbreviated by:

$$\begin{aligned} \|u\|_{q,\mu} &:= \Big(\int_{\mathbb{R}^d} |u(y)|^q \, d\mu\Big)^{1/q}, \quad \|u\|_{q,\mu,Y} := \Big(\int_Y |u(y)|^q \, d\mu\Big)^{1/q}, \\ \|u\|_{q,m} &:= \Big(\int_{\Omega \times Y} |u(x,y)|^q \, dm\Big)^{1/q}, \end{aligned}$$

and similarly for $q = \infty$ using the μ-essential supremum. By writing $\varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbb{T}))$ we mean that $\varphi = \varphi(x,y)$ is smooth in both its variables, compactly supported in Ω and Y-periodic in y. Finally note that by $p, p' \in [1, \infty]$ we always denote fixed conjugate exponents. Now for any $\varepsilon > 0$, we define the rescaled measure μ_ε as follows:

$$\int_\Omega \varphi(x) \, d\mu_\varepsilon(x) := \varepsilon^d \int_{\Omega/\varepsilon} \varphi(\varepsilon x) \, d\mu \quad \forall \varphi \in \mathcal{C}_0(\Omega), \tag{2.1}$$

where $\mathcal{C}_0(\Omega)$ is the space of continuous and compactly supported functions on Ω. Using (2.1), the periodicity and the normalization of μ we check

$$\mu_\varepsilon \rightharpoonup \mu(Y)\mathcal{L}^d \lfloor \Omega = \mathcal{L}^d \lfloor \Omega \tag{2.2}$$

in the vague topology of measures. Note that this implies $\mu_\varepsilon(\Omega) \to \mathcal{L}^d(\Omega)$ as $\varepsilon \to 0$, since Ω is bounded and $\mathcal{L}^d(\partial\Omega) = 0$. Periodic structures modeled by μ_ε will be considered in the subsequent chapters. We give an example of the relation between μ and μ_ε. Let $S \subset \mathbb{R}^d$ be a piecewise smooth and compact manifold of dimension one and $\mu = \mathcal{H}^1\lfloor S$. Then we have $\mu_\varepsilon = \varepsilon^{d-1}\mathcal{H}^1\lfloor\varepsilon S$:

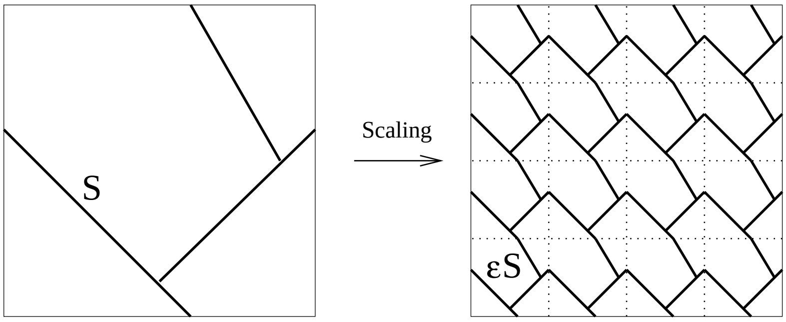


Figure 2.1: Support of μ and μ_ε.

The Lebesgue spaces with respect to μ_ε are, for $q \in [1,\infty]$, denoted by $L^q(\Omega, d\mu_\varepsilon)$, or shorter $L^q_{\mu_\varepsilon}(\Omega)$, and for the norms we set

$$\|u\|_{q,\varepsilon} := \left(\int_\Omega |u(x)|^q d\mu_\varepsilon(x)\right)^{1/q}, \quad \|u\|_{\infty,\varepsilon} := \mu_\varepsilon - \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|\,. \tag{2.3}$$

2.1 Two-scale convergence

The notion of two-scale convergence can be extended to the setting of periodic Radon measures, which may rescale nontrivially. This concept was first developed in [14, 62]. If μ is the Lebesque measure up to some density, the classical two-scale convergence introduced in [1, 50] is retained. We introduce a special class of measures, which covers the main application we have in mind, namely the homogenization of periodic multijunctions. However, we will also consider more general measures (cf. Example 2.2.5 below).

Definition 2.1.1. *We say that μ belongs to the class $J_\sharp$ of periodic multijunction measures, if $\mu\lfloor Y = \sum_{i=1}^n \mu_i$, where $\mu_i := m_i\mathcal{H}^{k_i}\lfloor S_i$. Here m_i are positive constants, k_i integers in $\{1,\dots,d\}$, and S_i are k_i-dimensional closed manifolds of class $\mathcal{C}^2$ contained in Y, such that $\mu_i(S_j) = 0$ for $i \neq j$.*

In this section we basically follow the lines of [14, Section 2]. Crucial is the weak compactness property of two-scale convergence.

Definition 2.1.2. *Let $v_\varepsilon \in L^p(\Omega, d\mu_\varepsilon)$ and $v \in L^p_m(\Omega\times\mathbb{T})$ for some $p \geq 1$. We say that the sequence $\{v_\varepsilon\}$ two-scale converges to v (with respect to μ and as $\varepsilon \to 0$) and write $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v$, if for each $\psi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbb{T}))$:*

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon(x)\psi(x, \tfrac{x}{\varepsilon})\, d\mu_\varepsilon(x) = \int_{\Omega\times Y} v(x,y)\psi(x,y)\, dm(x,y). \tag{2.4}$$

Occasionally we call (2.4) the *weak* two-scale convergence, in order to distinguish from the *strong* two-scale convergence introduced in Definition 2.1.11 below. Note that the two-scale limit takes into account the oscillations of the sequence $\{v_\varepsilon \mu_\varepsilon\}$ which have the same frequency as the test functions $\psi(x, \frac{x}{\varepsilon})$.

Example 2.1.3. *Let v be continuous in $\overline{\Omega} \times \mathbb{R}^d$ and Y-periodic in the last variable, and let $v_\varepsilon(x) = v(x, \frac{x}{\varepsilon})$. Then there holds*

$$v(x, \tfrac{x}{\varepsilon})\mu_\varepsilon \rightharpoonup \left(\int_Y v(x,y)\,d\mu(y)\right)\mathcal{L}^d\lfloor\Omega\,, \quad v_\varepsilon \rightharpoonup\!\!\rightharpoonup v\,. \tag{2.5}$$

The weak compactness property of two-scale convergence holds within the new setting of Definition 2.1.2. For the proof we refer to [14, Proposition. 2.3].

Proposition 2.1.4. *Let $p \in (1,\infty)$ and $v_\varepsilon \in L^p(\Omega, d\mu_\varepsilon)$ with $\|v_\varepsilon\|_{p,\varepsilon} \le C$ uniformly in ε. Then there exists a subsequence, still denoted by ε, and a function $v \in L^p_m(\Omega \times Y)$, such that $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v$.*

Enlarging the space of admissible test functions in (2.4) plays a significant role in the homogenization of nonlinear problems. The following definition generalizes the classical notion of admissibility (cf. [1, Definition 1.4]).

Definition 2.1.5. *Let $p \in [1,\infty)$. An m-measurable function $\varphi : \Omega\times\mathbb{R}^d \to \mathbb{R}$, $(x,y) \mapsto \varphi(x,y)$, Y-periodic in y, is called p-admissible, if $x \mapsto \varphi(x, \frac{x}{\varepsilon})$ is μ_ε-measurable on Ω for any $\varepsilon > 0$, and*

$$\lim_{\varepsilon\to 0} \|\varphi(x, \tfrac{x}{\varepsilon})\|_{p,\varepsilon} = \|\varphi(x,y)\|_{p,m}\,. \tag{2.6}$$

The following Lemma allows to find a sufficiently large class of admissible test functions. The proof can be found in [44, Theorem 2].

Lemma 2.1.6. *Let $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a function which satisfies*

(a) The function $x \mapsto \psi(x,y)$ is continuous for μ-almost every y.

(b) The function $y \mapsto \psi(x,y)$ is μ-measurable and Y-periodic for every x.

(c) The function $y \mapsto \sup_{x\in\Omega} |\psi(x,y)|$ belongs to $L^1_\mu(\mathbb{T})$.

Then the function $x \mapsto \psi(x, \frac{x}{\varepsilon})$ is μ_ε-measurable on Ω for any $\varepsilon > 0$, and there holds $\psi(x, \frac{x}{\varepsilon}) \rightharpoonup\!\!\rightharpoonup \psi(x,y)$ two-scale with respect to μ. In particular

$$\lim_{\varepsilon\to 0} \int_\Omega \psi(x, \tfrac{x}{\varepsilon})\,d\mu_\varepsilon = \int_{\Omega\times Y} \psi(x,y)\,dm\,. \tag{2.7}$$

We prove a corollary we will frequently use in the homogenization of nonlinear problems. Roughly speaking, a function φ being continuous in one of the variables has a chance to be admissible in the sense of Definition 2.1.5.

Corollary 2.1.7. *Let $p \in [1,\infty)$. Any $\varphi \in L^p_\mu(\mathbb{T}; \mathcal{C}(\overline{\Omega}))$ is p-admissible. Moreover, any $\varphi \in \mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T}))$ has a p-admissible representative.*

Proof. The assumptions of Lemma 2.1.6 hold for $\psi(x,y) := |\varphi(x,y)|^p$, where for (b) the arguments in the proof of [1, Lemma 5.3] can be carried over to a general measure μ. Again refering to [1, Lemma 5.6], one can show that there exists a representative $\tilde{\varphi}$ of $\varphi \in \mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T}))$, and a subset $U \subset Y$ independent of $x \in \overline{\Omega}$ with $\mu(U) = 0$, such that

$$x \mapsto \tilde{\varphi}(x,y) \text{ is continuous in } \overline{\Omega}, \text{ uniformly w.r.t. } y \in Y \setminus U\,,$$

$$|\tilde{\varphi}(x,y)| \leq C \text{ independent of } x \in \overline{\Omega},\, y \in Y \setminus U\,.$$

The function $\psi := |\tilde{\varphi}(x,y)|^p$ satisfies all the assumptions of Lemma 2.1.6. □

Remark 2.1.8. *It is evident that $\varphi(x, \frac{x}{\varepsilon}) \rightharpoonup\!\!\rightharpoonup \varphi(x,y)$ for any $\varphi \in L^p_\mu(\mathbb{T}; \mathcal{C}(\overline{\Omega}))$ or $\varphi \in \mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T}))$, as the proof of Corollary 2.1.7 shows.*

Weakening the regularity assumptions on a test function φ is delicate. As pointed out in [1, Proposition 5.8], even in the case $\mu = \mathcal{L}^d$ the second statement of Corollary 2.1.7 is sharp in the following sense:

Remark 2.1.9. *A function $\varphi \in \mathcal{C}(\overline{\Omega}; L^p_\mu(\mathbb{T})) \cap L^\infty_m(\Omega \times Y)$ with $p < \infty$ is in general not p-admissible.*

Note that the two-scale convergence is a stronger concept than the weak convergence of $\{v_\varepsilon \mu_\varepsilon\}$ in the sense of measures. Hence it is not surprising that the following lower semicontinuity property holds.

Proposition 2.1.10. *Let $p > 1$ and $v_\varepsilon \in L^p(\Omega, d\mu_\varepsilon)^d$ two-scale converge by components to $v \in L^p_m(\Omega \times Y)^d$. Then there holds*

$$\liminf_{\varepsilon \to 0} \int_\Omega |v_\varepsilon(x)|^p \, d\mu_\varepsilon(x) \geq \int_{\Omega \times Y} |v(x,y)|^p \, dm(x,y). \tag{2.8}$$

Proof. We refer to [14, Proposition 2.5] □

Since we are mainly concerned with nonlinear homogenization problems, a notion of *strong* two-scale convergence is required, which allows to pass to limits in nonlinear expressions. Proposition 2.1.10 suggests the following

Definition 2.1.11. *Let $v_\varepsilon \in L^p(\Omega, d\mu_\varepsilon)$ and $v \in L^p_m(\Omega \times \mathbb{T})$ for some $p > 1$. We say that $\{v_\varepsilon\}$ two-scale strongly converges to v (with respect to μ and as $\varepsilon \to 0$) and write $v_\varepsilon \twoheadrightarrow v$, if*

$$v_\varepsilon \rightharpoonup\!\!\rightharpoonup v \quad \textit{and} \quad \limsup_{\varepsilon \to 0} \int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon(x) \leq \int_{\Omega \times Y} |v|^p \, dm(x,y)\,. \tag{2.9}$$

If $v_\varepsilon \twoheadrightarrow v$, then the $L^p_{\mu_\varepsilon}$-norm of v_ε converges to the L^p_m-norm of v by (2.8). This means that the oscillations of the sequence $\{v_\varepsilon \mu_\varepsilon\}$ are captured by the two-scale limit. The next example directly follows from Corollary 2.1.7:

Example 2.1.12. *Let $v \in L^p_\mu(\mathbb{T}; \mathcal{C}(\overline{\Omega}))$ and $v_\varepsilon(x) := v(x, \frac{x}{\varepsilon})$. Then there holds $v_\varepsilon \twoheadrightarrow v$. The same statement is true for $v \in \mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T}))$.*

The following central result shows that one can pass to the limit given a product of a weakly and a strongly two-scale convergent sequence.

Proposition 2.1.13. *Let $p > 1$, $\{v_\varepsilon\} \subset L^p(\Omega, d\mu_\varepsilon)$ be a sequence that strongly two-scale converges to $v \in L^p_m$, and $\{w_\varepsilon\} \subset L^{p'}(\Omega, d\mu_\varepsilon)$ satisfy $w_\varepsilon \rightharpoonup\!\!\rightharpoonup w$ for some $w \in L^{p'}_m$ with $\|w_\varepsilon\|_{p',\varepsilon} \leq C$ uniformly in ε. Then there holds*

$$v_\varepsilon w_\varepsilon \mu_\varepsilon \rightharpoonup \left(\int_Y v(\cdot, y) w(\cdot, y)\, d\mu(y)\right) \mathcal{L}^d \lfloor \Omega\,. \tag{2.10}$$

Proof. Let $\{\varphi_\delta\} \subset \mathcal{D}(\Omega \times Y)$ be a sequence with $\varphi_\delta \to v$ in $L^p_m(\Omega \times Y)$ and extended by Y-periodicity to $\Omega \times \mathbb{R}^d$. For any function $\psi \in \mathcal{C}(\overline{\Omega})$ there holds

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon w_\varepsilon \psi\, d\mu_\varepsilon = \lim_{\delta\to 0} \lim_{\varepsilon \to 0} \left(\int_\Omega (v_\varepsilon - \varphi_\delta(x, \tfrac{x}{\varepsilon}) + \varphi_\delta(x, \tfrac{x}{\varepsilon})) w_\varepsilon \psi\, d\mu_\varepsilon\right).$$

Since $\varphi_\delta \cdot \psi$ is an admissible test function for the convergence $w_\varepsilon \rightharpoonup\!\!\rightharpoonup w$, the choice of $\{\varphi_\delta\}$ gives

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int_\Omega \varphi_\delta(x, \tfrac{x}{\varepsilon}) w_\varepsilon \psi\, d\mu_\varepsilon \;=\; \lim_{\delta \to 0} \int_{\Omega\times Y} \varphi_\delta w \psi\, dm \;=\; \int_{\Omega\times Y} v w \psi\, dm\,.$$

In order to prove (2.10), it is therefore enough to show

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int_\Omega (v_\varepsilon - \varphi_\delta(x, \tfrac{x}{\varepsilon})) w_\varepsilon \psi\, d\mu_\varepsilon \;=\; 0\,. \tag{2.11}$$

Applying the Hölder inequality, the uniform boundedness of w_ε yields

$$\left|\int_\Omega (v_\varepsilon - \varphi_\delta(x, \tfrac{x}{\varepsilon})) w_\varepsilon \psi\, d\mu_\varepsilon\right| \;\leq\; C \|\psi\|_\infty \|v_\varepsilon - \varphi_\delta(x, \tfrac{x}{\varepsilon})\|_{p,\varepsilon}.$$

In order to estimate the term on the right-hand side, we make use of the Clarkson inequalities leading to the following estimates for all $u, w \in L^p(\Omega, d\mu_\varepsilon)$:

1. $\forall p \in [2, \infty):\quad \|u + w\|^p_{p,\varepsilon} + \|u - w\|^p_{p,\varepsilon} \;\leq\; 2^{p-1}(\|u\|^p_{p,\varepsilon} + \|w\|^p_{p,\varepsilon}),$
2. $\forall p \in (1, 2]:\quad \|u + w\|^{p'}_{p,\varepsilon} + \|u - w\|^{p'}_{p,\varepsilon} \leq 2\,(\|u\|^p_{p,\varepsilon} + \|w\|^p_{p,\varepsilon})^{\frac{1}{p-1}}.$

Using the Clarkson inequalities we get respectively for $p \geq 2$ and $p \leq 2$:

$$\begin{aligned}
\|v_\varepsilon - \varphi_\delta(\tfrac{x}{\varepsilon})\|^p_{p,\varepsilon} \;&\leq\; 2^p \left(\tfrac{1}{2}\|v_\varepsilon\|^p_{p,\varepsilon} + \tfrac{1}{2}\|\varphi_\delta(\tfrac{x}{\varepsilon})\|^p_{p,\varepsilon} - \|\frac{v_\varepsilon + \varphi_\delta(\frac{x}{\varepsilon})}{2}\|^p_{p,\varepsilon}\right),\\
\|v_\varepsilon - \varphi_\delta(\tfrac{x}{\varepsilon})\|^{p'}_{p,\varepsilon} \;&\leq\; 2^{p'} \left([\tfrac{1}{2}\|v_\varepsilon\|^p_{p,\varepsilon} + \tfrac{1}{2}\|\varphi_\delta(\tfrac{x}{\varepsilon})\|^p_{p,\varepsilon}]^{\frac{1}{p-1}} - \|\frac{v_\varepsilon + \varphi_\delta(\frac{x}{\varepsilon})}{2}\|^{p'}_{p,\varepsilon}\right).
\end{aligned}$$

Note that Proposition 2.1.10 gives: $\liminf_{\varepsilon\to 0} \|v_\varepsilon + \varphi_\delta(\frac{x}{\varepsilon})\|^q_{p,\varepsilon} \geq \|v + \varphi_\delta\|^q_{p,m}$ for any $q \geq 1$. Example 2.1.12 can be applied to φ_δ, and the strong convergence of the sequence $\{v_\varepsilon\}$ gives

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \|v_\varepsilon - \varphi_\delta(x, \tfrac{x}{\varepsilon})\|^p_{p,\varepsilon} \;&\leq\; 2^p \left(\tfrac{1}{2}\|v\|^p_{p,m} + \tfrac{1}{2}\|\varphi_\delta\|^p_{p,m} - \|\frac{v + \varphi_\delta}{2}\|^p_{p,m}\right),\\
\limsup_{\varepsilon\to 0} \|v_\varepsilon - \varphi_\delta(x, \tfrac{x}{\varepsilon})\|^{p'}_{p,\varepsilon} \;&\leq\; 2^{p'} \left([\tfrac{1}{2}\|v\|^p_{p,m} + \tfrac{1}{2}\|\varphi_\delta\|^p_{p,m}]^{\frac{1}{p-1}} - \|\frac{v + \varphi_\delta}{2}\|^{p'}_{p,m}\right)
\end{aligned}$$

for $p \geq 2$ and $p \leq 2$ respectively. Taking the lim sup as $\delta \to 0$, by the choice of the sequence $\{\varphi_\delta\}$ the right-hand sides of the above inequalities tend to zero, so that (2.11) and hence the proposition is proved. $\square$

We show that the convergence in (2.10), if holding for every sequence $w_\varepsilon \rightharpoonup\!\!\rightharpoonup w$, is already sufficient for the strong two-scale convergence of v_ε. This turns out to be helpful for the homogenization of monotone operators.

Lemma 2.1.14. *Let $p \in (1, \infty)$ and $\{v_\varepsilon\} \subset L^p(\Omega, d\mu_\varepsilon)$ be a bounded sequence that admits the following property: There exists $v \in L^p_m(\Omega \times Y)$, such that*

$$v_\varepsilon w_\varepsilon \mu_\varepsilon \rightharpoonup \left(\int_Y v(\cdot, y) w(\cdot, y)\, d\mu(y) \right) \mathcal{L}^d \lfloor \Omega \tag{2.12}$$

whenever $w_\varepsilon \rightharpoonup\!\!\rightharpoonup w \in L^{p'}_m(\Omega \times Y)$ two-scale weakly for a sequence $\{w_\varepsilon\} \subset L^{p'}(\Omega, d\mu_\varepsilon)$. Then there holds $v_\varepsilon \twoheadrightarrow v$.

Proof. The prerequisites clearly imply $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v$, and $w_\varepsilon := |v_\varepsilon|^{p-2} v_\varepsilon$ is a bounded sequenced in $L^{p'}(\Omega, d\mu_\varepsilon)$. By Proposition 2.1.4 and (2.12) we get

$$\int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon(x) = \int_\Omega v_\varepsilon w_\varepsilon \, d\mu_\varepsilon(x) \to \int_{\Omega \times Y} vw \, dm(x, y) \tag{2.13}$$

for some $w \in L^{p'}_m(\Omega \times Y)$ possibly up to a subsequence. It suffices to show $w = |v|^{p-2} v$, which is nontrivial for $p \neq 2$. We introduce the continuous and monotone increasing function $f : t \mapsto |t|^{p-2} t$ on $\mathbb{R}$ and claim that

$$\int_{\Omega \times Y} [f(\psi(x, y)) - w(x, y)] \, [\psi(x, y) - v(x, y)] \, dm \geq 0 \tag{2.14}$$

for any $\psi \in L^p_m(\Omega \times Y)$. Indeed, let $\{\psi_\delta\} \subset \mathcal{D}(\Omega \times Y)$ be a sequence with $\psi_\delta \to \psi$ in $L^p_m(\Omega \times Y)$ and extended by Y-periodicity to $\Omega \times \mathbb{R}^d$. By the monotonicity of f we deduce

$$\int_\Omega [f(\psi_\delta(x, \tfrac{x}{\varepsilon})) - w_\varepsilon(x)] \, [\psi_\delta(x, \tfrac{x}{\varepsilon}) - v_\varepsilon(x)] \, d\mu_\varepsilon(x) \geq 0 \, , \tag{2.15}$$

where we used $f(v_\varepsilon) = w_\varepsilon$. As in the proof of Proposition 2.1.13, first pass to the limit $\varepsilon \to 0$ in (2.15) using (2.13), and then to the limit $\delta \to 0$ to obtain (2.14), where one has to use $f(\psi_\delta) \to f(\psi)$ strongly in $L^{p'}_m$. Now choose $\psi = v + t\varphi$ in (2.14) with $\varphi \in L^p_m(\Omega \times Y)$ arbitrary. Deviding by t for $t > 0$ and $t < 0$ respectively, we get

$$\int_{\Omega \times Y} [f(v + t\varphi) - w] \, \varphi \, dm \geq 0 \;\; (\leq 0). \tag{2.16}$$

It is obvious that $f(v + t\varphi) \to f(v)$ strongly in $L^{p'}_m$ for $t \to 0$. Therefore, passing to the limit in (2.16) in both cases yields $f(v) = w$ in $L^{p'}_m$. $\square$

2.2 Sobolev spaces

Sobolev spaces with respect to a Radon measure μ arise naturally in the weak formulation of elliptic problems posed on multidimensional structures. The well known fact that the gradient of a μ-Sobolev function is in general not unique will cause slight inconvenience. We first have to define a class $\tilde{H}^{1,p}_\mu$ of functions, whose elements may have many gradients. Using the concept of tangential gradients, which is strongly related to relaxation (see Proposition 2.2.11 below), we are able to extract the Banach spaces $H^{1,p}_\mu := H^{1,p}_\mu(\mathbb{R}^d)$. Recall that μ is a positive, Y-periodic Radon measure on $\mathbb{R}^d$ with $\mu(\partial Y) = 0$, and $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ the space of smooth, compactly supported functions on $\mathbb{R}^d$.

The Sobolev spaces $H^{1,p}_\mu$.

We follow the concepts in [17, 19]. For $p \in [1, \infty]$, let $V^p_\mu := V^p(\mathbb{R}^d, d\mu)$ be the subspace of $L^p_\mu \times (L^p_\mu)^d$ defined as follows:

$$(u, z) \in V^p_\mu \quad \text{if there exists a sequence } \{\varphi_n\} \subset \mathcal{D}, \text{ such that}$$

$$(\varphi_n, \nabla\varphi_n) \to (u, z) \quad \text{strongly in } (L^p_\mu)^{d+1}. \tag{2.17}$$

Obviously V^p_μ is a Banach space for the induced L^p_μ-norm. We define the class of Sobolev functions as the set of first components in V^p_μ:

$$\tilde{H}^{1,p}_\mu := \{u \in L^p_\mu : \exists z \in (L^p_\mu)^d \text{ such that } (u, z) \in V^p_\mu\}. \tag{2.18}$$

We often denote the vector z by ∇u and call it a gradient of u. Note that by now, $\tilde{H}^{1,p}_\mu$ is merely a linear subspace of L^p_μ. It is not clear how to define a norm in this space, since in general the gradient of a Sobolev function is not unique (see Example 2.2.4 below). It turns out that the set of gradients of $u = 0$, the vectors *normal* to the structure, plays an important role:

Observation 2.2.1. *The set $\Gamma^p_\mu := \{z \in (L^p_\mu)^d\colon\ (0, z) \in V^p_\mu\}$ of gradients of zero is a closed subspace of $(L^p_\mu)^d$ and satisfies the following stability property:*

$$z \in \Gamma^p_\mu,\ \psi \in \mathcal{D} \ \Rightarrow\ \psi z \in \Gamma^p_\mu\,. \tag{2.19}$$

Moreover if $u \in \tilde{H}^{1,p}_\mu$ with $(u, z), (u, \tilde{z}) \in V^p_\mu$, then there holds $z = \tilde{z} + z_0$ for some $z_0 \in \Gamma^p_\mu$.

Assume for a moment $p < \infty$. Then, thanks to the stability property (2.19), we can apply Lemma 6.12 in the appendix, which considers multifunctions associated with stable spaces. By (6.18), since Γ^p_μ is closed, there exists a μ-measurable multifunction $N^p_\mu : \mathbb{R}^d \to \mathrm{lin}(\mathbb{R}^d)$, where $\mathrm{lin}(\mathbb{R}^d)$ is the set of linear subspaces of $\mathbb{R}^d$, such that

$$z \in \Gamma^p_\mu \iff z(x) \in N^p_\mu(x) \quad \text{for } \mu\text{-a.e } x. \tag{2.20}$$

It is important to note that (2.20) gives a pointwise characterization of Γ_μ^p and hence, as we will see, of tangential and normal gradients. For μ-a.e. point $x \in \mathbb{R}^d$ we can now define the tangent space of μ at a point x by setting

$$\mathrm{lin}(\mathbb{R}^d) \ni T_\mu^p(x) := N_\mu^p(x)^\perp. \tag{2.21}$$

A more intrinsic way to define the tangent space $T_\mu(x)$ is to consider the orthogonal complement of Γ_μ^p in $(L_\mu^{p'})^d$, which will also work in the case $p = \infty$. We sketch this approach. Denote by $\mathcal{D}'$ the space of distributions on $\mathbb{R}^d$. Then for any $\sigma \in (L^1_{\mu,\mathrm{loc}})^d$, an element $\mathrm{div}(\sigma\mu) \in \mathcal{D}'$ is given by

$$\langle\!\langle \mathrm{div}(\sigma\mu), \varphi \rangle\!\rangle_{\mathcal{D}',\mathcal{D}} := -\int_{\mathbb{R}^d} \sigma \cdot \nabla\varphi \, d\mu \quad \forall \varphi \in \mathcal{D}. \tag{2.22}$$

Whenever $\mathrm{div}(\sigma\mu)$ is a measure absolutely continuous with respect to μ with a density belonging to $L_\mu^{p'}$, we write $\mathrm{div}(\sigma\mu) \in L_\mu^{p'}$ and denote by $\mathrm{div}_\mu\sigma$ the derivative $\frac{d}{d\mu}\mathrm{div}(\sigma\mu)$. For any pair of dual exponents $p, p' \in [1,\infty]$ one can define the class of all vector functions *tangent to* μ by

$$X_\mu^{p'} := \{\Phi \in (L_\mu^{p'})^d : \ \mathrm{div}(\Phi\mu) \in L_\mu^{p'}\}. \tag{2.23}$$

Then for any $p \in [1,\infty]$ the tangent space $T_\mu^p(x)$ of μ at x can be defined as (see [16, 14] for the details)

$$T_\mu^p(x) := \mu - \mathrm{ess}\bigcup\{\Phi(x) : \ \Phi \in X_\mu^{p'}\}, \quad x \in \mathbb{R}^d. \tag{2.24}$$

This definition coincides with (2.21) for $p < \infty$. As pointed out in [16], it is unknown whether there exists a positive Radon measure μ, such that T_μ depends on p. At least for all measures μ considered in this thesis, this is not the case and hence we will write $T_\mu(x) := T_\mu^p(x)$. In [17, Section 3.1] it was shown, similar to (2.19), that the following crucial stability property is satisfied:

$$\Phi \in X_\mu^{p'}, \ \varphi \in \mathcal{D} \ \Rightarrow \ \varphi\Phi \in X_\mu^{p'}. \tag{2.25}$$

It fact, for any $p \in [1,\infty]$ it turns out that $\Gamma_\mu^p = (X_\mu^{p'})^\perp$, and for any function $\Phi \in (L_\mu^{p'})^d$ there holds

$$\Phi \in \overline{X_\mu^{p'}} \implies \Phi(x) \in T_\mu(x) \ \text{ for } \mu\text{-a.e. } x, \tag{2.26}$$

with equivalence in the case $p' \in (1,\infty)$. We can now define the orthogonal projection P_μ from $\mathbb{R}^d$ onto T_μ independent of p, more precisely the μ-measurable, essentially bounded function

$$P_\mu : \mathbb{R}^d \to \mathbb{R}^{d\times d}, \, x \mapsto P_\mu(x)\,, \quad P_\mu(x)[v] = \tau \tag{2.27}$$

whenever $v = \tau + \eta$ is the unique orthogonal decomposition of $v \in \mathbb{R}^d$ with $\tau \in T_\mu(x)$, $\eta \in N_\mu(x)$ for μ-a.e. $x \in \mathbb{R}^d$. Thanks to Observation 2.2.1 and (2.20) the following definition of the μ-tangential gradient is well posed:

$$u \in \tilde{H}_\mu^{1,p} \ \leftrightarrow \ \nabla_\mu u := P_\mu[z] \quad \text{whenever } \ (u,z) \in V_\mu^p. \tag{2.28}$$

The notion of the gradient is pointwise, that means $\nabla_\mu u(x) = P_\mu(x)[z(x)]$ μ-almost everywhere, and clearly $P_\mu[z] = 0$ for each $z \in \Gamma^p_\mu$. Then it turns out that the linear operator

$$A : D(A) \subset L^p_\mu \to (L^p_\mu)^d, \ \varphi \mapsto \nabla_\mu \varphi \tag{2.29}$$

with dense domain $D(A) = \mathcal{D}$ is closable (see [17, Section 3.1] the for details), where one has to use (2.25) and (2.26). Hence the following definition:

Definition 2.2.2. *For any $p \in [1,\infty]$ we set $H^{1,p}_\mu := D(\bar{A})$, that means the Sobolev space is the domain of the unique closed extension $\bar{A}$ of the operator A in (2.29). In particular $H^{1,p}_\mu$ is a Banach space for the norm*

$$\|u\|_{H^{1,p}_\mu} := \|u\|_{p,\mu} + \|\nabla_\mu u\|_{p,\mu}\,, \tag{2.30}$$

and is separable for $p \in [1,\infty)$ and reflexive for $p \in (1,\infty)$.

We emphasize that $(u, \nabla_\mu u) \in V^p_\mu$ for any $u \in H^{1,p}_\mu$ (see [19, Section 2]). The different notation indicates the additional topological structure of $H^{1,p}_\mu$ as a Banach space. Examples of tangent and Sobolev spaces will be given in the next paragraphs, where we also discuss the necessity of considering both, $\tilde{H}^{1,p}_\mu$ and $H^{1,p}_\mu$ depending on the application. Let us also define

$$H^{1,p}_{\mu,\mathrm{loc}} := \{u \in L^p_{\mu,\mathrm{loc}} :\ u\varphi \in H^{1,p}_\mu \ \ \forall \varphi \in \mathcal{D}\}. \tag{2.31}$$

The periodic Sobolev spaces $H^{1,p}_\mu(\mathbb{T})$.

We need to introduce the periodic Sobolev spaces. The concepts are of course similar as above. For $p \in [1,\infty]$, let $V^p_\mu(\mathbb{T}) := V^p(\mathbb{T}, d\mu)$ be the subspace of $L^p_\mu(\mathbb{T}) \times L^p_\mu(\mathbb{T})^d$ defined as follows:

$$(u,z) \in V^p_\mu(\mathbb{T}) \quad \text{if there exists a sequence } \{\varphi_n\} \subset \mathcal{C}^\infty(\mathbb{T}), \text{ such that}$$

$$(\varphi_n, \nabla\varphi_n) \to (u,z) \ \text{ strongly in } L^p_\mu(Y)^{d+1}. \tag{2.32}$$

Obviously $V^p_\mu(\mathbb{T})$ is a Banach space with the induced L^p_μ-norm and is reflexive for $p \in (1,\infty)$. We define the class of periodic Sobolev functions as the set of first components:

$$\tilde{H}^{1,p}_\mu(\mathbb{T}) := \{u \in L^p_\mu(\mathbb{T}) : \exists z \in L^p_\mu(\mathbb{T})^d \text{ such that } (u,z) \in V^p_\mu(\mathbb{T})\}. \tag{2.33}$$

Again for $(u,z) \in V^p_\mu(\mathbb{T})$ we denote the vector z by ∇u and call it a gradient of u. The following statement corresponds to Observation 2.2.1:

Observation 2.2.3. *Let $\Gamma^p_\mu(\mathbb{T}) := \{z \in L^p_\mu(\mathbb{T})^d : (0,z) \in V^p_\mu(\mathbb{T})\}$ be the set of gradients of zero. If $(u,z), (u,\tilde{z}) \in V^p_\mu(\mathbb{T})$, then there holds*

$$z = \tilde{z} + z_0 \quad \textit{for some } z_0 \in \Gamma^p_\mu(\mathbb{T}). \tag{2.34}$$

As (2.34) shows, the problem of non-uniqueness of a gradient is closely related to the case $\{0\} \subsetneqq \Gamma^p_\mu(\mathbb{T})$. In what follows we give some examples.

Example 2.2.4. *Let $S := (0,1) \times \{\frac{1}{2}\} \subset (0,1)^2$ and $\mu = \mathcal{H}^1 \lfloor S$ the one-dimensional Hausdorff measure on S. Then for any $p \in [1,\infty)$ there holds*

$$\Gamma^p_\mu(\mathbb{T}) = \{(0,v) : \; v \in L^p(S,dx)\}. \tag{2.35}$$

Moreover any $u \in \tilde{H}^{1,p}_\mu(\mathbb{T})$ can be uniquely identified with a function $\hat{u} \in H^{1,p}_{\mathrm{per}}(S)$, and any vector $(\partial_x \hat{u}, v)$ with $v \in L^p(S)$ is a gradient of u.

Proof. By the definition of $\tilde{H}^{1,p}_\mu(\mathbb{T})$ there exists $\{\varphi_n\} \subset \mathcal{C}^\infty(\mathbb{T})$, such that

$$\int_S \left(|u - \varphi_n|^p + |z_1 - \partial_x \varphi_n|^p + |z_2 - \partial_y \varphi_n|^p\right) dx \;\to\; 0\,,$$

whenever $(u,z) \in V^p_\mu(\mathbb{T})$. This shows that $\hat{u} = u\,|_S$ belongs to $H^{1,p}_{\mathrm{per}}(S)$ with $z_1 = \partial_x \hat{u}$ in the classical sense of periodic Sobolev spaces on S. To show (2.35), let $v \in L^p(S)$ and $\{v_n\} \subset \mathcal{C}^\infty_{\mathrm{per}}(S)$ be a sequence with $v_n \to v$ strongly in $L^p(S)$. Then we easily deduce

$$\int_S \left(|\varphi_n|^p + |\partial_x \varphi_n|^p + |v - \partial_y \varphi_n|^p\right) dx \;\to\; 0\,,$$

for the $\mathcal{C}^\infty(\mathbb{T})$-function $\varphi_n(x,y) := \frac{1}{2\pi} v_n(x) \sin(\pi(2y-1))$. □

The next example is taken from [60, Section 5]. It shows that Γ^p_μ can be very large, i.e. there exist measures such that any L^p_μ-function is a gradient of zero, and hence $\tilde{H}^{1,p}_\mu(\mathbb{T}) = L^p_\mu(\mathbb{T})$. This case is called *total disconnectedness.*

Example 2.2.5. *Let $p \in (1,\infty), d = 1$ and $a : [0,1] \to [0,1]$ be a function with the following properties*

$$a(x) > 0 \;\; a.e. \; on \; [0,1]\,, \qquad \int_I a^{1/(1-p)} dx = \infty \quad for\ each\ interval\ I \subset [0,1].$$

Then if $d\mu = a(x)dx$, that means μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$ with density a, then there holds

$$\Gamma^p_\mu(\mathbb{T}) = L^p_\mu(\mathbb{T}), \quad \tilde{H}^{1,p}_\mu(\mathbb{T}) = L^p_\mu(\mathbb{T}). \tag{2.36}$$

Similar as above, periodic functions tangent to μ can be defined:

$$X^{p'}_\mu(\mathbb{T}) := \{\Phi \in L^{p'}_\mu(\mathbb{T})^d : \; \mathrm{div}(\Phi\mu) \in L^{p'}_{\mu,\mathrm{loc}}\}. \tag{2.37}$$

Since we always consider periodic measures μ with $\mu(\partial Y) = 0$, the tangent space results a periodic multifunction (see [14, Section 3])

$$T_\mu(x) = \mu - \mathrm{ess}\bigcup\{\Phi(x) : \; \Phi \in X^{p'}_\mu(\mathbb{T})\} \quad \mu\text{-a.e. on } Y. \tag{2.38}$$

As the first equality in (2.36) combined with (2.20) and (2.21) shows, T_μ can reduce to $\{0\}$. However, for the class $J_\sharp$ of multijunction measures we retain the classical notion of tangent spaces given by differential geometry.

Example 2.2.6. *Let $\mu \in J_\sharp$. Then for the tangent space there holds*

$$T_\mu(x) = T_{S_i}(x) \quad \text{for } \mu_i\text{-a.e. } x, \tag{2.39}$$

where T_{S_i} denotes the classical tangent space to the manifold S_i appearing in Definition 2.1.1.

It is clear that the orthogonal projection P_μ onto the tangent space T_μ belongs to $L^\infty_\mu(\mathbb{T};\mathbb{R}^{d\times d})$, and that the μ-tangential gradient can be defined in the same way as in (2.28). We can then consider the linear operator

$$A_\sharp : D(A_\sharp) \subset L^p_\mu(\mathbb{T}) \to L^p_\mu(\mathbb{T})^d,\ \varphi \mapsto \nabla_\mu \varphi \tag{2.40}$$

with dense domain $D(A_\sharp) = \mathcal{C}^\infty(\mathbb{T})$, which again turns out to be closable (see [14, Section 3] for the details). Hence the following

Definition 2.2.7. *For any $p \in [1,\infty]$ we set $H^{1,p}_\mu(\mathbb{T}) := D(\overline{A_\sharp})$, that means the periodic Sobolev space is the domain of the unique closed extension of the operator $A_\sharp$ in (2.40). In particular $H^{1,p}_\mu(\mathbb{T})$ is a Banach space for the norm*

$$\|u\|_{H^{1,p}_\mu(\mathbb{T})} := \|u\|_{p,\mu,Y} + \|\nabla_\mu u\|_{p,\mu,Y}\,. \tag{2.41}$$

We remark that $H^{1,p}_\mu(\mathbb{T})$ is a closed subspace of $H^{1,p}_{\mu,\mathrm{loc}}$, and that it is reflexive for any $p \in (1,\infty)$. For each $u \in H^{1,p}_\mu(\mathbb{T})$ the following useful integration by parts formula holds:

$$\forall \Phi \in X^{p'}_\mu(\mathbb{T}): \quad \int_Y \nabla_\mu u \cdot \Phi \, d\mu = -\int_Y u \operatorname{div}_\mu \Phi \, d\mu. \tag{2.42}$$

It is helpful to characterize the adjoint operator $A^\star_\sharp$ of $A_\sharp$ and its domain. This is done in the following remark. For a proof we refer to [14, Proposition 3.7].

Remark 2.2.8. *Let $Y^{p'}_\mu(\mathbb{T})$ denote the domain of $A^\star_\sharp$. Then there holds*

$$Y^{p'}_\mu(\mathbb{T}) = \{\sigma \in L^{p'}_\mu(\mathbb{T})^d :\ P_\mu\sigma \in X^{p'}_\mu(\mathbb{T})\}\,, \quad A^\star_\sharp\sigma = -\operatorname{div}_\mu(P_\mu\sigma). \tag{2.43}$$

For the class of multijunction measures there exists a natural relation between $H^{1,p}_\mu(\mathbb{T})$ and the classical spaces $H^{1,p}_{\mathrm{per}}(S_i)$ of periodic Sobolev functions (cf. Example 2.2.4) defined by local charts on S_i. For the proof of the following lemma we refer to [19, Lemma 2.2].

Lemma 2.2.9. *Let $\mu \in J_\sharp$ and denote by ∇_i the usual tangential gradient on the submanifold S_i. Then there holds*

$$u \in H^{1,p}_\mu(\mathbb{T}) \;\Rightarrow\; u \in H^{1,p}_{\mathrm{per}}(S_i) \quad \text{for all } i, \quad \nabla_\mu u = \nabla_i u \quad \mu_i\text{-a.e.} \tag{2.44}$$

We emphasize that the converse implication in (2.44) does not hold in general, and that its validity can depend on the exponent p. This is related to the notion of p-connectedness introduced in Section 2.3 below. We consider two examples. Let $F_1 := (0,1) \times \{1/2\}$, $F_2 := \{1/2\} \times (0,1)$ and

$$S_1 := \{(1/2, 1/2, z) :\ z \in (0,1)\},\ S_2 := \{(x, y, 0) :\ x, y \in (0,1)^2\},$$

$F_i \subset \mathbb{R}^2$ and $S_i \subset \mathbb{R}^3$ respectively, with intersection points $P := (1/2, 1/2)$ and $Q := (1/2, 1/2, 0)$. We define the multijunction measures $\mu, \tilde{\mu} \in J_\sharp$ by

$$\mu := c\left(\mathcal{H}^1 \lfloor F_1 + \mathcal{H}^1 \lfloor F_2\right), \quad \tilde{\mu} = \tilde{c}\left(\mathcal{H}^1 \lfloor S_1 + \mathcal{H}^2 \lfloor S_2\right),$$

where $c, \tilde{c}$ are normalizing constants. Then the following explicit characterization of the periodic Sobolev spaces with respect to μ and $\tilde{\mu}$ directly follows from Lemma 2.2.9 and the standard Sobolev embedding theorems.

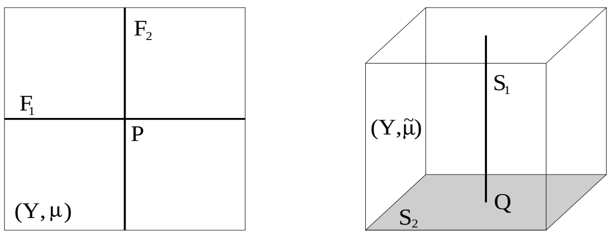


Figure 2.2: Multidimensional structures.

Observation 2.2.10. *Let $\mu, \tilde{\mu}$ and F_i, S_i for $i = 1, 2$ as above and assume $p \geq 1$. Then there holds*

1. $u \in H^{1,p}_\mu(\mathbb{T}) \Leftrightarrow \forall i :\ u \in H^{1,p}_{\text{per}}(F_i)$ *and u is continuous in P.*
2. *If $p \leq 2$:* $u \in H^{1,p}_{\tilde{\mu}}(\mathbb{T}) \Leftrightarrow \forall i :\ u \in H^{1,p}_{\text{per}}(S_i)$.
3. *If $p > 2$:* $u \in H^{1,p}_{\tilde{\mu}}(\mathbb{T}) \Leftrightarrow \forall i :\ u \in H^{1,p}_{\text{per}}(S_i)$ *and u is continuous in Q.*

We now discuss the relation between $\tilde{H}^{1,p}_\mu$ and $H^{1,p}_\mu$ and the necessity to distinguish the two. Once more we emphasize the pointwise characterization of the space $\Gamma^p_\mu(\mathbb{T})$, which follows from (2.20):

$$z \in \Gamma^p_\mu(\mathbb{T}) \iff z(y) \in T_\mu(y)^\perp \quad \mu\text{-a.e. in } Y. \tag{2.45}$$

We quote a central relaxation result [14, Proposition 3.8] and give an important application related to the study of elliptic equations.

Proposition 2.2.11. *For $p \in (1, \infty)$ consider the functional $J : L^p_\mu(\mathbb{T}) \to \hat{\mathbb{R}}$ defined by*

$$J(u) = \begin{cases} \int_Y j(y, \nabla u)\, d\mu & \text{if } u \in \mathcal{C}^\infty(\mathbb{T}), \\ +\infty & \text{else,} \end{cases}$$

where $j = j(y,z)$ is μ-measurable and Y-periodic in y, convex in z and satisfies for some positive constants c, C, the growth condition

$$c|z|^p \le j(y,z) \le C(1+|z|^p) \qquad \forall (y,z) \in Y \times \mathbb{R}^d. \tag{2.46}$$

Then the relaxed functional $\bar{J}$ of J on $L^p_\mu(\mathbb{T})$ is given by

$$\bar{J}(u) = \begin{cases} \int_Y j_\mu(y, \nabla_\mu u)\, d\mu & \text{if } u \in H^{1,p}_\mu(\mathbb{T}), \\ +\infty & \text{else,} \end{cases} \tag{2.47}$$

where $j_\mu(y,z) := \inf\{j(y, z+\xi) : \xi \in T_\mu(y)^\perp\}$ depends only on the component of z along $T_\mu(y)$.

Let us give an interpretation of this result. The functional J is not lower semicontinuous with respect to the L^p_μ-convergence, hence the relaxed functional $\bar{J}$ should be considered, to which the direct method can be applied. Since the relaxed integrand j_μ does not depend on the normal component of the gradient of some function $u \in \tilde{H}^{1,p}_\mu(\mathbb{T})$ (cf. (2.34) combined with (2.45)), it suffices to consider tangential gradients and find minimizers in the Banach space $H^{1,p}_\mu(\mathbb{T})$ (see also Lemma 2.3.13 below). For a given positive constant λ, consider the following elliptic problem on the torus

$$-\mathrm{div}\,(A(y)\nabla u(y)\mu) + \lambda u(y)\mu = f(y)\mu \quad \text{in } \mathbb{T}, \tag{2.48}$$

subject to a μ-measurable positive tensor $A \in L^\infty_\mu(\mathbb{T}; \mathcal{M}^d_{\mathrm{sym}})$ that satisfies

$$c|\xi|^2 \le \xi \cdot A(y)\xi \le C|\xi|^2 \quad \forall \xi \in \mathbb{R}^d,\ \forall y \in Y \tag{2.49}$$

for some constants $c, C > 0$, and $f \in L^2_\mu(\mathbb{T})$. We then call a Sobolev function $u \in \tilde{H}^{1,2}_\mu(\mathbb{T})$ a solution of problem (2.48), if the integral identity

$$\int_Y (A(y)\nabla u(y) \cdot \nabla\varphi(y) + \lambda u(y)\varphi(y))\, d\mu = \int_Y f(y)\varphi(y)\, d\mu \tag{2.50}$$

holds for every $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ and *some* gradient ∇u of u. It is easy to check that there exists a unique solution regarded as a pair $(u, \nabla u)$ in the Hilbert space $V^2_\mu(\mathbb{T})$. The uniqueness is twofold, regarding the function u in the Sobolev space and its unique gradient satisfying (2.50). By density, the identity also holds for any $\nabla\varphi \in \Gamma^2_\mu(\mathbb{T})$, and we deduce from (2.45) that

$$A(y)\nabla u(y) \in T_\mu(y) \quad \mu\text{-a.e. in } Y. \tag{2.51}$$

In other words, the relaxation process signifies to find a new (symmetric) matrix $\hat{A}$, such that the equation (2.48) with A replaced by $\hat{A}$ is solved by the same function $u \in \tilde{H}^{1,2}_\mu(\mathbb{T})$, but now with its tangential gradient. Then the problem can be studied in the Hilbert space $H^{1,2}_\mu(\mathbb{T})$, and in the sense of finding the unique solution $u \in \tilde{H}^{1,2}_\mu(\mathbb{T})$, (2.48) is equivalent to the problem

$$-\mathrm{div}\,(\hat{A}\nabla_\mu u) + \lambda u = f, \quad u \in H^{1,2}_\mu(\mathbb{T}). \tag{2.52}$$

Thanks to Proposition 2.2.11, the problem of finding the matrix $\hat{A}$ has a pointwise nature and we deduce

$$\eta \cdot \hat{A}(y)\eta = \min_{\xi \in T_\mu(y)^\perp} (\eta + \xi) \cdot A(y)(\eta + \xi), \tag{2.53}$$

where we applied Proposition 2.2.11 to the integrand $j(y, z) = z \cdot A(y)z$. We choose the symmetric part of $\hat{A}$, which is uniquely determined by (2.53).

Example 2.2.12. *As in Example 2.2.4, let μ be the 1D-Hausdorff measure supported on a line, such that $T_\mu(y) = \mathrm{span}\{\vec{e}_1\}$ μ-a.e., and let*

$$A(y) = \begin{pmatrix} a(y) & b(y) \\ b(y) & c(y) \end{pmatrix}$$

be a μ-measurable periodic matrix satisfying (2.49). Applying (2.53) we easily calculate

$$\hat{A}(y) = \begin{pmatrix} (a - \frac{b^2}{c})(y) & 0 \\ 0 & 0 \end{pmatrix}.$$

We conclude this paragraph by considering *potential* and *solenoidal* vectors on the torus. Such vectors play an important role in the study of fattened structures (see Chapter 5).

Definition 2.2.13. *For $p \in [1, \infty]$ we define the space $V^p_{\mathrm{pot}}(\mathbb{T}, d\mu)$ of potential vectors as the closure of the set $\{\nabla\varphi : \varphi \in \mathcal{C}^\infty(\mathbb{T})\}$ in $L^p_\mu(\mathbb{T})^d$, that means*

$$v \in V^p_{\mathrm{pot}}(\mathbb{T}, d\mu) \Leftrightarrow \exists\{\varphi_n\} \subset \mathcal{C}^\infty(\mathbb{T}) : \|v - \nabla\varphi_n\|_{p,\mu,Y} \to 0.$$

We call a vector $v \in L^p_\mu(\mathbb{T})^d$ solenoidal, and write $v \in V^p_{\mathrm{sol}}(\mathbb{T}, d\mu)$, if

$$\int_Y v \cdot \nabla\varphi \, d\mu = 0 \quad \textit{for each } \varphi \in \mathcal{C}^\infty(\mathbb{T}). \tag{2.54}$$

If there is no confusion about the underlying measure μ, we write $V^p_{\mathrm{pot}}(\mathbb{T})$ and $V^p_{\mathrm{sol}}(\mathbb{T})$. We remark that smooth functions are not necessarily dense in $V^p_{\mathrm{sol}}(\mathbb{T})$. There is a natural relation between the set of potential vectors and the class $\tilde{H}^{1,p}_\mu(\mathbb{T})$: Any gradient z of a pair $(u, z) \in V^p_\mu(\mathbb{T})$ is a potential vector by (2.32). On the other hand, any potential vector is a gradient of some Sobolev function provided the measure μ is sufficiently connected (see Section 2.3 below). For a subspace $X \subset Y$ of some Banach space Y, we denote by $X^\perp \subset Y'$ the annihilator of X with respect to the dual pairing.

Remark 2.2.14. *For any exponent $p \in (1, \infty)$ there holds $V^p_{\mathrm{pot}}(\mathbb{T})^\perp = V^{p'}_{\mathrm{sol}}(\mathbb{T})$ and $V^p_{\mathrm{sol}}(\mathbb{T})^\perp = V^{p'}_{\mathrm{pot}}(\mathbb{T})$. Moreover any solenoidal vector field is tangential:*

$$v \in V^p_{\mathrm{sol}}(\mathbb{T}) \Rightarrow v(y) \in T_\mu(y) \quad \textit{for } \mu\textit{-a.e. } y \in Y. \tag{2.55}$$

Proof. The first statement is a consequence of the relation $(X^{\perp})^{\perp} = \overline{X}$ and the fact that $V^p_{\mathrm{pot}}(\mathbb{T})$ and $V^p_{\mathrm{sol}}(\mathbb{T})$ are closed subspaces of $L^p_\mu(\mathbb{T})^d$. In particular

$$L^2_\mu(\mathbb{T})^d = V^2_{\mathrm{pot}}(\mathbb{T}) \oplus V^2_{\mathrm{sol}}(\mathbb{T}). \tag{2.56}$$

To show (2.55), observe that for any $z \in \Gamma^{p'}_\mu(\mathbb{T})$ and all $v \in V^p_{\mathrm{sol}}(\mathbb{T})$ we get $\int_Y v \cdot z \, d\mu = 0$ by approximation, which implies (2.55) by (2.45). □

If $\hat{V}^p_{\mathrm{pot}}(\mathbb{T})$ denotes the closure in $L^p_\mu(\mathbb{T})^d$ of the set $\{\nabla_\mu \varphi : \ \varphi \in \mathcal{C}^\infty(\mathbb{T})\}$, then any potential vector admits a unique decomposition

$$V^p_{\mathrm{pot}}(\mathbb{T}) \ni \nabla\varphi(y) = \nabla_\mu\varphi(y) + z(y), \quad z(y) =: \nabla^\perp_\mu \varphi(y)\,, \tag{2.57}$$

with $\nabla_\mu\varphi \in \hat{V}^p_{\mathrm{pot}}(\mathbb{T})$ and $z \in \Gamma^p_\mu(\mathbb{T})$. In particular (see also [62, §9]), for $p = 2$ we have the orthogonal decomposition

$$V^2_{\mathrm{pot}}(\mathbb{T}) \ = \ \hat{V}^2_{\mathrm{pot}}(\mathbb{T}) \oplus \Gamma^2_\mu(\mathbb{T})\,. \tag{2.58}$$

The Dirichlet spaces $H^{1,p}_0(\Omega, d\mu_\varepsilon)$.

We introduce the Sobolev spaces with respect to the rescaled measures. Since we have in mind elliptic problems with Dirichlet boundary conditions, we consider a suitable closure of the set $\mathcal{D}(\Omega)$ of smooth and compactly supported functions on Ω, thus obtaining μ_ε-Sobolev functions with zero trace.
For $p \in [1, \infty]$ and any $\varepsilon > 0$ we denote by $V^p(\Omega, d\mu_\varepsilon)$ the subspace of $L^p(\Omega, d\mu_\varepsilon) \times L^p(\Omega, d\mu_\varepsilon)^d$ defined as:

$(u, z) \in V^p(\Omega, d\mu_\varepsilon)$ if there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\Omega)$, such that

$$\lim_{n\to\infty} (\|u - \varphi_n\|_{p,\varepsilon} + \|z - \nabla\varphi_n\|_{p,\varepsilon}) \ = \ 0\,. \tag{2.59}$$

By its definition, $V^p(\Omega, d\mu_\varepsilon)$ is a Banach space with respect to the norm

$$\|(u, z)\|_{V^p(\Omega,d\mu_\varepsilon)} \ := \ \|u\|_{p,\varepsilon} + \|z\|_{p,\varepsilon}\,, \tag{2.60}$$

and is clearly reflexive for any $p \in (1, \infty)$. Similar as in (2.33), we define the space of Sobolev functions with zero trace as the set of first components:

$$\tilde{H}^{1,p}_0(\Omega, d\mu_\varepsilon) := \{u \in L^p_{\mu_\varepsilon}(\Omega) : (u, z) \in V^p(\Omega, d\mu_\varepsilon) \ \text{ for some } \ z \in L^p_{\mu_\varepsilon}(\Omega)^d\}. \tag{2.61}$$

We denote such a vector z by ∇u and call it a gradient of u. Again the set of gradients of zero plays an important role:

Observation 2.2.15. *If $(u, z), (u, \tilde{z}) \in V^p(\Omega, d\mu_\varepsilon)$, then there holds*

$$z = \tilde{z} + z_0 \quad \textit{for some } z_0 \in \Gamma^p(\Omega, d\mu_\varepsilon) := \{z : \ (0, z) \in V^p(\Omega, d\mu_\varepsilon)\}. \tag{2.62}$$

By the periodicity of μ and the definition of μ_ε, it is clear how the notion of a tangent space T_{μ_ε} can be adapted from the Y-periodic multifunction $T_\mu(y)$ defined in (2.38), together with the corresponding orthogonal projection and the concept of μ_ε-tangential gradients. We set

$$\operatorname{lin}(\mathbb{R}^d) \ni T_{\mu_\varepsilon}(x) := T_\mu(\tfrac{x}{\varepsilon}) \quad \text{for } \mu_\varepsilon\text{-a.e. } x \in \Omega. \tag{2.63}$$

In particular by rescaling the orthogonal projection $P_\mu \in L^\infty_\mu(\mathbb{T}; \mathbb{R}^{d\times d})$ we recover the notion of μ_ε-tangential gradients:

$$P_{\mu_\varepsilon}(x) := P_\mu(\tfrac{x}{\varepsilon}), \quad \nabla_{\mu_\varepsilon}\varphi(x) := P_{\mu_\varepsilon}(x)[\nabla\varphi(x)] \quad \text{for each } \varphi \in \mathcal{D}(\Omega). \tag{2.64}$$

Again, such as in (2.20) and (2.45), we get a local characterization of the set of gradients of zero:

$$z \in \Gamma^p(\Omega, d\mu_\varepsilon) \iff z(x) \in T_{\mu_\varepsilon}(x)^\perp \quad \text{for } \mu_\varepsilon\text{-a.e. } x \in \Omega. \tag{2.65}$$

We emphasize that by (2.62) and (2.65) the tangential gradient of a function in the class $\tilde{H}^{1,p}_0(\Omega, d\mu_\varepsilon)$ is uniquely determined, and hence, as in the last paragraphs, we can extract a Banach space by considering the linear operator

$$A_\varepsilon : D(A_\varepsilon) \subset L^p(\Omega, d\mu_\varepsilon) \to L^p(\Omega, d\mu_\varepsilon)^d,\ \varphi \mapsto \nabla_{\mu_\varepsilon}\varphi \tag{2.66}$$

with dense domain $D(A_\varepsilon) = \mathcal{D}(\Omega)$. Note that A_ε coincides with the operator A defined in (2.29), if μ is replaced by μ_ε and the functions $\varphi \in \mathcal{D}(\Omega)$ are extended trivially to the whole of $\mathbb{R}^d$ (in particular $A_1 = A$ in this sense). It follows that A_ε is closable, which justifies the following definition of the μ_ε-Sobolev spaces with zero trace, also called *Dirichlet spaces*:

Definition 2.2.16. *For any $p \in [1, \infty]$ we set $H^{1,p}_0(\Omega, d\mu_\varepsilon) := D(\overline{A_\varepsilon})$, i.e. the Dirichlet space is the domain of the unique closed extension of the operator A_ε in (2.66). In particular $H^{1,p}_0(\Omega, d\mu_\varepsilon)$ is a Banach space for the norm*

$$\|u\|_{1,p,\varepsilon} := \|u\|_{p,\varepsilon} + \|\nabla_{\mu_\varepsilon} u\|_{p,\varepsilon} \tag{2.67}$$

and there holds $(u, \nabla_{\mu_\varepsilon} u) \in V^p(\Omega, d\mu_\varepsilon)$, whenever $u \in H^{1,p}_0(\Omega, d\mu_\varepsilon)$.

The sets $\tilde{H}^{1,p}_0(\Omega, d\mu_\varepsilon)$ and $H^{1,p}_0(\Omega, d\mu_\varepsilon)$ occur in a natural way as solution spaces of elliptic boundary value problems on multidimensional structures (see Section 3.2 below for examples and more details). Consider the equation

$$-\operatorname{div}(A_\varepsilon(\mu, x)\nabla u_\varepsilon) + \lambda u_\varepsilon \mu_\varepsilon = f\mu_\varepsilon \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{2.68}$$

where $f \in \mathcal{C}(\overline{\Omega})$ and where $A_\varepsilon(\mu, \cdot) := A(\varepsilon^{-1}\cdot)\mu_\varepsilon$ is the rescaled tensor of period ε corresponding to a μ-measurable, Y-periodic matrix $A = A(y)$ that satisfies condition (2.49). We call a function $u_\varepsilon \in \tilde{H}^{1,2}_0(\Omega, d\mu_\varepsilon)$ a solution of problem (2.68), if the integral identity

$$\int_\Omega A(\tfrac{x}{\varepsilon})\nabla u_\varepsilon \cdot \nabla\varphi \, d\mu_\varepsilon(x) + \lambda \int_\Omega u_\varepsilon \varphi \, d\mu_\varepsilon(x) = \int_\Omega f\varphi \, d\mu_\varepsilon(x) \tag{2.69}$$

holds for all $\varphi \in \mathcal{D}(\Omega)$ and *some* gradient ∇u_ε of u_ε. Precisely as in the periodic setting, one can show that there exists a unique solution $(u_\varepsilon, \nabla u_\varepsilon) \in V^2(\Omega, d\mu_\varepsilon)$, and that the gradient satisfies

$$A(\tfrac{x}{\varepsilon})\nabla u_\varepsilon(x) \in T_{\mu_\varepsilon}(x) \quad \mu_\varepsilon\text{-a.e. in } \Omega. \tag{2.70}$$

Hence the natural solution space for equation (2.68) is $V^2(\Omega, d\mu_\varepsilon)$. In Section 3.2 we consider the case where A is a matrix diagonal with respect to the local coordinate system given by $T_{\mu_\varepsilon}(x)$. Due to (2.70) we can then consider solutions in the Hilbert space $H_0^{1,2}(\Omega, d\mu_\varepsilon)$ or calculate the relaxed matrix $\hat{A}$ first (cf. (2.53)) and investigate the equivalent equation

$$-\mathrm{div}\,(\hat{A}(\tfrac{x}{\varepsilon})\nabla_{\mu_\varepsilon} u_\varepsilon) + \lambda u_\varepsilon = f\,, \quad u_\varepsilon \in H_0^{1,2}(\Omega, d\mu_\varepsilon). \tag{2.71}$$

When we study the homogenization of equation (1.1), we have to deal with the composition of functions u in the Dirichlet space with functions b defined on the real line. It turns out that, when b is sufficiently regular, the composition $b \circ u$ is again a Sobolev function, and that a chain rule holds for the gradient. A systematic treatment on this problem, including an investigation of the minimal assumptions on the function b, was given by Marcus and Mizel [45]. Here we require that $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Recall that by Rademacher's theorem, the function b is then differentiable almost everywhere in $\mathbb{R}$ with (weak) derivative $b' \in L^\infty(\mathbb{R})$, and for every $t \in \mathbb{R}$ there holds

$$b(t) = b(0) + \int_0^t b'(s)\, ds\,.$$

Concerning the following statement, recall that μ is a positive periodic Radon measure on $\mathbb{R}^d$, where the rescaled measures μ_ε are defined in (2.1).

Lemma 2.2.17. *Let $p \geq 1$ and $b : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous with $b(0) = 0$. Then for any $u \in H_0^{1,p}(\Omega, d\mu_\varepsilon)$ there holds*

$$b \circ u =: a \in H_0^{1,p}(\Omega, d\mu_\varepsilon), \quad \nabla_{\mu_\varepsilon} a = b'(u)\nabla_{\mu_\varepsilon} u \quad \mu_\varepsilon\text{-a.e. in } \Omega. \tag{2.72}$$

Proof. Let us first assume $b \in \mathcal{C}^1(\mathbb{R})$ with $b' \in \mathcal{C}_\mathrm{b}(\mathbb{R})$. If $\varphi_n \in \mathcal{D}(\Omega)$ is an approximating sequence for u according to the definition of the Dirichlet space, then we clearly have

$$\psi_n := b \circ \varphi_n \in \mathcal{C}_0^1(\Omega), \quad \nabla_{\mu_\varepsilon}\psi_n = b'(\varphi_n)\nabla_{\mu_\varepsilon}\varphi_n \quad \mu_\varepsilon\text{-a.e. in } \Omega.$$

Using the Lipschitz condition on b it is easy to check that $\psi_n \to a$ strongly in $L^p(\Omega, d\mu_\varepsilon)$ as $n \to \infty$. On the other hand we can estimate

$$\begin{aligned} \int_\Omega |b'(u)\nabla_{\mu_\varepsilon} u - \nabla_{\mu_\varepsilon}\psi_n|^p\, d\mu_\varepsilon \;&=\; \int_\Omega |b'(u)\nabla_{\mu_\varepsilon} u - b'(\varphi_n)\nabla_{\mu_\varepsilon}\varphi_n|^p\, d\mu_\varepsilon \\ \leq\; C\int_\Omega \Big(|(b'(u) - b'(\varphi_n))\nabla_{\mu_\varepsilon} u|^p &+ |b'(\varphi_n)(\nabla_{\mu_\varepsilon} u - \nabla_{\mu_\varepsilon}\varphi_n)|^p\Big)\, d\mu_\varepsilon \;\to\; 0, \end{aligned}$$

where we used $b' \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the pointwise convergence of $b'(\varphi_n)$ and the Lebesgue convergence theorem. If b is merely Lipschitz, we do not have $b' \in \mathcal{C}(\mathbb{R})$. In this case, as proven in [10, Theorem 5.4], we use that

$$\forall u \in H_0^{1,p}(\Omega, d\mu_\varepsilon): \quad \nabla_{\mu_\varepsilon} u = 0 \quad \mu_\varepsilon\text{-a.e. on } u^{-1}(N) \tag{2.73}$$

for any $\mathcal{L}^1$-negligible set $N \subset \mathbb{R}$, and the proof for $b \in \mathcal{C}^1(\mathbb{R})$ can be adapted. □

2.3 Connectedness

In this section we introduce different notions of connectedness of a measure. The weakest concept is clearly the ordinary connectedness of the support of μ in the topological sense. In the last section (cf. Observation 2.2.10) we have seen that a Sobolev function can be nonconstant, although its gradient vanishes almost everywhere. Stronger concepts have to be introduced, in order to guarantee Poincaré-inequalities and Rellich-type embeddings.
The connectedness of the underlying measure μ is of fundamental importance for the homogenization of multidimensional structures. Following the lines of [14, Section 4] and [19, Section 4a], we first study properties of Sobolev functions on the torus related to the connectedness of μ, including the investigation of the relaxed functional $\bar{J}$ in (2.47), and the characterization of solenoidal vector fields (see Lemma 2.3.12 below). The following definition is valid for general positive Radon measures μ on $\mathbb{R}^d$, not necessarily periodic.

Definition 2.3.1. *We say that μ satisfies the doubling property, if there exists a constant $C > 0$, such that*

$$\mu(B_{2\varrho}(x)) \le C\mu(B_\varrho(x)) \quad \forall \varrho > 0, \text{ for } \mu\text{-a.e. } x\,. \tag{2.74}$$

We say that μ satisfies the p-Poincaré inequality for $p \ge 1$, if for μ-a.e. x and for every $\varrho > 0$ there exists a positive constant $C = C(\varrho)$, such that:

$$\int\limits_{B_\varrho(x)} |u|^p \, d\mu \le C \int\limits_{B_\varrho(x)} |\nabla_\mu u|^p \, d\mu \tag{2.75}$$

for all $u \in \mathcal{D}(\mathbb{R}^d)$ with $u = 0$ on $\partial B_\varrho(x)$ or $\int_{B_\varrho(x)} u \, d\mu = 0$.

We mention the work of Hajlasz and Koskela [36], where a systematic treatment on Poincaré-type inequalities in doubling spaces (metric spaces endowed with a doubling measure) can be found. Here we only note that any multijunction measure $\mu \in J_\sharp$ is doubling. The p-Poincaré inequality is closely related to the p-connectedness of μ on $\mathbb{R}^d$ defined below. We now derive an important compactness theorem for Sobolev functions on $\mathbb{R}^d$.

Lemma 2.3.2. *Let $p > 1$ and μ be a positive Radon measure on $\mathbb{R}^d$ satisfying the doubling property (2.74) and the p-Poincaré inequality (2.75). Then any bounded sequence $\{u_n\}$ in $H_\mu^{1,p}$ with $u_n \in \mathcal{C}_0^1(\mathbb{R}^d)$ and $\operatorname{spt} u_n \subset B$ for some ball B and all $n \in \mathbb{N}$, admits a strongly convergent subsequence in L_μ^p.*

Proof. If in addition to the prerequisites, μ is compactly supported on $\mathbb{R}^d$, then it is known that $H^{1,p}_\mu \subset\subset L^p_\mu$. To this end we refer to [19, Lemma 4.1] or [36, Theorem 8.3]. The lemma is then an obvious consequence. □

Now we define various notions of connectedness, which where first introduced in [14]. We will also discuss the differences and relations among them and give several examples, including the importance of each single property.

Definition 2.3.3. *We introduce the following notions of connectedness for a fixed exponent $p \in [1,\infty)$, where c and C are supposed to be real constants.*

- *μ is weakly p-connected on $\mathbb{T}$ if*

  (H1) $\quad u \in H^{1,p}_\mu(\mathbb{T}),\ \nabla_\mu u = 0\ \ \mu\text{-a.e.} \ \Rightarrow\ \exists c : u = c\ \ \mu\text{-a.e.};$

- *μ is weakly p-connected on $\mathbb{R}^d$ if*

  (H2) $\quad u \in H^{1,p}_{\mu,\mathrm{loc}},\ \nabla_\mu u = 0\ \ \mu\text{-a.e.} \ \Rightarrow\ \exists c : u = c\ \ \mu\text{-a.e.};$

- *μ is strongly p-connected on $\mathbb{T}$ if*

  (H3) $\quad \exists C : \ \|u\|_{p,\mu,Y} \le C\|\nabla_\mu u\|_{p,\mu,Y} \quad \forall u \in H^{1,p}_\mu(\mathbb{T})$ *with* $\int_Y u\,d\mu = 0;$

- *μ is strongly p-connected on $\mathbb{R}^d$ if*

  (H4) $\quad \exists C : \ \|u\|_{p,\mu,kY} \le Ck\|\nabla_\mu u\|_{p,\mu,kY} \quad \forall k \in \mathbb{N}_+, \forall u \in \mathcal{D}(\mathbb{R}^d)$ *with* $\int_{kY} u\,d\mu = 0$ *or* $u = 0$ *on* $\partial(kY)$.

Note that by the definition of $H^{1,p}_\mu(\mathbb{T})$, property $(H3)$ needs to be checked only on smooth functions. A similar statement is true for property $(H4)$.

Remark 2.3.4. *By density,* $(H4)$ *can be extended to all functions* $u \in H^{1,p}_{\mu,\mathrm{loc}}$ *with* $\int_{kY} u\,d\mu = 0$.

Proof. Given such u, choose $\psi \in \mathcal{D}$ with $\psi = 1$ in a neighbourhood of kY. Then there exists $\{\tilde\psi_n\} \subset \mathcal{D}$ with $\|\tilde\psi_n - u\psi\|_{H^{1,p}_\mu} \to 0$ and the function

$$\psi_n := \psi(\tilde\psi_n - c_n)\,, \quad c_n := \fint_{kY} \tilde\psi_n\,d\mu$$

is admissible in $(H4)$. Since $|c_n| \le \mu(kY)^{-1/p}\,\|\tilde\psi_n - u\|_{p,\mu,kY} \to 0$, it is easy to check that $\|\psi_n - u\|_{H^{1,p}_\mu(kY)} \to 0$. □

The following hierarchy among the different notions of connectedness is fairly easy to check, only the statement $(H2) \not\Rightarrow (H3)$ is not obvious. A counterexample can be found by taking μ as the Lebesgue measure weighted by a suitable degenerate density (see [14, Section 4]).

$$\begin{array}{ccccc} (H4) & \Rightarrow & (H2) & \Rightarrow & (H1)\,, \\ & & \not\Uparrow\,\not\Downarrow & & \\ (H4) & \Rightarrow & (H3) & \Rightarrow & (H1)\,. \end{array}$$

Property $(H1)$ In the special case when μ is the Lebesgue measure on an open periodic subset S of $\mathbb{R}^d$, the notion of (weak) connectedness on the torus was first introduced in [60]. As Observation 2.2.10 shows, property $(H1)$ depends in general on the size of p and is strictly stronger than the connectedness of the support of μ in the topological sense. Roughly speaking, $(H1)$ is a necessary condition in order to study the homogenization of multi-structures characterized by μ. More precisely, let a family $\{u_\varepsilon\}$ of solutions of equation (2.68) be given, endowed with an uniform a priori bound

$$\|u_\varepsilon\|_{2,\varepsilon} + \|\nabla u_\varepsilon\|_{2,\varepsilon} \le C\,. \tag{2.76}$$

Then the weak 2-connectedness of μ on the torus ensures (cf. Theorem 2.4.4 below) that the two-scale limit $u(x,y) \in L^2_m(\Omega\times Y)$ of $\{u_\varepsilon\}$ is independent of y, which leads to a single macroscopic equation for u. For measures that are absolutely continuous with respect to $\mathcal{L}^d$, we give a sufficient condition for $(H1)$ (see [44, Section 4]):

Example 2.3.5. *Let $p>1$ and $d\mu = a(x)dx$. Then μ is weakly p-connected on $\mathbb{T}$, provided*

$$\int_Y a(x)^{1/(1-p)}\,dx \;<\; \infty\,. \tag{2.77}$$

Properties $(H2)$ **and** $(H3)$ These two properties are mutually independent and strictly stronger than $(H1)$. It turns out that they ensure H^1-regularity for the two-scale limit u of the sequence in (2.76), and at the same time give a structure result for the corresponding sequence of gradients $\{\nabla u_\varepsilon\}$ (cf. Theorem 2.4.4 below). Further investigation shows that $(H3)$ guarantees the well-posedness of the cell problems associated with the homogenization of equation (2.68), whereas $(H2)$ ensures the ellipticity of the corresponding effective tensor (cf. Lemma 2.3.13 below). Property $(H2)$ is closely related to the p-Poincaré inequality on $\mathbb{R}^d$. The following statement can be found in [19, Remark 4.2].

