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Iwasawa Theory
of
p-adic Lie Extensions

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To Sofia,

> whose support and patience have been invaluable,

and to Federico,

> who has been providing all those refreshing wonderful distractions.

Contents

Introduction

The spirit of (classical) Iwasawa theory is imbued with Iwasawa's "revolutionary idea that previously inaccessible information about the arithmetic of a number field k can be obtained by investigating certain infinite towers of number fields lying above k" - as J. Coates mentioned in the introduction of [10]. The original example is the p-cyclotomic tower for a fixed rational prime p

$$k \subseteq k_0 \subseteq \cdots \subseteq k_n \cdots \subseteq k_\infty,$$

consisting of the fields $k_n = k(\mu_{p^{n+1}})$ obtained by adjoining the p^{n+1}-roots of unity - or more sophisticated the p^{n+1}-torsion points of the multiplicative group $\mathbb{G}_m$ - to k.
More precisely, the strategy of (the algebraic part of) Iwasawa theory consists in studying abelian Galois groups N over a $\mathbb{Z}_p$-extension k_∞ of k as a module over the completed group algebra, the Iwasawa algebra,

$$\Lambda(\Gamma) = \mathbb{Z}_p[\![\Gamma]\!] = \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma_n]$$

where $\Gamma = G(k_\infty/k) \cong \mathbb{Z}_p$, $\Gamma_n = p^n\Gamma$, and $\mathbb{Z}_p[\Gamma/\Gamma_n]$ denotes the group ring of Γ/Γ_n with coefficients in $\mathbb{Z}_p$. Since $\Lambda = \Lambda(\Gamma)$, which is (uncanonically) isomorphic to the ring of formal power series $\mathbb{Z}_p[[T]]$ in one indeterminate T with coefficients in $\mathbb{Z}_p$, is a commutative regular local ring of dimension 2, the general structure theory for such rings can be used to determine the Galois module structure of N. In particular, one associates to N its λ- resp. μ-invariant and its characteristic ideal, which bear a lot of information about the arithmetic of the ground field k or the fields k_n in the tower, e.g. if one takes for N the Λ-torsion module $X_{nr} := G(L_\infty/k_\infty)$ where L_∞ is the maximal unramified abelian p-extension of k_∞.

In [40] Mazur applied techniques similar to those of Iwasawa to study the Mordell-Weil group $\mathcal{A}(k_n)$ and the p-primary Shafarevich-Tate group $Ш(\mathcal{A}, k_n)$ of an abelian variety $\mathcal{A}$ over a number field k in a $\mathbb{Z}_p$-tower $k_n \subseteq k_\infty$ above k. For simplicity let us restrict to the case of an elliptic curve E. In this context the Pontryagin dual $\mathrm{Sel}(E,k_\infty)^\vee$ of the p-primary Selmer group $\mathrm{Sel}(E,k_\infty)$ over k_∞ plays a similar role as the module X_{nr} before. There is the long standing

Conjecture *(Mazur) If E has good, ordinary reduction at all places lying above p, then $\mathrm{Sel}(E,k_\infty)^\vee$ is a $\Lambda(\Gamma)$-torsion module.*

This conjecture has been proved recently by Kato [36] for $k|\mathbb{Q}$ abelian if E is already defined over $\mathbb{Q}$ (and hence modular due to Wiles et. al.). If E has supersingular reduction at some prime $\nu|p$ of k, then $\mathrm{Sel}(E,k_\infty)^\vee$ is definitely not Λ-torsion. The precise Λ-rank is predicted by Schneider [56]:

Conjecture *(Schneider)*

$$\mathrm{rk}_{\Lambda(\Gamma)}\mathrm{Sel}(E,k_\infty)^\vee = \sum_{\nu|p,\ ssg} [k_\nu : \mathbb{Q}_p],$$

where the sum varies over the primes $\nu|p$ where E has supersingular reduction.

Now assume that E has good, ordinary reduction at all places of k lying above p, that $\mathrm{Sel}(E,k_\infty)^\vee$ is $\Lambda(\Gamma)$-torsion and that $Ш(E,k_n)$ is finite for all $n \geq 0$. Then Mazur proves an asymptotic formula for the order of $Ш(E,k_n)$ in the cyclotomic tower involving the invariants of the $\Lambda(\Gamma)$-module $\mathrm{Sel}(E,k_\infty)^\vee$. Furthermore, its λ-invariant gives an upper bound for the $\mathbb{Z}$-rank of the Mordell-Weil groups $E(k_n)$. If, in addition, E is modular, there exists a Main Conjecture which links the characteristic ideal of $\mathrm{Sel}(E,k_\infty)^\vee$ to a p-adic L-function $\mathcal{L}_{MSD}$ constructed by Mazur and Swinnerton-Dyer ([41] and [42]). Roughly speaking it interpolates - up to the real period of E and a further factor - the values at 1 of the twisted Hasse-Weil L-series of E. For elliptic curves E over $\mathbb{Q}$, Kato proves "one half" of this conjecture: the characteristic ideal of $\mathrm{Sel}(E,k_\infty)^\vee$ at least contains $p^m\mathcal{L}_{MSD}$. Conjecturally, $\mathcal{L}_{MSD}$ should be in $\Lambda(\Gamma)$, but it is only known to be in $\Lambda(\Gamma)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ in general.

Since there is a beautiful Iwasawa theory for the field k_{cycl} obtained by adjoining the p-power roots of unity to a number field k, it seems natural to expect a reasonable analogous theory for the field $k_\infty = k(E(p))$ obtained by adjoining to k all p-power torsion points of an elliptic curve E - or more generally, abelian variety - defined over k.

If E admits complex multiplication (CM), the Galois group $G = G(k_\infty/k)$ is an open subgroup of $({\mathbb{Z}_p}^*)^2$ while otherwise - due to a celebrated theorem of Serre [58] - an open subgroup of $Gl_2(\mathbb{Z}_p)$. The last case is quite different as we will see in what follows.

The CM-case leads to the study of modules over the Iwasawa algebra $\Lambda(G)$ isomorphic to the ring of formal power series $\mathbb{Z}_p[[S,T]]$ in two indeterminates S and T. Therefore the structure theory for modules over regular local rings again applies again. In general, this theory is well-understood and almost complete thanks to the work of Coates-Wiles ([13], [14]) and Perrin-Riou [52]. For this situation a two-variable Main conjecture was formulated by Yager [65] and proved by Rubin [54].

In the non-CM case only little is known. It was M. Harris [25] in 1979 who began to study the Selmer group $\mathrm{Sel}(E,k_\infty)$ over the extension $k_\infty = k(E(p))$ in order to obtain information about the asymptotic growth of the Mordell-Weil group $E(k_n)$ in the "canonical tower"

$$k \subseteq k_0 \subseteq \cdots \subseteq k_\infty,$$

where $k_i = k(E[p^{i+1}])$ is the trivializing extension for the Galois module $E[p^{i+1}]$ of p^{i+1}-torsion points of E. Since $G = G(k_\infty/k)$ is an open subgroup of $Gl_2(\mathbb{Z}_p)$ and so a (compact) p-adic Lie group, Harris' thesis can be considered as the birth of the *Iwasawa theory of (non-commutative) p-adic Lie groups.* The underlying idea is that it is easier to compute the Selmer group over the trivializing extension k_∞ than for example over the cyclotomic k_{cycl}. Via descent to k_n - as in Mazur's theory - one hopes to get the desired information on $\mathrm{Sel}(E,k_n)$ provided that the "difference" between $\mathrm{Sel}(E,k_n)$ and $\mathrm{Sel}(E,k_\infty)^{G(k_\infty/k_n)}$ can be controlled. For asymptotic upper bounds of the Mordell-Weil rank over the intermediate fields in the above tower the following conjecture becomes crucial

Conjecture *(Harris) If E has good, ordinary reduction at all places lying above p, then* $\mathrm{Sel}(E,k_\infty)^\vee$ *is a $\Lambda(G)$-torsion module.*

But for the advantage of working over the trivializing extension one has to pay: The Iwasawa algebra $\Lambda = \Lambda(G)$ is a *non-commutative* ring and there exists no satisfactory theory for modules over such rings. We will come back to this problem below but we already would like to mention that $\Lambda(G)$ is at least a both left and right Noetherian ring without zero divisors if we assume that G is a torsion-free p-adic pro-p-group.
However, Harris' thesis contains some errors [1] and many questions were left untouched.

In the late 90^{th} J. Coates and S. Howson ([12], [11], [8], [29]) as well as Y. Ochi [49] revived the Iwasawa theory of p-adic Lie groups. First of all,

[1] See [27] for some corrections and a discussion which errors are irreparabel.

Coates-Howson refined Harris' conjecture to a precise statement of Λ-ranks - analogous to Schneider's conjecture - in case that E has good supersingular reduction at some places ν above p. Assume that E has good reduction at every prime above p and that $G = G(k(E(p))/k)$ is a torsion-free pro-p-group. Then they prove that

$$\sum_{\nu|p,\ ssg} [k_\nu : \mathbb{Q}_p] \leq \mathrm{rk}_{\Lambda(G)} \mathrm{Sel}(E, K_\infty)^\vee \leq [k : \mathbb{Q}],$$

where the sum varies over the primes $\nu|p$ where E has supersingular reduction, and they formulated the following

Conjecture *(Coates-Howson)*

$$\mathrm{rk}_{\Lambda(G)} \mathrm{Sel}(E, K_\infty)^\vee = \sum_{\nu|p,\ ssg} [k_\nu : \mathbb{Q}_p].$$

Assuming this latter conjecture, they proved a remarkable formula for the Euler characteristic

$$\chi := \chi(G, \mathrm{Sel}(E, k_\infty)) = \prod_i (\#\mathrm{H}^i(G, \mathrm{Sel}(E, k_\infty)))^{(-1)^i}$$

(if defined) for the Selmer group (see [11]).

Theorem *(Coates-Howson) Assume that $p \geq 5$ is a prime such that*

(i) *E has good ordinary reduction at all places ν of k dividing p,*

(ii) $\mathrm{Sel}(E, k)$ *is finite,*

(iii) $\mathrm{Sel}(E, k_\infty)^\vee$ *is $\Lambda(G)$-torsion.*

Then $\mathrm{H}^i(G, \mathrm{Sel}(E, k_\infty))$ is finite for $i = 0, 1$ and equal to 0 for $i \geq 2$, and

$$\chi = \frac{\#\text{Ш}(E, k) \cdot \prod_{\nu|p} |\#\widetilde{E}_\nu(k_\nu)|_p^{-2}}{(\#E(k)(p))^2} \times \prod_\nu |c_\nu|_p^{-1} \times \prod_{\nu\ \epsilon\ \mathcal{M}} |L_\nu(E, 1)|_p,$$

where the p-adic valuation $|\ |_p$ of $\mathbb{Q}$ is normalizes so that $|p|_p = p^{-1}$.

Here, for each prime ν of k, $c_\nu = |E(k_\nu) : E_0(k_\nu)|$ denotes the local Tamagawa factor at ν, where $E_0(k_\nu)$ is the subgroup of $E(k_\nu)$ consisting of the points with non-singular reduction at ν; $L_\nu(E, s)$ denotes the Euler factor of

E at ν. Finally, $\widetilde{E}_\nu$ is the reduction of E modulo ν while $\mathcal{M}$ is defined to be the finite set of places of k where the classical j-invariant of E is non-integral, i.e. where E has potential multiplicative reduction.

Y. Ochi, who studied more generally the properties of Iwasawa modules (over local and global p-adic Lie extensions) arising as cohomology groups from certain p-adic representations, determined the projective $\Lambda(G)$-dimension of the Pontryagin dual of the Selmer group under the assumption that the conjecture of Coates-Howson holds:

$$\mathrm{pd}_{\Lambda(G)}\mathrm{Sel}(E, k_\infty)^\vee = 2,$$

where $G = G(k(E(p)/k)$ as before.

All these results provide evidence that a deep and interesting Iwasawa theory also exists in the non-CM case and we hope that the present work will substantiate this even more.

One of the dominating themes of this thesis is the study of the Iwasawa algebra $\Lambda(G)$ for a p-adic analytic group G. Let us assume for simplicity that G is a torsion-free p-adic analytic pro-p-group. Then, as already mentioned, $\Lambda(G)$ is a (both left and right) Noetherian ring without zero-divisors. Furthermore, by results of Brumer it is known that $\Lambda(G)$ has finite projective dimension equal to

$$\mathrm{pd}(\Lambda(G)) = \dim(G) + 1,$$

where $\dim(G)$ denotes the dimension of G as p-adic analytic manifold and agrees with its p-cohomological dimension. So in some sense $\Lambda(G)$ should be considered as a “regular” ring and it is natural to ask

> Is there an analogous structure theory for $\Lambda(G)$-modules?

A more modest question but possibly fundamental for the first one is

> What is a good definition of *pseudo-null* resp. *pseudo-isomorphism* in the context of $\Lambda(G)$-modules?

We recall that for a commutative Noetherian ring R and a finitely generated R-module M the definition is the following: The dimension of M is defined to be the Krull dimension of the support of M in $\mathrm{Spec}(R)$ and M is said to be pseudo-null, if its codimension is greater than 1. M. Harris already proposed a vague definition of pseudo-null using a certain filtration of $\Lambda(G)$, which in general differs from the $\mathfrak{M}$-adic one, where $\mathfrak{M}$ denotes the maximal ideal of $\Lambda(G)$, and cannot be described easily. Besides some more or less trivial cases it turned out very difficult to verify whether a module is pseudo-null. In this thesis we give an answer to this question using a different philosophy to be explained now.

In [33] U. Jannsen proposed to use the homotopy theory for Λ-modules in order to study modules over the completed group algebra $\Lambda = \mathbb{Z}_p[\![G]\!]$ of a compact p-adic Lie group G. In this theory the "higher" Iwasawa adjoints $\mathrm{E}^r(M) := \mathrm{Ext}^r_\Lambda(M, \Lambda)$ play a crucial role and can be considered as a certain analogue of homotopy groups. In an absolutely different context and for an arbitrary (left and right Noetherian) associative ring Λ, Björk ([5]) analyzed a spectral sequence for such Ext-groups associated with the bidualizing complex. He shows that each finitely generated module over an *Auslander regular* or more generally *Auslander Gorenstein* ring (for the definitions see 1.5.3) is intrinsically equipped with a canonical filtration

$$\mathrm{T}_0(M) \subseteq \mathrm{T}_1(M) \subseteq \cdots \subseteq \mathrm{T}_{d-1}(M) \subseteq \mathrm{T}_d(M) = M.$$

Using this filtration he defines the dimension of a Λ-module M. It turns out that for a commutative regular local ring this dimension equals the Krull dimension and that $\mathrm{T}_i(M)$ is just the maximal submodule of M with dimension less or equal to i.

Thus the following theorem states a fundamental structure property of $\Lambda(G)$, which is crucial for the applications in Iwasawa theory we have in mind but is also interesting in its own right:

Theorem *(Theorem 1.5.27) $\Lambda(G)$ is an Auslander regular ring.*

For the purpose of studying the $\mathbb{Z}_p$-torsion part of $\Lambda(G)$-module the following consequence for the completed group algebra $\mathbb{Z}/p^n[\![G]\!] \cong \Lambda/p^n$ with coefficients in $\mathbb{Z}/p^n$ becomes very useful.

Theorem *(Theorem 1.5.28)*

(i) *$\mathbb{Z}_p/p^m[\![G]\!]$ is an Auslander-Gorenstein ring with injective dimension equal to $\mathrm{cd}_p(G)$.*

(ii) *$\mathbb{F}_p[\![G]\!]$ is an Auslander regular ring of dimension $\mathrm{cd}_p(G)$.*

Using these results and applying Björk's theory to Iwasawa theory, it is quite obvious how to define pseudo-null:

> A finitely generated Λ-module is called *pseudo-null* if and only if its co-dimension is greater or equal to 2.

In the case $G = \mathbb{Z}_p^d$ this is just the usual definition. So we are convinced that our definition is the right generalization to the non-commutative case.

Though we are far away from a satisfactory structure theorem for $\Lambda(G)$-modules, we should mention that at least the $\mathbb{Z}_p$-torsion part is uniquely determined in the quotient category Λ-mod$/\mathcal{PN}$ of the category Λ-mod of finitely generated $\Lambda = \Lambda(G)$-modules with respect to the Serre subcategory $\mathcal{PN}$ of pseudo-null Λ-modules.

Theorem *(Theorem 1.5.37) Assume that G is a p-adic analytic group without p-torsion such that both $\Lambda = \Lambda(G)$ and Λ/p are integral. Then, for any Λ-module M, there exist uniquely (up to order) determined natural numbers $n_1, \dots, n_r$ such that*

$$\mathrm{tor}_{\mathbb{Z}_p} M \equiv \bigoplus_{1 \le i \le r} \Lambda/p^{n_i} \mod \mathcal{PN}.$$

Since p lies in the center of $\Lambda(G)$ and generates a prime ideal, it is not surprising that we get this analogue for the p-primary part. Nevertheless the fact that at least this easiest case admits a "structure theorem" encourages to continue the investigation for a general structure theory. The first step might be to answer the following question

> What are the simple objects in Λ-mod$/\mathcal{PN}$ besides Λ/p and does any finitely generated Λ-torsion module have finite length in this category?

The answer would be very important in order to formulate a Main Conjecture over the trivializing extension of an elliptic curve E without CM.

Before passing over to apply these techniques in arithmetic geometry, we want to state two further results on the structure of $\Lambda(G)$, if G is a pro-p Poincaré group of finite cohomological dimension and such that $\Lambda = \Lambda(G)$ is Noetherian. The first result tells us that $\Lambda(G)$ "admits local duality à la Grothendieck", i.e. if local cohomology is defined in an natural way (see section 1.6), we obtain

Theorem *(Theorem 1.6.6) For any $M \epsilon \Lambda(G)$-mod, there are canonical isomorphisms*

$$\mathrm{E}^i(M) \cong \mathrm{Hom}_\Lambda(\mathrm{H}^{d-i}_{\mathfrak{M}}(M), \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) \cong \mathrm{H}^{d-i}_{\mathfrak{M}}(M)^\vee,$$

where $d = \mathrm{cd}_p(G) + 1$.

The second result generalizes the Auslander-Buchsbaum equality.

