Inaugural-Dissertation

zur Erlangung der Doktorwürde der
Naturwissenschaftlich-Mathematischen Gesamtfakultät der
Ruprecht-Karls-Universität Heidelberg

vorgelegt von
Diplom-Mathematiker Johannes Schmidt
aus Kreuztal

Tag der mündlichen Prüfung: ________________

Anabelian Aspects in the Étale Homotopy Theory of Brauer-Severi Varieties

Gutachter: Priv.-Doz. Dr. Jakob Stix,

Zusammenfassung

Wir untersuchen die étale Homotopie-Theorie von Brauer-Severi Varietäten über Körpern der Charakteristik 0. Wir zeigen, dass die auf geometrischen Homotopie-Invarianten (etwa ℓ-adischen Kohomologie- oder auch höheren Homotopiegruppen) induzierten Galois-Darstellungen nicht zwischen Brauer-Severi Varietäten gleicher Dimension unterscheiden können. Ist der Grundkörper sogar von kohomologischer Dimension ≤ 2, so können wir im Falle von Brauer-Severi Kurven noch mehr zeigen: Wir konstruieren einen Isomorphismus zwischen den Hochschild-Serre Spektral-Folgen zweier beliebiger Brauer-Severi Kurven, welche deren Kohomologie mit lokalen Koeffizienten berechnet. Weiter untersuchen wir homotopie-rationale- und homotopie Fixpunkte von Brauer-Severi Varietäten sowie deren Zusammenhang mit echten rationalen Punkten. Insbesondere werden wir ein Analogon der schwachen Schnittvermutung für Brauer-Severi Varietäten unter einer geeigneten Zusatzannahme an die erste pro-endliche Chernklassen Abbildung zeigen. Über p-adischen lokalen Körpern konstruieren wir ein Gegenbeispiel für dieses Analogon ohne besagte Zusatzannahme.

Abstract

We study the étale homotopy theory of Brauer-Severi varieties over fields of characteristic 0. We prove that the induced Galois representations on geometric homotopy invariants (e.g., ℓ-adic cohomology or higher homotopy groups) are all isomorphic for Brauer-Severi varieties of the same dimension. If the base field has cohomological dimension ≤ 2 then we can show more in the case of Brauer-Severi curves: There is even an isomorphism between the Hochschild-Serre spectral sequences computing cohomology with local coefficients. Further, we study homotopy rational and homotopy fixed points on Brauer-Severi varieties and their connections to genuine rational points. In particular, we show that under a suitable assumption on the first profinite Chern class map an analogue of the weak section conjecture for Brauer-Severi varieties turns out to be true. We can give a counter example to this analogue without the extra assumption over p-adic local fields.

Contents

Introduction

Background. In recent years there is a growing interest in homotopy theory for algebraic schemes over base schemes, more general than the spectrum of the complex or real numbers. Of particular interest is Voevodsky's motivic- or $\mathbb{A}^1$-homotopy theory he used in his celebrated proof of the Milnor- and later even motivic Bloch-Kato-conjecture. It comes with a good notion of homotopy groups (see [MV99] Chapt. 3 Prop. 2.14) as well as realization functors into other homotopy theories, e.g., the classical homotopy theory of the real or complex analytic realization (see [MV99] Chapt. 3.3) or the étale homotopy theory (see [Isa04] resp. [Sch12] for the Nisnevich resp. étale $\mathbb{A}^1$-homotopy theory). From an arithmetic point of view it has several nice properties, e.g., the existence of a rational point is invariant under $\mathbb{A}^1$-weak equivalences in the Nisnevich setting (see [MV99] Chapt. 3 Rem. 2.5), but also some rather unpleasant disadvantages, e.g., the base scheme is always contractible (in contrast to a notion of a $K(\Gamma_k, 1)$ for a base field k with absolute Galois group Γ_k) and projective spaces over algebraically closed fields are not simply connected (see [Mor12] Thm. 7.13 resp. Sect. 7.3).

Much earlier, Artin and Mazur developed étale homotopy theory, or more generally a homotopy theory for (pointed) connected sites. In contrast to the classical homotopy theory of topological spaces where the homotopy type of a space is its singular simplicial set, the homotopy type of a (pointed) connected site is a pro-object in the homotopy category of (pointed) simplicial sets, given levelwise by the simplicial set

$$\pi_0(\mathfrak{U}.)$$

of connected components of the various hypercoverings $\mathfrak{U}.$ of the site. Recall, that a hypercovering is a generalization of the Čech nerve of a covering, i.e., the simplicial object given by all the self intersections of the covering. By the Verdier hypercovering theorem the homotopy type of a (pointed) site captures the cohomology theory of the site with locally constant coefficients. Further, there are also the notions of homotopy (pro-) groups and the first homotopy (pro-) group classifies descent data of local isomorphisms. In particular, if for a certain class of fibres these descent data are all effective in our site (e.g., finite fibres in the étale case or arbitrary fibres in the classical topological case), this first (pro-) homotopy group classifies covering spaces with fibres in this class. Further, under relatively mild assumptions, the homotopy type of the classical topological site of a topological space is pro-discrete and isomorphic to its singular simplicial set.

But there are also some drawbacks in Artin and Mazur's notion of a homotopy theory: As mentioned earlier, the homotopy types of (pointed) connected sites live in the pro-homotopy category of simplicial sets

$$\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}),$$

yet a morphism inducing isomorphisms on all homotopy (pro-) groups is only an isomorphism up to a certain (non isomorphic) Postnikov replacement. Further, Artin and Mazur's notion of a homotopy type does not lift to a notion of a topological type living e.g., in the category of pro-simplicial sets.

These shortcomings where later fixed by Friedlander and Isaksen in the case of the small étale site of a scheme, or more generally of a locally connected site whose pointed neighbourhoods have a certain rigidity property (e.g., the Zariski or Nisnevich site of a scheme): First, Friedlander defined the notion of an étale topological type in the category of pro-simplicial sets inducing Artin and Mazur's étale homotopy type in the pro-homotopy category. Further, Isaksen defined a proper simplicial closed model structure on the pro-category of simplicial sets inducing a homotopy category on the nose

$$\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$$

whose weak equivalences between connected pointed pro-simplicial sets are precisely the morphisms inducing isomorphisms on all of Artin-Mazur's homotopy (pro-)groups. Let us also mention the related point of view of [Qui08]: In order to deal with homotopy types having profinite homotopy groups from the get-go, Quick defined a left proper closed model structure on the category of simplicial profinite sets.

The étale homotopy theory has several nice properties: First, The profinite completion of the first homotopy (pro-) group of a scheme is just Grothendieck's étale fundamental group. Next, there is a notion of a profinite completion of a pro-homotopy type analogue to the notion of the profinite completion of a group and the pro-homotopy types of schemes are already profinite complete under very mild conditions. Further, the profinite completion of the pro-homotopy type of a complex variety agrees with the profinite completion of the classical homotopy type of its complex analytification. Similar, the profinite completion of the pro-homotopy type of a real variety agrees with the profinite completion of the $\Gamma_{\mathbb{R}}$-orbit space of a good model of the classical homotopy type induced by its complex points. Finally, the étale homotopy type of the spectrum of a field k is just the classifying space $B\Gamma_k$ of the absolute Galois group Γ_k of k, i.e., an Eilenberg-MacLane space $K(\Gamma_k, 1)$.

Main Questions. As mentioned earlier, the existence of k-rational points is invariant under $\mathbb{A}^1$-weak equivalences in the (Nisnevich) motivic homotopy theory of smooth k-varieties: Indeed, the $\mathbb{A}^1$-homotopy type of $\mathrm{Spec}(k)$ is the point and the canonical map

$$X(k) \longrightarrow [\mathrm{pt}, X]_{\mathcal{H}^{\mathbb{A}^1}_{\mathrm{Nis}}(\underline{\mathrm{Sm}}_k)}$$

from the set of k-rational points of a k-variety X to the set of $\mathbb{A}^1$-homotopy classes of maps from the point to X is surjective (see [MV99] Chapt. 3 Rem. 2.5). Denote by $[-,-]_{\text{ét}}$ the homotopy classes of maps relative over $B\Gamma_k$ in Artin and Mazur's pro-homotopy category $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$, Isaksens homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$ or similar (pro-) homotopy categories in which the étale homotopy theory can be formulated. Unfortunately, the analogue statement about the canonical map

$$X(k) \longrightarrow [B\Gamma_k, X]_{\text{ét}}$$

is not at all clear for the étale homotopy theory of X. Let us call an element in the target a **homotopy rational point**. A very convenient setting to study this canonical map is the relative homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k)$. It comes equipped with a base extension functor

$$(-)^\natural \times_{B\Gamma_k} E\Gamma_k : \mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k) \longrightarrow \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_k}) ,$$

where $\underline{\mathrm{Sets}}_{\Gamma_k}$ is the category of discrete Γ_k-sets. We write $\bar{\mathfrak{X}}$ for $\mathfrak{X}^\natural \times_{B\Gamma_k} E\Gamma_k$. Further, write $[-,-]_{\Gamma_k}$ for the homotopy classes of maps in this or similar (pro-) equivariant homotopy categories and call an element in $[E\Gamma, \mathfrak{Y}]_{\Gamma_k}$ a **homotopy fixed point** of the equivariant pro-homotopy type $\mathfrak{Y}$. Thus, at least in the setting of the relative homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k)$ any homotopy rational point of a k-variety X induces a homotopy fixed point of $\bar{X}$ and $\bar{X}$ has the correct homotopy type of the base extension $X \otimes_k \bar{k}$ together with the induced Γ_k-action under some rather mild conditions.

This brings us to our main questions:

Main Questions: *Let X be an arbitrary element of an "interesting" class of k-varieties.*

(i) *Are the canonical maps $X(k) \to [B\Gamma_k, X]_{\text{ét}}$ and $X(k) \to [E\Gamma_k, \bar{X}]_{\Gamma_k}$ surjective, as well?*

(ii) *Suppose the targets are non empty. Does this imply that the source $X(k)$ is non empty, as well? In other words: Does the existence of a homotopy rational or fixed point already imply the existence of a genuine k-rational point?*

These questions are quite interesting. Consider them e.g., in the case of smooth projective curves of genus at least 1 over a field k of characteristic 0: From the perspective of étale homotopy theory such a (pointed) curve $(X, \bar{x})$ is a $K(\pi, 1)$. In particular, a (right-) splitting of the canonical map $X \to B\Gamma_k$ in Isaksens homotopy category or even in Artin and Mazur's pro-homotopy category is equivalent to a (right-) splitting of the induced map $\pi_1(X, \bar{x}) \to \Gamma_k$ (up to inner automorphisms), i.e., to a splitting of the exact sequence $\pi_1(X/k, \bar{x})$

$$\mathbf{1} \longrightarrow \pi_1(X \otimes_k \bar{k}, \bar{x}) \longrightarrow \pi_1(X, \bar{x}) \longrightarrow \Gamma_k \longrightarrow \mathbf{1} .$$

Now there is a negative answer to question (ii) for torsors under elliptic curves, i.e. genus 1 curves: E.g. over p-adic local fields there are plenty of non split torsors under a given elliptic curve whose fundamental group sequence admits a section (see [Sti12] Prop. 183 - see also loc. cit. Prop. 185 for counter examples over the rational numbers). In the case of projective anabelian curves, i.e. a smooth projective curve of genus at least 2 (over k a finitely generated field), our first resp. second question is equivalent to **Grothendieck's section conjecture**, resp. to the **weak section conjecture** (for details e.g., see [Qui11] Sect. 3.2).

An examples where Question (ii) has a positive answer is the case of real varieties: By [Cox79b] Thm. 2.1, a real variety X has an $\mathbb{R}$-rational point if and only if $\mathrm{cd}_2(X) = \infty$. Since each element in $[B\Gamma_{\mathbb{R}}, X]_{\text{ét}}$ resp. $[E\Gamma_{\mathbb{R}}, \bar{X}]_{\Gamma_{\mathbb{R}}}$ gives a (left-) splitting of the canonical map

$$H^\bullet(\Gamma_{\mathbb{R}}; \Lambda) \longrightarrow H^\bullet(X; \Lambda)$$

for Λ a discrete $\Gamma_{\mathbb{R}}$-module and $\mathbb{R}$ has infinite 2-cohomological dimension, X admits an $\mathbb{R}$-rational point.

Aim of the thesis. Our main interest during the work on this thesis was exactly the second question for the class of Brauer-Severi varieties over a field of characteristic 0. Recall, that a Brauer-Severi variety over a field k is a k-variety whose base extension to an algebraic closure $\bar{k}/k$ of k is $\bar{k}$-isomorphic to the projective space $\mathbb{P}^n$ for a suitable n. It is a basic fact that a Brauer-Severi variety has a k-rational point if and only if it is already isomorphic over k to a projective space. We say that X **splits** in this case. There is an equivalence relation on the set of isomorphism classes of Brauer-Severi varieties over a fixed base field k generated by basic relations given by twisted linear subvarieties, i.e., maps of Brauer-Severi varieties whose base extension along $\bar{k}/k$ is isomorphic to the inclusion of a linear subvariety of a projective space. It turns out that the set of equivalence classes is isomorphic to the Brauer group

$$\mathrm{Br}(k) = H^2(k; \mathbb{G}_{\mathrm{m}}),$$

where the trivial Brauer class corresponds to the equivalence class consisting of the projective spaces $\mathbb{P}^n$ over k. Thus, our second main question for a Brauer-Severi variety X over k is equivalent to the question if non emptiness of $[B\Gamma_k, X]_{\text{ét}}$ resp. $[E\Gamma_k, \bar{X}]_{\Gamma_k}$ implies the triviality of the Brauer class of X in $\mathrm{Br}(k)$. In other words, we have to show that the **period** of X, i.e., the order of its Brauer class, is 1.

Let us assume that the Brauer-Severi variety X admits a homotopy rational resp. fixed point s resp. $\bar{s}$. We want to know if X splits. Recall the first profinite Chern class map

$$\hat{c}_1 : \mathrm{Pic}(X) \longrightarrow H^2(X; \hat{\mathbb{Z}}(1)) .$$

It is not hard to see, that the first Chern class $\hat{c}_1[\mathcal{L}]$ of the positive degree generator of $\mathrm{Pic}(X)$ is not divisible by any integer $\neq \pm 1$. Thus, one could try to show that $\hat{c}_1[\mathcal{L}]$ is divisible by d the period of X, as soon as X admits a homotopy rational resp. fixed point s resp. $\bar{s}$. Any such "d^{th}-root" of $\hat{c}_1[\mathcal{L}]$ restricts to the Chern class of $\mathcal{O}_{X\otimes_k \bar{k}}(1)$ after base extension along $\bar{k}/k$. In abuse of notation, call an arbitrary such cohomology class in $H^2(X; \hat{\mathbb{Z}}(1))$ a $\boldsymbol{k}$**-structure of** $\boldsymbol{\hat{c}_1[\mathcal{O}(1)]}$.

There is a unique k-structure α_s of $\hat{c}_1[\mathcal{O}(1)]$ which is killed by the pullback s^*. If s is even given by a genuine rational point x of $\mathbb{P}^n$, the resulting k-structure α_x is the Chern class of $\mathcal{O}_{\mathbb{P}^n}(1)$ on the nose. In $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_{\Gamma_k}})$, cohomology with coefficients in discrete Γ_k-modules Λ is representable by the pro-discrete Eilenberg-MacLane spaces $K(\Lambda, q)$. Combining this with the Dold-Kan correspondence between simplicial discrete Γ_k-modules and chain complexes in simplicial discrete Γ_k-modules, we can give an explicit construction of α_s out of the suitable truncated canonical map between homology and reduced homology chains with resp. to a given homotopy rational point s of X. The class α_s behaves with respect to the twisted d-uple embedding

$$i : X \hookrightarrow \mathbb{P}^N$$

just as α_x would behave with respect to the genuine d-uple embedding, i.e.

$$i^*\alpha_{i_*s} = d.\alpha_s.$$

Since the Chern class of $\mathcal{L}$ is nothing but $i^*\alpha_y$ for y a rational point of $\mathbb{P}^N$, our class α_s is a good candidate for the desired "d^{th}-root" of $\hat{c}_1[\mathcal{L}]$.

Unfortunately α_s fails to be such a "root" in general: For any class $[A]$ in the p-torsion part of the Brauer group of a p-adic local field k, the corresponding Brauer-Severi variety X_A contains a homotopy rational point without being split. Indeed, a smooth projective curve whose relative Brauer group contains $[A]$ admits a map into X_A and it is well known to anabelian geometers that there are genus 1 curves C over k whose relative Brauer group contains $[A]$ and whose fundamental group sequence $\pi_1(C/k)$ splits, i.e. which admit homotopy rational points.

A closer analysis of our classes α_s shows, that two classes α_s and α_r given by two homotopy rational points s and r differ by the constant

$$s^*\alpha_r = -r^*\alpha_s$$

in $H^2(\Gamma_k;\hat{\mathbb{Z}}(1))$. In particular, α_{i_*s} differs from α_y by the constant $s^*\hat{c}_1[\mathcal{L}]$ and thus, $\hat{c}_1[\mathcal{L}]$ is divisible by d if and only if the class $s^*\hat{c}_1[\mathcal{L}]$ is divisible by d in $H^2(\Gamma_k;\hat{\mathbb{Z}}(1))$. This leads to our (partial) answer to the main question for Brauer-Severi varieties over fields of characteristic 0:

Theorem. *Let k be a field of characteristic 0 and X a Brauer-Severi variety over k of period d admitting a homotopy rational point s in $[B\Gamma_k, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma_k)}$ resp. a homotopy fixed point $\bar{s}$ in $[E\Gamma_k, \bar{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_k})}$.*

(i) *The composition $s^*\hat{c}_1$ resp. $\bar{s}^*\hat{c}_1$ is independent from the choice of s resp. $\bar{s}$ modulo d.*

(ii) *Suppose that s^* resp. $\bar{s}^*$ trivializes the first profinite Chern class map $\hat{c}_1$ modulo d. Then X splits over k, i.e., X admits a k-rational point.*

(iii) *Suppose that $\mathrm{scd}(k) \leq 2\dim(X)$. Then the homotopy fixed point $\bar{s}$ is induced by a genuine rational point of X if and only if $\bar{s}^*$ trivializes the first profinite Chern class map.*

Recall that p-adic local resp. totally imaginary number fields are of strict cohomological dimension 2, so the third statement of the theorem holds for all Brauer-Severi varieties over p-adic local resp. totally imaginary number fields.

Structure of the thesis. The thesis is organized as follows: In Sect. 1 we recall some basic facts on étale homotopy theory mainly taken from [AM69], [Fri82] and [Isa01].

In Sect. 2 we deal with some technicalities needed in the following two sections: First, we do some computation with the Čech topological type leading to base change theorems for étale homotopy types in Sect. 2.3. Next, in Sect. 2.4 we compare the Hochschild-Serre spectral sequence with the hypercohomology spectral sequence of a certain cochain complex computing the geometric cohomology with local coefficients. In Sect. 2.5 we deal with a universal coefficient theorem for pro-chain complexes and in Sect. 2.6 we express cohomology with locally constant coefficients of k-varieties in the pro-homotopy category $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_k})$.

In Sect. 3 we discuss some difficulties in dealing with our main question: First, in Sect. 3.1 we prove that the base extension along $\bar{k}/k$ of two Brauer-Severi varieties of the same dimension induces isomorphic Γ_k-objects in the homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$. Thus, the induced Galois representations on all geometric homotopy invariants (as e.g., ℓ-adic cohomology or higher homotopy groups) of these Brauer-Severi varieties agree. In Sect. 3.2 we define the notion of a homology and quasi homology fixed point and show that quasi homology fixed points are quite common for geometrically simply connected varieties over fields of cohomological dimension ≤ 2. From the existence of quasi homology fixed points we conclude in Sect. 3.3 that two arbitrary Brauer-Severi curves over such a field have isomorphic Hochschild-Serre spectral sequences computing étale cohomology with locally constant coefficients.

Finally, in Sect. 4 we discuss homotopy rational points and homotopy or (quasi) homology fixed points and their connections to genuine rational points of Brauer-Severi varieties over a field k of characteristic 0. In Sect. 4.1 we define and discuss homotopy rational and homotopy fixed points and their connections with (quasi) homology fixed points. In Sect. 4.2 we do some abstract nonsense constructions needed for the explicit construction of the classes α_s. We define the k-structures α_s of $\hat{c}_1[\mathcal{O}(1)]$ in Sect. 4.3 and give the explicit construction using the results of the preceding Sect. 4.2. In Sect. 4.4 we examine the maps induced by our classes α_s on homotopy fixed point sets. In particular we will see that these maps have trivial fibres at homotopy fixed points induced by homotopy rational points over fields of small strict cohomological dimensions (see Cor. 4.4.4 and Lem. 4.4.6). In Sect. 4.5 we will proof the above theorem. Part (i) is Prop. 4.5.10, part (ii) Thm. 4.5.11 and part (iii) Thm. 4.5.15. Finally, in Sect. 4.6 we explain a counter example to an affirmative answer of question (ii) in the case of Brauer-Severi varieties over p-adic local fields.

Notation. We collect a few notations used throughout the thesis. First, we distinguish between "degreewise" and "levelwise" properties: The degree usually refers to the simplicial degree of a simplicial object $X. \in \underline{SC}$ or $C_\bullet \in \mathrm{Ch}_*(\underline{A})$ for the $\underline{SC}$ the category of simplicial objects of a category $\underline{C}$ or $\mathrm{Ch}_*(\underline{A})$ for $* = +, -, b, \ldots$ the category of positive, negative, bounded,... chain complexes of an abelian category $\underline{A}$ (and similar for cochain complexes). The level usually refers to the pro-level of a representation $\underline{I} \to \underline{C}$ of an object $\mathfrak{X}$ in the pro-category $\mathrm{Pro}\underline{C}$ of a category $\underline{C}$. E.g., a levelwise map $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{Pro}\underline{C}$ is a map induced by a natural transformation between realizations of $\mathfrak{X}$ and $\mathfrak{Y}$ with a common index category.

For k is a field, fix a separable algebraic closure $\bar{k}/k$ and denote by Γ_k the corresponding absolute Galois group $\mathrm{Gal}(\bar{k}/k)$. In later parts of the thesis, if not otherwise stated the base field k will usually be a field of characteristic 0. Usually, we will refer to its absolute Galois group Γ_k just by Γ. If we work over $\mathbb{C}$, denote by $(-)^{\mathrm{an}}$ the analytification functor $\underline{\mathrm{Var}_{\mathbb{C}}} \to \underline{\mathrm{Top}}$ on the category of $\mathbb{C}$-varieties.

We fix the standard model structure on the category $\underline{\mathrm{SSets}}$ of simplicial sets resp. $\underline{\mathrm{SSets}_\bullet}$ of pointed connected simplicial sets. If we choose a functorial fibrant resolution $\mathrm{Ex}(-)$ in $\underline{\mathrm{SSets}}$, always choose one preserving simplicial discrete Γ-sets. This is possible e.g. via the small object argument (cf. Rem. 2.3.6 below). For a simplicial set $A. \in \underline{\mathrm{SSets}}$ and Λ an abelian group resp. a local system of abelian groups on $A.$ denote by $C_\bullet(A.;\Lambda)$ resp. $C^\bullet(A.;\Lambda)$ the standard chain- resp. cochain complex computing the homology $H_\bullet(A.;\Lambda)$ resp. cohomology $H^\bullet(A.;\Lambda)$. For a point $a \in A_0$ denote by $\tilde{C}_\bullet(A.,a;\Lambda)$ the reduced complex $C_\bullet(A.;\Lambda)/C_\bullet(a;\Lambda)$ computing the reduced homology $\tilde{H}_\bullet(A.,a;\Lambda)$. If $a : E. \to A.$ is more generally a morphism with contractible source, we denote by $\tilde{C}_\bullet(A.,a;\Lambda)$ the reduced complex given by the standard cone construction (cf. [GM03] Chapt. III Sect. 2) induced by a and by $\tilde{H}_\bullet(A.,a;\Lambda)$ its homology. If the coefficients Λ are just the integers, denote by $C_\bullet(A.)$, $\tilde{C}_\bullet(A.,a)$, $H_\bullet(A.)$ resp. $\tilde{H}_\bullet(A.,a)$ the complexes resp. groups $C_\bullet(A.;\mathbb{Z})$, $\tilde{C}_\bullet(A.,a;\mathbb{Z})$, $H_\bullet(A.;\mathbb{Z})$ resp. $\tilde{H}_\bullet(A.,a;\mathbb{Z})$.

If $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\underline{I}}$ is a pro-system of sheaves on (say the small étale site or whatever of) X, denote by $H^\bullet(X;\mathcal{F})$ the pro-abelian group $\{H^\bullet(X;\mathcal{F}_i)\}_{i\in\underline{I}}$. More general, if not otherwise explicitly stated, **never take limits of pro-systems, instead, work in the resp. pro-category** (e.g., $\mathrm{Pro}\underline{\mathrm{Ab}}$, $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ or $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}})$, etc.). Filtered colimits in contrast are exact in all the categories of our interest, so we usually will take colimits.

If $\underline{C}$ is a homotopy-, pro-homotopy, derived or pro-derived category and X and Y two objects in $\underline{C}$, write $[X,Y]_{\underline{C}}$ for the homset $\mathrm{Hom}_{\underline{C}}(X,Y)$. By a weak equivalence resp. quasi-isomorphism in a pro-homotopy resp. pro-derived category we will usually mean a morphism inducing isomorphisms on all homotopy resp. (co-)homology (pro-)groups. To stress this, we sometimes refer to these as weak equivalence resp. quasi-isomorphism in the pro-sense.

If G is a group, denote by EG the 0^{th}-coskeleton $\mathrm{cosk}_0 G$ in $\underline{\mathrm{SSets}}$. It comes equipped with the free diagonal G action from the left, denote by BG the corresponding quotient $G\backslash EG$. If $\Gamma = \{\Gamma_i\}_{i\in\underline{I}}$ is a pro-group denote by $E\Gamma$ resp. $B\Gamma$ the pro-simplicial set $\{E\Gamma_i\}_{i\in\underline{I}}$ resp. $\{B\Gamma_i\}_{i\in\underline{I}}$. By $K(\Gamma,1)$ we usually mean $B\Gamma$, by $K(\Lambda,n)$ for an abelian group (or more generally, a discrete Γ-module for a profinite group Γ, etc.) we usually mean the simplicial set (simplicial discrete Γ-set, etc.) corresponding to $\Lambda[-n]$ under the Dold-Kan correspondence. In abuse of notation we also say that $X.$ is a $K(\pi,n)$, if $X.$ is only weakly equivalent to $K(\pi,n)$. If we want to stress the difference between $X.$ and $K(\pi,n)$, we sometimes say that $X.$ has the $K(\pi,n)$-property.

Finally, if X is a scheme, we usually refer to its étale homotopy type $\acute{\mathrm{E}}\mathrm{t}(X)$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ or $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$ by X as well. Similar, we refer to $\acute{\mathrm{E}}\mathrm{t}(X)^\natural \times_{B\Gamma} E\Gamma$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$ just by $\bar{X}$.

Acknowledgments. Zu aller erst gilt mein Dank meinem Betreuer Jakob Stix sowohl für das interessante Thema als auch für die hervorragende Betreuung bei der Bearbeitung dieser Dissertation. Insbesondere bin ich dankbar für die Ermunterung sowie Freiheit meinen eigenen Ideen nachzugehen und dabei stets durch wertvolle Ratschläge und Hilfen Unterstützung erfahren zu haben.

Auch gilt mein Dank dem MAThematics Center Heidelberg (MATCH) welches mir die Arbeit an dieser Dissertation durch Gewährung eines Promotionsstipendiums erst ermöglichte.

Weiter gilt mein Dank der Fakultät für Mathematik und Informatik der Universität Heidelberg. Insbesondere danke ich dem gesamten Mathematischen Institut, an welchem ich mich dank der immer warmen und freundschaftlichen Atmosphäre stets wohl fühlen konnte. Mein besonderer Dank gilt dabei (in alphabetischer Ordnung): Jochen Gärtner, Armin Holschbach, Alexander Ivanov, Andreas Riedel, Alexander Schmidt (insbesondere für seine hilfreichen Bemerkungen zu dieser Arbeit), Ullrich Schmitt, Martin Sigl, Matthias Spiegel, Malte Witte sowie weiter Maximilian Böhrmann, Thorsten Heidersdorf, Thomas Krämer, und Mirko Rösner mit welchen ich mir (zeitweilig) ein Büro teilte.

Zuletzt will ich mich bei allen meinen Freunden außerhalb des mathematischen Instituts bedanken, ohne welche mir die Zeit meiner Promotion wohl wesentlich schwerer gefallen wäre.

1 Preliminaries: Étale homotopy theory

In this first chapter, we recall some basics on étale homotopy theory. These are mainly taken from [AM69], [Fri82], [Isa01] and [Isa04].

1.1 The pro-homotopy category. We start with some facts about Artin and Mazur's pro-homotopy category $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ resp. $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$ (recall that $\underline{\mathrm{SSets}_\bullet}$ is the category of connected pointed simplicial sets):

1.1.1 Definition. *Let $\mathfrak{X} = \{\mathfrak{X}_i\}_{i \in \underline{I}}$ be a pro-homotopy type in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ resp. $(\mathfrak{X}, x) = \{(\mathfrak{X}_i, x_i)\}_{i \in \underline{I}}$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$ and let Λ be an abelian group.*

(i) *Define the (pro-)set of connected components $\pi_0(\mathfrak{X})$ as the pro-set given by $\{\pi_0(\mathfrak{X}_i)\}_i$.*

(ii) *For $q > 0$ define the q^{th} homotopy (pro-)group $\pi_q(\mathfrak{X}, x)$ as the pro-group given by $\{\pi_q(\mathfrak{X}_i, x_i)\}_i$.*

(iii) *Define the standard (pro-)chain complex $C_\bullet(\mathfrak{X}; \Lambda)$ in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}})$ as the pro-object given by $\{C_\bullet(\mathfrak{X}_i; \Lambda)\}_i$. Denote its homology pro-groups by $H_\bullet(\mathfrak{X}; \Lambda)$.*

(iv) *Define the standard cochain complex $C^\bullet(\mathfrak{X}; \Lambda)$ in $\mathcal{D}^+(\underline{\mathrm{Ab}})$ as the complex $\mathrm{colim}_i\, C^\bullet(\mathfrak{X}_i; \Lambda)$ (filtered colimits are exact!). Denote its cohomology groups by $H^\bullet(\mathfrak{X}; \Lambda)$.*

Recall, that a local system on a simplicial set $A.$ is a representation of the fundamental groupoid $\Pi(A.)$ in the category of abelian groups, i.e., functor $\Pi(A.) \to \underline{\mathrm{Ab}}$. Similar one can also define local systems with values in other categories, e.g., in $\underline{\mathrm{Grps}}$, the category of groups. We define local systems on a pro-simplicial set as follows:

1.1.2 Definition. *Let $\mathfrak{X} = \{\mathfrak{X}_i\}_{i \in \underline{I}}$ be in $\mathrm{Pro}\underline{\mathrm{SSets}}$.*

(i) *Define the category $\underline{\mathrm{Loc}}(\mathfrak{X})$ of **local systems** on $\mathfrak{X}$ as the category consisting of*

$$\mathrm{Obj}(\underline{\mathrm{Loc}}(\mathfrak{X})) := \operatorname*{colim}_i \mathrm{Obj}(\underline{\mathrm{Loc}}(\mathfrak{X}_i))$$

as objects and for $\mathcal{L}'$ and $\mathcal{L}''$ in $\mathrm{Obj}(\underline{\mathrm{Loc}}(\mathfrak{X}))$ represented by the local systems $\mathcal{L}'_i \in \mathrm{Obj}(\underline{\mathrm{Loc}}(\mathfrak{X}_i))$ and $\mathcal{L}''_j \in \mathrm{Obj}(\underline{\mathrm{Loc}}(\mathfrak{X}_j))$

$$\mathrm{Hom}_{\underline{\mathrm{Loc}}(\mathfrak{X})}(\mathcal{L}', \mathcal{L}'') := \operatorname*{colim}_{\sigma: k \to i, \tau: k \to j} \mathrm{Hom}_{\underline{\mathrm{Loc}}(\mathfrak{X}_k)}(\mathfrak{X}_\sigma^* \mathcal{L}'_i, \mathfrak{X}_\tau^* \mathcal{L}''_j)$$

as morphisms.

(ii) *Let $\mathcal{L}$ be a local system (of abelian groups) on $\mathfrak{X}$, say represented by $\mathcal{L}_i \in \underline{\mathrm{Loc}}(\mathfrak{X}_i)$. Define the complex $C^\bullet(\mathfrak{X}; \mathcal{L})$ in $\mathcal{D}^+(\underline{\mathrm{Ab}})$ as the complex $\mathrm{colim}_{\sigma: k \to i}\, C^\bullet(\mathfrak{X}_k; \mathfrak{X}_\sigma^* \mathcal{L}_i)$. Denote its cohomology groups by $H^\bullet(\mathfrak{X}; \mathcal{L})$.*

We want to discuss the completion of a pro-homotopy type with resp. to a class of groups, parallel to the completion of a pro-group with resp. to a class of groups:

1.1.3 Definition. *In abuse of notation, a* ***class of groups*** *is a strictly full subcategory* $\underline{C} \hookrightarrow \underline{\text{Grps}}$ *containing the trivial group* $\mathbf{1}$*, stable under subquotients and extensions.*
A class $\underline{C}$ *is called* ***complete****, if moreover* $\underline{C}$ *contains* $\text{Hom}_{\underline{\text{Sets}}}(H, G)$ *(with the group structure induced by the group structure of* G*) for any* G *and* H *in* $\underline{C}$*.*

Let $\underline{C}$ be a class of groups. Recall that the inclusion of the full subcategory $\text{Pro}\underline{C} \hookrightarrow \text{Pro}\underline{\text{Grps}}$ has a left adjoint

$$(-)^{\wedge}_{\underline{C}} : \ \text{Pro}\underline{\text{Grps}} \longrightarrow \text{Pro}\underline{C} \ ,$$

the pro-$\underline{C}$-completion. Write just $(-)^{\wedge}$ in the case of $\underline{C}$ the complete class of finite groups.

We get a similar completion for pro-homotopy types: Let

$$\mathcal{H}(\underline{\text{SSets}_{\bullet}})_{\underline{C}} \hookrightarrow \mathcal{H}(\underline{\text{SSets}_{\bullet}})$$

be the full subcategory given by all the pointed simplicial sets whose homotopy groups all lie in the class of groups $\underline{C}$.

1.1.4 Theorem. *([AM69] Thm. 3.4) Let* $\underline{C}$ *be a class of groups. Then the inclusion of the full subcategory* $\text{Pro}\mathcal{H}(\underline{\text{SSets}_{\bullet}})_{\underline{C}} \hookrightarrow \text{Pro}\mathcal{H}(\underline{\text{SSets}_{\bullet}})$ *has a left adjoint*

$$(-)^{\wedge}_{\underline{C}} : \ \text{Pro}\mathcal{H}(\underline{\text{SSets}_{\bullet}}) \longrightarrow \text{Pro}\mathcal{H}(\underline{\text{SSets}_{\bullet}})_{\underline{C}} \ ,$$

the ***pro-$\underline{C}$-completion****. Write just* $(-)^{\wedge}$ *in the case of* $\underline{C}$ *the complete class of finite groups.*

Next, we want to discuss the Postnikov tower construction in the sense of [AM69]: Recall the following properties of the coskeleton functors

$$\text{cosk}_n(-) : \ \underline{\text{SSets}} \longrightarrow \underline{\text{SSets}} \ .$$

For $n > 0$ it preserves homotopy equivalences with respect to the standard cylinder object (since $\text{cosk}_n \Delta^1 = \Delta^1$), i.e., weak equivalences between fibrant simplicial sets. Thus, $\text{cosk}_n(-)$ defines an endofunctor on $\mathcal{H}(\underline{\text{SSets}})$ resp. on $\mathcal{H}(\underline{\text{SSets}_{\bullet}})$. Further, $\text{cosk}_n(-)$ preserves fibrant objects and

$$\pi_q(\text{cosk}_n A., a) = \begin{cases} \pi_q(A., a) & \text{for } q < n \\ \mathbf{1} & \text{for } q \geq n \end{cases}$$

for $A.$ a fibrant simplicial set.

1.1.5 Definition. *For* $(\mathfrak{X}, x) = \{(\mathfrak{X}_i, x_i)\}_{i \in \underline{I}}$ *in* $\text{Pro}\mathcal{H}(\underline{\text{SSets}_{\bullet}})$ *define the* ***Postnikov tower replacement***

$$(\mathfrak{X}, x) \longrightarrow (\mathfrak{X}^{\natural}, x)$$

as the canonical map to the pro-homotopy type given by

$$(\mathfrak{X}^{\natural}, x) := \{\text{cosk}_n \mathfrak{X}_i\}_{i \in \underline{I}, n > 0},$$

where the index category is just $\underline{I} \times \underline{\mathbb{N}}$ *for* $\underline{\mathbb{N}}$ *the category of natural numbers as objects and a unique morphism* $n \to m$ *for* $n \geq m$*.*

1.1.6 Remark. If we want to work in Pro$\underline{\mathrm{SSets}_\bullet}$ we have to choose a functorial fibrant resolution $\mathrm{Ex}(-)$ and set

$$(\mathfrak{X}^\natural, x) := \{\mathrm{cosk}_n \mathrm{Ex}(\mathfrak{X}_i)\}_{i\in \underline{I}, n>0}.$$

Clearly, $(\mathfrak{X}, x) \to (\mathfrak{X}^\natural, x)$ induces isomorphisms on all homotopy resp. homology pro-groups $\pi_q(-)$ resp. $H_q(-;\Lambda)$. But $(\mathfrak{X}, x) \to (\mathfrak{X}^\natural, x)$ is in general not an isomorphism in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$: Take a fibrant simplicial set $A.$ with non trivial homotopy groups in arbitrary high degrees. Then the map $A. \to \mathrm{cosk}_n A.$ can never have a left inverse $\mathrm{cosk}_n A. \to A.$, since $\mathrm{cosk}_n(-)$ kills the homotopy groups in degrees $\geq n$. In particular, $A. \to A.^\natural$ is not an isomorphism in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$, e.g., by [Isa01] Lem. 2.3. Since $(-)^\natural$ is an idempotent endofunctor, this makes $(\mathfrak{X}, x) \to (\mathfrak{X}^\natural, x)$ to the paradigm case of the following definition:

1.1.7 Definition. *A* ***$\natural$-isomorphism*** *is a morphism $f : (\mathfrak{X}, x) \to (\mathfrak{Y}, y)$ in* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$ *s.t. the induced map*

$$f^\natural : (\mathfrak{X}^\natural, x) \longrightarrow (\mathfrak{Y}^\natural, y)$$

is an isomorphism in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$.

For $\underline{C}$ a complete class of groups a **$\underline{C}$-local system** on $\mathfrak{X}$ is a local system $\mathcal{L} \in \underline{\mathrm{Loc}}(\mathfrak{X})$, say given by

$$\mathcal{L}_i : \Pi(\mathfrak{X}_i) \longrightarrow \underline{\mathrm{Ab}}\ ,$$

s.t. for all $x \in \mathfrak{X}_i$ both the stalk $\mathcal{L}_i(x)$ and the images of the induced map

$$\pi_1(\mathfrak{X}_i, x) \longrightarrow \mathrm{Aut}_{\underline{\mathrm{Ab}}}(\mathcal{L}_i(x))$$

lie in $\underline{C}$.

Let $(\mathfrak{X}, x)$ be a connected pointed pro-homotopy type. From $\mathfrak{X} \to \mathfrak{X}^\wedge_{\underline{C}}$ we get a canonical map

$$\pi_q(\mathfrak{X}, x)^\wedge_{\underline{C}} \longrightarrow \pi_q(\mathfrak{X}^\wedge_{\underline{C}}, x) \tag{1.1.1}$$

for $q > 0$. By the explicit construction of the levels of the pro-$\underline{C}$-completion of $\mathfrak{X}$ in [AM69] Cor. 3.6 we get an isomorphism for $q = 1$ (this is [AM69] Cor. 3.7). Thus, a $\underline{C}$-local system on $\mathfrak{X}$ is the same as a local system on $\mathfrak{X}^\wedge_{\underline{C}}$ with stalks in $\underline{C}$.

1.1.8 Theorem. *([AM69] Thm. 4.3 and Cor. 4.4) Let $\underline{C}$ be a complete class of groups. Then, for a map $f : (\mathfrak{X}, x) \to (\mathfrak{Y}, y)$ in* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$ *the following are equivalent:*

(i) $f^\wedge_{\underline{C}} : (\mathfrak{X}^\wedge_{\underline{C}}, x) \to (\mathfrak{Y}^\wedge_{\underline{C}}, y)$ *is a $\natural$-isomorphism.*

(ii) $\pi_q(f^\wedge_{\underline{C}}) : \pi_q(\mathfrak{X}^\wedge_{\underline{C}}, x) \to \pi_q(\mathfrak{Y}^\wedge_{\underline{C}}, y)$ *is an isomorphism for all q.*

(iii) $\pi_1(f)^\wedge_{\underline{C}} : \pi_1(\mathfrak{X}, x)^\wedge_{\underline{C}} \to \pi_1(\mathfrak{Y}, y)^\wedge_{\underline{C}}$ *is an isomorphism and for all* $\underline{C}$*-local systems* $\mathcal{L} \in \underline{\mathrm{Loc}}(\mathfrak{Y})$ *the induced map* $f^* : H^\bullet(\mathfrak{Y}; \mathcal{L}) \to H^\bullet(\mathfrak{X}; f^*\mathcal{L})$ *is an isomorphism.*

Note that the equivalence (iii) $\Leftrightarrow$ (i) implies that pro-$\underline{C}$-completion for a complete class of groups $\underline{C}$ preserves $\natural$-isomorphisms. Further, note that $\natural$-isomorphisms induce isomorphisms on integral (pro-)homology groups by implication (i) $\Rightarrow$ (iii): Indeed, by the universal coefficient theorem, it remains to check the cohomology groups with coefficients $\mathbb{Q}/\mathbb{Z}$.

1.1.9 Remark. An example for the usefulness of this theorem is the following pro-version of the Hurewicz-Theorem: Let $n > 0$ and suppose $(\mathfrak{X}, x) = \{(\mathfrak{X}_i, x_i)\}_i$ is $(n-1)$-connected in the pro-sense, i.e., $\pi_q(\mathfrak{X}, x) = 0$ for all $q < n$. We get natural homotopy fibre sequences

$$\mathfrak{F}_i \longrightarrow \mathfrak{X}_i \longrightarrow \mathrm{cosk}_n \mathfrak{X}_i$$

(see [AM69] the remarks following (2.5)) with the resulting homotopy fibre $\mathfrak{F}$ levelwise $(n-1)$-connected. Thus, we can apply the classical Hurewicz-Theorem levelwise to get isomorphisms

$$\pi_q(\mathfrak{F}) \longrightarrow H_q(\mathfrak{F})$$

for all $0 < q \leq n$. But with the help of Thm. 1.1.8 for $\underline{C}$ the class of all groups together with the long exact sequence of a fibration we get that $\mathfrak{F} \to \mathfrak{X}$ is a $\natural$-isomorphism, i.e., we get isomorphisms

$$\pi_q(\mathfrak{X}) \longrightarrow H_q(\mathfrak{X})$$

for all $0 < q \leq n$ for $\mathfrak{X}$, as well (see [AM69] Cor. 4.5 for details).

1.1.10 Remark. As we have seen above, (1.1.1) is an isomorphism in degrees ≤ 1 for a connected $(\mathfrak{X}, x)$ pro-homotopy type. Similar, (1.1.1) is an isomorphism in degrees $\leq n$ for an $(n-1)$-connected $(\mathfrak{X}, x)$ pro-homotopy type: As in Rem. 1.1.9, we may replace $\mathfrak{X}$ by $\mathfrak{F}$, i.e., we may assume, that $\mathfrak{X}$ is even levelwise $(n-1)$-connected. But then again by the explicit construction of the levels of the pro-$\underline{C}$-completion of $\mathfrak{X}$ in [AM69] Cor. 3.6 we see that

$$\pi_q(\mathfrak{X}, x)^\wedge_{\underline{C}} \longrightarrow \pi_q(\mathfrak{X}^\wedge_{\underline{C}}, x)$$

is an isomorphism for all $q \leq n$ (see [AM69] Cor. 6.2 for details).

For the rest of this section, we want to discuss a version of Rem. 1.1.10 for up-to-good-$\underline{C}$-groups $(n-1)$-connected pro-homotopy types:

1.1.11 Definition. *Let* $\underline{C}$ *be a complete class of groups and* Γ *a pro-group. Define* Γ *to be* ***$\underline{C}$-good***, *if for all* $\Gamma^\wedge_{\underline{C}}$*-modules* Λ *whose underlying abelian group lies in* $\underline{C}$, *the canonical map*

$$H^\bullet(\Gamma^\wedge_{\underline{C}}; \Lambda) \longrightarrow H^\bullet(\Gamma; \Lambda)$$

is an isomorphism.

1.1.12 Remark. Suppose $\underline{C}$ is the complete class of all finite groups. Then any group containing a subgroup of finite index which is solvable and moreover has only finitely generated subgroups is $\underline{C}$-good by [Sul74] Thm. 3.1. In particular, all finitely generated abelian groups are $\underline{C}$-good.

By the $\natural$-uniqueness of Eilenberg-MacLane spaces (see [AM69] Cor. 4.14), any pro-homotopy type $\mathfrak{X}$ having the $K(\Gamma,1)$- resp. $K(\Gamma^\wedge_{\underline{C}},1)$-property in the pro-sense is $\natural$-isomorphic to $B\Gamma$ resp. $B(\Gamma^\wedge_{\underline{C}})$. Thus, Γ is $\underline{C}$-good if and only if for any pro homotopy type $\mathfrak{X}$ having the $K(\Gamma,1)$-property the completion $\mathfrak{X}^\wedge_{\underline{C}}$ has the $K(\Gamma^\wedge_{\underline{C}},1)$-property.

Now the up-to-good-$\underline{C}$-groups version of Rem. 1.1.10 is the following theorem:

1.1.13 Theorem. *([AM69] Thm. 6.7) Let $\underline{C}$ be a complete class of groups contained in the complete class of all finite groups. Let $(\mathfrak{X},x)$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$ be simply connected with $\pi_q(\mathfrak{X},x)$ $\underline{C}$-good for all $q<n$. Then the canonical map*

$$\pi_q(\mathfrak{X},x)^\wedge_{\underline{C}} \longrightarrow \pi_q(\mathfrak{X}^\wedge_{\underline{C}},x)$$

is an isomorphism for all $q \leq n$.

1.1.14 Remark. Before giving a sketch of the proof, we need a technical result: In the following, let $\underline{C}$ be a complete class contained in the complete class of all finite groups. Let

$$f:\ (\mathfrak{X},x) \longrightarrow (\mathfrak{Y},y)$$

be a levelwise map in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$ with $\mathfrak{Y}$ levelwise fibrant. By [AM69] Lem. 5.3 we get a levelwise commutative square:

$$\begin{tikzcd}
(\mathfrak{X},x) \arrow[r, "f"] \arrow[d] & (\mathfrak{Y},y) \arrow[d] \\
(\mathfrak{X},x)^\wedge_{\underline{C}} \arrow[r, "f^\wedge_{\underline{C}}"] & (\mathfrak{Y},y)^\wedge_{\underline{C}}
\end{tikzcd}$$

By using a functorial factorization in $\underline{\mathrm{SSets}_\bullet}$, we may replace f and $f^\wedge_{\underline{C}}$ by levelwise fibrations and get a morphism of homotopy fibre sequences in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$:

$$\begin{tikzcd}
\mathfrak{F} \arrow[r] \arrow[d] & (\mathfrak{X},x) \arrow[r, "f"] \arrow[d] & (\mathfrak{Y},y) \arrow[d] \\
\bar{\mathfrak{F}} \arrow[r] & (\mathfrak{X},x)^\wedge_{\underline{C}} \arrow[r, "f^\wedge_{\underline{C}}"] & (\mathfrak{Y},y)^\wedge_{\underline{C}}
\end{tikzcd}$$

By the induced long exact sequence of homotopy groups we get

$$\bar{\mathfrak{F}} \in \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})_{\underline{C}},$$

i.e., we get a canonical map

$$\mathfrak{F}^\wedge_{\underline{C}} \longrightarrow \bar{\mathfrak{F}} \tag{1.1.2}$$

Suppose that $\mathfrak{Y}$ is levelwise simply connected in the above level representation of f. Then this canonical map is a $\natural$-isomorphism by [AM69] Thm. 5.9.

To prove Thm. 1.1.13 let us first assume that $\mathfrak{X}$ has the $K(\Delta, r)$-property for Δ a $\underline{C}$-good pro-abelian group. By the $\natural$-uniqueness of Eilenberg-MacLane spaces we may assume that $\mathfrak{X}$ is even a levelwise Eilenberg-MacLane space $K(\Delta, r)$ on the nose. After application of a functorial factorization in $\underline{\mathrm{SSets}_\bullet}$ we may replace the point $\mathrm{pt} \to K(\Delta, r)$ by a levelwise fibration

$$\mathfrak{E} \twoheadrightarrow K(\Delta, r) \ .$$

We get a levelwise homotopy fibre sequence

$$\mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow K(\Delta, r)$$

with $\mathfrak{E}$ levelwise contractible and $\mathfrak{F}$ again a levelwise pointed Eilenberg-MacLane space $K(\Delta, r-1)$. Thus, by induction on r using the $\natural$-isomorphism (1.1.2) we get that $K(\Delta, r)^\wedge_{\underline{C}}$ is $\natural$-isomorphic to a $K(\Delta^\wedge_{\underline{C}}, r)$ for all $r > 1$, as well. Thus, we get:

1.1.15 Proposition. *([AM69] Prop. 6.9) Let $\underline{C}$ be a complete class of groups contained in the complete class of all finite groups. A pro-abelian group Δ is $\underline{C}$-good if and only if for all $r > 0$ and and all pro-homotopy types $\mathfrak{X}$ having the $K(\Delta, r)$-property in the pro-sense, the completion $\mathfrak{X}^\wedge_{\underline{C}}$ has the $K(\Delta^\wedge_{\underline{C}}, r)$-property in the pro-sense.*

Next, let $\mathfrak{X}$ be any simply connected pro-homotopy type with $\pi_q(\mathfrak{X}, x)$ $\underline{C}$-good for all $q < n$. We have to show that

$$\pi_q(\mathfrak{X}, x)^\wedge_{\underline{C}} \longrightarrow \pi_q(\mathfrak{X}^\wedge_{\underline{C}}, x)$$

is an isomorphism for all $q \leq n$. Suppose $\mathfrak{X}$ is even $(r-1)$-connected. If $r \geq n$, then the claim follows by Rem. 1.1.10. If $1 < r < n$ we argue by descending induction on r: By assumption, $\mathrm{cosk}_{r+1}\mathfrak{X}$ is a $K(\pi_r(\mathfrak{X}, x), r)$ in the pro-sense. Application of a functorial factorization in $\underline{\mathrm{SSets}_\bullet}$ gives a levelwise fibration

$$\mathfrak{X}' \longrightarrow \mathrm{cosk}_{r+1}\mathfrak{X}$$

with $\mathfrak{X}'$ levelwise weakly equivalent to $\mathfrak{X}$. Thus, we get a levelwise homotopy fibre sequence

$$\mathfrak{F} \longrightarrow \mathfrak{X}' \longrightarrow \mathrm{cosk}_{r+1}\mathfrak{X}$$

with $\mathfrak{F}$ r-connected and

$$\pi_q(\mathfrak{F}) = \pi_q(\mathfrak{X})$$

for all $q > r$. By induction we get isomorphisms

$$\pi_q(\mathfrak{F})^\wedge_{\underline{C}} \longrightarrow \pi_q(\mathfrak{F}^\wedge_{\underline{C}})$$

for all $q \leq n$ so the claim of Thm. 1.1.13 follows for $\mathfrak{X} \cong \mathfrak{X}'$ using the $\natural$-isomorphism (1.1.2) of Rem. 1.1.14.