Remark 2.3.6. *Let $p\in[1,\infty)$ and $\mu\in J_\sharp$. Then μ is weakly p-connected on $\mathbb{R}^d$ if and only if it satisfies the p-Poincaré inequality (2.75).*

Property $(H3)$ is essential for the characterization of solenoidal (cf. Lemma 2.3.12 below) and potential vectors:

Lemma 2.3.7. *Let $p\in(1,\infty)$ and μ be strongly p-connected on $\mathbb{T}$. Then for any $v\in V^p_{\mathrm{pot}}(\mathbb{T})$ there exists a unique function $u\in\tilde{H}^{1,p}_\mu(\mathbb{T})$ with $\int_Y u\,d\mu = 0$, such that $(u,v)\in V^p_\mu(\mathbb{T})$.*

Proof. Let $\nabla\psi_n$ be an approximating sequence for v. Then by $(H3)$, the smooth sequence $\varphi_n := \psi_n - \bar{\psi}_n$, where $\bar{\psi}_n$ denotes the average of ψ_n over the cell Y, satisfies

$$\|\varphi_n\|_{p,\mu,Y} \le C\|\nabla_\mu\varphi_n\|_{p,\mu,Y} \le C\,, \quad \|\nabla\varphi_n - v\|_{p,\mu,Y}\to 0\,.$$

By reflexivity there exists $u \in L^p_\mu(\mathbb{T})$, such that $\varphi_n \rightharpoonup u$ weakly in $L^p_\mu(Y)$. By Mazur's lemma we find a sequence $\{\tilde{\varphi}_n\} \subset \mathcal{C}^\infty(\mathbb{T})$ of convex combinations of φ_n, such that

$$(\tilde{\varphi}_n, \nabla\tilde{\varphi}_n) \to (u, v) \quad \text{strongly in } L^p_\mu(Y)^{d+1}.$$

Finally if $(u, v), (\tilde{u}, v) \in V^p_\mu(\mathbb{T})$, then from the weak p-connectedness of μ on the torus we deduce that u and $\tilde{u}$ coincide up to an additive constant. □

Property $(H4)$ This is the strongest notion of connectedness. It requires that the Poincaré constant on kY is equal to k times the constant on Y, and hence, by a change of variables, that the Poincaré constant for each rescaled measure μ_ε on Ω does not explode (cf. Lemma 2.4.2 below). This allows to study the homogenization of equation (2.68) with $\lambda = 0$. We give some examples of $(H4)$-measures, which also highlight the dependence on the exponent p.

Example 2.3.8. *Let S be as in Example 2.2.4, $C \subset [0,1]$ the standard Cantor set and $\mathcal{P}$ the probability measure concentrated on C. Then for any $p \geq 1$, the following (normalized) measure is strongly p-connected on $\mathbb{R}^d$:*

$$\mu := \tfrac{1}{2}\mathcal{H}^1 \lfloor S + \tfrac{1}{2}(\mathcal{P}\lfloor C \otimes \mathcal{L}^1\lfloor(0,1)).$$

Proof. The measure μ on the reference cell is sketched in Figure 2.3 below on the left-hand side. Note that the condition $u \in H^{1,p}_\mu(\mathbb{T})$ implies

$$\lim_{y \to 1/2} u(x, y) = u(x, \tfrac{1}{2})$$

for $\mathcal{P}$-a.e. x, and hence $(H1)$. It is then also evident that μ satisfies $(H4)$. □

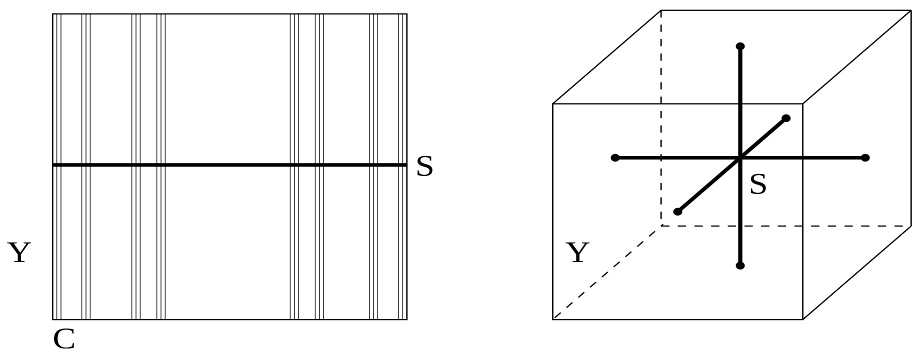


Figure 2.3: Strongly connected measures.

Example 2.3.9. *Let S be the union of three straight lines, which are parallel to the axis and meet at the center of the cube (see Figure 2.3). Then the* *normalized measure* $\mu = \frac{1}{3}\mathcal{H}^1\lfloor S$ *is strongly* p*-connected on* $\mathbb{R}^d$ *for any* $p > 1$*. On the other hand, the combined normalized measure*

$$\tilde{\mu} = \tfrac{1}{2}\mathcal{L}^3\lfloor Y + \tfrac{1}{6}\mathcal{H}^1\lfloor S \tag{2.78}$$

is strongly p*-connected on* $\mathbb{R}^d$ *if and only if* $p > 2$*.*

Proof. Let $S = \cup_i S_i$, where the sets S_i denote the straight lines. By Lemma 2.2.9, each Sobolev function $u \in H^{1,p}_\mu(\mathbb{T})$ with $\nabla_\mu u = 0$ μ-a.e. is constant on each set S_i. If $\{\varphi_n\} \subset \mathcal{C}^\infty(\mathbb{T})$ is an approximating sequence for u, then by a standard Sobolev embedding this sequence is bounded in $\mathcal{C}^{0,\alpha}(S_i)$ for some $\alpha > 0$ and each i, and hence the constants on S_i must be the same. This gives the crucial property $(H1)$, and $(H4)$ follows from the regular construction of μ. Considering the measure $\tilde{\mu}$ defined in (2.78), one can check that for any $p \leq 2$ the function

$$u(y) = \begin{cases} c_1 & \text{if } y \in S, \\ c_2 & \text{if } y \in Y \setminus S \end{cases} \tag{2.79}$$

belongs to $H^{1,p}_{\tilde{\mu}}(\mathbb{T})$ for an arbitrary choice of $c_1, c_2 \in \mathbb{R}$. Hence the measure $\tilde{\mu}$ does not even satisfy $(H1)$. By definition of $\tilde{\mu}$ we have $H^{1,p}_{\tilde{\mu}}(\mathbb{T}) \subset H^{1,p}_{\text{per}}(Y)$ in the classical sense. Let $\Gamma_i = \{y \in Y \mid y_i = \frac{1}{2}\}$ denote the three hyperplanes. Then if $p > 2$ we see that the restrictions $u\,|_{\Gamma_i}$ belong to $W^{s,q}(\Gamma_i)$ for some $s > \frac{1}{2}$ and $q > 2$ by a standard trace theorem [58, Theorem 11.2.3], and hence have a further trace on $S_j \subset \Gamma_i$, $j \neq i$. This shows that the function u defined in (2.79) belongs to $H^{1,p}_{\tilde{\mu}}(\mathbb{T})$ for $p > 2$, if and only if $c_1 = c_2$, and properties $(H1)$ and $(H4)$ follow. □

We conclude the discussion of property $(H4)$ by observing that it is sufficient to guarantee the p-Poincaré inequality (cf. Remark 2.3.6). Recall that we always consider positive, Y-periodic Radon measures μ on $\mathbb{R}^d$.

Remark 2.3.10. *Let* μ *be strongly* p*-connected on* $\mathbb{R}^d$ *for* $p \in [1, \infty)$*. Then* μ *satisfies the* p*-Poincaré inequality (2.75).*

Proof. We consider the case of functions vanishing at the boundary of a given ball (cf. Definition 2.3.1). Let $\varrho > 0$ and $x \in \mathbb{R}^d$ be given. We have to show that there exists a constant $C = C_\varrho$, such that

$$\int_{B_\varrho(x)} |u|^p \, d\mu \leq C_\varrho \int_{B_\varrho(x)} |\nabla_\mu u|^p \, d\mu \tag{2.80}$$

for every $u \in \mathcal{D}$ with $u = 0$ on $\partial B_\varrho(x)$. Note that it is no restriction to assume that $\mu(\partial B_\varrho(x)) = 0$. We choose a shift vector $v = \Sigma_i k_i \hat{e}_i \in \mathbb{R}^d$ with $k_i \in \mathbb{N}$ and an integer $k = k_\varrho$ depending on ϱ, such that $B_\varrho(x + v) \subset\subset k_\varrho Y$. For a given $u \in \mathcal{D}$ vanishing at the boundary of $B_\varrho(x)$ we define the function

$$\bar{u}(y) := \begin{cases} u(y - v) & \text{if } y \in B_\varrho(x + v), \\ 0 & \text{else.} \end{cases}$$

Clearly $\bar u \in H^{1,p}_\mu$, and we take a sequence $\{\varphi_n\} \subset \mathcal{D}$ with $\operatorname{spt}\varphi_n \subset\subset k_\varrho Y$ for each $n \in \mathbb{N}$ and

$$\|\varphi_n - \bar u\|_{p,\mu} + \|\nabla_\mu \varphi_n - \nabla_\mu \bar u\|_{p,\mu} \to 0$$

by the definition of the Sobolev space. Then by the definition of $\bar u$ and the Y-periodicity of μ we get

$$\int_{B_\varrho(x)} |u|^p \, d\mu = \int_{B_\varrho(x+v)} |u(y-v)|^p \, d\mu = \int_{k_\varrho Y} |\bar u|^p \, d\mu \leftarrow \int_{k_\varrho Y} |\varphi_n|^p \, d\mu. \tag{2.81}$$

On the other hand, if we set $C_\varrho := C k_\varrho^p$, where C denotes the constant occurring in $(H4)$, we get, using the periodicity of the projection P_μ and the assumption $\mu(\partial B_\varrho(x)) = 0$:

$$\int_{k_\varrho Y} |\varphi_n|^p \, d\mu \le C_\varrho \int_{k_\varrho Y} |\nabla_\mu \varphi_n|^p \, d\mu \;\to\; C_\varrho \int_{B_\varrho(x+v)} |\nabla_\mu \bar u|^p \, d\mu = C_\varrho \int_{B_\varrho(x)} |\nabla_\mu u|^p \, d\mu. \tag{2.82}$$

Combining (2.81) and (2.82) gives the desired Poincaré estimate (2.80). □

Under certain connectedness assumptions on the underlying measure μ, the annihilators of scalar functions in divergence form and of solenoidal vectors can be characterized. The statements are formulated with respect to the tangential operator ∇_μ and the Banach spaces $H^{1,p}_\mu$. For what follows, recall the definition $X^p_\mu(\mathbb{T})$ of periodic tangential vector fields given in (2.37).

Lemma 2.3.11. *Let $p \in (1,\infty)$ and $V := \{\operatorname{div}_\mu \Phi : \; \Phi \in X^{p'}_\mu(\mathbb{T})\}$. If μ satisfies $(H1)$, then the orthogonal space $V^\perp$ of V in $L^p_\mu(\mathbb{T})$ is given by the constant functions. Moreover the closure of V in $L^{p'}_\mu(\mathbb{T})$ is given by the functions with zero mean value.*

Proof. For the first statement we refer to [14, Lemma 4.3]. It follows that $V \subset \{u \in L^{p'}_\mu(\mathbb{T}) : \int_Y u \, d\mu = 0\} =: Y = \overline{Y} = (Y^\perp)^\perp = (V^\perp)^\perp = \overline{V}$. □

The next result is the *tangential version* of Remark 2.2.14. Note that by Lemma 2.3.7, if μ is strongly p-connected on $\mathbb{T}$, we have

$$\hat V^p_{\mathrm{pot}}(\mathbb{T}) = \{\nabla_\mu u : \; u \in H^{1,p}_\mu(\mathbb{T})\}\,. \tag{2.83}$$

Recall the characterization of $Y^{p'}_\mu(\mathbb{T})$ given by Remark 2.2.8. For the proof of the following statement we refer to [14, Lemma 4.6].

Lemma 2.3.12. *Let $p \in (1,\infty)$ and $V := \{\sigma \in Y^{p'}_\mu(\mathbb{T}) : \; \operatorname{div}_\mu(P_\mu \sigma) = 0\}$. If μ satisfies $(H3)$, then the orthogonal space $V^\perp$ of V in $L^p_\mu(\mathbb{T})^d$ is given by (2.83).*

The next lemma relies on the relaxation result in Proposition 2.2.11. It guarantees the well-posedness of the cell problems and the ellipticity of the effective equation related to the homogenization of equation (2.68).

Lemma 2.3.13. *For $p \in (1, \infty)$ we define the function $j : \mathbb{R}^d \to \mathbb{R}$ by*

$$j : z \mapsto \inf \left\{ \int_Y |z + \nabla u(y)|^p \, d\mu : \; u \in \mathcal{C}^\infty(\mathbb{T}) \right\}. \tag{2.84}$$

Then if $z_\mu(y) := P_\mu(y)[z]$ denotes the orthogonal projection of $z \in \mathbb{R}^d$ onto the tangent space of μ, then there holds

$$j(z) \;=\; \inf \left\{ \int_Y |z_\mu(y) + \nabla_\mu u(y)|^p \, d\mu : \; u \in H^{1,p}_\mu(\mathbb{T}) \right\}. \tag{2.85}$$

Moreover if μ satisfies $(H2)$ and $(H3)$, then there exists a positive constant $\hat{c} > 0$, such that

$$j(z) \;\geq\; \hat{c}|z|^p \quad \text{for all } z \in \mathbb{R}^d. \tag{2.86}$$

Proof. For the proof of (2.85) we refer to [14, Lemma 4.5]. By the connectedness assumption $(H3)$ on μ it is easy to check that for any fixed $z \in \mathbb{R}^d$ the infimum in (2.85) in attained on $H^{1,p}_\mu(\mathbb{T})$. Indeed the functional

$$J_z : X \to \mathbb{R}, \, u \mapsto \|z_\mu + \nabla_\mu u\|^p_{p,\mu,Y} \, , \quad X := \{u \in H^{1,p}_\mu(\mathbb{T}) : \int_Y u \, d\mu = 0\}$$

is clearly coercive and weakly lower semicontinuous due to $(H3)$, and hence by the direct method in the calculus of variations we see that the restriction

$$\hat{j} : \mathcal{S}^{d-1} \to \mathbb{R}, \, z \mapsto \min \{\|z_\mu + \nabla_\mu u\|^p_{p,\mu,Y} : \; u \in H^{1,p}_\mu(\mathbb{T})\}$$

is well defined. Assume that $\hat{j}(z) = 0$ for some $z \in \mathcal{S}^{d-1}$. Then it follows that $\nabla_\mu(z \cdot y + u) = 0$ μ-a.e. in Y for some $u \in H^{1,p}_\mu(\mathbb{T})$. Since μ is weakly p-connected on $\mathbb{R}^d$, the function

$$f : y \mapsto z \cdot y + u(y) \in H^{1,p}_{\mu,\mathrm{loc}}$$

is equal to a constant μ-a.e., and hence $z = 0$ by the periodicity of u. This contradicts $|z| = 1$, and so there exists a constant $\hat{c} > 0$, such that $\hat{j} \geq \hat{c}$ on $\mathcal{S}^{d-1}$. To show (2.86) we can assume $z \neq 0$. Then it is easy to check that

$$j(z) \geq \hat{j}(\hat{z})|z|^p \geq \hat{c}|z|^p \, ,$$

where $\hat{z}$ is the unit vector in z-direction. This completes the proof. □

The following advanced Poincaré estimate is known for the Lebesgue measure, whereas for a general Radon measure μ we have to require strong connectedness.

Lemma 2.3.14. *Let $p \in (1, \infty)$ and μ be strongly p-connected on $\mathbb{R}^d$ satisfying the doubling condition. Then there holds*

$$\forall k \in \mathbb{N}_+ \, \exists C > 0 : \quad \int_{kY} \left| u - \fint_{j+Y} u \, d\mu \right|^p d\mu \;\leq\; C \int_{kY} |\nabla_\mu u|^p d\mu \tag{2.87}$$

for all $u \in H^{1,p}_{\mu,\mathrm{loc}}$ and each multiindex $j = (j_1, \ldots, j_d)$, where $0 \leq j_i \leq k - 1$.

Proof. By density it suffices to show (2.87) for smooth functions $u \in \mathcal{D}$. Suppose the contrary, then there exists $k \in \mathbb{N}_+, j_0 \in \{0, \dots, k-1\}^d$ and a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$, such that

$$\forall n \in \mathbb{N}: \quad \int_{kY} |\nabla_\mu v_n|^p \, d\mu \; < \; \frac{1}{n} \int_{kY} |v_n - \int_{j_0+Y} v_n \, d\mu|^p \, d\mu \, , \tag{2.88}$$

where we used $\mu(j_0 + Y) = 1$. We can assume $\|\nabla_\mu v_n\|_{p,\mu,kY} = 1$ for each $n \in \mathbb{N}$ by estimate (2.88) and property $(H4)$. Setting

$$z_n := v_n - \fint_{kY} v_n \, d\mu \, ,$$

we get a sequence $\{z_n\} \subset \mathcal{C}^\infty(\mathbb{R}^d)$ with $\|z_n\|_{H^{1,p}_\mu(kY)} \leq Ck$ by $(H4)$ and Remark 2.3.4, satisfying

$$1 \; < \; \frac{1}{n} \int_{kY} |z_n - \int_{j_0+Y} z_n \, d\mu|^p \, d\mu \quad \text{for all } n \in \mathbb{N}. \tag{2.89}$$

If necessary, by a standard higher-order reflection technique [31, Sect. 5.4] at each facet of the cube kY, we can easily construct a sequence $\{w_n\} \subset \mathcal{C}^1_0(\mathbb{R}^d)$ which is bounded in $H^{1,p}_\mu$, such that $\operatorname{spt} w_n \subset B$ for each n, $w_n \equiv z_n$ in $\tilde{B}$, where $kY \subset\subset \tilde{B} \subset\subset B$ for suitable open balls $B, \tilde{B} \subset \mathbb{R}^d$. Combining Lemma 2.3.2 with the assumption on μ and Remark 2.3.10, we get the existence of a function $w \in L^p_\mu$, such that $w_n \to w$ strongly in L^p_μ for a subsequence. Hence, passing to the limit in (2.89) we get a contradiction. □

2.4 Compactness in variable Sobolev spaces

Studying the asymptotic behaviour of the spaces $\tilde{H}^{1,p}_0(\Omega, d\mu_\varepsilon)$ is delicate because of the moving geometry of the support of μ_ε. However, at least for connected measures there is a structure result (cf. Theorem 2.4.4 below) for all possible two-scale limits of bounded sequences in $\tilde{H}^{1,p}_0(\Omega, d\mu_\varepsilon)$ in the sense of (2.76), and we will prove (cf. Theorem 2.4.5 below) that such a sequence admits a two-scale strongly convergent subsequence in $L^p(\Omega, d\mu_\varepsilon)$. At least in this generality this result seems to be new, and is at the same time essential to study nonlinear homogenization problems within the measure setting.
We first show an embedding theorem for fixed ε, which relies on Lemma 2.3.2 and is especially important to show existence for equations of type (1.6). Recall that the Dirichlet space $H^{1,p}_0(\Omega, d\mu_\varepsilon)$ is reflexive for any $p \in (1, \infty)$. Its dual we denote by $H^{-1,p'}(\Omega, d\mu_\varepsilon)$, and for the dual pairing we write

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : H^{-1,p'}(\Omega, d\mu_\varepsilon) \times H^{1,p}_0(\Omega, d\mu_\varepsilon), \; (\lambda, u) \mapsto \langle\!\langle \lambda, u \rangle\!\rangle \, .$$

Lemma 2.4.1. *Let $p \in (1, \infty)$ and μ satisfy the p-Poincaré inequality and the doubling property. Then for any $\varepsilon > 0$, the following embeddings are compact:*

$$H^{1,p}_0(\Omega, d\mu_\varepsilon) \hookrightarrow L^p(\Omega, d\mu_\varepsilon) \, , \tag{2.90}$$

$$L^p(\Omega, d\mu_\varepsilon) \hookrightarrow H^{-1,p}(\Omega, d\mu_\varepsilon) \, . \tag{2.91}$$

Proof. To show (2.90), let $u_n \rightharpoonup u$ weakly in $H_0^{1,p}(\Omega, d\mu_\varepsilon)$. Note that by density it is not restrictive to assume $u_n \in \mathcal{D}(\Omega)$. If u_n is trivially extended to $\mathbb{R}^d$ outside Ω, we get

$$\{u_n\} \subset \mathcal{C}_0^1(\mathbb{R}^d)\,, \quad \|u_n\|_{H^{1,p}_{\mu_\varepsilon}(\mathbb{R}^d)} \leq C.$$

It is easy to check that with μ also the rescaled measure μ_ε satisfies the p-Poincaré inequality and the doubling condition. Hence (2.90) follows from Lemma 2.3.2. Let $\lambda_n \rightharpoonup \lambda$ weakly in $L^p(\Omega, d\mu_\varepsilon)$ and $u_n \rightharpoonup u$ weakly in $H_0^{1,p'}(\Omega, d\mu_\varepsilon)$. Thanks to Lemma 6.3 it suffices to show

$$\langle\!\langle \lambda_n, u_n \rangle\!\rangle \;=\; \int_\Omega \lambda_n u_n \, d\mu_\varepsilon \;\to\; \int_\Omega \lambda u \, d\mu_\varepsilon \;=\; \langle\!\langle \lambda, u \rangle\!\rangle\,.$$

But this is obvious, since $u_n \to u$ strongly in $L^{p'}(\Omega, d\mu_\varepsilon)$ by (2.90). □

Now we derive an uniform Poincaré estimate in the spaces $H_0^{1,p}(\Omega, d\mu_\varepsilon)$ for strongly connected measures μ on $\mathbb{R}^d$. Recall that we always assume Ω to be an open, bounded and smooth subset of $\mathbb{R}^d$.

Lemma 2.4.2. *Let μ be strongly p-connected on $\mathbb{R}^d$ for $p \in [1, \infty)$. Then there exists a constant c_p only depending on Ω, such that for any $\varepsilon > 0$:*

$$\|u\|_{p,\varepsilon} \;\leq\; c_p \, \|\nabla_{\mu_\varepsilon} u\|_{p,\varepsilon} \quad \textit{for all } \; u \in H_0^{1,p}(\Omega, d\mu_\varepsilon). \tag{2.92}$$

Proof. By density it suffices to prove (2.92) for $u \in \mathcal{D}(\Omega)$. For $\varepsilon > 0$ we choose an integer k_ε with $\Omega \subset\subset \varepsilon k_\varepsilon Y$ and $\varepsilon k_\varepsilon \leq M$ uniformly in ε. We get

$$\begin{aligned} \int_\Omega |u(x)|^p \, d\mu_\varepsilon(x) \;&=\; \varepsilon^d \int_{k_\varepsilon Y} |u(\varepsilon x)|^p \, d\mu(x) \;\leq\; C\varepsilon^d k_\varepsilon^p \int_{k_\varepsilon Y} |\nabla_\mu (u(\varepsilon x))|^p \, d\mu(x) \\ &\leq\; C M^p \int_\Omega |\nabla_{\mu_\varepsilon} u(x)|^p \, d\mu_\varepsilon(x)\,, \end{aligned}$$

after extending u trivially to the whole of $\mathbb{R}^d$ and using property $(H4)$. □

For the two-scale structure result below we need to introduce the class $L^p(\Omega; \tilde{H}^{1,p}_\mu(\mathbb{T}))$. This may not be completely obvious, since the set $\tilde{H}^{1,p}_\mu(\mathbb{T})$ is not a Banach space and its elements can have many gradients.

Definition 2.4.3. *We say that a function $u = u(x,y) \in L^p_m(\Omega \times \mathbb{T})$ belongs to the class $L^p(\Omega; \tilde{H}^{1,p}_\mu(\mathbb{T}))$ and $\nabla_y u \in L^p_m(\Omega \times \mathbb{T})^d$ is a gradient, if there exists a sequence $\varphi_n \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{T})$, such that*

$$\varphi_n \to u\,, \quad \nabla_y \varphi_n \to \nabla_y u \quad \textit{in } L^p_m(\Omega \times Y). \tag{2.93}$$

Note that each element φ_n of an approximating sequence is Y-periodic in y, and that $\nabla_y u$ belongs to $L^p(\Omega; V^p_{\text{pot}}(\mathbb{T}))$. As Lemma 2.3.7 shows, if μ is strongly p-connected on $\mathbb{T}$ for $p \in (1, \infty)$, then any vector $v \in L^p(\Omega; V^p_{\text{pot}}(\mathbb{T}))$ corresponds to a unique function $\hat{u} = \hat{u}(x,y)$ such that

$$\hat{u}(x, \cdot) \in \tilde{H}^{1,p}_\mu(\mathbb{T}), \; \int_Y \hat{u}(x,y) \, d\mu(y) = 0 \quad \text{and} \;\; v = \nabla_y \hat{u}\,. \tag{2.94}$$

It is also clear that $\hat{u} \in L^p(\Omega; \tilde{H}^{1,p}_\mu(\mathbb{T}))$ by Definition 2.4.3. Now we can prove the central two-scale structure result. In a different framework it was first proven in [41] for $p = 2$. We give a proof for arbitrary $p \in (1, \infty)$, based on the methods in [14], which rely on Lemma 2.3.11 and Lemma 2.3.12.

Theorem 2.4.4. *Let $p \in (1, \infty)$ and $\beta \geq 0$ a real number. We consider a sequence $(u_\varepsilon, \nabla u_\varepsilon) \in V^p(\Omega, d\mu_\varepsilon)$ subject to the uniform bound*

$$\|u_\varepsilon\|_{p,\varepsilon} + \varepsilon^\beta \|\nabla u_\varepsilon\|_{p,\varepsilon} \leq C\,. \tag{2.95}$$

Possibly passing to a subsequence, assume $u_\varepsilon \rightharpoonup\rightharpoonup u \in L^p_m(\Omega \times Y)$ and $\varepsilon^\beta \nabla u_\varepsilon \rightharpoonup\rightharpoonup \chi \in L^p_m(\Omega \times Y)^d$. Then there holds

1. *If $\beta \in [0, 1)$ and μ satisfies $(H1)$, then $u = u(x)$ is independent of y.*
2. *If $\beta = 0$ and μ satisfies $(H2)$ and $(H3)$, then additionally $u \in W^{1,p}_0(\Omega)$ and there exists $\tilde{u}_1 \in L^p(\Omega; \tilde{H}^{1,p}_\mu(\mathbb{T}))$, such that*
$$\chi(x, y) = \nabla u(x) + \nabla_y \tilde{u}_1(x, y)\,, \tag{2.96}$$
where $\nabla_y \tilde{u}_1 \in L^p(\Omega; V^p_{\mathrm{pot}}(\mathbb{T}))$ is a gradient of $\tilde{u}_1$ according to Definition 2.4.3.
3. *If $\beta = 1$ and μ satisfies $(H3)$, then $u \in L^p(\Omega; \tilde{H}^{1,p}_\mu(\mathbb{T}))$ and there holds $\chi(x, y) = \nabla_y u(x, y)$, where $\nabla_y u$ is a gradient of u.*

Proof. To prove the first statement, let $\beta \in [0, 1)$ and assume, by density, that $u_\varepsilon \in \mathcal{D}(\Omega)$. One can check that for any $\Phi \in X^{p'}_\mu(\mathbb{T})$ and $\psi \in \mathcal{C}^\infty(\overline{\Omega})$ there holds (cf. [14, Proof of Theorem 4.2])

$$\int_\Omega \psi u_\varepsilon (\mathrm{div}_\mu \Phi)(\tfrac{x}{\varepsilon})\, d\mu_\varepsilon = -\varepsilon \int_\Omega \big(\psi \Phi(\tfrac{x}{\varepsilon}) \cdot \nabla u_\varepsilon + u_\varepsilon \nabla \psi \cdot \Phi(\tfrac{x}{\varepsilon})\big)\, d\mu_\varepsilon. \tag{2.97}$$

The right-hand side of (2.97) clearly converges to zero for $\varepsilon \to 0$. For convenience we estimate the first term:

$$\varepsilon \left| \int_\Omega \psi\, \Phi(\tfrac{x}{\varepsilon}) \cdot \nabla u_\varepsilon\, d\mu_\varepsilon \right| \leq \varepsilon \|\psi\|_{\infty,\varepsilon} \|\Phi(\tfrac{\cdot}{\varepsilon})\|_{p',\varepsilon} \|\nabla u_\varepsilon\|_{p,\varepsilon} \leq C \varepsilon^{1-\beta} \to 0.$$

Note that by Corollary 2.1.7, both functions $\varphi(x, y) = \psi(x) \mathrm{div}_\mu \Phi(y)$ and $\varphi_i(x, y) = \partial_{x_i} \psi(x) \Phi_i(y)$ are p'-admissible. Hence by Proposition 2.1.13, passing to the limit in (2.97) yields

$$\int_\Omega \psi(x) \left[\int_Y u(x, y) \mathrm{div}_\mu \Phi(y)\, d\mu(y) \right] dx = 0.$$

Since Φ and ψ were arbitrary, we deduce that the function $u(x, \cdot)$ belongs to $V^\perp$ for $\mathcal{L}^d$-a.e. $x \in \Omega$ with V defined in Lemma 2.3.11. Since μ satisfies $(H1)$, the same result gives that $u(x, \cdot)$ is constant μ-a.e. on Y for $\mathcal{L}^d$-a.e. $x \in \Omega$, which shows the first assertion of the theorem.

Let us show that the two-scale limit $u \in L^p(\Omega)$ obtained above belongs to $W_0^{1,p}(\Omega)$ provided $\beta = 0$ and μ satisfies $(H2)$ and $(H3)$. To this end, let ψ and Φ as above with the additional assumption $\mathrm{div}_\mu \Phi = 0$. Then the integral on the right-hand side in (2.97) vanishes, and passing to the limit yields

$$\bar{\Phi} \cdot \int_\Omega u \nabla \psi \, dx \;=\; -\int_{\Omega \times Y} \psi \, \chi(x,y) \cdot \Phi(y) \, dm \;\leq\; \|\Phi\|_{p',\mu,Y} \, \|\chi\|_{p,m} \, \|\psi\|_{L^{p'}(\Omega)}, \tag{2.98}$$

where we have set $\bar{\Phi} := \int_Y \Phi(y) \, d\mu(y)$. In [14, Section 4] it was shown, that under the assumptions $(H2)$ and $(H3)$ on μ the convex subset

$$K := \{\bar{\Phi} : \; \Phi \in X_\mu^{p'}(\mathbb{T}), \; \mathrm{div}_\mu \Phi = 0, \; \|\Phi\|_{p',\mu,Y} \leq 1\}$$

of $\mathbb{R}^d$ has a nonempty interior. Since K is convex with $0 \in K$, it follows that $\overline{B_\delta(0)} \subset K$ for some $\delta > 0$. Hence choosing $\bar{\Phi} = \delta \hat{e}_i$, by (2.98) we get

$$\int_\Omega u(x) \partial_{x_i} \psi(x) \, dx \;\leq\; \delta^{-1} \|\chi\|_{p,m} \, \|\psi\|_{L^{p'}(\Omega)} \;\leq\; C \, \|\psi\|_{L^{p'}(\Omega)}$$

for any $i = 1, \ldots, d$ and all $\psi \in \mathcal{C}^\infty(\overline{\Omega})$, which implies $u \in W^{1,p}(\Omega)$. Since $\partial\Omega$ is smooth and the boundary values of ψ can be choosen arbitrarily, the trace of u on $\partial\Omega$ must vanish, hence $u \in W_0^{1,p}(\Omega)$. As a consequence, we can integrate by parts on the left-hand side in (2.98) and obtain

$$\int_{\Omega \times Y} \psi(x) [\chi(x,y) - \nabla u(x)] \cdot \Phi(y) \; dm \;=\; 0 \tag{2.99}$$

for all $\Phi \in X_\mu^{p'}(\mathbb{T})$ with $\mathrm{div}_\mu \Phi = 0$ and each $\psi \in \mathcal{C}^\infty(\overline{\Omega})$. In particular, if we choose $\Phi = P_\mu \sigma$ with $\sigma \in Y_\mu^{p'}(\mathbb{T})$, by Lemma 2.3.12 and the assumption on μ we get the existence of a function $u_1 \in L^p(\Omega; H_\mu^{1,p}(\mathbb{T}))$, such that

$$P_\mu(y)[\chi(x,y) - \nabla u(x)] \;=\; \nabla_{\mu,y} u_1(x,y) \quad m\text{-a.e. in } \Omega \times Y. \tag{2.100}$$

Using (2.45) it follows that there exists a function $\xi \in L^p(\Omega; \Gamma_\mu^p(\mathbb{T}))$, and hence a vector $v \in L^p(\Omega; V_{\mathrm{pot}}^p(\mathbb{T}))$ by (2.57), such that

$$\chi(x,y) = \nabla u(x) + \nabla_{\mu,y} u_1(x,y) + \xi(x,y) = \nabla u(x) + v(x,y) \, . \tag{2.101}$$

Since μ enjoys $(H3)$, by (2.94) there exists $\tilde{u}_1 \in L^p(\Omega; \tilde{H}_\mu^{1,p}(\mathbb{T}))$ with $\nabla_y \tilde{u}_1 = v$, where $\nabla_y \tilde{u}_1(x,\cdot) \in V_{\mathrm{pot}}^p(\mathbb{T})$ is a gradient of $\tilde{u}_1(x,\cdot) \in \tilde{H}_\mu^{1,p}(\mathbb{T})$. Now if $\beta = 1$, passing to the limit in (2.97) yields

$$\int_{\Omega \times Y} \psi(x) \chi(x,y) \cdot \Phi(y) \; dm \;=\; -\int_{\Omega \times Y} \psi(x) u(x,y) \mathrm{div}_\mu \Phi(y) \; dm \tag{2.102}$$

for all $\Phi \in X_\mu^{p'}(\mathbb{T})$ and $\psi \in \mathcal{C}^\infty(\overline{\Omega})$. Choosing $\Phi = P_\mu \sigma$ with $\sigma \in Y_\mu^{p'}(\mathbb{T})$ and $\mathrm{div}_\mu \Phi = 0$, precisely as above we deduce $\chi(x,y) \in L^p(\Omega; V_{\mathrm{pot}}^p(\mathbb{T}))$. Since μ is strongly p-connected on $\mathbb{T}$, we can consider the unique element $\hat{u}$ in the class

$L^p(\Omega;\tilde H^{1,p}_\mu(\mathbb{T}))$ that satisfies (2.94) with $v=\chi$. Since Φ is tangential, we can apply the integration by parts formula (2.42) in (2.102) and obtain

$$\int_Y [\hat u(x,y)-u(x,y)]\,\mathrm{div}_\mu\Phi(y)\,d\mu(y) \;=\; 0 \qquad \text{for } \mathcal{L}^d\text{-a.e. } x\in\Omega.$$

From Lemma 2.3.11 and (2.94) we deduce $\hat u(x,y) \;=\; u(x,y) - \int_Y u(x,y)d\mu(y)$. In particular, the function u belongs to $L^p(\Omega;\tilde H^{1,p}_\mu(\mathbb{T}))$, and there holds $\nabla_y u(x,y)=\nabla_y\hat u(x,y)=\chi(x,y)$, which completes the proof. □

Now we prove the *rescaled Rellich property* for strongly connected measures μ, which opens the door to the homogenization of quasilinear equations posed on associated multistructures. For the compactness result it suffices to control the μ_ε-tangential gradient of the Sobolev functions, so we consider the spaces $H^{1,p}_0(\Omega,d\mu_\varepsilon)$ endowed with the norm $\|\cdot\|_{1,p,\varepsilon}$ defined in (2.67).

Theorem 2.4.5. *Let $p\in(1,\infty)$ and μ be strongly p-connected on $\mathbb{R}^d$ satisfying the doubling condition. Let $\{w_\varepsilon\}$ be a sequence in $H^{1,p}_0(\Omega,d\mu_\varepsilon)$ endowed with the uniform bound*

$$\|w_\varepsilon\|_{1,p,\varepsilon}\;\le\; C\,. \tag{2.103}$$

Assume that $w_\varepsilon \rightharpoonup\!\rightharpoonup w_0\in L^p(\Omega)$. Then there holds $w_\varepsilon \twoheadrightarrow w_0$, that means

$$\int_\Omega |w_\varepsilon|^p\,d\mu_\varepsilon \;\to\; \int_\Omega |w_0|^p\,dx\,. \tag{2.104}$$

Proof. It is easy to check that we can assume $w_\varepsilon\in\mathcal{D}(\Omega)$ without loss of generality. The proof is then divided into two steps.

Step 1: We consider the sequence of piecewise constant functions comprising the averages of w_ε over the rescaled cells of size ε. In order to simplify notation we assume $\Omega=(0,1)^d$ and take the sequence $\varepsilon=\frac{1}{n}, n\in\mathbb{N}_+$. We define

$$I_\varepsilon := \{k\in\mathbb{Z}^d \mid \forall i:\ 0\le k_i<\tfrac{1}{\varepsilon}=n\}\,,\quad \overline{I_\varepsilon} := \{k\in\mathbb{Z}^d \mid \forall i:\ 0\le k_i\le n\}\,,$$
$$Y^k_\varepsilon := \varepsilon(k+Y)\ \text{ for } k\in I_\varepsilon\,.$$

By construction we have $\overline{\Omega}=\bigcup_{k\in I_\varepsilon}\overline{Y^k_\varepsilon}$ (see Figure 2.4 below). In the general case we cover Ω by a rectangular pavement Π of rescaled cells, extending w_ε to zero outside Ω (see Figure 2.5). Integrating respectively over Π instead, the proof goes completely analogue, so the assumptions on Ω are not restrictive.

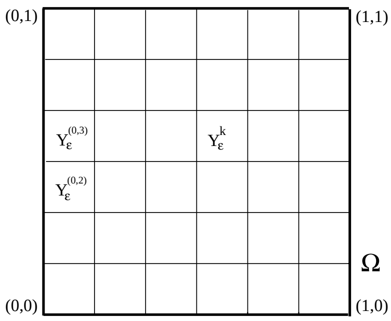


Figure 2.4: $\Omega=(0,1)^d$

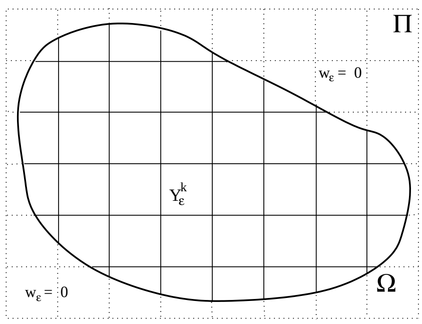


Figure 2.5: Ω arbitrary

We emphasize that, since $\mu(\partial Y) = 0$, we have

$$\int_\Omega d\mu_\varepsilon(x) \;=\; \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} d\mu_\varepsilon(x)\,, \quad \mu_\varepsilon(Y_\varepsilon^k) = \mathcal{L}^d(Y_\varepsilon^k) = \varepsilon^d\,. \tag{2.105}$$

Now we can define the sequence of auxiliary functions. Set

$$w_\varepsilon^k := \fint_{Y_\varepsilon^k} w^\varepsilon(x)\, d\mu_\varepsilon(x)\,, \quad \lambda_\varepsilon(x) := \sum_{k\in I_\varepsilon} w_\varepsilon^k \chi_\varepsilon^k(x)\,,$$

where χ_ε^k is the characteristic function of Y_ε^k. Observe that λ_ε is piecewise constant and uniformly bounded in $L^p(\Omega, d\mu_\varepsilon)$ with $\|\lambda_\varepsilon\|_{p,\varepsilon} = \|\lambda_\varepsilon\|_{L^p(\Omega)}$ by (2.105). Indeed

$$\begin{aligned} \|\lambda_\varepsilon\|_{p,\varepsilon}^p &= \varepsilon^{d(1-p)} \sum_{k\in I_\varepsilon} \left| \int_{Y_\varepsilon^k} w_\varepsilon(x)\, d\mu_\varepsilon(x) \right|^p \\ &\le \varepsilon^{d(1-p)} \sum_{k\in I_\varepsilon} \|w_\varepsilon\|_{p,\varepsilon,k}^p \, \|1\|_{p',\varepsilon,k}^p \;=\; \|w_\varepsilon\|_{p,\varepsilon}^p \;\le\; C\,, \end{aligned}$$

where we have set $\|u\|_{p,\varepsilon,k}^p := \int_{Y_\varepsilon^k} |u|^p\, d\mu_\varepsilon$. The main task in this step is to show the following two statements

$$\|w_\varepsilon\|_{p,\varepsilon} \;\le\; \|\lambda_\varepsilon\|_{L^p(\Omega)} \,+\, C\varepsilon\,, \quad \lambda_\varepsilon \rightharpoonup w_0 \;\text{ weakly in } L^p(\Omega)\,. \tag{2.106}$$

To this end, the following Poincaré type estimate based on the connectedness of μ is crucial:

$$\exists C \in \mathbb{R}\, \forall \varepsilon, k_\varepsilon \in I_\varepsilon : \quad \|w_\varepsilon - w_\varepsilon^k\|_{p,\varepsilon,k} \;\le\; C\varepsilon\, \|\nabla_{\mu_\varepsilon} w_\varepsilon\|_{p,\varepsilon,k}\,. \tag{2.107}$$

By Remark 2.3.4 the function $x \mapsto w_\varepsilon(\varepsilon(x+k)) - w_\varepsilon^k \in H^{1,p}_{\mu,\mathrm{loc}}$ is admissible in $(H4)$, since it has mean value zero over Y by construction. Thus

$$\begin{aligned} \int_{Y_\varepsilon^k} |w_\varepsilon(x) - w_\varepsilon^k|^p\, d\mu_\varepsilon &= \varepsilon^d \int_Y |w_\varepsilon(\varepsilon(x+k)) - w_\varepsilon^k|^p\, d\mu \\ &\le C\varepsilon^{d+p} \int_Y |\nabla_\mu w_\varepsilon(\varepsilon(x+k))|^p\, d\mu \;=\; C\varepsilon^p \int_{Y_\varepsilon^k} |\nabla_{\mu_\varepsilon} w_\varepsilon|^p\, d\mu_\varepsilon. \end{aligned}$$

This shows (2.107), which gives the first statement in (2.106) by a straightforward calculation using (2.103),(2.105) and $\|\lambda_\varepsilon\|_{p,\varepsilon} = \|\lambda_\varepsilon\|_{L^p(\Omega)}$. Now let $\phi \in \mathcal{D}(\Omega)$ be arbitrary. By the prerequisites of the theorem we have

$$\int_\Omega w_0 \phi\, dx \;\leftarrow\; \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} (w_\varepsilon - w_\varepsilon^k)\phi\, d\mu_\varepsilon + \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} w_\varepsilon^k \phi\, d\mu_\varepsilon \;=:\; J_\varepsilon^1 + J_\varepsilon^2\,.$$

Using (2.103) and (2.107) we obtain $|J_\varepsilon^1| \le C\varepsilon$. To estimate J_ε^2 we define the number $\phi_\varepsilon^k := \phi(x_\varepsilon^k)$, where x_ε^k is any point in the cube Y_ε^k. We have

$$\forall x \in Y_\varepsilon^k : \quad |\phi(x) - \phi_\varepsilon^k| \;\le\; \mathrm{lip}(\phi)|x - x_\varepsilon^k| \;\le\; C\varepsilon \tag{2.108}$$

by the definition of Y_ε^k and the smoothness of ϕ. An easy calculation gives

$$J_\varepsilon^2 - \int_\Omega \lambda_\varepsilon \phi \, dx \; = \; \sum_{k \in I_\varepsilon} w_\varepsilon^k \left(\int_{Y_\varepsilon^k} (\phi - \phi_\varepsilon^k) \, d\mu_\varepsilon - \int_{Y_\varepsilon^k} (\phi - \phi_\varepsilon^k) \, dx \right) =: s_\varepsilon .$$

Using (2.103) and (2.108) we easily deduce $|s_\varepsilon| \le C\varepsilon \|w_\varepsilon\|_{1,\varepsilon} \le C\varepsilon$, and finally $J_\varepsilon^1 + J_\varepsilon^2 = \int_\Omega \lambda_\varepsilon \phi \, dx + o(1)$ as $\varepsilon \to 0$, which completes the proof of (2.106).

Step 2: We choose a regular grid of tetrahedra with vertices $\{\varepsilon k \mid k \in \overline{I_\varepsilon}\}$. We set $\Theta_\varepsilon(\varepsilon k) = w_\varepsilon^k$ for all $k \in \overline{I_\varepsilon}$, where $w_\varepsilon^k := 0$ if $k_i = n$ for one i, and define $\Theta_\varepsilon \in \mathcal{C}(\overline{\Omega})$ as the piecewise linear interpolation of these values. We sketch the procedure in 2D, where we get a triangulation of Ω (see Figure 2.6). Each square Y_ε^k is composed of a lower right and an upper left triangle:

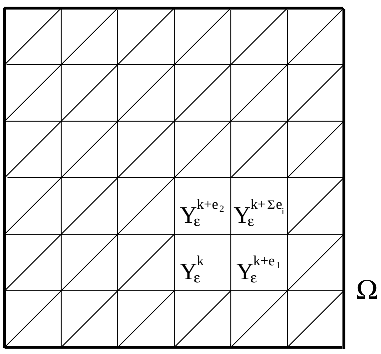


Figure 2.6: Regular grid

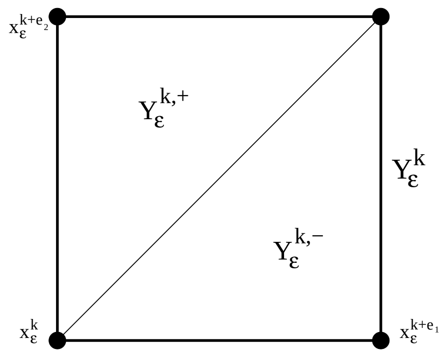


Figure 2.7: Linear interpolation

If we denote by e_i the i-th unit vector and by $x_\varepsilon^k = (y_\varepsilon^k, z_\varepsilon^k)$ the coordinates of the corresponding vertices (see Figure 2.7), then Θ_ε admits the following explicit form in $Y_\varepsilon^{k,-}$ and $Y_\varepsilon^{k,+}$ respectively:

$$\begin{aligned} \Theta_\varepsilon(y,z) &= w_\varepsilon^k + \tfrac{1}{\varepsilon}(w_\varepsilon^{k+e_1} - w_\varepsilon^k)(y - y_\varepsilon^k) + \tfrac{1}{\varepsilon}(w_\varepsilon^{k+\Sigma e_i} - w_\varepsilon^{k+e_1})(z - z_\varepsilon^{k+e_1}) , \\ \Theta_\varepsilon(y,z) &= w_\varepsilon^k + \tfrac{1}{\varepsilon}(w_\varepsilon^{k+e_2} - w_\varepsilon^k)(z - z_\varepsilon^k) + \tfrac{1}{\varepsilon}(w_\varepsilon^{k+\Sigma e_i} - w_\varepsilon^{k+e_2})(y - y_\varepsilon^{k+e_1}) . \end{aligned}$$

Obviously Θ_ε is an element of $\mathcal{C}(\overline{Y_\varepsilon^k})$ for each $k \in I_\varepsilon$. Putting these values together we get the piecewise linear interpolation $\Theta_\varepsilon \in \mathcal{C}(\overline{\Omega})$. Now with Θ_ε at hand we want to show

$$\|\Theta_\varepsilon - \lambda_\varepsilon\|_{L^p(\Omega)} \; = \; o(1) \quad \text{as } \varepsilon \to 0 \, , \tag{2.109}$$

$$\|\Theta_\varepsilon\|_{W^{1,p}(\Omega)} \; \le \; C \qquad \text{uniformly in } \varepsilon \, . \tag{2.110}$$

By the construction of Θ_ε, we deduce

$$\|\Theta_\varepsilon - \lambda_\varepsilon\|_{L^p(\Omega)}^p = \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} |\Theta_\varepsilon(x) - w_\varepsilon^k|^p \, dx \le C\varepsilon^d \sum_{k \in I_\varepsilon} \sum_{j \in \{0,1\}^d} |w_\varepsilon^{k,j} - w_\varepsilon^k|^p \tag{2.111}$$

with a constant only depending on d and p, and where for any $k \in I_\varepsilon$ and $j \in \{0,1\}^d$ we have set $w_\varepsilon^{k,j} := w_\varepsilon^{k+\Sigma j_i e_i}$. Similarly we define

$$\forall j \in \{0,1\}^d : \; Y_\varepsilon^{k,j} := Y_\varepsilon^{k+\Sigma j_i e_i}, \; Z_\varepsilon^k := \varepsilon(k + 2Y) \; \Rightarrow \; \overline{Z_\varepsilon^k} = \bigcup_{j \in \{0,1\}^d} \overline{Y_\varepsilon^{k,j}} \, .$$

As indicated in (2.111) we have to control the term $|w_\varepsilon^{k,j} - w_\varepsilon^k|^p$. Possibly extending w_ε to zero outside Ω if necessary, we can estimate

$$|w_\varepsilon^{k,j} - w_\varepsilon^k|^p \le C \left(\fint_{Z_\varepsilon^k} |w_\varepsilon - w_\varepsilon^k|^p \, d\mu_\varepsilon + \fint_{Z_\varepsilon^k} |w_\varepsilon - w_\varepsilon^{k,j}|^p \, d\mu_\varepsilon \right) \tag{2.112}$$

with a constant only depending on p. Due to Lemma 2.3.14, the terms on the right-hand side of (2.112) can be treated simultaneously for each $j \in \{0,1\}^d$:

$$\begin{aligned} \fint_{Z_\varepsilon^k} |w_\varepsilon - w_\varepsilon^{k,j}|^p \, d\mu_\varepsilon &= \fint_{2Y} |w_\varepsilon(\varepsilon(x+k)) - \fint_{j+Y} w_\varepsilon(\varepsilon(x+k))|^p \, d\mu \\ &\overset{(\dagger)}{\le} C \fint_{2Y} |\nabla_\mu [w_\varepsilon(\varepsilon(x+k))]|^p \, d\mu \le C\,\varepsilon^p \fint_{Z_\varepsilon^k} |\nabla_{\mu_\varepsilon} w_\varepsilon|^p \, d\mu_\varepsilon . \end{aligned}$$

In $(\dagger)$ we applied Lemma 2.3.14 to the function $x \mapsto w_\varepsilon(\varepsilon(x+k)) \in H^{1,p}_{\mu,\mathrm{loc}}$. Hence combining the last estimate with (2.111) and (2.112), we obtain

$$\begin{aligned} \|\Theta_\varepsilon - \lambda_\varepsilon\|^p_{L^p(\Omega)} &\le C 2^d \varepsilon^{d+p} \sum_{k\in I_\varepsilon} \fint_{Z_\varepsilon^k} |\nabla_{\mu_\varepsilon} w_\varepsilon|^p \, d\mu_\varepsilon \\ &\le C 4^d \varepsilon^p \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} |\nabla_{\mu_\varepsilon} w_\varepsilon|^p \, d\mu_\varepsilon = C\varepsilon^p \|\nabla_{\mu_\varepsilon} w_\varepsilon\|^p_{p,\varepsilon} , \end{aligned}$$

which gives (2.109) by (2.103). For the proof of (2.110), due to the second statement in (2.106) and (2.109), it suffices to control $\|\partial_l \Theta_\varepsilon\|_{L^p(\Omega)}$. Note that each rescaled cube Y_ε^k is composed of finitely many tetrahedra $T_{\varepsilon,i}^k$, such that $\overline{Y_\varepsilon^k} = \cup_{i=1}^{N_d} \overline{T_{\varepsilon,i}^k}$. For instance, we have $N_2 = 2, N_3 = 6$. It is well known that each Θ_ε possesses weak derivatives, which coincide with the derivatives of the polynomials on each tetrahedra (see [35, Section. 6.2]). We calculate

$$\int_\Omega |\partial_l \Theta_\varepsilon|^p \, dx = \sum_{k\in I_\varepsilon} \sum_{i=1}^{N_d} \int_{T_{\varepsilon,i}^k} |\partial_l \Theta_\varepsilon|^p \, dx = \sum_{k,i} \int_{T_{\varepsilon,i}^k} \left| \frac{w_\varepsilon^{k,j} - w_\varepsilon^{k,\tilde{j}}}{\varepsilon} \right|^p dx$$

for some $j, \tilde{j} \in \{0,1\}^d$ depending on i and on $l \in \{1,\dots,d\}$. Thus

$$\begin{aligned} \|\partial_l \Theta_\varepsilon\|^p_{L^p(\Omega)} &\le 2^{p-1} \varepsilon^{-p} \sum_{k,i} 2 \int_{T_{\varepsilon,i}^k} \Big(\sum_{j\in\{0,1\}^d} |w_\varepsilon^k - w_\varepsilon^{k,j}|^p \Big) \, dx \\ &= 2^p \varepsilon^{d-p} \sum_{k,j} |w_\varepsilon^k - w_\varepsilon^{k,j}|^p \le C \end{aligned}$$

uniformly in ε as shown above, which shows (2.110). We have $\Theta_\varepsilon \rightharpoonup w_0$ weakly in $L^p(\Omega)$ by (2.106) and (2.109). From (2.110) and the classical Rellich embedding theorem we deduce $\Theta_\varepsilon \to w_0$ strongly in $L^p(\Omega)$ and hence

$$\limsup_{\varepsilon\to 0} \|w_\varepsilon\|^p_{p,\varepsilon} \le \|w_0\|^p_{L^p(\Omega)}$$

by (2.106) and (2.109). By Proposition 2.1.10 this is sufficient. $\square$

3 Nonlinear elliptic problems

In this chapter we study the homogenization of quasilinear elliptic problems with respect to measures, in the most general setting the asymptotics of the equation

$$-\operatorname{div} a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon) + \lambda |u_\varepsilon|^{p-2} u_\varepsilon = f(\tfrac{x}{\varepsilon}, x, u_\varepsilon)\,, \quad u_\varepsilon \in \tilde{H}_0^{1,p}(\Omega, d\mu_\varepsilon), \quad (3.1)$$

where the flux a is monotone with respect to the gradient, $p > 1$ and $\lambda \geq 0$. This approach is quite flexible, since it contains various types of nonlinear elliptic problems. For instance, our structure conditions on the data (see Assumption 3.1.1 below) cover the following type of the p-Laplace equation

$$-\operatorname{div}\left(a(\tfrac{x}{\varepsilon}, u_\varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right) \;=\; f(\tfrac{x}{\varepsilon}, u_\varepsilon)\,, \quad p > 1. \quad (3.2)$$

Apart from the classical setting $\mu = \mathcal{L}^d$ included, we will also consider singular structures equipped with a nontrivial measure μ. For instance, for $p = 2$ equation (3.2) will be derived in Paragraph 3.2.1 by a model of single phase flow through a fractured porous medium, that contains a connected network of positive codimension.
We will be able to derive the homogenized equation, in particular an explicit characterization of the effective flux $a^\star = a^\star(u, \nabla u)$. This generalizes the analysis of Fusco and Moscariello [32], where in the case $\mu = \mathcal{L}^d$ the homogenization of equation (3.1) was investigated with $\lambda = 0$ and $f = f(x)$, as well as the recent result by Lukkassen and Wall [44], where the authors studied the asymptotics $\varepsilon \to 0$ of the problem

$$-\operatorname{div} a(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon) + \lambda |u_\varepsilon|^{p-2} u_\varepsilon = f_\varepsilon(x)\,, \quad (u_\varepsilon, \nabla u_\varepsilon) \in V^p(\Omega, d\mu_\varepsilon) \quad (3.3)$$

subject to a general p-connected Radon measure μ. The central point is that Theorem 2.4.5 yields the strong two-scale convergence of $\{u_\varepsilon\}$ *a priori* without relying on the equation, only using the uniform estimate (1.4). This is clearly necessary in order to pass to the limit in (3.1), whereas for the asymptotics of equation (3.3) it suffices to assume $f_\varepsilon \twoheadrightarrow f$ two-scale strongly in $L^{p'}(\Omega, d\mu_\varepsilon)$. Indeed, if $\{u_\varepsilon\}$ merely weakly two-scale converges to $u = u(x)$, this guarantees that

$$\int_\Omega f_\varepsilon(x) u_\varepsilon(x)\, d\mu_\varepsilon(x) \to \int_\Omega f u\, dx, \quad (3.4)$$

which, using the solution property of u_ε, turns out to be sufficient to derive the homogenized equation for (3.3) and *a posteriori* the strong two-scale convergence of $\{u_\varepsilon\}$ (see [44, Section 6] for the details). However, this approach does clearly not help if f, and even worse the principle part a depends on the unknown u_ε. This stresses once more the importance of the rescaled Rellich property, which allows to study the homogenization of equation (3.1).
We study the properties of the homogenized operator $A^\star u = -\operatorname{div} a^\star(u, \nabla u)$, which is essential in order to derive regularity, uniqueness and corrector results for the homogenized equations (Section 3.2). Finally we will consider a

nonlinear double porosity model (Section 3.3) associated with equation (3.2) for $p = 2$, where the coefficient a depends on the parameter ε in the following, more complicated way:

$$a(\tfrac{x}{\varepsilon}, s) = \begin{cases} a_1(\frac{x}{\varepsilon}, s) & x \in F_1^\varepsilon, \\ \varepsilon^\alpha a_2(\frac{x}{\varepsilon}, s) & x \in F_2^\varepsilon. \end{cases} \tag{3.5}$$

Here F_1^ε represents a singular structure, typically a connected lower dimensional network, and F_2^ε the bulk, where the permeability is of lower order ε^α with $\alpha > 0$. It turns out that different types of effective equations arise depending on the size of the parameter α. Thus we generalize the results in [62], where the author studied the corresponding linear model.

3.1 Homogenization of monotone operators

In this section we study the homogenization of second order elliptic monotone operators with respect to measures, more precisely the asymptotics of the quasilinear equation (3.1), where $p \in (1, \infty)$ and $\lambda \geq 0$ is a given parameter. For various applications we refer to Section 3.2 and the fattening approach in Chapter 5. We introduce the structure conditions on the data, where we set

$$r_0 := \min\{1, p-1\}, \ \alpha := \max\{p, 2\}. \tag{3.6}$$

Assumption 3.1.1. *Let $a : \mathbb{R}^d \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $(y, x, s, \xi) \mapsto a(y, x, s, \xi)$ be μ-measurable and Y-periodic in y, and continuous with respect to the x, s, ξ variables in $\mathbb{R}^d \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$. We assume that there exist constants $c_1, c_2 > 0$ and $r \in (0, r_0]$, such that for all $(y, x, s) \in \mathbb{R}^d \times \Omega \times \mathbb{R}$ and any $\xi_1, \xi_2 \in \mathbb{R}^d$:*

$$a(y, x, s, 0) = 0, \tag{3.7}$$

$$|a(y, x, s, \xi_1) - a(y, x, s, \xi_2)| \leq c_1(1 + |s| + |\xi_1| + |\xi_2|)^{p-1-r}|\xi_1 - \xi_2|^r, \tag{3.8}$$

$$(a(y, x, s, \xi_1) - a(y, x, s, \xi_2)) \cdot (\xi_1 - \xi_2) \geq c_2(1 + |s| + |\xi_1| + |\xi_2|)^{p-\alpha}|\xi_1 - \xi_2|^\alpha. \tag{3.9}$$

The source $f : \mathbb{R}^d \times \Omega \times \mathbb{R} \to \mathbb{R}$, $(y, x, s) \mapsto f(y, x, s)$ is μ-measurable and Y-periodic in y, continuous with respect to the x, s variables in $\mathbb{R}^d \times \overline{\Omega} \times \mathbb{R}$ and satisfies the following growth condition

$$\exists \beta \in [0, p-1): \quad |f(y, x, s)| \leq C(1 + |s|^\beta). \tag{3.10}$$

Let us first draw some simple conclusions. Using the assumptions (3.7), (3.8) and (3.10) we immediately derive

$$|a(y, x, s, \xi)| \leq C(1 + |s| + |\xi|)^{p-1}, \quad |f(y, x, s)| \leq C(1 + |s|)^{p-1} \tag{3.11}$$

for some universal constants. Given a pair $(u, \nabla u) \in V^p(\Omega, d\mu_\varepsilon)$ we see that $|u|^{p-2}u \in L^{p'}(\Omega, d\mu_\varepsilon)$ and

$$|a(\tfrac{x}{\varepsilon}, x, u, \nabla u)| \in L^{p'}(\Omega, d\mu_\varepsilon)^d, \quad |f(\tfrac{x}{\varepsilon}, x, u)| \in L^{p'}(\Omega, d\mu_\varepsilon). \tag{3.12}$$

Now we introduce the notion of a weak solution of equation (3.1). It coincides with the standard formulation in case μ is the Lebesgue measure. The well-posedness will be investigated in Theorem 3.1.4 below.

Definition 3.1.2. *The pair $(u_\varepsilon, \nabla u_\varepsilon) \in V^p(\Omega, d\mu_\varepsilon)$ is called a weak solution of equation (3.1), if the following integral identity holds for every $\varphi \in \mathcal{D}(\Omega)$:*

$$\int_\Omega a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\varphi + \lambda |u_\varepsilon|^{p-2} u_\varepsilon \varphi \, d\mu_\varepsilon = \int_\Omega f(\tfrac{x}{\varepsilon}, x, u_\varepsilon)\varphi \, d\mu_\varepsilon. \qquad (3.13)$$

We emphasize that both sides in (3.13) are well defined. In particular, due to the continuity of K and f with respect to the slow variables, the functions

$$x \mapsto a(\tfrac{x}{\varepsilon}, x, u(x), \nabla u(x)), \quad x \mapsto f(\tfrac{x}{\varepsilon}, x, u(x))$$

are μ_ε-measurable for any positive value of ε and any $(u, \nabla u) \in V^p(\Omega, d\mu_\varepsilon)$. Note that we could alternatively call $u_\varepsilon \in \tilde{H}_0^{1,p}(\Omega, d\mu_\varepsilon)$ a solution of (3.1), if the identity (3.13) holds for *some* gradient ∇u_ε of u_ε. It is important to notice that such a gradient is uniquely determined.

Observation 3.1.3. *If $(u_\varepsilon, z_\varepsilon), (u_\varepsilon, \tilde{z}_\varepsilon) \in V^p(\Omega, d\mu_\varepsilon)$ are solutions of equation (3.1) in the sense of Definition 3.1.2, then there holds $z_\varepsilon = \tilde{z}_\varepsilon$. Moreover the flux $a(\cdot, u_\varepsilon, z_\varepsilon)$ is tangential with respect to μ_ε.*

Proof. We have $z_\varepsilon = \tilde{z}_\varepsilon + z$ for some $z \in \Gamma^p(\Omega, d\mu_\varepsilon)$. By density, (3.13) also holds for $(0, z) \in V^p(\Omega, d\mu_\varepsilon)$ and using (3.9) we get

$$\begin{aligned} 0 &= \int_\Omega [a(u_\varepsilon, z_\varepsilon) - a(u_\varepsilon, \tilde{z}_\varepsilon)] \cdot [z_\varepsilon - \tilde{z}_\varepsilon] \geq c_2 \int_\Omega (1 + |u_\varepsilon| + |z_\varepsilon| + |\tilde{z}_\varepsilon|)^{p-\alpha} |z_\varepsilon - \tilde{z}_\varepsilon|^\alpha \\ &\geq c_2 \left(\int_\Omega (1 + |u_\varepsilon| + |z_\varepsilon| + |\tilde{z}_\varepsilon|)^p \right)^{\frac{p-\alpha}{p}} \left(\int_\Omega |z_\varepsilon - \tilde{z}_\varepsilon|^p \right)^{\frac{\alpha}{p}} \geq c \|z_\varepsilon - \tilde{z}_\varepsilon\|_p^\alpha \geq 0, \end{aligned}$$

where the integrals are taken with respect to $d\mu_\varepsilon$, and where for $p < 2$ we used the reversed Hölder inequality (see Theorem 6.6 below) with dual exponents $p/2 \in (0, 1)$ and $p/(p-2) < 0$. This proves the first statement. Similarly, for each $z \in \Gamma^p$ we obtain $\int_\Omega a(\cdot, u_\varepsilon, z_\varepsilon) \cdot z \, d\mu_\varepsilon = 0$. By (2.20) we can choose z as the normal component of the flux, which yields the second statement. □

In general we can not expect uniqueness for equations of type (3.1). Existence can be shown by freezing the function u_ε in the coefficients and applying the Schauder fixed point theorem to the corresponding solution operator. To this end we have to require the p-Poincaré inequality for μ. Since the gradient of a solution is in general not tangential, we have to use the artificial setting involving the Banach space $V^p(\Omega, d\mu_\varepsilon)$ endowed with the norm

$$\|(u, \nabla u)\|_{V^p(\Omega, d\mu_\varepsilon)} := \|u\|_{p,\varepsilon} + \|\nabla u\|_{p,\varepsilon}.$$

Theorem 3.1.4. *Let $\lambda > 0$, μ be doubling and satisfy the p-Poincaré inequality. Under Assumption 3.1.1 there exists a solution $(u_\varepsilon, \nabla u_\varepsilon) \in V^p(\Omega, d\mu_\varepsilon)$ of equation (3.1) in the sense of Definition 3.1.2 satisfying the uniform estimate*

$$\|u_\varepsilon\|_{p,\varepsilon} + \|\nabla u_\varepsilon\|_{p,\varepsilon} \leq C, \qquad (3.14)$$

where the constant C depends only on $|\Omega|, c_2, p, \beta, \lambda$, but not on ε.

Proof. To shorten the notation we write $a = a(s,\xi)$ and $f = f(s)$, use the abbreviations

$$V := V^p(\Omega, d\mu_\varepsilon)\,, \quad X := L^p(\Omega, d\mu_\varepsilon),$$

and denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the dual pairing between V' and V. We fix $v \in X$ and introduce the following operators $T_v : V \to V'$ and $V' \ni g_v : V \to \mathbb{R}$:

$$\langle\langle T_v(u,u_1),(\varphi,\varphi_1)\rangle\rangle := \int_\Omega a(v,u_1)\cdot\varphi_1 + \lambda|u|^{p-2}u\varphi\, d\mu_\varepsilon, \tag{3.15}$$

$$\langle\langle g_v,(\varphi,\varphi_1)\rangle\rangle := \int_\Omega f(v)\varphi\, d\mu_\varepsilon. \tag{3.16}$$

We want to show that there exists a unique element $(u,u_1) = w \in V$, that means u_1 is a gradient of u, such that $T_v(w) = g_v$. Note that V is reflexive, so we can apply the Browder-Minty theorem (see Theorem 6.5 below), provided T_v is strictly monotone, hemicontinuous and coercive.

• $2 \le p < \infty$: We need the following elementary estimate proven in Proposition 6.1 below, valid for some constants $c, C > 0$ and all $a, b \in \mathbb{R}$:

$$c|a-b|^p \le (|a|^{p-2}a - |b|^{p-2}b)(a-b) \le C(|a|+|b|)^{p-2}|a-b|^2. \tag{3.17}$$

In what follows, all the integrals over Ω are taken with respect to $d\mu_\varepsilon$:

(a) T_v strictly monotone:

$$\begin{aligned}
&\langle\langle T_v(u,u_1) - T_v(w,w_1),(u,u_1)-(w,w_1)\rangle\rangle\\
=\;& \int_\Omega (a(v,u_1)-a(v,w_1))\cdot(u_1-w_1) + \lambda(|u|^{p-2}u - |w|^{p-2}w)(u-w)\\
\ge\;& c_2\int_\Omega |u_1-w_1|^p + c\lambda\int_\Omega |u-w|^p \;>\; 0
\end{aligned}$$

for $(u,u_1) \ne (w,w_1)$, where c_2 and c are the positive constants ocurring in (3.9) and (3.17). Hence T_v is strictly monotone.

(b) T_v hemicontinuous:

$$\begin{aligned}
&|\langle\langle T_v((u,u_1)+t(w,w_1)) - T_v(u,u_1),(z,z_1)\rangle\rangle|\\
=\;& \left|\int_\Omega (a(v,u_1+tw_1)-a(v,u_1))\cdot z_1 + \lambda(|u+tw|^{p-2}(u+tw) - |u|^{p-2}u)z\right|\\
\le\;& c_1\int_\Omega (1+|v|+|u_1+tw_1|+|u_1|)^{p-1-r}|tw_1|^r|z_1|\\
&+C\lambda\int_\Omega (|u+tw|+|u|)^{p-2}|tw||z|\\
\le\;& c_1|t|^r\int_\Omega (1+|v|+|u_1+tw_1|+|u_1|+|w_1|)^{p-1}|z_1|\\
&+C\lambda|t|\,\|w\|_{p,\varepsilon}\|z\|_{p,\varepsilon}\left(\int_\Omega (|u+tw|+|u|)^p\right)^{\frac{p-2}{p}} \;\to\; 0
\end{aligned}$$

for $t \to 0$ and all $(u,u_1),(w,w_1),(z,z_1) \in V$, where c_1 and C are the positive constants ocurring in (3.8) and (3.17). Hence T_v is hemicontinuous.

(c) T_v coercive:

$$\langle T_v(u,u_1),(u,u_1)\rangle = \int_\Omega a(v,u_1)\cdot u_1 + \lambda|u|^p \geq \int_\Omega c_2|u_1|^p + \lambda|u|^p \geq c\|(u,u_1)\|_X^p$$

for a positive constant c and all $(u,u_1)\in V$. Since $p\geq 2$, T_v is coercive.

- $1<p<2$: In this case we need the second estimate of Proposition 6.1 below, where c is a strictly positive constant only depending on p:

$$c\,(|a|+|b|)^{p-2}(a-b)^2 \;\leq\; \left(|a|^{p-2}a-|b|^{p-2}b\right)(a-b) \;\leq\; 2\,|a-b|^p\,. \qquad (3.18)$$

(a) T_v strictly monotone:

$$\begin{aligned}
&\langle\!\langle T_v(u,u_1)-T_v(w,w_1),(u,u_1)-(w,w_1)\rangle\!\rangle\\
\geq\;& c_2\int_\Omega (1+|v|+|u_1|+|w_1|)^{p-2}|u_1-w_1|^2 + c\lambda\int_\Omega (|u|+|w|)^{p-2}|u-w|^2\\
\geq\;& c_2\left(\int_\Omega |u_1-w_1|^p\right)^{2/p}\cdot\left(\int_\Omega (1+|v|+|u_1|+|w_1|)^p\right)^{\frac{p-2}{p}}\\
&+c\lambda\left(\int_\Omega |u-w|^p\right)^{2/p}\cdot\left(\int_\Omega (|u|+|w|)^p\right)^{\frac{p-2}{p}} \;>\; 0
\end{aligned}$$

for $(u,u_1)\neq(w,w_1)$, where we used (3.9),(3.18) and the reverse Hölder inequality for the dual exponents $p/2\in(0,1)$ and $p/(p-2)<0$.

(b) T_v hemicontinuous: This can be shown precisely as in the case $p\geq 2$, where this time we use

$$\begin{aligned}
\lambda\left|\int_\Omega (|u+tw|^{p-2}(u+tw)-|u|^{p-2}u)z\,d\mu_\varepsilon\right| \;&\leq\; 2\lambda\int_\Omega |tw|^{p-1}|z|\,d\mu_\varepsilon\\
&\leq\; 2\lambda|t|^{p-1}\|z\|_{p,\varepsilon}\|w\|_{p,\varepsilon}^{p/p'} \;\to\; 0
\end{aligned}$$

for $t\to 0$ and all $(u,u_1),(w,w_1),(z,z_1)\in V$, where we used (3.18).

(c) T_v coercive:

$$\begin{aligned}
\langle\!\langle T_v(u,u_1),(u,u_1)\rangle\!\rangle \;&\geq\; c_2\int_\Omega ((1+|v|+|u_1|)^{p-2}|u_1|^2+\lambda|u|^p)\,d\mu_\varepsilon\\
&\geq\; c_2(\|1\|_{p,\varepsilon}+\|v\|_{p,\varepsilon}+\|u_1\|_{p,\varepsilon})^{p-2}\|u_1\|_{p,\varepsilon}^2+\lambda\|u\|_{p,\varepsilon}^p\,,
\end{aligned}$$

where we used (3.9), the reversed Hölder inequality as in (a), and the fact that $p<2$. If we set $c:=\min\{c_2,\lambda\}$ and $k:=\|1\|_{p,\varepsilon}+\|v\|_{p,\varepsilon}>0$, we get

$$\frac{\langle\!\langle T_v(u,u_1),(u,u_1)\rangle\!\rangle}{\|(u,u_1)\|_V} \geq c\left[\frac{\|u\|_{p,\varepsilon}^p}{\|(u,u_1)\|_V}+\frac{(k+\|u_1\|_{p,\varepsilon})^p}{\|(u,u_1)\|_V}\cdot\left(\frac{\|u_1\|_{p,\varepsilon}}{k+\|u_1\|_{p,\varepsilon}}\right)^2\right],$$

where the right-hand side converges to $+\infty$ as $\|(u,u_1)\|_V\to\infty$, since $p>1$.