Theorem *(Theorem 1.7.2) For any $M \epsilon \Lambda$-mod, it holds*

$$\mathrm{pd}_\Lambda(M) + \mathrm{depth}_\Lambda(M) = \mathrm{depth}_\Lambda(\Lambda).$$

The above, purely algebraic results, especially concerning the dimension theory, form the heart of chapter 1 and will be fundamental in our study of "arithmetic" Iwasawa modules in chapters 2 and 3, which we will outline now.

In *chapter* 2 we first recall what is known - due to S. Howson's and Y. Ochi's work - about the local Iwasawa modules coming from (local) Galois cohomology of p-adic representations. They calculated the ranks and Λ-torsion submodules in many cases. Here we follow closely Ochi's approach which uses Jannsen's homotopy theory of Λ-modules. Furthermore we generalize Wintenberger's result on the Galois module structure of local units. Let k be a finite extension of $\mathbb{Q}_p$ and assume that $k_\infty|k$ is a Galois extension with Galois group $G \cong \Gamma \rtimes_\rho \Delta$, where Γ is a pro-p Lie group of dimension 2 (e.g. $\Gamma = \mathbb{Z}_p \rtimes \mathbb{Z}_p$) and Δ is a profinite group of *possibly infinite* order prime to p, which acts on Γ via $\rho : \Delta \to Aut(\Gamma)$. Then we determine the $\Lambda(G)$-module structure of the Galois group $G^{ab}_{k_\infty}(p) = G(k_\infty(p)/k_\infty)$, where $k_\infty(p)$ is the maximal abelian p-extension of k_∞, see theorem 2.2.4.

In *section 2.3* we apply these results to the local study of elliptic curves *E with CM,* i.e. we determine the structure of local cohomology groups with certain division points of E as coefficients.

Chapter 3 is devoted to the study of "global" Iwasawa modules. For a finite set S of places of a number field k let $k_\infty|k$ be a Galois extension unramified outside S such that the Galois group $G(k_\infty/k)$ is a torsion-free p-adic Lie-group and let k_S be the maximal outside S unramified extension of k. In *section 3.1.1* we treat the Galois group

$$X_S = G(k_S/k_\infty)^{ab}(p)$$

of the maximal abelian p-extension of k_∞ unramified outside S. If $G \cong \mathbb{Z}_p^d$, there is a theorem of R. Greenberg [21], generalized by T. Nguyen-Quang-Do [47]:

Theorem *If the weak Leopoldt conjecture holds for k_∞, i.e. if $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, then the $\Lambda(G)$-module X_S does not contain any non-trivial pseudo-null submodule.*

If G is a (non-commutative) torsion-free p-adic Lie group, Greenberg still could prove that X_S does not contain any non-zero *finite* submodule but he suggested that there should be a stronger analogue of the above result. In fact, using our definition of pseudo-null, we prove

Theorem *(Theorem 3.1.1) If* $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, *then the* $\Lambda(G)$*-module* X_S *does not contain any non-trivial pseudo-null submodule.*

This result is a special case of the following. Let A be a p-divisible p-torsion abelian group of $\mathbb{Z}_p$-corank r with a continuous action by $G_S(k) = G(k_S/k)$. The Pontryagin dual of the cohomology group $\mathrm{H}^1(G_S(k_\infty), A)$ is denoted by $X_{S,A}$. Assuming the weak Leopoldt conjecture for k_∞ and A, i.e. the vanishing of $\mathrm{H}^2(G_S(k_\infty), A)$, Y. Ochi has shown that $X_{S,A}$ is a finitely $\Lambda(G)$ module of rank

$$\mathrm{rk}_{\Lambda(G)} X_{S,A} = r_2(k)r,$$

where $r_2(k)$ denotes the number of pairs of complex places of k. To this property we add

Theorem *(Theorem 3.0.3) Let G be a p-adic Lie group without p-torsion. If the "weak Leopoldt conjecture holds for A and k_∞", i.e.* $\mathrm{H}^2(G_S(k_\infty), A) = 0$, *then* $\mathrm{H}^1(G_S(k_\infty), A)^\vee$ *does not contain any non-zero pseudo-null submodule.*

The $\Lambda(G)$-module X_S is closely related to the module X_{nr} and to the module X_{cs}^S which denotes the Galois group of the maximal abelian unramified pro-p-extension of k_∞ in which every prime above S is completely decomposed. We will write $M \sim N$ if there exists a Λ-homomorphism $M \to N$ whose kernel and cokernel is pseudo-null.

Theorem *(Theorem 3.1.5) If* $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, $\mu_{p^\infty} \subseteq k_\infty$, *and* $\dim(G_\nu) \geq 2$ *for all* $\nu \,\epsilon\, S_f$, *then*

$$X_{nr}(-1) \sim X_{cs}^S(-1) \sim \mathrm{E}^1(\mathrm{tor}_\Lambda X_S).$$

If, in addition, $G \cong \mathbb{Z}_p^r$, $r \geq 2$, *then there is a pseudo-isomorphism*

$$X_{nr}(-1) \sim X_{cs}^S(-1) \sim (\mathrm{tor}_\Lambda X_S)^\circ,$$

where $^\circ$ *means that G acts via the involution* $g \mapsto g^{-1}$.

For the next result, which generalizes theorem 11.3.7 of [45], we need the notation of the μ-invariant for a Λ-module M: it is defined as the $\mathbb{F}_p[\![G]\!]$-rank of $\bigoplus_{i\geq 0} {}_{p^{i+1}}M/{}_{p^i}M$ in case the latter is well-defined, see (1.5.29).

Theorem *(Theorem 3.1.13) Let $k_\infty|k$ be a p-adic Lie extension such that G is without p-torsion and $\mathbb{F}_p[\![G]\!]$ is an integral ring. Then $\mathcal{G} = G(k_S(p)/k_\infty)$ is a free pro-p-group if and only if $\mu(X_S) = 0$ and the weak Leopoldt conjecture holds, i.e.* $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

In theorem 3.1.14 we describe how the weak Leopoldt conjecture and the vanishing of $\mu(X_S)$ - if considered simultaneously - behave under change of the base field. Furthermore, we get a formula for the μ-invariants for different S.

In *section 3.1.2* we study the norm-coherent S-units of k_∞

$$\mathbb{E}_S := \varprojlim_{k'} (\mathcal{O}_{k',S}^\times \otimes \mathbb{Z}_p)$$

by means of Jannsen's spectral sequence for Iwasawa adjoints. Using Kummer theory, we compare $\mathbb{E}_S$ to

$$\mathcal{E}_S(k_\infty) := (E_S(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p)^\vee,$$

where $E_S(k_\infty) = \varinjlim_{k'} E_S(k')$ denotes the (discrete module of) S-units of k_∞. In particular, we show that $\mathrm{E}^0(\mathbb{E}_S) \cong \mathrm{E}^0\mathrm{E}^0(\mathcal{E}_S(k_\infty))$ and thus

$$\mathrm{rk}_\Lambda \mathbb{E}_S = \mathrm{rk}_\Lambda \mathcal{E}_S = r_2(k)$$

under some assumptions, see corollary 3.1.22. If $\mathrm{E}^0(\mathbb{E}_S)$ is projective, its structure can be described more precisely. A criterion which tells us when this is the case is given in proposition 3.1.23.

In *section 3.2* we consider cohomology groups associated with abelian varieties. Let $\mathcal{A}$ be an abelian variety defined over k and $k_\infty = k(\mathcal{A}(p))$. Since $k(\mu_{p^\infty}) \subseteq k(\mathcal{A}(p))$ by the Weil-pairing and the fact that $\mathcal{A}$ is isogenous over k to its dual $\mathcal{A}^\vee$, the vanishing of $\mathrm{H}^2(G_S(k_\infty), \mathcal{A}(p))$ follows from the validity of the weak Leopoldt conjecture for the cyclotomic extension of any number field. Hence

Theorem *(Theorem 3.2.6) Let $k_\infty = k(\mathcal{A}(p))$ and assume that G does not have any p-torsion. Then $\mathrm{H}^1(G_S(k_\infty), \mathcal{A}(p))^\vee$ has no non-zero pseudo-null submodule.*

Then we draw our attention to the (p-)Selmer group $\mathrm{Sel}(\mathcal{A}, k_\infty)$ of $\mathcal{A}$ over $k_\infty = k(\mathcal{A}(p))$. First we generalize a result of P. Billot in the case of good, *supersingular* reduction, i.e. $\widetilde{\mathcal{A}_{k_\nu}}(p) = 0$, at any place dividing p. Over a $\mathbb{Z}_p$-extension an analogous statement was proved by K. Wingberg [63, cor. 2.5].

We shall write $\mathcal{A}^\vee$ for the dual abelian variety of $\mathcal{A}$. Assume that $G(k_\infty/k)$ is a pro-p-group without any p-torsion. Then the following holds (corollary 3.2.5):

$$X_{cs} \otimes_{\mathbb{Z}_p} (\mathcal{A}^\vee(p))^\vee \sim \mathrm{E}^1(\mathrm{tor}_\Lambda \mathrm{Sel}(\mathcal{A}, k_\infty)^\vee).$$

If, in addition, $\mathcal{A}$ has CM, then the following holds

$$X_{cs} \otimes_{\mathbb{Z}_p} (\mathcal{A}^\vee(p))^\vee \sim (\mathrm{tor}_\Lambda \mathrm{Sel}(\mathcal{A}, k_\infty)^\vee)^\circ.$$

The next two theorems, which are obtained in a joined work with Y. Ochi, concern the Selmer group of an elliptic curve E *without* CM.

Theorem *(Theorem 3.2.14) Assume that E has good reduction at any place dividing p and that the conjecture of Coates and Howson holds. Then* $\mathrm{Sel}(E, k_\infty)^\vee$ *has no non-zero pseudo-null* Λ*-submodule.*

There is a similar theorem due to Perrin-Riou on the non-existence of pseudo-null submodules in the CM case ([51], Theorem 2.4). Furthermore, there is a theorem of Greenberg ([23]) and the work of Hachimori-Matsuno on finite submodules of the Selmer group over the cyclotomic $\mathbb{Z}_p$-extension ([24]). In this case, it is known that non-zero finite submodules can occur in the dual of Selmer over the cyclotomic $\mathbb{Z}_p$-extension.

Concerning the "size" of the Selmer group over the trivializing extension, R. Greenberg had already remarked that $\mathrm{Sel}(E, k_\infty)^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has infinite dimension over $\mathbb{Q}_p$ for all $p \geq 5$ (see the appendix of [11]), and earlier M. Harris [26] had given examples where $E(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ has infinite dimension over $\mathbb{Q}_p$.

Recall that we denote by $k_{cycl} = k(\mu_{p^\infty})$ the extension obtained by adjoining the p-power roots of unity to k. Putting $H = G(k_\infty/k_{cycl})$ and $\Gamma = G(k_{cycl}/k)$, $\mathrm{Sel}(E, k_\infty)^\vee$ has a structure of $\Lambda(H)$-module by restriction. An observation of Coates and Howson [11] is that, if $\mathrm{Sel}(E, k_{cycl})^\vee$ is $\Lambda(\Gamma)$-torsion and its Iwasawa μ-invariant is zero, then $\mathrm{Sel}(E, k_\infty)^\vee$ is finitely generated over $\Lambda(H)$. It turns out that the $\Lambda(H)$-torsion submodule of it is a pseudo-null $\Lambda(G)$-submodule of $\mathrm{Sel}(E, k_\infty)^\vee$. Therefore the result above answers a question of John Coates positively as follows:

Theorem *(Theorem 3.2.15). Assume that G is pro-p and that* $\mathrm{Sel}(E, k_{cycl})^\vee$ *is a finitely generated $\mathbb{Z}_p$-module. Then* $\mathrm{Sel}(E, k_\infty)^\vee$ *is a finitely generated $\Lambda(H)$-module, whose $\Lambda(H)$-torsion submodule is zero.*

As a numerical example of this theorem, take E to be the modular elliptic curve $X_1(11)$, with equation

$$y^2 + y = x^3 - x^2 \ ,$$

take $p = 5$, $k = \mathbb{Q}(\mu_5)$ and $k_{cycl} = \mathbb{Q}(\mu_{5^\infty})$. Then $G = G(k_\infty/k)$ is a pro-5-group. Coates and Howson ([11]) showed that $\mathrm{Sel}(E, k_\infty)^\vee$ is a finitely generated $\Lambda(H)$-module of rank 4, where $H = G(k_\infty/\mathbb{Q}(\mu_{5^\infty}))$. The above theorem shows that the $\Lambda(H)$-torsion submodule of $\mathrm{Sel}(E, k_\infty)^\vee$ is zero.

Finally, we show that the Pontryagin dual of the 5-Selmer group of the elliptic curve $E = X_0(11)$ has a positive μ-invariant.

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Chapter 1

Λ-modules

1.1 Basic properties of Λ-modules

1.1.1 Preliminaries

The aim of the first part of this work is to give some complements to the theory of Λ-modules, where we denote by $\Lambda = \Lambda(G)$ the completed group algebra of a profinite group G over $\mathbb{Z}_p$

$$\Lambda(G) = \mathbb{Z}_p[\![G]\!] = \varprojlim_{U} \mathbb{Z}_p[G/U].$$

Here U runs through the open normal subgroups of G. We start by recalling some well-known facts concerning Λ, proofs of which can be found in [45, V§2]. By a (left) Λ-module M we understand a separated topological module, i.e M is a Hausdorff topological $\mathbb{Z}_p$-module with a continuous G-action. Since the involution of Λ given by passing to the inverses of group elements induces a natural equivalence between the categories of left and right Λ-modules, we will often ignore the difference without further mention. The category $\mathcal{C} = \mathcal{C}(G)$ of compact Λ-modules and the category $\mathcal{D} = \mathcal{D}(G)$ of discrete Λ-modules will be of particular interest. Both are abelian categories, and Pontryagin duality defines a contravariant equivalence of categories between them. Hence, while $\mathcal{C}$ has sufficiently many projectives and exact inverse limits the dual statement holds for $\mathcal{D}$.
By I_G we denote the augmentation ideal of Λ, i.e. the kernel of the canonical epimorphism

$$\mathbb{Z}_p[\![G]\!] \twoheadrightarrow \mathbb{Z}_p$$

and by

$$M_G = M/I_G M$$

the module of coinvariants of M. Then, the G-homology $\mathrm{H}_\bullet(G, M)$ of a compact Λ-module M can be defined as left derived functor of $-_G$ or alternatively as $\mathrm{Tor}_\bullet^\Lambda(\mathbb{Z}_p, M)$, where Tor denotes the left derived functor of the complete tensor product $-\widehat{\otimes}_\Lambda -$. We obtain a canonical isomorphism $\mathrm{H}_i(G, M) \cong \mathrm{H}^i(G, M^\vee)^\vee$, where $\mathrm{H}^\bullet(G, -)$ denotes the usual G-cohomology for a discrete Λ-module considered as a discrete abelian group and $^\vee$ is the Pontryagin dual.

In order to state the topological Nakayama lemma we define the radical Rad_G of Λ to be the intersection of all open left maximal ideals. It is a closed two-sided ideal and its powers define a topology on Λ which is called the R-topology. If a p-Sylow group G_p is of finite index in G, then this topology coincides with the canonical one [45, 5.2.16], Rad_G is an open ideal of Λ and all (left) maximal ideals are open. Furthermore, $\Lambda(G)$ is a local ring if and only if G is a pro-p-group. In this case the maximal ideal of Λ is equal to $p\Lambda + I_G$.

Lemma 1.1.1. *(Topological Nakayama Lemma)*

(i) *If* $M \in \mathcal{C}$ *and* $\mathrm{Rad}_G M = M$, *then* $M = 0$.

(ii) *Assume that G is a pro-p-group, i.e. Λ a local ring with maximal ideal $\mathfrak{M}$. Then the following holds:*

(a) *M is generated by $x_1, \ldots, x_r$ if and only if $x_i + \mathfrak{M}M$, $i = 1, \ldots, r$, generate $M/\mathfrak{M}M$ as $\mathbf{F}_p$-vector space.*

(b) *Let $P \in \mathcal{C}$ be finitely generated. Then P is Λ-free if and only if P is Λ-projective.*

Concerning the projective dimension $\mathrm{pd}_\Lambda M$, respectively global dimension $\mathrm{pd}(\Lambda)$ of Λ, which are both defined with respect to the category $\mathcal{C}$, there are the following results, where $\mathrm{cd}_p(G)$ denotes the p-cohomological dimension of G.

Proposition 1.1.2. (i) $\mathrm{pd}_\Lambda \mathbb{Z}_p = \mathrm{cd}_p(G)$ *and*

(ii) $\mathrm{pd}(\Lambda) = \mathrm{cd}_p(G) + 1$.

If Λ is Noetherian (e.g. if G is a p-adic Lie group), the forgetful functor from the category $\mathcal{C}$ of compact Λ-modules to the category Λ-Mod of abstract Λ-modules defines an equivalence between the full subcategory $\mathcal{C}_{fg}$ of finitely generated compact Λ-modules and the full subcategory Λ-mod of finitely generated abstract Λ-modules. In particular, the different definitions of the projective, respectively global dimension coincide in this case.

1.1.2 p-adic Lie groups

For the purposes of this work a special class of profinite groups is of particular interest: the (compact) p-adic Lie groups or also called (compact) p-adic analytic groups, i.e. the group objects in the category of p-adic analytic manifolds over $\mathbb{Q}_p$. There is a famous characterization of them due to Lazard [38] (see also [16] 9.36):

Theorem 1.1.3. *The following are equivalent for a topological group G :*

(i) *G is a compact p-adic Lie group.*

(ii) *G contains a normal open uniformly powerful pro-p-subgroup of finite index.*

Let us briefly recall the definitions: A pro-p-group G is called *powerful*, if $[G,G] \subseteq G^p$ for odd p, respectively $[G,G] \subseteq G^4$ for $p = 2$ holds. A (topologically) finitely generated powerful pro-p-group G is *uniform* if the p-power map induces isomorphisms

$$P_i(G)/P_{i+1}(G) \xrightarrow{\cdot p} P_{i+1}(G)/P_{i+2}(G), \quad i \geq 1,$$

where $P_i(G)$ denotes the lower central p-series given by

$$P_1(G) = G,\ P_{i+1}(G) = P_i(G)^p[P_i(G), G],\ i \geq 1,$$

(for finitely generated pro-p-groups). It can be shown that for a uniform group G the sets $G^{p^i} := \{g^{p^i} \mid g \in G\}$ form subgroups and in fact $G^{p^i} = P_{i+1}(G),\ i \geq 0$. For example, all the congruence kernels of $GL_n(\mathbb{Z}_p), SL_n(\mathbb{Z}_p)$ or $PGL_n(\mathbb{Z}_p)$ are uniform pro-p-groups for $p \neq 2$, in particular the lower central p-series of the first congruence kernel corresponds precisely to the higher congruence kernels.