1.2 The homotopy type of a connected site. Next, we want to discuss the homotopy type in Pro$\mathcal{H}$($\underline{\text{SSets}}$) resp. Pro$\mathcal{H}$($\underline{\text{SSets}_\bullet}$) of a connected resp. connected pointed site $\underline{S}$ (for convenience: complete under finite limits, finite coproducts and having enough points). Here, a site $\underline{S}$ is called **locally connected**, if every object is (in a necessarily unique way) isomorphic to the direct sum of connected objects, where an object of $\underline{S}$ is called **connected**, if it is not isomorphic to a non trivial direct sum of objects in $\underline{S}$. Thus, for $\underline{S}$ locally connected we get a functor

$$\pi_0 : \underline{S} \longrightarrow \underline{\text{Sets}}$$

mapping an object X isomorphic to $\bigoplus_{i\in I} X_i$ for connected X_i to the **set of connected components** $\{X_i | i \in I\}$. A locally connected connected site $\underline{S}$ is a **connected site**, if the final object $\text{pt}_{\underline{S}}$ is connected.

1.2.1 Remark. Let $U \to \text{pt}_{\underline{S}}$ be a covering of the final object of $\underline{S}$ and F a finite set or more generally, a set s.t. $\underline{S}$ is closed under direct sums indexed by F. A **descent datum** of the trivial covering

$$U \otimes F \longrightarrow U$$

with fibre F is an isomorphism

$$\varphi : (U \times U) \otimes F = \ (d_1^0)^*(U \otimes F) \longrightarrow (d_1^1)^*(U \otimes F) \ = (U \times U) \otimes F$$

in $\underline{S} \downarrow (U \times U)$ s.t.

$$(d_2^1)^*\varphi = (d_2^2)^*\varphi \circ (d_2^0)^*\varphi,$$

where the d_i^j's are just the face maps of the simplicial object $\text{cosk}_0^{\underline{S}} U$ in $\underline{S}$. Now an $\underline{S} \downarrow (U \times U)$-automorphism of $(U \times U) \otimes F$ is just an element of

$$\text{Hom}_{\underline{\text{Sets}}}(\pi_0(U \times U), S_F),$$

where S_F is the symmetric group on F. Thus, descend data of the trivial covering $U \otimes F \to U$ is classified by the pointed set

$$H^1(\pi_0(\text{cosk}_0^{\underline{S}} U); S_F).$$

On the other hand, for a simplicial set $A.$ the pointed set $H^1(A.; S_F)$ is isomorphic to $\text{Hom}_{\underline{\text{Cat}}}(P_*(A.), S_F)$, where $P_*(A.)$ is the path category of $A.$, i.e., to

$$\text{Hom}_{\underline{\text{Cat}}}(\Pi(A.), S_F).$$

Similar, for an abelian group L we get that the pointed set

$$H^1(\pi_0(\text{cosk}_0^{\underline{S}} U); \text{Aut}_{\underline{\text{Ab}}}(L))$$

classifies both descend data for the constant sheaf L on U as well as local systems on $\pi_0(\text{cosk}_0^{\underline{S}} U)$ with stalks L. Further, by the sheaf property, all of theses latter descent data are effective and $C^\bullet(\pi_0(\text{cosk}_0^{\underline{S}} U); \mathcal{L})$ for such a local system $\mathcal{L}$ is just the Čech complex $\Gamma(\text{cosk}_0^{\underline{S}} U; \mathcal{L})$ of the resulting locally constant sheaf.

Suppose that $\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}})$ is cofiltered, i.e.

$$\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}}) \longrightarrow \mathcal{H}(\underline{\mathrm{SSets}}) \ , \ \{U \to \mathrm{pt}_{\underline{S}}\} \longmapsto \pi_0(\mathrm{cosk}_0^{\underline{S}} U)$$

induces a well defined object of $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$. Then the reasoning of Rem. 1.2.1 suggests that this object is up to $\natural$-isomorphism the correct homotopy type of the site $\underline{S}$.

Yet, there are two problems with this: First, the category of coverings of the final object $\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}})$ is not cofiltered in general. Further, Čech cohomology does not agree with cohomology over $\underline{S}$ in general. We want to discuss the second problem more closely:

1.2.2 Remark. Let $\mathcal{F}$ be a sheaf of abelian groups over $\underline{S}$ and let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^\bullet$$

be an injective resolution in $\mathrm{Shv}(\underline{S})$. As

$$\mathbb{Z}[\mathrm{cosk}_0^{\underline{S}} U]_\bullet \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact (check this stalk wise and use that $\mathrm{cosk}_0 A$ is contractible for any set A), we have a canonical quasi-isomorphism

$$\mathbb{R}\Gamma(\mathrm{pt}_{\underline{S}}; \mathcal{F}) \xrightarrow{\sim} \mathrm{tot}^\bullet \Gamma((\mathrm{cosk}_0^{\underline{S}} U)_{\bullet_{\mathrm{II}}}; \mathcal{I}^{\bullet_{\mathrm{I}}}) \ .$$

Suppose that the colimit in $\underline{\mathrm{Ab}}$ over $\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}})$ is exact (this would be the case e.g., if $\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}})$ is cofiltered). By the second spectral sequence of a double complex, the colimit over $\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}})$ of the canonical map

$$\Gamma((\mathrm{cosk}_0^{\underline{S}} U)_\bullet; \mathcal{F}) \longrightarrow \mathrm{tot}^\bullet \Gamma((\mathrm{cosk}_0^{\underline{S}} U)_{\bullet_{\mathrm{II}}}; \mathcal{I}^{\bullet_{\mathrm{I}}})$$

is an isomorphism if the corresponding colimit of

$$H^p_{\mathrm{II}} H^q_{\mathrm{I}} \Gamma((\mathrm{cosk}_0^{\underline{S}} U)_{\bullet_{\mathrm{II}}}; \mathcal{I}^{\bullet_{\mathrm{I}}})$$

vanishes for $q > 0$. E.g., this would be the case, if for any covering $U \to \mathrm{pt}_{\underline{S}}$, any p and any $q > 0$ we could kill any class $\alpha \in H^q_{\underline{S}}((\mathrm{cosk}_0^{\underline{S}} U)_p; \mathcal{F})$ after a refinement of U in $\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}})$.

Cohomology in degrees > 0 vanishes locally, so we want to replace $\mathrm{cosk}_0^{\underline{S}}(-)$ of coverings in $\underline{\mathrm{Cov}}(\mathrm{pt}_{\underline{S}})$ by a certain class of simplicial objects $\mathfrak{U}.$ in $\underline{S}$ s.t. for all p and all coverings $V \to \mathfrak{U}_p$, there is a refinement $\mathfrak{V}. \to \mathfrak{U}.$ in this class whose p^{th}-degree $\mathfrak{V}_p \to \mathfrak{U}_p$ is a refinement of our original covering. This leads us to the definition of a hypercovering:

1.2.3 Definition. *A* ***hypercovering*** *in $\underline{S}$ is a simplicial object $\mathfrak{U}.$ in $\underline{S}$ s.t. the canonical maps*

$$\mathfrak{U}_n \longrightarrow (\mathrm{cosk}_{n-1}^{\underline{S}} \mathfrak{U}.)_n$$

are coverings for each $n \geq 0$ (where we define $\mathrm{cosk}_{-1}^{\underline{S}} \mathfrak{U}.$ to be $\mathrm{pt}_{\underline{S}}$ in the case $n = 0$).

Combining [AM69] Lem. 8.7, Lem. 8.8 and Lem. 8.9 we get the desired result:

1.2.4 Lemma. *(e.g., see the proof of [AM69] Thm. 8.16) Let* $\mathfrak{U}.$ *be a hypercovering of* $\underline{S}$ *and let* $V \to \mathfrak{U}_p$ *be a covering. Then there is a map of hypercoverings (i.e., of simplicial objects in* $\underline{S}$*)* $\mathfrak{V}. \to \mathfrak{U}.$ *s.t.* $\mathfrak{V}_p \to \mathfrak{U}_p$ *is a refinement of the original covering* $V \to \mathfrak{U}_p$*.*

Proof: Use [AM69] Lem. 8.7 and Lem. 8.8 to get a refinement $\mathfrak{U}.^{(p)} \to \mathfrak{U}.$ split up to level p and pull back V to a covering $V^{(p)} \to \mathfrak{U}_p^{(p)}$. Then apply [AM69] Lem. 8.9 to $V^{(p)} \to \mathfrak{U}_p^{(p)}$ and $\mathfrak{U}.^{(p)}$ as in the proof of [AM69] Thm. 8.16 to get the desired refinement. $\square$

For a simplicial set $A.$ and a simplicial object $X.$ of $\underline{S}$ define $X. \otimes A.$ degreewise as the direct sum of X_n over A_n. Note, that $\pi_0(X.) \times A.$ is just $\pi_0(X. \otimes A.)$. We say that two maps of hypercoverings $\mathfrak{U}. \rightrightarrows \mathfrak{V}.$ are **strictly homotopic**, if there is a commutative diagram of the form

$$\begin{tikzcd} & \mathfrak{V}. \\ \mathfrak{U}. \arrow[ur, shift left] \arrow[ur, shift right] \arrow[r, shift left] \arrow[r, shift right] & \mathfrak{U}. \otimes \Delta^1 \arrow[u] \end{tikzcd}$$

where $\mathfrak{U}. \rightrightarrows \mathfrak{U}. \otimes \Delta^1$ are induced by the canonical maps $\mathrm{pt} \rightrightarrows \Delta^1$. We say that two maps $\mathfrak{U}. \rightrightarrows \mathfrak{V}.$ are **homotopy equivalent**, if these maps are equivalent with resp. to the equivalence relation generated by strict homotopic maps.

1.2.5 Lemma. *([AM69] Cor. 8.13) Let* $\underline{\mathrm{HR}}(\underline{S})$ *be the category consisting of hypercoverings of* $\underline{S}$ *as objects and morphisms of simplicial objects of* $\underline{S}$ *modulo homotopy equivalence as morphisms. Then* $\underline{\mathrm{HR}}(\underline{S})$ *is cofiltered.*

Note, that $\pi_0(-)$ induces a functor $\mathrm{Ver}(\underline{S}) : \underline{\mathrm{HR}}(\underline{S}) \to \mathcal{H}(\underline{\mathrm{SSets}})$, the Verdier functor.

1.2.6 Definition. *Let* $\underline{S}$ *be a connected site. Define the* ***homotopy type of*** $\underline{S}$ *as the pro-homotopy type in* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ *given by the* ***Verdier functor***

$$\mathrm{Ver}(\underline{S}) : \ \underline{\mathrm{HR}}(\underline{S}) \longrightarrow \mathcal{H}(\underline{\mathrm{SSets}}) \ , \ \mathfrak{U}. \longmapsto \pi_0(\mathfrak{U}.) \ .$$

1.2.7 Remark. If we start with a pointed connected site $p : \underline{\mathrm{Sets}} \to \underline{S}$ we get a pointed homotopy type in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$: Instead of hypercoverings, we use pointed hypercoverings, i.e., hypercoverings $\mathfrak{U}. \in \underline{\mathrm{HR}}(\underline{S})$ with a distinguished point $\mathrm{pt} \to p^*\mathfrak{U}.$ in $\underline{\mathrm{SSets}}$. These form a cofiltered homotopy category $\underline{\mathrm{HR}}(\underline{S}, p)$, as well. There is a canonical natural transformation $p^* \to \pi_0$, i.e., π_0 induces a functor

$$\mathrm{Ver}(\underline{S}, p) : \ \underline{\mathrm{HR}}(\underline{S}, p) \longrightarrow \mathcal{H}(\underline{\mathrm{SSets}_\bullet}) \ .$$

From this, we get back $\mathrm{Ver}(\underline{S})$ if we forget the point in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$: Indeed, the canonical functor $\underline{\mathrm{HR}}(\underline{S}, p) \to \underline{\mathrm{HR}}(\underline{S})$ is cofinal (for details, see [AM69] Cor. 8.13).

Taking together Rem. 1.2.2 and Lem. 1.2.4 we get:

1.2.8 Theorem. *(Verdier-Hypercovering-Theorem for local systems). Let $\underline{S}$ be a connected site and let $\mathcal{L}$ be a locally constant sheaf on $\underline{S}$. Then there is a canonical quasi-isomorphism*

$$\mathbb{R}\Gamma(\mathrm{pt}_{\underline{S}};\mathcal{L}) \simeq C^\bullet(\mathrm{Ver}(\underline{S});\mathcal{L})$$

in the derived category $\mathcal{D}^+(\underline{\mathrm{Ab}})$.

1.2.9 Remark. Of course, the same arguments works for a corresponding statement for general sheaves of abelian groups on $\underline{S}$, as well.

Further, for $\mathfrak{U}.$ a hypercovering of $\underline{S}$ the canonical map

$$\mathfrak{U}. \longrightarrow \mathrm{cosk}_0^{\underline{S}}\mathfrak{U}.$$

induces an equivalence between descent data relative to $\mathrm{cosk}_0^{\underline{S}}\mathfrak{U}.$ and descent data relative to $\mathfrak{U}.$ by [AM69] Prop. 10.3. Thus, the discussion of Rem. 1.2.1 yields:

1.2.10 Proposition. *Let $(\underline{S}, p)$ be a pointed connected site. The fundamental group $\pi_1(\underline{\mathrm{Ver}}(\underline{S}, p))$ classifies descent data of locally trivial coverings in $(\underline{S}, p)$. Further, the fundamental group $\pi_1(\underline{\mathrm{Ver}}(\underline{S}, p)^\wedge)$ of the profinite completion of $\underline{\mathrm{Ver}}(\underline{S}, p)$ classifies descent data of locally trivial coverings in $(\underline{S}, p)$ with finite fibres.*

1.3 The étale homotopy type à la Artin-Mazur. In the following, we want to discuss a bit closer the étale homotopy type $\mathrm{Ver}(X_{\text{ét}})$ resp. $\mathrm{Ver}(X_{\text{ét}}, x)$ for a (local) Noetherian scheme X together with a geometric point $x \in X(\Omega)$: Recall, that for any étale covering $U \to X$ any descent datum for any finite trivial covering $U \otimes F \to U$ is effective. Thus, from Prop. 1.2.10 we get that the profinite completion $\pi_1(\mathrm{Ver}(X_{\text{ét}}, x))^\wedge$ is just the étale fundamental group $\pi_1^{\text{ét}}(X, x)$ (cf. [AM69] §10). Even more holds for X Noetherian, connected and geometrically unibranched: In this case $\mathrm{Ver}(X_{\text{ét}}, x)$ is already profinite complete, i.e., $\mathrm{Ver}(X_{\text{ét}}, x)$ has the correct homotopy type on the nose:

1.3.1 Theorem. *([AM69] Thm. 11.1) Let (X, x) be a pointed Noetherian connected and geometrically unibranched scheme. Then the étale homotopy type $\mathrm{Ver}(X_{\text{ét}}, x)$ is profinite complete.*

Sketch of proof: We have to show that $\pi_q(\pi_0(\mathfrak{U}.))$ is finite for all q and all hypercoverings $\mathfrak{U}.$ in a cofinal subcategory of $\underline{\mathrm{HR}}(X, x)$. To see this we may first assume that X is reduced. Since X is geometrically unibranched, each connected étale $U \to X$ is already irreducible. Let

$$\eta: \ \mathrm{Spec}(k(X)) \longrightarrow X$$

be the generic point. Then $\pi_0(\mathfrak{U}.)$ is just $\pi_0(\eta^*\mathfrak{U}.)$, i.e., we have reduced our problem to the case of a hypercovering $\mathfrak{U}.$ in $(k(X))_{\text{ét}}$ pointed by some extension

$\Omega/k(X)$ with Ω algebraically closed or more generally in the classifying site $\underline{B}\Gamma$ of a profinite group Γ (where the underling category is just $\underline{\mathrm{Sets}}^{\mathrm{f}}_{\Gamma}$, the category of finite discrete Γ-sets, cf. [Mil80] Chapt. II Rem. 1.11) pointed by the restriction to the trivial group. Here $\pi_0(-)$ is just the functor $(-)/\Gamma$ taking coinvariants, i.e., if we let Γ act trivially on $\pi_0(\mathfrak{U}.)$ we get a canonical map $\mathfrak{U}. \to \pi_0(\mathfrak{U}.)$ in $\underline{B}\Gamma$.

Denote by p the canonical point of $\underline{B}\Gamma$ with $p^* = \mathrm{res}^{\mathbf{1}}_{\Gamma}$. It is not hard to see that $p^*\mathfrak{U}. \to \mathrm{pt}$ is an acyclic fibration of simplicial sets (this holds for every pointed site!). Using this, we can solve any lifting problem of the form

$$\begin{tikzcd} \Lambda^2_k \arrow[r] \arrow[d, hook, "\mathrm{can.}"'] & \pi_0(\mathfrak{U}.) \\ \Delta^2 \arrow[ur] & \end{tikzcd}$$

(argue as in the proof of [AM69] Lem. 11.6), i.e., the path category $P_*(\pi_0(\mathfrak{U}.))$ has finite endomorphism sets and each roof in $P_*(\pi_0(\mathfrak{U}.))$ already has a solution in at least one direction. Thus, $\Pi(\pi_0(\mathfrak{U}.))$ has finite automorphism groups, i.e., $\pi_1(\pi_0(\mathfrak{U}.))$ is finite.

For the higher homotopy groups, it suffices to consider the total space of the universal covering

$$A. \to \pi_0(\mathfrak{U}.).$$

Since $\pi_1(\pi_0(\mathfrak{U}.))$ is finite, $A.$ has to be degreewise finite and hence $H_\bullet(A.)$ degreewise finitely generated. Thus, by the universal coefficients theorem together with an induction argument using [Ser53] Chap. III Thm. 1 we see that it suffices to show that $H^q(A.;\mathbb{Z})$ is finite for all $q > 0$. By our above description of π_0 as coinvariants, we get the simplicial object (not necessary a hypercovering!)

$$\mathfrak{V}. := \mathfrak{U}. \times_{\pi_0(\mathfrak{U}.)} A.$$

in $\underline{B}\Gamma$ (here Γ acts trivially on $A.$). By construction, $\pi_0(\mathfrak{V}.)$ is just $A.$ from which we started. Further, $p^*\mathfrak{V}.$ is $p^*\mathfrak{U}. \times_{\pi_0(\mathfrak{U}.)} A.$, since p^* preserves finite limits. In particular, $p^*\mathfrak{V}. \to p^*\mathfrak{U}.$ is a simplicial covering space of a contractible simplicial set, i.e., $p^*\mathfrak{V}.$ itself is contractible. By our description of π_0, both $\pi_0(-)$ and $p^*(-)$ preserves $\mathrm{sk}_{\mathrm{n}}(-)$. Further, Γ acts via a finite quotient $\Gamma \twoheadrightarrow \Gamma_n$ on $\mathrm{sk}^{\underline{B}\Gamma}_n\mathfrak{V}.$ for each n. To show that $H^q(A.;\mathbb{Z})$ is finite we choose $n \gg 0$ s.t.

$$H^q(\mathrm{sk}_n A.;\mathbb{Z}) = H^q(A.;\mathbb{Z})$$

and may thus assume that Γ is even a finite group.

For Γ finite and a Γ-module Λ (i.e., a sheaf of abelian groups on $\underline{B}\Gamma$), the composition

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{res}^{\mathbf{1}}_{\Gamma}"] \arrow[rr, bend right, "\cdot|\Gamma|"'] & p_*p^*\Lambda \arrow[r, "\mathrm{cor}^{\mathbf{1}}_{\Gamma}"] & \Lambda \end{tikzcd}$$

is just multiplication by the order of Γ. Now

$$C^\bullet(A.;\Lambda) = C^\bullet(\pi_0(\mathfrak{V}.);\Lambda) = \Gamma(\mathfrak{V}_\bullet;\Lambda).$$

But $p^*\mathfrak{V}.$ is contractible, so

$$\Gamma(\mathfrak{V}_\bullet, p_*p^*\Lambda) = C^\bullet(p^*\mathfrak{V}.; p^*\Lambda)$$

is quasi-isomorphic to $p^*\Lambda$ in degree 0. Thus, for $q > 0$ the cohomology group $H^q(\pi_0(\mathfrak{V}.); \Lambda)$ is killed by the order of Γ, i.e., is torsion. But since $A. = \pi_0(\mathfrak{V}.)$ is degreewise finite $H^q(A.; \Lambda)$ is finitely generated and hence finite, which completes the proof. $\square$

Next, we want to relate the étale homotopy type of a pointed connected $\mathbb{C}$-variety (X, x) in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$ to (the singular complex of) its analytification (X^{an}, x) in $\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$. Denote by $X^{\mathrm{an}}_{\mathrm{loc}}$ resp. $X^{\mathrm{an}}_{\mathrm{top}}$ the site consisting of local (with resp. to the source) isomorphisms $U \to X^{\mathrm{an}}$ resp. open subsets $U \hookrightarrow X^{\mathrm{an}}$ in $\underline{\mathrm{Top}}$. For any hypercovering in $X^{\mathrm{an}}_{\mathrm{loc}}$ there is a refinement even in $X^{\mathrm{an}}_{\mathrm{top}}$, i.e., the canonical map

$$\mathrm{Ver}(X^{\mathrm{an}}_{\mathrm{loc}}, x) \longrightarrow \mathrm{Ver}(X^{\mathrm{an}}_{\mathrm{top}}, x)$$

is an isomorphism in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$. On the other hand, the analytification of an étale map is a local isomorphism, i.e. we get a canonical morphism of sites $X^{\mathrm{an}}_{\mathrm{loc}} \to X_{\text{ét}}$. The resulting canonical morphism

$$\mathrm{Ver}(X^{\mathrm{an}}_{\mathrm{loc}}, x) \longrightarrow \mathrm{Ver}(X_{\text{ét}}, x)$$

induces isomorphisms on cohomology of local systems with finite stalks (see [SGA73] Exp. XVI Thm. 4.1) as well as on the profinite completion of the fundamental groups (see [SGA73] Exp. XI Thm. 4.3), i.e., its profinite completion is a $\natural$-isomorphism by Thm. 1.1.8.

Let $\mathfrak{U}.$ be a hypercovering in $(X^{\mathrm{an}}_{\mathrm{top}}, x)$ fine enough, s.t. each connected component of each degree is contractible. We get canonical morphisms from the diagonal simplicial set

$$\mathrm{diag}(\mathrm{Sing}.(\mathfrak{U}.)) \longrightarrow \pi_0(\mathfrak{U}.)$$

and

$$\mathrm{diag}(\mathrm{Sing}.(\mathfrak{U}.)) \longrightarrow \mathrm{Sing}.(X^{\mathrm{an}}) \simeq X^{\mathrm{an}}$$

in $\underline{\mathrm{SSets}_\bullet}$. It turns out that both morphisms are weak equivalences:

1.3.2 Theorem. *(see [AM69] Thm. 12.1) Let (X, x) be a pointed connected $\mathbb{C}$-variety and $\mathfrak{U}.$ a hypercovering in $(X^{\mathrm{an}}_{\mathrm{top}}, x)$ s.t. each connected component of each degree is contractible. Then the canonical morphisms*

$$\begin{tikzcd} & \mathrm{diag}(\mathrm{Sing}.(\mathfrak{U}.)) \arrow[dl] \arrow[dr] & \\ \pi_0(\mathfrak{U}.) & & (X^{\mathrm{an}}, x) \end{tikzcd}$$

are isomorphisms in $\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$. In particular, $\mathrm{Ver}(X^{\mathrm{an}}_{\mathrm{top}}, x)$ is pro-discrete and isomorphic to (X^{an}, x).

Summing up, we get a canonical $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$-morphism

$$(X^{\mathrm{an}}, x) \longrightarrow \mathrm{Ver}(X_{\text{ét}}, x) \tag{1.3.1}$$

whose profinite completion is a $\natural$-isomorphism. But even more is true:

1.3.3 Definition. *Let $\underline{C}$ be a class of groups. Then a site $\underline{S}$ has* ***local $\underline{C}$-dimension $\leq d$*** *if for any U in $\underline{S}$ there is a covering $V \to U$ s.t. $H^q(V;\mathcal{L})$ is trivial for any $q > d$ and any locally constant sheaf $\mathcal{L}$ on $\underline{S}$ with stalks in $\underline{C}$.*

Examples are the sites $X_{\text{ét}}$ and $X^{\mathrm{an}}_{\mathrm{loc}}$ for our $\mathbb{C}$-variety X: with resp. to $\underline{C}$ the complete class of all finite groups both sites have local $\underline{C}$-dimension $\leq 2\dim(X)$.

1.3.4 Theorem. *([AM69] Thm. 12.5) Let $f : \underline{S}' \to \underline{S}''$ be a morphism between two pointed sites of local $\underline{C}$-dimension $\leq d$ for $\underline{C}$ a complete class of groups. Suppose that the $\underline{C}$-completion $\mathrm{Ver}(f)^\wedge_{\underline{C}}$ of the induced map $\mathrm{Ver}(f)$ is a $\natural$-isomorphism. Then $\mathrm{Ver}(f)^\wedge_{\underline{C}}$ is even an $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$-isomorphism on the nose.*

The last theorem applies in particular for the above canonical morphism of sites

$$(X^{\mathrm{an}}_{\mathrm{loc}}, x) \longrightarrow (X_{\text{ét}}, x)\ ,$$

i.e., the profinite completion of (1.3.1) is a $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$-isomorphism on the nose. Thus, we get the generalized Riemann existence theorem from the usual descent arguments:

1.3.5 Theorem. *(see [AM69] Thm. 12.9, Cor. 12.11 and Cor. 12.12) Let k be a field together with an embedding $k \to \mathbb{C}$ and denote by $(-)^{\mathrm{an}}$ the induced analytification functor on $\underline{\mathrm{Var}_{\bar{k}}}$, as well. Then for (X,x) a proper pointed connected $\bar{k}$-variety resp. a pointed connected $\mathbb{C}$-variety, the canonical $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$-morphism*

$$(X^{\mathrm{an}}, x) \longrightarrow \mathrm{Ver}(X_{\text{ét}}, x)$$

is an isomorphism after profinite completion. In particular, if X is even geometrically unibranched, $(X^{\mathrm{an}}, x) \to \mathrm{Ver}(X_{\text{ét}}, x)$ is the profinite completion on the nose.

1.4 The étale homotopy type à la Friedlander. Recall that we solved the problem of $\underline{\mathrm{Cov}}(\underline{S})$ not being cofiltered in general by replacing the Čech nerves $\mathrm{cosk}_0^{\underline{S}} U$ of coverings $U \to \mathrm{pt}_{\underline{S}}$ by hypercoverings of $\underline{S}$ up to homotopy equivalence. The price we had to pay was that the Verdier functor could not factor over $\underline{\mathrm{SSets}}$, i.e., we did not get a notion of an étale topological type. In the following we want to discuss Friedlanders notion of an étale topological type fixing this disadvantage.

The central point of Friedlanders construction is the rigidity of pointed étale neighbourhoods:

1.4.1 Lemma. *(see [SGA71] Exp. I Cor. 5.4) Let X be a scheme, $U \to X$ connected étale and $V \to X$ étale separated. Let $f, g : U \rightrightarrows V$ two X-morphisms s.t. $fu = gu$ for a geometric point $u \in U(\Omega)$. Then f equals g.*

For simplicity we restrict our self to k-varieties. Fix a "big enough" algebraically closed field Ω containing k (see the preceding remarks of [Fri82] Def. 4.2). Without the restriction to k-varieties, we have to choose such a field Ω for each occurring characteristic of a residue field.

For X a k-variety and $x \in X(\Omega)$, the category $\underline{\mathrm{Cov}}(X_{\text{ét}}, x)^\circ$ of pointed connected étale separated neighbourhoods of the geometric point x in X is **rigid** in the sense that there is at most one morphism between two objects: This is just Lem. 1.4.1.

1.4.2 Definition. *Let X be a k-variety. Define the* ***category of rigid coverings*** *of X as the product category*

$$\underline{\mathrm{RC}}(X) := \prod_{x \in X(\Omega)} \underline{\mathrm{Cov}}(X_{\text{ét}}, x)^\circ .$$

We define the ***category of rigid coverings over k-varieties*** *$\underline{\mathrm{RC}}_k$ as the category consisting of k-varieties together with a rigid covering as objects with the obvious choice of morphisms.*

It is clear how to enlarge the definition of $\underline{\mathrm{RC}}(X)$ to ind-schemes of the form $X = \coprod_i X_i$ for arbitrary index sets.

1.4.3 Notation. Denote by $\coprod(-)$ the forgetful functor

$$\underline{\mathrm{RC}}(X) \longrightarrow \underline{\mathrm{Cov}}(X_{\text{ét}}) .$$

In abuse of notation, we usually will identify an object $\{(U_x, u_x) \to (X, x)\}_x$ of $\underline{\mathrm{RC}}(X)$, its induced covering $\coprod\{(U_x, u_x) \to (X, x)\}_x$ and the ind-scheme

$$U = \coprod\nolimits_{x \in X(\Omega)} U_x \longrightarrow X$$

and omit the distinguished points u_x of the pointed neighbourhoods (U_x, u_x) in our notation. If moreover the base space X is understood, we just write U for our rigid covering $\{(U_x, u_x) \to (X, x)\}_x$.

For a diagram of rigid coverings

$$\begin{array}{ccccc} U & \longrightarrow & W & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Z & \xleftarrow{g} & Y \end{array}$$

the fibre product in $\underline{\mathrm{RC}}_k$ exists and is given by the pointed neighbourhoods given by the connected components of $U_x \times_{W_z} V_y$ containing the distinguished points $u_x \otimes v_y$ for each $x \in X(\Omega)$ and $y \in Y(\Omega)$ with $fx = z = gy$.

1.4.4 Notation. *We refer to the fibre product in $\underline{\mathrm{RC}}_k$ as the* ***rigid product***

$$U \boxtimes^{\mathrm{rig}}_W V \longrightarrow X \times_Z Y .$$

If f and g are the identity we just write $U \times^{\mathrm{rig}}_W V$.

Thus, $\underline{\mathrm{RC}}_k$ is closed under finite limits. Combining this with Lem. 1.4.1 we get that $\underline{\mathrm{RC}}(X)$ is rigid and closed under finite limits. In particular, $\underline{\mathrm{RC}}(X)$ is cofiltered.

Let $f : Y \to X$ be a map of k-varieties and $U \to X$ a rigid covering in $\underline{\mathrm{RC}}(X)$. Define the **rigid pullback**

$$f^*U \longrightarrow U$$

in $\underline{\mathrm{RC}}_k$ by the pointed neighbourhoods given as the connected components of the naive pullback $U_{f(y)} \times_X Y$ containing the points $u_{f(y)} \otimes y$ together with the canonical projections $(f^*U)_y \to U_{f(y)}$.

Recall that a **cleavage** of a fibre category $\Phi : \underline{F} \to \underline{B}$ is the choice of an inverse image for each object X of $\underline{F}$ and each $\underline{B}$-morphism with target $\Phi(X)$. The chosen inverse images are called **transport morphisms** of the cleavage. If for all X the transport morphisms corresponding to $\mathrm{id}_{\Phi(X)}$ is just id_X, the cleavage is called **normalized**. Summing up, we get:

1.4.5 Lemma. *The canonical functor $\underline{\mathrm{RC}}_k \to \underline{\mathrm{Var}}_k$ mapping a rigid covering $U \to X$ to its base space X together with the rigid pullback is a fibred category with a normalized cleavage which is fibrewise rigid and closed under finite limits.*

Using the rigid pullback, we get a well defined functor

$$\check{C}(-) : \underline{\mathrm{Var}}_k \longrightarrow \mathrm{Pro}\underline{\mathrm{SSets}}$$

as follows:

1.4.6 Definition. *Let X be a k-variety. Define the* ***Čech topological type*** *of X as the pro-simplicial set given by the functor*

$$\check{\mathrm{C}}(X) : \underline{\mathrm{RC}}(X) \to \underline{\mathrm{SSets}},\ \{U \to X\} \mapsto \pi_0(\mathrm{cosk}_0^X U).$$

Suppose that every finite set of points in X is contained in an affine open. Then $H^\bullet(\check{\mathrm{C}}(X); \mathcal{L})$ agrees with $H^\bullet(X; \mathcal{L})$ for any local system $\mathcal{L}$ on $X_{\text{ét}}$ by [Art71] Cor. 4.2, i.e., Rem. 1.2.1 suggest that $\check{\mathrm{C}}(X)$ already has the right homotopy type in this special case. Thus we define:

1.4.7 Definition. *Let X be a k-variety. We say that X is of* ***Čech type****, if every finite set of points in X is contained in an affine open.*

Examples of k-varieties of Čech type include all quasi-projective varieties.

1.4.8 Remark. Let $U \to X$ étale and X of Čech type. Then U is of Čech type, as well: Indeed, every finite set of points of U lie over an affine open subset of X. But since the induced affine open subvariety of X is quasi-projective and an étale open of a quasi-projective k-varieties is still quasi-projective, the claim follows.

Going back to a general k-variety X we define the category of rigid hypercoverings:

1.4.9 Definition. *Let X be a k-variety. Define the category $\underline{\mathrm{HRR}}(X)$ of* ***rigid hypercoverings of X*** *as the subcategory of simplicial ind-schemes $\mathfrak{U}. \to X$ over X together with the structures of a rigid covering on each of the canonical maps*

$$\mathfrak{U}_n \longrightarrow (\operatorname{cosk}^X_{n-1}\mathfrak{U}.)_n$$

for $n \geq 0$ (recall, that we defined $\operatorname{cosk}^X_{-1}\mathfrak{U}.$ as X) together with morphisms of simplicial ind-schemes over X s.t. the induced morphisms between the above canonical maps come from maps of rigid coverings between the chosen structures of rigid coverings. We define the ***category of rigid hypercoverings over k-varieties*** *$\underline{\mathrm{HRR}}_k$ as the category consisting of k-varieties together with a rigid hypercovering as objects with the obvious choice of morphisms.*

1.4.10 Notation. *In abuse of notation, we usually will identify a rigid hypercovering in $\underline{\mathrm{HRR}}(X)$ and its underling morphism $\mathfrak{U}. \to X$ of simplicial ind-schemes, i.e. we will usually omit the extra structure on the maps*

$$\mathfrak{U}_n \longrightarrow (\operatorname{cosk}^X_{n-1}\mathfrak{U}.)_n$$

in our notation. If moreover the base space X is understood, we just write $\mathfrak{U}.$ for our rigid hypercovering.

For a diagram of rigid hypercoverings

$$\begin{tikzcd} \mathfrak{U}. \arrow[r] \arrow[d] & \mathfrak{W}. \arrow[d] & \mathfrak{V}. \arrow[l] \arrow[d] \\ X \arrow[r, "f"] & Z & Y \arrow[l, "g"'] \end{tikzcd}$$

the fibre product

$$\mathfrak{U}. \boxtimes^{\mathrm{rig}}_{\mathfrak{W}} \mathfrak{V} \longrightarrow X \times_Z Y$$

in $\underline{\mathrm{HRR}}_k$ exists and is given by induction on n via the n-truncated simplicial scheme given by the fibre product:

$$\begin{tikzcd} (\mathfrak{U}. \boxtimes^{\mathrm{rig}}_{\mathfrak{W}.} \mathfrak{V}.)_n \arrow[r] \arrow[d] & (\operatorname{cosk}^{X\times_Z Y}_{n-1}(\mathfrak{U}. \boxtimes^{\mathrm{rig}}_{\mathfrak{W}.} \mathfrak{V}.))_n \arrow[d, "\text{can.}"] \\ \mathfrak{U}_n \boxtimes^{\mathrm{rig}}_{\mathfrak{W}_n} \mathfrak{V}_n \arrow[r] & (\operatorname{cosk}^X_{n-1}\mathfrak{U}.)_n \times_{(\operatorname{cosk}^Z_{n-1}\mathfrak{W}.)_n} (\operatorname{cosk}^Y_{n-1}\mathfrak{V}.)_n \end{tikzcd}$$

For the simplicial structural maps, we refer to [Isa04] Rem. 3.31.

1.4.11 Notation. *We refer to the fibre product $\mathfrak{U}. \boxtimes^{\mathrm{rig}}_{\mathfrak{W}.} \mathfrak{V}.$ in $\underline{\mathrm{HRR}}_k$ as the* ***rigid product****. If f and g is the identity we just write $\mathfrak{U}. \times^{\mathrm{rig}}_{\mathfrak{W}.} \mathfrak{V}.$ for the rigid product in $\underline{\mathrm{HRR}}(X)$.*

Thus, $\underline{\mathrm{HRR}}_k$ is closed under finite limits. Again, combining this with Lem. 1.4.1 we get that $\underline{\mathrm{HRR}}(X)$ is rigid and closed under finite limits. In particular, $\underline{\mathrm{HRR}}(X)$ is cofiltered.

Let $f : Y \to X$ be a map of k-varieties and $\mathfrak{U}. \to X$ a rigid hypercovering. Define the **rigid pullback**

$$\begin{tikzcd} f^*\mathfrak{U}. \arrow[r] \arrow[d] & \mathfrak{U}. \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

of $\mathfrak{U}.$ in $\underline{\mathrm{HRR}}_k$ by induction on n via the corresponding diagram of n-truncated simplicial schemes given by the rigid pullback along the induced morphism on coskeletons $(\mathrm{cosk}^Y_{n-1} f^*\mathfrak{U}.)_n \to (\mathrm{cosk}^X_{n-1}\mathfrak{U}.)_n$:

$$\begin{tikzcd} (f^*\mathfrak{U}.)_n \arrow[r] \arrow[d] & (\mathrm{cosk}^Y_{n-1} f^*\mathfrak{U}.)_n \arrow[d] \\ \mathfrak{U}_n \arrow[r] & (\mathrm{cosk}^X_{n-1}\mathfrak{U}.)_n \end{tikzcd}$$

Again, for the simplicial structural maps, we refer to [Isa04] Sect. 3.5.

Summing up, we get:

1.4.12 Lemma. *The canonical functor $\underline{\mathrm{HRR}}_k \to \underline{\mathrm{Var}}_k$ mapping a rigid hypercovering $\mathfrak{U}. \to X$ to its base space X together with the rigid pullback is a fibred category with a normalized cleavage which is fibrewise rigid and closed under finite limits.*

Using the rigid pullback we get a well defined functor

$$\acute{\mathrm{E}}\mathrm{t}(-) : \ \underline{\mathrm{Var}}_k \longrightarrow \mathrm{Pro}\underline{\mathrm{SSets}}$$

as follows:

1.4.13 Definition. *Let X be a k-variety. Define the* ***étale topological type of X*** *as the pro-simplicial set given by the functor*

$$\acute{\mathrm{E}}\mathrm{t}(X) : \underline{\mathrm{HRR}}(X) \to \underline{\mathrm{SSets}}, \ \{\mathfrak{U}. \to X\} \mapsto \pi_0(\mathfrak{U}.).$$

We generalize the definition of the Čech resp. étale topological type of a k-variety to simplicial k-varieties. The essential two observations are collected in the following two remarks:

1.4.14 Remark. For a k-variety X we get back $\underline{\mathrm{RC}}(X)$ resp. $\underline{\mathrm{HRR}}(X)$ as the fibre category of $\underline{\mathrm{RC}}_k \to \underline{\mathrm{Var}}_k$ resp. $\underline{\mathrm{HRR}}_k \to \underline{\mathrm{Var}}_k$ over X (cf. Lem. 1.4.5 and 1.4.12).

1.4.15 Remark. Let $\Phi : \underline{F} \to \underline{B}$ be a fibred category together with a normalized cleavage. Suppose it is levelwise rigid and closed under finite limits. Then the induced functor $\Phi : \underline{SF} \to \underline{SB}$ is of the same type. In particular, each fibre category $\underline{SF}(X.)$ is cofiltered. Further, for a simplicial discrete object X in $\underline{B}$ the fiber categories $\underline{F}(X)$ and $\underline{SF}(X)$ agree.

By Rem. 1.4.15, we get a fibred category over simplicial k-varieties

$$\underline{\mathrm{SRC}}_k \longrightarrow \underline{\mathrm{SVar}}_k$$

resp.

$$\underline{\mathrm{SHRR}}_k \longrightarrow \underline{\mathrm{SVar}}_k$$

together with a normalized cleavage which is levelwise rigid and closed under finite limits.

1.4.16 Definition. *Let $X.$ be a simplicial k-variety.*

(i) *Define the category $\underline{\mathrm{RC}}(X.)$ of **rigid coverings over $X.$** as the fibre category over $X.$ of $\underline{\mathrm{SRC}}_k \to \underline{\mathrm{SVar}}_k$.*

(ii) *Define the category $\underline{\mathrm{HRR}}(X.)$ of **rigid hypercoverings over $X.$** as the fibre category over $X.$ of $\underline{\mathrm{SHRR}}_k \to \underline{\mathrm{SVar}}_k$.*

It is clear from the above, that $\underline{\mathrm{RC}}(X.)$ resp. $\underline{\mathrm{HRR}}(X.)$ is cofiltered.

For a map of simplicial k-varieties $Y. \to X.$ define $\mathrm{cosk}_0^{X.}Y.$ as the bisimplicial k-variety given in bidegree (m,n) as $(\mathrm{cosk}_0^{X_m}Y_m)_n$.

1.4.17 Definition. *Let $X.$ be a simplicial k-variety.*

(i) *Define the **Čech topological type of $X.$** as the pro-simplicial set given by the functor*

$$\check{\mathrm{C}}(X.) : \underline{\mathrm{RC}}(X.) \to \underline{\mathrm{SSets}},\ \{U. \to X.\} \mapsto \mathrm{diag}(\pi_0 \mathrm{cosk}_0^{X.}U.).$$

(ii) *Define the **étale topological type of $X.$** as the pro-simplicial set given by the functor*

$$\acute{\mathrm{E}}\mathrm{t}(X.) : \underline{\mathrm{HRR}}(X.) \to \underline{\mathrm{SSets}},\ \{\mathfrak{U}.. \to X.\} \mapsto \mathrm{diag}(\pi_0 \mathfrak{U}..).$$

1.4.18 Remark. By Rem. 1.4.15, we get back $\underline{\mathrm{RC}}(X)$ resp. $\underline{\mathrm{HRR}}(X)$ for a (simplicial discrete) k-variety X. Thus, our definition for the Čech resp. étale topological type for simplicial k-varieties restricts to our old definition in the simplicial discrete case.

As expected, for X any k-variety resp. for a k-variety of Čech type the étale topological type $\acute{\mathrm{E}}\mathrm{t}(X)$ resp. the Čech topological type $\check{\mathrm{C}}(X)$ has the right homotopy type. Moreover, for a simplicial k-variety levelwise of Čech type the Čech type and étale topological types are $\natural$-isomorphic:

1.4.19 Theorem. *(see [Fri82] Prop. 4.5, Prop. 8.2 and [Art71] Cor. 4.2) Let X be a connected k variety and $Y.$ a (possible simplicial discrete) simplicial k-variety, degreewise of Čech type.*

(i) *The pro-homotopy type in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ induced by the étale topological type $\acute{\mathrm{E}}\mathrm{t}(X)$ is isomorphic to $\mathrm{Ver}(X_{\acute{\mathrm{e}}\mathrm{t}})$.*

(ii) *The pro-homotopy type in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ induced by the Čech topological type $\check{\mathrm{C}}(Y.)$ is $\natural$-isomorphic to the pro-homotopy type induced by the étale topological type $\acute{\mathrm{E}}\mathrm{t}(Y.)$.*

1.5 Isaksens closed model structure on the category of pro-simplicial sets. A rather annoying drawback of working in the pro-homotopy category is the following: A Pro$\mathcal{H}$($\underline{\text{SSets}}_\bullet$)-morphisms inducing isomorphisms on π_0 and all pro-homotopy groups is only a $\natural$-isomorphism and not an isomorphism on the nose. To deal with this disadvantage, Isaksen defined a closed model structure on Pro$\underline{\text{SSets}}$, whose weak equivalences are a suitable generalization of the $\natural$-isomorphisms in Pro$\mathcal{H}$($\underline{\text{SSets}}$):

Let $A.$ be a simplicial set. Taking together all the homotopy groups $\pi_q(A., a)$ for the various points $a \in A_0$ together with the isomorphisms

$$\pi_q(A., d_1^1\alpha) \longrightarrow \pi_q(A., d_1^0\alpha)$$

for the various paths $\alpha \in A_1$ gives a local system (of not necessarily abelian groups in the case $q = 1$)

$$\Pi_q(A.) \in \underline{\text{Loc}}(A.).$$

For a map $f : A. \to B.$ the induced maps $\pi_q(A., a) \to \pi_q(B., f(a))$ induce a $\underline{\text{Loc}}(A.)$-morphism

$$\Pi_q(A.) \longrightarrow f^*\Pi_q(B.)\ .$$

Thus, for a pro-simplicial set $\mathfrak{X} = \{\mathfrak{X}_i\}_i$ each $\Pi_q(\mathfrak{X}_i)$ defines a local system in $\underline{\text{Loc}}(\mathfrak{X})$ and we get a well defined pro-local system

$$\Pi_q(\mathfrak{X}) := \{\Pi_q(\mathfrak{X}_i)\}_i$$

in Pro$\underline{\text{Loc}}(\mathfrak{X})$. Further, each Pro$\underline{\text{SSets}}$-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ induces a morphism in Pro$\underline{\text{Loc}}(\mathfrak{X})$

$$\Pi_q(\mathfrak{X}) \longrightarrow f^*\Pi_q(\mathfrak{Y})\ .$$

Now we can define Isaksens model structure on Pro$\underline{\text{SSets}}$:

1.5.1 Definition. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of pro-simplicial sets.*

(i) *Define f to be a **weak equivalence**, if $\pi_0(f)$ is an isomorphism in* Pro$\underline{\text{Sets}}$ *and $\Pi_q(\mathfrak{X}) \to f^*\Pi_q(\mathfrak{Y})$ is an isomorphism in* Pro$\underline{\text{Loc}}(\mathfrak{X})$ *for all $q > 0$.*

(ii) *Define f to be a **cofibration**, if it is isomorphic to a levelwise cofibration.*

(iii) *Define f to be a **fibration**, if it has the right lifting property with respect to all acyclic cofibrations.*

1.5.2 Remark. For all pro-simplicial sets $\mathfrak{X}$ the canonical map $\mathfrak{X} \to \mathfrak{X}^\natural$ is a weak equivalence (recall that we defined $(-)^\natural$ on Pro$\underline{\text{SSets}}$ using a fixed functorial fibrant replacement functor).

1.5.3 Theorem. *([Isa01] Thm. 6.4) Def. 1.5.1 gives a proper simplicial model structure on* Pro$\underline{\text{SSets}}$*.*

Denote the resulting homotopy category by $\mathcal{H}$(Pro$\underline{\text{SSets}}$).

1.5.4 Theorem. *(see [Isa01] Thm. 7.3) A map of pro-simplicial sets $f : \mathfrak{X} \to \mathfrak{Y}$ is a weak equivalence if and only if $\pi_0(f)$ is an isomorphism in* $\mathrm{Pro}\underline{\mathrm{Sets}}$*, the canonical map*

$$\Pi_1(\mathfrak{X}) \longrightarrow f^*\Pi_1(\mathfrak{Y})$$

is an isomorphism in $\mathrm{Pro}\underline{\mathrm{Loc}}(\mathfrak{X})$ *and for all $\mathcal{L} \in \underline{\mathrm{Loc}}(\mathfrak{Y})$ the map on cohomology*

$$H^\bullet(\mathfrak{Y}; \mathcal{L}) \longrightarrow H^\bullet(\mathfrak{X}; f^*\mathcal{L})$$

is an isomorphism.

There is also a direct description of the fibrations in this model structure: Define a map of pro-simplicial sets $f : \mathfrak{X} \to \mathfrak{Y}$ to be a **strong fibration**, if it is isomorphic to a levelwise map $\{\mathfrak{X}'_i \to \mathfrak{Y}'_i\}_{i \in \underline{I}}$ for $\underline{I}$ a cofinite directed set s.t. each level $\mathfrak{X}'_i \to \mathfrak{Y}'_i$ induces isomorphisms on $\pi_q(-)$ for all $q \gg 0$ (depending on i) and s.t. all the canonical maps

$$\mathfrak{X}'_i \longrightarrow \mathfrak{Y}'_i \times_{(\lim_{j<i} \mathfrak{Y}'_j)} (\lim_{j<i} \mathfrak{X}'_j)$$

are fibrations of simplicial sets.

1.5.5 Proposition. *([Isa01] Prop. 6.6) A map of pro-simplicial sets $f : \mathfrak{X} \to \mathfrak{Y}$ is a fibration if and only if it is a retract of a strong fibration in* $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow \mathfrak{Y}$.

As in the pro-discrete case, the classifying space $B\Gamma$ of a nice pro-group Γ is fibrant:

1.5.6 Remark. Let Γ be a pro-group induced by a tower $\{\Gamma_n\}_n$ of epimorphisms. Then $B\Gamma$ is fibrant in $\mathrm{Pro}\underline{\mathrm{SSets}}$:

We proof that $B\Gamma \to \mathrm{pt}$ is even a strong fibration. Since $\{\Gamma_n\}_n$ is a tower, we get

$$\lim_{m<n} B\Gamma_m = B\Gamma_{n-1}$$

for $n > 0$ resp. $=$ pt for $n = 0$. Now $B\Gamma_0$ is fibrant, so we have to show that

$$BG \longrightarrow BH$$

is a fibration for a group quotient $G \twoheadrightarrow H$. By adjointness of the nerve functor, it suffices to construct functor $\gamma : \underline{B} \to G$ in the category of internal groupoids in $\underline{\mathrm{Sets}}$ making

$$\begin{tikzcd} \underline{A} \arrow[r, "\alpha"] \arrow[d, "i"] & G \arrow[d] \\ \underline{B} \arrow[r, "\beta"] & H \end{tikzcd}$$

commutative for $i : \underline{A} \to \underline{B}$ a fully faithful embedding which is an equivalence of categories. To do this, we may assume $\underline{\mathrm{A}}$ is connected. Pick a base object $*$ in $\underline{\mathrm{A}}$ and fix a morphism

$$\varphi_b : \; b \longrightarrow *$$

for each object b in $\underline{B}$ with $\varphi_* = \mathrm{id}_*$. Further, choose a lift γ_b in G of $\beta(\varphi_b)$ s.t.

$$\gamma_a = \alpha(\varphi_a)$$

for all objects a of $\underline{A}$. Let $f : b' \to b''$ be any morphism in $\underline{A}$. We set

$$\gamma(f) := \gamma_{b''}^{-1} \circ \alpha(\varphi_{b''} \circ f \circ \varphi_{b'}^{-1}) \circ \gamma_{b'},$$

which gives a well defined functor $\gamma : \underline{B} \to G$ with the desired properties.

For $-1 \leq n \leq \infty$ a map $f : X. \to Y.$ of simplicial sets is an n-equivalence if for all $x \in X_0$ the induced maps $\pi_q(f, x)$ are isomorphisms for all $q < n$ and an epimorphisms for $q = n$. Similar, f is a co-n-equivalence if the induced maps $\pi_q(f, x)$ are isomorphisms for $q > n$ and a monomorphism for $q = n$. An n-cofibration is a cofibrations which is also an n-equivalence and a co-n-fibration is a fibration which is also a co-n-equivalence. By [Isa01] Prop. 3.3 we may factor any map of simplicial sets into an n-cofibration followed by a co-n-fibration. The following explicit factorization will turn out to be useful in Sect. 4.3:

1.5.7 Remark. Any Pro$\underline{\text{SSets}}$-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ can be factored as

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "i"] \arrow[rr, bend right, "f"'] & \mathfrak{X}' \arrow[r, "f'"] & \mathfrak{Y} \end{tikzcd}$$

for i an acyclic cofibration and f' a strong fibration: By [AM69] Appendix Cor. 3.2 we may write f as a levelwise map. Further, by [EH76] Thm. 2.1.6 we may assume that the index category $\underline{I}$ is cofinite (i.e. the set of isomorphism classes of $\underline{I}$ is cofinite with resp. to the ordering given by the rule: $[i] > [j]$ if and only if there is a non isomorphism $i \to j$). Thus, we get a well defined **height function**

$$h : \mathrm{Obj}(\underline{I}) \longrightarrow \mathbb{N}$$

with $h(i)$ the maximal length of a chain

$$i = i_0 \longrightarrow i_1 \longrightarrow \ldots \longrightarrow i_n$$

of non isomorphisms in $\underline{I}$.