By the Browder-Minty theorem, we get that T_v is bijective, hence for any $v\in X$ there exists a unique pair $(u,u_1)\in V$, including uniqueness of the gradient, such that

$$\int_\Omega a(v,u_1)\cdot\varphi_1+\lambda|u|^{p-2}u\varphi\,d\mu_\varepsilon = \int_\Omega f(v)\varphi\,d\mu_\varepsilon \quad \forall(\varphi,\varphi_1)\in V. \qquad (3.19)$$

In what follows we are looking for a fixed point of the solution operator

$$L : V \to V, (v, v_1) \mapsto (u, u_1),$$

which does not depend on v_1 and is well defined by the considerations above. It is clear by construction that such a fixed point is a solution of equation (3.1) in the sense of Definition 3.1.2. First of all we show that there exists a constant R, only depending on $|\Omega|, c_2, p, \beta, \lambda$, but not on ε, such that

$$L : \bar{B}_R \subset V \to \bar{B}_R\,, \quad \bar{B}_R := \overline{B_R(0)}. \tag{3.20}$$

We choose (u, u_1) as a test function in (3.19), and using (3.9) we get

$$c_2\left(\|1\|_{p,\varepsilon} + \|v\|_{p,\varepsilon} + \|u_1\|_{p,\varepsilon}\right)^{p-\alpha}\|u_1\|^{\alpha}_{p,\varepsilon} + \lambda\|u\|^p_{p,\varepsilon} \;\le\; \|u\|_{p,\varepsilon}\|f(v)\|_{p',\varepsilon}\,, \tag{3.21}$$

where for $p < 2$ we argued as in (c) above. To estimate the second factor on the right-hand side in (3.21), we use the fact that $|\mu_\varepsilon(\Omega)| \le C$ independent of ε by (2.2), and assumption (3.10), which also implies $\beta p' < (p-1)p' = p$:

$$\begin{aligned}
\|f(v)\|_{p',\varepsilon} &\le C(p)\|1 + |v|^\beta\|_{p',\varepsilon} \;\le\; C(p, |\Omega|)(1 + \|\,|v|^\beta\|_{p',\varepsilon}) \\
&\le C(p, \beta, |\Omega|)(1 + \|v\|^\beta_{p,\varepsilon}).
\end{aligned} \tag{3.22}$$

For $p \ge 2$ (i.e. $\alpha = p$) we combine (3.21) and (3.22), Young's inequality for p, p', and standard absorption techniques, to get

$$\|(u, u_1)\|^p_V \;\le\; C(1 + \|v\|^{\beta p'}_{p,\varepsilon}) \;\le\; \tilde{C} + \tfrac{1}{2}\|v\|^p_{p,\varepsilon} \;\le\; \tilde{C} + \tfrac{1}{2}\|(v, v_1)\|^p_V\,, \tag{3.23}$$

where the constant $\tilde{C}$ depends only on c_2, p, β, λ and $|\Omega|$. Hence choosing $R := (2\tilde{C})^{1/p}$ we get (3.20). For $p < 2$ we deduce from (3.21) and (3.22)

$$\|u_1\|^p_{p,\varepsilon}\left(\frac{\|u_1\|_{p,\varepsilon}}{\|1\|_{p,\varepsilon} + \|v\|_{p,\varepsilon} + \|u_1\|_{p,\varepsilon}}\right)^2 + \|u\|^p_{p,\varepsilon} \;\le\; C\,(1 + \|v\|^{\beta p'}_{p,\varepsilon})$$

for a constant C only depending on c_2, p, β, λ and $|\Omega|$. Note that for any positive constant $\hat{c} > 0$ we have, whenever $\|u_1\|_{p,\varepsilon} \ge 1$,

$$\left(\frac{\|u_1\|_{p,\varepsilon}}{\|1\|_{p,\varepsilon} + \hat{c}\|u_1\|_{p,\varepsilon}}\right)^2 \;\ge\; C(\hat{c}, p, |\Omega|) \;>\; 0,$$

since $\mu_\varepsilon(\Omega) \to |\Omega|$. After a simple distinction of cases $\hat{c}\|u_1\|_{p,\varepsilon} \ge \|v\|_{p,\varepsilon}$ and $\hat{c}\|u_1\|_{p,\varepsilon} \le \|v\|_{p,\varepsilon}$ for a suitable constant $\hat{c} > 0$, we obtain

$$\|(u, u_1)\|^p_V \;\le\; \tilde{C} + \tfrac{1}{2}\|(v, v_1)\|^p_V$$

as in (3.23) and find $R > 0$, such that (3.20) holds. If we show that L is compact (and continuous), we can apply Schauder's fixed point theorem, and Theorem 3.1.4 is proven, together with estimate (3.14) thanks to (3.20). We have to show that $L(\bar{B}_R)$ is precompact in V, so let (u^n, u^n_1) be a sequence in

$L(\bar{B}_R)$ and (v^n, v_1^n) a sequence in $\bar{B}_R$ with $(u^n, u_1^n) = L(v^n, v_1^n)$. Recall that by (2.62) and (2.65) we have the orthogonal decomposition

$$v_1^n(x) = \nabla_{\mu_\varepsilon} v^n(x) + \tau^n(x), \quad \tau^n \in \Gamma^p(\Omega, d\mu_\varepsilon)$$

μ_ε-almost everywhere in Ω. Hence we get $\|v^n\|_{p,\varepsilon} + \|\nabla_{\mu_\varepsilon} v^n\|_{p,\varepsilon} \leq R$, and consequently by the assumption on μ and Lemma 2.4.1

$$v^n \to v \quad \text{strongly in } X \tag{3.24}$$

for a subsequence and some $v \in X$. Since $L(\bar{B}_R) \subset \bar{B}_R$, with the same argument we get, after possibly passing to another subsequence, that

$$u^n \to z \quad \text{strongly in } X, \;\; u_1^n \rightharpoonup \xi \quad \text{weakly in } X^d \tag{3.25}$$

for some $z \in X, \xi \in X^d$. We emphasize that ξ is a gradient of z, since $\bar{B}_R$ is weakly sequentially closed in V. We set $(z, z_1) := (z, \xi)$. Note that since $\{v^n\}$ and $\{u_1^n\}$ are bounded sequences in X and X^d respectively, we have

$$\|(1 + |v^n| + |u_1^n| + |z_1|)\|_{p,\varepsilon}^{p-\alpha} \geq C > 0 \tag{3.26}$$

for a some constant C independent of n. For this constant we claim that

$$\begin{aligned} c_2 C \|u_1^n - z_1\|_{p,\varepsilon}^{\alpha} &\leq c_2 \|u_1^n - z_1\|_{p,\varepsilon}^{\alpha} \, \|(1 + |v^n| + |u_1^n| + |z_1|)\|_{p,\varepsilon}^{p-\alpha} \\ &\leq c_2 \int_\Omega (1 + |v^n| + |u_1^n| + |z_1|)^{p-\alpha} |u_1^n - z_1|^{\alpha} \\ &\leq \int_\Omega (a(v^n, u_1^n) - a(v^n, z_1)) \cdot (u_1^n - z_1) \overset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.27}$$

Indeed for the estimates in (3.27) we have used (3.9),(3.26) and the reversed Hölder inequality for $p < 2$. To show the convergence in (3.27), we use the solution property (3.19) and the first convergence in (3.25) to get

$$\int_\Omega a(v^n, u_1^n) \cdot (u_1^n - z_1) = \int_\Omega (f(v^n) - \lambda |u^n|^{p-2} u^n)(u^n - z) \to 0$$

for $n \to \infty$. On the other hand by (3.11),(3.24) and the continuity of a with respect to the s-variable, we deduce $a(v^n, z_1) \to a(v, z_1)$ strongly in X', hence by the second convergence in (3.25)

$$\int_\Omega a(v^n, z_1) \cdot (z_1 - u_1^n) \to 0 \quad \text{for } n \to \infty\,.$$

This shows (3.27), which together with (3.25) implies $(u^n, u_1^n) \to (z, z_1)$ strongly in V for a subsequence. Hence $L(\bar{B}_R)$ is precompact in V. To show the continuity of L, let

$$(v^n, v_1^n) \to (v, v_1) \text{ strongly in } \bar{B}_R, \quad (u^n, u_1^n) := L(v^n, v_1^n).$$

As shown above, for any subsequence there exists a further subsequence, still denoted by n, such that $(u^n, u_1^n) \to (z, z_1)$ strongly in $\bar{B}_R$. Using (3.11) and the continuity of a with respect to the slow variables, it is easy to check that $a(v^n, u_1^n) \to a(v, z_1)$ strongly in X'. Making use of the solution property (3.19), we get for any pair $(\varphi, \varphi_1) \in V$:

$$\begin{array}{rcl} \displaystyle\int_\Omega a(v^n, u_1^n) \cdot \varphi_1 + \lambda |u^n|^{p-2} u^n \varphi & = & \displaystyle\int_\Omega f(v^n)\varphi \\ \downarrow (n \to \infty) & & \downarrow \\ \displaystyle\int_\Omega a(v, z_1) \cdot \varphi_1 + \lambda |z|^{p-2} z \varphi & = & \displaystyle\int_\Omega f(v)\varphi \,, \end{array}$$

and therefore $(z, z_1) = L(v, v_1)$, since the operator T_v above is injective. It follows that $L : \bar{B}_R \to \bar{B}_R$ is continuous, which completes the proof. □

The following remark shows that the regularizing term $\lambda |u_\varepsilon|^{p-2} u_\varepsilon$ can be omitted for strongly connected measures μ. Recall that strong connectedness implies the p-Poincaré inequality for μ by Remark 2.3.10.

Remark 3.1.5. *If μ is strongly p-connected on $\mathbb{R}^d$, Theorem 3.1.4 still holds for $\lambda = 0$, that means there exists a weak solution $(u_\varepsilon, \nabla u_\varepsilon) \in V^p(\Omega, d\mu_\varepsilon)$ of the equation*

$$-\mathrm{div}\, a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon) = f(\tfrac{x}{\varepsilon}, x, u_\varepsilon)\,. \tag{3.28}$$

The uniform estimate (3.14) remains valid, only the constant C depends additionally on the Poincaré constant c_p of estimate (2.92).

Proof. Given a pair $(u, u_1) \in V$ we can uniquely identify u as an element of $H_0^{1,p}(\Omega, d\mu_\varepsilon)$, and by Lemma 2.4.2 we get $\|u\|_{p,\varepsilon} \le c_p \|u_1\|_{p,\varepsilon}$, in particular

$$\|u_1\|_{p,\varepsilon} \ \ge \ \frac{1}{1 + c_p} \|(u, u_1)\|_V \,, \tag{3.29}$$

which ensures the coercivity and the strict monotonicity of the operator T_v above for $\lambda = 0$. Estimate (3.29) is of course also sufficient to apply the standard absorption techniques that lead to (3.20). □

Now we turn to the homogenization of equation (3.1), that means we pass to the limit in the integral identity (3.13). To this end we heavily rely on the structure and compactness results proven in Section 2.4, especially on Theorem 2.4.4 and Theorem 2.4.5.

Proposition 3.1.6. *Let μ be strongly p-connected on $\mathbb{R}^d$ and doubling, and let $(u_\varepsilon, \nabla u_\varepsilon)$ be a weak solution of equation (3.1) for $\lambda \ge 0$. Then there exist functions $u \in W_0^{1,p}(\Omega)$, $\tilde{u}_1 \in L^p(\Omega; \tilde{H}_\mu^{1,p}(\mathbb{T}))$ and $a_0 \in L_m^{p'}(\Omega \times Y)$ such that*

$$u_\varepsilon \twoheadrightarrow u(x) \quad \textit{two-scale strongly in } L^p(\Omega, d\mu_\varepsilon), \tag{3.30}$$

$$\nabla u_\varepsilon \rightharpoonup\!\!\rightharpoonup \nabla u(x) + \nabla_y \tilde{u}_1(x, y) \quad \textit{two-scale in } L^p(\Omega, d\mu_\varepsilon)^d, \tag{3.31}$$

$$a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup\!\!\rightharpoonup a_0(x, y) \qquad \textit{two-scale in } L^{p'}(\Omega, d\mu_\varepsilon)^d \tag{3.32}$$

up to subsequences, where $\nabla_y \tilde{u}_1(x, \cdot) \in V_{\mathrm{pot}}^p(\mathbb{T}, d\mu)$ is a gradient of $\tilde{u}_1(x, \cdot)$.

Proof. The statements in (3.30) and (3.31) are an immediate consequence of the uniform a priori estimate (3.14) combined with Theorems 2.4.4–2.4.5, whereas (3.32) follows from (3.12),(3.14) and the weak compactness property of two-scale convergence. □

We mention one of the difficulties in the homogenization step. As suggested by (3.31), the monotonicity of a and Proposition 2.1.13, when passing to the limit in (3.13) we would like to show

$$a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \xi) \twoheadrightarrow a(y, x, u, \xi) \quad \text{two-scale strongly} \tag{3.33}$$

for any fixed $\xi \in \mathbb{R}^d$, where $u \in W_0^{1,p}(\Omega)$ is given by Proposition 3.1.6. Although we have $u_\varepsilon \twoheadrightarrow u$ for strongly connected measures, it is not clear that (3.33) can be expected, especially when a is merely continuous in u and does not separate in its variables (cf. Remark 2.1.9). Even if u is smooth and a separates in ξ, we can not expect an asymptotic behaviour of the form

$$\|a(\tfrac{x}{\varepsilon}, x, u_\varepsilon) - a(\tfrac{x}{\varepsilon}, x, u)\|_{q,\varepsilon} = o(1) \quad \text{as } \varepsilon \to 0\,, \tag{3.34}$$

for a suitable $q > 1$. We have a chance to show (3.34) if the flux a admits an uniform modulus of continuity with respect to the s-variable. To this end, for the case $\mu = \mathcal{L}^d$, it was assumed in [32] that a is locally Lipschitz continuous in s. However, we show that it suffices to require local Hölder continuity in s with arbitrary small exponent.

Assumption 3.1.7. *There exist $\gamma, \tilde{\gamma} \in (0, r_0]$, a function $\tilde{h} \in L^{\tilde{q}}_\mu(\mathbb{T}; \mathcal{C}(\overline{\Omega}))$ and a constant c_3, such that for all $(y, x, \xi) \in \mathbb{R}^d \times \Omega \times \mathbb{R}^d$ and $s_1, s_2 \in \mathbb{R}$:*

$$|a(y,x,s_1,\xi) - a(y,x,s_2,\xi)| \;\leq\; c_3\,(1 + |s_1| + |s_2| + |\xi|)^{p-1-\gamma}|s_1 - s_2|^\gamma, \tag{3.35}$$

$$|f(y,x,s_1) - f(y,x,s_2)| \;\leq\; |\tilde{h}(y,x)|\,|s_1 - s_2|^{\tilde{\gamma}}, \tag{3.36}$$

with $r_0 := \min\{1, p-1\}$ as in (3.6) and $\tilde{q} := p(p - \tilde{\gamma})^{-1} \in (1, p']$.

Under the additional Assumption 3.1.7 we are able to prove the main result of this chapter, the homogenization of quasilinear monotone elliptic operators on strongly connected multidimensional structures.

Theorem 3.1.8. *Let μ be strongly p-connected on $\mathbb{R}^d$ and doubling, and let $u \in W_0^{1,p}(\Omega)$, $\tilde{u}_1 \in L^p(\Omega, \tilde{H}^{1,p}_\mu(\mathbb{T}))$ and $a_0 \in L^{p'}_m(\Omega \times Y)^d$ as in Proposition 3.1.6. Then under Assumption 3.1.7 there holds*

$$a_0(x,y) = a(y, x, u(x), \nabla u(x) + \nabla_y \tilde{u}_1(x,y)) \tag{3.37}$$

and the pair $(u, \tilde{u}_1)$ is a solution of the two-scale homogenized problem

$$\int_{\Omega\times Y} a(y,x,u,\nabla u + \nabla_y \tilde{u}_1(y)) \cdot [\nabla\phi + \nabla_y \phi_1(y)]\,dm$$
$$+\lambda \int_\Omega |u|^{p-2} u\phi\,dx = \int_\Omega \bar{f}(x,u)\phi\,dx \tag{3.38}$$

for all $(\phi, \phi_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbb{T}))$, where $\bar{f}(\cdot) := \int_Y f(y,\cdot)\,d\mu(y)$.

Proof. For any pair (ϕ,ϕ_1) given in (3.38), we define a suitable test function $\varphi_\varepsilon \in \mathcal{D}(\Omega)$ in (3.13) by

$$\varphi_\varepsilon(x) := \phi(x) + \varepsilon\phi_1(x,\tfrac{x}{\varepsilon}), \quad \nabla\varphi_\varepsilon(x) = \nabla\phi(x) + \varepsilon\nabla_x\phi_1(x,\tfrac{x}{\varepsilon}) + \nabla_y\phi_1(x,\tfrac{x}{\varepsilon}), \quad (3.39)$$

and pass to the limit. We treat the three terms occurring in (3.13) separately. Our first claim is

$$\lim_{\varepsilon\to 0}\int_\Omega f(\tfrac{x}{\varepsilon},x,u_\varepsilon)\varphi_\varepsilon\,d\mu_\varepsilon = \int_\Omega \bar f(x,u)\phi\,dx. \quad (3.40)$$

Using (3.11) and (3.14) it is easy to check that $\|f(\tfrac{\cdot}{\varepsilon},\cdot,u_\varepsilon(\cdot))\|_{p',\varepsilon} \le C$ uniformly and, as a consequence,

$$\int_\Omega f(\tfrac{x}{\varepsilon},x,u_\varepsilon)\varphi_\varepsilon\,d\mu_\varepsilon = \int_\Omega f(\tfrac{x}{\varepsilon},x,u_\varepsilon)\phi\,d\mu_\varepsilon + o(1) \quad \text{as } \varepsilon\to 0 \quad (3.41)$$

by Proposition 2.1.4. Since $u \in W^{1,p}(\Omega)$ is in general not continuous, we need to choose a sequence of functions $\{\psi_\delta\} \subset \mathcal{D}(\Omega)$ with $\psi_\delta \to u$ strongly in $L^p(\Omega)$. By (3.36) and the choice of $\tilde q$ we get

$$\begin{aligned}\left|\int_\Omega (f(\tfrac{x}{\varepsilon},x,u_\varepsilon) - f(\tfrac{x}{\varepsilon},x,\psi_\delta))\phi\,d\mu_\varepsilon\right| &\le C\|\tilde h(\tfrac{\cdot}{\varepsilon},\cdot)\|_{\tilde q,\varepsilon}\,\|\,|u_\varepsilon-\psi_\delta|^{\tilde\gamma}\,\|_{\tilde q',\varepsilon}\\ &\le C\|\tilde h(\tfrac{\cdot}{\varepsilon},\cdot)\|_{\tilde q,\varepsilon}\,\|u_\varepsilon-\psi_\delta\|^{\tilde\gamma}_{p,\varepsilon}, \end{aligned} \quad (3.42)$$

where we used $\tilde\gamma\tilde q' = p$. If we fix $\delta > 0$, then precisely as in the proof of Proposition 2.1.13, we get using (3.30), the Clarkson inequalities and Proposition 2.1.10:

$$\limsup_{\varepsilon\to 0}\|u_\varepsilon-\psi_\delta\|^{\tilde\gamma}_{p,\varepsilon} \le \begin{cases} C\left(\frac12\|u\|_p^p + \frac12\|\psi_\delta\|_p^p - \|\frac{u+\psi_\delta}{2}\|_p^p\right)^{\tilde\gamma/p} & p\ge 2,\\ C\left([\frac12\|u\|_p^p + \frac12\|\psi_\delta\|_p^p]^{\frac{1}{p-1}} - \|\frac{u+\psi_\delta}{2}\|_p^{p'}\right)^{\tilde\gamma/p'} & p<2,\end{cases}$$

where $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$. Since $\psi_\delta \to u$ in $L^p(\Omega)$, combining the last estimate with (3.42) and applying Example 2.1.12 to the function $\tilde h$ yields:

$$\limsup_{\delta\to 0}\left(\limsup_{\varepsilon\to 0}\left|\int_\Omega (f(\tfrac{x}{\varepsilon},x,u_\varepsilon) - f(\tfrac{x}{\varepsilon},x,\psi_\delta))\phi\,d\mu_\varepsilon\right|\right) = 0. \quad (3.43)$$

Note that the function $(y,x) \mapsto f(y,x,\psi_\delta(x))$ belongs to $L^{p'}_\mu(\mathbb{T};\mathcal{C}(\overline\Omega))$ by (3.10) and the assumptions on f. By Remark 2.1.8, for any $\delta > 0$ we get

$$\lim_{\varepsilon\to 0}\int_\Omega f(\tfrac{x}{\varepsilon},x,\psi_\delta)\phi\,d\mu_\varepsilon = \int_\Omega \bar f(x,\psi_\delta)\phi\,dx \xrightarrow{\delta\to 0} \int_\Omega \bar f(x,u)\phi\,dx, \quad (3.44)$$

the latter convergence thanks to (3.11) and the continuity of f with respect to the last variable. Combining (3.41) with (3.43) and (3.44) we obtain (3.40). Using (3.30) and the monotonicity of the function $g : t \mapsto |t|^{p-2}t$, we can easily show

$$\lim_{\varepsilon\to 0}\lambda\int_\Omega |u_\varepsilon|^{p-2}u_\varepsilon\varphi_\varepsilon\,d\mu_\varepsilon = \lambda\int_\Omega |u|^{p-2}u\phi\,dx \quad (3.45)$$

with the methods from the proof of Lemma 2.1.14. Now (3.32) immediately gives

$$\int_\Omega a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \varphi_\varepsilon(x)\, d\mu_\varepsilon \to \int_{\Omega\times Y} a_0(x,y)\cdot [\nabla\phi(x) + \nabla_y \phi_1(x,y)]\, dm. \quad (3.46)$$

Hence if we show (3.37), the proof of the theorem is complete. To this end, for arbitrary $\psi \in \mathcal{C}(\overline{\Omega}\times\mathbb{T})^d$ and $t\in(-1,1)$ we define the vector functions

$$w_t(x,y) := \nabla\phi(x) + \nabla_y\phi_1(x,y) + t\psi(x,y), \quad w_{\varepsilon,t}(x) := w_t(x, \tfrac{x}{\varepsilon})$$

with (ϕ, ϕ_1) as above. Then by (3.9) we get

$$0 \le \int_\Omega [a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon) - a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, w_{\varepsilon,t})]\cdot[\nabla u_\varepsilon - w_{\varepsilon,t}]\, d\mu_\varepsilon\,. \quad (3.47)$$

We pass to the limit on the right-hand side in (3.47), treating each term separately. Using (3.12) and the definition of the space $V^p(\Omega, d\mu_\varepsilon)$, by density we can choose $\varphi = u_\varepsilon$ in (3.13) and obtain

$$\begin{aligned}\int_\Omega a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla u_\varepsilon\, d\mu_\varepsilon &= \int_\Omega f(\tfrac{x}{\varepsilon}, x, u_\varepsilon) u_\varepsilon\, d\mu_\varepsilon - \lambda\int_\Omega |u_\varepsilon|^p\, d\mu_\varepsilon \\ &\to \int_\Omega \bar f(x,u)u\, dx - \lambda \int_\Omega |u|^p\, dx, \quad (3.48)\end{aligned}$$

where we used (3.30),(3.40) and applied Proposition 2.1.13. For the second term in (3.47) we get

$$\int_\Omega a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, \nabla u_\varepsilon)\cdot w_{\varepsilon,t}\, d\mu_\varepsilon \to \int_{\Omega\times Y} a_0(x,y)\cdot w_t(x,y)\, dm \quad (3.49)$$

by (3.32) and the definition of $w_{\varepsilon,t}$. In order to treat the other two terms in (3.47) we have to use the arguments in the proof of (3.40). To shorten the notation, we set

$$I_\varepsilon^\delta := \int_\Omega [a(\tfrac{x}{\varepsilon}, x, \psi_\delta, w_{\varepsilon,t}) - a(\tfrac{x}{\varepsilon}, x, u_\varepsilon, w_{\varepsilon,t})]\cdot(\nabla u_\varepsilon - w_{\varepsilon,t})\, d\mu_\varepsilon\,.$$

Similar as in (3.42), using the a priori estimate (3.14) and the Hölder assumption (3.35) on a we get:

$$\begin{aligned}|I_\varepsilon^\delta| &\le C\left(\int_\Omega (1+|\psi_\delta| + |u_\varepsilon| + |w_{\varepsilon,t}|)^{(p-1-\gamma)p'} |u_\varepsilon - \psi_\delta|^{\gamma p'}\, d\mu_\varepsilon\right)^{1/p'} \\ &\le C\, \|(1+|\psi_\delta| + |u_\varepsilon| + |w_{\varepsilon,t}|)\|_{p,\varepsilon}^{p-\gamma p'}\, \|u_\varepsilon - \psi_\delta\|_{p,\varepsilon}^\gamma, \quad (3.50)\end{aligned}$$

where in the last estimate we applied the Hölder inequality with exponent $r = p/\gamma p'$ and its dual r'. With the same reasoning as in (3.43) we get

$$\limsup_{\delta\to 0}\left(\limsup_{\varepsilon\to 0} |I_\varepsilon^\delta|\right) = 0\,. \quad (3.51)$$

Using the regularity assumptions on a and the definition of ψ_δ and w_t, it is easy to check that the function $(y,x) \mapsto a(y,x,\psi_\delta(x), w_t(x,y))$ belongs to the class $L^{p'}_\mu(\mathbb{T}; \mathcal{C}(\overline{\Omega}))$. Hence by Example 2.1.12, Proposition 2.1.13 and (3.31) we get

$$\int_\Omega a(\tfrac{x}{\varepsilon}, x, \psi_\delta, w_{\varepsilon,t}) \cdot (w_{\varepsilon,t} - \nabla u_\varepsilon)\, d\mu_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega\times Y} a(y,x,\psi_\delta, w_t) \cdot (w_t - \nabla u - \nabla_y \tilde{u}_1)\, dm \tag{3.52}$$

for any fixed $\delta > 0$. Moreover by (3.11) and the continuity of a with respect to the third variable we get

$$\int_{\Omega\times Y} a(y,x,\psi_\delta, w_t) \cdot (w_t - \nabla u - \nabla_y \tilde{u}_1)\, dm \to \int_{\Omega\times Y} a(y,x,u,w_t) \cdot (w_t - \nabla u - \nabla_y \tilde{u}_1)\, dm \tag{3.53}$$

as $\delta \to 0$. Now if we combine the convergences (3.48)-(3.53) and pass to the limit in (3.47), we obtain

$$\begin{aligned} 0 &\le \int_\Omega (\bar{f}(x,u)u - \lambda |u|^p)\, dx - \int_{\Omega\times Y} (a_0 \cdot w_t - a(u,w_t) \cdot [w_t - \nabla u - \nabla_y \tilde{u}_1])\, dm \\ &= \int_{\Omega\times Y} [a_0(x,y) - a(y,x,u,w_t)] \cdot [\nabla u + \nabla_y \tilde{u}_1 - w_t]\, dm. \end{aligned} \tag{3.54}$$

Let us justify the equality in (3.54). If we combine (3.14),(3.40),(3.45) and (3.46) we deduce for any given pair (ϕ, ϕ_1) as above:

$$\int_{\Omega\times Y} a_0(x,y) \cdot [\nabla\phi + \nabla_y \phi_1]\, dm = \int_\Omega (\bar{f}(x,u) - \lambda |u|^{p-2} u)\phi\, dx. \tag{3.55}$$

In particular, if we choose an approximating sequence $\{\phi_\delta\} \subset \mathcal{D}(\Omega)$ and, by Definition 2.4.3 a sequence $\{\phi_{1,\delta}\} \subset \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{T})$ with

$$\phi_\delta \to u \ \text{ strongly in } H_0^{1,p}(\Omega), \quad \nabla_y \phi_{1,\delta} \to \nabla_y \tilde{u}_1 \ \text{ strongly in } L^p_m(\Omega \times Y), \tag{3.56}$$

we can pass to the limit $\delta \to 0$ in (3.55) and obtain, since $a_0 \in L^{p'}_m(\Omega \times Y)$ and $\bar{f}(x,u), |u|^{p-2}u \in L^{p'}(\Omega)$,

$$\int_{\Omega\times Y} a_0(x,y) \cdot [\nabla u + \nabla_y \tilde{u}_1]\, dm = \int_\Omega (\bar{f}(x,u)u - \lambda |u|^p)\, dx, \tag{3.57}$$

which shows (3.54). We can choose $\phi = \phi_\delta$ and $\phi_1 = \phi_{1,\delta}$ as above in the definition of w_t and pass to the limit $\delta \to 0$ in (3.54). This yields

$$0 \le t \int_{\Omega\times Y} [a(y,x,u,\nabla u + \nabla_y \tilde{u}_1 + t\psi) - a_0(x,y)] \cdot \psi(x,y)\, dm \tag{3.58}$$

by (3.12),(3.54),(3.56) and the continuity of a with respect to the last variable. Then dividing by t (for $t > 0$ and $t < 0$ respectively) and passing to the limit $t \to 0$, we obtain

$$\int_{\Omega\times Y} [a(y,x,u(x),\nabla u(x) + \nabla_y \tilde{u}_1(x,y)) - a_0(x,y)] \cdot \psi(x,y)\, dm = 0$$

for each $\psi \in \mathcal{C}(\overline{\Omega} \times \mathbb{T})^d$, which shows (3.37) and completes the proof. □

Note that the Hölder condition (3.35) on the flux a can not hold for $\gamma > 0$ if a separates in the ξ-variable, which however is the case in some applications discussed in Section 3.2 below. Hence we introduce an alternative type of local Hölder continuity similar to the one imposed on the source f in (3.36), which still guarantees that the term $|I_\varepsilon^\delta|$ can be estimated from above by $\|u_\varepsilon - \psi_\delta\|_{p,\varepsilon}^\gamma$ as in (3.50).

Remark 3.1.9. *Theorem 3.1.8 still holds if (3.35) is substituted by the assumption*

$$|a(y,x,s_1,\xi) - a(y,x,s_2,\xi)| \ \le\ |h(y,x)|\,(1+|\xi|)^{p-1}\,|s_1 - s_2|^\gamma$$

for some $\gamma \in (0, r_0)$ and a function $h \in L^q_\mu(\mathbb{T};\mathcal{C}(\overline{\Omega}))$ with $q = pp'(p-\gamma p')^{-1}$.

Now we show that the two-scale homogenized problem (3.38) can be decoupled to obtain a single effective equation for u. The information encoded in the corrector function $\tilde{u}_1$ leads to a cell problem, that determines the effective flux $a^\star$. For any $p \in [1,\infty)$ we consider the space

$$\tilde{V}^p := \tilde{V}^p(\mathbb{T}) = \{(v, v_1) \in V^p(\mathbb{T}, d\mu) : \int_Y v\,d\mu = 0\}\,, \tag{3.59}$$

which is a closed subspace of $V^p(\mathbb{T}, d\mu)$, in particular a reflexive Banach space for $p > 1$. We emphasize that if μ is strongly p-connected on $\mathbb{T}$, then the map

$$\|\cdot\|_{\sim,p} \colon \tilde{V}^p \to \mathbb{R},\ (v, v_1) \mapsto \|v_1\|_{p,\mu,Y} \tag{3.60}$$

defines a norm equivalent to the one induced by $V^p(\mathbb{T}, d\mu)$.

Lemma 3.1.10. *Let $p > 1$ and μ be strongly p-connected on $\mathbb{T}$. Then for any triple $(x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$ there exists a unique solution of the cell problem*

$$-\mathrm{div}\, a(y,x,s,\xi + v_1(y)) = 0 \quad \text{in } \tilde{V}^p\,, \tag{3.61}$$

more precisely there exists a unique element $(v, v_1)(\cdot, x, s, \xi) \in \tilde{V}^p$, such that the following integral identity holds:

$$\int_Y a(y,x,s,\xi + v_1(y,x,s,\xi)) \cdot \varphi_1(y)\,d\mu(y) = 0 \quad \text{for each } (\varphi, \varphi_1) \in \tilde{V}^p\,. \tag{3.62}$$

Moreover there exists a constant C independent of $(x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$, such that the following uniform estimate holds:

$$\|(v, v_1)(\cdot, x, s, \xi)\|_{\sim,p} = \|v_1(\cdot, x, s, \xi)\|_{p,\mu,Y} \le C(1 + |s| + |\xi|). \tag{3.63}$$

Proof. We fix $(x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$ and define the operator

$$T := T_{(x,s,\xi)} : \tilde{V}^p \to (\tilde{V}^p)',\ (v, v_1) \mapsto T(v, v_1)\,,$$

$$\langle T(v_1), \varphi_1 \rangle := \langle\!\langle T(v, v_1), (\varphi, \varphi_1) \rangle\!\rangle = \int_Y a(y,x,s,\xi + v_1(y)) \cdot \varphi_1(y)\,d\mu(y)\,.$$

Using (3.11) it is easy to check that T is well defined. The first statement of the lemma follows if T is bijective. To this end it suffices to check the prerequisites of the Browder-Minty theorem. To shorten the notation we also abbreviate $a(s,\xi+v_1) := a(y,x,s,\xi+v_1(y))$ and $\|v_1\|_p := \|v_1\|_{p,\mu,Y}$.

- T strictly monotone:

$$\langle T(v_1)-T(w_1), v_1-w_1\rangle \;\geq\; C(1+|s|+|\xi|+\|v_1\|_p+\|w_1\|_p)^{p-\alpha}\|v_1-w_1\|_p^{\alpha} \;>\; 0$$

for $\alpha=\max\{p,2\}$ by (3.9), when $(v,v_1)\neq(w,w_1)$ in $\tilde{V}^p$.

- T hemicontinuous:

$$\begin{aligned}
&|\langle T((v,v_1)+t(w,w_1))-T(v,v_1),(z,z_1)\rangle| \\
\leq\;& c_1\int_Y (1+|s|+|\xi+v_1+tw_1|+|\xi+v_1|)^{p-1-r}|tw_1|^r|z_1|\,d\mu \\
\leq\;& C|t|^r\int_Y (1+|s|+|\xi+v_1|+(1+|t|)|w_1|)^{p-1}|z_1|\,d\mu \;\to\; 0
\end{aligned}$$

for $t\to 0$ and all $(v,v_1),(w,w_1),(z,z_1)\in\tilde{V}^p$, where c_1 occurs in (3.8).

- T coercive:

$$\langle T(v_1),v_1\rangle \;\geq\; \tfrac{c_2}{2}\|v_1\|_p^p+\int_Y [a(s,\xi+v_1)-a(s,v_1)]\cdot v_1\,d\mu\,, \tag{3.64}$$

where for $p<2$ we used the reversed Hölder inequality and assumed without restriction that $\|v_1\|_p\geq 3(1+|s|)$. We abbreviate $\Theta := 1+|s|+|\xi|$. With the help of (3.8) we can further estimate:

$$\begin{aligned}
\left|\int_Y [a(s,\xi+v_1)-a(s,v_1)]\cdot v_1\right| \;&\leq\; C\,\Theta^{p-1}\int_Y (1+|v_1|)^{p-1-r}|v_1|\,d\mu \\
&\leq\; \tfrac{c_2}{4}\|v_1\|_p^p+C\,,
\end{aligned}\tag{3.65}$$

respectively by Hölder's and Young's inequality, where the constant C in (3.65) depends only on $r,p,c_1,c_2,|s|$ and $|\xi|$. Combining (3.64) and (3.65) we easily deduce:

$$\frac{\langle T(v_1),v_1\rangle}{\|v_1\|_p} \;\geq\; \tfrac{c_2}{4}\|v_1\|_p^{p-1}-\tfrac{C}{\|v_1\|_p} \;\to\; +\infty \quad \text{for } \|v_1\|_p\to\infty.$$

It remains to show estimate (3.63). Let $v_1 := v_1(\cdot,x,s,\xi)\in L^p_\mu(\mathbb{T})^d$ be the second component of the solution of the cell problem. We first consider the case $p\geq 2$. By (3.8),(3.9) and the solution property of v_1 we get, again using Young's inequality,

$$\begin{aligned}
c_2\|\xi+v_1\|_p^p \;\leq\; \int_Y a(s,\xi+v_1)\cdot\xi\,d\mu \;&\leq\; C|\xi|\cdot\|(1+|s|+|\xi+v_1|)\|_p^{p-1} \\
&\leq\; \tfrac{c_2}{2}\|\xi+v_1\|_p^p+C(1+|s|+|\xi|)^p
\end{aligned}$$

with a constant C independent of s,ξ and v_1. As a consequence, we get

$$\|\xi+v_1(\cdot,x,s,\xi)\|_{p,\mu,Y} \;\leq\; C(1+|s|+|\xi|)\,, \tag{3.66}$$

which shows (3.63). Now we assume $p \in (1,2)$. Using (3.9), the solution property of v_1, and the reversed Hölder inequality it is easy to check that

$$\int_Y a(s, \xi + v_1) \cdot \xi \, d\mu \ \geq \ \tfrac{c_2}{4}(1 + |s| + \|\xi + v_1\|_p)^p \tag{3.67}$$

whenever $\|\xi + v_1\|_p \geq 1 + |s|$. On the other hand, using (3.8) as well as Hölder's and Young's inequality we can easily show that

$$\int_Y a(s, \xi + v_1) \cdot \xi \, d\mu \ \leq \ \tfrac{c_2}{8}(1 + |s| + \|\xi + v_1\|_p)^p + C|\xi|^p, \tag{3.68}$$

again with C not depending on s, ξ and v_1. Combining (3.67) and (3.68) we get $\|\xi + v_1\|_p \leq \max\{1 + |s|; C|\xi|\}$, which gives (3.66) and hence (3.63). □

The unique solvability of the cell problem enables us to define the effective flux $a^\star$, and with it the homogenized equation (cf. Corollary 3.1.14 below).

Definition 3.1.11. *The effective flux $a^\star : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is given by*

$$a^\star(x, s, \xi) := \int_Y a(y, x, s, \xi + v_1(y, x, s, \xi)) \, d\mu(y) \, , \tag{3.69}$$

where $v_1(\cdot, x, s, \xi) \in V^p_{\mathrm{pot}}(\mathbb{T}, d\mu)$ is for given $(x, s, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$ the second component of the solution of problem (3.62).

The cell problem given by Definition 3.1.11 coincides with the one found in [44, Section 6] related to the homogenization of equation (3.3). In our case however, the effective flux $a^\star$ additionally depends on $(x, s) \in \Omega \times \mathbb{R}$ in a nonlinear way. We first investigate the most important properties of $a^\star$. In the situation of Lemma 3.1.10, we denote by

$$\Lambda : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \to \tilde{V}^p, \ (x, s, \xi) \mapsto (v, v_1)(\cdot, x, s, \xi) \tag{3.70}$$

the nonlinear cell solution operator. Also recall the definition of the numbers $r_0 := \min\{1, p-1\}$ and $\alpha := \max\{p, 2\}$ introduced in (3.6).

Lemma 3.1.12. *Under the assumptions of Lemma 3.1.10, let the flux a, in addition to Assumption 3.1.1 and Assumption 3.1.7, satisfy the following Hölder condition with respect to the x-variable:*

$$|a(y, x_1, s, \xi) - a(y, x_2, s, \xi)| \ \leq \ C(1 + |s| + |\xi|)^{p-1-\gamma}|x_1 - x_2|^\gamma \, , \tag{3.71}$$

where $\gamma \in (0, r_0]$ is the exponent in (3.35). Then the operators Λ and $a^\star$ are continuous, and there exist constants $c_1^\star > 0, r^\star \in (0, r_0]$, such that for all $(x, s) \in \overline{\Omega} \times \mathbb{R}$ and any $\xi_1, \xi_2 \in \mathbb{R}^d$:

$$a^\star(x, s, 0) = 0, \tag{3.72}$$

$$|a^\star(x, s, \xi_1) - a^\star(x, s, \xi_2)| \leq c_1^\star(1 + |s| + |\xi_1| + |\xi_2|)^{p-1-r^\star}|\xi_1 - \xi_2|^{r^\star}. \tag{3.73}$$

Moreover if μ is also weakly p-connected on $\mathbb{R}^d$, then there exists a constant $c_2^\star > 0$, such that for all $(x, s) \in \overline{\Omega} \times \mathbb{R}$ and any $\xi_1, \xi_2 \in \mathbb{R}^d$:

$$(a^\star(x, s, \xi_1) - a^\star(x, s, \xi_2)) \cdot (\xi_1 - \xi_2) \ \geq \ c_2^\star (1 + |s| + |\xi_1| + |\xi_2|)^{p-\alpha}|\xi_1 - \xi_2|^\alpha. \tag{3.74}$$

Proof. To prove the continuity of Λ we consider a sequence $(x_n, s_n, \xi_n) \to (x, s, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$ as $n \to \infty$ and use the abbreviations

$$v_{1,n} := v_1(\cdot, x_n, s_n, \xi_n),\ v_1 := v_1(\cdot, x, s, \xi) \in V^p_{\mathrm{pot}}(\mathbb{T}, d\mu),$$

$\Lambda_n := \Lambda(\cdot, x_n, s_n, \xi_n), \Lambda := (\cdot, x, s, \xi) \in \tilde{V}^p$ and $\|\cdot\|_p := \|\cdot\|_{p,\mu,Y}$. To proceed further we introduce the following auxiliary term:

$$\kappa_n := 1 + |s| + \|\xi_n + v_{1,n}\|_p + \|\xi + v_1\|_p, \quad 0 < \kappa \le (\kappa_n)^{p-\alpha} \le 1 \quad \forall n \in \mathbb{N},$$

where the uniform lower bound on κ_n is guaranteed by estimate (3.66). By the definition of the norm in (3.60) it is easy to check that

$$(\kappa_n)^{p-\alpha}\|\Lambda_n - \Lambda\|^p_{\sim,p} \le C\left(|\xi_n - \xi|^\alpha + (\kappa_n)^{p-\alpha}\|\xi_n + v_{1,n} - [\xi + v_1]\|^\alpha_p\right). \quad (3.75)$$

It suffices to estimate the second term on the right-hand side in (3.75). For $p < 2$ we use the definition of κ_n and the reverse Hölder inequality, and in any case (3.9), the Hölder conditions (3.35) and (3.71), estimate (3.66), and the solution property (3.62) of $v_{1,n}$ and v_1 for the definition of the integral I_n below, and obtain

$$\begin{aligned} &(\kappa_n)^{p-\alpha}\|\xi_n + v_{1,n} - [\xi + v_1]\|^\alpha_p \\ \le\ & \tfrac{1}{c_2}\int_Y [a(x, s, \xi_n + v_{1,n}) - a(x, s, \xi + v_1)] \cdot [(\xi_n + v_{1,n}) - (\xi + v_1)]\, d\mu \\ \le\ & C\left(\Theta^{p-\gamma}(|x_n - x|^\gamma + |s_n - s|^\gamma) + I_n\right), \end{aligned} \quad (3.76)$$

where we have set $\Theta := (1 + |s| + |s_n| + |\xi| + |\xi_n|)$, and where the integral I_n is defined and can be estimated as follows:

$$\begin{aligned} I_n &:= \int_Y [a(x_n, s_n, \xi_n + v_{1,n}) - a(x, s, \xi + v_1)] \cdot [\xi_n - \xi]\, d\mu \\ &\le |\xi_n - \xi|\, \|a(x_n, s_n, \xi_n + v_{1,n}) - a(x, s, \xi + v_1)\|_{p'} \le C\,\Theta^{p-1}|\xi_n - \xi|, \end{aligned}$$

where we have used (3.11) and estimate (3.66) again. Combining the last estimate with (3.75) and (3.76) yields the continuity of Λ and the continuity of $a^\star$ easily follows. Now property (3.72) is a trivial consequence of (3.7) and the unique solvability of the cell problem. We claim that (3.73) holds with

$$r^\star := \tfrac{r}{\alpha - r} \in (0, r_0]\,. \quad (3.77)$$

Fix $(x, s) \in \overline{\Omega} \times \mathbb{R}, \xi_1, \xi_2 \in \mathbb{R}^d$. We set $v_1(\cdot, \xi_i) := v_1(\cdot, x, s, \xi_i)$ and define the following auxiliary functions:

$$\lambda_1(y) := 1 + |s| + |\xi_1 + v_1(y, \xi_1)| + |\xi_2 + v_1(y, \xi_2)| \quad (3.78)$$

$$\lambda_2(y) := \xi_1 + v_1(y, \xi_1) - [\xi_2 + v_1(y, \xi_2)]. \quad (3.79)$$

To shorten the notation we omit the dependence of a and $a^\star$ on x. Now using the definition of $a^\star$, property (3.8) and the Hölder inequality, we get

$$\begin{aligned} |a^\star(s, \xi_1) - a^\star(s, \xi_2)| &\le c_1 \int_Y |\lambda_1|^{p-1-r}|\lambda_2|^r\, d\mu \\ &\le c_1 \left\| |\lambda_1|^{p-1-\frac{rp}{\alpha}} \right\|_{\frac{\alpha}{\alpha - r}} \left\| |\lambda_1|^{r(\frac{p}{\alpha}-1)}|\lambda_2|^r \right\|_{\frac{\alpha}{r}} =: c_1 I \cdot J. \end{aligned}$$

From (3.9), the solution property of $v_1(\cdot,\xi_i)$ and the definition of $a^\star$ we deduce

$$J=\left(\int_Y |\lambda_1|^{p-\alpha}|\lambda_2|^\alpha\,d\mu\right)^{r/\alpha}\le C\,|a^\star(s,\xi_1)-a^\star(s,\xi_2)|^{r/\alpha}\,|\xi_1-\xi_2|^{r/\alpha},\quad(3.80)$$

where the constant C depends only on r,p and c_2. In order to estimate the other factor I it is important to note that

$$\frac{\alpha}{\alpha-r}\left(p-1-\frac{rp}{\alpha}\right)\;=\;p-1-r^\star$$

by the definition of $r^\star$ in (3.77). We can then use (3.80) and apply Young's inequality with dual exponents α/r and $\alpha/(\alpha-r)$ and obtain

$$\begin{aligned}|a^\star(s,\xi_1)-a^\star(s,\xi_2)|\;&\le\;C\,|a^\star(s,\xi_1)-a^\star(s,\xi_2)|^{r/\alpha}\,\|\lambda_1\|_{p-1-r^\star}^{p-1-\frac{rp}{\alpha}}\,|\xi_1-\xi_2|^{r/\alpha}\\&\le\;\tfrac12|a^\star(s,\xi_1)-a^\star(s,\xi_2)|+C\,\|\lambda_1\|_{p-1-r^\star}^{p-1-r^\star}\,|\xi_1-\xi_2|^{r^\star},\end{aligned}$$

where the generic constant C depends only on r,p,c_1 and c_2. Hence, in order to complete the proof of (3.73), it suffices to observe

$$\|\lambda_1\|_{p-1-r^\star}^{p-1-r^\star}\;\le\;\|\lambda_1\|_p^{p-1-r^\star}\;\le\;C\,(1+|s|+|\xi_1|+|\xi_2|)^{p-1-r^\star},\qquad(3.81)$$

where we used estimate (3.66). Now assume that μ satisfies $(H2)$ and $(H3)$ for the $p\in(1,\infty)$ under consideration. For given $\xi_1,\xi_2\in\mathbb{R}^d$, $\xi_1\neq\xi_2$ we set

$$z:=\frac{\xi_1-\xi_2}{|\xi_1-\xi_2|}\in\mathcal{S}^{d-1},\quad w:=\frac{v_1(y,\xi_1)-v_1(y,\xi_2)}{|\xi_1-\xi_2|}\in V^p_{\mathrm{pot}}(\mathbb{T},d\mu).$$

Then using these abbreviations, the definition of $a^\star$, the solution property of $v_1(\cdot,\xi_i)$ and (3.9), we get

$$\begin{aligned}[a^\star(s,\xi_1)-a^\star(s,\xi_2)]\cdot(\xi_1-\xi_2)\;&\ge\;c_2\int_Y|\lambda_1|^{p-\alpha}|\lambda_2|^\alpha\,d\mu\\&\ge\;\tilde c(1+|s|+|\xi_1|+|\xi_2|)^{p-\alpha}|\xi_1-\xi_2|^\alpha\|z+w\|^\alpha_{p,\mu,Y}\end{aligned}$$

for a strictly positive constant $\tilde c$, where for $p<2$ we used the reversed Hölder inequality and the second estimate in (3.81). We can apply Lemma 2.3.13 due to the connectedness assumptions on μ. Recall the definition of the function

$$j:\mathbb{R}^d\to\mathbb{R},\quad z\mapsto\inf\left\{\int_Y|z+\nabla u(y)|^p\,d\mu:\;u\in\mathcal{C}^\infty(\mathbb{T})\right\}$$

in (2.84). Since by definition $\mathcal{C}^\infty(\mathbb{T})$ is dense in $V^p_{\mathrm{pot}}(\mathbb{T},d\mu)$ with respect to the L^p_μ-norm, we deduce from Lemma 2.3.13 and the last estimate

$$\begin{aligned}[a^\star(s,\xi_1)-a^\star(s,\xi_2)]\cdot(\xi_1-\xi_2)\;&\ge\;\tilde c(1+|s|+|\xi_1|+|\xi_2|)^{p-\alpha}|\xi_1-\xi_2|^\alpha\,(j(z))^{\alpha/p}\\&\ge\;\tilde c(\hat c)^{\alpha/p}(1+|s|+|\xi_1|+|\xi_2|)^{p-\alpha}|\xi_1-\xi_2|^\alpha\end{aligned}$$

since $|z|=1$, and hence (3.74) holds with $c_2^\star:=\tilde c(\hat c)^{\alpha/p}$, where $\hat c$ is the positive constant in (2.86). □

One can also show that $a^\star$ inherits Hölder continuity in s from the coefficient a. For simplicity, we will only consider the case $p = 2$.

Lemma 3.1.13. *Let $p = 2$ and hence (3.8) be satisfied for some $r \in (0, 1]$. Then $a^\star$ inherits local Hölder continuity from assumption (3.35), more precisely there holds*

$$|a^\star(x, s_1, \xi) - a^\star(x, s_2, \xi)| \le C(1 + |s_1| + |s_2| + |\xi|)^{1-r\gamma}|s_1 - s_2|^{r\gamma}. \quad (3.82)$$

Proof. Let $(x, \xi) \in \Omega \times \mathbb{R}^d$ and $s_1, s_2 \in \mathbb{R}$ be given. By $v_1 := v_1(\cdot, x, s_1, \xi)$ and $\tilde{v}_1 := v_1(\cdot, x, s_2, \xi)$ we denote the second components of the corresponding unique solutions of (3.61). Then we have

$$\begin{aligned} |a^\star(x, s_1, \xi) - a^\star(x, s_2, \xi)| \;&\le\; \int_Y |a(s_1, \xi + \tilde{v}_1) - a(s_2, \xi + \tilde{v}_1)| \\ &\quad + \int_Y |a(s_1, \xi + v_1) - a(s_1, \xi + \tilde{v}_1)| \;=:\; I_1 + I_2. \end{aligned}$$

We have to estimate the two terms I_j. To this end, using (3.35),(3.66) and Hölder's inequality we first get

$$I_1 \le C|s_1 - s_2|^\gamma(1 + |s_1| + |s_2| + |\xi|)^{1-\gamma}, \quad (3.83)$$

where the constant C depends on c_3. Moreover we need to control the term $\|v_1 - \tilde{v}_1\|_{2,\mu,Y}$. Using (3.9),(3.35) and the solution property (3.62) we get

$$\begin{aligned} \|v_1 - \tilde{v}_1\|^2_{2,\mu,Y} \;&\le\; c_2^{-1} \int_Y [a(s_1, \xi + v_1) - a(s_1, \xi + \tilde{v}_1)] \cdot [v_1 - \tilde{v}_1]\, d\mu \\ &=\; c_2^{-1} \int_Y [a(s_2, \xi + \tilde{v}_1) - a(s_1, \xi + \tilde{v}_1)] \cdot [v_1 - \tilde{v}_1]\, d\mu \\ &\le\; \tfrac{1}{2}\|v_1 - \tilde{v}_1\|^2_{2,\mu,Y} + C|s_1 - s_2|^{2\gamma}(1 + |s_1| + |s_2| + |\xi|)^{2(1-\gamma)}, \end{aligned}$$

where in the last step we used (3.66) for $p = 2$. Hence combining the last estimate with (3.8),(3.66) and $\mu(Y) = 1$ we deduce

$$\begin{aligned} I_2 \;&\le\; c_1\| \,(1 + |s_1| + |\xi + v_1| + |\xi + \tilde{v}_1|)\, \|^{1-r}_{1,\mu,Y}\, \|v_1 - \tilde{v}_1\|^r_{1,\mu,Y} \\ &\le\; C\,(1 + |s_1| + |s_2| + |\xi|)^{1-r\gamma}\, |s_1 - s_2|^{r\gamma} \end{aligned} \quad (3.84)$$

with a constant only depending on c_1, c_2 and c_3. Combining (3.83) with (3.84) and using $r \le 1$ completes the proof. □

Now we are able to derive the homogenized equation for the limit $u \in W^{1,p}_0(\Omega)$ in Proposition 3.1.6. Thanks to Lemma 3.1.12, existence can be derived precisely as in the proof of Theorem 3.1.4.

Corollary 3.1.14. *Under the asumptions of Theorem 3.1.8, any limit function u according to Proposition 3.1.6 is a solution of the homogenized problem*

$$-\mathrm{div}\, a^\star(x, u, \nabla u) + \lambda|u|^{p-2}u = \bar{f}(x, u) \quad , \, u \in W^{1,p}_0(\Omega)\,. \quad (3.85)$$

If $c_2^\star > 0$ denotes the ellipticity constant of $a^\star$, then any solution u of (3.85) satisfies the a priori estimate

$$\|u\|_{W^{1,p}(\Omega)} \leq \begin{cases} C\,(\min\{\lambda, c_2^\star\})^{-1/p} & (\lambda > 0), \\ C\,(1 + (c_2^\star)^{\tilde{\beta}}) & (\lambda = 0), \end{cases} \tag{3.86}$$

where $\tilde{\beta} := p'(\beta p' - p)^{-1} < 0$ and the finite constant C depends only on p, β, Ω and λ, respectively on the Poincaré constant $C_p = C_p(\Omega)$ in $W_0^{1,p}(\Omega)$ if $\lambda = 0$.

Proof. We first determine the corrector function $\tilde{u}_1$ given in (3.31) by setting $\phi = 0$ in (3.38). Using the assumption $(H1)$ on μ and the unique solvability of the cell problem (3.62), we check that $\nabla_y \tilde{u}_1$ is uniquely determined by

$$\nabla_y \tilde{u}_1(x, y) = v_1(y, x, u(x), \nabla u(x)) \in L^p(\Omega; V^p_{\text{pot}}(\mathbb{T})) \tag{3.87}$$

by (3.63), where $v_1(\cdot, x, u, \nabla u) \in V^p_{\text{pot}}(\mathbb{T})$ is the solution of (3.62) for $s = u(x)$ and $\xi = \nabla u(x)$. If we plug (3.87) into the two-scale homogenized problem (3.38) and set $\phi_1 = 0$, by the definition of $a^\star$ we immediately derive the standard weak formulation of equation (3.85). It remains to prove the estimate (3.86). Note that by (3.10)

$$|\bar{f}(x, s)| \leq C(1 + |s|^\beta) \quad \text{for all } (x, s) \in \Omega \times \mathbb{R}.$$

Testing equation (3.85) with u and using the strict monotonicity (3.74) of $a^\star$, with the same technique as in the proof of (3.20) we can show for $\lambda > 0$ that

$$c_2^\star \|\nabla u\|^p_{L^p(\Omega)} + \lambda \|u\|^p_{L^p(\Omega)} \leq C\,, \tag{3.88}$$

where the constant C depends only on p, β, λ and $|\Omega|$, and where we may assume $c_2^\star \leq 1$ for $p < 2$ without loss of generality. For $\lambda = 0$, testing equation (3.85) with u and using (3.72),(3.74), the Poincaré inequality in $W_0^{1,p}(\Omega)$, Young's inequality and standard absorption techniques, gives

$$\|\nabla u\|^p_{L^p(\Omega)} \leq C(c_2^\star)^{-p'} \|\bar{f}(\cdot, u)\|^{p'}_{L^{p'}(\Omega)} \leq C(c_2^\star)^{-p'}(1 + \|u\|^{\beta p'}_{L^p(\Omega)})$$

as in (3.22), where the constant C depends only on $p, |\Omega|$ and C_p, and where for $p < 2$ we assumed $\|\nabla u\|_{L^p(\Omega)} \geq 1$ without restriction. Applying Young's inequality with dual exponents $(\beta p')^{-1} p$ and $(p - \beta p')^{-1} p$ to the product of the right-hand side in the last inequality, we get by absorption

$$\|\nabla u\|^p_{L^p(\Omega)} \leq C(1 + (c_2^\star)^{\tilde{\beta} p}), \tag{3.89}$$

where we used the Poincaré inequality in $W_0^{1,p}(\Omega)$ again. Hence (3.88) and (3.89) yield (3.86) for $\lambda > 0$ and $\lambda = 0$ respectively. □

3.2 Quasilinear equations

In this section we study the homogenization of quasilinear elliptic Dirichlet problems of the form

$$-\mathrm{div}\,(K(\tfrac{x}{\varepsilon},u_\varepsilon)\nabla u_\varepsilon)+\lambda u_\varepsilon \;=\; f(\tfrac{x}{\varepsilon},u_\varepsilon),\quad u_\varepsilon\in \tilde{H}_0^{1,2}(\Omega,d\mu_\varepsilon)\,, \tag{3.90}$$

where $\lambda \geq 0$ is a given parameter and the coefficient K a positive, symmetric tensor depending in a nonlinear way on u_ε. The structure conditions on the special flux $a(\cdot,s,\xi)=K(\cdot,s)\xi$ considerably simplify ($p=2$ and $r=1$ in Assumption 3.1.1) or have to be modified (cf. Remark 3.1.9) respectively. However, our main motivation to dedicate an extra section to the study of equation (3.90) is its importance for many applications. Some of them we discuss in Paragraph 3.2.1 below. Moreover, equation (3.90) will alternatively be derived on lower dimensional singular structures by a measure fattening approach in Chapter 5.
We also investigate in more detail the regularity of the effective tensor $K^\star$ (cf. Lemma 3.2.12 below) and the related question of regularity and uniqueness for the homogenized equation (cf. Lemma 3.2.13 and Corollary 3.2.15 below). Moreover, we will prove new corrector results for equation (3.90) under comparatively low regularity assumptions on the corrector u_1, and will discuss some applications where these assumptions actually hold true. The results of this section are also relevant for the nonlinear double porosity model studied in Section 3.3 below.

3.2.1 Some model problems

Motivated by a model of single phase flow in singular networks, we study the relaxed (cf. Section 2.2, in particular (2.71)) version of equation (3.90), that means the homogenization of the equation

$$-\mathrm{div}\,(\hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon)\nabla_{\mu_\varepsilon} u_\varepsilon)+\lambda u_\varepsilon \;=\; f(\tfrac{x}{\varepsilon},u_\varepsilon),\quad u_\varepsilon\in H_0^{1,2}(\Omega,d\mu_\varepsilon) \tag{3.91}$$

subject to a tensor $\hat{K}$ that is positive definite with respect to the local coordinate system given by the tangent space $T_\mu(y)$. Let us make this more precise. Recall that for μ-almost every $y\in Y$ there exist an orthonormal basis

$$S(y) := \{\eta_\mu^1(y),\ldots,\eta_\mu^{i_y}(y),\eta_\perp^{i_y+1}(y),\ldots,\eta_\perp^d(y)\} \tag{3.92}$$

of $\mathbb{R}^d$, where $i_y=\dim T_\mu(y)\in\{1,\ldots,d\}$ depends on y and the vectors $\eta_\mu^k(y)$ and $\eta_\perp^l(y)$ form an orthonormal basis of $T_\mu(y)$ and $T_\mu^\perp(y)$ respectively. It is natural to assume the following structure conditions on the data, which will be kept throughout the whole Section 3.2.

Assumption 3.2.1. *Let $(\hat{K},f):\mathbb{R}^d\times\mathbb{R}\to(\mathcal{M}^d_{\mathrm{sym}},\mathbb{R})$, $(y,s)\mapsto(\hat{K},f)(y,s)$ be μ-measurable and Y-periodic in y, and satisfy the following properties:*

- *There exist functions $\Theta_i : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\,, (y,s) \mapsto \Theta_i(y,s)$ μ-measurable and Y-periodic in y and continuous in s, and constants $c_k, C_K > 0$, such that for all $(y,s) \in \mathbb{R}^d \times \mathbb{R}$:*

$$0 < c_k \le \Theta_i(y,s) \le C_K \qquad \text{for each } i \in \{1,\dots,d\}\,, \tag{3.93}$$

$$S(y)^t \hat{K}(y,s) S(y) = \operatorname{diag}\{\Theta_1(y,s),\dots,\Theta_{i_y}(y,s),0,\dots,0\}\,, \tag{3.94}$$

where $S(y)$ is the orthogonal transformation given in (3.92). In particular, the function $\hat{K}$ is continuous with respect to s.

- *f is continuous with respect to s and there exists $\beta \in [0,1)$ and a constant $c_f > 0$, such that*

$$\forall (y,s) \in \mathbb{R}^d \times \mathbb{R} : \quad |f(y,s)| \;\le\; c_f(1+|s|^\beta)\,. \tag{3.95}$$

Note that the tensor $\hat{K}(\cdot,s)$ is in general singular (cf. Example 2.2.12), but positive definite with respect to the subspace spanned by the tangential vectors $\eta^k_\mu(\cdot)$. We discuss several applications which justify the investigation of problem (3.91) and the structure conditions on the data given above.

Fulldimensional structures Our first example comprises the classical setting of perforated domains. A typical problem is temperature flow in a composite medium, where zones of different thermal conductivity are distributed. An example of such a medium, which consists of a periodic system of heterogeneities, is sketched in Figure 3.1 below. We consider a (not necessarily cubic) reference domain

$$Y = (0,m_1)\times\cdots\times(0,m_d)\,, \quad Y = \bigcup_i Y_i\,,$$

where in each subdomain Y_i we have a temperature u_i and, in the nonlinear case, a matrix valued conductivity $K_i(y,u_i)$.

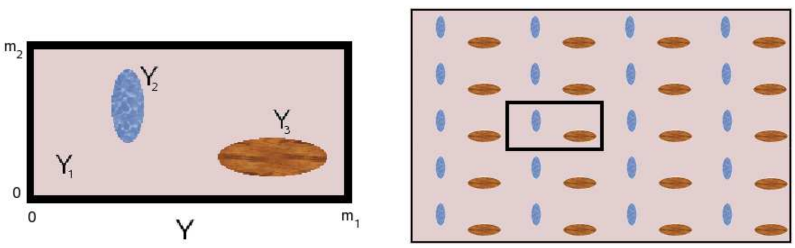


Figure 3.1: Full-dimensional structures.

We can then consider one global temperature $u(y)$ and one global conductivity $\hat{K}(y,u)$ as follows:

$$u(y) = u_i(y)\,, \quad \hat{K}(y,u) = K_i(y,u_i(y)), \;\text{ if } y \in Y_i.$$

The linear case $K_i(y,s) = k_i\mathbb{E}_d$, where $k_i > 0$ and $\mathbb{E}_d$ is the unit in $\mathcal{M}^d_{\mathrm{sym}}$, has widely been studied (see e.g. [26] and references therein). Under the standard assumptions comprising the continuity of u and the continuity of the fluxes across the interfaces, that means

$$u_i = u_j, \quad K_i(y,u_i)\nabla u_i \cdot n_i \;=\; -K_j(y,u_j)\nabla u_j \cdot n_j \quad \text{on } \partial Y_i \cap \partial Y_j, \tag{3.96}$$

where n_i, n_j are the corresponding outward unit normals, we get a family of nonlinear diffusion problems

$$-\mathrm{div}\,(\hat{K}(\tfrac{x}{\varepsilon}, u(x))\nabla u(x)) + \lambda u(x) \;=\; f(\tfrac{x}{\varepsilon}, u(x))\,, \quad u \in H^1_0(\Omega) := W^{1,2}_0(\Omega)\,, \tag{3.97}$$

where $y \leftrightarrow \varepsilon^{-1}x$ is the standard change of variables in the upscaling process and f a source term. If each tensor K_i is symmetric and positive definite, so is $\hat{K}$, and problem (3.97) clearly falls within the setting of equation (3.91)

$$\mu_\varepsilon \equiv \mu = \mathcal{L}^d\lfloor\Omega\,, \quad \nabla_{\mu_\varepsilon} u = \nabla u\,, \quad H^{1,2}_0(\Omega, d\mu_\varepsilon) = H^1_0(\Omega),$$

and Assumption 3.2.1 with $i_y = d$ for every $y \in Y$. Note that in the full-dimensional case the difference between (3.90) and the relaxed problem (3.91) does not make itself felt. Since the Lebesgue measure is strongly connected, an effective conductivity $K^\star = K^\star(u)$ can be derived along the lines of Lemma 3.1.10, where one has to solve cell problems with u as a parameter.

Multidimensional structures Now we turn to more sophisticated examples including lower dimensional structures. We introduce a model of single phase flow through a fractured porous medium (cf. Figure 1.1 in the introduction), which consists of two components: A set of isolated porous blocks F^ε_0 of low permeability, surrounded by a connected porous network F^ε (of codimension one) of high permeability. As usual, the parameter ε models the microscopic length scale associated with the period of the structure. The set F^ε_0 is sometimes called *matrix*, whereas F^ε is called the *fractures network*. In the nonlinear case this model is described by the following set of equations in the flow domain $\Omega \subset \mathbb{R}^d$:

$$\circledast \quad \left\{ \begin{array}{rcll} -\nabla\cdot(k_0(\frac{x}{\varepsilon}, u_0)\nabla u_0) &=& f_0 & \text{in } F^\varepsilon_0 \cap \Omega, \ \text{the matrix;} \\ -\nabla_\tau\cdot(k_1(\frac{x}{\varepsilon}, u_1)\nabla_\tau u_1) &=& [a_0\cdot\vec{n}] + f_1 & \text{in } F^\varepsilon \cap \Omega, \ \text{the network;} \\ \multicolumn{4}{c}{+ \ \text{conservation of the surface flux through the intersections}} \\ \multicolumn{4}{c}{\text{of the } (d-1)-\text{facets being faces of codimension two,}} \end{array} \right.$$

where ∇_τ is the nabla operator in the tangential variables of the fractures hypersurface, k_1 a quadratic matrix of codimension one,

$$a_0(x) := k_0(\tfrac{x}{\varepsilon}, u_0(x))\nabla u_0(x)\,, \quad x \in F^\varepsilon_0$$

the flux in the matrix, and $[a_0\cdot\vec{n}]$ the corresponding jump across the fractures hypersurface. The system $\circledast$ is the natural extension to nonlinear diffusion of

the corresponding linear model investigated in [20, Section 1]. The so called double porosity case, where $k_0 = \varepsilon^\alpha \hat{k}_0$ for some $\alpha > 0$ and $\hat{k}_0$ is of order one will be investigated in Section 3.3 below.
For now we study the case where the permeabilities k_i are of the same order with respect to ε, but can differ significantly in their dependence on x and u. We give an explicit example for which equation (3.91) can be derived from the system $\circledast$. Consider the thin cross $F := \{\frac{1}{2}\} \times (0,1) \cup (0,1) \times \{\frac{1}{2}\}$ together with the combined measure

$$\mu\lfloor Y \;=\; (\tfrac{1}{4}\mathcal{H}^1\lfloor F) + (\tfrac{1}{2}\mathcal{L}^2\lfloor F_0) \;=:\; \mu_1 + \mu_0\,, \quad F_0 := Y \setminus F. \tag{3.98}$$

It is easy to check that μ is doubling, normalized and strongly 2-connected on $\mathbb{R}^2$ with $\mu(\partial Y) = 0$. The sets F, F_0, their homothetic contractions $F^\varepsilon = \varepsilon F$ and $F_0^\varepsilon = \varepsilon F_0$, as well as the fractured medium are sketched below. The rescaled measure reads

$$\mu_\varepsilon = \tfrac{1}{4}\varepsilon\mathcal{H}^1\lfloor F^\varepsilon + \tfrac{1}{2}\mathcal{L}^2\lfloor F_0^\varepsilon. \tag{3.99}$$

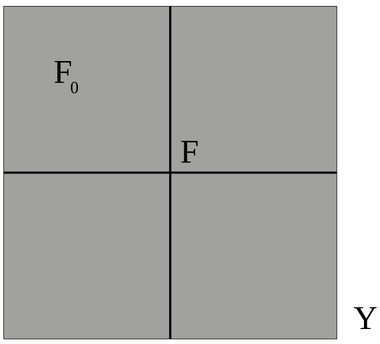


Figure 3.2: Reference cell

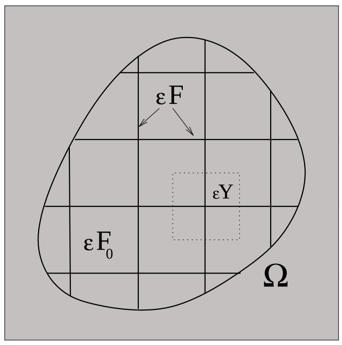


Figure 3.3: Fractured medium

We will show that the model problem $\circledast$ can be reformulated elegantly by equation (3.91) within the measure setting. It suffices to consider the problem on the unit cell Y with periodic boundary conditions. For the ε-problem we will choose a homogeneous Dirichlet condition on $\partial\Omega$. If we are given two Sobolev functions $u_i \in H^{1,2}_{\mu_i}(\mathbb{T})$ for $i = 0, 1$, note that

$$u_1 = \gamma(u_0) \;\text{ in } H^{1/2}(F) \;\Leftrightarrow\; u \in H^{1,2}_\mu(\mathbb{T}),\, u(y) = \begin{cases} u_0(y) & y \in F_0, \\ u_1(y) & y \in F, \end{cases} \tag{3.100}$$

where γ is the trace operator of a classical Sobolev function. Suppose that we are given sources $f_i \in L^2_{\mu_i}(\mathbb{T})$ and permeabilities $k_i \in L^\infty_{\mu_i}(\mathbb{T}; \mathcal{C}_\mathrm{b}(\mathbb{R}))$ with

$$\forall (y,s) \in Y \times \mathbb{R}: \quad 0 < c_k \le k_i(y,s) \le C_K\,, \;\; i = 0, 1. \tag{3.101}$$

Then if $u_0 \in H^{1,2}_{\mu_0}(\mathbb{T})$ is the unknown pressure in the matrix, testing the first equation in $\circledast$ with $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ we get

$$\int_{F_0} k_0(y,u_0)\nabla u_0 \cdot \nabla\varphi \, d\mu_0 \;=\; \int_{F_0} f_0\varphi \, d\mu_0 - \langle\!\langle [a_0 \cdot \vec{n}], \varphi \rangle\!\rangle_F\,, \tag{3.102}$$

where, as above, $[a_0 \cdot \vec{n}]$ denotes the jump of the matrix flux across F and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_F$ the dual pairing between $H^{-1/2}(F)$ and $H^{1/2}(F)$. If $u_1 \in H^{1,2}_{\mu_1}(\mathbb{T})$ denotes the pressure in the network F, the second equation in $\circledast$ can be interpreted as

$$-\mathrm{div}_{\mu_1}\left(k_1(y,u_1)\nabla_{\mu_1} u_1\right) = [a_0 \cdot \vec{n}] + f_1 \quad \text{in } \left(H^{1,2}_{\mu_1}(\mathbb{T})\right)'. \tag{3.103}$$

Testing (3.103) with the same $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ as above we get, using (3.102), the periodicity of k_1, u_1 and the Kirchhoff law of $\circledast$ in the intersection point:

$$\int_{F_0} k_0(y,u_0)\nabla u_0 \cdot \nabla\varphi \, d\mu_0 + \int_F k_1(y,u_1)\nabla_{\mu_1} u_1 \cdot \nabla_{\mu_1}\varphi \, d\mu_1 = \int_Y f\varphi \, d\mu\,, \tag{3.104}$$

where $f \in L^2_\mu(\mathbb{T})$ is defined as on the right-hand side in (3.100). We presume that u_1 coincides with the classical trace of u_0 in $H^{1/2}(F)$ (cf. [20, Lemma 1]), which corresponds to the assumption $u_i = u_j$ in (3.96). Then the function u defined in (3.100) belongs to $H^{1,2}_\mu(\mathbb{T})$, and (3.104) can be written as

$$\int_Y \hat{K}(y,u)\nabla_\mu u \cdot \nabla_\mu \varphi \, d\mu = \int_Y f\varphi \, d\mu \quad \forall \varphi \in \mathcal{C}^\infty(\mathbb{T})\,, \tag{3.105}$$

where the permeability $\hat{K} : Y \times \mathbb{R} \to \mathcal{M}^2_{\mathrm{sym}}$ and the tangential gradient $\nabla_\mu u \in L^2_\mu(\mathbb{T})$ are for μ-a.e. $y \in Y$ given by

$$\hat{K} = \begin{cases} \mathrm{diag}\,(k_0,k_0) & y \in F_0 \\ \mathrm{diag}\,(k_1,0) & y \in F_- \\ \mathrm{diag}\,(0,k_1) & y \in F_|\,, \end{cases} \qquad \nabla_\mu u = \begin{cases} \nabla u & y \in F_0 \\ (\partial_{y_1} u, 0) & y \in F_- \\ (0, \partial_{y_2} u) & y \in F_|\,, \end{cases}$$

where we decomposed $F = F_- \cup F_|$ into a horizontal and a vertical segment. Note that under the rescaling $y \leftrightarrow \varepsilon^{-1}x$ equation (3.105) leads to (3.91), if we require $u_\varepsilon = 0$ on $\partial\Omega$ and allow f to depend on u_ε. We emphasize that $\hat{K}$ satisfies all the prerequisites of Assumption 3.2.1, where we can choose

$$\Theta_1(y,s) = \begin{cases} k_0(y,s) & y \in F_0 \\ k_1(y,s) & y \in F, \end{cases} \qquad \Theta_2(y,s) = \begin{cases} k_0(y,s) & y \in F_0 \\ c_k & y \in F. \end{cases}$$

We point out that the special geometry of the support of μ_1 given in (3.98) was not relevant for the derivation of equation (3.91). Instead we can perfectly consider curvilinear structures as sketched in Figure 3.4 below, as long as the corresponding measure is strongly connected. In particular the matrix part F_0 need not be distributed over the whole domain Ω, as indicated in the left picture. Moreover we can consider the case $\mu = c\mathcal{H}^1\lfloor F$, where the flow takes place only in the singular network. Then the system $\circledast$ reduces to

$$\begin{cases} -\,\mathrm{div}_\mu\left(k(y,u)\nabla_\mu u\right) = f \quad , \, u \in H^{1,2}_\mu(\mathbb{T}) \\ +\ \text{conservation of the flux in the points } P_1, \ldots, P_4. \end{cases}$$

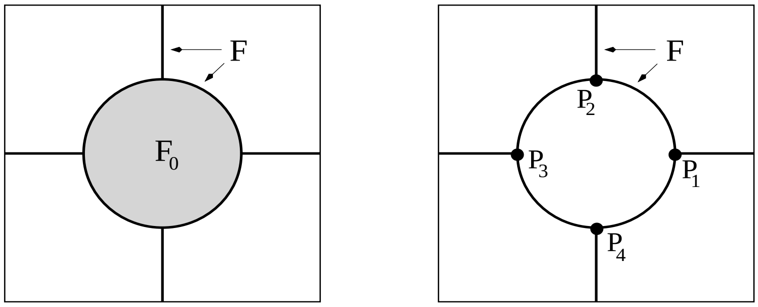


Figure 3.4: Curvilinear thin structures

Finally we note that our model is clearly valid also in higher space dimensions containing networks of codimension one. The simplest example in $\mathbb{R}^3$ is the standard cubic lattice, in which the fissures are the cubes faces. Then equation (3.91) can be derived similarly for $\mu = c_1\mathcal{L}^3\lfloor F_0 + c_2\mathcal{H}^2\lfloor F$.
For the rest of this section we will use the abbreviation $H_\varepsilon := H_0^{1,2}(\Omega, d\mu_\varepsilon)$ for the Hilbert space, which is equipped with the inner product

$$(u,v)_\varepsilon := \int_\Omega uv\,d\mu_\varepsilon + \int_\Omega \nabla_{\mu_\varepsilon} u \cdot \nabla_{\mu_\varepsilon} v\,d\mu_\varepsilon \tag{3.106}$$

and the induced norm. We introduce the weak formulation of problem (3.91):

Definition 3.2.2. *A function $u_\varepsilon \in H_\varepsilon$ is called a solution of problem (3.91), if the integral identity*

$$\int_\Omega \Big(\hat{K}(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla_{\mu_\varepsilon} u_\varepsilon \cdot \nabla_{\mu_\varepsilon}\varphi + \lambda u_\varepsilon \varphi\Big)\,d\mu_\varepsilon \;=\; \int_\Omega f(\tfrac{x}{\varepsilon}, u_\varepsilon)\varphi\,d\mu_\varepsilon \tag{3.107}$$

holds for each $\varphi \in \mathcal{D}(\Omega)$, by density for each $\varphi \in H_\varepsilon$ respectively.

