Decision Making under Ambiguity

Dissertation

zur Erlangung des akademischen Grades

Doctor Rerum Politicarum

an der

Fakultät für Wirtschafts- und Sozialwissenschaften

der Ruprecht-Karls-Universität Heidelberg

vorgelegt von

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Heidelberg, November 2010

Acknowledgments

This dissertation would not have been possible without the guidance, encouragement and support of many people. First of all, I am deeply indebted to my advisor, Jürgen Eichberger, who introduced me to the exciting world of decision making under ambiguity. He has guided me throughout the whole duration of my thesis. He gave me the freedom to develop my own ideas, encouraged me to search for answers and at the critical junctures, provided me the moral support without which I would not have been able to successfully finish this project. His influence on my development as researcher will remain substantial. I am also very grateful to Jörg Oechssler who kindly agreed to act as a second adviser. His helpful advice and valuable comments significantly deepened my understanding of experimental methodology.

I am also indebted to my colleagues Peter Dürsch, Jean-Philippe Lefort and Wendelin Schnedler, who co-authored papers on which this thesis is based. Working with them was both an enjoyable and valuable experience for me. Without them this thesis would not exist, at least not in this form.

The Alfred-Weber Institute of the University of Heidelberg provided, through their weekly seminars and invited guests, the unique possibility of communicating directly with internationally renowned scientists. In particular, I benefited a lot from the comments and suggestions of Ani Guerdijkova, Hans Haller, David Kelsey and David Schmeidler. I would also like to thank Phillipe Mongin, Simon Grant and Jacob Sagi for the illuminating discussions during their visits to the University of Heidelberg.

The Faculty of Economics and Social Sciences at the University of Heidelberg, as well as the Sonderforschungsbereich of the University of Mannheim provided financial support which allowed me to participate in many international conferences. This gave me the opportunity to meet Soo Hong Chew, Michèle Cohen, Daniel Ellsberg, Eran Hanany, Yoram Halevy, Mark Machina, Klaus Nehring, Sujoy Mukerji, Arno Riedel, Klaus Ritzberger, Tomasz Strzalecki, Peter Wakker and Jan Werner. I would like to thank all of them for illuminating discussions and for many helpful comments. A special thanks is extended to Peter Wakker who commented on two chapters of this thesis and who gave me fascinating recommendations on the relevant literature.

Thanks are also due to the participants of the Spring Meeting of Young Economists in Lille 2009 and the International Economic Meeting which took place in Warsaw 2008 and 2009. I am also grateful to the participants of the 5th Asia Pacific Meeting of the Economic Science Association in Haifa 2009, the 2nd Behavioral and Experimental Economics Symposium in Maastricht 2009. Participation at the Workshop on New Risks and Loss Aversion in honor of Daniel Kahnemann hosted by the University of Rotterdam 2009, the Workshop on Risk, Ambiguity, and Decisions in honor of Daniel Ellsberg hosted by the University of Vienna 2010, the 14th FUR conference in New Castle and the RUD conference in Paris 2010 allowed me to present and discuss my research results.

I would like to thank all my colleagues at the University of Heidelberg, especially Ute Schumacher and Gabi Rauscher, for the nice working atmosphere during these four years.

Last but not least, I would also like to thank my family. They have supported and encouraged me from the start with their best wishes.

Contents

Chapter 1

Introduction

Theories of decision making play a fundamental role in economic theory. Most economically relevant decisions have to be made in the presence of uncertainty. Uncertainty pertains to situations in which an agent, called a decision maker, faces the problem of choosing a course of action. The choice of a course of action, by itself, does not determine a unique outcome. The decision maker knows which circumstances affect the outcomes of her actions, but she is incapable of saying which of them she will obtain with certainty. The standard practice in economics when modeling decision making under uncertainty is to follow the Bayesian approach. In this approach it is assumed that the decision maker's subjective beliefs are quantifiable by a unique probability distribution and that these probabilistic beliefs are used in decision making, typically as a basis for expected utility maximization. Moreover, the arrival of new information affects the decision maker's beliefs, and posterior beliefs are obtained by updating the prior ones in accordance with Bayes' rule. The subjective expected utility theory of Savage (1954) is firmly established as the axiomatic underpinning of the Bayesian paradigm. Savage's theory offers an elegant and straightforward tool for modeling not only static and dynamic, but also interactive decision problems in the presence of uncertainty. However, ever since the contributions of Ellsberg (1961) and Aumann (1976) economists began to acknowledge that the Bayesian approach was too restrictive. Ellsberg pointed to the limitations of Bayesianism as a descriptive theory, while Aumann questioned the explanatory power of asymmetric information within Bayesian frameworks.

In his thought experiments, Daniel Ellsberg (1961) exemplified that Savage's theory cannot take into account the possibility that probabilities for some events are known, while for other ones they are not, and that such "ambiguity" may affect the decision makers' choice behavior. In particular, Ellsberg observed that most of his "non-experimental" subjects preferred to bet on events with known probabilities rather than on ones for which information about their likelihoods is missing. Such behavior, termed ambiguity aversion, has received ample empirical confirmation in recent years (see Camerer and Weber, 1992). For ambiguity-averse subjects it is impossible that their choices are based on a single probability distribution. This result implies that ambiguity-sensitive behavior cannot be modeled by the subjective expected utility theory of Savage (1954).

In his famous article on "agreeing to disagree", Robert Aumann (1976) challenged the role that asymmetric information plays in interactive decision problems. He showed that, under the assumption of common priors, differences in commonly known decisions cannot be explained solely by differences in decision makers' private information. In particular, if two decision makers share a common probability distribution, and their posteriors for some event are common knowledge, then these posteriors must coincide, although they may be conditioned on diverse information. Aumann's agreement on posterior beliefs has been extended to posterior expectations by Milgrom (1981) and Geanakoplos and Sebenius (1983). Based on these extensions, Milgrom and Stokey (1982) showed that in the absence of heterogeneous prior beliefs asymmetric information alone cannot generate any profitable trade opportunities among traders with the same risk attitudes. These results led to very puzzling consequences for economic theory. Within Bayesian frameworks, neither widely observed gambling behavior nor the existence of speculation in financial markets can be explained solely on the basis of asymmetric information. In this thesis I will provide an alternative solution to that "puzzle".

Essentially, the aim of the thesis is to investigate how access to "additional" or "new" information affects choice behavior under ambiguity. To scrutinize this issue four topics are suggested and explored by experimental as well as formal methods. Each topic can be viewed as focusing on a different "aspect" of information that may be seen as relevant for the decision maker when facing static, dynamic or interactive decision problems.

The first topic examines the relationship between ambiguity aversion and decision makers' attitudes towards objective randomization devices. To cope with the limitations of Bayesianism as pointed out by Ellsberg (1961), several alternatives to Savage's subjective expected utility theory have been proposed. The Choquet expected utility model of Schmeidler (1989), the multiple prior model of Gilboa and Schmeidler (1989), as well as the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005) are prominent examples. Many of these alternatives adopt Schmeidler's notion of ambiguity aversion which states that an ambiguity-averse decision maker should always prefer random mixtures between two ambiguous bets to each of the involved bet. Existing explanations for such a preference for mixtures often rely on the idea that access to an objective randomization device, such as a fair coin, mitigates the problem of lacking probabilistic information. In the words of Klibanoff (2001a, p.290), randomizing between two ambiguous bets have "[...] the effect of making the outcomes less subjective [...]". However, this explanation is controversial and the logic behind it depends upon the formal framework used to model uncertainty. When uncertainty is modeled in the two-stage setting of Anscombe and Aumann (1963), mixtures, indeed, have an intuitive effect of smoothing expected utilities across states and according to Schmeidler, an ambiguity-averse decision maker should always be randomization-loving. On the other hand, when uncertainty is modeled in the one-stage setting of Savage (1954), the effect of mixtures is not clear at all. Adopting the one-stage setting, Eichberger and Kelsey (1996b) showed that an ambiguity-averse decision maker with Choquet expected utility preferences will be randomization-neutral. Motivated by these competing predictions, we investigate this issue experimentally. In Chapter 4, we design and implement an experiment which allows an examination of the relationship between ambiguity and randomization attitudes.

The second topic focuses on dynamic choice behavior under ambiguity. In dynamic choice situations, a decision maker is informed sequentially which uncertain event has occurred. An important question that arises in this context is how preferences are updated to incorporate the receipt of new information. Many theories of updating preferences assume either dynamic consistency or consequentialism in order to axiomatically underpin the link between prior and posterior preferences. Dynamic consistency requires that prior choices are respected by updated preferences. According to consequentialism only outcomes that are still possible matter for updated preferences. It is well-known (see Ghirardato, 2002) that dynamic consistency together with consequentialism satisfied on all events implies that preferences admit subjective expected utility representation and that updated preferences are obtained by revising the decision maker's subjective beliefs according to Bayes' rule. This result implies that at least one of these axioms must be relaxed when extending ambiguity models to dynamic choice situations. An ambiguity-averse (resp. loving) decision maker must violate either dynamic consistency or consequentialism, or both. The existing theoretical literature has not yet reached consensus on which of these axiom is the more plausible assumption. In Chapter 5, we design a dynamic version of the classical 3-color experiment of Ellsberg (1961) which allows for differentiation between dynamic consistency and consequentialism. To test whether subjects facing ambiguity behave consistent with either of these two axioms, we conduct the dynamic 3-color experiment.

The third topic explores the link between the dynamic properties of Choquet expected utility preferences and existing notions of unambiguous events. The idea of unambiguous events is closely related to the idea of events which support some kind of probabilistic beliefs. Recently, Nehring (1999) and Zhang (2002) proposed two different notions of unambiguous events. In Chapter 6, we attempt to characterize these two notions of unambiguous events by imposing dynamic properties on Choquet preferences. When extending ambiguity-sensitive preferences to dynamic frameworks, Sarin and Wakker (1998a), Epstein and Schneider (2003) as well as Eichberger, Grant, and Kelsey (2005) showed that both axioms, dynamic consistency and consequentialism, can be maintained, however, at the cost of constraining the analysis to a fixed collection of events and by imposing restrictions on subjective beliefs. We follow this approach and ask whether, for Choquet expected utility preferences, dynamic consistency and consequentialism, constrained to a given collection of events, guarantees that these events are unambiguous in a peculiar sense and vice versa. The results we obtained allow us to answer this question in the affirmative.

The fourth topic scrutinizes the role that asymmetric information plays in the context of interactive decision problems à la Aumann (1976), Geanakoplos and Sebenius (1983) and Milgrom and Stokey (1982) under ambiguity. Many results on the impossibility of agreeing to disagree have been formulated in Bayesian frameworks. In Chapter 7, we generalize these results in a non-Bayesian setup. It is assumed that decision makers share a common, but not-necessarily-additive prior, and that their preferences admit Choquet expected utility representation. In this setting we characterize properties of decision makers' private information which are necessary and sufficient for the impossibility of agreeing to disagree to be true under ambiguity. The results obtained suggest that asymmetric information does matter and can explain differences in commonly known decisions due to the ambiguous (in a specific sense) character of the decision makers' private information. Thus, the existence of gambling behavior and speculative trade may be attributed to ambiguity of private information.

This thesis is organized as follows. Chapter 2 concentrates on Bayesian decision theory. First, historical roots of modern decision theory are reviewed. The main tenets of Savage's (1954) subjective expected utility theory are recalled in static as well as in dynamic setup. The notions of dynamic consistency and consequentialism are introduced. Chapter 3 starts with an explanation of the limitations of Bayesianism as pointed out by Ellsberg (1961). The most widely studied ambiguity models are briefly described. The chapter ends by presenting two alternative approaches used to define ambiguity attitudes. In Chapter 4, I report on results from experimental study examining relationship between ambiguity and randomization attitudes. In Chapter 5, first, the dynamic version of Ellsberg's 3-color experiment is introduced. The tension between dynamic consistency, consequentialism and ambiguity-sensitive behavior is explained. Finally, the data from the dynamic 3-color experiment is evaluated and discussed. In Chapters 6 and 7 the analysis is constrained to Schmeidler's (1989) Choquet expected utility model. For this reason, Chapters 6 starts with the definition of Choquet expected utility preferences. The notions of unambiguous events in the sense of Nehring (1999) and Zhang (2002) are introduced. Conditional Choquet preferences and the most widely used updating rules are defined. In the main part, Nehring's as well as Zhang's unambiguous events are characterized by imposing dynamic properties on Choquet preferences. In Chapter 7, the interpersonal decision model and the notion of common knowledge are introduced. Sufficient and necessary conditions are established for the well-known agreement theorems to hold under ambiguity. The no-trade theorem of Milgrom and Stokey (1982) is generalized for Choquet preferences. Finally, I summarize and conclude in Chapter 8.

Chapter 2

Bayesian Decision Theory

2.1 Historical Backgrounds

Modern decision theory as a branch of economic theory is mainly concerned with providing an axiomatic foundation of rational decision criteria under uncertainty. The principle of expected utility maximization is one such criterion. Whether this decision criterion is tenable for any type of uncertain situation has remained a topic of investigation for a long time.

When explaining economic phenomena, Knight (1921) emphasized distinguishing between two types of uncertainty: "measurable" and "unmeasurable". In his formulation, the measurable uncertainty, or simply *risk*, designates situations in which probabilities are known. That is, they can be deduced a priori or they can be reasonably approximated by relative frequencies. Games of chance in which outcomes are influenced by a randomizing device such as a fair roulette wheel or a fair coin, as well as insurance problems, are typical examples of risky situations. By contrast, the unmeasurable uncertainty, or simply *uncertainty*, refers to situations in which probabilities are not precisely known, in the sense that they can neither be calculated in an objective way nor they can be estimated from past data. Sporting events such as horse races, elections or most real investments involve such unmeasurable uncertainties. It took almost three hundred years before the expected utility maximization rule was axiomatically justified as a decision criterion for these two types of uncertainty.

Decision theory is as old as the notion of probability itself. The term probability emerged around 1660 (see Hacking, 1975). Interestingly, the person who is associated with the concept of probability, Blaise Pascal, also introduced several concepts of modern decision theory. In his famous “wager”, a thought designed to convince non-believers in God that they would be better off becoming believers, Pascal invented three arguments. These arguments were primarily designed to cope with decision problems in which experience and experimental data are not available. As Pascal said, “we are in the same *epistemological* position as someone who is gambling about a coin whose aleatory properties are unknown”(Hacking, 1975, p.70). His arguments were based on three decision criteria: first, if one action is better then another no matter which states of affairs occur, then one should perform a dominating action; second, if there is no dominating action and probability can be assessed to each state of affairs, then one should perform an action with highest expectation; third, if the probabilities of various states of affairs are not known, but instead the set of probability assignments is known, then one should perform an action of dominating expectation (see Hacking, 1972). However, even if Pascal’s technical terminology points to modern notions such as subjective probabilities or expected utility maximization, his arguments are consistent with the doctrine of gaming and chances which are characteristic at this time. “The only probability notion which Pascal uses is hazard, and there is no evidence that he interprets this notion in terms of degrees of belief”(Hacking, 1972, p.190). Nevertheless, it seems that Pascal was the first person who perceived that the structure of decision problem in which probabilities are “objectively” know is isomorphic to the decision problem in which probabilities are unknown.

In the 18th century, mathematicians and philosophers discovered mainly the mathematical aspects of probability theory. Jacob Bernoulli (1713) discovered the law of large numbers and Thomas Bayes (1763) introduced the idea of Bayesian updating of “prior” probabilities to “posterior” ones. During this time Daniel Bernoulli (1738/1954) is the only one who contributed to decision theory. To solve the famous St. Petersburg paradox he suggested using expected utilities of monetary outcomes rather than expected values when evaluating games of chance with known probabilities.[1]

After this inauguration, decision theory had been largely neglected until the beginning of 20th century. At this time it became commonplace for philosophers to interpret scientific theories as axiomatic calculi in which theoretical terms are related to observations. This conception of scientific theories is an intellectual achievement of Logical Positivism, the philosophical position developed by the Vienna Circle.[2] One of the main objectives of Logical Positivism was to eliminate metaphysical entities from philosophy and science. To avoid unverifiable concepts in science, the members of Vienna Circle advocated rigorous scientific standards. A scientific theory is to be axiomatized in the language of mathematical logic. Such theory consists of theoretical and observational terms. The axioms of the theory are formulations of scientific laws, and specify relations between theoretical terms. Theoretical terms are connected with observational ones by explicit definitions, called correspondence rules. Correspondence rules have three functions. First, they define explicitly theoretical terms by means of observational terms; second, they guarantee the "cognitive significance" of theoretical terms; and third, they specify the admissible experimental procedure for applying a theory to observations (see Suppe, 1974). For instance, if a correspondence rule defines a numerical quantity such as "mass" (the theoretical term) as the result of a particular measurement of an object under particular circumstances (observational terms), this specifies an empirical procedure for determining mass, that is, defines "mass" in terms of that procedure and does so in a way such as to guarantee the cognitive significance of the term "mass". Therefore, when the theoretical term "mass" is used for physical laws, one knows how

[1] St. Petersburg paradox is a situation in which a decision maker is willing to pay only a finite (and rather very small) amount of money to participate in a game with random outcomes, despite the fact that the expected value of such game is infinite.

[2] The movement of the Vienna Circle began in 1929 with the publication of the manifesto entitled "Wissenschaftliche Weltanschauung - Der Wiener Kreis", edited by Carnap, Neurath and Hahn.

the physical laws translate to observations. The notion of revealed preferences in economics "[...] is evidently an intellectual descendant of Logical Positivism" (Gilboa, 2009, p.60).

In the spirit of Logical Positivism, Ramsey (1926) proposed defining and measuring probabilities as a decision maker's subjective willingness to bet on the occurrence of an event. In his view, a reasonable decision maker will behave as if she had a subjective probability which guides her decisions, even if probabilities are not part of the description of a decision problem. In Ramsey's view (1926, p.71) subjective probabilities reflect "[...] the degree of beliefs [...], which we can express vaguely as the extent to which we are prepared to act on it". Consequently, one could look at pairwise choices between bets in order to measure the strength of belief: "The-old fashion method of measuring a person's belief is to propose a bet, and see what are the lowest odds which (s)he will accept. This method I regard as fundamentally sound" (Ramsey, 1926, p.73). Invoking the axiomatic approach and taking the existence of utilities as given, Ramsey sketched the proof of the existence of subjective probabilities. Independently, de Finetti (1937) also suggested using pairwise comparison of bets to measure degrees of belief. He offered an axiomatization of subjective probabilities in the context of maximization of expected monetary value (rather than expected utilities). Regarding the interpretation of probabilities, however, de Finetti had a more radical view than Ramsey. De Finetti claimed that "all" probabilistic beliefs are purely subjective. Even in the case of games of chance, where laws of chance can be deduced objectively, he criticized the view that these laws should be seen as a demonstration of the existence of an objective probability distribution. He argued that this "objectivity" might have its reasons in a common psychological perception of symmetry, which some people regard as reasonable, and which has nothing at all to do with objective considerations.

In their foundation of game theory, von Neumann and Morgenstern (1944) offered an axiomatic derivation of the notion of utility and the expected utility maximization rule. In their theory, objects of choice are probability distributions over outcomes, called lotteries. These probabilities are presupposed to be known in the sense that they are explicitly incorporated in the description of a decision problem. Von Neumann and Morgenstern showed that utilities over outcomes can be deduced from observable choices between such probability distributions. They established a set of simple and seemingly reasonable axioms imposed on the preferences between lotteries which are necessary and sufficient for the existence of a utility function over outcomes and for the expected utility maximization rule. That is, a decision maker, when confronted with any two lotteries, will choose the one with a higher expected utility. Moreover, such utility function is unique up to a positive linear transformations, i.e. "cardinal".

The achievements of Ramsey, de Finetti and von Neumann and Morgenstern culminated in the seminal work of Leonard Savage (1954). He proposed a novel analytical framework to model decision making under uncertainty without presupposing the existence of probabilities and utilities. Taking only states of nature and outcomes as primitives, Savage showed that *both* utility and probability, together with the expected utility maximization rule, can be deduced from observable choices among "acts". In this general framework, he established a set of axioms imposed on preferences among acts, which are sufficient and necessary for both the existence of a unique subjective probability distribution and the existence of a unique (up to a positive linear transformations) utility function. Moreover, decision makers' choices are guided by maximizing subjective expected utility. Thus, Savage showed that his axioms lead to the same representation of preferences as in situations in which probabilities are exogenously given. In other words, the decision problem with known probabilities is isomorphic to the decision problem with unknown probabilities.

Invoking the axiomatic approach in the dictum of Logical Positivism, the contributions of Ramsey, de Finetti, von Neumann and Morgenstern and Savage guaranteed that the theoretical terms, such as probability and utility, ware not merely metaphysical entities. They derived these theoretical terms from observable choices, specified a procedure to measure and interpret them, and thus endowed them with cognitive significance. The axiomatic foundation of the principle of expected utility maximization under risk and uncertainty had a tremendous impact on economic theory. It allowed for the formal incorporation of risk and uncertainly into economic theory and the application of an important number of results from probability theory. As Machina and Schmeidler (1992, p. 746) observe, "[...] it is hard to imagine where the theory of games, the theory of search and the theory of auctions would be without [them]".

2.2 Subjective Expected Utility Theory

In this section I depict the main components of subjective expected utility theory in the spirit of Savage (1954) and Anscombe and Aumann (1963). To model choice behavior under uncertainty Savage offered an analytical framework consisting of a set of states of nature Ω, a set of outcomes X, a set of acts $\mathcal{F}$ and a binary relation $\succcurlyeq$ on the set $\mathcal{F}$. The set Ω is a collection of mutually exclusive and exhaustive states representing all possible resolution of uncertainty. In Savage's words (1954, p.8) the state ω is "[...] a description of the world, leaving no relevant aspects undescribe". At ex-ante stage a decision maker does not know which state is the true one and has no influence upon the truth of the states. Ex-post exactly one state will be true and all uncertainty will be resolved. An uncertain event A is a subset of Ω. For all $A \subset \Omega$, we denote $\Omega \setminus A$, the complement of A, by A^c. The set X is a list of all possible consequences of any course of action that affects the decision maker's well-being. It captures "[...] everything that may happen to the decision maker"(Savage, 1954, p. 8). The elements of the set $\mathcal{F}$ represent possible courses of action and are called acts. An act f is a function from Ω to X, assigning to each state ω the outcome $f(\omega) \in X$ which would result if ω would be the true state and f would have been chosen. Since the decision maker is uncertain about which state is the true one, she is uncertain about which outcome will result from the chosen act. The binary relation $\succcurlyeq$ on $\mathcal{F}$ represents the decision maker's preferences. The preference relation $\succcurlyeq$ is viewed as governing the decision maker's choices. In the behavioristic tradition so ubiquitous in economics under the label of "revealed preferences", the preference relation $\succcurlyeq$ is partially observable. We can only observe the choices made by the decision maker. That is, if the act f and g are available to the decision maker and she chooses f rather than g then we can infer that she prefers f to g, i.e. $f \succcurlyeq g$.

Savage's idea is to impose some reasonable axioms on preferences and to show that if the decision maker's choices satisfy these axioms then she behaves as a person who possesses a single probability distribution over states and a utility function over outcomes, with respect to which she maximizes her subjective expected utility when choosing among acts. Savage postulated six axioms. Some of them, such as completeness and transitivity, are familiar. Other ones are technical and deal with different forms of separability and continuity property of preferences. Nevertheless, one postulate is the core axiom of Savage's theory and it merits the definition and a brief discussion. It is called the Sure-Thing-Principle. It requires that preferences between acts depend solely on the outcomes in states in which the outcomes of the two acts being compared are distinct. To state it formally, let us denote by $f_A g$ an act which assigns the outcome $f(\omega)$ to any state ω in A and the outcome $g(\omega)$ to any state ω in A^c.

Axiom 1 (Sure-Thing-Principle). *For any act $f, g, h, h' \in \mathcal{F}$ and any event A:*

$$f_A h \succcurlyeq g_A h \quad \Leftrightarrow \quad f_A h' \succcurlyeq g_A h'. \tag{2.1}$$

The Sure-Thing-Principle is a separability principle which has a practical meaning. Namely, when making choices between two acts, it is not necessary to consider states in which these acts yield the same outcomes. To illustrate the rationale behind this axiom, consider the following choice problem. A decision maker has to choose between two acts $f_A h$ and $g_A h$, where $f_A h$ states: "If Horse A wins the race, you will get a trip to France; if Horse A does not win, you will get a trip to Holland", and $g_A h$ states: "If Horse A wins the race, you will get a trip to Greece; if Horse A does not win, you will get a trip to Holland". These two acts offer the same trip if Horse A does not win the race, but different ones if Horse A does win. When facing the choice between these two acts the decision maker has only to decide whether she prefers the trip to France or the trip to Greece, regardless of what trip is offered by both acts in case Horse A does not win. Then, according to the Sure-Thing-Principle the common part of these two acts, here the trip to Holland, is immaterial for the choice between $f_A h$ and $g_A h$. If the decision maker prefers $f_A h$ to $g_A h$, then she also should prefer $f_A h'$ to $g_A h'$ whatever the trip h' offered in A^c is, let say to Hawaii or to the moon. Bacharach (1985, p.168) refers to the Sure-Thing-Principe as a "[...] fundamental principle of rational decision-making". Nevertheless, Ellsberg (1961) challenged the descriptive validity of this axiom (see Section 3.1).

Savage showed that his axioms are sufficient and necessary for the identification of both a utility function on the set of outcomes and a probability measure on the set of states that jointly characterize the decision maker's choices among acts by maximization of her subjective expected utility. Savage's theorem can be be expressed as follows. A preference relation $\succcurlyeq$ over $\mathcal{F}$ satisfies his six axioms (including the Sure-Thing-Principle) if and only if there exists a unique probability measure $\pi : \Omega \to [0, 1]$ and a unique (up to a positive linear transformation) utility function $u : X \to \mathbb{R}$ such that for any pair of acts $f, g \in \mathcal{F}$:

$$f \succcurlyeq g \quad \Leftrightarrow \quad \int_\Omega u(f(\omega)) \; d\pi(\omega) \geq \int_\Omega u(g(\omega)) \; d\pi(\omega). \tag{2.2}$$

In this formula, the measure π represents the decision maker's beliefs and is interpreted as a subjective probability distribution over states. The utility index u represents the decision maker's tastes, i.e. preferences over outcomes. In particular, when outcomes are monetary, the curvature of u is a measure of the decision maker's risk attitude (for given beliefs). Hence, Savage's subjective expected utility theory offers an attractive way to continue working with the expected utility approach even if the probabilities for uncertain events are unknown. This also means that a decision problem under uncertainty can be reduced in some sense to a decision problem under risk, with one important caveat. Beliefs are purely subjective and consequently two decision makers may have distinct subjective probability distributions even if they face the same decision problem.

Two key properties characterizing the subjective expected utility theory merit emphasis. First, probabilities assigned to mutually exclusive events are independent of the act being evaluated. This property is often referred as to "separability of beliefs from tastes" and it is implied by the Sure-Thing-Principle. Second, utilities assigned to outcomes are independent of the underlying state of nature. This property can be labelled as "separability of tastes from states" and has been criticized in a number of studies, e.g. Karni (1993).

In his original axiomatization, Savage allowed the set of outcomes to be an arbitrary set at the cost of assuming that the set of states is infinite. Anscombe and Aumann (1963) showed that if one accepts the existence of a physical (known, objective) randomization device such as roulette wheel then derivation of the subjective expected utility is also possible for a finite state space and with a more parsimonious set of axioms. In their setting the set of outcomes $Z = \Delta(X)$ is taken to be the set of all probability distributions (or simple lotteries) over some more primitive set of outcomes X with finite supports, i.e.:

$$\Delta(X) = \left\{ p : X \to [0,1] \;\middle|\; \begin{array}{c} \#\{x \in X \mid p(x) > 0\} < \infty, \\ \sum_{x \in X} p(x) = 1 \end{array} \right\}. \tag{2.3}$$

In the theory of Anscombe and Aumann objects of choice are "horse-race/roulette-wheel acts". An act f is a function from Ω to $\Delta(X)$ assigning to any state a simple lottery. To distinguish Savage's style acts from Anscombe and Aumann's ones we use $\mathcal{H}$ to denote the set of acts. Hence, there are two sources of uncertainty. All uncertainty is resolved sequentially. First, an outcome of the horse races, that is the state ω with unknown probability, is realized by determining the lottery $f(\omega)$ to be played out. Second, a roulette wheel is spun and the decision maker gets an outcome x with known probability $f(\omega)(x)$. Denote by $f(\omega)(x)$ the probability of outcome $x \in X$ in state

$\omega \in \Omega$ induced by the act f. Furthermore, the set $\Delta(X)$ is closed under a mixing operation, and so is the set $\mathcal{H}$. For every $f, g \in \mathcal{H}$ and any $\alpha \in [0,1]$, a mixture $\alpha f + (1-\alpha)g$ belongs to $\mathcal{H}$ and is performed state by state, that is:

$$\big(\alpha f + (1-\alpha)g\big)(\omega)(x) = \alpha f(\omega)(x) + (1-\alpha)g(\omega)(x), \tag{2.4}$$

for all $x \in X$ and all $\omega \in \Omega$. Adapting the independence axiom of von Neumann and Morgenstern, Anscombe and Aumann showed that their five axioms are sufficient and necessary for the existence of a unique subjective probability distribution μ and a unique (up to a positive linear transformation) utility function u such that for any pair of acts $f, g \in \mathcal{H}$:

$$f \succcurlyeq g \quad \Leftrightarrow \quad \int_\Omega \Big[\sum_{x\in X} f(\omega)(x)u(x)\Big]\; d\mu(\omega) \geq \int_\Omega \Big[\sum_{x\in X} g(\omega)(x)u(x)\Big]\; d\mu(\omega). \tag{2.5}$$

In the two-stage setting of Anscombe and Aumann, acts are evaluated by double integration. First, expectations are taken state by state with respect to known probabilities (over outcomes) dictated be the act and the states. This yields a function which assigns an expected utility to each state. In the second stage, an integral of this function is taken with respect to subjective probabilities for states in which the respective expected utilities materialize. The two-stage approach of Anscombe and Aumann has been established as a convenient setup in which alternative models for decision making under uncertainty have been developed.

More recently, Wakker (1989b), Nakamura (1990), Gul (1992) and Chew and Karni (1994) have shown that the subjective expected utility theory can also be derived in Savage's setting with a finite set of states by imposing topological restrictions on the set of outcomes. Sarin and Wakker (1997) simplified the axiomatization further by showing that the two-stage approach can be reduced to the one-stage approach while still allowing for a finite state space.

2.3 Dynamic Decision Problems

The subjective expected utility theory of Savage (1954) offers also an attractive tool for modeling dynamic decision problems. In dynamic choice situations, a decision maker receives new information in the form of an event A, and updates her preferences over acts in view of such information. The central question that arises in this context is how preferences are updated to incorporate the receipt of new information. Within Savage's framework the standard answer is to use Bayesian updating. That is, after being informed that the event A occurred, the decision maker updates her preferences by revising her prior beliefs according to Bayes's rule and by keeping her utility function unchanged.

There are two arguments used to justify Bayesian updating. The first one is based on the rationale behind the Sure-Thing-Principle and Savage's notion of conditional preferences. The second one relies on the idea of imposing dynamic restrictions on preferences. To discuss them briefly we limit our attention to events that the decision maker views as possible, i.e. non-null events. An event $A \subset \Omega$ is Savage-null if for any act $f, g, \in \mathcal{F}$ it is true that $f_A g \sim g$, otherwise it is non-null. Let $\mathcal{A}'$ be a collection of all non-null events. Before arrival of any information, the decision maker's preferences over acts are represented by $\succcurlyeq$, called unconditional (prior) preferences. After being informed that event A has occurred, the decision maker constructs her conditional (posterior) preferences over $\mathcal{F}$, represented by $\succcurlyeq_A$. Thus, in the dynamic setup the decision maker is characterized by a class of conditional preferences, $\{\succcurlyeq_A\}_{A\in\mathcal{A}'}$, one for each possible event. Such conditional preferences are viewed as governing future choice upon the realization of the event A.

Since conditional preferences govern future choices, it is important to know how conditional and unconditional preferences are linked to each other. It is a standard practice to underpin the link behaviorally by means of axioms, which are supposed to reflect dynamic properties of preferences. Two axioms are cornerstone in the theory of updating preferences. The first one, called *consequentialism*, concerns only the conditional preference relation. It requires that preferences conditional on a non-null event A are not affected by outcomes outside of the conditioning event, A^c. Intuitively, once the decision maker is informed that event A occurred, only the uncertainty about states in A matters for future choices. The uncertainty about the counterfactual states, i.e. these in A^c, is immaterial. Formally, whenever two acts f and g assign the same outcomes to states in A^c, then conditional on A^c, the decision maker should be indifferent between these two acts. The term consequentialism was introduced by Hammond (1988) in the presence of risk.[3]

Axiom 2 (Consequentialism). *For any non-null event A and all acts $f, g \in \mathcal{F}$, $f(\omega) = g(\omega)$ for each $\omega \in A$ implies $f \sim_A g$.*

The second axiom, called *dynamic consistency*, links directly conditional and unconditional preferences. It requires that choices made ex-ante be consistently implemented in the future. Essentially, dynamic consistency excludes reversals. As Machina (1989, p.1636-7) writes "[...] behavior [...] will be dynamically inconsistent, in the sense that [...] actual choice upon arriving at decision node would differ from [...] planned choice for that node". Formally, when the decision maker prefers f to g without any information regarding A, and f and g assign the same outcomes to states in A^c, she should also prefer f to g after being informed that A occurred, and vice versa. This requirement appears in a number of places in the literature, among others in Machina (1989), Epstein and LeBreton (1993) and in Ghirardato (2002).

Axiom 3 (Dynamic consistency). *For any non-null event A and all acts $f, g \in \mathcal{F}$ such that $f(\omega) = g(\omega)$ for each $\omega \in A^c$, $f \succcurlyeq g$ if and only if $f \succcurlyeq_A g$.*

Consequentialism and dynamic consistency are appealing and very useful properties of preferences. In particular, "[...] they provide the basis for backward induction and

[3]However, Hammond's notion is conceptually stronger than Axiom 2.

dynamic programming, two techniques whose practical importance simply cannot be overstated" (Siniscalchi, 2009a, p.339).

It is well-known that consequentialism and dynamic consistency are properties inherent in Savage's Sure-Thing-Principle. To illustrate this issue, let us first recall the notion of conditional preferences proposed by Savage (1954) in his book *The Foundations of Statistics*. When deriving a "static" version of the Sure-Thing-Principle, Savage begins with an informal, but in spirit, "dynamic" principle. It is, in essence, this: If a decision maker prefers some act f to another act g knowing that an event A occurred, and, if she prefers f to g knowing that an event A did not occur, then she definitely prefers f to g regardless of whether she knows if A occurred or not. When justifying this principle Savage (1954, p.22) asked himself: "What technical interpretation can we attach to the idea that f would be preferred to g, if $[A]$ were known to obtain". To answer this question, Savage (1954, p.22) made the following assumption: "Under any reasonable interpretation, the matter would seem not to depend on the values f and g assume at states outside of $[A]$". Then, one could modify f and g such that, for instance, they assign the same outcomes to states outside of A, i.e. $f(\omega) = g(\omega)$ for any $\omega \in A^c$, and as he continues "[...] f and g are surely to be regarded as equivalent given A^c; that is, they would be considered equivalent if it were known that A did not obtain". This is exactly the idea behind consequentialism. Assuming consequentialism, Savage derives his conditioning rule as follows. If f and g are modified such that they agree outside of A and f is preferred to g without knowing that A occurred, then f should also be preferred to g knowing that A occurred and vice versa. Savage's conditioning rule can be expressed formally as follows.[4]

Definition 1 (Savage's conditioning rule). *For any pair of acts $f, g \in \mathcal{F}$ and any non-null event $A \in \mathcal{A}'$ there is an act $h \in \mathcal{F}$ such that:*

$$f \succsim_A g \quad \text{if and only if} \quad f_A h \succsim g_A h. \tag{2.6}$$

[4]Savage's conditioning rule has been also suggested for updating more general preferences, see Gilboa and Schmeidler (1993); Ghirardato (2002).