We construct the factorization inductively with respect to the cofinite index category $\underline{I}$. Thus, assume we constructed a factorization in indices j with $i \to j$ a non isomorphism (we say $j < i$). We may factor the canonical map

$$\mathfrak{X}_i \longrightarrow \mathfrak{Y}_i \times_{\lim_{j<i} \mathfrak{Y}_j} \lim_{j<i} \mathfrak{X}'_j$$

into an $h(i)$-cofibration followed by a co-$h(i)$-fibration:

$$\mathfrak{X}_i \xrightarrow{i_i} \mathfrak{X}'_i \longrightarrow \mathfrak{Y}_i \times_{\lim_{j<i} \mathfrak{Y}_j} \lim_{j<i} \mathfrak{X}'_j$$

Then the $h(i)$-cofibration i_i together with the composition

$$\begin{tikzcd} \mathfrak{X}'_i \arrow[r] \arrow[rr, bend right, "f'_i :="'] & \mathfrak{Y}_i \times_{\lim_{j<i} \mathfrak{Y}_j} \lim_{j<i} \mathfrak{X}'_j \arrow[r, "\mathrm{pr}"] & \mathfrak{Y}_i \end{tikzcd}$$

defines our factorization in the index i. For details, see [Isa01] Prop. 15.1.

1.5.8 Remark. Let

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "i"] \arrow[rr, bend right, "f"'] & \mathfrak{X}' \arrow[r, "f'"] & \mathfrak{Y} \end{tikzcd}$$

be a factorization constructed as in Rem. 1.5.7 with index category $\underline{I}$. The full subcategory of $\underline{I}$ consisting of objects of heights $\geq n+1$ is cofinal. Thus, restricting to this subcategory gives an isomorphic factorization

$$\begin{tikzcd} \tilde{\mathfrak{X}} \arrow[r, "\tilde{i}"] \arrow[rr, bend right, "\tilde{f}"'] & \tilde{\mathfrak{X}}' \arrow[r, "\tilde{f}'"] & \tilde{\mathfrak{Y}} \end{tikzcd}$$

with $\tilde{f}'$ a strict fibration and $\tilde{i}$ an acyclic cofibration, which induces levelwise isomorphisms $\pi_q(\tilde{i})$ for all $q \leq n$.

There is a second model structure on Pro$\underline{\text{SSets}}$, the **strict model structure** of Edwards and Hastings:

1.5.9 Definition. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of pro-simplicial sets.*

(i) *Define f to be a **strict weak equivalence**, if it is isomorphic to a levelwise weak equivalence.*

(ii) *Define f to be a **strict cofibration**, if it is isomorphic to a levelwise cofibration.*

(iii) *Define f to be a **strict fibration**, if it has the right lifting property with respect to all acyclic strong cofibrations.*

1.5.10 Theorem. *([EH76] Thm. 3.3.3) Def. 1.5.9 gives a model structure on* Pro$\underline{\text{SSets}}$.

Denote the resulting homotopy category by $\mathcal{H}_{\text{strict}}(\text{Pro}\underline{\text{SSets}})$. Since each strict weak equivalence is a weak equivalence, we get the canonical localization functor

$$\mathcal{H}_{\text{strict}}(\text{Pro}\underline{\text{SSets}}) \longrightarrow \mathcal{H}(\text{Pro}\underline{\text{SSets}}) .$$

For nice pro-simplicial sets we even get:

1.5.11 Proposition. *([Isa01] Prop. 10.9) Let $\mathfrak{X}$ and $\mathfrak{Y} = \{\mathfrak{Y}_i\}_i$ be pro-simplicial sets and suppose for all i that $\mathfrak{Y}_i$ has only finitely many non trivial homotopy groups. Then the canonical morphism*

$$[\mathfrak{X}, \mathfrak{Y}]_{\mathcal{H}_{\text{strict}}(\text{Pro}\underline{\text{SSets}})} \longrightarrow [\mathfrak{X}, \mathfrak{Y}]_{\mathcal{H}(\text{Pro}\underline{\text{SSets}})}$$

is an isomorphism.

The last proposition applies in particular for $\mathfrak{Y} = \mathfrak{X}^\natural$. Since $\mathfrak{X} \to \mathfrak{X}^\natural$ is a weak equivalence and a map $f : X \to Y$ is a weak equivalence of a closed model category if and only if it induces an isomorphism in the homotopy category (this is standard homotopical algebra, e.g. see [DS95] Prop. 5.8), we get:

1.5.12 Corollary. *A map of pro-simplicial sets $f : \mathfrak{X} \to \mathfrak{Y}$ is a weak equivalence if and only if $f^\natural : \mathfrak{X}^\natural \to \mathfrak{Y}^\natural$ is a strict weak equivalence, i.e., is isomorphic to a levelwise weak equivalence.*

In particular, any weak equivalence in Pro$\underline{\text{SSets}}$ resp. in Pro$\underline{\text{SSets}_\bullet}$ induces a $\natural$-isomorphism in Pro$\mathcal{H}(\underline{\text{SSets}})$ resp. in Pro$\mathcal{H}(\underline{\text{SSets}_\bullet})$. Using the equivalent rather technical description of a weak equivalence in Pro$\underline{\text{SSets}}$ of [Isa01] Thm. 7.3, we get a converse statement in some sense, as well:

1.5.13 Proposition. *(see [Isa01] Cor. 7.5) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of pointed levelwise connected pro-simplicial sets inducing a $\natural$-isomorphism in* Pro$\mathcal{H}(\underline{\text{SSets}_\bullet})$. *Then f is a weak equivalence in* Pro$\underline{\text{SSets}}$ *resp. in* Pro$\underline{\text{SSets}_\bullet}$.

Thus, from now on we refer to $\natural$-isomorphisms just as weak equivalences.

1.6 Hypercover descent.

In this subsection we want to review the comparison between the topological types of a k-variety X, the topological type of one of its hypercoverings $\mathfrak{U}.$ and the homotopy colimit of the topological types of the simplicial degrees of $\mathfrak{U}.$ due to Cox, Friedlander and Isaksen.

Let $\underline{C}$ be a simplicial closed model category closed under arbitrary colimits and $\underline{SC}$ the category of simplicial objects in $\underline{C}$.

1.6.1 Definition. *Let $X.$ be a simplicial object in $\underline{C}$. Then its* ***realization*** *is defined as the $\underline{C}$-object*

$$\mathrm{Re}(X.) = \mathrm{Re}^{\underline{C}}(X.) := \mathrm{colim}\{ \textstyle\coprod_{\sigma\in\Delta^i_j}^{i,j} X_i \otimes \Delta^j \underset{\mathrm{id}\otimes\Delta^\sigma}{\overset{X_\sigma\otimes\mathrm{id}}{\rightrightarrows}} \coprod_i X_i \otimes \Delta^i \}.$$

Further, its ***m-truncated realization*** *is defined as the $\underline{C}$-object*

$$\mathrm{Re}_m(X.) = \mathrm{Re}^{\underline{C}}_m(X.) := \mathrm{colim}\{ \textstyle\coprod_{\sigma\in\Delta^i_j}^{i,j\le m} X_i \otimes \Delta^j \underset{\mathrm{id}\otimes\Delta^\sigma}{\overset{X_\sigma\otimes\mathrm{id}}{\rightrightarrows}} \coprod_{i\le m} X_i \otimes \Delta^i \}.$$

By the universal mapping property of colimits, we get a canonical $\underline{C}$-map

$$\mathrm{Re}_m(X.) \longrightarrow \mathrm{Re}(X.)$$

Some easy simplicial computations show:

1.6.2 Lemma. *The functor* $\mathrm{Re}(-) : \underline{SC} \to \underline{C}$ *is left adjoint to the functor*

$$\underline{C} \to \underline{SC},\ Y \mapsto Y^{\Delta^\bullet}$$

mapping Y to the path object $Y^{\Delta^\bullet}$ given by the simplicial structure on $\underline{C}$, while the functor $\mathrm{Re}_m(-) : \underline{SC} \to \underline{C}$ *is left adjoint to the functor*

$$\underline{C} \to \underline{SC},\ Y \mapsto \mathrm{cosk}^{\underline{C}}_m(Y^{\Delta^\bullet}).$$

Toying around with these adjointness properties and doing a few more simplicial computations we get the following lemma:

1.6.3 Lemma. *(see [Isa04] Prop. 4.7, [Hir03] Cor. 18.7.7) Let $X..$ be a bisimplicial set.*

(i) *The realization* $\mathrm{Re}(X..)$ *is nothing but the diagonal simplicial set* $\mathrm{diag}(X..)$.

(ii) *The realization* $\mathrm{Re}(X..)$ *and the homotopy colimit* $\mathrm{hocolim}_n X_n.$ *are weakly equivalent.*

(iii) *The canonical map* $\mathrm{sk}_m\mathrm{Re}_m(X..) \to \mathrm{sk}_m\mathrm{Re}(X..)$ *on the* m^{th} *skeletons is an isomorphism.*

For a pro-simplicial set $\mathfrak{X}$ we write $\mathrm{sk}_m\mathfrak{X}$ resp. $\mathrm{cosk}_m\mathfrak{X}$ for the levelwise m^{th} skeleton and coskeleton functors. For simplicial pro-simplicial sets we have:

1.6.4 Lemma. *(see [Isa04] Prop. 4.9 and Prop. 4.11) Let $\mathfrak{X}.$ be a simplicial pro-simplicial set.*

(i) *The realization* $\mathrm{Re}(\mathfrak{X}.)$ *and the homotopy colimit* $\mathrm{hocolim}_n \mathfrak{X}_n$ *are weakly equivalent.*

(ii) *The canonical map* $\mathrm{sk}_m\mathrm{Re}_m(\mathfrak{X}.) \to \mathrm{sk}_m\mathrm{Re}(\mathfrak{X}.)$ *on the levelwise* m^{th} *skeletons is an isomorphism.*

We want to compare levelwise truncated realizations with realizations of simplicial pro-simplicial sets. The following lemma is a straightforward generalization of [Isa04] Prop. 5.2. The proof essentially tells us how to compute the rather complicated colimit $\mathrm{Re}_m^{\mathrm{Pro}\underline{\mathrm{SSets}}}$ in special cases, so we give the proof for completeness:

1.6.5 Lemma. *(cf. [Isa04] Prop. 5.2) Let $\Phi : \underline{F} \to \underline{B}$ be a fibred category together with a normalized cleavage. Suppose it is fibrewise rigid and closed under finite limits. Let $\pi : \underline{F} \to \underline{\mathrm{SSets}}$ be a functor. Let $X.$ be a simplicial object in $\underline{B}$ and suppose that the canonical functor*

$$\underline{S}\underline{F}(X.) \longrightarrow \textstyle\prod_{0\le\nu\le m} \underline{F}(X_\nu)$$

mapping a simplicial object $U. \in \underline{S}\underline{F}(X.)$ to the first $m+1$ simplicial degrees is cofinal. Then the levelwise m-truncated realization of the induced simplicial pro-simplicial set $\pi_{|\underline{S}\underline{F}(X.)}$ agrees with its m-truncated realization in $\mathrm{Pro}\underline{\mathrm{SSets}}$*:*

$$(\mathrm{Re}_m^{\underline{\mathrm{SSets}}} \circ \pi)_{|\underline{S}\underline{F}(X.)} = \mathrm{Re}_m^{\mathrm{Pro}\underline{\mathrm{SSets}}}(\pi_{|\underline{S}\underline{F}(X.)}).$$

Proof: The truncated realization $\mathrm{Re}_m^{\mathrm{Pro}\underline{\mathrm{SSets}}}(\pi_{|\underline{S}\underline{F}(X.)})$ is isomorphic to the colimit of the diagram

$$\begin{array}{ccc} \pi_{|\underline{F}(X_i)} \times \Delta^j & \xrightarrow{\sigma^*\times\mathrm{id}} & \pi_{|\underline{F}(X_j)} \times \Delta^j \\ \downarrow{\scriptstyle \mathrm{id}\times\Delta^\sigma} & & \\ \pi_{|\underline{F}(X_i)} \times \Delta^i & & \end{array}, \text{ for } i,j \le m \text{ and } \sigma \in \Delta^i_j.$$

All the occurring maps are strict, so by [Isa02] Sect. 3.1 and [AM69] Appendix Prop. 4.1 we can write the truncated realization $\mathrm{Re}_m^{\mathrm{Pro}\underline{\mathrm{SSets}}}(\pi_{|\underline{\mathrm{S}F}(X.)})$ levelwise as

$$\operatorname{colim}\{\ \coprod_{\sigma\in\Delta^i_j}^{i,j\le m}\pi(U_i\times X^*_\sigma U_j)\times\Delta^j \underset{\pi(\mathrm{pr}_{U_i})\times\Delta^\sigma}{\overset{\pi(\mathrm{pr}_{U_j})\times\mathrm{id}}{\rightrightarrows}} \coprod_{i\le m}\pi(U_i)\times\Delta^i\ \}$$

for

$$(U_0,\ldots,U_m)\in\prod_{0\le\nu\le m}\underline{F}(X_\nu),$$

where pr_{U_j} is the composition of the canonical maps $\mathrm{pr}_{X^*_\sigma U_j}: U_i\times X^*_\sigma U_j\to X^*_\sigma U_j$ and $X^*_\sigma U_j\to U_j$. By assumption,

$$\underline{\mathrm{S}F}(X.)\longrightarrow\prod_{0\le\nu\le m}\underline{F}(X_\nu)$$

is cofinal, so we may replace the index category $\prod_{0\le\nu\le m}\underline{F}(X_\nu)$ by $\underline{\mathrm{S}F}(X.)$ and the U_ν by the ν^{th} simplicial degrees of $U.\in\underline{\mathrm{S}F}(X.)$.

Now for $\sigma\in\Delta^i_j$ as above

$$U_\sigma:\ U_i\longrightarrow U_j$$

uniquely factors over the the transport morphism

$$X^*_\sigma U_j\longrightarrow U_j$$

given by our cleavage. In particular, we get a (right-) splitting of the projection

$$\mathrm{pr}_{U_i}:\ U_i\times X^*_\sigma U_j\longrightarrow U_i\ .$$

By assumption, there is at most one morphism between two objects in $\underline{F}(X_i)$, so this projection has to be an isomorphism. Thus, we can write $\mathrm{Re}_m^{\mathrm{Pro}\underline{\mathrm{SSets}}}(\pi_{|\underline{\mathrm{S}F}(X.)})$ levelwise as

$$\operatorname{colim}\{\ \coprod_{\sigma\in\Delta^i_j}^{i,j\le m}\pi(U_i)\times\Delta^j \underset{\mathrm{id}\times\Delta^\sigma}{\overset{\pi(U_\sigma)\times\mathrm{id}}{\rightrightarrows}} \coprod_{i\le m}\pi(U_i)\times\Delta^i\ \}$$

for $U.\in\underline{\mathrm{S}F}(X.)$. But this is exactly the pro-object $(\mathrm{Re}_m^{\underline{\mathrm{SSets}}}\circ\pi)_{|\underline{\mathrm{S}F}(X.)}$, which completes the proof. $\square$

For a pro-simplicial set $\mathfrak{X}$ and $m\ge 2$ the canonical map $\mathrm{sk}_m\mathfrak{X}\to\mathfrak{X}$ induces isomorphisms on fundamental groupoids as well as on cohomology in degrees $<m$. Combining this with the isomorphisms

$$\mathrm{sk}_m\mathrm{Re}_m^{\underline{C}}\xrightarrow{\cong}\mathrm{sk}_m\mathrm{Re}^{\underline{C}}$$

of Lem. 1.6.3 resp. Lem. 1.6.4 for $\underline{C}$ the category of simplicial sets resp. of pro-simplicial sets means that the canonical map

$$\mathrm{Re}^{\mathrm{Pro}\underline{\mathrm{SSets}}}(\pi_{|\underline{\mathrm{S}F}(X.)})\longrightarrow(\mathrm{Re}^{\underline{\mathrm{SSets}}}\circ\pi)_{|\underline{\mathrm{S}F}(X.)}$$

is a weak equivalence in Pro$\underline{\text{SSets}}$ if and only if the same is true for the m-truncated versions of this map for arbitrary large m. But the latter statement holds by Lem. 1.6.5. Combining this with Lem. 1.6.4 (i) we get the following corollary:

1.6.6 Corollary. *Let $\Phi : \underline{F} \to \underline{B}$ be a fibred category and $X.$ a simplicial object of $\underline{B}$ as in Lem. 1.6.5. Then we have weak equivalences in* Pro$\underline{\text{SSets}}$

$$(\mathrm{Re}^{\underline{\mathrm{SSets}}} \circ \pi)_{|\underline{SF}(X.)} \simeq \mathrm{Re}^{\mathrm{Pro}\underline{\mathrm{SSets}}}(\pi_{|\underline{SF}(X.)}) \simeq \operatorname*{hocolim}_{n} \pi_{|\underline{F}(X_n)}.$$

We want to apply this to the special case of the Čech resp. étale topological type: By Lem. 1.6.3 the realization on $\underline{\text{SSets}}$ is nothing but the diagonal simplicial set. Thus, the Čech resp. étale topological type of a simplicial k-variety $X.$ is given by the functor

$$\underline{\mathrm{RC}}(X.) \to \underline{\mathrm{SSets}},\ \{U. \to X.\} \mapsto \mathrm{Re}(\pi_0 \mathrm{cosk}_0^{X.} U.)$$

resp.

$$\underline{\mathrm{HRR}}(X.) \to \underline{\mathrm{SSets}},\ \{\mathfrak{U}.. \to X.\} \mapsto \mathrm{Re}(\pi_0 \mathfrak{U}..).$$

This motivates the following definition:

1.6.7 Definition. *Let $X.$ be a simplicial k-variety.*

(i) *Define the **m-truncated Čech topological type** of the simplicial k-variety $X.$ as the pro-simplicial set given by the functor*

$$\check{\mathrm{C}}_m(X.) : \underline{\mathrm{RC}}(X.) \to \underline{\mathrm{SSets}},\ \{U. \to X.\} \mapsto \mathrm{Re}_m(\pi_0 \mathrm{cosk}_0^{X.} U.).$$

(ii) *Define the **m-truncated étale topological type** of the simplicial k-variety $X.$ as the pro-simplicial set given by the functor*

$$\acute{\mathrm{E}}\mathrm{t}_m(X.) : \underline{\mathrm{HRR}}(X.) \to \underline{\mathrm{SSets}},\ \{\mathfrak{U}.. \to X.\} \mapsto \mathrm{Re}_m(\pi_0 \mathfrak{U}..).$$

By definition we get canonical maps $\check{\mathrm{C}}_m(X.) \to \check{\mathrm{C}}(X.)$ and $\acute{\mathrm{E}}\mathrm{t}_m(X.) \to \acute{\mathrm{E}}\mathrm{t}(X.)$. As a direct consequence of Lem. 1.6.3 we get:

1.6.8 Corollary. *Let $X.$ be a simplicial k-variety. Then the canonical maps on the levelwise m^{th} skeletons*

$$\mathrm{sk}_m \check{\mathrm{C}}_m(X.) \longrightarrow \mathrm{sk}_m \check{\mathrm{C}}(X.)$$

$$\mathrm{sk}_m \acute{\mathrm{E}}\mathrm{t}_m(X.) \longrightarrow \mathrm{sk}_m \acute{\mathrm{E}}\mathrm{t}(X.)$$

are isomorphisms.

We conclude this section by comparing the topological types of a k-variety X, of one of its hypercoverings $\mathfrak{U}.$ and of the homotopy colimit over the resp. topological type of the simplicial degrees of $\mathfrak{U}.$:

1.6.9 Proposition. *(see [Cox79a] Thm. IV.2 and [Fri82] Prop. 8.1) Let $\mathfrak{U}. \to X$ be a (rigid or non rigid) hypercovering of a k-variety X. Then the canonical map*

$$\acute{\mathrm{E}}\mathrm{t}(\mathfrak{U}.) \longrightarrow \acute{\mathrm{E}}\mathrm{t}(X)$$

is a weak equivalence. If moreover X is of Čech type, then the corresponding statement holds for the Čech topological type, as well.

Proof: For the first statement see [Cox79a] Thm. IV.2 and [Fri82] Prop. 8.1. The last statement follows from the first one by Thm. 1.4.19 since $\mathfrak{U}.$ is levelwise étale over X and hence of Čech type, as well (see Rem. 1.4.8). □

Both $\underline{\mathrm{Var}}_k$ categories $\underline{\mathrm{RC}}_k \to \underline{\mathrm{Var}}_k$ and $\underline{\mathrm{HRR}}_k \to \underline{\mathrm{Var}}_k$ satisfy the assumptions of Lem. 1.6.5 by Lem. 1.4.5 and Lem. 1.4.12. Thus, combining Prop. 1.6.9 with Lem. 1.6.5 and Cor. 1.6.6 we get:

1.6.10 Theorem. *Let X be a k variety, $\mathfrak{U}. \to X$ a (rigid or non rigid) hypercovering and $Y.$ a simplicial k variety.*

(i) $\check{\mathrm{C}}_m(Y.)$ *resp.* $\acute{\mathrm{E}}\mathrm{t}_m(Y.)$ *is isomorphic to the realization*

$$\mathrm{Re}_m^{\mathrm{Pro}\underline{\mathrm{SSets}}}(n \mapsto \check{\mathrm{C}}(Y_n))$$

resp.

$$\mathrm{Re}_m^{\mathrm{Pro}\underline{\mathrm{SSets}}}(n \mapsto \acute{\mathrm{E}}\mathrm{t}(Y_n)).$$

(ii) *We have weak equivalences*

$$\check{\mathrm{C}}(Y.) \simeq \mathrm{Re}^{\mathrm{Pro}\underline{\mathrm{SSets}}}(n \mapsto \check{\mathrm{C}}(Y_n)) \simeq \operatorname*{hocolim}_n \check{\mathrm{C}}(Y_n)$$

resp.

$$\acute{\mathrm{E}}\mathrm{t}(Y.) \simeq \mathrm{Re}^{\mathrm{Pro}\underline{\mathrm{SSets}}}(n \mapsto \acute{\mathrm{E}}\mathrm{t}(Y_n)) \simeq \operatorname*{hocolim}_n \acute{\mathrm{E}}\mathrm{t}(Y_n).$$

(iii) *We have a canonical weak equivalence*

$$\mathrm{hocolim}_n \, \acute{\mathrm{E}}\mathrm{t}(\mathfrak{U}_n) \to \acute{\mathrm{E}}\mathrm{t}(X).$$

(iv) *If moreover X is of Čech type, then we have a canonical weak equivalence*

$$\mathrm{hocolim}_n \, \check{\mathrm{C}}(\mathfrak{U}_n) \to \check{\mathrm{C}}(X).$$

2 Some abstract nonsense

In this section we develop some abstract nonsense needed later in the thesis. In the first three subsections we do some explicit computation with the Čech topological type leading to base change theorems for étale homotopy types in Sect. 2.3. In Sect. 2.4, we compare the Hochschild-Serre spectral sequence with the Galois-hypercohomology spectral sequence of a certain cochain complex computing geometric cohomology with local coefficients. In Sect. 2.5 we prove a cohomological universal coefficient theorem for pro-chain complexes and finally, in Sect. 2.6 we link cohomology with locally constant coefficients of k-varieties to the corresponding Eilenberg-MacLane spaces in the pro-homotopy category $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$.

2.1 A variant of the Čech topological type. Throughout the thesis, we fix an algebraic closure $\bar{k}/k$ of k a field of characteristic 0. Let X be a geometrically unibranched k-variety. For computational reasons, we restrict the set $X(\Omega)$ to the set

$$X(\bar{k}/k) := \mathrm{Hom}_{\underline{\mathrm{Sch}}_k}(\mathrm{Spec}(\bar{k}), X),$$

i.e., we forget the non closed points and consider points always relative over k. Denote the resulting category of rigid coverings by $\underline{\mathrm{RC}}(X/k)$. Clearly, $\underline{\mathrm{RC}}(X/k)$ is still cofiltered. The gain is, that $\underline{\mathrm{RC}}(k/k)$ is the category of finite algebraic subextensions of $\bar{k}/k$ on the nose.

Fix an embedding $\iota : \bar{k} \hookrightarrow \Omega$. For $U \to X$ in $\underline{\mathrm{RC}}(X)$ define

$$U^{(\bar{k}/k)} \longrightarrow X$$

in $\underline{\mathrm{RC}}(X/k)$ using the induced map $\iota^* : X(\bar{k}/k) \to X(\Omega)$ as follows: For x a geometric point in $X(\bar{k}/k)$ the map

$$U_x^{(\bar{k}/k)} \longrightarrow X$$

is just $U_{\iota^* x} \to X$. We get a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\Omega) \arrow[r] \arrow[d, "\iota"] \arrow[rr, bend left, "u_{\iota^* x}"] & \mathrm{Spec}(k(u_{\iota^* x})) \arrow[r] \arrow[d] & U_x^{(\bar{k}/k)} \arrow[d] \\
\mathrm{Spec}(\bar{k}) \arrow[r] \arrow[rr, bend right, "x"'] & \mathrm{Spec}(k(x)) \arrow[r] & X
\end{tikzcd}$$

where $k(x)$ and $k(u_{\iota^* x})$ are the residue fields of the underling points. The left square corresponds to a map of $\bar{k}$-algebras

$$\bar{k} \otimes_{k(x)} k(u_{\iota^* x}) \longrightarrow \Omega.$$

Since the left hand side is a finite dimensional $\bar{k}$ algebra, it corresponds to a finite disjoint union of closed points. It follows that this map has to factor uniquely over the residue field of one of these points, i.e. over $\bar{k}$, since it is algebraically closed. As a result, we get a unique lift

$$\mathrm{Spec}(\bar{k}) \longrightarrow \mathrm{Spec}(k(u_{\iota^* x}))$$

in the left square, i.e. we get a canonical point $u_x^{(\bar{k}/k)}$ in $U_x^{(\bar{k}/k)}(\bar{k}/k)$ over x. This construction is functorial and the resulting functor

$$\underline{\mathrm{RC}}(X) \longrightarrow \underline{\mathrm{RC}}(X/k)$$

is cofinal. Arguing as in Sect. 1.6, we get a cofinal functor

$$\underline{\mathrm{RC}}(X.) \longrightarrow \underline{\mathrm{RC}}(X./k)$$

between categories of rigid coverings for a simplicial k-variety $X.$, as well.

Now define the **Čech topological type of $X./k$** as the pro-simplicial set given by the functor

$$\check{\mathrm{C}}(X./k) : \underline{\mathrm{RC}}(X./k) \to \underline{\mathrm{SSets}},\ \{U. \to X\} \mapsto \mathrm{Re}(\pi_0(\mathrm{cosk}_0^{X.} U.))$$

resp. the m-truncated Čech topological type of $X./k$ as the pro-simplicial set given by the functor

$$\check{\mathrm{C}}_m(X./k) : \underline{\mathrm{RC}}(X./k) \to \underline{\mathrm{SSets}},\ \{U. \to X\} \mapsto \mathrm{Re}_m(\pi_0(\mathrm{cosk}_0^{X.} U.)).$$

Again, the gain is that $\check{\mathrm{C}}(k/k)$ equals $B\Gamma_k$ on the nose. We define the **relative Čech topological type of $X./k$** as the canonical map

$$\check{\mathrm{C}}(X./k) \longrightarrow B\Gamma_k\ .$$

For $U \to X$ in $\underline{\mathrm{RC}}(X)$ we have the canonical embedding $U^{(\bar{k}/k)} \hookrightarrow U$ over X. Thus our cofinal functor $\underline{\mathrm{RC}}(X.) \to \underline{\mathrm{RC}}(X./k)$ defines a strict map

$$\check{\mathrm{C}}(X./k) \to \check{\mathrm{C}}(X.).$$

Both sides compute the étale fundamental group resp. the Čech cohomology of $X.$, so our strict map is in fact a weak equivalence. Thus, if $X.$ is degreewise of Čech type, $\check{\mathrm{C}}(X./k)$ is weakly equivalent to the étale topological type $\acute{\mathrm{E}}\mathrm{t}(X.)$ by Thm. 1.4.19.

However, to compare the Čech topological type of a k-variety with the Čech topological type of its base change along a non algebraic field extension (e.g., its generic fibre) it is convenient to drop our restriction to closed geometric points:

2.1.1 Remark. Let K/k be any field extension and fix a separable algebraic closure $\bar{K}/K$. The relative algebraic closure of the resulting extension $\bar{K}/k$ induces a canonical extension $\bar{K}/\bar{k}$ making the diagram

$$\begin{array}{ccc} K & \text{——} & \bar{K} \\ | & & | \\ k & \text{——} & \bar{k} \end{array}$$

commutative. For X a k-variety define $\mathrm{RC}(X/k)_K$ and $\check{\mathrm{C}}(X/k)_K$ similar as $\mathrm{RC}(X/k)$ and $\check{\mathrm{C}}(X/k)$ with resp. to the set of geometric points $X(\bar{K}/k)$. We get a canonical embedding

$$X(\bar{k}/k) \hookrightarrow X(\bar{K}/k)$$

together with the induced functor

$$\underline{\mathrm{RC}}(X/k)_K \longrightarrow \underline{\mathrm{RC}}(X/k)\ ,$$

which is an isomorphism resp. an equivalence for $\mathrm{Spec}(L)$ and L/k finite algebraic. In particular,

$$\check{\mathrm{C}}(k/k)_K = \check{\mathrm{C}}(k/k) = B\Gamma_k.$$

Clearly, the induced canonical $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k$-morphism

$$\check{\mathrm{C}}(X/k) \longrightarrow \check{\mathrm{C}}(X/k)_K$$

is a weak equivalence. Let x be a geometric point in

$$(X \otimes_k K)(\bar{K}/K) = X(\bar{K}/k)$$

and $(U, u) \to (X, x)$ a pointed connected étale neighbourhood. Now the connected component of the distinguished point in $U \times_X (X \otimes_k K)$ defines a canonical pointed connected étale neighbourhood of $(X \otimes_k K, x)$ together with a projection to (U, u). This induces a canonical functor

$$\mathrm{RC}(X/k)_K \longrightarrow \mathrm{RC}(X/K)$$

which in turn induces a canonical commutative square:

(2.1.1)
$$\begin{tikzcd} \check{\mathrm{C}}(X \otimes_k K/K) \arrow[r] \arrow[d] & B\Gamma_K \arrow[d, "\text{can.}"] \\ \check{\mathrm{C}}(X/k)_K \arrow[r] & B\Gamma_k. \end{tikzcd}$$

2.2 The Čech topological type of an étale covering. We want to study the Čech topological type of an étale covering space $f : Y \to X$.

First assume that $f : Y \to X$ is even Galois with group G. Let V be an arbitrary rigid covering of Y/k. We will construct a map of rigid coverings

$$\tilde{V} \longrightarrow V$$

functorial in $V \in \underline{\mathrm{RC}}(Y/k)$ with $\tilde{V}$ a G-equivariant rigid covering, universal with this property: For $y \in Y(\bar{k}/k)$ let $\tilde{V}_y$ be the connected component of the product (depending only on the G-orbit $[y]$ of y)

$$\tilde{V}_{[y]} := \prod_{\sigma \in G} \{\ \begin{tikzcd} V_{\sigma(y)} \arrow[r] \arrow[rr, bend right] & Y \arrow[r, "\sigma^{-1}"] & Y \end{tikzcd}\ \}$$

in $\underline{\mathrm{Sch}}_Y$ containing the distinguished point

$$\tilde{v}_y := \otimes_{\sigma \in G} v_{\sigma(y)}.$$

This gives a morphism

$$(\tilde{V}_y, \tilde{v}_y) \longrightarrow (V_y, v_y)$$

of pointed étale neighbourhoods of (Y, y). By the commutative diagram

$$\begin{tikzcd}
V_{(\tau\sigma)(y)} \arrow[r, equal] \arrow[d] & V_{\tau(\sigma(y))} \arrow[d] \\
Y \arrow[d, "(\tau\sigma)^{-1}"] & Y \arrow[d, "\tau^{-1}"] \\
Y \arrow[r, "\sigma"] & Y
\end{tikzcd}$$

the coordinate flip $\tilde{V}_{[y]} \to \tilde{V}_{[\sigma(y)]}$ given by right multiplication by σ^{-1} in the index set gives the commutative diagram:

$$\begin{tikzcd}
\tilde{V}_{[y]} \arrow[r] \arrow[d] & \tilde{V}_{[\sigma(y)]} \arrow[d] \\
Y \arrow[r, "\sigma"] & Y.
\end{tikzcd}$$

This coordinate flip restricts to an isomorphism

$$\tilde{V}_y \longrightarrow \tilde{V}_{\sigma(y)}$$

i.e., $\tilde{V} \to Y$ is indeed a G-equivariant rigid covering of Y/k. Thus,

$$U := \tilde{V}/G$$

is a rigid covering of X/k with $f^*U = \tilde{V}$.

Now let U be any rigid covering of X/k admitting a map

$$f^*U \longrightarrow V$$

Then there is a unique G-equivariant factorization over the above canonical map $\tilde{V} \to V$: Indeed, by the commutative diagram

$$\begin{tikzcd}
(f^*U)_y \arrow[r] \arrow[d, "\sigma"] & V_y \arrow[r] & Y \arrow[d, "\sigma"] \\
(f^*U)_{\sigma(y)} \arrow[r] & V_{\sigma(y)} \arrow[r] & Y
\end{tikzcd}$$

the compositions

$$\begin{tikzcd}
(f^*U)_y \arrow[r, "\sigma"] \arrow[rr, bend right] & (f^*U)_{\sigma(y)} \arrow[r] & V_{\sigma(y)}
\end{tikzcd}$$

induce a Y-morphism $(f^*U)_y \to \tilde{V}_y$. By the rigidity of maps between pointed étale neighbourhoods, the resulting map

$$f^*U \longrightarrow \tilde{V}$$

is unique and thus also G-equivariant.

In particular the rigid pullback

$$f^* : \underline{\mathrm{RC}}(X/k) \longrightarrow \underline{\mathrm{RC}}(Y/k)$$

is cofinal for f an étale Galois covering. For an arbitrary connected étale covering space $f : Y \to X$ we argue as follow: Let

$$\tilde{f} : \tilde{Y} \longrightarrow X$$

be a Galois covering dominating f, say with group G. The induced covering

$$g : \tilde{Y} \longrightarrow Y$$

is Galois, say with group $H \leq G$. By the above there is a rigid covering U of X/k with a map

$$\tilde{f}^*U \longrightarrow g^*V \ .$$

Again by rigidity of maps between pointed étale neighbourhoods, this map is H-equivariant and thus descends to a map

$$f^*U \longrightarrow V \ .$$

Thus, the rigid pullback

$$f^* : \underline{\mathrm{RC}}(X/k) \longrightarrow \underline{\mathrm{RC}}(Y/k)$$

is cofinal for a general connected étale covering space $f : Y \to X$, too. As a result, we get the following:

2.2.1 Lemma. *Let $f : Y \to X$ be a connected étale covering space. Then $\check{\mathrm{C}}(Y/k)$ is isomorphic to the pro-simplicial set given by the functor*

$$\underline{\mathrm{RC}}(X/k) \to \underline{\mathrm{SSets}},\ \{U \to X\} \mapsto \pi_0(\mathrm{cosk}_0^Y f^*U).$$

2.2.2 Remark. Of course, the analogue arguments give corresponding statements for $\check{\mathrm{C}}(Y)$ and $\acute{\mathrm{E}}\mathrm{t}(Y)$, as well.

Suppose our covering is of the form $X \otimes_k K \to X$ for K/k finite algebraic. Note, that the above construction of $\tilde{V} \to V$ is natural in rigid coverings of $X \otimes_k K/k$ for k-varieties X, i.e. for $f : X \to Y$ a map of k-varieties and a commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ X \otimes_k K \arrow[r, "f \otimes_k K"] & Y \otimes_k K \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} \tilde{V} \arrow[r] \arrow[d] & \tilde{U} \arrow[d] \\ V \arrow[r] & U \end{tikzcd}$$

functorial in the first diagram.

Thus, for K/k finite algebraic and $X.$ a simplicial k-variety, the rigid pullback

$$\underline{\mathrm{RC}}(X./k) \longrightarrow \underline{\mathrm{RC}}(X. \otimes_k K/k)$$

is cofinal and we get:

2.2.3 Corollary. *Let K/k finite algebraic and $X.$ a simplicial k-variety. Then $\check{\mathrm{C}}(X. \otimes_k K/k)$ is isomorphic to the pro-simplicial set given by the functor*

$$\underline{\mathrm{RC}}(X./k) \to \underline{\mathrm{SSets}},\ \{U. \to X.\} \mapsto \mathrm{Re}(\pi_0(\mathrm{cosk}_0^{X.\otimes_k K} p^*U.)),$$

where $p : X. \otimes_k K/k \to X.$ is the canonical projection.

2.3 Base change for étale homotopy types. In this subsection we want to relate the base change of a k-variety of Čech type along an algebraic extension K/k to the corresponding extension of the relative Čech topological type. Fix an algebraic extension K/k. Let $K(\bar{k}/k)$ be the set of k-algebra embeddings $K \to \bar{k}$ and write X_K for the base extension $X \otimes_k K$.

2.3.1 Remark. First note that

$$\Gamma_k \backslash (E\Gamma_k \times K(\bar{k}/k)) \cong \Gamma_K \backslash E\Gamma_k$$

over $B\Gamma_k$, where the structural map

$$\Gamma_k \backslash (E\Gamma_k \times K(\bar{k}/k)) \longrightarrow B\Gamma_k$$

is given by the projection onto $E\Gamma_k$. To see this choose an Γ_k-isomorphism $K(\bar{k}/k) \cong \Gamma_k/\Gamma_K$. Now

$$E\Gamma_k \times \Gamma_k/\Gamma_K \to \Gamma_K \backslash E\Gamma_k,\ (\underline{\sigma}, [\tau]) \mapsto [\tau^{-1}\underline{\sigma}]$$

is well defined, surjective and descents to a map

$$\Gamma_k \backslash (E\Gamma_k \times \Gamma_k/\Gamma_K) \longrightarrow \Gamma_K \backslash E\Gamma_k$$

over $B\Gamma_k$. It is easy to see that this map is injective, hence the claim.

2.3.2 Remark. From Rem. 2.3.1 we get that the canonical map

$$\Gamma_K \backslash E\Gamma_k \longrightarrow \lim_{K \supseteq L/k} (\Gamma_L \backslash E\Gamma_k)$$

is an isomorphism, where the limit runs over all finite subextensions L/k of K/k: First, note that any k-algebra embedding $K \to \bar{k}$ is given by a compatible choice of embeddings for the finite subextensions L/k, i.e.,

$$K(\bar{k}/k) = \lim_{K \supseteq L/k} L(\bar{k}/k).$$

Again, choose a Γ_k isomorphism $K(\bar{k}/k) \cong \Gamma_k/\Gamma_K$. For a finite subextension L/k the composition

$$\Gamma_k \twoheadrightarrow \Gamma_k/\Gamma_K \ \cong \ K(\bar{k}/k) \twoheadrightarrow L(\bar{k}/k)$$

factors over $\Gamma_k \to \Gamma_k/\Gamma_L$ and thus give compatible choices of Γ_k isomorphisms $L(\bar{k}/k) \cong \Gamma_k/\Gamma_L$. Representing a class $[\underline{\sigma}, \iota]$ by $\underline{\sigma} \in \Gamma_k^{n+1}$ with $\sigma_0 = 1$ gives a degreewise isomorphism

$$(\Gamma_k \backslash (E\Gamma_k \times L(\bar{k}/k)))_n \cong \Gamma_k^n \times L(\bar{k}/k),$$

natural in the algebraic extension L/k (but of course not in the simplicial degree n). But this implies our claim, since $K(\bar{k}/k) = \lim_{K \supseteq L/k} L(\bar{k}/k)$.

We need an easy but technical lemma: Let L_ν/k for $0 \leq \nu \leq n$ be a a family of finite algebraic extensions. Sending a tuple of geometric points ι_ν in $L_\nu(\bar{k}/k)$ to the underling point of the $\otimes$-product

$$\otimes_\nu \iota_\nu : \ L_0 \otimes_k \ldots \otimes_k L_n \longrightarrow \bar{k}$$

defines a surjection

$$\prod_{0 \leq \nu \leq n} L_\nu(\bar{k}/k) \twoheadrightarrow \pi_0(\mathrm{Spec}(L_0 \otimes_k \ldots \otimes_k L_n)) \ .$$

It is Γ_k-equivariant, where Γ_k act diagonally from the left on the source and trivially on the target.

2.3.3 Lemma. *Let L_ν/k for $0 \leq \nu \leq n$ be a family of finite algebraic extensions. Then the canonical epimorphism*

$$\Gamma_k \backslash (\prod_{0 \leq \nu \leq n} L_\nu(\bar{k}/k)) \twoheadrightarrow \pi_0(\mathrm{Spec}(L_0 \otimes_k \ldots \otimes_k L_n))$$

is an isomorphism.

Proof: Let L/k Galois containing L_0. Then L/L_0 is Galois, say with group G. It acts from the right on $L(\bar{k}/k)$ with orbit space

$$L(\bar{k}/k)/G = L_0(\bar{k}/k).$$

Note that this G-action commutes with the Γ_k-action from the left. It follows that

$$\Gamma_k \backslash (L(\bar{k}/k) \times \prod_{0 < \nu \leq n} L_\nu(\bar{k}/k))/G = \Gamma_k \backslash (\prod_{0 \leq \nu \leq n} L_\nu(\bar{k}/k)).$$

Further,

$$\mathrm{Spec}(L \otimes_k L_1 \otimes_k \ldots \otimes_k L_n) \longrightarrow \mathrm{Spec}(L_0 \otimes_k L_1 \otimes_k \ldots \otimes_k L_n)$$

is étale Galois with group G, so G acts on $\pi_0(\mathrm{Spec}(L \otimes_k L_1 \otimes_k \ldots \otimes_k L_n))$ with orbit space $\pi_0(\mathrm{Spec}(L_0 \otimes_k L_1 \otimes_k \ldots \otimes_k L_n))$. The canonical epimorphism

$$\Gamma_k \backslash (L(\bar{k}/k) \times \prod_{0<\nu\leq n} L_\nu(\bar{k}/k)) \twoheadrightarrow \pi_0(\mathrm{Spec}(L \otimes_k L_1 \otimes_k \ldots \otimes_k L_n))$$

is functorial and hence G-equivariant, i.e. it suffices to show that this epimorphism is an isomorphism. Enlarging L/k and arguing similar for the other factors L_i one by one, it suffices to show that the canonical epimorphisms

$$\Gamma_k \backslash G^{n+1} \to \pi_0(\mathrm{Spec}(L^{\otimes_k (n+1)}))$$

is an isomorphism. But the target has the same cardinality then the source, so this clearly holds. □

2.3.4 Remark. Let L_ν/k for $0 \leq \nu \leq n$ be a family of finite algebraic extensions. For $\underline{\iota}$ a tuple of embeddings in $\prod_\nu L_\nu(\bar{k}/k)$ let $L_{\underline{\iota}} \hookrightarrow \bar{k}$ be the embedding of the composition of subfields $\iota_\nu L_\nu$ of $\bar{k}$. The automorphism of $\bar{k}/k$ given by γ in Γ_k restricts to an isomorphism of fields

$$\gamma :\ L_{\underline{\iota}} \longrightarrow L_{\gamma\underline{\iota}}\ .$$

For $[\underline{\iota}]$ a class in $\Gamma_k \backslash (\prod_\nu L_\nu(\bar{k}/k))$ we get that the k-algebra

$$L_{[\underline{\iota}]} := (\prod_{\underline{\iota}\in[\underline{\iota}]} L_{\underline{\iota}})^{\Gamma_k}$$

is again a field, isomorphic to the fields $L_{\underline{\iota}}$ but without a distinguished embedding into $\bar{k}$. Now sending a generator $a_0 \otimes \ldots \otimes a_n$ to the product $\iota_0(a_0) \cdot \ldots \cdot \iota_n(a_n)$ in the $\underline{\iota}$-component induces a map of k-algebras

$$L_0 \otimes_k \ldots \otimes_k L_n \longrightarrow (\textstyle\prod_{\underline{\iota}} L_{\underline{\iota}})^{\Gamma_k} = \prod_{[\underline{\iota}]} L_{[\underline{\iota}]}\ .$$

Arguing similar as in Lem. 2.3.3 we can prove that this is an isomorphism: Again, we reduce the problem to $L_\nu = L$ finite Galois over k for each ν and then argue by induction on n.

Now we can prove:

2.3.5 Lemma. *Let $X.$ be a degreewise geometrically unibranched and geometrically irreducible simplicial k-variety with relative Čech type $\check{\mathrm{C}}(X./k) \to B\Gamma_k$. Then there is a canonical weak equivalence*

$$\check{\mathrm{C}}(X. \otimes_k K/k) \simeq \check{\mathrm{C}}(X./k) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)$$

for K/k algebraic. If K/k is finite algebraic, we even get an isomorphism in Pro$\underline{\mathrm{SSets}}$.

Proof: First, suppose K/k is a finite extension. Denote by $\tilde{K}/k$ its Galois hull. Let

$$p = p_K : \ X. \otimes_k K \longrightarrow X.$$

be the canonical projection. Let $U. \to X.$ be a rigid hypercovering s.t. each component of each degree $(U_n)_x$ has the structure of a $\tilde{K}$-variety. Then the rigid pullback

$$p^* U. \longrightarrow X. \otimes_k K$$

is just the naive pullback $U. \times_{X.} (X. \otimes_k K)$. The subcategory of such rigid coverings $U.$ is cofinal inside $\underline{\mathrm{RC}}(X./k)$. Thus by Cor. 2.2.3, $\check{\mathrm{C}}(X. \otimes_k K/k)$ is isomorphic to the system given by the functor mapping a rigid coverings $U.$ of such a type to the simplicial set

$$\mathrm{Re}(\pi_0(\mathrm{cosk}_0^{X. \otimes_k K}(U. \times_{X.} (X. \otimes_k K)))).$$

Let L/k be a finite Galois extension containing K and fix a point in $i \in L(\bar{k}/k)$. We restrict ourselves further to rigid coverings $U.$ whose components in each degree have the structure of an L-variety and whose distinguished points lie over i via the corresponding map $U. \to \mathrm{Spec}(L)$. Thus, we get a map of rigid hypercoverings

$$U. \longrightarrow (X. \to \mathrm{Spec}(k))^*(L/k, i) \ .$$

In particular we get a factorization of the underling map of $U. \to X.$ over the canonical map $X. \otimes_k L \to X.$ and thus, the cartesian square

$$\begin{array}{ccc} \mathrm{cosk}_0^{X. \otimes_k K}(U. \times_{X.} (X. \otimes_k K)) & \longrightarrow & \mathrm{cosk}_0^{X. \otimes_k K}((X. \otimes_k L) \times_{X.} (X. \otimes_k K)) \\ \downarrow & & \downarrow \\ \mathrm{cosk}_0^{X.} U. & \longrightarrow & \mathrm{cosk}_0^{X.}(X. \otimes_k L) \end{array}$$

We need to show that π_0 of the diagram stays cartesian. Since we may check this degreewise, we may assume that $X. = X$ is simplicial discrete. By assumption, X is geometrically unibranched so we may restrict our self to the corresponding cartesian square of generic points, i.e., of the Spec's of function fields.

Now what we have to prove is (new notation!) that for K/k finite algebraic, a family L_ν/k finite algebraic s.t. $K \subset L_\nu$ for every ν and L/k Galois inside the intersection $\bigcap_\nu L_\nu$ still containing K with the diagonal map $L \hookrightarrow \prod_\nu L_\nu$, the diagram

$$\begin{array}{ccc} \pi_0(\mathrm{cosk}_0^K \coprod_\nu \mathrm{Spec}(L_\nu \otimes_k K)) & \longrightarrow & \pi_0(\mathrm{cosk}_0^K \mathrm{Spec}(L \otimes_k K)) \\ \downarrow & & \downarrow \\ \pi_0(\mathrm{cosk}_0^k \coprod_\nu \mathrm{Spec}(L_\nu)) & \longrightarrow & \pi_0(\mathrm{cosk}_0^k \mathrm{Spec}(L)) \end{array} \tag{2.3.1}$$

is cartesian.

By assumption, each L_ν contains even the Galois hull $\tilde{K}$ of K/k. Using Lem. 2.3.3 we get natural isomorphisms

$$\Gamma_k \backslash \mathrm{cosk}_0(\coprod_\nu L_\nu(\bar{k}/k)) \xrightarrow{\sim} \pi_0(\mathrm{cosk}_0^k \coprod_\nu \mathrm{Spec}(L_\nu))$$

resp.

$$B\mathrm{Gal}(L/k) = \Gamma_k\backslash \mathrm{cosk}_0 L(\bar{k}/k) \xrightarrow{\sim} \pi_0(\mathrm{cosk}_0^k \mathrm{Spec}(L)) \ .$$

Further, $(L_\nu \otimes_k K)(\bar{k}/K)$ is just $L_\nu(\bar{k}/k)$, so we get a canonical epimorphism

$$\Gamma_K\backslash \mathrm{cosk}_0(\textstyle\coprod_\nu L_\nu(\bar{k}/k)) \twoheadrightarrow \pi_0(\mathrm{cosk}_0^K \textstyle\coprod_\nu \mathrm{Spec}(L_\nu \otimes_k K)) \tag{2.3.2}$$

similar to the above. This is an isomorphism, as well. We can check this degreewise and componentwise by comparing the cardinality of the source and target. Consider the canonical epimorphism

$$\Gamma_K\backslash \mathrm{cosk}_0(\textstyle\coprod_\nu L_\nu(\bar{k}/k))_n \twoheadrightarrow \Gamma_k\backslash \mathrm{cosk}_0(\textstyle\coprod_\nu L_\nu(\bar{k}/k))_n \ .$$

There are surjections from $\Gamma_K\backslash\Gamma_k$ to any Γ_k-orbit of any element in the source. Combining this with Lem. 2.3.3 we get

$$\begin{aligned}
|\Gamma_K\backslash \mathrm{cosk}_0(\coprod_\nu L_\nu(\bar{k}/k))_n| &\leq [K:k]|\Gamma_k\backslash \mathrm{cosk}_0(\coprod_\nu L_\nu(\bar{k}/k))_n| \\
&= [K:k]|\pi_0(\mathrm{cosk}_0^k \coprod_\nu \mathrm{Spec}(L_\nu))_n| \\
&= |\pi_0(\mathrm{cosk}_0^K \coprod_\nu \mathrm{Spec}(L_\nu \otimes_k K))_n|
\end{aligned}$$

in each simplicial degree n: Indeed, we have a degreewise isomorphism

$$(\mathrm{cosk}_0^K \coprod_\nu \mathrm{Spec}(L_\nu \otimes_k K))_n \cong ((\mathrm{cosk}_0^k \coprod_\nu \mathrm{Spec}(L_\nu)) \otimes_k K)_n$$

and $\tilde{K}$ is contained in every L_ν by assumption. Thus 2.3.2 is indeed an isomorphism. Similar, we get a canonical isomorphisms

$$\Gamma_K\backslash E\mathrm{Gal}(L/k) = \Gamma_K\backslash \mathrm{cosk}_0 L(\bar{k}/k) \xrightarrow{\sim} \pi_0(\mathrm{cosk}_0^K \mathrm{Spec}(L \otimes_k K)) \ .$$

In particular the diagram (2.3.1) is isomorphic to the diagram:

$$\begin{array}{ccc}
\Gamma_K\backslash \mathrm{cosk}_0(\coprod_\nu L_\nu(\bar{k}/k)) & \longrightarrow & \Gamma_K\backslash \mathrm{cosk}_0 L(\bar{k}/k) \\
\downarrow & & \downarrow \\
\Gamma_k\backslash \mathrm{cosk}_0(\coprod_\nu L_\nu(\bar{k}/k)) & \longrightarrow & \Gamma_k\backslash \mathrm{cosk}_0 L(\bar{k}/k)
\end{array}$$

It suffices to prove that this diagram is degree- and componentwise cartesian, i.e., that

$$\begin{array}{ccc}
\Gamma_K\backslash(\prod_{0\leq i\leq n} L_{\nu_i}(\bar{k}/k)) & \longrightarrow & \Gamma_K\backslash(L(\bar{k}/k)^{n+1}) \\
\downarrow & & \downarrow \\
\Gamma_k\backslash(\prod_{0\leq i\leq n} L_{\nu_i}(\bar{k}/k)) & \longrightarrow & \Gamma_k\backslash(L(\bar{k}/k)^{n+1})
\end{array}$$

is cartesian. Denote by S the pullback of the underling pullback datum. Clearly the canonical map

$$\Gamma_K\backslash(\textstyle\prod_{0\leq i\leq n} L_{\nu_i}(\bar{k}/k)) \longrightarrow S$$

is surjective. To see that it is injective, assume we have $\underline{i}, \underline{j} \in \prod_{0\leq i\leq n} L_{\nu_i}(\bar{k}/k)$, $\sigma \in \Gamma_k$ and $\tau \in \Gamma_K$ s.t. $\underline{j} = \sigma\underline{i}$ and $\mathrm{res}_L(\underline{j}) = \tau\mathrm{res}_L(\underline{i})$, where res_L is componentwise the restriction

$$L_\nu(\bar{k}/k) \longrightarrow L(\bar{k}/k)\ .$$

But then $\tau^{-1}\sigma$ acts trivially on a k-embedding of the Galois extension L/k, i.e., lies inside $\Gamma_L \leq \Gamma_K$. As a result σ itself lies in Γ_K, i.e., $[\underline{i}]$ and $[\underline{j}]$ agree in $\Gamma_K\backslash(\prod_{0\leq i\leq n} L_{\nu_i}(\bar{k}/k))$, which completes the proof of the injectivity.