The two problems (3.90) and (3.91) are equivalent in a rather obvious sense. Since (3.90) is formulated within the framework of Section 3.1, this equivalence gives a further motivation of studying monotone operators of type (3.1). By assumption (3.94) it is easy to check that for μ-almost every $y \in \mathbb{R}^d$ the symmetric matrix $\hat{K}(y,\cdot)$ vanishes on $T_\mu^\perp(y)$ and has $T_\mu(y)$ as its range:

$$\hat{K}(y,s)\xi \;=\; \hat{K}(y,s)\xi_\mu(y)\,, \quad \xi_\mu(y) := P_\mu(y)[\xi] \tag{3.108}$$

for any $\xi \in \mathbb{R}^d$, where P_μ is the pointwise orthogonal projection onto T_μ. On the other hand, for the matrix $S(y)$ and the functions Θ_i introduced in (3.92),(3.93), we check that $\hat{K}$ is the relaxed matrix of the positive tensor

$$\mathcal{M}^d_{\mathrm{sym}} \ni K(y,s) := S(y)\,\mathrm{diag}\,(\Theta_1(y,s),\ldots,\Theta_d(y,s))\,S(y)^t \tag{3.109}$$

according to formula (2.53). In addition, K commutes with the orthogonal projections P_μ and $P_\mu^\perp$, and hence for each $\xi \in \mathbb{R}^d$ we have:

$$(K(y,s)\cdot\xi) \in T_\mu(y) \;\Leftrightarrow\; \xi \in T_\mu(y) \tag{3.110}$$

for μ-a.e. point $y \in Y$ and all $s \in \mathbb{R}$, since K is positive definite. Recall that with this tensor K as a coefficient, a function $u_\varepsilon \in \tilde{H}_0^{1,2}(\Omega, d\mu_\varepsilon)$ is called a weak solution of problem (3.90), if

$$\int_\Omega \left(K(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi + \lambda u_\varepsilon \varphi\right) d\mu_\varepsilon = \int_\Omega f(\tfrac{x}{\varepsilon}, u_\varepsilon)\varphi \, d\mu_\varepsilon \tag{3.111}$$

for each $\varphi \in \mathcal{D}(\Omega)$ and *some* gradient ∇u_ε of u_ε. Under Assumption 3.2.1 we can deduce by density that such a gradient is uniquely determined and, additionally, that it is tangential due to (3.110). Hence, due to the fact that

$$\forall \xi \in T_\mu(y): \quad K(y,s)\xi = \hat{K}(y,s)\xi \tag{3.112}$$

for μ-a.e. $y \in Y$, we get the following statement:

Remark 3.2.3. *The function $u_\varepsilon \in H_\varepsilon$ is a solution of problem (3.91) if and only if the corresponding pair $(u_\varepsilon, \nabla_{\mu_\varepsilon} u_\varepsilon) \in V^2(\Omega, d\mu_\varepsilon)$ is a solution of problem (3.90) in the sense of (3.111). In particular,*

$$\|u_\varepsilon\|_{1,2,\varepsilon} = \|(u_\varepsilon, \nabla u_\varepsilon)\|_{V^2(\Omega, d\mu_\varepsilon)}, \quad \nabla u_\varepsilon = \nabla_{\mu_\varepsilon} u_\varepsilon, \tag{3.113}$$

with the norm $\|\cdot\|_{1,2,\varepsilon}$ of H_ε defined in (2.67) on page 28.

If μ is strongly 2-connected on $\mathbb{R}^d$, then by Lemma 2.4.2 the map

$$\|\cdot\|_{H_\varepsilon}: H_\varepsilon \to \mathbb{R}, \; u \mapsto \|\nabla_{\mu_\varepsilon} u\|_{2,\varepsilon} \tag{3.114}$$

defines norm on H_ε which is equivalent to the one induced by the scalar product in (3.106). Hence we can also allow $\lambda = 0$ when we study equation (3.91) (cf. Remark 3.1.5). The following existence result is an easy consequence of Theorem 3.1.4 (resp. Remark 3.1.5 for $\lambda = 0$) and Remark 3.2.3:

Corollary 3.2.4. *Let μ be strongly 2-connected on $\mathbb{R}^d$ and doubling. Then under the Assumption 3.2.1 there exists a solution $u_\varepsilon \in H_\varepsilon$ of problem (3.91) in the sense of Definition 3.2.2, fulfilling the uniform estimate*

$$\|u_\varepsilon\|_{H_\varepsilon} \leq C, \tag{3.115}$$

with a constant depending only on $c_k, c_f, \beta, \lambda, |\Omega|$ and, for $\lambda = 0$, on the uniform Poincaré constant occurring in (2.92), but not on ε.

3.2.2 Homogenization and regularity theory

The two-scale homogenized problem for equation (3.91) can be derived from Theorem 3.1.8. Due to the separation of the flux a in the gradient variable the unit cell problem defined in (3.61) is completely determined in terms of d linear problems, containing the two-scale limit u as a parameter. This enables us to define an effective permeability $K^\star = K^\star(u)$. Similar as in Section 3.1, for the homogenization step we need to require local Hölder continuity for $\hat{K}$ and f with respect to the s-variable.

Assumption 3.2.5. *There exist $\gamma, \tilde{\gamma} \in (0,1]$, a constant $C > 0$ and a function $\tilde{h} \in L^{\tilde{q}}_{\mu}(\mathbb{T})$, such that for all $y \in \mathbb{R}^d$ and $s_1, s_2 \in \mathbb{R}$:*

$$\|\hat{K}(y,s_1) - \hat{K}(y,s_2)\| \leq C\,(1+|s_1|+|s_2|)^{1-\gamma}\,|s_1-s_2|^{\gamma} \quad (3.116)$$

$$|f(y,s_1) - f(y,s_2)| \leq |\tilde{h}(y)|\,|s_1-s_2|^{\tilde{\gamma}}, \quad (3.117)$$

where $\|\cdot\|$ is some norm on $\mathbb{R}^{d\times d}$ and $\tilde{q} := 2(2-\tilde{\gamma})^{-1} \in (1,2]$.

Recall that we write $H^1_0(\Omega) := W^{1,2}_0(\Omega)$ for the classical Sobolev space, and that $\bar{f}(s)$ denotes the μ-average of $f(\cdot,s)$ over Y.

Lemma 3.2.6. *Under the assumptions of Corollary 3.2.4, let $\{u_\varepsilon\} \subset H_\varepsilon$ be a family of solutions of problem (3.91) fulfilling estimate (3.115). Then there exist $u \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega; H^{1,2}_\mu(\mathbb{T}))$ such that, up to a subsequence:*

$$u_\varepsilon \twoheadrightarrow u \quad \text{two-scale strongly in } L^2(\Omega, d\mu_\varepsilon)\,, \quad (3.118)$$

$$\nabla_{\mu_\varepsilon} u_\varepsilon \rightharpoonup\!\rightharpoonup P_\mu(y)[\nabla u] + \nabla_{\mu,y} u_1(y) \quad \text{two-scale in } L^2(\Omega, d\mu_\varepsilon)^d\,. \quad (3.119)$$

Under Assumption 3.2.5, the pair (u, u_1) is a solution of the two-scale homogenized problem

$$\int_{\Omega\times Y} \hat{K}(y,u)\,(P_\mu(y)[\nabla u] + \nabla_{\mu,y} u_1(y)) \cdot (\nabla\phi + \nabla_y \phi_1(y))\,dm$$
$$+\lambda \int_\Omega u\phi\,dx = \int_\Omega \bar{f}(u)\phi\,dx \quad (3.120)$$

for all $(\phi, \phi_1) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbb{T}))$.

Sketch of proof. Thanks to the a priori estimate (3.115), the convergences in (3.118) and (3.119) follow directly from Theorem 2.4.4 and Theorem 2.4.5, where for (3.119) we use the characterization (2.100) of the two-scale limit χ from the proof of Theorem 2.4.4. As the approximation method in the proof of Theorem 3.1.8 shows, we can assume that $u \in \mathcal{D}(\Omega)$. For the standard test function φ_ε introduced in (3.39), we can show

$$\int_\Omega \hat{K}(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla_{\mu_\varepsilon} u_\varepsilon \cdot \nabla\varphi_\varepsilon\,d\mu_\varepsilon = \int_\Omega \nabla_{\mu_\varepsilon} u_\varepsilon \cdot \hat{K}(\tfrac{x}{\varepsilon}, u)\nabla\varphi_\varepsilon\,d\mu_\varepsilon + o(1)$$

as $\varepsilon \to 0$ by the symmetry of $\hat{K}$. Indeed, similar as in the derivation of (3.51), we can use (3.115),(3.116) and (3.118) to deduce

$$\left|\int_\Omega [\hat{K}(u_\varepsilon) - \hat{K}(u)]\nabla_{\mu_\varepsilon} u_\varepsilon \cdot \nabla\varphi_\varepsilon\right| \leq C\,\|(1+|u_\varepsilon|+|u|)\|^{1-\gamma}_{2,\varepsilon}\|u_\varepsilon - u\|^{\gamma}_{2,\varepsilon}$$
$$\leq C\,\|u_\varepsilon - u\|^{\gamma}_{2,\varepsilon} \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

Passing to the limit in (3.107) with test function φ_ε, using (3.119) and the admissibility of the test function $(y,x) \mapsto \hat{K}(y, u(x))[\nabla\phi(x) + \nabla_y\phi_1(x,y)]$ for the two-scale convergence, we derive (3.120), where the source term can be treated precisely as in the proof of Theorem 3.1.8. □

The two-scale homogenized problem (3.120) comprising u and u_1 can clearly be decoupled again. In order to determine the effective coefficient, we introduce the Sobolev space of Y-periodic functions with zero mean value

$$W_\mu := \{u \in H^{1,2}_\mu(\mathbb{T}) \mid \bar{u} = 0\}\,, \quad \bar{u} := \fint_Y u(y)\, d\mu(y)\,,$$

which is obviously a closed subspace of $H^{1,2}_\mu(\mathbb{T})$. Note that if μ fulfills $(H3)$, then W_μ is a Hilbert space for the norm $\|u\|_{W_\mu} = \|\nabla_\mu u\|_{2,\mu,Y}$.

Lemma 3.2.7. *Let μ be strongly 2-connected on $\mathbb{T}$. Then for any $s \in \mathbb{R}$ there exists a unique weak solution $\Lambda_k(\cdot, s) \in W_\mu$ of the problem*

$$(C_k) \quad \begin{cases} -div_\mu\,(\hat{K}(y,s)[\nabla_\mu \Lambda_k(y,s) + \vec{e}_{k,\mu}(y)]) \;=\; 0 \;\; in \;Y \\ \qquad\qquad y \mapsto \Lambda_k(y,s) \;\; Y\text{-periodic} \quad , \quad \bar{\Lambda}_k = 0\,, \end{cases}$$

with $\vec{e}_{k,\mu}(y) := P_\mu(y)[\vec{e}_k]$, where $\vec{e}_k$ is the k-th unit vector. Moreover the following uniform estimate holds with a constant independent of k and $s \in \mathbb{R}$:

$$\|\Lambda_k(\cdot, s)\|_{W_\mu} \;=\; \|\nabla_\mu \Lambda_k(\cdot, s)\|_{2,\mu,Y} \;\le\; C. \tag{3.121}$$

Proof. The function $\Lambda_k(\cdot, s) \in W_\mu$ is called a weak solution of problem (C_k), if the following integral identity holds:

$$\int_Y \hat{K}(y,s)[\nabla_\mu \Lambda_k(y,s) + \vec{e}_{k,\mu}(y)] \cdot \nabla_\mu \varphi(y)\, d\mu(y) \;=\; 0 \quad \forall \varphi \in W_\mu\,. \tag{3.122}$$

Setting $a(y, s, \xi) := K(y, s)\xi$ with K defined as in (3.109), we deduce from Lemma 3.1.10 that for any $k = 1, \dots, d$ and $s \in \mathbb{R}$ there exists a unique pair $(v, v_1)(\cdot, s, \vec{e}_k) \in \tilde{V}^2$, where $\tilde{V}^2$ is defined in (3.59), such that

$$\int_Y K(y,s)[\vec{e}_k + v_1(y,s,\vec{e}_k)] \cdot \varphi_1(y)\, d\mu(y) = 0 \quad \text{for each } (\varphi, \varphi_1) \in \tilde{V}^2. \tag{3.123}$$

Note that we can take $(0, P^\perp_\mu(y)[\vec{e}_k + v_1(y, s, \vec{e}_k)]) \in \tilde{V}^2$ as a test function in (3.123), and hence from (3.93) and (3.109) deduce that $\vec{e}_k + v_1(y, s, \vec{e}_k)$ is tangential, since K commutes with $P^\perp_\mu$. In particular, due to (3.112) we can take $\hat{K}$ instead of K in (3.123) and choose W_μ as the space of test functions with $\varphi_1 = \nabla_\mu \varphi$. Hence the unique solution of problem (C_k) reads

$$\Lambda_k(\cdot, s) := v(\cdot, s, \vec{e}_k) \in W_\mu,$$

and estimate (3.121) follows by testing (3.122) with Λ_k and using the uniform estimates on the eigenvalues Θ_i of $\hat{K}$ from below and above in (3.94). □

With the help of the auxiliary functions Λ_k we can define the homogenized coefficient $K^\star$. Note that the symmetry of $K^\star$ follows from symmetry of $\hat{K}$.

Definition 3.2.8. *The effective coefficient $K^\star : \mathbb{R} \to \mathcal{M}^d_{\text{sym}}$ is given by*

$$K^\star_{ij}(s) \;=\; \int_Y \hat{K}(y,s)[\vec{e}_{i,\mu}(y) + \nabla_\mu \Lambda^s_i(y)] \cdot [\vec{e}_{j,\mu}(y) + \nabla_\mu \Lambda^s_j(y)]\, d\mu(y) \quad (3.124)$$

where Λ^s_k is for a given $s \in \mathbb{R}$ the unique solution of problem (C_k).

Clearly $K^\star$ is well defined by Lemma 3.2.7. We emphasize that the cell problems given by Definition 3.2.8 are basically the ones that arise in the asymptotics of the linear problem

$$\inf_{u \in \mathcal{C}^1_0} \left\{ \int_\Omega (j(\tfrac{x}{\varepsilon}, \nabla u) - fu)\, d\mu_\varepsilon \right\}, \quad j(y,z) = z \cdot K(y) z$$

with $K \in \mathcal{M}^d_{\text{sym}}$ studied in [14], but do now depend on the parameter $s \in \mathbb{R}$ in a nonlinear way. Before we derive the effective equation, we investigate the most important properties of the effective conductivity. In the situation of Lemma 3.2.7, we denote the nonlinear cell solution operator by

$$\Lambda_k : \mathbb{R} \to W_\mu,\ s \mapsto \Lambda_k(\cdot, s). \quad (3.125)$$

Lemma 3.2.9. *Under the assumptions of Lemma 3.2.7 there holds*

$$\Lambda_k \in \mathcal{C}(\mathbb{R}; W_\mu)\,,\ K^\star \in \mathcal{C}(\mathbb{R}; \mathbb{R}^{d\times d})\,. \quad (3.126)$$

Moreover if μ is also weakly 2-connected on $\mathbb{R}^d$, then $K^\star$ is uniformly positive definite and bounded, that means there exist constants $c_\star, C_\star > 0$, such that

$$\forall (s,\xi) \in \mathbb{R}\times\mathbb{R}^d : \quad c_\star|\xi|^2 \le \xi \cdot K^\star(s)\xi \le C_\star|\xi|^2. \quad (3.127)$$

Proof. Subject to the coefficient $a(y,s,\xi) := K(y,s)\xi$ with K defined in (3.109), we consider for any k the solution operator

$$F_k : \mathbb{R} \to \tilde{V}^2,\ s \mapsto (v, v_1)(\cdot, s, \vec{e}_k)\,, \quad k = 1, \dots, d,$$

according to Lemma 3.1.10, which is continuous by Lemma 3.1.12. As the proof of Lemma 3.2.7 shows, there holds

$$v_1(\cdot, s, \vec{e}_k) \;=\; \nabla_\mu \Lambda_k(\cdot, s) - \vec{e}^{\,\perp}_k(\cdot)\,, \quad \vec{e}^{\,\perp}_k(y) := P^\perp_\mu(y)[\vec{e}_k],$$

and (3.126) easily follows. The upper bound in (3.93) can be shown by using that neither the bounds on $\hat{K}$ nor the a priori estimate (3.121) on the cell solution depends on s. Due to its importance for the measure fattening approach investigated in Section 5.2 below, we concentrate on deriving the lower bound in (3.127). For $(s,\xi) \in \mathbb{R}\times\mathbb{R}^d$ given, we set

$$w := -\xi_\perp + \sum_{l=1}^d \xi_l \nabla_\mu \Lambda^s_l \in V^2_{\text{pot}}(\mathbb{T}, d\mu)\,, \quad \xi_\perp := P^\perp_\mu[\xi] \in \Gamma^2(\mathbb{T}, d\mu)\,, \quad (3.128)$$

where we used the orhogonal decomposition (2.58) of μ-potential vectors, and where Λ_l^s is the solution of the cell problem (C_l) for $s \in \mathbb{R}$. Using the definition of $K^\star$, the solution property of Λ_l^s and Assumption 3.2.1, we calculate

$$\begin{aligned} \xi \cdot K^\star(s)\xi &= \int_Y \langle \hat{K}(y,s)[\xi_\mu + \Sigma_l \xi_l \nabla_\mu \Lambda_l^s], \xi_\mu + \Sigma_l \xi_l \nabla_\mu \Lambda_l^s \rangle \, d\mu \\ &\geq c_k \int_Y |\xi_\mu + \Sigma_l \xi_l \nabla_\mu \Lambda_l^s|^2 \, d\mu \;=\; c_k \int_Y |\xi + w|^2 \, d\mu \,. \end{aligned}$$

Since $w \in V^2_{\mathrm{pot}}(\mathbb{T}, d\mu)$, by density we get, using Lemma 2.3.13 and the connectedness assumptions on μ,

$$\begin{aligned} \xi \cdot K^\star(s)\xi &\geq c_k \inf_{\varphi \in \mathcal{C}^\infty(\mathbb{T})} \int_Y |\xi + \nabla\varphi|^2 \, d\mu \\ &= c_k \inf_{\varphi \in H^{1,2}_\mu(\mathbb{T})} \int_Y |\xi_\mu + \nabla_\mu \varphi|^2 \, d\mu \;\geq\; c_\star |\xi|^2 \,, \end{aligned} \tag{3.129}$$

with $c_\star = c_k \hat{c} > 0$, where $\hat{c} = \hat{c}(\mu)$ is the positive constant occurring in (2.86) depending on μ in general. We wish to point out that the functional in (3.129) has a unique minimizer in the space W_μ, that means

$$\inf_{\varphi \in H^{1,2}_\mu(\mathbb{T})} \int_Y |\xi_\mu + \nabla_\mu \varphi|^2 \, d\mu \;=\; \|\xi_\mu + \nabla_\mu \varphi_\xi\|^2_{2,\mu,Y} \,, \tag{3.130}$$

where $\varphi_\xi \in W_\mu$ is uniquely determined as the solution of the cell problem (C_k) with $\hat{K}(y,s)$ replaced by $\mathbb{E}_d$ and $\vec{e}_{k,\mu}$ replaced by ξ_μ. □

With the standard procedure we can now determine the corrector function u_1 and thus are able to derive the effective equation.

Corollary 3.2.10. *Any limit function $u \in H^1_0(\Omega)$ according to Lemma 3.2.6 is a solution of the homogenized equation*

$$(P_0) \quad \left\{ \begin{aligned} -\mathrm{div}\,(K^\star(u)[\nabla u]) + \lambda u &= \bar{f}(u) \quad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \,. \end{aligned} \right.$$

For the corresponding subsequence there holds

$$u_\varepsilon \twoheadrightarrow u \quad \text{two-scale strongly in } L^2(\Omega, d\mu_\varepsilon) \,. \tag{3.131}$$

Proof. Similar as in the proof of Corollary 3.1.14, we check that the corrector function u_1 given in (3.119) is uniquely determined up to an arbitrary additive function $\tilde{u}_1 \in L^2(\Omega)$:

$$u_1(x,y) := \sum_{k=1}^d \partial_{x_k} u(x) \; \Lambda_k(y, u(x)) + \tilde{u}_1(x) \,, \tag{3.132}$$

where $\Lambda_k(y, u(x))$ is the unique solution of (C_k) for $s = u(x)$. Note that u_1 defined above belongs to $L^2(\Omega; H^{1,2}_\mu(\mathbb{T}))$ by (3.121). Inserting (3.132) and the definition of $K^\star$ in (3.120) with $\phi_1 = 0$, we obtain the weak formulation of problem (P_0), and the convergence in (3.131) is guaranteed by Lemma 3.2.6. $\square$

It is well known that nonlinear elliptic problems of type (P_0) can have unbounded solutions. However, we make use of Theorem 6.7 of the appendix to show maximum regularity for any *bounded* solution, provided the data are sufficiently smooth. To this end we first examine carefully, whether the effective coefficient $K^\star$ inherits regularity from the function $\hat{K}$. The following statement on the Hölder continuity of $K^\star$ can be proven precisely as in Lemma 3.1.13, where we use the trick $a(y, s, \xi) := K(y, s)\xi$ again and observe that

$$[a^\star(s, \vec{e}_k)]_m \;=\; K^\star_{km}(s) \quad \forall s \in \mathbb{R}, 1 \le k, m \le d.$$

Moreover one has to use that the a priori estimate (3.121) does neither depend on k nor on $s \in \mathbb{R}$.

Lemma 3.2.11. *Let μ be strongly 2-connected on $\mathbb{T}$ and the Hölder assumption (3.116) on $\hat{K}$ be satisfied with $\gamma \in (0, 1]$. Then there holds*

$$\|K^\star(s_1) - K^\star(s_2)\| \;\le\; C\,(1 + |s_1| + |s_2|)^{1-\gamma}\,|s_1 - s_2|^\gamma\,, \tag{3.133}$$

where C depends only on the constants in (3.93) and (3.116).

By the implicit function theorem, higher order regularity for the cell solution operator Λ_k and the coefficient $K^\star$ may be derived. Recall that $\mathcal{C}^k_{\mathrm{b}}(\mathbb{R})$ is the Banach space of k-times continuously differentiable functions on $\mathbb{R}$ with bounded derivatives up to order k.

Lemma 3.2.12. *Assume that $\hat{K} \in L^\infty_\mu(\mathbb{T}; \mathcal{C}^1_{\mathrm{b}}(\mathbb{R}; \mathbb{R}^{d\times d}))$ and let μ be strongly 2-connected on $\mathbb{T}$. Then there holds*

$$\Lambda_k \in \mathcal{C}^1(\mathbb{R}; W_\mu)\,,\; K^\star \in \mathcal{C}^1_{\mathrm{b}}(\mathbb{R}; \mathbb{R}^{d\times d})\,. \tag{3.134}$$

Proof. For fixed $k \in \{1, \dots, d\}$ we consider the map

$$F_k : W_\mu \times \mathbb{R} \to W_\mu^{'},\; (\Lambda, s) \mapsto -\mathrm{div}_\mu(\hat{K}(\cdot, s)[\nabla_\mu\Lambda(\cdot) + \vec{e}_{k,\mu}(\cdot)])\,.$$

Note that F_k is well defined by (3.93), acting on each $u \in W_\mu$ via the integration by parts formula (2.42). By Lemma 3.2.7, for any $s \in \mathbb{R}$ there exists a unique $\Lambda^s_k := \Lambda_k(\cdot, s) \in W_\mu$, such that $F_k(\Lambda^s_k, s) = 0$. We want to apply the implicit function theorem. In order to show that F_k is continuous, let $(\Lambda_n, s_n) \to (\Lambda, s)$ in $W_\mu \times \mathbb{R}$. Thanks to Assumption 3.2.1, in particular due to the continuity of $\hat{K}$ with respect to s, we get

$$\hat{K}_{ij}(y, s_n) \to \hat{K}_{ij}(y, s) \quad \text{strongly in } L^p_\mu(Y)$$

for any $p \in [1, \infty)$. Therefore, using the uniform estimates on the eigenvalues of $\hat{K}$, we easily deduce

$$\begin{aligned}\|F_k(\Lambda_n, s_n) - F_k(\Lambda, s)\|_{W'_\mu} \;&\le\; \sup_{\|\varphi\|\le 1}\left|\int_Y [\hat{K}(s_n) - \hat{K}(s)]\nabla_\mu\Lambda\cdot\nabla_\mu\varphi\right| + o(1)\\ &\le\; \|[\hat{K}(s_n) - \hat{K}(s)]\nabla_\mu\Lambda\|_{2,\mu,Y} + o(1) \;\to\; 0\end{aligned}$$

as $n \to \infty$, and hence that F_k is continuous. Note that by the regularity assumption on $\hat{K}$, there exists a subset $E \subset Y$ with $\mu(E) = 0$, such that for all $s, s_1, s_2 \in \mathbb{R}$ and $y \in Y \setminus E$:

$$\|\hat{K}(y, s_1) - \hat{K}(y, s_2)\| \leq C|s_1 - s_2|\,, \quad \|\partial_2 \hat{K}(y, s)\| \leq C \tag{3.135}$$

for some matrix norm $\|\cdot\|$ on $\mathbb{R}^{d\times d}$. The first Fréchet derivative

$$\partial_\Lambda F_k : W_\mu \times \mathbb{R} \to \mathcal{L}(W_\mu, W_\mu^{'}),\ (\Lambda, s) \mapsto -\mathrm{div}_\mu(\hat{K}(\cdot, s)\nabla_\mu \cdot) \tag{3.136}$$

does not depend on Λ and is continuous in $W_\mu \times \mathbb{R}$, which can easily be checked using the first inequality in (3.135). On the other hand, using the strong 2-connectedness of μ on the torus, the Lax-Milgram lemma gives that $\partial_\Lambda F_k(\Lambda, s)$ is an isomorphism for any $(\Lambda, s) \in W_\mu \times \mathbb{R}$. Using the estimates in (3.135), it is also straightforward to compute the other Fréchet derivative $\partial_s F_k : W_\mu \times \mathbb{R} \to \mathcal{L}(\mathbb{R}, W_\mu^{'})$, which is for any $\tau \in \mathbb{R}$ given by

$$\forall v \in W_\mu : \quad \partial_s F_k(\Lambda, s)(\tau)\langle v\rangle \;=\; \tau \int_Y \partial_2 \hat{K}(y, s)[\nabla_\mu \Lambda + \vec{e}_{k,\mu}] \cdot \nabla_\mu v \, d\mu\,.$$

Due to the regularity assumption on $\hat{K}$ it is also easy to check that $\partial_s F_k$ is continuous in $W_\mu \times \mathbb{R}$. Applying the implicit function theorem, we get that

$$\Lambda_k : \mathbb{R} \to W_\mu,\ s \mapsto \Lambda_k^s \quad \text{belongs to } \mathcal{C}^1(\mathbb{R}; W_\mu)\,. \tag{3.137}$$

We also have $\Lambda_k^{'}(s)\langle\tau\rangle = -\tau w_k^s$ for any $\tau \in \mathbb{R}$, where $w_k^s \in W_\mu$ is the unique solution of

$$\mathrm{div}_\mu\,(\hat{K}(\cdot, s)\nabla_\mu w_k^s) \;=\; \mathrm{div}_\mu\,(\partial_2 \hat{K}(\cdot, s)[\nabla_\mu \Lambda_k^s(\cdot) + \vec{e}_{k,\mu}(\cdot)]) \quad \text{in } W_\mu^{'}\,.$$

To show the regularity of $K^\star$, we calculate for $s \in \mathbb{R}$ and $|\tau|$ small:

$$\frac{K_{ij}^\star(s+\tau) - K_{ij}^\star(s)}{\tau} \;\overset{\tau\to 0}{\longrightarrow}\; \int_Y \partial_2 \hat{K}(s)[\vec{e}_{i,\mu} + \nabla_\mu \Lambda_k^s] \cdot \vec{e}_{j,\mu} + \hat{K}(s)\nabla_\mu w_k^s \cdot \vec{e}_{j,\mu} \, d\mu(y)\,,$$

which by the above considerations gives $K^\star \in \mathcal{C}_\mathrm{b}^1(\mathbb{R}; \mathbb{R}^{d\times d})$. □

Making use of Lemma 3.2.9 and Lemma 3.2.12, we will show that any bounded solution of problem (P_0) is automatically smooth by applying Theorem 6.7 of the appendix, which comprises an advanced regularity result [12, Theorem 2.25] for quasilinear scalar elliptic equations.

Lemma 3.2.13. *In addition to Assumption 3.2.1, let $\hat{K}_{ij} \in L_\mu^\infty(\mathbb{T}; \mathcal{C}_\mathrm{b}^1(\mathbb{R}))$ and let μ satisfy (H2) and (H3). Then a solution u of problem (P_0) in the class $H_0^1(\Omega) \cap L^\infty(\Omega)$ has the following maximum regularity property*

$$u \in W^{2,p}(\Omega) \quad \forall 1 \leq p < \infty\,. \tag{3.138}$$

Proof. We have to check the prerequisites (6.5)-(6.8) of Theorem 6.7 for

$$a : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d,\ (s, \xi) \mapsto K^\star(s)[\xi]\,.$$

Under the assumption on μ above, Lemma 3.2.9 gives $0 < c_\star \leq \eta \cdot K^\star(s)\eta$ for all $s \in \mathbb{R}, |\eta| = 1$. In particular, the functions

$$\frac{\partial a_i}{\partial s}(s, \xi) = \sum_j \partial_s K^\star_{ij}(s)\xi_j\,, \quad \frac{\partial a_i}{\partial \xi_k}(s, \xi) = K^\star_{ik}(s)$$

fulfill the prerequisites of Theorem 6.7 due to Lemma 3.2.9 and Lemma 3.2.12. Note that the source $\bar{f} : \mathbb{R} \to \mathbb{R}$ is continuous by Assumption 3.2.1. Hence for a solution u of problem (P_0) with $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$, we deduce

$$\operatorname{div} a(u, \nabla u) \;=\; \lambda u - \bar{f}(u) \in L^\infty(\Omega)\,,$$

which shows (6.8). Recall that we always assume that $\partial\Omega$ is smooth. Hence Theorem 6.7 can be applied, which guarantees (3.138). □

As indicated above, in general we do not have an uniqueness result for the homogenized equation. In particular, the two-scale limit u in (3.131) will in general depend on the chosen subsequence. However, we can take advantage of a comparison principle for elliptic operators L of the form

$$Lu = \operatorname{div} a(u, \nabla u) + b(u) \tag{3.139}$$

with suitable coefficients $a : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R} \to \mathbb{R}$. Recall that a function u, weakly differentiable in Ω, satisfies $Lu \geq 0\,(\leq 0)$ in Ω, if the functions $a_i(u, \nabla u), b(u)$ are locally integrable in Ω and

$$\int_\Omega \left(\langle a(u, \nabla u), \nabla\varphi\rangle - b(u)\varphi\right)\,dx \;\leq\; 0 \;\;(\geq 0)$$

for all non-negative $\varphi \in \mathcal{C}^1_0(\Omega)$. A proof of the following statement can be found in [33, Section 10.4].

Theorem 3.2.14. *Let $u, v \in \mathcal{C}^1(\overline{\Omega})$ satisfy $Lu \geq 0$ in Ω, $Lv \leq 0$ in Ω, where L is the operator in (3.139), and $u \leq v$ on $\partial\Omega$. Let L be elliptic in Ω, the functions a and b continuously differentiable, and b monotonically nonincreasing. Then there holds $u \leq v$ in Ω.*

Combining this result with Lemma 3.2.13 above, we can show that there exists at most one smooth solution of the homogenized equation. In particular, we obtain uniqueness provided there is no unbounded solution.

Corollary 3.2.15. *Under the assumptions of Lemma 3.2.13, let additionally $f(y, s)$ be continuously differentiable with respect to s and monotonically nonincreasing in s for fixed $y \in \mathbb{R}^d$. Then there exists at most one bounded solution of problem (P_0).*

Proof. Given a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, by Lemma 3.2.13 and a standard Sobolev embedding we deduce $u \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$. By Lemma 3.2.9, Lemma 3.2.12 and the assumptions of this corollary, Theorem 3.2.14 can be applied to the data

$$a(s,\xi) := K^\star(s)[\xi],\ b(s) := \bar{f}(s) - \lambda s\,, \quad \lambda \geq 0,$$

whence there exists at most one bounded solution of problem (P_0). □

3.2.3 Corrector results

In order to show corrector results in homogenization theory, it is common to presume that the limit function u and the corrector u_1 are sufficiently smooth in all their variables. In the linear case this is not a big deal, since u_1 separates in x and y. However, in the nonlinear case the situation is more complicated as can be seen from the representation

$$u_1(x,y) \;=\; \sum_{k=1}^{d} \partial_{x_k} u(x)\; \Lambda_k(y, u(x)). \tag{3.140}$$

We prove a first order corrector result under comparatively low regularity assumptions on the cell solutions and therefore on u_1. This is motivated by the statements in Corollary 2.1.7 and Remark 2.1.7, where we formulated sharp conditions on the admissibility of test functions for the notion of two-scale convergence. For simplicity we consider equation (3.91) with $\lambda = 0$. Since the trace of $x \mapsto u_1(x, \frac{x}{\varepsilon})$ on $\partial\Omega$ does in general not vanish, we need to consider the space $H^{1,2}(\Omega, d\mu_\varepsilon)$ obtained as the closure of $\mathcal{D}(\mathbb{R}^d)$ in the $\|\cdot\|_{1,2,\varepsilon}$-norm defined in (2.67) on page 28.

Theorem 3.2.16. *Let u_ε and u be solutions of problem (3.91) and of problem (P_0) respectively for $\lambda = 0$, with $u_\varepsilon \twoheadrightarrow u$ two-scale strongly according to Corollary 3.2.10. Assume that*

$$\Lambda_k : s \mapsto \Lambda_k(\cdot, s) \quad \textit{belongs to}\;\; \mathcal{C}^1(\mathbb{R}; L^\infty_\mu(\mathbb{T})) \cap \mathcal{C}(\mathbb{R}; H^{1,\infty}_\mu(\mathbb{T}))\,. \tag{3.141}$$

If in addition $u \in \mathcal{C}^2(\overline{\Omega})$, then there holds

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u - \varepsilon u_1(x, \tfrac{x}{\varepsilon})\|_{H^{1,2}(\Omega, d\mu_\varepsilon)} \;=\; 0\,. \tag{3.142}$$

Proof. Using (3.141) and the regularity assumption on u we get

$$F_k : x \;\mapsto\; \Lambda_k(\cdot, u(x)) \qquad \text{belongs to}\;\; \mathcal{C}(\overline{\Omega}; H^{1,\infty}_\mu(\mathbb{T})), \tag{3.143}$$

$$G_k : x \;\mapsto\; \partial_2\Lambda_k(\cdot, u(x)) \;\; \text{belongs to}\;\; \mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T})). \tag{3.144}$$

By virtue of (3.140) and (3.143), $u_1(x,y)$ is an element of $\mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T}))$. Hence by Corollary 2.1.7 we get

$$\int_\Omega |u_1(x, \tfrac{x}{\varepsilon})|^2\, d\mu_\varepsilon \;\to\; \int_{\Omega\times Y} |\sum_k \partial_k u(x) \Lambda_k(y, u(x))|^2\, dm \;<\; \infty\,. \tag{3.145}$$

Since u is smooth, by (3.131) and (3.145) it suffices to prove the convergence for the sequence of μ_ε-gradients in (3.142). By the regularity assumption on u it is clear that $u \in H^{1,2}(\Omega, d\mu_\varepsilon)$ for any ε. For u_1 we calculate

$$\begin{aligned}\nabla_{\mu_\varepsilon}[\varepsilon u_1(x,\tfrac{x}{\varepsilon})] &= \sum_k \partial_k u(x) P_\mu(\tfrac{x}{\varepsilon})[\nabla_y \Lambda_k(\tfrac{x}{\varepsilon},u)] + \varepsilon \sum_k \Lambda_k(\tfrac{x}{\varepsilon},u) P_\mu(\tfrac{x}{\varepsilon})[\nabla(\partial_k u)] \\ &\quad + \varepsilon \sum_k \partial_k u(x) \partial_2 \Lambda_k(\tfrac{x}{\varepsilon},u) P_\mu(\tfrac{x}{\varepsilon})[\nabla u] \ =: \ (I_1 + \varepsilon I_2 + \varepsilon I_3)(x,\tfrac{x}{\varepsilon}).\end{aligned}$$

By (3.141),(3.143) and (3.144) we have $I_j(x,y) \in \mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T}))$ and therefore $\|I_j(\cdot,\frac{\cdot}{\varepsilon})\|_{2,\varepsilon} \le C$ uniformly in ε by Corollary 2.1.7. It follows

$$\|u_\varepsilon - u - \varepsilon u_1(x,\tfrac{x}{\varepsilon})\|_{H^{1,2}(\Omega,d\mu_\varepsilon)} \ = \ \|\nabla_{\mu_\varepsilon}[u_\varepsilon - u] - I_1(x,\tfrac{x}{\varepsilon})\|_{2,\varepsilon} \ + \ o(1) \quad (3.146)$$

as $\varepsilon \to 0$. Thanks to Assumption 3.2.1 and the fact that the vector on the right-hand side in (3.146) is tangential, we get for $c_k > 0$:

$$c_k \|\nabla_{\mu_\varepsilon}[u_\varepsilon - u] - I_1(x,\tfrac{x}{\varepsilon})\|_{2,\varepsilon}^2 \ \le \ \int_\Omega \hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon) \left|\nabla_{\mu_\varepsilon}[u_\varepsilon - u] - I_1(x,\tfrac{x}{\varepsilon})\right|^2 d\mu_\varepsilon, \quad (3.147)$$

where here and for the rest of the proof we slightly abuse notation by writing $\hat{K}|v|^2$ for $v \cdot \hat{K}v$. We are done if we prove the convergence to zero of the right-hand side in (3.147), which comprises the following six terms:

$$\begin{aligned} J_1^\varepsilon &:= \int_\Omega \hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon)|\nabla_{\mu_\varepsilon} u_\varepsilon|^2\, d\mu_\varepsilon\,, & J_2^\varepsilon &:= -2\int_\Omega \hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon)\nabla_{\mu_\varepsilon} u_\varepsilon \cdot \nabla_{\mu_\varepsilon} u\, d\mu_\varepsilon\,, \\ J_3^\varepsilon &:= \int_\Omega \hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon)|\nabla_{\mu_\varepsilon} u|^2\, d\mu_\varepsilon\,, & J_4^\varepsilon &:= -2\int_\Omega \hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon)\nabla_{\mu_\varepsilon} u_\varepsilon \cdot I_1(x,\tfrac{x}{\varepsilon})\, d\mu_\varepsilon\,, \\ J_5^\varepsilon &:= \int_\Omega \hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon)|I_1(x,\tfrac{x}{\varepsilon})|^2\, d\mu_\varepsilon\,, & J_6^\varepsilon &:= 2\int_\Omega \hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon)\nabla_{\mu_\varepsilon} u \cdot I_1(x,\tfrac{x}{\varepsilon})\, d\mu_\varepsilon\,. \end{aligned}$$

We have to investigate the asymptotics of each term J_i^ε separately. Our first claim is

$$J_1^\varepsilon \ = \ \int_\Omega f(\tfrac{x}{\varepsilon},u_\varepsilon)u_\varepsilon\, d\mu_\varepsilon \ \to \ \int_\Omega \bar{f}(u)u\, dx\,. \quad (3.148)$$

We deduce the equality in (3.148) from Assumption 3.2.1 and a standard approximation argument, since u_ε is a solution of problem (3.91) for $\lambda = 0$. Using the Hölder continuity of f with respect to the second variable and the strong two-scale convergence of u_ε, we get

$$\int_\Omega [f(\tfrac{x}{\varepsilon},u_\varepsilon) - f(\tfrac{x}{\varepsilon},u)]\phi(x,\tfrac{x}{\varepsilon})\, d\mu_\varepsilon \ \to \ 0 \quad \forall \phi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbb{T})).$$

It follows $f(\frac{x}{\varepsilon},u_\varepsilon) \rightharpoonup\!\!\rightharpoonup f(y,u)$ two-scale weakly, and hence (3.148) by Proposition 2.1.13. As the proof of Lemma 3.2.6 shows, using the Hölder continuity of $\hat{K}$ with respect to s we get

$$\|\hat{K}(\tfrac{x}{\varepsilon},u_\varepsilon) - \hat{K}(\tfrac{x}{\varepsilon},u)\|_{2,\varepsilon} \ \to \ 0\,.$$

Choosing another representative of $(x,y) \mapsto \nabla_{\mu,y}\Lambda_k(y,u(x)) \in \mathcal{C}(\overline{\Omega}; L^\infty_\mu(\mathbb{T}))$ if necessary (cf. proof of Corollary 2.1.7), we get by (3.115),(3.143) and the regularity assumption on u:

$$\|\nabla_{\mu_\varepsilon} u_\varepsilon\|_{2,\varepsilon} + \|\nabla_{\mu_\varepsilon} u\|_{\infty,\varepsilon} + \|I_1(x, \tfrac{x}{\varepsilon})\|_{\infty,\varepsilon} \leq C$$

uniformly in ε. Hence it suffices to study the asymptotics of J_i^ε for $i \geq 2$ with $\hat{K}(\frac{x}{\varepsilon}, u_\varepsilon)$ replaced by $\hat{K}(\frac{x}{\varepsilon}, u)$. Then we easily check that

$$J_2^\varepsilon + J_3^\varepsilon \;\to\; -\int_{\Omega\times Y} \hat{K}(y,u)(P_\mu(y)[\nabla u] + 2\nabla_{\mu,y} u_1(y)) \cdot (P_\mu(y)[\nabla u])\, dm. \quad (3.149)$$

Again using Proposition 2.1.13 and inserting the two-scale limit of $\nabla_{\mu_\varepsilon} u_\varepsilon$ we get

$$\hat{K}(\tfrac{x}{\varepsilon}, u)[\nabla_{\mu_\varepsilon} u - \nabla_{\mu_\varepsilon} u_\varepsilon] \;\rightharpoonup\!\!\!\!\rightharpoonup\; -\hat{K}(y, u(x))[\nabla_{\mu,y} u_1(x,y)]$$

two-scale weakly. Note that $I_1(x, \frac{x}{\varepsilon}) \twoheadrightarrow \nabla_{\mu,y} u_1(x,y)$ two-scale strongly by the definition of I_1, (3.144) and Corollary 2.1.7. Therefore we obtain

$$J_4^\varepsilon + J_5^\varepsilon + J_6^\varepsilon \;\to\; -\int_{\Omega\times Y} \hat{K}(y,u)\nabla_{\mu,y} u_1(y) \cdot \nabla_{\mu,y} u_1(y)\, dm. \quad (3.150)$$

Using the symmetry of $\hat{K}$ and combining (3.148)-(3.150) we see that the term on the right-hand side in (3.147) converges as $\varepsilon \to 0$ to

$$\int_\Omega \bar{f}(u) u\, dx - \int_{\Omega\times Y} K(y,u)\, |P_\mu(y)[\nabla u] + \nabla_{\mu,y} u_1(y)|^2\, dm \quad (3.151)$$

However, the term in (3.151) vanishes, since by approximation the we can use (u, u_1) as a test function in the two-scale homogenized problem (3.120). □

We conclude this section by discussing some nontrivial examples, in which the cell solutions satisfy the required regularity. As Lemma 3.2.12 and Lemma 3.2.13 show, at least when $K(y,s)$ is smooth in s, we can expect $\Lambda_k \in \mathcal{C}^1(\mathbb{R}; L^\infty_\mu(\mathbb{T}))$ and $u \in \mathcal{C}^2(\Omega)$. The critical assumption in (3.141) is

$$\Lambda_k \in \mathcal{C}(\mathbb{R}; H^{1,\infty}_\mu(\mathbb{T}))\,. \quad (3.152)$$

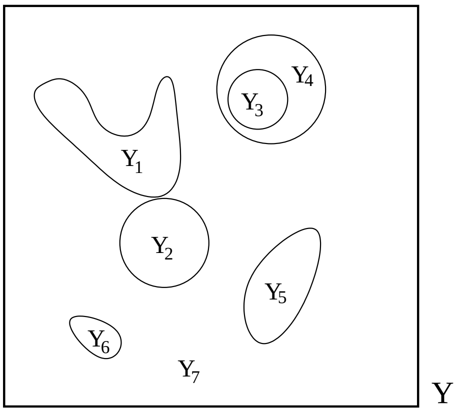
We emphasize that (3.152) is twofold: Does $\Lambda_k(\cdot, s)$ belong to $H^{1,\infty}_\mu(\mathbb{T})$ for any fixed $s \in \mathbb{R}$? If yes, does $\Lambda_k(\cdot,s) \in H^{1,\infty}_\mu(\mathbb{T})$ depend continuously on s? Concerning the first question, we sketch the regularity results available for energy solutions of the elliptic equation (with periodic b.c.)

$$-\mathrm{div}(K(y)[\nabla u(y) + \vec{e}_k]) = 0 \quad \text{in } Y \quad (3.153)$$

in the case $\mu = \mathcal{L}^d$, depending on the properties of the periodic coefficient K. To this end we refer to [7, 12, 23, 42, 43]:

$$\begin{aligned} K \in L^\infty(\mathbb{T}) \quad &\Rightarrow \quad u \in H^{1,2+\varepsilon}(\mathbb{T}) \quad \text{for some } \varepsilon = \varepsilon(d,K) > 0, \\ K \in \mathcal{C}(\mathbb{T}) \quad &\Rightarrow \quad u \in H^{1,p}(\mathbb{T}) \quad \text{for all } 1 \leq p < \infty, \\ K \in \mathcal{C}^{0,\alpha}(\mathbb{T}) \quad &\Rightarrow \quad u \in \mathcal{C}^{1,\tilde{\alpha}}(\mathbb{T}) \quad \text{for some } \tilde{\alpha} \leq \alpha. \end{aligned}$$

In particular we see that the answer to the first question can be negative, even for continuous (but not Hölder continuous) coefficients. However, we introduce two important applications for which we can expect (3.152), even for discontinuous coefficients.
The first example comprises diffusion in a composite medium occupying a bounded domain $D \subset \mathbb{R}^d$, whose physical characteristics, and hence the coefficients in the equation are smooth up to the boundary in some d-dimensional subdomains $D_n \subset D$, but not across the resulting interfaces. More precisely, for the periodic setting we assume that the reference cell Y contains $L-1$ disjoint subdomains $Y_1, \dots, Y_{L-1}$ of class $\mathcal{C}^{1,\alpha}, 0 < \alpha \le 1$, with $Y_n \subset\subset Y$, such that $Y = (\cup_{n=1}^L \overline{Y_n}) \setminus \partial Y$, where Y_L is the complement of the union of all $\overline{Y_n}$ for $n < L$ (cf. Figure 3.5 below). Moreover we assume that if a point in Y lies on some ∂Y_n, then the component of ∂Y_n containing the point is smooth. This implies that any point $y \in Y$ belongs to the boundaries of at most two of the Y_n (including ∂Y_L if any). However, refering to [42, Remark 1.2], we could also allow that some Y_n touch, as indicated in Figure 3.5.



Disjoint subdomains $Y_1, \dots, Y_6$ with ∂Y_n of class $\mathcal{C}^{1,\alpha}$ and

$$Y = (\bigcup_{n=1}^{7} \overline{Y_n}) \setminus \partial Y.$$

Figure 3.5: Composite material

The following statement follows from Theorem 6.8 of the appendix, which comprises $W^{1,\infty}$-estimates for solutions of divergence form elliptic equations with piecewise Hölder continuous coefficents [42, 43], posed in domains of the type defined above. We emphasize that the coefficients are allowed to be discontinuous across the interfaces. We consider here only full dimensional structures, so $\hat{K}$ coincides with the regular matrix $K(y,s)$ defined in (3.109).

Example 3.2.17. *Let $\mu = \mathcal{L}^d \lfloor Y$ and $Y = (\cup_{n=1}^L \overline{Y_n}) \setminus \partial Y$ as defined above. In addition to the positive definiteness (3.93), assume*

$$K(y,s) = \sum_{n=1}^{L} \chi_n(y) K_n(y,s) \quad \text{with } K_n(\cdot, s) \in \mathcal{C}^{0,\beta}(\overline{Y_n}; \mathbb{R}^{d\times d}) \tag{3.154}$$

for every $s \in \mathbb{R}$, where χ_n is the characteristic function of Y_n and $\beta \in (0,1)$. Then the cell solutions Λ_k introduced in Lemma 3.2.7 satisfy

$$\forall s \in \mathbb{R}: \quad \Lambda_k(\cdot, s) \in H^{1,\infty}_{\text{per}}(Y)\,. \tag{3.155}$$

Moreover, if $\|K_n(\cdot,s)\|_{\mathcal{C}^{0,\beta}(\overline{Y_n})} \le C$ *uniformly in* $s \in \mathbb{R}$*, then we have*

$$\Lambda_k \in L^\infty(\mathbb{R}; H^{1,\infty}_{\mathrm{per}}(Y)) \cap \mathcal{C}(\mathbb{R}; H^{1,p}_{\mathrm{per}}(Y)) \quad \forall\, p \in [1,\infty)\,. \tag{3.156}$$

Proof. We denote by $\tilde{\Lambda}^s_k \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ the Y-periodic extension of $\Lambda_k(\cdot,s)$ to the whole of $\mathbb{R}^d$. We can choose a bounded domain $D \subset \mathbb{R}^d$ of class $\mathcal{C}^{1,\alpha}$ with $Y \subset\subset D_\varepsilon$ for a suitable $\varepsilon > 0$, where $D_\varepsilon = \{x \in D : \mathrm{dist}(x,\partial D) > \varepsilon\}$, such that the assumptions of Theorem 6.8 are satisfied by the domain

$$D = \Big(\bigcup_{n=1}^{L} \overline{D_n}\Big) \setminus \partial D \quad \text{with } \overline{D_n} := \Big(\bigcup_{k\in\mathbb{Z}^d} (k + \overline{Y_n})\Big) \cap D.$$

Note that $\tilde{\Lambda}^s_k \in H^1(D)$ is a solution of $-\mathrm{div}\,(K(y,s)[\nabla\tilde{\Lambda}^s_k(y) + \vec{e}_k]) = 0$ in $\mathcal{D}'(D)$, and hence by (3.93) and (3.154) we can apply Theorem 6.8 to the tensor $A = K(\cdot,s)$ and the functions $g(x) = K(x,s)\vec{e}_k$ and $h \equiv 0$, and deduce

$$\|\nabla\Lambda_k(\cdot,s)\|_{L^\infty(Y)} \le \|\nabla\tilde{\Lambda}^s_k\|_{L^\infty(D_\varepsilon)} \le C\left(\|\tilde{\Lambda}^s_k\|_{L^2(D)} + \sum_{n=1}^{L} \|K_n(\cdot,s)\|_{\mathcal{C}^{0,\beta}(\overline{Y_n})}\right). \tag{3.157}$$

Since $\Lambda_k(\cdot,s)$ has mean value zero over Y, we get (3.155) by (3.121),(3.154) and (3.157). If the Hölder norm of $K_n(\cdot,s)$ does not depend on s, estimates (3.121) and (3.157) clearly show that $\|\Lambda_k(\cdot,s)\|_{H^{1,\infty}(Y)} \le C$ with a constant independent of $s \in \mathbb{R}$. This estimate combined with (3.126) and the Lebesgue dominated convergence theorem gives $\nabla\Lambda_k(\cdot,s_n) \to \nabla\Lambda_k(\cdot,s)$ strongly in $L^p(Y)$ for any $p < \infty$, whenever $s_n \to s$ in $\mathbb{R}$. This shows (3.156). $\square$

Note that (3.156) falls just short to guarantee (3.152), but indicates that we can expect the desired regularity of Λ_k for a large class of homogenization problems in composite media. Now we consider the case when μ is the one-dimensional Hausdorff measure on a regular thin network. As an example we take the normalized measures with support $S_1 \cup S_2$ and $S_1 \cup S$ respectively (cf. Figures 3.6–3.7), where $S_1 = (0,1) \times \{\frac{1}{2}\}, S_2 = \{\frac{1}{2}\} \times (0,1) \subset \mathbb{R}^2$ and

$$S := \{(y_1, f(y_1)) : \, y_1 \in (0,1)\} \subset \mathbb{R}^2\,, \quad f(y_1) = \tfrac{1}{4}(\sin(2\pi y_1) + 2)\,.$$

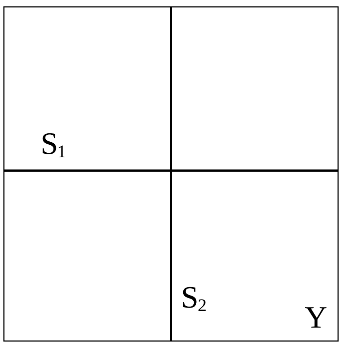


Figure 3.6: $\mu \propto (\mathcal{H}^1\lfloor S_1 + \mathcal{H}^1\lfloor S_2)$

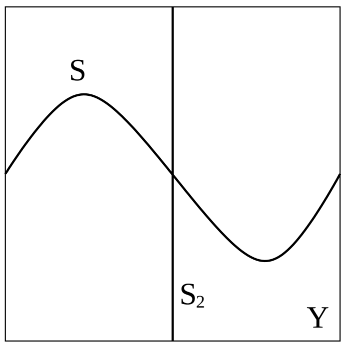


Figure 3.7: $\mu \propto (\mathcal{H}^1\lfloor S + \mathcal{H}^1\lfloor S_2)$

It is easy to check that both measures are strongly p-connected on $\mathbb{R}^d$. In both cases we get an explicit representation of the cell solution due to the one-dimensional nature of the problem and can show (3.152).

Example 3.2.18. *Let μ be one of the two measures defined above. In addition to Assumption 3.2.1, let $\hat{K} = \hat{K}(y,s)$ be Hölder continuous in s uniformly with respect to y, i.e. there exists $\gamma \in (0,1)$ and a constant $C \in \mathbb{R}$, such that*

$$\|\hat{K}(y,s) - \hat{K}(y,\tilde{s})\| \leq C|s-\tilde{s}|^\gamma \quad \forall y \in \mathbb{R}^d . \tag{3.158}$$

Then the cell solutions Λ_k satisfy (3.152), that means $\Lambda_k \in \mathcal{C}(\mathbb{R}; H^{1,\infty}_\mu(\mathbb{T}))$.

Proof. We consider the measure $\mu = \frac{1}{2}(\mathcal{H}^1\lfloor S_1 + \mathcal{H}^1\lfloor S_2)$ in Figure 3.6. In the other case the proof is slightly more involved, where one has to introduce tangential and normal coordinates on S. Then the cell problems can be solved by integration with respect to the tangential variable. Note that for the special measure μ in Figure 3.6 we have

$$\hat{K}(y,s) = \begin{cases} \operatorname{diag}\,(\Theta(y,s),0) & \text{if } y \in S_1 \\ \operatorname{diag}\,(0,\Theta(y,s)) & \text{if } y \in S_2 \end{cases}$$

according to Assumption 3.2.1, where $\Theta = \Theta_1$ is strictly positive and bounded from above uniformly in y and s. We have $\vec{e}_{1,\mu}(y) = 0$ on S_2 and vice versa. Hence for $k=1$ we can explicitly solve the cell problem (C_1) and obtain

$$\Lambda_1(y_1,y_2,s) = \int_0^{y_1} \left(\frac{M_1(s)}{K(\tau_1,\frac{1}{2},s)} - 1 \right) d\tau_1 + C\,, \tag{3.159}$$

where the constant C has to be chosen appropiately to ensure that Λ_1 has zero mean value over Y, and $M_1(s)$ is the harmonic mean of $K(\cdot,s)$ on S_1:

$$M_1(s) := \left(\int_0^1 \frac{d\tau_1}{K(\tau_1,\frac{1}{2},s)} \right)^{-1}, \quad \text{with } 0 < c_k \leq M_1(s) \leq C_K \tag{3.160}$$

for all $s \in \mathbb{R}$, which follows from (3.93). An easy calculation shows that $\Lambda_1(\cdot,s)$ belongs to $H^{1,\infty}_\mu(\mathbb{T})$. Now let $s_n \to s \in \mathbb{R}$ and denote by $\bar{y} := (y_1, \frac{1}{2})$ points on S_1. Then for μ-almost every $y \in S_1 \cup S_2$ we get

$$\begin{aligned} |\nabla_\mu \Lambda_1(y,s_n) - \nabla_\mu \Lambda_1(y,s)| &\leq C \sup_{\bar{y}\in S_1} \Big(|M(s_n) - M(s)| + |\hat{K}(\bar{y},s_n) - \hat{K}(\bar{y},s)| \Big) \\ &\leq C\left(|M(s_n) - M(s)| + |s_n - s|^\gamma\right) \end{aligned}$$

with a constant independent of y. This shows the statement for Λ_1. The same proof of course works for Λ_2, interchanging the role of y_1 and y_2. □

3.3 A nonlinear double porosity model

In the previous sections the rescaled permeabilty tensor K was of the form $K_\varepsilon = K(\frac{x}{\varepsilon})$, where $K(y)$ was a given Y-periodic matrix. We now consider the so called *double porosity* case, where the parameter ε is involved in a more complicated way. Our model is related to the behaviour of weakly compressible single phase flow through a fractured porous medium, occupying a domain $\Omega \subset \mathbb{R}^d$. Let us describe the main ingredients.
Assume that $\mathbb{R}^d$ is partitioned into two parts $F^\varepsilon = \varepsilon F$ and $F_0^\varepsilon = \varepsilon F_0$ of period ε. Each part is a separate porous medium, but the permeability coefficient in F^ε is of order 1, and in F_0^ε of order ε^α for some $\alpha > 0$. The Darcy law describing the filtration in such a composite system leads, in the linear case, to the equation

$$-\mathrm{div}\,(K_\varepsilon(x)\nabla u) = f, \quad K_\varepsilon(x) = \begin{cases} 1 & \text{in } F^\varepsilon, \\ \varepsilon^\alpha & \text{in } F_0^\varepsilon. \end{cases} \tag{3.161}$$

F^ε is sometimes called the *hard* phase, and is usually a connected subset of $\mathbb{R}^d$. F_0^ε is called the *soft* phase. In the physical literature one usually considers the *self-similar* case $\alpha = 2$. It turns out that this is the borderline case, in the sense that different effective equations arise for $\alpha < 2, \alpha = 2$ and $\alpha > 2$.
In the classical setting, the analysis of the double porosity model has been studied [6, 38, 59], however under fairly stringent restrictions on the smoothness of the phases and the correlation of fracture thickness and periodic length scale. Using the singular measure approach, we can also consider infintely thin structures, which has some applications in geohydrology and soil sciences [20]. As a consequence, we have to look for solutions of equation (3.161) in the spaces $H_0^{1,2}(\Omega, d\mu_\varepsilon)$. The asymptotics of equation (3.161) in the measure setting, including the linear non-stationary case, has been studied in [20, 62]. Using the methods developed in the previous sections, we can extend the analysis to some nonlinear cases (see equation (3.164) below).
Our methods apply to a variety of complex structures (see Figure 3.4 and the related discussion), but we content ourselves with the model problem $\circledast$ introduced in Paragraph 3.2.1 on the standard rectangular grid (cf. Figure 3.3). This already features the central aspects and main difficulties in the proofs. We set $F := \{\frac{1}{2}\} \times (0,1) \cup (0,1) \times \{\frac{1}{2}\}$, define $F_0 := Y \setminus F$ and consider the measure

$$\mu \lfloor Y \;=\; (\tfrac{1}{4}\mathcal{H}^1 \lfloor F) + (\tfrac{1}{2}\mathcal{L}^2 \lfloor F_0) \;=:\; \mu_1 + \mu_0\,, \tag{3.162}$$

which is normalized, doubling and strongly 2-connected on $\mathbb{R}^2$. Obviously, the measures μ_1 and μ_0 are mutually singular, that means $\mu_1(F_0) = 0$ and $\mu_0(F) = 0$. The measure μ, the sets F, F_0, the homothetic contractions $F^\varepsilon, F_0^\varepsilon$ and the fractured domain are sketched in Figures 3.2–3.3 on page 67. Recall that the rescaled measure reads

$$\mu_\varepsilon = \tfrac{1}{4}\varepsilon\mathcal{H}^1 \lfloor F^\varepsilon + \tfrac{1}{2}\mathcal{L}^2 \lfloor F_0^\varepsilon = (\mu_1)_\varepsilon + (\mu_0)_\varepsilon. \tag{3.163}$$

For a given number $\lambda > 0$ we study the quasilinear Dirichlet boundary value problem (cf. equation (3.91)) in the double porosity case, namely the equation

$$-\mathrm{div}\,(K_\varepsilon(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla_{\mu_\varepsilon} u_\varepsilon) + \lambda u_\varepsilon = f, \qquad u_\varepsilon \in H_0^{1,2}(\Omega, d\mu_\varepsilon) \tag{3.164}$$

and its asymptotics $\varepsilon \to 0$ subject to the following rescaled permeability tensor K_ε: For a given fixed $\alpha > 0$ and each $\varepsilon > 0$ we set

$$K_\varepsilon(\tfrac{x}{\varepsilon}, s) = \begin{cases} K_1(\tfrac{x}{\varepsilon}, s) & x \in F^\varepsilon, \\ \varepsilon^\alpha\, K_0(\tfrac{x}{\varepsilon}, s) & x \in F_0^\varepsilon, \end{cases} \tag{3.165}$$

where the functions $K_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, (y, s) \mapsto K_i(y, s)$ are μ_i-measurable and Y-periodic in y, Hölder continuous in s, and strictly positive and bounded:

$$\exists \gamma > 0\, \forall y \in \mathbb{R}^2\, \forall s, \tilde{s} \in \mathbb{R} : \quad |K_i(y, s) - K_i(y, \tilde{s})| \leq C|s - \tilde{s}|^\gamma, \tag{3.166}$$

$$\forall (y, s) \in \mathbb{R}^2 \times \mathbb{R} : \qquad 0 < c_k \leq K_i(y, s) \leq C_K. \tag{3.167}$$

For simplicity we assume $f \in \mathcal{C}(\overline{\Omega})$. We call a function $u_\varepsilon \in H_0^{1,2}(\Omega, d\mu_\varepsilon)$ a solution of the Dirichlet problem (3.164), if

$$\int_\Omega K_1(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi\, d\mu_\varepsilon^1 + \varepsilon^\alpha \int_\Omega K_0(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi\, d\mu_\varepsilon^0$$

$$+\lambda \int_\Omega u_\varepsilon \varphi\, d\mu_\varepsilon = \int_\Omega f\varphi\, d\mu_\varepsilon \qquad \forall \varphi \in \mathcal{D}(\Omega), \tag{3.168}$$

where we write $\mu_\varepsilon^i := (\mu_i)_\varepsilon$ for the ε-rescalings and denote for simplicity by ∇u_ε also the tangential gradient of u_ε with respect to μ_ε^1 on the singular domain $\Omega \cap F^\varepsilon$. Note that for $\alpha = 0$ we recover the setting of Paragraph 3.2.1. Existence and a priori estimates can be derived precisely in the same way:

Proposition 3.3.1. *For any $\varepsilon > 0$ there exists a solution of (3.164) in the sense of (3.168), which satisfies the estimate*

$$\int_\Omega |\nabla u_\varepsilon|^2\, d\mu_\varepsilon^1 + \varepsilon^\alpha \int_\Omega |\nabla u_\varepsilon|^2\, d\mu_\varepsilon^0 + \lambda \int_\Omega |u_\varepsilon|^2\, d\mu_\varepsilon \,\leq\, C < \infty \tag{3.169}$$

with a constant independent of ε. In particular, we get $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u(x, y)$ for some $u \in L_m^2(\Omega \times Y)$ and a subsequence, where $m = \mathcal{L}^2\lfloor\Omega \otimes \mu\lfloor Y$.

Proof. By the uniform lower bound (3.167) on the functions K_i we see that $K_\varepsilon(\tfrac{x}{\varepsilon}, s) \geq \varepsilon^\alpha c_k > 0$ in Ω, hence for fixed $\varepsilon > 0$ the existence of a solution can be derived exactly as in Corollary 3.2.4. The a priori estimate can then be obtained by testing (3.168) with the solution u_ε, applying standard absorption techniques and using (3.167) and the continuity of f up to the boundary. $\square$

The first step to determine the structure of the two-scale limit u is to study the asymptotics within the hard phase. Since its coefficient is of order one, we can take advantage of the results of the previous sections. The corresponding effective tensor (cf. Definition 3.2.8) we denote, for each $s \in \mathbb{R}$, by

$$(K_1^\star)_{ij}(s) := \int_Y K_1(y,s)[\vec{e}_{i,\mu_1}(y) + \nabla_{\mu_1}\Lambda_i^s(y)] \cdot (\vec{e}_{j,\mu_1}(y) + \nabla_{\mu_1}\Lambda_j^s(y))\, d\mu_1, \tag{3.170}$$

where for $k = 1, 2$ the function $\Lambda_k^s \in H^{1,2}_{\mu_1}(\mathbb{T})$ is the solution of the cell problem (C_k) defined in Lemma 3.2.7 with $\hat{K} = K_1$ and $\mu = \mu_1$. Note that $K_1^\star$ is symmetric. The following lemma will be frequently used and was proven in [62, Lemma 6.1] for the linear case.

Lemma 3.3.2. *There exists a function $\hat{u} \in H^1_0(\Omega)$, such that*

$$u(x,y) = \hat{u}(x) \quad \text{if } y \in F, \tag{3.171}$$

$$\int_\Omega K_1(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi\, d\mu_\varepsilon^1 \to \int_\Omega \langle K_1^\star(\hat{u})\nabla\hat{u}, \nabla\varphi\rangle\, dx \tag{3.172}$$

for each $\varphi \in \mathcal{D}(\Omega)$ and the subsequence selected in Proposition 3.3.1.

Proof. Let $\chi(y)$ be the characteristic function of F, which belongs to $L^\infty_\mu(\mathbb{T})$. Hence $\chi(\frac{x}{\varepsilon})u_\varepsilon(x) \rightharpoonup\!\!\rightharpoonup \chi(y)u(x,y)$ two-scale with respect to μ. Note that the restriction $u_\varepsilon\,|_{\Omega\cap F^\varepsilon}$ is uniformly bounded in $H_0^{1,2}(\Omega, d\mu_\varepsilon^1)$. Applying Theorems 2.4.4–2.4.5 to this restriction and the strongly connected measure μ_1, we get the existence of $\hat{u} \in H^1_0(\Omega)$ and $\hat{u}_1 \in L^2(\Omega, H^{1,2}_{\mu_1}(\mathbb{T}))$, such that

$$u_\varepsilon \twoheadrightarrow \hat{u}(x), \;\; \nabla_{\mu_\varepsilon^1} u_\varepsilon \rightharpoonup\!\!\rightharpoonup P_{\mu_1}(y)[\nabla\hat{u}(x)] + \nabla_{\mu_1,y}\hat{u}_1(x,y) \tag{3.173}$$

two-scale with respect to μ_1, possibly up to a further subsequence. But then, for arbitrary $\varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty(\mathbb{T}))$, we easily deduce

$$\begin{aligned} \int_\Omega u_\varepsilon(x)\chi(\tfrac{x}{\varepsilon})\varphi(\tfrac{x}{\varepsilon}, x)\, d\mu_\varepsilon(x) &= \int_\Omega u_\varepsilon(x)\varphi(\tfrac{x}{\varepsilon}, x)\, d\mu_\varepsilon^1(x) \\ &\to \int_{\Omega\times Y} \hat{u}(x)\chi(y)\varphi(x,y)\, dm\,, \end{aligned}$$

which implies $\chi(\frac{x}{\varepsilon})u_\varepsilon(x) \rightharpoonup\!\!\rightharpoonup \chi(y)\hat{u}(x)$ two-scale with respect to μ independent of the subsequence selected in (3.173), since $\chi(y)\hat{u}(x) = \chi(y)u(x,y)$. This shows (3.171). For $\psi(x) \in \mathcal{D}(\Omega)$ and $w(y) \in \mathcal{C}^\infty(\mathbb{T})$ we choose

$$\varphi(x) := \varepsilon\psi(x)w(\tfrac{x}{\varepsilon}), \;\; \varphi \in \mathcal{D}(\Omega)$$

as a test function in (3.168) and pass to the limit. Using (3.166),(3.173) and the same techniques as in the proof of Theorem 3.2.6, we deduce that the first term in (3.168) converges to

$$\int_{\Omega\times Y} K_1(y,\hat{u})[P_{\mu_1}(y)[\nabla\hat{u}] + \nabla_{\mu_1,y}\hat{u}_1(y)] \cdot \nabla_y w(y)\,\psi\, d\mu_1(y)dx. \tag{3.174}$$

We claim that all the other terms in (3.168) converge to zero. Let us estimate the only nontrivial term using (3.167):

$$\begin{aligned}\varepsilon^{\alpha}\left|\int_{\Omega} K_0(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla_y w(\tfrac{x}{\varepsilon})\,\psi\, d\mu_\varepsilon^0\right| &\le C\varepsilon^{\alpha}\|\nabla u_\varepsilon\|_{L^1(\Omega,d\mu_\varepsilon^0)} \\ &\le C\varepsilon^{\alpha/2}(\varepsilon^{\alpha}\|\nabla u_\varepsilon\|^2_{L^2(\Omega,d\mu_\varepsilon^0)}+1). \quad (3.175)\end{aligned}$$

Thanks to estimate (3.169), the right-hand side in (3.175) converges to zero. Since (3.168) holds as equality, we deduce that the term in (3.174) is actually equal to zero. Since ψ and w were arbitrary, we get that

$$\nabla_{\mu_1,y}\hat{u}_1(x,y) = \sum_k \partial_k \hat{u}(x)\nabla_{\mu_1}\Lambda_k(y,\hat{u}(x))$$

as in the proof of Corollary 3.2.10. Hence for any $\varphi \in \mathcal{D}(\Omega)$ we get

$$\begin{aligned}\int_{\Omega} K_1(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi\, d\mu_\varepsilon^1 &\to \int_{\Omega\times Y} K_1(y,\hat{u})(P_{\mu_1}(y)[\nabla\hat{u}] + \nabla_{\mu_1,y}\hat{u}_1)\cdot\nabla\varphi\, d\mu_1 dx \\ &= \int_{\Omega} K_1^\star(\hat{u})\nabla\hat{u}\cdot\nabla\varphi\, dx\end{aligned}$$

which proves (3.172). □

We will now distinguish the three cases $\alpha < 2, \alpha = 2$ and $\alpha > 2$, each one leading to a different effective problem. This generalizes the results for the linear setting studied in [62].

The case $\alpha < 2$ (high permeability) If the exponent is below the critical value (hence the permeability in the soft phase relatively high), only the flow in the hard phase is asymptotically relevant. In particular, the dependence of the permeability on the pressure does not make itself felt in the soft phase.