It is a remarkable fact that the analytic structure of a p-adic Lie group is already determined by its topological group structure. Also, the category of p-adic analytic groups is stable under the formation of closed subgroups, quotients and group extensions (See [16], chapter 10, for these facts). The following cohomological property is indispensable

Theorem 1.1.4. *A p-adic Lie group of dimension d (as p-adic analytic manifold) without p-torsion is a Poincaré group*[1] *at p of dimension d.*

[1] For the definition of Poincaré groups see [45].

With respect to the completed group algebra we know that $\Lambda(G)$ is Noetherian for any compact p-adic Lie group (see [38]V 2.2.4). If, in addition, G is uniform, then $\Lambda(G)$ is an integral domain, i.e. the only zero-divisor in $\Lambda(G)$ is 0 ([38]). In fact the latter property also holds for any p-adic analytic group without elements of finite order (see [46]). For instance, for $p \geq n+2$, the group $Gl_n(\mathbb{Z}_p)$ has no elements of order p, in particular, $GL_2(\mathbb{Z}_p)$ contains no elements of finite p-power order if $p \geq 5$ (see [29] 4.7) .

In this case (i.e. Λ is both left and right Noetherian and without zero-divisors) we can form a skew field $Q(G)$ of fractions of Λ (see [19]). This allows us to define the rank of a Λ-module:

Definition 1.1.5. The rank $\mathrm{rk}_\Lambda M$ is defined to be the dimension of $M \otimes_\Lambda Q(G)$ as a left vector space over $Q(G)$

$$\mathrm{rk}_\Lambda M = \dim_{Q(G)}(M \otimes_\Lambda Q(G)).$$

1.1.3 Minimal resolutions

Let $\Lambda = \mathbb{Z}_p[\![G]\!]$ the completed group algebra over $\mathbb{Z}_p$ of a finitely generated pro-p-group G and $k = \Lambda/\mathfrak{M} \cong \mathbb{F}_p$ its residue class field. We assume that Λ is Noetherian. For any finitely generated Λ-module M we have the minimal representation

$$\Lambda^{d_0} \overset{\varphi_0}{\twoheadrightarrow} M$$

with $d_0 = \dim_k M/\mathfrak{M}M$ by the Nakayama-Lemma. Proceeding in the same manner for $\ker(\varphi_0)$ and $d_1 = \dim_k \ker(\varphi_0)/\mathfrak{M}\ker(\varphi_0)$, we construct a *minimal free resolution*

$$F_\bullet : \cdots \longrightarrow \Lambda^{d_n} \xrightarrow{\varphi_n} \Lambda^{d_{n-1}} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow \Lambda^{d_1} \xrightarrow{\varphi_1} \Lambda^{d_0} \xrightarrow{\varphi_0} M \longrightarrow 0.$$

It is easily verified that $F_\bullet$ is determined by M up to isomorphism of complexes.

Proposition 1.1.6. *Let M be a finitely generated Λ-module and*

$$F_\bullet : \cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0.$$

a free resolution of M. Then the following are equivalent:

(i) *$F_\bullet$ is minimal,*

(ii) *$\varphi_i(F_i) \subseteq \mathfrak{M}F_{i-1}$ for all $i \geq 1$,*

(iii) $\mathrm{rk}_\Lambda(F_i) = \dim_k \mathrm{Tor}_i^\Lambda(M,k)$ *for all* $i \geq 0$,

(iv) $\mathrm{rk}_\Lambda(F_i) = \dim_k \mathrm{Ext}^i_\Lambda(M,k)$ *for all* $i \geq 0$.

Proof. The equivalence of (i) and (ii) follows easily from Nakayama's lemma. Since $\mathrm{Tor}_i^\Lambda(M,k) = \mathrm{H}_i(F_\bullet \otimes k)$, (iii) holds if and only if $\varphi_i \otimes k = 0$ for all $i \geq 0$, which is equivalent to (ii). Using $\mathrm{Ext}^i_\Lambda(M,k) = \mathrm{H}^i(\mathrm{Hom}_\Lambda(F_\bullet,k))$ the equivalence of (ii) and (iv) follows similarly. □

Corollary 1.1.7. *Let M be a finitely generated Λ-module. Then*

$$\begin{aligned} \mathrm{pd}(M) &= \max\{i \mid F_i \neq 0\} \\ &= \max\{i \mid \mathrm{Tor}_i^\Lambda(M,k) \neq 0\} \\ &= \max\{i \mid \mathrm{Ext}^i_\Lambda(M,k) \neq 0\} \end{aligned}$$

1.2 Homotopy theory of modules

In this section we briefly recall some definitions and results from the homotopy theory of modules for our special situation in the setting of U. Jannsen, who developed further the homotopy theory which was introduced by Eckmann and Hilton and extended by Auslander and Bridger [1]. The proofs can be found in [33, §1] or in [45, V§4]. Though this theory works in much larger generality, we restrict ourselves to the case where Λ is the completed group algebra over $\mathbb{Z}_p$ and even supposed to be Noetherian. Furthermore, *all Λ-modules considered are assumed to be finitely generated.*

Definition 1.2.1. A morphism $f : M \to N$ of Λ-modules is homotopic to zero, if it factors through a projective module P :

$$f : M \to P \to N.$$

Two morphisms f, g are homotopic ($f \simeq g$), if $f - g$ is homotopic to zero. Let $[M,N] = \mathrm{Hom}_\Lambda(M,N)/\{f \simeq 0\}$ be the group of homotopy classes of morphisms from M to N, and let $\mathrm{Ho}(\Lambda)$ be the category, whose objects are the objects of Λ-mod and whose morphism sets are given by $\mathrm{Hom}_{\mathrm{Ho}(\Lambda)}(M,N) = [M,N]$, i.e. the category of "$\Lambda$-modules up to homotopy."[2]

[2]The additive homotopy category of modules is not abelian in general. It can be shown that it forms a closed model category (for suitable definitions of (co)fibrations and weak equivalences). In general, it cannot be extended to a triangulated category: If it were a triangulated category in general there would have to exist for any module M a weak equivalence between M and ΩM, where Ω denotes the loop space functor which will be introduced below. But for a ring Λ with finite projective dimension this would imply that

It turns out that M and N are homotopy equivalent, $M \simeq N$, i.e isomorphic in $\mathrm{Ho}(\Lambda)$, if and only if $M \oplus P \cong N \oplus Q$ with projective Λ-modules P and Q. In particular, $M \simeq 0$ if and only if M is projective.

Definition 1.2.2. For $M \,\epsilon\, \Lambda$-mod we define the Iwasawa adjoints of M to be

$$\mathrm{E}^i(M) := \mathrm{Ext}^i_\Lambda(M, \Lambda), \quad i \geq 0,$$

which are a priori right Λ-modules by functoriality and the right Λ-structure of the bi-module Λ but will be considered as left modules via the involution of Λ. By convention we set $\mathrm{E}^i(M) = 0$ for $i < 0$. The Λ-dual $\mathrm{E}^0(M)$ will also be denoted by M^+.

It can be shown that for $i \geq 1$ the functor E^i factors through $\mathrm{Ho}(\Lambda)$ defining a functor

$$\mathrm{E}^i : \mathrm{Ho}(\Lambda) \to \Lambda\text{-mod}.$$

We just mention some functorial behaviour of E^i. For a closed subgroup $H \subseteq G$ we denote by $\mathrm{Ind}^H_G(M) = M \widehat{\otimes}_{\Lambda(H)} \Lambda(G)$ the compact induction of a $\Lambda(H)$-module to a $\Lambda(G)$-module.

Proposition 1.2.3. *Let H be a closed subgroup of G.*

(i) *For any $M \,\epsilon\, \Lambda(H)$-mod and any i we have an isomorphism of Λ-modules*

$$\mathrm{E}^i_{\Lambda(G)}(\mathrm{Ind}^H_G M) \cong \mathrm{Ind}^H_G \mathrm{E}^i_{\Lambda(H)}(M).$$

(ii) *If, in addition, H is an open subgroup, then there is an isomorphism of $\Lambda(H)$-modules*

$$\mathrm{E}^i_{\Lambda(G)}(M) \cong \mathrm{E}^i_{\Lambda(H)}(M).$$

Proof. The first statement is proved in [50, lemma 5.5] while the second one can be found in [33, lemma 2.3]. □

Now we will describe the construction of a contravariant duality functor, the transpose

$$\mathrm{D} : \mathrm{Ho}(\Lambda) \to \mathrm{Ho}(\Lambda).$$

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all modules in Λ-mod are projective.
However, if Λ is a quasi-Frobenius ring (for the definition and properties see [61, 4.2]), e.g. the group algebra of a finite group over a field $\Lambda = k[G]$, then its associated homotopy category is triangulated ([18, IV Ex. 4-8]).

For every $M \epsilon \Lambda$-mod choose a presentation $P_1 \to P_0 \to M \to 0$ of M by projectives and define the transpose $\mathrm{D}M$ by the exactness of the sequence

$$0 \longrightarrow M^+ \longrightarrow P_0^+ \longrightarrow P_1^+ \longrightarrow \mathrm{D}M \longrightarrow 0.$$

Then it can be shown that the functor D is well-defined and one has $\mathrm{D}^2 = \mathrm{Id}$. Furthermore, if $\mathrm{pd}_\Lambda M \leq 1$ then $\mathrm{D}M \simeq \mathrm{E}^1(M)$. The next result will be of particular importance:

Proposition 1.2.4. *For $M \epsilon \Lambda$-mod there is a canonical exact sequence*

$$0 \longrightarrow \mathrm{E}^1\mathrm{D}M \longrightarrow M \xrightarrow{\phi_M} M^{++} \longrightarrow \mathrm{E}^2\mathrm{D}M \longrightarrow 0,$$

where ϕ_M is the canonical map from M to its bi-dual. In the following we will refer to the sequence as "the" canonical sequence (of homotopy theory).

A Λ-module M is called *reflexive* if ϕ_M is an isomorphism from M to its bi-dual $M \cong M^{++}$.

As Auslander and Bridger suggest the module $\mathrm{E}^1\mathrm{D}M$ should be considered as torsion submodule of M. Indeed, if Λ is a Noetherian integral domain this submodule coincides exactly with the set [3] of torsion elements $\mathrm{tor}_\Lambda M$.

Definition 1.2.5. A Λ-module M is called Λ-*torsion module* if $\phi_M \equiv 0$, i.e. if $\mathrm{tor}_\Lambda M := \mathrm{E}^1\mathrm{D}M = M$. We say that M is Λ-*torsion-free* if $\mathrm{E}^1\mathrm{D}M = 0$.

For $\Lambda := \Lambda(G)$, where G is a p-adic Lie group, this definition can be interpreted as follows:

A finitely generated Λ-module M is a Λ-torsion module if and only if M is a $\Lambda(G')$-torsion module (in the strict sense) for some open pro-p subgroup $G' \subseteq G$ such that $\Lambda(G')$ is integral.

Indeed, for any open subgroup H of a p-adic Lie group G there is an isomorphism $\mathrm{E}^1_{\Lambda(G)}\mathrm{D}_{\Lambda(G)} \cong \mathrm{E}^1_{\Lambda(H)}\mathrm{D}_{\Lambda(H)}$ of $\Lambda(H)$-modules by lemma 1.2.3 (ii) and the analogue statement for $\mathrm{D}M$.

Since M^{++} embeds into a free Λ-module (just take the dual of an arbitrary surjection $\Lambda^m \twoheadrightarrow M^+$) the torsion-free Λ-modules are exactly the submodules of free modules. A different characterization of torsion(-free) modules will be given later using dimension theory, see 1.5.6.

Sometimes it is also convenient to have the notation of the 1^{st} syzygy or loop space functor $\Omega : \Lambda\text{-mod} \to \mathrm{Ho}(\Lambda)$ which is defined as follows (see [33, 1.5]): Choose a surjection $P \to M$ with P projective. Then ΩM is defined by the exact sequence

$$0 \longrightarrow \Omega M \longrightarrow P \longrightarrow M \longrightarrow 0.$$

[3] A priori it is not clear whether this sets forms a submodule if Λ is not commutative.

1.3 Some representation theory

We first recall some well-known facts from the representation theory of finite groups, whose proofs or references can be found in [45] 5.6.

Theorem 1.3.1. *Let R be complete discrete valuation ring and let G be a finite group. Assume that the quotient field K of R has characteristic* 0. *Furthermore, let L, M, N be finitely generated $R[G]$-modules. Then the following holds.*

(i) *(Krull-Schmidt) If $M \oplus L \cong N \oplus L$, then $M \cong N$.*

(ii) *(Swan) If M and N are projective and $M \otimes K \cong N \otimes K$ as $K[G]$-modules, then $M \cong N$.*

Proposition 1.3.2. *Let G be a profinite group. Let M and N be finitely generated Λ-modules such that*

(i) $M \simeq N$,

(ii) $M_U \otimes \mathbb{Q}_p \cong N_U \otimes \mathbb{Q}_p \oplus \mathbb{Q}_p[G/U]^m$ *for a basis of neighborhoods of* $1 \in G$ *consisting of open normal subgroups U.*

Then

$$M \cong N \oplus \mathbb{Z}_p[\![G]\!]^m.$$

In particular, a finitely generated projective $\mathbb{Z}_p[\![G]\!]$-module P is free if and only if $P \otimes \mathbb{Q}_p$ is $(\mathbb{Z}_p[\![G]\!] \otimes \mathbb{Q}_p)$-free.

In the next lemma we shall write $I(\Gamma)$ for the kernel of the canonical map $\mathbb{Z}_p[\![G]\!] \to \mathbb{Z}_p[G/\Gamma]$, where Γ is any closed normal subgroup of the profinite group G. By Rad_G we denote the radical of $\mathbb{Z}_p[\![G]\!]$, i.e. the intersection of all open maximal left (right) ideals of $\mathbb{Z}_p[\![G]\!]$.

Lemma 1.3.3. *Let $G = \Gamma \rtimes \Delta$ be the semi-direct product of a uniform pro-p-group Γ of dimension t and a finite group Δ of order k prime to p. If we write $U_n = \Gamma^{p^n} \trianglelefteq G$, then for any compact $\Lambda = \Lambda(G)$-module M, the following statements are equivalent:*

(i) $M \cong \Lambda^d$,

(ii) $M_\Gamma \cong \mathbb{Z}_p[\Delta]^d$ *as $\mathbb{Z}_p[\Delta]$-module and for all n*

$$\mathrm{rk}_{\mathbb{Z}_p} M_{U_n} = \mathrm{rk}_{\mathbb{Z}_p} \mathbb{Z}_p[G/U_n]^d = dkp^{tn},$$

(iii) *$M_\Gamma/p \cong \mathbb{F}_p[\Delta]^d$ as $\mathbb{F}_p[\Delta]$-module and for all n*

$$\mathrm{rk}_{\mathbb{F}_p} M_{U_n}/p = \mathrm{rk}_{\mathbb{F}_p}\mathbb{F}_p[G/U_n]^d = dkp^{tn}.$$

Proof. Obviously, (i) implies (ii) and (iii). For the converse let us first assume that (ii) holds and let $m-1,\dots,m_d \in M$ be lifts of a $\mathbb{Z}_p[\Delta]$-basis of M_Γ. Then the map $\phi : \bigoplus_{i=1}^d \Lambda e_i \to M$, which sends e_i to m_i, is surjective, because $I(\Gamma) \subseteq \mathrm{Rad}_G$ (compare to the proof of [45]. 5.2.14 (i), $d \Rightarrow b$) and therefore we can apply Nakayama's lemma [45], 5.2.16 (ii), (with Rad_G instead of $\mathfrak{M}$). Hence, the induced maps $\phi_{U_n} : \bigoplus_{i=1}^d \Lambda(G/U_n)e_i \to M_{U_n}$, are surjective, too. But since both modules have the same $\mathbb{Z}_p$-rank by assumption, these maps are isomorphisms and (i) follows. The implication (iii) $\Rightarrow$ (i) is proved analogously noting that $p\Lambda + I(\Gamma) \subseteq \mathrm{Rad}_G$. □

For a finite group G we denote by $K_0(\mathbb{Q}_p[G]) = K_0'(\mathbb{Q}_p[G])$ the Grothendieck group of finitely generated $\mathbb{Q}_p[G]$-modules (which are projective by Maschke's theorem). If G is a profinite group and $U \trianglelefteq G$ an open normal subgroup we define the Euler characteristic $h_U(M)$ of a finitely generated $\Lambda = \Lambda(G)$-module M to be the class

$$h_U(M) := \sum (-1)^i [\mathrm{H}_i(U,M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p] \in K_0(\mathbb{Q}_p[G/U]).$$

Before stating the next result we recall some facts about the representation theory of finite groups. So let Δ be a finite group of order n prime to p. Then, there is a decomposition

$$\mathbb{Z}_p[\Delta] \cong \prod \mathbb{Z}_p[\Delta]e_i, \; e_i = \frac{n_i}{n} \sum_{g \in \Delta} \chi_i(g^{-1})g$$

of $\mathbb{Z}_p[\Delta]$ in "simple" components (in the sense that they are simple algebras after tensoring with $\mathbb{Q}_p$), which induces a decomposition of $\Lambda = \prod \Lambda^{e_i}$, $\Lambda^{e_i} = \mathbb{Z}_p[\![\Gamma]\!][\Delta]e_i$ into a product of rings. Here $\{\chi_i\}$ is the set of irreducible $\mathbb{Q}_p$ characters ($\widehat{=}$ $\mathbb{F}_p$-characters because n is prime to p) of Δ and n_i are certain natural numbers associated with χ_i (see below). The simple algebras $\mathbb{Q}_p[\Delta]e_i$ decompose into the direct sum of their simple left ideals which all belong to the same isomorphism class, say N_i, i.e. there is a isomorphism of $\mathbb{Q}_p[\Delta]$-modules

$$\mathbb{Q}_p[\Delta] \cong N_i^{n_i}.$$

In particular, n_i is the length of $\mathbb{Q}_p[\Delta]e_i$ and can be expressed as $n_i = \chi(e_i)(\dim_{\mathbb{Q}_p} N_i)^{-1}$, where χ is the character of the left regular representation of $\mathbb{Q}_p[\Delta]$.

