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On Some Classes of Continuous Time Series Models and Their Use in Financial Economics

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to Tina

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Summary

This thesis considers continuous-time series processes defined by classical stochastic differential equations and investigates some of their applications to mathematical finance with a focus on analytical approximations for some important financial quantities like derivative prices or firm default probability in a more complex and realistic framework than the one used by Black and Scholes (1973) and Merton (1973, 1974).

In the first part some classes of continuous-time series models are introduced, which are further used to construct new financial models for asset and volatility dynamics. We illustrate some of them with the aid of simulation and estimation examples.

In the second part of this thesis, we derive new results for derivative pricing and credit risk problems in stochastic volatility models driven by some continuous-time series processes.

Kurzfassung

Die vorliegende Arbeit befasst sich mit kontinuierlichen Zeitreihen, welche mit Hilfe von klassischen stochastischen Differentialgleichungen definiert werden, und untersucht einige ihrer Anwendungen in der Finanzmathematik. Dabei wird auf analytische Approximationen von wichtigen Finanzvariablen fokussiert, wie z.B. Derivatenpreise oder die Ausfallwahrscheinlichkeit einer Firma, in einem komplexeren und realistischeren Rahmen als jener, der von Black und Scholes (1973) und Merton (1973, 1974) benutzt wurde.

Im ersten Teil werden neue Modelle für kontinuierliche Zeitreihen aufgestellt und für die Modellierung der Dynamik von Aktien und Volatilitäten weiterverwendet. Einige davon werden mit Hilfe von Simulations- und Schätzungsbeispielen illustriert.

Der zweite Teil der Arbeit befasst sich mit der Herleitung neuer Resultate für die Bewertung von Derivaten und für Kreditrisikoprobleme in stochastischen, von kontinuierlichen Zeitreihen gesteuerten Volatilitätsmodellen.

Contents

Introduction

Real world phenomena, including financial events, occur in continuous time and therefore realistic models should account for this feature. Even though the available data are discretely sampled, they are often obtained at unequal time intervals and they may also be recorded with high frequency. While it is not easy to include these issues in the framework of a discrete time model, its continuous time counterparts can implicitly handle dynamics over irregular time intervals in a consistent way, see e.g., Jones (1981, 1985), Jones and Ackerson (1990). Moreover, the continuous time approach can rely on the powerful tools of stochastic calculus, which is essential in financial mathematics, since it provides better insight into the involved phenomena. Therefore, for practically all financial applications it seems natural to choose a continuous-time model in order to get a good approximation of the reality.

In this sense, this thesis considers continuous-time series processes defined by classical stochastic differential equations and investigates some of their applications to financial mathematics with a focus on analytical approximations for some important financial quantities like derivative prices or default probabilities in a more complex and realistic framework than the one used by Black and Scholes (1973) and Merton (1973, 1974).

In the first part of this thesis we propose some new continuous-time series models with applications to financial mathematics and illustrate them with the aid of some simulation and estimation examples. The development of these models is motivated by the so-called stylized facts of financial data (such as volatility clustering, volatility mean reversion, leverage effects, fat tails, long range dependence) that cannot be easily explained by the classical models. In this context, for some subclass of the new introduced time-varying models, we derive a first generalization of the celebrated Black-Scholes formula for option prices.

Then, in the second part of this thesis, we derive new results for derivative pricing and credit risk problems in stochastic volatility (SV) models driven by some continuous-time series processes.

This thesis is organized in six chapters:

The introductory Chapter 1 provides a very short overview of existing continuous-time series processes. A prominent example from this class is the so-called continuous-time autoregressive moving average (CARMA) process, which is modelled with a linear system of stochastic differential equations. This linear specification gives a tractable likelihood and this is the reason why CARMA models have been intensively used in practice over the years.

These models are the continuous-time analogues of the well-known ARMA models and they have been introduced by Doob (1944).

Since then, the class of continuous-time series processes has been considerably enlarged. Among the most popular ones let us mention: the continuous-time threshold ARMA (shortly CTARMA) process, the Lévy driven CARMA processes introduced by Brockwell (2000, 2001b), the continuous-time analogues of the discrete-time GARCH (COGARCH) process introduced by Klüppelberg, Lindner and Maller (2004), and the continuous-time autoregressive fractionally integrated moving average (CARFIMA) process by Tsai and Chan (2005).

These are interesting theoretical generalizations of CARMA processes, but their estimation often raises serious challenges and their analytical tractability for certain financial applications like derivative pricing problems is relatively restrained, especially for those driven by some generalizations of the Brownian motion process.

Therefore, we improve in Chapter 2 the classical continuous-time series setting upon following other ideas, related to those presented in Dahlhaus, Neumann and Sachs (1999), Tsai and Chan (2000c), Chernov et al. (2003), Stărică and Granger (2005) and Rao (2006). The resulting models can still capture the well-known empirical features of financial time series and their estimation can be carried out for instance with the aid of some classical nonlinear filtering techniques.

The first part of Chapter 2 introduces some new nonstationary and/or nonlinear continuous-time series processes and discusses their modeling potential for financial applications.

One reason to extend the classical continuous-time series processes is the fact that a significant part of the information carried by the financial time series consists of non-stationarity. In order to model such effects we introduce a class of time-varying continuous-time autoregressive moving average processes (shortly tvCARMA) and then put in evidence their good mathematical tractability, which is the basis for many financial applications. However, their potential in modelling time series which exhibit heteroscedasticity is relatively restrained. In this sense, this class of models is then further improved e.g., by considering tvCARMA processes with linear state-dependent diffusion coefficients. This way we obtain a class of heteroscedastic bilinear tvCARMA processes which still have very good analytical properties.

Another reason to extend the classical continuous-time series processes is the nonlinear structure of many financial data. Following the recent developments in nonlinear discrete time series analysis, we first constructed continuous-time analogues of some nonlinear autoregressive processes with changing conditional variances, see e.g., Härdle, Tsybakov and Yang (1998) and Lütkepohl and Krätzig (2004).

The resulting class of models, generically denoted NLCAR, are extensions of the nonlinear continuous-time autoregressive models recently introduced in Tsai and Chan (2000). Moreover, the models of Tsai and Chan are homoscedastic and it is well known that financial and many other time series exhibit heteroscedasticity. This gives another strong motivation to develop models of NLCAR type, which are able to capture this empirical feature, e.g., NLCARMA and tvNLCARMA.

Most of the continuous-time series models are driven by a single Brownian motion and this leads, for a high order model, to an interpretation in terms of stochastic differential equations (SDE) with a degenerate diffusion matrix. For some financial applications, like derivative pricing, this can be a serious drawback. In order to avoid this, we propose some classes of perturbed continuous-time series processes which have an interpretation in terms of SDE with a non-degenerate diffusion matrix, however without changing the usual drift structure. For instance, in Chapter 3 we use the subclass of perturbed CARMA processes (shortly CARMA_ε) to model the asset volatility.

All these models are generalizations of many classical financial models and we put this in evidence by giving some examples from literature.

Next we introduce some new financial models based on the continuous-time series processes discussed above. Then we evidentiate some of them which are able to capture stylized facts like fat tails, jumps or long-memory properties and therefore offer an alternative to the financial models driven by diverse generalizations of the Brownian motion process. Moreover, in some financial application like derivative pricing problems, these models have the important advantage that, under relatively general assumptions, still allow to apply the standard no-arbitrage pricing theory. However, using financial models driven by fractional Brownian motions can affect the nonexistence of arbitrage in the market, see e.g., Biagini et. al. (2008).

Next we address the derivative pricing problem for a subclass of the new introduced time-varying models and we derive a first generalization of the Black-Scholes formula (see e.g., Appendix A).

In the second part of Chapter 2 we give a short overview of the prevalent estimation techniques for nonlinear and nonstationary continuous state-space models which can be applied in practice to carry out the estimation problem for our models of interest in the parametric case. These procedures can be combined with the general approaches in Dahlhaus, Neumann and Sachs (1999) (based on a nonlinear wavelet method) and Malliavin and Mancino (2002) (relying on a Fourier series method), in order to solve the estimation problem in the nonparametric case.

Finally, some of the models discussed in the first part of Chapter 2 are illustrated with the aid of some simulation and estimation examples.

The first part of Chapter 3 introduces stochastic volatility (SV) models based on CARMA_ε processes and derives the evolution of the pricing function for European derivatives in this framework. Again, some simulations are performed in order to illustrate these models. The volatility modeling is crucial in practically all financial applications, including derivatives pricing and risk management, and this topic has recently attracted great attention in the financial literature. Thus, a large number of different characterizations of the stochastic volatility have been suggested in the last decades. Among the most popular ones let us mention those developed by White (1987a), Scott (1982), Stein and Stein (1991), Heston (1993) and Ball and Roma (1994) (see e.g., Appendix B for a short description).

The idea to use a CARMA process to model the volatility has been recently proposed, see e.g., Brockwell (2009) and the references therein. In this chapter we further develop it in a rather different framework, where the asset volatility σ_t is given by $\sigma_t = f(U_t)$ with f being some positive smooth function, whereas $(U_t)_{t\geq 0}$ is a CARMA$_\varepsilon(p, q)$ process (i.e. a stochastic perturbation of a CARMA(p, q) process, as described in Chapter 2). When $p > 1$, the volatility process has a non-Markovian structure and we obtain in this way flexible extensions of the most SV models discussed in Fouqué et al. (2000). Moreover, in our framework, for p, $q > 1$ the volatility process is, in fact, a sum of short memory processes and, according to Granger (1980), this can describe long memory. Many other important stylized facts on financial data (such as volatility clustering, volatility mean reversion, leverage effects) can be easily explained with these SV models (further details in this direction are given in Chapter 4).

In this framework we also describe some important financial quantities for the pricing of derivatives, like integrated variance, spot volatility and effective volatility. From a probabilistic point of view, the integrated variance (or integrated volatility) is the quadratic variation of the log-asset process and the spot volatility can be recovered from the integrated variance by differentiation. Then the effective volatility is actually the square root of the mean of the stationary distribution for the spot volatility process.

Next we discuss in this setting the derivative pricing problem which is one of the most important issues in finance. Thereby we combine the risk neutral approach with this stochastic volatility framework in order to get the "fair price" of the derivative. Then the corresponding stochastic volatility derivative pricing partial differential equation is obtained and this result is the key to numerically compute the prices for such financial derivatives. However, solving numerically the corresponding partial differential equation for a high dimensional model is a nontrivial issue. An alternative way to overcome this difficulty for a subclass of the SV models addressed above is developed in Chapter 4.

In the context of high frequency data, the second part of Chapter 3 provides some estimation results for integrated variance, spot volatility and effective volatility. The availability of high frequency data on financial markets has motivated in the last years a large number of publications devoted to the measurement of the integrated variance, see e.g., Shephard (2005) and references therein.

Barndorff-Nielsen and Shephard (2002) proposed to use the so-called realized variation (or realized volatility), to estimate the integrated variance. The concept of realized variation traces back to an early idea of Merton (1980) and basically consists in the estimation of the daily variance via the sum of squared intraday returns, see e.g., Andersen et al. (2003).

Our first result in this context is concerned with the estimation of the integrated variance and it is related to the one presented by Barndorff-Nielsen and Shephard (2002). Based on a new proof, we focus on an enhanced description of the error bound, this being of great importance in practice. As a corollary we obtain a result concerning the estimation of the spot volatility. In a more complex setting, a related result can be found in Foster and Nelson (1996), however without achieving an analytical description as in our case. Finally, starting from the above results, we derive an estimator for the effective volatility.

Alternatively one can use the nonlinear filtering techniques described in Chapter 2 in order to estimate all financial quantities discussed above and this allows to do it in a far more general setting as the one we used in the context of high frequency data. However, it is no more possible to give an explicit form of the corresponding estimator.

Eventually we point out that, besides the above mentioned financial applications, these results can help simplifying the estimation problem of complex stochastic volatility models.

In Chapter 4 we derive an asymptotic analysis for European derivative prices in the framework offered by stochastic volatility models driven by a CAR_ε process of arbitrary order, under a fast mean-reverting regime. For this subclass of the SV models discussed in Chapter 3 we derive analytical approximations for European derivative prices which extend similar results of Fouqué et al. (2000). Thus, we obtain a "corrected price formula" which offers an alternative way to overcome the problems which arise when solving numerically the derivative pricing partial differential equation described in Chapter 3, especially in the case of a high dimensional model.

For this purpose, we use the asymptotic analysis developed in Fouqué et al. (2000), which is a modern and very powerful tool to obtain results similar to the Black-Scholes formula, in a generalized framework based on some fast mean-reverting SV models. The basic idea is to work on large intervals, where it can be assumed that the mean reversion property of the volatility process is fast and then the Black-Scholes model (with a correction to account for stochastic volatility) gives a good approximation.

Empirical evidence of a fast volatility factor was found in the analysis of high frequency S&P 500 data by Fouqué et al. (2000) and this factor has been modelled with a CAR(1) process, which induces a Markovian description of the volatility.

The models discussed in this chapter are illustrated with the aid of some simulation and estimation examples. These show that the clustering property (i.e. when the volatility is high, it tends to stay high for a few days or so, before dropping to a lower level where it tends to stay for a while, and so on) is closely related to the fast mean reversion property in such stochastic volatility models and the flexible non-Markovian structure of the volatility in our framework can help in practice to get a better description of financial data.

It is well known that for a multivariate setting where a large number of assets has to be analyzed it is extremely important, at least from the computational point of view, to have the possibility to work with models having very good analytical properties. The purpose of Chapter 5 is to find a class of multivariate stochastic volatility (MSV) models for which the correction constants appearing in the multivariate corrected price formula for European derivatives proposed by Fouqué et al. (2000) can be explicitly computed with respect to the parameters of the MSV model.

The class of models that we propose to this aim contains extensions of the univariate Scott stochastic volatility model (see e.g., Appendix B). Precisely, we construct a MSV where the logarithms of the volatilities are linear combinations of the components of a multivariate CAR(1) process (with the same rate of mean reversion for all its components). In order to capture the leverage effects, there have been introduced correlations between all involved stochastic processes.

In this context, the first result in this chapter gives an explicit solution to a multivariate Poisson equation and this is the key to the second result which derives the explicit computations for the correction constants appearing in the multivariate corrected price formula for European derivatives. These results are very expedient for practical purposes, since they allow the precise computation of relevant financial quantities upon avoiding computer intensive methods which would be otherwise needed for numerically solving the corresponding PDEs and/or for stochastic simulations, especially in higher dimensions.

By our knowledge this is the first general multivariate setting for incomplete markets with a comprehensive system of mutual correlations between the involved processes which is able to provide such type of analytical approximation for European derivative prices.

In Chapter 6 we derive a new analytical result for the firm default probability in a generalized Merton setting where the firm value evolves in a fast mean-reverting stochastic volatility scenario.

In the Merton (1974) approach, a firm defaults if, at the time of servicing the debt, its assets are lower than its outstanding debt. The non-observable value of a firm is assumed to follow a geometric Brownian motion. The Merton setting is attractive since it enables classical Black-Scholes option pricing theory to be used. However, it is well known that the underlying assumptions are quite unrealistic. The stochastic volatility models discussed in the previous chapters can be used to improve the classical Merton setting for modeling the risk of default.

For the case of a fast mean-reverting volatility, we obtain in this context a new analytical approximation for the firm default probability, based on an improved first correction term, when compared to the corresponding one in Fouqué et al. (2006). Moreover, unlike that result, our approximation also depends on the value of the volatility driven factor, which gives the chance to capture a larger amount of the relevant market informations.

Chapter 1

Classical Continuous-time Series Models

Continuous-time series processes are modelled by stochastic differential equations (SDE). For various accounts on stochastic calculus and SDEs we refer to Liptser & Shiryayev (1977), Karatzas and Shreve (1988), Kunita (1990), Protter (1995) and Øksendal (1998).

1.1 Continuous-time ARMA Models

The *continuous-time autoregressive moving average* (shortly CARMA) *processes* are modelled by linear SDEs and represent the continuous-time analogues of the well-known discrete-time ARMA processes.

They have been extensively studied in the last decades, see e.g., Doob (1944), Bartlett (1946), Phillips (1959), Durbin (1961), Dzhaparidze (1970,1971), Arató (1982), Brockwell (2001a).

The *formal description* of a CARMA(p,q) process $(Y_t)_{t\geq 0}$ with $0 \leq q < p$ ($p, q \in \mathbb{N}$) is given by

$$Y_t^{(p)} + a_1 Y_t^{(p-1)} + \cdots + a_p Y_t = \delta[b_0 W_t^{(1)} + b_1 W_t^{(2)} + \cdots + b_q W_t^{(q+1)} + c], \qquad (1.1)$$

where $(W_t)_{t\geq 0}$ is a Brownian motion, $a_1, ..., a_p, b_1, ..., b_q$, c, δ (the scale parameter) are constants and the superscript $\cdot^{(j)}$ denotes the j-fold differentiation with respect to t. It is assumed that $b_0 = 1$, $\delta > 0$ and the coefficients b_j satisfy $b_q \neq 0$ and $b_j = 0$ for $j > q$.

Since the derivatives $W_t^{(j)}$ do not exist in the usual sense, the pth order linear differential equation (1.1) is interpreted in terms of stochastic differential equations, as being equivalent to the *observation* and *state* equations

$$Y_t = \delta \mathbf{b}' \cdot \mathbf{X}(t), \quad t \geq 0, \qquad (1.2)$$

and

$$d\mathbf{X}(t) - \mathbf{A} \cdot \mathbf{X}(t)\, dt = \mathbf{e} \cdot (c\, dt + dW_t), \quad t \geq 0, \qquad (1.3)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{X}(t) = \begin{bmatrix} X_t \\ X_t^{(1)} \\ \vdots \\ X_t^{(p-2)} \\ X_t^{(p-1)} \end{bmatrix}, \tag{1.4}$$

$$\mathrm{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}, \quad (\mathrm{e}, \mathrm{b} \in \mathbb{R}^p). \tag{1.5}$$

The superscript $'$ denotes the transpose of a vector and $\mathbf{X}(0)$ is uncorrelated with the Brownian motion $(W_t)_{t\geq 0}$.

Because of the linearity of the Ito differential equation (1.3), its solution has the simple form

$$\mathbf{X}(t) = e^{\mathbf{A}t} \cdot \mathbf{X}(0) + \int_0^t e^{\mathbf{A}(t-u)} \cdot \mathrm{e}\, dW_u + c \int_0^t e^{\mathbf{A}(t-u)} \cdot \mathrm{e}\, du, \quad t \geq 0. \tag{1.6}$$

Hence $(\mathbf{X}(t))_{t\geq 0}$ is a stationary Gaussian diffusion process if and only if

$$\mathbf{X}(0) \sim N(a_p^{-1} \cdot c[1\, 0 \ldots 0]', \int_0^\infty e^{\mathbf{A}u} \mathrm{e}\mathrm{e}' e^{\mathbf{A}'u}\, du) \tag{1.7}$$

and the eigenvalues $\lambda_1, ..., \lambda_p$ of A all have negative real parts.

In the case where $q = 0$, the process $(Y_t)_{t\geq 0}$ reduces to a *continuous-time* AR(p) *process* (shortly CAR(p)). In particular, CAR(1) is the well-known Ornstein-Uhlenbeck (OU) process and this is one of the mostly used ingredients of modeling in mathematical finance.

Relations between a discrete-time ARMA process and a CARMA process have been discussed by many authors, e.g., Chan and Tong (1987), He and Wang (1989), Brockwell (1995), Brockwell and Brockwell (1999), Huzii (2004).

The CARMA processes have applications not only to economics. For instance, the random harmonic oscillator is the CAR(2) process

$$Y_t^{(2)} + b \cdot Y_t^{(1)} + \lambda^2 \cdot Y_t = \sigma \cdot W_t^{(1)}, \quad t \geq 0, \tag{1.8}$$

where b, λ are real constants, $\lambda^2 \cdot Y_t$ represents a linear, restoring force and $b \cdot Y_t^{(1)}$ is a frictional damping term.

The linearity and stationarity properties of CARMA processes are very attractive features from the estimation point of view, but to give a realistic description of financial data sets one needs to look beyond the stationary linear framework, and a number of generalizations of CARMA processes have been proposed in literature.

In the next section, a first nonlinear extension of CARMA processes will be discussed. Other nonstationary or/and nonlinear extensions will be introduced in chapter two.

1.2 Continuous-time Threshold ARMA Models

The *continuous-time threshold* ARMA (shortly CTARMA) *processes* are the continuous-time analogues of the discrete-time SETARMA processes of Tong (1990) and they have been used by several authors, e.g., Tong and Yeung (1991), Brockwell and Williams (1997) for the modeling of financial and other time series.

By definition, $(Y_t)_{t\geq 0}$ is a CTARMA(p, q) process with threshold r and constants $b_1, ..., b_q$, δ if it verifies (1.2) and (1.3), where $a_1, ..., a_p$, and c are this time allowed to depend on a linear function $f(\mathrm{X}(t))$ ($f(x) = \sum_{i=1}^{p} d_i \cdot x_i$, where d_i, $i = 1, \ldots, p$ are nonnegative constants) of the state vector $\mathrm{X}(t)$ in such a way that

$$a_i(f(\mathrm{X}(t))) = a_i^j, \quad i = 1, \ldots, p, \quad \text{and} \quad c(f(\mathrm{X}(t))) = c^j, \tag{1.9}$$

where $j = 1, 2$, according to whether $f(\mathrm{X}(t)) \leq r$ or $f(\mathrm{X}(t)) > r$. The extension to the case of more than one threshold is straightforward. In this case, the corresponding stochastic differential equation has no more an explicit solution, however an existence result was obtained in Brockwell, Stramer and Tweedie (1996), which have shown that (1.3) with coefficients as defined in (1.9) has a unique (in law) weak solution $\mathrm{X}(t)$.

Choosing $q = 0$ we obtain the class of *continuous-time threshold autoregressive processes* of order p (abbreviated CTAR(p)).

1.3 Continuous-time Series Models Driven by Some Generalized Brownian Motions

A usual way to extend the SDE models driven by Brownian motions is to replace the latter with one of their generalizations, e.g., Lévy processes or fractional Brownian motions. The Lévy processes are mainly used in order to model the assets in the presence of jumps, whereas fractional Brownian motions are a common ingredient of models for time series with long range dependence (LRD) properties. For a detailed description of Lévy processes we refer to Protter (1995), Bertoin (1996) and Sato (1999) and for fractional Brownian motions to Embrechts and Maejima (2002) and Mishura (2008).

Lévy driven CARMA models have been introduced by Brockwell (2000, 2001b) and further extended in Brockwell and Marquardt (2005) upon using more general Lévy processes. Such processes have been employed to model the volatility of asset returns in Todorov and Tauchen (2006).

A further class of Lévy driven models, denoted COGARCH(1,1), was introduced by Klüppelberg, Lindner and Maller (2004) and this is the continuous-time analogue of the discrete-time GARCH(1,1) (generalized autoregressive conditionally heteroscedastic) process.

The COGARCH(1,1) process is obtained by rewriting the explicit expression for volatility of the GARCH(1,1) process in such a way that it has a continuous-time interpretation. Higher-order continuous-time analogues of GARCH(p, q) processes have been introduced by Brockwell, Chadraa and Lindner (2006).

Some continuous-time analogues of the so-called fractionally integrated (or fractional) ARMA process have been introduced by Viano, Deniau and Oppenheim (1994) and Comte and Renault (1998). In this class of models there are many long-memory processes which can be used to model data with LRD properties. Another class of long-memory Levy-driven CARMA processes was introduced by Brockwell (2004) and Brockwell and Marquardt (2005).

Though the above mentioned models are interesting theoretical generalizations of CARMA processes, their estimation raises serious challenges and their analytical tractability for certain financial applications like derivative pricing problems is relatively small. Therefore, we tried to improve the classical continuous-time series setting upon following other ideas, related to those presented in Dahlhaus, Neumann and Sachs (1999), Tsai and Chan (2000c), Chernov et al. (2003), Stărică and Granger (2005) and Rao (2006).

Among the resulting models, we describe a subclass which can still capture the above enumerated empirical features of financial time series and therefore offer an alternative to the previously mentioned models. Details will be given in the next chapter.

Chapter 2

Some Extensions of Classical Continuous-time Series Models

The modeling of temporal variations of stock market prices has been the subject of intensive research for a long time now, starting with the famous random walk hypothesis of Bachelier (1900). A popular approach consists in specifying an explicit model for the dynamics of return series. In this sense, the seminal contribution of Engle (1982), who introduced the ARCH model, has been followed by a large number of variants, like GARCH models, regime switching models (see e.g., Hamilton (1989), Hamilton and Susmel (1994)), stochastic volatility models (which will be discussed later in this thesis), etc.

All these models have been developed to reflect the so-called *stylized facts* of financial time series. Among the most important ones are: volatility clustering, volatility mean reversion, leverage effects, fat tails, serial dependence without correlation.

These and many other empirical features of financial time series have been documented by many researchers: Bollerslev, Engle and Nelson (1994), Granger and Ding (1995), Pagan (1996), Ghysels, Harvey and Renault (1996), Guillaume et al. (1997), Cont (2001), Fryzlewicz, Sapatinas and Rao (2006).

However, there is no model able to capture the whole complexity of financial data and thus the field of statistical analysis of this type of data remains open for further investigations.

In this sense, we present firstly some extensions of the classical continuous-time series models and some procedures which can be applied in practice to solve the estimation problem for the newly introduced models.

Secondly, we put in evidence the potential of the new introduced continuous-time series processes in modeling the assets (or their stochastic volatility) and derivative prices, which are among the most important issues in mathematical finance.

From now on, the term *asset* will be used to describe any financial object whose value is known at present, but is liable to change in future, for instance shares in a company, commodities, currencies.

The *volatility* of a financial asset is the variance per unit time of the logarithm of the price of the asset. It is a crucial quantity for the determination of risk and in the valuation of derivatives. A short description of the classical derivative pricing theory and stochastic volatility models can be found in the Appendix.

For further accounts on mathematical finance we refer e.g., to Taylor (1986), Karatzas and Shreve (1998), Shiryaev (1999), Fouqué, Papanicolaou, and Sircar (2000), Duffie (2001), Brigo and Mercurio (2001), Björk (2004).

2.1 Linear and Nonlinear Time-varying Continuous-time Series Models

2.1.1 Linear Time-varying CARMA Models

It is well known that stationary time series models have nice theoretical properties but are not prone to realistically describe the financial data.

Several papers (see e.g., Ramsey (1999), Clémençon and Slim (2004) and the references therein) have put in evidence the presence of temporal inhomogeneities, which is a prominent characteristic of financial data. Mandelbrot (1963) emphasized:

"*Prices records do not look stationary, and statistical expressions such as the sample variance take very different values at different times; this nonstationarity seems to put a statistical model of price change out of the question.*"

Thus, it was and will still remain a difficult task to select a model which allows to deal properly with the time-inhomogeneous character of return series.

A significant part of the information carried by the financial time series consists in nonstationarity: beginning or end of some phenomena, ruptures due to shocks or structural change, drifts reflecting economical trends, business cycles etc.

Many discrete-time models have been introduced to model such nonstationary effects, see e.g., Priestley (1965), Cramer (1961) and Bibi (2003). However, the asymptotic results available for stationary time series are not immediately applicable to their nonstationary counterparts.

To circumvent this, Dahlhaus (1997) introduced the notion of locally stationary process in the context of discrete time series analysis.

The extension of such results to the context of continuous-time processes seems to remain further a nontrivial task, however one can define and analyze continuous-time processes which have properties similar to the discrete-time autoregressive processes with time-varying coefficients.

Here our objective is to introduce time-varying continuous-time processes which extend some classical models discussed in the previous chapter and to evidentiate their potential for financial applications.

tvCAR and tvCARMA processes

Following the definitions of discrete time-varying autoregressive model, we introduce the *time-varying continuous autoregressive process* of order p (shortly tvCAR(p)) as the solution of the pth order differential equation:

$$Y_t^{(p)} - \alpha_p(t) \cdot Y_t^{(p-1)} - \cdots - \alpha_1(t) \cdot Y_t - \alpha_0(t) = \sigma(t) \cdot W_t^{(1)}, \quad t \geq 0, \tag{2.1}$$

where $(W_t)_{t\geq 0}$ is a Brownian motion and $\alpha_0(.)$, $\ldots$ $\alpha_p(.)$ and $\sigma(.) > 0$ are continuous deterministic functions. As usual, we interpret (2.1) as being equivalent to the observation and state equations:

$$Y_t = \mathrm{b}' \cdot \mathbf{X}(t), \quad t \geq 0,$$
$$d\mathbf{X}(t) = (A(t) \cdot \mathbf{X}(t) + \alpha_0(t) \cdot \mathrm{e}) \cdot dt + \sigma(t) \cdot \mathrm{e}\, dW_t, \quad t \geq 0, \tag{2.2}$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1(t) & \alpha_2(t) & \alpha_3(t) & \cdots & \alpha_p(t) \end{bmatrix}, \quad \mathbf{X}(t) = \begin{bmatrix} X_t \\ X_t^{(1)} \\ \vdots \\ X_t^{(p-2)} \\ X_t^{(p-1)} \end{bmatrix}, \tag{2.3}$$

$$\mathrm{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad (\mathrm{e}, \mathrm{b} \in \mathbb{R}^p). \tag{2.4}$$

Whenever the vector **b** above is replaced by a vector of functions $\mathrm{b}_t = [b_0(t), b_1(t), \ldots, b_{p-1}(t)]'$ ($b_i(.) \in L^2(\mathbb{R}_+)$ for $i = 0, \ldots, (p-1)$), which for some $q < p$ ($q \in \mathbb{N}^*$) verify $||b_q(.)||_{L^2} > 0$ and $b_j(.) \equiv 0$ for $j > q$, we will denote the above model by tvCARMA(p, q) and we will say that $(Y_t)_{t\geq 0}$ is a *time-varying* CARMA *process*.

It is well known that (2.2) has an explicit solution (see e.g., Maybeck (1982)):

$$\mathbf{X}(t) = \Phi(t)\Big[\mathbf{X}(0) + \int_0^t \alpha_0(s)\Phi^{-1}(s) \cdot \mathrm{e}\, ds + \int_0^t \sigma(s)\Phi^{-1}(s) \cdot \mathrm{e}\, dW_s\Big], \tag{2.5}$$

for all $t \geq 0$, where $\Phi(t)$ is the fundamental matrix solution to the homogeneous equation

$$\dot{\xi}(t) = A(t) \cdot \xi(t), \quad \xi(0) = I_p, \tag{2.6}$$

where I_p denotes the $p \times p$ identity matrix. The solution of (2.6) exists uniquely and if

$$A(t)\Big(\int_0^t A(s)\,ds\Big) = \Big(\int_0^t A(s)\,ds\Big)A(t), \tag{2.7}$$

for all $t > 0$, then

$$\xi(t) = \exp\Big(\int_0^t A(s)\,ds\Big). \tag{2.8}$$

Suppose that $E||\mathrm{X}_0||^2 < \infty$, then the evolution of the mean vector and covariance matrix which are defined by

$$\mathbf{m}(t) = E\mathbf{X}(t), \tag{2.9}$$

$$\mathbf{V}(t) = E[(\mathbf{X}(t) - \mathbf{m}(t))(\mathbf{X}(t) - \mathbf{m}(t))^T], \tag{2.10}$$

for all $t \geq 0$, can be described by the following system of ordinary differential equations:

$$\dot{\mathbf{m}}(t) = A(t)\cdot\mathbf{m}(t) + \alpha_0(t)\cdot\mathrm{e}, \tag{2.11}$$

$$\dot{\mathbf{V}}(t) = A(t)\cdot\mathbf{V}(t) + \mathbf{V}(t)\cdot A(t)' + \mathbf{B}(t)\cdot\mathbf{B}(t)', \tag{2.12}$$

where $\mathrm{B}(t) = \mathrm{diag}(\sigma(t)\cdot\mathrm{e})$, for all $t \geq 0$.