Theorem 3.3.3. *For $\alpha < 2$ the sequence $\{u_\varepsilon\}$ of solutions of (3.164) converges, up to subsequences, two-scale strongly with respect to μ to the function $u = \hat{u}(x) \in H_0^1(\Omega)$ in (3.171), which is a solution of the homogenized problem*

$$-\mathrm{div}(K_1^\star(\hat{u})\nabla\hat{u}) + \lambda\hat{u} = f \ \ \text{in}\ \Omega, \quad \hat{u} = 0 \ \ \text{on}\ \partial\Omega. \qquad (3.176)$$

Proof. Estimate (3.169) gives $\varepsilon^{\beta}\|\nabla u_\varepsilon\|_{2,\varepsilon} \le C$ with $\beta = \alpha/2 < 1$. By Theorem 2.4.4 we get that the weak two-scale limit u does not depend on y and hence $u = u(x) = \hat{u}(x)$ by Lemma 3.3.2. We choose φ in (3.168) only depending on the slow variable. Passing to the limit and using (3.172) and (3.175), we see that u is a solution of (3.176). It remains to prove the strong two-scale convergence with respect to μ, since (3.173) gives it only with respect to μ_1. Let z_ε be the unique solution of

$$-\mathrm{div}(K_\varepsilon(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla_{\mu_\varepsilon} z_\varepsilon) + \lambda z_\varepsilon \;=\; u_\varepsilon\,, \quad z_\varepsilon \in H_0^{1,2}(\Omega, d\mu_\varepsilon), \qquad (3.177)$$

which obviously fulfills the same estimate (3.169) as u_ε. Applying the techniques from the proof of Lemma 3.3.2 and using $\alpha < 2$, we get $z_\varepsilon \rightharpoonup\!\!\rightharpoonup z(x)$ two-scale weakly with respect to μ, where $z \in H_0^1(\Omega)$ is the solution of

$$-\mathrm{div}(K_1^\star(\hat{u})\nabla z) + \lambda z = \hat{u} \ \text{ in } \Omega\,, \quad z = 0 \ \text{ on } \partial\Omega. \tag{3.178}$$

Testing (3.164) with z_ε and subtracting (3.177) tested with u_ε, we get

$$\int_\Omega |u_\varepsilon|^2\, d\mu_\varepsilon = \int_\Omega f z_\varepsilon\, d\mu_\varepsilon \to \int_\Omega f z\, dx = \int_\Omega |\hat{u}|^2\, dx\,, \tag{3.179}$$

where in the last equality we compared (3.176) tested with z and (3.178) tested with u. □

The case $\alpha = 2$ (self similar case) This is the borderline case and from the analytic point of view the most difficult one. Here the a priori estimate (3.169) only gives

$$\|u_\varepsilon\|_{2,\varepsilon} + \varepsilon\|\nabla u_\varepsilon\|_{2,\varepsilon} \le C\,, \tag{3.180}$$

which implies that the two-scale limit u depends in general on y (cf. Theorem 2.4.4). Hence we have to choose a test function of type $\varphi = \psi(x)w(\frac{x}{\varepsilon})$ in (3.168), in order to gain information about the structure of u. The critical term that arises, namely

$$\varepsilon \int_\Omega K_0(\tfrac{x}{\varepsilon}, u_\varepsilon)\langle \nabla u_\varepsilon, \nabla_y w(\tfrac{x}{\varepsilon})\rangle\, \psi\, d\mu_\varepsilon^0 \tag{3.181}$$

is merely bounded this time. Since we can not expect $u_\varepsilon \twoheadrightarrow u$ with respect to μ, we have to assume $K_0 = K_0(y)$, otherwise there is no chance to pass to the limit in (3.181).

Definition 3.3.4. *The limit problem in the self-similar case reads as follows: Find $u \in Z$, such that*

$$\int_\Omega \langle K_1^\star(\hat{u})\nabla\hat{u}, \nabla\hat{\varphi}\rangle\, dx + \int_{\Omega\times Y} K_0(y)\nabla_{\mu,y}u(y)\cdot\nabla_{\mu,y}\varphi(y)\, dm$$
$$+\lambda\int_{\Omega\times Y} u(y)\varphi(y)\, dm = \int_{\Omega\times Y} f\varphi(y)\, dm \quad \forall\varphi\in Z\,, \tag{3.182}$$

where $Z := \{u \in L^2(\Omega; H_\mu^{1,2}(\mathbb{T})) :\ \nabla_{\mu,y}u|_F = 0,\ u|_F =: \hat{u}(x) \in H_0^1(\Omega)\}$.

Me make sure that problem (3.182) is well defined. Note that Z is a linear subspace of $L^2H_\mu^{1,2}$. If we define the set of pairs $\tilde{Z} := \{(u, \nabla_{\mu,y}u) : u \in Z\}$, then we see that

$$A : \tilde{Z}\times\tilde{Z} \to \mathbb{R},\ [(u,\nabla_{\mu,y}u),(v,\nabla_{\mu,y}v)] \mapsto$$
$$\int_\Omega \langle\nabla\hat{u},\nabla\hat{v}\rangle\, dx + \int_{\Omega\times Y} K_0(y)\nabla_{\mu,y}u\cdot\nabla_{\mu,y}v\, dm + \lambda\int_{\Omega\times Y} uv\, dm \tag{3.183}$$

is a scalar product on $\tilde{Z}$, and that $\tilde{Z}$ is a Hilbert space with respect to the induced norm. The properties (3.126) and (3.127) of $K_1^\star$ proven in Lemma 3.2.9 and the Lax-Milgram lemma guarantee that for any $w \in L^2(\Omega)$ there exists a unique pair $L_w := (u, \nabla_{\mu,y} u) \in \tilde{Z}$, such that the equation in (3.182) holds for any $(\varphi, \nabla_{\mu,y}\varphi) \in \tilde{Z}$ with $K_1^\star(\hat{u})$ replaced by $K_1^\star(w)$. Precisely as in the proof of Theorem 3.1.4, we can then find a fixed point of the operator

$$L : B_R \subset \tilde{Z} \to B_R,\ (u, \nabla_{\mu,y} u) \mapsto L_{\hat{u}} \in \tilde{Z}$$

whose first component is a solution of the limit problem in the sense of Definition 3.3.4. We can now formulate the homogenization theorem.

Theorem 3.3.5. *If $\alpha = 2$ and $K_0 = K_0(y)$, then the sequence $\{u_\varepsilon\}$ of solutions of (3.164) converges, up to subsequences, two-scale strongly with respect to μ to a solution $u \in Z$ of the homogenized problem (3.182).*

Proof. From the a priori estimate (3.180) and the proof of Lemma 2.4.4 we deduce, since ∇u_ε is tangential, that

$$u_\varepsilon \rightharpoonup\rightharpoonup u(x,y) \in L^2(\Omega; H^{1,2}_\mu(\mathbb{T})), \quad \varepsilon \nabla u_\varepsilon \rightharpoonup\rightharpoonup \nabla_{\mu,y} u(x,y). \tag{3.184}$$

If $\chi(y)$ is the characteristic function of F, using estimate (3.169) and the same technique as in the proof of (3.171), we get $\chi(y)\nabla_{\mu,y}u(x,y) = 0$. Hence by Lemma 3.3.2 the function u belongs to Z. Now consider the following set of functions

$$W := \{\varphi = \varphi_1(x) + \varphi_0(x)w(y) \mid \varphi_i \in \mathcal{D}(\Omega), w \in \mathcal{C}^\infty(\mathbb{T}), w|_F = \nabla w|_F = 0\}.$$

It is easy to see that W is a subset of Z. For $\varphi = \varphi_1 + \varphi_0 w \in W$ we choose $\varphi_\varepsilon(x) := \varphi(x, \frac{x}{\varepsilon}) \in \mathcal{D}(\Omega)$ as a test function in (3.168). Note that

$$\varepsilon^2 \int_\Omega K_0(\tfrac{x}{\varepsilon}) \nabla u_\varepsilon \cdot \nabla \varphi_\varepsilon \, d\mu_\varepsilon^0 = \varepsilon \int_\Omega K_0(\tfrac{x}{\varepsilon}) \nabla u_\varepsilon \cdot \nabla_y w(\tfrac{x}{\varepsilon}) \varphi_0 \, d\mu_\varepsilon^0 + o(1) \tag{3.185}$$

as $\varepsilon \to 0$ by estimate (3.175) and $\nabla\varphi_\varepsilon |_{\Omega\cap F^\varepsilon} = \nabla\varphi_1 |_{\Omega\cap F^\varepsilon}$ by the definition of W. Hence passing to the limit in (3.168) and using (3.172),(3.184) and (3.185) we get

$$\int_\Omega \langle K_1^\star(\hat{u})\nabla\hat{u}, \nabla\varphi_1\rangle \, dx + \int_{\Omega\times Y} K_0(y)\nabla_{\mu,y}u(y) \cdot \nabla_y \varphi(y) \, dm$$

$$+\lambda \int_{\Omega\times Y} u(y)\varphi(y) \, dm = \int_{\Omega\times Y} f\varphi(y) \, dm \tag{3.186}$$

for any $\varphi \in W$. As pointed out in [62, Section 5], the linear span of pairs $(\varphi, \nabla_y\varphi)$ with $\varphi \in W$ is dense in $\tilde{Z}$ with respect to the norm induced by the bilinear form A introduced in (3.183), hence (3.186) is sufficient to show that $u \in Z$ solves equation (3.182). The strong two-scale convergence with respect to μ can be proved precisely as in Theorem 3.3.3. $\square$

The case $\alpha > 2$ (low permeability) Here the contribution of the soft phase to the total energy in (3.169) is comparatively large, and the two components of the two-scale limit $u(x,\cdot)$ in F and F_0 are mutually independent. It turns out that $u(x,\cdot)\,|_{F_0}$ coincides with the source term up to the factor λ, so more generally we assume that

$$f = f_\varepsilon = g(\tfrac{x}{\varepsilon}, x), \quad g = g(y,x) \in L^2_\mu(\mathbb{T}; \mathcal{C}(\overline{\Omega})), \tag{3.187}$$

such that $f_\varepsilon(x) \twoheadrightarrow g(y,x)$ by Example 2.1.12. For $\alpha > 2$, the estimate (3.169) is not good enough to get a structure result like (3.184), all we know is $u_\varepsilon(x) \rightharpoonup\rightharpoonup u(x,y) \in L^2_m(\Omega \times Y)$. We have to determine the restrictions

$$u_0(x,y) := u\,|_{\Omega\times F_0}, \quad \hat{u}(x) = u\,|_{\Omega\times F}\,. \tag{3.188}$$

Theorem 3.3.6. *If $\alpha > 2$, then the sequence $\{u_\varepsilon\}$ of solutions of (3.164) with right-hand side (3.187) converges, up to subsequences, two-scale strongly with respect to μ to a function $u \in L^2_m(\Omega \times Y)$ composed as in (3.188), where u_0 is uniquely determined by $\lambda u_0(x,y) = g(x,y)$ in $\Omega \times F_0$ and $\hat{u} \in H^1_0(\Omega)$ is a solution of the decoupled problem*

$$-\mathrm{div}\,(K_1^\star(\hat{u})\nabla\hat{u}) + \lambda\mu(F)\hat{u} \;=\; \overline{g\chi} \quad in\,\Omega\,, \qquad \hat{u} = 0 \;\; on\;\partial\Omega, \tag{3.189}$$

where $\overline{g\chi} = \int_Y g(\cdot,y)\chi(y)\,d\mu(y)$ and χ is the characteristic function of F.

Proof. For the same test function $\varphi_\varepsilon(x) \in \mathcal{D}(\Omega)$ as in the proof of Theorem 3.3.5, we observe this time

$$\begin{aligned}
\varepsilon^{\alpha-1}\left|\int_\Omega K_0(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla_y w(\tfrac{x}{\varepsilon})\varphi_0\, d\mu^0_\varepsilon\right| \;&\le\; C\varepsilon^{\alpha-1}\|\nabla u_\varepsilon\|_{L^1(\Omega,d\mu^0_\varepsilon)} \\
&\le\; C\varepsilon^{(\alpha-2)/2}(\varepsilon^\alpha\|\nabla u_\varepsilon\|^2_{L^2(\Omega,d\mu^0_\varepsilon)} + 1),
\end{aligned}$$

hence the term converges to zero since $\alpha > 2$. Passing to the limit in (3.168) and using Lemma 3.3.2 we obtain

$$\int_\Omega \langle K_1^\star(\hat{u})\nabla\hat{u}, \nabla\varphi_1\rangle\, dx + \lambda\int_{\Omega\times Y} u\varphi\, dm = \int_{\Omega\times Y} g\varphi\, dm. \tag{3.190}$$

First setting $\varphi_1 = 0$ we get $\lambda u_0(x,y) = g(x,y)$ in $\Omega \times F_0$, where we used $w\,|_F = 0$. Then, setting $\varphi_0 = 0$ we deduce

$$-\mathrm{div}\,(K_1^\star(\hat{u})\nabla\hat{u}) + \lambda\bar{u} \;=\; \bar{g}\,,$$

where $(\bar{u}, \bar{g})(x) = \int_Y (u,g)(x,y)\,d\mu(y)$. It is then straightforward that the statement of the theorem follows from the observation

$$\lambda\bar{u}(x) = \int_{F_0} g(x,y)\,d\mu_0(y) + \lambda\mu(F)\hat{u}(x), \;\; \bar{g}(x) = \int_{F_0} g(x,y)\,d\mu_0(y) + \overline{g\chi}(x),$$

where we also used $\mu_1(F) = \mu(F)$. The strong two-scale convergence follows as in the other proofs using (3.187). □

4 Nonlinear parabolic problems

In this chapter we study the homogenization of the quasilinear degenerate elliptic-parabolic equation

$$\partial_t b(u_\varepsilon) - \operatorname{div} a_\varepsilon(\mu, x, b(u_\varepsilon), \nabla u_\varepsilon) \;=\; f_\varepsilon(\mu, x, b(u_\varepsilon)) \;\text{ in } \Omega \times (0,T), \tag{4.1}$$

subject to suitable boundary conditions and a strongly connected Radon measure μ on $\mathbb{R}^d$ (cf. (1.1) and (1.2) in the introduction). If $b(z) = z$, equation (4.1) can be seen as the natural parabolic extension of the elliptic problems considered in Chapter 3. However, for the applications we have in mind, including the homogenization of Richards equation studied in Section 4.3 below, the function b depends in a nonlinear way on the unknown u_ε. Typically b is monotonically nondecreasing, and problem (4.1) degenerates to an elliptic equation if b has a vanishing derivative. In contrast to Section 3.1 we will consider only the exponent $p = 2$, and the case when the flux a_ε separates in the gradient, that means

$$a_\varepsilon(\mu, x, b(u_\varepsilon), \nabla u_\varepsilon) \;=\; \big(K(\tfrac{x}{\varepsilon}, b(u_\varepsilon))\nabla u_\varepsilon\big)\, \mu_\varepsilon \tag{4.2}$$

for some tensor $K = K(y,s)$ that is μ-measurable and Y-periodic in y and sufficiently smooth in s. In order to get rid of a principle part a_ε that does not separate in the gradient, equation (4.1) has to be tested with the solution u_ε in the homogenization step (cf. the proof of Theorem 3.1.8). The problem is to pass to the limit in the first resulting expression

$$\int_0^T \langle\!\langle \partial_t b(u_\varepsilon), u_\varepsilon \rangle\!\rangle \, dt \;=\; \int_\Omega (u_\varepsilon b(u_\varepsilon))(x,T)\, d\mu_\varepsilon(x) + I_\varepsilon, \tag{4.3}$$

where I_ε is a term of minor severity that can be controlled by the initial data. The main reason is that we have no uniform control on the norm $\|u_\varepsilon(t)\|_{q,\mu_\varepsilon}$ for some $q \geq 1$ and $t \in (0,T]$, and hence the first term on the right-hand side in (4.3) can hardly be dealt with. Only if b is *strictly* monotonically increasing (and hence invertible), we have a chance to show $u_\varepsilon \to u$ strongly in $L^q(Q)$ and pass to the limit in (4.3). This is essentially used to derive corrector results for the homogenization of Richards equation on perforated domains (cf. Theorem 4.3.3 below). For strictly monotone b we can also consider time-oscillating data in (4.1) as indicated in (1.1).

Notation and preliminaries We assume that $T > 0$ is a fixed real number and denote by $Q := \Omega \times (0,T)$ the space time cylinder, where Ω is an open, bounded and connected subset of $\mathbb{R}^d$ with smooth boundary. As usual, we will always presume that μ is a positive, normalized, Y-periodic Radon measure on $\mathbb{R}^d$ with $\mu(\partial Y) = 0$. For the nonstationary setting we introduce the product measures

$$\nu := (\mu\lfloor Y) \otimes (\mathcal{L}^1\lfloor(0,T))\,, \quad n := (\mathcal{L}^{d+1}\lfloor Q) \otimes (\mu\lfloor Y). \tag{4.4}$$

By its definition ν is Y-periodic in space and hence defined on $\mathbb{R}^d \times (0,T)$. Moreover for any $q \in [1,\infty)$ we set

$$L_n^q := L^q(Q \times Y, dn), \quad \|u\|_{q,n}^q := \int_{Q\times Y} |u(x,t,y)|^q \, dn, \tag{4.5}$$

and similar for $q = \infty$. Now we can introduce the rescaled measure ν_ε. It is clear that ν rescales trivially with respect to time, and hence we have

$$\nu_\varepsilon \lfloor Q \;=\; (\mu_\varepsilon \lfloor \Omega) \otimes \left(\mathcal{L}^1 \lfloor (0,T)\right) , \tag{4.6}$$

where μ_ε is defined in (2.1) on page 13. Again, by the periodicity of μ and the weak convergence of μ_ε in (2.2) it is easy to check that

$$\nu_\varepsilon \rightharpoonup \left(\mu(Y)\mathcal{L}^d \lfloor \Omega\right) \otimes \left(\mathcal{L}^1 \lfloor (0,T)\right) = \mathcal{L}^{d+1} \lfloor Q. \tag{4.7}$$

In particular, we have $0 < \nu_\varepsilon(Q) \le C$ uniformly by the boundedness of Ω. For $q \in [1,\infty]$ we denote the corresponding Lebesgue spaces by $L^q(Q, d\nu_\varepsilon)$, or shorter $L^q_{\nu_\varepsilon}(Q)$, and to distinguish the norm with respect to μ_ε we write

$$\|u\|_{q,\mu_\varepsilon}^q := \int_\Omega |u(x)|^q \, d\mu_\varepsilon(x), \quad \|u\|_{q,\nu_\varepsilon}^q := \int_Q |u(x,t)|^q \, d\nu_\varepsilon(x,t) \tag{4.8}$$

for finite q, and similar for $q = \infty$. Moreover, if X is a Banach space we abbreviate by $L^qX := L^q(0,T;X)$ the space of measurable functions $u : [0,T] \to X$ with, respectively, finite norm

$$\|u\|_{L^qX}^q := \int_0^T \|u(t)\|_X^q \, dt, \quad \|u\|_{L^\infty X} := \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_X . \tag{4.9}$$

4.1 Two-scale structure results

In this section we extend, in an obvious way, the notion of two-scale convergence to the measures ν_ε. Moreover, we claim that the central two-scale structure result (Theorem 2.4.4) can be saved for the time-dependent setting (see Theorem 4.1.7 below). The main reason is that the rescaling of ν is trivial with respect to the time variable.

Two-scale convergence We introduce the notion of two-scale convergence to the nonstationary setting involving the rescaled mesure ν_ε. We also carry over the main results from Section 2.1.

Definition 4.1.1. *Let $v_\varepsilon \in L^p(Q, d\nu_\varepsilon)$ and $v \in L_n^p(Q \times \mathbb{T})$ for some $p \ge 1$. We say that the sequence $\{v_\varepsilon\}$ two-scale converges to v (with respect to ν and as $\varepsilon \to 0$) and write $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v$, if*

$$\lim_{\varepsilon\to 0} \int_Q v_\varepsilon(x,t)\psi(x,t,\tfrac{x}{\varepsilon}) \, d\nu_\varepsilon \;=\; \int_{Q\times Y} v(x,t,y)\psi(x,t,y) \, dn \tag{4.10}$$

for all $\psi \in \mathcal{D}(Q; \mathcal{C}^\infty(\mathbb{T}))$.

It is obvious, at least as long as no gradients are involved, that the two-scale convergence with respect to the new measure ν_ε enjoys all the properties proven in Section 2.1 for μ_ε, such as the weak compactness and the weak lower semicontinuity property. The proofs remain almost completely unchanged.

Proposition 4.1.2. *Let $p \in (1,\infty)$ and $\{v_\varepsilon\}$ be a sequence in $L^p(Q, d\nu_\varepsilon)$ with $\|v_\varepsilon\|_{p,\nu_\varepsilon} \le C$ uniformly in ε. Then, up to subsequences, there exists a function $v \in L^p(Q \times Y, dn)$, such that $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v$.*

Recall the weak lower semicontinuity property of two-scale convergence: If $v_\varepsilon \rightharpoonup\!\!\rightharpoonup v$ for some $v \in L^p_n(Q \times Y)$ and $p \in (1,\infty)$, then there holds

$$\liminf_{\varepsilon\to 0} \int_Q |v_\varepsilon|^p \, d\nu_\varepsilon \ \ge \ \int_{Q\times Y} |v|^p \, dn \, . \tag{4.11}$$

Hence the following notion of strong two-scale convergence with respect to ν, which of course corresponds to Definition 2.1.11 from the stationary setting:

Definition 4.1.3. *Let $v_\varepsilon \in L^p(Q, d\nu_\varepsilon)$ and $v \in L^p_n(Q \times \mathbb{T})$ for some $p > 1$. We say that $\{v_\varepsilon\}$ two-scale strongly converges to v (with respect to ν and as $\varepsilon \to 0$) and write $v_\varepsilon \twoheadrightarrow v$, if*

$$v_\varepsilon \rightharpoonup\!\!\rightharpoonup v \ \ \textit{and} \ \ \limsup_{\varepsilon\to 0} \int_Q |v_\varepsilon|^p \, d\nu_\varepsilon \ \le \ \int_{Q\times Y} |v|^p \, dn \, . \tag{4.12}$$

It is important to find a sufficiently large class of functions $v = v(x,t,y)$ on $Q \times \mathbb{T}$, that converge strongly in the sense of (4.12) under the rescaling $y \leftrightarrow \frac{x}{\varepsilon}$. In most cases we need the time dependent version of Example 2.1.12:

Example 4.1.4. *Let $v \in L^p_\mu(\mathbb{T}; \mathcal{C}(\overline{Q}))$ and $v_\varepsilon(x,t) := v(x,t,\frac{x}{\varepsilon})$. Then there holds $v_\varepsilon \twoheadrightarrow v$.*

We have seen that passing to the limit in products of weakly and strongly two-scale convergent sequences is essential for the treatment of nonlinear problems. The proof of the following result coincides with the one of Proposition 2.1.13.

Proposition 4.1.5. *Let $p > 1$ and v_ε be a sequence in $L^p(\Omega, d\nu_\varepsilon)$ that strongly two-scale converges to $v \in L^p_n(Q \times Y)$. Let w_ε be a bounded sequence in $L^{p'}(Q, d\nu_\varepsilon)$ with $w_\varepsilon \rightharpoonup\!\!\rightharpoonup w$ for some $w \in L^{p'}_n(Q \times Y)$. Then there holds*

$$v_\varepsilon w_\varepsilon \nu_\varepsilon \ \rightharpoonup \ \left(\int_Y v(\cdot,y) w(\cdot,y) \, d\mu(y) \right) \mathcal{L}^{d+1} \lfloor Q \, . \tag{4.13}$$

Homogenization structure result We turn our attention to the central structure result for all possible two-scale limits of bounded sequences $\|\nabla u_\varepsilon\|_{p,\nu_\varepsilon}$, where the gradient is taken with respect to the space variable. Similar as in Definition 2.4.3, we first introduce the class of functions to which the correctors will belong:

Definition 4.1.6. *We say that a function $u = u(x,t,y) \in L^p_n(Q\times\mathbb{T})$ belongs to the class $L^p(Q;\tilde H^{1,p}_\mu(\mathbb{T}))$ and $\nabla_y u \in L^p_n(Q\times\mathbb{T})^d$ is one of its gradients, if*

$$\varphi_n \to u\,, \quad \nabla_y\varphi_n \to \nabla_y u \quad \textit{strongly in } L^p_n(Q\times Y) \tag{4.14}$$

for a sequence $\varphi_n \in \mathcal{C}^\infty(\overline{Q}\times\mathbb{T})$.

Recall from Section 2.4 that if μ is strongly p-connected on $\mathbb{T}$, then any vector $v \in L^p(Q;V^p_{\rm pot}(\mathbb{T}))$ corresponds to a unique function $\hat u = \hat u(x,t,y)$, such that

$$\hat u(x,t,\cdot) \in \tilde H^{1,p}_\mu(\mathbb{T}),\ \int_Y \hat u(x,t,y)\,d\mu(y) = 0 \quad\text{and}\quad v = \nabla_y\hat u\,. \tag{4.15}$$

For the stationary case we gave a rigorous proof (cf. Theorem 2.4.4) of the two-scale structure result, but we content ourselves with merely stating the result for the time dependent case, since all the nontrivial effects are related to the spatial variable.

Theorem 4.1.7. *Assume $p \in (1,\infty)$ and consider a sequence $(u_\varepsilon,\nabla u_\varepsilon) \in L^p(0,T;V^p(\Omega,d\mu_\varepsilon))$ subject to the uniform bound*

$$\|u_\varepsilon\|_{p,\nu_\varepsilon} + \|\nabla u_\varepsilon\|_{p,\nu_\varepsilon} \le C\,. \tag{4.16}$$

Let $u_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u \in L^p_n(Q\times Y)$ and $\nabla u_\varepsilon \rightharpoonup\!\!\!\rightharpoonup \chi \in L^p_n(Q\times Y)^d$ two-scale weakly with respect to ν. Then there holds

1. *If μ satisfies $(H1)$, then $u = u(x,t)$ is independent of y.*
2. *If μ satisfies $(H2)$ and $(H3)$, then additionally $u \in L^p(0,T;W^{1,p}_0(\Omega))$, and there exists $\tilde u_1 \in L^p(Q;\tilde H^{1,p}_\mu(\mathbb{T}))$, such that*

$$\chi(x,t,y) \;=\; \nabla_x u(x,t) + \nabla_y\tilde u_1(x,t,y)\,, \tag{4.17}$$

where $\nabla_y\tilde u_1 \in L^p(Q;V^p_{\rm pot}(\mathbb{T}))$ is a gradient of $\tilde u_1$ according to Definition 4.1.6. Moreover there holds

$$\nabla_{\mu_\varepsilon}u_\varepsilon \;\rightharpoonup\!\!\!\rightharpoonup\; P_\mu(y)[\nabla u(x,t)] + \nabla_{\mu,y}u_1(x,t,y)\,, \tag{4.18}$$

where $u_1(x,t,y)$ is the corresponding element of the Banach space $L^p(Q;H^{1,p}_\mu(\mathbb{T}))$.

4.2 Degenerate elliptic-parabolic equations

In this section we study the homogenization of the doubly nonlinear degenerate parabolic equation

$$(P_\varepsilon)\quad \begin{cases} \partial_t b(u_\varepsilon) - \operatorname{div}\big(K(\tfrac{x}{\varepsilon},b(u_\varepsilon))\nabla u_\varepsilon\big) &=& f(\tfrac{x}{\varepsilon},b(u_\varepsilon)) & \text{in } \Omega\times(0,T),\\ b(u_\varepsilon) &=& b^0_\varepsilon & \text{in } \Omega\times\{0\},\\ u_\varepsilon &=& 0 & \text{on } \partial\Omega\times(0,T) \end{cases}$$

on multidimensional structures associated with a measure μ and its periodic rescalings μ_ε (cf. (4.1) in combination with (4.2)). Our main structure conditions on the data comprise the monotonicity of b and the assumptions from Section 3.1 on μ, a and f. Recall that equation (P_ε) is degenerate in the sense that it changes type from parabolic to elliptic if b has a vanishing derivative. The following structure conditions will be kept, unless otherwise stated, throughout this section.

Assumption 4.2.1. *Let μ be strongly 2-connected on $\mathbb{R}^d$ and doubling, and assume the following structure conditions on the data:*

1. *$\Omega \subset \mathbb{R}^d$ is an open, bounded and connected set with smooth boundary, $Q := \Omega \times (0,T)$ is the space-time cylinder with $T > 0$ fixed.*
2. *$b : \mathbb{R} \to \mathbb{R}$ is monotone nondecreasing and continuous with $b(0) = 0$. The Legendre transform Ψ of the primitive of b is defined by*
$$\Psi : \mathbb{R} \to [0, +\infty], \quad s \mapsto \sup_{z\in\mathbb{R}} \left(zs - \int_0^z b(\tau)\, d\tau \right), \tag{4.19}$$
and therefore a convex and lower semicontinuous function on $\mathbb{R}$.
3. *The coefficient $K : \mathbb{R}^d \times \mathbb{R} \to \mathcal{M}^d_{\mathrm{sym}}$ is μ-measurable and Y-periodic in y, continuous in s, and there exist constants $c_k, C_K > 0$, such that for all $(y,s) \in \mathbb{R}^d \times \mathbb{R}$:*
$$c_k|\xi|^2 \le \xi \cdot K(y,s)\xi \le C_K|\xi|^2 \quad \forall \xi \in \mathbb{R}^d. \tag{4.20}$$
4. *The source $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}, (y,s) \mapsto f(y,s)$ is μ-measurable and Y-periodic in y, and continuous in s. Moreover there exist constants $c_3 > 0$ and $\beta \in [0,1)$, such that for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}$:*
$$|f(y, b(z))| \le c_3(1 + |b(z)|^\beta). \tag{4.21}$$
5. *For the initial data we assume $\psi(b^0_\varepsilon) \in L^1(\Omega, d\mu_\varepsilon)$, and that there exists a μ_ε-measurable function u^0_ε with $b^0_\varepsilon = b(u^0_\varepsilon)$ μ_ε-almost everywhere.*

Occasionally we will also require that b admits at most linear growth, which is automatically true if b is Lipschitz continuous:
$$\exists L \in \mathbb{R} : \quad |b(z)| \le L(1 + |z|) \quad \forall z \in \mathbb{R}. \tag{4.22}$$

We collect some important properties of the transformation Ψ defined above. For the following remark we refer to [2, 51].

Remark 4.2.2. *The function Ψ defined in (4.19) admits the following representation and superlinearity property:*
$$\forall z \in \mathbb{R} : \quad B(z) := \Psi(b(z)) = zb(z) - \int_0^z b(\tau)\, d\tau\,. \tag{4.23}$$
$$\forall \delta > 0\, \exists C_\delta \in \mathbb{R} : \quad |s| \le \delta\Psi(s) + C_\delta \quad \forall s \in \mathbb{R}\,. \tag{4.24}$$

Lemma 4.2.3. *Let b satisfy (4.22) with a constant $L > 0$. Then the function ψ enjoys the following coercivity property:*

$$\forall \eta \in b(\mathbb{R}) : \quad \Psi(\eta) \geq \frac{1}{2L}\eta^2 - |\eta|. \tag{4.25}$$

Moreover for any $(y, z) \in \mathbb{R}^d \times \mathbb{R}$ there holds $|f(y, b(z))| \leq C(1 + B(z)^{1/2})$, where C is a constant and B defined in (4.23).

Proof. It is easy to see that $\Psi(0) = 0$, and it suffices to show (4.25) for $\eta > 0$, the case $\eta < 0$ is completely analogue. We consider the inverse $\lambda := b^{-1}$ of b, more precisely the multivalued map $\lambda : b(\mathbb{R}) \to 2^{\mathbb{R}}$ defined by

$$\forall \tau \in b(\mathbb{R}) : \quad z \in \lambda(\tau) \Leftrightarrow b(z) = \tau.$$

It is easy to check (see [58, Section 2.1] for the details), that for any section $\tilde{\lambda}$ of λ, that means pointwise for arbitrary $\tilde{\lambda}(\tau) \in \lambda(\tau)$, there holds

$$\Psi(\eta) = \int_0^\eta \tilde{\lambda}(\tau)\, d\tau. \tag{4.26}$$

In particular, for any admissible $\tilde{\lambda}$ and $\tau \in (0, \eta)$ we have $\tilde{\lambda}(\tau) > 0$ by the monotonicity of b and $b(\tilde{\lambda}(\tau)) = \tau \leq L(1 + \tilde{\lambda}(\tau))$ by (4.22). Hence by (4.26) we get

$$\Psi(\eta) \geq \int_0^\eta (\frac{\tau}{L} - 1)\, d\tau = \frac{1}{2L}\eta^2 - \eta,$$

which proves (4.25) for $\eta > 0$. This combined with the superlinearity property (4.24) we get

$$b(z)^2 \leq 2L(|b(z)| + \Psi(b(z))) \leq C(2\Psi(b(z)) + C_1),$$

and hence $|b(z)| \leq C(1 + B(z)^{1/2})$. Then the second statement of the lemma directly follows from (4.21). $\square$

4.2.1 Existence

This paragraph is dedicated to show an existence result for problem (P_ε). To this aim we first introduce a natural solution space related to the measure ν_ε. The concept is similar from Definition 2.4.3, now with the time interval playing the role of a parameter set. Naturally, the solution space comprises the class of Sobolev functions with zero trace.

Definition 4.2.4. *We say that a function $u = u(t, x) \in L^2(Q, d\nu_\varepsilon)$ belongs to the class $L^2(0, T; \tilde{H}_0^{1,2}(\Omega, d\mu_\varepsilon))$ and $\nabla u \in L^2(Q, d\nu_\varepsilon)^d$ is a gradient, if*

$$\varphi_n \to u, \quad \nabla_x \varphi_n \to \nabla u \quad \textit{strongly in } L^2(Q, d\nu_\varepsilon) \tag{4.27}$$

for a sequence $\varphi_n \in \mathcal{C}^\infty([0, T]; \mathcal{D}(\Omega))$.

We emphasize that whenever $u \in L^2(0,T;\tilde{H}_0^{1,2}(\Omega,d\mu_\varepsilon))$ with gradient ∇u, then the function

$$U := (u, \nabla u), t \mapsto (u(t), \nabla u(t))$$

belongs to the space $L^2(0,T;V^2(\Omega,d\mu_\varepsilon))$. Conversely, by density of smooth functions in the corresponding Banach spaces, the first component u of a pair $(u,\nabla u) \in L^2(0,T;V^2(\Omega,d\mu_\varepsilon))$ belongs to the class given by Definition 4.2.4. We can now introduce the notion of weak solutions. Note that under Assumption 4.2.1.5, the initial value b_ε^0 belongs to $L^1(\Omega,d\mu_\varepsilon)$ by (4.24).

Definition 4.2.5. *Under Assumption 4.2.1.5 on the initial value b_ε^0, a function $u_\varepsilon \in L^2(0,T;\tilde{H}_0^{1,2}(\Omega,d\mu_\varepsilon))$ with $b(u_\varepsilon) \in L^2(Q,d\nu_\varepsilon)$ is called a weak solution of the initial boundary value problem (P_ε), if the integral identity*

$$\int_Q \big(-b(u_\varepsilon)\partial_t\varphi + K(\tfrac{x}{\varepsilon}, b(u_\varepsilon))\nabla u_\varepsilon \cdot \nabla\varphi\big)\, d\nu_\varepsilon \;=\; \int_Q f(\tfrac{x}{\varepsilon}, b(u_\varepsilon))\varphi\, d\nu_\varepsilon + \int_\Omega b_\varepsilon^0 \varphi(0)\, d\mu_\varepsilon \quad (4.28)$$

holds for some gradient ∇u_ε of u_ε and all functions $\varphi \in \mathcal{C}^\infty([0,T];\mathcal{D}(\Omega))$ with $\varphi(T) = 0$.

Since we need to show existence for fixed $\varepsilon > 0$, until the rest of this paragraph we will use the following abbreviations unless otherwise stated:

$$V := V^2(\Omega,d\mu_\varepsilon), \quad X := L^2(\Omega,d\mu_\varepsilon). \quad (4.29)$$

Note that after an obvious identification we have $X \subset\subset V'$ by Lemma 2.4.1 and Lemma 6.3. Let us give an interpretation of the weak formulation in (4.28) for the case that b satisfies a linear growth condition (4.22). By Lemma 4.2.3, this implies that $b_\varepsilon^0 \in X \subset V'$, which is needed for the identification (4.30) below. By (4.28) and the assumptions on the data we check

$$\frac{\partial b(u_\varepsilon)}{\partial t} - \operatorname{div}\left[K(\cdot, b(u_\varepsilon))\nabla u_\varepsilon\right] \;=\; f(\cdot, b(u_\varepsilon)) \quad \text{in } H^{-1}(0,T;V'),$$

whence $\partial_t b(u_\varepsilon) = f(\cdot,b(u_\varepsilon)) + \operatorname{div}\left[K(\cdot,b(u_\varepsilon))\nabla u_\varepsilon\right] \in L^2(0,T;V')$. It follows that $b(u_\varepsilon) \in H^1(0,T;V')$ and, by integrating (4.28) by parts in time,

$$b(u_\varepsilon)\,|_{t=0} = b_\varepsilon^0 \quad \text{in } V' \quad \text{(in the sense of traces of } H^1(0,T;V')\text{)}. \quad (4.30)$$

Let us also check that the gradient of a solution u_ε is unique:

Remark 4.2.6. *The gradient ∇u_ε of a solution u_ε according to Definition 4.2.5 is uniquely determined in $L^2(Q,d\nu_\varepsilon)$, and the flux $K(\cdot,b(u_\varepsilon))\nabla u_\varepsilon$ is tangential to the structure μ_ε for almost every $t \in (0,T)$.*

Proof. The first statement follows from the second, if we use the uniform lower bound on K in (4.20) and take into account that the difference of two possible gradients of u_ε is normal. Now let $\varphi_n \in \mathcal{D}(\Omega)$ be an approximating sequence for $(0, z) \in V$. For $\psi \in \mathcal{D}(0, T)$, use $\psi_n(t, x) := \psi(t)\varphi_n(x)$ as a test function in (4.28) as pass to the limit $n \to \infty$. This yields

$$\int_Q \psi(t) K(\tfrac{x}{\varepsilon}, b(u_\varepsilon)) \nabla u_\varepsilon \cdot z(x)\, d\nu_\varepsilon \;=\; 0.$$

Since $z \in \Gamma^2(\Omega, d\mu_\varepsilon)$ and ψ where arbitrary, the second statement follows. □

One could try to carry over the existence result of Alt and Luckhaus [2], where for the case $\mu = \mathcal{L}^d$ equations of type (P_ε) subject to Assumption 4.2.1 were investigated (see also Theorem 6.10). However, our proof will slightly simplify assuming that b is Lipschitz continuous, a condition we require in the homogenization step anyway (cf. Lemma 4.2.8 below). We use a Rothe method of time discretization similar as in the proof of [58, Theorem 4.2], where b was assumed to be strictly monotone. As we will see, the Lipschitz condition compensates for the lack of strict monotonicity, so we are safe to require that b is merely monotonically nondecreasing. Recall the definition of the norm $\|\cdot\|_{L^pX}$ in (4.9) and the abbreviations of the reflexive Banach spaces V and X in (4.29).

Theorem 4.2.7. *In addition to Assumption 4.2.1, let $b : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then there exists a solution u_ε of problem (P_ε) in the sense of Definition 4.2.5, which fulfills, after an obvious identification, the estimate*

$$\|\partial_t b(u_\varepsilon)\|_{L^2V'} + \|b(u_\varepsilon)\|_{L^\infty X} + \|(u_\varepsilon, \nabla u_\varepsilon)\|_{L^2V} \le C(1 + \|\Psi(b_\varepsilon^0)\|_{1,\mu_\varepsilon}), \tag{4.31}$$

with a constant C independent of ε.

Proof. It is important to note throughout the proof that each function $v \in X$ defines uniquely a continuous linear functional on V by

$$\langle\!\langle v, (\varphi, \nabla\varphi)\rangle\!\rangle := \int_\Omega v(x)\varphi(x)\, d\mu_\varepsilon(x), \quad v \in V', \tag{4.32}$$

and that the embedding $X \hookrightarrow V'$ is compact as argued above. We fix a large number $N \in \mathbb{N}$, the step margin $\Delta t := \frac{T}{N}$ and set $t_n := n\Delta t$ for all $n \in \{0, 1, \ldots, N\}$. Moreover for any $N \in \mathbb{N}$ we set

$$b_{\varepsilon,N}^0 := b_\varepsilon^0 \in X \tag{4.33}$$

by (4.25) and Assumption 4.2.1.5, since b is Lipschitz. Again, to shorten the notation we omit the dependence of the data on the spatial variable $\varepsilon^{-1}x$. For every time step $n \ge 1$ we have to solve the following, time discretized

Problem: Given $b_{\varepsilon,N}^{n-1} \in X$, find $b_{\varepsilon,N}^n \in X$ and $(u_{\varepsilon,N}^n, \nabla u_{\varepsilon,N}^n) \in V$ with $b_{\varepsilon,N}^n = b(u_{\varepsilon,N}^n)$ μ_ε–almost everywhere in Ω and

$$\int_\Omega \left(\frac{b_{\varepsilon,N}^n - b_{\varepsilon,N}^{n-1}}{\Delta t}\, \varphi \,+\, K(b_{\varepsilon,N}^n) \nabla u_{\varepsilon,N}^n \cdot \nabla\varphi \right) d\mu_\varepsilon \;=\; \int_\Omega f(b_{\varepsilon,N}^n)\varphi\, d\mu_\varepsilon \tag{4.34}$$

for any $\varphi \in \mathcal{D}(\Omega)$. We need to show that problem (4.34) has a solution. The proof is of course very similar to the one of Theorem 3.1.4, so we only highlight the few differences. Fix $v \in X$ and define the operators $T_v^n : V \to V'$ and $g_v^n : V \to \mathbb{R}$ by

$$\langle\langle T_v^n(u,u_1),(\varphi,\varphi_1)\rangle\rangle \;:=\; \int_\Omega \left(\tfrac{1}{\Delta t}\, b(u)\varphi + K(b(v))u_1\cdot\varphi_1\right)\, d\mu_\varepsilon\,, \quad (4.35)$$

$$\langle\langle g_v^n,(\varphi,\varphi_1)\rangle\rangle \;:=\; \int_\Omega \left(f(b(v)) + \tfrac{1}{\Delta t}\, b_{\varepsilon,N}^{n-1}\right)\varphi\, d\mu_\varepsilon\,. \quad (4.36)$$

Since $b_{\varepsilon,N}^{n-1} \in X$, we see that $g_v^n \in V'$ by (4.21),(4.22) and (4.32). We apply the Browder-Minty theorem in order to find a unique solution of the equation $T_v^n(u,u_1) = g_v^n$ in V'. The coercivity of T_v^n immediately follows from the Poincaré estimate (2.92) and the fact that the term $\int_\Omega b(u)u$ is nonnegative. The hemicontinuity and the strict monotonicity one can show exactly as in the proof of Theorem 3.1.4, where we have to use $\int_\Omega (b(u)-b(w))(u-w) \geq 0$ thanks to the monotonicity of b, and

$$\left|\int_\Omega \frac{1}{\Delta t}(b(u+tw)-b(u))z\right| \to 0 \quad \text{for } t\to 0$$

and all $(u,u_1),(w,w_1),(z,z_1) \in V$ by the Lipschitz continuity of b. It follows that the solution operator

$$L : V \to V,\ (v,v_1) \mapsto (u,u_1)\,,$$

which does not depend on v_1, is well defined. Testing the operator equation $T_v^n(u,u_1) = g_v^n$ with (u,u_1) and using (4.20),(4.21), the Poincaré estimate (2.92) and standard absorption techniques, we get precisely as in (3.21)-(3.23):

$$\|(u,u_1)\|_V^2 \;\leq\; C(1+\|f(b(v))\|_X^2) \;\leq\; \tilde{C} + \tfrac{1}{2}\|(v,v_1)\|_V^2\,,$$

where the constant $\tilde{C}$ depends only on $\Delta t, c_k, |\Omega|, \|b_{\varepsilon,N}^{n-1}\|_X$ and the Poincaré constant in (2.92). Hence we find a radius R, such that $L : \bar{B}_R \subset V \to \bar{B}_R$. To show the compactness of the operator L, let $(u^m,u_1^m) = L(v^m,v_1^m)$ be a sequence in $L(\bar{B}_R)$. As the proof of Theorem 3.1.4 shows, we get

$$v^m \to v,\ u^m \to u \quad \text{strongly in } X, \quad u_1^m \rightharpoonup u_1 \quad \text{weakly in } X^d \quad (4.37)$$

for a subsequence and some $v \in X$ and $(u,u_1) \in V$. By the continuity of K and b and estimate (4.20), we clearly get $K(b(v^m))u_1 \to K(b(v))u_1$ strongly in X^d. Using the solution property of (u^m,u_1^m) and (4.37) we get

$$\int_\Omega K(b(v^m))u_1^m\cdot(u_1^m-u_1) \;=\; \int_\Omega \left(f(b(v^m)) + \tfrac{1}{\Delta t}(b_{\varepsilon,N}^{n-1} - b(u^m))\right)(u^m-u) \;\to\; 0$$

for $m\to\infty$. Combining the results above with (4.20) we easily deduce

$$c_k\|u_1^m - u_1\|_X^2 \;\leq\; \int_\Omega K(b(v^m))(u_1^m-u_1)\cdot(u_1^m-u_1) \;\to\; 0\,,$$

which shows that $L(\bar{B}_R)$ is precompact in V, and the continuity of L can also easily be checked. It follows that L has a fixed point $(u, \nabla u) \in V$, which by construction is a solution of problem (4.34). To proceed further, we need the following identity, which follows from (4.23) and the monotonicity of b:

$$\forall n \geq 1: \quad (b(u^n_{\varepsilon,N}) - b^{n-1}_{\varepsilon,N}) u^n_{\varepsilon,N} \;\geq\; \Psi(b(u^n_{\varepsilon,N})) - \Psi(b^{n-1}_{\varepsilon,N}) \;\; \mu_\varepsilon\text{-a.e. in } \Omega. \quad (4.38)$$

By density, we see that the integral identity (4.34) also holds for any pair $(\varphi, \nabla\varphi) \in V$, which shows that $K(b^n_{\varepsilon,N}) \nabla u^n_{\varepsilon,N}$ is tangential and that the gradient of $u^n_{\varepsilon,N}$ is uniquely determined. In particular, testing (4.34) with the solution itself, we get by summing up to $m \leq N$:

$$\int_\Omega \Psi(b^m_{\varepsilon,N})\, d\mu_\varepsilon + \frac{c_k \Delta t}{2} \sum_{n=1}^m \|\nabla u^n_{\varepsilon,N}\|^2_X \;\leq\; \int_\Omega \Psi(b^0_\varepsilon)\, d\mu_\varepsilon + C\Delta t \sum_{n=1}^m \|f(b^n_{\varepsilon,N})\|^2_X,$$

where we have used (2.92),(4.20),(4.38) and standard absorption techniques. The constant C on the right-hand side depends only on c_k and the constant C_{pc} in (2.92). Using (4.25) and the positivity of Ψ, we get by absorption

$$\begin{aligned} \|b^m_{\varepsilon,N}\|^2_X + \Delta t \sum_{n=1}^m \|\nabla u^n_{\varepsilon,N}\|^2_X \quad &\leq \quad C(1 + \|\Psi(b^0_\varepsilon)\|_{1,\mu_\varepsilon} + \Delta t \sum_{n=1}^m \|f(b^n_{\varepsilon,N})\|^2_X) \\ &\leq \quad C(1 + \|\Psi(b^0_\varepsilon)\|_{1,\mu_\varepsilon} + \Delta t \sum_{n=1}^m \|u^n_{\varepsilon,N}\|^{2\beta}_X) \end{aligned}$$

with a constant C only depending on $\beta, |\Omega|, c_k, C_{\mathrm{pc}}, T$ and the constant L in (4.22), where for the last estimate we also used the sublinear growth condition (4.21) on f. Since $\beta < 1$, by (2.92) and absorption we derive the crucial estimate

$$\|b^m_{\varepsilon,N}\|^2_X + \Delta t \sum_{n=1}^m \|\nabla u^n_{\varepsilon,N}\|^2_X \;\leq\; C\,(1 + \|\Psi(b^0_\varepsilon)\|_{1,\mu_\varepsilon})\,, \qquad (4.39)$$

where the constant C does not depend on m, N and ε. We define the linear interpolation $b_{\varepsilon,N}$ and the piecewise constant interpolations $(\bar{b}_{\varepsilon,N}, \bar{u}_{\varepsilon,N})$ on the whole time interval with values in X by

$$\begin{aligned} b_{\varepsilon,N}(t) \quad &:= \quad \frac{b^n_{\varepsilon,N} - b^{n-1}_{\varepsilon,N}}{\Delta t}\,(t - t_{n-1}) + b^{n-1}_{\varepsilon,N} \qquad \text{if } t \in [t_{n-1}, t_n]\,, \\ (\bar{b}_{\varepsilon,N}, \bar{u}_{\varepsilon,N})(t) \quad &:= \quad (b^n_{\varepsilon,N}, u^n_{\varepsilon,N}) \qquad \text{if } t \in (t_{n-1}, t_n]\,. \end{aligned}$$

Observe that by construction we have $\bar{b}_{\varepsilon,N} = b(\bar{u}_{\varepsilon,N})$ ν_ε-a.e. in Q and also $\nabla \bar{u}_{\varepsilon,N}(t) = \nabla u^n_{\varepsilon,N}$ for $t \in (t_{n-1}, t_n]$. We claim that estimate (4.39) implies

$$\|\partial_t b_{\varepsilon,N}\|_{L^2V'} + \|(\bar{u}_{\varepsilon,N}, \nabla \bar{u}_{\varepsilon,N})\|_{L^2V} \;\leq\; C\,(1 + \|\Psi(b^0_\varepsilon)\|_{1,\mu_\varepsilon}) \qquad (4.40)$$

with a constant independent of N and ε. Indeed, using the Poincaré inequality (2.92), the definition of $\bar{u}_{\varepsilon,N}$, and (4.39) we easily calculate

$$\|(\bar{u}_{\varepsilon,N}, \nabla \bar{u}_{\varepsilon,N})\|^2_{L^2V} \;\leq\; C\Delta t \sum_{n=1}^N \|(u^n_{\varepsilon,N}, \nabla u^n_{\varepsilon,N})\|^2_V \;\leq\; C\,(1 + \|\Psi(b^0_\varepsilon)\|_{1,\mu_\varepsilon}). \qquad (4.41)$$

From the integral identity (4.34) we deduce $\partial_t b_{\varepsilon,N}(t) \in V'$ for almost every $t \in [0,T]$ subject to

$$\langle\langle \partial_t b_{\varepsilon,N}(t), (\varphi, \nabla\varphi) \rangle\rangle := \int_\Omega \big(f(\bar{b}_{\varepsilon,N})\varphi - K(\bar{b}_{\varepsilon,N})\nabla\bar{u}_{\varepsilon,N} \cdot \nabla\varphi \big)\, d\mu_\varepsilon \tag{4.42}$$

for any $(\varphi, \nabla\varphi) \in V$. Indeed, this follows in one step from the following estimate, for which we use (4.20),(4.21),(4.22),(4.41) and the same techniques that led to (4.39):

$$\begin{aligned}
\|\partial_t b_{\varepsilon,N}\|^2_{L^2V'} &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \sup_{\|(\varphi,\nabla\varphi)\|_V \le 1} \left| \int_\Omega f(b^n_{\varepsilon,N})\varphi - K(b^n_{\varepsilon,N})\nabla u^n_{\varepsilon,N} \cdot \nabla\varphi \right|^2 dt \\
&\le C + C\Delta t \sum_{n=1}^N \left(\|u^n_{\varepsilon,N}\|^2_X + \|\nabla u^n_{\varepsilon,N}\|^2_X \right).
\end{aligned} \tag{4.43}$$

Combining the estimates (4.41) and (4.43) we get (4.40). From (4.39),(4.40) and the uniform upper bound on K we also deduce

$$\|b_{\varepsilon,N}\|_{L^\infty X} + \|K(\bar{b}_{\varepsilon,N})\nabla\bar{u}_{\varepsilon,N}\|_{L^2X} \le C\,(1 + \|\Psi(b^0_\varepsilon)\|_{1,\mu_\varepsilon}) \tag{4.44}$$

with a constant independent of N and ε. Since (4.40) and (4.44) provide uniform estimates independent of N, we find a pair $(u_\varepsilon, \nabla u_\varepsilon) \in L^2V$ and functions $b_\varepsilon \in H^1V' \cap L^\infty X$, $K_0 \in (L^2X)^d$ and $f_0 \in L^2X$, such that

$$\begin{aligned}
(\bar{u}_{\varepsilon,N}, \nabla\bar{u}_{\varepsilon,N}) &\rightharpoonup (u_\varepsilon, \nabla u_\varepsilon) \text{ in } L^2(0,T;V)\,, && (4.45)\\
K(\bar{b}_{\varepsilon,N})\nabla\bar{u}_{\varepsilon,N} &\rightharpoonup K_0 \text{ in } L^2(0,T;X)^d\,, && (4.46)\\
f(\bar{b}_{\varepsilon,N}) &\rightharpoonup f_0 \text{ in } L^2(0,T;X)\,, && (4.47)\\
b_{\varepsilon,N} &\rightharpoonup b_\varepsilon \text{ in } H^1(0,T;V')\,, && (4.48)\\
b_{\varepsilon,N} &\overset{*}{\rightharpoonup} b_\varepsilon \text{ in } L^\infty(0,T;X) && (4.49)
\end{aligned}$$

up to a subsequence. For now we use the abbreviation $H_\varepsilon := H^{1,2}_0(\Omega, d\mu_\varepsilon)$ for the Dirichlet space. Since $K(\bar{b}_{\varepsilon,N})\nabla\bar{u}_{\varepsilon,N}$ is tangential (and therefore also its weak limit K_0), we can obviously identify $\partial_t b_{\varepsilon,N}(t)$ as an element of H'_ε for almost every t, and derive the same estimate

$$\|\partial_t b_{\varepsilon,N}\|_{L^2H'_\varepsilon} \le C\,(1 + \|\Psi(b^0_\varepsilon)\|_{1,\mu_\varepsilon}) \tag{4.50}$$

as above with a constant independent of ε, N. Hence given a smooth test function φ it makes no difference whether we take $\nabla\varphi$ or $\nabla_{\mu_\varepsilon}\varphi$ on the right-hand side in (4.42). Integrating this identity with respect to t and performing an integration by parts in time, passing to the limit $N \to \infty$ gives

$$\int_Q (-b_\varepsilon \partial_t\varphi + K_0 \cdot \nabla\varphi)\, d\nu_\varepsilon = \int_Q f_0\varphi\, d\nu_\varepsilon + \int_\Omega b^0_\varepsilon \varphi(0)\, d\mu_\varepsilon \tag{4.51}$$

for any $\varphi \in \mathcal{C}^\infty([0,T];\mathcal{D}(\Omega))$ with $\varphi(T)=0$, where we used (4.33) and applied the convergences in (4.45)-(4.49). It remains to identify the weak limits in Q:

$$b_\varepsilon(x,t) \quad = \quad b(u_\varepsilon)(x,t), \tag{4.52}$$

$$(K_0,f_0)(x,t) \quad = \quad (K(b_\varepsilon)\nabla u_\varepsilon, f(b_\varepsilon))(x,t). \tag{4.53}$$

Then by Definition 4.2.4, the first component $u_\varepsilon \in L^2(0,T;\tilde{H}_0^{1,2}(\Omega,d\mu_\varepsilon))$ of the weak limit in (4.45) is a solution of problem (P_ε) according to Definition 4.2.5, and the a priori estimate (4.31) is satisfied due to (4.40),(4.44), the convergences (4.45),(4.48),(4.49), and the weak lower semicontinuity of the norm. By Lemma 2.4.1 we have $X \subset\subset Z := H_\varepsilon^{'}$, so we can apply the statement (6.12) of Theorem 6.9 in the appendix thanks to the estimates (4.44),(4.50), and obtain

$$b_{\varepsilon,N} \to b_\varepsilon \ \text{ strongly in } \mathcal{C}([0,T];H_\varepsilon^{'})\,. \tag{4.54}$$

Fix $t\in(0,T]$. By the definition of $b_{\varepsilon,N}$ and $\bar{b}_{\varepsilon,N}$ we find $\tilde{t}\in(0,T]$ depending on N, with $0\le \tilde{t}-t\le \Delta t$ and $\bar{b}_{\varepsilon,N}(t)=\bar{b}_{\varepsilon,N}(\tilde{t})=b_{\varepsilon,N}(\tilde{t})$. Therefore if $\|\cdot\|$ denotes the norm in $H_\varepsilon^{'}$, we derive

$$\begin{aligned}\|\bar{b}_{\varepsilon,N}(t)-b_{\varepsilon,N}(t)\| \ &\le\ \|b_{\varepsilon,N}(\tilde{t})-b_\varepsilon(\tilde{t})\|+\|b_\varepsilon(\tilde{t})-b_\varepsilon(t)\|+\|b_\varepsilon(t)-b_{\varepsilon,N}(t)\| \\ &\le\ 2\sup_{s\in[0,T]}\|b_{\varepsilon,N}(s)-b_\varepsilon(s)\|+C(\Delta t)^{1/2}\ \to\ 0\end{aligned}$$

uniformly in $(0,T]$ for $N\to\infty$, where we used (4.54) and the fact that b_ε is Hölder continuous with values in $H_\varepsilon^{'}$ due to (4.50) and (4.54). It follows that

$$\bar{b}_{\varepsilon,N} \to b_\varepsilon \ \text{ strongly in } L^2(0,T;H_\varepsilon^{'})\,. \tag{4.55}$$

Moreover, from the Lipschitz continuity of b and the relation $\bar{b}_{\varepsilon,N}=b(\bar{u}_{\varepsilon,N})$ we also deduce $\bar{b}_{\varepsilon,N}\rightharpoonup b_\varepsilon$ weakly in L^2X by estimate (4.40). Clearly we can also consider $\{\bar{u}_{\varepsilon,N}\}$ as a bounded sequence in L^2H_ε, and hence deduce

$$\int_Q \bar{b}_{\varepsilon,N}\bar{u}_{\varepsilon,N}\,d\nu_\varepsilon \ = \int_0^T \langle\!\langle \bar{b}_{\varepsilon,N}(t),\bar{u}_{\varepsilon,N}(t)\rangle\!\rangle_{H_\varepsilon' H_\varepsilon}\,dt \ \stackrel{N\to\infty}{\longrightarrow} \int_Q b_\varepsilon u_\varepsilon\,d\nu_\varepsilon\,. \tag{4.56}$$

In order to prove (4.52), let $\delta>0$ and $\phi\in\mathcal{D}(Q)$ be arbitrary. By the monotonicity of b and due to the fact that $\bar{b}_{\varepsilon,N}=b(\bar{u}_{\varepsilon,N})$, we get

$$0\le \int_Q(\bar{b}_{\varepsilon,N}-b(u_\varepsilon-\delta\phi))(\bar{u}_{\varepsilon,N}-u_\varepsilon+\delta\phi)\to\delta\int_Q(b_\varepsilon-b(u_\varepsilon-\delta\phi))\phi$$

as $N\to\infty$, where we used (4.45),(4.56) and $b(u_\varepsilon-\delta\phi)\in L^2X$ by (4.22). Dividing by δ in the last inequality and passing to the limit $\delta\to 0$, we get

$$0\le\int_Q(b_\varepsilon-b(u_\varepsilon))\phi\,d\nu_\varepsilon \quad \forall\phi\in\mathcal{D}(Q) \tag{4.57}$$

by the continuity of b, which proves (4.52). Since b is Lipschitz continuous with $b(0) = 0$, we deduce from Lemma 2.2.17 that $\{\bar{b}_{\varepsilon,N}\}$ is a bounded sequence in L^2H_ε, in particular we get $\bar{b}_{\varepsilon,N} \rightharpoonup b_\varepsilon$ weakly in L^2H_ε. Since H_ε is a Hilbert space, it is well known that $(L^2H_\varepsilon)' = L^2H_\varepsilon'$ and it follows

$$\int_Q |\bar{b}_{\varepsilon,N}|^2 \, d\nu_\varepsilon = \int_0^T \langle\!\langle \bar{b}_{\varepsilon,N}(t), \bar{b}_{\varepsilon,N}(t)\rangle\!\rangle_{H_\varepsilon' H_\varepsilon} \, dt \overset{N\to\infty}{\longrightarrow} \int_Q |b_\varepsilon|^2 \, d\nu_\varepsilon \,, \tag{4.58}$$

which implies $\bar{b}_{\varepsilon,N} \to b_\varepsilon$ strongly in $L^2(Q, d\nu_\varepsilon)$. By the continuity of K and f with respect to the second variable, and due to the uniform estimates (4.20) and (4.21) on K and f, the equality in (4.53) easily follows. □

4.2.2 Homogenization

For the homogenization step we have to pass to the limit in the weak formulation (4.28) of problem (P_ε), relying on the a priori estimate (4.31). The following preparatory lemma guarantees the crucial strong two-scale convergence of the sequence $w_\varepsilon := b(u_\varepsilon)$. In what follows, recall that the spaces V, V' and X defined in (4.29) depend on ε.

Lemma 4.2.8. *Let μ be doubling and strongly 2-connected on $\mathbb{R}^d$. Consider two sequences $\{w_\varepsilon\} \subset H^1(0,T;V') \cap L^\infty(0,T;X)$ and $\{(u_\varepsilon, \nabla u_\varepsilon)\} \subset L^2(0,T;V)$ endowed with an uniform bound*

$$\|\partial_t w_\varepsilon\|_{L^2V'} + \|w_\varepsilon\|_{L^\infty X} + \|(u_\varepsilon, \nabla u_\varepsilon)\|_{L^2V} \leq C \,. \tag{4.59}$$

Possibly passing to a subsequence, assume that $w_\varepsilon \rightharpoonup\!\!\!\rightharpoonup w \in L^2_n(Q\times Y)$ and $u_\varepsilon \rightharpoonup\!\!\!\rightharpoonup u \in L^2(Q)$ two-scale with respect to ν by Theorem 4.1.7. Then we have

$$\int_Q w_\varepsilon u_\varepsilon \, d\nu_\varepsilon \to \int_Q \bar{w} u \, dxdt \,, \quad \bar{w} := \int_Y w(\cdot, y) \, d\mu(y). \tag{4.60}$$

If $w_\varepsilon = b(u_\varepsilon)$ with $b : \mathbb{R} \to \mathbb{R}$ continuous and monotonically nondecreasing, then there holds $\bar{w} = b(u)$ almost everywhere in Q. If additionally b is Lipschitz continuous with $b(0) = 0$, then $w = w(x,t) \in L^2(Q)$ and

$$\int_Q |w_\varepsilon|^2 \, d\nu_\varepsilon \to \int_Q |w|^2 \, dxdt \,. \tag{4.61}$$

Proof. Let $\delta > 0$ be given and denote by $\mathcal{S}$ the subsequence selected in the statement of the lemma. We need to show

$$\exists \varepsilon_0 > 0 \, \forall \varepsilon \leq \varepsilon_0, \, \varepsilon \in \mathcal{S} : \quad \left| \int_Q w_\varepsilon u_\varepsilon \, d\nu_\varepsilon - \int_Q \bar{w} u \, dxdt \right| \leq \delta \,. \tag{4.62}$$

Choose a function $\tilde{w} \in \mathcal{D}(Q)$ with $\|\bar{w} - \tilde{w}\|_{L^2(Q)} \leq \delta^2 \min(1, \|u\|^{-1}_{L^2(Q)})$, then

$$\begin{aligned} \left| \int_Q w_\varepsilon u_\varepsilon \, d\nu_\varepsilon - \int_Q \bar{w} u \right| &= \left| \int_Q (w_\varepsilon - \tilde{w}) u_\varepsilon \, d\nu_\varepsilon + \int_Q \tilde{w} u_\varepsilon \, d\nu_\varepsilon - \int_Q \bar{w} u \right| \\ &\leq C \|w_\varepsilon - \tilde{w}\|_{L^2V'} + \left| \int_Q \tilde{w} u_\varepsilon \, d\nu_\varepsilon - \int_Q \bar{w} u \right| , \end{aligned}$$

where we have used (4.32) and (4.59). We have $\int \tilde{w} u_\varepsilon \, d\nu_\varepsilon \to \int \tilde{w} u$ since $\tilde{w}$ is smooth, so inserting the last term we get by the choice of $\tilde{w}$:

$$\exists \varepsilon_1 > 0 \, \forall \varepsilon \leq \varepsilon_1, \, \varepsilon \in \mathcal{S} : \quad \left| \int_Q \tilde{w} u_\varepsilon \, d\nu_\varepsilon - \int_Q \bar{w} u \, dxdt \right| \leq 2\delta^2 \, . \tag{4.63}$$

Now we estimate the term $\|w_\varepsilon - \tilde{w}\|_{L^2 V'}$. Thanks to estimate (4.59) we get [31, Chapter 5.9] that $w_\varepsilon \in \mathcal{C}([0,T]; V')$ with

$$\max_{0 \leq t \leq T} \|w_\varepsilon(t)\|_{V'} \leq C \|w_\varepsilon\|_{H^1 V'} \leq C \, , \quad w_\varepsilon(t) = w_\varepsilon(s) + \int_s^t w_\varepsilon'(\tau) \, d\tau,$$

where the last equality holds for all $0 \leq s \leq t \leq T$. It follows that $v_\varepsilon := w_\varepsilon - \tilde{w}$ is uniformly bounded in $L^\infty(0,T;X) \cap \mathcal{C}([0,T]; V')$, and in addition

$$\forall s, t \in [0,T] : \quad \|v_\varepsilon(t) - v_\varepsilon(s)\|_{V'} \leq \tilde{C} \, |t-s|^{\frac{1}{2}} \, , \tag{4.64}$$

where the constant $\tilde{C}$ only depends on some Hölder norm of $\tilde{w}$. Given $\delta > 0$, choose an equidistant partition $0 = t_0, t_1, \ldots, t_{N_\delta} = T$ of the time interval with $t_i - t_{i-1} = c_\delta$ for all $i = 1, \ldots, N_\delta$ and c_δ to be chosen appropriately. Note that we have $N_\delta \cdot c_\delta = T$. We calculate

$$\tfrac{1}{2} \|v_\varepsilon\|^2_{L^2 V'} \leq \sum_{i=1}^{N_\delta} \int_{t_{i-1}}^{t_i} \left(\|v_\varepsilon(t_i)\|^2_{V'} + \|v_\varepsilon(t) - v_\varepsilon(t_i)\|^2_{V'} \right) dt =: I^\varepsilon_{1,\delta} + I^\varepsilon_{2,\delta}. \tag{4.65}$$

With the help of (4.64) the second term $I^\varepsilon_{2,\delta}$ is easy to estimate:

$$I^\varepsilon_{2,\delta} \leq \tilde{C}^2 \sum_{i=1}^{N_\delta} \int_{t_{i-1}}^{t_i} |t - t_i| \, dt = \tfrac{1}{2} \tilde{C}^2 \sum_{i=1}^{N_\delta} c_\delta^2 = \tfrac{1}{2} T \tilde{C}^2 c_\delta \, . \tag{4.66}$$

The main difficulty is to estimate the first term in (4.65). Let us denote the dual pairing between V' and V by $\langle\!\langle \Lambda, (\varphi, \nabla\varphi) \rangle\!\rangle_\varepsilon$ for $\Lambda \in V'$ and $(\varphi, \nabla\varphi) \in V$. We can assume $v_\varepsilon(t_i) \neq 0$ in V', so for every $i = 1, \ldots, N_\delta$ and $\varepsilon > 0$ there exists a pair $(\varphi^i_\varepsilon, \nabla\varphi^i_\varepsilon) \in V$ by Lemma 6.1, such that $\|(\varphi^i_\varepsilon, \nabla\varphi^i_\varepsilon)\|_V = 1$ and

$$\|v_\varepsilon(t_i)\|_{V'} = \langle\!\langle v_\varepsilon(t_i), (\varphi^i_\varepsilon, \nabla\varphi^i_\varepsilon) \rangle\!\rangle_\varepsilon = \int_\Omega v_\varepsilon(x, t_i) \varphi^i_\varepsilon(x) \, d\mu_\varepsilon \, ,$$

the last equality holding because for any t we have $v_\varepsilon(t) \in X$. This leads to

$$\begin{aligned} \|v_\varepsilon(t_i)\|_{V'} &= \frac{1}{T} \int_Q v_\varepsilon(x,t) \varphi^i_\varepsilon(x) \, d\nu_\varepsilon + \frac{1}{T} \int_Q [v_\varepsilon(x,t_i) - v_\varepsilon(x,t)] \varphi^i_\varepsilon(x) \, d\nu_\varepsilon \\ &=: \Lambda^{\varepsilon,i}_{1,\delta} + \Lambda^{\varepsilon,i}_{2,\delta} \, . \end{aligned} \tag{4.67}$$

The term $\Lambda^{\varepsilon,i}_{2,\delta}$ can be estimated with (4.64) and the normalization of φ^i_ε:

$$|\Lambda^{\varepsilon,i}_{2,\delta}| \leq \frac{1}{T} \sum_{i=1}^{N_\delta} \int_{t_{i-1}}^{t_i} \|v_\varepsilon(t_i) - v_\varepsilon(t)\|_{V'} \|(\varphi^i_\varepsilon, \nabla\varphi^i_\varepsilon)\|_V \, dt \leq \tfrac{2}{3} \tilde{C} \sqrt{c_\delta} \, , \tag{4.68}$$

where the estimate is independent of ε and i. To estimate the term $\Lambda_{1,\delta}^{\varepsilon,i}$ we first observe that

$$v_\varepsilon(x,t) \rightharpoonup\!\!\rightharpoonup w(x,t,y) - \tilde{w}(x,t)$$

for the whole sequence $\mathcal{S}$. Now let a subsequence $\mathcal{S}_1$ of $\mathcal{S}$ be given. By the uniform bound on φ_ε^1, there exists a function $\varphi_0^1 \in L^2(\Omega)$, such that $\varphi_\varepsilon^1 \twoheadrightarrow \varphi_0^1$ two-scale strongly with respect to μ for a subsequence $\mathcal{S}_{1,1}$ of $\mathcal{S}_1$ thanks to Theorem 2.4.4. Since φ_ε^1 does not depend on t, it is easy to check that also $\varphi_\varepsilon^1 \twoheadrightarrow \varphi_0^1$ two-scale strongly with respect to ν. By Proposition 2.1.13 we get

$$\Lambda_{1,\delta}^{\varepsilon,1} = \frac{1}{T}\int_Q v_\varepsilon \varphi_\varepsilon^i(x)\, d\nu_\varepsilon \xrightarrow{\mathcal{S}_{1,1}} \frac{1}{T}\int_Q [\bar{w} - \tilde{w}]\varphi_0^1(x)\, dxdt =: \Lambda_{1,\delta}^{0,1}.$$

Repeating the same argument finitely many (more precisely N_δ) times, we obtain a subsequence $\mathcal{S}_2$ of $\mathcal{S}_1$ and functions $\varphi_0^i \in L^2(\Omega)$, such that

$$\forall i = 1,\dots,N_\delta : \quad \Lambda_{1,\delta}^{\varepsilon,i} \xrightarrow{\mathcal{S}_2} \frac{1}{T}\int_Q [\bar{w} - \tilde{w}]\varphi_0^i(x)\, dxdt =: \Lambda_{1,\delta}^{0,i}.$$

Note that we have $\|\varphi_0^i\|_{L^2(\Omega)} \le 1$, and hence it follows

$$|\Lambda_{1,\delta}^{0,i}| \le \frac{1}{\sqrt{T}}\|\bar{w} - \tilde{w}\|_{L^2(Q)} \le \frac{1}{\sqrt{T}}\delta^2$$

for each i by the choice of $\tilde{w}$. Since the subsequence $\mathcal{S}_1$ of $\mathcal{S}$ was arbitrary, a standard contradiction argument yields

$$\exists \varepsilon_2 > 0\, \forall \varepsilon \le \varepsilon_2,\, \varepsilon \in \mathcal{S} : \; |\Lambda_{1,\delta}^{\varepsilon,i}| \le C\,\delta^2, \quad \forall i = 1,\dots,N_\delta, \tag{4.69}$$

with a constant C only depending on T. Combining (4.68) and (4.69) we get

$$\forall \varepsilon \le \varepsilon_2,\, \varepsilon \in \mathcal{S} : \; I_{1,\delta}^\varepsilon = \sum_{i=1}^{N_\delta} c_\delta \|v_\varepsilon(t_i)\|_{V'}^2 \le C(\tilde{C}\sqrt{c_\delta} + \delta^2)^2 \tag{4.70}$$

where the constant C depends only on T. Using the last estimate we deduce from (4.65) and (4.66):

$$\forall \varepsilon \le \varepsilon_2,\, \varepsilon \in \mathcal{S} : \; \|v_\varepsilon\|_{L^2V'} \le C\sqrt{\tilde{C}^2 c_\delta + \delta^4}. \tag{4.71}$$

Recall that $\tilde{C}$ depends on some norm of $\tilde{w}$ which can not be controlled by the $L^2(Q)$-norm of $\bar{w}$. To get rid of the dependence on $\tilde{C}$, we choose $c_\delta := \delta^4\tilde{C}^{-2}$. Setting $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ we deduce from (4.63) and (4.71):

$$\left|\int_Q w_\varepsilon u_\varepsilon\, d\nu_\varepsilon - \int_Q \bar{w}u\, dxdt\right| \le C\sqrt{\tilde{C}^2 c_\delta + \delta^4} + 2\delta^2 \le C\,\delta^2 \le \delta$$

for all $\varepsilon \le \varepsilon_0$, $\varepsilon \in \mathcal{S}$ and δ small, which proves (4.62) and hence (4.60). Now we show $\bar{w} = b(u)$, provided $w_\varepsilon = b(u_\varepsilon)$ for $b : \mathbb{R} \to \mathbb{R}$ monotonically nondecreasing and continuous. For such b we have the characterization

$$\lambda = b(r) \iff (\lambda - b(s)) \cdot (r - s) \ge 0 \quad \forall s \in \mathbb{R}. \tag{4.72}$$

Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{D}(Q), \varphi \geq 0$. From (4.60) and (4.72) we deduce

$$0 \leq \int_Q (b(u_\varepsilon) - b(s))(u_\varepsilon - s)\varphi \, d\nu_\varepsilon \to \int_Q (\bar{w} - b(s))(u - s)\varphi \, dxdt \, .$$

Since $\varphi \geq 0$ was arbitrary, the continuity of b and (4.72) yield $\bar{w} = b(u)$ almost everywhere in Q. If b is Lipschitz with $b(0) = 0$, Lemma 2.2.17 shows

$$b(u_\varepsilon) = w_\varepsilon \in L^2(0,T; H_0^{1,2}(\Omega, d\mu_\varepsilon)).$$

Thanks to the chain rule formula (2.72) on page 29 and the uniform estimate on u_ε in (4.59), $(w_\varepsilon, \nabla_{\mu_\varepsilon} w_\varepsilon)$ can be interpreted as a bounded sequence in L^2V, hence the two-scale limit w does not depend on y. The convergence in (4.61) follows from (4.60) by choosing $u_\varepsilon = w_\varepsilon$. $\square$

As discussed in the last chapter, for the type of quasilinear equations we are investigating we need local Hölder continuity of the data in s for the homogenization step. The following structure condition corresponds precisely to Assumption 3.2.5 of Section 3.2.

Assumption 4.2.9. *There exist $\gamma, \tilde{\gamma} \in (0,1]$, a function $\tilde{h} \in L_\mu^{\tilde{q}}(\mathbb{T})$ and a constant c_4, such that for all $y \in \mathbb{R}^d$ and all $s_1, s_2 \in \mathbb{R}$:*

$$\|K(y,s_1) - K(y,s_2)\| \leq c_4 \, (1 + |s_1| + |s_2|)^{1-\gamma} |s_1 - s_2|^\gamma, \tag{4.73}$$

$$|f(y,s_1) - f(y,s_2)| \leq |\tilde{h}(y)| \, |s_1 - s_2|^{\tilde{\gamma}}, \tag{4.74}$$

where $\| \cdot \|$ is some norm on $\mathbb{R}^{d \times d}$ and $\tilde{q} := 2(2 - \tilde{\gamma})^{-1} \in (1,2]$.