Now let G be again a p-adic Lie group and set $\Lambda := \Lambda(G)$. Recall that a finitely generated Λ-module M is a Λ-torsion module if and only if M is a $\Lambda(G')$-torsion module for some open pro-p subgroup $G' \subseteq G$ such that $\Lambda(G')$ is integral (1.2.5).

Proposition 1.3.4. *Let $G = \Gamma \times \Delta$ be the product of a pro-p Lie group Γ of finite cohomological dimension $\mathrm{cd}_p(\Gamma) = m$ and a finite group Δ of order n prime to p and let $U \trianglelefteq \Gamma$ be an open normal subgroup. Then, for any finitely generated Λ-torsion module M, it holds*

$$h_U(M) = 0.$$

Remark 1.3.5. For semi-direct products this statement is false in general. For example, it is easy to see that for $G = \mathbb{Z}_p \rtimes_\omega \Delta$ with non-trivial ω the Euler characteristic of $\mathbb{Z}_p$ is not zero: $h_U(\mathbb{Z}_p) = [\mathbb{Q}_p] - [\mathbb{Q}_p(\omega)] \neq 0$.

Proof. (of prop. 1.3.4) We claim that under the assumptions of the theorem M possesses a finite free resolution. Indeed, since the Noetherian ring Λ has finite global dimension $\mathrm{pd}\Lambda = m+1$, there is always a resolution of the form

$$0 \longrightarrow P \longrightarrow \Lambda^{d_m} \longrightarrow \cdots \longrightarrow \Lambda^{d_0} \longrightarrow 0,$$

with a projective module P. Since M^{e_i} is a $\Lambda(\Gamma)$-torsion module (it is even $\Lambda(\Gamma')$-torsion!) and since P^{e_i} is a free $\Lambda(\Gamma)$-module, it must hold that $P^{e_i} \cong (\Lambda(\Gamma))^{k_i d_{m+1}}$ as $\Lambda(\Gamma)$-modules, where $k_i = \chi(e_i)$ denotes the $\mathbb{Z}_p$-rank of $\mathbb{Z}_p[\Delta]e_i$ and $d_{m+1} = \sum_{i=0}^{m}(-1)^i d_{m-i}$. Consequently, $P_\Gamma^{e_i} \cong \mathbb{Z}_p^{k_i d_{m+1}}$ as $\mathbb{Z}_p$-modules, respectively $P_\Gamma^{e_i} \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^{k_i d_{m+1}}$ as $\mathbb{Q}_p$-modules holds. But $P_\Gamma^{e_i} \otimes \mathbb{Q}_p$ must be isomorphic to the direct sum of m copies of N_i for some m due to the semi-simplicity of $\mathbb{Q}_p[\Delta]$. Counting $\mathbb{Q}_p$-dimensions, we obtain $m = n_i d_{m+1}$ and hence $P_\Gamma^{e_i} \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[\Delta]e_i^{d_{m+1}}$. Since $P_\Gamma^{e_i}$ is a projective $\mathbb{Z}_p[\Delta]$-module, this implies $P_\Gamma^{e_i} \cong \mathbb{Z}_p[\Delta]e_i^{d_{m+1}}$, respectively $P^{e_i} \cong \Lambda(G)e_i^{d_{m+1}}$ (compare to the proof of lemma 1.3.3) and $P \cong \Lambda(G)^{d_{m+1}}$. This proves the claim.
Furthermore, we observe that $\sum(-1)^i d_i = 0$ and denote the resolution by $F^\bullet \to M$. Using the fact that the Euler characteristic of a bounded complex equals the Euler characteristic of its homology, we calculate

$$\begin{aligned}
\sum(-1)^i[\mathrm{H}_i(U, M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p] &= \sum(-1)^i[\mathrm{H}_i(F^\bullet \otimes_\Lambda \mathbb{Q}_p[G/U])] \\
&= \sum(-1)^i[F^\bullet \otimes_\Lambda \mathbb{Q}_p[G/U]] \\
&= \sum(-1)^i[\mathbb{Q}_p[G/U]^{d_i}] \\
&= (\sum(-1)^i d_i)[\mathbb{Q}_p[G/U]] = 0.
\end{aligned}$$

$\square$

Lemma 1.3.6. *Let G be a profinite group, $H \subseteq G$ a closed subgroup and $U \trianglelefteq G$ an open normal subgroup. Then for any compact $\mathbb{Z}_p[\![H]\!]$-module M the following is true:*

(i) $(\mathrm{Ind}_G^H(M))_U \cong \mathrm{Ind}_{G/U}^{HU/U}(M_{U\cap H})$ *and*

(ii) $\mathrm{H}_i(U, (\mathrm{Ind}_G^H(M))_U) \cong \mathrm{Ind}_{G/U}^{HU/U}\mathrm{H}_i(U\cap H, M)$ *for all* $i \geq 0$.

Proof. The dual statement of (i) is proved in [37] while (ii) follows from (i) by homological algebra. □

Lemma 1.3.7. *Let $G = \Gamma \times \Delta$ the product of of a pro-p-group Γ and a finite group Δ of order prime to p. Then, for any $\Lambda = \mathbb{Z}_p[\![\Gamma]\!][\Delta]$-module M and for any irreducible character χ of Δ with values in $\mathbb{Q}_p$, the following is true:*

(i) $\mathrm{Hom}_\Lambda(M^{e_\chi}, \Lambda) \cong \mathrm{Hom}_\Lambda(M, \Lambda)^{e_{\chi^{-1}}}$,

(ii) $\mathrm{E}^i_\Lambda(M^{e_\chi}) \cong \mathrm{E}^i_\Lambda(M)^{e_{\chi^{-1}}}$ *for any* $i \geq 0$.

Proof. While (ii) is a consequence of (i) by homological algebra the first statement can be verified at once using the decompositions $M \cong \bigoplus M^{e_\chi}$ and $\Lambda \cong \bigoplus \Lambda^{e_\chi}$:

$$\begin{aligned} \mathrm{Hom}_\Lambda(M, \Lambda)^{e_{\chi^{-1}}} &\cong \mathrm{Hom}_\Lambda(M, \Lambda^{e_\chi}) \\ &\cong \mathrm{Hom}_\Lambda(M^{e_\chi}, \Lambda^{e_\chi}) \\ &\cong \mathrm{Hom}_\Lambda(M^{e_\chi}, \Lambda). \end{aligned}$$

□

1.4 Spectral sequences for Iwasawa adjoints

For a profinite group G, we shall write $\mathcal{D}(G)$ and $\mathcal{C}(G)$ for the categories of discrete and compact Λ-modules, respectively, whereas we denote the full subcategories of cofinitely and finitely generated modules by $\mathcal{D}_{cfg}(G)$ and $\mathcal{C}_{fg}(G)$, respectively.

Now, let $G = H \times \Gamma$ be the product of profinite groups H and Γ. Assume that $\Lambda(G)$ is Noetherian and that Γ is separable, i.e. it possesses a countable ordered system of open normal subgroups Γ_n as a basis of open neighborhoods of $1 \in \Gamma$. Let $(\mathcal{D}_{cfg}(G))^{\mathbb{N}}$ be the category of inverse systems in $\mathcal{D}_{cfg}(G)$ and consider the left exact functor

$$T_\Gamma : \mathcal{D}_{cfg}(G) \to (\mathcal{D}_{cfg}(G))^{\mathbb{N}},$$

which sends B to the inverse system $\{B^{\Gamma_{n+1}} \overset{N_{\Gamma_n/\Gamma_{n+1}}}{\longrightarrow} B^{\Gamma_n}\}$, and

$$\varprojlim_n \mathrm{Hom}_{\Lambda(H)}(-^\vee, \Lambda(H)) : (\mathcal{D}_{cfg}(G))^{\mathbb{N}} \to \Lambda(G)\text{-Mod}.$$

Here the action of Γ on $f \in \mathrm{Hom}_{\Lambda(H)}(M, \Lambda(H))$ for $M \in \mathcal{C}(G)$ is defined by $(\gamma f)(m) := f(\gamma^{-1}m)$, whereas $h \in H$ acts by the rule $(hf)(M) := f(m)h^{-1}$ as usual.

Since the category $(\mathcal{D}_{cfg}(G))^{\mathbb{N}}$ has enough injectives, because $\mathcal{D}_{cfg}(G)$ has ([32], Prop. 1.1), we can form the right derived functors

$$R^iT_\Gamma(B) = \{\mathrm{H}^i(\Gamma_{n+1}, B) \xrightarrow{cor} \mathrm{H}^i(\Gamma_n, B)\}$$

and

$$R^i(\varprojlim_n \mathrm{Hom}_{\Lambda(H)}(B^\vee, \Lambda(H))),$$

which equals

$$\varprojlim_n R^i\mathrm{Hom}_{\Lambda(H)}(B^\vee, \Lambda(H))$$

(cf. [32] Prop. 1.2, 1.3), if we restrict ourselves to elements of the subcategory $(\mathcal{D}')^{\mathbb{N}}$ where $\mathcal{D}'$ is the abelian subcategory of $\mathcal{D}_{cfg}(G)$ consisting of $\Lambda(G)$-modules, which are cofinitely generated over $\Lambda(H)$. Indeed, in this case, the modules $\mathrm{Hom}_{\Lambda(H)}(B_n^\vee, \Lambda(H))$ are compact, i.e. the inverse limit functor is exact on the corresponding inverse systems. Since $R^i\mathrm{Hom}_{\Lambda(H)}(-, \Lambda(H))$ extends the functors $\mathrm{E}^i_{\Lambda(H)}(-)$ naturally from $\mathcal{C}(H)$ to $\mathcal{C}_{fg}(G)$, we will write also $\mathrm{E}^i_{\Lambda(H)}(-)$ for the first functor. Note that it is endowed with a natural Γ-action.

Lemma 1.4.1. *The functor T_Γ sends injectives to $\varprojlim_n \mathrm{Hom}_{\Lambda(H)}(-^\vee, \Lambda(H))$-acylics.*

Proof. It suffices to prove that $\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]$ is $\mathrm{Hom}_{\Lambda(H)}(-, \Lambda(H))$-acyclic. But, for any resolution of $\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]$ by $\Lambda(G)$-projectives

$$P^\bullet \to \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n],$$

the sequence

$$0 \to \mathrm{Hom}_{\Lambda(H)}(\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n], \Lambda(H)) \to \mathrm{Hom}_{\Lambda(H)}(P^\bullet, \Lambda(H))$$

is exact, because both, $\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]$ and the P^i, are projectives in $\mathcal{C}(H)$ (cf. [45] (5.3.13)). The result follows by taking homology. □

The Grothendieck spectral sequence for the composition of the above functors gives

Theorem 1.4.2. *With notation as above, there is a convergent cohomological spectral sequence*

$$\varprojlim_n \mathrm{E}^i_{\Lambda(H)}(\mathrm{H}^j(\Gamma_n, B)^\vee) \Rightarrow \mathrm{E}^{i+j}_{\Lambda(G)}(B^\vee)$$

for any B in $\mathcal{D}_{cfg}(G)$.

Note that all modules that occur in the spectral sequence are compact $\Lambda(G)$-modules.

Proof. The functor $\mathrm{E}^0_{\Lambda(G)}(-)$ is the composition of the functors T_Γ and $\varprojlim_n \mathrm{Hom}_{\Lambda(H)}(-^\vee, \Lambda(H))$, because by lemma 1.4.3 we have isomorphisms of $\Lambda(G)$-modules

$$\begin{aligned} \mathrm{E}^0_{\Lambda(G)}(B^\vee) &= \mathrm{Hom}_{\Lambda(G)}(B^\vee, \mathbb{Z}_p[\![H]\!][\![\Gamma]\!]) \\ &= \varprojlim_n \mathrm{Hom}_{\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]}((B^\vee)_{\Gamma_n}, \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]) \\ &= \varprojlim_n \mathrm{Hom}_{\Lambda(H)}((B^{\Gamma_n})^\vee, \Lambda(H)). \end{aligned}$$

Now the result follows by lemma 1.4.1.

□

Recall that there is a canonical $\Lambda(H)$-homomorphism

$$\pi_n : \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n] \to \mathbb{Z}_p[\![H]\!], \quad \sum_{g \,\in\, \Gamma/\Gamma_n} a_g \, g\Gamma_n \mapsto a_1,$$

and, for any $m \geq n$, a canonical $\Lambda(G)$-homomorphism $p_{m,n} : \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_m] \to \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]$ which sums up the coefficients of the same Γ_n-cosets.

Lemma 1.4.3. *The homomorphisms π_n and $p_{m,n}$ induce a commutative diagram of $\Lambda(G)$-modules:*

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_m]}(M_{\Gamma_m}, \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_m]) \arrow[r, "(\pi_m)_*", "\simeq"'] \arrow[d, "(p_{m,n})_*"] & \mathrm{Hom}_{\mathbb{Z}_p[\![H]\!]}(M_{\Gamma_m}, \mathbb{Z}_p[\![H]\!]) \arrow[d, "N_{\Gamma_n/\Gamma_m}"] \\ \mathrm{Hom}_{\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]}(M_{\Gamma_n}, \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]) \arrow[r, "(\pi_n)_*", "\simeq"'] & \mathrm{Hom}_{\mathbb{Z}_p[\![H]\!]}(M_{\Gamma_n}, \mathbb{Z}_p[\![H]\!]) \end{tikzcd}$$

Proof. It is easily checked that the diagram commutes and that the inverse of $(\pi_n)_*$ is given by $\psi \mapsto (m \mapsto \sum_{g\,\epsilon\,\Gamma/\Gamma_n} \psi(g^{-1}m)g\Gamma_n)$. (Note that the Γ-invariance of a homomorphism $\phi(m) = \sum \phi(m)_g g\Gamma_n$ is equivalent to $\phi(\gamma^{-1}m)_1 = \phi(m)_\gamma$ for all $\gamma \,\epsilon\, \Gamma$.) Recalling that $\gamma \,\epsilon\, \Gamma$ acts by $(\gamma\phi)(m) := \phi(\gamma^{-1}m)$ on $\mathrm{Hom}_{\mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n]}(M_{\Gamma_n}, \mathbb{Z}_p[\![H]\!][\Gamma/\Gamma_n])$, it is also immediate that $(\pi_n)_*$ is $\Lambda(G)$-invariant. □

Corollary 1.4.4. *If Γ contains an open subgroup of index prime to p and isomorphic to $\mathbb{Z}_p$, then there is a long exact sequence of $\Lambda(G)$-modules*

$$\varprojlim_n \mathrm{E}^i_{\Lambda(H)}(M_{\Gamma_n}) \to \mathrm{E}^i_{\Lambda(G)}(M) \to \varprojlim_n \mathrm{E}^{i-1}_{\Lambda(H)}(M^{\Gamma_n}) \to \varprojlim_n \mathrm{E}^{i+1}_{\Lambda(H)}(M_{\Gamma_n}) \to \mathrm{E}^{i+1}_{\Lambda(G)}(M)$$

Now we are going to present further spectral sequences due to U. Jannsen which were in some sense the models for the first one proved in this section. The next one describes the Iwasawa adjoints of certain cohomology groups associated with p-adic representations. So let G be a profinite group and G_∞ a closed subgroup, such that its quotient has a countable basis of neighbourhoods of identity, i.e. there is a countable family G_n, $G_\infty \subseteq G_n \subseteq G$, with $\bigcap_n G_n = G_\infty$. Furthermore, let $A = (\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \geq 1$ with some continuous action of G. We shall write

$$T_pA = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A) \cong \varprojlim_m {}_{p^m}A$$

for the Tate module of A. Then there is the following convergent spectral sequence ([34]):

Theorem 1.4.5. *(Jannsen)*

$$E_2^{p,q} = \mathrm{E}^p(\mathrm{H}^q(G_\infty, A)^\vee) \Rightarrow \varprojlim_n \mathrm{H}^{p+q}(G_n, T_pA)$$

Corollary 1.4.6. *Assume $\mathrm{cd}_p(G) \leq 2$. Then the exact sequence of low degrees degenerates to*

$$\begin{array}{l} 0 \to \mathrm{E}^1(A(k_\infty)^\vee) \to \varprojlim_n \mathrm{H}^1(G_n, T_pA) \to \mathrm{E}^0(\mathrm{H}^1(G_\infty, A)^\vee) \to \\ \mathrm{E}^2(A(k_\infty)^\vee) \to \ker(\varprojlim_n \mathrm{H}^2(G_n, T_pA) \to \mathrm{E}^0(\mathrm{H}^2(G_\infty, A)^\vee)) \to \\ \mathrm{E}^1(\mathrm{H}^1(G_\infty, A)^\vee) \to \mathrm{E}^3(A(k_\infty)^\vee) \to 0. \end{array}$$

The next result, which relates the (compact) Λ-modules $\mathrm{E}^i(M)$ to the discrete G-modules

$$D_i(M^\vee) := \varinjlim_{U \subseteq_o G} \mathrm{H}^i(U, M^\vee)^* \ , \ i \geq 0,$$

is derived by some spectral sequences, too, but we only state the associated long, respectively short, exact sequences:

Theorem 1.4.7. *Let G be a profinite group such that $\Lambda(G)$ is Noetherian. Then, for any finitely generated Λ-module M, there are functorial exact sequences*

(i)

$$0 \to D_i(M^\vee) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{E}^i(M)^\vee \to \mathrm{tor}_{\mathbb{Z}_p} D_{i-1}(M^\vee) \to 0,$$

for all i, where by definition $D_i(M^\vee) = 0$ for $i < 0$.

(ii)

$$\to \mathrm{E}^i(M)^\vee \to \varinjlim_m D_i({}_{p^m}(M^\vee)) \to \varinjlim_m D_{i-2}(M^\vee/p^m) \to \mathrm{E}^{i-1}(M)^\vee \to,$$

and the following isomorphisms

(iii) $\mathrm{E}^i(M/\mathrm{tor}_{\mathbb{Z}_p} M)^\vee \cong \varinjlim_m D_i({}_{p^m}(M^\vee))$,

(iv) $\mathrm{E}^i(\mathrm{tor}_{\mathbb{Z}_p} M)^\vee \cong \varinjlim_m D_{i-1}(M^\vee/p^m)$.