Bilinear tvCARMA models

Replacing in the definition of tvCARMA(p, q) the diffusion coefficient $\sigma(t)$ with a linear combination of the state variables

$$\sigma(t) = \tilde{\boldsymbol{\sigma}}_t'\cdot\mathrm{X}(t), \quad t \geq 0, \tag{2.13}$$

where the vector $\tilde{\boldsymbol{\sigma}}_t$ above is a vector of continuous functions $\tilde{\boldsymbol{\sigma}}_t = [\sigma_0(t),\ \sigma_1(t),\ \ldots,\ \sigma_{p-1}(t)]'$ ($\tilde{\boldsymbol{\sigma}}_t \in \mathbb{R}^p$) we obtain a class of *time-varying bilinear* CARMA *models*. The denomination suggests that their construction is inspired from the classical discrete (time-varying) bilinear models, see e.g., Bibi (2003).

For example, the following *time-varying continuous bilinear autoregressive* model

$$Y_t^{(1)} - \alpha_1(t)Y_t - \alpha_0(t) = (\sigma_0(t) + \sigma_1(t)\cdot Y_t)W_t^{(1)},\ t \geq 0, \tag{2.14}$$

where $\alpha_i(.)$, $\sigma_i(.)$, $i \in \{0,\ 1\}$ are some continuous deterministic functions, is a generalization of the Black-Scholes-Courtadon model for the volatility process (see Courtadon (1982)). In particular, if the coefficients are constants, where $\alpha_0(t) = \sigma_0(t) = 0$, $\alpha_1(t) = \mu > 0$, $\sigma_1(t) = \sigma > 0$, $t \geq 0$, one obtains the *geometric Brownian motion* (GBM) given by

$$Y_t^{(1)} = \mu\cdot Y_t + \sigma\cdot Y_t W_t^{(1)}, \quad t \geq 0. \tag{2.15}$$

This is also known as the *Black-Scholes-Merton model* (BS). For more details we refer to the Appendix or Black-Scholes (1973) and Merton (1973).

When the time series exhibit heteroscedasticity, but the nonlinear effects are still negligible, this class of models becomes very attractive, due to their nice analytical tractability.

Other examples from mathematical finance

• Lo and Wang (1995) considered for the dynamics of the logarithm of the stock price process a tvCAR(1) model of the form

$$Y_t^{(1)} - \alpha_1 \cdot Y_t - \alpha_0(t) = \sigma_0 \cdot W_t^{(1)}, \quad t \geq 0, \tag{2.16}$$

with $\alpha_0(t) = \mu(1 - \alpha_1 t)$, for all $t \geq 0$, $\alpha_1 < 0$ and $\mu, \sigma_0 > 0$ being constants.

• In Vasicek (1977) the short rate is modelled with a CAR(1) process. Applying this process to interest rate dynamics allows for closed-form solutions for bond prices and bond option prices. However, the analytical tractability of this model must be contrasted with the possibility of negative spot interest rate, but the mean reversion alleviates a great deal of the problem.

• Filipovic (2000) proposed generalizations of the Vasicek model of the form

$$Y_t^{(1)} - \alpha_1(t)Y_t - \alpha_0(t) = \sigma_0(t) \cdot W_t^{(1)}, \quad t \in [0, T], \tag{2.17}$$

where $T > 0$, $Y_0 = z_1 + z_2$, $\alpha_0(t) = z_3 e^{-z_5 t} + z_4 e^{-2z_5 t} + z_1 z_5$, $\alpha_1(t) = -z_5$, $\sigma_0(t) = \sqrt{z_4 z_5} e^{-z_5 t}$, for some appropriate constants $z_1, \ldots, z_5$. This tvCAR(1) model was further generalized in Ramponi and Lucca (2003). Similar models have been used in Ho and Lee (1986), Hull and White (1990), and Egorov, Li and Xu (2003).

Further extensions of classical continuous-time series and the above described models will be introduced in the next section.

2.1.2 Nonlinear Time-varying CARMA Models

The importance of non-linear models in time series analysis has been increasingly recognized over the past twenty years. There is much literature on nonlinear discrete-time series models, see Tong (1990, 1991), Granger and Teräsvirta (1993), Cox (1997), Straumann and Mikosch (2003), Rao (2006). However, the corresponding number of nonlinear continuous-time series models proposed in literature is relatively small, see e.g., Brockwell (1994), Tsai and Chan (2000c).

First, we introduce here a class of nonlinear heteroscedastic CAR processes which are the continuous-time analogues of the following conditional heteroscedastic autoregressive nonlinear model (see e.g., Härdle, Tsybakov and Yang (1998) and Lütkepohl and Krätzig (2004))

$$x_t = \phi(x_{t-1}, \ldots, x_{t-p}, \boldsymbol{\theta}) + \epsilon_t \cdot \sigma(x_{t-1}, \ldots, x_{t-\tilde{p}}, \boldsymbol{\theta}), \quad t \in \mathbb{N}, \tag{2.18}$$

where ϕ and σ are real-valued measurable functions on $\mathbb{R}^p$ and $\mathbb{R}^{\tilde{p}}$ ($\tilde{p} \leq p$), respectively, depending on the vector of model parameters $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^r$ ($r \in \mathbb{N}^*$) and $(\epsilon_t)_{t\in\mathbb{N}}$ is a sequence of $i.i.d.$ N$(0, 1)$ random variables.

For instance, Weiss (1984) proposed the model

$$X_t = \alpha X_{t-1} + \sqrt{a_0 + a_1 X_{t-1}^2} \cdot \epsilon_t, \tag{2.19}$$

and Franke, Kreiss and Mammen (2002) considered the model

$$x_t = 4\sin(x_{t-1}) + \epsilon_t \cdot \sqrt{0.5 + 0.25 x_{t-1}^2}. \tag{2.20}$$

NLCAR processes

Clearly, the continuous-time analogue of (2.18) can be formally described as the following p-th order nonlinear differential equation

$$Y_t^{(p)} = \phi(Y_t^{(p-1)}, \ldots, Y_t, \boldsymbol{\theta}) + \sigma(Y_t^{(p-1)}, \ldots, Y_t^{(p-\tilde{p})}, \boldsymbol{\theta}) \cdot W_t^{(1)}, \quad t \geq 0, \tag{2.21}$$

where $(W_t)_{t\geq 0}$ is a Brownian motion, $0 \leq \tilde{p} \leq p$ ($\tilde{p} = 0$ means σ is constant), $\tilde{p}$, $p \in \mathbb{N}$. For example, Tsai and Chan (2000c) considered a nonlinear continuous-time series model of the following form:

$$Y_t^{(p)} = (\alpha_0 + \alpha'\mathbf{X}(t) + e^{\lambda(\alpha_0 + \alpha'\mathbf{X}(t))^2} - 1)\, dt + \sigma \cdot W_t^{(1)}, \ \alpha = (\alpha_1, \ldots, \alpha_p)' \in \mathbb{R}^p, \tag{2.22}$$

for all $t \geq 0$, where λ, $\alpha_0 \in \mathbb{R}$, $\sigma_0 \in \mathbb{R}_+$ and $\mathbf{X}(t) = (Y_t, \ldots, Y_t^{(p-1)})' \in \mathbb{R}^p$.

The interpretation of (2.21) as observation and state equation can be given as follows:

$$Y_t = \mathrm{b}'\mathbf{X}(t), \quad t \geq 0, \tag{2.23}$$

$$d\mathbf{X}(t) = \mathbf{A}(\mathbf{X}(t), \boldsymbol{\theta})dt + \mathbf{B}(\mathbf{X}(t), \boldsymbol{\theta})\, d\mathbf{W}(t), \quad t \geq 0, \tag{2.24}$$

where $\mathrm{b} = [1, 0 \ldots, 0]' \in \mathbb{R}^p$, $\mathrm{e} = [0, 0, \ldots, 1]' \in \mathbb{R}^p$, $\mathbf{W}(t) = W_t \cdot \mathrm{e}$, for all $t \geq 0$, $\mathbf{A} : \mathbb{R}^p \to \mathbb{R}^p$, $\tilde{\sigma} : \mathbb{R}^p \to \mathbb{R}$ and for all $\mathbf{z} = (z_1, \ldots, z_p)' \in \mathbb{R}^p$

$$\mathbf{A}(\mathbf{z}, \boldsymbol{\theta}) = (z_2, \ldots, z_p, \phi(z_p, z_{p-1}, \ldots, z_1, \boldsymbol{\theta}))' \tag{2.25}$$

$$\tilde{\sigma}(\mathbf{z}, \boldsymbol{\theta}) = \sigma(z_p, z_{p-1}, \ldots, z_{p-\tilde{p}+1}, \boldsymbol{\theta}), \tag{2.26}$$

$$\mathbf{B}(\mathbf{z}, \boldsymbol{\theta}) = \mathrm{diag}\Big(\tilde{\sigma}(\mathbf{z}, \boldsymbol{\theta}\,) \cdot \mathrm{e}\Big). \tag{2.27}$$

We also assume that the functions ϕ and σ satisfy the usual conditions which ensure the existence of a strong solution to the above stochastic differential equations (see e.g., Protter (1995) or Karatzas and Shreve (1988)). Then $(Y_t)_{t\geq 0}$ is well-defined and we call it *nonlinear continuous-time autoregressive process* (shortly NLCAR$[p;\, \tilde{p}]$; when $\tilde{p} = p$ or $\tilde{p} = 0$ we abbreviate it NLCAR$[p]$, respectively NLCAR(p)).

The conditions under which the process $(\mathbf{X}(t))_{t\geq 0}$ is stationary and the partial differential equation for which the stationary density is a solution can be found in Hasminskii (1980).

NLCARMA processes and their time-varying extensions

Changing the observation equations (2.23) with

$$Y_t = \Psi(\mathbf{X}(t)), \quad t \geq 0, \tag{2.28}$$

where $\Psi(.)$ is a real-valued measurable function on $\mathbb{R}^p$ (possibly nonlinear), we obtain a class of *nonlinear* CARMA *processes* (shortly NLCARMA$([p;\,\tilde{p}], \Psi)$; when $\tilde{p} = p$ or $\tilde{p} = 0$ we abbreviate it NLCARMA$([p], \Psi)$, respectively NLCARMA(p, Ψ)).

If $\Psi(\mathbf{X}(t)) = \mathrm{b}' \cdot \mathbf{X}(t)$, with $\mathrm{b} = [b_0, b_1, \ldots, b_{p-1}]'$ and the real coefficients b_j satisfying $b_q \neq 0$ and $b_j = 0$ for $j > q$, with $0 < q < p$ ($p, \tilde{p}, q \in$ N*), then we denote this model by NLCARMA$([p;\,\tilde{p}], q)$ (when $\tilde{p} = p$ or $\tilde{p} = 0$ we abbreviate it NLCARMA$([p], q)$, respectively NLCARMA(p, q)).

The *time-varying extensions* of the above introduced models can be easily obtained by choosing the functions Ψ, ϕ and σ to depend of the supplementary time variable t, and we denote the resulting models by tvNLCARMA$([p;\,\tilde{p}], \Psi)$), tvNLCARMA$([p;\,\tilde{p}], q)$), etc.

The subclass of perturbed CARMA processes

This class of models can be further improved by choosing in the state equation (2.24) a nondegenerate diffusion matrix $\mathrm{B}(\mathrm{x}, \boldsymbol{\theta})$ and a multivariate p-dimensional Brownian motion $(\mathbf{W}(t))_{t\geq 0}$. We obtain in this way some classes of *perturbed continuous-time series processes* which are particularly useful in modeling the volatility process. For instance, in Chapter 3 we consider a stochastic volatility model driven by a *perturbed CARMA process* (shortly CARMA$_\varepsilon$) and we give a result concerning the evolution of the derivative pricing function in this context.

As an example, the formal description of a zero mean CAR$_\varepsilon$(2) process $(U_t)_{t\geq 0}$ can be given by

$$U_t^{(2)} + a_1 U_t^{(1)} + a_2 U_t = (a_1 \psi_1 W_1(t)^{(1)} + \psi_1 W_1(t)^{(2)}) + \psi_2 W_2(t)^{(1)}, \tag{2.29}$$

and this can be interpreted in terms of SDE as being equivalent to the observation and state equations

$$U_t = \mathrm{b}' \cdot \mathbf{U}(t), \quad t \geq 0, \tag{2.30}$$

$$d\mathbf{U}(t) = \mathbf{A} \cdot \mathbf{U}(t)\, dt + \Gamma_o \cdot d\mathbf{W}(t), \quad t \geq 0, \tag{2.31}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad \mathbf{U}_t = \begin{bmatrix} U_t \\ \tilde{U}_t \end{bmatrix}, \tag{2.32}$$

$$\mathrm{e} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.33}$$

$$\Gamma_o = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}, \quad \mathbf{W}(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}, \tag{2.34}$$

where a_1, a_2, ψ_1, ψ_2 are positive constants and $(\mathbf{W}(t))_{t\geq 0}$ is a bivariate Brownian motion.

Some non-financial examples

A small subclass of the above nonlinear continuous-time series models has been applied not only to finance, but also to mathematical biology and technical science. For instance, denoting with $(W_t)_{t\geq 0}$ a univariate Brownian motion:

• the stochastic analogue of the logistic growth model is

$$dX_t = \alpha X_t(1 - X_t/\beta)\, dt + \sigma X_t\, dW_t, \qquad X_0 = x_0, \quad t \in [0, T], \tag{2.35}$$

with $\alpha, \beta, \sigma \geq 0$ and this heteroscedastic NLCAR process is useful for modeling the growth of populations (see e.g., Bishwal (2008));

• Lasota & Mackey (1994) describe the dynamics of a particle moving in a potential which is a superposition of trigonometric functions by the following NLCAR process $(q_t)_{t\geq 0}$:

$$\begin{aligned} dq &= p\, dt, \\ dp &= (-\gamma p - \sum_{j=1}^{P} D_j \sin(q) \cos^{j-1}(q))dt + \sigma\, dW_t, \end{aligned} \tag{2.36}$$

with $P \in \mathbb{N}^*$, $D_j \in \mathbb{R}$, $j = 1, \ldots, P$, γ, $\sigma \geq 0$.

Some examples from mathematical finance

• Schwartz (1997) proposed to model the asset dynamics with the following heteroscedastic NLCAR$[1]$ process

$$dS_t = \alpha(\mu - \log S_t) \cdot S_t\, dt + \sigma S_t\, dW_t, \quad t \geq 0, \tag{2.37}$$

with α, μ, σ positive constants.

• Another general heteroscedastic NLCAR model for asset dynamics was introduced by Bibby and Sørensen (1997):

$$dS_t = S_t[\kappa + \frac{1}{2}\zeta^2(\log S_t - \kappa t)]\, dt + S_t \cdot \zeta(\log S_t - \kappa t)\, dW_t, \quad t \geq 0, \tag{2.38}$$

where $\kappa \in \mathbb{R}$, and $\zeta(.)$ is a positive function, e.g.,

$$\zeta(x) = \sigma \exp\left[\frac{v_1\sqrt{v_3^2 + (x - v_0)^2} - v_2(x - v_0)}{2}\right], \quad x \in \mathbb{R}, \tag{2.39}$$

with $\upsilon_1 > |\upsilon_2| \geq 0$, $\upsilon_3 > 0$, $\sigma > 0$ and $\upsilon_0 \in \mathbb{R}$. With this choice of $\zeta(.)$, after a sufficiently long time, the logarithm of the stock price will be approximately hyperbolically distributed. The density of such a distribution is proportional to

$$\exp\Big[-\upsilon_1\sqrt{\upsilon_3^2 + (x-\upsilon_0)^2} + \upsilon_2(x-\upsilon_0)\Big], \quad x \in \mathbb{R}. \tag{2.40}$$

This family of distributions was introduced by Barndorff-Nielsen (1977) in order to give a general framework for empirical studies in geology and other fields. Moreover, Barndorff-Nielsen (1978) noted that every hyperbolic distribution appears as a stationary distribution of a continuous-time Markov process described by a particular stochastic differential equation. One classical example in this sense is the following diffusion process

$$dX_t = -\theta\frac{X_t}{\sqrt{1+X_t^2}}dt + \sigma dW_t, \quad t \geq 0, \tag{2.41}$$

with σ, θ positive parameters (see e.g., Bishwal (2008)). This is a homoscedastic NLCAR process which has also been used to model stock returns. It has a stationary density proportional to $\exp(-\theta\sqrt{1+x^2}/\sigma)$, $x \in \mathbb{R}$.

• Hull and White (1988) and Heston (1993a) proposed the following NLCAR(1) model for the volatility process:

$$Y_t^{(1)} = (\frac{\alpha_0}{Y_t} - \alpha_1 Y_t)\, dt + \sigma \cdot W_t^{(1)}, \quad t \geq 0, \quad Y_0 > 0, \tag{2.42}$$

α_i, $\sigma > 0$, $i = 0,\, 1$.

• A model frequently used in literature is that of Cox-Ingersoll-Ross (CIR), also known as the *square-root model* defined by

$$dX_t = (\beta - \alpha \cdot X_t)\, dt + \delta\sqrt{X_t}\, dW_t, \quad t \geq 0, \quad X_0 = x_0 > 0, \tag{2.43}$$

where α, β, δ are real constants (α, $\delta > 0$, $\beta > \frac{1}{2}$). It is a heteroscedastic NLCAR$[1]$ model which has also been used to model the spot interest rate or the stochastic volatility for asset dynamics. The main advantage of this model is that it ensures that X_t always stays positive.

• The Chan-Karolyi-Longstaff-Sanders (CKLS) model is a generalization of CIR model and is defined by (see e.g., Chan et.al (1992) or Bishwal (2008))

$$dX_t = \theta(k - X_t)dt + \sigma X_t^{\gamma}\, dW_t, \quad t \geq 0, \quad X_0 = x_0 > 0, \tag{2.44}$$

with $k \in \mathbb{R}$, $\theta, \sigma \geq 0$ and $\gamma \in \mathbb{R}_+$. For $\gamma = \frac{3}{2}$ this is also known as the *inverse square-root model* and was studied by Ahn and Gao (1999), along with further NLCAR models with nonlinear drift.

• Ait-Sahalia (1996) proposed further generalizations of the following type:

$$dX_t = (\alpha + \beta_1 X_t + \beta_2 X_t^2 + \frac{\beta_3}{X_t})dt + (\sigma_1 + \sigma_2 X_t + \sigma_3 X_t^{\gamma})^{\frac{1}{2}}\, dW_t, \quad t \geq 0, \tag{2.45}$$

for the two previous models. Similar NLCAR models have been considered in Marsh and Rosenfeld (1982), Constantinides (1992) and Forman and Sørensen (2008).

• An extended Black-Scholes model proposed by Paulsen (2000) for the stock price:

$$dS_t = \mu \cdot S_t \, dt + \sigma(S_t) \cdot S_t \, dW_t, \quad t \geq 0, \tag{2.46}$$

where $\sigma : (0, \infty) \to (0, \infty)$ is a continuous function satisfying

$$0 < \inf_{x>0} \sigma(x) \leq \sup_{x>0} \sigma(x) < \infty. \tag{2.47}$$

As an example was considered the case

$$\sigma(x) = \gamma \sqrt{c(\frac{x-m}{x+m})^2 + 1}, \quad x > 0, \tag{2.48}$$

with γ, m, $c > 0$.

• The Larsen-Sørensen model (see e.g., Bishwal (2008)) reads

$$dX_t = -\theta \frac{\sin(\frac{1}{2}\pi(X_t - m)/z) - \rho}{\cos(\frac{1}{2}\pi(X_t - m)/z)} dt + \sigma \, dW_t, \quad t \geq 0, \tag{2.49}$$

with θ, z, $\sigma > 0$, $\rho \in (-1, 1)$, $m \in \mathbb{R}$. For $\rho = 0$, $m = 0$ and $z = \pi/2$ one obtains the Kessler-Sørensen model

$$dX_t = -\theta \tan(X_t) dt + \sigma \, dW_t, \quad t \geq 0. \tag{2.50}$$

These models have been used to characterize exchange rate values.

• Unlike the univariate models presented above, there are rather few classical financial models driven by a multivariate stochastic differential equation. Among them, the most popular ones are the *affine term structure models* (ATSMs) and the *quadratic term structure models* (QTSMs), used to characterize the dynamics of the short rate.

In the ATSMs the short rate r_t is an affine function of a multivariate square-root diffusion process $(\mathbf{X}_t)_{t \geq 0}$, i.e.

$$r_t = \delta_0 + \delta_x' \mathbf{X}_t, \quad t \geq 0, \tag{2.51}$$

where $\delta_0 \geq 0$, $\delta_x = (\delta_{ix}) \in \mathbb{R}^N$, $(\delta_{ix} \geq 0)$ and $\mathbf{X}_t$ is given by

$$d\mathbf{X}_t = \mu(\mathbf{X}_t) dt + \sigma(\mathbf{X}_t) \, d\mathbf{W}_t, \quad t \geq 0, \tag{2.52}$$

with μ and σ having the following *affine* structure:

$$\begin{aligned} \mu(\mathbf{x}) &= \theta + \mathcal{K}\mathbf{x}, \\ \sigma(\mathbf{x})\sigma(\mathbf{x})' &= h + \sum_{j=1}^{N} x_j H^{(j)}, \end{aligned}$$

where $\mathrm{x} = (x_i)_{1\leq i\leq N} \in \mathbb{R}^{N\times 1}$, $\theta \in \mathbb{R}^{N\times 1}$, $\mathcal{K} \in \mathbb{R}^{N\times N}$ and h and $H^{(j)}$ ($j = 1, \ldots, N$) are in $\mathbb{R}^{N\times N}$ and symmetric. For instance, a multivariate generalization of the above described CIR model results for $\sigma(\mathrm{x}) = \sqrt{\mathrm{x}}$, with $\sqrt{\mathrm{x}} = \mathrm{diag}((\sqrt{x_i})_{1\leq i\leq N})$.

For more accounts on ATSM models see e.g., Duffie and Kan (1996), Duffie, Pan, and Singleton (2000).

In the QTSMs the short rate r_t is a quadratic function of a multivariate linear and homoscedastic process $(\mathbf{X}_t)_{t\geq 0}$, i.e.

$$r_t = \delta_0 + \delta_1'\mathbf{X}_t + \mathbf{X}_t'\Upsilon\mathbf{X}_t, \ t \geq 0, \tag{2.53}$$

where δ_0 is a constant, δ_1 is a constant N-dimensional vector and Υ is a $N \times N$ positive definite matrix of constants, such that $\delta_0 - \frac{1}{4}\delta_1'\Upsilon^{-1}\delta_1 \geq 0$. The process $(\mathbf{X}_t)_{t\geq 0}$ is assumed to satisfy the stochastic differential equation

$$d\mathbf{X}_t = (K_0 + K_1\mathbf{X}_t)dt + \Sigma\, d\mathbf{W}_t, \ t \geq 0, \tag{2.54}$$

where K_0 is a constant N-dimensional vector and K_1, Σ are $N \times N$ matrices.

Ahn, Ditmar and Gallant (2002) have argued that the QTSMs are superior to ATSMs in that they are able to provide a better goodness of fit of term structure dynamics.

Most of the QTSMs can be extended by replacing $(\mathbf{X}_t)_{t\geq 0}$ in (2.53) by a multivariate process $\mathbf{Y}_t = (Y_i(t))_{1\leq i\leq N} \in \mathbb{R}^N$, where $(Y_i(t))_{t\geq 0}$ is one of the continuous time series processes introduced above, for all $i = 1, \ldots, N$. For instance, a class of *time-varying quadratic term structure models* (tvQTSMs) with deterministic time-varying coefficients $\delta_0(t)$, $\delta_1(t)$ and $\Upsilon(t)$ (such that $\delta_0(t) - \frac{1}{4}\delta_1(t)'\Upsilon^{-1}(t)\delta_1(t) \geq 0$, for all $t \geq 0$) can be introduced by

$$r_t = \delta_0(t) + \delta_1'(t)\mathbf{Y}_t + \mathbf{Y}_t'\Upsilon(t)\mathbf{Y}_t, \ t \geq 0, \tag{2.55}$$

where $\mathbf{Y}_t = (Y_i(t))_{1\leq i\leq N}$ and $(Y_i(t))_{t\geq 0}$ is a tvCARMA process, for all $i = 1, \ldots, N$.

• Finally, we recall some examples of nonlinear time-varying CAR processes from literature:

(i) the model of Hull and White (1990)

$$Y_t^{(1)} - \alpha_1(t)Y_t - \alpha_0(t) = \sigma_1(t) \cdot \sqrt{Y_t}W_t^{(1)}, \quad t \geq 0, \tag{2.56}$$

which was further generalized in Fan et al. (2003). In Egorov, Li and Xu (2003) this model was considered with the following parameterization of the time-dependent coefficients: $\alpha_0(t) = \frac{\sigma_0^2 d}{4}\exp(2\theta \cdot t)$, $\alpha_1(t) = -a$, $\sigma_1(t) = \sigma_0\exp(\theta \cdot t)$, with θ, a, d, σ_0 some appropriate constants and was called Extended Cox-Ingersoll-Ross (ECIR).

(ii) the model of Black and Karasinski (1991)

$$Y_t^{(1)} - \alpha_1(t)Y_t\log Y_t - \alpha_0(t)Y_t = \sigma_1(t) \cdot Y_tW_t^{(1)}, \quad t \geq 0, \tag{2.57}$$

where $\alpha_1(t) = \frac{d\log\sigma_1(t)}{dt}$. A similar model has been studied in Black, Derman and Toy (1990).

Further examples of econometric applications of continuous-time models can be found in the book of Bergstrom (1990).

The new class of models introduced above offers non-Markovian, nonlinear and nonstationary extensions for many classical models used in financial mathematics, as can be seen in the next section.

2.2 Financial Applications

Next we introduce some new financial models based on the classes of continuous-time series processes described above. The development of these models is motivated by stylized facts that are not easily explained by classical models.

Some comprehensive classes of financial models driven by continuous-time series processes

($\mathcal{M}_I$) Some of the previous models for asset dynamics are of the following type:

$$S_t = \hbar(t, Y_t), \quad t \geq 0, \tag{2.58}$$

with $\hbar : D \subseteq \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ and $(Y_t)_{t\geq 0}$ some continuous-time series process. For instance, most univariate models, like (2.37) or (2.38), are of the type (2.58) with $\hbar(t, y) = y$, for all t, $y \in \mathbb{R}_+$. Clearly, such a representation is not unique, since e.g., after a transformation with the Ito formula, the model (2.38) can be equivalently represented with $\hbar$ being the exponential function and with $(Y_t)_{t\geq 0}$ a nonlinear continuous-time series process given by

$$dY_t = \kappa\, dt + \zeta(Y_t - \kappa t)\, dW_t, \quad t \geq 0. \tag{2.59}$$

Many other classical financial models for the volatility or interest rate are of the form (2.58), however in order to also account for more general models like (2.55), one has to extend the class (2.58) by using a *multivariate continuous-time series process* $(\mathbf{Y}_t)_{t\geq 0}$, i.e.

$$S_t = \hbar(t, \mathbf{Y}_t), \quad t \geq 0, \tag{2.60}$$

with $\hbar : D \subseteq \mathbb{R}_+ \times \mathbb{R}^\ell \to \mathbb{R}_+$, $\mathbf{Y}_t = (Y_1(t), \ldots, Y_\ell(t))$ and $\{(Y_i(t))_{t\geq 0}\}_{1\leq i\leq \ell}$ being continuous-time series processes, possibly correlated.

Some examples

Many new interesting models can be found in the class (2.58) (or (2.60)). Among them we evidentiate in the following only two subclasses which are able to capture some of the so-called stylized facts like *fat tails* or *jumps*.

(i) It is well known that the distribution of stock returns typically has heavier tails than the normal distribution and this is often well fitted by a hyperbolic one, see e.g., Mandelbrot (1963), Fama (1965), Eberlein and Keller (1995), Küchler, Neumann, Sørensen and Streller (1994). Moreover, as already noted, the hyperbolic distributions appear as stationary distributions of some particular stochastic differential equations, like the one described in (2.38). This characterization opens the possibility to obtain new natural improvements of this distribution class and of the corresponding asset models. For instance, this leads to models of the form

$$S_t = Y_t^{\theta_1} S_{GH}^{1+\theta_2}(t), \quad t \geq 0, \tag{2.61}$$

where $\theta_i \in \mathbb{R}_+$, $i = 1, 2$, $(Y_t)_{t\geq 0}$ is some positive continuous-time series process with $E(Y_t) = 1$, $t \geq 0$, and $(S_{GH}(t))_{t\geq 0}$ is a process given by (2.38).

(ii) Another interesting subclass of (2.58) is the one adding jumps to the price process. For instance, choosing $\hbar$ of the form

$$\hbar(t, y) = 1_{\{\Bbbk_0(t,y)\geq B\}}\Bbbk_1(t, y) + 1_{\{\Bbbk_0(t,y)<B\}}\Bbbk_2(t, y), \tag{2.62}$$

where $\Bbbk_i(.)$, $i = 0, 1, 2$, are some nonlinear functions ($\Bbbk_i(.) \geq 0$, $i = 1, 2$), B is a real constant and $(Y_t)_{t\geq 0}$ is some positive continuous-time series process, we get a large class of *asset models with jumps*. Yet more complex jump models can be introduced by supplementary choosing the underlying process $(Y_t)_{t\geq 0}$ to be a CTARMA like process as in Chapter 1. Analogously, one can introduce jumps in the volatility process.

In figure 2.1 we illustrate two simulated trajectories of a jump process of type (2.58), where $(Y_t)_{t\in[0,1]}$ is a geometric Brownian motion given by (2.15) with the parameters $\mu = 0.1$, $\sigma = 0.07$ and $\hbar$ has the form (2.62) with

$$\Bbbk_0(t, y) = \cos(t(1-t)\log y) - \sin(t(1-t)\sqrt{y}), \tag{2.63}$$

$$\Bbbk_i(t, y) = \upsilon_3\Big(y + \Big(\upsilon_1(1 + (-1)^i \cdot \upsilon_2)\sqrt{y} + \sqrt{\frac{y_0 + y}{\upsilon_0}}\Big)^2\Big), \quad i = 1, 2, \tag{2.64}$$

for all $t \in [0, 1]$, $y \geq 0$ and parameters $B = 0$, $y_0 = 620$, $\upsilon_0 = 8$, $\upsilon_1 = \frac{1}{2}$, and $\upsilon_2 = \frac{1}{10}$, $\upsilon_3 = 0.05$.

The simulations have been done with the classical Euler-Maruyama scheme (see e.g., Kloeden and Platen (2001)). The corresponding stochastic differential equation (2.15) with the initial value $Y_0 = 620$ was discretized with a time step $\Delta = 0.0004$. The plotted values correspond to a temporal equidistant grid with a time step $\tilde{\Delta} = 0.004$.

Analogously, one can introduce jump processes of the type (2.60). We obtain in this way an alternative to the classical modeling of jumps with Lévy processes. One of the advantages of using this alternative is to provide models which are easier to estimate than the latter.

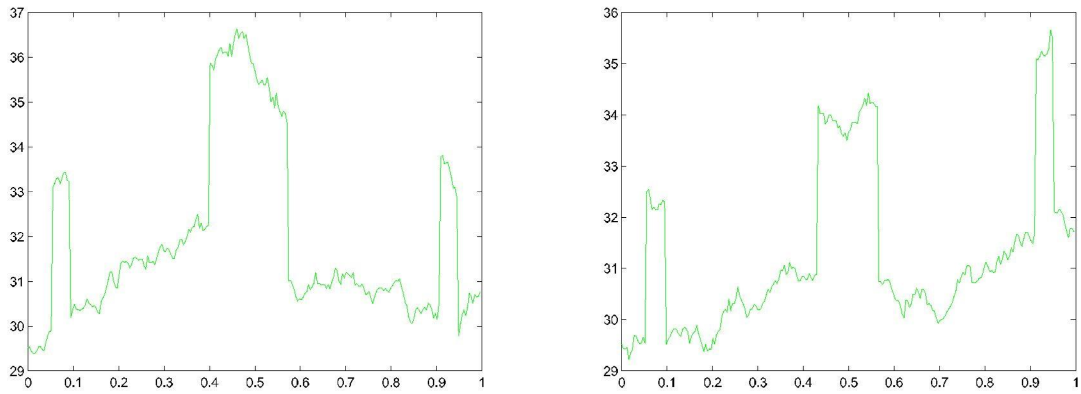

Figure 2.1: Two simulated scenarios for asset dynamics with jumps

($\mathcal{M}_{II}$) More sophisticated models for asset dynamics have also been proposed in literature. For instance, Bouchaud & Cont (1998) and Iino & Ozaki (2000) considered for the dynamics of log-asset $s_t = \ln(S_t)$ the following three dimensional model:

$$\begin{aligned} ds_t &= \phi_t \sigma_t^2 dt + \gamma_1 \sigma_t \, dW_t, \\ \sigma_t^2 &= \exp(\lambda_t), \\ d\lambda_t &= (\beta_1 + \alpha_1 \lambda_t) dt + \gamma_2 \, dZ_1(t), \\ d\phi_t &= (\beta_2 + \alpha_2 \phi_t) dt + \gamma_3 \, dZ_2(t), \end{aligned}$$

where α_i, β_i, γ_i are constants and $(W_t)_{t\geq 0}$, $(Z_1(t))_{t\geq 0}$ and $(Z_2(t))_{t\geq 0}$ are independent Brownian motions.