Concerning the following homogenization result we content ourselves with a sketch of the proof, since all the main aspects already occurred in the proof of Theorem 3.1.8 in the stationary setting.

Theorem 4.2.10. *Let u_ε be a solution of (P_ε) according to Theorem 4.2.7, and let $u_\varepsilon^0 \twoheadrightarrow u^0$ two-scale strongly with respect to μ for some $u^0 \in L_m^2(\Omega \times Y)$. Then there exist $u \in L^2(0,T; H_0^1(\Omega))$ and $\tilde{u}_1 \in L^2(Q; \tilde{H}_\mu^{1,2}(\mathbb{T}))$ such that, up to a subsequence:*

$$b(u_\varepsilon) \twoheadrightarrow b(u) \quad \textit{two-scale strongly in } L^2(Q, d\nu_\varepsilon)\, , \tag{4.75}$$

$$u_\varepsilon \rightharpoonup\rightharpoonup u \quad \textit{two-scale in } L^2(Q, d\nu_\varepsilon)\, , \tag{4.76}$$

$$\nabla u_\varepsilon \rightharpoonup\rightharpoonup \nabla_x u + \nabla_y \tilde{u}_1 \quad \textit{two-scale in } L^2(Q, d\nu_\varepsilon)^d \, . \tag{4.77}$$

If the data satisfy Assumption 4.2.9 and if we set $\bar{b}^0 := \int_Y b(u^0(x,y)) \, d\mu(y)$, then the pair $(u, \tilde{u}_1)$ is a solution of the two-scale homogenized problem

$$\int_{Q \times Y} K(y, b(u))(\nabla u + \nabla_y \tilde{u}_1) \cdot (\nabla \phi + \nabla_y \phi_1) \, dn = \int_Q (b(u)\partial_t \phi + \bar{f}(b(u))\phi) + \int_\Omega \bar{b}^0 \phi(0) \tag{4.78}$$

for all $(\phi, \phi_1) \in \mathcal{C}^\infty([0,T]; \mathcal{D}(\Omega)) \times \mathcal{D}(Q; \mathcal{C}^\infty(\mathbb{T}))$ with $\phi(T) = 0$.

Sketch of proof. The convergences in (4.75)-(4.77) directly follow from Theorem 4.1.7 and Lemma 4.2.8, provided $\|\psi(b_\varepsilon^0)\|_{1,\mu_\varepsilon} \le C$ uniformly in ε. However, this is guaranteed by Assumption 4.2.1.5 combined with (4.22),(4.23) and the fact that u_ε^0 is uniformly bounded in $L^2(\Omega, d\mu_\varepsilon)$. Given a pair (ϕ, ϕ_1) as required in (4.78), it is easy to check that

$$\varphi_\varepsilon(x,t) := \phi(x,t) + \varepsilon\phi_1(x,t,\tfrac{x}{\varepsilon}) \tag{4.79}$$

is an admissible test function in (4.28). Thanks to the strong two-scale convergence of $b(u_\varepsilon)$ and the Hölder continuity of K and f with respect to s, the integrals over Q in (4.28) can be treated precisely as in the proof of Theorem 3.1.8. The remaining convergence

$$\int_\Omega b_\varepsilon^0 \varphi_\varepsilon(0)\, d\mu_\varepsilon \;=\; \int_\Omega b(u_\varepsilon^0)\phi(0)\, d\mu_\varepsilon \;\to\; \int_\Omega \bar b^0 \phi(0)\, dx$$

can be verified by identifying $(b \circ u^0)(x,y)$ as the weak μ-two-scale limit of b_ε^0, where we use the strong two-scale convergence of u_ε^0 and the standard monotonicity trick. □

Using the methods developed in Chapter 3, it is easy to derive the homogenized equation for u. Similar as in Lemma 3.1.10, for any $s \in \mathbb{R}$ and $1 \le k \le d$ there exists a unique solution $v_k(\cdot, s) \in V^2_{\text{pot}}(\mathbb{T}, d\mu)$ of the variational problem

$$\int_Y K(y,s)[v_k(y,s) + \vec{e}_k] \cdot w(y)\, d\mu(y) \;=\; 0 \quad \forall w \in V^2_{\text{pot}}(\mathbb{T}, d\mu) \tag{4.80}$$

thanks to the uniform ellipticity of the tensor K presumed in (4.20).

Corollary 4.2.11. *Under the assumptions of Theorem 4.2.10, assume in addition that $u^0 = u^0(x) \in L^2(\Omega)$. Then the limit function $u \in L^2(0,T; H_0^1(\Omega))$ is a solution of the homogenized equation*

$$(P_0) \quad \begin{cases} \partial_t b(u) - \operatorname{div}(K^\star(b(u))\nabla u) &= \bar f(b(u)) \quad \text{in } Q, \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T) \end{cases}$$

in the sense of Definition 4.2.5 with initial value $b^0 = b(u^0) \in L^2(\Omega)$. The effective coefficient $K^\star : \mathbb{R} \to \mathcal{M}^d_{\text{sym}}$ is defined as

$$K^\star_{ij}(s) = \int_Y K(y,s)[\vec{e}_i + v_i(y,s)] \cdot (\vec{e}_j + v_j(y,s))\, d\mu(y), \tag{4.81}$$

where $v_i(\cdot, s) \in V^2_{\text{pot}}(\mathbb{T}, d\mu)$ is the solution of problem (4.80) for $k = i$.

Proof. First we determine the corrector term $\nabla_y \tilde u_1$ given in (4.77) by setting $\phi = 0$ in (4.78). As in the proof of Corollary 3.2.10 we deduce

$$\nabla_y \tilde u_1(x,t,y) \;=\; \sum_{k=1}^d \partial_{x_k} u(x,t)\, v_k(y, b(u)(x,t))\,, \tag{4.82}$$

where $v_k(\cdot, b(u)(x,t))$ is the unique solution of (4.80) for $s = b(u)(x,t)$. Inserting (4.82) in (4.78) with $\phi_1 = 0$ we obtain the weak formulation (4.28) of problem (P_0), where the requirements $\psi(b^0) \in L^1(\Omega)$ and $b^0 = b(u^0)$ easily follow from (4.22),(4.23) and the definition of $\bar{b}^0$. □

We will now prove uniqueness for the homogenized equation (P_0) derived in Corollary 4.2.11. To this end we introduce a slightly different notion of weak solutions of the problem

$$(\star)\quad \begin{cases} \partial_t b(u) - \operatorname{div}[a(b(u), \nabla u)] \;=\; g(b(u)) \quad \text{in } \Omega \times (0,T), \\ b(u) \;=\; b^0 \text{ in } \Omega \times \{0\}\,, \quad u \;=\; 0 \quad \text{on } \partial\Omega \times (0,T)\,, \end{cases}$$

which is adapted to the classical setting of the Lebesgue measure and, more importantly, to the weak formulation of Theorem 6.10 in the appendix, which has to be quoted for our uniqueness result. However, at least for $\mu = \mathcal{L}^d$, Lipschitz continous b and assumptions (4.20),(4.21) on the data, the new solution concept coincides with Definition 4.2.5 above for $a(s,\xi) = K(\cdot, s)\xi$ and $g(s) = f(\cdot, s)$, especially since Theorem 4.2.7 gives $b(u) \in L^\infty L^2$ (cf. Definition 4.2.12.1 below). Recall the standard assumptions on the initial data

$$\Psi(b^0) \in L^1(\Omega)\,, \quad b^0 = b(u^0) \tag{4.83}$$

for a measurable function u^0, and that Ω is an open, bounded and connected subset of $\mathbb{R}^d$ with smooth boundary. As usual we denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the dual pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Definition 4.2.12. *Assume (4.83). Then we call $u \in L^2(0,T; H_0^1(\Omega))$ a weak solution of the initial boundary value problem $(\star)$, if there holds:*

1. *$b(u) \in L^\infty(0,T; L^1(\Omega))$ and $\partial_t b(u) \in L^2(0,T; H^{-1}(\Omega))$ with*

$$\int_0^T \langle\!\langle \partial_t b(u), \zeta \rangle\!\rangle \;=\; \int_Q (b^0 - b(u))\partial_t \zeta \tag{4.84}$$

   for every $\zeta \in L^2 H_0^1 \cap W^{1,1} L^\infty$ with $\zeta(T) = 0$.

2. *$a(b(u), \nabla u)$, $g(b(u)) \in L^2(Q)$ and for every $\zeta \in L^2 H_0^1$:*

$$\int_0^T \langle\!\langle \partial_t b(u), \zeta \rangle\!\rangle + \int_Q a(b(u), \nabla u) \cdot \nabla \zeta \;=\; \int_Q g(b(u))\zeta\,. \tag{4.85}$$

Uniqueness for equation (P_0) can be derived from Theorem 6.10, if K and f are Lipschitz continuous in s, and if b is α-Hölder continuous on $\mathbb{R}$ with $\alpha \le 1/2$, which does not follow from the Lipschitz continuity of b as the example $b(s) = s$ shows.

Lemma 4.2.13. *In the situation of Corollary 4.2.11, let Assumption 4.2.9 be satisfied with $\gamma, \tilde{\gamma} = 1$, and b α-Hölder contiuous on $\mathbb{R}$ with $0 < \alpha \le 1/2$. Then the solution of problem (P_0) is unique, and there holds*

$$u_\varepsilon \rightharpoonup\!\!\rightharpoonup u\,, \quad b(u_\varepsilon) \twoheadrightarrow b(u) \quad \textit{in } L^2(Q, d\nu_\varepsilon) \tag{4.86}$$

respectively two-scale weakly and strongly for the whole sequence $\varepsilon \to 0$.

Proof. It suffices to show that the limit function $u \in L^2H_0^1$ given by Corollary 4.2.11 is the unique solution of problem $(\star)$ in the sense of Definition 4.2.12 with $a(s,\xi) = K^\star(s)\xi$ and $g = \bar{f}$. Indeed, (4.83) is clearly satisfied, and with the same reasoning as in the comment after Definition 4.2.5 we get $\partial_t b(u) \in L^2H^{-1}$. This also gives $b(u) \in \mathcal{C}([0,T];L^2)$, since $b(u) \in L^2H_0^1$ by Lemma 2.2.17, and (4.84) follows. Precisely as in Lemma 3.2.9 we obtain that $K^\star$ is continuous with

$$c_\star|\xi|^2 \;\le\; \xi \cdot K^\star(s)\xi \;\le\; C_\star|\xi|^2 \tag{4.87}$$

for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^d$ and some positive constants $c_\star, C_\star$. In particular, the second assertion of Definition 4.2.12 follows by the weak formulation of problem (P_0) according to (4.28), where we also use (4.21) and (4.22) for the requirement $\bar{f}(b(u)) \in L^2(Q)$. Also note that $\bar{f}$ is Lipschitz continuous by assumption, and that

$$|K^\star(b(z))\xi| + \left|\bar{f}(b(z))\right| \;\le\; C(1 + B(z)^{1/2} + |\xi|) \quad \forall (z,\xi) \in \mathbb{R} \times \mathbb{R}^d$$

by (4.87) and Lemma 4.2.3. It remains to verify for all $z_1, z_2 \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$:

$$|K^\star(b(z_1))\xi - K^\star(b(z_2))\xi|^2 \;\le\; C|z_1 - z_2|(1 + B(z_1) + B(z_2) + |\xi|^2). \tag{4.88}$$

Since B is nonnegative and $K^\star$ inherits the Lipschitz continuity from K (cf. Lemma 3.2.11), (4.88) directly follows from the combined Lipschitz- and Hölder continuity of b. Applying Theorem 6.10, we get that $u \in L^2H_0^1$ is the unique solution of problem $(\star)$, which completes the proof. □

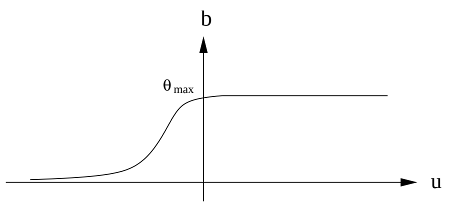
4.3 Richards equation

In this section we study the homogenization of Richards equation on perforated domains, which is of topical interest in (numerical) analysis [39, 41, 47, 48] and soil physics (see e.g. [28, 54]), and at the same time the most important application of the elliptic-parabolic problem (P_ε) investiagted in Section 4.2. The situation is simplified in the sense that we consider the measure $\mu = \mathcal{L}^d$, and a further reason why we dedicate an extra section is that in this case we will not require that b is Lipschitz continuous. This is especially relevant for applications, where typical functions b are merely Hölder continuous with small exponent α (cf. (4.89 below)). We highlight below our contribution to new homogenization and corrector results for Richards equation, but first discuss briefly some physical background.
Richards [55] formulated the dynamics of water movement through porous media by combining the conservation of water volume, $\partial_t\Theta + \mathrm{div}\vec{j}_w = f$, where Θ is the volume fraction of water and f the source term, with the empirical flux law $\vec{j}_w = -K(\Theta)[\nabla u - \rho\vec{g}]$, where u is the matric potential, ρ the mass density of water, and $\vec{g}$ the acceleration due to gravity. The flux law was proposed by Buckingham [22] as a generalization of Darcy's law to multiphase situations where all the fluid phases, except water, may be approximated as

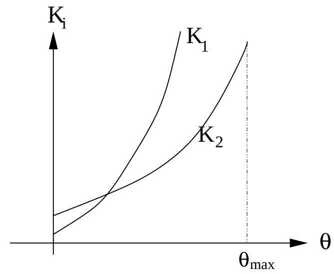
infinitely mobile. Richards equation $\partial_t \Theta - \mathrm{div}[K(\Theta)[\nabla u - \rho\vec{g}]] = f$ is widely used to numerically simulate the movement of soil moisture. We notice that this equation requires two constitutive relations, the soil water characteristic $\Theta(u)$ and the hydraulic conductivity $K(\Theta)$.
Natural porous media typically exhibit a hierarchical structure, hence $\Theta(u)$ and $K(\Theta)$ vary in space on various scales. The heuristic approach then is to consider a coarse-grained periodic composite $Y = \cup_i Y_i$, in which the Richards equation is valid in each component. Then K, and possibly the source f, jump on the interfaces and have the form

$$K(y,\Theta) = \sum_i \chi_i(y) K_i(\Theta)\,,\ f(y,\Theta) = \sum_i \chi_i(y) f_i(\Theta)\,,$$

where χ_i is the characteristic function of Y_i and K_i, f_i the individual characteristics of each component. On physical grounds one usually presumes the normalization $0 < \Theta_{\min} \le \Theta \le \Theta_{\max} < 1$, which corresponds to a bounded function b, if we set $\Theta = b(u)$ as in Section 4.2. Physical reasoning shows that b and K_i are monotonic. The qualitative form of the hydraulic functions is sketched below:



Soil-water charactristic



Hydraulic conductivity

Typical shapes of the function b relevant for the theory of flow in porous media or nonsteady filtration [2, 25, 51] are

$$b(z) = \max(0,z)^\alpha,\ b(z) = \mathrm{sgn}(z)|z|^\alpha \ \text{ or } \ b(z) = \min(e^{\alpha z}, 1) \tag{4.89}$$

with $\alpha > 0$ respectively, which implies that b is in general not Lipschitz (for $\alpha < 1$). As usual, in the upscaling process K and f become fast oscillating coefficients. If we abbreviate the gravity term by $\vec{e} := -\varrho\vec{g}$, Richards equation on a microscopic level with structure period ε gives rise to the family of doubly nonlinear parabolic equations of the form

$$(A_\varepsilon)\ \left\{\begin{array}{rcl} \partial_t b(u_\varepsilon) - \mathrm{div}\,(K(\frac{x}{\varepsilon}, b(u_\varepsilon))[\nabla u_\varepsilon + \vec{e}]) &=& f(\frac{x}{\varepsilon}, b(u_\varepsilon)) \ \text{ in } Q, \\ b(u_\varepsilon) &=& b^0_\varepsilon \ \text{ on } \Omega \times \{0\}, \\ u_\varepsilon &=& 0 \ \text{ on } \partial\Omega \times (0,T)\,, \end{array}\right.$$

which are familiar from Section 4.2. Extensions to time-oscillating data and more general boundary conditions will be discussed in Paragraph 4.3.3 below. We emphasize the degeneracy resulting from the fact that b can be flat. As the figure above and the examples in (4.89) show, this phenomenon occurs for physically reasonable data, which motivates not to restrict oneself to strictly monotone b. An additional degeneracy occurs if one allows $K(\cdot, \Theta_{\min}) = 0$, but as in (4.20) we will assume that K is strictly positive. Let us discuss the differences to the setting of Section 4.2, and the comparison with existing literature on the homogenization of Richards equation. The asymptotics of problem (A_ε) relies on the strong convergence

$$b(u_\varepsilon) \to b(u) \text{ in } L^1(Q)\,. \tag{4.90}$$

In contrast to Section 4.2, where we considered arbitrary Radon measures, we will not require that b is Lipschitz continuous, hence (4.90) can not be derived from the a priori estimates (cf. (4.31)) that one can expect for problem (A_ε). As shown in [2, Lemma 1.9], the crucial step in order to obtain (4.90) for a merely continuous function b is to prove an estimate of the form

$$\frac{1}{h}\int_0^{T-h}\int_\Omega (b(u_\varepsilon)(t+h) - b(u_\varepsilon)(t))(u_\varepsilon(t+h) - u_\varepsilon(t))\,dt \le C \tag{4.91}$$

uniformly in ε and $h > 0$. We are able to show (4.91) without presuming that the sequence $\{u_\varepsilon\}$ of solutions is uniformly bounded in $L^\infty(Q)$, thus generalizing the proof of (4.90) given in [41, Theorem 1.2]. The homogenized equation can then be derived as in Corollary 4.2.11, and we also investigate uniqueness, which is not done in the relevant papers [25, 39, 41, 47, 48]. In a second part, we are concerned with corrector results for problem (A_ε) or, more generally, for the equation (cf. (1.22) in the introduction)

$$\partial_t b(u_\varepsilon) - \operatorname{div} a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) = f. \tag{4.92}$$

In both cases we need the strong convergence $u_\varepsilon \to u$ in $L^1(Q)$, which in turn can only be expected for *strictly* monotone b. To this end, in [39, 41, 48] it was assumed that b enjoys the monotonicity condition

$$\exists r > 0\,\forall R > 0, \delta \in (0, R): \quad |b(s_1) - b(s_2)| > C(\delta, R)|s_1 - s_2|^r, \tag{4.93}$$

for all $s_1, s_2 \in [-R, R]$ and $|s_1| > \delta$, where $C(\delta, R) > 0$. However, using an argument from convex analysis (cf. Theorem 6.11 below), we can show the strong convergence of u_ε for any strictly monotone b (up to a linear growth condition) not necessarily satisfying (4.93) and, as mentioned above, without presuming any a priori bound on u_ε. Hence we make a new contribution to both, the homogenization and the derivation of corrector results for equations of type (A_ε) and (4.92). Let us briefly discuss the central aspects. The strong convergence of u_ε is essentially needed to justify the convergence

$$\int_0^T \langle\langle \partial_t b(u_\varepsilon), u_\varepsilon \rangle\rangle\,dt \to \int_0^T \langle\langle \partial_t b(u), u \rangle\rangle\,dt\,, \tag{4.94}$$

from which, as the proofs of Theorem 3.1.8 and Theorem 4.3.3 below show, the homogenization and the corrector result basically follow. However, the weak convergences

$$\partial_t b(u_\varepsilon) \overset{\star}{\rightharpoonup} \partial_t b(u) \text{ in } L^2H^{-1}, \quad u_\varepsilon \rightharpoonup u \text{ in } L^2H_0^1, \tag{4.95}$$

even combined with $u_\varepsilon \to u$ strongly in $L^p(Q)$ for some $p \geq 1$, are alone not sufficient to obtain (4.94), although this was suggested in [39, 41, 48]. The proof needs an additional argument, and can be saved by passing to the limit in the following identity proven in [2, Lemma 1.5], which essentially uses the solution property of u_ε (for the definition of B see (4.23)):

$$\int_0^\tau \langle\!\langle \partial_t b(u_\varepsilon), u_\varepsilon \rangle\!\rangle \, dt = \int_\Omega B(u_\varepsilon)(\tau) - \int_\Omega B(u_\varepsilon^0) \quad \text{for a.e. } \tau \in [0,T]\,. \tag{4.96}$$

In the case $b(s) = s, 2B(s) = s^2$, the convergence of the first term on the right-hand side precisely means $\|u_\varepsilon(\tau)\|_{L^2(\Omega)} \to \|u(\tau)\|_{L^2(\Omega)}$, which is guaranteed by (4.95) for almost every τ. However, for arbitrary b we have to work harder to get the desired convergence $B(u_\varepsilon)(\tau) \to B(u)(\tau)$ in $L^1(\Omega)$. To this end we have to prove the strong convergence of $\{u_\varepsilon\}$ first, where we have to use the strict monotonicity of b and Theorem 6.11 again.

Theorem 4.3.1. *Under Assumption 4.2.1 on the data, let b additionally satisfy the linear growth condition (4.22). Then there exists a solution u_ε of problem (A_ε) in the sense of Definition 4.2.12, fulfilling the a priori estimate*

$$\|\partial_t b(u_\varepsilon)\|_{L^2H^{-1}} + \|u_\varepsilon\|_{L^2H^1} + \|B(u_\varepsilon)\|_{L^\infty L^1} \leq C(1 + \|B(u_\varepsilon^0)\|_{L^1(\Omega)}) \tag{4.97}$$

with a constant independent of ε.

Proof. Note that we formulated problem $(\star)$ on page 108 for data a and f independent of $x \in \Omega$, and hence also the corresponding weak formulation in Definition 4.2.12 and the associated existence result in Theorem 6.10 of the appendix. However, as far as existence is concerned, the x-dependence makes no difference as pointed out in [2, Remark 1.10]. If for fixed $\varepsilon > 0$ we set

$$a(x,s,\xi) := K(\tfrac{x}{\varepsilon}, s)[\xi + \vec{e}], \; g(x,s) := f(\tfrac{x}{\varepsilon}, s)\,,$$

we easily check that all the prerequisites of Theorem 6.10 are satisfied, which guarantees the existence of a solution $u_\varepsilon \in L^2H_0^1$ of problem (A_ε) in the sense of Definition 4.2.12. To this end we use the uniform upper bound on K in (4.20) and the linear growth condition (4.22) combined with Lemma 4.2.3 to obtain

$$|K(\tfrac{x}{\varepsilon}, b(z))[\xi + \vec{e}]| + |f(\tfrac{x}{\varepsilon}, b(z))| \leq C(1 + B(z)^{1/2} + |\xi|). \tag{4.98}$$

Theorem 6.10 also gives $\partial_t b(u_\varepsilon) \in L^2H^{-1}$ and $B(u_\varepsilon) \in L^\infty L^1$ with

$$\int_\Omega B(u_\varepsilon)(\tau)\, dx - \int_\Omega B(u_\varepsilon^0)\, dx = \int_0^\tau \langle\!\langle \partial_t b(u_\varepsilon), u_\varepsilon \rangle\!\rangle \, dt \tag{4.99}$$

for almost every $\tau \in [0,T]$. Possibly choosing u_ε as a restriction of a solution on a larger time interval $(0, T+\delta)$, for the proof of estimate (4.97) we can assume that (4.99) holds for $\tau = T$. Then choosing u_ε in the weak formulation

(4.85), and using the positivity of B, the uniform bounds on K, the growth conditions on b and f, and standard absorption techniques, we get

$$\|\nabla u_\varepsilon\|^2_{L^2(Q)} \le C(1 + \|B(u^0_\varepsilon)\|_{L^1(\Omega)}) \tag{4.100}$$

where C only depends on the constants of Assumption 4.2.1 and the Poincaré constant in $H^1_0(\Omega)$. Similarly, by (4.85), the assumptions on K, f and b, and the definition of the norm in $L^2H^{-1} = (L^2H^1_0)'$ we get

$$\|\partial_t b(u_\varepsilon)\|_{L^2H^{-1}} \le C(1 + \|\nabla u_\varepsilon\|_{L^2(Q)}). \tag{4.101}$$

By (4.99)-(4.101) and the positivity of B it also easily follows that

$$\|B(u_\varepsilon)\|_{L^\infty L^1} \le C(1 + \|\nabla u_\varepsilon\|^2_{L^2(Q)}) + \|B(u^0_\varepsilon)\|_{L^1(\Omega)} \le C(1 + \|B(u^0_\varepsilon)\|_{L^1(\Omega)}),$$

which shows (4.97) and completes the proof of the theorem. □

4.3.1 Homogenization

The two-scale homogenized problem for (A_ε) and the homogenized equation for the weak limit u of u_ε can be derived as in Paragraph 4.2.2, if we manage to prove the strong convergence of $b(u_\varepsilon)$ in $L^1(Q)$. We emphasize that this time we do not require that b is Lipschitz continuous, but merely presume the linear growth condition (4.22), which suffices to derive the crucial estimate (4.91). As usual, for the homogenization step we require local Hölder continuity for the data K and f with respect to s. Recall the definition of the effective tensor $K^\star(s)$ in (4.81) on page 107, subject to the solutions $v_k(\cdot, s) \in V^2_{\mathrm{pot}}(\mathbb{T})$ of the standard cell problems given in (4.80) for $\mu = \mathcal{L}^d$.

Theorem 4.3.2. *Let $\{u_\varepsilon\} \subset L^2H^1_0$ be a family of solutions of problem (A_ε) according to Theorem 4.3.1. Assume that K and f satisfy Assumption 4.2.9, and that $u^0_\varepsilon \in L^2(\Omega)$ with $u^0_\varepsilon \to u^0$ strongly in $L^2(\Omega)$. Then up to subsequences there holds*

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0,T; H^1_0(\Omega)), \tag{4.102}$$

$$b(u_\varepsilon) \to b(u) \quad \text{strongly in } L^1(Q), \tag{4.103}$$

where $u \in L^2(0,T; H^1_0(\Omega))$ is a solution of the homogenized equation

$$(A_0) \quad \left\{ \begin{array}{rcll} \partial_t b(u) - \operatorname{div}(K^\star(b(u))[\nabla u + \vec{e}]) & = & \bar{f}(b(u)) & \text{in } Q, \\ u & = & 0 & \text{on } \partial\Omega \times (0,T) \end{array} \right.$$

in the sense of Definition 4.2.12 with initial value $b^0 = b(u^0)$. If Assumption 4.2.9 is satisfied with $\gamma, \tilde{\gamma} = 1$, and if b is Hölder continuous on $\mathbb{R}$ with exponent $1/2$, then the solution of problem (A_0) is unique.

Proof. Note that by the linear growth condition on b and the strong convergence of the initial values u_ε^0 we get $\|B(u_\varepsilon^0)\|_{L^1(\Omega)} \le C$ uniformly in ε. From the a priori estimate (4.97) of Theorem 4.3.1 we then deduce

$$\|\partial_t b(u_\varepsilon)\|_{L^2H^{-1}} + \|u_\varepsilon\|_{L^2H^1} + \|B(u_\varepsilon)\|_{L^\infty L^1} \le C, \tag{4.104}$$

which gives (4.102) for a subsequence. Using the superlinearity property of Ψ in (4.24), and then applying the coercivity property of Ψ in (4.25) gives

$$\|b(u_\varepsilon)\|_{L^\infty L^2} \le C \quad \text{uniformly in } \varepsilon, \tag{4.105}$$

where we also used the estimate on $B(u_\varepsilon)$ in (4.104). Hence by a standard approximation argument, any $\zeta \in H_0^1(Q)$ is an admissible test function in (4.84). Now let a small number $h > 0$ be given. We define the following auxiliary functions $\eta : \mathbb{R} \to \mathbb{R}$ and $\tilde{u}_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\eta(t) = \begin{cases} 1 & \text{if } t \in [0, T-h], \\ 0 & \text{else,} \end{cases} \qquad \tilde{u}_\varepsilon(x,t) = \begin{cases} u_\varepsilon(x,t) & \text{if } t \in (0,T), \\ 0 & \text{else.} \end{cases}$$

By the definition of η and $\tilde{u}_\varepsilon$ it is obvious that the following functions ζ_k belong to $H_0^1(Q)$:

$$\zeta_k : Q \to \mathbb{R},\ (x,t) \mapsto \frac{1}{h}\int_{t-h}^{t} \tilde{u}_\varepsilon(x, \tau + kh)\eta(\tau)\,d\tau\,, \quad k \in \{0,1\}\,.$$

Moreover, we claim that

$$\forall k = 0,1 : \quad \|\zeta_k\|_{L^2(Q)} + \|\nabla\zeta_k\|_{L^2(Q)} \le C \tag{4.106}$$

with a constant independent of ε and h. Indeed, by Jensen's inequality

$$\begin{aligned} \|\nabla\zeta_0\|_{L^2(Q)}^2 &\le \tfrac{1}{h}\int_0^T\int_\Omega\int_{t-h}^{t} |\nabla u_\varepsilon(x,\tau)|^2\eta(\tau)\,d\tau dx dt \\ &= \int_0^{T-h}\int_\Omega |\nabla u_\varepsilon(x,t)|^2\eta(t)\,dx dt \le C \end{aligned}$$

with a constant independent of h and ε by (4.104). Similarly we can estimate ζ_1, which shows (4.106). Now using the definition of η and a basic integral transformation an easy calculation gives

$$\begin{aligned} &\int_Q (b_\varepsilon^0 - b(u_\varepsilon))\partial_t(\zeta_1 - \zeta_0) \\ =\ & \tfrac{1}{h}\int_Q (b_\varepsilon^0 - b(u_\varepsilon)(t))\left[(\tilde{u}_\varepsilon(t+h) - \tilde{u}_\varepsilon(t))\eta(t) - (\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(t-h))\eta(t-h)\right] dt \\ =\ & \tfrac{1}{h}\int_0^{T-h}\int_\Omega (b(u_\varepsilon)(t+h) - b(u_\varepsilon)(t))(u_\varepsilon(t+h) - u_\varepsilon(t))\,dt\,, \end{aligned} \tag{4.107}$$

where we used that b_ε^0 does not depend on t. By the above considerations, $\zeta_1 - \zeta_0$ is an admissible test function both in (4.84) and (4.85). Thus the last term in (4.107) can be estimated using (4.20),(4.21),(4.22),(4.104) and (4.106), which yields estimate (4.91) on page 111 with a constant independent of ε and h. The latter combined with (4.102) and (4.104) allows to apply [2, Lemma 1.9], which gives the strong convergence of $b(u_\varepsilon) \to b(u)$ in (4.103). Moreover, from (4.103),(4.104) and (4.105) we immediately deduce

$$\partial_t b(u_\varepsilon) \stackrel{\star}{\rightharpoonup} \partial_t b(u) \quad \text{weak}^\star \text{ in } L^2(0,T;H^{-1}(\Omega))\,, \tag{4.108}$$

$$b(u_\varepsilon) \stackrel{\star}{\rightharpoonup} b(u) \quad \text{weak}^\star \text{ in } L^\infty(0,T;L^2(\Omega))\,. \tag{4.109}$$

In successively deriving all the conditions of Definition 4.2.12, we next observe that by the assumptions on the initial values u^0_ε and the continuity and the growth condition on b, we have $b^0 = b(u^0) \in L^2(\Omega)$ and $\Psi(b^0) \in L^1(\Omega)$. In particular, by (4.108) and (4.109) the identity in (4.84) holds for any admissible ζ. Note that (4.105) and the linear growth condition on b combined with the estimate (4.104) on u_ε yield that $b(u_\varepsilon)$ is also uniformly bounded in $L^{2+\delta}(Q)$ for some $\delta > 0$, whence $b(u_\varepsilon) \to b(u)$ strongly in $L^2(Q)$ by (4.103). As usual, the Hölder conditions of Assumption 4.2.9 on K and f then yield

$$\begin{aligned} \|K(\tfrac{x}{\varepsilon}, b(u_\varepsilon)) - K(\tfrac{x}{\varepsilon}, b(u))\|_{L^2(Q)} &\le C\,\|b(u_\varepsilon) - b(u)\|^{\gamma}_{L^2(Q)} = o(1), \\ \|f(\tfrac{x}{\varepsilon}, b(u_\varepsilon)) - f(\tfrac{x}{\varepsilon}, b(u))\|_{L^1(Q)} &\le C\,\|b(u_\varepsilon) - b(u)\|^{\tilde\gamma}_{L^2(Q)} = o(1) \end{aligned}$$

as $\varepsilon \to 0$ respectively. Choosing the standard test function φ_ε (cf. (4.79) on page 107) for ζ in the weak formulation (4.85) of problem (A_ε), and passing to the limit $\varepsilon \to 0$, we can derive the integral identity

$$\int_0^T \langle\!\langle \partial_t b(u), \phi \rangle\!\rangle + \int_Q K^\star(b(u))[\nabla u + \vec{e}] \cdot \nabla\phi = \int_Q \bar{f}(b(u))\phi \tag{4.110}$$

for any $\phi \in \mathcal{D}(Q)$ precisely as in the proof of Theorem 4.2.10 and Corollary 4.2.11 respectively. Since $K^\star(b(u))[\nabla u + \vec{e}]$ and $\bar{f}(b(u))$ belong to $L^2(Q)$, and due to (4.108), by density (4.110) also holds for any $\phi \in L^2 H^1_0$, which shows that u is a solution of problem (A_0) in the sense of Definition 4.2.12. For the uniqueness statement, we can argue precisely as in the proof of Corollary 4.2.13. □

4.3.2 Corrector results

Now we prove a classical first order corrector result (cf. (1.12)) under the assumption that b is strictly monotonically increasing, and provided the corrector function u_1 determined in (4.82) is sufficiently smooth. Since the crucial identity (4.96) on page 112 holds only for almost every τ, we possibly need to consider a smaller time interval.

Theorem 4.3.3. *Under the assumptions of Theorem 4.3.2, let $b : \mathbb{R} \to \mathbb{R}$ be strictly monotonically increasing, and the Hölder assumption (4.74) on f be satisfied for $\tilde{h} \in L^{\tilde q}_{\text{per}}(Y)$ with $\tilde q := 2(1-\tilde\gamma)^{-1} \in (2,\infty]$. Then there holds*

$$\forall q \in [1,2): \quad u_\varepsilon \to u \quad \textit{strongly in } L^q(0,T;L^2(\Omega)) \tag{4.111}$$

for the sequence in (4.103). If u and u_1 are sufficiently smooth, i.e. belong to $\mathcal{C}([0,T];\mathcal{C}^1(\overline{\Omega}))$ and $\mathcal{C}(\overline{Q};\mathcal{C}^1_{\mathrm{per}}(Y))$ respectively, then there holds

$$\nabla u_\varepsilon - \nabla u - \nabla_y u_1(x,t,\tfrac{x}{\varepsilon}) \;\to\; 0 \quad \textit{strongly in } L^2(\Omega\times(0,T-\delta)) \tag{4.112}$$

for any $\delta > 0$. If the functions u_ε are restrictions of solutions on a larger time interval, then we get the convergence in (4.112) for $\delta = 0$.

Proof. The central aspects of the proof have already been discussed in the introduction of Section 4.3. The main difficulty is to pass to the limit in the expression

$$\int_Q \langle K(\tfrac{x}{\varepsilon}, b(u_\varepsilon))\nabla u_\varepsilon, \nabla u_\varepsilon\rangle \;=\; \int_Q f(\tfrac{x}{\varepsilon}, b(u_\varepsilon))u_\varepsilon - \int_0^T \langle\!\langle \partial_t b(u_\varepsilon), u_\varepsilon\rangle\!\rangle\,. \tag{4.113}$$

The problem is twofold: The set of points where the crucial identity (4.96) does not hold depends on ε, and $t = T$ can be a point of exception. This is why in a first step we have to consider a smaller time interval. The second problem is the convergence of the right-hand side in (4.96). We need to show

$$u_\varepsilon \;\to\; u \qquad \text{strongly in } L^1(Q)\,, \tag{4.114}$$

$$B(u_\varepsilon) \;\to\; B(u) \quad \text{strongly in } L^1(Q)\,. \tag{4.115}$$

We apply Theorem 6.11 of the appendix to the strictly convex and continuous function $h(s) = \int_0^s b(z)\,dz$. Observe that $h(u_\varepsilon) < \infty$ pointwise in Q and

$$b(u_\varepsilon)(u_\varepsilon - u) \;\geq\; h(u_\varepsilon) - h(u) \quad \text{almost everywhere in } Q.$$

As the proof of Theorem 4.3.2 shows, we have $b(u_\varepsilon) \to b(u)$ strongly in $L^2(Q)$. Integrating the last inequality over Q, we obtain with (4.102):

$$0 \;\leftarrow\; \int_Q b(u_\varepsilon)(u_\varepsilon - u) \;\geq\; \int_Q h(u_\varepsilon) - \int_Q h(u) \;=:\; \Phi(u_\varepsilon) - \Phi(u)\,. \tag{4.116}$$

It follows $\limsup_{\varepsilon\to 0}\Phi(u_\varepsilon) \leq \Phi(u)$ and $\Phi(u) \neq +\infty$ since $h(u) \in L^1(Q)$ by the linear growth condition on b. Hence Theorem 6.11 can be applied which yields (4.114), and (4.115) follows from the definition of B and the strong convergence of $b(u_\varepsilon)$. The first corrector result (4.111) follows from (4.114) and (4.102). To prove (4.112), we set $Q_t := \Omega\times(0,t)$. First we get

$$\int_{Q_t} f(\tfrac{x}{\varepsilon}, b(u_\varepsilon))\,u_\varepsilon \;=\; \int_{Q_t} f(\tfrac{x}{\varepsilon}, b(u))\,u_\varepsilon + o(1) \quad \text{as } \varepsilon \to 0$$

thanks to the improved Hölder condition on f and the strong convergence of $b(u_\varepsilon)$. By the regularity assumption on b and u, $\Phi(x,t,y) := f(y, b(u(x,t)))$ is an admissible test function for the two-scale convergence and it follows

$$\forall t \in (0,T): \quad \int_{Q_t} f(\tfrac{x}{\varepsilon}, b(u_\varepsilon))\,u_\varepsilon \;\longrightarrow\; \int_{Q_t} \bar{f}(b(u))\,u\,. \tag{4.117}$$

To shorten the notation, we show (4.112) for $\vec{e} = 0$, which is clearly uncritical. Let $\delta > 0$ be given and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be an arbitrary subsequence. By (4.96) and (4.115) there exists $\tau \in (T-\delta, T)$, such that for every n:

$$\begin{aligned}
\int_0^\tau \langle\!\langle \partial_t b(u_{\varepsilon_n}), u_{\varepsilon_n} \rangle\!\rangle \quad &= \quad \int_\Omega B(u_{\varepsilon_n})(\tau) - \int_\Omega B(u^0_{\varepsilon_n}) \\
&\overset{n\to\infty}{\longrightarrow} \quad \int_\Omega B(u)(\tau) - \int_\Omega B(u^0) \; = \; \int_0^\tau \langle\!\langle \partial_t b(u), u \rangle\!\rangle \,,
\end{aligned}$$

the last equality holding, since u is a solution of the limit problem (A_0) with initial value $b^0 = b(u^0)$ (cf. Theorem 6.10 below). This combined with (4.113) and (4.117) gives

$$\begin{aligned}
\int_{Q_\tau} K(\tfrac{x}{\varepsilon_n}, b(u_{\varepsilon_n}))\nabla u_{\varepsilon_n} \cdot \nabla u_{\varepsilon_n} \quad &\to \quad \int_{Q_\tau} \bar{f}(b(u))u - \int_0^\tau \langle\!\langle \partial_t b(u), u \rangle\!\rangle \\
&= \quad \int_{Q_\tau} K^\star(b(u))\nabla u \cdot \nabla u \,. \qquad (4.118)
\end{aligned}$$

Exploiting the regularity assumptions on u and u_1, we define a test function $\phi_\varepsilon(x,t) := \phi(x,t,\frac{x}{\varepsilon})$ with $\phi(x,t,y) = \nabla u(x,t) + \nabla_y u_1(x,t,y) \in L^2_{\mathrm{per}}(Y;\mathcal{C}(\bar{Q}))$. We claim that

$$\begin{aligned}
\int_{Q_\tau} K(\tfrac{x}{\varepsilon_n}, b(u_{\varepsilon_n}))[2\nabla u_{\varepsilon_n} - \phi_{\varepsilon_n}] \cdot \phi_{\varepsilon_n} \quad &\to \quad \int_{Q_\tau \times Y} K(y, b(u))\{\nabla u + \nabla_y u_1\}^2 \\
&= \quad \int_{Q_\tau} K^\star(b(u))\nabla u \cdot \nabla u \,, \qquad (4.119)
\end{aligned}$$

where we used the notation $Kv^2 := v \cdot Kv$. Indeed, by construction we have $\|\phi_\varepsilon\|_{L^\infty(Q)} \le C$, so that the Hölder continuity of K with respect to s combined with the strong convergence of $b(u_\varepsilon)$ gives:

$$\int_{Q_\tau} \Big(K(\tfrac{x}{\varepsilon_n}, b(u_{\varepsilon_n})) - K(\tfrac{x}{\varepsilon_n}, b(u)) \Big) [-2\nabla u_{\varepsilon_n} + \phi_{\varepsilon_n}] \cdot \phi_{\varepsilon_n} \to 0 \,.$$

The function $K(y, b(u(x,t)))\phi(x,t,y) \in L^2_{\mathrm{per}}(Y;\mathcal{C}(\bar{Q}))$ is an admissible test function for the two-scale convergence, so by the definition of ϕ_ε and by the characterization (4.77) of the two-scale limit of $\{\nabla u_\varepsilon\}$ we get the convergence in (4.119). The equality in (4.119) is a straightforward calculation using the definition of $K^\star$. Recall that $c_k > 0$ denotes the uniform lower bound on K in (4.20). Combining (4.118) and (4.119) we get

$$c_k \, \|\nabla u_{\varepsilon_n} - \phi_{\varepsilon_n}\|^2_{L^2(Q_\tau)} \le \int_{Q_\tau} K(\tfrac{x}{\varepsilon_n}, b(u_{\varepsilon_n}))[\nabla u_{\varepsilon_n} - \phi_{\varepsilon_n}] \cdot (\nabla u_{\varepsilon_n} - \phi_{\varepsilon_n}) \to 0,$$

which proves (4.112) for a given $\delta > 0$. The additional statement is obvious. $\square$

4.3.3 Extensions and outlook

We briefly discuss some extensions of our analysis. For instance, we consider time oscillating coefficients and discuss more general boundary conditions.

- **Time oscillating data:**
  If b is strictly monotonically increasing, our results can with minor changes in the proofs be extended to equations with time-oscillating coefficients (cf. equation (1.1) in the introduction) of the type
  $$\partial_t b(u_\varepsilon) - \operatorname{div}\left(K(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, b(u_\varepsilon))[\nabla u_\varepsilon + \vec{e}]\right) \;=\; f(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, b(u_\varepsilon)), \qquad (4.120)$$
  where $K(y,\tau,s)$ and $f(y,\tau,s)$ are $Y \times (0,1)$-periodic in (y,τ) and sufficiently smooth in s. The modified assumption on K reads
  $$\forall (y,\tau,s) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : \quad c_k|\xi|^2 \;\le\; \xi \cdot K(y,\tau,s)\xi \;\le\; C_K|\xi|^2 .$$
  The central point is that for strictly monotone b we can deduce the strong convergence $u_\varepsilon \to u$ in $L^1(Q)$ as the proof of Theorem 4.3.3 shows, which implies that the two-scale limit $u_\varepsilon \rightharpoonup\!\!\rightharpoonup u(x,t,y,\tau)$, when testing with functions $\psi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ in the homogenization step, does neither depend on y nor on the fast time scale variable τ. Therefore the two-scale homogenized problem can be decoupled and we obtain
  $$K^\star_{ij}(s) \;=\; \int_{Y\times(0,1)} K(y,\tau,s)[\hat{e}_i + \nabla\Lambda^s_i(y,\tau)] \cdot (\hat{e}_j + \nabla\Lambda^s_j(y,\tau))\, dy d\tau\, ,$$
  where $\Lambda^s_k \in L^2((0,1); H^{1,2}_{\mathrm{per}}(Y))$ solves a cell problem in $Y \times (0,1)$.

- **Space oscillating saturation:**
  A natural question is whether we can consider an oscillating saturation $b = b(y,s)$, which is Y-periodic in y and continuous in s. In [48] it was shown that if b is continuous in y, there holds $b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u) \to 0$ strongly in $L^2(Q)$, at least under the assumption (4.93). If b is merely an element of $L^\infty_{\mathrm{per}}(Y; \mathcal{C}_{\mathrm{b}}(\mathbb{R}))$, which models a saturation that jumps between different characteristics, the analysis seems to be more involved.

- **Boundary conditions:**
  We assumed a homogeneous Dirichlet condition on the lateral boundary. However, as the analysis in [2, 41, 48] shows, with minor changes the corresponding results can be derived for the set of boundary conditions
  $$\begin{aligned} u_\varepsilon \;&=\; g \quad \text{on } \Gamma \times (0,T) \\ K(\tfrac{x}{\varepsilon}, b(u_\varepsilon))\nabla u_\varepsilon \cdot \nu \;&=\; 0 \quad \text{on } \partial\Omega \setminus \Gamma \times (0,T) \end{aligned}$$
  provided the regularity $g \in L^2H^1 \cap W^{1,1}L^\infty$ and compatible conditions on the initial data. Here $\Gamma \subset \partial\Omega$ is measurable with $\mathcal{H}^{d-1}(\Gamma) > 0$. For a recent study on Richards equation with an outflow boundary condition we refer to [57]. In this case the homogenization problem is open.

5 Two-parameter equations

In this chapter we consider the homogenization of fattened structures. To this end, in addition to the microscale parameter ε, another parameter δ has to be introduced which corresponds to the thickness of the reinforced singular structure. Technically speaking, for a given singular measure $\mu = \mathcal{H}^k \lfloor S$ with $k < d$, we consider approximating measures μ^δ that are absolutely continuous with respect to $\mathcal{L}^d$, such that

$$\mu^\delta \overset{\star}{\rightharpoonup} \mu \ \text{ in } \mathcal{C}(\mathbb{T}), \quad \mu_\varepsilon^\delta \overset{\star}{\rightharpoonup} \mu_\varepsilon \ \text{ in } \mathcal{C}_0(\Omega) \tag{5.1}$$

as $\delta \to 0$ respectively, where μ_ε^δ is the ε-periodization of μ^δ according to (1.3). Various examples will be introduced below (see also (1.8)). We investigate, whether the two-parameter diagram starting from the structure μ_ε^δ will commute in the sense that interchanging the order of passage to the limit in ε and δ leaves the homogenized equation invariant.

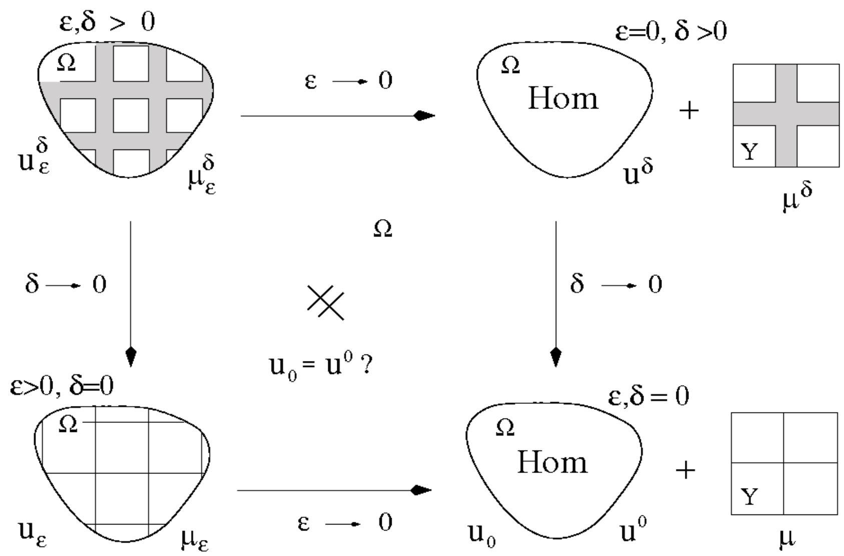


Figure 5.1: Commutativity of limits

Taking advantage of the results in Chapter 3, we are able to show commutativity for the class of quasilinear problems introduced in (1.13), provided the underlying measure μ is sufficiently connected: The limit functions u^0 and u_0 obtained respectively from the two limit processes

$$u_\varepsilon^\delta \ \overset{\varepsilon\to 0}{\longrightarrow} \ u^\delta \ \overset{\delta\to 0}{\longrightarrow} \ u^0 , \quad u_\varepsilon^\delta \ \overset{\delta\to 0}{\longrightarrow} \ u_\varepsilon \ \overset{\varepsilon\to 0}{\longrightarrow} \ u_0 \tag{5.2}$$

will be solutions of one and the same effective problem, and hence coincide provided we have uniqueness. We emphasize that commutativity can not be expected for nonconnected measures μ, and we will construct explicit counterexamples in Paragraph 5.2.3 below. We content ourselves with the stationary setting, in which the major nontrivial effects already occur. Our methods can be extended to the time-dependent case, in particular Richards equation (cf. problem (A_ε) on page 110) studied in Section 4.3 can perfectly be treated.

5.1 Convergence in variable L^p-spaces

To the aim of studying fattened structures we need to introduce a suitable notion of convergence of measures μ^δ supported on these structures to the singular limit measure μ. For the periodic setting this is clearly the weak*-convergence in the space of continuous functions on $\mathbb{T}$.

Definition 5.1.1. *Let $\{\mu^\delta\}$ be a sequence of positive, Y-periodic Radon measures on $\mathbb{R}^d$. We say that μ^δ converges weakly to μ in the sense of measures, and write $\mu^\delta \rightharpoonup \mu$, if*

$$\lim_{\delta\to 0} \int_Y \varphi(y)\, d\mu^\delta(y) \;=\; \int_Y \varphi(y)\, d\mu(y) \quad \forall \varphi \in \mathcal{C}(\mathbb{T})\,. \tag{5.3}$$

In particular, μ is also a positive, Y-periodic Radon measure on $\mathbb{R}^d$.

We usually assume that the measures μ^δ (and hence also the weak limit μ) are normalized, that means $\mu^\delta(Y) = 1$ for any $\delta > 0$, which is no restriction. An important example of a fattened structure and its singular limit is given in Example 5.1.2 below, which also serves as our model problem. Although the stated convergence $\mu^\delta \rightharpoonup \mu$ is rather obvious, for convenience we give a proof to get familiar with the techniques of this section. More sophisticated examples will be studied in the next section.

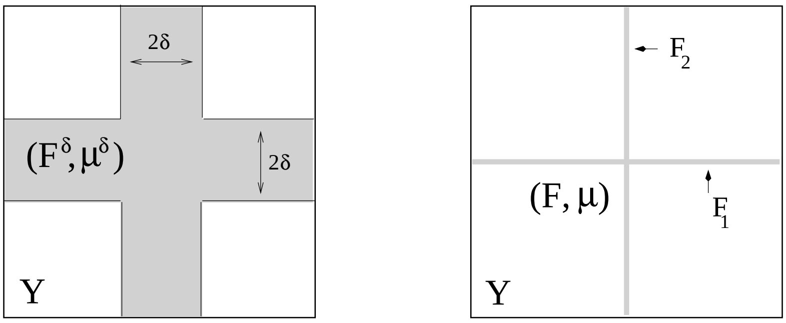


Figure 5.2: Fattened structure and limit

Example 5.1.2. *Let $F = \cup_i F_i$ and $F^\delta = \cup_i F_i^\delta$ as in Figure 5.2, that means*

$$F_i = \{y \in Y \mid y_j = \tfrac{1}{2},\, j \neq i\}\,, \quad F_i^\delta = \{y \in Y \mid |y_j - \tfrac{1}{2}| < \delta,\, j \neq i\}\,,$$

with corresponding measure $\mu^\delta := c_\delta \mathcal{L}^2 \lfloor F^\delta$, where $c_\delta = [4\delta(1-\delta)]^{-1}$ is the normalizing constant. Then there holds

$$\mu^\delta \rightharpoonup \mu := \tfrac{1}{2} \sum_i \mathcal{H}^1 \lfloor F^i = \tfrac{1}{2}\mathcal{H}^1 \lfloor F. \tag{5.4}$$

Proof. Let $\varphi \in \mathcal{C}(\mathbb{T})$ be given and set $\Box_\delta := (\frac{1}{2}-\delta, \frac{1}{2}+\delta)^2$. In order to calculate the limit of $\int_Y \varphi(y)\, d\mu^\delta(y)$ it suffices to estimate

$$\left| c_\delta \int_{\Box_\delta} \varphi(y)\, d\mathcal{L}^2(y) \right| \le C\delta^{-1}\delta^2 \|\varphi\|_\infty \to 0$$

and to pass, exemplary, to the limit in the following expression:

$$c_\delta \int_0^{\frac{1}{2}-\delta} \int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} \varphi(y)\, dy \;=\; 2\delta c_\delta \int_0^{\frac{1}{2}} \varphi(y_1, \xi_\delta(y_1))\, dy_1 + o(1) \;\to\; \tfrac{1}{2} \int_0^{\frac{1}{2}} \varphi(y_1, \tfrac{1}{2})\, dy_1$$

as $\delta \to 0$, where $\xi_\delta(y_1) \in (\frac{1}{2}-\delta, \frac{1}{2}+\delta)$ can be found by the mean value theorem of integration, and where we used the Lebesgue convergence theorem. □

We introduce a notion of weak convergence for bounded sequences $\{v_\delta\}$ in the variable Lebesgue spaces $L^p(Y, d\mu^\delta)$, subject to an underlying sequence of measures $\mu^\delta \rightharpoonup \mu$ in the sense of Definition 5.1.1. This new concept is adapted to the fattening approach and should not be confused with the two-scale convergence introduced in Definition 2.1.2, where we studied ε-rescalings of a fixed measure μ instead of a fattening approximation. Unless otherwise stated, we always assume $p \in (1, \infty)$.

Definition 5.1.3. *Let $\{v_\delta\}$ be a bounded sequence in $L^p(Y, d\mu^\delta)$, that means*

$$\|v_\delta\|_{p,\mu^\delta,Y} = \left(\int_Y |v_\delta|^p \, d\mu^\delta \right)^{1/p} \le C \tag{5.5}$$

with a constant independent of δ. We say that $\{v_\delta\}$ weakly converges to $v \in L^p(Y, d\mu)$ and write $v_\delta \rightharpoonup v$ in $L^p(Y, d\mu^\delta)$, if

$$\int_Y v_\delta \varphi \, d\mu^\delta \to \int_Y v\varphi \, d\mu \quad \textit{for each } \varphi \in \mathcal{C}(\mathbb{T}). \tag{5.6}$$

It is essential to verify that the weak convergence in the variable L^p-spaces defined above enjoys the weak compactness property:

Proposition 5.1.4. *Let $\{v_\delta\}$ be a sequence in $L^p(Y, d\mu^\delta)$ endowed with the uniform bound (5.5). Then, up to subsequences, there exists $v \in L^p(Y, d\mu)$, such that*

$$v_\delta \rightharpoonup v \quad \textit{in } L^p(Y, d\mu^\delta). \tag{5.7}$$

Proof. Let $\varphi \in \mathcal{C}(\mathbb{T})$ be given. We can assume that the measures μ^δ are normalized on Y. Since $|\varphi|^q$ belongs to $\mathcal{C}(\mathbb{T})$ we easily check

$$\forall q \in (1,\infty): \quad \|\varphi\|_{q,\mu,Y} \leftarrow \|\varphi\|_{q,\mu^\delta,Y} \le \|\varphi\|_\infty := \sup_{y\in Y} |\varphi(y)|, \tag{5.8}$$

where the inequality in (5.8) holds uniformly for any $\delta > 0$. Now we can define a sequence $T_\delta \in \mathcal{C}(\mathbb{T})'$ by

$$T_\delta : \mathcal{C}(\mathbb{T}) \to \mathbb{R}, \; \varphi \mapsto \int_Y v_\delta(y)\varphi(y)\, d\mu^\delta(y). \tag{5.9}$$

Let $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denote the dual pairing between $\mathcal{C}(\mathbb{T})'$ and $\mathcal{C}(\mathbb{T})$. Then using (5.5),(5.8) and Hölder's inequality we deduce

$$|\langle\!\langle T_\delta,\varphi\rangle\!\rangle| \le C\|\varphi\|_\infty .$$

Hence T_δ is uniformly bounded in $\mathcal{C}(\mathbb{T})'$, and by the separability of $\mathcal{C}(\mathbb{T})$ we get a subsequence and a measure $T_0 \in \mathcal{C}(\mathbb{T})'$, such that $\langle\!\langle T_\delta,\varphi\rangle\!\rangle \to \langle\!\langle T_0,\varphi\rangle\!\rangle$ for all $\varphi \in \mathcal{C}(\mathbb{T})$. Consequently

$$|\langle\!\langle T_0,\varphi\rangle\!\rangle| \le C\|\varphi\|_{p',\mu,Y} \quad \text{for each } \varphi \in \mathcal{C}(\mathbb{T})\,, \tag{5.10}$$

where we applied the convergence in (5.8) to $q = p'$. By the Hahn-Banach theorem, T_0 can be extended to a continuous linear functional on $L^{p'}(Y,d\mu)$, hence there exists a representative $v \in L^p(Y,d\mu)$, such that

$$\langle\!\langle T_0,\varphi\rangle\!\rangle = \int_Y v\varphi\,d\mu \leftarrow \int_Y v_\delta\varphi\,d\mu^\delta\,,$$

for each $\varphi \in \mathcal{C}(\mathbb{T})$, which completes the proof. □

It is easy to check, that the weak convergence in (5.7) enjoys the lower semicontinuity property:

Proposition 5.1.5. *Assume $v_\delta \rightharpoonup v$ in $L^p(Y,d\mu_\delta)$ for some $v \in L^p(Y,d\mu)$. Then there holds*

$$\liminf_{\delta\to 0}\int_Y |v_\delta|^p\,d\mu_\delta \ \ge\ \int_Y |v|^p\,d\mu\,. \tag{5.11}$$

As usual, the weak convergence it not sufficient to study nonlinear problems. We need to introduce a notion of strong convergence in the variable L^p-spaces, which is similar to the concept in Section 2.1:

Definition 5.1.6. *Let $\{v_\delta\}$ be a bounded sequence in $L^p(Y,d\mu^\delta)$ according to (5.5). We say that $\{v_\delta\}$ strongly converges to $v \in L^p(Y,d\mu)$ and write $v_\delta \to v$ in $L^p(Y,d\mu^\delta)$, if*

$$\lim_{\delta\to 0}\int_Y v_\delta w_\delta\,d\mu^\delta = \int_Y vw\,d\mu\,, \tag{5.12}$$

whenever $w_\delta \rightharpoonup w$ weakly in $L^{p'}(Y,d\mu^\delta)$.

In some situations (cf. Lemma 5.2.14 below) it is helpful to have an alternative characterization of the strong convergence in the variable Lebesgue spaces. It comprises the weak convergence combined with the convergence of the norms.

Lemma 5.1.7. *Assume $v_\delta \rightharpoonup v$ weakly in $L^p(Y,d\mu^\delta)$ for $p>1$ and some $v \in L^p(Y,d\mu)$. Then v_δ converges also strongly to v in $L^p(Y,d\mu^\delta)$, if and only if*

$$\lim_{\delta\to 0}\int_Y |v_\delta|^p\,d\mu^\delta \ =\ \int_Y |v|^p\,d\mu\,. \tag{5.13}$$

Proof. Since we apply the statement of Lemma 5.1.7 only for $p = 2$, we content ourselves with considering this case. For arbitrary $p \in (1, \infty)$ the proof is technically more involved and can be found in [63, Lemma 2.4]. So assume (5.13) holds and let $w_\delta \rightharpoonup w$ weakly in $L^2(Y, d\mu^\delta)$. Consider a sequence $\psi_\gamma \in \mathcal{C}(\mathbb{T})$ with $\psi_\gamma \to v$ strongly in $L^2(Y, d\mu)$ as $\gamma \to 0$. Precisely as in the proof of Proposition 2.1.13, we need to show

$$\lim_{\gamma\to 0} \lim_{\delta \to 0} \int_Y (v_\delta - \psi_\gamma) w_\delta \, d\mu^\delta \;=\; 0. \tag{5.14}$$

To this end, using the uniform boundedness of w_δ in $L^2(Y, d\mu^\delta)$ it suffices to control the term $\|v_\delta - \psi_\gamma\|_{2,\mu^\delta,Y}$. Using the Clarkson inequality for $p = 2$ and applying (5.13) for $p = 2$, we easily check

$$\limsup_{\delta\to 0} \|v_\delta - \psi_\gamma\|^2_{2,\mu^\delta,Y} \;\leq\; 2\left(\|v\|^2_{2,\mu,Y} + \|\psi_\gamma\|^2_{2,\mu,Y}\right) - \|v + \psi_\gamma\|^2_{2,\mu,Y} \overset{\gamma\to 0}{\longrightarrow} 0,$$

which shows (5.14) and ensures the convergence in (5.12), hence the strong convergence of v_δ in $L^2(Y, d\mu^\delta)$. □

Having in mind the homogenization of fattened structures, it is important to study sequences of potential and solenoidal vector fields subject to a sequence $\mu^\delta \rightharpoonup \mu$. Recall that for a general measure μ, a vector field $v \in L^p(\mathbb{T}, d\mu)^d$ is called *solenoidal*, and we write $v \in V^p_{\mathrm{sol}}(\mathbb{T}, d\mu)$, if

$$\int_Y v \cdot \nabla\varphi \, d\mu = 0 \quad \text{for each } \varphi \in \mathcal{C}^\infty(\mathbb{T}). \tag{5.15}$$

The *strong approximability* of solenoidal vectors on the singular structure, introduced below for the case $p = 2$, is crucial for the asymptotic behaviour of δ-fattened structures associated with a sequence $\mu^\delta \rightharpoonup \mu$:

Definition 5.1.8. *We say that a sequence $\mu^\delta \rightharpoonup \mu$ possesses the strong approximability property, if for any $v \in V^2_{\mathrm{sol}}(\mathbb{T}, d\mu)$ there exists a sequence $v_\delta \in V^2_{\mathrm{sol}}(\mathbb{T}, d\mu^\delta)$, such that*

$$v_\delta \to v \quad \textit{strongly in } L^2(Y, d\mu^\delta). \tag{5.16}$$

It is obvious that the weak limit of a sequence of solenoidal vectors is again solenoidal. In contrast, it is not at all clear that this stability property also holds for potential vectors. In fact, we essentially need the strong approximability property. Recall that for a general measure μ, a vector field $v \in L^2(\mathbb{T}, d\mu)^d$ is called *potential*, and we write $v \in V^2_{\mathrm{pot}}(\mathbb{T}, d\mu)$, if

$$\|v - \nabla\varphi_n\|_{2,\mu,Y} \to 0 \quad \text{for a sequence } \varphi_n \in \mathcal{C}^\infty(\mathbb{T}). \tag{5.17}$$

Lemma 5.1.9. *Let $\mu^\delta \rightharpoonup \mu$ possess the strong approximability property. Then for any family $w_\delta \in L^2(\mathbb{T}, d\mu^\delta)^d$ of potential (resp. solenoidal) vectors with*

$$w_\delta \rightharpoonup w \quad \textit{weakly in } L^2(Y, d\mu^\delta)^d \tag{5.18}$$

componentwise, the limit $w \in L^2(Y, d\mu)^d$ is also potential (resp. solenoidal).

Proof. Let $v \in V^2_{\mathrm{sol}}(\mathbb{T}, d\mu)$ be arbitrary and $v_\delta \in V^2_{\mathrm{sol}}(\mathbb{T}, d\mu^\delta)$ a strong approximating sequence according to Definition 5.1.8. Then by the orthogonal decomposition (2.56) on page 27 we get, using (5.12),(5.17) and (5.18):

$$0 = \lim_{\delta\to 0} \int_Y w_\delta \cdot v_\delta \, d\mu^\delta = \int_Y w \cdot v \, d\mu \,,$$

which implies $w \in V^2_{\mathrm{pot}}(\mathbb{T}, d\mu)$ by (2.56). In case of a sequence of solenoidal vector fields, simply choose $\nabla\varphi$ as a test function in (5.6) for $\varphi \in \mathcal{C}^\infty(\mathbb{T})$. □

Up to now we considered Radon measures on the torus. However, in the forthcoming section we also need to study sequences of nonperiodic measures $\{\mu_h\}$, that are supported on a bounded domain $\Omega \subset \mathbb{R}^d$. The adequate notion here is clearly the weak*-convergence in $\mathcal{C}_0(\Omega)$, that means $\mu_h \rightharpoonup \mu$ if

$$\lim_{h\to 0} \int_\Omega \varphi(x)\, d\mu_h(x) = \int_\Omega \varphi(x)\, d\mu(x) \quad \forall \varphi \in \mathcal{C}_0(\Omega). \tag{5.19}$$

For our applications, the prototype of such sequences is $\mu^\delta_\varepsilon \rightharpoonup \mu_\varepsilon$ as $\delta \to 0$ for ε fixed. Similar as in Definition 5.1.3, we can introduce the weak convergence in variable Lebesgue spaces $L^p(\Omega, d\mu_h)$: A bounded sequence v_h in $L^p(\Omega, d\mu_h)$ is weakly convergent to $v \in L^p(\Omega, d\mu)$, and we write $v_h \rightharpoonup v$, if

$$\lim_{h\to 0} \int_\Omega v_h(x)\varphi(x)\, d\mu_h(x) = \int_\Omega v(x)\varphi(x)\, d\mu(x) \quad \forall \varphi \in \mathcal{C}_0(\Omega). \tag{5.20}$$

The proof of Proposition 5.1.4 can be carried over to obtain the following important compactness result.

Proposition 5.1.10. *For $p > 1$, any bounded sequence in $L^p(\Omega, d\mu_h)$ contains a weakly convergent subsequence in the sense of (5.20).*

5.2 Homogenization of fattened structures

Let us introduce the general setting we consider, and with it the assumptions on the measure μ, that will be kept for the whole section unless otherwise stated. It merely comprises the standard conditions used in this thesis to handle nonlinear problems on multidimensional structures.

Assumption 5.2.1. *Let μ be a positive, normalized, Y-periodic Radon measure on $\mathbb{R}^d$, which is doubling and strongly 2-connected on $\mathbb{R}^d$, and satisfies $\mu(\partial Y) = 0$.*

In general, we have in mind measures that are supported on thin structures of codimension greater or equal to one. Unless otherwise stated, the δ-fattened structure will always be characterized by a positive, normalized, Y-periodic measure μ^δ, that is absolutely continuous with respect to the Lebesgue measure, that means

$$\mu^\delta(Y) = 1 \quad \text{and} \quad d\mu^\delta(y) = \varrho_\delta(y)dy \quad \text{with } \varrho_\delta \in L^1_{\mathrm{per}}(Y),\ \varrho_\delta \geq 0. \tag{5.21}$$

The fulldimensional approximation of the thin structure then signifies the weak convergence of measures $\mu^\delta \rightharpoonup \mu$ defined in (5.3). The measures in (5.21) clearly satisfy $\mu^\delta(\partial Y) = 0$, and let us draw some further consequences.

Remark 5.2.2. *Let μ satisfy Assumption 5.2.1 and let an approximation $\mu^\delta \rightharpoonup \mu$ of the type (5.21) be given. Then for the ε-rescalings there holds*

$$\mu_\varepsilon^\delta \rightharpoonup \mu_\varepsilon \quad \text{as } \delta \to 0 \tag{5.22}$$

for any $\varepsilon > 0$ in the sense of (5.19), and $0 < \mu_\varepsilon^\delta(\Omega) < C$ with a constant C independent of δ and ε.

Proof. Let $\varphi \in \mathcal{C}_0(\Omega)$ be given. We can assume that $\Omega = (0,1)^d$. We choose a suitable cutoff function $\psi_\eta \in \mathcal{D}(Y)$, which is equal to one a distance η away from ∂Y. With the same notation as in the proof of Theorem 2.4.5, using the periodicity of μ^δ and the fact that $\mu^\delta(\partial Y) = 0$, it suffices to observe that on each rescaled cell Y_ε^k:

$$\begin{aligned} \int_{Y_\varepsilon^k} \varphi(x)\, d\mu_\varepsilon^\delta(x) \quad &= \quad \varepsilon^d \int_Y [\psi_\eta(x) + (1 - \psi_\eta(x))]\varphi(\varepsilon(x+k))\, d\mu^\delta(x) \\ &\xrightarrow{\delta \to 0} \quad \int_{Y_\varepsilon^k} \varphi(x)\, d\mu_\varepsilon(x) + o(1) \quad \text{as } \eta \to 0, \end{aligned}$$

since $\mu^\delta \rightharpoonup \mu$ on the torus and due to the fact that the support of μ can not be concentrated on the boundary of Y by the assumption $\mu(\partial Y) = 0$. The second statement can easily be checked using the normalization of μ^δ and the definition of its ε-periodization. □

It is obvious that Assumption 5.2.1 and (5.21) alone do not guarantee that each μ^δ, although $\mu^\delta \rightharpoonup \mu$, is connected even in the topological sense. However, at least for connected periodic networks (F, μ) on $\mathbb{R}^d$ (cf. Definition 5.2.12 below), which we study in the next paragraphs, we can explicitly construct an approximating sequence of strongly connected measures μ^δ. We expect that such sequences can always be found given any strongly connected multijunction measure $\mu \in J_\sharp$ (cf. Definition 2.1.1), but a rigorous proof is beyond the scope of our investigation. This motivates the following additional assumption on the fattening process.

Assumption 5.2.3. *The approximating sequence $\mu^\delta \rightharpoonup \mu$ satisfies (5.21), and each measure μ^δ is doubling and strongly 2-connected on $\mathbb{R}^d$.*

We note that the approximating sequence μ^δ chosen in Example 5.1.2 satisfies Assumption 5.2.3. This follows from Lemma 5.2.13 below, where in a more general framework we consider such regular fattening of connected networks on $\mathbb{R}^2$, which consist of straight segments. However, one can of course consider more complex geometries including curvilinear structures as sketched in Figure 5.3 below. On the left-hand side we have a typical configuration of straight segments $F_1, \dots, F_4$ and a circle F_5 of the same dimension.

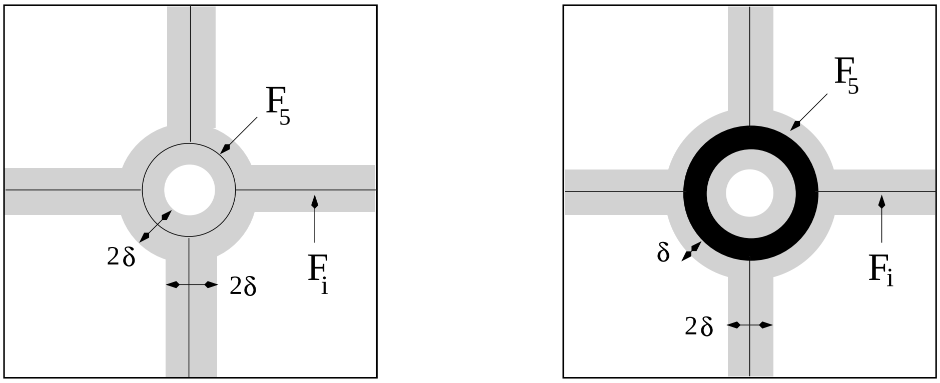


Figure 5.3: δ-fattened multistructures

Both, the fattened structure F^δ, which is grey shaded, and the thin limit structure are strongly 2-connected on $\mathbb{R}^2$. Figure 5.3 also shows that we have to be careful considering multijunctions of different dimensions. On the right-hand side, the component F_5 is a ring of finite width. Although any measure μ^δ supported on the fat structure is strongly connected, the limit measure μ fails to satisfy $(H1)$ for $p \leq 2$, since in this case a Sobolev function need not be continuous at the intersection points of F_5 with F_i. Such phenomena are closely related to the noncommutativity of the two-parameter diagram, for which we will find explicit counterexamples in Paragraph 5.2.3 below.

After this preparatory part we consider two-parameter equations, containing the microscale parameter ε and the fattening parameter δ. Note that for each δ, the ε-periodic rescaled measure μ_ε^δ corresponding to μ^δ is defined by

$$\mu_\varepsilon^\delta(B) := \varepsilon^d \mu^\delta(\tfrac{B}{\varepsilon}) \quad \text{for each Borel set } B \subset \mathbb{R}^d.$$

Recall that we exclusively consider measures μ^δ that have a positive density ϱ_δ with respect to the Lebesgue measure on the torus as in (5.21). Hence the Dirichlet space $H_0^{1,2}(\Omega, d\mu_\varepsilon^\delta)$ introduced in Definition 2.2.16 can be identified with the classical (weighted) Sobolev space up to the density $\varrho_\delta(\frac{x}{\varepsilon})$. In particular, the gradient of a Sobolev function with respect to μ_ε^δ is unique and coincides with the usual full gradient. Moreover we have

$$\int_\Omega \varphi(x)\, d\mu_\varepsilon^\delta(x) \;=\; \int_\Omega \varrho_\delta(\tfrac{x}{\varepsilon})\varphi(x)\, dx \quad \forall \varphi \in \mathcal{C}_0(\Omega).$$

Now for $\lambda > 0$ and suitable assumptions on the oscillating (with period ε) data K_δ and f_δ, we define as in (1.13) the following two-parameter family of quasilinear elliptic Dirichlet boundary value problems on the δ-fattened, ε-periodic structure:

$$(P_\varepsilon^\delta) \qquad \left\{ \begin{array}{rcll} -\mathrm{div}\,(K_\varepsilon^\delta(\mu^\delta, x, u_\varepsilon^\delta)\nabla u_\varepsilon^\delta) + \lambda u_\varepsilon^\delta \mu_\varepsilon^\delta &=& f_\varepsilon^\delta(\mu^\delta, x, u_\varepsilon^\delta) & \text{in } \Omega\,, \\ u_\varepsilon^\delta &=& 0 & \text{on } \partial\Omega. \end{array} \right.$$

As usual, we call $u_\varepsilon^\delta \in H_0^{1,2}(\Omega, d\mu_\varepsilon^\delta)$ a weak solution of problem (P_ε^δ), if

$$\int_\Omega \Big(K_\delta(\tfrac{x}{\varepsilon}, u_\varepsilon^\delta)\nabla u_\varepsilon^\delta \cdot \nabla\varphi + \lambda u_\varepsilon^\delta \varphi\Big)\, d\mu_\varepsilon^\delta \;=\; \int_\Omega f_\delta(\tfrac{x}{\varepsilon}, u_\varepsilon^\delta)\varphi\, d\mu_\varepsilon^\delta \tag{5.23}$$

for each $\varphi \in \mathcal{D}(\Omega)$. Now we impose the structure conditions on the data, which will be kept throughout this section unless otherwise stated. For simplicity, we will assume that K_δ is a scalar function.