Proof. See [33] 2.1, 2.2 or [45] theorem 5.4.12. □

Corollary 1.4.8. *Assume that G is a duality group at p of dimension n with dualizing module $D_n^{(p)} = \varinjlim_m D_n(\mathbf{Z}/p^m\mathbf{Z})$. Then the following holds:*

(i) *If M is Λ-module which is free of finite rank as $\mathbb{Z}_p$-module, then*

$$\mathrm{E}^i(M)^\vee \cong \begin{cases} \varinjlim_m D_n((M/p^m)^\vee) \cong M \otimes_{\mathbb{Z}_p} D_n^{(p)} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If N is a finite p-primary Λ-module, then*

$$\mathrm{E}^i(N)^\vee \cong \begin{cases} \mathrm{Hom}_{\mathbb{Z}_p}(N^\vee, D_n^{(p)}) & \text{if } i = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [33] 2.6 or [45] 5.4.14. □

1.5 Auslander regularity

1.5.1 Filtrations of Λ-modules

Since the completed group ring Λ of a p-adic Lie group without p-torsion is (both left and right) Noetherian and has finite global homological (and therefore finite injective) dimension we can apply the results of J.-E. Björk [5], which we will describe in this section.

Let Λ be a (not necessarily commutative) Noetherian ring with finite injective dimension d, i.e d is the minimal integer with respect to the property that $\mathrm{E}^j(M) = 0$ for all (left and right) Λ-modules M and integers $j > d$. Of course, this is equivalent to the condition that both the left and the right Λ-module Λ has (bounded) injective dimension d. It can be shown that these left and right injective dimensions are the same (see [66]). The analogous statement that the left and the right global homological dimension are the same is a consequence of the Tor-dimension theorem [61, 4.1.3].

In this section all Λ-modules are assumed to be finitely generated.

Since projective Λ-modules are reflexive, we get the equality

$$M = \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(M, \Lambda), \Lambda)$$

for left (or right) Λ-modules M in the derived category of complexes of Λ-modules (more generally, this equality holds for all perfect complexes). Calculating $\mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(M, \Lambda), \Lambda)$ by the bidualizing complex, the associated filtrations give rise to two convergent spectral sequences (see [39] for the convergence), the first of which degenerates. The second one becomes

$$E_2^{p,q} = \mathrm{E}^p(\mathrm{E}^{-q}(M)) \Rightarrow \mathrm{H}^{p+q}(\Delta^\bullet(M)),$$

where $\Delta^\bullet(M)$ is a filtered complex, which is exact in all degrees except zero: $\mathrm{H}^0(\Delta^\bullet) = M$, i.e. there is a canonical filtration

$$\mathrm{T}_0(M) \subseteq \mathrm{T}_1(M) \subseteq \cdots \subseteq \mathrm{T}_{d-1}(M) \subseteq \mathrm{T}_d(M) = M$$

on every module M. The convergence of the spectral sequence implies

$$E_\infty^{p,q} = \begin{cases} \mathrm{T}_{d-p}(M)/\mathrm{T}_{d-p-1}(M) & \text{if } p+q=0, \\ 0 & \text{otherwise.} \end{cases}$$

(By convention, $\mathrm{T}_i(M) = 0$ *for* $i < 0$).

Definition 1.5.1. (i) The number $\delta := min\{i \mid \mathrm{T}_i(M) = M\}$ is called the *dimension* $\delta(M)$ of a Λ-module M.

(ii) If M is a Λ-module we say that it has *pure* δ-dimension if $\mathrm{T}_{\delta-1}(M) = 0$, i.e. the filtration degenerates to a single term M.

(iii) A Λ-module M is called *pseudo-null*, if it is at least of codimension 2, i.e. if $\delta(M) \leq d - 2$.

By Grothendieck's local duality theorem, this definition coincides with the Krull dimension of $supp_\Lambda(M)$ if Λ is a commutative local Noetherian Gorenstein ring, see [7, Cor. 3.5.11].

First we want to state some basic facts on the δ-dimension. The functoriality of the spectral sequence implies

Proposition 1.5.2. (i) *If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is an exact sequence of* Λ*-modules then*

$$\mathrm{T}_i(M') \subseteq \mathrm{T}_i(M) \text{ for all } i$$

and $\delta(M'), \delta(M'') \leq \delta(M)$.

(ii) $\mathrm{T}_i(\bigoplus_k (M_k)) = \bigoplus_k \mathrm{T}_i(M_k)$ *and* $\delta(\bigoplus_k M_k) = \max_k \delta(M_k)$.

In order to analyze this spectral sequence more closely, the Auslander condition (for not necessarily commutative rings) is essential:

Definition 1.5.3. (i) If $M \neq 0$ is a Λ-module, then

$$j(M) := min\{i \mid \mathrm{E}^i(M) \neq 0\}$$

is called the *grade* of M.

(ii) A Noetherian ring Λ is called Auslander-Gorenstein ring if it has finite injective dimension and the following Auslander condition holds: For any Λ-module M, any integer m and any submodule N of $\mathrm{E}^m(M)$, the grade of N satisfies $j(N) \geq m$.

(iii) A Noetherian ring Λ is called Auslander regular ring if it has finite global homological dimension and the Auslander condition holds.

Remark 1.5.4. Let Λ be a commutative ring. Then Λ is Auslander-Gorenstein if and only if it is Gorenstein (in the usual sense). Similarly, Λ is Auslander regular if and only if it is regular (in the usual sense) and of finite Krull dimension. (The implications concerning the injective, respectively global homological dimensions are well known, for the Auslander condition see [1, Cor. 4.6,Prop. 4.21])

In the next section we will prove that $\Lambda = \Lambda(G)$ is Auslander regular for any p-adic Lie group without p-torsion. Generally, for this kind of rings we get the following properties:

Proposition 1.5.5. *Let Λ be an Auslander regular ring and M a Λ-module. Then*

(i) (a) *For all i, there is an exact sequence of Λ-modules*
$$0 \longrightarrow \mathrm{T}_i(M)/\mathrm{T}_{i-1}(M) \longrightarrow \mathrm{E}^{d-i}\mathrm{E}^{d-i}(M) \longrightarrow Q_i(M) \longrightarrow 0,$$
where $Q_i(M)$ is a subquotient of $\bigoplus_{k\geq 1} \mathrm{E}^{d-i+k+1}\mathrm{E}^{d-i+k}(M)$.

(b) $\mathrm{T}_0(M) = \mathrm{E}^d\mathrm{E}^d(M)$ *and* $\mathrm{T}_1(M)/\mathrm{T}_0(M) = \mathrm{E}^{d-1}\mathrm{E}^{d-1}(M)$.

(c) $\mathrm{T}_i(M)/\mathrm{T}_{i-1}(M) = 0$ *if and only if* $\mathrm{E}^{d-i}\mathrm{E}^{d-i}(M) = 0$.

(ii) $\delta(M) + j(M) = d$.

(iii) (a) $j(\mathrm{E}^i(M)) \geq i$, *i.e.* $\mathrm{E}^j\mathrm{E}^i(M) = 0$ *for all* $j < i$.

(b) $\delta(\mathrm{E}^i(M)) \leq d - i$.

(c) $\mathrm{E}^{j(M)}(M)$ *has pure δ-dimension* $\delta(M)$.

(iv) $\mathrm{E}^{k+j(M)+1}\mathrm{E}^{k+j(M)}\mathrm{E}^{j(M)}(M) = 0$ *for all* $k \geq 1$.

(v) (a) *For all $0 \leq i \leq d$, $\mathrm{E}^i\mathrm{E}^i(M)$ is either zero or of pure δ-dimension $d - i$.*

(b) *M has pure δ-dimension if and only if $\mathrm{E}^i\mathrm{E}^i(M) = 0$ for all $i > j(M)$.*

(vi) (a) $\delta(\mathrm{T}_i(M)) \leq i$.

(b) *$\mathrm{T}_i(M)$ is the maximal submodule of M with δ-dimension less or equal to i.*

(c) *The functor T_i is left exact.*

(d) $\mathrm{T}_i(M/\mathrm{T}_i(M)) = 0$.

(vii) *If $\delta(M) = 0$ then M has finite length.*

Proof. Except for (i) (a), (i) (b) and (vi), these properties are all proved in [5] or trivial: Prop. 1.21, 1.16, Prop. 1.18, Remark before 1.19, Cor. 1.20, Cor. 1.22 and 1.27., while (i)(a) is proved in [39, Cor. 4.3]

The assertion (i)(b) is clear, as $E_\infty^{i,-i} = E_2^{i,-i}$ because of (iii)(a). So let us prove (vi): By (iii), (a) is equivalent to $j(\mathrm{T}_i(M)) \geq d - i$ and this is true because of the Auslander condition using induction (cf. the proof of (iii)). Now let M be a Λ-module with $\delta(M) = \delta$ and assume that there is a submodule N of M with $\delta(N) \leq \delta - 1$ and $N \not\subseteq \mathrm{T}_{\delta-1}(M)$. Then the submodule $N + \mathrm{T}_{\delta-1}(M)$ of M has dimension $\leq \delta - 1$ and so also the quotient $(N + \mathrm{T}_{\delta-1}(M))/\mathrm{T}_{\delta-1}(M)$ by 1.5.2. Hence,

$$\begin{aligned} 0 \neq (N + \mathrm{T}_{\delta-1}(M))/\mathrm{T}_{\delta-1}(M) &= \mathrm{T}_{\delta-1}((N + \mathrm{T}_{\delta-1}(M))/\mathrm{T}_{\delta-1}(M)) \\ &\subseteq \mathrm{T}_{\delta-1}(\mathrm{E}^{d-\delta}\mathrm{E}^{d-\delta}(M)) = 0 \end{aligned}$$

by (v), which is a contradiction. So $\mathrm{T}_{\delta-1}(M)$ contains all submodules of dimension less or equal to $\delta - 1$ and (b) follows by induction.
Noting Prop. 1.5.2 (i), we only have to show $N \cap \mathrm{T}_i(M) \subseteq \mathrm{T}_i(N)$ in order to prove left exactness. Since the first module has dimension $\delta(N \cap \mathrm{T}_i(M)) \leq i$, this is a consequence of (b).
By (c) the exact sequence

$$0 \to \mathrm{T}_{i+1}(M)/\mathrm{T}_i(M) \to \mathrm{T}_{i+2}(M)/\mathrm{T}_i(M) \to \mathrm{T}_{i+2}(M)/\mathrm{T}_{i+1}(M) \to 0$$

induces the exact sequence

$$0 \to \mathrm{T}_i(\mathrm{T}_{i+1}(M)/\mathrm{T}_i(M)) \to \mathrm{T}_i(\mathrm{T}_{i+2}(M)/\mathrm{T}_i(M)) \to \mathrm{T}_i(\mathrm{T}_{i+2}(M)/\mathrm{T}_{i+1}(M)).$$

The first and third term are zero by (i) and (iii) as above. Hence the result follows by induction. □

We want to mention, that for an Auslander-Gorenstein ring, the result of proposition 1.5.2 (i) can be sharpened: $j(M) = \min\{j(M'), j(M'')\}$, $\delta(M) = \max\{\delta(M'), \delta(M'')\}$ respectively (cf. [6, Prop. 1.8]).

Remark 1.5.6. (i) Using the methods of [17], proposition 6, one can show the existence of the following exact sequences:

$$0 \to \mathrm{E}^{i+1}\mathrm{D}\Omega^i\mathrm{T}_{d-i}(M) \to \mathrm{T}_{d-i}(M) \to \mathrm{E}^i\mathrm{E}^i(M) \to \mathrm{E}^{i+2}\mathrm{D}\Omega^i\mathrm{T}_{d-i}(M) \to 0.$$

Hence, $\mathrm{T}_i(M)$ can also be obtained recursively by the formula $\mathrm{T}_{d-i-1}(M) = \mathrm{E}^{i+1}\mathrm{D}\Omega^i\mathrm{T}_{d-i}(M)$ and similarly, we get a description for $Q_{d-i}(M) \cong \mathrm{E}^{i+2}\mathrm{D}\Omega^i\mathrm{T}_{d-i}(M)$. The same arguments yield for a Λ-module M with $j(M) \geq j$ the isomorphisms

$$\mathrm{E}^{j+k}\mathrm{E}^j(M) \cong \mathrm{E}^{j+k+2}\mathrm{D}\Omega^j(M) \text{ for } k \geq 1.$$

(ii) In particular, $\mathrm{T}_{d-1}(M) = \mathrm{E}^1\mathrm{D}(M) = \mathrm{tor}_\Lambda M$, i.e. the torsion submodule of M is the maximal submodule of codimension greater or equal than 1. That means that M is Λ-torsion if and only if it is at least of codimension 1, and Λ-torsion-free if and only if M is of pure dimension d.

As in the commutative case we say that a homomorphism $\varphi : M \to N$ of Λ-modules is a *pseudo-isomorphism* if its kernel and cokernel are pseudo-null. A module M is by definition pseudo-isomorphic to a module N, denoted

$$M \sim N,$$

if and only if there exists a pseudo-isomorphism from M to N. In general, $\sim$ is not symmetric even in the $\mathbb{Z}_p$-case (cf. [45, V§3,ex.1]). While in the commutative case $\sim$ is symmetric at least for torsion modules (see the first remark after prop. 5.17 in [45]), we do not know whether this property still holds in the general case.
If we want to reverse pseudo-isomorphisms, we have to consider the quotient category Λ-mod$/\mathcal{PN}$ with respect to subcategory $\mathcal{PN}$ of pseudo-null Λ-modules, which is a Serre subcategory, i.e. closed under subobjects, quotients and extensions. By definition, this quotient category is the localization $(\mathcal{PI})^{-1}\Lambda$-mod of Λ-mod with respect to the multiplicative system $\mathcal{PI}$ consisting of all pseudo-isomorphisms (see [61, ex. 10.3.2]). Since Λ-mod is well-powered, i.e. the family of submodules of any module $M \in \Lambda$-mod forms a set[4], these localization exists, is an abelian category and the universal functor $q : \Lambda\text{-mod} \to \Lambda\text{-mod}/\mathcal{PN}$ is exact (see [60, p. 44ff]). Furthermore, $q(M) = 0$ in Λ-mod$/\mathcal{PN}$ if and only if $M \in \mathcal{PN}$.

Corollary 1.5.7. *Let Λ be an Auslander regular, integral domain with $d \geq 2$.*

(i) *Any torsion-free module M embeds into a reflexive module with pseudo-null cokernel.*

(ii) *Any torsion module M is pseudo-isomorphic to* $\mathrm{E}^1\mathrm{E}^1(M)$.

Proof. Observe that $\mathrm{T}_{d-1}(M)$ is the maximal torsion submodule in this case. Hence, the exact sequence in (i) (a) for $i = d$ respectively $i = d - 1$ proves both statements taking under consideration (iii)(b) and proposition 1.5.14. □

Following the structure theory for modules over a *commutative* regular local ring (see [45, 5.1.7,5.18]), it is natural to hope that at least the following question has an affirmative answer

[4] Any submodule of M can be represented by a (finite) subset of M, i.e. the family of submodules can be described as the quotientset of the power set M with respect to the equivalence relation which identifies subsets of M whose elements generate the same submodule.

Question 1.5.8. *Let Λ be an Auslander regular ring and $M \epsilon \Lambda$-mod. Does there exist an isomorphism in Λ-mod/$\mathcal{PN}$*

$$M \cong \mathrm{tor}_\Lambda M \oplus R \ mod\ \mathcal{PN},$$

where $R \cong M/\mathrm{tor}_\Lambda M\ mod\ \mathcal{PN}$ is a reflexive Λ-module?

Proposition 1.5.9. *Let Λ be an Auslander regular ring. For any Λ-module M it holds:*

$$\mathrm{E}^1(M) \sim \mathrm{E}^1(\mathrm{tor}_\Lambda M).$$

Proof. From the long exact Ext-sequence we get the exact sequence

$$\mathrm{E}^1(M/\mathrm{tor}_\Lambda M) \longrightarrow \mathrm{E}^1(M) \longrightarrow \mathrm{E}^1(\mathrm{tor}_\Lambda M) \longrightarrow \mathrm{E}^2(X/\mathrm{tor}_\Lambda M).$$

While the module on the right hand side is obviously pseudo-null the first one is so by the following argument: the long exact Ext-sequence of

$$0 \longrightarrow M/\mathrm{tor}_\Lambda M \longrightarrow \mathrm{E}^0\mathrm{E}^0(M) \longrightarrow \mathrm{E}^2\mathrm{D}(M) \longrightarrow 0$$

tells us that $\mathrm{E}^1(M/\mathrm{tor}_\Lambda M)$ fits into the exact sequence

$$\mathrm{E}^1\mathrm{E}^0\mathrm{E}^0(M) \longrightarrow \mathrm{E}^1(M/\mathrm{tor}_\Lambda M) \longrightarrow \mathrm{E}^2\mathrm{E}^2\mathrm{D}(M),$$

i.e. it suffices to show that $\mathrm{E}^1\mathrm{E}^0\mathrm{E}^0(M)$ is pseudo-null. But $\mathrm{E}^1\mathrm{E}^1\mathrm{E}^0\mathrm{E}^0(M) = 0$ by 1.5.5,(v), (and $\mathrm{E}^0\mathrm{E}^1\mathrm{E}^0\mathrm{E}^0(M) = 0$ anyway), i.e. $j(\mathrm{E}^1\mathrm{E}^0\mathrm{E}^0(M)) \geq 2$ respectively $\delta(\mathrm{E}^1\mathrm{E}^0\mathrm{E}^0(M)) \leq d-2$. □

As long as a structure theory which is comparable with that for *commutative* regular local rings is lacking it seems difficult to answer the following

Question 1.5.10. *Let G be a p-adic analytic pro-p Lie group without p-torsion and M a $\Lambda = \Lambda(G)$-torsion module. Does there exist an isomorphism in Λ-mod/$\mathcal{PN}$*

$$M^\circ \cong \mathrm{E}^1(M)\ mod\ \mathcal{PN},$$

where $^\circ$ means that G acts via $g \mapsto g^{-1}$.

The following class of Λ-modules satisfies some duality relations:

Definition 1.5.11. A Λ-module $M \neq 0$ is called Cohen-Macaulay if $j(M) = \mathrm{pd}_\Lambda(M)$ holds, i.e. if $\mathrm{E}^i(M) = 0$ for all $i \neq j(M)$.