We go back to our original notation. Note that $\pi_0(\mathrm{cosk}_0^X X_L)$ is nothing but $\pi_0(\mathrm{cosk}_0^k \mathrm{Spec}(L))$ and $\pi_0(\mathrm{cosk}_0^{X_K}(X_L \times_X X_K))$ just $\pi_0(\mathrm{cosk}_0^K \mathrm{Spec}(L\otimes_k K))$ (since X is geometrically connected). Thus, we have proven that

$$\begin{array}{ccc} \pi_0(\mathrm{cosk}_0^{X_K}(U \times_X X_K)) & \longrightarrow & \Gamma_K\backslash E\mathrm{Gal}(L/k) \\ \downarrow & & \downarrow \\ \pi_0(\mathrm{cosk}_0^X U) & \longrightarrow & B\mathrm{Gal}(L/k) \end{array}$$

is cartesian. Varying L/k Galois in $\underline{\mathrm{RC}}(k/k)$ over K/k and U in $\underline{\mathrm{RC}}(X/k)$ over the rigid pullback of L/k to X we finally get

$$\check{\mathrm{C}}(X_K/k) = \check{\mathrm{C}}(X/k) \times_{B\Gamma_k} (\Gamma_K\backslash E\Gamma_k),$$

where the map $\check{\mathrm{C}}(X/k) \to B\Gamma_k$ is just the relative Čech topological type and the projection to $\Gamma_K\backslash E\Gamma_k$ is just $\check{\mathrm{C}}(\{X_K \to \mathrm{Spec}(K)\}/k)$.

For K/k algebraic but not necessarily finite and $X. = X$ simplicial discrete note that $\check{\mathrm{C}}(X_K/k)$ is weakly equivalent to $\lim_{K\supseteq L/k} \check{\mathrm{C}}(X_L/k)$, where the limit runs over all finite subextensions L/k of K/k: Indeed, the π_0 part is trivial, π_1 and $\pi_1^{\text{ét}}$ preserve cofiltered limits and $\pi_1\check{\mathrm{C}}(X_L/k) = \pi_1^{\text{ét}}(X_L)$ by Thm. 1.3.1 (X is geometrically unibranched) and similar for X_K. Finally, the cohomology part follows in the same manner from the Verdier-Hypercovering-Theorem. But then the claim follows from the above canonical isomorphism $\Gamma_K\backslash E\Gamma_k = \lim_{K\supseteq L/k}(\Gamma_L\backslash E\Gamma_k)$.

For K/k algebraic but not necessarily finite and $X.$ not necessarily simplicial discrete we have to show that $\check{\mathrm{C}}(X. \otimes_k K/k)$ is still weakly equivalent to $\lim_{K\supseteq L/k} \check{\mathrm{C}}(X. \otimes_k L/k)$: Denote by

$$\underline{\mathrm{SubExt}}_f(K/k)$$

the category of finite subextensions of K/k. By Cor. 2.2.3, the cofiltered limit $\lim_{K\supseteq L/k} \check{\mathrm{C}}(X. \otimes_k L/k)$ is isomorphic to the levelwise realization of the simplicial pro-simplicial set induced by the functor

$$\underline{\mathrm{RC}}(X./k) \times \underline{\mathrm{SubExt}}_f(K/k) \longrightarrow \underline{\mathrm{SSSets}}$$

mapping rigid covering $U. \in \underline{\mathrm{RC}}(X./k)$ together with a finite subextension L/k of K/k to the bisimplicial set

$$\pi_0(\mathrm{cosk}_0^{X.\otimes_k L} p_L^* U.),$$

where $p_L : X. \otimes_k L \to X.$ is the canonical projection. Thus, Cor. 1.6.6 applies to both pro-simplicial sets $\lim_{K \supseteq L/k} \check{\mathrm{C}}(X. \otimes_k L/k)$ and $\check{\mathrm{C}}(X. \otimes_k K/k)$. But the n^{th} simplicial degrees are just $\lim_{K \supseteq L/k} \check{\mathrm{C}}(X_n \otimes_k L/k)$ resp. $\check{\mathrm{C}}(X_n \otimes_k K/k)$. These are weakly equivalent by the above, so our claim follows by the homotopy invariance of homotopy colimits (see [Hir03] Thm. 18.5.1). $\square$

Our next goal is to derive the base extension

$$(-) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) : \mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k \longrightarrow \mathrm{Pro}\underline{\mathrm{SSets}} \downarrow (\Gamma_K \backslash E\Gamma_k) .$$

First, we need a relative Postnikov replacement in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow \Gamma_k$:

2.3.6 Remark. If we choose a functorial fibrant resolution $\mathrm{Ex}(-)$ in $\underline{\mathrm{SSets}}$, always choose one preserving simplicial discrete Γ-sets throughout the rest of the thesis, e.g., the one given by the small object argument (cf. [Hir03] Prop. 10.5.16) with resp. to the inclusion of horns $\Lambda^n_k \hookrightarrow \Delta_n$: Indeed, the functorial construction of this factorization is made out of colimits in $\underline{\mathrm{Sets}}$, which preserves discrete Γ-sets.

2.3.7 Remark. Fix once and for all a functorial levelwise factorization

$$\begin{tikzcd} A. \arrow[r, hook] \arrow[rr, bend right, "f"'] & \mathrm{Ex}(f) \arrow[r, two heads] & B. \end{tikzcd}$$

in $\underline{\mathrm{SSets}}$. Let $f : \mathfrak{X} \to B\Gamma_k$ be in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k$. Using [AM69] Appendix Prop. 3.1 we write f as a levelwise map $\{f_i\}_i$. Then

$$\mathrm{Ex}(\mathfrak{X}) := \{\ \mathrm{Ex}(f_i) \longrightarrow B\Gamma_i\ \}_i$$

is levelwise weakly equivalent to $\mathfrak{X}$ in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k$ and even levelwise fibrant in $\mathrm{Pro}\underline{\mathrm{SSets}}$. Further, note that

$$\mathrm{cosk}_n B\underline{C} = B\underline{C}$$

for any small category $\underline{C}$ and any $n > 2$: Indeed, for such n and any simplicial set A.

$$P_*(\mathrm{sk}_n A.) = P_*(A.)$$

holds for the path categories, so the claim follows by adjointness. In particular,

$$(B\Gamma_k)^\natural = B\Gamma_k$$

holds for the naive Postnikov replacement of pro-systems of fibrant simplicial sets. Thus, our levelwise fibrant replacement in the definition of a Postnikov tower replacement (cf. Rem. 1.1.6) gives a Postnikov replacement relative over $B\Gamma_k$

$$(-)^\natural : \mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k \longrightarrow \mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k$$

together with a natural levelwise weak equivalence $\mathrm{id} \to (-)^\natural$. If we want to stress the difference to the naive Postnikov replacement of levelwise fibrant objects, we write $(-)^{h\natural}$ for the relative replacement and $(-)^{n\natural}$ for the naive replacement.

2.3.8 Lemma. *The canonical map*

$$\mathfrak{X} \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) \longrightarrow \mathfrak{X}^\natural \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)$$

is a weak equivalence in Pro$\underline{\text{SSets}}$.

Proof: Both $B\Gamma_k$ and $\Gamma_K \backslash E\Gamma_k$ are already levelwise fibrant. The naive Postnikov replacements $B\Gamma_k \simeq B\Gamma_k^{n\natural}$ and $\Gamma_K \backslash E\Gamma_k \simeq (\Gamma_K \backslash E\Gamma_k)^{n\natural}$ are levelwise weak equivalences if we restrict ourselves to $\operatorname{cosk}_n(-)$ for $n > 2$. Note that,

$$\Gamma_K \backslash E\Gamma_k \longrightarrow B\Gamma_k$$

is a levelwise simplicial covering, i.e., a levelwise fibration. Further, each such $\operatorname{cosk}_n(-)$ preserves fibrations, so the canonical map

$$(\Gamma_K \backslash E\Gamma_k)^{n\natural} \longrightarrow B\Gamma_k^{n\natural}$$

still is a levelwise fibration. In particular, we get the commutative diagram

$$\begin{tikzcd}
\mathfrak{X} \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) \arrow[r] \arrow[dd] & \mathfrak{X}^{h\natural} \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) \arrow[d] \\
& \mathfrak{X}^{h\natural} \times_{B\Gamma_k^{n\natural}} (\Gamma_K \backslash E\Gamma_k)^{n\natural} \arrow[d, equal] \\
\operatorname{Ex}(\mathfrak{X}) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) \arrow[r] & (\operatorname{Ex}(\mathfrak{X}) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k))^{n\natural}
\end{tikzcd}$$

whose vertical arrows are levelwise weak equivalences by the classical theory of homotopy cartesian diagrams (see e.g., [GJ99] Chap. II Cor. 8.13, for the equality in the bottom right, note that $\operatorname{cosk}_n(-)$ preserves limits). Since

$$\operatorname{Ex}(\mathfrak{X}) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)$$

is levelwise fibrant, the lower horizontal arrow is a weak equivalence in Pro$\underline{\text{SSets}}$, which finishes the proof. □

Recall that for Γ a profinite group, the category of simplicial discrete Γ-sets $\underline{\text{SSets}}_\Gamma$ is just the category of simplicial sheaves over the classifying site $\underline{B}\Gamma$. Thus, it carries the structure of a proper simplicial closed model structure by [Jar86] Cor. 2.7 (see also [Goe95] for a more elementary treatment) resp. [Jar87] Prop. 1.4 (note that properness for simplicial presheaves implies properness of simplicial sheaves: indeed, sheafification preserves weak equivalences).

2.3.9 Corollary. *The base extension functor*

$$(-) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) : \ \text{Pro}\underline{\text{SSets}} \downarrow B\Gamma_k \longrightarrow \text{Pro}\underline{\text{SSets}} \downarrow (\Gamma_K \backslash E\Gamma_k)$$

preserves weak equivalences. Moreover, the relative Postnikov replacement $(-)^\natural$ *composed with the induced functor into* Pro$\underline{\text{SSets}}$ *factors over* $\text{Pro}(\underline{\text{SSets}}_{\Gamma_k})$ *and maps a weak equivalence to a morphism isomorphic in (the morphism category of)* $\text{Pro}(\underline{\text{SSets}}_{\Gamma_k})$ *to a levelwise weak equivalence in* $\text{Pro}(\underline{\text{SSets}}_{\Gamma_k})$.

Proof: By Lem. 2.3.8 the first claim follows from the second. The factorization statement is trivial (use Rem. 2.3.6). Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a weak equivalence in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k$. By Cor. 1.5.12 $f^\natural$ is isomorphic to a levelwise weak equivalence. Again by the classical theory of homotopy cartesian diagrams, this property is preserved by our base change $(-) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)$, i.e. $f^\natural \times_{B\Gamma_k} (\Gamma_K\backslash E\Gamma_k)$ is isomorphic in $\mathrm{Pro}(\underline{\mathrm{SSets}}_{\Gamma_k})$ to a levelwise weak equivalence in $\mathrm{Pro}(\underline{\mathrm{SSets}}_{\Gamma_k})$, just as claimed. □

Fix a point $i_0 \in K(\bar{k}/k)$. Let $U.$ be a rigid covering in $\underline{\mathrm{RC}}(X./k)$. Restriction of the rigid pullback p^*U_n to the components of points

$$x \otimes i_0 \in (X_n \otimes_k K)(\bar{k}/K) = X_n(\bar{k}/k)$$

for $x \in X_n(\bar{k}/k)$ gives a rigid covering in $\underline{\mathrm{RC}}(X. \otimes_k K/K)$ together with a map of $(X. \otimes_k K)$-schemes from this rigid covering to $p^*U.$ in $\underline{\mathrm{RC}}(X. \otimes_k K/k)$. Thus, restriction to such coverings in $\underline{\mathrm{RC}}(X. \otimes_k K/K)$ defines a map

$$\check{\mathrm{C}}(X. \otimes_k K/K) \longrightarrow \check{\mathrm{C}}(X. \otimes_k K/k)$$

natural in $X. \in \underline{\mathrm{SVar}}_k$ which clearly is a weak equivalence. Note, that for $\mathrm{Spec}(k)$ this is just the canonical map

$$B\Gamma_K = \Gamma_K\backslash E\Gamma_K \longrightarrow \Gamma_K\backslash E\Gamma_k .$$

By [Ser02] Chap. I Prop. 1 there is a section

$$s : \Gamma_K\backslash\Gamma_k \longrightarrow \Gamma_k$$

in $\mathrm{Pro}\underline{\mathrm{Sets}}$ of the canonical map $\Gamma_k \to \Gamma_K\backslash\Gamma_k$. We may assume $s[\tau] = 1$ for $\tau \in \Gamma_K$ (otherwise, multiply s by the inverse of $s[\tau]$). Let r be the composition

$$\begin{tikzcd} \Gamma_k \arrow[r] \arrow[rr, bend right, "r"'] & \Gamma_K\backslash\Gamma_k \arrow[r, "s"] & \Gamma_k . \end{tikzcd}$$

For $\sigma \in \Gamma_k$ we get $r(\sigma) \equiv \sigma$ in $\Gamma_K\backslash\Gamma_k$, hence $\sigma r(\sigma)^{-1} \in \Gamma_K$. Further, for $\tau \in \Gamma_K$ we get $r(\tau\sigma) = r(\sigma)$ and $r(\tau) = 1$. It follows that the map

$$E\Gamma_k \longrightarrow E\Gamma_K$$

mapping a tuple $\underline{\sigma}$ componentwise to $\underline{\sigma} r(\underline{\sigma})^{-1}$ is compatible with the left diagonal action of Γ_K and thus gives a (left-) splitting

$$\Gamma_K\backslash E\Gamma_k \xrightarrow{\sim} \Gamma_K\backslash E\Gamma_K = B\Gamma_K$$

in $\mathrm{Pro}\underline{\mathrm{SSets}}$ of our canonical map $B\Gamma_K \to \Gamma_K\backslash E\Gamma_k$.

Summing up, we get the commutative diagram

$$\begin{tikzcd} \check{\mathrm{C}}(X. \otimes_k K/K) \arrow[r, "\sim"] \arrow[d] & \check{\mathrm{C}}(X. \otimes_k K/k) \arrow[r, "\sim"] \arrow[d] & \check{\mathrm{C}}(X./k) \times_{B\Gamma_k} (\Gamma_K\backslash E\Gamma_k) , \arrow[dl, "\mathrm{pr}"] \\ B\Gamma_K \arrow[r] \arrow[dr, "\mathrm{id}"'] & \Gamma_K\backslash E\Gamma_k \arrow[d] & \\ & B\Gamma_K & \end{tikzcd}$$

i.e., we get a weak equivalence in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_K$ between the relative Čech topological type $\check{\mathrm{C}}(X. \otimes_k K/K) \to B\Gamma_K$ and the composition

$$\begin{tikzcd} \check{\mathrm{C}}(X./k) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) \arrow[r, "\mathrm{pr}"] \arrow[rr, bend right] & \Gamma_K \backslash E\Gamma_k \arrow[r] & B\Gamma_K \end{tikzcd} .$$

Summing up we can prove:

2.3.10 Proposition. *Let K/k be (not necessarily finite) algebraic. Then we get a derived base change*

$$\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k) \longrightarrow \mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_K) ,$$

mapping a degreewise geometrically unibranched and geometrically irreducible simplicial k-variety of Čech type $X. \to B\Gamma_k$ to the scheme theoretical base extension $X. \otimes_k K \to B\Gamma_K$.

Proof: By Cor. 2.3.9 the functor

$$(-) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) : \ \mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k \longrightarrow \mathrm{Pro}\underline{\mathrm{SSets}} \downarrow (\Gamma_K \backslash E\Gamma_k)$$

preserves weak equivalences. Further, $\Gamma_K \backslash E\Gamma_k \to B\Gamma_K$ is a weak equivalence as splitting of a weak equivalence, i.e., the push forward

$$\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow (\Gamma_K \backslash E\Gamma_k) \longrightarrow \mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_K$$

preserves weak equivalences, as well. Thus the composition preserves weak equivalences, i.e., derives in the naive way to our desired derived base change

$$\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k) \longrightarrow \mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_K) .$$

As we have seen above, the derived base change of the relative Čech type $\check{\mathrm{C}}(X./k) \to B\Gamma_k$ is isomorphic to the relative Čech type $\check{\mathrm{C}}(X. \otimes_k K/K) \to B\Gamma_K$. But for $X.$ degreewise of Čech type these are the relative étale homotopy types $X. \to B\Gamma_k$ resp. $X. \otimes_k K \to B\Gamma_K$, hence the claim. $\square$

There is an equivariant version, as well:

2.3.11 Proposition. *Under the assumptions of Prop. 2.3.10 we get an equivariant derived base change*

$$(-)^{\natural} \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) : \ \mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma_k) \longrightarrow \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_k}) .$$

For a degreewise geometrically unibranched and geometrically irreducible simplicial k-variety of Čech type $X. \to B\Gamma_k$, the derived base change

$$X.^{\natural} \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)$$

is isomorphic to the Postnikov tower (à la [AM69]) $(X. \otimes_k K)^{\natural}$ of the scheme theoretical base extension as a Γ_k-object in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$. Further, the equivariant derived base change is compatible with the derived base change of Prop. 2.3.10.

Proof: Again, by Cor. 2.3.9, the composition of functors

$$\begin{tikzcd}
\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma \arrow[r, "(-)^\natural \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)"] \arrow[dr] & \mathrm{Pro}(\underline{\mathrm{SSets}}_{\Gamma_k}) \arrow[d] \\
& \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_k})
\end{tikzcd}$$

maps weak equivalences to isomorphisms, i.e., we get our equivariant derived base change in the naive way. Our weak equivalence

$$\check{\mathrm{C}}(X. \otimes_k K) \longrightarrow \check{\mathrm{C}}(X./k) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)$$

is Γ_k-equivariant, so the lower arrow in

$$\begin{tikzcd}
\check{\mathrm{C}}(X. \otimes_k K) \arrow[r, "\sim"] \arrow[d, hook, "\sim"] & \check{\mathrm{C}}(X./k) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k) \arrow[d, hook, "\sim"] \\
\mathrm{Ex}\check{\mathrm{C}}(X. \otimes_k K) \arrow[r] & \mathrm{Ex}\check{\mathrm{C}}(X./k) \times_{B\Gamma_k} (\Gamma_K \backslash E\Gamma_k)
\end{tikzcd}$$

is a Γ_k-equivariant weak equivalence in Pro$\underline{\mathrm{SSets}}$ and the second claim follows from Cor. 1.5.12.

For the compatibility claim note that both base changes were derived in a naive way, so it suffices for $L/K/k$ a tower of algebraic extensions and

$$\mathfrak{Y} := \mathfrak{X} \times_{B\Gamma_k} (\Gamma_K \setminus E\Gamma_k)$$

the Pro$\underline{\mathrm{SSets}} \downarrow B\Gamma_K$ object used in Prop. 2.3.10 to show that we have an isomorphism

$$\mathfrak{Y}^\natural \times_{B\Gamma_K} (\Gamma_L \setminus E\Gamma_K) \cong \mathrm{res}^{\Gamma_K}_{\Gamma_k} (\mathfrak{X}^\natural \times_{B\Gamma_k} (\Gamma_L \setminus E\Gamma_k))$$

in $\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_K})$. We have levelwise weak equivalences

$$\begin{tikzcd}
\mathfrak{Y} = \mathfrak{X} \times_{B\Gamma_k} (\Gamma_K \setminus E\Gamma_k) \arrow[r] \arrow[d] & \mathrm{Ex}(\mathfrak{X} \times_{B\Gamma_k} (\Gamma_K \setminus E\Gamma_k) \to B\Gamma_K) \\
\mathrm{Ex}(\mathfrak{X} \to B\Gamma_k) \times_{B\Gamma_k} (\Gamma_K \setminus E\Gamma_k) &
\end{tikzcd}$$

in Pro$\underline{\mathrm{SSets}} \downarrow B\Gamma_K$. Further, in Pro$\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_k})$ resp. Pro$\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_K})$ we have a levelwise isomorphism

$$\mathfrak{X}^{h\natural} \times_{B\Gamma_k} (\Gamma_L \setminus E\Gamma_k) \cong (\mathfrak{Z} \times_{B\Gamma_k} (\Gamma_L \setminus E\Gamma_k))^{n\natural}$$

for $\mathfrak{Z} := \mathrm{Ex}(\mathfrak{X} \to B\Gamma_k)$ as well as levelwise isomorphisms

$$\begin{aligned}
\mathfrak{Y}^{h\natural} \times_{B\Gamma_K} (\Gamma_L \setminus E\Gamma_K) &\cong (\mathrm{Ex}(\mathfrak{Y} \to B\Gamma_K) \times_{B\Gamma_K} (\Gamma_L \setminus E\Gamma_K))^{n\natural} \\
&\cong ((\mathfrak{Z} \times_{B\Gamma_k} (\Gamma_K \setminus E\Gamma_k)) \times_{B\Gamma_K} (\Gamma_L \setminus E\Gamma_K))^{n\natural}.
\end{aligned}$$

Thus, it suffices to show that the outer square of

$$\begin{tikzcd} \mathfrak{Z} \times_{B\Gamma_k} (\Gamma_L\backslash E\Gamma_k) \arrow[r, "\mathrm{pr}"'] \arrow[d] & \Gamma_L\backslash E\Gamma_k \arrow[r] \arrow[d] & \Gamma_L\backslash E\Gamma_K \arrow[d] \\ \mathfrak{Z} \times_{B\Gamma_k} (\Gamma_K\backslash E\Gamma_k) \arrow[r, "\mathrm{pr}"] & \Gamma_K\backslash E\Gamma_k \arrow[r] & B\Gamma_K \end{tikzcd}$$

is cartesian, where the horizontal maps in the right square are the maps induced by our fixed map

$$E\Gamma_k \longrightarrow E\Gamma_K$$

mapping a tuple $\underline{\sigma}$ to $\underline{\sigma} r(\underline{\sigma})^{-1}$. Since the left square clearly is cartesian, it suffices to show the same for the right square. First, suppose we have a tuple $\underline{\sigma} \in E\Gamma_k$ and $\underline{\tau} \in E\Gamma_K$ s.t.

$$\underline{\tau} \equiv (\underline{\sigma} r(\underline{\sigma})^{-1})$$

holds in $B\Gamma_K = \Gamma_K\backslash E\Gamma_K$, i.e. $\underline{\tau} = \gamma(\underline{\sigma} r(\underline{\sigma})^{-1})$ for a suitable γ in Γ_K. Since $r(\gamma\underline{\sigma}) = r(\underline{\sigma})$, the right side equals $(\gamma\underline{\sigma})r(\gamma\underline{\sigma})^{-1}$. But $\gamma\underline{\sigma} \equiv \underline{\sigma}$ in $\Gamma_K\backslash E\Gamma_k$, so the canonical map

$$\Gamma_L\backslash E\Gamma_k \longrightarrow (\Gamma_K\backslash E\Gamma_k) \times_{B\Gamma_K} (\Gamma_L\backslash E\Gamma_K)$$

is surjective. For injectivity assume we have two tuples $\underline{\sigma}', \underline{\sigma}'' \in E\Gamma_k$ agreeing in $\Gamma_K\backslash E\Gamma_k$ s.t.

$$\underline{\sigma}' r(\underline{\sigma}')^{-1} \equiv \underline{\sigma}'' r(\underline{\sigma}'')^{-1}$$

in $\Gamma_L\backslash E\Gamma_K$. Thus, there is a $\gamma \in \Gamma_K$ and a $\tau \in \Gamma_L$ s.t.

$$\sigma'' = \gamma\sigma'$$
$$\underline{\sigma}'' r(\underline{\sigma}'')^{-1} = \tau(\underline{\sigma}' r(\underline{\sigma}')^{-1}).$$

Again, $r(\gamma\underline{\sigma}') = r(\underline{\sigma}')$, so

$$\gamma(\underline{\sigma}' r(\underline{\sigma}')^{-1}) = \tau(\underline{\sigma}' r(\underline{\sigma}')^{-1}).$$

In particular, $\gamma = \tau$ already lies in Γ_L, i.e., $\underline{\sigma}'$ and $\underline{\sigma}''$ agree in $\Gamma_L\backslash E\Gamma_k$, which completes the injectivity. □

2.3.12 Notation. *For a relative pro-simplicial set $\mathfrak{X} \to B\Gamma$ in* Pro$\underline{\text{SSets}} \downarrow B\Gamma$ *set*

$$\bar{\mathfrak{X}} := \mathfrak{X}^\natural \times_{\mathrm{B}\Gamma} E\Gamma.$$

2.4 The Hochschild-Serre spectral sequence. Throughout the rest of the thesis denote the absolute Galois group Γ_k of k by Γ. Let X be a k-variety and let $p : X \otimes_k \bar{k} \to X$ be the canonical projection. Denote by $\mathrm{HS}_*^{\bullet,\bullet}(X;\mathcal{F})$ the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\Gamma; H^q(X \otimes_k \bar{k}; \mathcal{F})) \Rightarrow H^{p+q}(X; \mathcal{F})$$

for a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ given as the Grothendieck spectral sequence of the composition of functors

$$\mathbb{R}H^0(\Gamma; -) \circ \mathbb{R}\Gamma(X \otimes_k \bar{k}; p^*(-)).$$

Now $\Gamma(X \otimes_k \bar{k}; p^*(-))$ together with the induced Γ-action is just the pushforward f_* of $f : X \to \operatorname{Spec}(k)$ the canonical map. Thus, this spectral sequence is isomorphic to the Leray spectral sequence

$$E_2^{p,q} = H^p(\Gamma; \mathbb{R}^q f_*(\mathcal{F})) \Rightarrow H^{p+q}(X; \mathcal{F}).$$

Denote by $\mathbb{H}_*^{\bullet,\bullet}(C^\bullet)$ the Galois-hypercohomology spectral sequence

$$E_2^{p,q} = H^p(\Gamma; H^q C^\bullet) \Rightarrow \mathbb{H}^{p+q}(\Gamma; C^\bullet)$$

for $C^\bullet$ a complex of discrete Γ-modules.

In this section we want to show that the Hochschild-Serre spectral sequence $\mathrm{HS}_*^{\bullet,\bullet}(X; \Lambda)$ for $\Lambda \in \underline{\mathrm{Mod}}_\Gamma$ is isomorphic to the Galois-hypercohomology spectral sequence $\mathbb{H}_*^{\bullet,\bullet}(C^\bullet(X^\natural \times_{B\Gamma} E\Gamma; \Lambda))$:

2.4.1 Proposition. *Let X be a k-variety and Λ a Γ-module. There is a canonical isomorphism*

$$\mathrm{HS}_*^{\bullet,\bullet}(X; \Lambda) \cong \mathbb{H}_*^{\bullet,\bullet}(C^\bullet(X^\natural \times_{B\Gamma} E\Gamma; \Lambda))$$

between the Hochschild-Serre spectral sequence $\mathrm{HS}_*^{\bullet,\bullet}(X; \Lambda)$ and the hypercohomology spectral sequence $\mathbb{H}_*^{\bullet,\bullet}(C^\bullet(X^\natural \times_{B\Gamma} E\Gamma; \Lambda))$.

Again in abuse of notation, we will just write $\bar{X}$ for $X^\natural \times_{B\Gamma} E\Gamma$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$. Let

$$\mathrm{ex} : X(\Omega) = \textstyle\coprod_{X(\Omega)} \operatorname{Spec}(\Omega) \longrightarrow X$$

be the canonical map from the "exploded scheme" $X(\Omega)$ to X given as the disjoint union of all geometric points $X(\Omega)$. Call a sheaf $\mathcal{G}$ on $X_{\text{ét}}$ of **Godement type**, if $\mathcal{G}$ is isomorphic to a sheaf $\mathrm{ex}_* \underline{A}$ for $\underline{A}$ in

$$\operatorname{Shv}(X(\Omega)) = \prod_{X(\Omega)} \underline{\mathrm{Ab}}.$$

Before we will give the proof of Prop. 2.4.1, we will need a technical lemma. Recall that $\mathrm{ind}_\Gamma^{\mathbf{1}}(A)$ for A an abelian group is the induced discrete Γ-module

$$\mathrm{ind}_\Gamma^{\mathbf{1}}(A) := \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Sets}}}(\Gamma, A).$$

The induction $\mathrm{ind}_\Gamma^{\mathbf{1}}(-)$ is right adjoint to the restriction $\mathrm{res}_\Gamma^{\mathbf{1}}(-)$. Since the later functor is isomorphic to $\pi^*(-)$ for

$$\pi : \operatorname{Spec}(\bar{k}) \longrightarrow \operatorname{Spec}(k)$$

the canonical map given by our fixed choice of the separable closure $\bar{k}/k$, the induction $\mathrm{ind}_\Gamma^{\mathbf{1}}(-)$ is isomorphic to the push forward $\pi_*(-)$.

2.4.2 Lemma. *For a sheaf $\mathcal{F}$ on $(X \otimes_k \bar{k})_{\text{ét}}$ and $U \to X$ étale we have a canonical isomorphism*

$$\Gamma(U \otimes_k \bar{k}; p^* p_* \mathcal{F}) = \operatorname{ind}_\Gamma^{\mathbf{1}} \Gamma(U; p_* \mathcal{F})$$

of discrete Γ*-modules, natural in* $U \to X$ *and* $\mathcal{F}$*.*

Proof: In abuse of notation, we denote the canonical projection $U \otimes_k \bar{k} \to U$ by p, as well. We get the commutative diagram of canonical maps:

$$\begin{tikzcd} U \otimes_k \bar{k} \arrow[r, "\bar{f}"] \arrow[d, "p"] & \operatorname{Spec}(\bar{k}) \arrow[d, "\pi"] \\ U \arrow[r, "f"] & \operatorname{Spec}(k) \end{tikzcd}$$

The discrete Γ-module $\Gamma(U \otimes_k \bar{k}; p^* p_* \mathcal{F})$ is isomorphic to the discrete Γ-module $f_* p_* \mathcal{F}$. Now

$$f_* p_* \mathcal{F} = \pi_* \bar{f}_* \mathcal{F} = \pi_* \Gamma(U \otimes_k \bar{k}; \mathcal{F})$$

and

$$\Gamma(U \otimes_k \bar{k}; \mathcal{F}) = \Gamma(U; p_* \mathcal{F})$$

holds by definition of push forwards. Further, the functor $\pi_*(-)$ is isomorphic to $\operatorname{ind}_\Gamma^{\mathbf{1}}(-)$, which completes the proof. $\square$

For a field K denote by $K(\Omega)$ the set of embeddings $K \hookrightarrow \Omega$ ("we dropped the Spec in our notation"). Fix a splitting

$$s : \ k(\Omega) \longrightarrow \bar{k}(\Omega)$$

of the canonical surjection

$$\pi_* : \ \bar{k}(\Omega) \twoheadrightarrow k(\Omega) \ .$$

This in turn induces a splitting of the canonical surjection

$$p_* : \ (X \otimes_k \bar{k})(\Omega) \twoheadrightarrow X(\Omega) \ ,$$

i.e. we get a factorization of the exploded scheme map $\text{ex} : X(\Omega) \to X$:

$$\begin{tikzcd} & X \otimes_k \bar{k} \arrow[d, "p"] \\ X(\Omega) \arrow[ur, "\overline{\text{ex}}"] \arrow[r, "\text{ex}"] & X \end{tikzcd}$$

In particular, for each $\underline{A} \in \operatorname{Shv}(X(\Omega))$ we get

$$\text{ex}_* \underline{A} = p_*(\overline{\text{ex}}_* \underline{A}),$$

i.e. each sheaf of Godement type on $X_{\text{ét}}$ is a push forward along p. Thus, we get from Lem. 2.4.2:

2.4.3 Corollary. *Let $\mathcal{G}$ be a sheaf of Godement type on $X_{\text{ét}}$, say $\mathcal{G} = \text{ex}_* \underline{A}$ for $\underline{A} \in \text{Shv}(X(\Omega))$. Let $U \to X$ be étale. Then we have an isomorphism*

$$\Gamma(U \otimes_k \bar{k}; p^*\mathcal{G}) \cong \text{ind}_\Gamma^{\mathbf{1}} \Gamma(U; \mathcal{G})$$

of discrete Γ-modules, natural in $U \to X$ and $\underline{A}$. This morphism is canonical up to our choice of the splitting s of $\pi_* : \bar{k}(\Omega) \to k(\Omega)$.

We come back to the proof of Prop. 2.4.1: Let $\mathcal{G}$ be a sheaf of Godement type on $X_{\text{ét}}$. It is flasque, e.g., by [SGA73] Exp. XVII 4.2. Combining this with [SGA72] Exp. VII Cor. 5.8 we get

$$H^q(X \otimes_k \bar{k}; \mathcal{G}) = \operatorname*{colim}_{L/k} H^q(X \otimes_k L; \mathcal{G}) = 0$$

for $q > 0$, i.e. $\mathcal{G}$ is $\Gamma(X \otimes_k \bar{k}; p^*(-))$-acyclic. Thus, for

$$0 \longrightarrow \Lambda \longrightarrow \mathcal{G}^\bullet$$

the Godement resolution of Λ on $X_{\text{ét}}$, we get an isomorphism of spectral sequences

$$\text{HS}_*^{\bullet,\bullet}(X; \Lambda) \cong \mathbb{H}_*^{\bullet,\bullet}(\Gamma(X \otimes_k \bar{k}; \mathcal{G}^\bullet)).$$

Denote by $\tilde{\mathfrak{X}}$ the pro-object in $\mathcal{H}(\underline{\text{SSets}}_\Gamma)$ induced by the functor

$$\underline{\text{HR}}(X_{\text{ét}}) \to \underline{\text{SSets}}, \ \mathfrak{U}. \mapsto \pi_0(\mathfrak{U}. \otimes_k \bar{k}).$$

Then $\tilde{\mathfrak{X}}$ is weakly equivalent to $\bar{X}$ in $\text{Pro}\mathcal{H}(\underline{\text{SSets}}_\Gamma)$ and it remains to construct a canonical isomorphism

$$\Gamma(X \otimes_k \bar{k}, \mathcal{G}^\bullet) \simeq C^\bullet(\tilde{\mathfrak{X}}; \Lambda)$$

in $\mathcal{D}^+(\underline{\text{Mod}}_\Gamma)$. This will be done in the following two lemmata:

2.4.4 Lemma. *Let $\mathfrak{U}. \to X$ be a hypercovering in $\underline{\text{HR}}(X_{\text{ét}})$. There is a canonical weak equivalence in $\mathcal{D}^+(\underline{\text{Mod}}_\Gamma)$:*

$$\Gamma(X \otimes_k \bar{k}; \mathcal{G}^\bullet) \xrightarrow{\sim} \text{tot}^\bullet \Gamma(\mathfrak{U}_{\bullet_{\text{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\text{I}}}) \ .$$

2.4.5 Lemma. *The canonical map $\Gamma(\mathfrak{U}_\bullet \otimes_k \bar{k}; \Lambda) \to \text{tot}^\bullet \Gamma(\mathfrak{U}_{\bullet_{\text{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\text{I}}})$ induced by the resolution $\Lambda \to \mathcal{G}^{\bullet_{\text{I}}}$ gives a quasi-isomorphism in $\mathcal{D}^+(\underline{\text{Mod}}_\Gamma)$ after taking the colimit over all $\mathfrak{U}. \in \underline{\text{HR}}(X_{\text{ét}})$:*

$$C^\bullet(\tilde{\mathfrak{X}}; \Lambda) \xrightarrow{\sim} \text{colim}_{\mathfrak{U}. \in \underline{\text{HR}}(X_{\text{ét}})} \text{tot}^\bullet \Gamma(\mathfrak{U}_{\bullet_{\text{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\text{I}}}) \ .$$

Since filtered colimits are exact in $\underline{\text{Mod}}_\Gamma$, the colimit over $\mathfrak{U}. \in \underline{\text{HR}}(X_{\text{ét}})$ preserves the weak equivalence of Lem. 2.4.4, which completes the proof of Prop. 2.4.1. Thus, it remains to prove Lem. 2.4.4 and Lem. 2.4.5:

Proof of Lem. 2.4.4: The canonical map $\mathfrak{U}.\otimes_k \bar{k} \to X \otimes_k \bar{k}$ induces a morphism of double complexes

$$\Gamma(X \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}}) \longrightarrow \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}}) \ . \tag{2.4.1}$$

Using Cor. 2.4.3 we have an isomorphism

$$\Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}}) \cong \mathrm{ind}_{\Gamma}^{\mathbf{1}} \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}}; \mathcal{G}^{\bullet_{\mathrm{I}}}).$$

The induction $\mathrm{ind}_{\Gamma}^{\mathbf{1}}(-)$ is exact (since it is also left adjoint to $\mathrm{res}_{\Gamma}^{\mathbf{1}}(-)$), i.e. we get an isomorphism

$$H^q_{\mathrm{II}} \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}}) \cong \mathrm{ind}_{\Gamma}^{\mathbf{1}} H^q_{\mathrm{II}} \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}}; \mathcal{G}^{\bullet_{\mathrm{I}}}).$$

But $H^q\Gamma(\mathfrak{U}_\bullet; \mathcal{G})$ is trivial for any $q > 0$ and any sheaf $\mathcal{G} \cong \mathrm{ex}_* \underline{A}$ of Godement type: Indeed, $H^q\Gamma(\mathfrak{U}_\bullet; \mathcal{G})$ is isomorphic to $H^q(\prod_{x \in X(\Omega)} \mathrm{Hom}_{\underline{\mathrm{Sets}}}(x^*\mathfrak{U}_\bullet, A_x))$, the canonical map

$$H^q(\textstyle\prod_{x \in X(\Omega)} \mathrm{Hom}_{\underline{\mathrm{Sets}}}(x^*\mathfrak{U}_\bullet, A_x)) \hookrightarrow \prod_{x \in X(\Omega)} H^q \mathrm{Hom}_{\underline{\mathrm{Sets}}}(x^*\mathfrak{U}_\bullet, A_x)$$

is injective and the target vanishes, since $x^*\mathfrak{U}. \to \mathrm{pt}$ is a hypercovering in $\underline{\mathrm{Sets}}$, i.e., an acyclic fibration in $\underline{\mathrm{SSets}}$ (see [AM69] (8.5)).

Further, $H^0\Gamma(\mathfrak{V}_\bullet; \mathcal{F})$ equals $\Gamma(X \otimes_k \bar{k}; \mathcal{F})$ for any hypercovering $\mathfrak{V}_\bullet$ of $X \otimes_k \bar{k}$ and any sheaf $\mathcal{F}$ (this is just the sheaf property), so we get:

$$H^p_{\mathrm{I}} H^q_{\mathrm{II}} \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}}) = \begin{cases} 0 \text{ if } q > 0 \\ H^p\Gamma(X \otimes_k \bar{k}, \mathcal{G}^\bullet) = H^p(X \otimes_k \bar{k}; \Lambda) \text{ if } q = 0 \end{cases} .$$

It follows that (2.4.1) induces an isomorphism between the induced spectral sequences of double complexes and hence a quasi-isomorphism

$$\Gamma(X \otimes_k \bar{k}; \mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{tot}^\bullet \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}})$$

between the total complexes. □

Proof of Lem. 2.4.5: The canonical morphism $\Lambda \hookrightarrow \mathcal{G}^\bullet$ induces the morphism of double complexes

$$\Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \Lambda) \longrightarrow \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}}) \ .$$

Now $H^p_{\mathrm{II}} H^q_{\mathrm{I}} \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}})$ equals $H^p_{\mathrm{II}} H^q_{\mathrm{ét}}(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \Lambda)$ and we claim that the colimit over $\mathfrak{U}. \in \underline{\mathrm{HR}}(X)$ vanishes for $q > 0$:

We argue as in the proof of [AM69] Thm. 8.16: We have to kill the class $\alpha \in H^q_{\mathrm{ét}}(\mathfrak{U}_p \otimes_k \bar{k}; \Lambda)$ via a refinement $\mathfrak{V}. \to \mathfrak{U}.$ in $\underline{\mathrm{HR}}(X)$. The hypercovering $\mathfrak{U}.$ has a unique splitting

$$\mathfrak{U}_p \cong \coprod_\sigma N_\sigma,$$

where the coproduct runs over all surjective $\sigma \in \Delta^n_p$ (see the remark after loc. cit. Def. 8.1). Since $q > 0$, there are refinements $N'_\sigma \to N_\sigma$ in $X_{\mathrm{ét}}$ killing the restriction of α to $N_\sigma \otimes_k \bar{k}$ after base change to $X \otimes_k \bar{k}$. Thus we get the desired refinement $\mathfrak{V}. \to \mathfrak{U}.$ after iterative application of loc. cit. Lem. 8.9. Filtered colimits in $\underline{\mathrm{Mod}}_\Gamma$ are exact, so we get a quasi-isomorphism in $\mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$ between total complexes

$$\mathrm{colim}_{\mathfrak{U}.\in \underline{\mathrm{HR}}(X)} \Gamma(\mathfrak{U}_\bullet \otimes_k \bar{k}; \Lambda) \xrightarrow{\sim} \mathrm{colim}_{\mathfrak{U}.\in \underline{\mathrm{HR}}(X)} \mathrm{tot}^\bullet \Gamma(\mathfrak{U}_{\bullet_{\mathrm{II}}} \otimes_k \bar{k}; \mathcal{G}^{\bullet_{\mathrm{I}}})$$

again by the induced spectral sequence of double complexes. □

2.5 Universal coefficients. Fix a discrete Γ-module Λ. Recall that an inner hom object $\underline{\mathrm{Hom}}(-,-)$ exists in $\underline{\mathrm{Mod}}_\Gamma$ and is given by $\mathrm{colim}_\Delta \mathrm{Hom}_{\underline{\mathrm{Mod}}_\Delta}(-,-)$, where the colimit runs over open subgroups $\Delta \leq \Gamma$. In this subsection we want to derive a functor

$$\mathbb{R}\underline{\mathrm{Hom}}(-,\Lambda):\ (\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))^{\mathrm{op}}_{\mathrm{fgb}} \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$$

functorial in $\Lambda \in \underline{\mathrm{Mod}}_\Gamma$ for a suitable full subcategory $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))_{\mathrm{fgb}}$ of $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ containing $C_\bullet(X^\natural \times_{B\Gamma} E\Gamma)$ for all X geometrically unibranched, geometrically irreducible and of Čech type, s.t. $\mathbb{R}\underline{\mathrm{Hom}}(C_\bullet(X^\natural \times_{B\Gamma} E\Gamma), \Lambda)$ is quasi-isomorphic to $C^\bullet(X^\natural \times_{B\Gamma} E\Gamma; \Lambda)$.

Let I be an injective discrete Γ-module. For $\Delta \leq \Gamma$ open of finite index $\mathrm{res}^\Delta_\Gamma(-)$ is also right adjoint to $\mathrm{ind}^\Delta_\Gamma(-)$. In particular, restriction to Δ preserves injective objects, i.e., $\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Delta}(-, \mathrm{res}^\Delta_\Gamma(I))$ and hence $\underline{\mathrm{Hom}}(-, I)$ is an exact functor. Next, fix an injective resolution

$$0 \longrightarrow \Lambda \longrightarrow I^\bullet$$

in $\underline{\mathrm{Mod}}_\Gamma$. Then

$$H^p_{\mathrm{I}} H^q_{\mathrm{II}} \underline{\mathrm{Hom}}(C_{\bullet_{\mathrm{II}}}, I^{\bullet_{\mathrm{I}}}) = H^p_{\mathrm{I}} \underline{\mathrm{Hom}}(H^{\mathrm{II}}_q C_{\bullet_{\mathrm{II}}}, I^{\bullet_{\mathrm{I}}})$$

natural in $C_\bullet$, i.e., the total complex

$$\mathrm{tot}^\bullet \underline{\mathrm{Hom}}((-)_{\bullet_{\mathrm{II}}}, I^{\bullet_{\mathrm{I}}})$$

preserves quasi-isomorphisms by the first spectral sequences of a double complex. As a result, $\mathrm{tot}^\bullet \underline{\mathrm{Hom}}((-)_{\bullet_{\mathrm{II}}}, I^{\bullet_{\mathrm{I}}})$ defines a derived functor

$$\mathbb{R}\underline{\mathrm{Hom}}(-, I^\bullet):\ \mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)^{\mathrm{op}} \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$$

which satisfies $\mathbb{R}^q\underline{\mathrm{Hom}}(M, I^\bullet) = \underline{\mathrm{Ext}}^q(M, \Lambda)$ for all $M \in \underline{\mathrm{Mod}}_\Gamma$ by definition. Further, it comes equipped with a Grothendieck spectral sequence

$$E_2^{p,q} : \mathbb{R}^p\underline{\mathrm{Hom}}(H_q(-), I^\bullet) \Rightarrow \mathbb{R}^{p+q}\underline{\mathrm{Hom}}(-, I^\bullet). \tag{2.5.1}$$

We want to show that $\mathbb{R}\underline{\mathrm{Hom}}(-, I^\bullet)$ is independent of the choice of the injective resolution $0 \to \Lambda \to I^\bullet$ on the full subcategory

$$\mathcal{D}^{\mathrm{fgb}}_+(\underline{\mathrm{Mod}}_\Gamma)$$

of $\mathcal{D}_b(\underline{\mathrm{Mod}}_\Gamma)$ given by all complexes $C_\bullet$ with $H_q C_\bullet$ finitely generated as an abelian group for all q and trivial for all $q \gg 0$. First, we need the following

2.5.1 Lemma. *Any complex in $\mathcal{D}^{\mathrm{fgb}}_{+}(\underline{\mathrm{Mod}}_\Gamma)$ is quasi-isomorphic to a complex which is degreewise free and finitely generated as an abelian group.*

Proof: First assume that $C_\bullet$ is already degreewise finitely generated, say C_q is generated by $a_{q,1}, \ldots, a_{q,r_q}$. Since each C_q is a discrete Γ-module, we may assume that each of these sets of generators is already closed under the Γ-action. Define

$$F_{0,q} := \ \mathbb{Z}[a_{q,i}, da_{q+1,j} | 1 \leq i \leq r_q, 1 \leq j \leq r_{q+1}] \longrightarrow C_q$$

as the canonical Γ-equivariant map to C_q. Mapping each $a_{q,i}$ to $da_{q,i}$ and each $da_{q+1,j}$ to zero gives well defined differentials

$$d_q : \ F_{0,q} \longrightarrow F_{0,q-1}$$

i.e., we get a morphism of complexes

$$F_{0,\bullet} \longrightarrow C_\bullet \ .$$

By defining $F_{1,q}$ as the kernel of $F_{0,q} \to C_q$ and by setting $F_{p,q} = 0$ for $p > 1$ this extends to a morphism of double complexes

$$F_{\bullet_{\mathrm{I}},\bullet_{\mathrm{II}}} \longrightarrow C_{\bullet_{\mathrm{I}}}$$

inducing isomorphisms on $H^{\mathrm{I}}_p H^{\mathrm{II}}_q(-)$. Thus

$$\mathrm{tot}_\bullet(F_{\bullet_{\mathrm{I}},\bullet_{\mathrm{II}}}) \longrightarrow C_\bullet$$

is a quasi-isomorphism with $\mathrm{tot}_\bullet(F_{\bullet_{\mathrm{I}},\bullet_{\mathrm{II}}})$ degreewise free and finitely generated as an abelian group.

It remains to show that an arbitrary $C_\bullet \in \mathcal{D}^{\mathrm{fgb}}_{+}(\underline{\mathrm{Mod}}_\Gamma)$ is quasi-isomorphic to a complex degreewise finitely generated. By replacing $C_\bullet$ with the truncation $\tau_{\leq n} C_\bullet$ for $n \gg 0$ we may assume that C_q is trivial for all $q \gg 0$.

The homology $H_q C_\bullet$ is finitely generated in each degree q. Thus, we may find a $\underline{\mathrm{Mod}}_\Gamma$-subcomplex

$$D^{(0)}_\bullet \hookrightarrow C_\bullet$$

which is degreewise finitely generated and which induces epimorphisms on homology in each degree. Say $D^{(0)}_\bullet \hookrightarrow C_\bullet$ factors over a degreewise finitely generated $\underline{\mathrm{Mod}}_\Gamma$-subcomplex

$$D^{(n)}_\bullet \hookrightarrow C_\bullet$$

which induces epimorphisms on homology in each degree and even isomorphisms in degrees $< n$.

We want to find another degreewise finitely generated subcomplex $D^{(n+1)}_\bullet$ of $C_\bullet$ inducing epimorphisms on homology in each degree and isomorphisms in degrees $\leq n$. Denote by E_n the kernel of the epimorphism

$$\begin{tikzcd} \ker(d_n : D^{(n)}_n \to D^{(n)}_{n-1}) \arrow[r] \arrow[dr] & H_n D^{(n)}_\bullet \arrow[d] \\ & H_n C_\bullet \end{tikzcd} .$$

It is a finitely generated $\underline{\mathrm{Mod}}_\Gamma$-submodule of $D_n^{(n)}$ and lies inside $d_{n+1}C_{n+1}$ by definition. Thus, there is a finitely generated $\underline{\mathrm{Mod}}_\Gamma$-submodule F_{n+1} inside C_{n+1} s.t.

$$E_n = d_{q+1}F_{n+1}.$$

We set $D_q^{(n+1)} := D_q^{(n)}$ for $q \neq n+1$ and $D_{n+1}^{(n+1)}$ the Γ-submodule of C_{n+1} generated by the finitely generated submodules $D_{n+1}^{(n)}$ and F_{n+1}. It follows that $D_\bullet^{(n+1)}$ is still a degreewise finitely generated subcomplex of $C_\bullet$ inducing epimorphisms on homology in each degree but isomorphisms in all degrees $\leq n$.