Furthermore, as Savage argues, the ranking between f and g will not depend on which of the modification was actually carried out as long the modified acts are the same outside of A. In other words, we will never find two acts h and h' such that $f_A h \succcurlyeq g_A h$ and $f_A h' \prec g_A h'$. This is precisely the "static" version of the Sure-Thing-Principle (see Section 2.2) derived from the "dynamic" principle under the assumption of consequentialism. Equivalently, one could restate the Sure-Thing-Principle dynamically by saying that the above definition of conditional preferences is well-posed.[5]

In the light of this argument it is not surprising that consequentialism and dynamic consistency are properties of preferences characterized by the Sure-Thing-Principle and vice versa. That is, if we start with prior preferences which respect the Sure-Thing-Principle and conditional preferences are obtained from the prior ones by applying Savage's conditioning rule, then theses conditional preferences satisfy consequentialism and dynamic consistency. Conversely, if conditional preferences respect consequentialism and dynamic consistency, then prior preferences are consistent with the Sure-Thing-Principle and conditional preferences are obtained from prior ones by applying Savage's conditioning rule. This result has been celebrated as the "folk wisdom" of decision theory. It was formally proved, among others, by Ghirardato (2002, Lemma 2) in Savage's framework, and by Siniscalchi (2010, Proposition 1) in a more general framework, where preferences are defined on decision trees. The following lemma states the result formally.[6]

Lemma 1. *Let $\{\succcurlyeq_A\}_{A\in\mathcal{A}'}$ be the class of complete and transitive conditional preferences which satisfies consequentialism. Then $\{\succcurlyeq_A\}_{A\in\mathcal{A}'}$ satisfies dynamic consistency if and only if $\succcurlyeq$ satisfies the Sure-Thing-Principle, and $\{\succcurlyeq_A\}_{A\in\mathcal{A}'}$ satisfies Savage's conditioning rule.*

Finally, it is well-known (for instance, see Kreps, 1988; Ghirardato, 2002) that if

[5]This is exactly the restatement of the static Sure-Thing-Principle provided by Savage in the back cover of the Dover edition of his book.

[6]Note that Lemma 1 applies to a more general class of preferences than the class of subjective expected utility preferences.

prior preferences satisfy Savage's axioms, i.e. they are represented by maximization of subjective expected utility with respect to prior probability distribution π together with a utility function u, then the conditional preferences $\succcurlyeq_A$ are obtained from the prior ones by updating the prior π according to Bayes' rule and by maintaining the same utility function u. That is, the conditional preferences $\succcurlyeq_A$ are represented as follows: For all acts $f, g \in \mathcal{F}$ and any non-null event A:

$$f \succcurlyeq_A g \quad \Leftrightarrow \quad \int_\Omega u(f(\omega))\ d\pi_A(\omega) \geq \int_\Omega u(g(\omega))\ d\pi_A(\omega), \tag{2.7}$$

where π_A is the Bayesian update of π conditional on A, i.e.:

$$\pi_A(B) = \frac{\pi(B \cap A)}{\pi(A)} \ \text{ for any } \ B \subset \Omega. \tag{2.8}$$

Thus, in view of Lemma 1, dynamic consistency and consequentialism offer an equivalent way to justify Bayesian updating. This issue was scrutinized by Ghirardato (2002). He showed that, if the class of conditional preferences $\{\succcurlyeq_A\}_{A \in \mathcal{A}'}$ satisfies all Savage's axioms except the Sure-Thing Principle, plus consequentialism and dynamic consistency, then the prior preference relation $\succcurlyeq$ admits subjective expected utility representation with respect to a unique prior probability distribution π and a unique (up to a positive linear transformation) utility function u, and $\{\succcurlyeq_A\}_{A \in \mathcal{A}'}$ is a result of Bayesian updating of $\succcurlyeq$. That is, for any non-null event A, $\succcurlyeq_A$ is represented as in Equation 2.7 with π_A defined as in Equation 2.8. Conversely, if the prior preference relation $\succcurlyeq$ admits subjective expected utility representation and the class of conditional preferences $\{\succcurlyeq_A\}_{A \in \mathcal{A}'}$ is the result of Bayesian updating, then $\{\succcurlyeq_A\}_{A \in \mathcal{A}'}$ satisfies dynamic consistency and consequentialism.

Thus, the subjective expected utility theory of Savage provides a complete theory of behavior in dynamic decision problems. It underpins Bayesian updating and ensures that dynamic behavior respect two key properties, dynamic consistency and consequentialism.

Chapter 3

Non-Bayesian Decision Theory

3.1 Ellsberg's Experiments

In his famous article titled "Risk, Ambiguity, and the Savage's axioms", Daniel Ellsberg (1961) questioned the descriptive validity of the subjective expected utility theory. He exemplified that a special type of uncertainty, called *ambiguity*, will affect subjects' choice behavior in a way such that it cannot be explained by Savage's theory. Ellsberg associated ambiguity with the lack of information about likelihoods in situations characterized neither by "measurable" nor "unmeasurable" uncertainty. Intuitively, ambiguity designates situations in which probabilities for some events are known, whereas for other ones they are not. "What is at issue might be called *ambiguity* of information, a quality depending on the amount, type, reliability and 'unanimity' of information giving rise to one's 'degree of confidence' in an estimate of relative likelihoods" (Ellsberg, 1961, p.657). Following Frisch and Baron (1988), Camerer and Weber (1992, p.330) conceptualized further the notion of ambiguity as "[. . .] uncertainty about probability, created by missing information that is relevant and could be known". Ellsberg conjectured that a majority of subjects facing ambiguity will prefer to bet on events with known probabilities rather than on events for which probabilistic information is missing. To test his claim he designed two experiments incorporating decision problems under ambiguity. The first experiment, called the 3-color experiment, involves one urn filled with balls of three distinct colors.

3-color Experiment. *There is an urn containing 30 balls, 10 of which are known to be yellow and 20 of which are somehow divided between blue and green, with no further information on the distribution. One ball will be drawn at random from the urn. Subjects face two choice situations, I and II, in which they have to choose between bets paying out 4 or nothing, depending on the color of the randomly drawn ball. In the first choice situation, I, the subjects are asked to choose between two bets:* f_1*, "You receive 4 if a yellow ball is drawn and nothing otherwise"; and* f_2*, "You receive 4 if a blue ball is drawn and nothing otherwise". Similarly, in the second choice situation, II, the subjects have to choose one of two following bets:* f_3*, "You receive 4 if a yellow or green is drawn and nothing otherwise"; or* f_4*, "You receive 4 if a blue or green is drawn and nothing otherwise. Table 3.1 summarizes the two relevant choice problems in the 3-color experiment of Ellsberg.*

		Yellow	*Blue*	*Green*
Choice I	f_1	4	0	0
	f_2	0	4	0
Choice II	f_3	4	0	4
	f_4	0	4	4

Table 3.1: Ellsberg's 3-color experiment

Denote by Y, B and G the event that the ball drawn is yellow, blue and green, respectively. For the moment, we describe an event to be unambiguous if its probability is known, i.e. deducible from the available information. Events for which probabilities are unknown are termed to be ambiguous. According to the available information it is natural to assume that subjects view the events Y and $B \cup G$ as unambiguous. The probabilities for their respective occurrence, one third and two thirds, are known. The remaining events are ambiguous. For instance, the probabilities for the event B as well as for the event G are only known to be somewhere between nil and two third. Thus, the observable choices in the 3-color experiment can be viewed as revealing subjects' attitudes towards ambiguity. When considering subjects with strong preferences, there are four possible patterns of preferences. Of course subjects can also be indifferent between two alternatives, but then it would not be valid to infer the "ambiguity attitude" from their choices. Each column in Table 3.2 depicts the chosen bet in each of the two relevant choice problems.

	Ambiguity Attitude			
	Averse	*Neutral*		*Loving*
Choice I	f_1	f_1	f_2	f_2
Choice II	f_4	f_3	f_4	f_3

Table 3.2: Ambiguity attitudes in Ellsberg's 3-color experiment

The choices depicted in the first and fourth column reflect subjects' sensitive attitude towards ambiguity. For instance, consider the first column in which the subjects prefer f_1 to f_2 and f_4 to f_3. The subjects displaying such preferences are called *ambiguity-averse*, since they are reluctant to bet on events with unknown probabilities. Conversely, in the last column, the subjects prefer f_2 to f_1 and f_3 to f_4. These subjects are said to exhibit *ambiguity-loving* behavior, since they favor to bet on events with unknown probabilities.

These two patterns of choices are inconsistent with Savage's Sure-Thing-Principle. To see this illustrated, consider an ambiguity-averse subject with choices f_1 and f_4. In the first choice situation, the subject faces two bets f_1 and f_2 paying off the same amount, 0, if the ball drawn is green. On the other hand, in the second choice situation she faces exactly the same bets labelled f_3 and f_4, but now with the common payoff of 4 instead of 0 if in the case the ball drawn is green. According to the Sure-Thing-Principle these common payoffs should not affect her choices. Preferences between bets depend only on the payoffs in states in which the payoffs of the two bets being compared are distinct. Thus, if the subject prefers to bet on yellow rather than on blue in the first choice situation then she should consequently make the same choices in the second situation. But, this is exactly the logic being violated by the ambiguity-averse subject. She reverses the choices in the second choice situation. One could hypothesize that, the ambiguity-averse subject cannot decide between f_3 and f_4 without paying attention to the common payoff of these two bets. Then, looking on all states in which the payoff of 4 is possible, she recognizes that bet f_3 offers 4 with a probability from the interval between one third and one, whereas bet f_4 pays the same amount with the exact probability of one third. Since she does not know which probability is the correct one she decides to choose the bet f_4 with known probability of getting 4. Thus, the subject exhibiting aversion towards ambiguity violates the separability property inherent in the Sure-Thing-Principle. One explanation for this could be, for instance, that she perceives the events B and G as complementary in the sense that information given on their union cannot be further elaborated.

Moreover, for the ambiguity-averse subject there is no probability distribution that can adequately represent her subjective beliefs. If to the contrary, we assume that she has a subjective probability distribution π, then preferring f_1 to f_2 implies that she has a higher probability for a yellow ball than for a blue ball to be drawn, i.e. $\pi(Y) > \pi(B)$. But, the fact that she prefers f_4 to f_3 implies that she assigns a higher probability to the event blue or green to be be drawn than to the event yellow or green, i.e. $\pi(Y \cup G) < \pi(B \cup G)$. Thus, by additivity $\pi(Y) < \pi(B)$. These two deductions are contradictory. Choices revealing ambiguity-averse behavior violate not only the subjective expected utility theory of Savage, but also any other theory based on probabilistic sophistication in the sense of Machina and Schmeidler (1992, 1995), Grant (1995) or Chew and Sagi (2006, 2008). According to this theory, subjects' subjective beliefs are represented by a unique and additive probability distribution, but preferences do not need to have expected utility representation.

The second experiment, called 2-color experiment, was originated by Knight (1921). It involves two urns each of them filled with balls of two possible colors.

2-color Experiment. *There are two urns, K and U, each containing 100 balls. Each ball is either yellow or white. Urn K contains 50 yellow and 50 white balls. In Urn U the proportion of yellow and white balls is unknown. One ball will be drawn at random from each urn. Subjects face four bets paying out* 4 *or* 0*, depending on the urn and the color of the ball drawn:* f_1*, "You receive* 4 *if a yellow ball is drawn from Urn K and nothing otherwise";* f_2*, "You receive* 4 *if a white ball is drawn from Urn K and nothing otherwise";* f_3*, "You receive* 4 *if a yellow ball is drawn from Urn U and nothing otherwise";* f_4 *"You receive* 4 *if a white ball is drawn from Urn U and nothing otherwise". In the first choice situation,* I*, subjects are asked for each urn respectively which color do they prefer to bet on. In the second choice situation,* II*, they are asked for each color respectively which urn do they prefer to bet on. Table 3.3 summarizes the relevant bets in the 2-color experiment.*

Urn K

	Yellow	*White*
f_1	4	0
f_2	0	4

Urn U

	Yellow	*White*
f_3	4	0
f_4	0	4

Table 3.3: Ellsberg's 2-color experiment

Denote by Y^I and W^I the event that a randomly drawn ball from Urn $I \in \{K, U\}$ is yellow and white, respectively. Since the probability for a yellow ball, as well as a white one, to be drawn from Urn K is known to be one half, the events Y^K and W^K are unambiguous. Conversely, the probability for a yellow ball, respectively a white one, to be drawn from Urn U is only known to be in the interval between nil and one. Thus, the events Y^U and W^U and the whole Urn U are purely ambiguous. Again, subjects' choices in the 2-color experiment allow us to draw conclusions about their ambiguity attitude. For instance, as Ellsberg observed, a majority of his "non-experimental" subjects are indifferent between betting on yellow and on white when facing Urn K and Urn U, respectively. That is, $f_1 \sim f_2$ and $f_3 \sim f_4$. But, when asked whether they prefer that the ball be drawn from Urn K or from Urn U for each color separately, they strictly prefer the unambiguous Urn U. That is, $f_1 \succ f_3$ and $f_2 \succ f_4$. Such choices can be seen as revealing aversion towards ambiguity. Moreover, Ellsberg also observed a small minority of subjects who prefer a ball to be drawn form Urn U rather than from Urn K. These subjects seems to be ambiguity-loving. Again, these two pattern of choices violate the Sure-Thing-Principle and they can not be rationalized by a single probability distribution (see Gilboa, 2009, Chapter 12).

Ever since the contribution of Ellsberg there has been overwhelming empirical evidence confirming ambiguity-sensitive behavior as a systematic and robust phenomenon. Camerer and Weber (1992) provide a comprehensive survey of the literature on experimental studies of decision making under ambiguity. They note a number of stylized facts which emerged from these studies. It is worth mentioning a few of them: "Ambiguity aversion is found consistently in variants of the Ellsberg problems [...] (fact 1). Ambiguity aversion persists when preference is strict, excluding indifference (fact 2), and when ambiguity is reduced by drawing samples from ambiguous urns (fact 3). Ambiguity averters have generally not been swayed in experiments that offered written arguments against their paradoxical choices (fact 4)" (Camerer and Weber, 1992, p.340).

As a part of this thesis we also ran the two experiments inspired by Ellsberg (1961). Our results unequivocally confirm the previous observations. A majority of subjects exhibit ambiguity-sensitive behavior. In the 2-color experiment conducted to examine the relationship between ambiguity and randomization attitudes (see Chapter 4) we observe that: 54.5% of subjects are ambiguity-averse, 11.3% are ambiguity-loving, while 37.5% exhibit neutral attitude towards ambiguity. In the 3-color experiment conducted to test dynamic choice behavior under ambiguity (see Chapter 5) we find a similar pattern: 54.8% of subjects prefer to bet on events with known probabilities, 7.4% prefer to bet on events with unknown probabilities, while 38.1% are ambiguity-neutral. These observations are true for all subjects with strong preferences.

3.2 Ambiguity Models

In the wake of strong empirical evidence confirming ambiguity-sensitive behavior, several generalizations of Savage's subjective expected utility theory have been suggested. These generalizations seek to provide an alternative representation of preferences apt to explain the Ellsberg-type behavior. The leading idea is to abandon the main tenet of Bayesianism, namely that subjective beliefs are representable by a single probability distribution. When information about probabilities is too scarce, as in Ellsberg's experiments, it seems to be more plausible to assume that the decision maker behaves as though she had many possible priors in mind, rather than a single one. There are three widely studied approaches adopting this view. They differ from each other with regard to the notion of subjective beliefs. In the first approach subjective beliefs are represented by a non-additive prior called a capacity, in the second one, by a set of priors called multiple priors and in the third one, by a prior over the set of priors called a second order prior. Note, in these approaches, we discussed below, subjective beliefs are represented uniquely, but not necessarily by a single prior probability distribution.

Non-Additive Prior. Historically, the first axiomatically sound theory of decision making under ambiguity is the Choquet expected utility theory developed by Schmeidler (1989). In this theory subjective beliefs are represented by a capacity. That is, a normalized and monotone, but non-necessarily-additive, set function. The concept of capacity generalizes the notion of probability by weakening the additivity property. The only requirement is that they must satisfy the usual monotonicity property. In other words, the lack of information about likelihoods hinders the decision maker from forming beliefs which satisfy all mathematical properties of probabilities. The decision maker, though, is able to assign weights to uncertain events in such a way that "larger" events (with respect to set inclusion) are "more likely". By weakening the separability property inherent in Savage's Sure-Thing-Principle, Schmeidler establish a set of axioms which are sufficient and necessary for the existence of a unique capacity ν and a unique (up to a positive linear transformation) utility function u such that for any pair of acts $f, g \in \mathcal{F}$:

$$f \succcurlyeq g \quad \Leftrightarrow \quad \int_{\Omega} u(f(\omega))\; d\nu(\omega) \geq \int_{\Omega} u(g(\omega))\; d\nu(\omega). \tag{3.1}$$

In the presence of non-additive beliefs, expected utilities are computed by means of Choquet integrals, introduced by Gustave Choquet (1954). For this reason this theory is called the theory of Choquet expected utility maximization. In Section 6.1 the notion of capacities and Choquet integrals is defined and discussed in detail. Within the Choquet expected utility theory, the decision maker's attitude towards ambiguity is mainly captured by the mathematical properties of capacity (see Section 3.3). However, one issue needs to be clarified. How are subjective beliefs, represented uniquely by a capacity, linked to the idea of a non-single prior? Roughly speaking, the technique of Choquet integration provides an answer. The Choquet integral of an act f with respect to the capacity ν can been written as an expected utility with respect to an additive probability measure m. However, the probability measure m depends on the act f being evaluated. More precisely, the measure m depends on how events are ranked with respect to the attractiveness of their outcomes assigned by the act f. In other words, the weight ascribed to an event by m depends not only on the event, but also on how good the outcome yielded by the event under f is in comparison with the outcomes yielded by the other events. In general, two non-comonotonic acts, i.e. acts generating distinct ranking position of mutually exclusive events, will be evaluated with respect to distinct probability measures. Acts generating the same ranking position of states, so-called comonotonic acts, are evaluated with respect to the same probability measure. This is the way in which the independence of beliefs from tastes, the key property of the subjective expected utility theory, is generalized in Schmeidler's theory. The separability of beliefs from tastes is respected only among comonotonic acts. One could term this property, "comonotonic separability of beliefs from tastes". For these reasons, the concept of capacities can be seen as a mathematical tool "summarizing" all such “rank-dependent” probabilistic scenarios.

Multiple Priors. The second prominent theory known as the maxmin expected utility theory, or “multiple prior” model, was pioneered by Gilboa and Schmeidler (1989). This approach relies on the idea that subjective beliefs are represented by a set C of probability distributions. Intuitively, one can think of each prior in C as describing a possible probabilistic scenario that the decision maker has in mind. With multiple priors in mind the decision maker can still compute expected utility of an act as usual, but now with one expected value per prior. When evaluating the act, the decision maker considers only the probabilistic scenario in which she gets the lowest expected utility. To decide between two acts she compares their lowest expected utilities (which may be obtained with respect to distinct priors) and chooses the one that yields the maximum, among the lowest, expected utility. Gilboa and Schmeidler (1989) provided an axiomatic justification for this decision rule. They established a set of axioms that are necessary and sufficient for the existence of a unique convex and closed set of probability measures C and a unique (up to a positive linear transformation) utility function u such that for any pair of acts f and g:

$$f \succcurlyeq g \quad \Leftrightarrow \quad \min_{\pi \in C} \int_{\Omega} u(f(\omega)) \; d\pi(\omega) \geq \min_{\pi \in C} \int_{\Omega} u(g(\omega)) \; d\pi(\omega). \tag{3.2}$$

As in the Savage's theory, the set C is also purely subjective and represents the decision maker's perception of ambiguity. When $C = \{\pi\}$ is a singleton then the decision maker behaves as a subjective expected utility maximizer with a single prior π. By taking the minimum of the set of all possible expected utility values of an act f the decision maker reveals her reluctance towards ambiguity. The cautious attitude of the decision maker featured by the multiple priors model is often viewed as the result of a malevolent “Nature” which can influence the occurrence of events to her disadvantage. That is, Nature chooses a probability π from the set C with the objective of minimizing her expected utilities conditional on her choices. Under this view, Maccheroni, Marinacci, and Rustichini (2006) extended the multiple prior model by generalizing Nature's constraint. In their extension, called theory of variational preferences, the constraint on Nature is given by a cost function associated with the choice of a particular probability distribution. The cost function is then supposed to capture the decision maker's attitude towards ambiguity. In other extensions, Ghirardato, Maccheroni, and Marinacci (2004) derive the α-maxmin expected utility representation. In this representation an act is evaluated as a linear combination of maxmin expected utility and maxmax expected utility in which not the worst, but the best expected utility is considered. The maxmin expected utility is weighted with a coefficient $\alpha \in [0, 1]$ while the maxmax expected utility is weighted with the coefficient $1 - \alpha$. Both maxmin and maxmax expected utilities are taken with respect to a set of set of priors C, which is a part of the representation. That is, the set of priors C and the coefficient α are uniquely defined and may be interpreted as ambiguity and ambiguity attitude respectively. For instance, $\alpha = 1$ indicates an extreme aversion towards ambiguity (as in the multiple prior model). By contrast, $\alpha = 0$ reflects purely ambiguity-loving behavior. Furthermore, modeling α-maxmin preferences in setups with a finite set of states of nature imposes additional restrictions on the parameter α. For such setups, Eichberger, Grant, Kelsey, and Koshevoy (2010) showed that preferences over acts satisfy the axioms of Ghirardato, Maccheroni, and Marinacci (2004) if and only if $\alpha = 0$ or $\alpha = 1$. Thus, within finite setups, α-maxmin preferences may only exhibit the two extreme attitudes towards ambiguity.

Second Order Prior. The third approach, proposed by Nau (2006) and further developed by Klibanoff, Marinacci, and Mukerji (2005), is often referred to as the second order prior model, or the smooth ambiguity model. In this approach the decision process is modeled as a two-stage process, however, differently than in the tradition of Anscombe and Aumann (1963). A decision maker starts with a set of all probabilistic scenarios captured by the set $\Delta(\Omega)$.[1] In addition to it, she comes up with a probability

[1]That is, $\Delta(\Omega) = \left\{p : \Omega \to [0, 1] \mid \sum_{\omega\in\Omega} p(\omega) = 1\right\}$.

distribution μ over the set of priors, called a second order prior. Intuitively, one can think of the prior μ as describing the probabilistic belief of the decision maker that a particular probabilistic scenario will occur. For instance, consider the set-up of Ellsberg's 3-color experiment described in Section 3.1. The decision maker may think that the urn has different compositions, each one representing a possible probabilistic scenario. Altogether there are twenty-one possible scenarios (from all 20 balls being blue to all 20 balls being green). If the decision maker considers all twenty-one scenarios as possible they will be in the support of her subjective probability distribution μ over $\Delta(\Omega)$. Furthermore, she can distinguish which of the probabilistic scenarios is subjectively more likely to occur than the other. The decision process can be summed up as follows. In the first stage a composition of the urn is drawn among a set of hypothetical compositions according to probability μ. In the second stage, a ball is drawn from the urn whose composition has been determined by the realisation of μ. Klibanoff, Marinacci, and Mukerji (2005) showed that under the seemingly mild assumption imposed on preferences over $\mathcal{F}$, the decision maker choices are characterized by the following decision rule. For any pair of acts f and g, $f \succcurlyeq g$ if and only if:

$$\int_{\Delta(\Omega)} \phi\Big(\int_{\Omega} u(f(\omega))\ dp(\omega)\Big)\ d\mu(p) \geq \int_{\Delta(\Omega)} \phi\Big(\int_{\Omega} u(g(\omega))\ dp(\omega)\Big)\ d\mu(p), \tag{3.3}$$

where ϕ is function from $\mathbb{R}$ to $\mathbb{R}$. The function ϕ expresses the decision maker's attitude towards ambiguity. The evaluation of an act f goes as follows. Once the probabilistic scenario is fixed, the decision maker faces a situation with known probability p and evaluates the act f by its expected utility. Since there are many possible scenarios, the decision maker gets a set of expected utilities of f, one for each probabilistic scenario. Then, instead of taking the minimum of these expected utilities, as the multiple priors approach does, an expectation of the function ϕ distorting the expected utilities is taken. The role of the distortion ϕ is crucial. If ϕ is linear then the decision criterion in Equation (3.3) reduces to the expected utility maximization rule with respect to a "reduced" probability distribution obtained by the combination of μ and all possible

p's. If the distortion ϕ is not linear, the μ and all p's cannot be combined to construct a reduced probability distribution. In this case, the decision maker takes the expected utility with respect to μ and ϕ of the expected u-utilities with respect to p's. A concave distortion ϕ reflects aversion towards ambiguity, in the sense that it places larger weights on lower expected u-utilities. By contrast, a convex distortion ϕ reflects ambiguity-loving behavior.

These three prominent models tackle the problem raised by Ellsberg. In particular, they allow ambiguity and the decision maker's attitude towards it to play role in decision making. Nevertheless, they are not immune to criticism. In a recent article Machina (2009) proposed a counterexample which points out the limitations of the Choquet expected utility theory of Schmeidler (1989) in the same spirit as Ellsberg's experiments point out the limitations of Savage's theory. He constructed a 4-color experiment and conjectured that a majority of subjects will exhibit an intuitive pattern of preferences which cannot be explained by the Choquet expected utility theory. L'Haridon and Placido (2010) provided experimental evidence which confirms Machina's conjecture. Moreover, Baillon, L'Haridon, and Placido (2010) showed that Machina's paradox poses difficulties not only for the Choquet expected utility theory, but also for the other ambiguity models discussed in this section (namely, the multiple priors model, the variational preferences, the α-maxmin model and the smooth ambiguity model).

3.3 Ambiguity Attitudes

In Section 3.1, we presumed a decision maker to be ambiguity-averse whenever she is reluctant to bet on events with unknown probabilities. A decision maker who is prone to bet on such events is called ambiguity-loving. In this section we recall two methods used in the literature which attempt to operationalize the notion of ambiguity attitudes in a formal fashion.

Many models incorporating ambiguity-sensitive behavior adopt Schmeidler's (1989) notion. It states that the decision maker is ambiguity-averse if for any two acts she is indifferent between, she prefers a random mixture between these two acts. The idea of using randomizing devices in the presence of ambiguity was originally proposed by Raiffa (1961) in his comment on Ellsberg's paradox.[2] To elucidate the relevant part of Raiffa's argument consider again the 2-color experiment by Ellsberg as described in Section 3.1. A subject who was indifferent between betting on white and on yellow when facing each urn separately (i.e. $f_1 \sim f_2$ and $f_3 \sim f_4$) was called to be ambiguity-averse when she strictly preferred to bet on the urn with known proportion of balls (i.e. $f_1 \succ f_3$ and $f_2 \succ f_4$). Now, suppose that the subject has access to a fair coin and she flips this coin to decide on which color to bet. The decision maker declares to choose f_3 if heads appears and f_4 otherwise. When evaluating such a strategy, Raiffa (1961), but also Ellsberg (2001, Chapter 8) in his reply to Raiffa, argued that this "mixed strategy" yields exactly the same chance of getting 4 or 0 as the unambiguous bets f_1 and f_2, namely $50 : 50$. For the moment, call this strategy a mixture between f_3 and f_4 and denote it by h. The argument of Raiffa is the following: Imagine that the ball has already been drawn from Urn U, but the subject has not yet seen its color. If, on the one side, the ball drawn is yellow, then the mixture h offers exactly a $50 : 50$ chance of getting 4 or 0; if heads turns up, the subject chooses f_3 and gets 4 when the color of the ball drawn is revealed, and if it is tails she chooses f_4 and gets nothing. Likewise if the ball is white, then h offers again a $50 : 50$ chance of 4 or 0; if heads turns up, the subject chooses f_3 and gets nothing when the color is revealed, and if it is tails she chooses f_4 and gets 4. Thus, the subject is "guaranteed" a $50 : 50$ chance of 4 or 0 regardless whether the ball drawn is yellow or white. For this reason, she should be indifferent between h and either unambiguous bets f_1 and f_2. In view of this argument, the ambiguity-averse subject should always display a preference

[2]Raiffa (1961) argued that subjective expected utility is a "normative prescription" to which the decision maker should aspire. Accordingly, as long there is access to an objective randomizing device such as a coin, there is no reason to behave in an ambiguity-averse manner.

for mixtures between bets whose outcomes depend upon the realization of ambiguous events. Note, an implicit assumption in this argument is the fact that first the ball is drawn and then the coin is flipped to determine the outcome of h. This is exactly the same way uncertainty resolves in the two-stage approach of Anscombe and Aumann (1963). In this setting Schmeidler, formulated his definition of ambiguity aversion. As mentioned in Section 2.2, in the formal framework of Anscombe and Aumann, objects of choice are horse-race/roulette-wheel acts, and random mixtures between two such acts are well-defined. Adapting their framework, the bet f_3 (resp. f_4) can be seen as a function assigning a degenerate lottery yielding 4 for sure, if the ball drawn from Urn U is yellow (resp. white); otherwise, a degenerate lottery yields 0 for sure. Consequently, the mixture between f_3 and f_4, based an a coin flip, yields a random mixture between the two degenerate lotteries ascribed to each color by the respective acts f_3 and f_4. That is, the mixture $h = (\frac{1}{2})f_3 + (\frac{1}{2})f_4$ yields a lottery which pays 4 with a probability of one half, and 0 with probability of one half given that the ball drawn from Urn U is yellow; and it yields exactly the same lottery given that the ball drawn from Urn U is white. Table 3.4 depicts these three acts.

	Urn U	
	Yellow	*White*
f_3	4	0
f_4	0	4
$h = (\frac{1}{2})f_3 + (\frac{1}{2})f_4$	$(\frac{1}{2})4 + (\frac{1}{2})0$	$(\frac{1}{2})4 + (\frac{1}{2})0$

Table 3.4: Random mixture in an Anscombe and Aumann setting

Thus, in the setting of Anscombe and Aumann, the mixture $\alpha f + (1 - \alpha)g$ can be seen as randomization over f and g with known probabilities α and $1 - \alpha$, respectively. Under this interpretation, preference for mixtures implies preference for randomization. Thus, the decision maker is ambiguity-averse in the sense of Schmeidler if she exhibits preference for randomization (or is *randomization-loving*). Conversely, the decision maker is ambiguity-loving if she dislikes randomization (or is *randomization-averse*). Neutrality towards ambiguity is associated with neutrality towards randomization.

Axiom 4 (Ambiguity attitudes). *For any $f, g \in \mathcal{H}$ and any $\alpha \in (0,1)$ a decision maker with $\succcurlyeq$ over $\mathcal{H}$ is said to be:*

(i) ambiguity-averse if $f \sim g \implies \alpha f + (1-\alpha)g \succcurlyeq f$ *(or g),*

(ii) ambiguity-loving if $f \sim g \implies \alpha f + (1-\alpha)g \preccurlyeq f$ *(or g),*

(iii) ambiguity-neutral if both (i) and (ii) hold.

When evaluating acts in the formal setting of Anscombe and Aumann, first, expectations are taken state by state with respect to known probabilities (over outcomes). This yields a function which assigns an expected utility to each state. In the second stage, an integral of this function is taken with respect to state for which probabilities are unknown. Therefore, random mixtures between two acts are often said to have the effect of "smoothing" utilities of outcomes across states. In the words of Schmeidler (1989, p.582), ambiguity aversion "[...] means that "smoothing" or averaging utility distributions makes the decision maker better off".

In the Choquet expected utility model, ambiguity aversion is mainly described by the properties of the capacity characterizing the decision maker's beliefs. Schmeidler (1989) showed that a decision maker with Choquet expected utility preferences exhibits ambiguity aversion if and only if her capacity is convex (see Section 6.1). In the maxmin expected utility model of Gilboa and Schmeidler (1989) and in its generalization (the variational preferences of Maccheroni, Marinacci, and Rustichini, 2006), the ambiguity aversion axiom is imposed *directly* on preferences. Whether the preferences in the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005) display ambiguity aversion in the sense of Schmeidler has been hotly debated. While Epstein (2010) excludes this possibility, Klibanoff, Marinacci, and Mukerji (2009) defend the opposite position and argue that the decision maker with a strictly concave function ϕ will also exhibit preference for randomization.

The preference for randomization as an expression of ambiguity aversion is less convincing when one evaluates the mixtures between bets conditional on the realisation of a random device. Suppose that the ambiguity-averse subject has a linear utility function and values either of the two bets f_3 and f_4 at prize 1, due to ambiguity. Then, if the coin comes up heads the subject chooses f_3, which she valued at 1, and if the coin comes up tails she chooses f_4, which has the same value as f_3. Ellsberg (2001, Chapter 8) argues that it is plausible to assume that the subject also values the equiprobable mixture h between these two bets at the same prize, 1. Even when the coin flip and the draw of a ball from Urn U is carried out simultaneously it is not clear at all why the ambiguity-averse subject should assign a higher value to the random mixture between f_3 and f_4. This issue was scrutinized by Eichberger and Kelsey (1996b). They argue that the decision maker with non-additive beliefs will display a strict preference for randomization in the Anscombe and Aumann framework, but she will not do so in the setting of Savage. In the setting of Anscombe and Aumann, the randomizing device is incorporated in the structure of the outcome space. In the setting of Savage, the set of outcomes is an arbitrary set and does not need to be closed under convex combinations. Therefore, to model a randomizing device, one needs, for instance, to expand the original state space by forming a product space. One ordinate in that space describes all possible realisations of the randomizing device. Adopting such framework, Eichberger and Kelsey (1996b) showed that the ambiguity-averse decision maker with convex capacities will never display a strict preference for randomization. That is, in the 2-color experiment, the ambiguity-averse subject should be indifferent between all three acts h, f_3 and f_4. Sarin and Wakker (1992), Ghirardato (1997) and Klibanoff (2001b) have already observed that the choice of the one-stage setting of Savage as opposed to the two-stage setting of Anscombe and Aumann may lead to different predictions regarding Choquet preferences. Sarin and Wakker (1992) attribute these differences to distinction between one- and two-stage Choquet integrals, Ghirardato (1997) to failure of Fubini-theorem, while Klibanoff (2001b) attributes it to violation of stochastic independence. In the next chapter we explore the relationship between different randomization and ambiguity attitudes in an experimental study.