Proposition 1.5.12. *Let Λ be an Auslander regular ring.*

(i) *Let M be a Cohen-Macaulay module of dimension j. Then*

$$\mathrm{E}^j\mathrm{E}^j(M) = M.$$

(ii) *In particular, if $\delta(M) = 0$, then*

$$\mathrm{E}^d\mathrm{E}^d(M) = M.$$

Proof. In both cases the spectral sequence degenerates. □

One could hope that any Λ-module M can be decomposed into Cohen-Macaulay modules in the following sense: there is an filtration of M such that the ith subquotient is Cohen-Macaulay of dimension i. But it is easy to construct counterexamples which show that in general such a filtration does not exist. Nevertheless, there is a different type of filtration: Auslander and Bridger proved the existence of a *spherical filtration* (up to homotopy, i.e. after adding a projective summand P)

$$M_d \subseteq M_{d-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M \oplus P,$$

the subquotients of which form *spherical* or *Eilenberg-MacLane* modules of type $\mathrm{E}^i(M)$, i.e. for $1 \leq i \leq d$

$$\mathrm{E}^j(M_{i-1}/M_i) \cong \begin{cases} \mathrm{E}^i(M) & \text{if } j = i \\ 0 & \text{if } j \neq i, 0 \end{cases}.$$

Fossum [17] compared the spherical filtration to the filtration $\mathrm{T}_i(M)$ for a commutative Gorenstein ring and proved ([17], prop. 9) that their "torsion parts" agree for $i < d$

$$\begin{aligned} \mathrm{T}_i(M) &\cong \mathrm{T}_{d-1}(M_{d-i-1}) \\ &\cong \mathrm{T}_i(M_k) \text{ for all } k < d - i. \end{aligned}$$

The proof generalizes at once to the non-commutative case.

Proposition 1.5.13. *Let Λ be an Auslander regular ring. A Λ-module M with projective dimension $\mathrm{pd}_\Lambda(M) = k$ has no non-trivial submodule of dimension less or equal to $d - k - 1$, i.e. $\mathrm{T}_{d-k-1}(M) = 0$.*

Proof. See prop. 1.5.5, (i) (b). □

The next result extends a well known result for commutative regular rings (see for example [45], cor. 5.1.3).

Proposition 1.5.14. *Let Λ be an Auslander regular ring.*

(i) *For any Λ-module M, the module $\mathrm{E}^0(M)$ is reflexive:*

$$\mathrm{E}^0(M) \cong \mathrm{E}^0\mathrm{E}^0\mathrm{E}^0(M).$$

(ii) *Assume that $d \geq 2$ and $\delta(M) = d - i$. Then*

$$\mathrm{E}^i(M) \cong \mathrm{E}^i\mathrm{E}^i\mathrm{E}^i(M).$$

Proof. Let $N := \mathrm{E}^0(M)$ and apply proposition 1.5.5 (iv) to conclude that $\bigoplus_{k\geq 1} \mathrm{E}^{k+1}\mathrm{E}^k(N) = \bigoplus_{k\geq 1} \mathrm{E}^{k+1}\mathrm{E}^k\mathrm{E}^0(M) = 0$, i.e. $Q_d(N) = 0$. Since we already know by (iii)(c) that N is of pure dimension d (if $N \neq 0$), the statement (i) follows considering (i)(a). The proof of (ii) is analogous. □

Corollary 1.5.15. *For any i it holds*

(i) $\mathrm{E}^i\mathrm{E}^i\mathrm{E}^i\mathrm{E}^i(M) \cong \mathrm{E}^i\mathrm{E}^i(M)$ *and*

(ii) $\mathrm{E}^i\mathrm{E}^i\mathrm{T}_{d-i}(M) \cong \mathrm{E}^i\mathrm{E}^i(M)$.

Proof. To prove (i) assume first that $\delta(\mathrm{E}^i(M) = d - i$. Applying the proposition to the module $\mathrm{E}^i(M)$ gives the result while in the second case, i.e. $j(\mathrm{E}^i(M) > i$, the module $\mathrm{E}^i\mathrm{E}^i(M)$ is zero anyway. Noting that $j(Q_i(M)) \geq i + 2$, the second assertion follows at once calculating the long exact E^i-sequence of

$$0 \longrightarrow \mathrm{T}_{d-i-1}(M)/\mathrm{T}_{d-i}(M) \longrightarrow \mathrm{E}^i\mathrm{E}^i(M) \longrightarrow Q_i \longrightarrow 0.$$

□

Proposition 1.5.16. *Let Λ be an Auslander regular ring. For any Λ-module M such that* $\mathrm{pd}_\Lambda \mathrm{E}^0(M) \leq 1$ *(e.g. if* $\mathrm{pd}\,\Lambda = 3$ *or if* $\mathrm{pd}\,\Lambda = 4$ *and* $\mathrm{E}^4\mathrm{E}^1(M) = 0$*) its double dual* $\mathrm{E}^0\mathrm{E}^0(M)$ *is a 2-syzygy of* $\mathrm{E}^1\mathrm{E}^0(M)$*, i.e. there is an exact sequence*

$$0 \longrightarrow \mathrm{E}^0\mathrm{E}^0(M) \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \mathrm{E}^1\mathrm{E}^0(M) \longrightarrow 0$$

with projective modules P_0 and P_1. Furthermore, in the case of $\mathrm{pd}\,\Lambda = 3$ *or* 4*, it holds that* $\mathrm{E}^1\mathrm{E}^0(M) \cong \mathrm{E}^3\mathrm{E}^1(M)$*. If, in addition, M itself is reflexive and* $\mathrm{pd}\,\Lambda = 3$*, then* $\mathrm{E}^3\mathrm{E}^1 M \cong \mathrm{E}^1(M)^\vee$.

Proof. First observe that $\mathrm{E}^0(M)$ is a 2-syzygy of $\mathrm{D}(M)$ due to the definition of the latter module, i.e. $\mathrm{pd}_\Lambda \mathrm{E}^0(M) \leq \mathrm{pd}\ \Lambda - 2 = 1$, if $\mathrm{pd}\ \Lambda = 3$. In the case of $\mathrm{pd}\ \Lambda = 4$ it holds $\mathrm{E}^3\mathrm{E}^0(M) = \mathrm{E}^4\mathrm{E}^0(M) = 0$ and $\mathrm{E}^2\mathrm{E}^0(M) \cong \mathrm{E}^4\mathrm{E}^1(M)$ due to Björk's spectral sequence. Hence, if $\mathrm{E}^4\mathrm{E}^0(M)$ vanishes, it follows that $\mathrm{pd}_\Lambda \mathrm{E}^0(M) \leq 1$. Now, choosing a projective resolution of $E^0(M)$

$$0 \longrightarrow \mathrm{E}^0(P_1) \longrightarrow \mathrm{E}^0(P_0) \longrightarrow \mathrm{E}^0(M) \longrightarrow 0,$$

we derive the exact sequence

$$0 \longrightarrow \mathrm{E}^0\mathrm{E}^0(M) \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \mathrm{E}^1\mathrm{E}^0(M) \longrightarrow 0.$$

But $\mathrm{E}^1\mathrm{E}^0(M) \cong \mathrm{E}^3\mathrm{E}^1(M)$ due to Björk's spectral sequence for $\mathrm{pd}\ \Lambda \leq 4$. If M itself is reflexive and $\mathrm{pd}\ \Lambda = 3$, then $\mathrm{E}^1\mathrm{E}^1(M) = \mathrm{E}^2\mathrm{E}^1(M) = 0$, i.e. $\mathrm{E}^1(M)$ is finite, respectively $\mathrm{E}^3\mathrm{E}^1(M) \cong \mathrm{E}^1(M)^\vee$. □

Proposition 1.5.17. *Let G be a compact p-adic analytic group without p-torsion, $H \subseteq G$ a closed subgroup and M a finitely generated $\Lambda(H)$-module. If $d_{\Lambda(G)}$ (resp. $d_{\Lambda(H)}$) denotes the (projective or δ-) dimension of $\Lambda(G)$ (resp. $\Lambda(H)$), then the following holds:*

(i) $j_{\Lambda(G)}(\mathrm{Ind}_G^H M) = j_{\Lambda(H)}(M)$,

(ii) $\delta_{\Lambda(G)}(\mathrm{Ind}_G^H M) = \delta_{\Lambda(H)}(M) + d_{\Lambda(G)} - d_{\Lambda(H)}$,

(iii) $\mathrm{pd}_{\Lambda(G)}(\mathrm{Ind}_G^H M) = \mathrm{pd}_{\Lambda(H)}(M)$.

Proof. This is a consequence of 1.2.3, 1.5.5, (ii), and 1.7.3. □

Lemma 1.5.18. *Assume that $G = H \times \Gamma$ is a p-adic Lie group without p-torsion where Γ contains an open subgroup of index prime to p which is isomorphic to $\mathbb{Z}_p$. Let $M \in \mathcal{C}(G)$ be finitely generated and torsion as $\Lambda(H)$-module. Then M is a pseudo-null $\Lambda(G)$-module.*

Proof. By the corollary 1.4.4, there is an exact sequence

$$0 \longrightarrow \varprojlim_n \mathrm{E}^1_{\Lambda(H)}(M_{\Gamma_n}) \longrightarrow \mathrm{E}^1_{\Lambda(G)}(M) \longrightarrow \varprojlim_n \mathrm{E}^0_{\Lambda(H)}(M^{\Gamma_n}) = 0.$$

So, if we can show that the left term vanishes, we are done, because then $\mathrm{E}^1\mathrm{E}^1(M) = 0 = \mathrm{E}^0\mathrm{E}^0(M)$. Consider the commutative exact diagram

$$\begin{tikzcd} M \arrow[r, "\omega_n"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\frac{\omega_m}{\omega_n}"] & M_{\Gamma_n} \arrow[r] \arrow[d, "\frac{\omega_m}{\omega_n}"] & 0 \\ M \arrow[r, "\omega_m"] & M \arrow[r] & M_{\Gamma_m} \arrow[r] & 0, \end{tikzcd}$$

where $\omega_n = \gamma^{p^n} - 1$ for some generator γ of $\mathbb{Z}_p \subseteq \Gamma$. Since M is assumed to be $\Lambda(H)$-torsion, we get the commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{E}^1_{\Lambda(H)}(M_{\Gamma_m}) \arrow[r] \arrow[d, "\frac{\omega_m}{\omega_n}"] & \mathrm{E}^1_{\Lambda(H)}(M) \arrow[r, "\omega_m"] \arrow[d, "\frac{\omega_m}{\omega_n}"] & \mathrm{E}^1_{\Lambda(H)}(M) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{E}^1_{\Lambda(H)}(M_{\Gamma_n}) \arrow[r] & \mathrm{E}^1_{\Lambda(H)}(M) \arrow[r, "\omega_n"] & \mathrm{E}^1_{\Lambda(H)}(M).
\end{tikzcd}$$

Passing to the limit, we obtain

$$\varprojlim_n \mathrm{E}^1_{\Lambda(H)}(M_{\Gamma_n}) \subseteq \varprojlim_n \mathrm{E}^1_{\Lambda(H)}(M) = \varprojlim_n \bigcap_{m \geq n} \frac{\omega_m}{\omega_n} \mathrm{E}^1_{\Lambda(H)}(M) = 0,$$

because $\frac{\omega_m}{\omega_n}$ tends to zero.

$\square$

Remark 1.5.19. The same arguments show that $\delta_G(M) \leq \delta_H(M)$ for any finitely generated $\Lambda(H)$-module M.

Besides the case $G = \mathbb{Z}_p^d$ these results apply also to the following situation where G is an open subgroup of $Gl_d(\mathbb{Z}_p)$, d is prime to p, such that the determinant takes values in $\Gamma := \det(G) \subseteq \mathbb{Z}_p \subseteq \mathbb{Z}_p^*$. at least if . Indeed, we have the following exact commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & Sl_d(\mathbb{Z}_p) \arrow[r] & Gl_d(\mathbb{Z}_p) \arrow[r, "\det"] & \mathbb{Z}_p^* \arrow[r] & 1 \\
1 \arrow[r] & Sl_d(\mathbb{Z}_p) \cap G \arrow[r] \arrow[u, hook] & G \arrow[r] \arrow[u, hook] & \Gamma \arrow[r] \arrow[u, hook] & 1,
\end{tikzcd}$$

in which the lower sequence possesses the following splitting

$$s : \Gamma \cong \mathbb{Z}_p \to G, \ a \mapsto \begin{pmatrix} a^{1/d} & & & \\ & a^{1/d} & & \\ & & \ddots & \\ & & & a^{1/d} \end{pmatrix}$$

(Note that $\Gamma \cong \mathbb{Z}_p$ is considered as subgroup of the units and that $\mathbb{Z}_p$ is uniquely d-divisible. Furthermore, if the image of this homomorphism is not contained in G, we just apply the theory to an open subgroup U of G which fulfills this condition with respect to $\det(U)$ and contains $H := Sl_d(\mathbb{Z}_p) \cap G$. Such U always exists because $Gl_d(\mathbb{Z}_p)$ is p-adic analytic, i.e. the lower p-series forms a basic of neighborhoods of the neutral element. Hence at least for some m the image of $p^m\Gamma$ is contained in $G : s(a^{p^m}) = s(a)^{p^m} \epsilon P_m(Gl_d) \subseteq G$. Take $U := \det^{-1}(p^m\Gamma) \cap G$.) Since the splitting takes values in the center of G, we get a presentation of G as the direct product $G = H \times \mathbb{Z}_p$.

1.5.2 The graded ring $gr(\Lambda)$

An important method to verify the Auslander condition of a ring Λ consists of endowing Λ with a suitable filtration and studying the associated graded ring $gr(\Lambda)$. By a filtration on a ring Λ we mean an increasing (!) sequence of additive subgroups $\Sigma_{i-1} \subseteq \Sigma_i \subseteq \Sigma_{i+1}$ satisfying $\bigcup \Sigma_i = \Lambda$ and $\bigcap \Sigma_i = 0$ and the inclusions $\Sigma_i \Sigma_k \subseteq \Sigma_{i+k}$ hold for all pairs of integers i and k. The main example on a local ring is the $\mathfrak{M}$ -adic filtration with $\Sigma_{-i} = \mathfrak{M}^i$ for all $i \geq 0$ (by convention, $\mathfrak{M}^0 = \Lambda$). For our aim the *closure condition* will be crucial:

Definition 1.5.20. The filtration Σ satisfies the *closure condition* if the additive subgroups $\Sigma_{m_1} u_1 + \cdots + \Sigma_{m_s} u_s$ and $u_1 \Sigma_{m_1} + \cdots + u_s \Sigma_{m_s}$ are closed with respect to the topology induced by Σ for any finite subset $u_1, \ldots, u_s$ in Λ and all integers $m_1, \ldots, m_s$.

Lemma 1.5.21. *Let G be a p-adic analytic pro-p-group. Then the $\mathfrak{M}$ -adic filtration on $\Lambda(G)$ satisfies the closure condition.*

Proof. Note that the $\mathfrak{M}$-adic topology on Λ coincides with the $(\mathfrak{m}, I)$-topology (cf. [45, (5.2.15)]). Since $\mathfrak{M}$ is a two-sided ideal of Λ the subgroup $\mathfrak{M}^{i-m_1} u_1 + \cdots + \mathfrak{M}^{i-m_s} u_s$, $u_1 \mathfrak{M}^{i-m_1} + \cdots + u_s \mathfrak{M}^{i-m_s}$ is a finitely generated left, right ideal, respectively. Hence, these subgroups are compact as continuous images of the compact module Λ^n for some n. □

Put $gr(\Lambda) = \bigoplus \Sigma_i / \Sigma_{i-1}$, which is called the associated graded ring of Λ with respect to the filtration Σ. The above lemma admits applying the following theorem of Björk to certain completed group rings:

Theorem 1.5.22 (Björk). (i) *Assume that $gr(\Lambda)$ is a Auslander regular ring and that Σ satisfies the closure condition. Then Λ is a Auslander regular ring.*

(ii) *In the situation of* (i), *the equality $j(M) = j(gr(M))$ holds. If, in addition, $gr(\Lambda)$ is commutative and of pure dimension d, then also $\delta(M) =$* $\dim(gr(M))$ *holds, where* $\dim(gr(M)) = \dim(supp_{gr(\Lambda)}(gr(M)))$ *is the Krull dimension of $gr(M)$.*

Proof. See [5, Theorems 4.1,4.3] and also [6, Thm. 3.9. and Remark]. For the last equality note that

$$\begin{aligned}
\dim(gr(M)) &= \max\{\dim(gr(M)_{\mathfrak{p}} \mid \mathfrak{p} \textit{ maximal ideal of } gr(\Lambda)\} \\
&= d - \min\{j(gr(M)_{\mathfrak{p}}) \mid \mathfrak{p} \textit{ maximal ideal of } gr(\Lambda)\} \\
&= d - j(gr(M) \\
&= d - j(M) \\
&= \delta(M),
\end{aligned}$$

where we used prop. 1.5.5, (ii), and the fact that localization commutes with Ext-groups, if all objects are Noetherian. □

Our task will be to determine the structure of $gr(\Lambda(G))$. Before stating the next, result we recall that a pro-p-group G is called extra-powerful, if the relation $[G,G] \subseteq G^{p^2}$ holds. Furthermore, note that $gr(\mathbb{Z}_p) \cong \mathbb{F}_p[X_0]$ if $\mathbb{Z}_p$ is endowed with the $\mathfrak{m}$ -adic filtration.

Theorem 1.5.23. *Let G be a uniform and extra-powerful pro-p-group of dimension* $\dim(G) = r$. *Then there is a $gr(\mathbb{Z}_p)$-algebra-isomorphism*

$$gr(\Lambda(G)) \cong \mathbb{F}_p[X_0, \ldots, X_r].$$

In particular, $gr(\Lambda(G))$ is a commutative regular Noetherian ring.

A consequence of Lazard's results is the

Remark 1.5.24. Any compact p-adic analytic group contains an open characteristic subgroup, which is an uniform and extrapowerful pro-p-group (cf. [16, Cor. 9.36] and [62, Prop. 8.5.3])

For the proof of the theorem we need some more terminology. Let G be an uniform pro-p-group with a minimal system of (topological) generators $\{x_1, \ldots, x_r\}$, in particular $\dim(G) = r$. Then the lower p-series is given by $P_1(G) = G,\ P_{i+1}(G) = (P_i(G))^p,\ i \geq 1$. This filtration defines a p-valuation $\omega : G \longrightarrow \mathbb{N}_{>0} \cup \{\infty\} \subseteq \mathbb{R}_{>0} \cup \{\infty\}$ of G in the sense of Lazard via $\omega(g) := \sup\{i \mid g \in P_i(G)\}$, which induces a filtration on the group algebra $\mathbb{Z}_p[G]$ of the underlying abstract group of G, too (cf. [38, Chap. III, 2.3.1.2]).