These suggest considering the following general class of models for log-asset dynamics:

$$ds_t = a(t, \mathbf{Y}_t) dt + b(t, \mathbf{Y}_t) \, d\mathbf{W}_t, \quad t \geq 0, \tag{2.65}$$

where $(\mathbf{Y}_t)_{t\geq 0}$ is a multivariate continuous-time series process, $(\mathbf{W}_t)_{t\geq 0}$ is a multivariate Brownian motion and a, b are some nonlinear functions. These represent an alternative to models of type (2.58) and include many classical stochastic volatility models.

Choosing $(\mathbf{Y}_t)_{t\geq 0}$ to be a multivariate tvCARMA process in (2.65), we obtain a class of non-stationary models which can be seen as continuous analogues to the discrete-time models of Stărică and Granger (2005). These authors argue that modeling the returns as a non-stationary sequence of independent random variable with time-varying unconditional variance can describe the dynamics of S&P 500 log-returns better than GARCH-type or long-memory-type models. This suggests that long range dependence (LRD) properties of financial time series can be captured with the time-varying stochastic volatility models of type (2.65).

Alternatively, it is also possible to give a good approximation of the long memory feature within the subclass of time-homogeneous models with a, b depending only on the sum of the components of a multivariate CARMA process $(Y_t)_{t\geq 0}$. Such type of models have been already proposed in literature, see e.g., Fouqué, Papanicolaou, Sircar and Sølna (2003b) or Chernov et al. (2003). This is not surprising given the result of Granger (1980) which shows that the sum of short memory processes can describe long memory. Moreover, these models are easier to estimate than the classical ones driven by fractional Brownian motions.

Another important advantage of using models of type (2.65) in many financial applications like derivative pricing problems comes from the fact that, under relatively general assumptions, they still allow to apply the standard no-arbitrage pricing theory. To let fractional Brownian motion simply replace the classical Brownian motion in the Black-Scholes model would affect the nonexistence of arbitrage in the market, see e.g., Biagini et. al. (2008).

Many other interesting models can be found among the class (2.65), e.g., those with jumps in the drift and/or in the volatility dynamics. Since these are processes of the type (2.60), jumps can be introduced as explained above. Bates (1996) argues that jumps should be included in a stochastic volatility model, at least when the volatility is Markovian. A large non-Markovian subclass of stochastic volatility models of type (2.65) will be discussed in the next chapters. For more accounts on stochastic volatility models and their potential for capturing the stylized facts see e.g., Scott (1982), Andersen et al. (2003), Shephard (2005), Shephard and Andersen (2009).

Some flexible extensions of CIR and BS models

Generalizations of the CIR model

Stein and Stein (1991) modelled the instantaneous volatility with a CAR(1) process, which implies that the volatility may become negative. This suggests that the positivity condition on $\hbar$ may be dropped out in modeling the volatility process. Then, as in Gallant and Tauchen (1997a), we can use the signed square root function (i.e. $\sqrt{z} = \text{sign}(z) \cdot \sqrt{|z|}$, for all z real) to introduce a class of non-Markovian generalizations of the CIR model (2.43) obtained by choosing ϕ and σ in (2.25)-(2.26) such that ϕ is a linear function and

$$\sigma(x_1, \ldots, x_{\tilde{p}}) = (d_0 + d_1 x_1 + \ldots + d_{\tilde{p}} x_{\tilde{p}})^{\frac{1}{2}}, \tag{2.66}$$

with d_i real constants, $i = 0, \ldots, \tilde{p}$ ($\tilde{p} \leq p$ and $p > 1$). Alternatively, one can choose the parameterization:

$$\sigma(x_1, \ldots, x_{\tilde{p}}) = (d_0^2 + d_1^2 x_1^2 + \ldots + d_{\tilde{p}}^2 x_{\tilde{p}}^2)^{\frac{1}{4}}. \tag{2.67}$$

Similar non-Markovian generalizations of CKLS model (2.44) can be obtained by choosing $\hbar = |.|$ in (2.58) and for σ one of the following parameterizations:

$$\sigma(x_1, \ldots, x_{\tilde{p}}) = (d_0 + d_1 x_1 + \ldots + d_{\tilde{p}} x_{\tilde{p}})^{\frac{2}{\gamma}}, \tag{2.68}$$

(here we interpret z^θ as the signed power function, i.e. $z^\theta = \text{sign}(z) \cdot |z|^\theta$, for all z real and $\theta \in \mathbb{R}_+$) or

$$\sigma(x_1, \ldots, x_{\tilde{p}}) = (d_0^2 + d_1^2 x_1^2 + \ldots + d_{\tilde{p}}^2 x_{\tilde{p}}^2)^{\frac{1}{\gamma}}, \tag{2.69}$$

with $\gamma \geq 2$ and d_i real constants, $i = 0, \ldots, \tilde{p}$ $(1 \leq \tilde{p} \leq p)$. For $\gamma = 2$, this is similar to the parameterization used in (2.19), (2.20) or (2.48).

A time-varying Black-Scholes model

Above we have introduced some classes of models which are able to capture very well many of the stylized features of financial data. Among them we discuss in the following a particular subclass of time-varying models which enable us to obtain a first analytical generalization of the Black-Scholes formula (see e.g., the Appendix A).

Consider now a market model with two assets $(\beta_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$, where $(\beta_t)_{t\geq 0}$ is a riskless asset given by

$$d\beta_t = r(t) \cdot d\beta_t, \quad t \geq 0, \quad \beta_0 = 1, \tag{2.70}$$

with a time-varying deterministic instantaneous interest rate $r(\cdot) \in C^1(\mathbb{R}_+)$, and $(S_t)_{t\geq 0}$, the risky asset, modelled with a tvNLCARMA(p, Ψ) with Ψ given by

$$\Psi(t, \mathbf{x}) = \exp(\mathbf{b}_t' \mathbf{x}), \quad t \geq 0, \quad \mathbf{x} \in \mathbb{R}^p, \tag{2.71}$$

and the state equation described by (2.2), where $\mathrm{b}_t = [b_0(t), b_1(t), \ldots, b_{p-1}(t)]'$ $b_i(.) \in C^1(\mathbb{R}_+)$ for $i = 0, \ldots, (p-1)$, $\sigma(.) \in C^1(\mathbb{R}_+)$ and $b_{p-1}(\cdot), \sigma(\cdot)$ bounded positive functions, bounded away from zero.

Let us denote $S(0) = s_0$,

$$d_t = \mathrm{b}'(t)(A(t) \cdot \mathbf{X}(t) + \alpha_0(t) \cdot \mathrm{e}) + \frac{d\mathbf{b}'}{dt} \mathbf{X}(t) + \frac{1}{2} b_{p-1}^2(t) \sigma(t)^2 \tag{2.72}$$

and

$$\theta_t^{(0)} = \frac{d_t - r(t)}{b_{p-1}(t)\sigma(t)}, \tag{2.73}$$

for all $t \geq 0$ (the superscript $'$ denotes the transpose of a vector).

Then for a European derivative with the payoff function $h(S_T)$ (h is some nonnegative bounded C^2 function) at the maturity date T ($T > 0$) we have the following *time-varying Black-Scholes formula*:

Theorem 2.2.1
With the above notations assume that

$$E\left[\exp\left(\frac{1}{2}\int_0^T (\theta_\tau^{(0)})^2 \, d\tau\right)\right] < \infty. \tag{2.74}$$

If $S_t = s > 0$*, then the no arbitrage price at time* t *for a European derivative with the payoff function* $h(S_T)$ *at the maturity date* T *is given by*

$$P_{tvBS}(t,s) = \frac{e^{-\int_t^T r(\tau)\,d\tau}}{\sqrt{2\pi\int_t^T b_{p-1}^2(\tau)\sigma^2(\tau)\,d\tau}} \int_{\mathbb{R}} h(s\exp(x + \int_t^T (r(\tau) - \frac{1}{2}b_{p-1}^2(\tau)\sigma^2(\tau))\,d\tau))\times$$

$$\times \exp(-\frac{x^2}{2\int_t^T b_{p-1}^2(\tau)\sigma^2(\tau)\,d\tau})\,dx \tag{2.75}$$

for all $t \in [0,T)$.

Proof. It is well-known that the so-called Novikov criterion (2.74) gives a sufficient condition for the validity of Girsanov's Theorem (see e.g., Novikov (1972), Kallianpur (1980)). The equivalent martingale measure under which the market has no arbitrage is given by

$$\frac{dP^*}{dP} = \exp\left(-\frac{1}{2}\int_0^T (\theta_s^{(0)})^2 ds - \int_0^T \theta_s^{(0)} dW_s\right).$$

Moreover, the market is complete and defining

$$W_t^* = W_t + \int_0^t \theta_s^{(0)} ds, \quad t \in [0,T],$$

we have that $(W_t^*)_{t\geq 0}$ is a Brownian motion under P^* (by the Girsanov Theorem) and the asset dynamics under this risk-neutral measure becomes

$$dS_t = r(t)S_t dt + b_{p-1}(t)\sigma(t)S_t\,dW_t^*, \quad t \in [0,T]. \tag{2.76}$$

Thus, the price for the European derivative with the payoff $h(S_T)$ is given by

$$P_{tvBS}(t,s) = E^*\{e^{-\int_t^T r(\tau)\,d\tau}h(S_T)|S_t = s\}, \tag{2.77}$$

where E^* denotes the expectation with respect to the risk-neutral measure P^*. Now using the fact that $\int_t^T b_{p-1}(\tau)\sigma(\tau)dW_\tau^*$ is normally distributed, we get the above explicit formula.

■

2.3 The Estimation Problem

The estimation of continuous time series models is actually an estimation problem for a continuous state-space model, and, away from the linear case, it is known to be one of the most difficult problems in statistics.

Ideally, the estimation problem should be solved with the maximum likelihood procedure, but excepting some few models, the likelihood function is not available analytically and there is a large class of alternative methods proposed in the literature, according to each concrete context in which the estimation problem should be solved.

For continuous autoregressive time series processes where all components of the state variable vector are observed there is a well established statistical estimation theory, both for the parametric and nonparametric cases, see e.g., Prakasa Rao (1999). Among many references in this field, let us quote Kutoyants (1984), Dacunha-Castelle and Florens-Zmirou (1986), Genon-Catalot and Jacod (1993), Bibby and Sorensen (1995), Kessler and Sørensen (1999), Aït Sahalia (2002) and Reiß (2006).

These results do not give an answer to the estimation problem raised by the above discussed general class of continuous time series models of higher order and with latent components, however, for some particular instances, they can be adapted to cope with such cases, too (see, e.g., Aït Sahalia and Kimmel (2007)).

Here we shortly review some procedures proposed in the literature for the case of partially observed systems and which can be applied in practice to carry out the estimation problem for our models of interest. These procedures can be combined with the general approaches in Dahlhaus, Neumann and Sachs (1999) (based on a nonlinear wavelet method) and Malliavin and Mancino (2002) (relying on a Fourier series method), in order to solve the estimation problem in the nonparametric case.

2.3.1 Simulation Free Estimation Procedures

Estimation based on nonlinear filters

This approach can be used not only to estimate the parameters of a general continuous-time state-space model with smooth coefficients using discrete-time observations, but also to solve the corresponding filtering problem, which is very important for many financial applications (e.g., when we are interested in estimating the volatility process).

For nonlinear systems the *extended Kalman filter* (EKF) provides an approximate solution to the estimation and filtering problem (see e.g., Jazwinski (1970)), but for SDE models with state-dependent diffusion coefficient, higher order filters are needed (see e.g., Maybeck (1982), Nielsen and Vestergaard (2000) and Nørgaard et al. (2004)).

Consider a general continuous-time state-space model

$$\mathrm{y}_{t_k} = \mathrm{g}(\mathrm{x}_{t_k}, \boldsymbol{\theta}) + \boldsymbol{\epsilon}(t_k), \quad k = 0, 1, \ldots, N, \tag{2.78}$$

$$d\mathrm{x}_t = \boldsymbol{\alpha}(\mathrm{x}_t, \mathrm{u}_t, \boldsymbol{\theta})\, dt + \boldsymbol{\sigma}(\mathrm{x}_t, \mathrm{u}_t, \boldsymbol{\theta})\, d\mathbf{W}_t, \quad t \geq 0, \tag{2.79}$$

where $0 \leq t_0 < t_1 < \cdots < t_N$ are the times of observation, $\mathrm{y}_t \in \mathbb{R}^m$, $\mathrm{u}_t \in \mathbb{R}^l$ is a deterministic input, $\mathrm{x}_t \in \mathbb{R}^n$, $\mathrm{x}(t_0) = \mathrm{x}_0$ (with $E[|\mathrm{x}_0|^2] < \infty$), $\mathbf{W}_t = (W_t^1, \ldots, W_t^d)^T$

is a d-dimensional Brownian motion with covariance Q_t and the observational noise $\epsilon(t_k)$ is Gaussian with mean zero and covariance R_{t_k}. The smooth functions $\boldsymbol{\alpha} = (\alpha_i)_{1\leq i\leq n}$, $\boldsymbol{\sigma} = (\sigma^{il}) \in \mathbb{R}^{n\times d}$ and $\mathrm{g} = (g_j)_{1\leq j\leq m}$ are known up to the unknown parameter vector $\boldsymbol{\theta} \in \Theta$ and satisfy the conditions which ensure the existence of a unique solution to (2.79).

Basically, the estimation problem reduces to maximizing the following conditional (quasi) log-likelihood function (conditioned on y_{t_0}) of observations:

$$\ln \mathcal{L}(\boldsymbol{\theta}; \mathcal{F}_{t_N}) = -\frac{1}{2}\sum_{k=1}^{N}\Big[\ln\det(\mathrm{S}_{t_k|t_{k-1}}) + \boldsymbol{\delta}^T_{t_k}(\boldsymbol{\theta})\mathrm{S}^{-1}_{t_k|t_{k-1}}\boldsymbol{\delta}_{t_k}(\boldsymbol{\theta}) + m\ln 2\pi\Big], \tag{2.80}$$

where $\mathcal{F}_k$ (or $\mathcal{F}_{t_k}$) is the σ-algebra generated by the observations up to time t_k,

$$\hat{\mathrm{x}}_{t_k|t_{k-1}} := E[\mathrm{x}_{t_k}|\mathcal{F}_{k-1};\boldsymbol{\theta}], \quad \hat{\mathrm{y}}_{t_k|t_{k-1}} = E[\mathrm{y}_{t_k}|\mathcal{F}_{k-1};\boldsymbol{\theta}], \quad \boldsymbol{\delta}_{t_k}(\boldsymbol{\theta}) := \mathrm{y}_{t_k} - \hat{\mathrm{y}}_{t_k|t_{k-1}}$$

and

$$\mathrm{S}_{t_k|t_{k-1}} = E\{[\mathrm{y}_t - \hat{\mathrm{y}}_{t_k|t_{k-1}}][\mathrm{y}_t - \hat{\mathrm{y}}_{t_k|t_{k-1}}]^T|\mathcal{F}_{k-1};\boldsymbol{\theta}\}.$$

These quantities can be determined upon using a second order filter, in which

- the propagation equations are

$$\begin{aligned}
\frac{d}{dt}\hat{\mathrm{x}}_{t|t_{k-1}} &= \boldsymbol{\alpha}(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta}) + E_{k-1}[\mathrm{b}_{t|t_{k-1}}] &\quad (2.81)\\
\frac{d}{dt}\mathrm{P}_{t|t_{k-1}} &= \mathrm{F}(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta})\mathrm{P}_{t|t_{k-1}} + \mathrm{P}_{t|t_{k-1}}\mathrm{F}^T(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta}) &\quad (2.82)\\
&+ E_{k-1}[\boldsymbol{\sigma}(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta})\mathrm{Q}_t\boldsymbol{\sigma}^T(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta})],
\end{aligned}$$

  with the initial conditions $\hat{\mathrm{x}}_{t_{k-1}|t_{k-1}}$ and $\mathrm{P}_{t_{k-1}|t_{k-1}}$, where the bias-correction term $E_{k-1}[\mathrm{b}_{t|t_{k-1}}]$ is an n-dimensional vector whose jth component is

$$E^j_{k-1}[\mathrm{b}_{t|t_{k-1}}] = \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2\alpha_j}{\partial\mathrm{x}^2}(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta})\mathrm{P}_{t|t_{k-1}}\right\} \tag{2.83}$$

  and $\mathrm{F}(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta})$ is the $n\times n$ matrix

$$\mathrm{F}(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta}) = \frac{\partial\boldsymbol{\alpha}}{\partial\mathrm{x}}(\hat{\mathrm{x}}_{t|t_{k-1}};\boldsymbol{\theta}). \tag{2.84}$$

  The last term in (2.82) is a $n\times n$ symmetric matrix, whose element (i,j) is given by

$$E^{ij}_{k-1}[\boldsymbol{\sigma}\mathrm{Q}_t\boldsymbol{\sigma}^T] = \sum_{l,r=1}^{d}\boldsymbol{\sigma}^{il}\mathrm{Q}^{lr}_t(\boldsymbol{\sigma}^T)^{rj} + \,\mathrm{tr}\left\{\Big((\frac{\partial\boldsymbol{\sigma}^{il}}{\partial\mathrm{x}})^T\mathrm{Q}^{lr}_t\frac{\partial(\boldsymbol{\sigma}^T)^{rj}}{\partial\mathrm{x}}\Big)\mathrm{P}\right\}$$

$$+\frac{1}{2}\boldsymbol{\sigma}^{il}\mathrm{Q}^{lr}_t\,\mathrm{tr}\left\{\frac{\partial^2(\boldsymbol{\sigma}^T)^{rj}}{\partial\mathrm{x}^2}\mathrm{P}\right\} + \frac{1}{2}\,\mathrm{tr}\left\{\mathrm{P}\frac{\partial^2\boldsymbol{\sigma}^{il}}{\partial\mathrm{x}^2}\right\}\mathrm{Q}^{lr}_t(\boldsymbol{\sigma}^T)^{rj}.$$

  where a superscript rl denotes the element (r,l) of the corresponding matrix and the dependence on $\hat{\mathrm{x}}_{t|t_{k-1}}$, $t|t_{k-1}$ and $\boldsymbol{\theta}$ has been dropped for convenience.

- the updating equations are

$$\begin{aligned}
\mathbf{A}_{t_k} &= \mathbf{G}(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta})\mathbf{P}_{t_k|t_{k-1}}\mathbf{G}^T(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta}) - E_{k-1}[\tilde{\mathbf{b}}_{t_k|t_{k-1}}]E^T_{k-1}[\tilde{\mathbf{b}}_{t_k|t_{k-1}}] + \mathbf{R}_{t_k}\\
\mathbf{K}_{t_k} &= \mathbf{P}_{t_k|t_{k-1}}\mathbf{G}^T(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta})\mathbf{A}^{-1}_{t_k}\\
\hat{\mathbf{x}}_{t_k|t_k} &= \hat{\mathbf{x}}_{t_k|t_{k-1}} + \mathbf{K}_{t_k}\{\mathbf{y}_{t_k} - \mathbf{g}(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta}) - E_{k-1}[\tilde{\mathbf{b}}_{t_k|t_{k-1}}]\}\\
\mathbf{P}_{t_k|t_k} &= \mathbf{P}_{t_k|t_{k-1}} - \mathbf{K}_{t_k}\mathbf{G}(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta})\mathbf{P}_{t_k|t_{k-1}},
\end{aligned}$$

where

$$\mathbf{G}(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta}) := \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta}) \in \mathbb{R}^{m\times n}$$

and the bias-correction term $E_{k-1}[\tilde{\mathbf{b}}_{t_k|t_{k-1}}]$ is an m-dimensional vector having the jth component

$$E^j_{k-1}[\tilde{\mathbf{b}}_{t_k|t_{k-1}}] = \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 g_j}{\partial \mathbf{x}^2}(\hat{\mathbf{x}}_{t_k|t_{k-1}};\boldsymbol{\theta})\mathbf{P}_{t_k|t_{k-1}}\right\}.$$

To start the algorithm one has to choose $\hat{\mathbf{x}}_{t_0|t_0}$ and $\mathbf{P}_{t_0|t_0}$, however the initialization is a non-trivial issue (see e.g., Koopman (1997) and Durbin and Koopman (2001) for a discussion). The resulting estimator is usually called *quasi maximum likelihood estimator* (QMLE). More about the QML estimator and other improvements of the EKF procedure can be found for instance in Lund (1997). All these algorithms can be easily adaptated to the case when the functions α, σ and g depend on the supplementary time variable t.

Besides being a simulation free method, this has another great advantage, as it offers as a byproduct an easy way to compute classical model selection criteria.

The two most common information criteria are AIC (Akaike's Information Criterion) and BIC (Bayesian Information Criterion), where the model order can be found through minimizing the following expressions (see e.g., Burnham and Anderson (2002)):

$$\begin{aligned}
\text{AIC} &= 2\cdot \dim\boldsymbol{\theta} - 2\ln\mathcal{L}(\boldsymbol{\theta};\mathcal{F}_{t_N}), &(2.85)\\
\text{BIC} &= \dim\boldsymbol{\theta}\cdot\ln(N+1) - 2\ln\mathcal{L}(\boldsymbol{\theta};\mathcal{F}_{t_N}). &(2.86)
\end{aligned}$$

In the context of state-space models, AIC was used among others by Kitagawa (1981) and Harvey (1989).

It is well known that BIC generally gives a consistent estimate of the model order dim $\boldsymbol{\theta}$, whereas AIC is most often not a consistent estimator, since it has the tendency to overestimate the model order. More on this topic can be found for instance in Cavanaugh and Shumway (1997), Basak, Chan and Lee (2003), Berg, Meyer and Yu (2004), Bengtsson and Cavanaugh (2006).

The class of models (2.78)-(2.79) includes many of the continuous time series models discussed above and this estimation method applies (at least in principle) to nonlinear and non-stationary models of higher dimension.

In practice, the QMLE procedure described above remains computationally intensive, especially for high dimensional models, since one has to solve numerically a large set of differential equations for every evaluation of the log-likelihood function.

2.3.2 Estimation Procedures Based on Monte Carlo Simulations

Estimation by means of an auxiliary model

There are two main methods proposed within this framework: the *Indirect Inference Method*, proposed by Gourieroux et al. (1993) and the *Efficient Method of Moments* ($\mathbb{EMM}$) proposed by Gallant and Tauchen (1996). The objective is to estimate parameters efficiently when maximum likelihood is infeasible. These methods are very flexible and can be applied essentially whenever simulations from the model are possible and there is a suitable auxiliary model available.

The main idea is the following: let $\{\tilde{Y}_t\}_{t=1}^n$ denote the actual observed data for the estimation of $\boldsymbol{\rho}$, the unknown parameter vector of the structural model. Given the sequence of densities for the auxiliary model $\{f_t(y_t|y_{t-L},\ldots,y_{t-1},\boldsymbol{\theta})\}_{t\geq 1}$, $L \geq 1$, $\boldsymbol{\theta} \in \Theta$, the first step is to compute the maximum likelihood estimator of $\boldsymbol{\theta}$, i.e.

$$\tilde{\boldsymbol{\theta}}_n = \operatorname*{argmax}_{\boldsymbol{\theta}\in\Theta} \frac{1}{n}\sum_{t=1}^{n} \log f_t(\tilde{Y}_t|\tilde{Y}_{t-L},\ldots,\tilde{Y}_{t-1},\boldsymbol{\theta}). \tag{2.87}$$

For the second step, the moment criterion is

$$m_n(\boldsymbol{\rho},\tilde{\boldsymbol{\theta}}_n) = \frac{1}{N}\sum_{\tau=1}^{N}\frac{\partial}{\partial\boldsymbol{\theta}}\log f_\tau(\hat{Y}_\tau(\boldsymbol{\rho})|\hat{Y}_{\tau-L}(\boldsymbol{\rho}),\ldots,\hat{Y}_{\tau-1}(\boldsymbol{\rho}),\tilde{\boldsymbol{\theta}}_n), \tag{2.88}$$

which is computed by averaging over a long simulation $\{\hat{Y}_t(\boldsymbol{\rho})\}_{t=1}^N$ from the structural model. Then the estimator of the structural parameter vector is

$$\hat{\boldsymbol{\rho}}_n = \operatorname*{argmin}_{\boldsymbol{\rho}\in R} m'(\boldsymbol{\rho},\tilde{\boldsymbol{\theta}}_n)(\tilde{\mathcal{I}}_n)^{-1}m(\boldsymbol{\rho},\tilde{\boldsymbol{\theta}}_n), \tag{2.89}$$

where $(\tilde{\mathcal{I}}_n)^{-1}$ is a weighting matrix. If the auxiliar model is a good statistical approximation to the data generating process, then one can use

$$\tilde{\mathcal{I}}_n = \frac{1}{n}\sum_{t=1}^{n}\left[(\frac{\partial}{\partial\boldsymbol{\theta}})\log f_t(\tilde{Y}_t|\tilde{Y}_{t-L},\ldots,\tilde{Y}_{t-1},\tilde{\boldsymbol{\theta}}_n)\right]\left[(\frac{\partial}{\partial\boldsymbol{\theta}})\log f_t(\tilde{Y}_t|\tilde{Y}_{t-L},\ldots,\tilde{Y}_{t-1},\tilde{\boldsymbol{\theta}}_n)\right]'. \tag{2.90}$$

The estimators obtained with these methods are consistent and asymptotically normal (see e.g., Gallant and Tauchen (1996) and the references therein).

One of the main drawbacks of the Indirect Inference and $\mathbb{EMM}$ estimators is that none of them provide filtering and smoothing solutions for the associated latent variables. Furthermore, these procedures are very expensive in computational terms.

More about the problems with $\mathbb{EMM}$ can be read e.g., in Alankar (2003).

Estimation based on the Bayesian approach

The Bayesian approach assumes the unknown parameter as random and then uses methods like *Markov Chain Monte Carlo* (MCMC) or *particle filtering* to approximate the densities of interest for parameter estimation.

For discretely observed diffusions the MCMC methods have been discused by Eraker (2001) and Elerian et al. (2001). The unknown model parameter is treated as a missing data point and MCMC methods are used for simulating from the posterior distribution of the parameter. These methods also apply to time-homogeneous diffusion models which are partially observed, however are computationally very expensive.

Moreover, as noticed in Doucet, Godsil and West (2004), for models with significant degree of nonlinearity and non-Gaussianity it is not always straightforward to construct an effective MCMC sampler and the danger then is that the MCMC will be slowly mixing and may never converge to the target distribution within a realistic time scale.

Such difficulty will increase, of course, when trying to treat the case of nonlinear time-inhomogeneous partially observed diffusion models, which is the most interesting one in the class of non-stationary and nonlinear models.

More appropriate to deal with the nonstationary and nonlinear models seem to be the particle filtering methods. They estimate the densities of interest by a swarm of weighted particles and are among the most used methods for analyzing partially-observed models.

Particle filters, also referred in the literature as bootstrap filters, interacting particle filters, condensation algorithms and Monte Carlo filters perform sequential Monte Carlo (SMC) estimation based on point mass (or particle) representation for the probability densities of interest. The basic SMC ideas in the form of sequential importance sampling have been introduced in statistics back in the 1950s, by Hammersley and Morton. Although these ideas continued to be explored sporadically during the 1960s and 1970s, they were largely overlooked and ignored, most likely because of the modest computational power available at that time. But in the last years the research activity in this field has dramatically increased, resulting in many improvements of the particle filtering methods and their numerous applications.

For a comprehensive state of the art see Doucet, de Freitas and Gordon (2001). Here, instead of estimating the parameter vector $\boldsymbol{\theta}$ from the discrete time observations $\{\mathrm{y}(t_k)\}$, $k = 0, \ldots, N$ by the maximum likelihood method (which for nonlinear continuous time series models is infeasible) the Bayesian estimation is considered by augmenting the state vector. That is, consider a different model in which $\boldsymbol{\theta}$ is replaced by $\boldsymbol{\theta}(t_k)$ and simply include $\boldsymbol{\theta}(t_k)$ in an augmented state vector $L(t_k)$, for all $k \geq 1$. The other components of $L(t_k)$ are obtained by the principle of 'data augmentation' applied between t_{k-1} and t_k, which in our context means to substitute the continuous state-space model with a discrete one, using for instance an Euler discretization schema for diffusion processes. The model parameters are viewed as if they were time-evolving, $\boldsymbol{\theta}(t_k) = \boldsymbol{\theta}(t_{k-1}) + \eta_k$, , $k = 1, \ldots, N$, with η_k *i.i.d.*

Gaussian random variables involving some specified variance matrix Υ_k.

Denoting $\mathcal{F}_k$ the σ-algebra generated by the observations up to time t_k, the optimal filtering problem is solved by the sequence of conditional densities, $p(L(t_k)|\mathcal{F}_k)$, for $k = 1, \dots, N$. By Bayes' rule

$$p(L(t_k)|\mathcal{F}_k) = \frac{p(\mathrm{y}(t_k)|L(t_k))p(L(t_k)|\mathcal{F}_{k-1})}{p(\mathrm{y}(t_k)|\mathcal{F}_{k-1})}, \tag{2.91}$$

however the computation of these densities is difficult, as none of the distributions is analytical. For example,

$$p(L(t_k)|\mathcal{F}_{k-1}) = \int p(L(t_k)|L(t_{k-1})p(L(t_{k-1})|\mathcal{F}_{k-1})\; dL(t_{k-1}) \tag{2.92}$$

is a high-dimensional integral, usually computable in practice only with Monte Carlo simulation.

The particle filter method approximates the filtering density $p(L(t_k)|\mathcal{F}_k)$ with a discrete probability distribution $p^{\mathcal{J}}(L(t_k)|\mathcal{F}_k)$ given by

$$p^{\mathcal{J}}(L(t_k)|\mathcal{F}_k) = \sum_{i=1}^{\mathcal{J}} \delta_{L(t_k)^{(i)}} \pi_{t_k}^{(i)},$$

where δ is the Dirac function, $\{L(t_k)^{(i)}\}$ is a set of $\mathcal{J}$ particles and $\{\pi_{t_k}^{(i)}\}$ are the corresponding probabilities associated to these particles.

Once the distribution is discretized, the integrals become sums. For example, the estimate of the predictive distribution is

$$p^{\mathcal{J}}(L(t_k)|\mathcal{F}_{k-1}) = \sum_{i=1}^{\mathcal{J}} p\Big(L(t_k)|L(t_{k-1})^{(i)}\Big)\pi_{t_{k-1}}^{(i)}$$

and then for the filtering density at time t_k we have

$$p^{\mathcal{J}}(L(t_k)|\mathcal{F}_k) \propto p(\mathrm{y}(t_k)|L(t_k)) \sum_{i=1}^{\mathcal{J}} p\Big(L(t_k)|L(t_{k-1})^{(i)}\Big)\pi_{t_{k-1}}^{(i)}.$$

Following the approach described in Gordon, Salmond and Smith (1993), in order to update the particles one needs to propagate the states forward by drawing

$$L(t_k)^{(i)} \sim p\Big(L(t_k)|L(t_{k-1})^{(i)}\Big),$$

for all $i = 1, \dots \mathcal{J}$ and then use a resampling procedure for these states $\{L(t_k)^{(i)}\}$ with weights $\pi_{t_k}^{(i)} \propto p(\mathrm{y}(t_k)|L(t_k))$.