Recently, Epstein (1999) and Ghirardato and Marinacci (2002) suggested a comparative approach to define ambiguity attitudes. In this approach ambiguity attitudes are defined without requiring the existence of auxiliary concepts such as randomizing devices. For instance, when defining ambiguity aversion, first a comparative notion of ambiguity aversion is established, and then an absolute definition is derived. The comparative definition is based on the following idea: if a subject prefers an unambiguous bet to an ambiguous one, then a more ambiguity-averse subject will do the same. For the absolute definition, a class of ambiguity-neutral preferences is chosen. Then, a subject is ambiguity-averse if there is a benchmark preference order in this class such that the subject is more ambiguity-averse than this benchmark. Epstein (1999) and Ghirardato and Marinacci (2002) differ in their assumptions about the benchmark and what can be regarded as an unambiguous bet. For Epstein, unambiguous bets are bets for which payoffs depend on exogenously given unambiguous events. That is, events for which randomness is objectively known (for instance a fair coin, an urn with known proportion of balls, etc.). Ghirardato and Marinacci consider only constant acts as unambiguous acts. An act is constant, if it assigns the same outcome to all states of nature. Let $\mathcal{F}^{ua}$ be the set of unambiguous bets. Consider two preference relations $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{F}$. Then, $\succcurlyeq_2$ is said to be more ambiguity-averse than $\succcurlyeq_1$, if for any unambiguous bet $h \in \mathcal{F}^{ua}$ and any bet $e \in \mathcal{F}$:

$$h \succcurlyeq_1 (\succ_1) e \;\Rightarrow\; h \succcurlyeq_2 (\succ_2) e. \tag{3.4}$$

Let $\succcurlyeq^B$ be a benchmark order in a class of ambiguity-neutral preferences. Then, $\succcurlyeq$ is said to be ambiguity-averse if there exists a benchmark preference relation $\succcurlyeq^B$ such that for any $h \in \mathcal{F}^{ua}$ and any bet $e \in \mathcal{F}$:

$$h \succcurlyeq^B (\succ^B) e \;\Rightarrow\; h \succcurlyeq (\succ) e. \tag{3.5}$$

Conversely, $\succcurlyeq$ is said to be ambiguity-loving if there exists a benchmark preference relation $\succcurlyeq^B$ such that for any $h \in \mathcal{F}^{ua}$ and any bet $e \in \mathcal{F}$:

$$h \preccurlyeq^B (\prec^B) e \;\Rightarrow\; h \preccurlyeq (\prec) e. \tag{3.6}$$

If $\succcurlyeq$ is both ambiguity-averse and ambiguity-loving then $\succcurlyeq$ is ambiguity-neutral. With regard to the class of benchmark orders, Epstein (1999) assumes preferences to be probabilistically sophisticated in the sense of Machina and Schmeidler (1992). Ghirardato and Marinacci (2002) take subjective expected utility preferences in the sense of Savage (1954) as a benchmark.

In Section 4.2 we apply the comparative approach to derive subjects' ambiguity attitudes in the 2-color experiment without presupposing that they are indifferent between betting on white and on yellow when facing each Urn K and Urn U, respectively.

Chapter 4

Ambiguity and Randomization Attitudes

In this chapter we examine the relationship between different attitudes towards ambiguity and randomization in an experimental study.[1] In Section 3.3 we discussed this relationship from a theoretical point of view and showed that there are different predictions about randomization attitudes depending on how the randomization device is modeled. In view of such competing predictions it would be reasonable to look at real behavior. As Klibanoff (2001b, p.618) writes: “Any discussion of behavior, such as preference for randomization, that departs from what is considered standard raises some natural questions. First, descriptively, do actual decision makers behave in this way? Unfortunately, there are no studies that I am aware of that examine this issue”. To answer this question, we designed and implemented an experiment suited to study the relationship between different attitudes towards ambiguity and randomization.

The rest of this chapter is organized as follows. The next section describes our experimental design. In Section 4.2, we formally define randomization and ambiguity attitudes and derive our main hypotheses. Section 4.3 deals with the implementation. Section 4.4 presents the results. We conclude in Section 4.5.

[1]The content of this chapter is based on the article Dominiak and Schnedler (2010).

4.1 Experimental Design

In order to examine the relationship between ambiguity and randomization attitude, information about both attitudes from the same subject is required. We elicited the value of various bets which are based on three random devices. This section describes the random devices, the bets, and the elicitation mechanism.

4.1.1 Random Devices and Tickets

During the experiment, we use three different random devices: an urn with 20 table tennis balls of which half were white and the other half yellow (urn with known proportions or short: Urn K), an urn with 20 table tennis balls with an unknown proportion of yellow and white balls (short: Urn U), and a coin.

Subjects were informed that only white and yellow balls are used in the experiment. Urn K's contents were shown to the subjects before the experiment, while Urn U's contents were only revealed after the experiment. During the experiment, both urns were placed on a table in view of the subjects to demonstrate to them that the contents cannot be manipulated. For similar reasons, the coin was volunteered by one of the subjects and not by us.

In the experiment bets were called tickets and outcomes were expressed in Taler, our experimental currency unit. While subjects knew that they would be offered different tickets involving the three random devices, they did not know which or how many tickets they would face. In order to later identify subjects who regard the coin as fair, we introduced the following tickets.

1. **Head ticket,** h**:** 100 Taler are paid if the coin lands heads up and nothing otherwise.

2. **Tails ticket,** t**:** 100 Taler are paid if the coin lands tails up and nothing otherwise.

To elicit ambiguity attitude, we ask the subjects to evaluate the following tickets for Urn K.

3. **White ticket for Urn K**, w^K: 100 Taler are paid if the ball drawn from Urn K is white and nothing otherwise.

4. **Yellow ticket for Urn K**, y^K, 100 Taler are paid if the ball drawn from Urn K is yellow and nothing otherwise.

Ambiguity attitude is then detected by comparing the subject's certainty equivalent for these tickets with that of the following similar tickets for Urn U.

5. **Yellow ticket for Urn U**, y^U: 100 Taler are paid if the ball drawn from Urn U is yellow and nothing otherwise.

6. **White ticket for Urn U**, w^U: 100 Taler are paid if the ball drawn from Urn U is white and nothing otherwise.

The next ticket involves two random devices: the coin and Urn U. The subject always receives a ticket for Urn U. Whether this ticket will be yellow or white is determined by flipping the coin. Since the color of the ticket changes with the outcome of the coin toss, we use the name *chameleon ticket*.

7. **Chameleon ticket for Urn U,** c^U**:** If the coin lands heads up, the subject receives a yellow ticket for Urn U. If the coin lands tails up the subject receives a white ticket for Urn U.[2]

By comparing the certainty equivalent for the chameleon ticket with that of a yellow or white ticket for Urn U, we can infer whether a subject is randomization-loving.

For our predictions later, it must be possible to identify whether subjects are indifferent between yellow and white tickets on Urn U. This necessitates that subjects are asked about both tickets, which in principle allows them to hedge against ambiguity. The danger of hedging against ambiguity is that subjects no longer exhibit ambiguity

[2]Put differently, the subject receives 100 Taler in two cases: if the coin lands heads up and the ball drawn from Urn U is yellow and if the coin lands tails up and the ball drawn from Urn U is white. In the other two cases, the subject receives nothing.

aversion. We tried to reduce this danger by not informing subjects about the number and types of bets and switching the order in which tickets are presented for Urn U. Consequently, subjects do not know that there will be a hedging opportunity when evaluating the yellow ticket for Urn U. As we will see later, our method was successful in the sense that the proportion of ambiguity-averse subjects in our experiment is in line with that of similar experiments.

4.1.2 Eliciting Ticket Values

In order to elicit ticket values, we employ the following procedure. For each ticket, the subject had to make twenty choices. The first choice was between a ticket and a payment of 2.5 Taler. The second was between a ticket and a payment of 7.5 Taler etc. The payments offered to the subject increased in steps of 5 Taler until the last choice, in which the subject had to choose between a ticket and 97.5 Taler. The point at which the subject switches from the ticket to the payment then reveals the value of the ticket to the subject (up to 5 Taler). All of the subject's choices were implemented and affected the subject's payoff. To ensure independence, a separate draw was carried out for each ticket. The draws took place after all choices were made to avoid wealth effects.

Many experiments employ less time-consuming and laborious methods of paying subjects by combining choices over bets with additional randomization (see e.g. Holt and Laury, 2002; Becker, DeGroot, and Marschak, 1964). Such methods have also been used in experiments on ambiguity. For example, Hey, Lotito, and Maffioletti (2010) randomly select only one of the subjects' choices to be payoff-relevant, while Halevy (2007) employs the mechanism by Becker, DeGroot, and Marschak (1964). In the Becker-DeGroot-Marschak mechanism, the subject receives a ticket and states the certainty equivalent. Then, a random offer is generated and the subject has to sell the ticket if the offer exceeds the stated value.

Despite the considerable effort involved, we decided to pay all decisions rather than employing a mechanism that relies on additional randomization. We do so for two reasons. First, as Karni and Safra (1987) point out, a method based on additional randomization, such as the Becker-DeGroot-Marschak mechanism, is no longer guaranteed to elicit the true (subjective) value for subjects who violate the independence axiom.[3] Since ambiguity-averse subjects violate the independence axiom and we are interested in their valuations, we cannot use this mechanism.[4]

Second, had we introduced another source of randomness, all bets faced by the subject would have been compounded; none would have been purely based on the three devices that we are interested in (Urn K, Urn U, coin). By implementing all choices, we avoid that randomization attitude interacts with other sources of randomness.

4.2 Ambiguity and Randomization Attitude

In this section, we define randomization and ambiguity attitude, relate them to concepts from the literature, and derive empirical predictions. Let $\mathcal{L}$ be the set of tickets faced by subjects in our experiment. The binary relation $\succcurlyeq$ represents subjects preferences over $\mathcal{L}$. Denote by $\mu(l)$ a subject's certainty equivalent or value of ticket l in $\mathcal{L}$. For any two tickets k and l in $\mathcal{L}$, we say that subjects weakly prefer k to l, written $k \succcurlyeq l$, if and only if $\mu(k) \geqslant \mu(l)$.

4.2.1 Empirical Definitions

Comparing the certainty equivalents for the white and yellow ticket for Urn U with that for the chameleon ticket, we can classify subjects according to their randomization attitudes. Consider a subject who favors the yellow ticket y^U to the white ticket w^U for

[3] A similar observation has been made by Holt (1986). The Becker-DeGroot-Marschak mechanism also fails to elicit true valuations if the compound lottery axiom is violated (Segal, 1988).

[4] Apart from the theoretical argument, there is empirical evidence that preference reversals in measurements of ambiguity aversion occur when using the Becker-DeGroot-Marschak mechanism—see Trautmann, Vieider, and Wakker (2009).

urn U, i.e., $y^U \succcurlyeq w^U$. Such a subject is *randomization-averse* if she values the chameleon ticket even less than the white ticket w^U. Conversely, this subject is *randomization-loving* if she values the chameleon ticket even more than the yellow ticket y^U. If a subject values the chameleon ticket weakly more than the white ticket w^U but weakly less than the yellow ticket w^U, we say she is *randomization-neutral.* The next definition formalizes this idea, where s^U and t^U stands for the favorite and least favorite ticket on Urn U.

Definition 2 (**Randomization attitude**). *A subject with* $s^U \succcurlyeq t^U$*, where* $s^U, t^U \in \{y^U, w^U\}$*, is:* (i) *randomization-averse if* $s^U \succcurlyeq t^U \succ c^U$,
(ii) *randomization-neutral if* $s^U \succcurlyeq c^U \succcurlyeq t^U$,
(iii) *randomization-loving if* $c^U \succ s^U \succcurlyeq t^U$.

As will become clear later, this definition coincides with the idea of a preference for convex combinations embodied in Schmeidler's ambiguity aversion axiom (1989) for subjects who are indifferent between the yellow and white ticket on Urn U.

Subjects are typically regarded to be ambiguity-averse if they prefer betting on the urn with known proportions of yellow and white balls. Let us be more precise about this statement by considering a subject who weakly prefers the yellow to the white ticket on both urns ($y^K \succcurlyeq w^K$ and $y^U \succcurlyeq w^U$). Suppose this subject compares her two favorite tickets (y^K and y^U) and her two least favorite tickets (w^K and w^U) across Urn K and Urn U. Then this subject is *ambiguity-averse* if she weakly prefers the tickets on Urn K to those on Urn U for her favorite as well as least favorite tickets and her preference is strict in at least one case: $y^K \succcurlyeq y^U$ and $w^K \succcurlyeq w^U$ with at least one strict preference ($\succ$). Conversely, she is *ambiguity-loving* if she weakly prefers the tickets based on Urn U in both cases and strictly in at least one case: $y^U \succcurlyeq y^K$ and $w^U \succcurlyeq w^K$ with at least one strict preference ($\succ$). Finally, she is *ambiguity-neutral* if she either prefers another urn for her favorite tickets than for her least favorite tickets or if she is indifferent between urns for the favorite as well as least favorite tickets: $y^U \succ y^K$ but $w^K \prec w^U$, or $y^U \succ y^K$ but $w^K \succ w^U$, or $y^U \sim y^K$ and $w^U \sim w^K$. The following definition generalizes this idea to arbitrary preferences, where q^K and r^K stands for the favorite and least favorite ticket on Urn K, and s^U and t^U for the favorite and least favorite ticket on Urn U. The defined order is complete: each preference is either ambiguity-averse, ambiguity-loving, or ambiguity-neutral.

Definition 3 (**Ambiguity attitude**). *A subject with* $q^K \succcurlyeq r^K$ *and* $s^U \succcurlyeq t^U$*, where* $q^K, r^K \in \{y^K, w^K\}$ *and* $s^U, t^U \in \{y^U, w^U\}$*, is:*

(i) *ambiguity-averse if* $q^K \succcurlyeq s^U$ *and* $r^K \succcurlyeq t^U$ *with at least one* $(\succ)$,

(ii) *ambiguity-loving if* $q^K \preccurlyeq s^U$ *and* $r^K \preccurlyeq t^U$ *with at least one* $(\prec)$,

(iii) *ambiguity-neutral otherwise, i.e.,*

$$\begin{aligned} &\text{if} \quad q^K \sim s^U \quad \text{and} \quad r^K \sim t^U, \\ &\text{or} \quad q^K \succ s^U \quad \text{and} \quad r^K \prec t^U, \\ &\text{or} \quad q^K \prec s^U \quad \text{and} \quad r^K \succ t^U. \end{aligned}$$

In Section 3.3 we briefly described the comparative approach to define ambiguity attitudes proposed by Epstein (1999) and Ghirardato and Marinacci (2002). It can be shown that the comparative approach yields the above definition of ambiguity attitudes when the class subjective expected utility preferences is taken as a benchmark and when bets defined on the urn with known proportions of balls are regarded as unambiguous bets (see Section 4.6 for the proof).

Proposition 4.1. *Given that yellow and white ticket defined on urn K are viewed as unambiguous,* $y^K, w^K \in \mathcal{F}^{ua}$*, and taking subjective expected utility preferences as the benchmark, the two-stage approach yields the definition of ambiguity attitude from Definition 3.*

4.2.2 Predictions

The general definitions allow for any combination of ambiguity and randomization attitude. For example, a subject may in principle be ambiguity-neutral but like randomization or it may be averse to ambiguity and randomization. This section uses existing theoretical models to restrict the relationship between ambiguity and randomization attitude and derive predictions.

In order to represent ambiguity aversion, a large class of models appeals to Schmeidler's notion (1989) that a mixture between two bets is preferred to each of the bets itself. In the specific framework used by Schmeidler, bets are mappings from events to probability distributions over the set of payoffs, so that the convex combination of two bets, f and g: $\alpha f + (1-\alpha)g$ with $\alpha \in (0,1)$ is well defined. Schmeidler calls a subject with $f \succcurlyeq g$ ambiguity-averse if

$$\alpha f + (1-\alpha)g \succcurlyeq g. \tag{4.1}$$

Intuitively, smoothing utility across ambiguous events makes an ambiguity-averse subject better off. In perfect analogy, subjects are ambiguity-loving if $\alpha f + (1-\alpha)g \preccurlyeq f$. For subjects who violate the independence axiom, preferences are strict.

Taking 'yellow' and 'white' to be events and the probability distribution in each event to result from the coin flip, the chameleon ticket is a convex combination in the sense of Schmeidler. The axiom then means that ambiguity-averse subjects strictly prefer the chameleon ticket, i.e., the mixture of two bets, to the least favorite ticket on Urn U. Likewise, ambiguity-loving subjects should prefer their favorite ticket on Urn U to the chameleon ticket.

For subjects who are indifferent between white and yellow on Urn U, $y^U \sim w^U$, Schmeidler's notion fully coincides with our definition of randomization attitude (see Definition 2). Based on the various models that appeal to this notion in order to explain ambiguity attitude, we hence predict ambiguity-averse subjects to be randomization-loving and ambiguity-loving subjects to be randomization-averse (given $y^U \sim w^U$).

Hypothesis 1

For subjects who are indifferent between the yellow and white ticket on Urn U, $y^U \sim w^U$, ambiguity and randomization attitude are negatively associated: ambiguity-averse subjects are randomization-loving and vice versa.

As the null hypothesis, we consider that ambiguity and randomization attitude are not associated.

If ambiguity aversion is modeled using Choquet expected utility models with convex capacities, the relationship between ambiguity and randomization attitude depends on whether the randomization device is modeled as part of the consequence space (C-approach) or as a part of an extended state space (S-approach). Eichberger and Kelsey (1996b) show that ambiguity-averse decision makers who are indifferent between two bets based on an uncertain urn, $y^U \sim w^U$, and who regard the randomization device as fair, $h \sim t$, are randomization-loving in the C-approach but are randomization-neutral in the S-approach. This directly leads to the hypotheses.

Hypothesis $\mathbf{2_C}$

Ambiguity-averse subjects with $y^U \sim w^U$ *and* $h \sim t$ *are randomization-loving.*

Hypothesis $\mathbf{2_S}$

Ambiguity-averse subjects with $y^U \sim w^U$ *and* $h \sim t$ *are randomization-neutral.*

We test these two alternatives against the null hypothesis that ambiguity-averse subjects with $y^U \sim w^U$ and $h \sim t$ are equally likely to be randomization-neutral and randomization-loving.

4.3 Implementation

We ran a total of 5 sessions with 90 subjects. All sessions were conducted in the experimental laboratory at the University of Mannheim in September 2008. Subjects were primarily students who were randomly recruited from a pool of approximately 1000 subjects using an e-mail recruitment system. Each subject only participated in one of the sessions. Ticket values were elicited electronically using the software z-tree (Fischbacher, 2007).

After the subjects' arrival at the laboratory, they were randomly seated at the computer terminals. Instructions were read out loud and ticket types were practically explained. Then, the subjects were given time to study the instructions (see appendix for a translation). Finally, they were asked to answer a series of questions to test their understanding of the instructions. During all this time, subjects could ask the experimenters clarifying questions. This part lasted about 30 minutes. It was followed by the evaluation of the tickets. In order to simplify the input for subjects, we programmed a slider that allowed them to specify their value for each ticket. The program then automatically selected choices that were consistent with this ticket value. Using the slider was not obligatory and a subject could arbitrary alter its choice until he or she decided to finish evaluation of a specific ticket (see Figure 1 in the appendix for a screen shot). After the evaluation of tickets, we asked subjects questions about their demographics and attitudes towards ambiguity. We also gave them some problems to test their statistics knowledge and cognitive ability. Subjects took about 30 minutes for this second part. The last and final part required drawing balls and flipping coins in order to determine payoffs. With 8 types of tickets and twenty choices between ticket and fixed payment for each type, subjects could obtain up to 160 tickets. This last part required roughly 30 minutes so that the whole experiment lasted about 90 minutes.

At the end of the experiment, we paid each subject privately in cash. All payoffs were initially explained in Taler that were later converted using the rate of 100 Taler=10 cents. Subjects earned on average 11.35 Euro.

4.4 Results

Two subjects violated transitivity in their choices, which leaves us with 88 independent observations. In line with previous experimental studies (see Camerer and Weber, 1992), many subjects exhibit the Ellsberg paradox: a share of about 55% prefer betting on the urn with known proportions, while ca. 9% prefer betting on the urn with unknown proportions, and roughly 36% are indifferent.

4.4.1 Main findings

In order to formally check Hypothesis 1, we restrict our sample to subjects who value white and yellow ticket on Urn U equally, so that Schmeidler's notion of mixture preference co-incides with the definition of randomization attitude. Since about a third of the subjects prefer a ticket of one color on Urn U, the analysis is based on 53 observations.

Result 1. *For subjects who value white and yellow tickets on Urn U equally, ambiguity and randomization attitude are not negatively associated.*

From the literature, we expect ambiguity-averse subjects to be randomization-loving and ambiguity-loving subjects to be randomization-averse. Accordingly, observations should lie on the diagonal from the top-left to the bottom-right in Table 4.1. While 19 out of the 53 observations exhibit this relationship, about two thirds of the observations lie off the diagonal. Using Fisher's exact test, we cannot reject the null hypothesis that there is no association at any conventional level (p-value=0.118).[5] Moreover, the number of observations that lie on the other diagonal and are consistent with a positive relationship is higher (25 out of 53). Accordingly, Goodman and Kruskal's γ as well as Kendall's τ_b, which can be used to measure the association between the two ordinal scaled attitudes, are both positive. If there is any tendency to reject independence it is hence in favor of a positive rather than a negative relationship.

Recall that S- and C-approach lead to diverging predictions about the randomization attitude of ambiguity-averse subjects, regard the coin as fair, and value white and yellow ticket on Urn U equally. This concerns 29 subjects in our sample. The C-approach predicts these subjects to like randomization, while the S-approach predicts them to be randomization-neutral. The following result is based on the 20 observations that are in line with one of these predictions.

[5]Neither Pearson's χ^2 (p-value=0.163) nor the likelihood ratio test (p-value=0.083) are significant.

		Ambiguity Attitude			
		Averse	*Neutral*	*Loving*	*Total*
Randomization Attitude	*Loving*	6	0	1	7
	Neutral	17	12	2	31
	Averse	12	2	1	15
	Total	35	14	4	53

Table 4.1: Ambiguity and randomization attitude for subjects who value white and yellow ticket on Urn U equally

Result 2. *Consider ambiguity-averse subjects who regard the coin as fair and value the yellow and white ticket on Urn U equally. These subjects are more likely to be randomization-neutral rather than randomization-loving.*

Sixteen of the 20 subjects are randomization-neutral, while four prefer randomization—see Figure 4.1. The hypothesis that subjects are equally likely to be randomization-loving or neutral can be rejected at any conventional level (The respective binomial test has a p-value below 0.01): a significantly larger fraction of subjects is randomization-neutral.

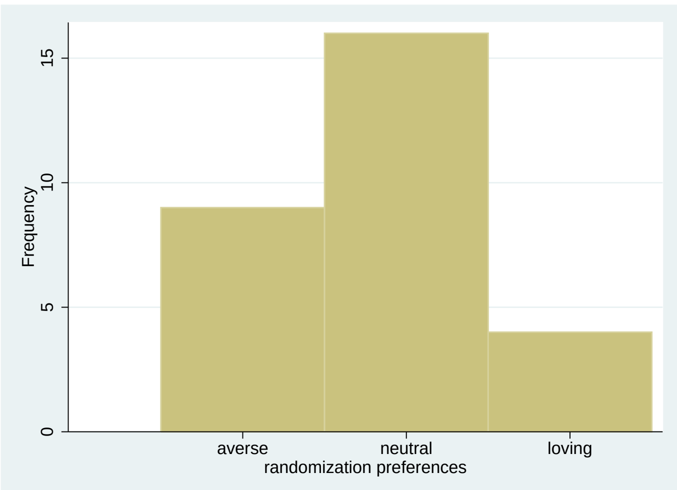

Figure 4.1: Randomization attitudes of ambiguity-averse subjects who regard the coin as fair and value white and yellow ticket on Urn U equally

This result can be extended to ambiguity-loving subjects, who are supposed to dislike randomization according to the C-approach and to be randomization-neutral according to the S-approach. Two ambiguity-loving subjects are randomization-neutral and one is randomization-averse. Overall, 18 of 23 observations are in line with the S-approach and only 5 with the C-approach. Again, a uniform distribution of randomization attitudes can be rejected in favor of the predictions consistent with the S-approach at any conventional level (p-value below 0.01).

4.4.2 Robustness

The theoretical results, which underpin Hypothesis 1 and 2, only apply to subjects with specific preferences. Consequently, Result 1 and 2 are based on a selected sample of subjects, which may not only differ by their preferences but by other characteristics.

We check whether any selection on observables has taken place by running two probit regressions. Hypothesis 1 requires subjects to be indifferent between the yellow and white ticket on Urn U. This indifference, however, does not seem to be related to observables: the null hypothesis that no observable affects the probability of being indifferent cannot be rejected (p-value of the likelihood ratio test: 0.43, see Table 2 in the appendix). For Hypothesis 2, subjects must additionally regard the coin as fair. This time there is some indication that observables affect selection (p-value for the likelihood ratio test: 0.04). More specifically, subjects who correctly compute the probability of two independently thrown dice (variable: stats knowledge 2) are significantly more likely to be in the sample (see Table 3 in the appendix). There is, however, no reason why statistically more literate subjects should be less inclined to prefer randomization.

The subjects on which we test our hypotheses may also differ in unobservable ways from our full sample. The independence between ambiguity and randomization attitude could, for example, be driven by the fact that subjects who are indifferent between white and yellow tickets on Urn U systematically differ from other subjects. In order to refute this idea, we re-examine the relationship between ambiguity and randomization attitude without restricting attention to certain preferences. Of course, Hypotheses 1 and 2 no longer apply in this case. If, however, results are similar, we can be confident that they do not hinge on an alternative explanation such as a general trait to value tickets equally. Table 4.2 exhibits the attitudes when all subjects are considered. Both findings are confirmed. First, the null hypothesis that ambiguity aversion and random preference are unrelated cannot be rejected (p-value of Fisher's exact test: 0.18). As before, the data suggests that ambiguity aversion is associated positively with randomization aversion. Second, ambiguity-averse subjects tend to be randomization-neutral rather than randomization-loving and ambiguity-loving subjects are more likely to be randomization-neutral than to be randomization-averse (p-value for the two-sided binomial test is below 0.01). This robustness of results gives us some confidence that they are not driven by selection effects.

4.4.3 Other Findings

In addition to these results, which directly relate to our hypotheses, we also want to report on two additional and unexpected findings.

The first finding concerns randomization- and ambiguity-averse subjects. We expected to find very few of them because they are not backed by the most prevalent models of ambiguity-aversion.

Result 3. *A non-negligible fraction of ambiguity-averse subjects dislikes randomization.*

Of the 48 ambiguity-averse subjects, 14 express a dislike for randomization (see Table 4.2). If we restrict attention to subjects for whom behavior can be predicted using the S- or C-approach because they regard the coin as fair and have no color preference on Urn U, a similar picture emerges: 9 out of 29 ambiguity-averse subjects prefer the pure tickets over the mixture—see Figure 4.1. In both cases, the share is statistically not distinguishable at any conventional level from the naive prediction by someone who does not know any of these theories and expects randomization aversion to occur in a

		Ambiguity Attitude			
		Averse	*Neutral*	*Loving*	*Total*
Randomization Attitude	*Loving*	10	2	2	14
	Neutral	24	23	6	53
	Averse	14	5	2	21
	Total	48	30	10	88

Table 4.2: Ambiguity and randomization attitude: all subjects

third of the cases. The observed combination of randomization and ambiguity aversion is puzzling. The respective subjects prefer to know whether the ticket, which they receive, is white or yellow—although they are indifferent between receiving a white and a yellow ticket. Possible reasons are that knowing the color has a value in itself to these subjects, that they assign lower values to tickets when complexity is involved, or that they dislike the loss of control associated with the coin.[6]

In order to accommodate the behavior of these subjects, one would need a more general model which does not exogenously assume a specific relationship between ambiguity and randomization attitude. Classes of preferences that do not engender such specific relationship are the source-dependent preferences axiomatized by Chew and Sagi (2008), the vector expected utility preferences by Siniscalchi (2009b) and the monotonic, Bernoullian and continuous preferences by Ghirardato and Siniscalchi (2010).

Our second finding is related to a theoretical result by Klibanoff (2001b). Klibanoff shows that if a randomizing device is stochastically independent and Choquet-expected utility preferences are modeled in the S-approach, preferences cannot exhibit ambiguity-aversion. This implies for our context that subjects whose preferences can be modeled

[6]Keren and Teigen (2008) argue that such decision makers like to maintain control. Dittmann, Kübler, Maug, and Mechtenberg (2008) find that experimental subjects are willing to pay a premium for exerting the right to vote even if the probability that this affects the outcome is very low. On the other hand, Cettolin and Riedl (2008) observe that subjects with incomplete preferences prefer a random draw when having to decide.

using the S-approach because they are ambiguity-averse and randomization-neutral should regard the coin to be correlated with Urn U. In order to test this, we constructed a bet in which a ball is drawn from Urn U; the subject then receives a head ticket if the ball is yellow and its certainty equivalent of a head ticket if the ball is white. Subjects who view coin and ball draw as independent should attach the same value to this bet, which we call *combination ticket*, and a head ticket. We restrict attention to subjects who regard the coin as fair, value white and yellow tickets on Urn U equally, and are randomization-neutral. Following Klibanoff's argument, we expect these subjects to be less likely to attach different values to the combination and head ticket if they are ambiguity-neutral. Indeed, the respective share of subjects is lower among ambiguity-neutral subjects (20%) than amongst other ambiguity-averse subjects (31%); however, the difference is not significant at any conventional level (p-value of one-sided two-sample test of proportion: 0.26). More surprising, the proportion of *all* subjects who value the head ticket more than the combination ticket is 37%. Put differently, these subjects prefer a head ticket to a mixture of head ticket and its certainty equivalent. While a possible explanation is that subjects regard coin throw and ball draw as correlated, there is an interesting link between this finding and randomization aversion: subjects who favor the heads to the combination ticket also tend to favor tickets of a specific color to the chameleon ticket (Kendall's τ_b=0.1966, p-value: 0.0559). A first tentative conclusion may thus be that both results are driven by the same explanation, e.g., a contempt for complexity.

4.5 Summary

We started our analysis with the classical observation from the two-color experiment by Ellsberg (1961): individuals prefer to bet in situations about which they are better informed. Existing explanations for such behavior often rely on the idea that access to an objective randomization device mitigates the problem of lacking information.

Accordingly, ambiguity-averse individuals are supposed to prefer randomization. The data from our experiment, however, does not support this view: there is no negative association between ambiguity and randomization attitude. Ambiguity-averse subjects are more likely to be randomization-neutral than randomization-loving. This behavior can be explained within Choquet-expected utility theory, when the randomization device is modeled within the Savage setup rather than using the consequence space in the tradition of Anscombe-Aumann. However, we also observe a considerable number of ambiguity-averse subjects who exhibit a contempt for randomization. This observation indicates that for many subjects, the randomization device does not reduce but enhances the problem of missing information.

4.6 Proof

Proof of Proposition 4.1. Let $\mathcal{L}$ be the set of tickets faced by subjects in our experiment and let $\succcurlyeq$ be a binary relation that represents subjects' preferences over $\mathcal{F}$. For any bet $k, l, m \in \mathcal{L}$ we write $k \succcurlyeq \{l, m\}$ to denote $k \succcurlyeq l$ and $k \succcurlyeq m$. Throughout, we consider a subject with the following preferences:

$$q^K \succcurlyeq r^K \;\text{ and }\; s^U \succcurlyeq t^U, \tag{4.2}$$

where $q^K, r^K \in \{y^K, w^K\}$ and $s^U, t^U \in \{y^U, w^U\}$. Let $Q^K, R^K \in \{Y^K, W^K\}$ be the corresponding (unambiguous) events to which subjects assign probabilities $\pi[Y^K]$ and $\pi[W^K]$, while , $S^U, T^U \in \{Y^U, W^U\}$ are the ambiguous events.

The preferences of a subjective expected utility maximizer fall into one of the following three sets:

$$q^K \sim^B s^U \succcurlyeq^B t^U \sim^B r^K, \tag{4.3}$$

$$q^K \succ^B s^U \succcurlyeq^B t^U \succ^B r^K, \tag{4.4}$$

$$s^U \succ^B q^K \succcurlyeq^B r^K \succ^B t^U. \tag{4.5}$$

These three sets provide the benchmark preferences. For the proof, we decompose all preferences into three classes: (i) $q^K \succcurlyeq s^U$ and $r^K \succcurlyeq t^U$ with at least one strict preference relation ($\prec$) , (ii) $q^K \preccurlyeq s^U$ and $r^K \preccurlyeq t^U$ with at least one strict preference relation ($\prec$), and (iii) $q^K \sim s^U$ and $r^K \sim t^U$, or $q^K \succ s^U$ and $r^K \prec t^U$, or $q^K \prec s^U$ and $r^K \succ t^U$. We now show that the two-stage approach implies ambiguity-aversion for the first class (Step 1), ambiguity-love for the second class (Step 2), and ambiguity-neutrality for the last class (Step 3). These are exactly the ambiguity attitudes from Definition 3.

Step 1. $q^K \succcurlyeq s^U$ and $r^K \succcurlyeq t^U$ with at least one strict preference relation ($\succ$). In this step, we examine two cases: $q^K \sim r^K$ and $q^K \succ r^K$.

Case 1: $q^K \sim r^K$. In this case, we obtain:

$$q^K \sim r^K \succcurlyeq s^U \succcurlyeq t^U, \tag{4.6}$$

with at least one strict preference. Take $\succcurlyeq^B$ as in (4.3) with $\pi^B(Q^K) = \pi(Q^K)$ and:

$$q^K \sim^B s^U \sim^B t^U \sim^B r^K. \tag{4.7}$$

Comparing $\succcurlyeq$ from (4.6) with $\succcurlyeq^B$ as in (4.7), we get:

$$\begin{aligned} q^K \sim^B \{r^K, s^U, t^U\} \;&\Rightarrow\; q^K \sim \{r^K\} \succcurlyeq \{s^U\} \succcurlyeq \{t^U\}, \\ r^K \sim^B \{q^K, s^U, t^U\} \;&\Rightarrow\; r^K \sim \{q^K\} \succcurlyeq \{s^U\} \succcurlyeq \{t^U\}, \end{aligned}$$

where at least one of the weak preference is strict in each row. Thus, there exists $\succcurlyeq^B$ such that $\succcurlyeq$ is more ambiguity-averse then $\succcurlyeq^B$ according to (3.5). Furthermore, there does not exist $\succcurlyeq^B$ such that $\succcurlyeq$ is more ambiguity-loving then $\succcurlyeq^B$.

Case 2: $q^K \succ r^K$. In this case, one of the following can occur:

$$q^K \succ r^K \succcurlyeq s^U \succcurlyeq t^U, \text{ or} \tag{4.8}$$

$$q^K \succcurlyeq s^U \succ r^K \succcurlyeq t^U, \tag{4.9}$$

with at least one strict preference in each case. As a benchmark, take $\succcurlyeq^B$ as in (4.3) with $\pi^B(Q^K) = \pi(Q^K)$ and: $q^K \sim^B s^U \succ^B t^U \sim^B r^K$. Comparing this $\succcurlyeq^B$ with $\succcurlyeq$ as in (4.8), we get:

$$\begin{aligned} q^K \sim^B \{s^U\} \succ^B \{r^K, t^U\} \;&\Rightarrow\; q^K \succ \{r^K, s^U, t^U\}, \\ r^K \sim^B \{t^U\} \;&\Rightarrow\; r^K \succcurlyeq \{s^U, t^U\}. \end{aligned}$$

Analogously, the comparison with $\succcurlyeq$ as in (4.9), yields:

$$q^K \sim^B \{s^U\} \succ^B \{r^K, t^U\} \;\; \Rightarrow \;\; q^K \succcurlyeq \{s^U\} \succcurlyeq \{r^K, t^U\},$$
$$r^K \sim^B \{t^U\} \;\; \Rightarrow \;\; r^K \succ \{t^U\}.$$

Thus, for preference ordering $\succcurlyeq$ as in (4.8) and as in (4.9), there exists $\succcurlyeq^B$ such that $\succcurlyeq$ is more ambiguity-averse then $\succcurlyeq^B$ according to (3.5) and again, there is no such $\succcurlyeq^B$ for which $\succcurlyeq$ is more ambiguity-loving then $\succcurlyeq^B$. Summarizing both cases, we have seen that for $q^K \succcurlyeq s^U$ or $r^K \succcurlyeq t^U$ with at least one strict preference relation ($\succ$), $\succcurlyeq$ is ambiguity-averse, which coincides with (i) in Definition 3.