By assumption C_q is trivial for all $q \gg 0$, i.e. this construction becomes stationary after finitely many steps. In particular,

$$D_\bullet^{(n)} \hookrightarrow C_\bullet$$

is a quasi-isomorphism for all $n \gg 0$, which completes the proof. □

Using this we get:

2.5.2 Corollary. *The restriction of* $\mathbb{R}\underline{\mathrm{Hom}}(-, I^\bullet)$ *to* $\mathcal{D}_+^{\mathrm{fgb}}(\underline{\mathrm{Mod}}_\Gamma)$ *is independent of the choice of the injective resolution* $0 \to \Lambda \to I^\bullet$*. Thus we get*

$$\mathbb{R}\underline{\mathrm{Hom}}(-, \Lambda) : \ \mathcal{D}_+^{\mathrm{fgb}}(\underline{\mathrm{Mod}}_\Gamma)^{\mathrm{op}} \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma),$$

functorial in the discrete Γ*-module* Λ*.*

Proof: Let $C_\bullet \in \mathrm{Ch}_+(\underline{\mathrm{Mod}}_\Gamma)$ be a complex with bounded and degreewise finitely generated homology as abelian groups. By Lem. 2.5.1 and the Grothendieck spectral sequence (2.5.1) we may assume that $C_\bullet$ is degreewise free and finitely generated as an abelian group. But then

$$\underline{\mathrm{Hom}}(C_p, -) = \mathrm{Hom}_{\underline{\mathrm{Ab}}}(C_p, -)$$

is exact and thus

$$\mathrm{tot}^\bullet \underline{\mathrm{Hom}}(C_{\bullet_{\mathrm{II}}}, (-)^{\bullet_{\mathrm{I}}})$$

preserves quasi-isomorphisms by the second spectral sequence of a double complex, which finishes the proof. □

Composing the induced functor on ind-categories

$$(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))^{\mathrm{op}} \longrightarrow \mathrm{Ind}\mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$$

with the exact functor

$$\mathrm{colim} : \ \mathrm{Ind}\mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma) \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$$

gives a functor

$$(2.5.2) \qquad \mathbb{R}\underline{\mathrm{Hom}}(-, I^\bullet) : \ (\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))^{\mathrm{op}} \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma) \ ,$$

whose restriction to the full subcategory $(\mathrm{Pro}\mathcal{D}_+^{\mathrm{fgb}}(\underline{\mathrm{Mod}}_\Gamma))^{\mathrm{op}}$ is independent from the injective resolution $0 \to \Lambda \to I^\bullet$ and functorial in Λ. Let

$$(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))_{\mathrm{fgb}}$$

be the full subcategory of $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ give by (pro-)complexes being quasi-isomorphic in the pro-sense (i.e., connected by a roof of quasi-isomorphisms in the pro-sense) to an object in $\mathrm{Pro}\mathcal{D}_+^{\mathrm{fgb}}(\underline{\mathrm{Mod}}_\Gamma)$.

2.5.3 Corollary. $\mathbb{R}\underline{\mathrm{Hom}}(-, I^\bullet)$ *maps quasi-isomorphisms in the pro-sense to quasi-isomorphisms. Thus, the restriction of* $\mathbb{R}\underline{\mathrm{Hom}}(-, I^\bullet)$ *to* $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))_{\mathrm{fgb}}$ *is independent from the choice of the injective resolution* $0 \to \Lambda \to I^\bullet$ *and we get*

$$\mathbb{R}\underline{\mathrm{Hom}}(-, \Lambda) : \ (\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))_{\mathrm{fgb}}^{\mathrm{op}} \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma),$$

functorial in the discrete Γ*-module* Λ.

Proof: The Grothendieck spectral sequence (2.5.1) induces a functor from the ind-category $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))^{\mathrm{op}}$ to the ind-category of spectral sequences in $\underline{\mathrm{Ab}}$. Since the canonical functor

$$\underline{\mathrm{Func}}(\underline{I}, \underline{\mathrm{Ab}}) \longrightarrow \mathrm{Ind}\underline{\mathrm{Ab}}$$

for a small filtered category $\underline{I}$ is exact (the dual statement holds by [AM69] Appendix Prop. 4.1), we get a canonical functor from the ind-category of spectral sequences in $\underline{\mathrm{Ab}}$ to the category of spectral sequences in $\mathrm{Ind}\underline{\mathrm{Ab}}$. Composition with the functor induced by the exact functor $\mathrm{colim}(-)$ thus gives a spectral sequence of abelian groups

$$E_2^{p,q} : \mathbb{R}^p\underline{\mathrm{Hom}}(H_q C_\bullet, I^\bullet) \Rightarrow \mathbb{R}^{p+q}\underline{\mathrm{Hom}}(C_\bullet, I^\bullet).$$

natural in the (pro-)complex $C_\bullet$. Here $\mathbb{R}^p\underline{\mathrm{Hom}}(-, I^\bullet)$ is just the induced functor (2.5.2) composed with $H^p(-)$. Thus, any quasi-isomorphism of (pro-)complexes induces an isomorphism between these spectral sequences. In particular, quasi-isomorphic (pro-)complexes have quasi-isomorphic images under $\mathbb{R}\underline{\mathrm{Hom}}(-, I^\bullet)$, which completes the proof. $\square$

2.5.4 Remark. For an arbitrary (pro-)chain complex $C_\bullet = \{C_\bullet(i)\}_i$ in the pro-derived category $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ we define the (pro-)chain complex

$$C_\bullet^\sharp := \{\tau_{\leq n} C_\bullet(i)\}_{i,n>0}$$

in $\mathrm{Pro}\mathcal{D}_b(\underline{\mathrm{Mod}}_\Gamma)$. Note that there is a canonical natural quasi-isomorphism in the pro-sense

$$C_\bullet \longrightarrow C_\bullet^\sharp \, .$$

Let X be a geometrically unibranched, geometrically irreducible k-variety of Čech type. Again, we just write $\bar{X}$ for $X^\natural \times_{B\Gamma} E\Gamma$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$. Recall that we defined $\tilde{\mathfrak{X}}$ as the object in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$ given by the functor mapping an étale hypercovering $\mathfrak{U}. \in \underline{\mathrm{HR}}(X)$ à la [AM69] to the set of Zariski connected components $\pi_0(\mathfrak{U}. \otimes_k \bar{k})$ in $\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$. Then $\bar{X}$ is weakly equivalent to $\tilde{\mathfrak{X}}$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$, i.e., $C_\bullet(\bar{X})$ is quasi-isomorphic in the pro-sense to $C_\bullet(\tilde{\mathfrak{X}})$. Now $\pi_0(\mathfrak{U}. \otimes_k \bar{k})$ is levelwise finite since $\mathfrak{U}.$ is levelwise of finite type. In particular, $C_\bullet(\tilde{\mathfrak{X}})^\sharp$ lies in $\mathrm{Pro}\mathcal{D}_+^{\mathrm{fgb}}(\underline{\mathrm{Mod}_\Gamma})$. But then both $C_\bullet(\tilde{\mathfrak{X}})$ and $C_\bullet(\bar{X})$ lie in $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}_\Gamma}))_{\mathrm{fgb}}$.

Now $\mathbb{R}\underline{\mathrm{Hom}}(C_\bullet(\bar{X}), \Lambda)$ is quasi-isomorphic to the colimit over $\underline{\mathrm{HR}}(X)$ of the total complex of $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(C_{\bullet_{\mathrm{II}}}(\tilde{\mathfrak{X}}), I^{\bullet_{\mathrm{I}}})$: Indeed, $C_\bullet(\tilde{\mathfrak{X}})$ is degreewise finitely generated, so $\underline{\mathrm{Hom}}(C_\bullet(\tilde{\mathfrak{X}}), -)$ is just $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(C_\bullet(\tilde{\mathfrak{X}}), -)$. But as in the last subsection, this colimit is quasi-isomorphic to $C^\bullet(\bar{X}; \Lambda)$. Thus, taking everything together, we get:

2.5.5 Corollary. *Let X be a geometrically unibranched and geometrically irreducible k-variety of Čech type. Then the complex $C_\bullet(X^\natural \times_{B\Gamma} E\Gamma)$ lies in the full subcategory $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}_\Gamma}))_{\mathrm{fgb}}$. In particular,*

$$\mathbb{R}\underline{\mathrm{Hom}}(C_\bullet(X^\natural \times_{B\Gamma} E\Gamma), \Lambda)$$

is well defined and natural in $\Lambda \in \underline{\mathrm{Mod}_\Gamma}$. Further, $\mathbb{R}\underline{\mathrm{Hom}}(C_\bullet(X^\natural \times_{B\Gamma} E\Gamma), \Lambda)$ is quasi-isomorphic to $C^\bullet(X^\natural \times_{B\Gamma} E\Gamma, \Lambda)$ in $\mathcal{D}^+(\underline{\mathrm{Mod}_\Gamma})$.

2.6 Eilenberg-MacLane spaces in the pro-homotopy category of simplicial discrete Γ-sets. In this subsection we want to prove that in some sense cohomology of k-varieties with coefficients $\Lambda \in \underline{\mathrm{Mod}_\Gamma}$ is representable in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$:

2.6.1 Lemma. *Let X be a geometrically unibranched, geometrically irreducible k-variety of Čech type and Λ a discrete Γ-module. Then*

$$[X^\natural \times_{B\Gamma} E\Gamma, K(\Lambda, n)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})} = H^n(X; \Lambda).$$

Proof: First, let $C_\bullet$ be an arbitrary complex in $\mathrm{Ch}_+(\underline{\mathrm{Mod}_\Gamma})$. Further, let $I^\bullet$ be an exact and degreewise injective complex in $\mathrm{Ch}^+(\underline{\mathrm{Mod}_\Gamma})$. Then $\mathrm{tot}^\bullet\underline{\mathrm{Hom}}(C_{\bullet_{\mathrm{II}}}, I^{\bullet_{\mathrm{I}}})$ is still exact, since $\underline{\mathrm{Hom}}(C_p, -)$ is left exact. Thus we get the derived functor

$$\mathbb{R}\underline{\mathrm{Hom}}(C_\bullet, -) : \mathcal{D}^+(\underline{\mathrm{Mod}_\Gamma}) \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}_\Gamma}) .$$

By construction, $\mathbb{R}\underline{\mathrm{Hom}}(D_\bullet, -)(\Lambda)$ agrees with $\mathbb{R}\underline{\mathrm{Hom}}(-, \Lambda)(D_\bullet)$ for any $D_\bullet$ in the full triangulated subcategory $\mathcal{D}_+^{\mathrm{fgb}}(\underline{\mathrm{Mod}_\Gamma})$.

Now let $I^\bullet$ be any degreewise injective complex in $\mathrm{Ch}^+(\underline{\mathrm{Mod}_\Gamma})$. Any C_q sits in a resolution

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C_q \longrightarrow 0$$

in $\underline{\mathrm{Mod}_\Gamma}$ with F_i free as an abelian group. But we have already seen that $\underline{\mathrm{Hom}}(-, I^p)$ is exact and $\underline{\mathrm{Hom}}(F_i, I^p)$ is $H^0(\Gamma; -)$-acyclic since $\underline{\mathrm{Hom}}(F_i, -)$ is right adjoint to the exact functor $(-) \otimes_{\mathbb{Z}} F_i$. Thus, $\underline{\mathrm{Hom}}(C_q, I^p)$ is $H^0(\Gamma; -)$-acyclic for all p, q, i.e., $\mathrm{tot}^\bullet\underline{\mathrm{Hom}}(C_{\bullet_{\mathrm{II}}}, I^{\bullet_{\mathrm{I}}})$ is degreewise $H^0(\Gamma; -)$-acyclic. In particular,

$$\mathbb{R}H^0(\Gamma; -) \circ \mathbb{R}\underline{\mathrm{Hom}}(C_\bullet, -) = \mathbb{R}\mathrm{Hom}_{\underline{\mathrm{Mod}_\Gamma}}(C_\bullet, -)$$

holds by [KS06] Prop. 10.3.5 and we get a Grothendieck spectral sequence (the local-to-global spectral sequence)

$$E_2^{p,q} : H^p(\Gamma; \mathbb{R}^q \underline{\mathrm{Hom}}(C_\bullet, -)) \Rightarrow \mathbb{R}^{p+q}\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(C_\bullet, -).$$

Let $C_\bullet$ be a (pro-)complex in $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))_{\mathrm{fgb}}$. Summing up and arguing as in the proofs of Cor. 2.5.3 we get a spectral sequence

$$E_2^{p,q} : H^p(\Gamma; \mathbb{R}^q \underline{\mathrm{Hom}}(C_\bullet, \Lambda)) \Rightarrow \mathbb{R}^{p+q}\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(C_\bullet, \Lambda), \tag{2.6.1}$$

where $\mathbb{R}^q\underline{\mathrm{Hom}}(-, \Lambda)$ is the induced functor

$$(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))_{\mathrm{fgb}} \longrightarrow \mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$$

of the preceding subsection and $\mathbb{R}^q\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(-, \Lambda)$ is the colimit of the respective derived functor applied levelwise. Again, just write $\bar{X}$ for $X^\natural \times_{B\Gamma} E\Gamma$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$. For $C_\bullet = C_\bullet(\bar{X})$, this is just the hypercohomology spectral sequence $\mathbb{H}_*^{\bullet,\bullet}(C^\bullet(\bar{X}, \Lambda))$ (use Cor. 2.5.5), which in turn is isomorphic to the Hochschild-Serre spectral sequence $\mathrm{HS}_*^{\bullet,\bullet}(X; \Lambda)$ by Prop. 2.4.1. As a result,

$$H^q(X; \Lambda) = \mathbb{R}^q\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(C_\bullet(\bar{X}), \Lambda).$$

But the latter group is just

$$[\bar{X}, K(\Lambda, q)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$$

by [Goe95] Lem. 3.13, which completes the proof. □

Taking limits, we get for pro-Eilenberg-MacLane spaces:

2.6.2 Corollary. *Let X be a geometrically unibranched and geometrically irreducible k-variety of Čech type and Λ in $\mathrm{Pro}(\underline{\mathrm{Mod}}_\Gamma)$ a pro-discrete Γ-module. Then*

$$[X^\natural \times_{B\Gamma} E\Gamma, K(\Lambda, n)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} = \lim H^n(X; \Lambda).$$

2.6.3 Notation. *For a cohomology class a in $\lim H^n(X; \Lambda)$ let*

$$\varphi_a : \bar{X} \longrightarrow K(\Lambda, n)$$

be the corresponding morphism in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$.

Combining [Goe95] Lem. 3.13 with the local-to-global spectral sequence (2.6.1) and Lem. 2.5.3 we get the following corollary:

2.6.4 Corollary. *Let Λ be a pro-discrete Γ-module and let $f : \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ be a weak equivalence in the pro-sense. Further, assume that $H_q(\mathfrak{X})$ and $H_q(\mathfrak{Y})$ are finitely generated as abelian groups in each degree q. Then*

$$[f, K(\Lambda, n)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$$

is an isomorphisms between abelian groups.

3 Why it is hard to distinguish étale homotopy types of Brauer-Severi varieties

Let k be a field of characteristic 0. In this section, we want to discuss some of the more easily accessible homotopy invariants of Brauer-Severi varieties.

As an example, cohomology with locally constant coefficients can distinguish between two (Brauer-Severi) varieties of different dimension:

3.0.1 Remark. Let X be a geometrically unibranched and geometrically irreducible proper k-variety of Čech-type. Then the étale homotopy type can detect the dimension of X: Indeed, by the derived base change of Prop. 2.3.10, the étale homotopy type of X can detect the cohomological dimension of the small étale site $(X \otimes_k \bar{k})_{\text{ét}}$. This cohomological dimension is always $\leq 2\dim(X)$ by [SGA73] Exp. X Cor. 4.3. But since X is moreover proper, it is exactly $2\dim(X)$, e.g., by [Mil80] Chapt. VI Lem. 11.3.

Yet, it turns out that many of the more easily accessible homotopy invariants of k-varieties can not distinguish between non isomorphic Brauer-Severi varieties of the same dimension, e.g. it will turn out, that over fields of cohomological dimension ≤ 2 étale cohomology with locally constant coefficients can not distinguish between different Brauer-Severi curves at all (see Cor. 3.3.5 below).

3.1 Galois representations on geometric homotopy invariants. Let X be a Brauer-Severi variety over k. The multiplicative structure on the graded ℓ-adic cohomology ring

$$H^{\bullet}_{\mathrm{T}}(X, \mathbb{Z}_\ell) := \bigoplus_{q \geq 0} H^{2q}(X \otimes_k \bar{k}; \mathbb{Z}_\ell(q))$$

is compatible with the Galois action (the integral Tate realization of the motive of X). As a ring, it is generated by $\hat{c}_1[\mathcal{O}_{\bar{X}}(1)]$ and Γ acts trivially on

$$\mathrm{Pic}(X \otimes_k \bar{k}) = \mathbb{Z},$$

i.e. the Galois structure on $H^{\bullet}_{\mathrm{T}}(X, \mathbb{Z}_\ell)$ is trivial. It follows, that ℓ-adic cohomology can not distinguish between two Brauer-Severi varieties of the same dimension.

In this subsection we will show that this is true for all Γ-representations on geometric homotopy invariants (as e.g. higher homotopy groups): By a geometric homotopy invariant we mean a functor

$$\underline{\mathrm{Var}}_{\bar{k}} \longrightarrow \underline{C}$$

factoring over the homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$. Composition with base extension along $\bar{k}/k$ induces an abstract Γ-representation

$$\underline{\mathrm{Var}}_k \longrightarrow \underline{C}_\Gamma$$

factoring over the functor mapping a k-variety X to the geometric étale homotopy type of X (i.e. the étale homotopy type of $X \otimes_k \bar{k}$) together with the induced

Γ-action as a Γ-object in $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$. We will show that two Brauer-Severi varieties of the same dimension have isomorphic geometric étale homotopy types in the category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})_\Gamma$ of Γ-objects in $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$ (one can see this as the universal Γ-representations on a geometric homotopy invariant).

First we need a few more technicalities:

3.1.1 Lemma. *Let X be a geometrically unibranched proper and simply connected $\bar{k}$-variety and let Y be any normal $\bar{k}$-variety. Then the canonical map*

$$\acute{\mathrm{E}}\mathrm{t}(X \times Y) \longrightarrow \acute{\mathrm{E}}\mathrm{t}(X) \times \acute{\mathrm{E}}\mathrm{t}(Y)$$

is a weak equivalence in $\mathrm{Pro}\underline{\mathrm{SSets}}$.

Proof: The morphism induced on π_q by our canonical morphism sits in the commutative diagram

$$\begin{tikzcd}
\pi_q \acute{\mathrm{E}}\mathrm{t}(X \times Y) \arrow[dr] \arrow[dd] & \\
& \pi_q \acute{\mathrm{E}}\mathrm{t}(X) \times \pi_q \acute{\mathrm{E}}\mathrm{t}(Y) \\
\pi_q(\acute{\mathrm{E}}\mathrm{t}(X) \times \acute{\mathrm{E}}\mathrm{t}(Y)) \arrow[ur] &
\end{tikzcd}$$

and it suffices to show that the other two morphisms in the diagram are isomorphisms.

First, note that all the occurring homotopy groups are already profinite complete: This follows from [Gro65] Prop. 16.15.10 and Thm. 1.3.1 since X resp. Y is geometrically unibranched resp. normal.

For $\pi_q \acute{\mathrm{E}}\mathrm{t}(X \times Y) \to \pi_q \acute{\mathrm{E}}\mathrm{t}(X) \times \pi_q \acute{\mathrm{E}}\mathrm{t}(Y)$ we argue as follows: Denote by F_y the fibre of the projection $X \times Y \to Y$ over $y \in Y(\Omega)$. Thus, the canonical map from F_y to X is just the projection

$$X \otimes_{\bar{k}} \Omega \longrightarrow X\ ,$$

which induces isomorphisms on each $\pi_q(-)$ since X is proper (see [AM69] Cor. 12.12). In particular, F_y is simply connected. Further,

$$\mathrm{pr}_Y :\ X \times Y \longrightarrow Y$$

is proper and thus, from [Fri73] Cor. 4.8 we get the exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_{q+1}(Y) \arrow[r] & \pi_q(F_y) \arrow[r] \arrow[d] & \pi_q(X \times Y) \arrow[r, "\mathrm{pr}_Y"] \arrow[dl, "\mathrm{pr}_X"] & \pi_q(Y) \arrow[r] & \ldots . \\
& & \pi_q(X) & & &
\end{tikzcd}$$

The above isomorphism $\pi_q(F_y) \cong \pi_q(X)$ together with the map induced by pr_X gives a (left-) splitting of the canonical map

$$\pi_q(F_y) \longrightarrow \pi_q(X \times Y) ,$$

i.e.,

$$\pi_q \acute{\mathrm{E}}\mathrm{t}(X \times Y) \longrightarrow \pi_q \acute{\mathrm{E}}\mathrm{t}(X) \times \pi_q \acute{\mathrm{E}}\mathrm{t}(Y)$$

is an isomorphism.

For the isomorphism $\pi_q(\acute{\mathrm{E}}\mathrm{t}(X) \times \acute{\mathrm{E}}\mathrm{t}(Y)) \to \pi_q \acute{\mathrm{E}}\mathrm{t}(X) \times \pi_q \acute{\mathrm{E}}\mathrm{t}(Y)$, note that the composition of the geometric realization $| - |$ as functor $\underline{\mathrm{SSets}} \to \underline{\mathrm{Kel}}$ (where $\underline{\mathrm{Kel}}$ denotes the full subcategory of $\underline{\mathrm{Top}}$ consisting of Kelley spaces) with the singular functor $\mathrm{Sing.}(-)$ gives a functorial fibrant replacement in $\underline{\mathrm{SSets}}$ preserving finite limits (see [GZ67] Chap. III the Thm. in Sect. 3.1). Thus, we may replace $\acute{\mathrm{E}}\mathrm{t}(X)$ and $\acute{\mathrm{E}}\mathrm{t}(Y)$ by levelwise fibrant models and can argue similar to the above using the homotopy sequence of a fibration in $\underline{\mathrm{SSets}}$. $\square$

3.1.2 Lemma. *Let Y be a non empty irreducible $\bar{k}$-variety. Then the canonical map*

$$Y(\bar{k}) \longrightarrow [E\Gamma, Y]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})}$$

is trivial.

3.1.3 Remark. Note that the claim is trivial, if we replace the homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$ with the pro-homotopy category $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$: Indeed, in the latter case we have

$$[\mathrm{pt}, \mathfrak{Y}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})} = \lim \pi_0(\mathfrak{Y}),$$

which is trivial for $\mathfrak{Y} = Y$ by assumption. For a fibrant pro-simplicial set $\mathfrak{Y}$,

$$[\mathrm{pt}, \mathfrak{Y}]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})} = \pi_0(\lim \mathfrak{Y})$$

and more generally for $\mathfrak{Y}$ not necessarily fibrant, at least

$$[\mathrm{pt}, \mathfrak{Y}]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})} = \pi_0(\mathrm{holim}\, \mathfrak{Y})$$

by [Isa01] Prop. 8.4. But π_0 does not preserve (even cofiltered) limits in general.

Thus, the difficulty of the lemma lies in the difference between $\pi_0(\mathrm{holim}\, \mathfrak{Y})$ and $\lim \pi_0(\mathfrak{Y})$, which can be made more precise for $\mathfrak{Y}$ levelwise fibrant using the spectral sequence

$$E_2^{p,q} = \lim{}^p \pi_{-q}(\mathfrak{Y}) \Rightarrow \pi_{-(p+q)}(\mathrm{holim}\, \mathfrak{Y})$$

(with differentials in the usual directions!) in the case of complete convergence (see [BK72] Chapt. XI 7.1). Unfortunately, the π_0 part lies on the "fringed" line of this spectral sequence, so we will avoid using this spectral sequence and make explicit computations with suitable models of Y instead.

Proof: Since Y is connected and quasi-compact, any two $\bar{k}$-points y' and y'' can be connected via a finite chain of open affine subschemes. Thus, we may even assume that y' and y'' factor over an open affine irreducible subscheme. Replacing Y by the normalization of this open affine, we may assume that Y is even geometrically unibranched and of Čech type. In particular, we may work with the Čech topological type $\check{\mathrm{C}}(Y/k)$.

Denote by $S.(-,-)$ the mapping space functor of $\underline{\mathrm{SSets}}$. The canonical map

$$\mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{SSets}}}(\mathrm{pt}, \check{\mathrm{C}}(Y/k)) \longrightarrow [\mathrm{pt}, \check{\mathrm{C}}(Y/k)]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})}$$

factors over the set of homotopy classes of maps $\pi_0(\lim S.(\mathrm{pt}, \check{\mathrm{C}}(Y/k)))$. Thus, a splitting of the canonical weak equivalence $E\Gamma \to \mathrm{pt}$ gives a factorization:

$$\begin{tikzcd} Y(\bar{k}) \arrow[r] \arrow[d] & {[E\Gamma, Y]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})}} \\ \pi_0(\lim S.(\mathrm{pt}, \check{\mathrm{C}}(Y/k))) \arrow[ur] & \end{tikzcd}$$

But $\pi_0(\lim S.(\mathrm{pt}, \check{\mathrm{C}}(Y/k)))$ is just $\pi_0(\lim \check{\mathrm{C}}(Y/k))$ so it suffices to show that the limit of $\check{\mathrm{C}}(Y/k)$ is a connected simplicial set.

First, note that

$$\pi_0(A.) := \mathrm{colim}\{\ A_1 \underset{d_1^1}{\overset{d_1^0}{\rightrightarrows}} A_0\ \}.$$

We get that $\pi_0(\check{\mathrm{C}}(Y/k))$ is levelwise trivial: Indeed, since Y is connected and geometrically unibranched, for arbitrary $y', y'' \in Y(\bar{k}/k)$ and $U \in \underline{\mathrm{RC}}(Y/k)$ there are points $y' = y_0, y_1, \ldots, y_n = y''$ s.t. $U_{y_i} \otimes_Y U_{y_{i+1}}$ is nonempty for all i. Further, the system $\check{\mathrm{C}}(Y/k)$ is constant $Y(\bar{k}/k)$ in degree 0, so we are done if we can show that $\lim \check{\mathrm{C}}(Y/k)_1 \to \pi_0(\mathrm{cosk}_0^Y U)_1$ is surjective for any rigid covering U of Y/k.

We have to show that for any refinement $U \to V$ in $\underline{\mathrm{RC}}(Y/k)$ the induced map

$$\pi_0(U \times_Y U) \longrightarrow \pi_0(V \times_Y V)$$

is surjective. Since Y is geometrically unibranched by assumption, all components of $U \to V$ are dominant. But the composition of dominant morphisms as well as the base change of a dominant morphism along an étale hence flat morphism stays dominant, so

$$U \times_Y U \longrightarrow V \times_Y V$$

is dominant and thus $\pi_0(U \times_Y U) \to \pi_0(V \times_Y V)$ surjective. □

Taking together the last two lemmata we can prove:

3.1.4 Corollary. *Let G be a connected, quasi-projective group scheme over $\bar{k}$. Let $\mu : X \times G \to X$ be a proper, simply connected and geometrically unibranched G-space in $\underline{\mathrm{Var}}_{\bar{k}}$. Then the induced $G(\bar{k})$-action on X is trivial in $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$.*

Proof: Recall that the $G(\bar{k})$-action on X is given by

$$\begin{tikzcd} X = X \times \mathrm{Spec}(\bar{k}) \arrow[r, "\mathrm{id}\times g"] \arrow[rr, bend right, "(-).g"'] & X \times G \arrow[r, "\mu"] & X \end{tikzcd}$$

for $g \in G(\bar{k})$. As a connected, quasi-projective group scheme, G is smooth and hence normal. By Lem. 3.1.1, application of $\acute{\mathrm{E}}\mathrm{t}(-)$ gives

$$\begin{tikzcd} \acute{\mathrm{E}}\mathrm{t}(X) = \acute{\mathrm{E}}\mathrm{t}(X \times \mathrm{Spec}(\bar{k})) \arrow[r, "\acute{\mathrm{E}}\mathrm{t}(\mathrm{id}\times g)"] \arrow[d, "\sim"'] & \acute{\mathrm{E}}\mathrm{t}(X \times G) \arrow[r, "\acute{\mathrm{E}}\mathrm{t}(\mu)"] \arrow[d, "\sim"'] & \acute{\mathrm{E}}\mathrm{t}(X) . \\ \acute{\mathrm{E}}\mathrm{t}(X) \times \acute{\mathrm{E}}\mathrm{t}(\bar{k}) \arrow[r, "\mathrm{id}\times \acute{\mathrm{E}}\mathrm{t}(g)"'] & \acute{\mathrm{E}}\mathrm{t}(X) \times \acute{\mathrm{E}}\mathrm{t}(G) & \end{tikzcd}$$

Now $\acute{\mathrm{E}}\mathrm{t}(g)$ equals $\acute{\mathrm{E}}\mathrm{t}(1)$ by Lem. 3.1.2, so the claim follows. □

The composition of the canonical functor $\mathrm{Pro}\underline{\mathrm{SSets}} \to \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ with the functorial Postnikov tower $(-)^\natural$ à la [AM69] maps weak equivalences to isomorphisms (see Cor 1.5.12), i.e., factors over the homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$. In particular we get:

3.1.5 Corollary. *The* $\mathrm{PGL}_{n+1}(\bar{k})$*-action on* $\bar{\mathbb{P}}^n$ *resp. on* $(\bar{\mathbb{P}}^n)^\natural$ *is trivial in the homotopy category* $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$ *resp. in* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$.

Let X be a Brauer-Severi variety over k of dimension n. Two trivializations

$$f, g : \bar{X} \xrightarrow{\sim} \bar{\mathbb{P}}^n$$

over $\bar{k}$ differ by the element $h = gf^{-1}$ in $\mathrm{PGL}_{n+1}(\bar{k})$. It follows from the last corollary that they agree in $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$ and thus define a canonical isomorphism in $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$. But for any such trivialization f and $\gamma \in \Gamma$, the translate $\gamma \circ f \circ \gamma^{-1}$ is another such trivialization, so our canonical isomorphism is even Γ-equivariant. Thus we have proven:

3.1.6 Proposition. *Let* X, Y *be two Brauer-Severi varieties over* k *of the same dimension. Then the induced* Γ*-objects in the homotopy category* $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})$ *are canonical isomorphic in* $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}})_\Gamma$*. In particular, all Galois representations on geometric homotopy invariants of* X *and* Y *(e.g.,* ℓ*-adic cohomology, higher homotopy groups, …) are canonically isomorphic.*

As we will see in the next subsections, at least for Brauer-Severi curves over characteristic 0 fields of cohomological dimension 2 the situation is even worse: Not only do the Γ-modules $H^\bullet(\bar{X};\Lambda)$ for $\Lambda \in \underline{\mathrm{Mod}}_\Gamma$ turn out to be independent from the Brauer-Severi curve X, but also the much finer datum of the underling complex $C^\bullet(\bar{X};\Lambda)$ in the derived category $\mathcal{D}^b(\underline{\mathrm{Mod}}_\Gamma)$. Note the difference to the much more crude invariant of the corresponding complex $C^\bullet(\bar{X};\Lambda)$ in the derived category $\mathcal{D}^b(\underline{\mathrm{Ab}})_\Gamma$ (of which we already know by the last proposition that it can not distinguish between different Brauer-Severi varieties of the same dimension): An isomorphism between the much finer invariant $C^\bullet((-)\otimes_k \bar{k};\Lambda)$ in $\mathcal{D}^b(\underline{\mathrm{Mod}}_\Gamma)$ e.g. induces an isomorphism of Hochschild-Serre spectral sequences and thus even an isomorphism on the cohomology $H^\bullet(-;\Lambda)$.

3.2 (Quasi) homology fixed points. Let k be a field of characteristic 0 with absolute Galois group Γ and let X be an arbitrary k-variety. As above, we write just $\bar{X}$ for $X^\natural \times_{B\Gamma} E\Gamma$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$.

3.2.1 Definition. *Denote by*

$$\int : C_\bullet(X^\natural \times_{B\Gamma} E\Gamma) \longrightarrow C_\bullet(E\Gamma) = \mathbb{Z}$$

the morphism induced by the canonical map $X \to B\Gamma$. We call a (right-) splitting of $\int$ in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}_\Gamma})$ a ***homology fixed point*** *of X. We call a (right-) splitting up to quasi-isomorphisms in the pro-sense in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}_\Gamma}) \downarrow C_\bullet(E\Gamma)$ a* ***quasi homology fixed point****.*

3.2.2 Remark. If the focus of our interest is the cohomology of X, a quasi homology fixed point is as good as a homology fixed point: Indeed, $\mathbb{R}\underline{\mathrm{Hom}}(-,\Lambda)$ maps quasi-isomorphisms in the pro-sense to quasi-isomorphisms in $\mathcal{D}^+(\underline{\mathrm{Mod}_\Gamma})$ by Cor. 2.5.3.

As usually, we study the set of morphisms $[C_\bullet(E\Gamma), C_\bullet]_{\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}_\Gamma})}$ for a (pro-) chain complex $C_\bullet \in \mathrm{Pro}\mathcal{D}_b(\underline{\mathrm{Mod}_\Gamma})$ by looking at the limit of the (pro-)Galois-hypercohomology of the (pro-)cochain complex $C_{-\bullet}$ in $\mathrm{Pro}\mathcal{D}^b(\underline{\mathrm{Mod}_\Gamma})$. In abuse of notation, we still write $C_\bullet$ for that cochain complex. Now a (right-) splitting of a map $C_\bullet \to C_\bullet(E\Gamma)$ corresponds to a lift of the element 1 in the target of the induced map

$$\lim \mathbb{H}^0(\Gamma; C_\bullet) \longrightarrow \lim \mathbb{H}^0(\Gamma; C_\bullet(E\Gamma)) = \mathbb{Z}.$$

A quasi-isomorphism in the pro-sense induces an isomorphism on (pro-)Galois-hypercohomology groups $\mathbb{H}^\bullet(\Gamma; -)$: Indeed, the induced (pro-) hypercohomology spectral sequences induces spectral sequences in $\mathrm{Pro}\underline{\mathrm{Ab}}$ still computing the pro-Galois-hypercohomology groups (cf. the spectral sequence in the proof of Cor. 2.5.3 for the dual case). But any quasi-isomorphism in the pro-sense in $\mathrm{Pro}\mathcal{D}^b(\underline{\mathrm{Mod}_\Gamma})$ induces an isomorphism between the latter spectral sequences. Thus we get:

3.2.3 Lemma. *A geometrically unibranched k-variety X has a quasi homology fixed point if and only if the canonical map*

$$\int_* : \lim \mathbb{H}^0(\Gamma; C_\bullet(X^\natural \times_{B\Gamma} E\Gamma)^\sharp) \longrightarrow \lim \mathbb{H}^0(\Gamma; C_\bullet(E\Gamma)) = \mathbb{Z}$$

is an epimorphism. If $\mathrm{cd}(\Gamma) < \infty$ this in turn is equivalent to the surjectivity of the canonical map

$$\lim \mathbb{H}^0(\Gamma; \tau_{\leq n} C_\bullet(X^\natural \times_{B\Gamma} E\Gamma)) \longrightarrow \lim \mathbb{H}^0(\Gamma; C_\bullet(E\Gamma))$$

for any $n \geq \mathrm{cd}(\Gamma)$ (here $\tau_{\leq n}$ means the truncation in $\mathcal{D}_+(\underline{\mathrm{Mod}_\Gamma})$ in contrast to the truncation $\tau^{\leq n}$ in $\mathcal{D}^+(\underline{\mathrm{Mod}_\Gamma})$).

3.2.4 Notation. *We call a class $\bar{s}$ in $\lim \mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^\sharp)$ satisfying $\int_* \bar{s} = 1$ the* ***class of a quasi homology fixed point*** *or just quasi homology fixed point. We denote the* ***set of all quasi homology fixed points of X*** *by*

$$\mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^\sharp)_{\int_*=1}.$$

In abuse of notation, we denote the ***set of all homology fixed points of X*** *by*

$$\mathbb{H}^0(\Gamma; C_\bullet(\bar{X}))_{\int_*=1}.$$

Denote the (pro-) hypercohomology spectral sequence $\mathbb{H}_*^{\bullet,\bullet}(C_\bullet(X^\natural \times_{B\Gamma} E\Gamma)^\sharp)$ by $\mathbb{H}_*^{\bullet,\bullet}(X)$ and similar for other Γ-equivariant pro-homotopy types in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$. In abuse of notation, we denote the induced spectral sequence in $\mathrm{Pro}\underline{\mathrm{Ab}}$ computing the (pro-) hypercohomology groups by $\mathbb{H}_*^{\bullet,\bullet}(X)$, as well and similar for other Γ-equivariant pro-homotopy types in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$.

For the remaining section, suppose $\mathrm{cd}(\Gamma) \leq 2$. Consider the induced map of spectral sequences

$$\int_* : \mathbb{H}_*^{\bullet,\bullet}(X) \longrightarrow \mathbb{H}_*^{\bullet,\bullet}(B\Gamma) .$$

To see that a geometrically unibranched k-variety X has a quasi homology fixed point we have to show that this map is an epimorphism on the limits of the abutments. To do this, we may as well replace $\bar{X}$ by the weakly equivalent Γ-equivariant pro-homotopy type $\tilde{\mathfrak{X}}$ given by the functor

$$\underline{\mathrm{HR}}(X_{\text{ét}}) \longrightarrow \mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$$

mapping a hypercovering $\mathfrak{U}. \to X$ to the Zariski connected components of the hypercovering $\mathfrak{U}. \otimes_k \bar{k}$. By the proof of Thm. 1.3.1 our replacement $\tilde{\mathfrak{X}}$ enjoys the advantage of levelwise finite homotopy groups $\pi_q(\tilde{\mathfrak{X}})$ on the nose. Thus, for $q > 0$ the homology groups $H_q(\tilde{\mathfrak{X}})$ are levelwise finite on the nose by [Ser53] Chap. III Thm. 1. Using this replacement, we work with the level representation of the induced pro-spectral sequence $\mathbb{H}_*^{\bullet,\bullet}(\tilde{\mathfrak{X}})$. Now

$$\mathbb{H}^0(\Gamma; C_\bullet(E\Gamma)) = \mathbb{H}_\infty^{0,0}(B\Gamma) = H^0(\Gamma; H_0(E\Gamma))$$

since $E\Gamma \simeq \mathrm{pt}$. In particular $\mathbb{H}^0(\Gamma; C_\bullet(\int))$ factors over $\mathbb{H}_\infty^{0,0}(X)$.

Suppose that $\bar{X}$ is simply connected. For a hypercovering $\mathfrak{U}. \to X$ the fundamental group $\pi_1^{\mathrm{simpl}}(\pi_0^{\mathrm{Zar}}(\mathfrak{U}. \otimes_k \bar{k}))$ classifies étale coverings of $\bar{X}$, trivial over $\mathfrak{U}_0 \otimes_k \bar{k}$. Thus, this fundamental group has to be trivial, i.e., $\tilde{\mathfrak{X}}$ is even levelwise simply connected. In particular, $H^2(\Gamma; \pi_1^{\mathrm{ab}}(\tilde{\mathfrak{X}}))$ is even levelwise trivial on the nose. Further, $H_q(\tilde{\mathfrak{X}})$ for $q > 0$ are levelwise finite and $\mathrm{cd}(\Gamma) \leq 2$, so for $q \geq 2$ $H^{q+1}(\Gamma; H_q(\tilde{\mathfrak{X}}))$ is levelwise trivial on the nose, as well.

Thus, we get that $\mathbb{H}_\infty^{0,0}(\tilde{\mathfrak{X}})$ equals $H^0(\Gamma; H_0(\tilde{\mathfrak{X}}))$. In particular, we get that our map $\mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^\sharp) \to \mathbb{Z}$ is at least isomorphic to the canonical levelwise surjection

$$\mathbb{H}^0(\Gamma; C_\bullet(\tilde{\mathfrak{X}})^\sharp) \longrightarrow \mathbb{H}_\infty^{0,0}(\tilde{\mathfrak{X}}) . \tag{3.2.1}$$

We apply this to prove the following proposition:

3.2.5 Proposition. *Let k be a field of characteristic 0 and cohomological dimension ≤ 2 and let X be geometrically unibranched and geometrically simply connected proper k-variety. Then $\bar{X}$ has a quasi homology fixed point.*

Proof: We have to show that the limit of the map of pro-abelian groups

$$\int_* : \mathbb{H}^0(\Gamma; C_\bullet(\tilde{\mathfrak{X}})^\sharp) \longrightarrow \mathbb{Z}$$

is an epimorphism. As we have just seen, this is equivalent to the surjectivity of the limit over the canonical levelwise surjection (3.2.1). Thus, we are done if we can show that the kernel of the map (3.2.1) is a Mittag-Leffler system.

First, note that

$$\mathbb{H}_\infty^{1,-1}(\tilde{\mathfrak{X}}) = \mathbb{H}_2^{1,-1}(\tilde{\mathfrak{X}})$$

is trivial, since $\tilde{\mathfrak{X}}$ is levelwise simply connected and $H_2(\tilde{\mathfrak{X}})$ levelwise finite. Further,

$$\mathbb{H}_\infty^{2,-2}(\tilde{\mathfrak{X}}) = \mathbb{H}_2^{2,-2}(\tilde{\mathfrak{X}}) = H^2(\Gamma; H_2(\tilde{\mathfrak{X}})).$$

Indeed, $H^0(\Gamma; \pi_1^{\mathrm{ab}}(\tilde{\mathfrak{X}}))$ vanishes again since $\tilde{\mathfrak{X}}$ is levelwise simply connected and $H^4(\Gamma; H_3(\tilde{\mathfrak{X}}))$ vanishes since $\mathrm{cd}(\Gamma) \leq 2$. For the same reason plus the levelwise finiteness of $H_3(\tilde{\mathfrak{X}})$

$$\mathbb{H}_\infty^{q,-q}(\tilde{\mathfrak{X}}) = \mathbb{H}_2^{q,-q}(\tilde{\mathfrak{X}})$$

vanishes for $q > 2$, as well. Hence

$$0 \longrightarrow H^2(\Gamma; H_2(\tilde{\mathfrak{X}})) \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(\tilde{\mathfrak{X}})^\sharp) \longrightarrow \mathbb{H}_\infty^{0,0}(\tilde{\mathfrak{X}}) \longrightarrow 0$$

is levelwise exact, i.e. it suffices to see that $H^2(\Gamma; H_2(\tilde{\mathfrak{X}}))$ is isomorphic to a Mittag-Leffler pro-system. To do this, we may again replace $H^2(\Gamma; H_2(\tilde{\mathfrak{X}}))$ by the isomorphic original pro-system $H^2(\Gamma; H_2(\bar{X}))$.

We claim that $H_2(\bar{X})$ is the profinite completion of a finitely generated group. As X can be defined over a finitely generated field extension over $\mathbb{Q}$, it can be defined over $\mathbb{C}$, as well. Combining this with [AM69] Cor. 12.12, we may assume that k is $\mathbb{C}$. As X^{an} is a compact topological manifold, $H_2(X^{\mathrm{an}})$ is finitely generated and our claim follows by Lem. 3.2.6 below applied to the profinite completion $X^{\mathrm{an}} \to X$ (see Thm. 1.3.5).

Profinite completion is a left adjoint and thus preserves colimits. In particular, we can write $H_2(\bar{X})$ as a system with surjective transition maps. But $\mathrm{cd}(\Gamma) \leq 2$, so $H^2(\Gamma; -)$ is right exact on torsion modules. Thus, $H^2(\Gamma; H_2(\bar{X}))$ can still be written as a system with surjective transition maps, i.e., is a Mittag-Leffler system, which completes the proof. $\square$

3.2.6 Lemma. *Let $\mathfrak{X}$ be a pro-homotopy type in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\bullet})$. Then $H_q(\mathfrak{X}^\wedge)$ is profinite for $q > 0$ and the map $H_q(\mathfrak{X}) \to H_q(\mathfrak{X}^\wedge)$ induced by the profinite completion of pro-homotopy types $\mathfrak{X} \to \mathfrak{X}^\wedge$ is the profinite completion of pro-groups.*

Proof: We may assume that each level of $\mathfrak{X}^\wedge$ has finite homotopy groups. We claim that any connected simplicial set $A.$ having this property has finite homology $H_q(A.)$ in degrees $q > 0$ (in particular, $H_q(\mathfrak{X}^\wedge)$ is profinite in these degrees). If $A.$ is simply connected, this follows from [Ser53] Chap. III Thm. 1. Thus, for $A.$ not necessary simply connected at least the universal covering space $\tilde{A}.$ has finite homology $H_q(\tilde{A}.)$ in degrees $q > 0$. But then the Serre spectral sequence

$$E^2_{p,q} = H_p(\pi_1(A., a), H_q(\tilde{A}.)) \Rightarrow H_{p+q}(A.)$$

together with the finiteness of higher homology of finite groups with finitely generated coefficients (see e.g. [Wei94] Cor. 6.5.10) give the claim.

We have a canonical isomorphism $H^q(A.; \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\underline{\mathrm{Ab}}}(H_q(A.), \mathbb{Q}/\mathbb{Z})$ for any simplicial set $A.$ by the universal coefficients theorem ($\mathbb{Q}/\mathbb{Z}$ is injective in $\underline{\mathrm{Ab}}$). The profinite completion $\mathfrak{X} \to \mathfrak{X}^\wedge$ induces the commutative diagram

$$\begin{tikzcd} H^q(\mathfrak{X}^\wedge; \mathbb{Q}/\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X}^\wedge), \mathbb{Q}/\mathbb{Z}) \arrow[d] \\ H^q(\mathfrak{X}; \mathbb{Q}/\mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X}), \mathbb{Q}/\mathbb{Z}) \end{tikzcd}$$

where the left vertical map is an isomorphism by Thm. 1.1.8. Now $H_q(\mathfrak{X}^\wedge)$ is profinite by the above, so its Pontryagin dual $\mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X}^\wedge), \mathbb{Q}/\mathbb{Z})$ is torsion. Thus, each $\alpha \in \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X}), \mathbb{Q}/\mathbb{Z})$ factors over $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ for suitable $n \neq 0$. As a result

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X}), \mathbb{Q}/\mathbb{Z}) &= \mathrm{colim}_n \, \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X}), \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \\ &= \mathrm{colim}_n \, \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X})^\wedge, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \ , \\ &= \mathrm{Hom}_{\mathrm{Pro}\underline{\mathrm{Ab}}}(H_q(\mathfrak{X})^\wedge, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

i.e., $H_q(\mathfrak{X})^\wedge \to H_q(\mathfrak{X}^\wedge)$ induces an isomorphism of Pontryagin duals, i.e., is itself an isomorphism of profinite abelian groups. □

3.3 Quasi homology fixed points of Brauer-Severi varieties.

We apply Prop. 3.2.5 to Brauer-Severi varieties. All the statements of this sections are trivial for $\mathrm{cd}(\Gamma) < 2$ (since the Brauer group $\mathrm{Br}(k)$ is trivial in this case), so we assume $\mathrm{cd}(\Gamma) = 2$.

3.3.1 Corollary. *Let k be a field of characteristic 0 and cohomological dimension 2. Then every Brauer-Severi variety over k admits a quasi homology fixed point.*

In the case of Brauer-Severi curves we get even more. Note that the pullback along $X \to B\Gamma$ for X a Brauer-Severi variety induces an equivalence between $\underline{\mathrm{Mod}}_\Gamma$ and the category of local systems on X.

3.3.2 Theorem. *Let k be a field of characteristic 0 and cohomological dimension 2. Let X and Y be two Brauer-Severi curves over k and let Λ be a Γ-module.Then we get an isomorphism*

$$C^\bullet(\bar{X}; \Lambda) \simeq C^\bullet(\bar{Y}; \Lambda)$$

in the derived category $\mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$ *which agrees in* $\mathcal{D}^+(\underline{\mathrm{Ab}})_\Gamma$ *with the canonical isomorphism given by any* $\underline{\mathrm{Var}}_{\bar{k}}$*-isomorphism* $\bar{X} \cong \bar{Y}$ *(see Prop. 3.1.6).*

Before we give the proof, we need another technical lemma:

3.3.3 Lemma. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be two objects in* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ *with* $\mathrm{res}^{\mathbf{1}}_\Gamma \mathfrak{X}$ *and* $\mathrm{res}^{\mathbf{1}}_\Gamma \mathfrak{Y}$ *profinite complete. Further, assume that there is a* Γ*-equivariant isomorphism of pro-abelian groups*

$$\mathrm{res}^{\mathbf{1}}_\Gamma H_n(\mathfrak{X}) \cong \mathrm{res}^{\mathbf{1}}_\Gamma H_n(\mathfrak{Y})$$

for an $n > 0$*. Then this is even an isomorphism in* $\mathrm{Pro}(\underline{\mathrm{Mod}}_\Gamma)$*. In particular, we can check if a* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ *morphism*

$$f: \mathfrak{X} \longrightarrow \mathfrak{Y}$$

induces isomorphism on higher homology pro-groups after forgetting the Γ*-action.*

Proof: By assumption, we even have an isomorphism $\mathrm{res}^{\mathbf{1}}_\Gamma H_n(\mathfrak{X}) \cong \mathrm{res}^{\mathbf{1}}_\Gamma H_n(\mathfrak{Y})$ in $(\mathrm{Pro}\underline{\mathrm{Ab}}^{\mathrm{fin}})_\Gamma$. Pontryagin duality applied twice gives the commutative diagram of functors

$$\begin{tikzcd}
\mathrm{Pro}(\underline{\mathrm{Mod}}^{\mathrm{fin}}_\Gamma) \arrow[r, "\mathrm{can}"] \arrow[d, leftrightarrow, "\sim"] & (\mathrm{Pro}\underline{\mathrm{Ab}}^{\mathrm{fin}})_\Gamma \arrow[d, leftrightarrow, "\sim"] \\
\underline{\mathrm{Mod}}^{\mathrm{tors}}_\Gamma \arrow[r, "\mathrm{can}"] & (\underline{\mathrm{Ab}}^{\mathrm{tors}})_\Gamma
\end{tikzcd}$$

where the lower horizontal arrow is the canonical full embedding of torsion discrete Γ-modules in arbitrary torsion Γ-modules. In particular, the upper horizontal arrow is a full embedding, as well. But the Pontryagin dual of $H_n(\mathfrak{X})$ resp. $H_n(\mathfrak{Y})$ is just $H^n(\mathfrak{X};\mathbb{Q}/\mathbb{Z})$ resp. $H^n(\mathfrak{Y};\mathbb{Q}/\mathbb{Z})$ and thus already lies in $\underline{\mathrm{Mod}}^{\mathrm{tors}}_\Gamma$, which completes the proof. $\square$

Proof of Thm. 3.3.2: X has a quasi homology fixed point by the Cor. 3.3.1, i.e., we get a $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$-morphism

$$\mathbb{Z} \cong C_\bullet(E\Gamma) \longrightarrow \tau_{\leq 2} C_\bullet(\bar{X})$$

by Lem. 3.2.3. The induced morphism is an isomorphism on the 0th homology and is trivial on all higher (pro-)homologie groups. The canonical morphism

$$\tau_{\geq 2}\tau_{\leq 2} C_\bullet(\bar{X}) \longrightarrow \tau_{\leq 2} C_\bullet(\bar{X})$$

induces an isomorphism on the 2nd homology and is trivial on all the other homologie (pro-)groups. Further, by Prop. 3.1.6 any choice of a $\underline{\mathrm{Var}}_{\bar{k}}$-isomorphism

$$X \otimes_k \bar{k} \cong \mathbb{P}^1 \otimes_k \bar{k}$$

gives a canonical isomorphism

$$H_2(\bar{X}) = H_2(X \otimes_k \bar{k}) \longrightarrow H_2(\mathbb{P}^n \otimes_k \bar{k}) = \hat{\mathbb{Z}}(1)$$

in $(\mathrm{Pro}\underline{\mathrm{Ab}}^{\mathrm{fin}})_\Gamma$. Thus, $H_2(\bar{X})$ is even canonical isomorphic to $\hat{\mathbb{Z}}(1)$ in $\mathrm{Pro}(\underline{\mathrm{Mod}}_\Gamma)$ by Lem. 3.3.3, i.e. in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ we get

$$\tau_{\geq 2}\tau_{\leq 2}C_\bullet(\bar{X}) = \hat{\mathbb{Z}}(1)[-2].$$

All in all we have a zig-zag

$$\begin{tikzcd} & C_\bullet(E\Gamma) \oplus \tau_{\leq 2}\tau_{\geq 2}C_\bullet(\bar{X}) \arrow[dl] \arrow[dr, "\cong"] & \\ \tau_{\leq 2}C_\bullet(\bar{X}) & & \mathbb{Z} \oplus \hat{\mathbb{Z}}(1)[-2] \end{tikzcd} \tag{3.3.1}$$

in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ which induces isomorphisms on all the (pro-) homology groups (it induces even an isomorphism in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ by Rem. 3.3.4 below - but we will not need this). Clearly we get a similar zig-zag for Y as well.