Assumption 5.2.4. *Let $(K_\delta, f_\delta) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $(y,s) \mapsto (K_\delta, f_\delta)(y,s)$ be μ^δ-measurable and Y-periodic in y, continuous in s, and satisfy the following properties:*

1. *There exist positive constants c_1, c_2, c_3 and $\beta \in [0,1)$, such that*
$$0 < c_1 \le K_\delta(y,s) \le c_2, \quad |f_\delta(y,s)| \le c_3(1+|s|^\beta) \tag{5.24}$$
respectively for all $\delta > 0$ and $(y,s) \in \mathbb{R}^d \times \mathbb{R}$.

2. *There exist $\gamma, \tilde\gamma \in (0,1]$, a sequence $\{\tilde h_\delta\} \subset L^{\tilde q}(\mathbb{T}, d\mu^\delta)$ with*
$$\tilde h_\delta \to \tilde h \quad \text{strongly in } L^{\tilde q}(Y, d\mu^\delta) \tag{5.25}$$
for some $\tilde h \in L^{\tilde q}_\mu(\mathbb{T})$, and a positive constant c_4, such that
$$|K_\delta(y,s_1) - K_\delta(y,s_2)| \;\le\; c_4\,(1+|s_1|+|s_2|)^{1-\gamma}\,|s_1-s_2|^\gamma, \tag{5.26}$$
$$|f_\delta(y,s_1) - f_\delta(y,s_2)| \;\le\; |\tilde h_\delta(y)|\,|s_1-s_2|^{\tilde\gamma} \tag{5.27}$$
respectively for all $y \in \mathbb{R}^d$ and $s_i \in \mathbb{R}$, where $\tilde q = 2(2-\tilde\gamma)^{-1} \in (1,2]$.

3. *There exist functions $(K,f) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $(y,s) \mapsto (K,f)(y,s)$ that are μ-measurable and Y-periodic in y, continuous in s, such that*
$$K_\delta(\cdot,s) \to K(\cdot,s) \qquad \text{strongly in } L^2(Y, d\mu^\delta) \tag{5.28}$$
$$f_\delta(\cdot,s) \rightharpoonup f(\cdot,s) \qquad \text{weakly in } L^2(Y, d\mu^\delta) \tag{5.29}$$
for any fixed $s \in \mathbb{R}$ in the sense of (5.12) and (5.6) respectively.

Let us comment on the structure conditions above, which are familiar from Section 3.2. Estimate (5.24) is needed for existence and δ-independent a priori estimates for the (δ,ε)-problem. Properties (5.25),(5.28) and (5.29) are introduced for the asymptotics $\delta \to 0$ for fixed ε, whereas (5.26) and (5.27) are required for the homogenization process $\varepsilon \to 0$ and fixed δ. Concerning the following statement, recall that $\bar f$ denotes the average over the unit cell Y with respect to the y-variable.

Remark 5.2.5. *Assumption 5.2.4 implies that the data K and f satisfy respectively for μ-almost every $y \in \mathbb{R}^d$ and all $s \in \mathbb{R}$:*

$$0 < c_1 \le K(y,s) \le c_2, \quad |f(y,s)| \le c_3(1+|s|^\beta), \tag{5.30}$$

$$\begin{aligned} |K(y,s_1)-K(y,s_2)| &\leq c_4\,(1+|s_1|+|s_2|)^{1-\gamma}\,|s_1-s_2|^{\gamma}, &(5.31)\\ |f(y,s_1)-f(y,s_2)| &\leq |\tilde h(y)|\,|s_1-s_2|^{\tilde\gamma} &(5.32)\end{aligned}$$

with the same constants c_i and $\gamma, \tilde\gamma$ as in Assumption 5.2.4, where $\tilde h \in L^{\tilde q}_\mu(\mathbb{T})$ is given by (5.25). Moreover using (5.24),(5.27) and (5.29) we see that

$$\bar f_\delta \to \bar f \quad \textit{locally uniformly in}\, \mathbb{R}. \tag{5.33}$$

Sketch of proof. We prove exemplary the Hölder estimate for f in (5.32), which is the least obvious statement, since we require only the weak convergence of f_δ in (5.29). The latter combined with the strong convergence of $\tilde h_\delta$ and the weak lower semicontinuity property (5.11) gives

$$0 \leq \int_Y \left(|\tilde h(y)|^{\tilde q}|s_1-s_2|^{\tilde q\tilde\gamma} - |f(y,s_1)-f(y,s_2)|^{\tilde q}\right)|\psi(y)|^{\tilde q}\,d\mu(y),$$

for any $\psi \in \mathcal{C}(\mathbb{T})$ and s_1, s_2 fixed, where we also used $\tilde q > 1$ and the characterization of strong convergence according to Lemma 5.1.7. Hence there exists a set $E \subset Y$ with $\mu(E) = 0$, such that (5.32) holds for any $y \in Y \setminus E$ and all $(s_1, s_2) \in \mathbb{Q}^2$. However, since f is continuous in s by Assumption 5.2.4.3, the inequality also holds for any $y \notin E$ and all $(s_1, s_2) \in \mathbb{R}^2$. □

It is now easy to carry over the existence results from Chapter 3 to the equation (P^δ_ε). In what follows we use the abbreviations $H^\delta_\varepsilon := H^{1,2}_0(\Omega, d\mu^\delta_\varepsilon)$ for the Dirichlet space and for the norm

$$\|u\|_{H^\delta_\varepsilon} := \|u\|_{2,\varepsilon,\delta} + \|\nabla u\|_{2,\varepsilon,\delta}\,, \quad \|v\|^2_{2,\varepsilon,\delta} := \int_\Omega |v|^2\,d\mu^\delta_\varepsilon\,,\ v \in L^2(\Omega, d\mu^\delta_\varepsilon). \tag{5.34}$$

Since we assume $\lambda > 0$ in (5.23), for an uniform a priori estimate we do not need to show that the Poincaré constant on Ω with respect to H^δ_ε is independent of δ, which however seems to be true whenever μ and μ^δ satisfy Assumption 5.2.1 and Assumption 5.2.3 respectively.

Corollary 5.2.6. *Let μ^δ satisfy Assumption 5.2.3 and the data K_δ and f_δ satisfy Assumption 5.2.4.1. Then for any $\varepsilon, \delta > 0$ there exists a solution $u^\delta_\varepsilon \in H^\delta_\varepsilon$ of equation (P^δ_ε) in the sense of (5.23), fulfilling the uniform estimate*

$$\|u^\delta_\varepsilon\|_{H^\delta_\varepsilon} \leq C \tag{5.35}$$

with a constant independent of ε and δ.

Proof. With $\mu = \mu^\delta$ and $a(y,s,\xi) = K_\delta(y,s)\xi$, we are precisely in the situation of Theorem 3.1.4 with $p = 2$, which provides existence. As usual, the a priori estimate (5.35) can be derived by testing the equation with the solution u^δ_ε, where one has to use that the constants in (5.24) do not depend on δ, and the fact that

$$0 \leq \mu^\delta_\varepsilon(\Omega) \leq C \tag{5.36}$$

with a constant independent of ε and δ, which was shown in Remark 5.2.2. □

Studying the commutativity of the two-parameter diagramm, the comparatively easiest step is to pass to the limit $\varepsilon \to 0$ in (5.23) for fixed $\delta > 0$. In light of Assumption 5.2.4 it is clear that we can rely completely on the homogenization procedure of Paragraph 3.2.2. To this end, we introduce the Hilbert space

$$W_\mu^\delta := \{u \in H^{1,2}(\mathbb{T}, d\mu^\delta) \mid \bar{u} = 0\}\,, \quad \bar{u} := \fint_Y u(y)\, d\mu^\delta(y)\,, \tag{5.37}$$

and the δ-fattened cell problems (C_k^δ). Their unique solvability, together with an uniform a priori estimate, can be proven exactly as in Lemma 3.2.7:

Corollary 5.2.7. *Under the assumptions of Corollary 5.2.6, for any $\delta > 0$ and $s \in \mathbb{R}$ there exists a unique weak solution $\Lambda_{\delta,k}(\cdot, s) \in W_\mu^\delta$ of the problem*

$$(C_k^\delta) \quad \begin{cases} -div\,(K_\delta(y,s)[\nabla\Lambda_{\delta,k}(y,s) + \vec{e}_k]) &= 0 \;\; in\ Y \\ y \mapsto \Lambda_{\delta,k}(y,s)\;\; Y\text{-}periodic \quad , & \bar{\Lambda}_{\delta,k} = 0\,, \end{cases}$$

satisfying the following uniform estimate with a constant independent of δ and $s \in \mathbb{R}$:

$$\|\Lambda_{\delta,k}(\cdot,s)\|_{W_\mu^\delta} \;=\; \|\nabla\Lambda_{\delta,k}(\cdot,s)\|_{2,\mu^\delta,Y} \;\le\; C. \tag{5.38}$$

Similar as in Definition 3.2.8, we can now introduce the δ-fat effective tensor $K_\delta^\star$ depending on the parameter s:

Definition 5.2.8. *The effective, δ-fat coefficient $K_\delta^\star : \mathbb{R} \to \mathcal{M}_{\mathrm{sym}}^d$ is given by*

$$(K_\delta^\star)_{ij}(s) \;=\; \int_Y K_\delta(y,s)[\vec{e}_i + \nabla\Lambda_{\delta,i}^s(y)] \cdot (\vec{e}_j + \nabla\Lambda_{\delta,j}^s(y))\, d\mu^\delta(y), \tag{5.39}$$

where $\Lambda_{\delta,k}^s$ is for given $\delta > 0$ and $s \in \mathbb{R}$ the solution of the cell problem (C_k^δ).

Motivated by Corollary 3.2.10, we can define the ε-homogenized, δ-fat problem (P^δ), which is well defined by Lemma 3.2.9 and (5.24) for any $\delta > 0$:

$$(P^\delta) \qquad \begin{cases} -\mathrm{div}\,(K_\delta^\star(u)\nabla u) + \lambda u &= \bar{f}_\delta(u) \quad \text{in}\ \Omega\,, \\ \qquad\qquad\qquad\qquad u &= 0 \qquad\;\; \text{on}\ \partial\Omega. \end{cases}$$

Recall that $\bar{f}_\delta(\cdot)$ is the average of $f_\delta(y,\cdot)$ over Y with respect to μ^δ. As usual, we call $u^\delta \in H_0^1(\Omega)$ a weak solution of problem (P^δ), if

$$\int_\Omega \Big(K_\delta^\star(u^\delta)\nabla u^\delta \cdot \nabla\varphi + \lambda u^\delta\varphi\Big)\, dx \;=\; \int_\Omega \bar{f}_\delta(u^\delta)\varphi\, dx \qquad \forall\varphi \in \mathcal{D}(\Omega). \tag{5.40}$$

We can now pass to the limit $\varepsilon \to 0$ in (5.23) for fixed δ, where μ^δ plays the role of μ. Thanks to Assumption 5.2.4.2, Lemma 3.2.6 and Corollary 3.2.10 it is evident that a sequence $\{u_\varepsilon^\delta\}$ of solutions of (P_ε^δ) converges, as $\varepsilon \to 0$, to a solution of (P^δ) up to subsequences.

Lemma 5.2.9. *Let the data K_δ and f_δ satisfy Assumption 5.2.4.2, and let $\{u_\varepsilon^\delta\}_{\varepsilon>0}$ be a sequence of solutions of (P_ε^δ) according to Corollary 5.2.6. Then, up to subsequences, there holds*

$$u_\varepsilon^\delta \twoheadrightarrow u^\delta \quad \text{two-scale strongly in } L^2(\Omega, d\mu_\varepsilon^\delta) \text{ as } \varepsilon \to 0, \tag{5.41}$$

where $u^\delta \in H_0^1(\Omega)$ is a solution of problem (P^δ) in the sense of (5.40).

Proof. Since μ^δ is strongly 2-connected on $\mathbb{R}^d$, and thanks to Assumption 5.2.4.2 and the uniform a priori estimate (5.35), the proof of Lemma 3.2.6 can be carried over to derive the two-scale homogenized problem first. The homogenized equation can then be derived from it by determining the corrector function $u_1^\delta \in L^2(\Omega; H^{1,2}(\mathbb{T}, d\mu^\delta))$ as in (3.132). □

The next step is of course to pass to the limit in (5.40). Note that it is not at all obvious that the sequence $\{u^\delta\}$ of solutions of problem (P^δ) is bounded in $H_0^1(\Omega)$. Indeed, although we can apply Lemma 3.2.9 for each fixed $\delta > 0$, it could happen that the positive lower bound $c_\star = c_\star(\delta)$ on the effective coefficient $K_\delta^\star$ degenerates to zero in the limit $\delta \to 0$. It turns out that the strong approximability property introduced in Definition 5.1.8 plays a crucial role. Let us first define the effective coefficient $K^\star$ of the singular structure (cf. Lemma 3.2.7 and Definition 3.2.8).

Definition 5.2.10. *Let μ satisfy Assumption 5.2.1 and $K : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ as in Assumption 5.2.4.3. Then for any given $s \in \mathbb{R}$ we denote by $\Lambda_k^s \in W_\mu$ the unique solution of the cell problem*

$$(C_k^0) \quad \begin{cases} -div_\mu \left(K(y,s)[\nabla_\mu \Lambda_k(y,s) + \vec{e}_{k,\mu}(y)]\right) & = \ 0 \ \text{ in } Y \\ \qquad y \mapsto \Lambda_k(y,s) \ \text{ Y-periodic} & , \quad \bar{\Lambda}_k = 0\,, \end{cases}$$

where the Hilbert space W_μ is defined as in Paragraph 3.2.2 on page 72. The effective tensor $K^\star : \mathbb{R} \to \mathcal{M}_{\mathrm{sym}}^d$ is then defined by

$$K_{ij}^\star(s) := \int_Y K(y,s)[\vec{e}_{i,\mu}(y) + \nabla_\mu \Lambda_i^s(y)] \cdot (\vec{e}_{j,\mu}(y) + \nabla_\mu \Lambda_j^s(y))\, d\mu(y). \tag{5.42}$$

Note that by (5.30) and Lemma 3.2.7, Λ_k^s and $K^\star$ are well defined. Now we can show that the sequence $\{u^\delta\}$ converges, up to subsequences, weakly in $H^1(\Omega)$ to a solution of the homogenized problem with effective coefficient $K^\star$, provided the sequence $\mu^\delta \rightharpoonup \mu$ enjoys the strong approximability property.

Lemma 5.2.11. *In the situation of Corollary 5.2.6, let additionally $\mu^\delta \rightharpoonup \mu$ satisfy the strong approximability property, μ satisfy Assumption 5.2.1, and the data K_δ, K, f_δ, f fulfill all the prerequisites of Assumption 5.2.4. Then any sequence $\{u^\delta\}_{\delta>0}$ of solutions of problem (P^δ) is bounded in $H_0^1(\Omega)$ and, up to subsequences, there holds*

$$u^\delta \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega)\,, \tag{5.43}$$

where $u^0 \in H_0^1(\Omega)$ *is a solution of the uniformly elliptic homogenized equation*

$$(P^0) \qquad \begin{cases} -\mathrm{div}\,(K^\star(u)\nabla u) + \lambda u &= \bar f(u) \quad \text{in } \Omega\,, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Proof. Note that for any fixed $s \in \mathbb{R}$, from the uniform estimate (5.38) and the weak compactness property of bounded sequences with respect to $\mu^\delta \rightharpoonup \mu$ we deduce, up to a subsequence,

$$\nabla \Lambda^s_{\delta,k} \rightharpoonup \Phi^s_k \in V^2_{\mathrm{pot}}(\mathbb{T}, d\mu) \tag{5.44}$$

weakly in $L^2(Y, d\mu^\delta)$, where we used the strong approximability property and Lemma 5.1.9. By the connectedness assumption on μ and Lemma 2.3.7, there exists a unique function $\Theta^s_k \in W_\mu$, such that

$$P_\mu(y)[\Phi^s_k(y)] = \nabla_\mu \Theta^s_k(y), \tag{5.45}$$

where $P_\mu(y)$ is the orthogonal projection onto the tangent space $T_\mu(y)$ defined in (2.27) on page 20. The solution property of $\Lambda^s_{\delta,k}$ and the strong convergence $K_\delta(\cdot, s) \to K(\cdot, s)$ in $L^2(Y, d\mu^\delta)$ according to assumption (5.28) yield

$$0 = \int_Y K_\delta(y,s)[\nabla\Lambda^s_{\delta,k}(y) + \vec e_k]\cdot\nabla\varphi\, d\mu^\delta \to \int_Y K(y,s)[\Phi^s_k(y) + \vec e_k]\cdot\nabla\varphi\, d\mu \tag{5.46}$$

for all $\varphi \in \mathcal{C}^\infty(\mathbb{T})$. Since K is a scalar function, as usual we deduce that the vector $\Phi^s_k(y) + \vec e_k$ is tangential with respect to μ, and hence

$$\vec e_k + \nabla\Lambda^s_{\delta,k} \rightharpoonup \vec e_{k,\mu} + \nabla_\mu\Lambda^s_k\,, \quad \vec e_{k,\mu}(y) := P_\mu(y)[\vec e_k] \tag{5.47}$$

in the sense of (5.6), where we used (5.45) and the unique solvability of the cell problem (C^0_k) in W_μ. Note that by the solution property of each $\Lambda^s_{\delta,k}$, the term $\nabla\Lambda^s_{\delta,j}$ can also be omitted in the definition of $K^\star_\delta$. Hence by (5.47) and the strong convergence of $K_\delta(\cdot, s)$ we get

$$\forall s \in \mathbb{R}: \quad K^\star_\delta(s) \to K^\star(s) \ \text{ in } \mathbb{R}^{d\times d}. \tag{5.48}$$

We claim that there exist positive constants $c_5, c_6, c_7 > 0$ independent of δ, such that

$$\forall (s,\xi) \in \mathbb{R}\times\mathbb{R}^d: \quad c_5|\xi|^2 \le \xi\cdot K^\star_\delta(s)\xi \le c_6|\xi|^2, \tag{5.49}$$

$$\forall s_i \in \mathbb{R}: \quad \|K^\star_\delta(s_1) - K^\star_\delta(s_2)\| \le c_7(1 + |s_1| + |s_2|)^{1-\gamma}\,|s_1 - s_2|^\gamma. \tag{5.50}$$

The uniform upper bound on $K^\star_\delta$ in (5.49) immediately follows from (5.24) and (5.38). As the proof of Lemma 3.2.9 shows, we have

$$\xi\cdot K^\star_\delta(s)\xi \ge c_1 \inf_{\varphi\in H^{1,2}(\mathbb{T},d\mu^\delta)} \int_Y |\xi + \nabla\varphi|^2\, d\mu^\delta = c_1\|\xi + \nabla\varphi^\delta_\xi\|^2_{2,\mu^\delta,Y}\,, \tag{5.51}$$

where c_1 is the constant in (5.24) and, for the given $\xi \in \mathbb{R}^d$, the function $\varphi_\xi^\delta \in W_\mu^\delta$ the unique solution of the periodic cell problem

$$-\mathrm{div}\,(\xi + \nabla\varphi_\xi^\delta) = 0 \quad \text{in } Y.$$

Similar as in the proof of (5.47), we can then show that $\xi + \nabla\varphi_\xi^\delta \rightharpoonup \xi_\mu + \nabla_\mu\varphi_\xi$, with $\varphi_\xi \in W_\mu$ as in (3.130). As the proof of Lemma 3.2.9 shows, the weak lower semicontinuity of the norm with respect to the convergence $\mu^\delta \rightharpoonup \mu$ (cf. Proposition 5.1.5) yields

$$\liminf_{\delta\to 0} \|\xi + \nabla\varphi_\xi^\delta\|_{2,\mu^\delta,Y}^2 \;\geq\; \|\xi_\mu + \nabla_\mu\varphi_\xi\|_{2,\mu,Y}^2 \;\geq\; \hat{c}|\xi|^2\,, \tag{5.52}$$

where $\hat{c}$ is the positive constant in estimate (2.86) of Lemma 2.3.13 only depending on μ. Combining (5.51) and (5.52) we get the existence of a positive constant c_5 in (5.49) by a simple contradiction argument. As Lemma 3.2.11 shows, the effective tensor $K_\delta^\star$ inherits local Hölder continuity from K_δ, and the constant c_7 only depends on the numbers c_1, c_2 in (5.24) and c_4 in (5.26). This proves (5.50). Combining (5.33),(5.48),(5.49) and (5.50) we get

$$(K_\delta^\star)_{ij} \to K_{ij}^\star\,, \quad \bar{f}_\delta \to \bar{f} \quad \text{locally uniformly in } \mathbb{R}. \tag{5.53}$$

Now given a sequence $\{u^\delta\}_{\delta>0}$ of solutions of problem (P^δ), testing the integral identity (5.40) with $\varphi = u^\delta$ and using (5.49) and the second inequality in (5.24), we see that $\{u^\delta\}$ is bounded in $H^1(\Omega)$, and hence, up to a subsequence

$$u^\delta \rightharpoonup u^0 \quad \text{weakly in } H^1(\Omega)\,, \quad u^\delta \to u^0 \quad \text{strongly in } L^2(\Omega) \tag{5.54}$$

for some $u^0 \in H_0^1(\Omega)$. Now combining (5.24) and (5.27) with the estimate (5.50) and (5.53) and (5.54), we easily deduce

$$K_\delta^\star(u^\delta) \to K^\star(u^0)\,, \quad \bar{f}_\delta(u^\delta) \to \bar{f}(u^0) \quad \text{strongly in } L^2(\Omega)$$

by components. Hence passing to the limit in (5.40), we see that u^0 is a weak solution of problem (P^0), which is well defined by the growth condition on f in (5.30) and the uniform ellipticity of $K^\star$ proven in Lemma 3.2.9. □

5.2.1 Networks in 2D

In this paragraph we exclusively study connected 1D-networks in $\mathbb{R}^2$, which are made up of infinitely thin, straight segments, and are therefore modeled by a sum of one-dimensional Hausdorff measures. We will show that on such a structure the two-parameter diagram commutes, at least for a large class of quasilinear equations. As pointed out in [24], our methods also apply to sufficiently regular curved structures. In the next paragraph we will consider networks embedded in $\mathbb{R}^3$.

Definition 5.2.12. *We call the pair (F, μ) a connected periodic network on $\mathbb{R}^2$, if $F = \operatorname{spt}\mu$ is a Y-periodic subset of $\mathbb{R}^2$, and $F \cap Y$ the finite union of straight segments F_k contained in Y, such that*

$$\mu\lfloor Y \;=\; c\sum_k \mathcal{H}^1\lfloor F_k \tag{5.55}$$

is strongly 2-connected on $\mathbb{R}^2$, where c is the normalizing constant.

Definition 5.2.12 makes sure that the measure μ corresponding to a connected network (F, μ) satisfies Assumption 5.2.1. In particular, as pointed out in Section 2.3, such a measure is always doubling, since it belongs to the class $J_\sharp$ of multijunction measures. The normalizing constant in (5.55) reads

$$c = \left(\sum_k \mathcal{H}^1(F_k)\right)^{-1}. \tag{5.56}$$

It is obvious that the segments of a connected network defined above can not be arranged in an arbitrary manner. Admissible structures can be found in Figure 5.4 and Figure 5.5 below. The former also comprises a counterexample.

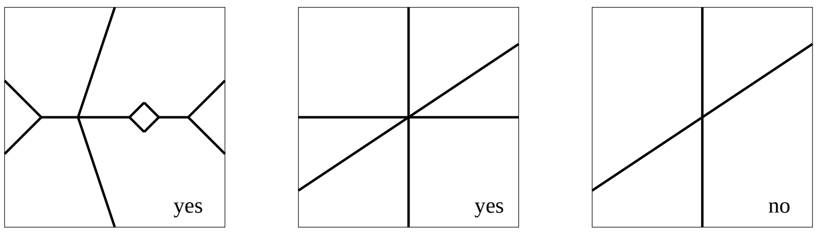


Figure 5.4: Connected networks and a counterexample

We introduce the natural fattened structure (F^δ, μ^δ) that approximates a given connected network (F, μ) on $\mathbb{R}^2$ with m segments F_k, where $m \geq 2$, by

$$F^\delta = \bigcup_{l\in\mathbb{Z}^2}\left(l + \cup_{k=1}^m F_k^\delta\right), \quad F_k^\delta := \{y \in \mathbb{R}^2 : \operatorname{dist}(y, F_k) < \delta\} \cap Y, \tag{5.57}$$

where the distance is measured in the Euclidean norm on $\mathbb{R}^2$. The measure corresponding to the fattened structure F^δ is, on the unit cell, chosen as

$$\mu^\delta\lfloor Y = c_\delta \mathcal{L}^2\lfloor(F^\delta \cap Y), \quad c_\delta = |F^\delta \cap Y|^{-1}, \quad F^\delta \cap Y = \bigcup_{k=1}^m F_k^\delta, \tag{5.58}$$

and periodically extended to $\mathbb{R}^2$. Note that μ^δ is doubling and of type (5.21) with a density proportional to the characteristic function of the set $F^\delta \cap Y$. In light of Lemma 5.2.11, we need to verify that the sequence $\{\mu^\delta\}$ does not only satisfy Assumption 5.2.3, but also the strong approximability property introduced in Definition 5.1.8. In order to show $\mu^\delta \rightharpoonup \mu$, we set $l_k := \mathcal{H}^1(F_k)$ for $k = 1, \ldots, m$ and easily check

$$\begin{aligned}\int_Y \varphi\, d\mu^\delta &= c_\delta \sum_k \int_{F_k^\delta \setminus (\cup_{j=1}^{k-1} F_j^\delta)} \varphi(y)\, dy = \sum_k \left(\frac{l_k}{\Sigma_j l_j}\right) ⨍_{F_k^\delta \cap Y} \varphi\, dy + o(1) \\ &\to c \sum_k \int_{F_k} \varphi\, d\mathcal{H}^1 = \int_Y \varphi\, d\mu\end{aligned}$$

for any $\varphi \in \mathcal{C}(\mathbb{T})$, where the constant c is given in (5.56). In addition to Example 5.1.2, more complex admissible networks (F, μ) are sketched in Figure 5.5 below, together with the corresponding fattened structure (F^δ, μ^δ) according to (5.57) and (5.58).

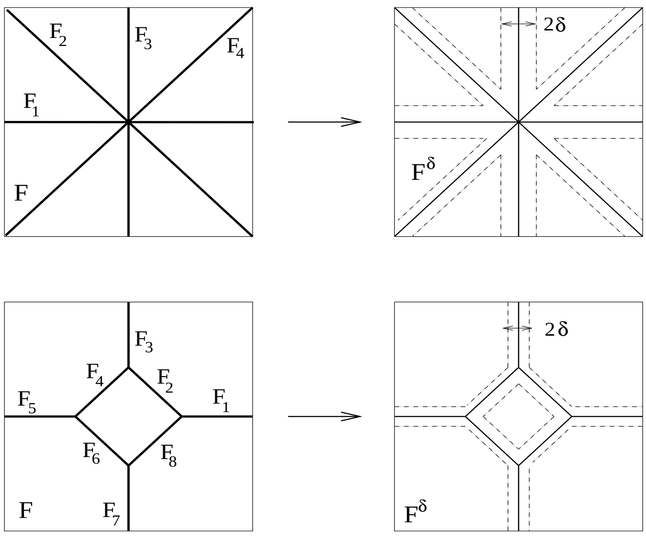


Figure 5.5: Connected networks with suitable fattening

Now we show that the measures defined in (5.58) are strongly 2-connected on $\mathbb{R}^2$. This is not trivial, since in general it is not clear how the Poincaré constant behaves and rescales on periodic nonconvex domains (cf. definition of $(H4)$). However, with a similar technique as in the proof of Theorem 2.4.5 we can show the strong connectedness for each μ^δ that characterizes the comparatively simple structures considered in this section. It is essential that the thickness of the connected, open subset of $F^\delta \subset \mathbb{R}^d$ is uniformly minorized (by the parameter δ).

Lemma 5.2.13. *The measures μ^δ defined in (5.58), corresponding to a connected periodic network (F, μ), are strongly 2-connected on $\mathbb{R}^2$.*

Proof. By the definition of (F, μ) and F^δ it is obvious that μ^δ is at least weakly connected on $\mathbb{R}^2$. Moreover, it is strongly connected on $\mathbb{T}$, since the support $F^\delta \cap Y$ is an open connected Lipschitz domain, and by a standard contradiction argument using the classical Rellich theorem, the Poincaré inequality on the torus can be obtained, however with no control on the size of the constant. In order to show property $(H4)$, for a given rescaling integer $k \in \mathbb{N}$ we introduce the following notation:

$$\Omega = \bigcup_{i,j=1}^{k} Z_{ij}^k\,, \quad Z_{ij}^k = \tfrac{1}{k}\left(Y + (i-1, j-1)\right)\,, \quad i,j = 1, \ldots, k,$$

$$\Omega_k = \bigcup_{i,j=1}^{k} Y_{ij}^k\,, \quad Y_{ij}^k = \tfrac{1}{k}\Big((F^\delta \cap Y) + (i-1, j-1)\Big)\,, \quad i,j = 1, \ldots, k.$$

Note that we do not label the sets with δ since it is fixed, whereas the integer k varies in $\mathbb{N}$. In each rescaled cell the thickness of the structure is then of order δ/k (see Figure 5.6 below). Upon cancelling the fixed, k-independent constant c_δ on both sides, the strong connectedness of μ^δ on $\mathbb{R}^2$ is, after rescaling, equivalent to the statement

$$\exists C : \int_{\Omega_k} |u|^2\,dy \leq C \int_{\Omega_k} |\nabla u|^2\,dy \quad \forall k \in \mathbb{N}, \forall u \in \mathcal{D} \text{ with } \int_{\Omega_k} u\,dy = 0. \tag{5.59}$$

In Figure 5.6 we sketched the domain Ω_k for two different values of k and for the special measure of Example 5.1.2. For the structures (F^δ, μ^δ) under consideration we can expect (5.59) to hold true, since the intersecting bars are thinning in a regular manner as $k \to \infty$.

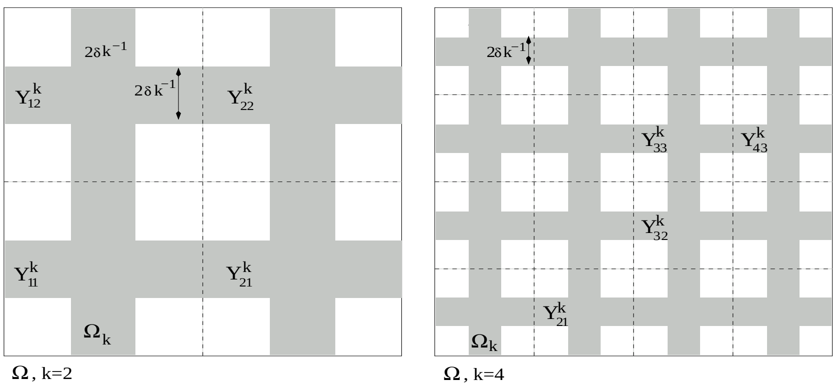


Figure 5.6: Rescaled k-structure

So let us suppose the contrary, then there exists a sequence $\{u_k\}$ of smooth functions, such that

$$\int_{\Omega_k} u_k\,dy = 0, \quad \int_{\Omega_k} |\nabla u_k|^2\,dy = 1, \quad \text{but} \quad \int_{\Omega_k} |u_k|^2\,dy \geq k \tag{5.60}$$

for all $k \in \mathbb{N}$. Similar as in the proof of Theorem 2.4.5, we first compare the function u_k with the piecewise constant function $\bar{u}_k$ defined on Ω, which comprises the mean values over the small cells. More precisely, we define

$$\bar{u}_k : \Omega \to \mathbb{R},\ y \mapsto \sum_{i,j=1}^{k} \bar{u}_{ij}^k \chi_{ij}^k(y)\,, \quad \bar{u}_{ij}^k := \fint_{Y_{ij}^k} u_k(y)\,dy\,, \tag{5.61}$$

where χ_{ij}^k is the characteristic function of the cube Z_{ij}^k. Now using the strong connectedness on one cell, we get a constant depending only on c_δ and the (δ-dependent) constant in $(H3)$, but not on k, such that

$$\begin{aligned}\int_{\Omega_k} |u_k|^2\,dy &= \sum_{ij}^{k} \int_{Y_{ij}^k} \left(|u_k(y) - \bar{u}_{ij}^k|^2 + |\bar{u}_{ij}^k|^2 \right) dy \\ &\le C\left(k^{-2} \int_{\Omega_k} |\nabla u_k|^2 + \int_\Omega |\bar{u}_k|^2 \right) \le C(1 + \int_\Omega |\bar{u}_k|^2), \end{aligned} \tag{5.62}$$

where in the last estimate we used (5.60). In the next step we construct a sequence of continuous functions $\hat{u}_k$ on $\overline{\Omega}$, that interpolate a suitable arrangement of mean values of u_k. More precisely, in each knot $(i/k, j/k)_{i,j=0,\ldots,k}$ of the grid we introduce the real number

$$\hat{u}_{ij}^k := \fint_{\bigcup_{l=i}^{i+1} \bigcup_{n=j}^{j+1} Y_{l,n}^k} u_k(y)\,dy\,, \quad Y_{i,0}^k = Y_{0,j}^k = Y_{i,k+1}^k = Y_{k+1,j}^k := \emptyset$$

as the mean value of u_k over all cells attached to this knot, and define $\hat{u}_k$ on each square Z_{ij}^k as the bilinear interpolation of the four corner values $\hat{u}_{ln}^k$, $(l,n) \in \{i-1, i\} \times \{j-1, j\}$. It is then easy to check that by construction we get a sequence

$$\hat{u}_k \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})\,, \quad \int_\Omega \hat{u}_k(y)\,dy = 0\,, \tag{5.63}$$

where we essentially use that the mean value of u_k over Ω_k vanishes. From estimate (5.62) we deduce

$$\int_{\Omega_k} |u_k|^2\,dy \le C\left(1 + \int_\Omega |\hat{u}_k|^2\,dy + \int_\Omega |\bar{u}_k - \hat{u}_k|^2\,dy \right) \tag{5.64}$$

with a constant independent of k. Using the definition of $\bar{u}_{ij}^k$ and $\hat{u}_{ij}^k$, straightforward calculation shows that there exists a constant C independent of k, such that

$$\int_\Omega |\nabla \hat{u}_k|^2\,dy + \int_\Omega |\bar{u}_k - \hat{u}_k|^2\,dy \le C(1+k^{-2}) \sum_{i,j=1}^{k} \sum_{N(i,j)} |\bar{u}_{ij}^k - \bar{u}_{N(i,j)}^k|^2, \tag{5.65}$$

where $N(i,j)$ runs over the five-point stencil with center (i,j), and where we may set $\bar{u}_{ln}^k := 0$ whenever $l, n \in \{0, k+1\}$. In order to estimate the term on the right-hand side in (5.65), we first observe

$$|\bar{u}_{i+1,j}^k - \bar{u}_{ij}^k|^2 \le C \fint_{Y_{ij}^k \cup Y_{i+1,j}^k} (|u_k(y) - \fint_{Y_{ij}^k} u_k|^2 + |u_k(y) - \fint_{Y_{i+1,j}^k} u_k|^2)\,dy. \tag{5.66}$$

Note that by the definition of μ and μ^δ, the support of the periodic measure μ^δ on two neighbouring cells of order one is a connected, full-dimensional Lipschitz domain, and hence for each smooth function v, we get

$$\int_{Y_\delta \cup (Y_\delta + \vec{e}_j)} |v(y) - \fint_{Y_\delta} v|^2 \, dy \;\le\; C \int_{Y_\delta \cup (Y_\delta + \vec{e}_j)} |\nabla v(y)|^2 \, dy \,, \quad Y_\delta := Y \cap F^\delta, \tag{5.67}$$

with a constant only depending on δ, which can be proven with the same technique as in the proof of Lemma 2.3.14. As a consequence, combining (5.65),(5.66) and (5.67), we obtain

$$\int_\Omega |\nabla \hat{u}_k|^2 \, dy + \int_\Omega |\bar{u}_k - \hat{u}_k|^2 \, dy \;\le\; C \sum_{i,j=1}^{k} \int_{Y_{ij}^k} |\nabla u_k|^2 \, dy \;\le\; C \tag{5.68}$$

with a constant depending on δ but not on k, and where we used (5.60). Finally, from (5.63),(5.64) and (5.68) we deduce $\|u_k\|_{L^2(\Omega_k)} \le C$ with a constant independent of k, where we applied the standard Poincaré inequality for functions with zero mean value on the unit square Ω. This is a contradiction to (5.60). □

For the commutativity of the two-parameter diagram it is essential to verify (cf. Lemma 5.2.11) that the sequence $\mu^\delta \rightharpoonup \mu$ corresponding to the network structures F, F^δ defined above satisfies the strong approximability property introduced in Definition 5.1.8:

Lemma 5.2.14. *Let (F,μ) be a connected periodic network on $\mathbb{R}^2$ and (F^δ, μ^δ) the corresponding fattened structure according to (5.57) and (5.58). Then the sequence $\mu^\delta \rightharpoonup \mu$ enjoys the strong approximability property.*

Proof. For our reference measure μ and its approximating sequence μ^δ given by Example 5.1.2, the statement is easy to prove. However, we directly consider a general network (F,μ) admissible in (5.55). So let a solenoidal vector $v \in V^2_{\mathrm{sol}}(\mathbb{T}, d\mu)$ be given. Recall that v is μ-almost everywhere tangential to the segments by Remark 2.2.14. Using suitable test functions along each segment, we check that v is also constant on each segment F_k, that means

$$v|_{F_k} = \lambda_k \tau_k \,, \quad \lambda_k \in \mathbb{R}, \tag{5.69}$$

where τ_k is a unit vector directed along F_k. Since μ is connected, it is clear that at least one end point of each segment belongs to at least one further segment. On the other hand, a solenoidal vector v vanishes on segments with a free end point. This includes free end points at the boundary of Y, i.e. where no segment is attached at the corresponding point of the opposite face of Y. Therefore we consider sements F_{ij}, whose end points P_i, P_j are intersection points of two or more segments, and lie inside Y (see top of Figure 5.7 below). The case when P_i belongs to ∂Y (see bottom of Figure 5.7) will be investigated afterwards. In order to construct a strong approximating sequence $v_\delta \in V^2_{\mathrm{sol}}(\mathbb{T}, d\mu^\delta)$ subject to the structure defined in (5.57) and (5.58), we label by

$$\tau_l^1, \ldots, \tau_l^{N_l},\ l \in \{i,j\},\ N_l \in \mathbb{N}$$

the unit vectors in the knot P_l directed along the segments F_l^m attached to this knot. The upper configuration in Figure 5.7 comprises an example with $N_l = 4,\ \tau_i^1 = -\tau_j^3$ and $F_{ij} = F_i^1 = F_j^3$.

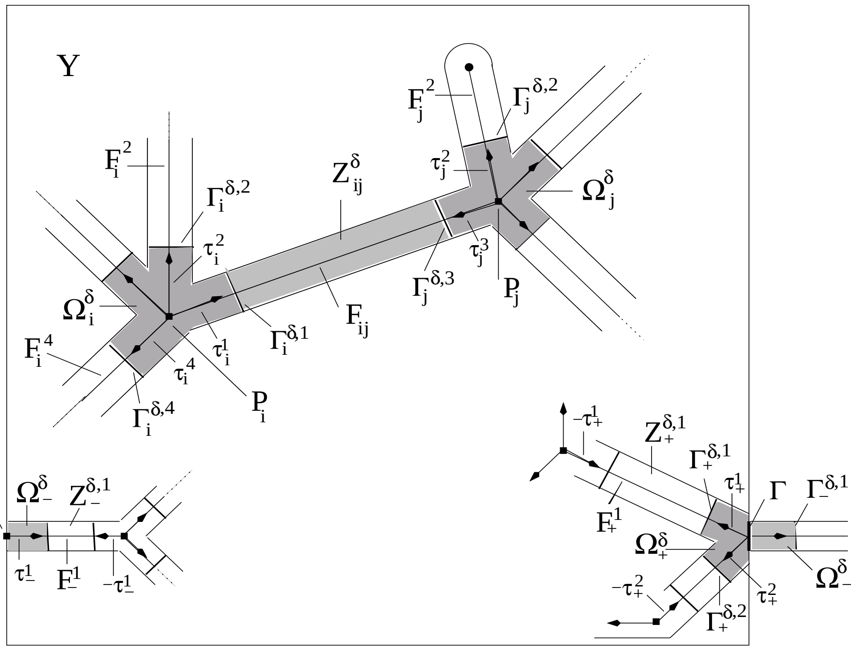


Figure 5.7: Strong approximability for complex networks

It is important to note that, in addition to (5.69), in each knot the Kirchhoff law holds for solenoidal vectors v with respect to μ, that means

$$\sum_{m=1}^{N_l} \lambda_l^m = 0 \quad \text{for } v|_{F_l^m} = \lambda_l^m \tau_l^m\,, \quad l \in \{i,j\}. \tag{5.70}$$

Clearly in Figure 5.7 we have $\lambda_i^1 = -\lambda_j^3$. In a neighbourhood of P_l we define the Lipschitz domain Ω_l^δ as the union of the disc of radius δ with center P_l, and N_l rectangles of width 2δ and length $C\delta$ with middle line F_l^m. By $\Gamma_l^{\delta,m}$ we denote the outer facets of these rectangles, which are perpendicular to the segment F_l^m, in particular

$$\operatorname{dist}(P_l, \Gamma_l^{\delta,m}) = C\delta\,, \quad m = 1, \ldots, N_l, \tag{5.71}$$

where the constant C will be chosen appropriately. Moreover by Z_{ij}^δ we denote the rest of the fattened segment joining P_i and P_j, that means

$$Z_{ij}^\delta := F_{ij}^\delta \setminus (\Omega_i^\delta \cap \Omega_j^\delta)\,, \quad |Z_{ij}^\delta| = 2\delta \cdot (\mathcal{H}^1(F_{ij}) - 2C\delta)\,, \tag{5.72}$$

where F_{ij}^δ is defined as in (5.57). We emphasize that, since only a finite number of segments is involved, we can always choose the constant C in (5.71) such that $Z_{ij}^\delta \cap Z_{ln}^\delta = \emptyset$ for $(i,j) \neq (l,n)$, which is relevant for the construction of v_δ below. Also note that when δ is sufficiently small, the domain Ω_i^δ corresponding to a knot $P_i \subset Y$ is also strictly contained in Y. Figure 5.7 illustrates all the quantities defined above.

If two or more segments meet in a knot P that belongs to ∂Y (this is the case at least once on each facet of ∂Y, since μ is connected on $\mathbb{R}^2$), we make a similar construction. To this end, we distinguish the direction with respect to ∂Y, in which the unit vectors belonging to the attached segments point away from P, i.e. we introduce the notation

$$(F_+^m, \tau_+^m),\ m = 1, \ldots, N_+\,, \quad (F_-^m, \tau_-^m),\ m = 1, \ldots, N_-\,,$$

respectively for the segments and vectors pointing away from P. Figure 5.7 shows a typical configuration with $N_+ = 2$ and $N_- = 1$. It is important to note that a vector $v \in V^2_{\mathrm{sol}}(\mathbb{T}, d\mu)$ has the following shape in this part:

$$v|_{F_\pm^m} = \lambda_\pm^m \tau_\pm^m \ \text{ with } \sum_{m=1}^{N_+} \lambda_+^m + \sum_{m=1}^{N_-} \lambda_-^m = 0\,. \tag{5.73}$$

Precisely as above we construct Lipschitz domains $\Omega_\pm^\delta$ with respect to the segments $F_\pm^m$ on each side of ∂Y, we only have to take additionally the intersection with Y. The sets $Z_\pm^{\delta,m}$ and $\Gamma_\pm^{\delta,m} \subset \partial\Omega_\pm^\delta$ are then defined in an obvious way similarly as above, where in addition we denote by $\Gamma \subset \partial Y$ the segment of length 2δ with center in P (see bottom right in Figure 5.7). Note that Γ is in general strictly contained in $\partial\Omega_\pm^\delta \cap \partial Y$ by the definition of the structure $F^\delta \cap Y$. We emphasize that with the above defined sets

$$\Omega_i^\delta, \Omega_j^\delta, Z_{ij}^\delta \ \text{ and } \ \Omega_\pm^\delta, Z_\pm^{\delta,m}$$

respectively, we precisely exhaust the support of $\mu^\delta \lfloor Y$, in particular avoiding double coverage. Now we can introduce the following auxiliary problems:

$$\begin{cases} \Delta u_i^\delta &= 0 \quad \text{in } \Omega_i^\delta, \\ \partial_n u_i^\delta &= \lambda_i^m \quad \text{on } \Gamma_i^{\delta,m}, \end{cases} \qquad \begin{cases} \Delta u_\pm^\delta &= 0 \quad \text{in } \Omega_\pm^\delta, \\ \partial_n u_\pm^\delta &= \lambda_\pm^m \quad \text{on } \Gamma_\pm^{\delta,m}, \\ \partial_n u_\pm^\delta &= \sum_m \lambda_\mp^m \quad \text{on } \Gamma, \end{cases}$$

where on those parts of $\partial\Omega_i^\delta$ and $\partial\Omega_\pm^\delta$, which are not explicitly specified, we choose a homogeneous Neumann condition. Note that both Neumann problems have, up to an additive constant, a unique solution in H^1, since the compatibility conditions

$$\int_{\partial\Omega_i^\delta} \partial_n u_i^\delta \, d\sigma = \sum_{m=1}^{N_i} \lambda_i^m |\Gamma_i^{\delta,m}| = 2\delta \sum_{m=1}^{N_i} \lambda_i^m = 0,$$

$$\int_{\partial\Omega_\pm^\delta} \partial_n u_\pm^\delta \, d\sigma = \sum_{m=1}^{N_\pm} \lambda_\pm^m |\Gamma_\pm^{\delta,m}| + |\Gamma| \sum_{m=1}^{N_\mp} \lambda_\mp^m = 0$$

are satisfied by (5.70) and (5.73). Due to the regular geometry of the domains Ω_i^δ and $\Omega_\pm^\delta$ we can use a simple rescaling argument to show that the solutions of the corresponding Neumann problems satisfy

$$\int_{\Omega_i^\delta} |\nabla u_i^\delta|^2 \, dy \le C\delta^2 \,, \quad \int_{\Omega_\pm^\delta} |\nabla u_\pm^\delta|^2 \, dy \le C\delta^2 \tag{5.74}$$

with constants independent of δ. By construction, it is easy to check that the μ^δ-measurable function $v_\delta : F^\delta \cap Y \to \mathbb{R}^2$ defined by

$$v_\delta(y) := \begin{cases} \nabla u_i^\delta & \text{in } \Omega_i^\delta, \\ \nabla u_\pm^\delta & \text{in } \Omega_\pm^\delta, \end{cases} \qquad v_\delta(y) := \begin{cases} \lambda_i \tau_i & \text{in } Z_{ij}^\delta, \\ \lambda_\pm^m \tau_\pm^m & \text{in } Z_\pm^{\delta,m} \end{cases}$$

belongs to the class $V^2_{\mathrm{sol}}(\mathbb{T}, d\mu^\delta)$, where τ_i denotes the unit vector along the segment F_{ij} that joins two inner points P_i and P_j. Using (5.74) and the same technique with which we proved $\mu^\delta \rightharpoonup \mu$, it is also straightforward to show that $v_\delta \rightharpoonup v$ weakly in $L^2(Y, d\mu^\delta)$ in the sense of (5.6). To this end we take into account the shape of a solenoidal vector v on the thin structure according to (5.69). Finally, using the estimates in (5.74), we check $\|v_\delta\|_{2,\mu^\delta,Y} \to \|v\|_{2,\mu,Y}$, which implies the strong convergence of v_δ by Lemma 5.1.7. $\square$

Now we prove the commutativity of the two limit processes $\delta \to 0$ and $\varepsilon \to 0$ starting from the quasilinear problem (P_ε^δ) on page 127, subject to connected periodic networks on $\mathbb{R}^2$. Note that if the data satisfy Assumption 5.2.4, thanks to Lemma 5.2.13 and Lemma 5.2.14 the path

$$(P_\varepsilon^\delta, u_\varepsilon^\delta) \overset{\varepsilon\to 0}{\longrightarrow} (P^\delta, u^\delta) \overset{\delta\to 0}{\longrightarrow} (P^0, u^0)\,.$$

is already covered by Lemma 5.2.9 and Lemma 5.2.11. The crucial step in showing the commutativity is now to investigate the asymptotics

$$(P_\varepsilon^\delta, u_\varepsilon^\delta) \overset{\delta\to 0}{\longrightarrow} (P_\varepsilon, u_\varepsilon) \tag{5.75}$$

for each fixed $\varepsilon > 0$, where the ε-microscale problem (P_ε) on the singular structure is given by

$$(P_\varepsilon) \qquad -\operatorname{div}\left(K(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla u_\varepsilon\right) + \lambda u_\varepsilon = f(\tfrac{x}{\varepsilon}, u_\varepsilon), \quad u_\varepsilon \in \tilde{H}_0^{1,2}(\Omega, d\mu_\varepsilon),$$

and where the data $(K, f) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are given by Assumption 5.2.4. Since K is a scalar, it is natural to call a function $u_\varepsilon \in H_\varepsilon := H_0^{1,2}(\Omega, d\mu_\varepsilon)$ a solution of problem (P_ε), if

$$\int_\Omega \left(K(\tfrac{x}{\varepsilon}, u_\varepsilon)\nabla_{\mu_\varepsilon} u_\varepsilon \cdot \nabla\varphi + \lambda u_\varepsilon \varphi\right) d\mu_\varepsilon = \int_\Omega f(\tfrac{x}{\varepsilon}, u_\varepsilon)\varphi \, d\mu_\varepsilon \tag{5.76}$$

for each $\varphi \in \mathcal{D}(\Omega)$. Thanks to the assumptions on μ and Remark 5.2.5, the problem (P_ε) is well defined as also Corollary 3.2.4 shows. We can now prove the central statement corresponding to the limit process in (5.75).

Lemma 5.2.15. *Let (F, μ) be a connected periodic network on $\mathbb{R}^2$ with fattened structure (F^δ, μ^δ) according to (5.57),(5.58), and assume the data satisfy Assumption 5.2.4. Let $\{u_\varepsilon^\delta\}_{\delta>0}$ be a sequence of solutions of (P_ε^δ) according to Corollary 5.2.6. Then, up to subsequences, there holds*

$$u_\varepsilon^\delta \rightharpoonup u_\varepsilon \,, \quad \nabla u_\varepsilon^\delta \rightharpoonup \nabla_{\mu_\varepsilon} u_\varepsilon \tag{5.77}$$

as $\delta \to 0$ in the sense of (5.20), where $u_\varepsilon \in H_\varepsilon$ is a solution of problem (P_ε) in the sense of (5.76).

Proof. Recall that by Remark 5.2.2, for any fixed ε there holds $\mu_\varepsilon^\delta \rightharpoonup \mu_\varepsilon$ as $\delta \to 0$ in the sense of (5.19). Thanks to the a priori bound (5.35), by Proposition 5.1.10 there exists a subsequence, still denoted by δ, and functions $u_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)$, $\Phi_\varepsilon \in L^2(\Omega, d\mu_\varepsilon)^2$, such that

$$u_\varepsilon^\delta \rightharpoonup u_\varepsilon \,, \quad \nabla u_\varepsilon^\delta \rightharpoonup \Phi_\varepsilon \quad \text{as } \delta \to 0 \ \text{ in } L^2(\Omega, d\mu_\varepsilon^\delta). \tag{5.78}$$

We need to show that $u_\varepsilon \in H_\varepsilon$ and that Φ_ε is its μ_ε-tangential gradient. Without loss of generality we can assume $u_\varepsilon^\delta \in \mathcal{D}(\Omega)$. Indeed, by the definition of the Dirichet space H_ε^δ there holds

$$\forall \delta > 0 \, \exists \psi_\varepsilon^\delta \in \mathcal{D}(\Omega): \quad \|u_\varepsilon^\delta - \psi_\varepsilon^\delta\|_{2,\varepsilon,\delta} + \|\nabla u_\varepsilon^\delta - \nabla \psi_\varepsilon^\delta\|_{2,\varepsilon,\delta} \le \delta. \tag{5.79}$$

Since $\mu_\varepsilon^\delta(\Omega) \le C$ uniformly, we can then replace u_ε^δ by ψ_ε^δ in (5.78). The same substitution can be made when passing to the limit in the weak formulation (5.23) of problem (P_ε^δ), where one has to use the assumptions (5.24)-(5.27) on the data. As an example, we estimate the following error term using (5.26):

$$\begin{aligned}
&\left| \int_\Omega [K_\delta(\tfrac{x}{\varepsilon}, u_\varepsilon^\delta) - K_\delta(\tfrac{x}{\varepsilon}, \psi_\varepsilon^\delta)] \nabla u_\varepsilon^\delta \cdot \nabla \varphi \, d\mu_\varepsilon^\delta \right| \\
\le \;& c_4 \, \|\nabla \varphi\|_\infty \|\nabla u_\varepsilon^\delta\|_2 \left(\int_\Omega (1 + |u_\varepsilon^\delta| + |\psi_\varepsilon^\delta|)^{2(1-\gamma)} |u_\varepsilon^\delta - \psi_\varepsilon^\delta|^{2\gamma} \right)^{1/2} \\
\le \;& C \, \|(1 + |u_\varepsilon^\delta| + |\psi_\varepsilon^\delta|)\|_2^{1-\gamma} \|u_\varepsilon^\delta - \psi_\varepsilon^\delta\|_2^\gamma \;\to\; 0
\end{aligned}$$

as $\delta \to 0$ by the uniform a priori estimate (5.35) and (5.79), where we have set $\|\cdot\|_q := \|\cdot\|_{q,\varepsilon,\delta}$. Similar, the other error terms can be estimated. Hence from now on we assume $u_\varepsilon^\delta \in \mathcal{D}(\Omega)$. As a consequence, precisely as in the proof of Theorem 2.4.5, by the boundedness of Ω we can assume that

$$\Omega = (0, l)^2 \,, \quad l \in \mathbb{N}_+, \tag{5.80}$$

after possibly extending u_ε^δ trivially to the whole of $\mathbb{R}^2$. In order to simplify the notation further, we prove the lemma for the special measure μ given by Example 5.1.2. It is obvious that the analysis below can, up to more complicated notation, be carried over for the case of an arbitrary measure μ admissible in (5.55), since there are only straight segments involved with a similar, regular fattening structure. We introduce the following notation:

$$\varepsilon = \tfrac{1}{n}, \; n \in \mathbb{N}, \; m := nl, \; z := (x, y), \; dz = dxdy,$$

where $z \in \Omega$ is the 2D-variable and l the integer in (5.80). For the measure μ^δ under consideration we define rows $(R_i^\delta)_{i=1,\dots,m}$ and columns $(C_i^\delta)_{i=1,\dots,m}$ depending on the thickness parameter δ by

$$R_i^\delta := [0,l] \times (y_i - \tfrac{\delta}{n}, y_i + \tfrac{\delta}{n}), \quad C_i^\delta := (x_i - \tfrac{\delta}{n}, x_i + \tfrac{\delta}{n}) \times [0,l], \tag{5.81}$$

where the points (x_i, y_j) are the centers of the rescaled unitary cubes. The thin limit structure comprises a Cartesian network with rows $(R_i)_{i=1,\dots,m}$ and columns $(C_i)_{i=1,\dots,m}$:

$$R_i := [0,l] \times \{y_i\}, \quad C_i := \{x_i\} \times [0,l]. \tag{5.82}$$

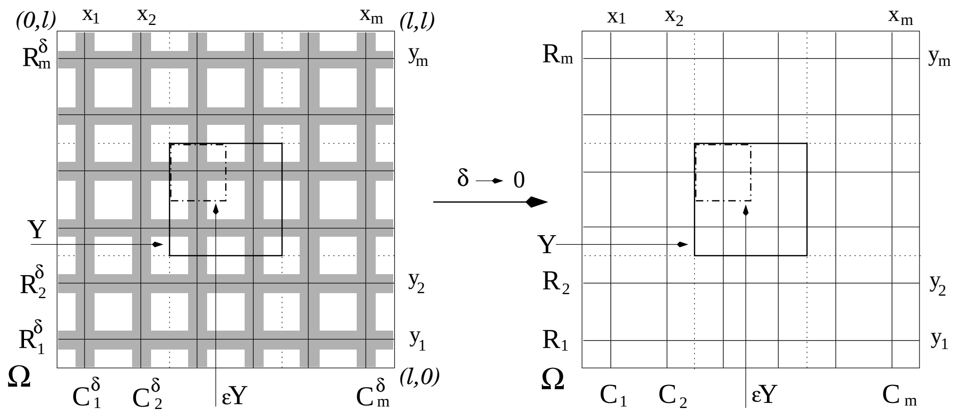


Figure 5.8: Fattening approach for fixed $\varepsilon > 0$.

The setting is sketched in Figure 5.8 for the data $n = 2, l = 3, m = 6$ and $x_i, y_i = \frac{1}{4}(2i-1)$ for $i = 1\dots,6$. Recall the definition of μ and μ^δ in Example 5.1.2. Within this framework it is then easy to check that for any $\varphi \in \mathcal{C}_0(\Omega)$ there holds

$$\begin{aligned}\int_\Omega \varphi\, d\mu_\varepsilon^\delta &= c_\delta \sum_{i=1}^m \left(\int_{R_i^\delta} \varphi(z)\, dz + \int_{C_i^\delta} \varphi(z)\, dz - \sum_{j=1}^m \int_{R_j^\delta \cap C_i^\delta} \varphi(z)\, dz \right) \\ &\to \frac{1}{2n} \sum_{i=1}^m \left(\int_{R_i} \varphi(x, y_i)\, dx + \int_{C_i} \varphi(x_i, y)\, dy \right) = \int_\Omega \varphi\, d\mu_\varepsilon. \end{aligned} \tag{5.83}$$

To proceed further, we define for $i = 1,\dots,m$ a family of auxiliary functions $v_i^\delta \in H_0^1(R_i)$ and $w_i^\delta \in H_0^1(C_i)$ by

$$v_i^\delta : x \mapsto ⨍_{y_i - \frac{\delta}{n}}^{y_i + \frac{\delta}{n}} u_\varepsilon^\delta(x,y)\, dy, \quad w_i^\delta : y \mapsto ⨍_{x_i - \frac{\delta}{n}}^{x_i + \frac{\delta}{n}} u_\varepsilon^\delta(x,y)\, dx. \tag{5.84}$$

We claim that v_i^δ and w_i^δ are bounded sequences in $H_0^1(R_i)$ and $H_0^1(C_i)$ respectively. Indeed, there holds

$$\int_0^l |\partial_x v_i^\delta(x)|^2\, dx \le \frac{n}{2\delta} \int_{R_i^\delta} |\partial_x u_\varepsilon^\delta(z)|^2\, dz \le Cn \int_\Omega |\nabla u_\varepsilon^\delta|^2\, d\mu_\varepsilon^\delta \le Cn \tag{5.85}$$

with a constant independent of δ by (5.35). Similarly we can treat $\partial_y w_i^\delta$. As a consequence, upon taking a further subsequence of the sequence selected in (5.78), we get

$$v_i^\delta \rightharpoonup v_i \quad \text{weakly in } H_0^1(R_i)\,, \quad w_i^\delta \rightharpoonup w_i \quad \text{weakly in } H_0^1(C_i)\,. \tag{5.86}$$

Let us show that $v_i(x_j) = w_j(y_i)$ for all $i, j = 1, \ldots, m$ in the sense of continuous representatives. It suffices to consider $i, j = 1$. Note that

$$|v_1(x_1) - w_1(y_1)| \;\le\; |v_1^\delta(x_1) - w_1^\delta(y_1)| + o(1) \quad \text{as } \delta \to 0\,, \tag{5.87}$$

since $v_1^\delta \to v_1$, $w_1^\delta \to w_1$ uniformly in R_i and C_i respectively. We can estimate the term on the right-hand side in (5.87) further and get

$$|v_1^\delta(x_1) - w_1^\delta(y_1)| \;\le\; \left| v_1^\delta(x_1) - \fint_{x_1-\frac{\delta}{n}}^{x_1+\frac{\delta}{n}} v_1^\delta\,dx \right| + \left| w_1^\delta(y_1) - \fint_{y_1-\frac{\delta}{n}}^{y_1+\frac{\delta}{n}} w_1^\delta\,dy \right| \;\le\; C\sqrt{\delta}$$

with a constant depending on n but not on δ, where we used the continuous embedding of H^1 in $\mathcal{C}^{0,1/2}$ in one space dimension. It follows that $v_i(x_j) = w_j(y_i)$ and we conclude that the function $\bar u_\varepsilon$ defined by

$$\bar u_\varepsilon(x,y) := \begin{cases} v_i(x) & \text{if } y = y_i \\ w_i(y) & \text{if } x = x_i \end{cases}, \quad \nabla_{\mu_\varepsilon} \bar u_\varepsilon(x,y) := \begin{cases} (v_i'(x), 0) & \text{if } y = y_i \\ (0, w_i'(y)) & \text{if } x = x_i \end{cases}$$

belongs to H_ε (cf. Observation 2.2.10.1). We want to show that $u_\varepsilon = \bar u_\varepsilon$ in $L^2(\Omega, d\mu_\varepsilon)$, which implies $u_\varepsilon \in H_\varepsilon$ in (5.78). To this end consider a family of functions

$$L^2(\Omega, d\mu_\varepsilon^\delta) \ni \bar u_\varepsilon^\delta(x,y) := \begin{cases} v_i^\delta(x) & \text{if } (x,y) \in R_i^\delta, \\ w_i^\delta(y) & \text{if } (x,y) \in C_i^\delta \setminus \cup_j R_j^\delta. \end{cases}$$

We show that for the subsequence in (5.86), we get $\bar u_\varepsilon^\delta \rightharpoonup \bar u_\varepsilon$ as $\delta \to 0$ in the sense of (5.20). Indeed, for arbitrary $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} \int_\Omega \bar u_\varepsilon^\delta \varphi\,d\mu_\varepsilon^\delta \;&=\; c_\delta \sum_{i=1}^m \left(\int_{R_i^\delta} v_i^\delta(x)\varphi(z)\,dz + \int_{C_i^\delta} w_i^\delta(y)\varphi(z)\,dz - \sum_{j=1}^m I_{i,j}^\delta \right) \\ &\to\; \frac{1}{2n} \sum_{i=1}^m \left(\int_{R_i} (v_i\varphi)(x,y_i)\,dx + \int_{C_i} (w_i\varphi)(x_i,y)\,dy \right) \;=\; \int_\Omega \bar u_\varepsilon \varphi\,d\mu_\varepsilon, \end{aligned}$$

where we have used

$$I_{i,j}^\delta \;:=\; \int_{C_i^\delta \cap R_j^\delta} w_i^\delta(y)\varphi(x,y)\,dxdy \;=\; o(\delta) \quad \text{as } \delta \to 0,$$

which can easily be shown by using $\sup_{y \in C_i} |w_i^\delta(y)| \le C$ with a constant independent of δ by (5.86). Hence, by (5.78), in order to obtain $\bar u_\varepsilon = u_\varepsilon$ it suffices to show

$$\|u_\varepsilon^\delta - \bar u_\varepsilon^\delta\|_{2,\varepsilon,\delta}^2 \;=\; c_\delta \sum_{i=1}^m (I_i^\delta + J_i^\delta - \sum_{j=1}^m L_{ij}^\delta) \;=\; o(1) \quad \text{as } \delta \to 0, \tag{5.88}$$

where one can easily check that the first equality in (5.88) holds for

$$
\begin{aligned}
I_i^\delta &= \int_{R_i^\delta} \left| u_\varepsilon^\delta(z) - \fint_{y_i-\frac{\delta}{n}}^{y_i+\frac{\delta}{n}} u_\varepsilon^\delta(x,y)\,dy \right|^2 dz\,, \qquad (5.89)\\
J_i^\delta &= \int_{C_i^\delta} \left| u_\varepsilon^\delta(z) - \fint_{x_i-\frac{\delta}{n}}^{x_i+\frac{\delta}{n}} u_\varepsilon^\delta(x,y)\,dx \right|^2 dz\,,\\
L_{i,j}^\delta &= \int_{C_i^\delta\cap R_j^\delta} |u_\varepsilon^\delta(z) - w_i^\delta(y)|^2\,dz.
\end{aligned}
$$

In order to prove (5.88), the three terms above can be treated similarly. For convenience, we estimate the first term:

$$
\begin{aligned}
\sum_{i=1}^m I_i^\delta &= \sum_{i=1}^m \int_0^l \left(\int_{y_i-\frac{\delta}{n}}^{y_i+\frac{\delta}{n}} \left| u_\varepsilon^\delta(x,y) - \fint_{y_i-\frac{\delta}{n}}^{y_i+\frac{\delta}{n}} u_\varepsilon^\delta(x,\tilde y)\,d\tilde y \right|^2 dy \right) dx\\
&\le C\,\frac{4\delta^2}{n^2} \sum_{i=1}^m \int_{R_i^\delta} |\partial_y u_\varepsilon^\delta(z)|^2\,dz \;\le\; C_n\delta^3 \int_\Omega |\nabla u_\varepsilon^\delta|^2\,d\mu_\varepsilon^\delta \;\le\; C_n\delta^3,
\end{aligned}
$$

where for fixed $x \in (0,l)$ we applied the Poincaré inequality on the interval $(y_i - \frac{\delta}{n}, y_i + \frac{\delta}{n})$ to the smooth function $u_\varepsilon^\delta(x,\cdot)$ and used the linear dependence of the Poincaré constant on the diameter of this interval. This shows (5.88), which implies $\bar u_\varepsilon = u_\varepsilon \in H_\varepsilon$. Let us now show that the pair $(u_\varepsilon, \Phi_\varepsilon)$ with Φ_ε given in (5.78) belongs to $V^2(\Omega, d\mu_\varepsilon)$. We use a standard localization argument and consider, as an example, the segment

$$
I_i^j := (x_i, x_{i+1}) \times \{y_j\} \subset R_j, \quad i \in \{0,1,\dots,m\}, \;\; x_0 := 0,\; x_{m+1} := l.
$$

Consider test functions $\varphi_1(x) \in \mathcal{D}(x_i, x_{i+1})$ and $\varphi_2(y) \in \mathcal{D}(0,l)$, the latter with the additional property that there exist sufficiently small neighbourhoods U, V of y_j with $U \subset V$, such that $\varphi_2 = 1$ in U and $\varphi_2 = 0$ in $(0,l) \setminus V$. Then if we choose the vector $\varphi(z) = \varphi_1(x)\varphi_2(y)\vec e_1$ as a test function in the second convergence in (5.78), we get

$$
c_\delta^{-1} \int_\Omega \nabla u_\varepsilon^\delta \cdot \varphi\,d\mu_\varepsilon^\delta = \int_{R_j^\delta} \partial_x u_\varepsilon^\delta(z)\varphi_1(x)\,dz = -\int_{I_{i,\delta}^j} u_\varepsilon^\delta(x,y)\partial_x\varphi_1(x)\,dxdy \qquad (5.90)
$$

for δ small enough, where we have set $I_{i,\delta}^j := (x_i, x_{i+1}) \times (y_j - \frac{\delta}{n}, y_j + \frac{\delta}{n})$ and extended φ_1 trivially to $(0,l)$. Multiplying by c_δ and passing to the limit in (5.90), we get

$$
\frac{1}{2n} \int_{x_i}^{x_{i+1}} (\Phi_\varepsilon)_1(x,y_j)\varphi_1(x)\,dx = -\frac{1}{2n} \int_{x_i}^{x_{i+1}} u_\varepsilon(x,y_j)\partial_x\varphi_1(x)\,dx\,,
$$

which shows that $(\Phi_\varepsilon)_1 = \partial_x u_\varepsilon = \partial_x v_j$ in I_i^j. This proof can of course be carried over to the vertical subsegments $J_i^j := \{x_i\} \times (y_j, y_{j+1}) \subset C_i$, and we obtain

$$
P_\mu(\tfrac{z}{\varepsilon})[\Phi_\varepsilon(z)] = \nabla_{\mu_\varepsilon} u_\varepsilon(z) \quad \text{for } \mu_\varepsilon\text{-a.e. } z \in \Omega. \qquad (5.91)
$$

Later we will see in addition, that Φ_ε is already tangential and therefore $\Phi_\varepsilon = \nabla_{\mu_\varepsilon} u_\varepsilon$. Now we have to pass to the limit in (5.23) as $\delta \to 0$ for fixed ε. We consider a suitable extension $\hat{u}_\varepsilon$

$$\hat{u}_\varepsilon : \Omega \to \mathbb{R}\,, \quad \hat{u}_\varepsilon = u_\varepsilon \;\text{ on } F_n := \bigcup_{i=1}^{m} (R_i \cup C_i) \tag{5.92}$$

of the limit function u_ε, which is only defined on the skeleton F_n, to the whole of Ω. We claim that we can assume $\hat{u}_\varepsilon$ to be Hölder continuous. Indeed, if we use for $0 \le i, j \le m$ the notation

$$\Omega_i^j := (x_i, x_{i+1}) \times (y_j, y_{j+1})\,, \quad \Gamma_i^j := \partial\Omega_i^j = I_i^j \cup I_i^{j+1} \cup J_i^j \cup J_{i+1}^j\,,$$

we can apply standard embedding and trace extension theorems on Sobolev spaces [58, Theorems 11.2.1,11.2.3] and get, since we are in two space dimensions, a sequence of continuous embeddings and extension operators

$$W^{1,2}(\Gamma_i^j) \hookrightarrow W^{1/2,4}(\Gamma_i^j) \to W^{3/4,4}(\Omega_i^j) \hookrightarrow \mathcal{C}^{0,1/4}(\overline{\Omega_i^j}). \tag{5.93}$$

Since u_ε is continuous in the intersection points (x_i, y_j), we can apply (5.93) and find a function $\hat{u}_\varepsilon \in \mathcal{C}^{0,1/4}(\overline{\Omega})$ in (5.92). Using such an extension, we claim that

$$\int_\Omega K_\delta(\tfrac{z}{\varepsilon}, u_\varepsilon^\delta) \nabla u_\varepsilon^\delta \cdot \nabla\varphi \, d\mu_\varepsilon^\delta = \int_\Omega K_\delta(\tfrac{z}{\varepsilon}, \hat{u}_\varepsilon) \nabla u_\varepsilon^\delta \cdot \nabla\varphi \, d\mu_\varepsilon^\delta + o(1), \tag{5.94}$$

$$\int_\Omega f_\delta(\tfrac{z}{\varepsilon}, u_\varepsilon^\delta) \varphi \, d\mu_\varepsilon^\delta = \int_\Omega f_\delta(\tfrac{z}{\varepsilon}, \hat{u}_\varepsilon) \varphi \, d\mu_\varepsilon^\delta + o(1), \tag{5.95}$$

for each $\varphi \in \mathcal{D}(\Omega)$ as $\delta \to 0$. We already have some routine in estimating the error terms by using Assumption 5.3.2 and the a priori estimate (5.35):

$$\begin{aligned} &\left| \int_\Omega [K_\delta(\tfrac{z}{\varepsilon}, u_\varepsilon^\delta) - K_\delta(\tfrac{z}{\varepsilon}, \hat{u}_\varepsilon)] \nabla u_\varepsilon^\delta \cdot \nabla\varphi \, d\mu_\varepsilon^\delta \right| \\ &\le C \, \|(1 + |u_\varepsilon^\delta| + |\hat{u}_\varepsilon|)\|_{2,\varepsilon,\delta}^{1-\gamma} \, \|u_\varepsilon^\delta - \hat{u}_\varepsilon\|_{2,\varepsilon,\delta}^{\gamma} \le C \, \|u_\varepsilon^\delta - \hat{u}_\varepsilon\|_{2,\varepsilon,\delta}^{\gamma} \\ &\le C \, (\|u_\varepsilon^\delta - \bar{u}_\varepsilon^\delta\|_{2,\varepsilon,\delta}^{\gamma} + \|\bar{u}_\varepsilon^\delta - \hat{u}_\varepsilon\|_{2,\varepsilon,\delta}^{\gamma}) \end{aligned}$$

with a constant not depending on δ. Hence by (5.88) it suffices to control $\|\bar{u}_\varepsilon^\delta - \hat{u}_\varepsilon\|_{2,\varepsilon,\delta}$ in order to show (5.94). Precisely as in the proof of (5.88), we can subdivide this term into sums of integrals over R_i^δ, C_i^δ and $C_i^\delta \cap R_j^\delta$ respectively, with u_ε^δ substituted by $\hat{u}_\varepsilon$. Let us exemplary estimate the term corresponding to (5.89):

$$\begin{aligned} c_\delta \sum_{i=1}^{m} \int_{R_i^\delta} |\bar{u}_\varepsilon^\delta - \hat{u}_\varepsilon|^2 \, dz &= c_\delta \sum_{i=1}^{m} \int_0^l \int_{y_i - \frac{\delta}{n}}^{y_i + \frac{\delta}{n}} |v_i^\delta(x) - \hat{u}_\varepsilon(x, y)|^2 \, dy dx \\ &= \frac{1}{2n(1-\delta)} \sum_{i=1}^{m} \int_0^l |v_i^\delta(x) - \hat{u}_\varepsilon(x, y_x^\delta)|^2 \, dx \stackrel{\delta \to 0}{\longrightarrow} 0, \end{aligned}$$

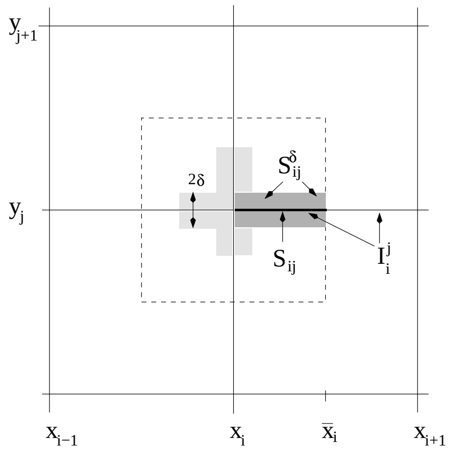
where $y_x^\delta \in (y_i - \frac{\delta}{n}, y_i + \frac{\delta}{n})$ depending on δ and x can be found by the mean value theorem of integration, and where in the last step we have used (5.86), the continuity of $\hat{u}_\varepsilon$ and $y_x^\delta \to y_i$ for $\delta \to 0$ and each fixed $x \in (0, l)$. This proves (5.94), and (5.95) can be shown in the same way using the assumptions (5.25) and (5.27). In order to pass to the limit on the right-hand side in (5.94), consider the sequence of vector functions

$$\Theta^\delta : z \mapsto K_\delta(\tfrac{z}{\varepsilon}, \hat{u}_\varepsilon(z))\nabla u_\varepsilon^\delta(z)\,,$$

which is uniformly bounded in $L^2(\Omega, d\mu_\varepsilon^\delta)^2$ by (5.24) and (5.35). Hence by Proposition 5.1.10, possibly up to another subsequence, there holds $\Theta^\delta \rightharpoonup \Theta$ in $L^2(\Omega, d\mu_\varepsilon^\delta)$ componentwise as $\delta \to 0$, and therefore we need to show

$$\Theta^\delta \rightharpoonup \Theta \overset{!}{=} K(\tfrac{z}{\varepsilon}, u_\varepsilon(z))\Phi_\varepsilon(z) \quad \mu_\varepsilon\text{ - a.e. in } \Omega. \tag{5.96}$$

The pointwise equality in (5.96) has, with the notations above, to be verified on any subsegment $I_i^j \subset R_j$ and $J_i^j \subset C_i$ respectively.