Step 2. $q^K \preccurlyeq s^U$ and $r^K \preccurlyeq t^U$ with at least one strict preference relation ($\prec$). Again, we consider two cases: $q^K \sim r^K$ and $q^K \succ r^K$.

Case 1: $q^K \sim r^K$. In this case, we obtain:

$$s^U \succcurlyeq t^U \succcurlyeq q^K \sim r^K, \tag{4.10}$$

with at least one strict preference. Take $\succcurlyeq^B$ with $\pi^B(Q^K) = \pi(Q^K)$ as in (4.3) such that:

$$q^K \sim^B s^U \sim^B t^U \sim^B r^K. \tag{4.11}$$

Comparing the respective $\succcurlyeq^B$ with $\succcurlyeq$ from (4.10), we obtain:

$$q^K \sim^B \{r^K, s^U, t^U\} \;\; \Rightarrow \;\; q^K \sim \{r^K\} \preccurlyeq \{s^U\} \preccurlyeq \{t^U\},$$
$$r^K \sim^B \{q^K, s^U, t^U\} \;\; \Rightarrow \;\; r^K \sim \{q^K\} \preccurlyeq \{s^U\} \preccurlyeq \{t^U\},$$

where at least one of the weak preference is strict in each row. Thus, there exists $\succcurlyeq^B$ such that $\succcurlyeq$ is more ambiguity-loving then $\succcurlyeq^B$ and there exist no $\succcurlyeq^B$ such that $\succcurlyeq$ is more ambiguity-averse then $\succcurlyeq^B$. Hence, $\succcurlyeq$ is ambiguity-loving according to (3.6).

Case 2: $q^K \succ r^K$. In this case, one of the following can occur:

$$s^U \succcurlyeq t^U \succcurlyeq q^K \succ r^K, \text{ or} \tag{4.12}$$

$$s^U \succcurlyeq q^K \succ t^U \succcurlyeq r^K, \tag{4.13}$$

with at least one strict preference in each case. Take $\succcurlyeq^B$ with $\pi^B(Q^K) = \pi(Q^K)$ as in (4.3) such that:

$$q^K \sim^B s^U \succ^B t^U \sim^B r^K. \tag{4.14}$$

Comparing this $\succcurlyeq^B$ with $\succcurlyeq$ from (4.12), we obtain:

$$\begin{aligned} q^K \sim^B \{s^U\} &\Rightarrow q^K \preccurlyeq \{t^U\} \preccurlyeq \{s^U\}, \\ r^K \sim^B \{t^U\} \prec \{s^U, q^K\} &\Rightarrow r^K \prec \{q^K, t^U, s^U\}. \end{aligned}$$

Comparing the same benchmark with $\succcurlyeq$ from (4.13), we get:

$$\begin{aligned} q^K \sim^B \{s^U\} &\Rightarrow q^K \preccurlyeq \{s^U\}, \\ r^K \sim^B \{t^U\} \prec \{s^U, q^K\} &\Rightarrow r^K \preccurlyeq \{t^U\} \prec \{q^K, s^U\}. \end{aligned}$$

Thus, in both cases, there exists $\succcurlyeq^B$ such that $\succcurlyeq$ is more ambiguity-loving then $\succcurlyeq^B$ and there exists no such $\succcurlyeq^B$ for which $\succcurlyeq$ is more ambiguity-averse then $\succcurlyeq^B$. Thus, we conclude that $\succcurlyeq$ is ambiguity-loving according to (3.6). Hence, if $q^K \preccurlyeq s^U$ or $r^K \preccurlyeq t^U$ with at least one strict preference relation ($\prec$), then $\succcurlyeq$ is ambiguity-loving, which coincides with (ii) in Definition 3.

Step 3. $q^K \sim s^U$ and $r^K \sim t^U$, or $q^K \succ s^U$ and $r^K \prec t^U$, or $q^K \prec s^U$ and $r^K \succ t^U$.

Suppose now that $q^K \sim s^U$ and $r^K \sim t^U$, or $q^K \succ s^U$ and $r^K \prec t^U$, or $q^K \prec s^U$ and $r^K \succ t^U$. Then one of the following can occur:

$$q^K \sim s^U \succcurlyeq t^U \sim r^K, \tag{4.15}$$

$$q^K \succ s^U \succcurlyeq t^U \succ r^K, \tag{4.16}$$

$$s^U \succ q^K \succcurlyeq r^K \succ t^U. \tag{4.17}$$

Take $\succcurlyeq^B$ with $\pi^B(Q^K) = \pi(Q^K)$ as in (4.3), in (4.4) and in (4.5). Any $\succcurlyeq$ as in (4.15), in (4.16) and in (4.17) is order equivalent with $\succcurlyeq^B$ as in (4.3), in (4.4) and in (4.5), respectively. Thus, for any $\succcurlyeq$ as in (4.15), in (4.16) and in (4.17) there exists $\succcurlyeq^B$ such that both is true: $\succcurlyeq$ is more ambiguity-averse than $\succcurlyeq^B$ and $\succcurlyeq$ is more ambiguity-loving than $\succcurlyeq^B$. Therefore, $\succcurlyeq$ is ambiguity-neutral according to (3.5) and (3.6).

$\square$

Chapter 5

Dynamic Ellsberg Urn Experiment

In this chapter we explore experimentally dynamic choice behavior under ambiguity.[1] In order to apply ambiguity models to dynamic decision problems, a theory of how to update preferences in the face of new information is needed. Recently, several approaches for updating ambiguity-sensitive preferences have been proposed. As opposed to the subjective expected utility theory, in which Bayesian updating is a logical consequence of Savage's Sure-Thing-Principle, updating of non-expected utility preferences is rather a complicated task. The reason is the following one: On the one hand, sensitivity towards ambiguity, as manifested in the Ellsberg-type choices, entails violation of the Sure-Thing-Principle (see Section 3.1). On the other hand, the Sure-Thing-Principle is a property of preferences implied by dynamic consistency and consequentialism (see Lemma 1, Section 2.3). Consequently, if one is interested in a theory of updating ambiguity-sensitive preferences, then either consequentialism or dynamic consistency (or both) must be relaxed in some respect. The existing theoretical literature on dynamic extensions of subjective ambiguity models has not yet reached consensus on whether dynamic consistency or consequentialism is the more plausible assumption. Sarin and Wakker (1998a), Epstein and Schneider (2003) and Eichberger, Grant, and Kelsey (2005) show that it is possible to maintain both, however, at the cost of constraining the analysis to a fixed collection of events and by imposing restrictions on

[1]The content of this chapter is based on the article Dominiak, Dürsch, and Lefort (2010).

subjective beliefs. Other theories focus on one property. For instance, Gilboa and Schmeidler (1993), Pires (2002), Eichberger, Grant, and Kelsey (2007) and Siniscalchi (2010) drop dynamic consistency and maintain consequentialism, whereas Hanany and Klibanoff (2007) and Eichberger and Kelsey (1996a) advocate dynamic consistency and drop consequentialism.[2] More recently, Al-Najjar and Weinstein (2009) and Siniscalchi (2009a) have provided a very insightful discussion on the normative appeal of these different approaches to dynamic choices under ambiguity. In this chapter we conduct a dynamic version of the classical 3-color Ellsberg experiment which allows us to differentiate between consequentialism and dynamic consistency and test whether subjects behave consistently with either of these two properties.

The first experimental evidence on dynamically inconsistent behavior in the presence of risk was reported by Tversky and Kahneman (1981). More recently, the properties of dynamic consistency and consequentialism were tested by Cubitt, Starmer, and Sugden (1998) and by Busemeyer, Weg, Barkan, Li, and Ma (2000) in a framework with exogenously given probability distributions. In the presence of ambiguity, there are two other contributions which investigate dynamic choice behavior. Cohen, Gilboa, Jaffray, and Schmeidler (2000) test the Full-Bayesian versus the Maximum-Likelihood updating rule using a design very similar to ours. However, in contrast to our paper, they assume that subjects always maintain consequentialism. Maher and Kashima (1997) run a series of six differently framed Ellsberg urns to test behavior of subjects who display ambiguity aversion. They use questions similar to those in this study, but do not test for dynamic consistency or consequentialism (they assume "separability", which is close to consequentialism, throughout most of the paper). They also use some strong implicit assumptions, e.g. Bayesian updating for ambiguity-averse subjects and a very

[2]There are other approaches relaxing consequentialism. Lehrer (2005) characterizes updating rules using geometric properties of non-additive conditional expectations. Ozdenoren and Peck (2007) elucidate the concept of dynamic consistency by interpreting the dynamic Ellsberg experiment as a game against a malevolent nature.

strong form of ambiguity aversion, when conducting their analysis. Both studies do not incentivize decisions through payment to subjects. We do incentivize and conduct the analysis in a model-free setup.

The rest of this chapter is organized as follows. In the section, the dynamic extension of Ellsberg's 3-color experiment is presented. In Section 5.2 the experimental design is described. In Section 5.3 the empirical results are presented and discussed. Finally, we conclude in Section 5.4.

5.1 Dynamic 3-color experiment

In this section we show that it is impossible for ambiguity-sensitive preferences to satisfy both consequentialism and dynamic consistency on all events. For this purpose we consider a dynamic version of the classical 3-color Ellsberg. In the dynamic version, there is an interim stage at which subjects are informed whether or not the ball drawn is green. Conditional on the revealed information subjects are allowed to update their preferences. As a mind experiment it was described by Ghirardato, Maccheroni, and Marinacci (2008) and Siniscalchi (2010).

Dynamic 3-color Experiment. *Consider Ellsberg's 3-color experiment described in Section 3.1. There are two stages, ex-ante and interim stage. At the ex-ante stage subject face two choice situations, I and II, with the same information about the composition of the urn as it was described initially. At the interim stage subjects face again two choice problems, III and VI, with an additional information about the outcome of a random draw from the urn. Bets in the third choice situation, III, are identical to those in I and bets in the fourth choice problem, IV, are identical to those in II with one exception, namely that subjects are informed that the randomly drawn ball is not green, i.e. $Y \cup B$. For the sake of completeness we summarize the two relevant choice problems III and IV in Table 5.1.*

Depending on the choices subjects made at ex-ante and interim stage, one can conclude whether subjects behave consistently with either dynamic consistency or consequen-

		Ball is not Green!		
		Yellow	*Blue*	*Green*
Choice III	f_1	4	0	0
	f_2	0	4	0
Choice IV	f_3	4	0	4
	f_4	0	4	4

Table 5.1: Ellsberg's 3-color experiment

tialism. Table 5.2 depicts implications on dynamic consistency and consequentialism resulting from choices made ex-ante and choices made on the interim stage. The columns refer to choices made in the static Ellsberg experiment. Correspondingly, rows refer to choices made after being informed that the ball drawn is not green.

		Ambiguity Attitude			
		Averse	*Neutral*		*Loving*
		$(f_1; f_4)$	$(f_1; f_3)$	$(f_2; f_4)$	$(f_1; f_4)$
Interim Choices	$(f_1; f_4)$	$DC, \neg C$	$\neg DC, \neg C$	$\neg DC, \neg C$	$\neg DC, \neg C$
	$(f_1; f_3)$	$\neg DC, C$	DC, C	$\neg DC, C$	$\neg DC, C$
	$(f_2; f_4)$	$\neg DC, C$	$\neg DC, C$	DC, C	$\neg DC, C$
	$(f_2; f_3)$	$\neg DC, \neg C$	$\neg DC, \neg C$	$\neg DC, \neg C$	$DC, \neg C$

Table 5.2: Dynamic consistency and consequentialism in the dynamic 3-color experiment

Consider for instance an ambiguity-averse subject (first column with choices $(f_1; f_4)$). At the interim stage there are again four possible patterns of conditional (strict) preferences, $Y \cup B$. As we will see some of them respect dynamic consistency, (DC), and other ones respect consequentialism, (C), but not both.

(DC) Since $f_1 = f_2$ and $f_3 = f_4$ on the event G and they differ only on states in the conditional event $Y \cup B$, dynamic consistency requires that the conditional preferences have to respect the choices made ex-ante, i.e.:

$$f_1 \succ f_2 \implies f_1 \succ_{Y \cup B} f_2, \text{ and}$$
$$f_4 \succ f_3 \implies f_4 \succ_{Y \cup B} f_2.$$

(C) Since $f_1 = f_3$ and $f_2 = f_4$ on the event $Y \cup B$ and they differ only outside of that event, consequentialism requires that the subject must be conditionally indifferent between them, i.e. $f_1 \sim_{Y \cup B} f_3$ and $f_2 \sim_{Y \cup B} f_4$. Furthermore, it implies that either (*i*) or (*ii*) must be true:

$$(i) \quad f_1 \succ_{Y \cup B} f_2 \implies f_3 \succ_{Y \cup B} f_4 \text{ and vice versa, or}$$
$$(ii) \quad f_1 \prec_{Y \cup B} f_2 \implies f_3 \prec_{Y \cup B} f_4 \text{ and vice versa.}$$

It can be immediately seen that in the dynamic 3-color experiment the ambiguity-averse subject must violate either the property of dynamic consistency or consequentialism (or both). Then, if conditional preferences respect dynamic consistency (first row with $f_1; f_4$) then the property of consequentialism is violated (henceforth $\neg C$). On the other hand, if the conditional preferences remain consistent with consequentialism (as in second and third row with $f_1; f_3$ and $f_2; f_4$ respectively) then exactly one of the ex-ante preferences is reversed, what violates dynamic consistency (henceforth $\neg DC$). Finally, if conditionally on the event $Y \cup B$ the ambiguity-averse subject reverses both ex-ante preferences (as in fourth row with $f_2; f_3$) then the interim choices are neither consistent with dynamic consistency nor with consequentialism (henceforth $\neg DC$ and $\neg C$).

5.2 Experimental Design

The experiment was conducted in December 2008 in Mannheim in the experimental lab of SFB504. A total of 90 subjects participated in four sessions, with each subject participating only once. 46 participants were male, 44 female; all but one subject were students from various majors. Subjects were recruited via ORSEE (Greiner (2004)) and paid in private and cash directly after the experiment. On average they earned 14.00 Euro in about 60 minutes.

The urn was represented by a bucket with white table tennis balls (with yellow, blue or green stickers on them). Before making their choices, subjects were shown one ball of each color, taken from the bucket. So subjects were informed that the urn included at least one ball of each color. The bucket remained in the room for the whole experiment and after the drawings were finished, subjects had the opportunity to look at the balls inside the bucket. After receiving and reading the instructions detailing the complete experiment, all subjects were handed the decision sheet, on which they marked their bets.

To implement the choice problem described above, subjects were asked to make 4 decisions. The first two decisions were equivalent to choices I and II in Table 3.1. The third and forth decision where designed to test the conditional preferences as described in section three. Choice III was identical to choice I and choice IV identical to choice II, with one exception: at the end of each question, we added the sentence "if you come to know that the ball drawn is not green".[3] Dubois and Prade (1994) distinguish between dynamic choice situations which they call "focusing" and those they call "learning". In the terminology of Dubois and Prade, the situation we implement is focusing. utility maximizer will update according to Bayes rule to express her preferences.

A particular problem in ambiguity related experiments is how to deal with indifference. One possible solution is to force subjects to make a choice, the drawback being that some data points will reflect indifferent subjects, such that inferences from the Ellsberg decisions could be wrong (e.g. what looks like a preference reversal is not inconsistent with subjective expected utility theory if the subject was indifferent). On the other hand, including an explicit *indifferent* option raises problems in incentivized experiments: How will the subjects marking indifferent be paid? Choosing any rule, such as "the experimenter flips a coin" turns the problem into a decision with three

[3]See the appendix for complete instructions.

alternatives, the coin flip being one of them. Subjects who prefer the coin flip need not be identical with those who are indifferent in the original two alternative decision. To solve this problem, we did not offer an indifferent option. However, additionally to each decision, subjects were asked to mark "How strong is your liking for the alternative you choose?" on a scale ranging from 0 (nil) to 5 (very strong).[4] We interpret subjects who marked zero as having no confidence that their choices are better than the alternatives, that is, as being indifferent. These subjects where paid according to their decisions, but discarded from the analysis.

When everyone had finished their decisions, subjects took part in a timed 10 minute statistics and cognitive ability test, with 9 questions in total (3 questions from Shane Frederick's cognitive ability test (Frederick, 2005), the Wason selection task (Wason and Shapiro, 1971) and 5 simple statistics questions). Each correct answer was paid with 1 Euro. Finally, subjects were asked to answer an unpaid questionnaire which included demographics.

The draws took place at the end of the experiment. A randomly selected subject blindly drew a ball for each question. The balls were returned to the bucket after being shown to all subjects, so that all drawings were with replacement. Regarding question three and four, the following was stated in the instructions and implemented if needed: "If the first ball drawn happens to be green, we will continue drawing balls till a non-green ball is drawn." After the drawings were done, each subject was paid according to his/her decisions (each winning bet paid 4 Euro) and answers and the experiment ended.

[4]Our question is similar to the one used by Chen, Katuscak, and Ozdenoren (2007). Curley, Young, and Yates (1989) test three methods to measure ambiguity in an experiment and find that a question about confidence in the decision performs best.

5.3 Results

Out of our 90 subjects, 6 marked a confidence of nil for at least one of their choices. We interpret these subjects as indifferent and drop them from the following analysis since we are interested in strict preferences, leaving us with 84 data points.[5]

First, we look at the choices in the first two questions, which replicate the static Ellsberg experiment. The last row in Table 5.3 shows the proportion of ambiguity-averse, neutral and loving subjects. We confirm previous observations (see Camerer and Weber, 1992) that a majority of people are ambiguity-averse in this decision task: 54.8% prefer to bet on colors with known probabilities; 7.1% are ambiguity loving, while 38.1% exhibit ambiguity-neutral behavior.

According to the responses in the third and forth question, we can classify 21 subjects as both dynamically consistent and consequentialist, 44 as not dynamically consistent, but consequentialist, 6 as dynamically consistent but not consequentialist and 13 as neither dynamically consistent, nor consequentialist.[6] Taken together, 32.1% are dynamically consistent, while 77.4% are consequentialist. This difference is highly significant using a McNemar test. This result does not change when we look only at subjects who are ambiguity-averse or ambiguity-loving according to the first two questions.

The two bold numbers in Table 5.3 highlight subjects who would be classified as ambiguity-neutral in the static Ellsberg urn, yet who turn out to be not Bayesian in the dynamic urn. Thus, we find additional violations of subjective expected utility theory in the dynamic experiment.

[5] For us, only those subjects are truly indifferent who mark 0 in the confidence question. However, when we use a wider definition of indifference and also exclude subjects who marked 1 in at least one of their choices, our results stay qualitatively the same for all results and for the main results in table 5 the significance levels are unchanged as well (or, in one case, even stronger).

[6] Note that in our experiment, it is not possible for subjects to be ambiguity-averse/loving, dynamically consistent and consequentialist at the same time. Similar, there are no choice combinations that allow subjects to be ambiguity neutral, dynamically consistent, but not consequentialist.

	Ambiguity Attitude			
	Averse	*Neutral*	*Loving*	*Total*
DC, C	-	21	-	21
$\neg DC, C$	35	**3**	6	44
$DC, \neg C$	6	-	0	6
$\neg DC, \neg C$	5	**8**	0	13
Total	46	32	6	84

Table 5.3: Distribution of dynamically consistent/consequentialist and ambiguity-averse/neutral/loving subjects

The results in Table 5.3 suggest that when subjects are not both dynamically consistent and consequentialist, they rather drop dynamic consistency than consequentialism. However, due to the design of the urn, there are more combinations of choices which are consequentialist than dynamically consistent.

		Random	*Observed*	*Binomial test two − sided*
All subjects	*DC*	25%	32%	.132
	C	50%	77%	.000
Non − neutral	*DC*	25%	12%	.024
	C	50%	79%	.000

Table 5.4: Fraction of dynamically consistent and consequentialist subjects

To check this result for robustness, we list in Table 5.4 the hypothetic distributions we would expect if all our subjects chose purely randomly and compare them to the observed results. Looking at all subjects, there are more consequentialist and dynamically consistent choices than under a random distribution. However this result is significant only for consequentialism. The difference is even more pronounced when we restrict the analysis to subjects who are non-neutral towards ambiguity. Now significantly *less* subjects than under random choice are dynamically consistent, while, still, there are significantly more consequentialist ones.

Result 1: More subjects than expected under random choice are consequentialist. Among the non-neutral subjects, less than expected are dynamically consistent.

Regarding the way subjects update preferences, Dubois and Prade (1994) distinguish two different approaches, learning and focusing, which coincide in the additive case thank to the Bayes rule, but need not coincide outside of subjective expected utility. They consider two different updating rules: Maximum-Likelihood updating and Full-Bayesian updating.[7] Intuitively, in the case of learning, the decision maker learns something about the composition of the urn. In this case, Dubois and Prade (1994) argue for the use of the Maximum Likelihood rule. On the other hand, focusing is a situation in which no information is provided regarding the composition of the urn, as it is the case in our experiment. Dubois and Prade (1994) argue that in this situation of focusing the Full-Bayesian rule should be used. In their paper, Cohen, Gilboa, Jaffray, and Schmeidler (2000) test whether subjects follow the Full-Bayesian or the Maximum-Likelihood updating rule using a design very similar to ours. The questions they use are identical to our questions one, two and four. Then, ambiguity averse agents using the Maximum-Likelihood rule would choose blue in question four and while those updating according to Full-Bayesian updating would choose yellow. However, Cohen et al. assume that subjects are consequentialist. We repeat their test using only our consequentialist subjects. Similar to their results, we find significantly more support for the Full-Bayesian updating rule (p-value below 0.001, chi square test) among ambiguity-averse subjects. The result for ambiguity-loving subjects is not significant, very likely due to the small number of ambiguity-loving subjects in our experiment.

Result 2: More subjects who are ambiguity-averse and consequentialist use the Full-

[7]Roughly speaking the Full-Bayesian updating rule is a rule where the decision maker updates all the probabilistic scenarios she has in mind and derives the conditional preference relation from these updated probabilities. According to the Maximum-Likelihood updating rule the decision maker updates only the probabilities that maximize the event which has occurred.

Bayesian updating rule than the Maximum-Likelihood updating rule.

	Averse	*Loving*
Full − Bayesian	82.9%	66.7%
Maximum − Likelihood	17.1%	33.3%

Table 5.5: Full-Bayesian vs Maximum-Likelihood

Moreover, we asked all subjects about their confidence in their choices for each question. Apart from using these responses to discard indifferent subjects from the analysis, it is also interesting to look at the different levels of confidence for each question. Again, we start by looking at the first two questions, the static Ellsberg case. As Figure 5.1

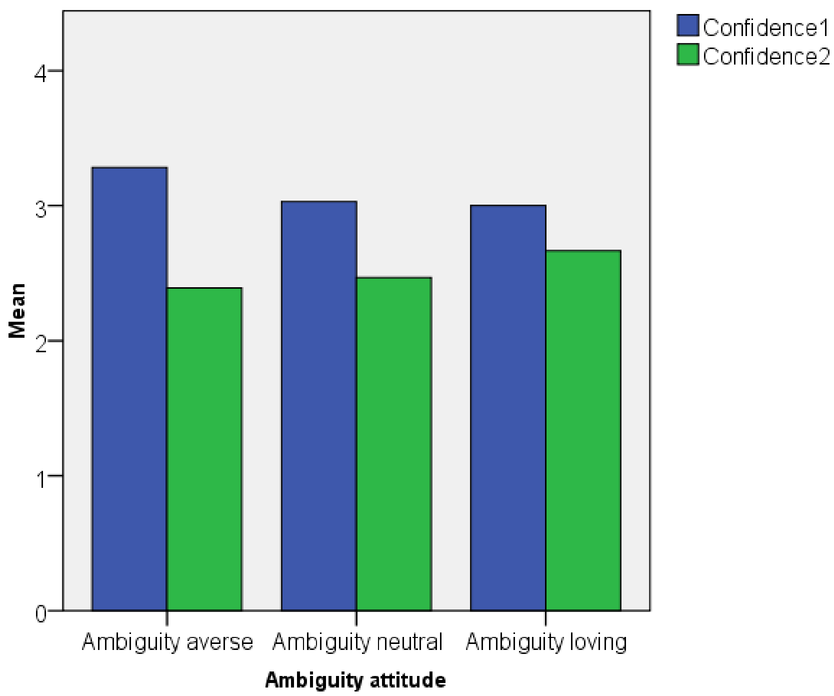

Figure 5.1: Confidence and ambiguity attitudes

shows, all subjects are less confident in their second answer compared to the first one. This difference is significant at the 1% level for ambiguity-averse and ambiguity-neutral subjects, but not significant for ambiguity loving subjects in a Wilcoxon test. However, the "amount" of confidence that subjects lose depends on their choices: ambiguity averse subjects lose more confidence than ambiguity-neutral ones.[8]

Result 3: Ambiguity averse subjects report a higher loss of confidence in their second choice compared to ambiguity-neutral ones.

Next, we turn to confidence levels for all four answers. Figure 5.2 depicts the confidence levels for subjects depending on their adherence to dynamic consistency and consequentialism. To evaluate the impact of going from a static to a dynamic Ellsberg urn, we look

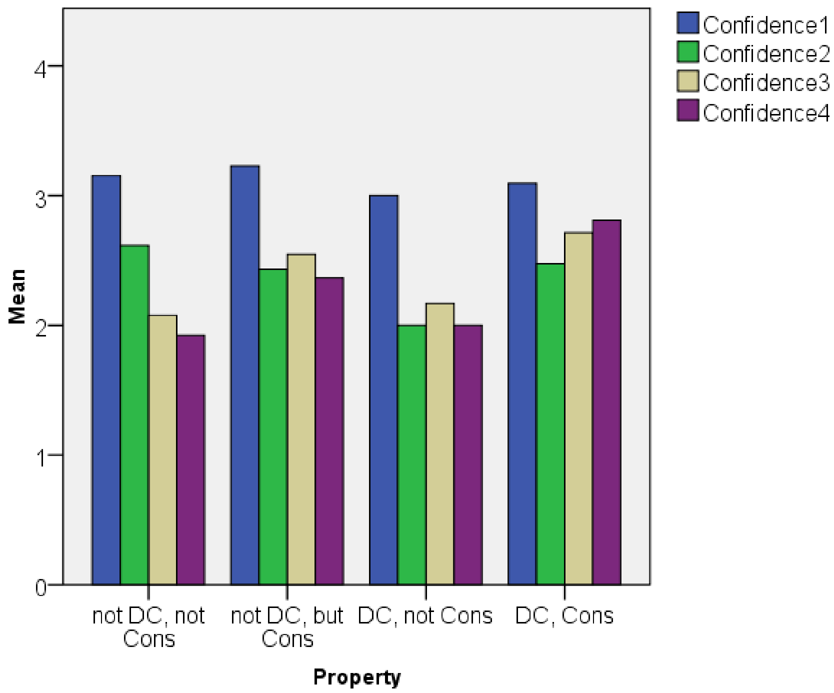

Figure 5.2: Confidence in the dynamic 3-color experiment

at the difference in average confidence in the first two compared to the last two questions: $confidence\ loss = (confidence1 + confidence2) - (confidence3 + confidence4)$. The first impression that subjects who adhere to the rationality arguments lose less confidence in the dynamic case is confirmed. As Table 5.6 shows, they have a signif-

[8]The two-sided p-value of a Mann-Whitney-U-Test on $confidence1 - confidence2$ comparing ambiguity-averse with ambiguity-neutral subjects is 0.032. No comparison with ambiguity loving subjects is significant. In both cases, the insignificant results for ambiguity-loving subjects might be due to their small number in our experiment.

icantly lower confidence loss than those subjects who violate one or both properties.

	DC, C	$\neg DC, C$	$DC, \neg C$	$\neg DC, \neg C$
DC, C	-	-	-	-
$\neg DC, C$	0.024	-	-	-
$DC, \neg C$	0.011	0.371	-	-
$\neg DC, \neg C$	0.000	0.455	0.01	-

Table 5.6: Significance levels from two-sided MW test on updating confidence loss

In a multinomial logistic regression that controls for demographics and subjects' score in our cognitive ability questions (see Table 4 in the appendix) we also find significantly lower confidence for subjects who do not behave according to dynamic consistency and consequentialism, compared to those who do. Our results for subjects' confidence can be explained with the assumption that subjects are more confident in their choice if they know of a way to rationally argue in favor of that choice. The probabilistic Bayesian theory is the most mathematically simple and arguably the only one which our subjects might *consciously* use in the experiment. We find the highest levels of confidence for choices two to four exactly for those subjects who behave probabilistic Bayesian. We do not test this assumption, so other explanations are possible. However, we do not find an effect of the demographics and the cognitive ability.[9]

Result 4: Subjects who adhere to both dynamic consistency and consequentialism loose less confidence in the dynamic choice situation, compared to those who violate one or both of these properties.

[9]The variable cognitive ability codes the number of a subject's correctly answered questions in questionnaire 1.

5.4 Summary

People who display the Ellsberg paradox can not be dynamically consistent and consequentialist at the same time. We conduct a dynamic extension of Ellsberg's 3-color experiment and find that, in our setup, significantly more subjects behave in accordance with consequentialism rather than with dynamic consistency. As such, our results can be seen as support for theories which retain consequentialism.

We observe that being ambiguity-neutral when facing the static Ellsberg urn does not necessarily imply that subjects always behave Bayesian. Several subjects who are classified as ambiguity neutral in the static choice situation can not be described by subjective expected utility theory in the dynamic extension.

Furthermore, we measure confidence. While all subjects are more confident in their first choice, ambiguity-neutral subjects lose less confidence in later choices than ambiguity-averse ones. Similarly, Bayesian subjects lose less confidence compared to those who violate dynamic consistency or consequentialism.

The dynamic Ellsberg urn experiment provides a tool to test both static and dynamic properties of decision making under uncertainty. We hope that our results will be informative for future theoretical work.

Chapter 6

Dynamic Choquet Preferences and Unambiguous Events

The objective of this chapter is to explore the link between updating Choquet expected utility preferences and two existing notions of unambiguous events.[1] In particular, we ask whether, for Choquet expected utility preferences, the property of dynamic consistency, constrained to a given collection of events, guarantees that its elements are unambiguous and vice versa. The results we have obtained allow us to answer this question in the affirmative.

Recently, a number of extensions of Choquet expected utility preferences to dynamic decision problems have been proposed (see Sarin and Wakker (1998a), Eichberger, Grant, and Kelsey (2005), Eichberger, Grant, and Kelsey (2007)). Here, we constrain the analysis to some fixed collection of events and characterize properties of these events on which Choquet preferences respect dynamic consistency and consequentialism. Natural candidates for such events on which both axioms are satisfied are events which support some kind of probabilistic beliefs, for instance events, with known probabilities, i.e. Y and $B \cup G$ in the 3-color experiment presented in Section 3.1. The idea of events characterized by probabilistic beliefs is closely related to the recently suggested notions of *unambiguous* events by Nehring (1999), Epstein and Zhang (2001),

[1]The content of this chapter is based on the article Dominiak and Lefort (2011b).

Zhang (2002) and Ghirardato, Maccheroni, and Marinacci (2004).

First, we focus on the definition Nehring (1999) provides, since it mimics the desirable separability property of expected utility theory.[2] His definition is based on the idea, which stems from Sarin and Wakker (1998b), to interpret capacities in terms of *rank-dependent probability assignments*. According to this interpretation, subjective probabilities used for evaluating acts depend on the rank ordering of their consequences. In general, two acts generating distinct ranks are evaluated with respect to different subjective probabilities. Thus, the separability of preferences and beliefs may be achieved for acts that generate the same rank. Such acts are called comonotonic. In the instance that the subjective likelihood of an event is unaffected by changing its position, it must be viewed as unambiguous. Correspondingly, Nehring calls an event unambiguous, henceforth *N-unambiguous*, if the subjective probability attached to the event does not depend on the ranking position of states.

We argue that conditional on N-unambiguous events, the Bayes updating rule for capacities is the most appropriate updating rule. The reason is twofold. First, because updating on N-unambiguous events according to the Bayes revision rule is the only way to retain dynamic consistency. Second, when conditioning on N-unambiguous events, the Bayesian updating rule coincides with other popular updating rules. These include the Full-Bayesian updating rule introduced by Jaffray (1992) and all the h-Bayesian updating rules as axiomatized by Gilboa and Schmeidler (1993). Motivated by this rationale we show that consequentialist Choquet expected utility preferences satisfy dynamic consistency on a fixed filtration if and only if the algebra generated by the smallest elements in the filtration belongs to the algebra generated by N-unambiguous events. This result on its own may be viewed as an alternative characterization of

[2]Separability of preferences and beliefs is a key property of expected utility theory. It means that subjective probabilities assigned to uncertain events are not affected by outcomes that are associated to these events. This property is also satisfied by the more general class of preferences, called probabilistically sophisticated preferences axiomatized by Machina and Schmeidler (1992).

N-unambiguous events in a conditional decision problem.

Furthermore, Nehring (1999) emphasized the restrictiveness of Choquet expected utility preferences, since the collection of N-unambiguous events must be always an algebra. However, there may be potentially interesting ambiguity situations, as exemplified by Zhang (2002) in his 4-color example, in which the candidates for unambiguous events form a weaker structure. By departing from the intuition behind Savage's key axiom, called the Sure-Thing-Principle, Zhang (2002) suggested a weaker definition of unambiguous events, henceforth *Z-unambiguous*. Thus, it is impossible to maintain dynamic consistency on events that are Z-unambiguous. An illustrative dynamic extension of the 4-color example is given in Section 5. Adopting an axiom, called *conditional certainty equivalence consistency* and constraining the dynamic consistency to partition measurable acts, we provide a dynamic characterization of Z-unambiguous events in a conditional decision problem.

This chapter is organized as follows. Section 6.1.1 presents the Choquet expected utility. First, the necessary notation is introduced. In Section 6.1.2, the definitions of N-unambiguous events and Z-unambiguous events are provided. Section 6.1.3 presents the main concepts regarding the conditional Choquet preferences. In Section 6.2.1 we provide a characterization of N-unambiguous events in a conditional decision problem. Moreover, we make some remarks on the related literature. In Section 6.2.2, we provide an illustrative example and establish a dynamic characterization of Z-unambiguous events. Finally, we conclude in Section 6.3.

6.1 Choquet Expected Utility Theory

6.1.1 Static Choquet Preferences

Let Ω a finite set of states of nature. An event A is a subset of Ω. The algebra generated by Ω is denoted by $\mathcal{A}$. Let X be the set of outcomes. An act f is a function from Ω to X. For instance, an act $f = (A_1, x_1; \ldots; A_n, x_n)$ assigns the outcome x_j to each

$\omega \in A_j$, $j = 1, \ldots, n$, where $A_1, \ldots, A_n$ are events partitioning Ω. Let $f_A g$ be an act that assigns the outcome $f(\omega)$ to each $\omega \in A$ and the outcome $g(\omega)$ to each $\omega \in A^c$. An act f that assigns a constant outcome $f(\omega) = x$ to each $\omega \in \Omega$ is called a constant act. Denote the set of all acts by $\mathcal{F}$. In Schmeidler's (1989) theory subjective beliefs are represented by *capacities*.

Definition 4. *A capacity* $\nu : \mathcal{A} \to \mathbb{R}$ *is a normalized and monotone set function, i.e.:*

(*i*) $\nu(\emptyset) = 0,\ \nu(\Omega) = 1,$

(*ii*) $\nu(E) \leq \nu(F)$ *for any* $E \subset F \subset \Omega$.