In terms of Galois-hypercohomology the forgetful map on Hom-sets is just the restriction map from Γ to the trivial group:

$$\begin{tikzcd} {[C_\bullet(E\Gamma), \tau_{\leq 2}C_\bullet(\bar{X})]_{\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[r, equal] \arrow[d] & \lim \mathbb{H}^0(\Gamma; \tau_{\leq 2}C_\bullet(\bar{X})) \arrow[d, "\mathrm{res}^\Gamma_{\mathbf{1}}"] \\ {[C_\bullet(E\Gamma), \tau_{\leq 2}C_\bullet(\bar{X})]_{(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}}))_\Gamma}} \arrow[r, equal] & \lim \mathbb{H}^0(\mathbf{1}, \tau_{\leq 2}C_\bullet(\bar{X})) \end{tikzcd}$$

Now $\mathbb{H}^0(\mathbf{1}, \tau_{\leq 2}C_\bullet(\bar{X}))$ is isomorphic to the $E_\infty^{0,0}$-term in the hypercohomology spectral sequence and thus there is a unique (right-) splitting of the canonical morphism

$$\tau_{\leq 2}C_\bullet(\bar{X}) \longrightarrow C_\bullet(E\Gamma)$$

in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}})$. In particular this splitting is even Γ-equivariant and coincides with the $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}}))_\Gamma$-morphism induced by any choice of a quasi homology fixed point of X. All in all we get that all choices of quasi homology fixed points of X and Y are compatible in $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}}))_\Gamma$ with the canonical isomorphism given by any choice of a $\mathrm{Var}_{\bar{k}}$-isomorphism $X \otimes_k \bar{k} \cong Y \otimes_k \bar{k}$. A consequence of this is that the above zig-zags of X and Y are compatible with this canonical isomorphism as well.

The above zig-zags lie in the full subcategory $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma))_{\mathrm{fgb}}$ and their right ends are independent from X resp. Y. Thus we get an isomorphism between $C^\bullet(\bar{X};\Lambda)$ and $C^\bullet(\bar{Y};\Lambda)$ in $\mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$ by Cor. 2.5.3 and Cor. 2.5.5 (the truncation $\tau_{\leq 2}C_\bullet(\bar{X})$ is quasi-isomorphic in the pro-sense to $C_\bullet(\bar{X})$ and similar for Y). Further, the last constructions are compatible with the forgetful functor

$$\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma) \longrightarrow (\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}}))_\Gamma$$

resp.

$$\mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma) \longrightarrow \mathcal{D}^+(\underline{\mathrm{Ab}})_\Gamma$$

and our zig-zags are compatible in $(\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Ab}}))_\Gamma$ with our canonical isomorphism given by Lem. 3.1.6. As a result our $\mathcal{D}^+(\underline{\mathrm{Mod}}_\Gamma)$-isomorphism

$$C^\bullet(\bar{X};\Lambda) \simeq C^\bullet(\bar{Y};\Lambda)$$

agrees in $\mathrm{Pro}\mathcal{D}^{+}(\underline{\mathrm{Ab}})_{\Gamma}$ with the canonical isomorphism induced by any $\underline{\mathrm{Var}}_{\bar{k}}$-isomorphism

$$X \otimes_k \bar{k} \cong Y \otimes_k \bar{k}$$

(see Prop. 3.1.6), which completes the proof. □

3.3.4 Remark. It follows from Lem. 4.3.4 below, that $\bar{X}$ is isomorphic to a suitable levelwise simply connected pro-simplicial discrete Γ-set $\bar{\mathfrak{X}}$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_{\Gamma}})$. It follows that

$$C_{\bullet}(E\Gamma) \oplus \tau_{\geq 2}\tau_{\leq 2}C_{\bullet}(\bar{\mathfrak{X}}) \longrightarrow \tau_{\leq 2}C_{\bullet}(\bar{\mathfrak{X}})$$

is even an isomorphism in $\mathrm{Pro}\mathcal{D}_{+}(\underline{\mathrm{Mod}_{\Gamma}})$. As a result, the analog zig-zag for $\bar{\mathfrak{X}}$ isomorphic to (3.3.1) is even a $\mathrm{Pro}\mathcal{D}_{+}(\underline{\mathrm{Mod}_{\Gamma}})$-isomorphism, i.e.

$$\tau_{\leq 2}C_{\bullet}(\bar{X}) \cong \mathbb{Z} \oplus \hat{\mathbb{Z}}(1)[-2].$$

By Prop. 2.4.1 we get:

3.3.5 Corollary. *Let k be a field of characteristic 0 and cohomological dimension 2. Let X and Y be two Brauer-Severi curves over k and let Λ be a Γ-module. Then we get a canonical isomorphism of Hochschild-Serre spectral sequences*

$$\mathrm{HS}_{*}^{\bullet,\bullet}(X;\Lambda) = \mathrm{HS}_{*}^{\bullet,\bullet}(Y;\Lambda)$$

in which the isomorphisms on the E_2-tableau are just the canonical isomorphisms induced by any $\underline{\mathrm{Var}}_{\mathrm{k}}$-isomorphism $\bar{X} \cong \bar{Y}$. In Particular $H^{\bullet}(X;\Lambda)$ is isomorphic to $H^{\bullet}(Y;\Lambda)$ under $H^{\bullet}(k;\Lambda)$.

Proof: Combining Prop. 2.4.1 and Thm. 3.3.2 we get an isomorphism

$$\mathrm{HS}_{*}^{\bullet,\bullet}(X;\Lambda) \cong \mathrm{HS}_{*}^{\bullet,\bullet}(Y;\Lambda)$$

canonical up to the choice of the quasi homology fixed points of X and Y. But since the isomorphisms on the E_2-tableau are just the canonical isomorphisms given by any $\underline{\mathrm{Var}}_{\mathrm{k}}$-isomorphism $\bar{X} \cong \bar{Y}$, our isomorphism is independent from this choice, which completes the proof. □

4 Homotopy rational points and homotopy or (quasi) homology fixed points of Brauer-Severi varieties

In this section we want to discuss homotopy rational, fixed and (quasi) homology fixed points of Brauer-Severi varieties and their connection with rational points.

4.1 Homotopy rational points and homotopy fixed points. Somewhat similar to a (quasi-) homology fixed points we define homotopy rational points and homotopy fixed points:

4.1.1 Definition. *Let X be a k-variety and $\mathfrak{X}$ an equivariant pro-homotopy type in* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$.

(i) *A **homotopy rational point** of X is a morphisms $B\Gamma \to X$ in the relative homotopy category $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma)$.*

(ii) *A **homotopy fixed point** of $\mathfrak{X} \in \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ is an element of*

$$[\mathrm{pt}, \mathfrak{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} = [E\Gamma, \mathfrak{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}.$$

(iii) *A **homotopy fixed point** of X is a homotopy fixed point of*

$$\bar{X} = X^\natural \times_{B\Gamma} E\Gamma.$$

4.1.2 Remark. In [Goe95] Def. 2.1 Goerss defines a homotopy fixed point functor

$$(-)^{h\Gamma} : \mathcal{H}(\underline{\mathrm{SSets}}_\Gamma) \longrightarrow \mathcal{H}(\underline{\mathrm{SSets}}) .$$

Levelwise application induces the homotopy fixed point functor

$$(-)^{h\Gamma} : \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma) \longrightarrow \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}) .$$

Unraveling the definitions, one sees that the limit over $\pi_0(\mathfrak{X}^{h\Gamma})$ is nothing but $[E\Gamma, \mathfrak{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$, the set of homotopy fixed points of $\mathfrak{X}$. Thus, we treat $\pi_0(\mathfrak{X}^{h\Gamma})$ as $[E\Gamma, \mathfrak{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$ enriched with the structure of a pro-set.

4.1.3 Remark. The equivariant derived base change in Prop. 2.3.11 induces a canonical map from homotopy rational points to homotopy fixed points:

$$[B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)} \longrightarrow [E\Gamma, \bar{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} .$$

Further, mapping a pro-homotopy type $\mathfrak{X} \in \mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ to the (pro-)chain complex $C_\bullet(\mathfrak{X})$ defines a canonical map from homotopy fixed points to homology fixed points:

$$[E\Gamma, \mathfrak{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(\mathfrak{X}))_{\int_*=1} .$$

Finally, $(-)^\sharp$ induces a canonical map from homology fixed points to quasi homology fixed points:

$$\mathbb{H}^0(\Gamma; C_\bullet(\mathfrak{X}))_{\int_*=1} \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(\mathfrak{X})^\sharp)_{\int_*=1} .$$

Let us also mention the canonical map of pro-sets

$$\pi_0(\mathfrak{X}^{h\Gamma}) \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(\mathfrak{X})^\sharp) ,$$

whose limit factors over the canonical map from the set of homotopy fixed points $[E\Gamma, \mathfrak{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$ to the set of quasi homology fixed points $\mathbb{H}^0(\Gamma; C_\bullet(\mathfrak{X})^\sharp)_{\int_*=1}$.

4.1.4 Remark. For $\mathrm{cd}(\Gamma) \leq n$ and Λ a (pro-)finite module in $\underline{\mathrm{Mod}}_\Gamma$ there is a nice description of the canonical map

$$\pi_0(K(\Lambda, n)^{h\Gamma}) \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(K(\Lambda, n))^\sharp) .$$

Note that there is a homotopy fixed point of $K(\Lambda, n)$ admitting a model

$$\bar{r} : \ E\Gamma \longrightarrow K(\Lambda, n)$$

in $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma)$ (e.g., take the actual fixed point corresponding to the zero map to $\Lambda[-n]$). Using the functorial standard cone in $\mathrm{Ch}_+(\underline{\mathrm{Mod}}_\Gamma)$, we can define the reduced homology (pro-) chains

$$\tilde{C}_\bullet(K(\Lambda, n), \bar{r}).$$

The truncation $\tau_{\leq n}\tilde{C}_\bullet(K(\Lambda, n), \bar{r})$ is canonically levelwise quasi-isomorphic to the image of $K(\Lambda, n)$ under the Dold-Kan correspondence, i.e. we get a canonical isomorphisms of pointed (pro-)sets

$$\begin{aligned}\pi_0(K(\Lambda, n)^{h\Gamma}, \bar{r}) &= (\mathrm{Hom}_{\mathcal{D}_b(\underline{\mathrm{Mod}}_\Gamma)}(\mathbb{Z}, \tau_{\leq n}\tilde{C}_\bullet(K(\Lambda, n), \bar{r})), 0)\\ &= (\mathbb{H}^0(\Gamma; \tilde{C}_\bullet(K(\Lambda, n), \bar{r})^\sharp), 0)\end{aligned}$$

from the left adjointness of $\mathbb{Z}[-]$ and the Dold-Kan correspondence. By the Galois-hypercohomology sequence of $\tilde{C}_\bullet(K(\Lambda, n), \bar{r})^\sharp$ we get

$$\mathbb{H}^0(\Gamma; \tilde{C}_\bullet(K(\Lambda, n), \bar{r})^\sharp) = \mathbb{H}^0(\Gamma; \tau_{\leq n}\tilde{C}_\bullet(K(\Lambda, n), \bar{r})).$$

Combining this with the long exact sequence linking the hypercohomology of $\tilde{C}_\bullet(K(\Lambda, n), \bar{r})^\sharp$ to the one of $C_\bullet(K(\Lambda, n))^\sharp$, it is not hard to see that

$$\mathbb{H}^0(\Gamma; C_\bullet(K(\Lambda, n))^\sharp) = \pi_0(K(\Lambda, n)^{h\Gamma}, \bar{r}) \oplus \mathbb{H}^0(\Gamma; C_\bullet(E\Gamma))$$

and that the map $\pi_0(K(\Lambda, n)^{h\Gamma}, \bar{r}) \to \mathbb{H}^0(\Gamma; C_\bullet(K(\Lambda, n))^\sharp)$ is just $\mathrm{id} \oplus \mathbf{1}$. In particular, we get:

$$[E\Gamma, K(\Lambda, n)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} = \mathbb{H}^0(\Gamma; C_\bullet(K(\Lambda, n))^\sharp)_{\int_*=1}.$$

Rem. 4.1.4 shows the importance of a model of a homotopy fixed point in $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma)$. The existence of such a model is not at all clear for an arbitrary homotopy fixed point in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$. The problem is that we work in a pro-homotopy category instead of a genuine homotopy category. In particular, a map in such a pro-homotopy category is given by a compatible system of homotopy commutative diagrams. Since the index category of these compatible systems usually are infinite, there is no reason for a compatible system of commutative lifts of these diagrams to exist. Thus, the following simple observation will be convenient in the following subsection:

4.1.5 Remark. Let $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$ be a homotopy rational point of a k-variety X. Replacing $X \to B\Gamma$ by a fibrant replacement $\mathfrak{X} \twoheadrightarrow B\Gamma$, we see that s has a model

$$\mathfrak{s}: \ B\Gamma \longrightarrow \mathfrak{X}$$

in the model category $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma$. In particular, the induced homotopy fixed point $\bar{s} \in [E\Gamma, \bar{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$ has a model

$$\bar{\mathfrak{s}}: \ E\Gamma \longrightarrow \bar{\mathfrak{X}}$$

in $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma)$. See Cor. 4.3.5 below for a refinement of this observation for a careful choice of the fibrant replacement $\mathfrak{X} \twoheadrightarrow B\Gamma$ of X moreover a geometrically simply connected k-variety.

4.1.6 Remark. Let Λ be a discrete Γ-module and $f: X \to Y$ a morphisms of geometrically connected and geometrically unibranched k-varieties of Čech type. Using Lem. 2.6.1, the adjointness properties of both the free abelian group functor $\mathbb{Z}[-]$ and the truncation $\tau_{\leq n}(-)$ for $n \geq q$ together with the Dold-Kan correspondence gives the commutative diagram

$$\begin{tikzcd}
H^q(Y;\Lambda) \arrow[r, "f^*"] \arrow[d, equal] & H^q(X;\Lambda) \arrow[d, equal] \\
{[\bar{Y}, K(\Lambda,q)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}} \arrow[r, "f^*"] \arrow[d, equal] & {[\bar{X}, K(\Lambda,q)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}} \arrow[d, equal] \\
{[C_\bullet(\bar{Y}), \Lambda[-q]]_{\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[r, "f^*"] \arrow[d, equal] & {[C_\bullet(\bar{X}), \Lambda[-q]]_{\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[d, equal] \\
{[C_\bullet(\bar{Y})^\sharp, \Lambda[-q]]_{\mathrm{Pro}\mathcal{D}_b(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[r, "f^*"] & {[C_\bullet(\bar{X})^\sharp, \Lambda[-q]]_{\mathrm{Pro}\mathcal{D}_b(\underline{\mathrm{Mod}}_\Gamma)}}
\end{tikzcd}$$

In particular, any homotopy or (quasi) homology fixed point $\bar{s}$ induces a (left-) splitting

$$\bar{s}^*: \ H^q(X;\Lambda) \longrightarrow H^q(\Gamma;\Lambda)$$

of the canonical map

$$H^q(\Gamma;\Lambda) \longrightarrow H^q(X;\Lambda) \ ,$$

which particularly, is a monomorphism.

Finally, let us also mention the following observation:

4.1.7 Remark. Let A and B be central simple algebras over k. We get the twisted Segre embedding

$$s_{A,B}: \ X_A \times X_B \longrightarrow X_{A\otimes B}$$

between the corresponding Brauer-Severi varieties (cf. [Art82] (4.1)). Consider the composition

$$X_A \xrightarrow{\mathrm{diag}} X_A^{\times n} \xrightarrow{\mathrm{id}^{\times(n-2)} \times s_{A,A}} X_A^{\times(n-2)} \times X_{A^{\otimes 2}} \xrightarrow{\mathrm{id}^{\times(n-3)} \times s_{A,A^{\otimes 2}}} \dots \xrightarrow{s_{A,A^{\otimes(n-1)}}} X_{A^{\otimes n}} \ .$$

Since this composition induces maps on homotopy rational points and homotopy or (quasi) homology fixed points, $X_{A^{\otimes n}}$ has a homotopy rational, homotopy or (quasi) homology fixed point as soon as X_A has one.

4.2 The first non trivial step of the Postnikov tower and homotopy fixed points. In this subsection we want to give a Γ-equivariant construction of the first non trivial step in the Postnikov tower of an m-connected (pro-) simplicial discrete Γ-set for $m \geq 1$ adapted to a given (model of a) homotopy fixed point. Note that analogue constructions work for a (model of a) (quasi) homology fixed point, as well.

4.2.1 Lemma. *Let $A.$ be an m-connected and $E.$ a contractible simplicial discrete Γ-set in $\underline{\mathrm{SSets}}_\Gamma$ for $m \geq 1$ together with a model $a : E. \to A.$ of a homotopy fixed point in $\pi_0(A.^{h\Gamma})$. Then the canonical map*

$$A. \longrightarrow \mathrm{cosk}_{m+2}\mathrm{Ex}(A.)$$

for $\mathrm{Ex}(-)$ a functorial fibrant replacement in $\underline{\mathrm{SSets}}$ is isomorphic in $\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ to a canonical map

$$\varphi_a : \; A. \longrightarrow K(\tilde{H}_{m+1}(A., a), m+1)$$

functorial in models of homotopy fixed points $a : E. \to A.$ such that $\varphi_a \circ a$ factors over the zero map into $K(\tilde{H}_{m+1}(A., a), m+1)$.

Proof: Using the functorial standard construction of a cone in $\mathrm{Ch}_+(\underline{\mathrm{Mod}}_\Gamma)$ (e.g., see [GM03] Chapt. III Sect. 2), we get a distinguished triangle

$$C_\bullet(E.) \xrightarrow{a_*} C_\bullet(A.) \longrightarrow \tilde{C}_\bullet(A., a) \xrightarrow{-1} C_\bullet(E.)[-1]$$

in $\mathrm{Ch}_+(\underline{\mathrm{Mod}}_\Gamma)$ functorial in morphisms $E. \to A.$ (i.e. models of homotopy fixed points). As usually we identify the category of simplicial discrete Γ-modules with the category of chain complexes in $\underline{\mathrm{Mod}}_\Gamma$ via the Dold-Kan correspondence (for the classical case see e.g. [GJ99] Chapt. III.2). We get the composition of canonical maps

$$\begin{tikzcd} A. \arrow[r, "\text{can.}"] \arrow[drrr] & \mathbb{Z}[A.] \arrow[r, "\cong"] & C_\bullet(A.) \arrow[r] & \tilde{C}_\bullet(A., a) \arrow[d] \\ & & & \tau_{\leq m+1}\tilde{C}_\bullet(A., a) \end{tikzcd}$$

in $\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$. Since $A.$ is m-connected, the truncation

$$\tilde{H}_{m+1}(A., a)[-(m+1)] = \; \tau_{\geq m+1}\tau_{\leq m+1}\tilde{C}_\bullet(A., a) \longrightarrow \tau_{\leq m+1}\tilde{C}_\bullet(A., a)$$

is an isomorphism in $\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ by Hurewicz (e.g., see [GJ99] Chapt. III Sect. 3). Let φ_a be the resulting $\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$-map

$$A. \longrightarrow K(\tilde{H}_{m+1}(A., a), m+1) \; ,$$

i.e. the functoriality claim and $a^*\varphi_a = 0$ hold by construction.

Again by Hurewicz, $\pi_q(\varphi_a)$ is an isomorphism for all $q \leq m$. In particular, φ_a is a $\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$-isomorphism for $A.$ an Eilenberg-MacLane space $K(\pi, m+1)$. As $A.$ is m-connected, $\mathrm{cosk}_{m+2}\mathrm{Ex}(A.)$ is an Eilenberg-MacLane space. Thus by functoriality of φ_a, we get a commutative diagram

$$\begin{tikzcd}
 & E. \arrow[dl, "a"'] \arrow[dr, "=:b"] & \\
A. \arrow[rr] \arrow[d, "\varphi_a"'] & & \mathrm{cosk}_{m+2}\mathrm{Ex}(A.) \arrow[d, "\varphi_b"', "\simeq"] \\
K(\tilde{H}_{m+1}(A., a), m+1) \arrow[rr, "\mathrm{can.}"] & & K(\tilde{H}_{m+1}(\mathrm{cosk}_{m+2}\mathrm{Ex}(A.), b), m+1)
\end{tikzcd}$$

in $\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$. The lower horizontal arrow is an isomorphism, which gives the desired isomorphism between the canonical map $A. \to \mathrm{cosk}_{m+2}\mathrm{Ex}(A.)$ and φ_a. $\square$

Let $b : E. \to A.$ be a second model of a homotopy fixed point in $\pi_0(A.^{h\Gamma})$. We get a commutative diagram

(4.2.1)

$$\begin{tikzcd}
\tau_{\leq m+1}\tilde{C}_\bullet(A., a) & & \tau_{\leq m+1}C_\bullet(A.) \arrow[ll] \arrow[rr] \arrow[dll, "\varphi_a"'] \arrow[drr, "\varphi_b"] & & \tau_{\leq m+1}\tilde{C}_\bullet(A., b) \\
\tau_{\geq m+1}\tau_{\leq m+1}\tilde{C}_\bullet(A., a) \arrow[u, "\simeq", "\mathrm{can.}"'] & & \tau_{\geq m+1}\tau_{\leq m+1}C_\bullet(A) \arrow[ll, "\simeq"', "h_a :="] \arrow[rr, "\simeq", "h_b :="'] \arrow[u, "c :=", "\mathrm{can.}"'] & & \tau_{\geq m+1}\tau_{\leq m+1}\tilde{C}_\bullet(A., b) \arrow[u, "\simeq", "\mathrm{can.}"']
\end{tikzcd}$$

in $\mathcal{D}_+(\underline{\mathrm{Mod}_\Gamma})$. Further, denote by $\mathrm{pr}_{A.}$ the canonical map $A. \to \mathrm{pt}$.

4.2.2 Lemma. *Let $A.$ be an m-connected and $E.$ a contractible simplicial discrete Γ-set in $\underline{\mathrm{SSets}_\Gamma}$ for $m \leq 1$ together with two models $a, b : E. \rightrightarrows A.$ of homotopy fixed points in $\pi_0(A.^{h\Gamma})$. Then we get*

$$\varphi_b = (h_b \circ h_a^{-1})_*(\varphi_a) + (\varphi_b \circ a \circ \mathrm{pr}_{A.}).$$

In particular, φ_a and φ_b are isomorphic via a $\mathcal{H}(\underline{\mathrm{SMod}_\Gamma})$-morphism, if

$$a^*\varphi_b = b^*\varphi_a = 0$$

and we get $(h_b^{-1})_*(\varphi_b)$ out of $(h_a^{-1})_*(\varphi_a)$ via an isomorphic affine linear cohomology operation on $H^{m+1}(-; H_{m+1}(A.))$ with constant $a^*((h_b^{-1})_*(\varphi_b))$ induced by the canonical map of $H^{m+1}(\Gamma; H_{m+1}(A.))$ into $H^{m+1}(-; H_{m+1}(A.))$ in general.

Proof: The second and third claims follow directly from the first claim. It remains to proof the first claim. Using adjointness properties of both the free abelian group functor $\mathbb{Z}[-]$ and the truncation $\tau_{\leq m+1}(-)$ together with the Dold-Kan correspondence translates our problem to the corresponding problem for the maps in (4.2.1).

First, we claim that

$$c \circ h_a^{-1} \circ \varphi_a = \mathrm{id} - (a \circ \mathrm{pr}_{A.}) \tag{4.2.2}$$

holds on $\tau_{\leq m+1}C_\bullet(A.)$ in $\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$. To see this, recall the distinguished triangle

$$\mathbb{Z} \xrightarrow{a} C_\bullet(A.) \longrightarrow \tilde{C}_\bullet(A., a) \xrightarrow{-1} \mathbb{Z}[-1]$$

in $\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$. From this we get the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[d, "-1"] & \\
{[\tau_{\leq m+1}C_\bullet(A.), \mathbb{Z}]_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[r, "a_*"] & {[\tau_{\leq m+1}C_\bullet(A.), \tau_{\leq m+1}C_\bullet(A.)]_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[d, "(\varphi_a)_*"] \\
{[\tau_{\leq m+1}C_\bullet(A.), \mathbb{Z}[-1]]_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[d, "a_*"] & {[\tau_{\leq m+1}C_\bullet(A.), \tau_{\leq m+1}\tilde{C}_\bullet(A., a)]_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}} \arrow[l, "-1"'] \\
\cdots &
\end{tikzcd}$$

Now the maps induced by the canonical map $\mathrm{pr}_{A.} : A. \to \mathrm{pt}$ split the maps induced by a. Further, note that

$$\mathrm{Hom}_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}(\tau_{\leq m+1}C_\bullet(A.), \mathbb{Z}[-1]) = \mathbb{R}^1\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(\tau_{\leq m+1}C_\bullet(A.), \mathbb{Z})$$

is trivial: Indeed, this follows from the hypercohomology spectral sequences

$$E_2^{p,q} = \mathbb{R}^p\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(H_q(\tau_{\leq m+1}C_\bullet(A.)), \mathbb{Z}) \Rightarrow \mathbb{R}^{p+q}\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(\tau_{\leq m+1}C_\bullet(A.), \mathbb{Z})$$

since

$$\mathrm{Hom}_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}(H_1(A.), \mathbb{Z}) = 0$$

($A.$ is simply connected so $H_1(A.) = 0$) and

$$\mathbb{R}^1\mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(H_0(A.), \mathbb{Z}) = H^1(\Gamma; \mathbb{Z}) = 0$$

(Γ is profinite and $\mathbb{Z}$ torsion free). Summing up we get a direct sum decomposition of $[\tau_{\leq m+1}C_\bullet(A.), \tau_{\leq m+1}C_\bullet(A.)]_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}$ into

$$[\tau_{\leq m+1}C_\bullet(A.), \mathbb{Z}]_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)} \oplus [\tau_{\leq m+1}C_\bullet(A.), \tau_{\leq m+1}\tilde{C}_\bullet(A., a)]_{\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)}$$

via $(\mathrm{pr}_{A.})_* \oplus (\varphi_a)_*$. Thus, it suffices to show (4.2.2) after application of $(\mathrm{pr}_{A.})_*(-)$ resp. $(\varphi_a)_*(-)$.

We start with $(\mathrm{pr}_{A.})_*(-)$: We have the commutative diagram

$$\begin{tikzcd}
\tau_{\geq m+1}\tau_{\leq m+1}C_\bullet(A.) \arrow[r] \arrow[d, "c"'] & \tau_{\geq m+1}\mathbb{Z} \arrow[d, "\mathrm{can.}"] \\
\tau_{\leq m+1}C_\bullet(A.) \arrow[r, "\mathrm{pr}_{A.}"] & \mathbb{Z}
\end{tikzcd}$$

and $\tau_{\geq m+1}\mathbb{Z}$ is trivial, so $\mathrm{pr}_{A.} \circ c \circ h_a^{-1} \circ \varphi_a$ is trivial. But $(\mathrm{pr}_{A.})_*(-)$ applied to the right hand side of (4.2.2) is trivial, as well, i.e. $(\mathrm{pr}_{A.})_*(-)$ applied to (4.2.2) holds.

On the other hand $\varphi_a \circ c$ is just h_a, so $(\varphi_a)_*(-)$ applied to the left hand side of (4.2.2) is just φ_a. But $\varphi_a \circ a$ factors over the zero map, so $(\varphi_a)_*(-)$ applied to the right hand side of (4.2.2) is φ_a, as well and $(\varphi_a)_*(-)$ applied to (4.2.2) holds.

Using (4.2.2) we finally can compute:

$$\begin{aligned} h_b \circ h_a^{-1} \circ \varphi_a &= \varphi_b \circ c \circ h_a^{-1} \circ \varphi_a \\ &= \varphi_b \circ (\mathrm{id} - (a \circ \mathrm{pr}_{A.})) \\ &= \varphi_b - (\varphi_b \circ a \circ \mathrm{pr}_{A.}), \end{aligned}$$

which completes the proof of the first claim. $\square$

4.2.3 Remark. There is an alternative way to proof Lem. 4.2.2: Recall that

$$[A., K(\Lambda, n)]_{\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} = \mathbb{R}^n \mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(C_\bullet(A.), \Lambda)$$

(see [Goe95] Lem. 3.13). Arguing similar as in the proof of Lem. 4.3.1 below using the hypercohomology spectral sequences

$$E_2^{p,q} = \mathbb{R}^p \mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(H_q(A.), \Lambda) \Rightarrow \mathbb{R}^{p+q} \mathrm{Hom}_{\underline{\mathrm{Mod}}_\Gamma}(C_\bullet(A.), \Lambda)$$

we get the direct sum decomposition

$$[A., K(\Lambda, n)]_{\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} = [E., K(\Lambda, n)]_{\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} \oplus [A., K(\Lambda, n)]_{\mathcal{H}(\underline{\mathrm{SSets}})},$$

where the projections onto the two summands are given by a^* resp. $\mathrm{res}_\Gamma^{\mathbf{1}}(-)$. Now it is not hard to see that

$$\mathrm{res}_\Gamma^{\mathbf{1}}(h_a^{-1} \circ \varphi_a) = \mathrm{res}_\Gamma^{\mathbf{1}}(h_b^{-1} \circ \varphi_b) :$$

Indeed, as morphisms in $\underline{\mathrm{SSets}}$ with target $K(H_{m+1}(A.), m+1)$, it suffices to check this after application of $H^{m+1}(-, H_{m+1}(A.))$. But since $A.$ is m-connected, this follows from

$$H_{m+1}(\mathrm{res}_\Gamma^{\mathbf{1}}(h_a^{-1} \circ \varphi_a)) = H_{m+1}(\mathrm{res}_\Gamma^{\mathbf{1}}(h_b^{-1} \circ \varphi_b)),$$

which in turn holds by construction of φ_a resp. φ_b. In particular, $h_a^{-1} \circ \varphi_a$ differs from $h_b^{-1} \circ \varphi_b$ by

$$(a \circ \mathrm{pr}_{\mathrm{A.}})^*((h_b^{-1} \circ \varphi_b) - (h_a^{-1} \circ \varphi_a)) = h_b^{-1} \circ \varphi_b \circ a \circ \mathrm{pr}_{A.},$$

which completes the proof.

4.2.4 Remark. Let $A.$ and $E.$ be as in Lem. 4.2.1 and 4.2.2. Say we have a commutative diagram

$$\begin{tikzcd} E. \arrow[r, "a'"] \arrow[d, "f"] & A. \arrow[d, "g"] \\ E. \arrow[r, "a''"] & A. \end{tikzcd}$$

in $\underline{\mathrm{SSets}}_\Gamma$ with $f \simeq \mathrm{id}_{E.}$ resp. $g \simeq \mathrm{id}_{A.}$. By the functionality of our cone construction, we get a map between distinguished triangles

$$\begin{tikzcd} \tau_{\leq m+1}C_\bullet(E.) \arrow[r, "a'_*"] \arrow[d, "f_*"] & \tau_{\leq m+1}C_\bullet(A.) \arrow[r] \arrow[d, "g_*"] & \tau_{\leq m+1}\tilde{C}_\bullet(A., a') \arrow[r, "-1"] \arrow[d, "=:h"] & \tau_{\leq m+1}C_\bullet(E.)[-1] \arrow[d, "f_*[-1]"] \\ \tau_{\leq m+1}C_\bullet(E.) \arrow[r, "a''_*"] & \tau_{\leq m+1}C_\bullet(A.) \arrow[r] & \tau_{\leq m+1}\tilde{C}_\bullet(A., a'') \arrow[r, "-1"] & \tau_{\leq m+1}C_\bullet(E.)[-1] \end{tikzcd}$$

in $\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$, with $f_* \equiv \mathrm{id}_{C_\bullet(E.)}$ resp. $g_* \equiv \mathrm{id}_{C_\bullet(A.)}$. Further, we get a commutative diagram

$$\begin{tikzcd} \tau_{\leq m+1}\tilde{C}_\bullet(A., a') \arrow[d, "h"] & \tau_{\geq m+1}\tau_{\leq m+1}\tilde{C}_\bullet(A., a') \arrow[l, "\text{can.}", "\simeq"'] \arrow[d, "h"] & H^{m+1}(A.)[-(m+1)] \arrow[l, "h_{a'}", "\simeq"'] \arrow[d, "H_{m+1}(g)"] \\ \tau_{\leq m+1}\tilde{C}_\bullet(A., a'') & \tau_{\geq m+1}\tau_{\leq m+1}\tilde{C}_\bullet(A., a'') \arrow[l, "\text{can.}", "\simeq"'] & H^{m+1}(A.)[-(m+1)] \arrow[l, "h_{a''}", "\simeq"'] \end{tikzcd}$$

in $\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ and $H_{m+1}(g)$ is just the identity on $H_{m+1}(A.)$ by assumption. As a result, $h_a^{-1} \circ \varphi_a$ is independent from homotopy equivalences. In particular, we may replace $A.$ by a fibrant resolution and thus may assume that a model of a given homotopy fixed point of $A.$ always exists.

Levelwise application of Lem. 4.2.1 and Lem. 4.2.2 gives the corresponding statements in the pro-sense (we use analog notation as in the pro-discrete case above):

4.2.5 Corollary. *Let $\bar{s} : E\Gamma \to \mathfrak{X}$ be a model of a homotopy fixed point in* $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma)$ *with $\mathfrak{X}$ levelwise m-connected for an $m \geq 1$. Then there is a levelwise morphism*

$$\varphi_{\bar{s}} : \mathfrak{X} \longrightarrow K(H_{m+1}(\mathfrak{X}), m+1)$$

in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ *inducing levelwise isomorphisms $\pi_q(\varphi_{\bar{s}})$ for all $q \leq m+1$. This morphism is natural in the category* $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma) \uparrow E\Gamma$ *of models of homotopy fixed points. Further, $\varphi_{\bar{s}}$ is isomorphic to the canonical map*

$$\mathfrak{X} \longrightarrow \mathrm{cosk}_{m+2}\mathrm{Ex}(\mathfrak{X})$$

for $\mathrm{Ex}(-)$ *a functorial fibrant replacement in* $\underline{\mathrm{SSets}}$ *and the pull back $\bar{s}^*\varphi_{\bar{s}}$ factors over the zero map into $K(H_{m+1}(\mathfrak{X}), m+1)$.*

4.2.6 Corollary. *Let $\bar{s}, \bar{r} : E\Gamma \rightrightarrows \mathfrak{X}$ be two models of homotopy fixed points in* $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma)$ *with $\mathfrak{X}$ levelwise m-connected for an $m \geq 1$. Then we get*

$$\varphi_{\bar{r}} = (h_{\bar{r}} \circ h_{\bar{s}}^{-1})_*(\varphi_{\bar{s}}) + (\varphi_{\bar{r}} \circ \bar{s} \circ \mathrm{pr}_{\mathfrak{X}}).$$

In particular, $\varphi_{\bar{s}}$ and $\varphi_{\bar{r}}$ are isomorphic via a $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SMod}}_\Gamma)$*-morphism, if*

$$\bar{s}^*\varphi_{\bar{r}} = \bar{r}^*\varphi_{\bar{s}} = 0$$

and we get $(h_{\bar{r}}^{-1})_*(\varphi_{\bar{r}})$ out of $(h_{\bar{s}}^{-1})_*(\varphi_{\bar{s}})$ via an isomorphic affine linear cohomology operation on $H^{m+1}(-; H_{m+1}(\mathfrak{X}))$ with constant $\bar{s}^*((h_{\bar{r}}^{-1})_*(\varphi_{\bar{r}}))$ induced by the canonical map of $H^{m+1}(\Gamma; H_{m+1}(\mathfrak{X}))$ into $H^{m+1}(-; H_{m+1}(\mathfrak{X}))$ in general.

4.2.7 Remark. Note that the proof of Cor. 4.2.5 fails if we start with $\bar{s}$ only a homotopy fixed point or even a (quasi) homology fixed point without a model in $\mathrm{Pro}(\underline{\mathrm{Mod}}_\Gamma)$ resp. $\mathrm{ProCh}_+(\underline{\mathrm{Mod}}_\Gamma)$: Indeed, there is no functorial way to get a cone in $\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$, so there is no reason for the levelwise cones of our map

$$C_\bullet(E\Gamma) \longrightarrow C_\bullet(\mathfrak{X})$$

to determine a pro-object $\tilde{C}_\bullet(\mathfrak{X}, \bar{s})$ in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ for complicated index categories. But even if the index category of $\mathfrak{X}$ is simple enough to permit the construction of a (pro-) chain complex $\tilde{C}_\bullet(\mathfrak{X})$ (e.g., if $\mathfrak{X}$ is induced by a tower of simplicial sets), there is no reason to expect functoriality in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma) \uparrow E\Gamma$ to hold.

4.3 k-structures of $\hat{c}_1[\mathcal{O}(1)]$.

The connecting homomorphisms

$$\delta_{\mathrm{Kum}}: \; H^1(Y; \mathbb{G}_{\mathrm{m}}) \longrightarrow H^2(Y; \mu_m)$$

induced by the Kummer sequences for various m form a compatible system and thus give a natural map

$$\hat{c}_1 = \hat{c}_1(Y): \; \mathrm{Pic}(Y) \longrightarrow H^2(Y; \hat{\mathbb{Z}}(1)) \; ,$$

the **profinite first Chern class map**. For $[\mathcal{L}] \in \mathrm{Pic}(Y)$ we obtain a morphism

$$\varphi_{\hat{c}_1[\mathcal{L}]}: \bar{Y} = \; Y^\natural \times_{B\Gamma} E\Gamma \longrightarrow K(\hat{\mathbb{Z}}(1), 2)$$

in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ via Lem. 2.6.1.

Let X be a Brauer-Severi variety over k. In abuse of notation, we call a class of $H^2(X; \hat{\mathbb{Z}}(1))$ resp. of $\lim H^2(X; \hat{\mathbb{Z}}(1))$ mapping to the Chern class of $\mathcal{O}_{X \otimes_k \bar{k}}(1)$ in $H^2(\bar{X}; \hat{\mathbb{Z}}(1))$ resp. $\lim H^2(\bar{X}; \hat{\mathbb{Z}}(1))$ a ***k*-structure of $\hat{\mathbf{c}}_\mathbf{1}[\boldsymbol{\mathcal{O}}(\mathbf{1})]$**. Thus, a k-structure of $\hat{c}_1[\mathcal{O}(1)]$ is nothing but a compatible system of maps

$$\bar{X} \longrightarrow K(\mu_m, 2)$$

resp. a map

$$\bar{X} \longrightarrow K(\hat{\mathbb{Z}}(1), 2)$$

in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$, whose restriction to $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$ is just the compatible system of maps resp. the map given by $\hat{c}_1[\mathcal{O}_{X \otimes_k \bar{k}}(1)]$.

4.3.1 Lemma. *Let X be a Brauer-Severi variety over k admitting a homotopy or (quasi) homology fixed point $\bar{s}$. Let Λ be a pro-object of $\underline{\mathrm{Mod}}_\Gamma$. Then we get a direct sum decomposition of $[\bar{X}, K(\Lambda, 2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$ into*

$$[E\Gamma, K(\Lambda, 2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} \oplus [\bar{X}, K(\Lambda, 2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})},$$

where the projections onto the two summands correspond to $\bar{s}^*$ resp. $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$.

Proof: Let $p: \bar{X} \to X$ be the canonical projection. Consider the Hochschild-Serre spectral sequence $\mathrm{HS}_*^{\bullet,\bullet}(X;\Lambda)$ for $\Lambda \in \underline{\mathrm{Mod}}_\Gamma$ pro-discrete: We claim that the differential

$$\partial_3^{0,2}: \mathrm{HS}_3^{0,2}(X;\Lambda) \longrightarrow \mathrm{HS}_3^{3,0}(X;\Lambda)$$

is trivial: Indeed, the canonical map $H^3(\Gamma;\Lambda) \to H^3(X;\Lambda)$ factors as follows:

$$\begin{tikzcd} H^3(\Gamma;\Lambda) \arrow[r] \arrow[d, equal] & H^3(X;\Lambda) \\ \mathrm{HS}_2^{3,0}(X;\Lambda) \arrow[r, two heads] & \mathrm{HS}_\infty^{3,0}(X;\Lambda) \arrow[u, hook] \end{tikzcd}$$

By Rem. 4.1.6, any homotopy or (quasi) homology fixed point $\bar{s}$ gives a (left-) splitting of this map, i.e. the lower horizontal arrow is an isomorphism which forces the image of $\partial_3^{0,2}$ to be trivial (for general k-varieties, this argument would give $\partial_2^{1,1} = 0$, as well).

Thus, for Λ not necessarily pro-discrete we get from the Hochschild-Serre spectral sequence the levelwise exact sequence of pro-abelian groups

$$(4.3.1) \qquad 0 \longrightarrow H^2(\Gamma;\Lambda) \longrightarrow H^2(X;\Lambda) \xrightarrow{p^*} H^2(\bar{X};\Lambda) \longrightarrow 0$$

and any homotopy or (quasi) homology fixed point $\bar{s}$ gives levelwise compatible splittings, i.e. a compatible levelwise direct sum decomposition:

$$H^2(X;\Lambda) = H^2(\Gamma;\Lambda) \oplus H^2(\bar{X};\Lambda),$$

where the projections onto the two summands are given by $\bar{s}^*$ resp. p^*. Finite products and finite sums agree in $\underline{\mathrm{Ab}}$, so this decomposition is preserved by limits (use [AM69] Appendix Prop. 4.1). Thus, by taking limits and translating this back via Lem. 2.6.1 we get a decomposition of $[\bar{X}, K(\Lambda,2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})}$ into

$$[E\Gamma, K(\Lambda,2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})} \oplus [\bar{X}, K(\Lambda,2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})}$$

and the projections onto the two summands correspond to $\bar{s}^*$ resp. $\mathrm{res}_\Gamma^{\mathbf{1}}(-)$. $\square$

4.3.2 Remark. By Lem. 4.3.1, any Brauer-Severi variety admitting a homotopy or (quasi) homology fixed point admits k-structures of $\hat{c}_1[\mathcal{O}(1)]$. If k is even of cohomological dimension ≤ 2, then $\mathrm{HS}_\infty^{3,0}(X;\Lambda)$ is trivial and $\partial_3^{0,2}$ vanishes for any (Brauer-Severi) variety. Thus for $\mathrm{cd}(k) \leq 2$, any Brauer-Severi variety admits k-structures of $\hat{c}_1[\mathcal{O}(1)]$.

4.3.3 Notation. *Let X be a Brauer-Severi variety admitting a homotopy or (quasi) homology fixed point $\bar{s}$. Denote by $\alpha_{\bar{s}}$ the unique k-structure of $\hat{c}_1[\mathcal{O}(1)]$ in $H^2(X;\hat{\mathbb{Z}}(1))$ killed by $\bar{s}^*$. We just write α_s if $\bar{s}$ is even induced by the homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$.*

Say X admits a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. In the following we want to relate α_s to the classes constructed in Sect. 4.2.

First, we have to make sure that Cor. 4.2.5 and Cor. 4.2.6 applies to suitable models of homotopy fixed points induced by homotopy rational points of a Brauer-Severi variety X resp. a geometrically unibranched and geometrically simply connected k-variety Y of Čech-type in general (cf. Rem. 4.1.5). Thus, we have to make sure that there is a fibrant replacement for $Y \to B\Gamma$ whose base change to $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma)$ is levelwise simply connected:

4.3.4 Lemma. *Let Y be a geometrically unibranched and geometrically simply connected k-variety of Čech-type. Then there is a fibrant replacement*

$$\begin{tikzcd} Y \arrow[rr, hook, "\sim"] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\ & B\Gamma & \end{tikzcd}$$

in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma$ *with* $\bar{\mathfrak{Y}} = \mathfrak{Y}^\natural \times_{B\Gamma} E\Gamma$ *levelwise simply connected.*

Proof: We may assume that Y is reduced. Since Y is of Čech type, we may work with the relative Čech topological type

$$\check{\mathrm{C}}(Y/k) \longrightarrow B\Gamma \ .$$

Recall that $\check{\mathrm{C}}(Y/k)$ is levelwise connected. Let U be a rigid covering in $\underline{\mathrm{RC}}(Y/k)$. Note that $\pi_1(\pi_0(\mathrm{cosk}_0^{\mathrm{Y}}\mathrm{U}), y)$ is finite for any $y \in Y(\bar{k}/k)$: Indeed, since Y is geometrically unibranched, we may apply [AM69] Lem. 11.6 to see that every zig-zag from U_y to U_y in the path category $P_*(\pi_0(\mathrm{cosk}_0^{\mathrm{Y}}\mathrm{U}))$ can be represented by one of the finitely many connected components of $U_y \times_Y U_y$.

Denote by L_y the relative algebraic closure of k inside $k(U_y)$. Then U_y admits a L_y structure

$$U_y \longrightarrow \mathrm{Spec}(L_y) \ .$$

Indeed, this is true if and only if the étale covering space

$$U \otimes_k L_y \longrightarrow U$$

has a section. But this clearly holds for the open normal locus $V \hookrightarrow U$ and $V \otimes_k L_y \hookrightarrow U \otimes_k L_y$ is dominant since Y is geometrically unibranched. It follows, that the restriction of $U \otimes_k L_y \to U$ to at least one of the connected components is an isomorphism, so we get our section.

The distinguished geometric point u_y in $U_y(\bar{k}/k)$ gives L_y/k the structure of a rigid covering in $\underline{\mathrm{RC}}(k/k)$. Note, that this is the maximal subextension L/k of $\bar{k}/k$ such that U_y admits an L-structure. Let L_U be the intersection in $\bar{k}$ of all the rigid coverings L_y/k over all the geometric points y in $Y(\bar{k}/k)$. This gives a rigid covering L_U/k in $\underline{\mathrm{RC}}(k/k)$ together with a canonical map of rigid coverings

$$U \longrightarrow f^*(L_U/k)$$

for $f : Y \to \mathrm{Spec}(k)$ the structural map. In particular, we get a canonical map

$$\pi_0(\mathrm{cosk}_0^Y U)) \longrightarrow \pi_0(\mathrm{cosk}_0^k L_U) \tag{4.3.2}$$

representing our map $f : \check{\mathrm{C}}(Y/k) \to B\Gamma$. By construction, this map is maximal among all the maps

$$\pi_0(\mathrm{cosk}_0^Y U)) \longrightarrow \pi_0(\mathrm{cosk}_0^k L)$$

in our representation of the strict morphism f.

Let $L^{(0)}/k$ is the maximal Galois subextension of an algebraic extension L/k. For K/k finite algebraic $L \otimes_k K$ splits completely if and only if K is contained in $L^{(0)}$. Further, all the $L_y^{(0)}$ contain $L_U^{(0)}$ and the intersection of all the $L_y^{(0)}$ is again Galois over k, so this intersection is just $L_U^{(0)}$.

Let $Z \to Y$ be any étale covering space. Since Y is geometrically simply connected, it is just the pull back of a finite extension K/k. In particular, U trivializes Z (i.e. $U \times_Y Z \cong U \otimes F$ for a finite set F) if and only if K is contained inside $L_y^{(0)}$ for any $y \in Y(\bar{k}/k)$, i.e. if and only if K lies inside $L_U^{(0)}$. But this holds if and only if $L_U \otimes_k K$ splits completely. Thus, our canonical map (4.3.2) induces an isomorphism on the first non abelian cohomology

$$H^1(\pi_0(\mathrm{cosk}_0^k L_U); S_F) \longrightarrow H^1(\pi_0(\mathrm{cosk}_0^Y U); S_F)$$

for the symmetric group S_F for F any finite set (see Rem. 1.2.1). But both $\pi_0(\mathrm{cosk}_0^k L_U)$ and $\pi_0(\mathrm{cosk}_0^Y U)$ have finite fundamental groups, so the induced map

$$\pi_1(\pi_0(\mathrm{cosk}_0^Y U))) \longrightarrow \pi_1(\pi_0(\mathrm{cosk}_0^k L_U))$$

is an isomorphism, as well. Thus, we can write the canonical map $\check{\mathrm{C}}(Y/k) \to B\Gamma$ as a levelwise map whose levels induce isomorphisms on fundamental groups. Using [EH76] Thm. 2.1.6 we may assume that the index category is even cofinite. Let $\underline{I}$ be the corresponding index category and write the corresponding levelwise map as

$$\check{\mathrm{C}}(Y/k)' \longrightarrow B\Gamma.$$

Thus, Rem. 1.5.8 gives a fibrant replacement

$$\begin{tikzcd} \check{\mathrm{C}}(Y/k)' \arrow[rr] \arrow[dr] & & \mathfrak{Y} \arrow[dl, two heads] \\ & B\Gamma & \end{tikzcd}$$

in Pro$\underline{\mathrm{SSets}} \downarrow B\Gamma$ with $\mathfrak{Y} \to B\Gamma$ a levelwise map inducing isomorphisms on $\pi_q(-)$ for $q \leq 1$ on each level. In particular,

$$\bar{\mathfrak{Y}} = \mathfrak{Y}^\natural \times_{B\Gamma} E\Gamma = (\mathfrak{Y} \times_{B\Gamma} E\Gamma)^\natural$$

is levelwise simply connected, which finishes the proof. $\square$

Using Lem. 4.3.4, it is easy to show the following refinement of Rem. 4.1.5:

4.3.5 Corollary. *Let X and Y be two geometrically unibranched and geometrically simply connected k-varieties of Čech-type. Then for any morphism*

$$f : X \longrightarrow Y$$

in $\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma)$ *the base change* $\bar{f}$ *has a model*

$$\bar{\mathfrak{f}}: \bar{\mathfrak{X}} \longrightarrow \bar{\mathfrak{Y}}$$

in $\mathrm{Pro}\underline{\mathrm{SSets}}_\Gamma$ *with* $\bar{\mathfrak{X}}$ *and* $\bar{\mathfrak{Y}}$ *both levelwise simply connected. If moreover* X *admits a homotopy rational point* $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$ *we even get a map in* $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma) \uparrow E\Gamma$ *between models of the induced homotopy fixed points* $\bar{s}$ *and* $f_*\bar{s}$.