The above described particle filter requires only that the likelihood function $p(\mathrm{y}(t_k)|L(t_k))$ can be evaluated and the states can be sampled from $p(L(t_k)|L(t_{k-1}))$.

Under these two mild assumptions, the particle filter can be applied to a large class of models (nearly all state space models of practical interest). Moreover, in practice, this procedure can be improved using additional sampling methods such as those described e.g., in Pitt and Shephard (1999).

The particle filter methods have already proven their usefulness in practice for many examples including highly non-linear models that are not easily implemented, upon using standard MCMC (see Doucet, Godsil and West (2004) and references therein).

2.4 Simulation and Estimation Results

In figure 2.2 we plotted two simulated trajectories of the geometric Brownian motion $(Y_t)_{t\geq 0}$ given by (2.15) with the parameters $\mu = 0.1$, $\sigma = 0.07$. The simulations have been done with the classical Euler-Maruyama scheme (see e.g., Kloeden and Platen (2001)). The corresponding stochastic differential equation with initial value $Y_0 = 620$ was discretised with a time step $\Delta = 0.0004$. The plotted values correspond to a temporal equidistant grid with a time step $\tilde{\Delta} = 0.004$.

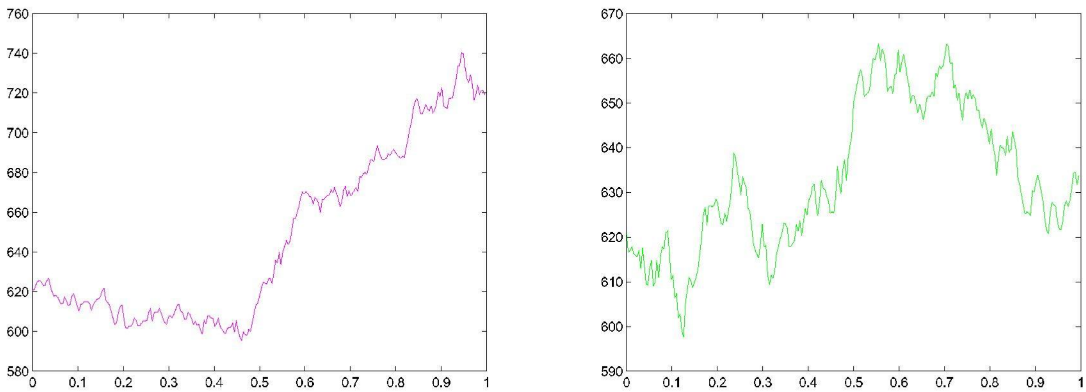

Figure 2.2: Simulated trajectories of Y_t

As observed above, most financial time-varying models studied in literature are actually continuous autoregressive models of order one.

Here we give an example of a tvNLCARMA$(2, \Psi)$ process along with the corresponding simulation and estimation results. The proposed model for asset prices is

$$S_t = \Psi(t, \mathbf{X}(t)) = \exp(\theta^2(-2 + \exp(\theta^2 t)) \cdot X_t + \exp(\theta^2 t) \cdot X_t^{(1)}), \tag{2.93}$$

$$d\mathbf{X}(t) = (A \cdot \mathbf{X}(t) + \alpha_0(t) \cdot \mathrm{e})\, dt + \sigma(t) \cdot \mathrm{e}\, dW_t, \tag{2.94}$$

for all $t \in [0, 1]$, where

$$A = \begin{bmatrix} 0 & 1 \\ \alpha_2 & \alpha_1 \end{bmatrix}, \quad \mathbf{X}(t) = \begin{bmatrix} X_t \\ X_t^{(1)} \end{bmatrix}, \tag{2.95}$$

$e = [0, 1]'$, $\alpha_0(t) = (1 - 5\theta) \cdot \exp(-\theta^2 t)$, $\alpha_1 = -2 \cdot \theta^2$, $\alpha_2 = -\theta^4$, $\sigma(t) = \sigma_0 \cdot \exp(-\theta^2 t)$, θ and σ_0 are real constants ($\sigma_0 > 0$).

Remark 2.4.1
Notice that this model was defined on $[0, 1]$ and thus the diffusion coefficient is here always positive, $\sigma(t) \geq \sigma_0 \cdot \exp(-\theta^2) > 0$, for all $t \in [0, 1]$; however, the definition can be easily extended to other compact time intervals by an appropriate time rescaling.

The construction of this model was inspired from (2.17), studied in Filipovic (2000), and from the asymptotic analysis developed in one of the next chapters for higher order stochastic volatility models.

For the estimation procedure we used daily Standard & Poor's 500 (S&P 500) stock index[1] closures for the year 1996 plotted in figure 2.3 after the usual rescaling on $[0, 1]$. Here we do not neglect the presence of a microstructure noise in the data set (see e.g., Ait-Sahalia, Mykland and Zhang (2006)).

Using the QML estimation method described above, the estimated values are $\hat{\theta} = -1.2$ and $\hat{\sigma_0} = 0.01$. Initially, at $t_0 = 0$, the algorithm was started with $\hat{x}_{t_0|t_0} = [0, 6.4]'$ and $P_{t_0|t_0} = 10^{-4} I_2$, where I_2 is the identity matrix. The computations have been done with MATLAB using implementation ideas similar to those in Grewal and Andrews (2001).

With the previously estimated parameters we plotted in figure 2.4 two simulated trajectories of daily values of the process $(S_t)_{t \geq 0}$ (i.e. the values corresponding to the same observation times as for the S&P 500 data), given by (2.93). As usual, the simulations have been done with the Euler-Maruyama scheme. The corresponding stochastic differential equations were discretised with a time step $\Delta = 0.0004$ with the initial values $[0, 6.4]'$.

Transforming the classical Black-Scholes model in a state-space form and repeating the above described estimation procedure it can be seen that the smallest values for both criteria described in (2.85)-(2.86) are attained by the newly introduced model.

[1] freely available on the web e.g., at `http://finance.yahoo.com/`

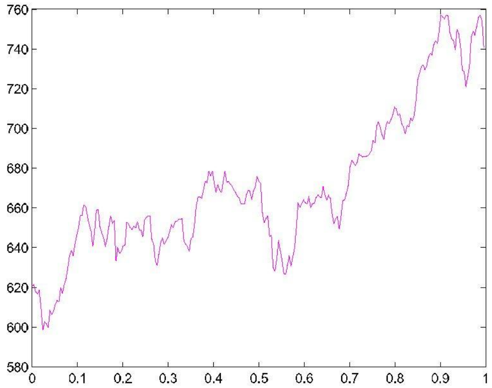

Figure 2.3: Daily S&P 500 index for the year 1996.

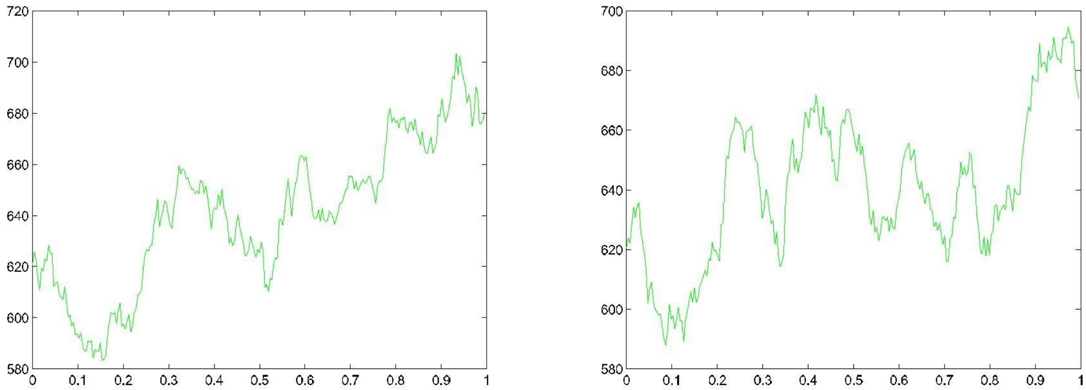

Figure 2.4: Simulated trajectories of S_t

Chapter 3

Extensions of Classical Stochastic Volatility Models

The accurate modeling of volatility is important for derivative pricing and risk management. Volatility as a measure of risk is seen by traders as the most important variable (after the price of the underlying asset itself) in deriving the probabilities of profit or loss.

The celebrated Black-Scholes option pricing formula (see Appendix A) is based on a model for asset prices which assumes constant volatility. However, such a model is not flexible enough to reproduce some stylized facts observed in derivative prices such as the smile effect, which is a U-shaped relationship between the implied Black-Scholes volatility and the strike price, for any given residual maturity. To overcome this difficulty, the model has to be extended.

One way to create such a smile effect is to introduce a stochastic volatility in the Black-Scholes model. Since the late 1980s, this approach has been developed among others by Wiggins (1987), Hull and White (1987a), Scott (1982), Stein and Stein (1991), Heston (1993), Ball and Roma (1994) and Fouqué, Papanicolaou and Sircar (2000). For more accounts on stochastic volatility, see e.g., Ghysels et al. (1996) and Shephard (2005).

The stochastic volatility (SV) model describes a much more complex market behavior than the Black-Scholes model, however the corresponding derivative pricing problem to be presented in the following becomes more difficult.

The purpose of this chapter is to evidentiate the potential of CARMA$_\varepsilon$ processes in modeling the volatility. Thus, first we extend some classical SV models and then we give a result concerning the evolution of the derivative pricing function in this framework. Some simulations are performed in order to illustrate these models. Finally, in this general SV setting and in the context of high frequency (HF) data, we discuss a few estimation results for some important financial quantities for the pricing of derivatives, like integrated variance, spot volatility and effective volatility.

3.1 Stochastic Volatility Models Driven by CARMA Type Processes

Stein & Stein (1991) considered a case where the volatility followed a CAR(1) process. Fouqué et al. (2000) extended the above settings and presented a new approach to stochastic volatility modeling by exploiting the mean-reverting behavior of volatility and the empirical fact that it is persistent.

Here we extend the stochastic volatility models discussed in Fouqué et al. (2000) in order to increase the flexibility for modeling financial time series. For characterizing the volatility we use in the following a CARMA$_\varepsilon$ process (i.e. a perturbed CARMA process as described in Subsection 2.1.2). Thus we consider the volatility to be a function of a multivariate linear diffusion process of order p which is a special case of (2.65). In this way the volatility becomes for $p > 1$ a non-Markovian structure and this allows to capture the main characteristics of financial series of returns better than the classical SV models shortly described in the Appendix B.

The idea of modeling the volatility with CARMA processes has been recently proposed in the literature (see e.g., Brockwell (2009) and the references therein).

The model

Let the asset price $(S_t)_{t\geq 0}$ satisfy the SDE

$$dS_t = \mu \cdot S_t\, dt + \sigma_t \cdot S_t\, dW_t, \quad t \geq 0, \tag{3.1}$$

where the volatility process $(\sigma_t)_{t\geq 0}$ is given by $\sigma_t = f(U(t))$; $f \in C^2(\mathbb{R})$ is some positive bounded function which is also bounded away from zero, and $(U(t))_{t\geq 0}$ is a stationary CARMA$_\varepsilon(p, q)$ process given by the observation and state equations

$$U(t) = \mathrm{b}' \cdot \mathbf{U}(t), \quad t \geq 0, \tag{3.2}$$

$$d\mathbf{U}(t) = (\mathbf{A} \cdot \mathbf{U}(t) + \delta c \cdot \mathrm{e})\, dt + \Gamma_o \cdot d\hat{\mathbf{Z}}(t), \quad t \geq 0, \tag{3.3}$$

where

$$\mathrm{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{U}_t = \begin{bmatrix} U_0(t) \\ U_1(t) \\ \vdots \\ U_{p-2}(t) \\ U_{p-1}(t) \end{bmatrix}, \tag{3.4}$$

$$\mathrm{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} \psi_1 & 0 & \cdots & 0 & 0 \\ 0 & \psi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \psi_{p-1} & 0 \\ 0 & 0 & \cdots & 0 & \psi_p \end{bmatrix}, \quad \hat{\mathbf{Z}}(t) = \begin{bmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_{p-1}(t) \\ \hat{Z}(t) \end{bmatrix}, \tag{3.5}$$

$$\hat{Z}(t) = \rho W(t) + \sqrt{1-\rho^2} Z_p(t), \quad t \geq 0, \tag{3.6}$$

where $a_1, ..., a_p, b_1, ..., b_q$, c, δ, $\psi_1, \ldots, \psi_p$ are constants ($\delta = \psi_p > 0$, $\psi_k \geq 0$, $k = 1, \ldots, p-1$), W, $Z_1, \ldots, Z_p$ are $p+1$ independent Brownian motions, $\rho \in [-1, 1]$, $\mathrm{e}, \mathrm{b} \in \mathbb{R}^p$, $b_0 = 1$, $q < p$, $b_q \neq 0$ and $b_j = 0$ for $j > q$.

In the following we will use for the above introduced model the shortcut SV CARMA type model or SV CAR type model if $q = 0$.

Remark 3.1.1

(a) In order not to complicate the exposition we only considered for the components of $\hat{\mathbf{Z}}(t)$ the simplified correlation scheme (3.6) in order to allow for a *leverage effect* (see e.g., Appendix B), however the corresponding results can be easily extended to allow for more general correlations.

(b) We let $\langle . \rangle$ denote the averaging with respect to the invariant distribution ϕ of the process $(\mathbf{U}_t)_{t\geq 0}$, i.e.

$$\langle g \rangle = \int_{\mathbb{R}^p} g(\mathbf{u})\phi(\mathbf{u})\, d\mathbf{u}, \tag{3.7}$$

for all $g : \mathbb{R}^p \to \mathbb{R}$ for which the above integral exists.

In this context, the *spot volatility* process of the above SV model is $(\sigma_t^2)_{t\geq 0}$ and the *effective volatility* $\overline{\sigma}$ is given by $\overline{\sigma}^2 = \langle \tilde{g} \rangle$, where $\tilde{g}(\mathrm{u}) = f^2(\mathrm{b}'\mathrm{u})$, for all $\mathrm{u} \in \mathbb{R}^p$. This is a crucial quantity in pricing problems for European derivatives, as discussed in the next chapter. Analogously to Fouqué et al. (2000), we have for the ergodic process $(U(t))_{t\geq 0}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f^2(U(s))ds \to \overline{\sigma}^2 \qquad a.s. \tag{3.8}$$

Consequently, when T is large

$$\overline{\sigma^2} = \frac{1}{T-t} \int_t^T f^2(U(s))ds \approx \overline{\sigma}^2. \tag{3.9}$$

Clearly, this kind of approximation of the *mean-square time-averaged volatility* $\overline{\sigma^2}$ holds also when the volatility is bursty (i.e. its well-known mean reversion property is fast) for any $T > t$. These facts lead to the asymptotic computations in the next chapter.

(c) The quantity $\sigma_t^* := \int_0^t f^2(U(s))ds$, for all $t \geq 0$ is called the *integrated variance* (or *integrated volatility*) and it is the quadratic variation of the log-asset process at time t.

Simulation results

In figures 3.1 and 3.2 we show some sample paths or realizations of some SV CARMA(2,1) and SV CARMA(3,2) type models with $f = \exp$. These are generalizations of the classical Scott SV model, see Appendix B. For the involved CARMA$_\varepsilon$ processes we used the decomposition $U(t) = m + \tilde{U}(t)$, where:

- for the SV CARMA(2,1) type model $S_0 = 620$, $\mu = 0.06$, $m = -2.1$ and $\tilde{U}(t)$ is a zero mean CARMA$_\varepsilon$ process with the corresponding state variable starting from $(0,0)'$, with parameters $\mathrm{b} = (-0.5, 1)'$, $\psi_1 = 0.02$, $\delta = 4$, $c = 0$, $\rho = -0.2$,
$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ -0.4 & -246.5 \end{bmatrix}.$$
- for the SV CARMA(3,2) type model $S_0 = 972$, $\mu = 0.02$, $m = -2.5$, and $\tilde{U}(t)$ is a zero mean CARMA$_\varepsilon$ process with the corresponding state variable starting from $(0,0,0)'$, with parameters $\mathrm{b} = (1.2, 4, 1)'$, $\psi_1 = 0.02$, $\psi_2 = 0.02$, $\delta = 4.7$, $c = 0$, $\rho = -0.2$,
$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.9 & -3.6 & -246.2 \end{bmatrix}.$$

The simulations are done with the classical Euler-Maruyama scheme, upon using a discretisation with time step $\Delta = 0.002$.

These figures show that many important stylized facts of financial data (such as volatility clustering, volatility mean reversion, or leverage effects) can be captured with the SV CARMA type models. Further simulation experiments will be performed and discussed in the next chapter for the subclass of SV CAR type models.

3.2 Derivative Pricing in SV CARMA Type Models

One of the important issues in finance is to price derivatives based on the dynamics of asset prices. Next we characterize the evolution of the pricing function for European derivatives (see Appendix A) when the asset dynamics is described with the stochastic volatility model given by (3.1)- (3.3) under the assumption that the usual conditions for the validity of Girsanov's Theorem are satisfied in the above framework (a sufficient condition is the well-known Novikov criterion, see e.g., Kallianpur (1980)).

We consider the case of a European derivative with the maturity date T ($T > 0$) and the payoff function $h(S_T)$ (h is some nonnegative bounded C^2 function). The fundamental problem in mathematical finance is to find the "fair price" $\tilde{P}(t, s, \mathrm{u})$ of such derivative at a time t

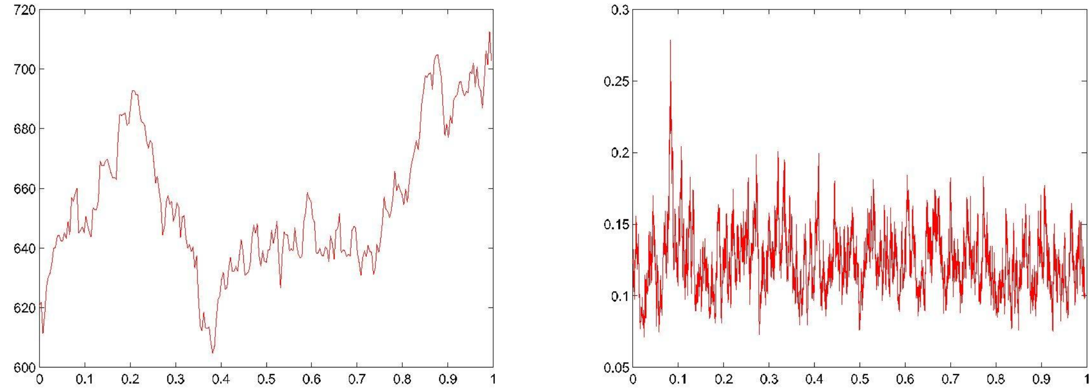

Figure 3.1: Simulated SV CARMA(2,1) type process. Left: asset values. Right: volatility

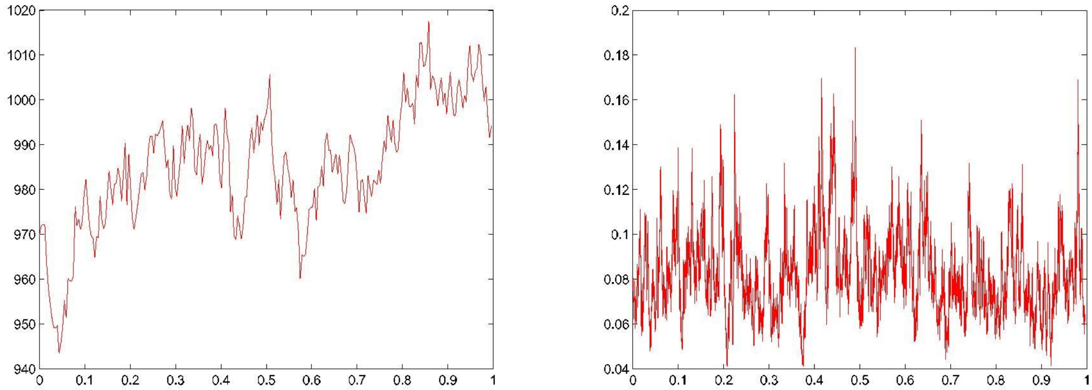

Figure 3.2: Simulated SV CARMA(3,2) type process. Left: asset values. Right: volatility

prior to expiry, if $S_t = s \in \mathbb{R}_+$ and $\mathbf{U}_t = \mathbf{u} \in \mathbb{R}^p$. Following the classical no-arbitrage approach (see e.g., Fouqué et al. (2000)), one can derive a partial differential equation (PDE) for $\tilde{P}$. For various accounts on PDEs see e.g., Evans (1998), Karatzas and Shreve (1988) or Øksendal (1998).

Let us define for $t \in [0, T]$

$$W_t^* = W_t + \int_0^t \frac{(\mu - r)}{f(U_s)} ds, \tag{3.10}$$

$$Z_k^*(t) = Z_k(t) + \int_0^t \gamma_k(s) ds, \quad k = 1, \dots p, \tag{3.11}$$

where $\boldsymbol{\gamma}_t = (\gamma_1(t), \dots, \gamma_p(t))'$ are any adapted (and suitably regular) processes. Then by the classical Girsanov Theorem, $(W_t^*)_{t\in[0,T]}$, $(Z_k^*(t))_{t\in[0,T]}^{k=1,\dots,p}$ are independent standard Brownian motions under the measure $P^{*(\gamma)}$ defined by

$$\frac{dP^{*(\gamma)}}{dP} = \exp\left(-\frac{1}{2}\int_0^T (\sum_{k=0}^{p+1}(\theta_s^{(k)})^2)ds - \int_0^T \theta_s^{(0)} dW_s - \sum_{k=1}^{p}\int_0^T \theta_s^{(k)} dZ_k(s) \right), \tag{3.12}$$

$$\theta_t^{(0)} = \frac{(\mu - r)}{f(U_t)}, \quad \theta_t^{(k)} = \gamma_k(t),\ k = 1, \dots, p. \tag{3.13}$$

The components of the process $(\gamma_t)_{t\in[0,T]}$ are called *risk premium factors* or *market prices of volatility risk.*

We will consider $\gamma_t = \gamma(\mathbf{b}'\mathbf{U}_t)$, with $\gamma = (\gamma_1, \dots, \gamma_p)'$, $\gamma_k \in C_c^2(\mathbb{R})$ (i.e. $\gamma_k \in C^2(\mathbb{R})$ with compact support), $k = 1, \dots, p$.

Any allowable choice of γ leads to an *equivalent martingale measure* (EMM) $P^{*(\gamma)}$ and the corresponding *no arbitrage price* for a European derivative with the payoff function $h(S_T)$ at maturity time T is

$$\tilde{P}(t, S_t, \mathbf{U}_t) = E^{*(\gamma)}\{e^{-r(T-t)}h(S_T)|S_t, \mathbf{U}_t\}, \quad t \in [0, T], \tag{3.14}$$

or

$$\tilde{P}(t, s, \mathbf{u}) = E^{*(\gamma)}\{e^{-r(T-t)}h(S_T)|S_t = s, \mathbf{U}_t = \mathbf{u}\}, \tag{3.15}$$

where r is the instantaneous interest rate, $s \in \mathbb{R}_+$, $\mathbf{u} = (u_0, u_1, \dots, u_{p-1})' \in \mathbb{R}^p$ and $E^{*(\gamma)}$ denotes the expectation with respect to the probability measure $P^{*(\gamma)}$.

Proposition 3.2.1
In the above framework the pricing function $\tilde{P}(t, s, u_0, u_1, \dots, u_{p-1})$ satisfies the partial differential equation

$$\frac{\partial \tilde{P}}{\partial t} + L\,\tilde{P} - r\tilde{P} = 0, \tag{3.16}$$
$$0 \le t < T, \quad s \in \mathbb{R}_+, \quad \mathbf{u} \in \mathbb{R}^p,$$

with $\tilde{P}(T, s, \mathbf{u}) = h(s)$, for all $s \in \mathbb{R}_+$, $\mathbf{u} \in \mathbb{R}^p$, $\mathbf{u} = (u_0, u_1, \dots, u_{p-1})'$, where

$$L = \frac{1}{2}\Big(f^2(\mathbf{b}'\mathbf{u})s^2\frac{\partial^2}{\partial s^2} + 2\delta\rho f(\mathbf{b}'\mathbf{u})s\frac{\partial^2}{\partial s\,\partial u_{p-1}} + \delta^2\frac{\partial^2}{\partial u_{p-1}^2} + \sum_{k=0}^{p-2}\psi_{k+1}^2\frac{\partial^2}{\partial u_k^2}\Big) + r\cdot s\frac{\partial}{\partial s}$$
$$+\Big(\mathbf{A}\cdot\mathbf{u} + c\delta\cdot\mathbf{e} - \Gamma_o\cdot\mathbf{\Lambda}(\mathbf{u})\Big)'\frac{\partial}{\partial\mathbf{u}}, \quad \frac{\partial}{\partial\mathbf{u}} = \left[\frac{\partial}{\partial u_0}, \cdots \frac{\partial}{\partial u_{p-1}}\right]', \tag{3.17}$$

$$\mathbf{\Lambda}(\mathbf{u}) = \rho\frac{(\mu - r)}{f(\mathbf{b}'\mathbf{u})}\mathbf{e} + \mathcal{D}(\sqrt{1-\rho^2})\cdot\gamma(\mathbf{b}'\mathbf{u}), \quad (\forall)\,\mathbf{u} \in \mathbb{R}^p, \tag{3.18}$$

with

$$\mathcal{D}(x) = (x-1)\cdot\mathbf{e}\mathbf{e}' + \mathbf{I}_p = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & x \end{bmatrix}, \quad (\forall)\, x \in \mathbb{R}. \tag{3.19}$$

Proof. Under $P^{*(\gamma)}$, the above model becomes

$$dS_t = rS_t\,dt + f(U_t)S_t dW_t^*, \quad U(t) = \mathbf{b}' \cdot \mathbf{U}(t), \tag{3.20}$$

$$d\mathbf{U}(t) = \Big[\mathbf{A} \cdot \mathbf{U}(t) + \Big(\delta c\,\mathbf{e} - \Gamma_o \cdot \Lambda(\mathbf{U}(t))\Big)\Big]\,dt + \Gamma_o \cdot d\hat{\mathbf{Z}}^*(t), \tag{3.21}$$

for all $t \geq 0$, where

$$\hat{\mathbf{Z}}^*(t) = \begin{bmatrix} Z_1^*(t) \\ Z_2^*(t) \\ \vdots \\ Z_{p-1}^*(t) \\ \hat{Z}^*(t) \end{bmatrix}, \tag{3.22}$$

$$\hat{Z}^*(t) = \rho \cdot W_t^* + \sqrt{1-\rho^2} \cdot Z_p^*(t). \tag{3.23}$$

Then, clearly the operator L in the Proposition is the generator of the above multivariate diffusion process (see e.g., Karatzas and Shreve (1988) or Øksendal (1998)) and the result follows from standard arguments like in Fouqué et al. (2000).

■

Remark 3.2.1

(a) The result presented above offers the possibility to compute the fair price of the derivative by solving a partial differential equation and this is very useful in practice, especially for SV models with lower order CARMA$_\varepsilon$ processes. It can be easily extended to the case when the volatility is modelled with more general continuous-time series processes, like those introduced in Chapter 2. However, solving numerically the corresponding partial differential equation for a high dimensional model is a nontrivial issue.

We present in the next chapter an alternative way to overcome this difficulty. It is based on the fast mean-reversion asymptotic theory developed in Fouqué et al. (2000).

(b) The above regularity and boundedness assumptions on the payoff function or the asset volatility driving function have been made in order to keep the exposition simple, however many of them can be relaxed upon employing more technical results from the theory of partial differential equations, see e.g., Karatzas and Shreve (1988). This remark also applies to the results in the subsequent chapters. A detailed analysis of these issues would be beyond the purpose of this thesis.

3.3 Some Estimation Results for the High Frequency Data Case

This is the case where on a fixed interval of time h (for instance a day or a week) there are M observations (with M very large). The observed process will be in our case $X_t = \log S_t$

(i.e. the logarithmic asset price) and the main financial quantities of interest in this section are the integrated variance, spot volatility and effective volatility.

One purpose of using high frequency (HF) financial time series is to estimate more precisely and more directly the volatility and its derivatives. There is by now a substantial literature on the use of HF financial data, see, e.g., Andersen, Bollerslev and Diebold (2002) for a survey.

We consider now for the dynamics of the asset price a stochastic volatility model driven by a CARMA$_\varepsilon$ process $(U(t))_{t\geq 0}$ like above, however our results concerning the estimation of the integrated variance and spot volatility hold as well for more general volatility processes.

Here we assume that we are in the no leverage case, meaning that there is no correlation between the Brownian motions in the model. Using Ito's formula and (3.1) we have for $X_t = \log S_t$

$$dX(t) = d\alpha^*(t) + f(U(t))dW(t), \quad t \geq 0, \tag{3.24}$$

where

$$\alpha^*(t) = \int_0^t (\mu - \frac{1}{2}f^2(U(s)))ds, \quad t \geq 0. \tag{3.25}$$

The high frequency observations are $\{X((i-1)h+(j-1)\frac{h}{M})\}$, $i = i_A, \ldots, i_B, j = 1, \ldots, M$, $(i_A, i_B \in \mathrm{N}^*$, $i_B \geq i_A)$. We will denote

$$\Delta = \frac{h}{M}, \quad t_0 = (i_A - 1) \cdot h, \tag{3.26}$$

$$t_{j,i} = \left[(i-1) + \frac{j-1}{M}\right] \cdot h, \quad (\forall)\, i = i_A, \ldots i_B, \quad j = 1, \ldots, M. \tag{3.27}$$

Denote the j-th intra-h return for the i-th period by

$$x_{j,i} = X(t_{j+1,i}) - X(t_{j,i}) \in \mathbb{R}, \tag{3.28}$$

for all $i = i_A, \ldots, i_B$, $j = 1, \ldots, M$.

The *realized variation* for the i-th period is defined as:

$$[X_M]_i = \sum_{j=1}^{M} x_{j,i}^2, \quad i = i_A, \ldots i_B. \tag{3.29}$$

Recall that the spot volatility of the SV model is $\sigma_t^2 = f^2(U(t))$ and the quadratic variation of X at time t is $[X](t) = \sigma_t^* = \int\limits_0^t f^2(U(s))ds$, for all $t \geq 0$ (see also Remark 3.1.1, (b), (c)).

Let us consider $t \in \{i \cdot h | i = i_A, \ldots, i_B\}$ and denote by $\hat{\sigma^2}_{h,M}(t)$ the realized variation corresponding to the period $[t-h, t]$, divided by the length h of this period.

Our next result is concerned with the estimation of the integrated variance and it is related to the one presented by Barndorff-Nielsen and Shephard (2002) or Barndorff-Nielsen et al. (2005). However, we give here a new proof and, moreover, we deduce a better description of the error bound.