Thus, the capacity is not required to be additive, although it must satisfy a monotonicity property that has natural interpretation in terms of qualitative beliefs: "larger" events are "more likely". A capacity ν, that satisfies an additional condition (*iii*) $\nu(A)+\nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$ for all $A, B \in \mathcal{A}$, is called a *convex* capacity. In a behavioral context Schmeidler (1989) showed that convex capacities reflect agents' aversion towards ambiguity. Moreover, if ν satisfies the condition (*iii*) with equality then ν is a probability distribution.

Let $\succcurlyeq$ be a preference relation defined on the set of acts $\mathcal{F}$. A decision maker is said to have Choquet expected utility preferences if there exists a utility function u and a capacity ν such that for any $f, g \in \mathcal{F}$:

$$f \succcurlyeq g \Leftrightarrow \int_\Omega u \circ f \, d\nu \geq \int_\Omega u \circ g \, d\nu. \tag{6.1}$$

The expectations are taken in the following sense. For a given act f let $A_1, A_2, \ldots, A_n$ be a partition ordered from least to most favorable events, i.e. such that $u(f(A_1)) \leq u(f(A_2)) \leq \cdots \leq u(f(A_n))$. Then, the *Choquet integral* of f with respect to a capacity ν and an utility function u is defined as:

$$\int_\Omega u \circ f \, d\nu = u(f(A_1)) + \sum_{i=2}^{n} \left[u(f(A_i)) - u(f(A_{i-1})) \right] \nu(A_i, \ldots, A_n).$$

Choquet Expected Utility preferences have been justified axiomatically by Schmeidler (1989), Gilboa (1987) and Sarin and Wakker (1992) for an infinite state space. Imposing some richness conditions on the set of outcomes and allowing for a finite state space Choquet expected utility preferences has been axiomatized by Wakker (1989a), Nakamura (1990) and Chew and Karni (1994).

Throughout the chapter we assume that preferences admit Choquet expected utility representation. Additionally we restrict the set of outcomes X and preferences $\succcurlyeq$ on $\mathcal{F}$ by assuming that:

Assumption 1. (Continuity) The utility function $u : X \to \mathbb{R}$ is continuous.

Assumption 2. (Solvability) For any $f \in \mathcal{F}$ there exists $x \in X$ such that $f \sim x$.

Solvability serves as a richness condition on $\succcurlyeq$ and X. It is satisfied in all axiomatizations of Choquet expected utility theory in finite state space set-up. For instance, Nakamura (1990) and Chew and Karni (1994) impose it directly on $\succcurlyeq$, while Wakker (1989b) requires X to be a connected and separable topological space.

6.1.2 Unambiguous Events

This section provides a behavioral characterization of unambiguous events. In the context of ambiguity it is important to localize events that a decision maker perceive as unambiguous, i.e. events on which she has some kind of probabilistic beliefs. We begin with the characterization suggested by Nehring (1999), who interprets capacities in terms of *rank-dependent probability assignments*. Let ρ be a bijection $\rho : \Omega \to \{n, \ldots, 1\}$. The mapping ρ expresses the *ranking* position of states associated with an act f, i.e. the favorableness of their outcome relative to the outcomes obtained under other states. Let $\mathcal{R}$ be a set of such rankings and let Δ^{Ω} be a set of probability distributions over Ω. Two ranks ρ and ρ', for which at most two adjacent states swapped their ranking position, are said to be neighboring ranks. Formally, we say that ρ is a neighbor of ranking ρ', written $\rho N \rho'$, if and only if for at most two states $\omega \in \Omega$, $\rho(\omega) = \rho'(\omega)$, and for all $\omega \in \Omega$, $|\rho(\omega) - \rho'(\omega)| \leq 1$. A mapping $m : \mathcal{R} \to \Delta^{\Omega}$ is called *rank-dependent probability assignment* if and only if for all $\rho, \rho' \in \mathcal{R}$ such that $\rho N \rho'$, and all $\omega \in \Omega$ such that $\rho(\omega) = \rho'(\omega)$: $m_{\rho}(\omega) = m_{\rho'}(\omega)$. For a given capacity ν on Ω the rank-dependent probability assignment m_ρ may be defined as follows $m_\rho(\omega) = \nu(\omega' : \rho(\omega') \leq \rho(\omega)) - \nu(\omega' : \rho(\omega') < \rho(\omega))$.[3] The mapping m_ρ may be interpreted as the marginal capacity contribution of the state ω to all states yielding better outcomes. The Choquet integral of an act f with respect to ν and u can be written as the Choquet integral with respect to m_ρ and

$$\begin{aligned}\int_\Omega u \circ f \, d\nu &= \int_\Omega u \circ f \, dm_\rho \\ &= u(f(A_1)) + \sum_{i=2}^{n} [u(f(A_i)) - u(f(A_{i-1}))] \, m_\rho(A_i, \ldots, A_n).\end{aligned}$$

By abuse of notation, we denote a measure $m_{\rho(f)}$, such that $m_{\rho(f)}(A_i, \ldots, A_n) = v(A_i, \ldots, A_n)$ with $1 \leq i \leq n$, as the rank-dependent probability assignment m_ρ associated with an act f. Thus, throughout the paper we write the Choquet expectation of f, taken with respect the measure $m_{\rho(f)}$, as:

$$\int_\Omega u \circ f \, d\nu = \int_\Omega u \circ f \, dm_{\rho(f)}.$$

Call a pair of acts f and g *comonotonic*, if there are no two states ω, ω' such that $f(\omega) < f(\omega')$and $g(\omega) > g(\omega')$. For any act g, comonotonic with f and measurable with respect to f, the Choquet integral of g with respect to ν and u is equal to the expectation of g with respect to $m_{\rho(f)}$and u.

According to this view, Nehring (1999) associates ambiguity of events with their rank dependence. In particular, he calls an event A unambiguous, henceforth N-unambiguous, if the probability attached to the event does not depend on the ranking position of A.

Definition 5. *Fix an event $A \in \mathcal{A}$. A is N-unambiguous if $m_\rho(A) = \nu(A)$ for all $\rho \in \mathcal{R}$, otherwise A is N-ambiguous.*

Let $\mathcal{A}_N^U$ be the set of all N-unambiguous events. Nehring (1999) proves that for any capacity ν the set $\mathcal{A}_N^U$ is an algebra. Moreover, any capacity ν is always additively

[3]Nehring (1999) showed that there is a one-to-one relation between capacities and rank-dependent probability assignments, m_ρ. In his definition the superscript ν is used for m_ρ^ν. We drop it for notational simplicity.

separable across its unambiguous events. That is, let $A \in \mathcal{A}_N^U$ be a N-unambiguous event, then for all $B \in \mathcal{A}$:

$$\nu(B) = \nu(B \cap A) + \nu(B \cap A^c). \tag{6.2}$$

An alternative way to characterize N-unambiguous events is to use Savage's Sure-Thing-Principle. However, since the Sure-Thing-Principle, applied to the whole algebra of events $\mathcal{A}$, implies that beliefs are probabilistic, we have to constrain its domain to some events. Thus, we say that the Sure-Thing-Principle holds at A and A^c if: (i) For any $f, g, h, h' \in \mathcal{F}$:

$$f_A h \succcurlyeq g_A h \;\Leftrightarrow\; f_A h' \succcurlyeq g_A h', \tag{6.3}$$

and (ii) The condition (i) is also satisfied if A is everywhere replaced by A^c. The Sure-Thing-Principle constrained to A and A^c guarantees that the ranking of pairs of acts remains unchanged whatever the common parts are. Thus, an event A is N-unambiguous if and only if the Sure-Thing-Principle holds at A and A^c.

Proposition 6.1. *Fix an event $A \in \mathcal{A}$. The following two statements are equivalent:*

i) A is N-unambiguous, i.e. $A \in \mathcal{A}_N^U$.

ii) The Sure-Thing-Principle at A and A^c is satisfied.

Ghirardato, Maccheroni, and Marinacci (2004) provide the behavioral counterpart to N-unambiguous events in a different setup, assuming a convex structure on the set of consequences. In particular, an event A is N-unambiguous if for any $x, x' \in X$ bets of the form $x_A x'$ cannot not be used for hedging other acts. According to their Proposition 10 all such bets (called crisp acts) are evaluated with respect to the same probability distribution. Thus, the measure of an event A, $m_{\rho(x_A x')}(A)$, is independent of the rank ρ, meaning that A is N-unambiguous event.

Zhang (2002) suggested an alternative definition of unambiguous events by weakening the Sure-Thing-Principle. He calls an event A to be unambiguous, henceforth Z-unambiguous, if replacing a constant outcome x outside of A by any other constant outcome x' does not change the ranking of the pair of acts being compared.

Definition 6. *Fix an event $A \in \mathcal{A}$. A is Z-unambiguous if: (i) For any $f, g \in \mathcal{F}$ and for any outcome $x, x^{'} \in X$*

$$f_A x \succcurlyeq g_A x \;\Leftrightarrow\; f_A x' \succcurlyeq g_A x'$$

and (ii) The condition (i) is also satisfied if A is everywhere replaced by A^c. Otherwise A is Z-ambiguous.

Let $\mathcal{A}_Z^U$ be the collection of all Z-unambiguous events. In terms of capacities Zhang (2002) showed that $A \in \mathcal{A}_Z^U$ if and only if for all $B \in \mathcal{A}$ such that $B \subset A^c$:

$$\nu(A \cup B) = \nu(A) + \nu(B). \tag{6.4}$$

Thus, the additive separability property of ν is satisfied only on subevents of their unambiguous complements. It is worth to mention that $\mathcal{A}_N^U \subset \mathcal{A}_N^U$, since $\mathcal{A}_Z^U$ does not need to be an algebra. It is a λ-system, a collection of events that contains Ω and that is closed under complements and disjoint unions, but not under intersections.[4]

6.1.3 Updating Choquet Preferences

We limit our attention to updating on events that the decision maker views as possible to occur, i.e. non-null events. An event $A \in \mathcal{A}$ is non-null if $\nu(A) > 0$.[5] As time progresses the decision maker is informed that the true state of the nature ω is an element of an event A, i.e. $\omega \in A$. A natural way to model information is by means of event trees represented by a filtration. We assume that time is discrete, finite and goes over the index set $\mathcal{T} = \{0, \ldots, T\}$. Let $\mathcal{P}_t$ be a partition of the state space Ω. A filtration $\mathcal{P} = \{\mathcal{P}_t\}_{t \in \mathcal{T}}$ is a collection of partitions such that $\mathcal{P}_0 = \{\Omega\}$, any $\mathcal{P}_{t+1}$ is finer than $\mathcal{P}_t$ for all $t < T$, and $\mathcal{P}_T = \{\{\omega\} : \omega \in \Omega\}$. A filtration is given and

[4]For more general preferences than Choquet preferences Kopylov (2007) showed that $\mathcal{A}_Z^U$ is weaker than originally claimed λ- systems, it is a mosaic. A mosaic is a collection of events closed under complements but not under all disjoint unions.

[5]When an event A is either N-unambiguous or Z-unambiguous this definition of null events is equivalent to the stronger notion, A is null if $\nu(A \cup B) = \nu(B)$ for any $B \in \mathcal{A}$.

fixed throughout. Let $\mathcal{A}_{\mathcal{P}}$ be the algebra generated by the smallest elements of a given filtration $\mathcal{P}$.

At the ex-ante stage, $t = 0$, the decision maker formulates a complete contingent plan of action. When no information is given, the relation $\succcurlyeq$ represents the decision maker's unconditional preferences, that is $\succcurlyeq$ is equivalent to $\succcurlyeq_\Omega$. At any interim stage, $t < T$, the decision maker faces new information and has a chance to review the contingent plan for the remaining time periods. We denote by $\succcurlyeq_A$ the Choquet expected utility preferences over $\mathcal{F}$ conditional on $A \in \mathcal{P}_t$, i.e. for all $f, g \in \mathcal{F}$,

$$f \succcurlyeq_A g \Leftrightarrow \int_\Omega u \circ f d\nu_A \geq \int_\Omega u \circ g d\nu_A$$

with ν_A a capacity conditional on A. In the conditional decision problem the information available at time of the single choice is finer then just the knowledge of Ω. In the spirit of Ghirardato (2002) we reduce conditional decision problems to static ones.

Throughout the chapter, we assume that preferences satisfy consequentialism (see Axiom 2 in Section 2.3). To underpin the link between conditional and unconditional preferences we consider two axioms, dynamic consistency and conditional certainty equivalent consistency. Dynamic consistency has been already defined and discussed in Section 2.3 (see Axiom 3).

The second property, called *conditional certainty equivalent consistency*, is adopted from Pires (2002).[6] This property is a weaker version of dynamic consistency. It states: if and only if conditional on a non-null event A, the decision maker is indifferent between the act f and the constant payment x, then the unconditional preferences should also express indifference between the outcome x and the act $f_A x$, which agrees with the act f on A and otherwise assigns the constant outcome x.

Axiom 5 (Conditional certainty equivalent consistency)**.** *For any non-null $A \in \mathcal{A}$ any outcome $x \in X$ and any $f \in \mathcal{F}$, $f \sim_A x \Leftrightarrow f_A x \sim x$.*

[6] In her paper Pires (2002) axiomatizes the Full-Bayesian updating rule for the multiple prior preferences of Gilboa and Schmeidler (1989).

At the interim stage, the revealed information is taken into account by updating the decision maker's subjective beliefs. For Choquet expected utility preferences, there are several ways of defining the conditional capacity ν_A. The most common updating rules used to revise capacities are: the Bayes updating rule, the Maximum-Likelihood updating rule and the Full-Bayesian updating rule. For the sake of completeness, we recall the respective definitions.

Definition 7. *Let ν be a capacity on Ω and let $A \subset \Omega$. If A is observed and $B \subset A$, then:*

i) the Bayes updating rule (B) is given by

$$\nu_A(B) = \frac{\nu(B \cap A)}{\nu(A)},$$

ii) the Maximum-Likelihood updating rule (ML) is given by

$$\nu_A^{ML}(B) = \frac{\nu((B \cap A) \cup A^c) - \nu(A^c)}{1 - \nu(A^c)},$$

ii) the Full-Bayesian updating rule (FB) is given by

$$\nu_A^{FB}(B) = \frac{\nu(B)}{1 - \nu(B \cup A^c) + \nu(B \cap A)}.$$

Gilboa and Schmeidler (1993) characterize behaviorally the Maximum-Likelihood updating rule, introduced by Dempster (1968) and Shafer (1976). Eichberger, Grant, and Kelsey (2007) provide an axiomatic characterization of the Full-Bayesian updating rule for Choquet expected utility preferences. Moreover, the Maximum-Likelihood and the Bayes updating rule belong to the class of so called *h-Bayesian* updating rules introduced by Gilboa and Schmeidler (1993).

Definition 8 (h-Bayesian updating rule)**.** *There is an act $h \in \mathcal{F}$ such that for all $f, g \in \mathcal{F}$ and all $A \in \mathcal{A}$, $f \succcurlyeq_A g \Leftrightarrow f_A h \succcurlyeq g_A h$.*

When preferences admit a Choquet expected utility representation then for the Maximum-Likelihood (or *pessimistic*) updating rule, the act $h = x^*$ is a constant act yielding the most preferred outcome in X. That is, under the Maximum-Likelihood updating rule, the conditionally null event, A^c, is associated with the best outcome possible. For the Bayes (or *optimistic*) updating rule the act $h = x_*$ is a constant act associating the worst possible outcome in X (note that w.l.o.g. we suppose that such x^* and x_* exist). According to Gilboa and Schmeidler (1993) the decision maker exhibits "happiness" that an event A occurred and decisions are made as if she were always in "the best of all possible worlds" (happiness comes from the fact that the event A^c, which did not occur, was associated by the decision maker with the worst outcomes). All the h-Bayesian updating rules satisfy consequentialism but not necessarily dynamic consistency.

6.2 Dynamic Characterization of Unambiguous Events

6.2.1 N-Unambiguous Events

The objective of this section is to establish the necessary and sufficient conditions for Choquet expected utility preferences to be dynamically consistent on events in a fixed filtration. We begin by looking for an appropriate updating rule on the filtration $\mathcal{P}$ made up of N-unambiguous events, i.e. $\mathcal{A}_{\mathcal{P}} \subset \mathcal{A}_N^U$. It turns out that the Bayes revision rule for capacities is the only way to ensure dynamic consistency on the filtration $\mathcal{P}$, whose elements are N-unambiguous events. Moreover, when the conditional event is N-unambiguous, then the property of conditional certainty equivalent consistency implies that beliefs are revised according to the Bayes updating rule. These observations are summarized in the following proposition.

Proposition 6.2. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_N^U$ be a N-unambiguous event, then the following three statements are equivalent:*

i) Conditional certainty equivalent consistency is satisfied.

ii) The capacity ν is updated according to Bayes updating rule.

iii) Dynamic consistency is satisfied.

Remark 6.1. *Ghirardato, Maccheroni, and Marinacci (2008) provide a similar result for a larger class of preferences than the class of Choquet expected utility preferences, the invariant biseparable preferences. However, the properties that they obtain are not available for all acts but only for acts which are unambiguous (i.e. acts measurable with respect to the unambiguous partition).*

As next we state that the Full-Bayesian updating rule and all the h-Bayesian updating rules coincide with the Bayes revision rule when the conditional event A belongs to the algebra generated by N-unambiguous events, i.e. $A \in \mathcal{A}_N^U$.

Proposition 6.3. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_N^U$ be a N-unambiguous event, then the Full-Bayesian updating rule and all the h-Bayesian updating rules coincide with the Bayes updating rule.*

Now we are ready to state our first theorem. It claims that Choquet expected utility preferences satisfy dynamic consistency on events in a fixed filtration if and only if the algebra generated by the events from that filtration belongs to the algebra generated by N-unambiguous events. Intuitively, Choquet expected utility preferences respect dynamic consistency on a fixed collection of events, which are not affected by ambiguity.

Theorem 6.1. *Let $\mathcal{P} = \{\mathcal{P}_t\}_{t \in \mathcal{T}}$ be a fixed filtration on Ω and let $\mathcal{A}_{\mathcal{P}}$ be an algebra generated by $\mathcal{P}$. If the decision maker has Choquet expected utility preferences then the following conditions are equivalent:*

i) The decision maker is dynamically consistent with respect to $\mathcal{P}$.

ii) $\mathcal{A}_{\mathcal{P}}$ belongs to $\mathcal{A}_N^U$ and ν is updated according to the Bayes updating rule.

Some remarks regarding the theorem and the related literature are in order.

Remark 6.2. *Our result extends the theorem of Eichberger, Grant, and Kelsey (2005), which is true only for convex capacities, to all capacities. Then, for a capacity ν being convex, the additivity on $A \in \mathcal{A}$, i.e. $\nu(A^c) + \nu(A) = 1$, is equivalent to A being N-unambiguous. The proof relies on their Lemma 2.1 stating that if $\nu(A^c) + \nu(A) = 1$,* *then for any $B \in \mathcal{A}$, $\nu(B) = \nu(A^c \cap B) + \nu(A \cap B)$. Instead of assuming the Bayesian updating rule as in Eichberger, Grant, and Kelsey (2005) we show that it is actually the only way to retain the property of dynamic consistency.*

Remark 6.3. *Sarin and Wakker (1998a) show in their Theorem 3.2 that dynamic consistency is equivalent to the additivity of the Choquet functional. Our theorem strengthens this result by showing that dynamic consistency on fixed filtration actually implies that the algebra generated by this filtration belongs to the algebra of N-unambiguous events and vice versa.*

6.2.2 Z-Unambiguous Events

We begin this section by presenting the 4-color experiment, suggested by Zhang (2002), and extend it to a dynamic framework. In particular it illustrates that conditionally on a Z-unambiguous event (which is not N-unambiguous) it is impossible that a consequentialist decision maker satisfies the property of dynamic consistency.

Example 6.1. *Consider an urn containing 100 balls. The color of each ball may be black (B), red (R), gray (G) or white (W). The decision maker is supposed to rank six acts, $f, f', g, g', h, h' \in \mathcal{F}$, which are defined as below. At the ex-ante stage ($t = 0$) the decision maker is told that that the sum of black and red balls is 50 and the sum of black and gray is also 50. At interim stage ($t = 1$) one ball is drawn at random from the urn and the decision maker is informed the event $\{B, R\}$ occurred.*

Suppose that at the ex-ante stage ($t = 0$) the decision maker is ambiguity averse and displays the following pattern of preferences:

$$f = \begin{pmatrix} 1 & \textit{if } \omega \in B \\ 100 & \textit{if } \omega \in R \\ 0 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} \succ \begin{pmatrix} 100 & \textit{if } \omega \in B \\ 0 & \textit{if } \omega \in R \\ 0 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} = f'$$

$$g = \begin{pmatrix} 1 & \textit{if } \omega \in B \\ 100 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} \prec \begin{pmatrix} 100 & \textit{if } \omega \in B \\ 0 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 0 & \textit{if } \omega \in W \end{pmatrix} = g'$$

$$h = \begin{pmatrix} 1 & \textit{if } \omega \in B \\ 100 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 100 & \textit{if } \omega \in W \end{pmatrix} \succ \begin{pmatrix} 100 & \textit{if } \omega \in B \\ 0 & \textit{if } \omega \in R \\ 100 & \textit{if } \omega \in G \\ 100 & \textit{if } \omega \in W \end{pmatrix} = h'$$

The decision maker prefers f to f' and she also prefers h and h', because the chance of getting 100 *by choosing f is the same as by choosing f', but also with additional chance of getting* 1 *under f. The same way of reasoning holds for the preference relation between the act h and the act h'. Furthermore, the decision maker prefers g' to g. Choosing the act g' leads to the payment of* 100 *with probability of one half, since the probability of the event $\{B, G\}$ is known to be one half, whereas the act g pays* 100 *only with probability in the range between null and one half. Moreover, changing the outcome on the event $\{G, W\}$ in the pair of acts $\{f, f'\}$ and $\{h, h'\}$ leaves the preference relation between these acts unchanged. Thus, the event $\{B, R\}$ is Z-unambiguous. In particular the collection of all Z-unambiguous events, $\mathcal{A}_Z^U = \{\emptyset, \{B, R\}, \{G, W\}, \{B, G\}, \{R, W\}, \Omega\}$, is not an algebra, since it is not closed under intersections. However, as mentioned before, it is a λ-system.*

Consider now the filtration $\mathcal{P} = \{\mathcal{P}_0, \mathcal{P}_1\}$, with $\mathcal{P}_0 = \Omega$ and $\mathcal{P}_1 = \{\{B, R\}, \{G, W\}\}$. At the interim stage ($t = 1$) *the decision maker is informed that the event $\{B, R\}$ occurred. Since all acts $a, b \in \{f, g, h\}$ and all acts $a', b' \in \{f', g', h'\}$ are the same on the event $\{B, R\}$, $a = b$ and $a' = b'$, and differ only outside of that event, consequentialism requires that $a \sim_{\{B,R\}} b$ and $a' \sim_{\{B,R\}} b'$ and furthermore $a \succ_{\{B,R\}} a'$ (or $a \prec_{\{B,R\}} a'$ respectively). But this is possible only by reversing the conditional preference relation between g and g'. Thus, it is impossible for the ambiguity-averse decision maker to respect dynamic consistency on the fixed filtration $\mathcal{P}$ made up of Z-unambiguous events.*

We maintain dynamic consistency for all acts measurable with respect to the filtration $\mathcal{P}$. That is for all $f \in \mathcal{F}$ such that for any $x \in X$, $f^{-1}(x) \in \mathcal{P}$. Denote by $\mathcal{F}_{\mathcal{P}}$

the set of all acts measurable with respect to the filtration $\mathcal{P}$. We say that an event $A \in \mathcal{A}$ is $\mathcal{P}$ measurable if the indicator function of A is measurable with respect to the filtration $\mathcal{P}$.

Axiom 6 ($\mathcal{P}$-Dynamic Consistency)**.** *For any non-null event $A \in \mathcal{A}$ which is $\mathcal{P}$ measurable and for any $f, g \in \mathcal{F}_{\mathcal{P}}$, $f \sim_A g \Leftrightarrow f_A g \sim g$.*

In the same spirit as for N-unambiguous events, we look for the most natural revision rule to update capacities conditionally on Z-unambiguous events. According to the next result, applying the Bayes revision rule is the only way to ensure that the conditional certainty equivalent consistency and the $\mathcal{P}$-dynamic consistency are satisfied.

Proposition 6.4. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_Z^U$ be a Z-unambiguous event, then the following two statements are equivalent:*

i) The capacity ν is updated according to the Bayes updating rule.

ii) Conditional certainty equivalent consistency and $\mathcal{P}$-dynamic consistency are satisfied.

Next we show that conditional on Z-unambiguous events the Bayes revision rule coincides with all the h-Bayesian updating rules, whenever h is a constant act, and with the Full-Bayesian updating rule.

Proposition 6.5. *Let ν be a capacity on Ω and let $A \in \mathcal{A}_Z^U$ be an Z-unambiguous event, then the Full-Bayesian updating rule and all the h-Bayesian updating rules, with $h = x$ for some $x \in X$, coincide with the Bayes updating rule.*

In the following, we assume that the finest partition in $\mathcal{P}$ contains at least three elements. Then we provide a necessary and sufficient condition for Z-unambiguous events in a conditional decision problem.

Theorem 6.2. *Let $\mathcal{P} = \{\mathcal{P}_t\}_{t \in \mathcal{T}}$ be a fixed filtration on Ω. If a decision maker has Choquet expected utility preferences then the following conditions are equivalent:*

i) Conditional certainty equivalent consistency and $\mathcal{P}$-dynamic consistency are satisfied on $\mathcal{P}$.

ii) $\mathcal{A}_{\mathcal{P}}$ belongs to $\mathcal{A}_Z^U$ and ν is updated according to the Bayes updating rule.

Remark 6.4. *If conditional certainty equivalent consistency is satisfied but not $\mathcal{P}$-dynamic consistency, then the event fails to be Z-unambiguous. When updated according to the Full-Bayes updating rule, the capacities known as ϵ-contamination respect conditional certainty equivalent consistency. For a characterization of capacities which satisfies the conditional certainty equivalent consistency on all events see Eichberger, Grant, and Lefort (2010).*

Remark 6.5. *This characterization of Z-unambiguous events through conditional certainty equivalent consistency is a specific property of Choquet expected utility preferences. For instance when preferences admit the multiple prior representation, then according to the result of Pires (2002) conditional certainty equivalent consistency holds on all events whenever the Full-Bayesian updating rule is used.*

6.3 Summary

In this chapter the notion of unambiguous events is related to conditional decision problems. We consider a consequentialist decision maker with Choquet expected utility preferences. We look for a fixed collection of events on which the decision maker respects dynamic consistency. It turns out that dynamic consistency satisfied on a fixed filtration guarantees that its elements are N-unambiguous events. The converse is also true, when the capacity is updated according to the Bayes updating rule. As an implication, the decision maker will in general violate dynamic consistency on events which are Z-unambiguous (but not N-unambiguous). However, when the fixed filtration is made up of Z-unambiguous events, the decision makers's preferences respect an axiom called conditional certainty equivalence consistency and dynamic consistency constrained to partition measurable acts.

On the one side, the tight structure of Choquet expected utility models can be seen as a drawback of these models. On the other, side it allows to characterize sharply the usual dynamic properties of preferences from the static point of view. We hope that these results on their own may give some new insights into the nature of dynamic Choquet expected utility preferences and may also contribute to the existing debate regarding the suitable notion of unambiguous events.

6.4 Proofs

Proof of Proposition 6.1. First we show that *i)* $\Rightarrow$ *ii)* is true. Let A be a N-unambiguous event. Suppose that there are four acts $f, f', g, g' \in \mathcal{F}$ such that $f_A g \succcurlyeq f'_A g$, but $f_A g' \prec f'_A g'$. By computing the Choquet expectations of $f_A g$ we get

$$\begin{aligned}
\int_\Omega u \circ (f_A g)\ d\nu &= u(x_1) + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\nu(A_j, \dots, A_n) \\
&= u(x_1)(\nu(A) + \nu(A^c)) \\
&+ \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})](\nu((A_j, \dots, A_n) \cap A) + \nu((A_j, \dots, A_n) \cap A^c)) \\
&= \int_A u \circ f\ d\nu + \int_{A^c} u \circ g\ d\nu.
\end{aligned}$$

Furthermore, after computing the Choquet expectations of $f'_A g$, $f_A g'$, and $f'_A g'$ we obtain

$$\int_A u \circ f\ d\nu \geq \int_A u \circ f'\ d\nu,$$

and

$$\int_A u \circ f\ d\nu < \int_A u \circ f'\ d\nu.$$

Thus, we get a contradiction.

Now, we show that *ii)* $\Rightarrow$ *i)* is true.

Step 1. Fix an event $A \in \mathcal{A}$. For any act $f \in \mathcal{F}$ take an outcome $x \in X$ such that $f_A x \sim x$. Let $m_{\rho(f_A x)}$ be a rank-dependent probability assignment for rank ρ generated by $f_A x$. Hence, $\int_\Omega u \circ (f_A x)\ d\nu = \int_\Omega u \circ (f_A x)\ dm_{\rho(f_A x)}$. Take any $y \in X$ such that $f_A x$ and $f_A y$ are comonotonic. By the Sure-Thing-Principle we have $f_A y \sim x_A y$. After computing the Choquet integral we obtain

$$\int_A u \circ f \; dm_{\rho(f_A x)} + u(x) m_{\rho(f_A x)}(A^c) = u(x),$$

thus,

$$\int_A u \circ f \; dm_{\rho(f_A x)} = u(x) m_{\rho(f_A x)}(A).$$

Furthermore, whenever $u(x) < u(y)$ we have

$$u(x) m_{\rho(f_A x)}(A) + u(y) m_{\rho(f_A x)}(A^c) = u(y)\nu(A^c) + u(x)(1 - \nu(A^c)).$$

By continuity of u there are infinitely many such outcomes y and therefore we get

$$\nu(A^c) = m_{\rho(f_A x)}(A^c).$$

Let now $B \in \mathcal{A}$ be an event such that $B = \{\omega | f(\omega) \succ x\}$, then

$$m_{\rho(f_A x)}(A^c) = \nu(A^c \cup B) - \nu(B)$$

and

$$\nu(A^c) = \nu(A^c \cup B) - \nu(B).$$

This holds for any $B \in \mathcal{A}$ such that $B \cap A^c = \emptyset$. Since the Sure-Thing-Principle is satisfied at A^c as well, then

$$\nu(A) = \nu(A \cup C) - \nu(C)$$

for any $C \in \mathcal{A}$ such that $A \cap C = \emptyset$.

Step 2. For any $x, z \in X$ such that $u(x) < u(z)$, there exists $y \in X$ with $u(x) < u(y) < u(z)$ such that $f_A g \sim f'_A g$ where the acts $f_A g$ and $f'_A g$ are defined as follows

$$f_A g = \begin{pmatrix} z & \text{if } \omega \in A \cap B \\ x & \text{if } \omega \in A \cap B^c \\ x & \text{if } \omega \in A^c \end{pmatrix} \quad \text{and} \quad f'_A g = \begin{pmatrix} y & \text{if } \omega \in A \cap B \\ y & \text{if } \omega \in A \cap B^c \\ x & \text{if } \omega \in A^c \end{pmatrix}.$$

By the Sure-Thing-Principle $f_A g \sim f'_A g \;\Rightarrow\; f_A h \sim f'_A h$ for any $f_A h$ and $f'_A h$ defined as

$$f_A h = \begin{pmatrix} z & \text{if } \omega \in A \cap B \\ x & \text{if } \omega \in A \cap B^c \\ z & \text{if } \omega \in A^c \cap B \\ x & \text{if } \omega \in A^c \cap B^c \end{pmatrix} \quad \text{and} \quad f'_A h = \begin{pmatrix} y & \text{if } \omega \in A \cap B \\ y & \text{if } \omega \in A \cap B^c \\ z & \text{if } \omega \in A^c \cap B \\ x & \text{if } \omega \in A^c \cap B^c \end{pmatrix}.$$

Now by computing the Choquet integrals, we get

$$\begin{aligned} f_A g &= u(x)(1-\nu(A\cap B)) + u(z)\nu(A\cap B) \\ f'_A g &= u(x)(1-\nu(A)) + u(y)\nu(A) \\ f_A h &= u(x)(1-\nu(B)) + u(z)\nu(B) \\ f'_A h &= u(x)(1-\nu(A\cup(A^c\cap B))) + u(y)(\nu(A\cup(A^c\cap B)) - \nu(A^c\cap B)) \\ &\quad + u(z)\nu(A^c\cap B). \end{aligned}$$

Since $f_A g \sim f'_A g$ we obtain

$$\begin{aligned} u(x)(1-\nu(A\cap B)) + u(z)\nu(A\cap B) &= u(x)(1-\nu(A)) + u(y)\nu(A) \\ u(x)(\nu(A) - \nu(A\cap B)) &= u(y)\nu(A) - u(z)\nu(A\cap B). \end{aligned}$$

From Step 1 we have $\nu(A) = \nu(A\cup(A^c\cap B)) - \nu(A^c\cap B)$ and since $f_A h \sim f'_A h$ we obtain

$$u(x)(\nu(A\cup(A^c\cap B)) - \nu(B)) = u(y)\nu(A) - u(z)(\nu(A^c\cap B) - \nu(B)).$$

Since this equation is true for any $x, z \in X$, then $\nu(B) = \nu(B\cap A) + \nu(B\cap A^c)$ for any $B \in \mathcal{A}$ and we conclude that A is a N-unambiguous event, i.e. $A \in \mathcal{A}^U_N$.