For $i, j = 1, \ldots, m-1$:

$$\bar{x}_i := \tfrac{1}{2}(x_i + x_{i+1}),$$

$$S_{ij}^\delta := (x_i, \bar{x}_i) \times (y_j - \tfrac{\delta}{n}, y_j + \tfrac{\delta}{n}),$$

$$S_{ij} := (x_i, \bar{x}_i) \times \{y_j\} \subset I_i^j.$$

It suffices to consider one half S_{ij} of I_i^j defined in the figure above, the other half and all the other segments can of course be treated equally. As a test function on the left-hand side in (5.96) we choose

$$\psi_\eta(z) = \varphi_1(x)\varphi_2(y)\vec{e}_\eta,\ \eta = 1, 2,$$

where $\varphi_1 \in \mathcal{D}(x_i, \bar{x}_i)$ is arbitrary and φ_2 defined precisely as in the proof of (5.91), supported in a small neighbourhood of y_j. For δ sufficiently small we obtain

$$c_\delta \int_{S_{ij}^\delta} K_\delta(nz, \hat{u}_\varepsilon)\langle \nabla u_\varepsilon^\delta, \vec{e}_\eta\rangle \varphi_1\, dz \;=\; \int_\Omega \Theta^\delta(z)\cdot\psi_\eta(z)\; d\mu_\varepsilon^\delta(z) \tag{5.97}$$

$$\to\; \frac{1}{2n}\int_{x_i}^{\bar{x}_i} \Theta_\eta(x, y_j)\varphi_1\, dx\,, \tag{5.98}$$

where Θ_η is the corresponding component of the vector function Θ. In order to pass to the limit on the left-hand side in (5.97), we choose an equidistant decomposition of the interval $[x_i, \bar{x}_i]$ into subintervals $[\tau_{k-1}, \tau_k]_{k=1,\dots,N}$, where N is a large integer to be chosen and

$$x_i = \tau_0, \tau_1, \dots, \tau_N = \bar{x}_i\,, \quad |\tau_k - \tau_{k-1}| = \tfrac{1}{2nN}\,, \ \ k = 1, \dots, N.$$

Subject to this decomposition, we can write the term on the left-hand side in (5.97) as a sum of the following two terms:

$$\begin{aligned} I^\delta_{1,N} &:= c_\delta \sum_{k=1}^N \int_{S^{\delta,k}_{ij}} K_\delta(nz, \hat{u}_\varepsilon(\tau_k, y_j)) \langle \nabla u^\delta_\varepsilon, \vec{e}_\eta \rangle \varphi_1 \, dz, \\ I^\delta_{2,N} &:= c_\delta \sum_{k=1}^N \int_{S^{\delta,k}_{ij}} [K_\delta(nz, \hat{u}_\varepsilon(z)) - K_\delta(nz, \hat{u}_\varepsilon(\tau_k, y_j))] \langle \nabla u^\delta_\varepsilon, \vec{e}_\eta \rangle \varphi_1 \, dz, \end{aligned}$$

where we have set $S^{\delta,k}_{ij} := (\tau_{k-1}, \tau_k) \times (y_j - \frac{\delta}{n}, y_j + \frac{\delta}{n})$. Using the uniform Hölder continuity of K_δ in (5.26) and the one of $\hat{u}_\varepsilon$, the second term can be estimated by the uniform a priori estimate on u^δ_ε:

$$|I^\delta_{2,N}| \le C \max\{\tfrac{1}{N}, \delta\}^{\gamma/4} \left(c_\delta \int_{S^\delta_{ij}} |\langle \nabla u^\delta_\varepsilon, \vec{e}_\eta \rangle| \, dz \right) \le C \max\{\tfrac{1}{N}, \delta\}^{\gamma/4}, \tag{5.99}$$

where the constant depends only on n, and $\gamma > 0$ is the Hölder constant in (5.26). Now let $\chi_k \in L^\infty(0, l)$ be the characteristic function of the interval $[\tau_{k-1}, \tau_k]$ and set $s_k := \hat{u}_\varepsilon(\tau_k, y_j)$. Then we can rewrite the term $I^\delta_{1,N}$ as

$$\begin{aligned} I^\delta_{1,N} &= c_\delta \sum_k \int_{S^\delta_{ij}} \chi_k(x) K_\delta(nz, s_k) \langle \nabla u^\delta_\varepsilon(z), \vec{e}_\eta \rangle \varphi_1(x) \, dz \\ &= \sum_k \frac{1}{n^2} \int_Y \chi_k(\tfrac{x}{n}) \chi_\delta(y) K_\delta(z, s_k) \langle \nabla u^\delta_\varepsilon(\tfrac{z}{n}), \vec{e}_\eta \rangle \varphi_1(\tfrac{x}{n}) \, d\mu^\delta(z), \end{aligned} \tag{5.100}$$

where we extended φ_1 trivially to $(0, l)$ and denoted by χ_δ the characteristic function of the interval $U_\delta(1/2)$. Also note that for simplicity we have chosen $i, j = 1$ in (5.100). In the general case we get, using the periodicity of K_δ and μ^δ, the same integral over the reference cell up to the translations

$$x \to x + (i-1), \ y \to y + (j-1), \ z \to (x + (i-1), y + (j-1))$$

and proceed precisely as below. Using the uniform boundedness of $K_\delta(y, s)$ and the strong convergence $K_\delta(\cdot, s_k) \to K(\cdot, s_k)$ in $L^2(Y, d\mu^\delta)$ according to Assumption 5.2.4, it is easy to check that for any $k = 1, \dots, N$:

$$\chi_k(\tfrac{z_1}{n}) \chi_\delta(z_2) K_\delta(z, s_k) \ \to \ \chi_k(\tfrac{z_1}{n}) \chi^-(z_2) K(z, s_k) \quad \text{strongly in } L^2(Y, d\mu^\delta), \tag{5.101}$$

where χ^- denotes the characteristic function of $\{z_2 = 1/2\}$. Recall that we have chosen $i, j = 1$ for simplicity, so we claim

$$\varrho^\delta(z) := \langle \nabla u^\delta_\varepsilon(\tfrac{z}{n}), \vec{e}_\eta \rangle \varphi_1(\tfrac{z_1}{n}) \ \rightharpoonup \ \langle \Phi_\varepsilon(\tfrac{z_1}{n}, y_1), \vec{e}_\eta \rangle \varphi_1(\tfrac{z_1}{n}) \tag{5.102}$$

weakly in $L^2(Y, d\mu^\delta)$ in the sense of (5.6). Indeed, given $\phi(z) \in \mathcal{C}^\infty(\mathbb{T})$ we get, since φ_1 is compactly supported in $(x_1, \bar{x}_1)$, for sufficiently small δ:

$$\begin{aligned}\int_Y \varrho^\delta(z)\phi(z)\,d\mu^\delta(z) &= n^2\int_\Omega \langle \nabla u_\varepsilon^\delta(z), \psi_\eta(z)\rangle \phi(nz)\,d\mu_\varepsilon^\delta(z)\\ &\to n^2\int_\Omega \langle \Phi_\varepsilon(z), \psi_\eta(z)\rangle \phi(nz)\,d\mu_\varepsilon(z)\\ &= \int_Y \langle \Phi_\varepsilon(\tfrac{z_1}{n}, y_1), \vec{e}_\eta\rangle \varphi_1(\tfrac{z_1}{n})\phi(z)\,d\mu(z).\end{aligned}$$

We set $S_{ij}^k = (\tau_{k-1}, \tau_k) \times \{y_j\}$. Passing to the limit $\delta \to 0$ in (5.100) and using (5.101) and (5.102) we obtain, now for arbitrary i, j,

$$\begin{aligned}I_{1,N}^\delta &\to \frac{1}{2n}\sum_k \int_{S_{ij}^k} K(n(x,y_j), s_k)\langle \Phi_\varepsilon(x,y_j), \vec{e}_\eta\rangle \varphi_1\,dx\\ &= \frac{1}{2n}\int_{x_i}^{\bar{x}_i} K(n(x,y_j), \hat{u}_\varepsilon(x,y_j))\langle \Phi_\varepsilon(x,y_j), \vec{e}_\eta\rangle \varphi_1\,dx + o(1) \qquad (5.103)\end{aligned}$$

as $N \to \infty$, where in (5.103) we argued precisely as in (5.99), using the Hölder continuity of both $\hat{u}_\varepsilon$ and $K(y,\cdot)$ by Remark 5.2.5. Hence combining (5.98),(5.99) and (5.103) we obtain, since $\varphi_1 \in \mathcal{D}(x_i, \bar{x}_i)$ was arbitrary,

$$\Theta(z) = K(nz, \hat{u}_\varepsilon(z))\Phi_\varepsilon(z) = K(\tfrac{z}{\varepsilon}, u_\varepsilon(z))\Phi_\varepsilon(z) \quad \mu_\varepsilon\text{ - a.e. on each } S_{ij}$$

by (5.92), and hence as argued above on the whole of Ω, which shows (5.96). The lower order source term in (5.95) can of course be treated completely analogue using Assumption 5.2.4 on the data f_δ and f, and we deduce

$$\lim_{\delta\to 0}\int_\Omega f_\delta(\tfrac{z}{\varepsilon}, \hat{u}_\varepsilon)\varphi\,d\mu_\varepsilon^\delta = \int_\Omega f(\tfrac{z}{\varepsilon}, u_\varepsilon)\varphi\,d\mu_\varepsilon \quad \text{for each } \varphi \in \mathcal{D}(\Omega). \qquad (5.104)$$

Now we can pass to the limit $\delta \to 0$ in the weak formulation (5.23) of the (δ, ε)-problem, where we have to use (5.78), the asymptotics in (5.94) and (5.95), as well as the limit identifications in (5.96) and (5.104). This yields

$$\int_\Omega \big(K(\tfrac{x}{\varepsilon}, u_\varepsilon)\langle \Phi_\varepsilon, \nabla\varphi\rangle + \lambda u_\varepsilon\varphi\big)\,d\mu_\varepsilon = \int_\Omega f(\tfrac{x}{\varepsilon}, u_\varepsilon)\varphi\,d\mu_\varepsilon \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Above we have shown $(u_\varepsilon, \Phi_\varepsilon) \in V^2(\Omega, d\mu_\varepsilon)$, and hence by density, where we have to use (5.30), the last integral identity also holds for any pair $(\varphi, \nabla\varphi) \in V^2(\Omega, d\mu_\varepsilon)$. Since K is a scalar, choosing $\varphi = 0$ yields that Φ_ε is tangential, and hence we deduce $\Phi_\varepsilon = \nabla_{\mu_\varepsilon} u_\varepsilon$ almost everywhere in Ω with respect to μ_ε, which completes the proof of (5.77) and of Lemma 5.2.15. □

Now we formulate the central theorem of this paragraph, the commutativity of the two-parameter diagram for quasilinear elliptic equations posed on periodic connected networks. To this end, we will implicitly assume that one chooses a regular fattening such as introduced in (5.57) and (5.58).

Theorem 5.2.16. *On connected periodic networks (F,μ) on $\mathbb{R}^2$, the two-parameter diagram starting from problem (P_ε^δ) commutes, more precisely: Under Assumtion 5.2.4 on the data, the functions $u^0, u_0 \in H_0^1(\Omega)$ obtained respectively from the two limit processes in (5.2) are a solution of one and the same effective problem*

$$(P) \qquad \begin{cases} -\mathrm{div}\,(K^\star(u)\nabla u) + \lambda u &= \bar{f}(u) \quad \text{in}\ \Omega\,, \\ u &= 0 \qquad \text{on}\ \partial\Omega, \end{cases}$$

where the effective tensor $K^\star$ is defined in (5.42) on page 130. Moreover if K and f are $\mathcal{C}^1$ with respect to s, and f is monotonically nonincreasing in s, then there holds $u^0 = u_0$, provided both functions belong to $L^\infty(\Omega)$.

Proof. Note that Assumption 5.2.4 guarantees that all the prerequisites of Lemma 3.2.6 on page 71 are fulfilled, where we can neglect the artificial difference between $\hat{K}$ and $K = K{\cdot}\mathbb{E}_2$. Using the standard set of test functions we can then pass to the limit in (5.76) and obtain, thanks to Corollary 3.2.10, the same effective coefficient $K^\star$ as in (5.42). Hence by Lemma 5.2.9 and Lemma 5.2.11, we obtain the same effective problem (P) for both limit procedures in (5.2). The additional statement is a direct consequence of Lemma 3.2.13 and Corollary 3.2.15. □

5.2.2 Networks in 3D

In this paragraph we consider thin networks embedded in three space dimensions. It is intuitively clear that the results of the last paragraph can be carried over. In particular, the commutativity of limit processes on such (regular fattened) networks can also be proven for Richards equation in the application relevant 3D-case, which justifies to study equation (1.1) on singular structures.

However, our main motivation to study 1D-networks in 3D is the fact that they form, although strongly connected for themselfes, combined with the surrounding Lebesgue measure a nonconnected structure in $\mathbb{R}^3$ due to their vanishing capacity. In this case the methods of the last paragraph do not apply and we will construct explicit counterexamples, for which the two limit functions u_0 and u^0 obtained in Theorem 5.2.16 solve respectively equations with different effective coefficients. Our prototype of a singular network in three space dimensions is the one of Example 2.3.9. In this case the structure (F,μ) is given by

$$\mu = \frac{1}{3}\mathcal{H}^1\lfloor F\,, \quad F = \bigcup_{k=1}^{3} F_k\,, \quad F_k := \{y \in Y:\ y_i = \tfrac{1}{2},\ i \neq k\}. \tag{5.105}$$

We already know that the measure μ defined in (5.105) is strongly 2-connected on $\mathbb{R}^3$ and doubling. Figure 5.9 below comprises one way of approximating the thin structure, namely the cubic fattening introduced in (5.107) below, comprising the parallel-epipeds F_k^δ centered around F_k with cross section $4\delta^2$. The cubic fattening has some advantages in notation, we could as well consider structures of cylindrical shape.

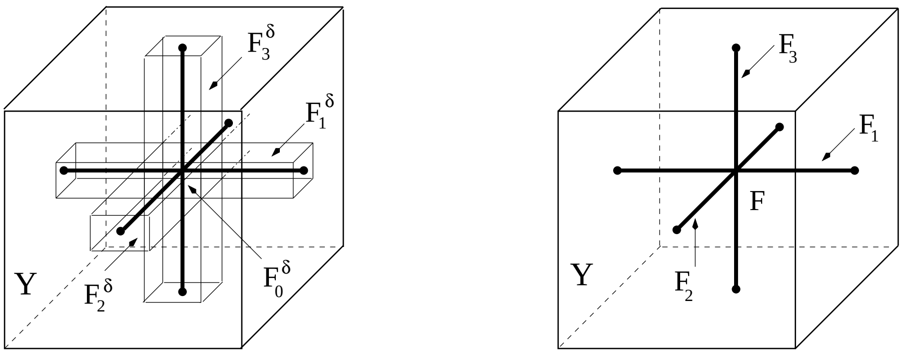


Figure 5.9: Network in 3D and fattened structure.

Note that by $F_0^\delta := \cap_i F_i^\delta$ we have denoted the intersection cube whose size is of order δ^3. More generally, we can define connected 1D-networks embedded in $\mathbb{R}^3$ as follows:

Definition 5.2.17. *We call (F,μ) a connected periodic network on $\mathbb{R}^3$, if $F = \operatorname{spt}\mu$ is a Y-periodic subset of $\mathbb{R}^3$, and $F \cap Y$ the finite union of straight segments F_k contained in Y, such that*

$$\mu\lfloor Y \;=\; c\sum_k \mathcal{H}^1\lfloor F_k \tag{5.106}$$

is strongly 2-connected on $\mathbb{R}^3$, where c is the normalizing constant.

The measure defined in (5.105) is a perfect example of such a connected network, more complex admissible structures are sketched in Figure 5.10:

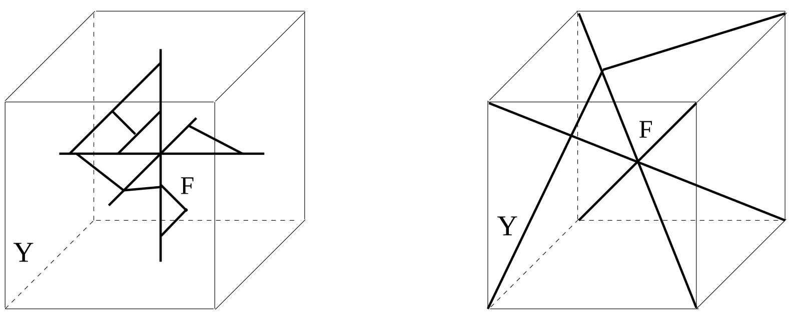


Figure 5.10: Connected periodic networks in $\mathbb{R}^3$.

Similar as in the last paragraph, given a connected network (F,μ) on $\mathbb{R}^3$ according to Definiton 5.2.17 with m segments F_k, where $m \geq 3$, we can define a suitable fattened structure (F^δ, μ^δ) by

$$F^\delta = \bigcup_{l \in \mathbb{Z}^3} \left(l + \cup_{k=1}^m F_k^\delta \right), \quad F_k^\delta := \{ y \in \mathbb{R}^3 : \text{ dist } (y, F_k) < \delta \} \cap Y, \quad (5.107)$$

where this time the distance is measured in the maximium norm on $\mathbb{R}^3$, that means

$$\text{dist } (y, z) := \max_i |y_i - z_i| \quad \text{for } z \in F_k.$$

This results a cubic channel structure as sketched in Figure 5.9. The measure μ^δ corresponding to the fattened set F^δ is, on the unit cell, chosen as

$$\mu^\delta \lfloor Y = c_\delta \mathcal{L}^3 \lfloor (F^\delta \cap Y), \quad c_\delta = |F^\delta \cap Y|^{-1}, \qquad (5.108)$$

and periodically extended to $\mathbb{R}^3$ with support F^δ as in (5.107). In particular, μ^δ is of type (5.21) and doubling, and there holds $\mu^\delta \to \mu$, which can be checked precisely as in the 2D-case. The following crucial properties of the sequence μ^δ can also be saved:

Lemma 5.2.18. *The measures μ^δ defined in (5.108) are strongly 2-connected on $\mathbb{R}^3$ and the sequence $\mu^\delta \to \mu$ enjoys the strong approximability property.*

Sketch of proof. The strong connectedness for fixed $\delta > 0$ can be proven precisely as in Lemma 5.2.13. Then we characterize a given vector function $v \in V^2_{\text{sol}}(\mathbb{T}, d\mu)$ as to be tangential to the singular structure and constant on each segment, satisfying the Kirchhoff law in each knot. The strong approximability property can then be shown by solving the Laplace equation with Neumann boundary values on suitable Lipschitz domains in a neighbourhood of each knot, just as in the proof of Lemma 5.2.14. We content ourselves with a proof for the measure given by (5.105). In this case a solenoidal vector v is of the form $v|_{F_k} = \lambda_k \vec{e}_k$ with arbitrary $\lambda_k \in \mathbb{R}$. Then with the notation of Figure 5.9, the function

$$v_\delta = \begin{cases} \lambda_k \vec{e}_k & \text{on} \quad F_k^\delta \setminus F_0^\delta \\ \sum_k \lambda_k \vec{e}_k & \text{on} \quad F_0^\delta \end{cases}$$

belongs to $V^2_{\text{sol}}(\mathbb{T}, d\mu^\delta)$ and there holds $v_\delta \to v$ strongly in $L^2(Y, d\mu^\delta)$. □

Thanks to Lemma 5.2.18, the commutativity of the two-parameter diagram also holds in the 3D-case. Again we implicitly presume a regular fattening ansatz, such as in (5.107).

Corollary 5.2.19. *The statement of Theorem 5.2.16 is valid for connected periodic networks (F, μ) on $\mathbb{R}^3$, that means the limit functions $u^0, u_0 \in H_0^1(\Omega)$ obtained in (5.2) both solve the effective problem (P) defined above.*

Sketch of proof. By Lemma 5.2.18 and the previous investigations it clearly suffices to consider the asymptotics $\delta \to 0$ for fixed $\varepsilon > 0$. To this end we have to adapt the proof of Lemma 5.2.15 to the 3D-case. In order to characterize the weak two-scale limits in (5.78), we can again assume that u_ε^δ is smooth and that $\Omega = (0,1)^3$, as well as $\varepsilon = n^{-1}$ for some $n \in \mathbb{N}_+$. It also suffices to consider the measure μ defined in (5.105). If the knots of the skeleton are labeled by $(x_i, y_j, z_k)_{i,j,k=1,\ldots,n}$, we can introduce the segments and the corresponding fattened cubes parallel to the first coordinate axis by

$$R_{jk} := (0,1) \times \{y_j\} \times \{z_k\}, \tag{5.109}$$

$$R_{jk}^\delta := (0,1) \times \{y_j - \tfrac{\delta}{n}, y_j + \tfrac{\delta}{n}\} \times \{z_k - \tfrac{\delta}{n}, z_k + \tfrac{\delta}{n}\}, \tag{5.110}$$

and similar C_{ik} and Z_{ij} for the segments parallel to the other two axises, together with C_{ik}^δ and Z_{ij}^δ. Introducing, as in (5.84), the auxiliary functions

$$v_{jk}^\delta \in H_0^1(R_{jk}), \quad v_{jk}^\delta : x \mapsto \fint_{y_j-\frac{\delta}{n}}^{y_j+\frac{\delta}{n}} \fint_{z_k-\frac{\delta}{n}}^{z_k+\frac{\delta}{n}} u_\varepsilon^\delta(x,y,z)\, dydz, \tag{5.111}$$

we can argue precisely as in the proof of Lemma 5.2.15 and deduce that $u_\varepsilon \in H_\varepsilon$ and that $(u_\varepsilon, \Phi_\varepsilon) \in V^2(\Omega, d\mu_\varepsilon)$. Using the fact that u_ε is in H^1 on the 1D-skeleton, and combining the sequence of continuous embeddings and trace extension operators in (5.93) with the further sequence

$$W^{3/4,4}(\partial\mathcal{O}) \hookrightarrow W^{5/8,4}(\partial\mathcal{O}) \to W^{7/8,4}(\mathcal{O}) \hookrightarrow \mathcal{C}^{0,1/8}(\overline{\mathcal{O}}), \tag{5.112}$$

where $\mathcal{O} := (x_i, x_{i+1}) \times (y_j, y_{j+1}) \times (z_k, z_{k+1})$ denote the 3D-cubes contained in Ω, we see that we can find a Hölder continuous extension

$$\hat{u}_\varepsilon : \Omega \to \mathbb{R}, \quad \hat{u}_\varepsilon = u_\varepsilon \ \text{ on } F_n := \bigcup_{i,j,k=1}^n (R_{jk} \cup C_{ik} \cup J_{ij}) \tag{5.113}$$

of u_ε to the whole of Ω and complete the proof as in the 2D-case. □

5.2.3 Counterexamples to noncommutativity

We have seen that one can expect the commutativity of the fattening process for a large class of quasilinear elliptic equations, provided the underlying singular structure is strongly connected on $\mathbb{R}^d$ (and sufficiently regular). In this final paragraph we will construct a counterexample, where the limits do not commute. To this end we choose a 1D-structure (S, μ_1) of vanishing capacity embedded in $\mathbb{R}^3$, and consider the nonconnected measure $\mu = \mu_1 \lfloor (S \cap Y) + \mathcal{L}^3 \lfloor Y$. Roughly speaking, the reason for noncommutativity is that the energy stored in the δ-thickened connected structure S_δ survives in the cell problems corresponding to (P_{hom}^δ), whereas it can get lost in the singular problem $(P_\varepsilon^{\text{sing}})$ comprising the nonconnected structure. For simplicity, we will consider the semilinear equation

$$(Q_\varepsilon^\delta) \qquad -\operatorname{div}(\nabla u_\varepsilon^\delta) + \lambda u_\varepsilon^\delta = f_\delta(\tfrac{x}{\varepsilon}, u_\varepsilon^\delta), \quad u_\varepsilon^\delta \in H_\varepsilon^\delta := H_0^{1,2}(\Omega, d\mu_\varepsilon^\delta)$$

subject to a suitable source term, where $\lambda > 0$ is fixed and the parameter δ characterizes the approximating connected structure. Again we consider the asymptotics $\varepsilon \to 0$ and $\delta \to 0$ independently with the other parameter being fixed, and then compare the two functions obtained by the different limit procedures in (5.2). An alternative approach is to consider the simultaneous limit $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and investigate the dependence of the effective problem on the velocity of the convergence to zero of $\delta(\varepsilon)$. The latter is done in [14], where the authors showed that the limit energy for the sequence of functionals

$$J(u_\varepsilon^\delta) = \frac{1}{p}\int_\Omega |\nabla u_\varepsilon^\delta|^p \, d\mu_\varepsilon^\delta$$

can in the nonconnected case in fact depend on this velocity. Now we introduce the nonconnected structure comprising the 1D-wire of Figure 5.9 combined with the surrounding 3D-Lebesgue measure, that means we define

$$\mu\lfloor Y = \tfrac{1}{2}\mathcal{L}^3\lfloor Y + \tfrac{1}{6}\mathcal{H}^1\lfloor F\,, \quad \mu(Y) = 1\,, \tag{5.114}$$

where F is the skeleton defined in (5.105). As discussed in Example 2.3.9 on page 33, the measure μ is not even weakly 2-connected on $\mathbb{T}$, and hence our methods used in the previous paragraphs do not apply. We consider the cubic fattening F^δ of the skeleton F defined in (5.107). We choose μ^δ to be absolutely continuous with respect to the $\mathcal{L}^3$-measure by setting

$$d\mu^\delta = \varrho_\delta \, dy\,, \quad \varrho_\delta(y) = \begin{cases} c_\delta^{\mathcal{L}} & y \in Y \setminus F^\delta, \\ c_\delta^{\mathcal{H}} & y \in F^\delta, \end{cases} \qquad \mu^\delta(Y \setminus F^\delta) = \mu^\delta(F^\delta) = \frac{1}{2}, \tag{5.115}$$

where in order to guarantee the normalization the constants defining the density ϱ_δ are chosen as

$$(c_\delta^{\mathcal{L}})^{-1} = 2 - 8\delta^2(3 - 4\delta)\,, \quad (c_\delta^{\mathcal{H}})^{-1} = 8\delta^2(3 - 4\delta).$$

It is easy to check that $\mu^\delta \rightharpoonup \mu$, and that each μ^δ is doubling and strongly 2-connected on $\mathbb{R}^3$, however with the $(H4)$-constant exploding as $\delta \to 0$. For the sources we choose the μ^δ-measurable and Y-periodic (with respect to the y-variable) functions

$$f_\delta : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}\,, \quad f_\delta(y, s) = \begin{cases} f_{\mathcal{L}}(s) & y \in Y \setminus F^\delta, \\ f_{\mathcal{H}}(s) & y \in F^\delta, \end{cases} \tag{5.116}$$

where $f_{\mathcal{L}}, f_{\mathcal{H}} : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, monotonically nonincreasing, and satisfy the sublinear growth condition

$$|f_i(s)| \le C(1 + |s|^\beta)\,, \quad \beta \in [0, 1)\,, \quad i = \mathcal{L}, \mathcal{H}. \tag{5.117}$$

As a consequence, the prerequisites of Assumption 5.2.4 on the sequence of sources $\{f_\delta\}$ is satisfied, and there holds, for any fixed $s \in \mathbb{R}$:

$$f_\delta(\cdot, s) \to f(\cdot, s) \text{ strongly in } L^2(Y, d\mu^\delta)\,, \quad f(y, s) := \begin{cases} f_{\mathcal{L}}(s) & y \in Y \setminus F, \\ f_{\mathcal{H}}(s) & y \in F. \end{cases} \tag{5.118}$$

Remark 5.2.20. *For any $\varepsilon, \delta > 0$ there exists a unique solution $u_\varepsilon^\delta \in H_\varepsilon^\delta$ of problem (Q_ε^δ) satisfying the uniform a priori estimate*

$$\|u_\varepsilon^\delta\|_{H_\varepsilon^\delta} \le C \tag{5.119}$$

with a constant independent of δ and ε, where the Hilbert space H_ε^δ and its norm are defined in (5.34) on page 128.

Proof. Existence and estimate (5.119) can be shown precisely as in Corollary 5.2.6, where one has to use $\mu_\varepsilon^\delta(\Omega) \le C$ uniformly. Uniqueness follows by testing the weak formulation

$$\forall \varphi \in \mathcal{D}(\Omega): \quad \int_\Omega (\nabla u_\varepsilon^\delta \cdot \nabla \varphi + \lambda u_\varepsilon^\delta \varphi)\, d\mu_\varepsilon^\delta \;=\; \int_\Omega f_\delta(\tfrac{x}{\varepsilon}, u_\varepsilon^\delta)\varphi \, d\mu_\varepsilon^\delta \tag{5.120}$$

respectively with the difference of two solutions and using the monotonicity of f_δ with respect to the second variable. $\square$

We first consider the asymptotics $\varepsilon \to 0$ for fixed δ. This is not a big deal, since μ^δ is strongly 2-connected on $\mathbb{R}^3$. It turns out to be essential to find an explicit lower bound on the effective coefficient $K_\delta^\star$. Also note that for the averaged sources $\bar{f}_\delta, \bar{f} : \mathbb{R} \to \mathbb{R}$ there holds

$$\bar{f}_\delta(s) = \bar{f}(s) = \tfrac{1}{2}(f_{\mathcal{L}}(s) + f_{\mathcal{H}}(s)). \tag{5.121}$$

Lemma 5.2.21. *Let $\{u_\varepsilon^\delta\}_{\varepsilon>0}$ be the sequence of solutions of problem (Q_ε^δ). Then there holds $u_\varepsilon^\delta \twoheadrightarrow u^\delta$ two-scale strongly in $L^2(\Omega, d\mu_\varepsilon^\delta)$ as $\varepsilon \to 0$, where $u^\delta \in H_0^1(\Omega)$ is the unique solution of the problem*

$$(Q^\delta) \qquad -\operatorname{div}(K_\delta^\star \nabla u^\delta) + \lambda u^\delta \;=\; \bar{f}(u^\delta) \quad \text{in } \Omega\,, \quad u^\delta \;=\; 0 \quad \text{on } \partial\Omega.$$

Moreover $K_\delta^\star = a_\delta^\star\, \mathbb{E}_3$, where $\mathbb{E}_3$ denotes the unit in $\mathcal{M}_{\mathrm{sym}}^3$, and there holds

$$\frac{2}{3} \le \frac{1}{2(3-4\delta)} + \frac{1-4\delta^2}{2(1-4\delta^2(3-4\delta))} \le a_\delta^\star \le 1 \quad \forall\, \delta \in (0, \tfrac{1}{4}]. \tag{5.122}$$

Proof. The first statement exactly coincides with Lemma 5.2.9, where in the definition of problem (Q^δ) we have used (5.121). The unique solvability will follow from (5.122) and the monotonicity of $\bar{f}$. In order to determine the effective coefficient we have to consider the cell problems

$$(A_k^\delta) \qquad \operatorname{div}(\nabla \Lambda_{\delta,k}(y) + \vec{e}_k) \;=\; 0 \quad \text{in } Y, \quad \Lambda_{\delta,k} \in W_\mu^\delta\,, \tag{5.123}$$

where the Hilbert space W_μ^δ is defined as in (5.37) on page 129. Again, by the definition on μ^δ, there exists a unique solution of problem (A_k^δ), satisfying the estimate

$$\|\Lambda_{\delta,k}\|_{W_\mu^\delta} \;=\; \|\nabla \Lambda_{\delta,k}\|_{2,\mu^\delta,Y} \;\le\; 1 \tag{5.124}$$

for each δ, which can be obtained by testing (5.123) with the solution $\Lambda_{\delta,k}$. As usual, the effective coefficient $K^\star_\delta \in \mathcal{M}^3_{\rm sym}$ reads

$$(K^\star_\delta)_{ij} = \int_Y (\vec{e}_i + \nabla\Lambda_{\delta,i}(y)) \cdot (\vec{e}_j + \nabla\Lambda_{\delta,j}(y))\, d\mu^\delta(y), \quad i,j = 1,\dots,3. \quad (5.125)$$

By the symmetry of ϱ_δ it is easy to check that $\tilde{\Lambda}_{\delta,1}(y) := \Lambda_{\delta,1}(y_1, 1-y_2, y_3)$ is a solution of problem (A^δ_1) as well, and hence the functions $\tilde{\Lambda}_{\delta,1}$ and $\Lambda_{\delta,1}$ coincide. It follows that

$$\begin{aligned}(K^\star_\delta)_{12} &= \int_Y \partial_2\Lambda_{\delta,1}\, \varrho_\delta\, dy = \int_Y \partial_2\Lambda_{\delta,1}(y_1, 1-y_2, y_3)\varrho_\delta(y_1, 1-y_2, y_3)\, dy \\ &= -\int_Y \partial_2\tilde{\Lambda}_{\delta,1}\, \varrho_\delta\, dy = -\int_Y (\vec{e}_1 + \nabla\tilde{\Lambda}_{\delta,1}) \cdot \vec{e}_2\, d\mu^\delta = -(K^\star_\delta)_{12},\end{aligned}$$

and hence $(K^\star_\delta)_{12} = 0$. On the other hand, it is easy to check that

$$(K^\star_\delta)_{11} = \int_Y (\vec{e}_1 + \nabla\Lambda_{\delta,1}(y)) \cdot \vec{e}_1\, d\mu^\delta = 1 - \|\nabla\Lambda_{\delta,1}\|^2_{2,\mu^\delta,Y} \le 1,$$

where we used the normalization of μ^δ and the solution property of $\Lambda_{\delta,1}$. By the symmetry of the problem we obtain

$$K^\star_\delta = a^\star_\delta\, \mathbb{E}_3\,, \quad 0 \le a^\star_\delta \le 1\,. \quad (5.126)$$

It remains to show the lower bound in (5.122). To this end we define an auxiliary measure $\tilde{\mu}^\delta$ by

$$d\tilde{\mu}^\delta = \tilde{\varrho}_\delta(y)\, dy\,, \quad \tilde{\varrho}_\delta(y) = \begin{cases} c^{\mathcal{L}}_\delta & y \in Y \setminus F^\delta_1, \\ c^{\mathcal{H}}_\delta & y \in F^\delta_1. \end{cases}$$

An easy calculation shows that $\tilde{\varrho}_\delta \le \varrho_\delta$ everywhere in Y whenever $\delta \in (0, \frac{1}{4}]$. Using the variational formulation of the first cell problem corrresponding to the measure $\tilde{\mu}^\delta$, we obtain

$$\min_{\phi \in W^\delta_{\tilde{\mu}}} \|\vec{e}_1 + \nabla\phi\|^2_{2,\tilde{\mu}^\delta,Y} = \|\vec{e}_1\|^2_{2,\tilde{\mu}^\delta,Y} = \frac{1}{2(3-4\delta)} + \frac{1-4\delta^2}{2(1-4\delta^2(3-4\delta))}\,.$$

It is important to note that by adding a suitable constant C_δ, the function $\Lambda_{\delta,1} + C_\delta$ belongs to $W^\delta_{\tilde{\mu}}$, and hence by the above equality we obtain

$$\begin{aligned} a^\star_\delta &= \int_Y \varrho_\delta(y)|\vec{e}_1 + \nabla\Lambda_{\delta,1}(y)|^2\, dy \ge \int_Y \tilde{\varrho}_\delta(y)|\vec{e}_1 + \nabla\Lambda_{\delta,1}(y)|^2\, dy \\ &\ge \int_Y \tilde{\varrho}_\delta(y)|\vec{e}_1|^2\, dy = \frac{1}{2(3-4\delta)} + \frac{1-4\delta^2}{2(1-4\delta^2(3-4\delta))} =: g(\delta),\end{aligned}$$

and it is easy to calculate that g is monotonically increasing on $[0, \frac{1}{4}]$ with $g(0) = 2/3$. $\square$

Thanks to the uniform estimate (5.122) and the growth condition (5.117) imposed on the source f, we can show that the sequence $\{u^\delta\}$ of solutions of problem (Q^δ) is bounded in $H^1(\Omega)$, where we refer to the absorption techniques in the proof of (3.22) on page 50. Moreover there exists a subsequence, still denoted by δ, such that

$$a^\star_\delta \to a^\star \in [\tfrac{2}{3}, 1] \quad \text{as } \delta \to 0. \tag{5.127}$$

The selection of a subsequence is necessary, since we have not shown a monotone dependence of $a^\star_\delta$ on δ. It suffices to consider from now on only the subsequence selected in (5.127). The following statement easily follows.

Corollary 5.2.22. *Let u^δ be the solution of problem (Q^δ) and $a^\star \in [\frac{2}{3}, 1]$ the limit in (5.127). Then there holds $u^\delta \rightharpoonup u^0$ weakly in $H^1(\Omega)$, where the function $u^0 \in H^1_0(\Omega)$ is the unique solution of the problem*

$$(Q^0) \qquad -a^\star \Delta u^0 + \lambda u^0 \;=\; \tfrac{1}{2}\left(f_{\mathcal{L}}(u^0) + f_{\mathcal{H}}(u^0)\right) \quad \textit{in } \Omega\,, \quad u^0 \;=\; 0 \quad \textit{on } \partial\Omega.$$

Now we consider the asymptotics $\delta \to 0$ for fixed $\varepsilon > 0$. It is easy to check that there holds

$$\mu^\delta_\varepsilon \rightharpoonup \mu_\varepsilon = \tfrac{1}{2}\mathcal{L}^3\lfloor\Omega \;+\; \tfrac{1}{6}\varepsilon^2\mathcal{H}^1\lfloor(\Omega \cap \varepsilon F) \quad \text{as } \delta \to 0, \tag{5.128}$$

where we have extended F by Y-periodicity to $\mathbb{R}^3$. Using the uniform a priori estimate (5.119) and the techniques familiar from the proofs of Lemma 5.2.15 and Corollary 5.2.19 respectively, we can show that there exists a function $u_\varepsilon \in H_\varepsilon := H^{1,2}_0(\Omega, d\mu_\varepsilon)$, such that

$$u^\delta_\varepsilon \rightharpoonup u_\varepsilon\,, \quad \nabla u^\delta_\varepsilon \rightharpoonup \nabla_{\mu_\varepsilon} u_\varepsilon \quad \text{as } \delta \to 0 \tag{5.129}$$

in the sense of (5.20). As usual, the fact that the limit gradient is tangential on the singular structure will be justified a posteriori using the solution property of u_ε below. Moreover, the selection of a subsequence is not necessary, since we will show that u_ε is the unique solution of the problem

$$(Q_\varepsilon) \qquad -\operatorname{div}(\nabla_{\mu_\varepsilon} u_\varepsilon) + \lambda u_\varepsilon \;=\; f(\tfrac{x}{\varepsilon}, u_\varepsilon)\,, \quad u_\varepsilon \in H^{1,2}_0(\Omega, d\mu_\varepsilon).$$

Indeed, as far as uniqueness is concerned we can test the weak formulation of the problem

$$\int_\Omega (\nabla_{\mu_\varepsilon} u_\varepsilon \cdot \nabla\varphi + \lambda u_\varepsilon \varphi)\, d\mu_\varepsilon \;=\; \int_\Omega f(\tfrac{x}{\varepsilon}, u_\varepsilon)\varphi\, d\mu_\varepsilon \quad \forall \varphi \in \mathcal{D}(\Omega) \tag{5.130}$$

with the difference of two solutions $u_\varepsilon, v_\varepsilon \in H_\varepsilon$ respectively, and use the monotonicity of $f(\cdot, s)$ to show that $u_\varepsilon = v_\varepsilon$. For what follows we introduce, in the classical setting, the semilinear equation

$$(A_\varepsilon) \qquad -\Delta v_\varepsilon + \lambda v_\varepsilon \;=\; f_{\mathcal{L}}(v_\varepsilon) \quad \text{in } \Omega, \quad v_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

Lemma 5.2.23. *The weak limit $u_\varepsilon \in H_\varepsilon$ of the sequence of solutions $\{u_\varepsilon^\delta\}_{\delta>0}$ of problem (Q_ε^δ) according to (5.129) is the unique weak solution of problem (Q_ε) in the sense of (5.130). In particular, if $f_{\mathcal{H}}(0) = 0$, then there holds*

$$H_\varepsilon \ni u_\varepsilon = \begin{cases} v_\varepsilon(x) & \text{if } x \in \Omega \setminus \varepsilon F \\ 0 & \text{if } x \in \Omega \cap \varepsilon F, \end{cases} \tag{5.131}$$

where $v_\varepsilon \in H_0^1(\Omega)$ is the unique solution of problem (A_ε).

Proof. The function u_ε defined in (5.131) belongs indeed to the Dirichlet space H_ε (cf. Example 2.3.9) due to the vanishing capacity of $\Omega \cap \varepsilon F$. When passing to the limit $\delta \to 0$ in the weak formulation (5.120), it suffices to show by (5.129) that

$$f_\delta(\tfrac{x}{\varepsilon}, u_\varepsilon^\delta) \rightharpoonup f(\tfrac{x}{\varepsilon}, u_\varepsilon) \quad \text{weakly in } L^2(\Omega, d\mu_\varepsilon^\delta) \tag{5.132}$$

as $\delta \to 0$. For convenience, we sketch the proof of (5.132) for the critical term corresponding to the skeleton structure, where we use the notation introduced in (5.109)-(5.111) on page 152, and where $w = (x, y, z)$ denotes a point in $\mathbb{R}^3$:

$$\begin{aligned} c_\delta^{\mathcal{H}} \int_{R_{jk}^\delta} f_{\mathcal{H}}(u_\varepsilon^\delta)(w)\varphi(w)\,dw \quad &= \quad c_\delta^{\mathcal{H}} \int_{R_{jk}^\delta} f_{\mathcal{H}}(v_{jk}^\delta)(x)\varphi(w)\,dw \;+\; o(1) \\ &\to \quad \tfrac{1}{6}\varepsilon^2 \int_{R_{jk}} f_{\mathcal{H}}(u_\varepsilon)(x)\varphi(x, y_j, z_k)\,dx \end{aligned}$$

as $\delta \to 0$, where we use the Lipschitz continuity of $f_{\mathcal{H}}$ and the techniques familiar from the proof of Lemma 5.2.15. $\square$

The following result is a direct consequence of the characterization of u_ε in (5.131) and the application of Theorems 2.4.4– 2.4.5 to the $\mathcal{L}^3$-component of μ. We also have to use that the sequence $\{v_\varepsilon\}$ of solutions of problem (A_ε) is uniformly bounded in $H^1(\Omega)$, which is guaranteed by the assumptions on the source $f_{\mathcal{L}}$. We introduce the second limit problem

$$(Q_0) \qquad -\Delta u_0 + \lambda u_0 \;=\; f_{\mathcal{L}}(u_0) \;\text{ in } \Omega\,, \quad u_0 \;=\; 0 \;\text{ on } \partial\Omega.$$

Also recall the definition $m = (\mathcal{L}^d \lfloor \Omega) \otimes (\mu \lfloor Y)$ of the product measure.

Corollary 5.2.24. *If $f_{\mathcal{H}}(0) = 0$, then the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of solutions of problem (Q_ε) two-scale strongly converges to the function*

$$L_m^2(\Omega \times Y) \ni u(x, y) = \begin{cases} u_0(x) & y \in Y \setminus F, \\ 0 & y \in F, \end{cases} \tag{5.133}$$

where $u_0 \in H_0^1(\Omega)$ is the unique solution of problem (Q_0).

Lemma 5.2.23 and Corollary 5.2.24 illustrate that whenever the underlying (nonconnected) measure μ is a sum of m strongly connected periodic measures μ_i with $\operatorname{spt}\mu_i = F_i$ and $\mu_i(F_j) = \delta_{ij}$, the two-scale limit $u(x,y)$ of the sequence $\{u_\varepsilon\}$ of solutions of the singular ε-microscale problem is of the form

$$u(x,y) = \sum_{i=1}^{m} \chi_{F_i}(y)\, u_i(x)\,, \quad u \in L^2_m(\Omega \times Y). \tag{5.134}$$

In the investigations above we assumed for simplicity that $f_{\mathcal{H}}(0) = 0$. In general, the component $u(\cdot, y)$ of u belonging to $y \in F$ in (5.133) solves a nontrivial semilinear equation as well. As long as we require $f_{\mathcal{L}} \neq f_{\mathcal{H}}$ and $f_{\mathcal{L}}(0) \neq 0$ it is obvious that $u^0 \neq u_0$, even for the case $f_{\mathcal{H}} \equiv 0$ since $2a^\star > 1$. However, in order to state the noncommutativity we have to compare the function $u^0(x)$ obtained from the first limit procedure in (5.2) with the two-scale limit u in (5.134). The point is that we should make sure that u^0 is not a convex (or linear) combination of the components $u_i(x)$ of $u(x,y)$, which in our case reduces to u^0 not being a scalar multiple of u_0. This result is clearly stronger than the simple observation $u^0 \neq u_0$.

Theorem 5.2.25. *If the singular limit structure is not (weakly)* 2*-connected, in the semilinear case the two-parameter diagram does in general not commute. Moreover, if we choose* $f_{\mathcal{H}}(0) = 0$ *in our counterexample, there exist admissible functions* $f_{\mathcal{L}}$*, such that*

$$\nexists s \in \mathbb{R} : \quad u_0 = s u^0 \quad \text{in } \Omega. \tag{5.135}$$

Proof. We can assume $f_{\mathcal{H}} \equiv 0$ and choose $f_{\mathcal{L}} \in \mathcal{C}^1(\mathbb{R})$ to be strictly positive. This implies that both functions u^0, u_0 belong to $\mathcal{C}(\overline{\Omega})$, are nonnegative and not identical to zero. Due to the boundary condition it is easy to calculate that the equality in (5.135) can only hold for $s = 2a^\star$. However, in this case choosing λ sufficiently small and $f_{\mathcal{L}}(s) = 1 - s$ in a neighbourhood of zero, we deduce $u^0 \equiv 0$ in Ω, which is a contradiction. $\square$

6 Appendix

Before we prove or merely quote some theorems and technical lemmas needed in the text, we first review in telegram style some notation, which is valid throughout the whole thesis.

6.1 Notation

(i) **Notation for matrices**

- We write $A = (A_{ij})$ to mean that A is a $d \times d$ matrix with $(i,j)^{\text{th}}$ entry A_{ij}. The transpose of a matrix A we denote by A^t.
- A diagonal matrix is denoted by $\operatorname{diag}(\Theta_1, \ldots, \Theta_d)$, and we set $\mathbb{E}_d := \operatorname{diag}(1, \ldots, 1)$.
- $\mathcal{M}^d_{\text{sym}}$ is the space of real symmetric $d \times d$ matrices.
- If $A \in \mathcal{M}^d_{\text{sym}}$ and, as below, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we write for the corresponding quadratic form $x \cdot Ax = \sum_{i,j=1}^d A_{ij} x_i x_j$.

(ii) **Geometric notation**

- By $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ we denote respectively the set of natural, integer-valued, rational and real numbers, $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$.
- By $\mathbb{R}^d$ we denote the d-dimensional real Euclidean space. A typical point in $\mathbb{R}^d$ is $x = (x_1, \ldots, x_d)$ and $\mathcal{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$.
- $\vec{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the k^{th} unit vector in $\mathbb{R}^d$.
- Ω and D usually denote open subsets of $\mathbb{R}^d$. We write $D \subset\subset \Omega$, if $D \subset \overline{D} \subset \Omega$ and $\overline{D}$ is compact.
- $B_R(x_0) = \{x \in X : \|x - x_0\| < R\}$ denotes the open ball with center $x_0 \in X$ and radius $R > 0$ in a normed vector space $(X, \|\cdot\|)$.
- We write $x \cdot y = \sum_{i=1}^d x_i y_i$ for the inner product in $\mathbb{R}^d$.

(iii) **Notation for measures and functions**

- By $\mathcal{B} = \mathcal{B}(X)$ we denote the σ-algebra of Borel sets of a metric space X. A locally finite measure $\mu : \mathcal{B} \to [0, \infty]$ is called a (positive) Borel measure.
- An inner regular Borel measure is called a (positive) Radon measure. A Radon measure defined on a domain $\Omega \subset \mathbb{R}^d$ can be uniquely identified with a (positive) linear functional on $\mathcal{C}_0(\Omega)$.
- By $\operatorname{spt} \mu$ we denote the support of a Radon measure μ, and by $\mu \lfloor \Omega$ the restriction of μ to the set Ω.
- By $\mathcal{L}^d$ and $\mathcal{H}^k$ we denote respectively the Lebesgue measure and the k-dimensional Hausdorff measure on $\mathbb{R}^d$.
- If $u : \Omega \to \mathbb{R}$, we write $u(x) = u(x_1, \ldots, x_d)$, $x \in \Omega$. We set $u := v$ to define u as equaling v. The support of u is denoted by $\operatorname{spt} u$.

- For the sign function and the indicator function χ_E we write

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases} \qquad \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

- For a Radon measure μ, we denote the average of a μ-measurable function f over a μ-measurable set E by

$$\fint_E f(x)\, d\mu(x) \;=\; \frac{1}{\mu(E)} \int_E f(x)\, d\mu(x).$$

(iv) **Classical function spaces**

- For $\Omega \subset \mathbb{R}^d$ open, we write $\mathcal{C}^k(\Omega), k \in \mathbb{N}$ for the space of k-times continuously differentiable functions on Ω, where $\mathcal{C}(\Omega) := \mathcal{C}^0(\Omega)$ is the space of continuous functions and $\mathcal{C}^\infty(\Omega) = \cap_{k=0}^\infty \mathcal{C}^k(\Omega)$.
- $\mathcal{C}^k(\overline{\Omega})$ comprises the functions $u \in \mathcal{C}^k(\Omega)$ with $\partial^\alpha u$ uniformly continuous on bounded subsets of Ω for all $|\alpha| \leq k$, $\mathcal{C}(\overline{\Omega}) := \mathcal{C}^0(\overline{\Omega})$.
- The Hölder space $\mathcal{C}^{k,\beta}(\overline{\Omega})$ for $k \in \mathbb{N}$ and $\beta \in (0, 1]$ consists of all functions $u \in \mathcal{C}^k(\overline{\Omega})$ with finite norm

$$\|u\|_{\mathcal{C}^{k,\beta}(\overline{\Omega})} := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha u(x)| + \sum_{|\alpha| = k} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\beta}.$$

- The space $\mathcal{C}_0^k(\Omega)$ denotes those functions in $\mathcal{C}^k(\Omega)$ with compact support, where $\mathcal{C}_0(\Omega) := \mathcal{C}_0^0(\Omega)$ and $\mathcal{D}(\Omega) := \mathcal{C}_0^\infty(\Omega)$. By $\mathcal{D}'(\Omega)$ we denote the space of distributions on Ω.
- $\mathcal{C}_{\mathrm{b}}^k(\mathbb{R})$ denotes the Banach space of k-times continuously differentiable functions on $\mathbb{R}$ with finite norm

$$\|u\|_{\mathcal{C}_{\mathrm{b}}^k} := \sum_{i=0}^k \sup_{x \in \mathbb{R}} |f^{(i)}(x)|.$$

(v) **Lebesgue and Sobolev spaces**

- For $p \in [1, \infty]$ and a Radon measure μ we denote by $L^p(\Omega, d\mu)$ the standard Lebesgue spaces, and $u \in L^p_{\mathrm{loc}}(\Omega, d\mu)$ if $u \in L^p(D, d\mu)$ for any $D \subset\subset \Omega$. For $\mu = \mathcal{L}^d$ we simply write $L^p(\Omega), L^p_{\mathrm{loc}}(\Omega)$.
- For $k \in \mathbb{N}, p \in [1, \infty]$ the Sobolev space $W^{k,p}(\Omega)$ comprises those functions in $L^p(\Omega)$, whose α-th weak derivatives belong to $L^p(\Omega)$ for all $|\alpha| \leq k$. $W_0^{k,p}(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$.
- For the fractional Sobolev spaces we write $W^{s,p}(\Omega), s > 0, p \geq 1$, and $H^s(\Omega) := W^{s,2}(\Omega)$, $H^{-s}(\Omega) := (H_0^s(\Omega))'$ for any $s > 0$.
- By $H_\mu^{1,p}(\mathbb{R}^d)$ and $H_0^{1,p}(\Omega, d\mu)$ we denote respectively the Sobolev spaces on $\mathbb{R}^d$ and on $\Omega \subset \mathbb{R}^d$ (with zero trace) with respect to a Radon measure μ, introduced in Section 2.2.

(vi) **Periodic spaces**

- Y is the unit cube of $\mathbb{R}^d$ and $\mathbb{T} := \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus. We identify functions on $\mathbb{T}$ with Y-periodic functions on $\mathbb{R}^d$.
- $\mathcal{C}^k(\mathbb{T}), k \in \mathbb{N}$ denotes the space of k-times continuously differentiable functions on $\mathbb{T}$, with $\mathcal{C}(\mathbb{T}) := \mathcal{C}^0(\mathbb{T})$. We also write $\mathcal{C}^k_{\mathrm{per}}(Y)$.
- For $p \in [1,\infty]$ and a Radon measure μ we denote by $L^p_\mu(\mathbb{T}), H^{1,p}_\mu(\mathbb{T})$ the Lebesgue and Sobolev spaces on the torus introduced in Section 2.2. For $\mu = \mathcal{L}^d$ we also write $L^p_{\mathrm{per}}(Y)$ and $H^{1,p}_{\mathrm{per}}(Y)$.

(vi) **Spaces involving time**

- For a Banach space X, the space $\mathcal{C}([0,T];X)$ consists of all continuous functions $u : [0,T] \to X$ with finite norm $\max_{0\le t\le T} \|u(t)\|_X$.
- The space $L^p(0,T;X)$ consists of all strongly measurable functions $u : [0,T] \to X$, with respectively finite norm

$$\|u\|^p_{L^pX} := \int_0^T \|u(t)\|^p_X \, dt\,, \quad \|u\|_{L^\infty X} := \operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\|_X .$$

- The space $W^{1,p}(0,T;X)$ consists of all functions $u \in L^p(0,T;X)$, such that $\partial_t u$ exists in the weak sense and belongs to $L^p(0,T;X)$. We write $H^1(0,T;X) := W^{1,2}(0,T;X)$.

(vii) **Miscellanea**

- For the dual pairing between a Banach space X and its dual X' we usually write $\langle\!\langle \cdot,\cdot \rangle\!\rangle$. For a subspace $X \subset Z$ of a Banach space Z, the annihilator is denoted by $X^\perp := \{\lambda \in Z' : \langle\!\langle \lambda, x \rangle\!\rangle = 0 \;\forall x \in X\}$.
- For a Radon measure μ and a function $f : Y \times X \to \mathbb{R}$ that is μ-measurable and Y-periodic in $y \in Y = (0,1)^d$, we write

$$\bar f : X \to \mathbb{R},\; x \mapsto \fint_Y f(y,x)\, d\mu(y).$$

- Given dual exponents $p, p' \in (1,\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$ we refer to Young's inequality as

$$\forall a,b > 0 : \quad ab \,\le\, \tfrac{1}{p} a^p + \tfrac{1}{p'} b^{p'}$$

- We write $f = o(g)$ as $x \to x_0$, provided $\lim_{x\to x_0} \frac{|f(x)|}{|g(x)|} = 0$.

6.2 Toolbox

We collect several results from functional analysis and the (regularity) theory of quasilinear elliptic and parabolic equations, to which we refered in the text. First we prove a useful inequality in $\mathbb{R}$, needed in Section 3.1.

Proposition 6.1. *The following inequalities hold for all $a, b \in \mathbb{R}$. If $p \geq 2$, then there holds*

$$2^{1-p}|a-b|^p \;\leq\; \left(|a|^{p-2}a - |b|^{p-2}b\right)(a-b) \;\leq\; (p-1)(|a|+|b|)^{p-2}(a-b)^2\,. \quad (6.1)$$

If $p \in (1,2)$ and $c := \min\{p-1, 2^{1-p}\} > 0$, then there holds

$$c\,(|a|+|b|)^{p-2}(a-b)^2 \;\leq\; \left(|a|^{p-2}a - |b|^{p-2}b\right)(a-b) \;\leq\; 2\,|a-b|^p\,. \quad (6.2)$$

Proof. We prove the statement for $p \geq 2$, the case $p < 2$ requires only minor changes. Concerning the second inequality in (6.1), note that the function $f : x \mapsto |x|^{p-2}x$ is differentiable with $f^{'}(x) = (p-1)|x|^{p-2}$ for each $x \in \mathbb{R}$. Given $a, b \in \mathbb{R}$, we obtain

$$|f(a) - f(b)| \leq \left(\sup_{x\in[a,b]} |f^{'}(x)|\right)|a-b| \leq (p-1)(|a|+|b|)^{p-2}|a-b|,$$

and hence the second estimate in (6.1). On the other hand, it is easy to check that

$$\forall x < 1 : \quad 2^{1-p}(1-x)^p \leq (1-|x|^{p-2}x)(1-x)\,. \quad (6.3)$$

It is no restriction to assume $a > b$ in order to prove the first inequality in (6.1). If $a > 0$, apply (6.3) to $x := a^{-1}b$ and multiply both sides with a^p, if $a < 0$, choose $x := b^{-1}a$ and multiply with $|b|^p$. □

Now we quote some well known facts from functional analysis, so we can renounce to give a reference. Recall that the dual pairing between a Banach space X and its dual $X^{'}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Lemma 6.2. *Let X be a Banach space and $X^{'}$ its dual. Then there holds*

- *For any $x \in X, x \neq 0$ there exists $x^{'} \in X^{'}$, such that*

$$\|x^{'}\|_{X^{'}} = 1 \;\text{ and }\; \langle\!\langle x^{'}, x \rangle\!\rangle = \|x\|_X\,.$$

- *If X is reflexive, then for any $x^{'} \in X^{'}$, $x^{'} \neq 0$ there exists $x \in X$, such that*

$$\|x\|_X = 1 \;\text{ and }\; \langle\!\langle x^{'}, x \rangle\!\rangle = \|x^{'}\|_{X^{'}}\,.$$

Lemma 6.3. *Let X be a reflexive Banach space and $u_n \rightharpoonup u$ weakly in $X^{'}$. Then the following two statements are equivalent:*

(i) $u_n \to u$ strongly in $X^{'}$.

(ii) $\varphi_n \rightharpoonup \varphi$ weakly in X $\Rightarrow$ $\langle\!\langle u_n, \varphi_n \rangle\!\rangle \to \langle\!\langle u, \varphi \rangle\!\rangle$.

Theorem 6.4 (Schauder)**.** *Let X be a Banach space, $M \subset X$ a bounded, closed and convex subset, and $T : M \to M$ continuous. If T is compact, that means $T(M)$ precompact in X, then T has a fixed point.*

Theorem 6.5 (Browder-Minty)**.** *Let X be a real reflexive Banach space with dual X'. An operator $T : X \to X'$ is bijective, provided it is*

- **strictly monotone**, *that means* $\langle\langle Tx_1 - Tx_2, x_1 - x_2\rangle\rangle > 0$ *for* $x_1 \neq x_2$,
- **hemicontinuous**, *that means* $\lim_{t\to 0}\langle\langle T(x_1 + tx), x_2\rangle\rangle = \langle\langle Tx_1, x_2\rangle\rangle$ *for all* $x, x_1, x_2 \in X$,
- *and* **coercive**, *that means* $\lim_{\|x\|\to\infty} \frac{\langle\langle Tx,x\rangle\rangle}{\|x\|} = +\infty$.

The reversed Hölder inequality introduced below may not be standard. It was essentially used when dealing with monotone elliptic operators subject to an exponent $p \in (1, 2)$. For a proof we refer to [37, Theorem 13.6].

Theorem 6.6. *Let $(\mathcal{S}, \mathcal{B}, \mu)$ be a complete, σ-finite measure space, $p \in (0, 1)$ and $q < 0$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f and g be nonnegative, μ-measurable functions satisfying $f \in L^p_\mu(\mathcal{S})$ and $g^{-1} \in L^{-q}_\mu(\mathcal{S})$. Then there holds*

$$\int_{\mathcal{S}} fg\, d\mu \;\geq\; \left(\int_{\mathcal{S}} f^p\, d\mu\right)^{1/p} \left(\int_{\mathcal{S}} g^q\, d\mu\right)^{1/q} \tag{6.4}$$

provided $\int g^q\, d\mu \neq 0$.

The following maximum regularity result for quasilinear scalar elliptic equations is taken from [12, Theorem 2.25].

Theorem 6.7. *Let $a : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, (s, \xi) \mapsto (a_i(s, \xi))_{i=1,\dots,d}$ be measurable and set*

$$a_{i,s} := \frac{\partial a_i}{\partial s}\,,\; a_{i,k} := \frac{\partial a_i}{\partial \xi_k}\,.$$

We assume that all $a_{i,s}$ are measurable, that all $a_{i,k}$ are globally continuous in all arguments, and that

$$|a_{i,s}(s, \xi)| \;\leq\; C(1 + |\xi|)\,, \tag{6.5}$$

$$|a_{i,k}(s, \xi)| \;\leq\; C, \quad \forall (s, \xi), \;\; \textit{with}\; |s| \leq R\,, \tag{6.6}$$

$$a_{i,k}(s, \xi)\lambda_i\lambda_k \;\geq\; \alpha|\lambda|^2, \quad \forall \lambda \in \mathbb{R}^d,\; \alpha > 0\,. \tag{6.7}$$

If in addition Ω is of class $\mathcal{C}^{2,\delta}$, then for any $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with

$$\operatorname{div} a(u, \nabla u) \in L^1(\Omega)\,, \quad |\operatorname{div} a(u, \nabla u)| \leq C(1 + |\nabla u|^2) \tag{6.8}$$

there holds $u \in W^{2,p}(\Omega)$ for any $1 \leq p < \infty$.

In Section 3.2 we investigated the regularity of cell solutions, taking advantage of the following result (we refer to [42, 43]) on $W^{1,\infty}$-estimates for divergence form elliptic equations with piecewise Hölder continuous coefficients.

Theorem 6.8. *Let $D \subset \mathbb{R}^d$ be a bounded domain containing L disjoint subdomains $D_1, \dots, D_L$ with $D = (\cup_n \overline{D}_n) \setminus \partial D$. Assume that every D_n is of class $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha \leq 1$ and that, whenever a point in D lies on ∂D_n, then the component of ∂D_n containing this point is smooth. Let $u \in H^1(D)$ be a weak solution of the elliptic equation*

$$-\mathrm{div}\,(A\nabla u) \;=\; h + \mathrm{div}\, g \quad in\ \mathcal{D}^{'}(D), \tag{6.9}$$

subject to the following assumptions on the data: $A \in L^\infty(D; \mathcal{M}^d_{\mathrm{sym}})$, $(h, g) \in L^\infty(D; \mathbb{R}^{d+1})$, and there exist $\beta \in (0, 1)$ and constants $c_1, c_2 > 0$, such that

$$c_1|\xi|^2 \;\leq\; \xi \cdot A(x)\xi \;\leq\; c_2|\xi|^2 \quad \forall x \in D, \xi \in \mathbb{R}^d\,,$$

$$(A, g)(x) = \sum_{n=1}^{L} \chi_{D_n}(x)(A_n, g_n)(x)\,, \quad where\ (A_n, g_n) \in \mathcal{C}^{0,\beta}(\overline{D}_n; \mathbb{R}^{d\times d} \times \mathbb{R}^d).$$

Set $\alpha' := \min\{\beta, \frac{\alpha}{2(\alpha+1)}\}$. Then for any $\varepsilon > 0$ there exists a constant C only depending on $L, d, \alpha, \beta, \varepsilon$, c_1, c_2, $\|A\|_{\mathcal{C}^{0,\alpha'}(\overline{D_n})}$ and the $\mathcal{C}^{1,\alpha'}$-norms of the D_n, such that

$$\|\nabla u\|_{L^\infty(D_\varepsilon)} \;\leq\; C\left(\|u\|_{L^2(D)} + \|h\|_{L^\infty(D)} + \sum_{n=1}^{L} \|g\|_{\mathcal{C}^{0,\alpha'}(\overline{D_n})}\right), \tag{6.10}$$

where $D_\varepsilon := \{x \in D :\ \mathrm{dist}(x, \partial D) > \varepsilon\}$.

Now we prove an embedding theorem related to spaces involving time. It was essentially used for the existence proof in Paragraph 4.2.1. Recall that for two Banach spaces X, Y the expression $X \subset\subset Y$ means that $X \subset Y$, and that any bounded subset of X is relatively strongly compact in Y.

Theorem 6.9. *Let X, Y, Z be Banach spaces, X and Z reflexive with $X \subset\subset Y \subset Z$ with continuous injections, and $p, q \in (1, \infty)$. Then there holds*

$$L^p(0, T; X) \cap W^{1,q}(0, T; Z) \quad \subset\subset \quad L^p(0, T; Y)\,, \tag{6.11}$$

$$L^p(0, T; X) \cap W^{1,q}(0, T; Z) \quad \subset\subset \quad \mathcal{C}([0, T]; Z)\,. \tag{6.12}$$

Proof. The first statement is known as the Lions-Aubin lemma, and we refer to [58, Theorem 11.3.5]. To prove (6.12), let (v_n) be a bounded sequence in $L^p(0, T; X) \cap W^{1,q}(0, T; Z)$. By choosing $Y = Z$ in (6.11) we get

$$v_n \to v \quad \text{strongly in } L^p(0, T; Z)\,, \quad v_n \rightharpoonup v \quad \text{weakly in } W^{1,q}(0, T; Z) \tag{6.13}$$

for a subsequence and a function $v \in L^p(0, T; Z) \cap W^{1,q}(0, T; Z)$. Since $q > 1$, there exists a positive Hölder exponent $\alpha > 0$, such that

$$\|v_n\|_{\mathcal{C}^{0,\alpha}([0,T];Z)} + \|v\|_{\mathcal{C}^{0,\alpha}([0,T];Z)} \;\leq\; C \tag{6.14}$$

with a constant C independent of n. Now for $\varepsilon > 0$ given and a positive number c_ε to be chosen appropriately, there exists a positive integer N_ε and finitely many closed intervals $I_i \subset \mathbb{R}$ with $|I_i| \leq c_\varepsilon$ for $i = 1, \ldots, N_\varepsilon$, such that

$$[0,T] \subset \bigcup_{i=1}^{N_\varepsilon} I_i\,,$$

and points $t_i \in I_i$, such that $\|v_n(t_i) - v(t_i)\|_Z \to 0$ for any i by (6.13). Hence we find $M \in \mathbb{N}$, such that

$$\forall n \geq M\ \forall i = 1, \ldots, N_\varepsilon : \quad \|v_n(t_i) - v(t_i)\|_Z < \frac{\varepsilon}{2}. \tag{6.15}$$

Now for $t \in [0,T]$ given, we find an index $i \in \{1, \ldots, N_\varepsilon\}$ depending on t, such that $t \in I_i$ and $|t - t_i| \leq c_\varepsilon$. Hence we get

$$\begin{aligned} \|v_n(t) - v(t)\|_Z &\leq \|v_n(t) - v_n(t_i)\|_Z + \|v_n(t_i) - v(t_i)\|_Z + \|v(t_i) - v(t)\|_Z \\ &\leq 2C(c_\varepsilon)^\alpha + \frac{\varepsilon}{2} \ < \ \varepsilon \end{aligned}$$

for all $n \geq M$ uniformly in $[0,T]$, if we choose $c_\varepsilon := (\frac{\varepsilon}{8C})^{1/\alpha}$, where α and C are the constants occurring in (6.14). □

For the classical case $\mu = \mathcal{L}^d$ we summarize, adapted to our framework, the existence and uniqueness results of [2] and [51] for the quasilinear elliptic-parabolic equation

$$(\star) \quad \left\{ \begin{aligned} \partial_t b(u) - \operatorname{div}[a(b(u), \nabla u)] &= g(b(u)) \quad \text{in } \Omega \times (0,T), \\ b(u) = b^0 \text{ in } \Omega \times \{0\}, \quad u &= 0 \quad \text{on } \partial\Omega \times (0,T)\,, \end{aligned} \right.$$

that are available under certain structure conditions on b, the nonlinear flux a and the source g. As usual we assume that $\Omega \subset \mathbb{R}^d$ is open, bounded and connected with smooth boundary. For the definition and the characterization of the positive functions Ψ and $B := \Psi \circ b$ we refer to (4.19) and (4.23).

Theorem 6.10. *Let $b : \mathbb{R} \to \mathbb{R}$ be monotone nondecreasing and continuous, let $\psi(b^0) \in L^1(\Omega)$ with $b^0 = b(u^0)$ for some measurable function u^0, and let $a : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy the following properties:*

1. *Natural growth: $\exists C < \infty$, such that for all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^d$:*

$$|a(b(z), \xi)| + |g(b(z))| \ \leq \ C(1 + B(z)^{\frac{1}{2}} + |\xi|).$$

2. *Strict monotonicity in ξ: $\exists c > 0$, such that for all $s \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^d$:*

$$(a(s, \xi_1) - a(s, \xi_2)) \cdot (\xi_1 - \xi_2) \ \geq \ c\,|\xi_1 - \xi_2|^2.$$

Then there exists a solution $u \in L^2(0,T; H^1_0(\Omega))$ of problem $(\star)$ in the sense of Definition 4.2.12 on page 108. For any solution there holds

$$B(u) \in L^\infty(0,T; L^1(\Omega))\,, \tag{6.16}$$

$$\int_\Omega B(u(\tau)) - \int_\Omega B(u^0) \; = \; \int_0^\tau \langle\!\langle \partial_t b(u), u \rangle\!\rangle \, dt \qquad \textit{for a.e. } \tau \in [0,T] \,, \tag{6.17}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ *denotes the dual pairing between* $H^{-1}(\Omega)$ *and* $H_0^1(\Omega)$. *If in addition* g *is Lipschitz continuous and* a *fulfills the following*

3. *Hölder condition:* $\exists C < \infty$, *such that for all* $z_1, z_2 \in \mathbb{R}, \xi \in \mathbb{R}^d :$

$$|a(b(z_1), \xi) - a(b(z_2), \xi)|^2 \; \le \; C|z_1 - z_2|(1 + |\xi|^2 + B(z_1) + B(z_2))$$

then the solution u *of* $(\star)$ *is unique.*

We quote an advanced result from convex analysis. It was used to proof the corrector results for Richards equation in Section 4.3. For a proof of the statement we refer to [58, Section 10.2].

Theorem 6.11. *Let* $h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be proper (i.e.* $h \not\equiv +\infty$*), strictly convex and lower semicontinuous. Define the functional*

$$\Phi : L^1(Q) \to \hat{\mathbb{R}} \,, \; v \mapsto \begin{cases} \int_Q h(v) \, dxdt & \textit{if } h(v) \in L^1(Q), \\ +\infty & \textit{otherwise,} \end{cases}$$

where Q *is a typical space-time cylinder. Then* Φ *is weakly lower semicontinuous and proper. For a sequence* $v_\varepsilon \rightharpoonup v$ *weakly in* $L^1(Q)$*, there holds*

$$\lim_{\varepsilon \to 0} \Phi(v_\varepsilon) = \Phi(v) \neq +\infty \;\Rightarrow\; h(v_\varepsilon) \to h(v) \;\textit{ and }\; v_\varepsilon \to v \;\textit{ strongly in } L^1(Q).$$

The last result (we refer to [19, Lemma A1]) considers multifunctions associated with stable spaces. It was used to construct tangent spaces associated with a positive Radon measure μ.

Lemma 6.12. *Let* μ *be a Radon measure on* $\mathbb{R}^d, p \in [1, \infty)$ *and* $\mathcal{V}$ *a linear subspace of* $(L^p_\mu)^d$. *Assume that the following stability property holds*

$$z \in \mathcal{V}, \; \psi \in \mathcal{D}(\mathbb{R}^d) \;\Rightarrow\; \psi z \in \mathcal{V} \,.$$

Then for any countable dense set $\{z_n : n \in \mathbb{N}\} \subset \mathcal{V}$, *we have*

$$\overline{\mathcal{V}} \; = \; \{z \in (L^p_\mu)^d : \, z(x) \in V(x) \; \mu\text{-a.e.}\}, \tag{6.18}$$

$$\mathcal{V}^\perp \; = \; \{z \in (L^{p'}_\mu)^d : \, z(x) \in V(x)^\perp \; \mu\text{-a.e.}\} \,, \tag{6.19}$$

where we set $V(x) := \overline{\{z_n(x) : \, n \in \mathbb{N}\}}$, *which is a linear subspace of* $\mathbb{R}^d$.

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