Theorem 3.3.1
Under the above conditions,

$$M \cdot \left(\hat{\sigma^2}_{h,M}(t) - \frac{1}{h}\int_{t-h}^{t} f^2(U(s))ds\right) = O(1) \quad \text{in } L^2, \text{ w.r.t. } M, \tag{3.30}$$

or, more precisely, we have

$$E\left(\hat{\sigma^2}_{h,M}(t,h) - \frac{1}{h}\int_{t-h}^{t} f^2(U(s))ds\right)^2 \leq \frac{3h}{M}C(h,M)G(t,h), \tag{3.31}$$

for all $M \in \mathbb{N}^*$, where $C(h,M) = \frac{h}{M}(1 + \frac{6\sqrt{c_2}}{h}) + \frac{6\sqrt{2c_2}}{h}$, with c_2 the positive constant in the classical Burkholder-Davis-Gundy (BDG) inequality for stochastic integrals, and

$$G(t,h) = \sqrt{\frac{1}{h}\int_{t-h}^{t} E(\frac{d\alpha^*}{ds})^4\, ds} + \sqrt{\frac{1}{h}\int_{t-h}^{t} E(f^4(U(s)))\, ds}. \tag{3.32}$$

Proof. Let us make the following notations:

$$s_j = s_j(t,h) = t - h + (j-1)\Delta, \quad j = 1, \ldots, M+1, \tag{3.33}$$

$$x_{0j} = x_{0j}(t,h) = \int_{s_j}^{s_{j+1}} f(U(s))\, dW(s), \quad j = 1, \ldots, M, \tag{3.34}$$

$$\alpha_j = \alpha_j(t,h) = \alpha^*(s_{j+1}) - \alpha^*(s_j), \quad j = 1, \ldots, M, \tag{3.35}$$

$$x_j = x_j(t,h) = \alpha_j + x_{0j}, \quad j = 1, \ldots, M, \tag{3.36}$$

$$\mathbf{T}_{1,h,M}(t,h) = \frac{1}{h}\sum_{j=1}^{M} \alpha_j^2 \tag{3.37}$$

$$\mathbf{T}_{2,h,M}(t,h) = \frac{2}{h}\sum_{j=1}^{M} \alpha_j x_{0j} \tag{3.38}$$

$$\mathbf{T}_{3,h,M}(t,h) = \frac{1}{h}\sum_{j=1}^{M} x_{0j}^2. \tag{3.39}$$

We have then $x_j^2 = \alpha_j^2 + 2\alpha_j x_{0j} + x_{0j}^2$ for all $j = 1, ..., M$ and thus

$$\hat{\sigma^2}_{h,M}(t,h) = \mathbf{T}_{1,h,M}(t,h) + \mathbf{T}_{2,h,M}(t,h) + \mathbf{T}_{3,h,M}(t,h). \tag{3.40}$$

Now applying the classical Cauchy-Schwarz and Jensen inequalities (in discrete and also in integral form), we get

$$E\left(\hat{\sigma^2}_{h,M}(t) - \frac{1}{h}\int_{t-h}^{t} f^2(U(s))ds\right)^2 \leq 3 \cdot E\mathbf{T}_{1,h,M}^2(t)$$

$$+3 \cdot E\mathbf{T}^2_{2,h,M}(t) + 3E\left(\mathbf{T}_{3,h,M}(t) - \frac{1}{h}\int\limits_{t-h}^{t} f^2(U(s))ds\right)^2. \tag{3.41}$$

Next let us analyze each of the terms in (3.41).

For the first term:

$$3 \cdot E\mathbf{T}^2_{1,h,M}(t) = \frac{3}{h^2}E(\sum_{j=1}^{M}\alpha_j^2)^2 \leq \frac{3\Delta^2}{h^2}E(\int\limits_{t-h}^{t}(\frac{d\alpha^*}{dt})^2 dt)^2 \leq \frac{3\Delta^2}{h}\int\limits_{t-h}^{t}E(\frac{d\alpha^*}{dt})^4 dt.$$

Thus,

$$\frac{M}{h}\mathbf{T}_{1,h,M}(t) = O(1) \quad \text{in } L^2 \text{ w.r.t. } M.$$

The analysis of the third term of (3.41): since we are in the no leverage case, we have

$$3E\left(\mathbf{T}_{3,h,M}(t) - \frac{1}{h}\int\limits_{t-h}^{t} f^2(U(s))\,ds\right)^2$$

$$= 3E\left[\left(\frac{1}{h}\sum_{j=1}^{M}x_{0j}^2 - \frac{1}{h}\int\limits_{t-h}^{t} f^2(U(s))\,ds\right)^2\right] = \frac{3}{h^2}E\left(\sum_{j=1}^{M}(x_{0j}^2 - \int\limits_{s_j}^{s_{j+1}} f^2(U(s))\,ds)\right)^2$$

$$= \frac{3}{h^2}\sum_{j=1}^{M}E\left(x_{0j}^2 - \int\limits_{s_j}^{s_{j+1}} f^2(U(s))\,ds\right)^2. \tag{3.42}$$

But

$$x_{0j}^2 - \int\limits_{s_j}^{s_{j+1}} f^2(U(s))\,ds = 2\int\limits_{s_j}^{s_{j+1}} X_{0,(j)}(u)\,dX_{0,(j)}(u),$$

where $dX_{0,(j)}(s) = f(U(s))\,dW(s)$, $X_{0,(j)}(s_j) = 0$, for all $j = 1, \ldots, M$.

It follows that

$$E(x_{0j}^2 - \int\limits_{s_j}^{s_{j+1}} f^2(U(s))\,ds)^2 = 4E(\int\limits_{s_j}^{s_{j+1}} X_{0,(j)}(u)\,dX_{0,(j)}(u))^2. \tag{3.43}$$

Now applying the Hölder and BDG inequalites we obtain

$$E(\int\limits_{s_j}^{s_{j+1}} X_{0,(j)}(u)\,dX_{0,(j)}(u))^2 = \int\limits_{s_j}^{s_{j+1}} E[X^2_{0,(j)}(u) \cdot f^2(U(s))]\,ds \tag{3.44}$$

$$\leq \int\limits_{s_j}^{s_{j+1}} \sqrt{E(X^4_{0,(j)}(u))} \cdot \sqrt{E(f^4(U(s)))}\,ds \leq \sqrt{\int\limits_{s_j}^{s_{j+1}} E(X^4_{0,(j)}(u))\,ds}\sqrt{\int\limits_{s_j}^{s_{j+1}} E(f^4(U(s)))\,ds}$$

$$\leq \sqrt{\int_{s_j}^{s_{j+1}} c_2(s-s_j)ds \cdot \int_{s_j}^{s_{j+1}} E(f^4(U(s)))\,ds} = \Delta\sqrt{\frac{c_2}{2}} \cdot \int_{s_j}^{s_{j+1}} E(f^4(U(s)))\,ds,$$

where c_2 is the constant in the BDG inequality. Therefore,

$$E\left(x_{0j}^2 - \int_{s_j}^{s_{j+1}} f^2(U(s))\,ds\right)^2 \leq C_3 \cdot \Delta \int_{s_j}^{s_{j+1}} E(f^4(U(s)))\,ds, \tag{3.45}$$

for all $j = 1, \dots, M$ and $i = i_A \dots, i_B$, where $C_3 = 2\sqrt{2c_2}$.

Thus, by (3.42) and (3.45) we obtain:

$$3E\left(\mathbf{T}_{3,h,M}(t) - \frac{1}{h}\int_{t-h}^{t} f^2(U(s))\,ds\right)^2 = \frac{3C_3}{M}\frac{1}{h}\int_{t-h}^{t} E(f^4(U(s)))\,ds \tag{3.46}$$

Thus,

$$\sqrt{M}\left(\mathbf{T}_{3,h,M}(t) - \frac{1}{h}\int_{t-h}^{t} f^2(U(s))\,ds\right) = O(1) \quad \text{in } M.$$

The analysis of the second term of (3.41): using again the no leverage assumption and applying the Cauchy-Schwarz inequality it follows that

$$3 \cdot E\mathbf{T}_{2,h,M}^2(t) = \frac{12}{h^2}E\left[\sum_{j=1}^{M} \alpha_j x_{0j}\right]^2 = \frac{12}{h^2}\sum_{j=1}^{M} E(\alpha_j^2 x_{0j}^2)$$

$$\leq \Delta\frac{12}{h^2}\sum_{j=1}^{M} E\Big[\Big(\int_{t-h+(j-1)\Delta}^{t-h+j\Delta} (\frac{d\alpha^*}{dt})^2 dt\Big)x_{0j}^2\Big] \leq \Delta\frac{12}{h^2}\sum_{j=1}^{M}\sqrt{E\Big[\int_{s_j}^{s_{j+1}}(\frac{d\alpha^*}{dt})^2 dt\Big]^2}\sqrt{Ex_{0j}^4}$$

$$\leq \sqrt{c_2}\Delta^2\frac{12}{h^2}\sum_{j=1}^{M}\sqrt{\int_{s_j}^{s_{j+1}} E(\frac{d\alpha^*}{dt})^4 dt}\sqrt{\int_{s_j}^{s_{j+1}} Ef^4(U(s))\,ds}$$

$$\leq \sqrt{c_2}\Delta^2\frac{12}{h^2}\sqrt{\Big(\sum_{j=1}^{M}\int_{s_j}^{s_{j+1}} E(\frac{d\alpha^*}{dt})^4 dt\Big)\Big(\sum_{j=1}^{M}\int_{s_j}^{s_{j+1}} Ef^4(U(s))\,ds\Big)}$$

$$= \sqrt{c_2}\Delta^2\frac{12}{h^2}\sqrt{\Big(\int_{t-h}^{t} E(\frac{d\alpha^*}{ds})^4 ds\Big)\Big(\int_{t-h}^{t} Ef^4(U(s))\,ds\Big)}.$$

Thus,

$$\frac{M}{h}\mathbf{T}_{2,h,M}(t) = O(1) \quad \text{in } L^2 \text{ w.r.t. } M.$$

Thus, for (3.41) we obtain eventually

$$E\left(\hat{\sigma^2}_{h,M}(t) - \frac{1}{h}\int_{t-h}^{t} f^2(U(s))ds\right)^2 \leq 3\Delta^2 \frac{1}{h}\int_{t-h}^{t} E(\frac{d\alpha^*}{dt})^4 dt \tag{3.47}$$

$$+\sqrt{c_2}\Delta^2 \frac{12}{h^2}\sqrt{\Big(\int_{t-h}^{t} E(\frac{d\alpha^*}{ds})^4 ds\Big)\Big(\int_{t-h}^{t} Ef^4(U(s))\,ds\Big)} + \frac{3C_3}{M}\frac{1}{h}\int_{t-h}^{t} E(f^4(U(s)))ds$$

$$\leq 3\Delta\Big[(1+\frac{6\sqrt{c_2}}{h})\Delta + \frac{3C_3}{h}\Big]\Big(\sqrt{\frac{1}{h}\int_{t-h}^{t} E(\frac{d\alpha^*}{ds})^4 ds} + \sqrt{\frac{1}{h}\int_{t-h}^{t} E(f^4(U(s)))ds}\Big)^2.$$

■

The next result is related to the one presented in Foster and Nelson (1996) concerning the estimation of the spot volatility.

Corolary 3.3.1
With the previous notations,

$$\hat{\sigma^2}_{h,M}(t,h) \to f^2(U(t)) \text{ in } L^2 \text{ as } M \to \infty \text{ and } h \to 0, \text{ for all } t > 0,$$

or, more precisely, we have

$$E\Big(\hat{\sigma^2}_{h,M}(t,h) - f^2(U(t))\Big)^2 \leq \frac{6h}{M}\sqrt{G(t,h)} + \frac{2}{h}\int_{t-h}^{t} E\Big(f^2(U(s)) - f^2(U(t))\Big)^2 ds \tag{3.48}$$

for all $M \in \mathbb{N}^*$, $h \in (0,t)$.

Proof. Using the Cauchy-Schwarz inequality and the Theorem 3.3.1 we have

$$E\Big(\frac{1}{h}\int_{t-h}^{t} f^2(U(s))ds - f^2(U(t))\Big)^2 \leq \frac{1}{h}\int_{t-h}^{t} E\Big(f^2(U(s)) - f^2(U(t))\Big)^2 ds$$

and then

$$E\Big(\hat{\sigma^2}_{h,M}(t,h) - f^2(U(t))\Big)^2 \leq 2E\Big(\hat{\sigma^2}_{h,M}(t,h) - \frac{1}{h}\int_{t-h}^{t} f^2(U(s))ds\Big)^2$$

$$+2E\Big(\frac{1}{h}\int_{t-h}^{t} f^2(U(s))ds - f^2(U(t))\Big)^2$$

$$\leq 6\Delta C(h,M)\sqrt{G(t,h)} + \frac{2}{h}\int_{t-h}^{t} E\Big(f^2(U(s)) - f^2(U(t))\Big)^2 ds$$

for all $M \in \mathbb{N}^*$, $h \in (0,t)$.

■

The above results open the possibility to estimate the effective volatility in the context of HF data, for instance by using the estimator

$$\hat{\bar{\sigma}}_M = \sqrt{\frac{\sum_{i=i_A}^{I_M}[X_M]_i}{(1+n_0 M)\cdot h}}, \tag{3.49}$$

where $I_M := i_A + n_0 \cdot M$, n_0, $M \in \mathbb{N}^*$. Since the mean-square time-averaged volatility offers an approximation to the square of the effective volatility, we obtain the following

Theorem 3.3.2
Under the above conditions we have

$$\sqrt{\frac{M}{h}}\Big(\hat{\bar{\sigma}}_M^2 - \frac{1}{t_M - t_0}\int_{t_0}^{t_M} f^2(U(s))\,ds\Big) \to 0 \quad \textit{in } L^2, \tag{3.50}$$

as $M \to \infty$, *where* $t_0 = (i_A - 1)\cdot h$ *and* $t_M = t_0 + (1 + n_0 M)\cdot h = I_M \cdot h$.

The proof is similar to the one of Theorem 3.3.1.

These results can be relatively easily extended to the multivariate setting to be described in Chapter 5. For several approaches to modeling high frequency data, however in a different context, we refer to Haug (2006) and Dacorogna et al. (2001).

Remark 3.3.1
Alternatively one can use the nonlinear filtering techniques described in Chapter 2 in order to estimate all financial quantities discussed above and these methods allow to do this in a far more general setting as the one we used in the context of high frequency data.

Beside the above mentioned financial applications, these results can help simplifying the estimation problem of complex stochastic volatility models. For instance, if the spot volatility of a SV model is of the form $g(U_t)$ with g some nonlinear positive bijective smooth function and U_t a CARMA$_\varepsilon$ process, then using the above result about the estimation of the spot volatility we can, in principle, reduce the estimation problem of this SV model to the simple problem of estimating a CARMA$_\varepsilon$ process.

Chapter 4

Pricing European Derivatives under an FMR Volatility Regime

The valuation of derivative securities constitutes one of the main topics in modern mathematical finance. The PDE characterization of the pricing function of European derivatives given in the previous chapter is numerically infeasible in practice for a high dimensional model. We present in this chapter an alternative way for a class of SV CAR type models with *fast mean-reverting* volatility processes. In such a SV scenario, the volatility level fluctuates randomly around its mean level and the epochs of high/low volatility are relatively short. Under this regime we analyze in the following the price of European derivatives. Empirical evidence of a fast volatility factor was found in the analysis of high frequency S&P 500 data in Fouqué et al. (2000) and this factor has been modelled with a CAR(1) process. This key empirical remark leads to a very effective and practical way of correcting the prices computed in the classical Black-Scholes setting, which is based on the fast mean-reversion (FMR) asymptotic theory developed in Fouqué et al. (2000). This is a method to construct an approximate derivative pricing formula for the case of fast mean-reverting volatility by performing an expansion of the price in powers of the characteristic mean-reversion time of volatility. Then it can be shown that the leading order term corresponds to a Black-Scholes price computed under a constant effective volatility and the first correction involves derivatives of this Black-Scholes price. Thus the corrected price is the leading order plus the first correction.

In this chapter we extend this result to the class of SV CAR type models with fast mean-reverting regime. These models are illustrated with the aid of some simulation and estimation examples. These show that the observed "clustering" property of asset volatility (i.e. a large volatility tends to stay so for a certain timespan, before dropping to a lower level where it tends to stay for another while, and so on) is closely related to the fast-mean reversion property in such SV models and the non-Markovian structure of the volatility can help in practice to get a better description of financial data.

4.1 The Rescaled Stochastic Volatility Model

Here we describe the dynamics of a risky asset with a SV CAR type model obtained from (3.1)- (3.3) by choosing $b_0 = 1$, $b_j = 0$ for $j > 0$. In the following we also preserve the assumptions made in Section 3.2.

The main idea of the asymptotic analysis by Fouqué et al. (2000) is to consider that the volatility has the fast mean-reverting property which is modeled with the aid of a CAR(1) process $(Y_t)_{t\geq 0}$:

$$dY_t = \alpha(m - Y_t)dt + \beta\, d\tilde{Z}_t, \quad t \geq 0, \tag{4.1}$$

with the positive mean-reversion rate α large and $(\tilde{Z}_t)_{t\geq 0}$ a Brownian motion. Thus the aim is to compare the characteristic mean reverting time $\epsilon = \frac{1}{\alpha}$ to the time scale $T - t$ of a corresponding derivative pricing problem in a SV CAR(1) framework of the type described in the previous chapter. It is also assumed that $\epsilon \ll T - t$, or equivalently $\alpha \gg \frac{1}{T-t}$. Notice that this is the asymptotic framework

$$\epsilon \to 0, \quad m,\, \nu^2 = \text{constants}, \tag{4.2}$$

where $\nu^2 = \frac{\beta^2}{2\alpha}$, which means $\beta = \nu\sqrt{2\alpha} = \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \overset{\epsilon\to 0}{\to} \infty$.

In order to extend this method to the above higher order SV CAR type model, the main challenge is to find an appropriate reparameterization of its coefficients. As a possible choice we propose for example

$$a_k = \alpha^k \xi_k, \quad k = 1, \dots p, \quad (p > 1), \tag{4.3}$$

with ξ_k, $k = 1, \dots, p$, fixed real constants, $\xi_1 = 1$ and $\xi_p \neq 0$ and $\alpha > 0$,

$$\delta = \alpha^{p-\frac{1}{2}}\psi_{o,p}, \qquad c = \frac{ma_p}{\delta} = \sqrt{\alpha}\cdot m \cdot \frac{\xi_p}{\psi_{o,p}}, \tag{4.4}$$

$$\psi_k = \alpha^{k-\frac{1}{2}}\psi_{o,k},\ \forall k = 1, \dots, p \tag{4.5}$$

with $m \in \mathbb{R}$ a constant and $\psi_{o,k}$, , $k = 1, \dots p$, fixed nonnegative real constants, $\psi_{o,p} > 0$.

We also assume in the following that the above parameters are chosen such as to ensure the stationarity of the corresponding CAR$_\varepsilon$ process. Then, as above, the volatility is fast mean-reverting when $\epsilon = \frac{1}{\alpha}$ is small. Under this regime we give in the following an approximate derivative pricing formula by performing an expansion of the price in powers of $\sqrt{\epsilon}$.

In this sense, we rescale our SV CAR type model with respect to ϵ and then, under the risk-neutral probability $P^{*(\Upsilon)}$, this model becomes

$$dS_t^\epsilon = rS_t^\epsilon dt + f(U_t^\epsilon)S_t^\epsilon\, dW_t^*, \quad t \geq 0, \tag{4.6}$$

$$U^\epsilon(t) = \mathbf{b}'\mathbf{U}^\epsilon(t), \tag{4.7}$$

$$d\mathbf{U}^\epsilon(t) = \Big[\mathbf{A}_\epsilon(\mathbf{U}^\epsilon(t) \ - m\mathbf{b})dt - \Gamma_o(\epsilon)\mathbf{\Lambda}(\mathbf{U}^\epsilon(t))\Big]dt + \Gamma_o(\epsilon)\, d\hat{\mathbf{Z}}^*(t), \tag{4.8}$$

where $a_1(\epsilon), ..., a_p(\epsilon)$, $\mathbf{A}_\epsilon$, c_ϵ and δ_ϵ are given by

$$a_k = a_k(\epsilon) = \frac{1}{\epsilon^k}\xi_k, \quad k = 1, \ldots p, \tag{4.9}$$

with ξ_k, $k = 1, \ldots, p$, fixed real constants, $\xi_1 = 1$ and $\xi_p \neq 0$,

$$\delta = \delta(\epsilon) = \frac{1}{\epsilon^{p-\frac{1}{2}}}\psi_{o,p}, \quad c = c(\epsilon) = \frac{1}{\sqrt{\epsilon}} \cdot m \cdot \frac{\xi_p}{\psi_{o,p}}, \tag{4.10}$$

$$\psi_k = \psi_k(\epsilon) = \frac{1}{\epsilon^{k-\frac{1}{2}}}\psi_{o,k}, \ \forall k = 1, \ldots, p, \tag{4.11}$$

with $\psi_{o,k}$, , $k = 1, \ldots p$, fixed nonnegative real constants, $\psi_{o,p} > 0$, and

$$\mathbf{A} = \mathbf{A}_\epsilon = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{1}{\epsilon^p}\xi_p & -\frac{1}{\epsilon^{p-1}}\xi_{p-1} & -\frac{1}{\epsilon^{p-2}}\xi_{p-2} & \cdots & -\frac{1}{\epsilon} \end{bmatrix}. \tag{4.12}$$

Analogously, we obtain $\Gamma_o(\epsilon)$ from (3.5) and (4.11). Recall that Λ is given by (3.18).

In this context, observe that the invariant distribution of the volatility driven process $(U^\epsilon(t))_{t\geq 0}$ is independent of ϵ. Moreover, this process decorrelates exponentially fast on the time scale ϵ and thus we refer to $U^\epsilon(t)$ as the fast volatility factor.

The pricing function and the corresponding pricing equations for a European derivative with the payoff function $h(S_T^\epsilon)$ at the maturity date T (where, like in the previous chapter, h is some nonnegative bounded C^2 function) can be rescaled in an analogous way. Thus, the pricing function is (see Chapter 3)

$$\tilde{P}^\epsilon(t, S_t^\epsilon, \mathbf{U}_t^\epsilon) = E^{*(\Upsilon)}\{e^{-r(T-t)}h(S_T^\epsilon)|S_t^\epsilon, \mathbf{U}_t^\epsilon\}, \tag{4.13}$$

for all $t \in [0, T]$, where r denotes the instantaneous interest rate and $E^{*(\Upsilon)}$ denotes the expectation with respect to the risk-neutral measure $P^{*(\Upsilon)}$.

Now rescale (3.16) and denote the outcome by

$$\frac{\partial \tilde{P}^\epsilon}{\partial t}(t, s, \mathbf{u}) + (L_\epsilon \tilde{P}^\epsilon)(t, s, \mathbf{u}) - r\tilde{P}^\epsilon(t, s, \mathbf{u}) = 0, \tag{4.14}$$

$$0 \leq t < T, \quad s \in \mathbb{R}_+, \quad \mathbf{u} \in \mathbb{R}^p,$$

with $\tilde{P}^\epsilon(T, s, \mathbf{u}) = h(s)$, for all $s \in \mathbb{R}_+$, $\mathbf{u} = (u_0, u_1, \ldots, u_{p-1})' \in \mathbb{R}^p$.

4.2 The Corrected Price Formula

Expanding in powers of $\sqrt{\epsilon}$ a transformation of $\tilde{P}^\epsilon$ (to be specified in (4.27) below), leads to the approximation

$$\tilde{P}^\epsilon \approx \tilde{P}_0 + \sqrt{\epsilon}\tilde{P}_1, \tag{4.15}$$

where $\tilde{P}_0$, $\tilde{P}_1$ are solutions of some Black-Scholes type equations.

Let us denote

$$\mathcal{L}_0 = \frac{1}{2}\Big(\psi_{o,p}^2\frac{\partial^2}{\partial u_{p-1}^2} + \sum_{k=0}^{p-2}\psi_{o,k+1}^2\frac{\partial^2}{\partial u_k^2}\Big) + \Big(\mathbf{A}_1\mathbf{u} + m\cdot\xi_p\mathbf{e}\Big)'\frac{\partial}{\partial \mathbf{u}}, \tag{4.16}$$

$$\mathcal{L}_{BS}(\bar{\sigma}) = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2\frac{\partial^2}{\partial s^2} + r\cdot\Big(s\frac{\partial}{\partial s} - \cdot\Big) \tag{4.17}$$

$$< g >= \int_{\mathbb{R}^p} g(\mathbf{u})\phi(\mathbf{u})\, d\mathbf{u}, \tag{4.18}$$

where $\frac{\partial}{\partial \mathbf{u}} = \left[\frac{\partial}{\partial u_0}, \dots \frac{\partial}{\partial u_{p-1}}\right]'$, $\bar{\sigma}$ is the effective volatility (see Chapter 3), $\mathbf{A}_1 = \mathbf{A}_\epsilon\,|_{\epsilon=1}$, ϕ is the density of the invariant distribution of $(\mathbf{U}_t)_{t\geq 0}$.

Theorem 4.2.1
With the above notations we have:

1. $\tilde{P}_0$ *does not depend on the volatility and it is given by the following Black-Scholes equation*

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_0 = 0, \tag{4.19}$$

with the terminal condition $\tilde{P}_0(T, s) = h(s)$.

2. $\tilde{P}_1$ *does not depend on the volatility and denoting*

$$\tilde{Q}_1(t, s) = \sqrt{\epsilon}\tilde{P}_1(t, s), \tag{4.20}$$

then $\tilde{Q}_1$ *is the solution of*

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{Q}_1 = \tilde{H}(t, s), \tag{4.21}$$

with the terminal condition $\tilde{Q}_1(T, s) = 0$, *where the source term* $\tilde{H}$ *is given by*

$$\tilde{H}(t, s) = V_2 s^2\frac{\partial^2\tilde{P}_0}{\partial s^2} + V_3 s^3\frac{\partial^3\tilde{P}_0}{\partial s^3}, \tag{4.22}$$

with V_2 *and* V_3 *two small coefficients, given in terms of* $\alpha = 1/\epsilon$ *by*

$$V_2 = \frac{1}{\sqrt{\alpha}}\Big(\frac{1}{2}\rho\psi_{o,p}\big\langle f(\mathbf{b}'\mathbf{u})\frac{\partial\varphi}{\partial u_{p-1}}\big\rangle - \frac{1}{2}\big\langle\Lambda(\mathbf{u})'\frac{\partial\varphi}{\partial \mathbf{u}}\big\rangle\Big) \tag{4.23}$$

$$V_3 = \frac{1}{\sqrt{\alpha}}\rho\psi_{o,p}\big\langle f(\mathbf{b}'\mathbf{u})\frac{\partial\varphi}{\partial u_{p-1}}\big\rangle, \tag{4.24}$$

where φ satisfies the Poisson equation

$$\mathcal{L}_0\varphi = f^2(\mathbf{b}'\mathbf{u}) - \bar{\sigma}^2, \tag{4.25}$$

and is such that V_2 and V_3 above are well defined.

Thus, "the corrected price" is:

$$\tilde{P}_0 - (T-t)(V_2 s^2 \frac{\partial^2 \tilde{P}_0}{\partial s^2} + V_3 s^3 \frac{\partial^3 \tilde{P}_0}{\partial s^3}). \tag{4.26}$$

Proof. We make the change of variables

$$P^\epsilon(t,s,\mathbf{v}) = \tilde{P}^\epsilon(t,s,D_p(\epsilon)\mathbf{v}), \tag{4.27}$$

for all t, $s \in \mathbb{R}_+$ and $\mathbf{v} \in \mathbb{R}^p$, where

$$D_p(\epsilon) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{\epsilon} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\epsilon^{p-1}} \end{bmatrix}. \tag{4.28}$$

Then $P^\epsilon(t,s,\mathbf{v})$ satisfies the following PDE:

$$\frac{\partial P^\epsilon}{\partial t}(t,s,\mathbf{v}) + (\mathcal{L}^\epsilon P^\epsilon)(t,s,\mathbf{v}) - rP^\epsilon(t,s,\mathbf{v}) = 0, \tag{4.29}$$

$$0 \le t < T, \quad s \in \mathbb{R}_+, \quad \mathbf{v} \in \mathbb{R}^p,$$

with $P^\epsilon(T,s,\mathbf{v}) = h(s)$, for all $s \in \mathbb{R}_+$, $\mathbf{v} \in \mathbb{R}^p$, $\mathbf{v} = (v_0, v_1, \ldots, v_{p-1})'$, where

$$\mathcal{L}^\epsilon = \frac{1}{2}\Big(f^2(\mathbf{b}'\mathbf{v})s^2\frac{\partial^2}{\partial s^2} + 2\rho\frac{\psi_{o,p}}{\sqrt{\epsilon}}f(\mathbf{b}'\mathbf{v})s\frac{\partial^2}{\partial s\,\partial v_{p-1}} + \frac{\psi_{o,p}^2}{\epsilon}\frac{\partial^2}{\partial v_{p-1}^2}$$

$$+\frac{1}{\epsilon}\sum_{k=0}^{p-2}\psi_{o,k+1}^2\frac{\partial^2}{\partial v_k^2}\Big) + r\cdot s\frac{\partial}{\partial s} + \Big(\frac{1}{\epsilon}\mathbf{A}(1)\mathbf{v} + \frac{1}{\epsilon}m\cdot\xi_p\mathbf{e} - \frac{1}{\sqrt{\epsilon}}\Gamma_o(1)\mathbf{\Lambda}(\mathbf{v})\Big)'\frac{\partial}{\partial \mathbf{v}}, \tag{4.30}$$

where $\frac{\partial}{\partial \mathbf{v}} = \left[\frac{\partial}{\partial v_0}, \cdots \frac{\partial}{\partial v_{p-1}}\right]'$.

Let us denote

$$\mathcal{L}_1 = \rho\psi_{o,p}f(\mathbf{b}'\mathbf{v})s\frac{\partial^2}{\partial s\,\partial v_{p-1}} - \Big(\psi_{o,p}\rho\frac{(\mu - r)}{f(\mathbf{b}'\mathbf{v})} + \Gamma_o(1)\,\mathcal{D}(\sqrt{1-\rho^2})\cdot\gamma(\mathbf{b}'\mathbf{v})\Big)'\frac{\partial}{\partial \mathbf{v}}, \tag{4.31}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f^2(\mathbf{b}'\mathbf{v})s^2\frac{\partial^2}{\partial s^2} + r\cdot\Big(s\frac{\partial}{\partial s} - .\Big) = \mathcal{L}_{BS}(f(\mathbf{b}'\mathbf{v})). \tag{4.32}$$

Using the operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$, the above PDE becomes

$$\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big)P^\epsilon = 0, \tag{4.33}$$

with the terminal condition $P^\epsilon(T, s, \mathrm{v}) = h(s)$, for all $s \in \mathbb{R}_+$, $\mathrm{v} \in \mathbb{R}^p$, $\mathrm{v} = (v_0, v_1, \ldots, v_{p-1})'$.
Now expand P^ϵ in powers of $\sqrt{\epsilon}$:

$$P^\epsilon = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3 + ..., \tag{4.34}$$

where $P_0, P_1, ...$ are functions of (t, s, v), of which the first two ones can be determined with the same ideas as in Fouqué et al. (2000). Then using (4.27), it is easy to see that $\tilde{P}_0$, $\tilde{P}_1$ and $\tilde{Q}_1$ satisfy the properties stated in the hypotheses of the theorem.

■

Remark 4.2.1

(a) The asymptotic approximations performed above and the corrected price (4.26) have the same theoretical properties (for instance, the accuracy of the approximation or the region of validity) as those described in Fouqué et al. (2000). Thus, such type of asymptotic approximations cannot be expected to be valid for all (t, s, u) values. In particular, the approximations perform poorly close to the expiration date or to the other frontiers of the corresponding domain for s. Actually, they will not be so accurate in all situations where the mean-reverting process which models the volatility does not have sufficient time to fluctuate.

(b) The above result can be easily adapted to the case of multiple time scales (see e.g., Fouqué et al. (2003b)) when the volatility is driven by two CAR$_\varepsilon$ processes, one fluctuating on a fast time scale and the other varying on a slow time scale.

(c) The parameters of the corrected price (4.26) only capture a small amount of information about the volatility process, therefore it provides a less-than-perfect description of the real world. The nonlinear filtering techniques resumed in Chapter 2 give us the chance not only to calibrate more accurately these parameters (than with the classical calibration techniques described in Fouqué et al. (2000)), but also to obtain information about all parameters of the SV model and the dynamics of the spot volatility.

This supplementary information can contribute to further improvements of (4.26) and a natural way to take this into account is to look for a corrected formula based on a higher order approximation which also depends on the variable u. The developments that we present in Chapter 6 for the special case of computing the firm default probability in a SV scenario suggest the possibility to derive such type of improvements of (4.26), where the dependence on u is introduced through a solution φ of the Poisson equation (4.25). Of course, this makes the calibration problem much more complicated as in Fouqué et al. (2000), however it can still be solved with the nonlinear filtering methods mentioned in Chapter 2. In this way, the corrected price formula has a stronger dependence on the corresponding SV model (i.e. the same type of corrected price formula performs better in practice for the SV model which gives the best description to real data; for this purpose one can use the classical statistical criteria described in Chapter 2, like AIC, BIC).