□

Proof of Proposition 6.2. *i)* ⇒ *ii)* Let us suppose that conditional certainty equivalent consistency is satisfied. Let $f = y_B x$ be a simple bet with $u(x) < u(y)$. By solvability, there is $z \in X$ such that $f \sim_A z$. Thus, by conditional certainty equivalent consistency, we have $f_A z \sim z$. After rearranging terms, we get

$$\begin{aligned} u(z) &= u(x)(1-\nu_A(B)) + u(y)\nu_A(B) \\ u(z) &= u(x)(1-\nu(A^c\cup B)) + u(z)(\nu(A^c\cup(B\cap A)) - \nu(B)) + u(y)\nu(B\cap A). \end{aligned}$$

Thus,

$$u(z) = \frac{u(x)(1 - \nu(A^c \cup B)) + u(y)\nu(A \cap B)}{1 - \nu(A^c \cup B) + \nu(A \cap B)}.$$

Since A is a N-unambiguous event, then by the property of additive separability, we get $1 - \nu(A^c \cup B) + \nu(B) = 1 - \nu(A^c) - \nu(A \cap B) + \nu(A \cap B) = \nu(A)$. Thus, for any outcomes $x, y \in X$ such that $u(x) < u(y)$ the following is true

$$\begin{aligned} u(z) &= \frac{u(x)(1 - \nu(A^c \cup B)) + u(y)\nu(A \cap B)}{\nu(A)} \\ &= u(x)(1 - \nu_A(B)) + u(y)\nu_A(B). \end{aligned}$$

Therefore, we have

$$\nu_A(B) = \frac{\nu(A \cap B)}{\nu(A)}.$$

ii) $\Rightarrow$ *iii)* Now, suppose that the capacity ν is updated according to the Bayes updating rule. Let the events A and B be N-unambiguous. Consider acts $f, g \in \mathcal{F}$ with the following conditional preference relation: $f \prec_A g$ and $f \prec_B g$. By computing the conditional Choquet expected utilities we get

$$\begin{aligned} \int_\Omega u \circ f d\nu_A &= u(x_1) + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\nu_A(A_j, \ldots, A_n) \\ &= u(x_{1)} + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\frac{\nu((A_j, \ldots, A_n) \cap A)}{\nu(A)}, \\ \int_\Omega u \circ f d\nu_{A \cup B} &= u(x_1) + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\nu_{A \cup B}(A_j, \ldots, A_n) \\ &= u(x_1) + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\frac{\nu((A_j, \ldots, A_n) \cap (A \cup B))}{\nu(A \cup B)}. \end{aligned}$$

Since the event $A \cup B$ is N-unambiguous we have $\nu((A_j, \ldots, A_n) \cap (A \cup B)) = \nu((A_j, \ldots, A_n) \cap A) + \nu((A_j, \ldots, A_n) \cap B)$ for any $j = 2, \ldots, n$. Hence, the conditional Choquet integral $\int_\Omega u \circ f d\nu_{A \cup B}$ is proportional to the sum of $\int_\Omega u \circ f d\nu_A$ and $\int_\Omega u \circ f d\nu_B$. Therefore, we obtain $f \prec_{A \cup B} g$.

iii) $\Rightarrow$ *i)* Dynamic consistency directly implies conditional certainty equivalent consistency. □

Proof of Proposition 6.3. Consider the Full-Bayesian updating rule,

$$\nu_A^{FB}(B) = \frac{\nu(A \cap B)}{1 - \nu(A^c \cup B) + \nu(A \cap B)}.$$

Since the conditional event A is N-unambiguous, $\nu(A^c \cup B) = \nu(A^c) + \nu(A \cap B)$ and $\nu(A^c) + \nu(A) = 1$, therefore we have

$$\nu_A^{FB}(B) = \frac{\nu(A \cap B)}{\nu(A)}.$$

Consider now the Maximum-Likelihood updating rule,

$$\nu_A^{ML}(B) = \frac{\nu((A \cap B) \cup A^c) - \nu(A^c)}{\nu(A)}.$$

Since A is a N-unambiguous event, $\nu((A \cap B) \cup A^c) - \nu(A^c) = \nu(A \cap B) + \nu(A^c) - \nu(A^c)$, therefore we have

$$\nu_A^{ML}(B) = \frac{\nu(A \cap B)}{\nu(A)}.$$

Since A is N-unambiguous event, then for any $f \in \mathcal{F}$ we get

$$\begin{aligned}
\int_\Omega u \circ f d\nu &= u(x_1) + \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})]\nu(A_j, \ldots, A_n) \\
&= u(x_1)(\nu(A) + \nu(A^c)) \\
&+ \sum_{j=2}^{n} [u(x_j) - u(x_{j-1})](\nu((A_j, \ldots, A_n) \cap A) + \nu((A_j, \ldots, A_n) \cap A^c)) \\
&= \int_A u \circ f \; d\nu + \int_{A^c} u \circ f \; d\nu.
\end{aligned}$$

Thus, by definition of the h-Bayesian updating rules: $f \preceq_A g$ iff $f_A h \preceq g_A h$. For a N-unambiguous event this is equivalent to $\int_A u \circ f \; d\nu \leq \int_A u \circ g \; d\nu$ which is independent of h. So all the h-Bayesian updating rules coincide when the conditional event A is N-unambiguous. □

Proof of Theorem 6.1. *i)* $\Rightarrow$ *ii)* Let $A \in \mathcal{A}$ be an event on which dynamic consistency is satisfied. It is well known (see Ghirardato, Maccheroni, and Marinacci (2008)) that dynamic consistency implies that the utility functions u and u_A are the same up to an affine transformation. Let $f = (A_1, x_1; \ldots; A_n, x_n)$ be an act such that $u(x_i) < u(x_{i+1})$ with $1 \leq i \leq n-1$. The Choquet expectation of f is taken with respect to a rank-dependent probability assignment $m_{\rho(f)}$ with rank ρ given the act f, i.e.

$$\int_\Omega u \circ f \; d\nu = \int_\Omega u \circ f \; dm_{\rho(f)}.$$

By solvability, there is an outcome $x \in X$ such that $f \sim_A x$. Without loss of generality, we assume that f does not take the value x, i.e. $x \neq x_i$ with $i = 1, \ldots, n$. Consider acts

$f_A y$ for any $y \in X$. Let $m_{\rho(f_A y)}$ be a rank-dependent probability assignment associated with the act g. Let ν be a capacity such that $\nu(A) + \nu(A^c) = 1$ and let ν_A be a conditional capacity given A. In the first step we prove that

$$\frac{1}{\nu(A)} \int_A u \circ f \; dm_{\rho(f_A y)} = \int_A u \circ f \; d\nu_A.$$

In the second step, it is shown that for any act $f \in \mathcal{F}$

$$\frac{1}{\nu(A)} \int_A u \circ f \; dm_{\rho(f)} = \int_A u \circ f \; d\nu_A.$$

In the third step we conclude that that $m_{\rho(f)}(A) = \nu(A)$ for any act $f \in \mathcal{F}$. Thus, for any ranking position of states, that is for all ranks $\rho \in \mathcal{R}$, $m_\rho(A) = \nu(A)$ and therefore A is a N-unambiguous event.

Step 1. Since $f \sim_A x$, by dynamic consistency we get $f_A y \sim x_A$ for any $y \in X$.

i) Let y be an outcome such that $u(y) < u(x)$. Since $\int_\Omega u \circ g \; d\nu = \int_\Omega u \circ (f_A y) \; dm_{\rho(f_A y)}$ we have

$$\int_A u \circ f dm_{\rho(f_A y)} + u(y) m_{\rho(f_A y)}(A^c) = u(y)(1 - \nu(A)) + u(x)\nu(A).$$

This equality is true for any such outcome y for which the ranking ρ given the act $f_A y$ and the ranking ρ' given the act $x_A y$ are the same, i.e. $\rho = \rho'$. Thus, we get the following equality $u(y) m_{\rho(f_A y)}(A^c) = u(y)(1 - \nu(A))$, which implies that

$$m_{\rho(f_A y)}(A) = v(A). \qquad (1)$$

Therefore, we conclude that $\int_A u \circ f dm_{\rho(f_A y)} = u(x) m_{\rho(f_A y)}(A)$.

ii) Let y^* be an outcome such that $u(x) < u(y^*)$. Again, since $\int_\Omega u \circ (f_A y^*) \; d\nu = \int_\Omega u \circ (f_A y^*) \; dm_{\rho(f_A y^*)}$ we have

$$\int_A u \circ f \; dm_{\rho(f_A y^*)} + u(y^*) m_{\rho(f_A y^*)}(A^c) = u(y^*)(1 - \nu(A)) + u(x)\nu(A).$$

This equality is true for all outcomes y^* which keep the same ranking. Namely, the rank ρ associated with the act $f_A y^*$ and the rank ρ' associated with the act $x_A y^*$ are the same, i.e. $\rho(\omega) = \rho'(\omega)$ for all $\omega \in \Omega$. So we have the equality $u(y^*) m_{\rho(f_A y^*)}(A^c) = u(y^*)\nu(A^c)$, which implies that

$$m_{\rho(f_A y^*)}(A^c) = \nu(A^c). \qquad (2)$$

Therefore, we have $\int_A u \circ \; f dm_{\rho(f_A y^*)} = u(x)(1 - \nu(A^c))$.

$iii)$ Consider now an act $f_A x$. Let $m_{\rho(f_A x)}$ be a rank-dependent probability assignment with rank ρ given the act $f_A x$. Since the act f does not take the value x, there is an outcome $y \in X$ such that $u(y) = u(x) - \epsilon$ and there is an outcome $y^* \in X$ such that $u(y^*) = u(x) + \epsilon$ and such that the act $f_A y$ and the act $f_A y^*$ are comonotonic acts. This is possible by continuity of u. By applying (1) and (2) to $m_{\rho(f_A x)}$ we can deduce that $m_{\rho(f_A x)}(A) = \nu(A)$ and $m_{\rho(f_A x)}(A^c) = \nu(A^c)$ and therefore $\nu(A) + \nu(A^c) = 1$.

Thus, for any outcome $y \in X$ and for any rank-dependent probability assignment $m_{\rho(f_A y)}$ with rank ρ given the act $f_A y$ we have

$$u(x) = \frac{1}{v(A)} \int_A u \circ f \ dm_{\rho(f_A y)} = \int_A u \circ f \ dv_A.$$

Step 2. Since $f \sim_A x$, dynamic consistency implies that $f \sim x_A f$. Let $m_{\rho(x_A f)}$ be a rank-dependent probability assignment for a rank ρ given the act $x_A f$. Thus, we have

$$\int_A u \circ f \ dm_{\rho(f)} + \int_{A^c} u \circ f \ dm_{\rho(f)} = \int_{A^c} u \circ f \ dm_{\rho(x_A f)} + u(x) m_{\rho(x_A f)}(A).$$

Let us consider an act $f^* \in \mathcal{F}$ such that $f(\omega) = f^*(\omega)$ for any $\omega \in A$, but $f(\omega) \neq f^*(\omega)$ for at least one $\omega \in A^c$. Moreover, let f^* be comonotonic with f and let $x_A f$ be comonotonic with $x_A f^*$. According to dynamic consistency we have $f_A f^* \sim x_A f^*$. Therefore, we obtain the following equality

$$\int_A u \circ f \ dm_{\rho(f)} + \int_{A^c} u \circ f^* \ dm_{\rho(f)} = \int_{A^c} u \circ f^* \ dm_{\rho(x_A f)} + u(x) m_{\rho(x_A f)}(A^c),$$

which implies that $\int_A u \circ f \ dm_{\rho(f)} = u(x) m_{\rho(x_A f)}(A)$. Since dynamic consistency is satisfied on the event A, it is also satisfied on the complementary event A^c. Thus, applying Step 1 to A^c we get $m_{\rho(x_A f)}(A) = \nu(A)$.

Step 3. From Step 2 we have

$$u(x) = \frac{1}{v(A)} \int_A u \circ f \ dm_{\rho(f)}.$$

From Step 1 we have for any $y \in X$

$$u(x) = \frac{1}{v(A)} \int_A u \circ f \ dm_{\rho(f_A y)}.$$

Therefore, we have for any $y \in X$

$$\int_A u \circ f \; dm_{\rho(f)} = \int_A u \circ f \; dm_{\rho(f_A y)}.$$

Let us consider an act g that is f measurable and comonotonic with the act f. Then, $\int_\Omega u \circ f \; dm_{\rho(f)} = \int_\Omega u \circ g \; dm_{\rho(g)}$. For any outcome y^* there is an outcome y such that $g_A y^*$ is $f_A y$ measurable and comonotonic with the act $f_A y$. By applying the same way of reasoning for act g as for act f in Step 1 and in Step 2 we obtain

$$\int_A u \circ g \; dm_{\rho(g)} = \int_A u \circ f \; dm_{\rho(f)} = \int_A u \circ g \; dm_{\rho(g_a y^*)} = \int_A u \circ f \; dm_{\rho(f_A y)}.$$

This implies that on the algebra on A generated by f we obtain $m_{\rho(f)} = m_{\rho(f_A y)}$. From Step 1 we have that $\nu(A) = m_{\rho(f_A y)}(A)$. Therefore, we get $\nu(A) = m_{\rho(f)}(A)$ for any act $f \in \mathcal{F}$.

ii) $\Rightarrow$ *i)* See Proposition 5.1. *ii)* $\Rightarrow$ *iii)*.

□

Proof of Proposition 6.4. *i)* $\Rightarrow$ *ii)* $\mathcal{P}$-Dynamic Consistency follows directly: the capacity on the filtration constructed from Z-unambiguous events is additive. Applying the Bayes updating rule on it ensures dynamic consistency for filtration measurable acts. $f \sim_A x \Leftrightarrow f_A x \sim x$ is satisfied if the updating rule is h-Bayesian with $h = x$. In Proposition 6.2. we prove that all the h-Bayesian updating rules with h constant coincide on Z-unambiguous events. Since the Bayes updating rule corresponds to h-Bayesian updating rule with $h = x$, such that x is the worst possible outcome in X, the property of conditional certainty equivalent consistency holds on Z-unambiguous events, when applying this updating rule.

ii) $\Rightarrow$ *i)* Let us suppose that conditional certainty equivalent consistency is satisfied. Let $f = y_B x$ be a simple bet with $u(x) < u(y)$. By solvability there is $z \in X$ such that $f \sim_A z$. Thus, by conditional certainty equivalent consistency we have $f_A z \sim z$. After some computations we get

$$\begin{aligned} u(z) &= u(x)(1 - \nu_A(B)) + u(y)\nu_A(B) \\ u(z) &= u(x)(1 - \nu(A^c \cup B)) + u(z)(\nu(A^c \cup (B \cap A)) - \nu(B)) + u(y)\nu(B \cap A). \end{aligned}$$

Thus,

$$u(z) = \frac{u(x)(1 - \nu(A^c \cup B)) + u(y)\nu(A \cap B)}{1 - \nu(A^c \cup B) + \nu(A \cap B)}.$$

Since A is a Z-unambiguous event, then by the characterization of Z-unambiguous events, we get $1 - \nu(A^c \cup B) + \nu(B) = 1 - \nu(A^c) - \nu(A \cap B) + \nu(A \cap B) = \nu(E)$. Thus, for any outcomes $x, y \in X$ such that $u(x) < u(y)$ the following is true:

$$\begin{aligned} u(z) &= \frac{u(x)(1 - \nu(A^c \cup B)) + u(y)\nu(A \cap B)}{\nu(A)} \\ &= u(x)(1 - \nu_A(B)) + u(y)\nu_A(B). \end{aligned}$$

Therefore, we have

$$\nu_A(B) = \frac{\nu(A \cap B)}{\nu(A)}.$$

□

Proof of Proposition 6.5. From the definition of Z-unambiguous events it follows directly that all the h-Bayesian updating rules with h being constant act coincide with the Bayes updating rule. If A is observed and $B \subset A$ then the Full-Bayesian updating rule is given by

$$\nu_A^{FB}(B) = \frac{\nu(B)}{1 - \nu(B \cup A^c) + \nu(B \cap A)}.$$

Since A is Z-unambiguous then $v(A \cup E^c) = v(A) + v(E^c)$. Thus,

$$\nu_A^{FB}(B) = \frac{\nu(B)}{\nu(A)}.$$

□

Proof of Theorem 6.2. *(i)* $\Rightarrow$ *(ii)*. Let $\mathcal{P}$ be the fixed filtration and A_j the atoms of this filtration with $1 \leq j \leq n$. From Eichberger, Grant, and Kelsey (2007) we know that conditional certainty equivalent consistency guarantees that the same utility index u is used for conditional and unconditional preference relation. Let $f = (A_1, x_1; \ldots; A_n, x_n)$ be a $\mathcal{P}$-measurable act such that $u(x_j) < u(x_{j+1})$ with $1 \leq j \leq n - 1$. The Choquet expectation of f is taken with respect to a rank-dependent probability assignment $m_{\rho(f)}$ associated with the act f, i.e.

$$\int_\Omega u \circ f \; d\nu = \int_\Omega u \circ f \; dm_{\rho(f)}.$$

Let us assume that A_i^c, with $i \neq 1$ and $i \neq n$, has occurred. In the first Step we show that

$$\frac{1}{m_{\rho(f)}(A_i^c)} \int_{A_i^c} u \circ f \; dm_{\rho(f)} = \int_{A_i^c} u \circ f \; dv_{A_i^c}.$$

Step 1. By solvability there is an outcome $y \in X$ such that $f \sim_{A_i^c} y$. As next we construct an act g that is comonotonic with the act f. The construction is conducted as follows. If $u(y) \leq u(x_{i-1})$, we define g on A_i^c as $g = z$ on A_n with $z \in X$ and $g = f$ otherwise. By choosing z properly, that is, such that $u(z) > u(x_n)$, we obtain g such that $g \sim_{A_i^c} x$ with $u(x_{i-1}) < u(x) < u(x_{i+1})$. By continuity of u this is possible. On the other hand, if $u(x_{i+1}) \leq u(y)$ we define another act g by decreasing x_1, such that $g \sim_{A_i^c} x$ with $u(x_{i-1}) < u(x) < u(x_{i+1})$. Then the acts f and g are comonotonic, because g is different of f only on the lowest value of f, and this lowest value of g can only be lower than the lowest value of f, or the highest value of f, and this highest value of g can only be higher than the highest value of f. Therefore, we get

$$\int_{A_i^c} u \circ g \; dv_{A_i^c} = \int_{A_i^c} u \circ g \; m_{\rho(g)},$$

where $m_{\rho(g)}$ is the rank-dependent probability assignment associated with the act g. Now, we apply conditional certainty equivalent consistency and get $g_{A_i^c} x \sim x$. Since $u(x_{i-1}) < u(x) < u(x_{i+1})$, the act f and the act $g_{A_i^c} x$ are comonotonic. Thus, their Choquet integrals are computed with respect to the same measure $m_{\rho(f)}$, namely $\int_\Omega u \circ (g_{A_i^c} x) \; dv = \int_\Omega u \circ (g_{A_i^c} x) \; dm_{\rho(f)}$. Thus, we have $\int_\Omega u \circ (g_{A_i^c} x) \; dv = u(x)$. Therefore, we get

$$u(x) = \int_{A_i^c} u \circ g \; dm_{\rho(f)} + m_{\rho(f)}(A_i) u(x).$$

Finally, we obtain

$$u(x) = \frac{1}{m_{\rho(f)}(A_i^c)} \int_{A_i^c} u \circ g \; dm_{\rho(f)} = \int_{A_i^c} u \circ g \; dv_{A_i^c},$$

which is also true for the act f

$$\frac{1}{m_{\rho(f)}(A_i^c)} \int_{A_i^c} u \circ f \; dm_{\rho(f)} = \int_{A_i^c} u \circ f \; dv_{A_i^c}.$$

Step 2. We show that the above result is true for any possible permutation of the indexes $\{2, \ldots, n-1\}$ of the atoms $\{A_2, \ldots, A_{n-1}\}$. That is for any such $\mathcal{P}$-measurable act f^* the rank-dependent probability assignment $m_{\rho(f^*)}$ associated with the act f^* is independent of the ranking position of the event A_i provided that $i \neq 1$ and $i \neq n$. Consider an act $f^* = (A_1, x_1^*; \ldots; A_n, x_n^*)$ such that $f^* \sim_{A_i^c} y$ for some outcome $y \in X$ and such that $u(x_i)$ is between $u(x_j^*)$ and $u(x_{j+1}^*)$. Consider also an another act $f^{**} = (A_1, x_1^{**}; \ldots; A_n, x_n^{**})$ with different rearrangements of atoms, such that $u(x_i)$ is between $u(x_j^{**})$ and $u(x_{j+1}^{**})$ and such that $f^{**} \sim_{A_i^c} y$. Let $m_{\rho(f^*)}$ and $m_{\rho(f^{**})}$ be a rank-dependent probability assignment associated with the act f^*, respectively with f^{**}. By applying Step 1 we obtain

$$\frac{1}{m_{\rho(f^*)}(A_i^c)} \int_{A_i^c} u \circ f \; dm_{\rho(f^*)} = \frac{1}{m_{\rho(f^{**})}(A_i^c)} \int_{A_i^c} u \circ f \; dm_{\rho(f^{**})}. \qquad (1)$$

Now, we can vary the values of x_1^* and x_1^{**}, equality (1) remains true, provided that f^* and f^{**} have still the same certainty equivalent conditional on the A_i^c, i.e. there is some z such that $f^* \sim_{A_i^c} z$ and $f^{**} \sim_{A_i^c} z$. Thus, it must be true that $m_{\rho(f^*)}(A_i^c) = m_{\rho(f^{**})}(A_i^c)$. Then, we have

$$m_{\rho(f^*)}(A_i^c) = 1 - m_{\rho(f^*)}(A_i) = 1 - v(A_i \cup A_{j+1}^*, \ldots, A_n) + v(A_{j+1}^*, \ldots, A_n), \qquad (2)$$

and

$$m_{\rho(f^{**})}(A_i^c) = 1 - m_{\rho(f^{**})}(A_i) = 1 - v(A_i \cup A_{j+1}^{**}, \ldots, A_n) + v(A_{j+1}^{**}, \ldots, A_n). \qquad (3)$$

Equations (2) and (3) lead to the following

$$v(A_i \cup A_{j+1}^*, \ldots, A_n) - v(A_{j+1}^*, \ldots, A_n) = v(A_i \cup A_{j+1}^{**}, \ldots, A_n) - v(A_{j+1}^{**}, \ldots, A_n).$$

The last equation is true for any f. Let $A_i = E$ with $i \neq 1$ and $i \neq n$. Moreover, let $F = (A_{j+1}^*, \ldots, A_n)$ and let $G = (A_{j+1}^{**}, \ldots, A_n)$. The left hand side of the equation is true if $(A_i \cup A_{j+1}^*, \ldots, A_n) - (A_{j+1}^*, \ldots, A_n) \neq 1$, i.e. $v(F) \neq 0$ and $v(F \cup E) \neq 1$. The right hand side of the equation is true if $(A_i \cup A_{j+1}^{**}, \ldots, A_n) - (A_{j+1}^{**}, \ldots, A_n) \neq 1$, i.e. $v(G) \neq 0$ and $v(G \cup E) \neq 1$. Thus, we get

$$v(F \cup E) - v(F) = v(G \cup E) - v(G).$$

Step 3. Since $\mathcal{P}$-dynamic consistency holds on the algebra generated by the filtration $\mathcal{P}$ the capacity ν is additive on this algebra.

Case 1. There exists an event $F \in \mathcal{P}$ such that $v(F) \neq 0$ and $v(F \cup E) \neq 1$. Thus, by additivity of v on $\mathcal{P}$ we get $v(F \cup E) - v(F) = v(E)$. Then from the result in Step 1 we conclude that $v(A \cup E) = v(A) + v(E)$ for all $A \subset E^c$.

Case 2. Suppose that there exists no such event F and then let us assume at least three atoms in $\mathcal{P}$. There exists E' and E'' in $\mathcal{P}$ such that $E = E' \cup E''$ and the complements of E' and E'' are not atoms in $\mathcal{P}$. Therefore, we can apply case 1 to them obtaining

$$\begin{aligned}\nu(F \cup E) - \nu(F) &= \nu(F \cup E' \cup E'') - \nu(F \cup E') + \nu(F \cup E') - \nu(F) \\ &= \nu(E') + \nu(E'') \\ &= \nu(E).\end{aligned}$$

Therefore, we have $v(A \cup E) = v(A) + v(E)$ for all $A \subset E^c$.

By applying Step 1, Step 2 and Step 3 to the complementary event, E^c, we can conclude that E and E^c are Z-unambiguous events.

(ii) $\Rightarrow$ *(i)*. The converse follows immediately from the Proposition 6.4. □

Chapter 7

"Agreeing to Disagree" Type Results Under Ambiguity

In this chapter we apply Schmeidler's (1989) Choquet expected utility theory to interpersonal decision problems.[1] There is a finite group of agents. Each agent is characterized by her private information represented by a partition over a finite set of states of nature. The agents share identical prior beliefs over states. Conditional on her private information, each agent generates her posterior beliefs by updating the prior ones. These posterior beliefs are used by the agents as the basis for making individual decisions. An interesting question that arises here is, which role does asymmetric information play in the context of interpersonal decision problems? In particular, suppose that at some state the agents make distinct decisions which are common knowledge among them. That is, each agent knows the decisions of the other agents, and each agent knows that each agent knows the decisions of the other agents, . . . , and so on, ad infinitum. Can asymmetric information *alone* explain the differences in agent's decisions? Surprisingly, within Bayesian frameworks, the answer is "No". This negative answer is due to Aumann's (1976) celebrated result, known as Agreement Theorem. Aumann showed that, if two agents share a common additive probability distribution, and their posteriors for some event are common knowledge, then these posteriors must

[1]The content of this chapter is based on the article Dominiak and Lefort (2011a).

coincide, despite the fact that they may be conditioned on diverse information. Aumann's impossibility of "agreeing to disagree" in posterior beliefs has been extended by Geanakoplos and Sebenius (1983) to posterior expectations, and by Bacharach (1985) to actions maximizing conditional expectation. Following this line of research, Milgrom (1981) and Milgrom and Stokey (1982) established an even more puzzling result. In a simple exchange economy under uncertainty, they showed that differences in traders' private information alone cannot generate any profitable trade opportunities. That is, given an ex-ante Pareto-efficient allocation, after the receipt of private information, there will be no transaction with the property that it is common knowledge among the traders (with the same risk attitudes) that each of them is willing to carry it out. This result is often interpreted as establishing the impossibility of "speculative" trade. To paraphrase Werlang (1989, p.83): "Their result is a problem for the theory of speculative markets: asymmetric information alone cannot be responsible for the existence of large stock exchanges. A very important research project in the finance literature is to find where Milgrom-Stokey's model departs from reality. It is a point that is crucial for the understanding of the very complex speculative markets we see nowadays". In this chapter we pursue Werlang's desiderata.

Aumann's Agreement Theorem as well as Milgrom-Stokey's no-trade theorem rely on two main assumptions which can be questioned. It is assumed that the traders share common prior beliefs and that the priors are represented by additive probability distribution. Morris (1994) advocated weakening the "commonness" assumption, while still assuming that the traders are subjective expected utility maximizers. Essentially, he identified which types of heterogeneous prior beliefs will lead to speculative trade in the presence of asymmetric information. Here, we suggest an alternative approach. We maintain the assumption of common priors, but weaken the "additivity" requirement by allowing the traders to be Choquet expected utility maximizers. We assume that the agents share a common-but-not-necessarily-additive prior beliefs which are represented by a capacity. Each agent incorporates the receipt of new information by updating the prior capacity conditional on her private information. The posterior capacities are used as a basis for individual decisions. Our objective is to characterize the properties of events in each agent's information partition which guarantee that disagreement in commonly known decisions is impossible. In turns out that, whenever each agent's information partition is made up of unambiguous events in the sense of Nehring (1999), then it is impossible that they disagree on their commonly known decisions, whatever these decisions are, whether posterior capacities or conditional Choquet expectations. Conversely, an agreement in conditional expectations, but not in posterior beliefs, implies that each agent's private information consists of Nehring-unambiguous events. Based on these results, we can generalize the no-trade theorem of Milgrom and Stokey (1982) in the context of ambiguity. It is shown that, whenever each agent's information partition is made up of Nehring-unambiguous events there will be no-trade among Choquet expected utility maximizers. The results obtained suggest that within non-Bayesian frameworks asymmetric information does matter and can explain differences in commonly known decisions. In particular, the existence of gambling behavior and speculative trade may be attributed to differences in agents' private and ambiguous information.

The rest of this chapter is organized as follows. In Section 7.1.1, the partitional information structure is introduced and the notion of common knowledge is presented. In Section 7.1.2, we define an interpersonal decision model. In Section 7.2, sufficient as well as necessary conditions are established for agreement theorems to be true in the presence of ambiguity. In Section 7.3, Milgrom-Stokey's no-trade theorem is generalized for Choquet expected utility preferences.

7.1 Preliminaries

7.1.1 Knowledge Structure

We consider a finite set Ω of states. An event E is a subset of Ω. Let $\mathcal{A} = 2^{\Omega}$ be the set of all subsets of Ω. For any $E \subset \Omega$ we denote $\Omega \setminus E$, the complement of E, by E^c. There is a finite group of agents I indexed by $i = 1, \ldots, N$. Each agent i is endowed with a partition $\mathcal{P}_i$ of Ω, which represents i's private information in the following sense. If the true state is ω, then i is informed of the atom $\mathcal{P}_i(\omega)$ of $\mathcal{P}_i$ to which ω belongs. Intuitively, $\mathcal{P}_i(\omega)$ is the set of all states that agent i considers possible at ω. In other words: if the true state is ω, then the agent i does not know that, but knows only that the true state is a member of $\mathcal{P}_i(\omega)$ containing ω. Given this information structure it is said that the agent i knows an event E at ω if $\mathcal{P}_i(\omega) \subset E$. The event that i knows E, denoted by K_iE, is a set of all states in which i knows E. Thus, an operator $K_i : \mathcal{A} \to \mathcal{A}$, defined as:

$$K_iE = \{\omega \in \Omega : \mathcal{P}_i(\omega) \subset E\}. \tag{7.1}$$

is called i's knowledge operator. An event E is common knowledge at a state ω if everyone knows E at ω, everyone knows that everyone knows E at ω, and so on, ad infinitum. The event that everyone knows an event E is captured by an operator $K^1 : \mathcal{A} \to \mathcal{A}$ defined as:

$$K^1 = K_1E \cap \cdots \cap K_nE = \bigcap_{i=1}^{N} K_iE. \tag{7.2}$$

A common knowledge operator $CK : \mathcal{A} \to \mathcal{A}$ is defined as an infinite application of the operator K^1, i.e.:

$$CKE = K_1E \cap K_1K_1E \cap K_1K_1K_1K_1E \cdots = \bigcap_{m=1}^{\infty} K^m(E). \tag{7.3}$$

Then, E is *commonly known* at ω if $\omega \in CKE$. The concept of common knowledge can be expressed equivalently in the following way. Let $\mathcal{M} = \wedge_{i=1}^{N} \mathcal{P}_i$ be the meet (i.e. finest common coarsening) and $\mathcal{J} = \vee_{i=1}^{N} \mathcal{P}_i$ the joint (i.e. coarsest common refinement) of all agents' partitions. Denote by $\mathcal{M}(\omega)$ the member of $\mathcal{M}$ that contains ω. Then, E is commonly known at ω if and only if $\mathcal{M}(\omega) \subset E$ (see Aumann, 1976; Milgrom, 1981).

7.1.2 Interpersonal Decision Model

Let X be a set of outcomes. In this chapter we refer to mappings $f : \Omega \to X$ as actions. Let $\mathcal{F}$ be a set of all actions. For any $f_1, f_2, \ldots, f_n \in \mathcal{F}$ denote by $f = (f_1, E_1; f_2, E_2; \ldots; f_n, E_n)$ an action that assigns the outcome $f(\omega) = f_j(\omega)$ to any state ω in E_j where the collection of events $E_1, E_2, \ldots, E_n$ form a partition of Ω. An action $f = x_E y$ that assigns the same outcome $f(\omega) = x$ to all states in E and $f(\omega) = y$ to all states in E^c, is called a bet. If the true state is ω, each agent makes a decision. Let $\mathcal{D}$ be a non-empty set of possible decisions. Decisions are determined by i's *decision function* $d_i : \Omega \to \mathcal{D}$ which is a function of i's private information, i.e. $d_i(\omega) = d_i(\mathcal{P}_i(\omega))$. A collection $\mathcal{I} = (I, \Omega, \mathcal{F}, (\mathcal{P}_i, d_i)_{i \in I})$ where I is the set of agents, Ω the set of states, $\mathcal{F}$ the set of actions, $(\mathcal{P}_i)_{i\in I}$ the agents' information partitions, and $(d_i)_{i\in I}$ the agents' decision functions is called an *interpersonal decision model*. An interpersonal decision model $\mathcal{I}$ can be viewed as a formal setup to study the role of common knowledge and of private information in interactive decision problems.

Essentially, an Agreement Theorem states that if at some state agents' decisions are common knowledge then they must be the same the same. For a given interpersonal decision problem $\mathcal{I}$ let $D_i(\xi_i) = \{\omega : d(\mathcal{P}_i(\omega)) = \xi_i\}$ be the event that the agent i makes a decision ξ_i. We say that at some state ω the agents' decisions are commonly known among them (or common knowledge) if and only if $\mathcal{M}(\omega^*) \subseteq D_1(\xi_1) \cap \cdots \cap D_I(\xi_I)$. The impossibility of "agreeing to disagree" on commonly known decisions can be stated formally as follows.

Agreement Theorem. *Let $\mathcal{I}$ be an interpersonal decision model and let $D_i(\xi_i) = \{\omega : d(\mathcal{P}_i(\omega)) = \xi_i\}$ be the event that the agent i makes a decision ξ_i. If at some state ω^* the event $\bigcap_{i\in I} D_i(\xi_i)$ is common knowledge, i.e. $\mathcal{M}(\omega^*) \subset \bigcap_{i\in I} D_i(\xi_i)$, then $\xi_1 = \xi_2 = \cdots = \xi_N$.*

Many famous Agreement Theorems have been formulated within a Bayesian framework.

That is, it is assumed that agents share a common prior probability distribution π over Ω, where $\pi(\mathcal{P}_i(\omega)) > 0$ for any ω and all $i \in I$. If the true state is ω, then the agent i is informed of the atom $\mathcal{P}_i(\omega)$ of her partition $\mathcal{P}_i$ to which ω belongs and revises the prior π given $\mathcal{P}_i(\omega)$ according to Bayes' rule. The posterior probability $\pi(\cdot \mid \mathcal{P}_i(\omega))$ is then used as a basis for agents' decisions. Usually, the decision function $d(\cdot)$ is either:

i) a conditional probability (posterior belief) for some event E in $\mathcal{A}$:

$$d_i(\omega) = \pi(E \mid \mathcal{P}_i(\omega)), \tag{7.4}$$

or *ii*) a conditional expectation (posterior expectation) of some action f in $\mathcal{F}$:

$$d_i(\omega) = \int_\Omega u \circ f \; d\pi(\cdot \mid \mathcal{P}_i(\omega)). \tag{7.5}$$

For a given set of feasible actions $\mathcal{B} \subset \mathcal{F}$ the decision function $d_i(\cdot)$ may also be defined as a mapping, choosing an action f maximizing conditional expectations from the feasible set $\mathcal{B}$. When the decision function $d_i(\cdot)$ is defined as a conditional probability (7.4) and the Agreement Theorem holds, we designate this situation as an *Agreement in Beliefs*. When the decision function $d_i(\cdot)$ is defined as a conditional expectation (7.5) and the Agreement Theorem holds, we term this situation an *Agreement in Expectations*. Under the common prior assumption, an Agreement in Beliefs was proved by Aumann (1976) and Bacharach (1985), and an Agreement in Expectations by Milgrom (1981), Geanakoplos and Sebenius (1983), Bacharach (1985) and Rubinstein and Wolinsky (1990). All "agreeing to disagree" type results established within Bayesian setting are referred to as *probabilistic Agreement Theorems*. In the next section we extend these results to non-Bayesian setups in which agents' subjective beliefs are represented by a common-but-non-necessarily-additive prior distribution.

7.2 Agreement Theorems under Ambiguity

7.2.1 Sufficient Condition

Throughout our study we consider an interpersonal decision model $\mathcal{I}$ with a finite group of agents. Each agent is endowed with Choquet expected utility preferences. Furthermore, it is assumed that the agents share a common capacity distribution ν on the state space Ω where $\nu(\mathcal{P}_i(\omega)) > 0$ for all states $\omega \in \Omega$ and for all $i \in I$. If the true state is ω, each agent i revises the prior capacity ν given her private information $\mathcal{P}_i(\omega)$ by applying one of the possible updating rules (see Section 6.1.3). Again, the updated capacity $\nu(\cdot \mid \mathcal{P}_i(\omega))$ serve as a basis for agents' decisions. As in the Bayesian framework we mainly consider two types of decision functions:

i) a conditional capacity for some event $E \in \mathcal{A}$,

$$d_i(\omega) = \nu(E \mid \mathcal{P}_i(\omega)), \tag{7.6}$$

or *ii)* a conditional Choquet expectation for some action $f \in \mathcal{F}$,

$$d_i(\omega) = \int_\Omega u \circ f \; d\nu(\cdot \mid \mathcal{P}_i(\omega)). \tag{7.7}$$

In the existing non-Bayesian extensions of probabilistic Agreement Theorems, established by, among others Cave (1983) and Bacharach (1985), the nature of agents' subjective beliefs is inessential and the decision function may be an arbitrary function. To guarantee that the Agreement Theorem holds it is required, that agents are "like-minded", i.e. they would make the same decisions if they had the same information, and that the decision function $d(\cdot)$ satisfies the *Sure-Thing-Condition* (STC).