Lemma 1.5.25. *The filtration on $\mathbb{Z}_p[G]$, induced by ω, is the $\mathfrak{M}_d$-adic one, where $\mathfrak{M}_d = \mathfrak{m} + I_d(G)$ with the augmentation ideal $I_d(G)$ of $\mathbb{Z}_p[G]$.*

Proof. Conferring the proof of Lemma III, (2.3.6) in [38], the induced filtration is given by the following ideals in $\mathbb{Z}_p[G]$, $n \in \mathbb{N} : A_n$ is generated as $\mathbb{Z}_p$-module by the elements $p^l(g_1 - 1) \cdots (g_m - 1)$ where $l, m \in \mathbb{N},\ g_i \in G$ and $l + \omega(g_1) + \ldots + \omega(g_m) \geq n$, whereas the $\mathfrak{M}_d$-adic filtration is defined by the ideals $\mathfrak{M}_d^n$, which are generated (over $\mathbb{Z}_p[G]$) by the elements $p^l(g_1 - 1) \cdots (g_m - 1)$, where $l, m \in \mathbb{N},\ g_i \in G$ and $l + m = n$. Since $\omega(g) \geq 1$ for all $g \in G$ the ideal $\mathfrak{M}_d^n$ is contained in A_n. The converse is a consequence of the following

Claim: Let $g \in G$ with $\omega(g) = t \geq 1$, then $g - 1 \in \mathfrak{M}_d^t$.

Since G is uniform, the map $G \longrightarrow P_t(G)$, which assigns $g^{p^{t-1}}$ to g, is surjective (cf. [16, lemma 4.10]), i.e. there exists an element $h \in G$ with

$g = h^{p^{t-1}}$. Writing

$$g - 1 = (1 + (h-1))^{p^{t-1}} - 1 = \sum_{k\geq 1} \begin{pmatrix} p^{t-1} \\ k \end{pmatrix} (h-1)^k,$$

one verifies that $g - 1 \,\epsilon\, \mathfrak{M}_d^t$, because $v_p(\begin{pmatrix} p^{t-1} \\ k \end{pmatrix}) = t - 1 - v_p(k) \geq t - k$, i.e. $\begin{pmatrix} p^{t-1} \\ k \end{pmatrix} (h-1)^k \,\epsilon\, \mathfrak{M}_d^t$. □

Lemma 1.5.26. *The $\mathfrak{M}_d$-adic filtration on $\mathbb{Z}_p[G]$ induces the $\mathfrak{M}$-adic filtration on $\mathbb{Z}_p[\![G]\!]$.*

Proof. The ideals defining the induced filtration are just the closure $\overline{\mathfrak{M}_d^n}$ of $\mathfrak{M}_d^n \subseteq \mathbb{Z}_p[G] \subseteq \mathbb{Z}_p[\![G]\!]$ with respect to the $\mathfrak{M}$-adic topology on $\mathbb{Z}_p[\![G]\!]$. Since they contain the elements $p^l(g_1-1)\cdots(g_m-1)$ with $l, m \,\epsilon\, \mathbb{N}$, $g_i \,\epsilon\, \{x_1, \ldots, x_r\}$ and $l + m = n$, which generate $\mathfrak{M}^n$ as ideal of $\mathbb{Z}_p[\![G]\!]$, they contain $\mathfrak{M}^n$, too. On the other hand $\mathfrak{M}^n$ is closed and contains all the generators of the $\mathbb{Z}_p[G]$-ideal $\mathfrak{M}_d^n : p^l(g_1 - 1)\ldots(g_m - 1),\ l, m \,\epsilon\, \mathbb{N},\ g_i \,\epsilon\, G$. This proves the lemma. □

Now we can prove theorem 1.5.23.

Proof. Since $gr(G) = \bigoplus P_i(G)/P_{i+1}(G)$ is a Lie algebra, which is free of rank r as $gr(\mathbb{Z}_p)$-module, we get the following inclusion:

$$\begin{aligned} gr(G) \subseteq Ugr(G) &\cong gr(\mathbb{Z}_p[G]) \\ &\cong gr(\mathbb{Z}_p[\![G]\!]), \end{aligned}$$

where the first equation holds by [38, Chap. III, 2.3.3] and $Ugr(G)$ is the enveloping algebra of the Lie algebra $gr(G)$, whereas the second one is a consequence of lemma 1.5.26. According to [62, Theorem 8.7.7], the graded ring $gr(\mathbb{Z}_p[\![G]\!])$ is commutative (G is assumed to be extra-powerful), i.e.

$$Ugr(G) \cong gr(\mathbb{Z}_p)[X_1, \ldots, X_r] \cong \mathbb{F}_p[X_0, \ldots, X_r].$$

□

As an important consequence we obtain the

Theorem 1.5.27. *Let G be a compact p-adic analytic group without p-torsion. Then the completed group ring $\Lambda(G)$ is an Auslander regular ring.*

Proof. G posses an open characteristic subgroup N which is an uniform, extra-powerful pro-p-group. By the theorem of Björk and theorem 1.5.23, $\Lambda(N)$ is an Auslander regular ring, because $gr(\mathbb{Z}_p[\![N]\!])$ has this property as a regular commutative Noetherian ring (cf. [4, pp. 65-69]). But $\mathrm{E}^i_{\Lambda(G)}(M) \cong \mathrm{E}^i_{\Lambda(N)}(M)$ as $\Lambda(N)$-modules for any $\Lambda(G)$-module M, by which the Auslander condition is easily verified. □

1.5.3 The μ-invariant

For the purpose to study the p-torsion part $\mathrm{tor}_\Lambda M$ of a Λ-module M we are also interested in the rings $\mathbb{Z}/p^m[\![G]\!] \cong \Lambda(G)/p^m$, especially the ring $\mathbb{F}_p[\![G]\!]$, and will consider the change of rings $\Lambda(G) \to \Lambda/p^m$. For a Λ/p^m-module M there exists a convergent spectral sequence (see [61, Ex. 5.6.3])

$$\mathrm{Ext}^i_{\Lambda/p^m}(M, \mathrm{Ext}^j_\Lambda(\Lambda/p^m, \Lambda)) \Rightarrow \mathrm{Ext}^{i+j}_\Lambda(M, \Lambda).$$

Using the free resolution

$$0 \longrightarrow \Lambda \xrightarrow{p^m} \Lambda \longrightarrow \Lambda/p^m \longrightarrow 0,$$

it is easy to calculate that

$$\mathrm{Ext}^j_\Lambda(\Lambda/p^m, \Lambda) \cong \begin{cases} \Lambda/p^m & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the spectral sequence degenerates to

$$\mathrm{E}^i_{\Lambda/p^m}(M) \cong \mathrm{E}^{i+1}_\Lambda(M)$$

for any Λ/p^m-module M and any integer i. We obtain the following

Theorem 1.5.28. *Let G be a compact p-adic analytic group without p-torsion and m any natural number. Then*

(i) $\mathbb{Z}_p/p^m[\![G]\!]$ *is an Auslander-Gorenstein ring of injective dimension* $\mathrm{cd}_p(G)$.

(ii) $\mathbb{F}_p[\![G]\!]$ *is an Auslander regular ring of dimension* $\mathrm{cd}_p(G)$.

Proof. From the above formula we derive that Λ/p^m has finite injective dimension $\mathrm{cd}_p(G)$. On the other hand it is well known that the projective dimension of $\mathbb{F}_p[\![G]\!]$ is equal to $\mathrm{cd}_p(G)$ (see [45, V§2Ex.5]). Hence it suffices to verify the Auslander condition: For a Λ/p^m- module M let $N \subseteq \mathrm{E}^i_{\Lambda/p^m}(M)$ be a Λ/p^m-submodule which we will also consider as Λ-submodule of $\mathrm{E}^{i+1}_\Lambda(M)$. Applying again the above isomorphism, we see that $\mathrm{E}^j_{\Lambda/P^m}(N) \cong \mathrm{E}^{j+1}_\Lambda(N) = 0$ for any integer $j < i$ because Λ fulfills the Auslander condition. □

A different possibility to prove (ii) of the previous theorem would be to imitate the proof of theorem 1.5.27 using the analogue of theorem 1.5.23: if G is a uniform pro-p-group of dimension d, then there is an isomorphism

$$gr(\mathbb{F}_p[\![G]\!]) \cong \mathbb{F}_p[X_1, \ldots, X_r],$$

where $\mathbb{F}_p[\![G]\!]$ is endowed with its $\mathfrak{M}$-adic filtration (see [62, 8.7.10]). In particular, $\mathbb{F}_p[\![G]\!]$ has no zero divisors for uniform G ([62, 8.7.9]).

In order to measure the size of the p-torsion part of a Λ-module we have (as usual) the μ-invariant which is defined as follows.

Definition 1.5.29. [5] Assume that G is a p-adic Lie group without p-torsion such that $\mathbb{F}_p[\![G]\!]$ is integral. For any $\Lambda(G)$-module M we define its μ-invariant $\mu(M)$ as

$$\mu(M) = \mathrm{rk}_{\mathbb{F}_p[\![G]\!]} \bigoplus_{i\geq 0} {}_{p^{i+1}}M/{}_{p^i}M,$$

where ${}_{p^0}M = 0$ by convention. Observe that the sum is finite because Λ is Noetherian.

Note that the μ-invariant only depends on the Λ- resp. $\mathbb{Z}_p$-torsion-submodule: $\mu(M) = \mu(\mathrm{tor}_\Lambda M) = \mu(\mathrm{tor}_{\mathbb{Z}_p} M) = \mu({}_{p^m}M)$ for m sufficiently large. With respect to the vanishing we have the following characterization:

Remark 1.5.30. Since ${}_{p^{i+1}}M/{}_{p^i}M \overset{p^i}{\hookrightarrow} {}_pM$ the following is equivalent

$$\begin{aligned} \mu(M) = 0 \ &\Leftrightarrow\ \mu({}_pM) = 0 \\ &\Leftrightarrow\ {}_pM \text{ is } \mathbb{F}_p[\![G]\!]\text{-torsion} \\ &\Leftrightarrow\ {}_pM \text{ is a pseudo-null } \Lambda\text{-module.} \end{aligned}$$

For the latter equivalence we used again the above isomorphism.

The next proposition shows that the μ-invariant is in fact an invariant "up to pseudo-isomorphism", i.e. it factors through the quotient category $\Lambda\text{-mod}/\mathcal{PN}$. In particular, our definition of μ generalizes the usual definition via the structure theory if G is isomorphic to $\mathbb{Z}_p^r$ for some r.

Proposition 1.5.31. *Let G be a p-adic analytic group without p-torsion such that both $\Lambda = \Lambda(G)$ and Λ/p are integral. Then*

$$M \sim N \textit{implies } \mu(M) = \mu(N).$$

Proof. The statement will follow if it holds in the two special cases of exact sequences

(a) $0 \longrightarrow Q \longrightarrow M \longrightarrow N \longrightarrow 0,$

[5] I am grateful to Susan Howson for inspiring this definition: Her original suggestion was to take $\mathrm{rk}_{\mathbb{F}_p[\![G]\!]} \bigoplus_{i\geq 0} p^i \mathrm{tor}_{\mathbb{Z}_p} M/p^{i+1}\mathrm{tor}_{\mathbb{Z}_p} M$. Though this will turn out to be equivalent it seemed to be more convenient to take the above definition. Of course, we also could have defined the μ-invariant via proposition 1.5.32.

(b) $\quad 0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0,$

where Q is pseudo-null. More generally, we consider a short exact sequence of Λ-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

The snake lemma implies the exactness and commutativity of the following diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_{p^n}X \arrow[r] \arrow[d, hook] & {}_{p^n}Y \arrow[r] \arrow[d, hook] & {}_{p^n}Z \arrow[r] \arrow[d, hook] & X/p^n \arrow[d, "p"] \\
0 \arrow[r] & {}_{p^{n+1}}X \arrow[r] & {}_{p^{n+1}}Y \arrow[r] & {}_{p^{n+1}}Y \arrow[r] & X/p^{n+1}.
\end{tikzcd}$$

Again by the snake lemma we obtain the exact sequences

$$0 \longrightarrow {}_{p^{n+1}}X/{}_{p^n}X \longrightarrow {}_{p^{n+1}}Y/{}_{p^n}Y \longrightarrow A_{n+1}/A_n \longrightarrow 0,$$

$$0 \longrightarrow K_n \longrightarrow A_{n+1}/A_n \longrightarrow {}_{p^{n+1}}Z/{}_{p^n}Z \longrightarrow B_{n+1}/B_n \longrightarrow 0,$$

where A_i denotes the image of ${}_{p^i}Y$ in ${}_{p^i}Z$ with cokernel B_i, the latter module considered as submodule of X/p^i, and $K_n := \ker(B_n \to B_{n+1})$. In case (b) A_{n+1}/A_n is a pseudo-null Λ-module because $A_{n+1} \subseteq Z$. Hence $\mathrm{rk}_{\mathbb{F}_p[\![G]\!]} A_{n+1}/A_n = 0$ by remark 1.5.30. In case (a) $\mathrm{rk}_{\mathbb{F}_p[\![G]\!]} {}_{p^{n+1}}X/{}_{p^n}X = 0$ by the same argument. Furthermore, $K_n \subseteq X/p^n$, $B_{n+1} \subseteq X/p^{n+1}$ and finally B_{n+1}/B_n are pseudo-null, too. $\square$

By Λ-mod(p) we shall write the plain subcategory of Λ-mod consisting of $\mathbb{Z}_p$-torsion modules while by $\mathcal{PN}(p)$ "$= \mathcal{PN} \cap \Lambda\text{-mod}(p)$" we denote the Serre subcategory of Λ-mod(p) the objects of which are pseudo-null Λ-modules. In other words M belongs to $\mathcal{PN}(p)$ if and only if it is a Λ/p^n-module for an appropriate n such that $\mathrm{E}^0_{\Lambda/p^n}(M) = 0$. Recall that there is a canonical exact functor $q : \Lambda\text{-mod}(p) \to \Lambda\text{-mod}(p)/\mathcal{PN}(p)$. For the description of the p-torsion part the following result will be crucial.

Proposition 1.5.32. *Assume that G is a p-adic analytic group without p-torsion such that both $\Lambda = \Lambda(G)$ and Λ/p are integral. Then the following holds:*

(i) *$q(\Lambda/p)$ is simple object in Λ-mod$(p)/\mathcal{PN}(p)$, i.e. does not contain any proper subobject.*

(ii) *Every object A in Λ-mod$(p)/\mathcal{PN}(p)$ has a finite composition series*

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{i+1} = A$$

of subobjects A_j of A such that $A_{j+1}/A_j \cong q(\Lambda/p)$ for every $i \geq j \geq 0$. In particular, $q(\Lambda/p)$ is the unique simple object of Λ-mod$(p)/\mathcal{PN}(p)$.

(iii) *Any $q(M)$ in Λ-mod$(p)/\mathcal{PN}(p)$ has finite length equal to $\mu(M)$. Thus, $[q(M)] \mapsto \mu(M)$ induces an isomorphism*

$$K_0(\Lambda\text{-}mod(p)/\mathcal{PN}(p)) \cong \mathbb{Z}.$$

We need the following lemma which can be proved literally as [29, lem 2.25] because $\mathbb{F}_p[\![G]\!]$ is both left and right Noetherian ring without zero divisors and thus it has a skew field of fractions.

Lemma 1.5.33. *With the assumptions of the proposition let M be a torsion-free Λ/p-module of rank $\mathrm{rk}_{\Lambda/p}(M) = m$. Then there exist free Λ/p-modules F, F' such that $F \subseteq M$, $M \subseteq F'$ and both M/F and F'/M are Λ/p-torsion, i.e. pseudo-null considered as Λ-module. In particular, for any Λ/p-module of rank m there is an isomorphism*

$$q(M) \cong q(\Lambda/p)^m.$$

Proof. Let $h : q(M) \hookrightarrow q(\Lambda/p)$ be a monomorphism in the quotient category. By [60, I 2.9] there exists a diagram

$$\begin{tikzcd} M & & \Lambda/p \\ & M' \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

in Λ-mod(p) with f a pseudo-isomorphism in Λ-mod such that

$$\begin{tikzcd} q(M) \arrow[rr, "h"] & & q(\Lambda/p) \\ & q(M') \arrow[ul, "q(f)"] \arrow[ur, "q(g)"'] & \end{tikzcd}$$

commutes. Since h is a monomorphism and $q(f)$ an isomorphism, $\ker(g)$ must be in $\mathcal{PN}(p)$. Since $M'/\ker(g) \subseteq \Lambda/p$, we can consider its Λ/p-rank which can be either 1 or 0. In the first case we conclude that g is a pseudo-isomorphism, i.e. $q(g)$ is an isomorphism, while in the second case $M'/\ker(g)$ and hence M' is pseudo-null, thus $q(M') = 0$. This proves (i).
For any $M \in \Lambda$-mod(p), the canonical decomposition

$$0 \subseteq {}_pM \subseteq {}_{p^2}M \subseteq \cdots \subseteq {}_{p^m}M = M$$

for some m, induces a decomposition

$$0 \subseteq q({}_pM) \subseteq q({}_{p^2}M) \subseteq \cdots \subseteq q({}_{p^m}M) = q(M)$$

with

$$q({}_{p^{j+1}}M)/q({}_{p^j}M) \cong q({}_{p^{j+1}}M/{}_{p^j}M) \cong q(\Lambda/p)^{d_j},$$

where $d_j = \mathrm{rk}_{\Lambda/p}({}_{p^{j+1}}M/{}_{p^j}M)$ by the previous lemma. Since this filtration can be refined easily to a decomposition series of the desired kind, we are done. □

Corollary 1.5.34. *The invariant μ is additive on short exact sequences of Λ-torsion modules.*

Proof. Since μ is additive on short exact sequences of p-torsion modules by the proposition it suffices to reduce the general statement to this case. Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a short exact sequence of Λ-torsion modules. Choosing a number n such that the p-torsion parts of X, Y and Z are annihilated by p^n, we obtain an exact sequence

$$0 \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}X \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}Y \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}Z \longrightarrow X/p^n \xrightarrow{\varphi} Y/p^n.$$

Considering the exact, commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{tor}_{\mathbb{Z}_p}X \arrow[r] \arrow[d, hook] & X/p^n \arrow[r] \arrow[d] & (X/\mathrm{tor}_{\mathbb{Z}_p})/p^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{tor}_{\mathbb{Z}_p}Y \arrow[r] & Y/p^n \arrow[r] & (Y/\mathrm{tor}_{\mathbb{Z}_p})/p^n \arrow[r] & 0,
\end{tikzcd}$$

we see that $\ker(\varphi)$ is pseudo-null by the following lemma, i.e.

$$0 \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}X \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}Y \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}Z \longrightarrow 0$$

is exact mod $\mathcal{PN}$. □

Lemma 1.5.35. *Assume that G is a p-adic analytic group without p-torsion such that both $\Lambda = \Lambda(G)$ and Λ/p are integral. Let M be a (not necessarily torsion) Λ-module. Then the following holds*

$$\mu(M/p) = 0 \quad \Rightarrow \quad \mu({}_pM) = 0.$$

Proof. Since $(\mathrm{tor}_{\mathbb{Z}_p}M)/p \subseteq M/p$ by the snake lemma, it suffices to deal with the case that M is Λ-torsion. But then the additivity of the μ-invariant shows immediately that $\mu({}_pM) = \mu(M/p)$. □

Lemma 1.5.36. *Assume that G is a p-adic analytic group without p-torsion such that both $\Lambda = \Lambda(G)$ and Λ/p are integral. Let M be a Λ-torsion module with* $\mathrm{tor}_{\mathbb{Z}_p} M = 0$. *Then*

(i) *for any integer $n \geq 1$, the module M/p^n is pseudo-null.*

(ii) $\mathrm{tor}_{\mathbb{Z}_p} \mathrm{E}^1(M) = 0$.