The simulation and estimation examples in the next section show that, as expected, a higher order SV CAR type model gives usually a better description to the observed volatility clustering and the corresponding value of α is bigger than the one need for the SV CAR(1) model. Thus, the class of SV CAR type models is particularly appropriate for the FMR asymptotic analysis, especially when one can work with a higher order model. However, despite its universality, the corrected price formula (4.26), can only capture a few of such structural properties of the underlying model for the volatility.

4.3 Simulation and Estimation Results

Figures 4.1-4.5, respectively 4.6-4.10 show simulated paths of asset and volatility processes under various parameter settings for SV CAR(1), respectively SV CAR(2) type models over the course of a year. Again, the simulations are done with the classical Euler-Maruyama scheme, upon using a discretisation with time step $\Delta = 0.002$.

We choose $\alpha \in \{1, 50, 100, 300, 500\}$, $\mu = 0.02$, $m = -2.2$, $\rho = -0.1$, $\xi_1 = 1$, $f = \exp$ for both models and the rest of parameters are $\psi_{o,1} = 0.2$, $S_0 = 620$ (for the SV CAR(1) type model), respectively $\xi_2 = 0.001$, $\psi_{o,1} = 0.0001$, $\psi_{o,2} = 0.001$, $S_0 = 970$, (for the SV CAR(2) type model). For the CAR type processes the starting values are chosen by sampling from their stationary distributions. Then using the parameterization (4.3)-(4.5) we computed each time all parameters of the corresponding SV CAR type model (recall that these models are obtained from (3.1)- (3.3) by choosing $b_0 = 1$, $b_j = 0$ for $j > 0$).

Maybe the most important stylized fact on volatility is its clustering or persistence. This idea can be found already in Mandelbrot (1963) or Fama (1965). These simulations show how the volatility 'clusters' with the increasing value of the parameter α. Notice that increasing the order of the model triggers the necessity of increasing the parameter α in order to obtain similar behaviors for the volatility. Thus, in this context one can observe that the clustering property is closely related to the fast-mean reversion property.

Let us now consider the following SV CAR(3) type model:

$$dS_t = \mu S_t dt + S_t \sigma_t \, dW_t, \quad t \geq 0, \tag{4.35}$$

$$\sigma_t = \exp(U_t), \quad U(t) = \mathbf{b}'(m + \mathbf{U}(t)), \quad t \geq 0, \tag{4.36}$$

$$d\mathbf{U}(t) = \mathbf{A} \cdot \mathbf{U}(t) \, dt + \Gamma_o \, d\hat{\mathbf{Z}}(t), \quad t \geq 0, \tag{4.37}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \quad \mathbf{U}_t = \begin{bmatrix} U_0(t) \\ U_1(t) \\ U_2(t) \end{bmatrix}, \tag{4.38}$$

$$\mathrm{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{4.39}$$

$$\Gamma_o = \begin{bmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & \psi_3 \end{bmatrix}, \quad \hat{\mathbf{Z}}(t) = \begin{bmatrix} \hat{Z}_1(t) \\ \hat{Z}_2(t) \\ \hat{Z}_3(t) \end{bmatrix}, \quad \tilde{\mathbf{Z}}(t) = \begin{bmatrix} \tilde{Z}_1(t) \\ \tilde{Z}_2(t) \\ \tilde{Z}_3(t) \end{bmatrix}, \tag{4.40}$$

$$\hat{Z}_1(t) = \tilde{Z}_1(t), \quad \hat{Z}_j(t) = \rho_{1j}\tilde{Z}_1(t) + \sqrt{1-\rho_{1j}^2}\tilde{Z}_j(t), \quad j = 2, 3, \tag{4.41}$$

with $\rho_{1j} \in [-1, 1]$, $j = 2, 3$ and $(\tilde{\mathbf{Z}}(t))_{t\geq 0}$ a standard multivariate Brownian motion independent of $(W_t)_{t\geq 0}$.

Much attention has been paid in literature to the estimation of various Markovian models for σ_t. After an appropriate reparameterization, we estimate in the following the above described non-Markovian CAR$_\varepsilon$(3) model for σ_t.

The parsimonious subclass of models which will be described below is inspired from the asymptotic analysis developed previously in this chapter for SV CAR type models. Further, we consider A in the class of matrices with at most two distinct and negative eigenvalues and assume that there is a leverage effect between any two consecutive components of the multivariate process $(\mathrm{U}(t))_{t\geq 0}$.

Let α, $\tilde{\beta}$ be positive parameters and denote

$$\tilde{\omega} = \frac{\tilde{\beta}^2}{1+\tilde{\beta}^2}. \tag{4.42}$$

Then an appropriate reparameterization for this subclass is the following:

- for the elements of A:
$$a_k = \alpha^k \xi_k, \; k = 1, 2, 3, \tag{4.43}$$
with $\xi_1 = 1$, $\xi_2 = \frac{2\tilde{\omega}}{1+4\tilde{\omega}}$, $\xi_3 = \frac{4\tilde{\omega}^2}{(1+4\tilde{\omega})^3}$,
- for the elements of Γ_o: $\psi_1 = \sqrt{\tilde{\omega}}\alpha\tilde{\beta}$, $\psi_2 = \tilde{\omega}\alpha\psi_1$, $\psi_3 = \tilde{\omega}^2\alpha^2\psi_1$,
- for the correlations between the components of $\hat{\mathbf{Z}}(t)$:

$$\rho_{12} = -\exp(-5\tilde{\omega}) \text{ and } \rho_{13} = -\rho_{12}. \tag{4.44}$$

We transformed the S&P 500 daily closing index values between 1990-1999 as it was done in Molina, Han and Fouqué (2004), then extracted the set of data corresponding to 1996 in order to perform the classical QML estimation method based on Kalman filtering. This estimation strategy for stochastic volatility models was suggested by Harvey et al. (1994).

Starting the algorithm with $\hat{\mathrm{x}}_{t_0|t_0} = [0, 0, 0]'$ and $\mathrm{P}_{t_0|t_0} = 10^{-4}\mathrm{I}_3$, where $t_0 = 0$ and I_3 is the identity matrix, we obtained $\hat{m} = -2.2$, $\hat{\alpha} = 446.9$ and $\hat{\tilde{\beta}} = 0.3$.

Clearly, a similar subclass of models can be easily introduced for any other SV CAR type models (of any order). Moreover, repeating the above described estimation procedure in these subclasses when the model order is varying from 1 to 4, it can be seen that the smallest values for both criteria described in (2.85)-(2.86) are met for the SV CAR(3) type model discused above.

These results and the similar ones by Fouqué et al. (2000) confirm our simulation-based observation that in order to explain the volatility clustering a higher value of parameter α is needed when increasing the model order. Thus, these SV CAR type models are particularly appropriate for the asymptotic analysis of financial derivatives.

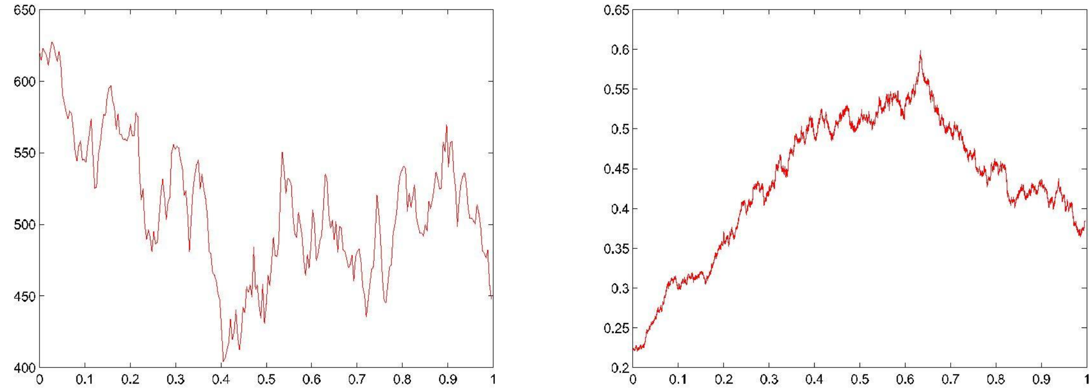

Figure 4.1: Simulated SV CAR(1) type process for $\alpha = 1$. Left: asset values. Right: volatility

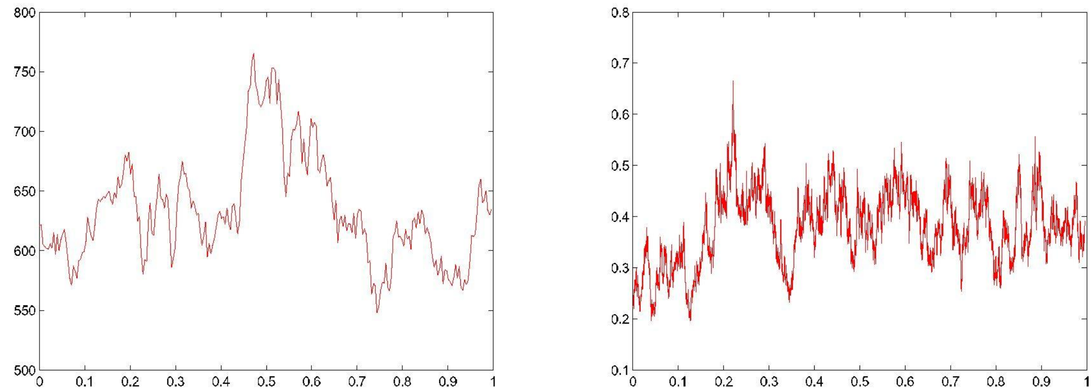

Figure 4.2: Simulated SV CAR(1) type process for $\alpha = 50$. Left: asset values. Right: volatility

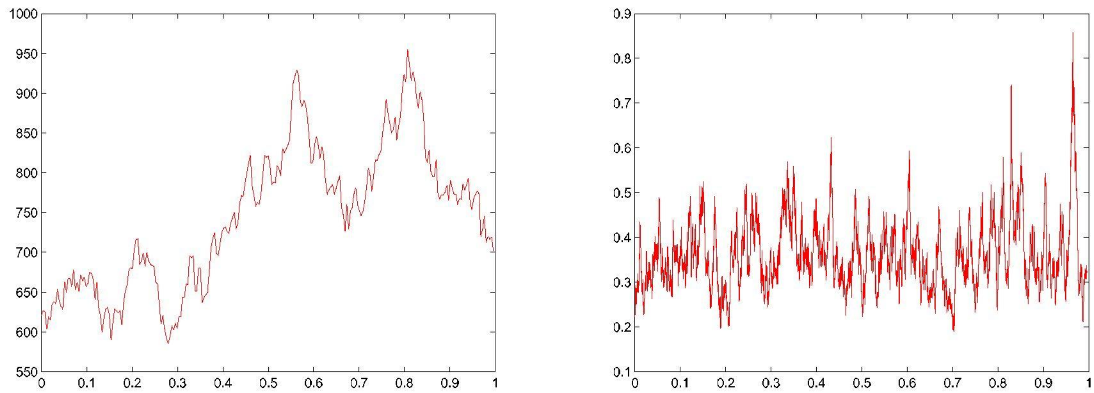

Figure 4.3: Simulated SV CAR(1) type process for $\alpha = 100$. Left: asset values. Right: volatility

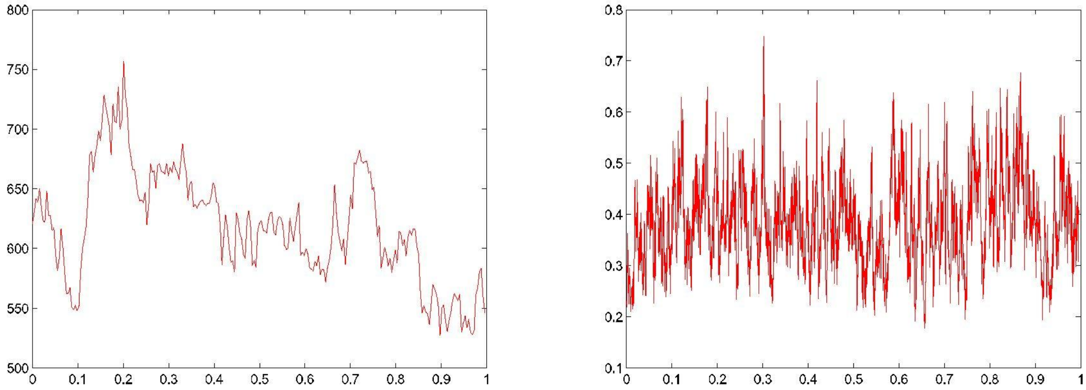

Figure 4.4: Simulated SV CAR(1) type process for $\alpha = 300$. Left: asset values. Right: volatility

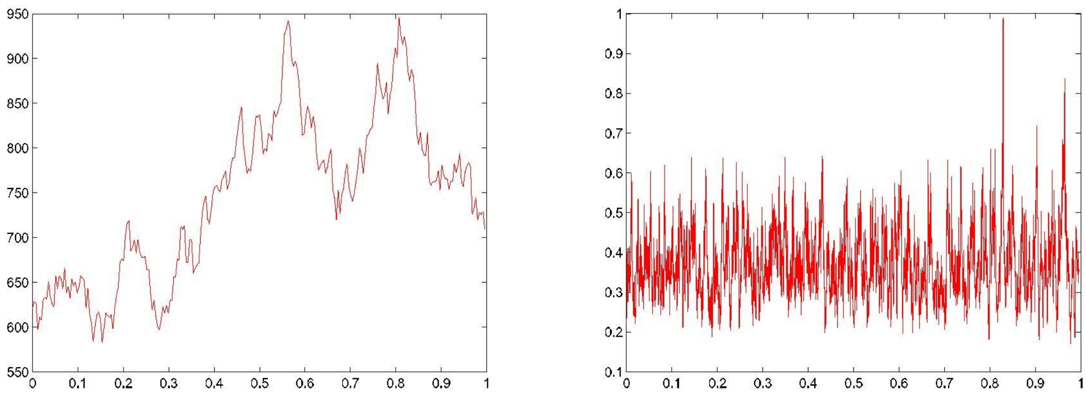

Figure 4.5: Simulated SV CAR(1) type process for $\alpha = 500$. Left: asset values. Right: volatility

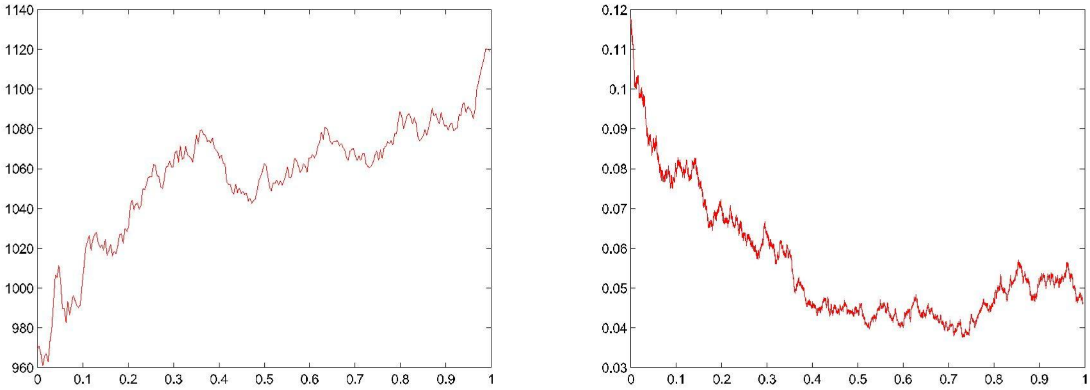

Figure 4.6: Simulated SV CAR(2) type process for $\alpha = 1$. Left: asset values. Right: volatility

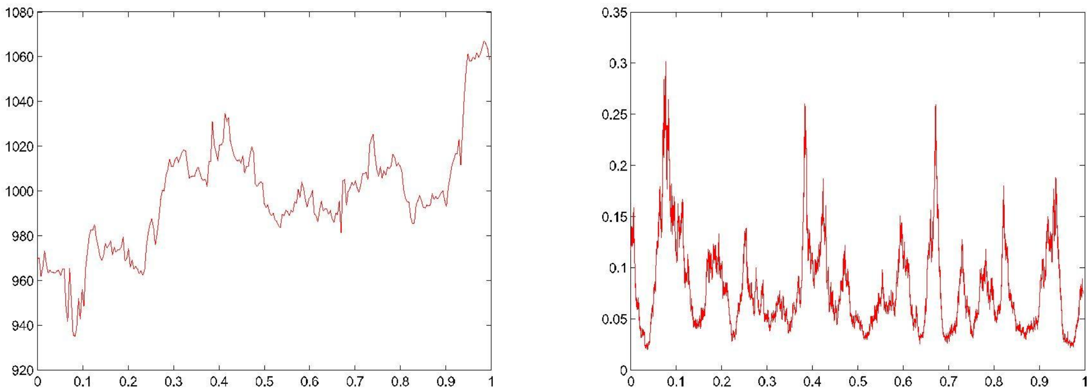

Figure 4.7: Simulated SV CAR(2) type process for $\alpha = 50$. Left: asset values. Right: volatility

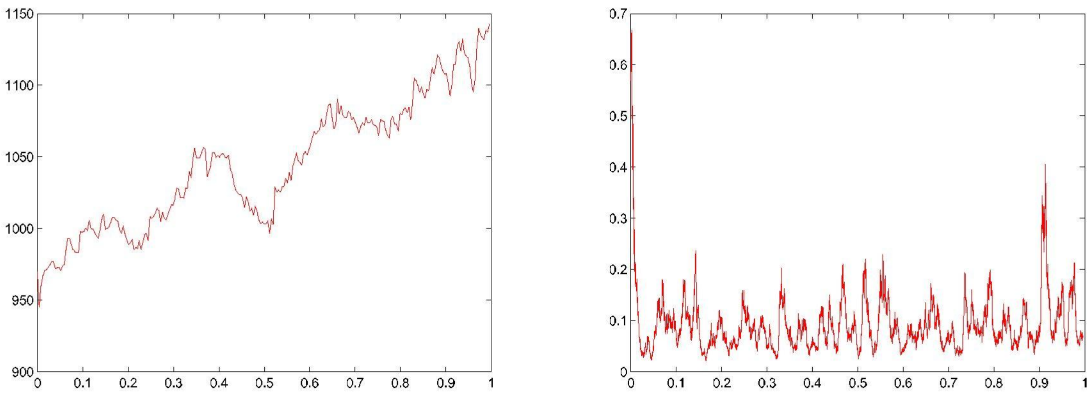

Figure 4.8: Simulated SV CAR(2) type process for $\alpha = 100$. Left: asset values. Right: volatility

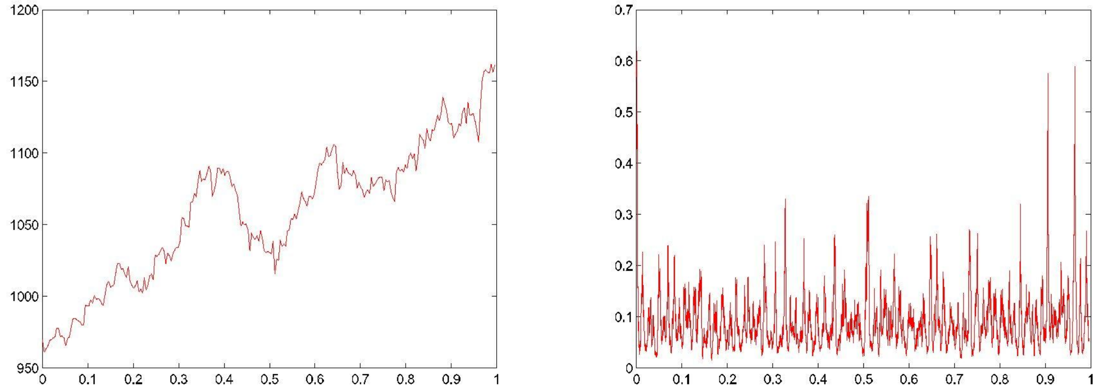

Figure 4.9: Simulated SV CAR(2) type process for $\alpha = 300$. Left: asset values. Right: volatility

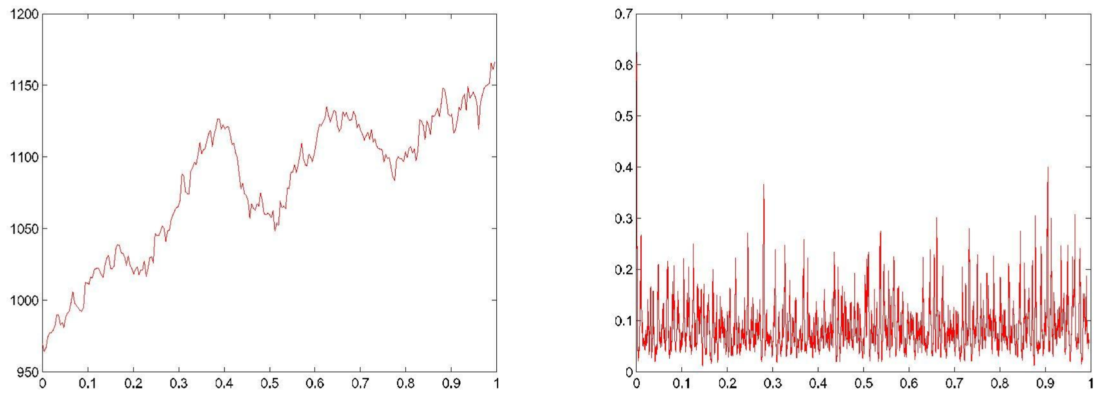

Figure 4.10: Simulated SV CAR(2) type process for $\alpha = 500$. Left: asset values. Right: volatility

Chapter 5

Analytical Developments for the Pricing of Securities in a Multivariate Case

The success and longevity of the Black-Scholes approach depends on two main factors: firstly, the mathematical tractability of the model, and secondly the fact that in many circumstances the model provides a simple approximation to the observed market behavior.

We saw in the previous chapters that a modification of the Black-Scholes model was required in order to account for some stylized facts observed on derivative prices and that stochastic volatility models have provided a potential explanation.

However, there is a price to be paid, namely that the new models no more allow the same mathematical tractability as the Black-Scholes model.

For the univariate case, these problems are not so dramatical, since one can use computer power to solve them in a relatively small amount of time. Unfortunately, this idea does no longer work in practice for the multivariate setting, where a large number of assets and series of derivative prices have to be analyzed.

Thus, for the multivariate setting, it is extremely important to deal with stochastic volatility models which have very good analytical properties.

In this chapter we focus on finding a class of multivariate stochastic volatility (MSV) models for which the correction constants appearing in the multivariate corrected price formula derived by Fouqué et al. (2000) can be explicitly computed with respect to the parameters of the MSV model.

The class of models that we propose to this aim contains extensions of the univariate Scott stochastic volatility model (see e.g., Appendix B). In this context, we derive first an explicit solution to a multivariate Poisson equation and this is the key to the explicit computations for the other components of the corrected price formula.

These results are very expedient for practical purposes, since they allow the precise computation of relevant financial quantities in a general multivariate setting for incomplete markets upon avoiding the usual computer intensive methods.

5.1 The Multivariate Model

Let us consider N assets with the following dynamics:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{j=1}^{N} \sigma_{ij}(\mathbf{Y}(t))dW_j(t), \quad (\forall)\, i = 1, \dots, N, \tag{5.1}$$

where $\sigma_{ij}(\mathbf{y}) = a_{ij} \exp(\sum_{l=1}^{N} \lambda_{ijl} y_l)$, $\forall \mathbf{y} = (y_1, \dots, y_N) \in \mathbb{R}^N$, $a_{ij} \in \mathbb{R}_+$, $\lambda_{ijl} \geq 0$, for all $i, j, l = 1, \dots, N$ and $\mathbf{Y}(t) = (Y_1(t), \dots, Y_N(t))'$ is a multivariate CAR(1) process given by

$$dY_k(t) = \alpha_k(m_k - Y_k(t))dt + \beta_k\, d\hat{Z}_k(t), \tag{5.2}$$

with $\alpha_k > 0$, $\beta_k > 0$, $m_k \in \mathbb{R}$ and

$$\hat{Z}_k(t) = \sum_{j=1}^{N} \rho_{jk} W_j(t) + \sqrt{1 - \sum_{j=1}^{N} \rho_{jk}^2} Z_k(t), \tag{5.3}$$

with $\sum_j \rho_{jk}^2 \leq 1$, for all $k = 1, \dots, N$, and where $\mathbf{W}(t) = (W_1(t), \dots, W_N(t))'$ and $\mathbf{Z}(t) = (Z_1(t), \dots, Z_N(t))'$ are independent Brownian motions. Thus, $\mathbf{W}(t)$ and $\hat{\mathbf{Z}}(t) = (\hat{Z}_1(t), \dots, \hat{Z}_N(t))'$ are correlated Brownian motions in $\mathbb{R}^N$ and

$$E(dW_j(t)\, d\hat{Z}_k(t)) = d\langle W_j, \hat{Z}_k \rangle_t = \rho_{jk} dt, \quad \forall j, k = 1, \dots, N.$$

The model described above is a multivariate extension of the classical Scott's stochastic model and in the following we put in evidence its good analytical properties in the context of asymptotic analysis for derivative pricing.

Let us denote the corresponding *spot covolatility matrix* in this model by $\Sigma = (\Sigma_{kl})_{k,l=1,\dots,N}$, where

$$\Sigma_{kl}(.) = \sum_{a=1}^{N} \sigma_{ka}(.)\sigma_{la}(.). \tag{5.4}$$

The invariant distribution of the multivariate CAR process $(\mathbf{Y}_t)_{t\geq 0}$ is $N(\mathrm{m}, \nu^2)$, where $\mathrm{m} \in \mathbb{R}^N$ has components m_k and $\nu^2 = (s_{kl})_{1\leq k,l \leq N}$ is the covariance matrix with diagonal entries

$$s_{kk} = \nu_k^2 = \frac{\beta_k^2}{2\alpha_k}, \quad k = 1, \dots, N \tag{5.5}$$

and off-diagonals

$$s_{kl} = cov(Y_k, Y_l) = \frac{\beta_k \beta_l}{\alpha_k + \alpha_l} \sum_{j=1}^{N} \rho_{jk}\rho_{jl}, \quad k \neq l. \tag{5.6}$$

For the financial applications, the matrix $\Sigma_{eff} = (\Sigma_{eff;(k,l)})_{k,l=1,\dots,N}$ is of paramount importance. Its components with respect to the invariant density $\Phi(.)$ of $(\mathbf{Y}_t)_{t\geq 0}$ are given by

$$\Sigma_{eff;(k,l)} = \langle \Sigma_{kl} \rangle = \int_{\mathbb{R}^N} \Sigma_{kl}(y)\Phi(y)dy, \quad k,l = 1,\dots,N. \tag{5.7}$$

where, as usual, $\langle g \rangle$ denotes the average with respect to the invariant density. In this case an *effective volatility matrix* σ_{eff} is a square matrix which satisfies $\sigma_{eff} * \sigma'_{eff} = \Sigma_{eff}$.

5.2 Asymptotics for Pricing European Derivatives

As it is usually done for derivative pricing problems, one chooses an equivalent martingale $P^{(*\Lambda)}$, under which the dynamics of assets and volatility processes becomes

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sum_{j=1}^{N} \sigma_{ij}(\mathbf{Y}(t))dW_j^*(t), \tag{5.8}$$

$$dY_k(t) = [\alpha_k(m_k - Y_k(t)) - \beta_k\tilde{\Lambda}_k]dt + \beta_k\, d\hat{Z}_k^*(t) \tag{5.9}$$

for some volatility risk premium $\tilde{\Lambda} = (\tilde{\Lambda}_k)_{1\leq k\leq N}$, chosen by the market and where, as usual, r is the instantaneous interest rate. Mostly it is assumed to be a function of the multivariate CAR process $\mathbf{Y}_t$, for instance $\tilde{\Lambda}_k = \Lambda_k(\mathbf{Y}_t)$, $k = 1,\dots,N$, where

$$\Lambda_k(.) = \sum_{i,j=1}^{N} b_{kij}\sigma_{ij}(.), \quad i,j,k = 1,\dots,N, \tag{5.10}$$

with b_{kij} real constants. Similar parameterizations of the risk premium have been suggested in literature, see e.g., Stein and Stein (1991) and Dai and Singleton (2000).

We only stated our results for the parameterization (5.10), however they can be extended to include more general ones, as can be seen by following the respective proof lines.

In this context, for a European contract with payoff function $h(\mathrm{x})$ (where h is some multivariate nonnegative bounded C^2 function) and maturity date T the price can be given by $P(t,\mathrm{x},\mathrm{y}) = E^{*(\Lambda)}\{e^{-r(T-t)}h(\mathbf{S}_T)|\mathbf{S}_t = \mathrm{x}, \mathbf{Y}_t = \mathrm{y}\}$, where the expectation is taken with respect to $P^{(*\Lambda)}$. The pricing function $P(t,\mathrm{x},\mathrm{y})$ satisfies

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \frac{1}{2}\sum_{i,j=1}^{N} \Sigma_{ij}x_ix_j\frac{\partial^2 P}{\partial x_i\partial x_j} + \sum_{i,j,k=1}^{N} \rho_{jk}\beta_k\sigma_{ij}x_i\frac{\partial^2 P}{\partial x_i \partial y_k}\\ &+ \frac{1}{2}\sum_{k=1}^{N}\beta_k^2\frac{\partial^2 P}{\partial y_k^2} + \frac{1}{2}\sum_{k\neq l}\beta_k\beta_l(\sum_j \rho_{jk}\rho_{jl})\frac{\partial^2 P}{\partial y_k\partial y_l}\\ &+ \sum_{k=1}^{N}[\alpha_k(m_k - y_k) - \beta_k\Lambda_k]\frac{\partial P}{\partial y_k} + r(\sum_{j=1}^{N} x_j\frac{\partial P}{\partial x_j} - P) = 0,\end{aligned} \tag{5.11}$$

with terminal condition $P(T, \mathrm{x}, \mathrm{y}) = h(\mathrm{x})$. Introducing the usual scaling to model fast mean reversions in the volatilities,

$$\beta_k = \nu_k \sqrt{2\alpha_k}, \quad \alpha_k = \frac{c_k}{\epsilon}, \quad \epsilon > 0, \quad (\forall)\, k = 1, \ldots, N \tag{5.12}$$

with c_k some positive constants, the pricing problem can be rewritten as

$$\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big) P^\epsilon = 0, \tag{5.13}$$

where

$$\mathcal{L}_0 = \sum_{k=1}^{N} c_k\big(\nu_k^2 \frac{\partial^2}{\partial y_k^2} + (m_k - y_k)\frac{\partial}{\partial y_k}\big) + \sum_{k \neq l} \nu_k \nu_l \sqrt{c_k c_l}\big(\sum_{j=1}^{N} \rho_{jk}\rho_{jl}\big)\frac{\partial^2}{\partial y_k \partial y_l}, \tag{5.14}$$

$$\mathcal{L}_1 = \sum_{i,j,k} \nu_k \sqrt{2c_k}\rho_{jk}\sigma_{ij}x_i \frac{\partial^2}{\partial x_i \partial y_k} - \sum_{k} \nu_k \sqrt{2c_k}\Lambda_k \frac{\partial}{\partial y_k}, \tag{5.15}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{N} \Sigma_{ij}x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + r\big(\sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} - \cdot\big). \tag{5.16}$$

Then we get

$$P^\epsilon = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \ldots \tag{5.17}$$

where P_0 is the Black-Scholes price satisfying

$$\mathcal{L}_{BS}^{N}(\Sigma_{eff})P_0 = 0, \tag{5.18}$$

with $P(T, \mathrm{x}) = h(\mathrm{x})$ and $\mathcal{L}_{BS}^{N}(\Sigma_{eff})$ is the Black-Scholes operator

$$\mathcal{L}_{BS}^{N}(\Sigma_{eff}) = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{N} \Sigma_{eff;(i,j)}x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + r\big(\sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} - \cdot\big). \tag{5.19}$$

5.3 The Corrected Price

With the above notations, under fast mean reversions in the volatilities, the corrected price for the multivariate case is (see Fouqué et al. (2000))

$$P_0 - (T-t)\Big(\sum_{i,m} V_{im}^{(2)} \frac{\partial^2 P_0}{\partial x_i \partial x_m} + \sum_{i,m,l} V_{ilm}^{(3)} \frac{\partial^3 P_0}{\partial x_i \partial x_l \partial x_m}\Big), \tag{5.20}$$

where the first order correction constants $V_{im}^{(2)}$ and $V_{ilm}^{(3)}$ are given by

$$V_{im}^{(2)} = \sum_{k=1}^{N} \frac{\nu_k}{\sqrt{2\alpha}}\Big(\sum_{j=1}^{N} \rho_{jk}\big\langle \sigma_{ij}\frac{\partial \phi_{im}}{\partial y_k}\big\rangle - \big\langle \Lambda_k \frac{\partial \phi_{im}}{\partial y_k}\big\rangle\Big) \tag{5.21}$$

$$V_{ilm}^{(3)} = \sum_{k=1}^{N} \frac{\nu_k}{\sqrt{2\alpha}} \sum_{j=1}^{N} \rho_{jk} \langle \sigma_{ij} \frac{\partial \phi_{lm}}{\partial y_k} \rangle, \tag{5.22}$$

where $\alpha = \alpha_1$ and for all $i, j = 1, \ldots, N$, ϕ_{ij} satisfies the Poisson equation

$$\mathcal{L}_0 \phi_{ij} = \sum_{k=1}^{N} \sigma_{ik} \cdot \sigma_{jk} - \Sigma_{eff;(i,j)}, \tag{5.23}$$

and is such that the correction constants above are well defined.