(STC). *The decision function d_i satisfies the Sure-Thing Condition if and only if, for any partition $E_1, \ldots, E_n$ of Ω it is true that:*

$$d_i(E_1) = \cdots = d_i(E_n) = \xi_i \quad \Rightarrow \quad d(\bigcup_{j=1}^{n} E_j) = \xi_i. \tag{7.8}$$

Let $E_1, \ldots, E_n$ be a partition of Ω. The Sure-Thing-Condition requires that, if an agent i makes the same decision ξ_i knowing which of the mutually exclusive events E_j has occurred, then she also should make the same decision ξ_i without knowing which one occurred, i.e. $E_1 \cup \ldots \cup E_n$. Bacharach (1985) refers to the condition (7.8) as a"[...] fundamental principle of rational decision-making". Cave (1983) and Bacharach (1985) showed that if the agents follow the same decision function satisfying the Sure-Thing-Condition then the Agreement Theorem holds. Note, in the class of probabilistic models, decision functions such as conditional probabilities, conditional expectations, as well as actions maximizing conditional expectations, satisfy the Sure-Thing-Condition on *any* partition. In non-probabilistic models, however, the decision function may satisfy the Sure-Thing-Condition on some fixed partitions, but not on others.

For this reason our first objective is to fix a partition and to look at properties of events of that partition which are sufficient for a decision function $d(\cdot)$ to satisfy the Sure-Thing-Condition on it. It turns out that the decision function $d_i(\cdot)$, defined as a conditional capacity or a conditional Choquet expectation or an action maximizing conditional expectations, satisfies the Sure-Thing-Condition on partitions made up of N-unambiguous events. This condition on its own is a sufficient condition for Agreement Theorem to hold under ambiguity. That is, if each agent i's private information is represented by a partition $\mathcal{P}_i$ made up of N-unambiguous events, then the agents cannot disagree on their commonly known decisions, whatever these decisions are: whether conditional capacities, conditional Choquet expectations or actions maximizing conditional Choquet expectations. In other words, the unambiguous character of agents' private information precludes the possibility of agreeing to disagree on their decisions despite the fact that these decisions are based on diverse information. In view of this result, it seems that asymmetries in private information *do* matter and that they can explain differences in agents' commonly known decisions due to ambiguity of their private information. This result is formally stated and proved in Theorem 1.

Theorem 7.1. *Let ν be a common capacity distribution on Ω and let $\mathcal{A}_N^U \subset \mathcal{A}$ be a* *collection of N-unambiguous events. Let $P_1^i, \ldots, P_k^i, \ldots, P_K^i$ be the events in i's partition $\mathcal{P}_i$. If $P_k^i \in \mathcal{A}_N^U$ for all $k = 1, \ldots, K$ and all agents $i \in I$, then the following statements are true:*

(i) Agreement in Beliefs holds,

(ii) Agreement in Expectations holds.

How strong is the sufficiency condition in Theorem 7.1? In particular, suppose that we adapt a weaker notion of unambiguous events, for instance, the one proposed by Zhang (2002). Is the claim still true that a disagreement in commonly known decisions is impossible? Example 7.1 answers this question negatively. Even a small departure from Nehring's notion of unambiguous events may create disagreement opportunities. That is, if for an agent i her information partition $\mathcal{P}_i$ is made up of Z-unambiguous events, which are not N-unambiguous, then her decision function may violate the Sure-Thing-Condition on $\mathcal{P}_i$. Consequently, we may construct information partitions for other agents and find a state in which agents' decisions are common knowledge and do not coincide after all. Example 7.1 demonstrates a possibility of a disagreement on posterior beliefs among two agents, where one agent is endowed with information partition consisting of Z-unambiguous events.

Example 7.1 (*Disagreement in Beliefs*)**.** *Consider an interpersonal decision model $\mathcal{I}$ with two agents $I = \{A, B\}$, called Anna and Bob, the set of states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, the set of decisions $\mathcal{D} = [0, 1]$ and the decision function defined as in (7.6). Let $\mathcal{P}_A = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\mathcal{P}_B = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be the agents' information partitions. Anna and Bob face the following capacity distribution on $\mathcal{A}$:*

$$
\begin{aligned}
\nu(\omega_j) &= \tfrac{1}{10}, && \text{for any} \quad j = 1, \ldots, 4,\\
\nu(\omega_j, \omega_k) &= \tfrac{1}{2}, && \text{for any} \quad j + k \neq 5,\\
\nu(\omega_j, \omega_k) &= \alpha, && \text{for any} \quad j + k = 5 \text{ where } \alpha \in [\tfrac{1}{10}; \tfrac{1}{2}),\\
\nu(\omega_j, \omega_k, \omega_l) &= \tfrac{6}{10}, && \text{for any} \quad j, k, l = 1, \ldots, 4.
\end{aligned}
$$

Note, all events $\{\omega_j, \omega_k\}$ with $j + k \neq 5$ are Z-unambiguous, but not N-unambiguous. To see this, consider the event $\{\omega_1, \omega_2\}$ and its complement $\{\omega_3, \omega_4\}$. On this partition the capacity sums up to one. Now, if these events were N-unambiguous, then *according to the additive separability property (6.2) the capacity for the event* $\{\omega_1, \omega_3\}$ *were* $\nu(\omega_1, \omega_3) = \nu(\omega_1) + \nu(\omega_3) = \frac{1}{5}$, *but not* $\frac{1}{2}$. *One can verify that the capacity* ν *satisfies the additive separability property (6.4) only on subevents of its unambiguous complements. For instance,* $\nu(\omega_1, \omega_2, \omega_3) = \nu(\omega_1, \omega_2) + \nu(\omega_3) = \frac{6}{10}$. *Accordingly,* $\mathcal{A}_Z^U = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}, \Omega\}$ *is the collection of* Z*-unambiguous events.*

Therefore, Anna's partition is made up of Z*-unambiguous events which are not* N*-unambiguous. Consider the event* $E = \{\omega_1, \omega_3\}$. *At any state Anna and Bob announce their posterior beliefs for the occurrence of* E *given their private information. Given Bob's private information he announces* $d_B(\omega) = \nu(E \mid \mathcal{P}_B(\omega)) = \frac{1}{2}$ *at any state* $\omega \in \Omega$. *Anna has finer information than Bob and therefore her decision equals the conditional capacity, i.e.* $d_A(\omega) = \nu(E \mid \mathcal{P}_A(\omega)) = \frac{1}{5}$ *for all* $\omega \in \Omega$. *Note, Anna's decision function* $d_A(\cdot)$ *violates the Sure-Thing Condition on her partition. Furthermore, since* $\mathcal{M} = \Omega$, *the event that Anna's decision is* $\frac{1}{5}$ *and that Bob's decision is* $\frac{1}{2}$ *is commonly known at any state. That is,* $\mathcal{M}(\omega) = D_A(\frac{1}{5}) \cap D_B(\frac{1}{2}) = \Omega$ *for all* $\omega \in \Omega$. *But, these decisions are in fact not the same. This shows that, if for one agent her private information is made up of* Z*-unambiguous events, which are not* N*-unambiguous, than the Sure-Thing Condition is violated and it is possible that the agents end up agreeing to disagree after all!*

Suppose now Anna's partition $\mathcal{P}_A = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ *were made up of* N*-unambiguous events. In this case, the capacity for the event* E *must be equal to* $\frac{1}{5}$ *due to the additive separability property (6.2). Now,* $\frac{1}{5}$ *is Bob's decision which he announces in all states. Therefore, agents decisions are commonly known at any state and in fact they are the same. Thus, when agents' private information is made up of* N*-unambiguous events it is impossible for them to agree to disagree on their posterior capacities.*

7.2.2 Agreement Theorem - The Converse Result

In this section we address the following issue. Suppose that agents' decisions satisfy the Sure-Thing Condition on their information partitions and that the agents cannot disagree on their commonly known decisions. Can we infer something about the nature of agents' private information? In principle, the answer is "yes". However, what we may infer observing an agreement depends on the type of decisions on which agents agree to agree. There are situations in which Agreement in Beliefs is present and nothing can be said about the nature of agents' private information. Example 7.2 illustrates this point. We consider an interpersonal decision model $\mathcal{I}$ in which agents' decisions are posterior capacities for some event E. Then, it is possible to find a common capacity distribution and an updating rule such that the conditional capacities for E satisfy the Sure-Thing-Condition on their private information partitions, and such that at some state Agreement in Beliefs holds, but agents' partitions are neither made up of N-unambiguous, nor of Z-unambiguous events.

Example 7.2. *Consider the interpersonal decision model $\mathcal{I}$ as it was described in Example 7.1. Suppose now, Anna and Bob face the following capacity distribution on $\mathcal{A}$:*

$$\nu(\omega_j) = \tfrac{1}{9}, \quad \nu(\omega_j, \omega_k) = \tfrac{1}{3}, \quad \nu(\omega_j, \omega_k, \omega_l) = \tfrac{4}{9},$$

where $j, k, l \in \{1, \ldots, 4\}$ are distinct indexes. Consider the event $E = \{\omega_j, \omega_k\}$ where $j+k = 5$. Suppose that at any state agents announce posterior capacities for E. To estimate their posteriors, the agents apply the Bayesian updating rule given their private information. Note that the Bayesian update coincides here with the Maximum-Likelihood and the Full-Bayesian update. Then, given Anna's information her announcement is $\frac{1}{3}$ at any state, i.e. $d_A(\omega, B) = \nu(E \mid \mathcal{P}_A(\omega)) = \frac{1}{3}$ for all ω. Given Bobs's information, his announcement is also $\frac{1}{3}$ at any state, i.e. $d_B(\omega, B) = \nu(E \mid \mathcal{P}_B(\omega)) = \frac{1}{3}$ for all ω. Anna's as well as Bobs's decision function satisfy the Sure-Thing-Condition on their individual partitions. Thus, it is impossible that at some state agents' posteriors for the event E are commonly known and not the same. However, this Agreement in Beliefs does not indicate that Anna's or Bob's private information is in some sense unambiguous. Events in Anna's partition are made up neither of N-unambiguous, nor of Z-unambiguous events. On Anna's partition the capacity ν does not even add up to one.

Such examples for Agreement in Beliefs can be constructed easily when one constrains the analysis to a particular class of capacities. For instance, in Dominiak and Lefort

(2010) we consider the class of neo-additive capacities axiomatized by Chateauneuf, Eichberger, and Grant (2007).[2] We characterize the family of updating rules for neo-additive capacities which are necessary and sufficient for Aumann's Agreement Theorem to hold in the context of such beliefs. The neo-additive capacities, by construction, are not suitable to model unambiguous events, neither in the sense of Nehring (1999) nor in the one of Zhang (2002). This observation provides an additional argument for the claim that an Agreement in Beliefs is too "weak" to infer something about the nature of agents' private information.

Furthermore, it turns out that if at some state ω it impossible that the agents agree to disagree on conditional capacities for some event E then it is also impossible at ω that they agree to disagree on conditional Choquet expectations of binary actions defined on the event E. A binary action or bet $b = x_E y$ is a function which assigns the constant outcome $f(\omega) = x \in X$ to each state ω in E and the constant outcome $f(\omega) = y \in X$ to each ω in E^c. The next proposition states this observation formally.

Proposition 7.1. *Let ν be a common capacity distribution ν on Ω. Let $\mathcal{P}_i$ be i's information partition and let $d_i(\cdot)$ be i's conditional capacity for some event $E \in \mathcal{A}$. Suppose that at some state ω^* Agreement in Beliefs holds for E. Consider a bet $b = x_E y$ defined on the event E with $x, y \in X$. Let $\tilde{d}_i$ be i's conditional Choquet expectation of b. Then, Agreement in Expectations holds at ω^* for b.*

Thus, knowing that agents cannot agree to disagree on expectations for some bet, nothing can be said about the nature of events representing their private information. Then, in the view of Example 7.2 we may find a common capacity distribution and an updating rule such that for some event E an Agreement in Beliefs holds. According to Proposition 7.1, agents will also reach an Agreement in Expectations for bets on the

[2]A neo-additive capacity ν is defined as follows: for any $\emptyset \subsetneq E \subsetneq \Omega$, $\nu(E) = \delta\alpha + (1-\delta)\pi(E)$, where π is a probability measure with support equal to Ω, and $\alpha, \delta \in [0,1]$ are parameters. A neo-additive capacity describes situations in which an agent behaves as if she had an additive probability distribution, but she doubts whether this distribution is the correct one. The parameter $1-\delta$ measures the agent's confidence in π. The parameter α can be viewed as reflecting the agent's ambiguity attitude.

event E and the agents' partitions will neither be made up of N-unambiguous, nor of Z-unambiguous events.

For this reason we constrain our attention to the whole set of possible actions $\mathcal{F}$ and ask again whether it is possible to infer something about the nature of events in an agent's partition knowing that the agents reached Agreement in Expectations for more general action f. Theorem 7.2 answers this question in the affirmative. Agreement in Expectations for an action implies that agents' information partitions are made up of N-unambiguous events.

Theorem 7.2. *Let ν be a common capacity distribution on Ω. Let $\mathcal{A}'$ be a sub-algebra of $\mathcal{A}$. Let $d_i(\cdot)$ be the Choquet conditional expectation for some action f in $\mathcal{F}$. If for any information partition $\mathcal{P}_i = P^i_1, \ldots, P^i_k, \ldots, P^i_K$ such that $P^i_k \in \mathcal{A}'$ for all $k = 1, \ldots, K$ and all agents $i \in I$, $d_i(\cdot)$ satisfies the Sure-Thing-Condition on $\mathcal{P}_i$, then $\mathcal{A}'$ is the algebra made up of N-unambiguous events.*

7.3 Speculative Trade under Ambiguity

In this section the no-trade theorem of Milgrom and Stokey (1982) is generalized within the class of Choquet expected utility preferences. In the view of the aforementioned results, we are able to characterize the properties of agents' private information which are sufficient to guarantee that asymmetric information alone cannot generate any profitable trade opportunities under ambiguity.

We interpret an interpersonal decision model $\mathcal{I}$ as a pure exchange economy with as a single commodity. That is, let $X = \mathbb{R}_+$ be the commodity space and call elements of $\mathcal{F}$ contingent consumption bundles. An allocation a is a family $a = [a_1, \ldots, a_N]$ where each $a_i \in \mathcal{F}$ represents i's contingent consumption bundle. An initial allocation is denoted by $e = [e_1, \ldots, e_N]$, where each $e_i \in \mathcal{F}$ is referred to as i's endowment. As in the previous sections, it is assumed that the agents share an identical capacity distribution ν on $\mathcal{A}$. Moreover, each agent i is characterized by her preferences over $\mathcal{F}$ which are supposed to admit Choquet expected utility representation, an initial endowment $e_i \in \mathcal{A}$, and her private information $\mathcal{P}_i$. A trade $t = [t_1, \ldots, t_N]$ is an N-tuple of functions $t_i : \Omega \to \mathbb{R}$. If the true state is ω, $t_i(\omega)$ corresponds to i's net trade of the single commodity. We say that the trade t is *feasible*, if:

$$\begin{aligned} \sum_{i=1}^{N} t_i(\omega) \leq 0 \quad \forall\, \omega \in \Omega, \\ e_i(\omega) + t_i(\omega) \geq 0 \quad \forall\, \omega \in \Omega,\ \forall i \in I. \end{aligned} \tag{7.9}$$

An initial allocation e is called *ex-ante efficient* if there does not exist a feasible trade t such that at ex-ante stage each agent i prefers the contingent consumption bundle $e_i + t_i$ to her endowment e_i, i.e.:

$$\int_\Omega u \circ (e_i + t_i)\ d\nu \geq \int_\Omega u \circ e_i\ d\nu \quad \forall i \in I. \tag{7.10}$$

Suppose that the agents trade to an ex-ante efficient allocation e before any information is revealed. After the receipt of private information the market is reopened and the agents have the chance to reallocate the initial allocation e through a feasible trade t. That is, when the true state is ω, each agent i observes $\mathcal{P}_i(\omega)$ and then the feasible trade t is proposed. We call the feasible trade t *acceptable* at ω (or weakly preferable to a zero trade) if each agent i prefers the contingent consumption bundle $e_i + t_i$ to her endowment e_i given $\mathcal{P}_i(\omega)$, i.e.:

$$\int_\Omega u \circ (e_i + t_i)\ d\nu(\cdot \mid \mathcal{P}_i(\omega)) \geq \int_\Omega u \circ e_i\ d\nu(\cdot \mid \mathcal{P}_i(\omega)). \tag{7.11}$$

In Bayesian frameworks, where all uncertainty is quantifiable by a common additive probability distribution, the receipt of private information can not create any incentives to re-trade an ex-ante efficient allocation, even though the information the agents receive may be distinct. What are the conditions on agents private information which are sufficient to ensure that the no-trade theorem still holds in the presence of common, but non-additive priors? It turns out that as long as agents' information partitions are made up of N-unambiguous events, at interim stage the agents will not find it advantageous to re-trade an initially efficient allocation. In other words, when each agent's private information is free from ambiguity it is impossible that purely speculative trade occurs only due to differences in their private information. This result is stated in the following theorem.

Corollary 7.1 (No-Trade Theorem)**.** *Let ν be a common capacity distribution on Ω and let $\mathcal{A}_N^U \in \mathcal{A}$ be a collection of N-unambiguous events. Let $P_1^i, \ldots, P_k^i, \ldots, P_K^i$ be the events in i's partition $\mathcal{P}_i$. Suppose that $P_k^i \in \mathcal{A}_N^U$ for all $k = 1, \ldots, K$ and for all agents $i \in I$. Suppose the initial allocation $e = [e_1, \ldots, e_N]$ is ex-ante efficient. Let $t = [t_1, \ldots, t_N]$ be a trade proposed at interim stage. If it is common knowledge at ω^* that t is feasible and acceptable, then $t_1(\omega^*) = \ldots = t_N(\omega^*) = 0$.*

Corollary 7.1 provides an intuitive explanation for the existence of speculative trade. As was already stipulated by Knight (1921), it is the presence of ambiguity, or what he called "unmeasurable uncertainty", that generates profitable trade opportunities. When agents' private information is ambiguous, then, conditional on different information agents may expect gains from re-trading an initially efficient allocation. Example 7.3 illustrates how gains from trade may occur even when one agent's private information partition is made up of Z-unambiguous events, which are not N-unambiguous.

Example 7.3. *Let $X = \mathbb{R}_+$ be the set of outcomes. Consider an interpersonal decision model $\mathcal{I}$ with the set of contingent consumption bundles $\mathcal{F} = \{a \mid a : \Omega \to \mathbb{R}_+\}$ and the same information structure and the same capacity distribution as in Example 7.1. Let $e = [e_A = (2, 0, 2, 0), e_B = (1, 2, 1, 0)]$ be the initial allocation. Suppose Anna and Bob are risk neutral. By computing the Choquet expectations of e_A and e_B with respect to u and ν for both agents, we get:*

$$\int u \circ e_A \; d\nu = 2\frac{1}{2} + 0[1 - \frac{1}{2}] = 1. \tag{7.12}$$

$$\int u \circ e_B \; d\nu = 2\frac{1}{10} + 1[\frac{6}{10} - \frac{1}{10}] + 0[1 - \frac{6}{10}] = \frac{7}{10}, \tag{7.13}$$

At ex-ante stage there is no feasible trade t that would make both agents better off. In fact, the contingent consumption bundle e_A makes Bob better off, but any feasible trade would make Anna worse off. Hence, e is ex-ante efficient. Now, let ω_1 be the true state. Because of Bob's information at ω_1, i.e. $\mathcal{P}_B(\omega_1) = \Omega$, his evaluation of e_A and e_B does not change. Given Anna's information at ω_1, i.e. $\mathcal{P}_A(\omega_1) = \{\omega_1, \omega_2\}$, she updates her *preferences by taking into account the conditional capacities $\nu(\omega_1 \mid \mathcal{P}_A(\omega_1)) = \nu(\omega_2 \mid \mathcal{P}_A(\omega_1)) = \frac{2}{10}$ and calculates the conditional Choquet expectations of e_A and e_B:*

$$\int u \circ e_A \, d\nu(\cdot \mid \mathcal{P}_A(\omega_1)) = 2\frac{2}{10} + 0[1 - \frac{2}{10}] = \frac{4}{10}. \tag{7.14}$$

$$\int u \circ e_B \, d\nu(\cdot \mid \mathcal{P}_A(\omega_1)) = 2\frac{2}{10} + 1[1 - \frac{2}{10}] = \frac{12}{10}, \tag{7.15}$$

Now, consider the trade $t := [t_A = (-1, 2, -1, 0), t_B = (1, -2, 1, 0)]$ proposed at the interim stage. Note, since $e_A + t_A = e_B$ and $e_B + t_B = e_A$ the trade t is feasible. By (7.14) and (7.15) Anna prefers e_B to e_A and by (7.12) and (7.13) Bob prefers e_A to e_B making the trade t acceptable at ω_1. At ω_1 it is commonly known between Anna and Bob that the trade t is feasible and acceptable and t is not the null-trade. The events in Anna's partition are Z-unambiguous, but not N-unambiguous; due to this fact differences in agents' private information matter and make a profitable trade possible.

A few remarks with regard to the related literature are in order. Close to our approach are the contributions of Rubinstein and Wolinsky (1990) and Dow, Madrigal, and Werlang (1990). Their results are obtained without constraining the analysis to a particular class of ambiguity-sensitive preferences. Rubinstein and Wolinsky (1990) argued that Milgrom-Stokey's result is valid for any theory of decision making under uncertainty as long as preferences satisfy dynamic consistency. Dow, Madrigal, and Werlang (1990) showed that the no-trade theorem is true if and only if preferences are representable by a state-additive utility function. Corollary 7.1 can be viewed as characterizing those properties of events in information partitions on which dynamic consistency as well as state-additivity of Choquet preferences are satisfied. Then, if a fixed partition is made up of N-unambiguous events, then Choquet expected utility preferences respect dynamic consistency on that partition (see Section 6.2.1). Furthermore, Choquet preferences respect dynamic consistency on a fixed partition if and only if the Choquet integral satisfies the additivity property constrained to that partition (see Sarin and Wakker, 1998a). In two other related works, Ma (2002) and Halevy (2004) attempt to establish sufficient condition for the no-trade theorem to be true for the class of preferences violating consequentialism in some respect.

7.4 Proofs

Proof of Theorem 7.1. First we show that (ii) is true.

Step 1 Consider an agent $i \in I$. Let $P_1, \dots, P_k, \dots, P_K$ be the events in the agent i's partition $\mathcal{P}_i$. That is $\mathcal{P}_i(\omega) = \mathcal{P}_i(\omega^{'})$ for all states $\omega, \omega' \in P_k$. Suppose that the i's information partition $\mathcal{P}_i$ is made up off N-unambiguous events, i.e. $P_k \in \mathcal{A}_N^U$ for any $k = 1, \dots, K$. Fix an action $f \in \mathcal{F}$. Let d_i be the Choquet decision rule defined as in (7.7). Furthermore, we assume that the agent i computes the posterior capacity $\nu(\cdot \mid P_k)$ conditional on P_k by applying Bayes' rule. This assumption is reasonable, since all other updating rules, among others those defined in Section 6.1.3, coincide with Bayes' rule when conditioning on partitions made up off N-unambiguous events (see Proposition 6.3). Suppose that for any index $k = 1, \dots, K$ the conditional Choquet expectation of f given P_k is equal to ξ:

$$d_i(P_1) = \dots d_i(P_k) = \dots = d_i(P_K) = \xi, \tag{7.16}$$

where:

$$\begin{aligned} d_i(P_k) &= \int_\Omega u \circ f \; d\nu(\cdot \mid P_k) \\ &= \sum_{j=1}^{n-1} [u(x_j) - u(x_{j+1})] \; \frac{\nu(E_1, \dots, E_j \;\cap P_k)}{\nu(P_k)} \\ &= \xi. \end{aligned}$$

By the additive separability condition (6.2) of N-unambiguous events the Choquet expected value of f with respect to the prior capacity ν can be written as:

$$\int_\Omega u \circ f \; d\nu \;\; = \;\; \sum_{k=1}^{n} \int_{P_k} u \circ f \; d\nu. \tag{7.17}$$

Thus, we obtain:

$$\int_\Omega u \circ f \; d\nu(\cdot) \;\; = \;\; \sum_{k=1}^{n} \nu(P_k) \int_{P_k} u \circ f \; d\nu(\cdot \mid P_k) = \sum_{k=1}^{n} \nu(P_k) \; \xi = \xi.$$

Therefore, $d_i(\bigcup_{j=1}^{K} P_j) = \xi$ shows that the Choquet conditional expectations of f satisfy the Sure-Thing-Condition on partitions made up off N-unambiguous events.

Step 2 Fix an agent i. Let $D_i(\xi_i) = \{\omega : d(\mathcal{P}_i(\omega)) = \xi_i\}$ be the event that the i's decision is ξ_i. Suppose at some state ω^* the event $\bigcap_{i \in I} D_i(x_i)$ is common knowledge, i.e. $\mathcal{M}(\omega^*) \subseteq \bigcap_{i \in I} D_i(x_i)$. Denote by $Q = \mathcal{M}(\omega^*)$ the member of $\mathcal{M}$ that contains ω^*. Let $Q_1, \ldots, Q_l, \ldots, Q_L$ be events in i's partition $\mathcal{P}_i$ such that $Q = \bigcup_{l=1}^{L} Q_l$. By assumption, $\mathcal{M}(\omega^*) \subseteq D_i(\xi_i)$ and $d_i(\mathcal{P}_i(\omega)) = \xi_i$ for any $\omega \in Q_l$ with $l = 1, \ldots, L$. Furthermore, since each event Q_l is N-unambiguous the decision function $d_i(\cdot)$ satisfies the Sure-Thing-Condition by Step 1. Thus, $d_i(Q) = \xi_i$. The same argument is true for any agent $j \in I \setminus \{i\}$. That is, $d_j(Q) = \xi_j$. Thus, $\xi_1 = \ldots = \xi_N$. The fact that the Sure-Thing-Principle is sufficient for Agreement Theorem to be true has been proved, among others, by Bacharach (1985, Theorem 3, p.182).

□

Proof of Proposition 7.1. Fix an event E. Let $D_i(\alpha_i) = \{\omega : \nu(E \mid \mathcal{P}_i(\omega)) = \alpha_i\}$ be the event that i's conditional capacity of E is α_i. Suppose that at some state ω^* the agents reached Agreement in Beliefs. That is, the event $\bigcap_{i \in I} D_i(\alpha_i)$ is common knowledge at ω^* and agents' conditional capacities for E are the same, $\alpha_1 = \ldots = \alpha_N$.

For any $x, y \in X$ such that $x \succ y$ let $b = x_E y$ be a bet. Fix an agent i. Let $P_1, \ldots, P_k, \ldots, P_n$ be events in i's information partition $\mathcal{P}_i$. Let $d_i(P_k)$ be the i's conditional Choquet expectation of b given P_k. Suppose that $d_i(P_k) = \beta_i$ for any $k = 1, \ldots, K$, i.e.:

$$\begin{aligned} d_i(P_k^i) &= \int_\Omega u \circ b \; d\nu(\cdot \mid P_k) \\ &= [u(x) - u(y)]\nu(E \mid P_k^i) + u(y) \\ &= \beta_i, \end{aligned}$$

Rearranging the above equation we get for any $k = 1, \ldots, n$:

$$\begin{aligned} \nu(E \mid P_k^i) &= \frac{\beta - u(y)}{u(x) - u(y)} \\ &= \alpha_i. \end{aligned}$$

Thus, since Agreement in Beliefs holds it follows that:

$$\begin{aligned} \nu(E \mid \bigcup_{k=1}^{K} P_k) &= \frac{\beta - u(y)}{u(x) - u(y)} \\ &= \alpha_i. \end{aligned}$$

Therefore:

$$\begin{aligned} d_i(\bigcup_{k=1}^{K} P_k^i) &= \int_\Omega u \circ b \; d\nu(\cdot) \\ &= [u(x) - u(y)]\nu(E) + u(y) \\ &= \beta_i. \end{aligned}$$

Let $D_i(\beta_i) = \{\omega : d(\mathcal{P}_i(\omega)) = \beta_i\}$ be the event that i's conditional Choquet expectation of b is β_i. Since the Sure-Thing Condition holds, the event $\bigcap_{i \in I} D_i(\beta_i)$ is common knowledge at ω^* and in fact $\beta_1 = \ldots = \beta_N$. Therefore, we conclude that an Agreement in Beliefs implies an Agreement in Expectations for binary actions. □

Proof of Theorem 7.2. Let $\mathcal{A}'$ be a sub-algebra of $\mathcal{A}$. In Step 1 we show that for any event $E \in \mathcal{A}'$ and all events $F, G \in \mathcal{A}$ such that $\emptyset \not\subseteq F, G \not\subseteq E^c$, the capacity ν has the following property:

$$\nu(E \cup G) - \nu(G) = \nu(E \cup F) - \nu(F). \tag{7.18}$$

In Step 2 it is shown that for any event $E \in \mathcal{A}'$ the capacity ν is separable among all subevents of E^c, i.e. for any $F \subset E^c$:

$$\nu(E) = \nu(E \cup F) - \nu(F) = 1 - \nu(E^c). \tag{7.19}$$

Step 1. Let $A_1, A_2, A_3 \in \mathcal{A}$ be a collection of disjoint events partitioning the event E^c. Consider an action $f = (x_1 A_1, x_2 A_2, x_3 A_3)$ with outcomes $x_1, x_2, x_3 \in X$ such that $x_1 < x_2 < x_3$. Suppose that the Choquet expected utility of f conditional on E^c equals x, i.e.:

$$f \sim_{E^c} x. \tag{7.20}$$

By computing the conditional Choquet expectation of f we get:

$$\int u \circ f \; d\nu(\cdot \mid E^c) = u(x_1) \left[1 - \nu(A_2, A_3 \mid E^c)\right] + u(x_2) \left[\nu(A_2, A_3 \mid E^c) - \nu(A_3 \mid E^c)\right] + u(x_3) \; \nu(A_3 \mid E^c) = x. \tag{7.21}$$

Now, consider an action $g = f_{E^c} x$. By the assumption (7.20) the conditional Choquet expectation of g satisfies the Sure-Thing-Condition on the partition E, E^c, i.e.:

$$\int u \circ g \; d\nu(\cdot \mid E^c) = x \quad \text{, and} \quad \int u \circ g \; d\nu(\cdot \mid E) = x$$

implies

$$\int u \circ g \; d\nu(\cdot \mid \Omega) = x. \tag{7.22}$$

When computing the unconditional Choquet integral (7.22) of g with respect to ν we consider two cases. In Case 1 we consider any x such that $x_2 < x < x_3$. In Case 2, we consider any x such that $x_1 < x < x_2$.