Proof: Choose $\mathfrak{X}$ and $\mathfrak{Y}$ as in Lem. 4.3.4 Since $\mathfrak{Y} \to B\Gamma$ is fibrant and any $\mathfrak{X} \to B\Gamma$ is cofibrant, we get a model

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\mathfrak{f}"] \arrow[dr] & \mathfrak{Y} \arrow[d] \\ & B\Gamma \end{tikzcd}$$

of f in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma$. Since the derived base change of Prop. 2.3.10 works in the naive way without the need of any fibrant or cofibrant replacement,

$$\bar{\mathfrak{f}} = \mathfrak{f} \times_{B\Gamma} E\Gamma : \bar{\mathfrak{X}} \longrightarrow \bar{\mathfrak{Y}}$$

is a model of $\bar{f}$ in $\mathrm{Pro}(\underline{\mathrm{SSets}}_\Gamma)$, which completes the proof of the first claim. For the second claim, use that $\mathfrak{X}$ is fibrant in $\mathrm{Pro}\underline{\mathrm{SSets}} \downarrow B\Gamma$, as well. □

Before going on, let us first recall the fundamental algebraic topology of a Brauer-Severi variety:

4.3.6 Remark. Let X be a Brauer-Severi variety over the characteristic 0 field k. Using the homotopy fibre sequence

$$\bar{X} \longrightarrow X \longrightarrow B\Gamma$$

we may assume that k is algebraically closed and hence X is a projective space $\mathbb{P}^n$. Using [AM69] Cor. 12.12, we may even assume $k = \mathbb{C}$. Then $\mathbb{P}^n$ is the profinite completion of the complex analytification $(\mathbb{P}^n)^{\mathrm{an}}$ by Thm. 1.3.5. This analytification $(\mathbb{P}^n)^{\mathrm{an}}$ in turn sits in the homotopy fibre sequence

$$S^1 \longrightarrow S^{2n+1} \longrightarrow (\mathbb{P}^n)^{\mathrm{an}}$$

given by the Hopf-fibration. The homotopy groups of spheres are finitely generated (use [Ser51] Chap. V.2 Prop. 1) and hence good with respect to the class of all finite groups by Rem. 1.1.12. Thus from Thm. 1.1.13 we get for the prohomotopy groups (without Γ-action):

$$\pi_q(X) = \begin{cases} \Gamma & \text{if } q = 1 \\ \hat{\mathbb{Z}} & \text{if } q = 2 \\ \pi_q(S^{2n+1})^\wedge & \text{if } q \neq 1, 2 \end{cases}$$

As Γ-module we have at least:

$$\pi_2(X) = \hat{\mathbb{Z}}(1).$$

Indeed, by Hurewicz $\pi_2(X)$ is just $H_2(\bar{X})$, which is canonically isomorphic to $H_2(\bar{\mathbb{P}}^n)$ by Prop. 3.1.6 and Lem. 3.3.3, i.e. we may again assume that X is a projective space $\mathbb{P}^n$.

Now for $\mathbb{P}^n$ we claim even more: $\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}$ induces $\mathrm{Pro}(\underline{\mathrm{Mod}_\Gamma})$-isomorphisms on homology in degrees $\leq 2n$. This statement is trivial in degree 0 so we may restrict our self to $H_q(-)$ for $q > 0$. Arguing as above, we may again assume $k = \mathbb{C}$. Over $\mathbb{C}$, our map $\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}$ is the profinite completion of $\varphi_{c_1[\mathcal{O}_{\mathbb{P}^n}(1)]}$ for

$$c_1 : \operatorname{Pic}((\mathbb{P}^n)^{\mathrm{an}}) \longrightarrow H^2((\mathbb{P}^n)^{\mathrm{an}}; \mathbb{Z})$$

the classical first profinite Chern class map given by the homotopy equivalence $(\mathbb{P}^\infty)^{\mathrm{an}} \simeq K(\mathbb{Z}, 2)$. Thus, profinite completion induces a commutative diagram

$$\begin{tikzcd}
C_\bullet((\mathbb{P}^n)^{\mathrm{an}}) \arrow[r, "\varphi_{c_1[\mathcal{O}_{\mathbb{P}^n}(1)^{\mathrm{an}}]}"] \arrow[d] & C_\bullet(K(\mathbb{Z}, 2)) \arrow[d] \\
C_\bullet(\mathbb{P}^n) \arrow[r, "\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}"] & C_\bullet(K(\hat{\mathbb{Z}}, 2))
\end{tikzcd}$$

The vertical maps in the induced diagram

$$\begin{tikzcd}
H_q((\mathbb{P}^n)^{\mathrm{an}}) \arrow[r, "\varphi_{c_1[\mathcal{O}_{\mathbb{P}^n}(1)^{\mathrm{an}}]}"] \arrow[d] & H_q(K(\mathbb{Z}, 2)) \arrow[d] \\
H_q(\mathbb{P}^n)) \arrow[r, "\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}"] & H_q(K(\hat{\mathbb{Z}}, 2))
\end{tikzcd}$$

are just the profinite completions by Lem. 3.2.6. But under the identification $(\mathbb{P}^\infty)^{\mathrm{an}} \simeq K(\mathbb{Z}, 2)$, the canonical map $\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}$ corresponds to the analytification of the standard embedding

$$\mathbb{P}^n \hookrightarrow \mathbb{P}^\infty ,$$

which induces an isomorphism on homology in each degree $\leq 2n$.

Combining the isomorphism $H_2(\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]})$ with Prop. 3.1.6 and the constructions of Sect. 4.2 we can give an explicit construction of a class $\tilde{\alpha}_s$ in $H^2(X; \hat{\mathbb{Z}}(1))$ which will turn out to be α_s in Cor. 4.3.10 below:

4.3.7 Remark. Let X be a Brauer-Severi variety over k admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. From Cor. 4.3.5 we get a good model

$$\bar{\mathfrak{s}} : E\Gamma \longrightarrow \bar{\mathfrak{X}} \tag{4.3.3}$$

in $\mathrm{Pro}(\underline{\mathrm{SSets}_\Gamma})$ of the induced homotopy fixed point $\bar{s}$ of X with $\bar{\mathfrak{X}}$ levelwise simply connected. Application of Cor. 4.2.5 gives a class

$$(h_{\bar{\mathfrak{s}}}^{-1})_*(\varphi_{\bar{\mathfrak{s}}}) \in H^2(X; H_2(\bar{X})).$$

By Rem. 4.3.6, the second homology $H_2(\bar{X})$ is canonically isomorphic to $\hat{\mathbb{Z}}(1)$ in $\mathrm{Pro}(\underline{\mathrm{Mod}}_\Gamma)$, so we finally get a class

$$\tilde{\alpha}_s \in H^2(X; \hat{\mathbb{Z}}(1)).$$

By construction, the class $\tilde{\alpha}_s$ satisfies $s^*\tilde{\alpha}_s = 0$.

4.3.8 Lemma. *Let X be a Brauer-Severi variety over k admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. Then the class $\tilde{\alpha}_s$ in $H^2(X; \hat{\mathbb{Z}}(1))$ is well defined, i.e. independent from the choice of the model (4.3.3) of the induced homotopy fixed point $\bar{s}$ of X.*

Proof: To show that $\tilde{\alpha}_s$ is independent from the choice of the good model (4.3.3) of the induced homotopy fixed point $\bar{s}$ of X, we argue similar as in Rem. 4.2.3: Recall from Lem. 4.3.1 that we get a direct sum decomposition of $[\bar{X}, K(\hat{\mathbb{Z}}(1), 2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})}$ into

$$[E\Gamma, K(\hat{\mathbb{Z}}(1), 2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})} \oplus [\bar{X}, K(\hat{\mathbb{Z}}(1), 2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})},$$

where the projections onto the two summands correspond to $\bar{s}^*$ resp. $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$. Thus, we have to check that both $\bar{s}^*(\tilde{\alpha}_s)$ and $\mathrm{res}^{\mathbf{1}}_\Gamma(\tilde{\alpha}_s)$ is independent from the choice of (4.3.3).

Since $\bar{s}^*(\tilde{\alpha}_s)$ is trivial by construction, the first statement clearly holds. For the independence of $\mathrm{res}^{\mathbf{1}}_\Gamma(\tilde{\alpha}_s)$ we argue as follows: We have to show, that $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$ applied to the composition given by the diagram

$$\begin{tikzcd}
\bar{\mathfrak{X}} \arrow[r, "\varphi_{\bar{\mathfrak{s}}}"] & K(\tilde{H}_2(\bar{\mathfrak{X}}, \bar{\mathfrak{s}}), 2) \arrow[r, "h_{\bar{\mathfrak{s}}}^{-1}", "\simeq"'] & K(H_2(\bar{\mathfrak{X}}), 2) \arrow[dr] & \\
\bar{X} \arrow[u] \arrow[rr] & & K(H_2(\bar{X}), 2) \arrow[u, "\simeq"'] \arrow[r, "\mathrm{can.}", "\simeq"'] & K(\hat{\mathbb{Z}}(1), 2)
\end{tikzcd}$$

is independent from the choice of (4.3.3). Since the target $\mathrm{res}^{\mathbf{1}}_\Gamma K(\hat{\mathbb{Z}}(1), 2)$ represents $H^2(-; \mathbb{Z}(1))$ on $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$, it suffices to check this after application of $H^2(-; \mathbb{Z}(1))$. Further, all the pro-spaces involved are simply connected, so it suffices to check this even after application of the second integral homology $H_2(-)$. But $H_2(\varphi_{\bar{\mathfrak{s}}})$ is nothing but $H_2(h_{\bar{\mathfrak{s}}})$, so $H_2(\mathrm{res}^{\mathbf{1}}_\Gamma(\alpha_s))$ is just the canonical isomorphism $H_2(\bar{X}) \cong \hat{\mathbb{Z}}(1)$ of Rem. 4.3.6, i.e. independent from the choice of (4.3.3). □

Both classes $\tilde{\alpha}_s$ and α_s are killed by s^*. Further, $\mathrm{res}^{\mathbf{1}}_\Gamma(\tilde{\alpha}_s)$ equals $\tilde{\alpha}_{\bar{x}}$ for $\bar{x}$ a geometric point in $X(\bar{k})$: Indeed, by Cor. 4.2.6 these two classes differ by a constant in the trivial group $H^2(\Gamma_{\bar{k}}; \hat{\mathbb{Z}}(1))$. Thus, to show

$$\tilde{\alpha}_s = \alpha_s$$

we can argue as in the proof of Lem. 4.3.8 and have to show

$$\tilde{\alpha}_{\bar{x}} = \hat{c}_1[\mathcal{O}_{X\otimes_k \bar{k}}(1)].$$

4.3.9 Lemma. *Let y be a k-rational point of $\mathbb{P}^n$. Then*

$$\tilde{\alpha}_y = \hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)].$$

Proof: Let

$$\bar{\mathfrak{y}}:\ E\Gamma \longrightarrow \bar{\mathfrak{Y}}$$

be a good model of the homotopy fixed point $\bar{y}$ of $\mathbb{P}^n$ induced by y. We have to show that the diagram

(4.3.4)
$$\begin{tikzcd}
 & K(\hat{\mathbb{Z}}(1),2) & \\
\bar{\mathbb{P}}^n \arrow[ur, "\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}"] \arrow[rr] \arrow[d, "\simeq"] & & K(H_2(\bar{\mathbb{P}}^n),2) \arrow[ul, "H_2(\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]})"'] \arrow[d, "\simeq"] \\
\bar{\mathfrak{Y}} \arrow[r, "\varphi_{\bar{\mathfrak{y}}}"] & K(\tilde{H}_2(\bar{\mathfrak{Y}},\varphi_{\bar{\mathfrak{y}}}),2) \arrow[r, "h_{\bar{\mathfrak{y}}}^{-1}"] & K(H_2(\bar{\mathfrak{Y}}),2)
\end{tikzcd}$$

commutes in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$, where the lower square commutes by definition of the upper horizontal arrow.

We argue similar as in the proof of Lem. 4.3.8: Using the direct sum decomposition of $[\bar{\mathbb{P}}^n, K(\hat{\mathbb{Z}}(1),2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$ into

$$[E\Gamma, K(\hat{\mathbb{Z}}(1),2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)} \oplus [\bar{\mathbb{P}}^n, K(\hat{\mathbb{Z}}(1),2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})},$$

via the projections $\bar{y}^*$ and $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$, we have to show that both $\bar{y}^*(-)$ and $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$ applied to (4.3.4) commutes in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ resp. $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$.

For $\bar{y}^*(-)$ this holds, since $\bar{\mathfrak{y}}^*\varphi_{\bar{\mathfrak{y}}}$ is trivial by construction, resp. since

$$y^*\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)] = \hat{c}_1[y^*\mathcal{O}_{\mathbb{P}^n}(1)]$$

is trivial by Hilbert 90. To check if $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$ applied to (4.3.4) commutes, we may again check this after application of $H_2(-)$ (cf. the proof of Lem. 4.3.8). But $H_2(\mathrm{res}^{\mathbf{1}}_\Gamma(-))$ applied to (4.3.4) commutes again since $H_2(\varphi_{\bar{\mathfrak{y}}})$ is nothing but $H_2(h_{\bar{\mathfrak{y}}})$, which completes the proof of the commutativity of $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$ applied to (4.3.4). □

Thus we have shown:

4.3.10 Corollary. *Let X be a Brauer-Severi variety over k admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. Then the class $\tilde{\alpha}_s$ in $H^2(X;\hat{\mathbb{Z}}(1))$ is the unique k-structure α_s of $\hat{c}_1[\mathcal{O}(1)]$ satisfying $s^*\alpha_s = 0$.*

4.4 Maps on homotopy fixed point sets induced by k-structures of $\hat{c}_1[\mathcal{O}(1)]$. Let X be a Brauer-Severi variety over k admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$ and let $\alpha \in H^2(X;\hat{\mathbb{Z}}(1))$ be a k-structure of $\hat{c}_1[\mathcal{O}(1)]$. In this subsection we want to study the map induced by

$$\varphi_\alpha:\ \bar{X} \longrightarrow K(\hat{\mathbb{Z}}(1),2)$$

on homotopy fixed point sets.

First, we go back to arbitrary levelwise m-connected models of homotopy fixed points:

4.4.1 Lemma. *For $m \geq 1$, $m+1 \geq \mathrm{scd}(\Gamma)$ let $A.$ be an m-connected simplicial discrete Γ-set in $\underline{\mathrm{SSets}}_\Gamma$ together with a homotopy fixed point a in $\pi_0(A.^{h\Gamma})$. Then the canonical map φ_a induces an injection of homotopy fixed point sets*

$$\pi_0(\varphi_a^{h\Gamma}):\ \pi_0(A.^{h\Gamma}) \hookrightarrow \pi_0(K(\tilde{H}_{m+1}(A.,a), m+1)^{h\Gamma})\ .$$

Proof: Let $b \in \pi_0(A.^{h\Gamma})$ be a homotopy fixed point. We may assume that b has a model $b: E. \to A.$ in $\underline{\mathrm{SSets}}_\Gamma$ (cf. Rem. 4.2.4). We claim that $\pi_0(\varphi_a^{h\Gamma})$ has trivial fibre at b. To see this, we may assume that b factors even over an actual fixed point $* \in A_0^\Gamma$: If not, replace $A.$ by the weakly equivalent cone C_b.

Recall from Lem. 4.2.1 that φ_a is isomorphic in $\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ to the canonical map

$$A. \longrightarrow \mathrm{cosk}_{m+2}\mathrm{Ex}(A.)$$

for $\mathrm{Ex}(-)$ a functorial fibrant replacement in $\underline{\mathrm{SSets}}$. As $\underline{\mathrm{SSets}}_\Gamma$ is the category of simplicial sheaves over the classifying site $\underline{B}\Gamma$, it has functorial factorization, as well. In particular, it has a functorial fibrant replacement $\mathrm{Ex}_\Gamma(-)$. From the composition

$$\begin{tikzcd} A. \arrow[r] \arrow[rr, bend right] & \mathrm{cosk}_{m+2}\mathrm{Ex}(A.) \arrow[r, hook, "\sim"] & \mathrm{Ex}_\Gamma(\mathrm{cosk}_{m+2}\mathrm{Ex}(A.)) \end{tikzcd} =: B.$$

we get a fibrant factorization (with respect to $\underline{\mathrm{SSets}}_\Gamma$)

$$\begin{tikzcd} & (A,*) \arrow[d, hook, "\sim"] \arrow[dr] & \\ (F.,*) \arrow[r] & (A.',*) \arrow[r, two heads] & (B.,*) \end{tikzcd}$$

with pointed fibre $(F.,*)$. By Lem. 4.4.2 below together with the long exact homotopy sequence, $F.$ is $m+1$-connected. As a fibre of a fibration, $F.$ is fibrant in $\underline{\mathrm{SSets}}_\Gamma$, as well. In particular, $F.^{h\Gamma} = F.^\Gamma$. The corresponding statement holds for the fibrant simplicial discrete Γ-sets $A.'$ and $B.$, too. Further, $(F.^\Gamma,*)$ is still the fibre of

$$((A.')^\Gamma,*) \longrightarrow (B.^\Gamma,*)\ .$$

This map is a fibration in $\underline{\mathrm{SSets}}_\bullet$, since $(-)^\Gamma: \underline{\mathrm{SSets}}_\Gamma \to \underline{\mathrm{SSets}}$ is right adjoint to the functor mapping a simplicial set to itself together with trivial Γ-action which preserves acyclic cofibrations. We get a homotopy fibre sequence in $\mathcal{H}(\underline{\mathrm{SSets}}_\bullet)$

$$(F.^{h\Gamma},*) \longrightarrow (A.^{h\Gamma},*) \longrightarrow (B.^{h\Gamma},*)\ ,$$

i.e. we have to show that $F.^{h\Gamma}$ is connected.

To see this, note that $F.$ is $m+1$ connected and $m+1 \geq \mathrm{scd}(\Gamma)$ by assumption. Recall from [Goe95] Thm. 4.8 that there is a spectral sequence

$$E_2^{p,q} = H^p(\Gamma; \pi_{-q}(F.,*)) \Rightarrow \pi_{-(p+q)}(F.^{h\Gamma},*)$$

fringed along the line $p + q = 0$. Since $m + 1 \geq \mathrm{scd}(\Gamma)$, all the entries

$$H^q(\Gamma; \pi_q(F., *))$$

along this line are trivial, i.e. $F.^{h\Gamma}$ is indeed connected by the Connectivity Lemma of [BK72] Chapt. IX 5.1. $\square$

4.4.2 Lemma. *Let $f : A. \twoheadrightarrow B.$ be a fibration in $\underline{\mathrm{SSets}}_\Gamma$ with fibre $F.$ over a base point $* \in B_0$. Then the restriction of*

$$(F., *) \longrightarrow (A., *) \longrightarrow (B., *)$$

to $\underline{\mathrm{SSets}}_\bullet$ is a homotopy fibre sequence in $\underline{\mathrm{SSets}}_\bullet$.

Proof: Choosing a functorial factorization in $\underline{\mathrm{SSets}}$ (cf. Rem. 2.3.6), we can factor f sectionwise in $\underline{\mathrm{SSets}}_\Gamma$ as

$$\begin{tikzcd} A. \arrow[r, "f"] \arrow[d, hook, "i"', "\sim"] & B. \\ A.' \arrow[ur, "f'"'] & \end{tikzcd}$$

for i an acyclic cofibration in $\underline{\mathrm{SSets}}$ (hence also in $\underline{\mathrm{SSets}}_\Gamma$) and f' a local fibration. Denote by $F.'$ the fibre of $*$ with res. to f'. Then we have to show that the induced map

$$i_{|F} : \ F. \longrightarrow F.'$$

is a weak equivalence in $\underline{\mathrm{SSets}}$.

The weak equivalence i induces an isomorphism in the relative homotopy category $\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma \downarrow B.)$. Since f is a fibration in $\underline{\mathrm{SSets}}_\Gamma$, the inverse has a model

$$j : \ A.' \longrightarrow A.$$

in $\underline{\mathrm{SSets}}_\Gamma \downarrow B.$ and both compositions $j \circ i$ resp. $i \circ j$ are homotopy equivalent in $\underline{\mathrm{SSets}}_\Gamma \downarrow B.$ resp. $\underline{\mathrm{SSets}} \downarrow B.$ to the respective identities. We may find a homotopy with resp. to any good cylinder object of $A.$ resp. $A.'$ in $\underline{\mathrm{SSets}}_\Gamma \downarrow B.$ resp. $\underline{\mathrm{SSets}} \downarrow B.$, since f resp. f' is a fibration in $\underline{\mathrm{SSets}}_\Gamma$ resp. $\underline{\mathrm{SSets}}$. We choose the standard cylinder object $A. \times \Delta^1$ resp. $A.' \times \Delta^1$. In particular, our homotopy equivalences over $B.$ restrict to homotopy equivalences

$$j_{|F.'} \circ i_{|F.} \simeq \mathrm{id}_{F.}$$
$$i_{|F.} \circ j_{|F.'} \simeq \mathrm{id}_{F.'},$$

which completes the proof. $\square$

Levelwise application of 4.4.1 gives the corresponding statement in the pro-sense:

4.4.3 Corollary. *For $m \geq 1$, $m + 1 \geq \mathrm{scd}(\Gamma)$ let $\bar{s} : E\Gamma \to \mathfrak{X}$ be a levelwise m-connected model of a homotopy fixed point in* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$. *Then the canonical map $\varphi_{\bar{s}}$ induces an injection of homotopy fixed point sets*

$$\pi_0(\varphi_{\bar{s}}^{h\Gamma}) : \ \pi_0(\mathfrak{X}^{h\Gamma}) \hookrightarrow \pi_0(K(\tilde{H}_{m+1}(\mathfrak{X}, \bar{s}), m+1)^{h\Gamma}) \ .$$

At least for fields of strict cohomological dimension ≤ 2 this gives us:

4.4.4 Corollary. *Let X be a Brauer-Severi variety admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$ over k a field of strict cohomological dimension ≤ 2. Then our k-structure α_s of $\hat{c}_1[\mathcal{O}(1)]$ induces an injection of homotopy fixed point sets*

$$\pi_0(\varphi_{\alpha_s}^{h\Gamma}):\ \pi_0(\bar{X}^{h\Gamma}) \hookrightarrow \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma})\ = H^2(\Gamma;\hat{\mathbb{Z}}(1)).$$

4.4.5 Remark. Cor. 4.4.4 applies e.g. for p-adic local resp. totally imaginary number fields: a p-adic local resp. totally imaginary number field has strict cohomological dimension 2 by [NSW08] Cor. 7.2.5 resp. [Hab78] Prop. 12.

For base fields of larger cohomological dimension at least we get:

4.4.6 Lemma. *Let X be a Brauer-Severi variety over k admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. Assume that* $\mathrm{scd}(\Gamma) \leq 2\mathrm{dim}(X)$. *Then the map*

$$\pi_0(\varphi_{\alpha_s}^{h\Gamma}):\ \pi_0(\bar{X}^{h\Gamma}) \longrightarrow \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma})\ = H^2(\Gamma;\hat{\mathbb{Z}}(1)).$$

induced by α_s has trivial fibres at all homotopy fixed points induced by homotopy rational points of X.

Proof: Let r be a homotopy rational point in $[B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. We have to show that that $\pi_0(\varphi_{\alpha_s}^{h\Gamma})$ has trivial fibre at $\bar{r}$. Take a good model

$$\bar{\mathfrak{r}}:\ E\Gamma \longrightarrow \bar{\mathfrak{X}}$$

of $\bar{r}$ as in Cor. 4.3.5. This good model allows us to form the cone $C_{\bar{\mathfrak{r}}}$ of $\bar{\mathfrak{r}}$. Thus, doing all the steps of the corresponding part of the proof of Lem. 4.4.1 levelwise starting with the canonical map

$$\bar{\mathfrak{X}} \longrightarrow \mathrm{cosk}_3\mathrm{Ex}(\bar{\mathfrak{X}})$$

and our good model of $\bar{\mathfrak{r}}$, we have to proof the following: Let $n \geq \mathrm{scd}(\Gamma)$ and $\mathfrak{Y}$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$ weakly n-connected and even levelwise connected with $* \in \lim \mathfrak{Y}_0^\Gamma$. Then the homotopy fixed point set $\pi_0(\mathfrak{Y}^{h\Gamma}, *)$ is the point (our $\mathfrak{Y}$ corresponds to the fibre $F.$ in the proof of Lem. 4.4.1).

We prove this by descending induction on m, where $\mathfrak{Y}$ is even levelwise m-connected: First, assume $m \geq n$ ($\geq \mathrm{scd}(\Gamma)$). Again, we make use of the spectral sequence

$$E_2^{p,q} = H^p(\Gamma; \pi_{-q}(A., a)) \Rightarrow \pi_{-(p+q)}(A^{h\Gamma}, a)$$

fringed along the line $p + q = 0$ for $A. \in \underline{\mathrm{SSets}}_\Gamma$ simply connected and $a \in A_0^\Gamma$: Since $m \geq \mathrm{scd}(\Gamma)$, all the entries $H^q(\Gamma; \pi_{-q}(\mathfrak{Y}, *))$ are levelwise trivial, i.e. $\mathfrak{Y}^{h\Gamma}$ is levelwise connected by the Connectivity Lemma of [BK72] Chapt. IX 5.1.

For $0 \leq m < n$ arbitrary, we argue as follows: Arguing levelwise as in the proof of Lem. 4.4.1, we get a levelwise fibrant replacement

$$\begin{tikzcd} & (\mathfrak{Y}, *) \arrow[d, hook, "\sim"] \arrow[dr] & \\ (\mathfrak{F}, *) \arrow[r] & (\mathfrak{Y}', *) \arrow[r, two heads] & (\mathrm{Ex}_\Gamma(\mathrm{cosk}_{m+2}\mathfrak{Y}), *) \end{tikzcd}$$

with fibre $\mathfrak{F}$. Now $\mathrm{Ex}_\Gamma(\mathrm{cosk}_{m+2}\mathfrak{Y})$ has levelwise the $K(\pi_{m+1}(\mathfrak{Y}, *), m+1)$-property and $\mathfrak{F}$ is levelwise $m+1$ connected. As a result, it suffices to prove the triviality of any $\mathfrak{Z}$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$ admitting a fixed point $* \in \mathfrak{Z}^\Gamma$ and having levelwise the $K(\pi, m)$-property for π a pro-Γ-group isomorphic to the trivial pro-Γ-group and $m > 0$. Note, that this is trivial in the non equivariant case ($\mathfrak{Z} \cong \mathfrak{Z}^\natural$ by assumption), so the only difficulty is to write down an Γ-equivariant isomorphism between $\mathfrak{Z}$ and the point.

If $m = 1$, we get levelwise isomorphisms

$$\begin{tikzcd} (\mathfrak{Z}, *) \arrow[r, "\sim"] & (B\Pi(\mathfrak{Z}), *) \\ & (B\pi_1(\mathfrak{Z}, *), *) \arrow[u, "\sim"'] \end{tikzcd}$$

in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$, where the horizontal arrow is the canonical map from $\mathfrak{Z}$ to the nerve of its fundamental groupoid $\Pi(\mathfrak{Z})$. But the pro-Γ-group $\pi_1(\mathfrak{Z}, *)$ is isomorphic to the trivial pro-Γ-group, i.e.

$$B\pi_1(\mathfrak{Z}) \simeq \mathrm{pt}$$

holds in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})$ by the functoriality of the nerve functor

$$B(-): \ \mathrm{Pro}(\underline{\mathrm{Grps}_\Gamma}) \longrightarrow \mathrm{Pro}(\underline{\mathrm{SSets}_\Gamma}) \ .$$

Finally, for $m > 1$ we argue as follows: As usually, we identify $\underline{\mathrm{SMod}_\Gamma}$ with $\mathrm{Ch}_+(\underline{\mathrm{Mod}_\Gamma})$ via the Γ-equivariant version of the Dold-Kan correspondence. We apply Cor. 4.2.5 to the fixed point $* \in \mathfrak{Z}^\Gamma$ to get a levelwise map

$$\varphi_* : \ \mathfrak{Z} \longrightarrow K(\tilde{H}_m(\mathfrak{Z}, *), m) \ = \tilde{H}_m(\mathfrak{Z}, *)[-m]$$

inducing levelwise isomorphisms $\pi_q(\varphi_*)$ for each $q \leq m$. But $\mathfrak{Z}$ has levelwise the $K(\pi, m)$-property, so this map is even a levelwise isomorphism in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{Mod}_\Gamma})$. Now $\tilde{H}_m(\mathfrak{Z}, *)[-m]$ is isomorphic to 0 by assumption and Hurewicz, i.e. $\mathfrak{Z}$ is indeed isomorphic to the point in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{Mod}_\Gamma})$. $\square$

The corresponding statement for quasi homology fixed points is less complicated. First, we need to generalize the statement for $\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}$ of Rem. 4.3.6 to φ_{α_s} for arbitrary homotopy rational points s or even more generally, to arbitrary k-structures α of $\hat{c}_1[\mathcal{O}(1)]$ of arbitrary Brauer-Severi varieties over k:

4.4.7 Lemma. *Let X be a Brauer-Severi variety over k and $\alpha \in H^2(X;\hat{\mathbb{Z}}(1))$ a k-structure of $\hat{c}_1[\mathcal{O}(1)]$. Then the canonical* $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$*-morphism φ_α induces a quasi-isomorphism in the pro-sense*

$$\tau_{\leq 2n} C_\bullet(\bar{X}) \longrightarrow \tau_{\leq 2n} C_\bullet(K(\hat{\mathbb{Z}}(1),2))$$

in $\mathrm{Pro}\mathcal{D}_b(\underline{\mathrm{Mod}}_\Gamma)$ *for n the dimension of X.*

Proof: We have to check if $H_q(\varphi_\alpha)$ is an isomorphism in $\mathrm{Pro}\underline{\mathrm{Mod}}_\Gamma$ for $q \leq 2n$. This is clearly the case for $q = 0$. For $q > 0$ we argue as follows: Using Lem. 3.3.3 we can check this after restriction $\mathrm{res}^{\mathbf{1}}_\Gamma(-)$. But

$$\mathrm{res}^{\mathbf{1}}_\Gamma(\varphi_\alpha) = \varphi_{\hat{c}_1[\mathcal{O}_{X\otimes_k \bar{k}}(1)]}$$

by the definition of a k-structure of $\hat{c}_1[\mathcal{O}(1)]$, i.e. $H_q(\mathrm{res}^{\mathbf{1}}_\Gamma(\varphi_\alpha))$ is indeed an isomorphism by Rem. 4.3.6. □

Combining Lem. 4.4.7 with the hypercohomology spectral sequence $\mathbb{H}^{\bullet,\bullet}_*(-)$ we get the following corollary:

4.4.8 Corollary. *Let X be a Brauer-Severi variety over k and $\alpha \in H^2(X;\hat{\mathbb{Z}}(1))$ a k-structure of $\hat{c}_1[\mathcal{O}(1)]$. Assume that* $\mathrm{cd}(\Gamma) \leq 2\dim(X)$*. Then the push forward along the canonical map φ_α induces an isomorphism*

$$(\varphi_\alpha)_* :\ \mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^\sharp)_{\int_*=1} \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(K(\bar{\mathbb{Z}}(1),2))^\sharp)_{\int_*=1}$$

between the sets of quasi homology fixed points of X and $K(\hat{\mathbb{Z}}(1),2)$.

4.4.9 Remark. Let s be a homotopy rational point of a Brauer-Severi variety X over k a field of cohomological dimension ≤ 2. We get a commutative diagram

$$\begin{tikzcd}
\pi_0(\bar{X}^{h\Gamma}) \arrow[r, "(\varphi_{\alpha_s})_*"] \arrow[d, "\text{can.}"] & \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma}) \arrow[d, "\text{can.}"] \\
\mathbb{H}^0(\Gamma; C_\bullet(\bar{X}))_{\int_*=1} \arrow[r, "(\varphi_{\alpha_s})_*"] \arrow[d, "\text{can.}"] & \mathbb{H}^0(\Gamma; C_\bullet(K(\hat{\mathbb{Z}}(1),2)))_{\int_*=1} \arrow[d, "\text{can.}"] \\
\mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^\sharp)_{\int_*=1} \arrow[r, "(\varphi_{\alpha_s})_*"] & \mathbb{H}^0(\Gamma; C_\bullet(K(\hat{\mathbb{Z}}(1),2))^\sharp)_{\int_*=1}
\end{tikzcd}$$

where the horizontal maps are just the canonical maps given by Rem. 4.1.3. Now the lower horizontal arrow is an isomorphism by Cor. 4.4.8 while the composition of the vertical right arrows is an isomorphism by Rem. 4.1.4. Thus, Lem. 4.4.6 resp. Cor. 4.4.4 implies that the canonical map

$$\pi_0(\bar{X}^{h\Gamma}) \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^{(\sharp)})_{\int_*=1}$$

mapping a homotopy fixed point to its induced (quasi) homology fixed point has trivial fibres at homotopy fixed points induced by homotopy rational points if $\dim(X) \geq 2$ (since $\mathrm{scd}(\Gamma) \leq \mathrm{cd}(\Gamma) + 1$, see e.g. [NSW08] Prop. 3.3.3) resp. is injective if k is even of strict cohomological dimension ≤ 2 (e.g. k a p-adic local resp. totally imaginary number field).

Let α be an arbitrary k-structure of $\hat{c}_1[\mathcal{O}(1)]$. We still get a commutative diagram

$$\begin{tikzcd} \pi_0(\bar{X}^{h\Gamma}) \arrow[r, "(\varphi_\alpha)_*"] \arrow[d, "\text{can.}"] & \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma}) \arrow[d, "\text{can.}"] \\ \mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^\sharp)_{\int_*=1} \arrow[r, "(\varphi_\alpha)_*"] & \mathbb{H}^0(\Gamma; C_\bullet(K(\hat{\mathbb{Z}}(1),2))^\sharp)_{\int_*=1} \end{tikzcd}$$

and the lower horizontal arrow is still an isomorphism by Cor. 4.4.8. Thus, Rem. 4.1.4 together with Rem. 4.4.9 implies:

4.4.10 Corollary. *Let X be a Brauer-Severi variety over k a field of cohomological dimension ≤ 2 admitting a homotopy rational point. Suppose $\dim(X) \geq 2$ resp. k is even of strict cohomological dimension ≤ 2 (e.g. k a p-adic local resp. totally imaginary number field). Then the map*

$$\pi_0(\varphi_\alpha^{h\Gamma}):\ \pi_0(\bar{X}^{h\Gamma}) \hookrightarrow \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma})\ = H^2(\Gamma;\hat{\mathbb{Z}}(1)).$$

induced by α on homotopy fixed points has trivial fibres at all homotopy fixed points of X induced by homotopy rational points resp. is injective.

Let k be a totally imaginary number field. For a finite place ν of k denote by res_ν the restriction to the absolute Galois group Γ_ν of the henselization k_ν^h and by X_ν the base extension of X along k_ν^h/k. We get a canonical map

$$\prod_\nu \mathrm{res}_\nu:\ \pi_0(\bar{X}^{h\Gamma}) \longrightarrow \textstyle\prod_\nu \pi_0(\bar{X}_\nu^{h\Gamma_\nu})\ ,$$

where ν runs through all finite places of k. Let us mention the following observation:

4.4.11 Corollary. *Let X be a Brauer-Severi variety over k a totally imaginary number field admitting a homotopy rational point. Then the canonical map*

$$\prod_\nu \mathrm{res}_\nu:\ \pi_0(\bar{X}^{h\Gamma}) \longrightarrow \textstyle\prod_\nu \pi_0(\bar{X}_\nu^{h\Gamma_\nu})$$

is injective.

Proof: Let s be a homotopy rational point. Using Prop. 2.3.10 we get homotopy rational points s_ν of X_ν over k_ν^h. The induced homotopy fixed point of s_ν is just the restriction $\mathrm{res}_\nu(\bar{s})$ of the induced homotopy fixed point $\bar{s}$ of s (cf. the compatibility claim of Prop. 2.3.11). Thus, the restriction $\mathrm{res}_\nu(\alpha_s)$ is nothing but α_{s_ν} and we get a commutative diagram

$$\begin{tikzcd}
\pi_0(\bar{X}^{h\Gamma}) \arrow[r, "\pi_0(\varphi^{h\Gamma}_{\alpha_s})"] \arrow[d, "\prod_\nu \mathrm{res}_\nu"] & H^2(k;\hat{\mathbb{Z}}(1)) \arrow[d, "\prod_\nu \mathrm{res}_\nu"] \\
\prod_\nu \pi_0(\bar{X}_\nu^{h\Gamma_\nu}) \arrow[r, "\prod_\nu \pi_0(\varphi^{h\Gamma_\nu}_{\alpha_{s_\nu}})"] & \prod_\nu H^2(k_\nu^h;\hat{\mathbb{Z}}(1))
\end{tikzcd}$$

Now both horizontal arrows are injective by Cor. 4.4.4 while the vertical right arrow is a monomorphism by Brauer-Hasse-Noether: Indeed, this is just the map on Tate-modules induced by the monomorphism

$$\mathrm{Br}(k) \hookrightarrow \prod_\nu \mathrm{Br}(k_\nu^h)$$

and the functor taking an abelian group to its Tate-module is left exact (since it is entirely build out of limits). As a result, the left vertical arrow is an injection, just as claimed. □

4.5 An analogue for the weak section conjecture for Brauer-Severi varieties. We come back to the questions raised in the introduction. Recall the following observation:

4.5.1 Remark. Let X be a geometrically connected and geometrically unibranched $\mathbb{R}$-variety of Čech type. By Rem. 4.1.6 a homotopy rational, homology or (quasi) homology fixed point splits the canonical map

$$H^\bullet(\mathbb{R};\Lambda) \longrightarrow H^\bullet(X;\Lambda)$$

for any $\Gamma_\mathbb{R}$-module Λ. In particular, $\mathrm{cd}_2(X)$ is infinite and so $X(\mathbb{R})$ in non empty by [Cox79b] Thm. 2.1. Thus, $X(\mathbb{R})$ is non empty if and only if X has a homotopy rational point, homotopy fixed point resp. a (quasi) homology fixed point, i.e. an analogue of the weak section conjecture holds for X.

4.5.2 Remark. Let k be a number field admitting a real place ν. In abuse of notation refer to the real closure of k with resp. to the ordering given by the induced embedding $k \hookrightarrow \mathbb{R}$ as the henselization k_ν^h. We claim that the analogue of Rem. 4.5.1, i.e. an analogue of the weak section conjecture, holds for X a proper geometrically connected and geometrically unibranched k_ν^h-variety of Čech type, as well: Indeed,

$$\mathrm{cosk}_0^X(X \otimes_k \bar{k}) \longrightarrow X$$

resp. its base extension

$$(\mathrm{cosk}_0^X(X \otimes_k \bar{k})) \otimes_{\bar{k}} \bar{k}_\nu = \mathrm{cosk}_0^{X \otimes_{k_\nu^h} k_\nu}(X \otimes_{k_\nu^h} \bar{k}_\nu) \longrightarrow X_{k_\nu}$$

is a hypercovering, i.e. a weak equivalence by Prop. 1.6.9. Further, the canonical map

$$(\mathrm{cosk}_0^X(X \otimes_k \bar{k})) \otimes_{\bar{k}} \bar{k}_\nu \longrightarrow \mathrm{cosk}_0^X(X \otimes_k \bar{k})$$

induces an isomorphism on cohomology by [Fri82] Prop. 2.4 and [AM69] Cor. 12.12. As a result,

$$\mathrm{cd}_2(X \otimes_{k_\nu^h} k_\nu) = \mathrm{cd}_2(X),$$

i.e., if X admits a homotopical rational, homotopy or (quasi) homology fixed point point over k_ν^h, the base change $X \otimes_{k_\nu^h} k_\nu$ admits a k_ν-rational point, again by [Cox79b] Thm. 2.1. But k_ν/k_ν^h is an extension of real closed fields, i.e. X admits a k_ν^h rational point by [Pre84] Cor. 5.2, as well. If X is a Brauer-Severi variety over k_ν^h there is also an algebraic reason for this: The canonical map

$$\mathrm{Br}(k_\nu^h) \longrightarrow \mathrm{Br}(k_\nu)$$

is an isomorphism, i.e., X splits over k_ν^h if and only if $X \otimes_{k_\nu^h} k_\nu$ splits over k_ν.

We try to generalize Rem. 4.5.2 in this sections: Let X be a Brauer-Severi variety over k an arbitrary field of characteristic 0 admitting a homotopy rational point or at least a homotopy or (quasi) homology fixed point. Our aim is to show an analogue of the weak section conjecture for X, i.e., that under some possible (reasonable) extra assumptions X admits a rational point. This in turn is equivalent to X being isomorphic to a projective space over k.

It turns out that the analogue of the weak section conjecture is wrong for Brauer-Severi varieties in general (see Sect. 4.6 below). In this subsection we try to find a reasonable extra assumption under which the weak section conjecture does hold for Brauer-Severi varieties.

Fix a Brauer-Severi variety X over k. An easy application of the Hochschild-Serre spectral sequence shows that $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$ and that the canonical map

$$p^*: \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \otimes_k \bar{k})$$

corresponds to the multiplication by the period d of X (i.e. the degree of its Brauer class in $\mathrm{Br}(k)$). Let $[\mathcal{L}_X]$ be the positive degree generator of $\mathrm{Pic}(X)$, i.e. the unique generator satisfying

$$p^*[\mathcal{L}_X] = [\mathcal{O}_{X\otimes_k \bar{k}}(d)].$$

Further, let

$$i: X \hookrightarrow \mathbb{P}^N$$

be the **twisted *d*-uple embedding**, i.e. the unique embedding for minimal N with $i^*[\mathcal{O}_{\mathbb{P}^N}(1)] = [\mathcal{L}_X]$. Thus its base change $i \otimes_k \bar{k}$ is isomorphic to the usual d-uple embedding.

4.5.3 Lemma. *Let X be a Brauer-Severi variety over k of period d admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. Then the twisted d-uple embedding $i: X \to \mathbb{P}^N$ satisfies*

$$i^*\alpha_{i_*s} = d.\alpha_s.$$

Proof: We claim that under the canonical isomorphisms

$$H_2(\bar{X}) \cong \hat{\mathbb{Z}}(1) \cong H_2(\bar{\mathbb{P}}^N)$$

given by Rem. 4.3.6 the induced map $H_2(\bar{i})$ is isomorphic to the multiplication by d on $\hat{\mathbb{Z}}(1)$. If this holds, the Lemma follows from the functoriality claim of Cor. 4.2.5.

To prove the claim, note that $H_2(\bar{i})$ is Pontryagin dual to the map between torsion Γ-modules

$$\bar{i}^* : \; H^2(\mathbb{P}^N \otimes_k \bar{k}; \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(X \otimes_k \bar{k}; \mathbb{Q}/\mathbb{Z}) \, .$$

Further, the Brauer groups of both $X \otimes_k \bar{k}$ and $\mathbb{P}^N \otimes_k \bar{k}$ vanish, since for any field extension L, the Brauer groups $\mathrm{Br}(\mathbb{P}^n \otimes_k L)$ and $\mathrm{Br}(L)$ agree. By twisting we therefore get the commutative diagram

$$\begin{tikzcd} H^2(\mathbb{P}^N \otimes_k \bar{k}; \mathbb{Q}/\mathbb{Z}(1)) \arrow[r, "\bar{i}^*"] & H^2(X \otimes_k \bar{k}; \mathbb{Q}/\mathbb{Z}(1)) \\ \mathrm{Pic}(\mathbb{P}^N \otimes_k \bar{k}) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r, "\bar{i}^*"] \arrow[u, "\cong"'] & \mathrm{Pic}(X \otimes_k \bar{k}) \otimes \mathbb{Q}/\mathbb{Z} \arrow[u, "\cong"'] \end{tikzcd}$$

But $\bar{i}^*[\mathcal{O}_{\mathbb{P}^N \otimes_k \bar{k}}(1)] = [\mathcal{O}_{X \otimes_k \bar{k}}(d)]$, i.e. $\bar{i}^*$ and thus also $\bar{i}_*$ on $H_2(-)$ is just multiplication by d, which completes the proof. □

4.5.4 Remark. Let X be a Brauer-Severi variety over k of period d admitting a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$. If we expect X to split, we should at least expect the existence of a homotopy rational point r in $[B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$ whose image under i_* is the homotopy rational point induced by a genuine rational point $y \in \mathbb{P}^n(k)$. By Lem. 4.5.3 we get

$$i^*\alpha_y = d.\alpha_r.$$

Further, Cor. 4.2.6 implies

$$\begin{aligned} \alpha_r &= \alpha_s + s^*\alpha_r \\ \alpha_y &= \alpha_{i_*s} + (i_*s)^*\alpha_y, \end{aligned}$$

i.e. we get

$$(i_*s)^*\alpha_y = d.s^*\alpha_r,$$

again by Lem. 4.5.3. But $(i_*s)^*\alpha_y$ is just the pullback along s of the first profinite Chern class of $[\mathcal{L}_X]$ by Lem. 4.3.9, i.e. $s^*\hat{c}_1[\mathcal{L}_X]$ is divisible in $H^2(\Gamma; \hat{\mathbb{Z}}(1))$ by the period d of X.

Rem. 4.5.4 suggests that our extra assumption for an analogue of the weak section conjecture for s resp. $\bar{s}$ a homotopy rational resp. homotopy or (quasi) homology fixed point of a Brauer-Severi variety X should be the divisibility of the class $s^*\hat{c}_1[\mathcal{L}_X]$ resp. $\bar{s}^*\hat{c}_1[\mathcal{L}_X]$ in $H^2(\Gamma; \hat{\mathbb{Z}}(1))$ by d the period of X. We want to show that this extra assumption is independent from the choice of s resp. $\bar{s}$ (see Prop. 4.5.10 below):

Let A be a central simple algebra over k of period d and A^{op} its opposite algebra. Denote by X_A resp. $X_{A^{\mathrm{op}}}$ the corresponding Brauer-Severi varieties, by $\mathcal{O}_A(q)$ resp. $\mathcal{O}_{A^{\mathrm{op}}}(q)$ the line bundles $\mathcal{O}_{X_A \otimes_k \bar{k}}(q)$ resp. $\mathcal{O}_{X_{A^{\mathrm{op}}} \otimes_k \bar{k}}(q)$ and by $[\mathcal{L}_A]$ resp. $[\mathcal{L}_{A^{\mathrm{op}}}]$ the positive degree generator of $\mathrm{Pic}(X_A)$ resp. $\mathrm{Pic}(X_{A^{\mathrm{op}}})$. We get the twisted Segre embedding

$$s_{A,A^{\mathrm{op}}} : X_A \times X_{A^{\mathrm{op}}} \longrightarrow X_{A \otimes A^{\mathrm{op}}} \cong \mathbb{P}^N$$

(see e.g., [Art82] Sect. 4.1). Since

$$\bar{s}^*_{A,A^{\mathrm{op}}} \mathcal{O}_{\mathbb{P}^N \otimes_k \bar{k}}(1) = \mathcal{O}_A(1) \boxtimes \mathcal{O}_{A^{\mathrm{op}}}(1)$$

we get from the induced commutative diagram of the Picard groups

$$\begin{tikzcd} \mathrm{Pic}(\mathbb{P}^N \otimes_k \bar{k}) \arrow[r, "\bar{s}^*_{A,A^{\mathrm{op}}}"] & \mathrm{Pic}((X_A \times X_{A^{\mathrm{op}}}) \otimes_k \bar{k}) = \mathrm{Pic}(X_A \otimes_k \bar{k}) \oplus \mathrm{Pic}(X_{A^{\mathrm{op}}} \otimes_k \bar{k}) \\ \mathrm{Pic}(\mathbb{P}^N) \arrow[u, "\cong"] \arrow[r, "s^*_{A,A^{\mathrm{op}}}"] & \mathrm{Pic}(X_A \times X_{A^{\mathrm{op}}}) \arrow[u, hook] \end{tikzcd}$$

that $\mathcal{O}_A(1) \boxtimes \mathcal{O}_{A^{\mathrm{op}}}(1)$ descents to $X_A \times X_{A^{\mathrm{op}}}$ and

$$s^*_{A,A^{\mathrm{op}}} \mathcal{O}_{\mathbb{P}^N}(1) = (\mathcal{O}_A(1) \boxtimes \mathcal{O}_{A^{\mathrm{op}}}(1))/\Gamma. \tag{4.5.1}$$

In abuse of notation, we just write $s^*_{A,A^{\mathrm{op}}} \mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_A(1) \boxtimes \mathcal{O}_{A^{\mathrm{op}}}(1)$.

Now let $\bar{s}$ be a homotopy fixed point of X_A and $\bar{r}$ a homotopy fixed point of $X_{A^{\mathrm{op}}}$. The canonical map

$$\overline{X_A \times X_{A^{\mathrm{op}}}} \longrightarrow \bar{X}_A \times \bar{X}_{A^{\mathrm{op}}}$$

is a weak equivalence by Lem. 3.1.1. By Cor. 2.6.4

$$[-, K(\hat{\mathbb{Z}}(1), 2)]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}$$

maps weak equivalences to isomorphisms. In particular, we get a unique map $\bar{X}_A \times \bar{X}_{A^{\mathrm{op}}} \to K(\hat{\mathbb{Z}}(1), 2)$ in $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$, making the square

$$\begin{tikzcd} \overline{X_A \times X_{A^{\mathrm{op}}}} \arrow[r, "\bar{s}_{A,A^{\mathrm{op}}}"] \arrow[d, "\sim"] & \bar{\mathbb{P}}^N \arrow[d, "\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^N}(1)]}"] \\ \bar{X}_A \times \bar{X}_{A^{\mathrm{op}}} \arrow[r] & K(\hat{\mathbb{Z}}(1), 2) \end{tikzcd}$$

commutative. Further, for $\mathrm{Ex}_\Gamma(-)$ a functorial fibrant replacement in $\underline{\mathrm{SSets}}_\Gamma$,

$$\bar{X}_A \times \bar{X}_{A^{\mathrm{op}}} \longrightarrow \mathrm{Ex}_\Gamma(\bar{X}_A) \times \mathrm{Ex}_\Gamma(\bar{X}_{A^{\mathrm{op}}})$$

is a levelwise fibrant replacement (since $\pi_q(-)$ preserves finite products in $\underline{\mathrm{SSets}}$, use the existence of a fibrant replacement preserving finite limits and the long exact homotopy sequence of a fibration). In particular, we get a Pro$\mathcal{H}$(SSets)-morphism

$$\bar{X}_A^{h\Gamma} \times \bar{X}_{A^{\mathrm{op}}}^{h\Gamma} = (\bar{X}_A \times \bar{X}_{A^{\mathrm{op}}})^{h\Gamma} \longrightarrow K(\hat{\mathbb{Z}}(1),2)^{h\Gamma} ,$$

i.e. we get a canonical pairing of pro-sets

$$(4.5.2) \quad \langle -,-\rangle : \pi_0(\bar{X}_A^{h\Gamma}) \times \pi_0(\bar{X}_{A^{\mathrm{op}}}^{h\Gamma}) \longrightarrow \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma}) = H^2(\Gamma;\hat{\mathbb{Z}}(1)).$$

4.5.5 Remark. Let k be a field of cohomological dimension ≤ 2 and assume that X_A resp. $X_{A^{\mathrm{op}}}$ admits a homotopy rational point. We want to study the left- and right-kernels of our pairing (4.5.2), i.e. the fibres of the maps $\langle -,\bar{r}\rangle$ resp. $\langle \bar{s},-\rangle$ for $\bar{r} \in \pi_0(\bar{X}_{A^{\mathrm{op}}}^{h\Gamma})$ resp. $\bar{s} \in \pi_0(\bar{X}_A^{h\Gamma})$.