It is well known that in practice for large N's it is very difficult to accurately evaluate the above correction constants with the classical computer intensive methods. Thus, it is desirable to have some analytical results for the multivariate case and this is what we do in the following.

As a first step in this direction we put in evidence an analytical formula for the components of the matrix Σ_{eff} which is of course crucial for the computation of the classical Black-Scholes price P_0. With the above notations, for all $k, l = 1, \ldots, N$ we can write

$$\Sigma_{eff;(k,l)} = \sum_{j=1}^{N} a_{kj} a_{lj} \exp\{\sum_{p=1}^{N} (\lambda_{kjp} + \lambda_{ljp}) m_p + \frac{1}{2} \sum_{p,q=1}^{N} (\lambda_{kjp} + \lambda_{ljp})(\lambda_{kjq} + \lambda_{ljq}) s_{pq}\}. \tag{5.24}$$

Having a closed form for Σ_{eff}, the next step is to find explicit solutions to the above high dimensional PDE (5.23).

5.4 An Explicit Solution to the Multidimensional Poisson Equation

Some basic facts about the univariate Poisson equation are resumed in the Appendix C. Here we give a generalization to the multidimensional case, under the assumption $c_k = c > 0$, for all $k = 1, \ldots, N$ in (5.12) and (5.14).

Our next result gives an explicit solution to the PDE (5.23) and this opens the possibility of explicitly computing the correction constants (5.21), (5.22).

Theorem 5.4.1
Assume that the coefficients $\{\lambda_{ijl}\}$ from the definition of the volatility process satisfy the following nondegeneracy condition:

$$\sum_{l,q=1}^{N} (\lambda_{ikl} + \lambda_{jkl})(\lambda_{ikq} + \lambda_{jkq}) s_{lq} > 0, \tag{5.25}$$

for all $i, j, k = 1 \ldots, N$. Then a classical solution $\phi(\mathbf{y}) = (\phi_{ij}(\mathbf{y}))_{i,j=1,\ldots,N}$ for (5.23) can be explicitly given by

$$\phi_{ij}(\mathbf{y}) = \frac{1}{c}\sum_{k=1}^{N} \int_{-\infty}^{\sum_{l=1}^{N}(\lambda_{ikl}+\lambda_{jkl})y_l} \Big(\Phi_{ijk}(z)\sum_{l,q=1}^{N}(\lambda_{ikl}+\lambda_{jkl})(\lambda_{ikq}+\lambda_{jkq})s_{lq}\Big)^{-1}$$

$$\times \int_{-\infty}^{z} (a_{ik}a_{jk}e^{x} - \langle \sigma_{ik}\sigma_{jk}\rangle)\Phi_{ijk}(x)\,dx\,dz, \tag{5.26}$$

for all $\mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{R}^N$, where $\Phi_{ijk}(x)$ is the density of the univariate normal distribution

$$N\Big(\sum_{l=1}^{N}(\lambda_{ikl}+\lambda_{jkl})m_l, \sum_{l,q=1}^{N}(\lambda_{ikl}+\lambda_{jkl})(\lambda_{ikq}+\lambda_{jkq})s_{lq}\Big). \tag{5.27}$$

Proof. For all $i, j, k = 1, \ldots, N$, let us denote by h_{ijk} a solution of the univariate Poisson equation

$$\nu_{ijk}^2 h_{ijk}''(y) + (M_{ijk} - y)h_{ijk}'(y) = \frac{1}{c}a_{ik}a_{jk}e^y - \frac{1}{c}\langle\sigma_{ik}\sigma_{jk}\rangle, \ y \in \mathbb{R}, \tag{5.28}$$

where

$$M_{ijk} = \sum_{l=1}^{N}(\lambda_{ikl}+\lambda_{jkl})m_l, \tag{5.29}$$

$$\nu_{ijk}^2 = \sum_{l,q=1}^{N}(\lambda_{ikl}+\lambda_{jkl})(\lambda_{ikq}+\lambda_{jkq})s_{lq}. \tag{5.30}$$

Then, observe that

$$\phi_{ij}(\mathbf{y}) = \sum_{k=1}^{N} h_{ijk}\Big(\sum_{l=1}^{N}(\lambda_{ikl}+\lambda_{jkl})y_l\Big) \tag{5.31}$$

is a solution for (5.23).

■

Now, this result enables us to perform the next step of our developments, which will eventually lead to the closed form of the correction constants.

5.5 Analytical Computation of the Correction Constants

In order to derive explicit formulae for the correction constants $V_{im}^{(2)}$ and $V_{ilm}^{(3)}$ we have first to compute analytically the quantities $\langle\sigma_{ij}\frac{\partial\phi_{lm}}{\partial y_k}\rangle$ and then the derivation of $\langle\Lambda_k\frac{\partial\phi_{im}}{\partial y_k}\rangle$ follows easily upon using the parameterization (5.10) for the risk premium.

Let us denote

$$P_{lm;ij} := \sum_{p,p'=1}^{N} s_{pp'}\lambda_{lmp}\lambda_{ijp'}, \tag{5.32}$$

for all l, m, i, $j = 1, \ldots, N$.

Theorem 5.5.1
Assume that $\sum_{p=1}^{N} \lambda_{ijp} > 0$ *and* $P_{lm;ij} > 0$ *for all* $i, j, l, m = 1, \ldots, N$. *Then*

$$\left\langle\sigma_{ij}\frac{\partial\phi_{lm}}{\partial y_k}\right\rangle = \sum_{\vartheta=1}^{N} \frac{\lambda_{l\vartheta k} + \lambda_{m\vartheta k}}{c(P_{l\vartheta;ij} + P_{m\vartheta;ij})}\Big(\langle\sigma_{l\vartheta}\sigma_{m\vartheta}\rangle\langle\sigma_{ij}\rangle - \langle\sigma_{ij}\sigma_{l\vartheta}\sigma_{m\vartheta}\rangle\Big), \tag{5.33}$$

for all $i, j, l, m, k = 1, \ldots, N$ *where*

$$\langle\sigma_{ij}\rangle = a_{ij}\exp\{\sum_{p=1}^{N}\lambda_{ijp}m_p + \frac{1}{2}\sum_{p,p'=1}^{N} s_{pp'}\lambda_{ijp}\lambda_{ijp'}\}, \tag{5.34}$$

$$\begin{aligned}\langle\sigma_{lq}\sigma_{mq}\rangle &= a_{lq}a_{mq}\exp\{\sum_{p=1}^{N}(\lambda_{lqp} + \lambda_{mqp})m_p \\ &+ \frac{1}{2}\sum_{p,p'=1}^{N} s_{pp'}(\lambda_{lqp} + \lambda_{mqp})(\lambda_{lqp'} + \lambda_{mqp'})\},\end{aligned} \tag{5.35}$$

$$\begin{aligned}\langle\sigma_{ij}\sigma_{lq}\sigma_{mq}\rangle &= a_{ij}a_{lq}a_{mq}\exp\{\sum_{p=1}^{N}(\lambda_{ijp} + \lambda_{lqp} + \lambda_{mqp})m_p \\ &+ \frac{1}{2}\sum_{p,p'=1}^{N} s_{pp'}(\lambda_{ijp} + \lambda_{lqp} + \lambda_{mqp})(\lambda_{ijp'} + \lambda_{lqp'} + \lambda_{mqp'})\}\end{aligned} \tag{5.36}$$

for all $i, j, l, m, q = 1, ..., N$.

Proof. For all $l, m, \vartheta = 1, \ldots, N$ and $\mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{R}^N$, let

$$D_{lm\vartheta}(\mathbf{y}) := \sum_{k=1}^{N}(\lambda_{l\vartheta k} + \lambda_{m\vartheta k})y_k. \tag{5.37}$$

From the proof of Theorem 5.4.1 we notice that ϕ_{lm} has the representation

$$\phi_{lm}(\mathbf{y}) = \sum_{\vartheta=1}^{N} h_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y})).$$

Then we have

$$\frac{\partial\phi_{lm}}{\partial y_k}(\mathbf{y}) = \sum_{\vartheta=1}^{N} h'_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))(\lambda_{l\vartheta k} + \lambda_{m\vartheta k}),$$

and

$$\left\langle\sigma_{ij}\frac{\partial\phi_{lm}}{\partial y_k}\right\rangle = \sum_{\vartheta=1}^{N}(\lambda_{l\vartheta k}+\lambda_{m\vartheta k})\left\langle\sigma_{ij}h'_{lm\vartheta}(D_{lm\vartheta}(.))\right\rangle.$$

Thus, to find the correction constants it is enough to compute

$$\left\langle\sigma_{ij}h'_{lm\vartheta}(D_{lm\vartheta}(.))\right\rangle = \int_{\mathbb{R}^N}\Phi(\mathbf{y})\sigma_{ij}(\mathbf{y})h'_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))d\mathbf{y} \tag{5.38}$$

and this is what we do for the rest of the proof. From the proof of Theorem 5.4.1 we have that

$$\left(h'_{lm\vartheta}(y)\Phi_{lm\vartheta}(y)\right)' = \frac{1}{c\cdot\nu^2_{lm\vartheta}}\Phi_{lm\vartheta}(y)(a_{l\vartheta}a_{m\vartheta}e^y - <\sigma_{l\vartheta}\sigma_{m\vartheta}>),$$

for all $\vartheta = 1,\dots,N$ and for all $y\in\mathbb{R}$, where upon using the same notations as in the proof of Theorem 5.4.1 we obtain

$$\Phi_{lm\vartheta}(y) = \frac{1}{\nu_{lm\vartheta}\sqrt{2\pi}}\exp\{-\frac{(y-M_{lm\vartheta})^2}{2\nu^2_{lm\vartheta}}\},\ \forall y\in\mathbb{R}. \tag{5.39}$$

Thus

$$\begin{aligned}
&\frac{\partial}{\partial y_q}\left(h'_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))\right)\\
&= \frac{a_{l\vartheta}a_{m\vartheta}}{c\nu^2_{lm\vartheta}}\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))\cdot(e^{D_{lm\vartheta}(\mathbf{y})}-e^{M_{lm\vartheta}+\frac{1}{2}\nu^2_{lm\vartheta}})\frac{\partial}{\partial y_q}D_{lm\vartheta}(\mathbf{y})\\
&= \frac{a_{l\vartheta}a_{m\vartheta}}{c\nu^2_{lm\vartheta}}(\lambda_{l\vartheta q}+\lambda_{m\vartheta q})\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))(e^{D_{lm\vartheta}(\mathbf{y})}-e^{M_{lm\vartheta}+\frac{1}{2}\nu^2_{lm\vartheta}})
\end{aligned}$$

for all $\vartheta, q = 1,\dots,N$. Since $\sum_{p=1}^{N}\lambda_{ijp} > 0$, $\forall i,j$, it follows that for all l, m, $\vartheta = 1,\dots,N$ there exists $q\in\{1,\dots,N\}$ such that $\gamma_{lm\vartheta q} := \lambda_{l\vartheta q}+\lambda_{m\vartheta q} > 0$. Using this index q we have

$$\begin{aligned}
\left\langle\sigma_{ij}h'_{lm\vartheta}(D_{lm\vartheta}(.))\right\rangle &= \int_{\mathbb{R}^N}\frac{\Phi(\mathbf{y})}{\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))}\sigma_{ij}(\mathbf{y})h'_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))d\mathbf{y}\\
&= \int_{\mathbb{R}^N}\left(\int_{-\infty}^{y_q}\sigma_{ij}(y_1,\dots,z,\dots,y_N)\frac{\Phi(y_1,\dots,z,\dots,y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1,\dots,z,\dots,y_N))}dz\right)\times\\
&\qquad\times\frac{\partial\left(h'_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))\right)}{\partial y_q}d\mathbf{y}\\
&= -\frac{a_{l\vartheta}a_{m\vartheta}\gamma_{lm\vartheta q}}{c\cdot\nu^2_{lm\vartheta}}\int_{\mathbb{R}^N}\left(\int_{-\infty}^{y_q}\sigma_{ij}(y_1,\dots,z,\dots,y_N)\frac{\Phi(y_1,\dots,z,\dots,y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1,\dots,z,\dots,y_N))}dz\right)\times\\
&\qquad\times\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))(e^{D_{lm\vartheta}(\mathbf{y})}-B_{lm\vartheta})d\mathbf{y} = \frac{a_{l\vartheta}a_{m\vartheta}\gamma_{lm\vartheta q}}{c\cdot\nu^2_{lm\vartheta}}(I_1-I_2),
\end{aligned}$$

where z is on the position q and

$$B_{lm\vartheta} := e^{M_{lm\vartheta}+\frac{1}{2}\nu^2_{lm\vartheta}}, \tag{5.40}$$

for all $l, m, \vartheta = 1, \ldots, N$. I_1, I_2 denote the following integrals:

$$I_1 := B_{lm\vartheta} \int_{\mathbb{R}^N} \Big(\int_{-\infty}^{y_q} \sigma_{ij}(y_1, \ldots, z, \ldots, y_N) \frac{\Phi(y_1, \ldots, z, \ldots, y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1, \ldots, z, \ldots, y_N))} dz \Big) \times$$
$$\times \Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y})) d\mathbf{y},$$

and

$$I_2 := \int_{\mathbb{R}^N} \Big(\int_{-\infty}^{y_q} \sigma_{ij}(y_1, \ldots, z, \ldots, y_N) \frac{\Phi(y_1, \ldots, z, \ldots, y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1, \ldots, z, \ldots, y_N))} dz \Big) \times$$
$$\times \Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y})) e^{D_{lm\vartheta}(\mathbf{y})} d\mathbf{y}.$$

The computation of I_1:

By using the Fubini theorem we have

$$I_1 = B_{lm\vartheta} \int_{\mathbb{R}^N} \Big(\int_{-\infty}^{0} \sigma_{ij}(y_1, \ldots, t + y_q, \ldots, y_N) \frac{\Phi(y_1, \ldots, t + y_q, \ldots, y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1, \ldots, t + y_q, \ldots, y_N))} dt \Big) \times$$
$$\times \Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y})) d\mathbf{y}$$
$$= B_{lm\vartheta} \int_{-\infty}^{0} \Big(\int_{\mathbb{R}^N} \sigma_{ij}(y_1, \ldots, t + y_q, \ldots, y_N) \frac{\Phi(y_1, \ldots, t + y_q, \ldots, y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1, \ldots, t + y_q, \ldots, y_N))} \times$$
$$\times \Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y})) d\mathbf{y} \Big) dt.$$

With the change of variables $z_i = y_i + \delta_{qi} t$, $i = 1, \ldots, N$, t fixed, where δ_{qi} is Kronecker's symbol, we have

$$\int_{\mathbb{R}^N} \sigma_{ij}(y_1, \ldots, t + y_q, \ldots, y_N) \frac{\Phi(y_1, \ldots, t + y_q, \ldots, y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1, \ldots, t + y_q, \ldots, y_N))} \Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y})) d\mathbf{y}$$
$$= \int_{\mathbb{R}^N} \sigma_{ij}(\mathbf{z}) \frac{\Phi(\mathbf{z})}{\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{z}))} \Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{z}) - t\gamma_{lm\vartheta q}) d\mathbf{z}$$

$$= \int_{\mathbb{R}^N} \sigma_{ij}(\mathbf{z}) \exp\{ \frac{1}{2\nu_{lm\vartheta}^2} t\gamma_{lm\vartheta q}(2D_{lm\vartheta}(\mathbf{z}) - 2M_{lm\vartheta} - t\gamma_{lm\vartheta q}) \} \Phi(\mathbf{z}) d\mathbf{z}$$

$$= \exp\{ -\frac{1}{2\nu_{lm\vartheta}^2} t\gamma_{lm\vartheta q}(2M_{lm\vartheta} + t\gamma_{lm\vartheta q}) \} \int_{\mathbb{R}^N} \sigma_{ij}(\mathbf{z}) \exp\{ \frac{1}{\nu_{lm\vartheta}^2} t\gamma_{lm\vartheta q} D_{lm\vartheta}(\mathbf{z}) \} \Phi(\mathbf{z}) d\mathbf{z}$$

$$= a_{ij} \exp\{ -\frac{1}{2\nu_{lm\vartheta}^2} t\gamma_{lm\vartheta q}(2M_{lm\vartheta} + t\gamma_{lm\vartheta q}) \} \int_{\mathbb{R}^N} \exp\{ \sum_{p=1}^{N} \lambda_{ijp} z_p + \frac{1}{\nu_{lm\vartheta}^2} t\gamma_{lm\vartheta q} D_{lm\vartheta}(\mathbf{z}) \} \Phi(\mathbf{z}) d\mathbf{z}$$

$$= a_{ij} \exp\{ -\frac{1}{2\nu_{lm\vartheta}^2} t\gamma_{lm\vartheta q}(2M_{lm\vartheta} + t\gamma_{lm\vartheta q}) \} \int_{\mathbb{R}^N} \exp\{ \sum_{p=1}^{N} \lambda_{ijp} z_p$$

$$+\frac{1}{\nu_{lm\vartheta}^2}t\gamma_{lm\vartheta q}\sum_{p=1}^{N}(\lambda_{l\vartheta p}+\lambda_{m\vartheta p})z_p\}\Phi(\mathbf{z})d\mathbf{z}=a_{ij}\exp\{-\frac{1}{2\nu_{lm\vartheta}^2}t\gamma_{lm\vartheta q}(2M_{lm\vartheta}+t\gamma_{lm\vartheta q})\}\times$$

$$\times\int_{\mathbb{R}^N}\exp\{\sum_{p=1}^{N}\Big(\lambda_{ijp}+\frac{1}{\nu_{lm\vartheta}^2}t\gamma_{lm\vartheta q}(\lambda_{l\vartheta p}+\lambda_{m\vartheta p})\Big)z_p\}\Phi(\mathbf{z})d\mathbf{z}$$

$$=a_{ij}\exp\{-\frac{1}{2\nu_{lm\vartheta}^2}t\gamma_{lm\vartheta q}(2M_{lm\vartheta}+t\gamma_{lm\vartheta q})\}\exp\{\sum_{p=1}^{N}\Big(\lambda_{ijp}+\frac{1}{\nu_{lm\vartheta}^2}t\gamma_{lm\vartheta q}(\lambda_{l\vartheta p}+\lambda_{m\vartheta p})\Big)m_p$$

$$+\frac{1}{2}\sum_{p,p'=1}^{N}s_{pp'}v_p(t)v_{p'}(t)\}$$

$$=a_{ij}\exp\{-\frac{1}{2\nu_{lm\vartheta}^2}t^2\gamma_{lm\vartheta q}^2\}\exp\{\sum_{p=1}^{N}\lambda_{ijp}m_p+\frac{1}{2}\sum_{p,p'=1}^{N}s_{pp'}v_p(t)v_{p'}(t)\},$$

where

$$v_p(t)=\lambda_{ijp}+\frac{1}{\nu_{lm\vartheta}^2}t\gamma_{lm\vartheta q}(\lambda_{l\vartheta p}+\lambda_{m\vartheta p}),\quad \text{for all } p=1,\dots,N.$$

But

$$\sum_{p,p'=1}^{N}s_{pp'}v_p(t)v_{p'}(t)=\sum_{p,p'=1}^{N}s_{pp'}\Big(\lambda_{ijp}\lambda_{ijp'}+\frac{1}{\nu_{lm\vartheta}^4}t^2\gamma_{lm\vartheta q}^2(\lambda_{l\vartheta p}+\lambda_{m\vartheta p})(\lambda_{l\vartheta p'}+\lambda_{m\vartheta p'})$$

$$+\frac{1}{\nu_{lm\vartheta}^2}t\gamma_{lm\vartheta q}[\lambda_{ijp}\lambda_{ijp'}+(\lambda_{l\vartheta p}+\lambda_{m\vartheta p})\lambda_{ijp'}+(\lambda_{l\vartheta p'}+\lambda_{m\vartheta p'})\lambda_{ijp}]\Big)$$

$$=\frac{t^2\gamma_{lm\vartheta q}^2}{\nu_{lm\vartheta}^2}+\frac{2t\gamma_{lm\vartheta q}}{\nu_{lm\vartheta}^2}(P_{l\vartheta;ij}+P_{m\vartheta;ij})+P_{ij;ij}$$

and thus

$$\int_{\mathbb{R}^N}\sigma_{ij}(y_1,\dots,t+y_q,\dots,y_N)\frac{\Phi(y_1,\dots,t+y_q,\dots,y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1,\dots,t+y_q,\dots,y_N))}\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathbf{y}))d\mathbf{y}$$

$$=a_{ij}\exp\{\sum_{p=1}^{N}\lambda_{ijp}m_p+\frac{1}{2}P_{ij;ij}+\frac{t\gamma_{lm\vartheta q}}{2\nu_{lm\vartheta}^2}\sum_{p,p'=1}^{N}s_{pp'}[\lambda_{ijp}\lambda_{ijp'}+(\lambda_{l\vartheta p}+\lambda_{m\vartheta p})\lambda_{ijp'}$$

$$+(\lambda_{l\vartheta p'}+\lambda_{m\vartheta p'})\lambda_{ijp}]\}=a_{ij}\exp\{\sum_{p=1}^{N}\lambda_{ijp}m_p+\frac{1}{2}P_{ij;ij}+\frac{t\gamma_{lm\vartheta q}}{\nu_{lm\vartheta}^2}(P_{l\vartheta;ij}+P_{m\vartheta;ij})\}$$

$$=\langle\sigma_{ij}\rangle\exp\{\frac{t\gamma_{lm\vartheta q}}{\nu_{lm\vartheta}^2}(P_{l\vartheta;ij}+P_{m\vartheta;ij})\}.$$

Finally, we obtain

$$I_1=B_{lm\vartheta}\int_{-\infty}^{0}\Big(\int_{\mathbb{R}^N}\sigma_{ij}(y_1,\dots,t+y_q,\dots,y_N)\frac{\Phi(y_1,\dots,t+y_q,\dots,y_N)}{\Phi_{lm\vartheta}(D_{lm\vartheta}(y_1,\dots,t+y_q,\dots,y_N))}\times$$

$$\times\Phi_{lm\vartheta}(D_{lm\vartheta}(\mathrm{y}))d\mathrm{y}\Big)dt$$

$$= B_{lm\vartheta}\big\langle\sigma_{ij}\big\rangle \int_{-\infty}^{0} \exp\{\frac{t\gamma_{lm\vartheta q}}{\nu^2_{lm\vartheta}}(P_{l\vartheta;ij} + P_{m\vartheta;ij})\}dt$$

$$= B_{lm\vartheta}\big\langle\sigma_{ij}\big\rangle \frac{\nu^2_{lm\vartheta}}{\gamma_{lm\vartheta q}(P_{l\vartheta;ij} + P_{m\vartheta;ij})},$$

with $B_{lm\vartheta}$ given in (5.40). The calculation of I_2 is similar, step by step, with the above computation of I_1 and we obtain

$$I_2 =< \sigma_{ij}\sigma_{l\vartheta}\sigma_{m\vartheta} > \frac{\nu^2_{lm\vartheta}}{\gamma_{lm\vartheta q}(P_{l\vartheta;ij} + P_{m\vartheta;ij})}.$$

Thus (5.38) becomes

$$\big\langle\sigma_{ij}h'_{lm\vartheta}(D_{lm\vartheta}(.))\big\rangle = \frac{a_{l\vartheta}a_{m\vartheta}}{c\nu^2_{lm\vartheta}}(\lambda_{l\vartheta q} + \lambda_{m\vartheta q})(I_1 - I_2) = \frac{a_{l\vartheta}a_{m\vartheta}\gamma_{lm\vartheta q}}{c\nu^2_{lm\vartheta}}(I_1 - I_2)$$

$$= \frac{a_{l\vartheta}a_{m\vartheta}\gamma_{lm\vartheta q}}{c\nu^2_{lm\vartheta}}I_1 - \frac{\gamma_{lm\vartheta q}}{c\nu^2_{lm\vartheta}}\big\langle\sigma_{ij}\sigma_{l\vartheta}\sigma_{m\vartheta}\big\rangle \frac{\nu^2_{lm\vartheta}}{\gamma_{lm\vartheta q}(P_{l\vartheta;ij} + P_{m\vartheta;ij})}$$

$$= \frac{a_{l\vartheta}a_{m\vartheta}\gamma_{lm\vartheta q}}{c\nu^2_{lm\vartheta}}B_{lm\vartheta}\big\langle\sigma_{ij}\big\rangle \frac{\nu^2_{lm\vartheta}}{\gamma_{lm\vartheta q}(P_{l\vartheta;ij} + P_{m\vartheta;ij})} - \frac{\gamma_{lm\vartheta q}}{c\nu^2_{lm\vartheta}}\big\langle\sigma_{ij}\sigma_{l\vartheta}\sigma_{m\vartheta}\big\rangle \frac{\nu^2_{lm\vartheta}}{\gamma_{lm\vartheta q}(P_{l\vartheta;ij} + P_{m\vartheta;ij})}$$

$$= \frac{1}{c(P_{l\vartheta;ij} + P_{m\vartheta;ij})}\big\langle\sigma_{l\vartheta}\sigma_{m\vartheta}\big\rangle\big\langle\sigma_{ij}\big\rangle - \frac{1}{c(P_{l\vartheta;ij} + P_{m\vartheta;ij})}\big\langle\sigma_{ij}\sigma_{l\vartheta}\sigma_{m\vartheta}\big\rangle$$

$$= \frac{1}{c(P_{l\vartheta;ij} + P_{m\vartheta;ij})}\Big(\big\langle\sigma_{l\vartheta}\sigma_{m\vartheta}\big\rangle\big\langle\sigma_{ij}\big\rangle - \big\langle\sigma_{ij}\sigma_{l\vartheta}\sigma_{m\vartheta}\big\rangle\Big),$$

which completes the proof.

■

Remark 5.5.1

(a) The above results are very useful in practice, since they allow the precise computation of the corrected price formula upon avoiding costly methods which would be otherwise needed for numerically solving the corresponding PDEs and/or for stochastic simulations, especially in higher dimensions.

(b) As usual in a multivariate context, the calibration problem is very difficult, but in principle still abordable with the simulation-free nonlinear filtering procedure described in Chapter 2. The main difficulty to carry this out in practice relies on the large number of parameters to be estimated, which raises serious problems with the usual optimization procedures.

Chapter 6

A New Analytical Approximation for Default Probability in a Generalized Merton Setting

Merton (1974) proposed the first model of default which is also considered to be the first structural one (i.e. which uses the value of a firm to characterize the default). Thereby, the non-observable value of a firm is assumed to follow a geometric Brownian motion and it is well known that this is quite unrealistic.

The fast mean-reverting stochastic volatility models open the possibility to improve the classical Merton setting above. Applying in this special framework the same asymptotic theory as in the previous chapters, we obtain a new analytical approximation for the default probability (PD) in an incomplete market setting. This is based on an improved first correction term, when compared with the one used in Fouqué et al. (2006). Moreover, unlike similar results present in literature, our approximation also depends on the value of the volatility driven factor, which allows to account for more market information.

For more accounts on credit risk modeling we refer to Duffie and Singleton (2003) and Elizalde (2003, 2005). Various extensions of Ito formula and generalized Feynman-Kac type results can be found for instance in Alsmeyer and Jaeger (2002) or Karatzas and Shreve (1988).

6.1 Default Risk in Stochastic Volatility Models

There are two primary types of models that attempt to describe default processes in the literature: structural and reduced form models.

In contrast to the structural ones, the default in reduced form models is not determined via the value of the firm, but it is simply characterized with a hazard rate process describing the instantaneous probability of default, see e.g., Jarrow and Turnbull (1995). Thus, defaults are exogenously given in the reduced approach instead of being endogenously generated like in the former models.

The structural models are particularly useful for practitioners in the credit portfolio and credit risk management fields. Their intuitive economic interpretation facilitates consistent discussions regarding a variety of credit risk exposures.

Merton (1974) firstly builds a model based on the capital structure of the firm, which became the basis of the structural approach. In this setting a company defaults if, at the time of servicing the debt, its assets are lower than its outstanding debt. Black and Cox (1976) extended the Merton model to a first passage one, whereby bondholders can force the reorganization of the bankruptcy of the firm if its value falls under some barrier. Other extensions based on mean reverting SV models have been recently considered in Fouqué et al. (2006), in the context of pricing defaultable bonds.

In a similar framework we address some issues which are relevant to credit risk modeling, upon showing how to determine default probabilities under a fast mean-reversion volatility regime and how different they are when compared to those computed in the Merton setting. Concerning the latter issue, this difference is too small if trying to translate the recent results in Fouqué et al. (2006) to the context of approximating the default probabilities (in the nonleverage case there is actually no difference). This is due to the fact that in this case one has to work under the subjective probability measure and the order of the asymptotic expansion in Fouqué et al. (2006) is too small.

The model

We describe in the following the evolution of the firm value $S(t)$ with the SV CAR(1) process

$$dS(t) = \mu S(t)\,dt + \sigma_t S(t)\,dW(t), \quad \sigma_t = f(Y(t)), \tag{6.1}$$

$$dY(t) = \alpha(m - Y(t))dt + \beta\,dZ(t), \quad t \geq 0, \tag{6.2}$$

for all $t \geq 0$, where $(W(t), Z(t))_{t\geq 0}$ is a standard bivariate Brownian motion, $\alpha, \beta > 0$, and f is some bounded positive C^2 function, which is also bounded away from zero.

Following the Merton model, the capital structure of the firm comprises equity and a zero-coupon bond with maturity T and face value B. Under these assumptions, equity represents a call option on the firm's assets with maturity T and strike price B. Then the firm will default at time T if $S_T < B$. The variable $B > 0$ is the default barrier. In this sense, we use $PD(T, B|t, x, y)$ to denote the corresponding default probability of the firm, i.e. $PD(T, B|t, x, y) = E\{1_{\{S_T<B\}}|S(t) = x, Y(t) = y\}$, ($t < T$), where E denotes the expectation with respect to the subjective probability measure.

It is well known that under a fast mean-reverting volatility regime the above SV model can be approximated with the following Merton model:

$$d\tilde{S}(t) = \mu\tilde{S}(t)\,dt + \overline{\sigma}\tilde{S}(t)\,dW(t), \quad t \geq 0, \tag{6.3}$$

where $\overline{\sigma}$ is the effective volatility (see e.g., Chapter 3).

Denoting $PD_0(T, B|t, x)$ the corresponding default probability in this Merton model, we have

$$PD_0(T, B|t, x) = \Phi_{NS}\Big(\frac{\log B - \log x - (T-t)\cdot(\mu - \frac{1}{2}\overline{\sigma}^2)}{\sqrt{T-t}\cdot\overline{\sigma}}\Big), \tag{6.4}$$

for all $(t, x) \in [0, T) \times [B, \infty)$, where Φ_{NS} is the cdf of a standard normal random variable.

Such a simple description for $PD(T, B|t, x, y)$ is of course not possible, however we obtained a new analytical approximation in the context of the asymptotic theory of Fouqué et al. (2000). This result shows which kind of correction should be made to $PD_0(T, B|t, x)$ under a fast mean-reverting stochastic volatility setting.