Case 1. For any x such that $x_2 < x < x_3$ the unconditional Choquet integral of g yields:

$$\begin{aligned} \int u \circ g \; d\nu &= u(x_1) \left[1 - \nu(A_2, E, A_3)\right] + u(x_2) \left[\nu(A_2, E, A_3) - \nu(E, A_3)\right] \\ &\quad + u(x) \left[\nu(E, A_3) - \nu(A_3)\right] + u(x_3) \; \nu(A_3) \\ &= x. \end{aligned} \tag{7.23}$$

Solving Equation (7.23) for x we get:

$$\begin{aligned} \int u \circ g \; d\nu(\cdot \mid E) &= \frac{1}{1 - \nu(E, A_3) + \nu(A_3)} \Big\{ u(x_1) \left[1 - \nu(A_2, E, A_3)\right] \\ &\quad + u(x_2) \left[\nu(A_2, E, A_3) - \nu(E, A_3)\right] + \; u(x_3) \; \nu(A_3) \Big\} \\ &= x. \end{aligned} \tag{7.24}$$

Equation (7.24) is true for any x_1, x_2, x_3 such that $x_1 < x_2 < x_3$ and any $g = f_{E^c} x$ with x such that $x_2 < x < x_3$. Thus, when fixing the values x_1, x_2 and varying the value of x_3 we get from Equation (7.21) and (7.24):

$$\nu(A_3 \mid E) = \frac{\nu(A_3)}{1 - \nu(E, A_3) + \nu(A_3)} \tag{7.25}$$

Case 2. For x such that $x_1 < x < x_2$ computing the unconditional Choquet integral of g yields:

$$\begin{aligned} \int u \circ g \; d\nu &= u(x_1) \left[1 - \nu(E, A_2, A_3)\right] + u(x) \left[\nu(E, A_2, A_3) - \nu(A_2, A_3)\right] \\ &\quad + u(x_2) \left[\nu(A_2, A_3) - \nu(A_3)\right] + u(x_3) \; \nu(A_3) \\ &= x. \end{aligned} \tag{7.26}$$

Solving the above Equation (7.26) for x'' we get:

$$\begin{aligned} \int u \circ g \; d\nu(\cdot \mid E) &= \frac{1}{1 - \nu(E, A_2, A_3) + \nu(A_2, A_3)} \Big\{ u(x_1) \left[1 - \nu(E, A_2, A_3)\right] \\ &\quad + u(x_2) \left[\nu(E, A_2, A_3) - \nu(A_2, A_3)\right] + \; u(x_3) \; \nu(A_3) \Big\} \\ &= x. \end{aligned} \tag{7.27}$$

Again, Equation (7.27) is true for any x_1, x_2, x_3 such that $x_1 < x_2 < x_3$ and any $g = f_{E^c}x$ with x such that $x_1 < x < x_2$. Thus, when fixing the values x_1, x_2 and varying the value of x_3 we get from Equations (7.21) and (7.27):

$$\nu(A_3 \mid E) = \frac{\nu(A_3)}{1 - \nu(E, A_2, A_3) + \nu(A_2, A_3)}. \tag{7.28}$$

From Equations (7.25) and (7.28) we conclude that:

$$\nu(E, A_3) - \nu(A_3) = \nu(E, A_2, A_3) - \nu(A_2, A_3). \tag{7.29}$$

Now, we repeat the same argument for an action $h = (y_1 A_1, y_2 A_2, y_3 A_3)$ with outcomes $y_1, y_2, y_3 \in X$ such that $y_1 < y_3 < y_2$. Suppose that $h \sim_{E^c} y$ and construct an action $k = h_{E^c}y$. By construction, the conditional Choquet expectation of k satisfies the Sure-Thing-Condition on the partition E, E^c. After having considered two cases, Case 1 in which y is such that $y_3 < y < y_2$ and in Case 2 in which y is such that $y_1 < y < y_3$, we conclude:

$$\nu(E, A_2) - \nu(A_2) = \nu(E, A_2, A_3) - \nu(A_2, A_3). \tag{7.30}$$

From Equation (7.29) and Equation 7.30 it follows then that:

$$\nu(E, A_2) - \nu(A_2) = \nu(E, A_3) - \nu(A_3). \tag{7.31}$$

Therefore, it is true that for any event $E \in \mathcal{A}'$ for all events $F, G \in \mathcal{A}$ such that $\emptyset \not\subseteq F, G \not\subseteq E^c$:

$$\nu(E \cup G) - \nu(G) = \nu(E \cup F) - \nu(F). \tag{7.32}$$

Step 2. Let $A_1, A_2 \in \mathcal{A}$ be two disjoint events partitioning the event E and $A_3, A_4 \in \mathcal{A}$ two events partitioning E^c. Consider an action $f = (x_1 A_1, x_2 A_2, x_3 A_3, x_4 A_4)$ with outcomes $x_1, x_2, x_3, x_4 \in X$ such that $x_1 < x_3 < x_4 < x_2$. Suppose the conditional Choquet expectation of f satisfies the Sure-Thing-Condition on the partition E, E^c, i.e.:

$$\int u \circ f \; d\nu(\cdot \mid E^c) = x \quad \text{and} \quad \int u \circ f \; d\nu(\cdot \mid E) = x$$

implies

$$\int u \circ f \; d\nu(\cdot \mid \Omega) = x. \tag{7.33}$$

By computing the respective conditional Choquet integrals of f we have:

$$\int u \circ f \; d\nu(\cdot \mid E) \;=\; x_1 \left[1 - \nu(A_2 \mid E)\right] + x_2 \; \nu(A_2 \mid E) = x, \tag{7.34}$$

$$\int u \circ f \; d\nu(\cdot \mid E^c) \;=\; x_3 \left[1 - \nu(A_4 \mid E^c)\right] + x_4 \; \nu(A_4 \mid E^c) = x. \tag{7.35}$$

The unconditional Choquet integrals of f is:

$$\begin{aligned}\int u \circ f \; d\nu \;=\; & x_1 \left[1 - \nu(A_3, A_4, A_2)\right] + x_3 \left[\nu(A_3, A_4, A_2) - \nu(A_4, A_2)\right] \\ & + x_4 \left[\nu(A_4, A_2) - \nu(A_2)\right] + x_2 \; \nu(A_2).\end{aligned} \tag{7.36}$$

From Step 1 and Equation (7.34) we obtain the following equation:

$$x_1 \left[1 - \nu(A_3, A_4, A_2)\right] + x_4 \; \nu(A_4) = x \left[1 - \nu(A_4, A_3, A_2) + \nu(A_4)\right], \tag{7.37}$$

and thus:

$$x_1 \left[1 - \nu(A_3, A_4, A_2)\right] + x \left[\nu(A_4, A_3, A_2) - \nu(A_4)\right] + x_4 \; \nu(A_4) = x. \tag{7.38}$$

From Equation (7.36) and (7.38) we get:

$$x_2 \left[\nu(A_3, A_4, A_2) - \nu(A_3, A_4)\right] + x_3 \left[\nu(A_3, A_4) - \nu(A_4)\right] = x \left[\nu(A_3, A_4, A_2) - \nu(A_4)\right]. \tag{7.39}$$

and thus:

$$x_2 \frac{\left[\nu(A_3, A_4, A_2) - \nu(A_3, A_4)\right]}{\left[\nu(A_3, A_4, A_2) - \nu(A_4)\right]} + x_3 \frac{\left[\nu(A_4, A_2) - \nu(A_4)\right]}{\left[\nu(A_3, A_4, A_2) - \nu(A_4)\right]} = x. \tag{7.40}$$

Recall, in Equation (7.35) we had:

$$x_3 \left[1 - \nu(A_4 \mid E^c)\right] + x_4 \; \nu(A_4 \mid E^c) = x, \tag{7.41}$$

Therefore, for any $x_3, x_4 \in X$ such that $x_3 < x_4$ we have:

$$\nu(A_4 \mid E^c) = \frac{\nu(A_4, A_2) - \nu(A_4)}{\nu(A_3, A_4, A_2) - \nu(A_4)}, \tag{7.42}$$

and

$$1 - \nu(A_4 \mid E^c) = \frac{\nu(A_3, A_4, A_2) - \nu(A_3, A_4)}{\nu(A_3, A_4, A_2) - \nu(A_4)}. \tag{7.43}$$

Now, let us consider an action $g = (x_1 A_1, x_2 A_2, x_3 A_3, x_4 A_4)$ with outcomes x_1, x_2, x_3, $x_4 \in X$ such that $x_1 < x_4 < x_3 < x_2$. The same argument as above leads to the conclusion that:

$$1 - \nu(A_3 \mid E^c) = \frac{\nu(A_3, A_4, A_2) - \nu(A_3, A_4)}{\nu(A_3, A_4, A_2) - \nu(A_4)}. \tag{7.44}$$

After applying Step 1 to the partition A_4, A_4^c we get:

$$\nu(A_3, A_4, A_2) - \nu(A_3, A_2) = \nu(A_4, A_2) - \nu(A_2). \tag{7.45}$$

Thus, by Equation (7.42), (7.44) and (7.45) we have:

$$\nu(A_4 \mid E^c) = 1 - \nu(A_3 \mid E^c). \tag{7.46}$$

After applying Step 1 to the partition E, E^c we get:

$$\nu(A_4 \mid E^c) = \frac{\nu(A_4)}{1 + \nu(A_4) - \nu(A_4, E)}, \tag{7.47}$$

and

$$\nu(A_3 \mid E^c) = \frac{\nu(A_3)}{1 + \nu(A_3) - \nu(A_3, E)}. \tag{7.48}$$

Thus, by Equation (7.46), (7.47) and (7.48) we obtain:

$$\frac{\nu(A_4)}{1 + \nu(A_4) - \nu(A_4, E)} = \frac{1 - \nu(A_3, E)}{1 + \nu(A_3) - \nu(A_3, E)}, \tag{7.49}$$

From Step 1 we know that:

$$\nu(A_4, E) - \nu(A_4) = \nu(A_3, E) - \nu(A_3). \tag{7.50}$$

and therefore:

$$\nu(A_4) + \nu(A_4, E) \quad = \quad 1, \tag{7.51}$$

$$\nu(A_4) + \nu(A_4^c) \quad = \quad 1. \tag{7.52}$$

Step 3. Let $A_1, A_2, A_3 \in \mathcal{A}$ be events partitioning the event E and Let $A_4, A_5 \in \mathcal{A}$ be events partitioning the complementary event E^c. By applying the argument from Step 1 when deriving the updating rule we obtain:

$$\nu(A_2, A_3 \mid E) - \nu(A_3 \mid E) = \frac{\nu(A_2, A_3)}{1 + \nu(A_2, A_3) - \nu(E^c, A_2, A_3)} - \frac{\nu(A_2)}{1 + \nu(A_2) - \nu(E^c, A_2)}.$$

From the property of the capacity ν derived in Step 1 we get:

$$\nu(A_2, A_3 \mid E) - \nu(A_3 \mid E) = \frac{\nu(A_2, A_3) - \nu(A_2)}{1 + \nu(A_1) - \nu(E^c, A_1)}. \tag{7.53}$$

Furthermore, from Step 2 we get:

$$\nu(A_2, A_3 \mid E) = \frac{\nu(A_2, A_3, A_4) - \nu(A_4)}{\nu(A_1, A_2, A_3, A_4) - \nu(A_4)}, \tag{7.54}$$

$$\nu(A_3 \mid E) = \frac{\nu(A_2, A_4) - \nu(A_4)}{\nu(A_1, A_2, A_3, A_4) - \nu(A_4)}. \tag{7.55}$$

Some computations yield:

$$\nu(A_2, A_3 \mid E) - \nu(A_3 \mid E) = \frac{\nu(A_2, A_3, A_4) - \nu(A_4)}{\nu(A_1, A_2, A_3, A_4) - \nu(A_2, A_4)}, \tag{7.56}$$

$$= \frac{\nu(A_2, A_3) - \nu(A_3)}{\nu(A_1, A_2, A_3, A_4) - \nu(A_4)}, \tag{7.57}$$

$$= \frac{\nu(A_2, A_3) - \nu(A_3)}{1 + \nu(A_1) - \nu(E^c, A_4)}. \tag{7.58}$$

and thus:

$$\nu(A_1, A_2, A_3, A_4) - \nu(A_4) = 1 + \nu(A_1) - \nu(E^c, A_4). \tag{7.59}$$

Again, from Step 1 and 2 we get the following equality:

$$\nu(A_1 \mid E) = \frac{\nu(A_1, A_4) - \nu(A_4)}{\nu(A_1, A_2, A_3, A_4) - \nu(A_4)} = \frac{\nu(A_1)}{1 + \nu(A_1) - \nu(E^c, A_4)}. \tag{7.60}$$

By Equation (7.59) the denominators are the same and thus:

$$\nu(A_1) = \nu(A_1, A_4) - \nu(A_4), \tag{7.61}$$

and the capacity ν is updated according to Bayes' rule, i.e.:

$$\nu(A_1 \mid E) = \frac{\nu(A_1)}{\nu(E)}. \tag{7.62}$$

Step 4. Fix an event $E \in \mathcal{A}'$ and let $A \in \mathcal{A}$ be an event such that $E \cap A \neq \emptyset$ and $E^c \cap A \neq \emptyset$. Suppose that:

$$\nu(A \mid E) = \alpha, \tag{7.63}$$

$$\nu(A \mid E^c) = \beta < \alpha. \tag{7.64}$$

Let x be an outcome for which Choquet conditional expectation of the action $f = x_A 0$ is equal to α, i.e.:

$$\int u \circ f \; d\nu(\cdot \mid E^c) = x \; \nu(A \mid E^c),$$

$$= \alpha. \tag{7.65}$$

Now, consider an action $g = (x, E^c \cap A; 1, E \cap A; 0)$. Suppose the conditional Choquet expectation of g satisfies the Sure-Thing-Condition on the partition E, E^c, i.e.:

$$\int u \circ g \; d\nu(\cdot \mid E^c) = \alpha \quad \text{and} \quad \int u \circ g \; d\nu(\cdot \mid E) = \alpha$$

implies

$$\int u \circ g \; d\nu(\cdot \mid \Omega) = \alpha. \tag{7.66}$$

The unconditional Choquet expectation of g is:

$$\begin{aligned} \int u \circ g \; d\nu &= 1\Big[\nu(A) - \nu(A \cap E^c)\Big] + x\nu(A \cap E^c), \\ &= \alpha. \end{aligned} \tag{7.67}$$

From Step 3 we know that the updating rule is Bayes' rule:

$$\nu(A \mid E^c) = \frac{\nu(E^c \cap A)}{\nu(E^c)},$$

and thus:

$$x \; \nu(A \mid E^c) = \alpha \; \nu(E^c). \tag{7.68}$$

From Equation (7.67) and (7.68) we have:

$$\begin{aligned} \int u \circ g \; d\nu &= 1\Big[\nu(A) - \nu(A \cap E^c)\Big] + \alpha \; \nu(E^c), \\ &= \alpha. \end{aligned} \tag{7.69}$$

From Equation (7.68) and Step 2 we obtain:

$$\begin{aligned} \int u \circ g \; d\nu = 1\Big[\nu(A) - \nu(A \cap E^c)\Big] &= \alpha(1 - \nu(E^c)), \\ &= \alpha \; \nu(E). \end{aligned} \tag{7.70}$$

Thus we have:

$$\frac{\nu(A) - \nu(A \cap E^c)}{\nu(E)} = \frac{\nu(A \cap E)}{\nu(E)} = x, \tag{7.71}$$

showing that E is N-unambiguous event, that is for any $A \in \mathcal{A}$ the capacity ν is additive separable:

$$\nu(A) = = \nu(A \cap E^c) + \nu(A \cap E). \tag{7.72}$$

□

Proof of Corollary 7.1. Follows directly from Theorem 6.1 and 7.2 □

Chapter 8

Conclusion

The goal of this thesis is to examine how new information affects choice behavior under ambiguity. We focused on static, dynamic, and interpersonal decision problems. Our first experimental results unequivocally confirm that ambiguity-sensitive behavior is a robust phenomenon. In each of the two Ellsberg experiments, run as a part of this thesis, we observed that fifty percent of subjects exhibit ambiguity-averse behavior and about ten percent of subjects are ambiguity-loving. Neither of these two attitudes towards ambiguity can be modeled by the subjective expected utility theory of Savage (1954). To accommodate ambiguity-sensitive behavior, several alternatives to Savage's theory have been proposed in the literature. However, to make these alternatives attractive for economic applications it is important to know first how well they perform descriptively, and second, which economic facts can they explain in contrast to the orthodox expected utility theory.

Our first investigation concerned static decision problems. We tested the descriptive validity of the widely accepted methodology used to formalize the notion of different ambiguity attitudes, namely, that ambiguity-averse subjects are randomization-loving, while ambiguity-loving subjects are randomization-averse. Our experimental data do not support this view. Ambiguity-averse subjects are more likely to be randomization-neutral rather than randomization-loving. This behavior can be explained by Choquet expected utility theory within Savage's framework when the randomization device is modeled as part of an extended state space, but not in the Anscombe-Aumann framework. Furthermore, we also observe a considerable number of ambiguity-averse subjects who exhibit a contempt for randomization. These observations suggest that ambiguity models which do not exogenously assume a specific relationship between ambiguity and randomization attitudes would be better suited to describe real behavior in the presence of ambiguity.

Next, we focused on dynamic decision problems. In a dynamic version of the classical 3-color experiment of Ellsberg (1961) we tested whether subjects behave consistently with either dynamic consistency or consequentialism. We find that more subjects act in line with consequentialism rather than with dynamic consistency and that this result is even stronger among ambiguity-averse subjects. This evidence can be seen as support for theories of updating ambiguity-sensitive preferences which maintain consequentialism and relax dynamic consistency. This approach is pursued, for instance, to justify behaviorally the Full-Bayesian updating rule for the Choquet expected utility preferences by Eichberger, Grant, and Kelsey (2007) and for the maxmin expected utility preferences by Pires (2002). Furthermore, we find additional violation of the subjective expected utility theory in the dynamic experiment. Several subjects who are classified as ambiguity-neutral in the static choice situation do not exhibit Bayesian behavior in the dynamic extension. They violate either dynamic consistency or consequentialism. Therefore, the dynamic version of the 3-color experiment can also be seen as a tool to test Bayesianism and to make the observation from static experiment more robust.

In the second part we continued to study dynamic choice problems, but constrained our attention to the class of Choquet expected utility preferences. We argued that this class of preferences has a very attractive feature. Namely, it is possible to characterize dynamic properties of Choquet preferences from a static point of view by constraining the analysis to a fixed collection of events. Assuming consequentialism, we showed that Choquet expected utility preferences respect dynamic consistence on a fixed collection of events if and only if these events are unambiguous in the sense of Nehring

(1999). Accordingly, one can apply the same techniques used in expected utility theory to solve optimization problems, e.g. backward induction and dynamic programming, presupposed that the events in a fixed decision tree are Nehring-unambiguous events.

In the last part of this thesis we applied the Choquet expected utility theory to interpersonal decision problems. We showed that for this class of models asymmetric information matters and can explain differences in commonly known decisions. Under the common capacity assumption it was shown that whenever agents' private information partitions are made up of unambiguous events in the sense of Nehring (1999) then it is impossible that the agents disagree on commonly known decisions, whatever these decisions are, whether conditional beliefs or conditional expectations. Consequently, the possibility of speculative trade is precluded only if private information is made up of unambiguous events in this peculiar sense. Even a small departure from that notion of unambiguous events creates profitable trade opportunities due to differences in agents' private and ambiguous information. The presence of ambiguity offers an intuitive explanation for the existence of gambling behavior and of speculative trade. This explanation seems to be less radical than the heterogeneous priors approach of Morris (1994).

Our experimental results strengthen the evidence that subjects facing ambiguity behave in a manner inconsistent with Bayesianism. Non-Bayesian decision theory offers attractive tools to incorporate such behavior into economic theory. I argue that the Choquet expected utility theory is a particularly interesting and promising approach. First, the Choquet expected utility theory makes accurate predictions with regards to ambiguity and randomization attitudes. Second, it allows for characterization of dynamic properties of preferences from static point of view. Finally, Choquet expected utility theory makes it possible to gain new insights into the role that ambiguity plays in economic decisions. One such insight is that differences in private information matter due to ambiguity and can explain, unlike the subjective expected utility theory, the existence of purely speculative behavior.

Instructions

Welcome to our experiment! These instructions are the same for all participants. During the experiment, we ask you to remain silent and not to talk with other participants. Please switch off your mobile phones and leave them switched off until the end of the experiment. If you have any questions, please raise your hand and one of the experimentators will come to you.

Aim and structure of the experiment

This experiment is about decisions under uncertainty. You will be presented with different tickets and asked to value these tickets. To do so, you get a choice between the ticket and different fix payments. There are no „right“ or „wrong“ answers. Only your preferences count. Depending on your preferences, it may well be that you find this easy. Respond truthfully whether you prefer the ticket or the fix payment because these alternatives are real and not only hypothetical. So, if you decide for a ticket, you will actually get this ticket. If you decide for a fix payment, you will receive this payment.

Throughout the experiment, Taler are used as a currency unit, which are later converted at a rate of 100 Talern = 10 Cent. The amount will be rounded up to full cents and paid out. The deciscions of other participants have no effect on your payoff.

Uncertainty

Three sources of uncertainty play a role for the tickets.

- A **coin** will be thrown and the payoff depends on whether it shows tails or heads up. We will ask you or another participant to lend us the coin.
- A **Ball** will be drawn from a bucket and the payoff depends on whether the ball is yellow or white. There are two buckets. In both buckets there are 20 table tennis balls. We only use table tennis balls that are either white or yellow.
    - **Bucket H**: Half of the balls is white, the other is yellow.
    - **Bucket U**: It is not known how many of the balls are white and how many are yellow.

  This is the only difference between bucket H and bucket U.

There are the following simple tickets:

Coin tickets

- **Head ticket:** A head ticket pays 100 Taler if the coin lands heads up and nothing else.

- **Tail ticket:** A tail ticket pays 100 Taler if the coin lands tails up and nothing else.

Color tickets

- **White ticket:** A white ticket pays 100 Taler if the drawn ball is white and nothing else.

- **Yellow ticket:** A yellow ticket pays 100 Taler if the drawn ball is yellow and nothing else.

- **Chameleon ticket:** The color of the chameleon ticket is determined by a coin throw.
    - If the coin lands heads up, the chameleon ticket becomes a yellow ticket.
    - If the coin lands tails up, the chameleon ticket becomes a white ticket.

For color tickets, it will be specified to which bucket they apply: H or U. A yellow ticket for bucket U thus means that a ball is drawn from the bucket with unknown proportions and that 100 Taler are paid if this ball is yellow..

Apart from these tickets there will be other variations that you will get to know during the experiment.

Decitions and the value of tickets

For each ticket there will be a set of questions. For example:

	Head ticket	Fix payment of ...
Question 1	( )	...68 Taler ( )
Question 2	( )	...96 Taler ( )

For Question 1 you have to decide between a head ticket or a fix payment of 68 Taler. For Question 2 between a head ticket and 96 Taler.

For each question concerning the same color ticket, a new ball will be drawn; already drawn balls are replaced. For each question concerning a coin ticket, the coin is thrown. All draws are hence *completely independent* of each other. Your payoff is hence maximized if you answer according to the value of the ticket.

If for example the ticket is worth 80 Taler to you, then you should prefer the ticket to a fix payment of 68 Taler (otherwise you lose 12 Taler). If you have the choice between the ticket and 96 Talern, you should choose 96 Taler (otherwise you lose sixteen Taler).

Input assistant

The close relationship between the value of a ticket and your decisions is used by the program to facilitate the input. You have the possibility to directly specify the value of a ticket in steps of 5 Taler using a slider. The program then automatically marks the corresponding decisions. If you want to you can change these decisions. The program then adjusts the value of the ticket. Note that the value of the ticket cannot always be computed. For example, if you select a fix payment of 58 Taler rather than the ticket but also choose the ticket rather than a fix payment of 63 Taler, this means that the ticket is worth less than 58 Taler to you but also more than 63 Taler. In this case, it is impossible to determine the value of the ticket to you.

Sequence

The experiment starts with a few problems, which should help you to acquaint yourself with the different types of questions. Moreover, we want to ensure that you have not misunderstood the instructions. Decisions during this part do not affect your payoffs. After the understanding part, the main part of the experiment begins. The decision during this part are for real. They hence affect your payoffs. Finally, we ask you some general questions. Altogether the experiment will take 90 minutes. You have enough time for your answers since the draws only start if all participants are ready.

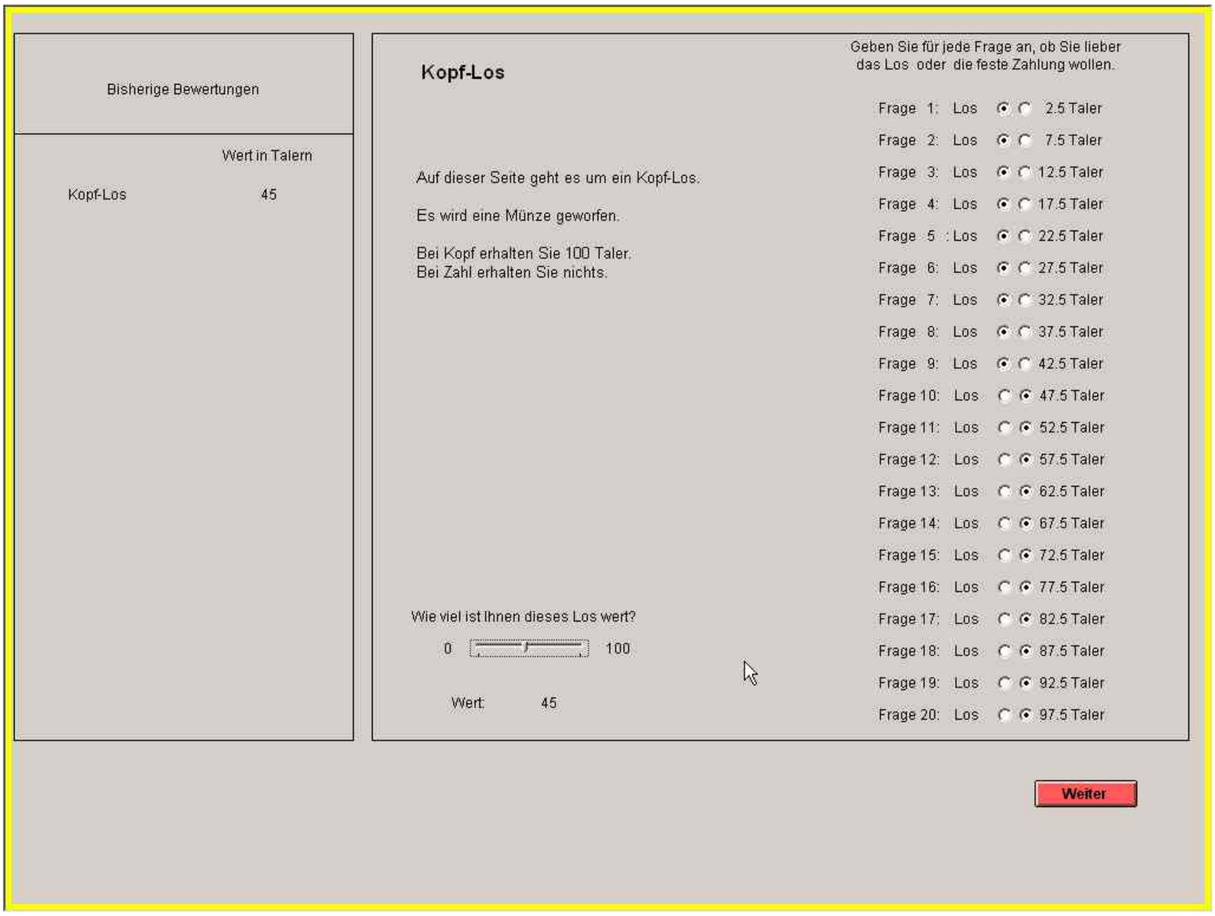

Figure 1: Valuation screen for head ticket (in German)

Variable name	Dummy variables which take the value one if...
no color preference	subject indifferent between white and yellow ticket for Urn U
coin fair	subject regards coin as fair
male	subject male
economics student	subject studies economics
business student	subject studies business administration
stats knowledge 1	Prob(10-sided fair dice shows 2 or less) computed correctly
stats knowledge 2	Prob(two 10-sided fair dice show two ones) computed correctly
stats knowledge 3	Prob(10-sided fair dice shows 4| even number)* computed correctly
stats knowledge 4	Prob(10-sided fair dice shows 4| odd number) computed correctly
stats knowledge 5	average payoff of two bets, one which pays 100 in case of even the other pays 100 in case of odd computed correctly
cognitive ability 1	correct answer to... A bat and a ball cost $1.10. The bat costs $1.00 more than the ball. How much does the ball cost?
cognitive ability 2	correct answer to... 5 machines need 5 min to produce 5 pieces. How long do 100 machines need to produce 100 pieces?
cognitive ability 3	correct answer to... A lake is covered by sea roses. The covered surface doubles every day. If 48 days are needed until the lake is entirely covered, how long does it take until half the lake is covered?
Variable name	Subjective agreement with following statements on a scale from 1 (totally disagree) to 7 (totally agree)
superstition	There are unlucky numbers.
god	God is important in my life.
religion	Religion gives me strength and support.
fate	What one achieves in life depends on fate and luck.

* Prob($A|B$) denotes the conditional probability of event A to occur after the occurrence of B.

Table 1: Variable definitions

Table 2: Selection on observables: Hypothesis 1

Dependent variable: No color preference on Urn U ($y^U \sim w^U$)			
Variable	**Coefficient**	**Stand. Error**	**p − value**
male	0.150	0.116	0.196
economics student	-0.214	0.194	0.271
business student	0.188	0.130	0.147
stats knowledge 1	-0.021	0.181	0.906
stats knowledge 2	0.143	0.131	0.275
stats knowledge 3	0.243	0.146	0.094
stats knowledge 4	-0.124	0.177	0.480
stats knowledge 5	-0.046	0.143	0.747
cognitive ability1	-0.108	0.131	0.409
cognitive ability2	-0.009	0.140	0.947
cognitive ability3	0.022	0.150	0.883
superstitious	-0.017	0.042	0.681
god	0.083	0.069	0.222
religion	-0.104	0.072	0.144
fate	-0.004	0.038	0.923
Number of Obs. = 88		$Prob > \chi^2$ = 0.43	
Log likelihood = -51.495		*Pseudo* R^2 = 0.129	
Significance levels:(5%), **(2%), ***(1%)*			

Table 3: Selection on observables: Hypothesis 2

Dependent variable: fair coin ($t \sim h$) and no color preference ($y^U \sim w^U$)			
Variable	**Coefficient**	**Stand. Error**	**p − value**
male	0.067	0.125	0.593
economics student	-0.252	0.188	0.180
business student	0.183	0.144	0.205
reference group: other fields of study (mostly: teaching, law, languages)			
stats knowledge 1	0.028	0.214	0.895
stats knowledge 2	0.293*	0.131	0.025
stats knowledge 3	0.189	0.157	0.228
stats knowledge 4	-0.320	0.189	0.090
stats knowledge 5	-0.027	0.162	0.869
cognitive ability1	-0.094	0.138	0.493
cognitive ability2	0.131	0.144	0.364
cognitive ability3	0.204	0.150	0.175
superstitious	-0.006	0.047	0.899
god	0.050	0.073	0.492
religion	-0.064	0.076	0.401
fate	-0.041	0.040	0.305
Number of Obs. = 88		$Prob > \chi^2$ = 0.042	
Log likelihood = -48.160		*Pseudo* R^2 = 0.210	
Significance levels:(5%), **(2%), ***(1%)*			

All instructions translated from German. Original instructions are available from the authors upon request.

Instructions

Welcome to our Experiment! Please read these instructions carefully. The instruction is identical for all participants. During the entire experiment, we want to ask you to be quiet and not to talk with the other participants. Please turn your mobile phone off and keep it turned off till the end of the experiment. If you have any questions, please raise your hand and one of the experimenters will come to you.

Goal of the experiment

This experiment includes decisions under uncertainty. In the decision phase, there are no “right” or “wrong” decisions. Only your personal preferences count. Depending on your preferences, it could well be that the decision will be very easy for you. The alternatives are real and not only hypothetical. Every participant will be privately paid in cash. The decisions of the other participants have no influence on your payment.

Structure of the experiment

At the start of the experiment, we will answer questions regarding the instructions. Afterwards we start the decision phase. Decisions in this phase are real. They do have an impact on your payment. Please take your time in answering, the experiment only continues once all participant are done. At the end, the payments for the decision phase will be determined and all participants are paid.

Overall, the experiment will take approximately 60 minutes.

Bucket

The bucket contains 30 table tennis balls. Every table tennis ball has a colored sticker, which determines the color of the ball. There are 10 yellow table tennis balls. The other 20 table tennis balls are either blue or green. The exact number of the blue and green table tennis balls is unknown. However, taken together, there are exactly 20 blue and green balls.

Decision phase

At the end of the experiment, 4 independent draws (with replacement) will be taken from the bucket – one draw for each of the 4 questions, which you answer on the decision sheet. Your payment depends on your answers and on the result of the draws.

On the decision sheet, you have to choice 4 times between 2 alternatives. The alternatives are as follows:

- Alternative W: You receive a payment of 4€, if a *yellow or green* ball is drawn.
- Alternative X: You receive a payment of 4€, if a *blue or green* ball is drawn.
- Alternative Y: You receive a payment of 4€, if a *yellow* ball is drawn.
- Alternative Z: You receive a payment of 4€, if a *blue* ball is drawn.

Questionnaire 1

The decision phase is followed by questionnaire 1. Here right and wrong answers exist! In total, you have 10 minutes to answer all questions. For each correct answer, you will be paid 1€ at the end of the experiment.

Questionnaire 2

Questionnaire 2 collects some personal data. This information will only be used for the evaluation of this experiment. The answers in questionnaire 2 do have no influence on your payment.

Draws

In the end, there will be 4 draws, one for each question from the decision phase. After each draw, the table tennis ball will be put back into the bucket. The draws will be taken by a randomly chosen participant.

It it happens that the first drawn ball is green for question 3 or question 4, there will be additional draws till the drawn ball is not green.

Payment

For each draw, you receive a payment if and only if the color of the drawn table tennis ball matches the color of the answer you marked. Additionally, you receive 1€ for each correctly answered question in questionnaire 1.

Decision Sheet ID: _______

- Alternative W: You receive a payment of 4€, if a *yellow or green* ball is drawn.
- Alternative X: You receive a payment of 4€, if a *blue or green* ball is drawn.
- Alternative Y: You receive a payment of 4€, if a *yellow* ball is drawn.
- Alternative Z: You receive a payment of 4€, if a *blue* ball is drawn.

Question 1

What do you like more?:

☐ W ☐ X

How strong is your liking for the alternative you choose?

Nil ☐ ☐ ☐ ☐ ☐ Very strong

Question 2

What do you like more?:

☐ Y ☐ Z

How strong is your liking for the alternative you choose?

Nil ☐ ☐ ☐ ☐ ☐ Very strong

Question 3

What do you like more, *if you come to know that the drawn ball is not green*:

☐ W ☐ X

How strong is your liking for the alternative you choose?

Null ☐ ☐ ☐ ☐ ☐ Very strong

Question 4

What do you like more, *if you come to know that the drawn ball is not green*:

☐ Y ☐ Z

How strong is your liking for the alternative you choose?

Null ☐ ☐ ☐ ☐ ☐ Very strong

Questionnaire 1 ID: _______

Page 1: 5 minutes maximum

Please assume for all questions that dice are six-sided and fair. Answer

Question 1: What is the probability that the number in a throw of a die is smaller or equal 2?	
Question 2: What is the probability that in two throws, the number is both times equal to 4?	
Question 3: Look at a single throw. Assume that the result is an even number. What is the probability that the number is equal to 2?	
Question 4: Assume that the number 3 was thrown 5 times in a row. What is the probability that the next throw will result in a 3?	
Question 5: Assume 4 dice are thrown and the numbers added. What is the total number on average?	

Questionnaire 1

Page 2: 5 minutes maximum

<table>
<tr><td colspan="4"><u>Question 6:</u> A bat and a ball cost $1.10 in total. The bat costs $1.00 more than the ball. How much does the ball cost?</td><td></td></tr>
<tr><td colspan="4"><u>Question 7:</u> If it takes 5 machines 5 minutes to make 5 widgets, how long would it take 100 machines to make 100 widgets?</td><td></td></tr>
<tr><td colspan="4"><u>Question 8:</u> In a lake, there is a patch of lily pads. Every day, the patch doubles in size. If it takes 48 days for the patch to cover the entire lake, how long would it take for the patch to cover half of the lake?</td><td></td></tr>
<tr><td colspan="4"><u>Question 9:</u> Assume you see 4 double sided cards in front of you. Each card has a number on one side and a letter on the other side. Which card or cards do you have to turn around to test whether the following assertion is true: “If there is a vowel (A,E,I,O,U) on one side, there is an even number on the other side.”</td><td rowspan="3"></td></tr>
<tr><td>E</td><td>K</td><td>4</td><td>7</td></tr>
<tr><td>Card 11</td><td>Card 12</td><td>Card 13</td><td>Card 14</td></tr>
</table>

Questionnaire 2 ID: _______

The questions on this questionnaire are not payoff relevant.

Question 1: Please give an estimate, how many balls are in the urn:

______ blue balls _______ yellow balls _______ green balls

Question 2: What is your gender? ☐ male ☐ female

Question 3: How tall are you? _______ cm

Question 5: What is your major? ________________________ ☐ not a student

Question 6: Did you participate in a statistics course before? ☐ yes ☐ no

Question 7: Would you call yourself politically left wing or right wing?

Left ☐ ☐ ☐ ☐ ☐ Right

Question 8: Are you religious? ☐ yes ☐ no

Question 9: Which of the following game do you play occasionally?

☐ Lottery
☐ Roulette
☐ Poker
☐ Sports bets
☐ Lottery scratch tickets
☐ others: ______________________

	Variable	Coef.	Std. Err.	z	P > |z|	95% Conf. Interval	
not DC, C	*Religious*	-0.271	0.718	-0.38	0.705	-1.679	1.136
	Male	-0.802	1.058	-0.76	0.448	-2.877	1.272
	Size	-0.009	0.057	-0.17	0.869	-0.122	0.103
	Gambling	0.127	0.467	0.27	0.786	-0.789	1.043
	Cog. Ability	0.203	0.270	0.75	0.451	-0.325	0.732
	Conf. loss	-0.630	0.302	-2.09	0.037	-1.222	-0.038
	Cons.	3.772	10.292	0.37	0.714	-16.402	23.945
DC, not C	*Religious*	-1.055	1.203	-0.88	0.381	-3.414	1.304
	Male	0.924	1.690	0.55	0.585	-2.389	4.237
	Size	0.077	0.095	0.81	0.419	-0.110	0.264
	Gambling	1.329	0.628	2.12	0.034	0.098	2.559
	Cog. Ability	0.099	0.481	0.21	0.836	-0.843	1.042
	Conf. loss	-0.668	0.523	-1.28	0.201	-1.693	0.357
	Cons.	-15.310	17.173	-0.89	0.373	-48.970	18.349
DC, C	*Religious*	-0.548	0.843	-0.65	0.516	-2.199	1.104
	Male	-0.583	1.211	-0.48	0.630	-2.957	1.791
	Size	0.033	0.068	0.50	0.618	-0.099	0.166
	Gambling	0.473	0.515	0.92	0.359	-0.537	1.483
	Cog. Ability	-0.229	0.325	-0.70	0.481	-0.865	0.407
	Conf. loss	-1.092	0.373	-2.93	0.003	-1.823	-0.361
	Cons.	-3.833	12.124	-0.32	0.752	-27.595	19.929

Number of Obs. = 84 *Log likelihood* = -82.151742	$LR\ \chi^2$ (18) = 31.01 $Prob > \chi^2$ = 0.0287 $Pseudo\ R^2$ = 0.1588

not DC, not C *is the base outcome. Cog. Ability = avg. score in questionnaire 1.*

Table 4: Multinomial Logistic Regression

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