By the symmetry of our construction it suffices to discuss the left-kernels. We get a map

$$\bar{X}_A \simeq \bar{X}_A \times E\Gamma \xrightarrow{\mathrm{id}\times\bar{r}} \bar{X}_A \times \bar{X}_{A^{\mathrm{op}}} \longrightarrow K(\hat{\mathbb{Z}}(1),2) ,$$

i.e. a cohomology class $\beta_{\bar{r}}$ in $H^2(X_A;\hat{\mathbb{Z}}(1))$, s.t. $\langle \bar{s},\bar{r}\rangle$ is nothing but $\bar{s}^*\beta_{\bar{r}}$. It follows that $\langle -,\bar{r}\rangle$ is nothing but the induced map

$$\pi_0(\varphi_{\beta_{\bar{r}}}^{h\Gamma}) : \pi_0(\bar{X}_A^{h\Gamma}) \longrightarrow \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma}) = H^2(\Gamma;\hat{\mathbb{Z}}(1)).$$

Now $\mathrm{res}_\Gamma^{\mathbf{1}}(\bar{r})$ is just the homotopy fixed point given by any geometric point $\bar{y}$ of $X_{A^{\mathrm{op}}}(\bar{k})$. It follows that $\mathrm{res}_\Gamma^{\mathbf{1}}(\varphi_{\beta_{\bar{r}}})$ corresponds to the composition

$$\begin{tikzcd}
X_A \otimes_k \bar{k} \arrow[r, "\mathrm{id}\times\bar{y}"] \arrow[drr] & (X_A \otimes_k \bar{k}) \times (X_{A^{\mathrm{op}}} \otimes_k \bar{k}) \arrow[r, "s_{A,A^{\mathrm{op}}}\otimes_k\bar{k}"] & \mathbb{P}^N \otimes_k \bar{k} \arrow[d, "\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^N\otimes_k\bar{k}}(1)]}"] \\
 & & K(\hat{\mathbb{Z}}(1),2)
\end{tikzcd}$$

i.e. $\mathrm{res}_\Gamma^{\mathbf{1}}(\beta_{\bar{r}})$ is the class

$$\begin{aligned}(\mathrm{id}\times\bar{y})^*(s_{A,A^{\mathrm{op}}}\otimes_k\bar{k})^*\hat{c}_1[\mathcal{O}_{\mathbb{P}^N\otimes_k\bar{k}}(1)] &= \hat{c}_1[\mathcal{O}_{X_A\otimes_k\bar{k}}(1) \boxtimes \bar{y}^*\mathcal{O}_{X_{A^{\mathrm{op}}}\otimes_k\bar{k}}(1)] \\ &= \hat{c}_1[\mathcal{O}_{X_A\otimes_k\bar{k}}(1)].\end{aligned}$$

Thus, $\beta_{\bar{r}}$ is a k-structure of $\hat{c}_1[\mathcal{O}(1)]$ and we get from Cor. 4.4.10 that $\langle -,\bar{r}\rangle$ has trivial fibres at all homotopy fixed points of $\bar{X}_A$ induced by homotopy rational points if $\mathrm{rk}(A) \geq 3$ and is injective if k is of strict cohomological dimension ≤ 2 (e.g. k a p-adic local resp. totally imaginary number field).

4.5.6 Remark. Suppose A is a matrix algebra over k, i.e. $\bar{X}_A$ and $\bar{X}_{A^{\mathrm{op}}}$ are projective spaces. The spectral sequence of [Goe95] Thm. 4.8 suggests that $\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}^n}(1)]}$ induces an isomorphism between homotopy fixed point (pro-)sets

$$(4.5.3) \qquad \pi_0((\bar{\mathbb{P}}^n)^{h\Gamma}) \longrightarrow \pi_0(K(\hat{\mathbb{Z}}(1),2)^{h\Gamma}) = H^2(\Gamma;\hat{\mathbb{Z}}(1))$$

and even a weak equivalence between homotopy fixed points

$$(\bar{\mathbb{P}}^\infty)^{h\Gamma} \longrightarrow K(\hat{\mathbb{Z}}(1),2)^{h\Gamma}$$

(unfortunately, it is fringed along the line $s = t$, so it does not necessarily compute the homotopy fixed point (pro-)set). Under this later weak equivalence, the additive structure of the target would correspond to the infinite Segre embedding

$$s_\infty : \mathbb{P}^\infty \times \mathbb{P}^\infty \longrightarrow \mathbb{P}^\infty$$

(since $s_\infty^*\mathcal{O}(1)$ equals $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$). This suggests, that our pairing corresponds to the one given by the abelian group structure of $H^2(\Gamma; \hat{\mathbb{Z}}(1))$ in the case of a trivial Brauer-Severi variety.

What we do know is that (4.5.3) is injective for $\mathrm{scd}(k) \leq 2$ (see Cor. 4.4.4) and has at least trivial fibers at all homotopy fixed points induced by homotopy rational points if $\mathrm{scd}(k) \leq 2n$ (Lem. 4.4.6). Further in the case $\mathrm{cd}(k) \leq 2$, arguing as in Rem. 4.4.9 we see that (4.5.3) is surjective if and only if the canonical map

$$\pi_0((\bar{\mathbb{P}}^n)^{h\Gamma}) \longrightarrow \mathbb{H}^0(\Gamma; C_\bullet(\bar{\mathbb{P}}^n)^\sharp)_{\int_*=1}$$

is surjective.

We go back to general Brauer-Severi varieties resp. general central simple algebras. By the same arguments as above together with the integral Eilenberg-Zilber Theorem the diagram

$$\begin{array}{ccccc} C_\bullet(E\Gamma) \xrightarrow{\Delta_*} C_\bullet(E\Gamma \times E\Gamma) & & C_\bullet(\bar{X}_A \times \bar{X}_{A^{\mathrm{op}}})^{(\sharp)} \longrightarrow \hat{\mathbb{Z}}(1)[-2] \\ \downarrow \cong & & \downarrow \cong \\ C_\bullet(E\Gamma) \otimes C_\bullet(E\Gamma) & \xrightarrow{\bar{s}\otimes\bar{r}} & C_\bullet(\bar{X}_A)^{(\sharp)} \otimes C_\bullet(\bar{X}_{A^{\mathrm{op}}})^{(\sharp)} \end{array}$$

in $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$ induces canonical pairings for (quasi) homology fixed point sets

$$\langle -,- \rangle : \mathbb{H}^0(\Gamma; C_\bullet(\bar{X}_A)^{(\sharp)})_{\int_*=1} \times \mathbb{H}^0(\Gamma; C_\bullet(\bar{X}_{A^{\mathrm{op}}})^{(\sharp)})_{\int_*=1} \longrightarrow \lim H^2(\Gamma; \hat{\mathbb{Z}}(1)) ,$$

as well. Combining these pairings with Rem. 4.1.3 we also get "mixed" pairings of the form

$$\langle -,- \rangle : [E\Gamma, \bar{X}_A^{h\Gamma}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}_\Gamma})} \times \mathbb{H}^0(\Gamma; C_\bullet(\bar{X}_{A^{\mathrm{op}}})^\sharp)_{\int_*=1} \longrightarrow \lim H^2(\Gamma; \hat{\mathbb{Z}}(1)) .$$

Note, that these pairings are compatible with the limit of the above pairing of pro-sets (4.5.2) by construction.

4.5.7 Remark. Let us also mention, that the latter "mixed" pairing can be enriched to a pairing of a pro-set with a set

$$\langle -,- \rangle : \pi_0(\bar{X}_A^{h\Gamma}) \times \mathbb{H}^0(\Gamma; C_\bullet(\bar{X}_{A^{\mathrm{op}}})^\sharp)_{\int_*=1} \longrightarrow H^2(\Gamma; \hat{\mathbb{Z}}(1)) .$$

Indeed, fix a quasi homology fixed point $\bar{r}$ in the set $\mathbb{H}^0(\Gamma; C_\bullet(\bar{X}_{A^{\mathrm{op}}})^\sharp)_{\int_*=1}$. By the Eilenberg-Zilber theorem we get the $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$-morphism

$$\mathrm{id} \otimes \bar{r} : C_\bullet(\bar{X}_A) = C_\bullet(\bar{X}_A \times E\Gamma) \longrightarrow C_\bullet(\bar{X}_A \times \bar{X}_{A^{\mathrm{op}}}) ,$$

i.e. a canonical $\mathrm{Pro}\mathcal{D}_+(\underline{\mathrm{Mod}}_\Gamma)$-morphism

$$C_\bullet(\bar{X}_A) \longrightarrow \hat{\mathbb{Z}}(1)[-2] .$$

By adjointness this gives a canonical $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)$-morphism

$$\bar{X}_A \longrightarrow K(\hat{\mathbb{Z}}(1), 2) ,$$

i.e. a $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}})$-morphism between homotopy fixed points

$$\bar{X}_A^{h\Gamma} \longrightarrow K(\hat{\mathbb{Z}}(1), 2)^{h\Gamma} .$$

4.5.8 Lemma. *Let $\bar{s}$ be a homotopy or (quasi) homology fixed point of X_A and $\bar{r}$ a homotopy or (quasi) homology fixed point of $X_{A^{\mathrm{op}}}$. Then*

$$d.\langle \bar{s}, \bar{r}\rangle = \bar{s}^*\hat{c}_1[\mathcal{L}_A] + \bar{r}^*\hat{c}_1[\mathcal{L}_{A^{\mathrm{op}}}]$$

Proof: It follows from the definition and (4.5.1) that

$$\langle \bar{s}, \bar{r}\rangle = (\bar{s} \otimes \bar{r})^*\hat{c}_1[\mathcal{O}_A(1) \boxtimes \mathcal{O}_{A^{\mathrm{op}}}(1)]. \tag{4.5.4}$$

By the functoriality of the profinite first Chern class map we get

$$\bar{s}^*\hat{c}_1[\mathcal{L}_A] = (\bar{s} \otimes \bar{r})^*\hat{c}_1[\mathrm{pr}^*_{X_A}\mathcal{L}_A],$$
$$\bar{r}^*\hat{c}_1[\mathcal{L}_{A^{\mathrm{op}}}] = (\bar{s} \otimes \bar{r})^*\hat{c}_1[\mathrm{pr}^*_{X_{A^{\mathrm{op}}}}\mathcal{L}_{A^{\mathrm{op}}}].$$

But in $\mathrm{Pic}(X_A \times X_{A^{\mathrm{op}}})$ we have

$$d.[\mathcal{O}_A(1) \boxtimes \mathcal{O}_{A^{\mathrm{op}}}(1)] = [\mathrm{pr}^*_{X_A}\mathcal{L}_A] + [\mathrm{pr}^*_{X_{A^{\mathrm{op}}}}\mathcal{L}_{A^{\mathrm{op}}}],$$

so the claim follows from (4.5.4). □

4.5.9 Remark. Note that we could replace X_A and $X_{A^{\mathrm{op}}}$ by Brauer-equivalent Brauer-Severi varieties to get similar pairings still satisfying all of the last arguments. We use this observation as follows: For $n+1$ divisible by the period of $[A]$, the Brauer class $[A^{\otimes n}]$ is just the inverse of $[A]$ in $\mathrm{Br}(k)$, i.e. equals $[A^{\mathrm{op}}]$. Thus, if X_A admits a homotopy or (quasi) homology fixed point we replace $X_{A^{\mathrm{op}}}$ by $X_{A^{\otimes n}}$ and get a pairings between non empty sets of homotopy or (quasi) homology fixed points using Rem. 4.1.7.

Now let $\bar{s}'$ be a second homotopy or (quasi) homology fixed point of X_A and $\bar{r}$ a quasi homology fixed point of $X_{A^{\mathrm{op}}}$ (which always exists for $\mathrm{cd}(\Gamma) \leq 2$ by Cor. 3.3.1). Applying Lem. 4.5.8 twice we get

$$(\bar{s}')^*\hat{c}_1[\mathcal{L}_A] = \bar{s}^*\hat{c}_1[\mathcal{L}_A] + d.(\langle \bar{s}', \bar{r}\rangle - \langle \bar{s}, \bar{r}\rangle)$$

in $H^2(\Gamma; \bar{\mathbb{Z}}(1))$. If no such quasi homology fixed point $\bar{r}$ exists, we use Rem. 4.5.9 and argue similar to get:

4.5.10 Proposition. *Let A be a central simple algebra of period d and s resp. $\bar{s}$ a homotopy rational resp. homotopy or (quasi) homology fixed point of the Brauer-Severi variety X_A. Then the classes $s^*\hat{c}_1[\mathcal{L}_A]$ resp. $\bar{s}^*\hat{c}_1[\mathcal{L}_A]$ modulo d are independent from the choice of s resp. $\bar{s}$. In particular, the property that $s^*\hat{c}_1[\mathcal{L}_A]$ resp. $\bar{s}^*\hat{c}_1[\mathcal{L}_A]$ is divisible by d in $H^2(\Gamma;\hat{\mathbb{Z}}(1))$ is independent from the choice of s resp. $\bar{s}$.*

Rem. 4.5.4 and Prop. 4.5.10 motivates our analogue of the weak section conjecture for Brauer-Severi varieties:

4.5.11 Theorem. *Let k be a field of characteristic 0 and X a Brauer-Severi variety over k of period d admitting a homotopy rational point resp. a homotopy or (quasi) homology fixed point s resp. $\bar{s}$. Suppose that the class $s^*\hat{c}_1[\mathcal{L}_X]$ resp. $\bar{s}^*\hat{c}_1[\mathcal{L}_X]$ is divisible by d in $H^2(\Gamma;\hat{\mathbb{Z}}(1))$ for $[\mathcal{L}_X]$ the (positive degree) generator of $\mathrm{Pic}(X)$. Then X is isomorphic over k to a projective space $\mathbb{P}^n$, i.e., X admits a k-rational point.*

We need two more easy technicalities for the proof:

4.5.12 Lemma. *Let A be an abelian group with (pro-) Tate module*

$$\mathrm{T}(A) := \{A[m]\}_m,$$

where the structural maps are given by the multiplication maps. Then $\mathrm{T}(A)$ is torsion free in $\mathrm{Pro}\underline{\mathrm{Ab}}$.

Proof: Multiplication by n is a levelwise map on $\mathrm{T}(A)$, i.e. its kernel $\mathrm{T}(A)[n]$ is given as the levelwise kernels $A[m][n]$ by [AM69] Appendix Prop. 4.1. The induced structural maps

$$A[km][n] \xrightarrow{k.(-)} A[m][n]$$

are trivial for all k divided by n, i.e. $\mathrm{T}(A)[n]$ is a Mittag-Leffler null system in $\mathrm{Pro}\underline{\mathrm{Ab}}$, i.e. trivial. □

As a consequence we get:

4.5.13 Corollary. *Let Y be a k-variety and let $\mathcal{L}$ be a line bundle on Y. Then the Chern class $\hat{c}_1[\mathcal{L}]$ is divisible by an integer r in $H^2(Y;\hat{\mathbb{Z}}(1))$ if and only if the class $[\mathcal{L}]$ is divisible by r in $\mathrm{Pic}(Y)$.*

Proof: Let $\{\mathbb{Z}/m\mathbb{Z}\}_m$ be the pro-system with the obvious transfer maps (i.e. this is just $\hat{\mathbb{Z}}$). We get the pro-system $\{\mathrm{Pic}(Y)\otimes\mathbb{Z}/m\mathbb{Z}\}_m$ satisfying

$$\mathrm{Pic}(Y)\otimes\mathbb{Z}/r\mathbb{Z} = \{\mathrm{Pic}(Y)\otimes\mathbb{Z}/m\mathbb{Z}\}_m\otimes\mathbb{Z}/r\mathbb{Z}$$

(if $\mathrm{Pic}(Y)$ is torsion free and finitely generated, this is just the profinite completion $\mathrm{Pic}(Y)^\wedge$). Thus, it suffices to show that $[\mathcal{L}]$ is divisible by r in the pro-system $\{\mathrm{Pic}(Y)\otimes\mathbb{Z}/m\mathbb{Z}\}_m$.

From the Kummer sequences for various m we get the exact sequence of pro-abelian groups

$$0 \longrightarrow \{\mathrm{Pic}(Y) \otimes \mathbb{Z}/m\mathbb{Z}\}_m \xrightarrow{\hat{c}_1} H^2(Y; \hat{\mathbb{Z}}(1)) \longrightarrow \mathrm{T}H^2(Y; \mathbb{G}_\mathrm{m}) \longrightarrow 0\ .$$

By assumption, the Chern class $\hat{c}_1[\mathcal{L}]$ is divisible by r, say

$$\hat{c}_1[\mathcal{L}] = r.\alpha$$

for a suitable class α in $H^2(Y; \hat{\mathbb{Z}}(1))$. It follows, that the image of α in the (pro-)Tate module $\mathrm{T}H^2(Y; \mathbb{G}_\mathrm{m})$ is killed by r. But $\mathrm{T}H^2(Y; \mathbb{G}_\mathrm{m})$ is torsion free by Lem. 4.5.12, so α lies even in the image of the monomorphism $\hat{c}_1$, i.e. $[\mathcal{L}]$ is divisible by r in $\{\mathrm{Pic}(Y) \otimes \mathbb{Z}/m\mathbb{Z}\}_m$, which finishes the proof. □

Proof of Thm. 4.5.11: As a generator, $[\mathcal{L}_X]$ is not divisible in $\mathrm{Pic}(X)$ by any other integer except ± 1. Thus, by Cor. 4.5.13 it suffices to show that $\hat{c}_1[\mathcal{L}_X]$ is divisible by d in $H^2(X; \hat{\mathbb{Z}}(1))$.

For the sake of our motivation in Rem. 4.5.4, let us first discuss the special case of a homotopy rational point: By Lem. 4.5.3 we have

$$i^*\alpha_{i_*s} = d.\alpha_s$$

and by Lem. 4.3.9 together with Cor. 4.2.6

$$\hat{c}_1[\mathcal{O}_{\mathbb{P}^N}(1)] = \alpha_y = \alpha_{i_*s} + (i_*s)^*\hat{c}_1[\mathcal{O}_{\mathbb{P}^N}(1)]$$

for y any k-rational point of $\mathbb{P}^N$. Combining this, we therefore get

$$\begin{aligned}\hat{c}_1[\mathcal{L}_X] = i^*\hat{c}_1[\mathcal{O}_{\mathbb{P}^N}(1)] &= i^*\alpha_{i_*s} + (i_*s)^*\hat{c}_1[\mathcal{O}_{\mathbb{P}^N}(1)] \\ &= d.\alpha_s + s^*\hat{c}_1[\mathcal{L}_X],\end{aligned}$$

i.e. $\hat{c}_1[\mathcal{L}_X]$ is divisible by d in $H^2(X; \hat{\mathbb{Z}}(1))$ if and only if $s^*\hat{c}_1[\mathcal{L}_X]$ is divisible by d in $H^2(\Gamma; \hat{\mathbb{Z}}(1))$.

This of course is just an explicit version of the following easy argument working for a general homotopy or (quasi) homology fixed point $\bar{s}$, as well: By the proof of Lem. 4.3.1, $\bar{s}$ gives us a compatible levelwise direct sum decomposition

$$H^2(X; \hat{\mathbb{Z}}(1)) = H^2(\Gamma; \hat{\mathbb{Z}}(1)) \oplus H^2(\bar{X}; \hat{\mathbb{Z}}(1)),$$

where the projections onto the two summands are given by $\bar{s}^*$ resp. p^*. Now

$$p^*\hat{c}_1[\mathcal{L}_X] = \hat{c}_1[\mathcal{O}_{X \otimes_k \bar{k}}(d)] = d.\hat{c}_1[\mathcal{O}_{X \otimes_k \bar{k}}(1)],$$

so $\hat{c}_1[\mathcal{L}_X]$ is divisible by d in $H^2(X; \hat{\mathbb{Z}}(1))$ if and only if $\bar{s}^*\hat{c}_1[\mathcal{L}_X]$ is divisible by d in $H^2(\Gamma; \hat{\mathbb{Z}}(1))$. □

Next, we want to give a better characterization of the images of the canonical maps

$$\begin{tikzcd} X(k) \arrow[r] \arrow[dr] \arrow[ddr] \arrow[dddr] & {[B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}} \arrow[d] \\ & {[E\Gamma, \bar{X}]_{\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_\Gamma)}} \arrow[d] \\ & \mathbb{H}^0(\Gamma; C_\bullet(\bar{X}))_{\int_*=1} \arrow[d] \\ & \mathbb{H}^0(\Gamma; C_\bullet(\bar{X})^\sharp)_{\int_*=1} \end{tikzcd} \tag{4.5.5}$$

4.5.14 Remark. Consider the canonical upper horizontal map

$$X(k) \longrightarrow [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}\ . \tag{4.5.6}$$

First, we discuss the actual image of this map if $X(k)$ is nonempty, i.e. for X a projective space $\mathbb{P}^n$. By functoriality of the first profinite Chern class map $\hat{c}_1$, the composition $x^*\hat{c}_1$ is trivial for any k-rational point x of $\mathbb{P}^n$. Thus, for a homotopy rational point s of $\mathbb{P}^n$ to come from a k-rational point of X, it is necessary that the composition $s^*\hat{c}_1$ is trivial.

On the other hand, it is not hard to see that (4.5.6) is in fact the constant map for $\mathbb{P}^n$: Indeed, two arbitrary k-rational points x and y in $\mathbb{P}^n(k)$ factor over a suitable k-morphisms $\mathbb{A}^1 \to \mathbb{P}^n$. But $\mathbb{A}^1 \to B\Gamma$ is a weak equivalence, since $\mathbb{A}^1 \times_{B\Gamma} E\Gamma$ is contractible in characteristic 0.

On the other hand for X an arbitrary Brauer-Severi variety and s a homotopy rational point trivializing $\hat{c}_1$, obviously $s^*\hat{c}_s[\mathcal{L}_X]$ is divisible by d, i.e. $X(k)$ is nonempty by Thm. 4.5.11. Thus, the set S of homotopy rational points s in $[B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$ trivializing the first profinite Chern class map $\hat{c}_1$ is either empty or X splits and the canonical map

$$X(k) \longrightarrow S \subseteq [B\Gamma, X]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}$$

is the constant map.

Since all the non vertical maps in (4.5.5) factor through the canonical map (4.5.6), either $X(k)$ is empty or all these maps are constant by Rem. 4.5.14. We want to give a different description of the resulting **rational homotopy resp. quasi homology fixed point**: Again, for a homotopy or (quasi) homology fixed point $\bar{s}$ of X to come from a k-rational point of X, it is necessary that the composition $\bar{s}^*\hat{c}_1$ is trivial. We will show that the converse statement holds for homotopy resp. quasi homology fixed points over base fields of small (strict) cohomological dimensions, as well:

4.5.15 Theorem. *Let k be a field of characteristic 0. Let X be a Brauer-Severi variety over k. Assume $\mathrm{scd}(\Gamma) \leq 2\mathrm{dim}(X)$ resp. $\mathrm{cd}(\Gamma) \leq 2\mathrm{dim}(X)$. Then the set* *of homotopy fixed points resp. quasi homology fixed points trivializing the first profinite Chern class map is either empty or consists of the unique homotopy fixed point resp. quasi homology fixed point induced by any k-rational point of X. In particular X is isomorphic to a projective space $\mathbb{P}^n$ in the latter case.*

In short: a homotopy or quasi homology fixed point of a Brauer-Severi variety is rational if and only if it trivializes $\hat{c}_1$.

Proof: We proof the claim for homotopy fixed points. Using Cor. 4.4.8 instead of Lem. 4.4.6 the proof is similar in the case of homology fixed points.

Let $\bar{s}$ be a homotopy fixed point of $\mathbb{P}^n$ trivializing the first profinite Chern class map $\hat{c}_1$. Since $H^2(\Gamma; \hat{\mathbb{Z}}(1))$ is the (pro-)Tate module of the Brauer group $\mathrm{Br}(k)$ which is torsion free by Lem. 4.5.12, this is equivalent to the triviality of the class $\bar{s}^*\hat{c}_1[\mathcal{O}_{\mathbb{P}_n}(1)]$ in $H^2(\Gamma; \hat{\mathbb{Z}}(1))$. But under the identification of Lem. 2.6.1, $\bar{s}^*\hat{c}_1[\mathcal{O}_{\mathbb{P}_n}(1)]$ corresponds to the homotopy fixed point $(\varphi_{\hat{c}_1[\mathcal{O}_{\mathbb{P}_n}(1)]})_*\bar{s}$ of $K(\hat{\mathbb{Z}}(1), 2)$. By triviality of $\bar{s}^*\hat{c}_1[\mathcal{O}_{\mathbb{P}_n}(1)]$ this homotopy fixed point lies in the fibre of

$$\pi_0(\varphi_{\alpha_y}^{h\Gamma}) : \pi_0((\bar{\mathbb{P}}^n)^{h\Gamma}, \bar{y}) \longrightarrow \pi_0(K(\hat{\mathbb{Z}}(1), 2)^{h\Gamma}, *)$$

for any k-rational point $y \in \mathbb{P}^n(k)$, which is trivial by Lem. 4.4.6, i.e. $\bar{s}$ and $\bar{y}$ agree in $\pi_0((\bar{\mathbb{P}}^n)^{h\Gamma})$, which finishes the proof for homotopy fixed points. □

4.5.16 Remark. Again, Thm. 4.5.15 applies to all Brauer-Severi varieties over p-adic local resp. totally imaginary number fields: a p-adic local resp. totally imaginary number field has strict cohomological dimension 2 by [NSW08] Cor. 7.2.5 resp. [Hab78] Prop. 12.

As a Corollary we get a nice reformulation of our rationality condition on the first profinite Chern class map given in Thm. 4.5.15:

4.5.17 Corollary. *Let X be a Brauer-Severi variety over k admitting a homotopy resp. quasi homology fixed point $\bar{s}$ and let*

$$f : X \longrightarrow \mathbb{P}^N$$

be a non constant morphism of k-varieties. Further, suppose $\mathrm{scd}(\Gamma) \leq 2\mathrm{dim}(X)$ *resp.* $\mathrm{cd}(\Gamma) \leq 2\mathrm{dim}(X)$. *Then $\bar{s}$ is rational if and only if the push forward $f_*\bar{s}$ is rational. If we only have* $\mathrm{scd}(\Gamma) \leq 2N$ *resp.* $\mathrm{cd}(\Gamma) \leq 2N$, *then X splits if $f_*\bar{s}$ is rational.*

Proof: Use that $H^2(\Gamma; \hat{\mathbb{Z}}(1))$ is torsion free as the Tate-module of $\mathrm{Br}(k)$. □

Let L/k be a finite extension of a local p-adic or a totally imaginary number field k and $\bar{s}$ a homotopy or quasi homology fixed point of a Brauer-Severi variety X over k. Restricting $\bar{s}$ to $\mathrm{Pro}\mathcal{H}(\underline{\mathrm{SSets}}_{\Gamma_L})$ gives a homotopy or quasi homology fixed point $\bar{s}_L := \mathrm{res}_\Gamma^{\Gamma_L}(\bar{s})$ of $X \otimes_k L$. We get the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(X \otimes_k L) \arrow[r, "\hat{c}_1"] & H^2(X \otimes_k L; \hat{\mathbb{Z}}(1)) \arrow[r, "\bar{s}_L^*"] & H^2(\Gamma_L; \hat{\mathbb{Z}}(1)) \\
\mathrm{Pic}(X) \arrow[u, "\cdot a"] \arrow[r, "\hat{c}_1"] & H^2(X; \hat{\mathbb{Z}}(1)) \arrow[u] \arrow[r, "\bar{s}^*"] & H^2(\Gamma; \hat{\mathbb{Z}}(1)) \arrow[u, "\mathrm{res}_\Gamma^{\Gamma_L}"']
\end{tikzcd}$$

for a a suitable divisor of the period of X. The Galois cohomology group $H^2(\Gamma; \hat{\mathbb{Z}}(1))$ resp. $H^2(\Gamma_L; \hat{\mathbb{Z}}(1))$ is just the Tate-module of $\mathrm{Br}(k)$ resp. $\mathrm{Br}(L)$. Since Tate-modules are torsion free (Lem. 4.5.12) and the restriction $\mathrm{res}_\Gamma^{\Gamma_L}$ is just multiplication by the degree $[L : k]$ for k a p-adic local field ([NSW08] Cor. 7.1.4), the vertical right map is a monomorphism in the p-adic local case. The same is true in the case of a totally imaginary number field, as well: Apply the p-adic local case to the commutative diagram

$$\begin{tikzcd}
H^2(k; \hat{\mathbb{Z}}(1)) \arrow[r, "\mathrm{res}_\Gamma^{\Gamma_L}"] \arrow[d] & H^2(L; \hat{\mathbb{Z}}(1)) \arrow[d] \\
\prod_\nu H^2(k_\nu; \hat{\mathbb{Z}}(1)) \arrow[r] & \prod_\nu \prod_{\omega|\nu} H^2(L_\omega; \hat{\mathbb{Z}}(1))
\end{tikzcd}$$

where ν runs through the places of k and ω through the places of L and use Brauer-Hasse-Noether and the left exactness of the Tate-module functor (since it is entirely build out of limits). Thus

$$a.\bar{s}_L^* \hat{c}_1[\mathcal{L}_{X_L}] = \mathrm{res}_\Gamma^{\Gamma_L}(\bar{s}^* \hat{c}_1[\mathcal{L}_X])$$

is trivial if and only if $\bar{s}^* \hat{c}_1[\mathcal{L}_X]$ is trivial, i.e. Thm. 4.5.15 together with Rem. 4.5.16 imply:

4.5.18 Corollary. *Let L/k be a finite extension of a local p-adic or totally imaginary number field k and X a Brauer-Severi variety over k admitting a homotopy or quasi homology fixed point $\bar{s}$. Then $\bar{s}$ is rational if and only if the base extension* $\mathrm{res}_\Gamma^{\Gamma_L}(\bar{s})$ *is rational.*

Arguing similar, we can proof a "local-to-global-principle" for the rationality of homotopy fixed points of Brauer-Severi varieties over totally imaginary number fields (compare this with Cor. 4.4.11):

4.5.19 Corollary. *Let k be a totally imaginary number field and X a Brauer-Severi variety over k admitting a homotopy or quasi homology fixed point $\bar{s}$. Then $\bar{s}$ is rational if and only if for all finite places ν of k the restriction* $\mathrm{res}_\nu(\bar{s})$ *is rational.*

Proof: The "only if" part is obvious, so let us assume that $\mathrm{res}_\nu(\bar{s})$ is rational for all finite places ν of k. Similar as above, there are positive integers a_ν s.t.

$$\mathrm{res}_\nu(\bar{s}^* \hat{c}_1[\mathcal{L}_X]) = a_\nu.\mathrm{res}_\nu(\bar{s})^* \hat{c}_1[\mathcal{L}_{X_\nu}].$$

Since the right hand side is trivial by assumption, Hasse-Brauer-Noether implies

$$\bar{s}^* \hat{c}_1[\mathcal{L}_X] = 0,$$

i.e. $\bar{s}$ is rational by Thm. 4.5.15 and Rem. 4.5.16. $\square$

4.6 A counter example. By the general weak section conjecture for Brauer-Severi varieties we mean the statement of Thm. 4.5.11 without the assumption on the first profinite Chern class map. Unfortunately, such a general weak section conjecture does not hold for Brauer-Severi varieties in general: In this section we will prove the existence of non split Brauer-Severi varieties over p-adic local fields admitting a homotopy rational point.

Let Y be an arbitrary geometrically connected k-variety together with a geometric point $\bar{y}$. Refer to the short exact sequence

$$\mathbf{1} \longrightarrow \pi_1^{\text{ét}}(Y \otimes_k \bar{k}, \bar{y}) \longrightarrow \pi_1^{\text{ét}}(Y, \bar{y}) \longrightarrow \Gamma \longrightarrow \mathbf{1}$$

given by [SGA71] Exp. IX Thm. 6.1 as $\pi_1(Y/k, \bar{y})$. If Y is proper and geometrically unibranched, these are just the first terms of the long exact homotopy sequence given by the fibre sequence

$$\bar{Y} \longrightarrow Y \longrightarrow B\Gamma$$

(see [Fri73] Cor. 4.8). By a **section of $\boldsymbol{\pi_1(Y/k, \bar{y})}$** we mean a section of the canonical map $\pi_1^{\text{ét}}(Y, \bar{y}) \to \Gamma$. Finally, by the **relative Brauer group of $\boldsymbol{Y}$** we mean the kernel

$$\mathrm{Br}(Y/k) := \ker(\ H^2(\Gamma; \mathbb{G}_{\mathrm{m}}) \xrightarrow{\text{can.}} H^2(Y; \mathbb{G}_{\mathrm{m}})\ ).$$

Note, that the relative Brauer group of a Brauer-Severi variety X_A corresponding to the Brauer class $[A] \in \mathrm{Br}(k)$ is generated by $[A]$: This is Amitsur's Theorem (e.g. [GS06] Thm. 5.4.1) together with the injectivity of the canonical map

$$H^2(X; \mathbb{G}_{\mathrm{m}}) \longrightarrow H^2(k(X); \mathbb{G}_{\mathrm{m}})$$

induced by the generic point (see e.g. [Mil80] Chap. III Ex. 2.22).

4.6.1 Remark. Let X be a smooth projective curve over a characteristic 0 field k of genus ≥ 1 (e.g., a projective anabelian curve) admitting a section of $\pi_1(X/k, \bar{x})$ for a suitable geometric point $\bar{x}$ of X. Recall that X has the $K(\pi, 1)$ property (see e.g., [Sti02] Prop. A.4.1). Thus, the roof of canonical morphisms

$$\begin{tikzcd} & B\Pi(X) & \\ X \arrow[ur, "\sim"] & & B\pi_1(X, \bar{x}) \arrow[ul, "\sim"'] \end{tikzcd}$$

induces an isomorphism $X \cong B\pi_1(X, \bar{x})$ in $\mathcal{H}(\underline{\mathrm{SSets}} \downarrow B\Gamma)$. In particular, our section of $\pi_1(X/k, \bar{x})$ induces a homotopy rational point

$$s \in [B\Gamma, X]_{\mathcal{H}(\underline{\mathrm{SSets}} \downarrow B\Gamma)}$$

by the functoriality of the nerve functor $B(-)$.

4.6.2 Remark. Let X be a connected smooth projective curve over a characteristic 0 field k. Further, let $[A]$ be a class in the relative Brauer group $\mathrm{Br}(X/k)$ given by a central simple algebra A over k and let X_A be the corresponding Brauer-Severi variety. The central simple algebra A splits over the function field $k(X)$ of X, i.e. X_A admits a $k(X)$-rational point. This $k(X)$-rational point extents to a non constant rational map

$$\begin{tikzcd} X \arrow[r, dashed, "f"] & X_A \\ \mathrm{Spec}(k(X)) \arrow[u, "\text{can.}"] \arrow[ur] & \end{tikzcd}$$

which is even regular, since X_A is proper over k: Say f restricts to a regular function $f_{|U}$ on $U \hookrightarrow X$ open. The local rings of the closed points of X are valuation rings so $f_{|U}$ extents uniquely to a regular function on X by the Valuative Criterion of Properness.

Say, X is even a curve of genus ≥ 1 admitting a section of $\pi_1(X/k, \bar{x})$ for a suitable geometric point $\bar{x}$ of X. By Rem. 4.6.1 this section induces a homotopy rational point $s \in [B\Gamma, X]_{\mathcal{H}(\underline{\mathrm{SSets}}\downarrow B\Gamma)}$, i.e. the push forward $f_* s$ is a homotopy rational point of X_A.

For the rest of this subsection, let k be a p-adic local field. Recall from [NSW08] Cor. 7.1.4 that

$$\mathrm{Br}(k) = \mathbb{Q}/\mathbb{Z}.$$

Further, recall that the **index** $\mathrm{ind}(Y)$ of a smooth projective curve Y over k is the greatest common divisor of the degrees $[k(y) : k]$ for all closed points y of Y. Over p-adic local fields k, the index of Y equals the order of the relative Brauer group $\mathrm{Br}(Y/k)$ (see [Lic69] Thm. 3), i.e.

$$\mathrm{Br}(Y/k) = \frac{1}{\mathrm{ind}(Y)}\mathbb{Z}/\mathbb{Z}.$$

4.6.3 Remark. Let X be a smooth projective curve over the p-adic local field k of genus ≥ 1 (e.g., a projective anabelian curve) admitting a section s of $\pi_1(X/k, \bar{x})$ for a suitable geometric point $\bar{x}$ of X. Let $[A]$ be the generator of $\mathrm{Br}(X/k)$ and $f : X \to X_A$ a morphism as in Rem. 4.6.2. Then

$$[\mathcal{L}_f^{(0)}] := (f \otimes_k \bar{k})^*[\mathcal{O}_{X \otimes_k \bar{k}}(1)]$$

is a Γ-invariant class of a line bundle in $\mathrm{Pic}(X \otimes_k \bar{k})$ with Brauer-obstruction $[A]$ (i.e. the differential $\partial_2^{1,1}$ in the Hochschild-Serre spectral sequence $\mathrm{HS}_*^{\bullet,\bullet}(X; \mathbb{G}_{\mathrm{m}})$): Indeed, $[A]$ is the Brauer-obstruction of $[\mathcal{O}_{X \otimes_k \bar{k}}(1)]$ since $\mathrm{Br}(X_A/k)$ is generated by $[A]$. Thus, $[\mathcal{L}_f^{(0)}]$ is a geometric d^{th}-root of

$$[\mathcal{L}_f] := f^*[\mathcal{L}_{X_A}]$$

for $[\mathcal{L}_{X_A}]$ the positive degree generator of $\mathrm{Pic}(X_A)$ and d the period of $[A]$.

Now for $\mathrm{Br}(X/k)$ to be trivial, it would suffices to check if $s^*\hat{c}_1[\mathcal{L}]$ is divisible by d in $H^2(\Gamma; \hat{\mathbb{Z}}(1))$: Indeed, then the induced homotopy rational point f_*s of X_A would satisfy the condition of Thm. 4.5.11, i.e. X_A would split. Unfortunately we do not know how to show this divisibility of $s^*\hat{c}_1[\mathcal{L}]$ on genus ≥ 2 curves. For genus 1 curves, this is wrong in general: See Cor. 4.6.6 below.

To get a counter example of the general weak section conjecture for Brauer-Severi varieties it would therefore suffice to show the existence of a smooth projective curve X of genus ≥ 1 with non trivial relative Brauer group $\mathrm{Br}(X/k)$ admitting a section s of $\pi_1(X/k, \bar{x})$ for a suitable geometric point $\bar{x}$ of X. Granting the local weak section conjecture, this should be impossible for smooth projective curves of genus > 1. Thus we restrict our search to genus 1 curves, i.e. torsors under elliptic curves.

4.6.4 Lemma. *Let X be a torsor under an elliptic curve E over k a p-adic local field. Then X splits if and only if the relative Brauer group $\mathrm{Br}(X/k)$ is trivial.*

Proof: As a torsor under an elliptic curve, the canonical map

$$X \longrightarrow \mathrm{Alb}^1(X)$$

into the Albanese-torsor of X is an isomorphism (cf. [Gro62] Thm. 3.3). As X is a curve, $\mathrm{Alb}^1(X)$ is just $\underline{\mathrm{Pic}}^1_X$ and the above isomorphism is an isomorphism of $E = \underline{\mathrm{Pic}}^0_X$-torsors. Thus, X splits if and only if it has index 1, i.e. if and only if $\mathrm{Br}(X/k)$ is trivial. $\square$

Thus, any non split torsor under an elliptic curve over a local p-adic field admitting a section of its fundamental group sequence would produce non split Brauer-Severi varieties admitting homotopy rational points. Now it is well known to anabelian geometers that such torsors do exist:

4.6.5 Lemma. *(see [Sti12] Prop. 183) Let E be an elliptic curve over a local p-adic field k. Then there are non split E-torsors X whose fundamental group sequence $\pi_1(X/k, \bar{x})$ splits for a suitable geometric point $\bar{x}$ of X.*

Sketch of proof: By [Sti12] Cor. 177 there is an exact sequence

$$0 \longrightarrow \mathrm{Div}(H^1(k; E)) \longrightarrow H^1(k; E) \xrightarrow{\delta} H^2(k; \pi_1(E \otimes_k \bar{k})) ,$$

where $\mathrm{Div}(H^1(k; E))$ is the maximal divisible subgroup of $H^1(k; E)$ and δ maps the class of an E-torsor to the class of its fundamental group sequence $[\pi_1(X/k)]$ (see [Sti12] Prop. 174). But

$$\mathrm{Div}(H^1(k; E)) = (\mathbb{Q}_p/\mathbb{Z}_p)^{[k:\mathbb{Q}_p]},$$

which could easily be computed using [Mil86] Cor. 3.4 and Lem. 3.3. Thus, there are plenty of non split E-torsors X with a split exact fundamental group sequence $\pi_1(X/k)$. $\square$

From [Sti10] Thm. 15 we know that the relative Brauer group $\mathrm{Br}(X/k)$ of a torsor as in Lem. 4.6.5 is p-torsion. Combining this with Lem. 4.6.4 we get:

4.6.6 Corollary. *Let k be a local p-adic field and $[A]$ a non trivial Brauer class in the p-torsion part $\mathrm{Br}(k)[p^\infty]$ of the Brauer group. Then X_A admits a non rational homotopy rational point*

$$s \in [B\Gamma, X_A]_{\mathcal{H}(\mathrm{Pro}\underline{\mathrm{SSets}}\downarrow B\Gamma)}.$$

Proof: We have to show that any p-torsion class $[A]$ of $\mathrm{Br}(k)$ lies in the relative Brauer group of a suitable "bad" genus 1 curves as in Lem. 4.6.5.

Start with any elliptic curve E over k and X a non split E-torsor admitting a section s of $\pi_1(X/k, \bar{x})$ for a suitable geometric point $\bar{x}$ of X. Let

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ B\Gamma \arrow[ur, "r"] \arrow[r, "s"] & X \end{tikzcd}$$

be a neighbourhood of the section s, i.e. h finite étale and r a section of $\pi_1(Y/k, \bar{y})$ for a suitable geometric point $\bar{y}$ of Y, compatible with s under h. By the Riemann-Hurwitz formula, Y is still a genus 1 curve, i.e. a torsor under an elliptic curve. As

$$\mathrm{Br}(X/k) \leq \mathrm{Br}(Y/k),$$

Y is still a non split torsors under an elliptic curve whose fundamental group sequence $\pi_1(Y/k, \bar{y})$ splits. Now $\mathrm{Br}(X/k)$ is non trivial by Lem. 4.6.4 so

$$\operatorname*{colim}_{(Y,r)} \mathrm{Br}(Y/k) \leq \mathrm{Br}(k)[p^\infty]$$

is unbounded by [Sti12] Prop. 122, where (Y, r) runs through all neighbourhoods of the section s of X. But $\mathrm{Br}(k)[p^\infty]$ is just $\mathbb{Q}_p/\mathbb{Z}_p$, i.e. this colimit is already $\mathrm{Br}(k)[p^\infty]$, which finishes the proof. □

4.6.7 Remark. Using Cor. 4.5.18 we get that the homotopy rational point s of X_A given by Cor. 4.6.6 will never become rational after a finite extension L/k. But $X_K \otimes_k L$ splits for sufficiently large L/k, i.e. admits at least two homotopy rational points: the rational and at least one non rational one.

Let k be a local p-adic field and $[A]$ a (non trivial) Brauer class in the p-torsion part $\mathrm{Br}(k)[p]$ of the Brauer group. We want to compare the homotopy rational points of X_A given by Cor. 4.6.6 resp. the induced homotopy or quasi homology fixed points:

4.6.8 Proposition. *Let k be a local p-adic field and $[A]$ a (non trivial) Brauer class in the p-torsion part $\mathrm{Br}(k)[p]$ of the Brauer group. Further, for $\nu = 1, 2$ let X_ν be two genus 1 curves admitting a section s_ν of the resp. fundamental group sequences. Suppose $[A]$ is contained in both $\mathrm{Br}(X_\nu/k)$. We get two non constant morphisms*

$$f_\nu :\ X_\nu \longrightarrow X_A$$

with induced classes of line bundles $[\mathcal{L}_{f_\nu}] = f_\nu^*[\mathcal{L}_{X_A}]$, i.e. two homotopy rational points $(f_\nu)_* s_\nu$. Then the two induced homotopy or quasi homology fixed points

$(f_\nu)_*\bar{s}_\nu$ *agree in* $\pi_0(\bar{X}_A^{h\Gamma})$ *resp.* $\mathbb{H}^0(\Gamma; C_\bullet(\bar{X}_A)^\sharp)_{\int_*=1}$ *if and only if the two classes* $s_\nu^*\hat{c}_1[\mathcal{L}_{f_\nu}]$ *agree in* $H^2(\Gamma;\hat{\mathbb{Z}}(1))$.

Proof: We proof the statement for homotopy fixed points, the proof of the statement for quasi homology fixed points is similar.

Let r_ν be $(f_\nu)_*s_\nu$ and $r = r_1$. By the injectivity of $\pi_0(\varphi_{\alpha_r}^{h\Gamma})$ (see Cor. 4.4.4), it suffices to compare the classes

$$r_\nu^*\alpha_r.$$

Let d be the period of $[A]$ and $i : X_A \to \mathbb{P}^n$ the twisted d-uple embedding. From Lem. 4.5.3 we get

$$i^*\alpha_{i_*r} = d.\alpha_r$$

in $H^2(X_A;\hat{\mathbb{Z}}(1))$. Further, as the Tate module of the Brauer group, $H^2(\Gamma;\hat{\mathbb{Z}}(1))$ is torsion free. Thus it suffices to compare the classes

$$(i_*r_\nu)^*\alpha_{i_*r}.$$

Let y be any k-rational point of $\mathbb{P}^N$. From Cor. 4.2.6 we get

$$\alpha_y = \alpha_{i_*r} + (i_*r)^*\alpha_y$$

in $H^2(\mathbb{P}^N;\hat{\mathbb{Z}}(1))$, where $(i_*r)^*\alpha_y$ is a constant coming from $H^2(\Gamma;\hat{\mathbb{Z}}(1))$. Thus, pullback of this constant along a homotopy fixed point is again the class $(i_*r)^*\alpha_y$ in $H^2(\Gamma;\hat{\mathbb{Z}}(1))$, i.e. is independent form the choice of this homotopy fixed point. It follows that it suffices to compare the classes

$$(i_*r_\nu)^*\alpha_y.$$

But α_y is just $\hat{c}_1[\mathcal{O}_{\mathbb{P}^N}(1)]$ by Lem. 4.3.9. Thus, unraveling the definitions we see that in fact it suffices to compare the classes

$$s_\nu^*\hat{c}_1[\mathcal{L}_{f_\nu}]$$

in $H^2(\Gamma;\hat{\mathbb{Z}}(1))$, which was exactly our claim. $\square$

References

[AM69] Michael Artin and Barry Mazur. *Etale Homotopy*, volume 100 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[Art71] Michael Artin. On the Joins of Hensel Rings. *Advances in Mathematics*, 7: 282–296, 1971.

[Art82] Michael Artin. Brauer-Severi Varieties (Notes by A. Verschoren). In *Brauer Groups in Ring Theory and Algebraic Geometry*, volume 917 of *Lecture Notes in Mathematics*, pages 194–210. Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[BK72] Aldridge K. Bousfield and Daniel M. Kan. *Homotopy Limits, Completions and Localizations*, volume 304 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[Cox79a] David A. Cox. Homotopy Theory of Simplicial Schemes. *Compositio Mathematica*, 39(3): 263–296, 1979.

[Cox79b] David A. Cox. The Etale Homotopy Type of Varieties over $\mathbb{R}$. *Proceedings of the American Mathematical Society*, 76(1): 17–22, 1979.

[DS95] William G. Dwyer and Jan Spalinski. Homotopy Theories and Model Categories. In *Handbook of Algebraic Topology*, pages 73–126. Elsevier, Amsterdam, 1995.

[EH76] David A. Edwards and Harold M. Hastings. *Čech and Steenrod Homotopy Theories with Applications to Geometric Topology*, volume 542 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[Fri73] Eric M. Friedlander. Fibrations in Etale Homotopy Theory. *Puplications mathématiques de l' I.H.É.S.*, 42: 5–46, 1973.

[Fri82] Eric M. Friedlander. *Etale Homotopy of Simplicial Schemes*, volume 104 of *Annals of Mathematics Studies*. Princeton University Press, 1982.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel-Boston-Berlin, 1999.

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of Homological Algebra*. Springer Monographs in Mathematics. Springer, Berlin-Heidelberg-New York, second edition, 2003.

[Goe95] Paul G. Goerss. Homotopy Fixed Points For Galois Groups. *Contemporary Mathematics*, 181: 187–224, 1995.

[Gro62] Alexander Grothendieck. Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard : propriétés générales. *Séminaire Bourbaki*, 7: 221–243, 1961-1962.

[Gro65] Alexander Grothendieck. Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): IV. Étude locale des schémas et des morphismes de schémas, Seconde partie. *Publ. Math. IHES*, 24: 5–231, 1965.

[GS06] Philippe Gille and Tamás Szamuely. *Central Simple Algebras and Galois Cohomology*, volume 101 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 2006.

[GZ67] Peter Gabriel and Michel Zisman. *Calculus of Fractions and Homotopy Theory*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[Hab78] Klaus Haberland. *Galois Cohomology of Algebraic Number Fields*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[Hir03] Philip S. Hirschhorn. *Model Categories and Their Localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2003.

[Isa01] Daniel C. Isaksen. A Model Structure on the Category of Pro-Simplicial Sets. *Transactions of the American Mathematical Society*, 353(7): 2805–2841, 2001.

[Isa02] Daniel C. Isaksen. Calculating Limits and Colimits in Pro-Categories. *Fundamenta Mathematicae*, 175: 175–194, 2002.

[Isa04] Daniel C. Isaksen. Etale Realization on the $\mathbb{A}^1$-Homotopy Theory of Schemes. *Advances in Mathematics*, 184: 37–63, 2004.

[Jar86] John F. Jardine. Simplicial Presheaves. *Journal of Pure and Applied Algbra*, 47(1): 35–87, 1986.

[Jar87] John F. Jardine. Stable Homotopy Theory of Simplicial Presheaves. *Canadian Journal of Mathematics*, 39(3): 733–747, 1987.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer-Verlag, Berlin-Heidelberg-New York, 2006.

[Lic69] Stephen Lichtenbaum. Duality Theorems for Curves over p-adic Fields. *Inventiones Mathematicae*, 7(2): 120–136, 1969.

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton mathematical series*. Univ. Pr., Princeton, N.J., 1980.

[Mil86] James S. Milne. *Arithmetic Duality Theorems*, volume 1 of *Perspectives in Mathematics*. Academic Press, Inc., Boston-Orlando-San Diego-..., 1986.

[Mor12] Fabien Morel. $\mathbb{A}^1$-*Algebraic Topology over a Field*, volume 2052 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 2012.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-Homotopy Theory of Schemes. *Puplications mathématiques de l' I.H.É.S.*, 90: 45–143, 1999.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields, Second Edition*, volume 323 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-Heidelberg-New York, 2008.

[Pre84] Alexander Prestel. *Lectures on Formally Real Fields*, volume 1093 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[Qui08] Gereon Quick. Profinite Homotopy Theory. *Documenta Mathematica*, 13: 585–612, 2008.

[Qui11] Gereon Quick. Continuous Group Actions on Profinite Spaces. *Journal of Pure and Applied Algbra*, 215(5): 1024–1039, 2011.

[Sch12] Alexander Schmidt. Motivic Aspects of Anabelian Geometry. In *Galois-Teichmüller Theory and Arithmetic Geometry*, volume 63 of *Advanced Studies in Pure Mathematics*, pages 503–517. 2012.

[Ser51] Jean-Pierre Serre. Homologie Singulière des Espaces Fibrés. *Annals of Mathematics*, 54(3): 425–505, 1951.

[Ser53] Jean-Pierre Serre. Groupes d'Homotopie et Classes de Groupes Abéliens. *Annals of Mathematics*, 58(2): 258–294, 1953.

[Ser02] Jean-Pierre Serre. *Galois Cohomology*. Springer monographs in mathematics. Springer, Berlin-Heidelberg-New York, corr. 2. print. of the 1. engl. ed. edition, 2002.

[SGA71] *Revêtements Etales et Group Fondamental (SGA 1)*, volume 224 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1971. Un séminaire dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud.

[SGA72] *Théorie de Topos et Cohomologie Etale des Schémas (SGA 4), Tome 2*, volume 270 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1972. Un séminaire dirigé par M. Artin, A. Grothendieck, J.L. Verdier avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat.

[SGA73] *Théorie de Topos et Cohomologie Etale des Schémas (SGA 4), Tome 3*, volume 305 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1973. Un séminaire dirigé par M. Artin, A. Grothendieck, J.L. Verdier avec la collaboration de P. Deligne, B. Saint-Donat.

[Sti02] Jakob Stix. *Projective Anabelian Curves in Positive Characteristic and Descent Theory for Log-Etale Covers*, volume 354 of *Bonner Mathematische Schriften*. 2002.

[Sti10] Jakob Stix. On the period-index problem in light of the section conjecture. *American Journal of Mathematics*, 132(1): 157–180, 2010.

[Sti12] Jakob Stix. *Rational Points and Arithmetic of Fundamental Groups, Evidence for the Section Conjecture*, volume 2054 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 2012.

[Sul74] Dennis Sullivan. Genetics of Homotopy Theory and the Adams Conjecture. *The Annals of Mathematics, Second Series*, 100(1): 1–79, 1974.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1994.