In the following we restrict our attention to the case $x > B$, $t < T$, since this is the most interesting one for practical purposes.

The result given below can be easily extended to approximate all the financial quantities which can be represented as $E\{h(S_T)|S(t) = x, Y(t) = y\}$, where h is some nonnegative bounded piecewice continuous function.

6.2 A Corrected Default Probability Formula

In the following we assume that the usual conditions for the asymptotic theory of Fouqué et al. (2000) are satisfied.

Let $\mathcal{L}_0$ be the infinitesimal generator of the process $(Y_t)_{t\geq 0}$, and ϕ a classical solution of the Poisson equation:

$$\mathcal{L}_0\phi = f^2 - \langle f^2\rangle, \tag{6.5}$$

with $\langle|\phi|\rangle < \infty$. For some elementary facts about the above Poisson equation we refer to the Appendix C.

We recall that $\langle g\rangle = \int_{\mathbb{R}} g(y)\Phi(y)\,dy$, for all g (for which the integral exists), where Φ is the density of the invariant distribution[1] of $(Y_t)_{t\geq 0}$. Then, under a fast mean-reverting volatility regime, we have the following

Theorem 6.2.1
A corrected default probability formula under the above stochastic volatility setting can be explicitly given by

$$\min\{1, \frac{\tilde{PD}(T, B|t, x, y) + |\tilde{PD}(T, B|t, x, y)|}{2}\}, \tag{6.6}$$

[1]Note that the invariant distribution for this CAR(1) process is $N(m, \nu^2)$, where $\nu = \frac{\beta}{\sqrt{2\alpha}}$.

where

$$\tilde{PD}(T,B|t,x,y) = PD_0(T,B|t,x) + \frac{1}{\alpha}PD_1(T,B|t,x,y),$$

with $PD_0(T,B|t,x)$ is the default probability in the Merton setting with constant volatility $\overline{\sigma}$ and

$$PD_1(T,B|t,x,y) = -\frac{1}{2}(\phi(y) - \langle\phi\rangle)\cdot x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2} - \frac{1}{4}(T-t)(\langle f^2\phi\rangle - \langle f^2\rangle\langle\phi\rangle)\cdot$$

$$\cdot\Big(2x^2\frac{\partial^2 PD_0}{\partial x^2} + 4x^3\frac{\partial^3 PD_0}{\partial x^3} + x^4\frac{\partial^4 PD_0}{\partial x^4}\Big), \tag{6.7}$$

for all $(t,x,y) \in [0,T) \times (B,\infty) \times \mathbb{R}$.

Proof. Considering the pricing problem in Fouqué et al. (2000), the default probability function $PD(T,B|t,x,y)$ satisfies

$$\frac{\partial PD}{\partial t} + \frac{1}{2}f^2(y)x^2\frac{\partial^2 PD}{\partial x^2} + \frac{1}{2}\beta^2\frac{\partial^2 PD}{\partial y^2} + \alpha(m-y)\frac{\partial PD}{\partial y} + \mu x\frac{\partial PD}{\partial x} = 0, \tag{6.8}$$

on $(t,x,y) \in [0,T) \times (B,\infty) \times \mathbb{R}$ with terminal condition $PD(T,B|T,x,y) = h_{PD}(x) := \frac{1}{2}(1 + sgn(B-x))$, $x \geq B$, $y \in \mathbb{R}$.

With the usual rescaling method for modeling fast mean reversion in the volatilities

$$\alpha = \frac{1}{\epsilon}, \quad \beta = \nu\frac{\sqrt{2}}{\sqrt{\epsilon}}, \quad \epsilon > 0,$$

(m and ν fixed constants, $\nu > 0$) we can rewrite the above problem as

$$\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1\Big)PD^\epsilon = 0, \tag{6.9}$$

where PD^ϵ is the rescaled default probability.

$$\mathcal{L}_0 = \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \tag{6.10}$$

$$\mathcal{L}_1 = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y)\cdot x^2 \cdot \frac{\partial^2}{\partial x^2} + \mu x \cdot \frac{\partial}{\partial x}. \tag{6.11}$$

The idea is to expand PD^ϵ in powers of ϵ:

$$PD^\epsilon = PD_0 + \epsilon PD_1 + \epsilon^2 PD_2 + \ldots, \tag{6.12}$$

where PD_k, $k = 0,1,\ldots$, are function of (t,x,y) to be determined.

As usual in the asymptotic theory of Fouqué et al. (2000), we are primarily interested in the first two terms $PD_0 + \epsilon PD_1$.

Substituting (6.12) in (6.9), we get that PD_0 is the default probability in the Merton setting with constant volatility $\overline{\sigma}$ and PD_1 is a solution of the following PDE problem:

$$\mathcal{L}_0 PD_1 = -\mathcal{L}_1 PD_0, \quad \text{on } [0,T) \times (B,\infty) \times \mathbb{R}, \tag{6.13}$$

$$PD_1(T,B|T,x,y)=0, \quad x>B, \quad \lim_{t\nearrow T}\langle PD_1(T,B|t,x,\cdot)\rangle=0, \quad x\geq B, \tag{6.14}$$

and with the centering condition $\langle \mathcal{L}_1 PD_1\rangle=0$.

Hence, to prove our assertion it is sufficient to show that the function PD_1 described in the Theorem is a solution to the above problem.

Firstly observe that

$$\langle \mathcal{L}_1(x^n\cdot\frac{\partial^n PD_0}{\partial x^n})\rangle=0, \quad (\forall)\, n\in\mathbb{N}. \tag{6.15}$$

Using property (6.15) and the fact that ϕ is a solution of the above Poisson equation, we obtain

$$\mathcal{L}_1 PD_0=\frac{1}{2}(f^2(y)-\langle f^2\rangle)\cdot x^2\cdot\frac{\partial^2 PD_0}{\partial x^2},$$

and

$$\mathcal{L}_0 PD_1=-\frac{1}{2}(f^2(y)-\langle f^2\rangle)\cdot x^2\cdot\frac{\partial^2 PD_0}{\partial x^2}.$$

By summing up both equations, we obtain the PDE (6.13) and it only remains to verify the centering condition, which results as follows:

$$\begin{aligned}\langle \mathcal{L}_1 PD_1\rangle &= \frac{1}{2}\langle (f^2-\langle f^2\rangle)\cdot x^2\cdot\frac{\partial^2 PD_1}{\partial x^2}\rangle-\frac{1}{2}\langle\phi\rangle\langle\mathcal{L}_1\Big(x^2\cdot\frac{\partial^2 PD_0}{\partial x^2}\Big)\rangle\\ &+\langle\mathcal{L}_1\Big(PD_1+\frac{1}{2}\phi x^2\cdot\frac{\partial^2 PD_0}{\partial x^2}\Big)\rangle.\end{aligned}$$

The last term can be rewritten as

$$\begin{aligned}&\langle\mathcal{L}_1\Big(PD_1+\frac{1}{2}\phi x^2\cdot\frac{\partial^2 PD_0}{\partial x^2}\Big)\rangle\\ &\quad=\frac{1}{4}(\langle f^2\phi\rangle-\langle f^2\rangle\langle\phi\rangle)\Big(2x^2\frac{\partial^2 PD_0}{\partial x^2}+4x^3\frac{\partial^3 PD_0}{\partial x^3}+x^4\frac{\partial^4 PD_0}{\partial x^4}\Big).\end{aligned}$$

From property (6.15) we have $\langle\mathcal{L}_1\Big(x^2\cdot\frac{\partial^2 PD_0}{\partial x^2}\Big)\rangle=0$. Hence, it remains to show that

$$\begin{aligned}&\langle (f^2-\langle f^2\rangle)\cdot x^2\cdot\frac{\partial^2 PD_1}{\partial x^2}\rangle\\ &\quad=-\frac{1}{2}(\langle f^2\phi\rangle-\langle f^2\rangle\langle\phi\rangle)\Big(2x^2\frac{\partial^2 PD_0}{\partial x^2}+4x^3\frac{\partial^3 PD_0}{\partial x^3}+x^4\frac{\partial^4 PD_0}{\partial x^4}\Big).\end{aligned} \tag{6.16}$$

For this purpose observe first that

$$\begin{aligned}&\langle (f^2-\langle f^2\rangle)PD_1\rangle\\ &=-\frac{1}{2}(\langle f^2\phi\rangle-\langle f^2\rangle\langle\phi\rangle)x^2\frac{\partial^2 PD_0}{\partial x^2}-\frac{1}{4}(T-t)(\langle f^2\phi\rangle-\langle f^2\rangle\langle\phi\rangle)\langle(f^2-\langle f^2\rangle)\rangle\\ &\cdot\Big(2x^2\frac{\partial^2 PD_0}{\partial x^2}+4x^3\frac{\partial^3 PD_0}{\partial x^3}+x^4\frac{\partial^4 PD_0}{\partial x^4}\Big)\\ &=-\frac{1}{2}(\langle f^2\phi\rangle-\langle f^2\rangle\langle\phi\rangle)x^2\frac{\partial^2 PD_0}{\partial x^2}.\end{aligned}$$

Finally, using the fact that

$$\frac{1}{2}\langle (f^2 - \langle f^2 \rangle) \cdot x^2 \cdot \frac{\partial^2 PD_1}{\partial x^2} \rangle = \frac{1}{2} x^2 \cdot \frac{\partial^2}{\partial x^2}\Big(\langle (f^2 - \langle f^2 \rangle) PD_1 \rangle \Big)$$

we obtain (6.16), which completes the proof.

■

Remark 6.2.1

(i) As already mentioned in Chapter 4, this type of approximations performs poorly close to T or to the other frontiers of the corresponding domain for x. However, our approximation is more accurate than the corresponding one in Fouqué et al. (2006), since we performed the expansion in powers of ϵ, instead of $\sqrt{\epsilon}$, while preserving the same number of terms. Moreover, unlike Fouqué et al. (2006), our approximation also depends on y, which gives the chance to capture with this analytical formula a larger amount of the relevant market informations.

(ii) Upon averaging w.r.t. y, we get the following y-independent version of the previous corrected default probability formula

$$\min\{1, \frac{\overline{\tilde{PD}}(T, B|t, x) + |\overline{\tilde{PD}}(T, B|t, x)|}{2}\},$$

where

$$\overline{\tilde{PD}}(T, B|t, x) := PD_0(T, B|t, x) + \frac{1}{\alpha}\overline{PD_1}(T, B|t, x),$$

$$\overline{PD_1}(T, B|t, x) = -\frac{1}{4}(T-t)(\langle f^2\phi\rangle - \langle f^2\rangle\langle\phi\rangle) \cdot \Big(2x^2\frac{\partial^2 PD_0}{\partial x^2} + 4x^3\frac{\partial^3 PD_0}{\partial x^3} + x^4\frac{\partial^4 PD_0}{\partial x^4}\Big),$$

for all $(t, x) \in [0, T) \times (B, \infty)$.

(iii) In practice, all the parameters of the above presented formulae can be estimated from equity prices data using the nonlinear filtering techniques discussed in Chapter 2. In this way, supplementary asymptotic expansions for the parameter calibration can be alleviated.

Denoting $\tilde{PD}(T, B|t, x, y) = PD_0(T, B|t, x) + \frac{1}{\alpha}PD_1(T, B|t, x, y)$, we immediately obtain

$$\lim_{\alpha \to \infty} \tilde{PD}(T, B|t, x, y) = PD_0(T, B|t, x).$$

When the rate of mean reversion becomes very large, the SV model converges to the Merton model with a constant volatility. Thus, the difference between the SV and Merton settings can be studied by examining

$$\tilde{PD}(T, B|t, x, y) - PD_0(T, B|t, x), \tag{6.17}$$

as a function of α, which will be done in the following.

For fixed T, B, t, x, y we introduce

$$\delta(\alpha) = \tilde{PD}(T, B|t, x, y) - PD_0(T, B|t, x) = \frac{1}{\alpha} PD_1(T, B|t, x, y), \quad (\forall)\, \alpha > 0. \tag{6.18}$$

Then, by observing that

$$\frac{d^n \delta}{d\alpha^n} = (-1)^n \cdot n! \frac{1}{\alpha^{n+1}} PD_1(T, B|t, x, y), \quad (\forall)\, n \in \mathbb{N}, \tag{6.19}$$

we obtain the following

Corolary 6.2.1

A.) If $PD_1(T, B|t, x, y) \neq 0$*, then*

$$sgn(PD_1(T, B|t, x, y)) \cdot (\tilde{PD}(T, B|t, x, y) - PD_0(T, B|t, x)) > 0, \quad (\forall)\, \alpha > 0$$

and the function $\alpha \to |\tilde{PD}(T, B|t, x, y) - PD_0(T, B|t, x)|$ *is strictly monotonically decreasing.*

B.) If $PD_1(T, B|t, x, y) = 0$*, then*

$$\tilde{PD}(T, B|t, x, y) = PD_0(T, B|t, x)), \quad (\forall)\, \alpha > 0.$$

The above result concerning the difference between the default probability in our SV model and the default probability in the corresponding Merton setting will be graphically illustrated and discussed in the next section.

6.3 Examples

In this section we use the analytical formula derived above in order to assess the difference between default probabilities in the two models (the SV versus the Merton one).

As a basic case, we adopt the Scott stochastic volatility model (see e.g., Appendix B) and we use the following set of parameters: $t = 0$, $T = 1$, $\mu = 0.1$, $\nu = 0.26$, $m = -0.5358$, $Y_0 = -0.5$, and the current value of the first firm is taken to be $S_0 = 100$ monetary units. Recall that $\nu = \frac{\beta}{\sqrt{2\alpha}}$.We now vary the face value of debt B and the rate of mean reversion α and compare the resulting (approximated) default probabilities under the SV model and under the corresponding Merton model.

For all following figures we fix the default barrier B. The x-axis represents the values for α, which we vary between 0 and 1200. The y-axis stands for the default probability. The straight line in each figure is the default probability under the Merton setting, as the default risk does not depend on α. The other curve shows how the default probability depends on the rate of mean reversion under the stochastic volatility model. Figures 6.1 and 6.2 have been drawn for $B = 40, 55, 60, 65$.

One can observe that, as expected, for lower rates of mean reversion the default probabilities in the SV model substantially differ from that of the Merton model. If the default barrier (and thus the default risk) is not too large then the probability of default is higher under stochastic volatility. The opposite behavior can be observed for large values of the default barrier B.

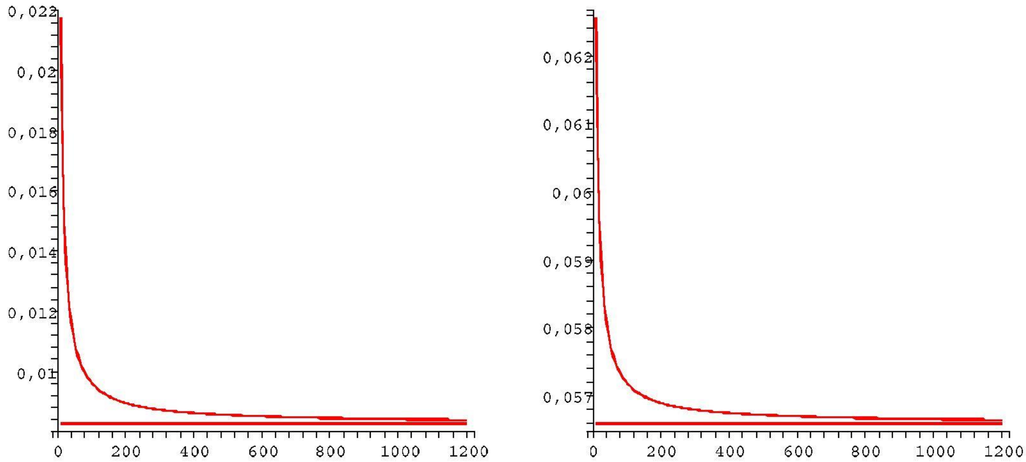

Figure 6.1: Default barrier: left: $B = 40$; right: $B = 55$

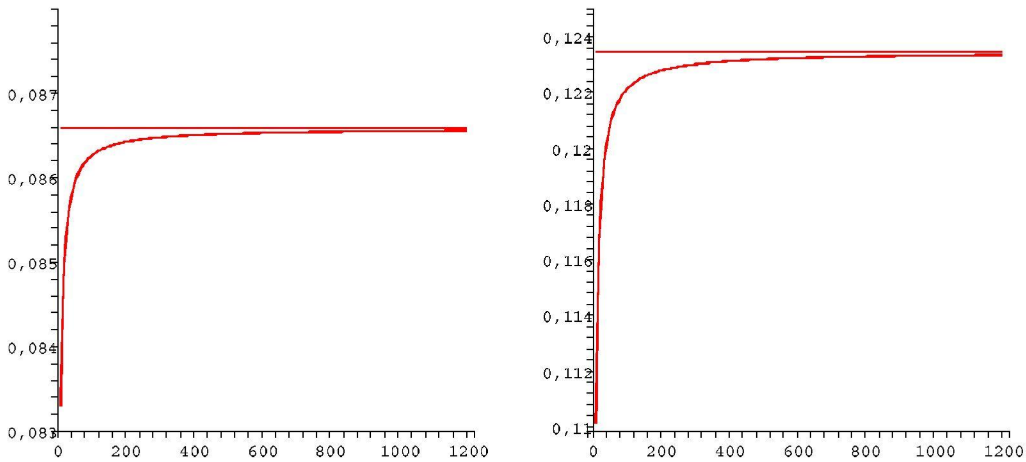

Figure 6.2: Default barrier: left: $B = 60$; right: $B = 65$

Conclusion and Outlook

Continuous-time methods have become nowadays an integral part of research in financial economics. This field has left an indelible mark on several core areas of mathematical finance such as derivative pricing theory and risk management.

The celebrated paper of Black-Scholes (1973), together with the works of Merton marked the birth of modern mathematical finance. However, the approach by Black, Scholes and Merton uses relatively stringent assumptions which do not account for many relevant phenomena of modern financial markets. Therefore during the last decades a great diversity of financial models has been developed. However, there is no model able to capture the whole complexity of financial data, which preserves the strong motivation to develop new financial models.

In this sense, we proposed in this thesis some new continuous-time series models with applications to financial mathematics and illustrated them with the aid of some simulation and estimation examples. The development of these models was motivated by the so-called stylized facts of financial data, but we also aimed to strike a balance between our ambition to make the models as realistic as possible and the need to keep them simple enough.

The classes of models proposed in this work include time varying and/or nonlinear extensions of existing continuous time series models, based on which we introduced some new financial models, with a focus on non-Markovian extensions, e.g., a time varying Black-Scholes model or SV models where the volatility is driven by a higher order continuous-time series processes. Furthermore, some of them are able to accommodate other interesting features like jumps, fat tails and long-memory properties. Moreover, unlike the models relying on fractional Brownian motion, our models are still able to use the standard no-arbitrage pricing theory (compare for this Biagini et. al. (2008)) and they are easier to estimate with the classical nonlinear filtering methods.

Using S&P 500 data and a corresponding QMLE procedure, we obtained parameter estimates for some of these financial models. Classical model selection criteria have then shown that the new introduced models perform better than the classical ones. Moreover, these results confirm the non-Markovian character for both asset and volatility dynamics. In the context of high frequency data we also provided some theoretical estimation results for some important financial quantities in derivative pricing, like integrated variance, spot volatility and effective volatility.

Since analytical results are seldom available for realistic financial models, many companies use simulations or numerical solutions of an adequate PDE characterization when pricing various financial contracts. However, the size of the market and the complex dynamics of the stock market volatility are calling for better techniques, a need which becomes even more stringent in the context of incomplete markets. Relying on the asymptotic techniques developed by Fouqué et al. (2000) we proposed an alternative way to overcome this difficulty, based on an analytical approximation of the pricing function. Such results are even more important in the multivariate case.

Starting from the multivariate corrected price formula for European derivatives proposed by Fouqué et al. (2000), we have been able to explicitly obtain the corresponding correction constants with respect to the parameters of a multivariate Scott SV model. To our awareness this is the first general multivariate setting for incomplete markets with a comprehensive system of mutual correlations between the involved processes which is able to provide such type of analytical approximation for European derivative prices and for any number of companies.

Finally, we proposed a new corrected default probability formula when the volatility of the firm dynamics follows a fast mean-reverting regime. Our result is based on an improved first correction term when compared to the corresponding one in Fouqué et al. (2006). This gives the chance to capture a larger amount of the relevant market informations.

The models introduced in this thesis are far from perfect – no model ever is. However, we do believe that their ability to improve the classical models without complicating too much the framework deserves the attention of researchers seeking to model in continuous time complex time series data. Although this subject is mature, it still presents many challenges to the mathematician and the financial engineer alike.

Future research will focus on further developing and implementing these models in order to make them as parsimonious and flexible as possible, to test them against the established ones, and to use them in finance for valuing and hedging complex derivative instruments and transactions. In this sense, it would then be desirable to further extend our estimation results and to assess the sensitivity of our models w.r.t. several different data sets, especially for the case of high frequency data, in order to increase the accuracy of the corresponding estimation and model selection procedures. Further studies are needed to assess (higher order) extensions of the analytical developments derived in this thesis and these are currently under investigation.

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Appendix A

Black-Scholes Formula

Black and Scholes have used in their theory a simple market model with two assets. One of the assets is a riskless asset (bond) with price β_t at time t, described by the ordinary differential equation

$$d\beta_t = r\beta_t \, dt, \quad t \geq 0, \tag{A.1}$$

where r, a nonnegative constant, is the instantaneous interest rate for lending or borrowing money. The price X_t of the other asset, the risky stock or stock index, evolves according to the stochastic differential equation

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad t \geq 0, \tag{A.2}$$

where μ is a constant mean return rate, $\sigma > 0$ is a constant volatility and $(W_t)_{t \geq 0}$ is a standard Brownian motion. The process $(X_t)_{t \geq 0}$ is called *geometric Brownian motion.*

Derivatives are contracts based on the underlying asset price X_t. They are also called *contingent claims.*

A *European call option* is a contract that gives its holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined *strike price* K on the *maturity* date T.

If X_T is the price of the underlying asset at maturity time T, then the value of this contract at maturity, its *payoff,* is

$$h(X_T) = (X_T - K)^+ = \begin{cases} X_T - K & \text{if } \; X_T > K, \\ 0 & \text{if } \; X_T \leq K, \end{cases} \tag{A.3}$$

since in the first case the holder will exercise the option and make a profit $X_T - K$ by buying the stock for K and selling it immediately at the market price X_T. In the second case the option is not exercised, since the market price of the asset is less than the strike price.

Similarly, a *European put option* is a contract that gives its holder the right to sell a unit of the asset for a strike price K at the maturity date T. Its payoff is

$$h(X_T) = (K - X_T)^+ = \begin{cases} K - X_T & \text{if } X_T < K, \\ 0 & \text{if } X_T \geq K, \end{cases}. \tag{A.4}$$

In the first case, buying the stock at the market price and exercising the put option yields a profit $K - X_T$. In the second case the option is simply not exercised.

More generally, the European derivatives are defined by their maturity time T and their nonnegative payoff function $h(x)$. This will be a contract that pays $h(X_T)$ at maturity time T when the stock price is X_T.

At time $t < T$ this contract has a value, known as the *derivative price*, which will vary with t and the observed stock price X_t. This option price at time t for a stock price $X_t = x$ is denoted by $P(t, x)$ and the problem of *derivative pricing* is to determine this pricing function.

The pricing function $P(t, x)$ is the solution of the *Black-Scholes partial differential equation*

$$\mathcal{L}_{BS}(\sigma)P = 0, \tag{A.5}$$

with the final condition $P(t, x) = h(x)$, where

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\Big(x\frac{\partial}{\partial x} - \cdot\Big). \tag{A.6}$$

For European call options the Black-Scholes PDE (A.5) is solved with the final condition $h(x) = (x - K)^+$. Prices of European calls at time t and for an observed risky asset price $X_t = x$ will be denoted by $C_{BS}(t, x)$. In this particular case, there is a closed-form solution known as the *Black-Scholes formula*:

$$C_{BS}(t, x) = xN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{A.7}$$

where

$$d_1 = \frac{\log(\frac{x}{K}) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \tag{A.8}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}, \tag{A.9}$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}} dy. \tag{A.10}$$

Appendix B

Classical Stochastic Volatility Models

One popular way to improve the Black-Scholes model is to let the volatility to be an Ito process satisfying a SDE driven by a second Brownian motion. This leads to an incomplete market and that there is no *unique* equivalent martingale measure.

One well-known feature of the volatility process is *mean reversion*. The term "mean reverting" refers to the characteristic (typical) time it takes for a process to get back to the mean level of its invariant distribution (the long-run distribution of the process).

In pure mean-reverting stochastic volatility models, the asset price $(S_t)_{t\geq 0}$ satisfies the SDE

$$dS_t = \mu S_t\, dt + \sigma_t S_t\, dW_t, \quad t \geq 0, \tag{B.1}$$

and the volatility process $(\sigma_t)_{t\geq 0}$ is given by

$$\sigma_t = f(Y_t), \quad t \geq 0, \tag{B.2}$$

$$dY_t = \alpha(m - Y_t)dt + \cdots d\hat{Z}_t, \tag{B.3}$$

where f is some (most often a positive) function and $(\hat{Z}_t)_{t\geq 0}$ is a Brownian motion correlated with $(W_t)_{t\geq 0}$. It is convenient to write

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t, \; t \geq 0, \tag{B.4}$$

where $\rho \in [-1, 1]$ and $(Z_t)_{t\geq 0}$ is a Brownian motion independent of $(W_t)_{t\geq 0}$.

Here α is called *the rate of mean reversion* and m is the long-run mean level of Y.

It is often found from financial data that $\rho < 0$ and there are economic arguments for a negative correlation or *leverage effect* between stock price and volatility shocks. From common experience and empirical studies asset prices tend to go down when volatility goes up.

Some common driving processes $(Y_t)_{t\geq 0}$ are:

1. lognormal (LN)

$$dY_t = c_1 Y_t dt + c_2 Y_t d\hat{Z}_t, \tag{B.5}$$

2. Ornstein-Uhlenbeck (OU)

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t, \tag{B.6}$$

3. Feller or Cox-Ingersoll-Ross (CIR),

$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{Y_t}d\hat{Z}_t, \tag{B.7}$$

Note that the lognormal is not mean-reverting.

Some models studied in the literature are listed below.

Models of Volatility

Authors	Correlation	$f(y)$	Y Process
Hull-Withe	$\rho = 0$	$f(y) = \sqrt{y}$	Lognormal
Scott	$\rho = 0$	$f(y) = e^y$	Mean-reverting OU
Stein-Stein	$\rho = 0$	$f(y) = y$	Mean-reverting OU
Ball-Roma	$\rho = 0$	$f(y) = \sqrt{y}$	CIR
Heston	$\rho \neq 0$	$f(y) = \sqrt{y}$	CIR

More details on this topic can be found e.g., in Fouqué, Papanicolaou, and Sircar (2000), Shephard (2005) and Shephard and Andersen (2009).

Appendix C

Basic Facts about the Poisson Equation

The univariate Poisson equation has the following form:

$$\mathcal{L}_0 \phi = g, \tag{C.1}$$

where $\mathcal{L}_0$ is the second order differential operator

$$\mathcal{L}_0 = \nu^2 \frac{\mathbf{d}^2}{\mathbf{d}y^2} + (m - y)\frac{\mathbf{d}}{\mathbf{d}y}, \quad m, \nu \in \mathbb{R} \ (\nu > 0). \tag{C.2}$$

Let $\Phi(y)$ be the density of the normal distribution $N(m, \nu^2)$ and denote for all integrable g

$$\langle g \rangle := \int_{\mathbb{R}} \Phi(y) g(y) dy. \tag{C.3}$$

If $\langle g \rangle = 0$, the Poisson equation has a solution whose first derivative can be explicitly given by

$$\phi'(y) = \frac{1}{\nu^2 \Phi(y)} \int_{-\infty}^{y} g(u) \cdot \Phi(u)\, du, \quad y \in \mathbb{R}. \tag{C.4}$$

This solution satisfies the following growth property:

if $|g(y)| \leq C_1(1 + |y|^n),\ n \in \mathbb{N}^*,$ then $|\phi(y)| \leq C_2(1 + |y|^n),$

where C_1 and C_2 are some positive constants. This ensures that all terms involving ϕ in our asymptotic developments are well defined.

For further details we refer to Fouqué, Papanicolaou, and Sircar (2000).

Appendix D

Notation

- A', A^T – the transpose of a matrix.
- I_p – the $p \times p$ identity matrix.
- $\mathbb{N} = \{0, 1, 2, \dots, \}$.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
- $\mathbb{R}_+$ – the nonnegative real numbers.
- $\mathbb{R}^n$ – the n-dimensional Euclidean space; $\mathbb{R}^1 = \mathbb{R}$.
- C; $C(\mathbb{R})$ – space of continuous functions.
- C^k; $C^k(\mathbb{R})$ – space of k times continuously differentiable functions.
- C_c^k; $C_c^k(\mathbb{R})$ – space of k times continuously differentiable functions with compact support.
- C^∞ – space of smooth functions.
- C_b; $C_b(\mathbb{R})$ – space of bounded continuous functions.
- L^1 – space of integrable functions.
- L^2 – space of square integrable functions.
- $N(m, \nu^2)$ – normal distribution with expectation m and variance ν^2.
- $a_n = O(b_n)$ if $\frac{a_n}{b_n}$ is bounded.
- $a_n = o(b_n)$ if $\frac{a_n}{b_n} \to 0$ as $n \to \infty$.
- $sgn(x) := \begin{cases} 1; & x > 0, \\ 0; & x = 0, \\ -1; & x < 0. \end{cases}$
- $1_A(x) := \begin{cases} 1; & x \in A, \\ 0; & x \notin A. \end{cases}$
- $a.e.$ – almost everywhere.
- AIC – Akaike's Information Criterion.
- a.o. – among others.
- AR – discrete-time autoregressive process.
- ARMA – discrete-time autoregressive moving average process.
- $a.s.$ – almost surely.
- BIC – Bayesian Information Criterion.
- BS – Black-Scholes model.
- BDG – Burkholder-Davis-Gundy inequality.
- CAR – continuous-time autoregressive process.
- CAR$_\varepsilon$ – a CAR type process with a non-degenerate diffusion matrix.
- CARMA – continuous-time autoregressive moving average process.

- CARMA$_\varepsilon$ – a CARMA type process with a non-degenerate diffusion matrix.
- cdf – cummulative probability density function.
- CIR – Cox-Ingersoll-Ross model.
- CKLS – Chan-Karolyi-Longstaff-Sanders model.
- COGARCH – continuous-time generalized autoregressive conditionally heteroscedastic process.
- CTARMA – continuous-time threshold autoregressive moving average process.
- EKF – extended Kalman filter.
- EMM – equivalent martingale measure.
- FMR – fast mean-reversion.
- GARCH – discrete-time generalized autoregressive conditionally heteroscedastic process.
- GBM – geometric Brownian motion.
- HF – high frequency (data).
- $i.i.d.$ – independent and identically distributed.
- LRD – long range dependence.
- MC – Monte Carlo (simulations).
- MCMC – Markov Chain Monte Carlo.
- MSV – multivariate stochastic volatility model.
- NLCAR – nonlinear continuous-time autoregressive process.
- NLCARMA – nonlinear continuous-time autoregressive moving average process.
- OU – Ornstein-Uhlenbeck process.
- PD – probability of default.
- PDE – partial differential equation.
- pdf – (transition) probability density function.
- QMLE – quasi maximum likelihood estimation.
- r.v. – random variable.
- S&P 500 – Standard & Poor's 500 stock index.
- SDE – stochastic differential equation.
- SV – stochastic volatility.
- SV CAR – stochastic volatility model driven by a CAR$_\varepsilon$ process.
- SV CARMA – stochastic volatility model driven by a CARMA$_\varepsilon$ process.
- tvAR – time-varying AR process.
- tvBS – time-varying Black-Scholes model.
- tvCAR – time-varying CAR process.
- tvCARMA – time-varying CARMA process.
- tvNLCARMA – time-varying NLCARMA process.
- w.r.t. – with respect to.