We will denote the annihilator in Λ of an element $m \,\epsilon\, M$ by $ann_\Lambda(m) := \{\lambda \,\epsilon\, \Lambda | \lambda m = 0\}$.

Proof. Since there is a surjection

$$\bigoplus \Lambda/ann_\Lambda(m_i) \twoheadrightarrow M$$

for a finite set of generators m_i of M, it suffices to prove (i) in the case $M := \Lambda/I$, where I is a non-zero left ideal of Λ. As M/p^n is Λ-torsion we are done once we have shown the vanishing of $\mathrm{E}^1_\Lambda(M/p^n)$. But

$$\begin{aligned} \mathrm{E}^1_\Lambda(M/p^n) &\cong \mathrm{E}^0_{\Lambda/p^n}(M/p^n) \\ &\cong \mathrm{Hom}_{\Lambda/p^n}(M, \Lambda/p^n) \\ &\cong \mathrm{Hom}_\Lambda(\Lambda/I, \Lambda/p^n) = 0. \end{aligned}$$

Indeed, the vanishing of the latter module can be seen as follows: let $\varphi : \Lambda \to \Lambda/p^n$ be a non-trivial homomorphism of Λ-modules which factors through Λ/I, i.e. $I \subseteq ann_\Lambda(x \bmod p^n)$ with $x \equiv \varphi(1) \bmod p^n$.

Claim: $ann_\Lambda(x \bmod p^n) \subseteq p\Lambda$.

Let $\lambda \,\epsilon\, ann_\Lambda(x \bmod p^n)$, i.e. $\lambda x = p^n y$ for some $y \,\epsilon\, \Lambda$ and let n_o be the maximal integer with $x \,\epsilon\, p^{n_0}\Lambda$, i.e. $x = p^{n_0} x_0$ for some $x_0 \,\epsilon\, \Lambda \backslash p\Lambda$ and $n_0 < n$. Since the multiplication by p^{n_0} is injective we obtain $\lambda x_o = p^{n-n_0} y \equiv 0 \bmod p$. Hence $\lambda \,\epsilon\, p\Lambda$ because Λ/p is integral. This proves the claim.
The fact that ${}_pM = 0$, implies $I \cap p\Lambda = pI$ and regarding the claim it holds

$$I = I \cap p\Lambda = pI = \ldots = p^m I$$

for any $m \geq 0$. Since p^m tends to zero if m goes to infinity the ideal I must be zero, a contradiction.
The second statement results from the first one regarding the exact sequence

$$0 = \mathrm{E}^1(M/p) \longrightarrow \mathrm{E}^1(M) \xrightarrow{p} \mathrm{E}^1(M).$$

$\square$

We finish this section with a "structure theorem for the p-torsion part of Λ-modules."

Theorem 1.5.37. *Assume that G is a p-adic analytic group without p-torsion such that both $\Lambda = \Lambda(G)$ and Λ/p are integral. Let M be in Λ-mod(p). Then there exist uniquely determined natural numbers $n_1, \dots, n_r$ and an isomorphism in Λ-mod(p)/$\mathcal{PN}(p)$*

$$M \equiv \bigoplus_{1 \le i \le r} \Lambda/p^{n_i} \mod \mathcal{PN}(p).$$

Proof. Let m be minimal with the property: ${}_{p^{m+1}}M = M$. The theorem is proved using induction with respect to m. The case $m = 0$ is just lemma 1.5.33, so let m be arbitrary. Again by lemma 1.5.33 we are in the following situation:

$$\begin{tikzcd}
 & & & (\Lambda/p)^d \arrow[d, hook, "\iota"] & \\
0 \arrow[r] & {}_{p^m}M \arrow[r] & M \arrow[r] & M/{}_{p^m}M \arrow[r] & 0,
\end{tikzcd}$$

where d is the Λ/p-rank of $M/{}_{p^m}M$ and the cokernel of ι is pseudo-null. Replacing M by the pull-back with ι, we may assume that $M/{}_{p^m}M \cong (\Lambda/p)^d$. Since $(\Lambda/p^{m+1})^d$ is free in the category of Λ/p^{m+1}-modules, we obtain easily the following exact and commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\Lambda/p^m)^d \arrow[r] \arrow[d, "\varphi"] & (\Lambda/p^{m+1})^d \arrow[r] \arrow[d, "\psi"] & (\Lambda/p)^d \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & {}_{p^m}M \arrow[r] \arrow[d] & M \arrow[r, "pr"] \arrow[d] & (\Lambda/p)^d \arrow[r] & 0 \\
 & N \arrow[r, equal] & N & &
\end{tikzcd}$$

where N is by definition the cokernel of ψ respectively φ. First we will show that ψ and hence also φ is injective. Since $(\Lambda/p^{m+1})^d$ - being of projective Λ-dimension 1 - does not contain any proper pseudo-null Λ-submodule, it suffices to prove that $\ker(\psi)$ is pseudo-null. Assuming the contrary, i.e. that $\mu(\ker(\psi)) \neq 0$, it follows that $\mu({}_{p^{m+1}}K/{}_{p^m}K) < d$ for the image K of ψ because for an arbitrary p-torsion Λ-module N $\mathrm{rk}_{\Lambda/p}({}_{p^{i+1}}N/{}_{p^i}N) \geq \mathrm{rk}_{\Lambda/p}({}_{p^{i+2}}N/{}_{p^{i+1}}N)$ holds for any $i \geq 0$. But this contradicts the surjectivity of $pr \circ \psi$.

To prove the theorem we only have to show that φ has a co-section in Λ-mod(p)/$\mathcal{PN}(p)$, i.e. that the short exact sequence in the left column splits.

Indeed, then a section $N \hookrightarrow {}_{p^m}M$ would give rise to a section $N \hookrightarrow M$, i.e. $M \cong N \oplus (\Lambda/p^{m+1})^d$, and by the assumption of the induction N is already of the desired form. *Here and in what follows we are arguing in the quotient category Λ-mod(p)$/\mathcal{PN}(p)$, though we omit the functor q in the notation of maps and objects for simplicity.*
Again by this assumption, the module ${}_{p^m}M$ is isomorphic to a module of the form $(\Lambda/p^m)^{d'} \oplus \bigoplus_i \Lambda/p^{n_i}$, where $n_i < m$. Assume first that $d = 1$. We claim that the image of φ is mapped surjectively onto one of the factors Λ/p^m under the correspondent projection. Indeed, it is easy to see that otherwise the image would be contained in ${}_{p^{m-1}}M$, which contradicts the injectivity of φ. Counting μ-invariants, we see that φ followed by the projection onto such a factor gives an isomorphism and therefore induces the desired co-section. If $d > 1$ we make the same procedure iteratively for every factor of $(\Lambda/p^m)^d$ after first splitting up the image of the previous factor(s). The theorem follows because the uniqueness can be deduced easily from the decomposition

$$0 \subseteq {}_pM \subseteq {}_{p^2}M \subseteq \cdots \subseteq {}_{p^m}M = M$$

counting Λ/p-ranks. □

1.6 Local Duality

In this and the following section let $\Lambda = \Lambda(G) = \mathbb{Z}_p[\![G]\!]$ be the completed group algebra over $\mathbb{Z}_p$, *where G is a pro-p Poincaré group, such that Λ is Noetherian,* $\mathfrak{M}$ the maximal ideal of Λ and $k = \Lambda/\mathfrak{M} \cong \mathbb{F}_p$ its finite residue class field. It is well known that the global homological dimension of Λ is $d = \mathrm{cd}(G) + 1$. By Λ-Mod we denote the category of (abstract) modules over the (abstract) ring Λ and we write Λ-mod for the full subcategory of finitely generated modules. In the sequel we will use frequently the equivalence of the latter category with the category of finitely generated compact modules.

Definition 1.6.1. For a finitely generated Λ-module M, we define the depth by

$$\mathrm{depth}(M) := \min\{i \mid \mathrm{Ext}^i_\Lambda(k, M) \neq 0\}.$$

Recall that for a commutative Noetherian ring Λ the I-depth $depth_I(M)$ of a finitely generated Λ-module M with respect to an ideal I is the maximal length of a M-regular sequence in I. For a local ring the depth(M) is $depth_{\mathfrak{M}}(M)$, while the grade defined in 1.5.3 is $j(M) = depth_{ann(M)}(\Lambda)$, where $ann(M)$ is the annihilator of M in Λ.

We consider the additive functor $\Gamma_{\mathfrak{M}}(-) : \Lambda\text{-Mod} \to \Lambda\text{-Mod}$ defined by $\Gamma_{\mathfrak{M}}(M) := \{x \in M \mid \mathfrak{M}^l x = 0 \text{ for some l }\}$ and state some basic properties:

Lemma 1.6.2. (i) $\Gamma_{\mathfrak{M}}(M) = \varinjlim_{l} \mathrm{Hom}_\Lambda(\Lambda/\mathfrak{M}^l, M)$,

in particular, the functor $\Gamma_{\mathfrak{M}}(-)$ *is left exact.*

(ii) *The restriction of* $\Gamma_{\mathfrak{M}}$ *to* Λ*-mod equals* T_0*, i.e.* $\Gamma_{\mathfrak{M}}(M)$ *is the maximal finite submodule of* M*, if the latter module is finitely generated.*

Proof. Since $\mathrm{Hom}_\Lambda(\Lambda/\mathfrak{M}^l, M) = \{x \in M \mid \mathfrak{M}^l x = 0\}$, the first statement is obvious. If M is finitely generated, there is some l such that $\mathfrak{M}^l \mathrm{T}_0(M) = 0$, i.e. $\mathrm{T}_0(M) \subseteq \Gamma_{\mathfrak{M}}(M)$. On the other hand $\Lambda/\mathfrak{M}^l$ is a finite ring. Therefore $\Lambda x \subseteq \mathrm{T}_0(M)$ holds for any $x \in \Gamma_{\mathfrak{M}}(M)$. □

Since Λ-Mod has sufficiently many injectives, we can form the right derived functors

$$\mathrm{H}^i_{\mathfrak{M}}(-) = R^i\Gamma_{\mathfrak{M}}(-) = \varinjlim_{l} \mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M}^l, M)$$

(noting the exactness of direct limits in Λ-Mod). We write

$$\Lambda\text{-Mod}_{\mathfrak{M}},\ \Lambda\text{-mod}_{\mathfrak{M}}$$

for the full subcategory of Λ-Mod , Λ-mod respectively, consisting of those modules M, for which $\mathrm{H}^0_{\mathfrak{M}}(M) = M$ holds.

$$\mathcal{D}(\Lambda\text{-Mod})\ (\text{resp. } \mathcal{C}(\Lambda\text{-Mod}))$$

means the category of discrete (resp. compact) Λ-modules, where Λ is endowed with its canonical $(\mathfrak{m}, I)$-topology.

Lemma 1.6.3. $\mathrm{H}^i_{\mathfrak{M}}(-)$ *commutes with direct limits.*

Proof. Choose a resolution $P_\bullet$ of $\Lambda/\mathfrak{M}^l$ by finitely generated projectives in order to calculate $\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M}^l, M)$. Since $\mathrm{Hom}_\Lambda(P_j, -)$ commutes with direct limits (as P_j is finitely generated, i.e. any homomorphism $\phi : P_j \to \varinjlim_{i} M_i$ factors over some M_i), $\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M}^l, -)$ does also and the lemma follows. □

Proposition 1.6.4. *The forgetful functor defines an equivalence of categories*

$$\mathcal{D}(\Lambda\textit{-Mod}) \cong \Lambda\textit{-Mod}_{\mathfrak{M}}.$$

Proof. Both categories consists exactly of direct limits of finite modules (cf. [45, Prop. (5.2.4)] for $\mathcal{D}(\Lambda\text{-Mod})$). □

Lemma 1.6.5. (i) $\mathrm{H}^i_{\mathfrak{M}}(\Lambda\text{-}Mod) \subseteq \Lambda\text{-}Mod_{\mathfrak{M}} \cong \mathcal{D}(\Lambda\text{-}Mod)$ *for all* $i \geq 0$.

(ii) *For any* $M \in \Lambda\text{-}mod$, *it holds* $\mathrm{depth}(M) = \min\{i \mid \mathrm{H}^i_{\mathfrak{M}}(M) \neq 0\}$.

(iii) $\mathrm{depth}(\Lambda) = d$ *and* $\mathrm{H}^d_{\mathfrak{M}}(\Lambda) = \Lambda^\vee$.

(iv) $\mathrm{Hom}_\Lambda(M, \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) \cong M^\vee$ *for all* M *in* $\Lambda\text{-}Mod_{\mathfrak{M}}$ *or in* $\Lambda\text{-}mod$ *, in particular,* $\mathrm{H}^d_{\mathfrak{M}}(\Lambda)$ *is an injective* Λ*-module.*

Proof. Since $\mathrm{H}^i_{\mathfrak{M}}(-)$ are the derived functors of $\mathrm{H}^0_{\mathfrak{M}}(-)$, it suffices to prove (i) for the latter functor. But in this case the statement holds just by definition.

Now we will prove (ii) and set $k = \min\{i \mid \mathrm{H}^i_{\mathfrak{M}}(M) \neq 0\}$. Since $\mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M}^l, M) = 0$ for all $i < \mathrm{depth}(M)$ (note that $\Lambda/\mathfrak{M}^l$ has a finite composition series with subquotients isomorphic to k), it holds $\mathrm{depth}(M) \leq k$. So we only have to prove that $\mathrm{H}^j_{\mathfrak{M}}(M) \neq 0$ for $j = \mathrm{depth}(M) < \infty$. But the short exact sequences

$$0 \longrightarrow \mathfrak{M}/\mathfrak{M}^l \longrightarrow \Lambda/\mathfrak{M}^l \longrightarrow k \longrightarrow 0$$

induce the long exact sequences

$$0 = \mathrm{Ext}^{j-1}_\Lambda(\mathfrak{M}/\mathfrak{M}^l, M) \longrightarrow \mathrm{Ext}^j_\Lambda(k, M) \longrightarrow \mathrm{Ext}^j_\Lambda(\Lambda/\mathfrak{M}^l, M) \longrightarrow \cdots,$$

i.e. $0 \neq \mathrm{Ext}^j_\Lambda(k, M) \subseteq \mathrm{H}^j_{\mathfrak{M}}(M)$.

Using 1.4.8 and denoting the character of the dualizing module by χ, we calculate

$$\mathrm{H}^i_{\mathfrak{M}}(\Lambda) = \varinjlim_l \mathrm{E}^i(\Lambda/\mathfrak{M}^l) = \begin{cases} \varinjlim_l (\Lambda/\mathfrak{M}^l(\chi))^\vee = \Lambda^\vee & \text{if } i = d \\ 0 & \text{otherwise,} \end{cases}$$

whence (iii) follows. In order to prove (iv) first let M be in $\Lambda\text{-Mod}_{\mathfrak{M}}$, i.e. $M = \varinjlim_i M_i$ for some finite Λ-modules M_i. Then, noting that M_i is a $\Lambda/\mathfrak{M}^{l(i)}$-module for some $l(i)$ and using the adjunction of "Hom and ⊗",

$$\begin{aligned}
\mathrm{Hom}_\Lambda(M, \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) &= \mathrm{Hom}_\Lambda(\varinjlim_i M_i, \varinjlim_l (\Lambda/\mathfrak{M}^l)^\vee) \\
&= \varprojlim_i \mathrm{Hom}_\Lambda(M_i, \varinjlim_l (\Lambda/\mathfrak{M}^l)^\vee) \\
&= \varprojlim_i \mathrm{Hom}_\Lambda(M_i, \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda/\mathfrak{M}^{l(i)}, \mathbb{Q}_p/\mathbb{Z}_p)) \\
&= \varprojlim_i \mathrm{Hom}_{\mathbb{Z}_p}(M_i, \mathbb{Q}_p/\mathbb{Z}_p) \\
&= M^\vee.
\end{aligned}$$

Now let M be in Λ-mod . Then, noting that $\mathrm{Hom}_\Lambda(M,-)$ commutes with direct limits, because M is finitely generated,

$$\begin{aligned}
\mathrm{Hom}_\Lambda(M, \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) &= \mathrm{Hom}_\Lambda(M, \varinjlim_l (\Lambda/\mathfrak{M}^l)^\vee) \\
&= \varinjlim_l \mathrm{Hom}_\Lambda(M, (\Lambda/\mathfrak{M}^l)^\vee) \\
&= \varinjlim_l \mathrm{Hom}_\Lambda(M/\mathfrak{M}^l, \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda/\mathfrak{M}^l, \mathbb{Q}_p/\mathbb{Z}_p)) \\
&= \varinjlim_l \mathrm{Hom}_{\mathbb{Z}_p}(M/\mathfrak{M}^l, \mathbb{Q}_p/\mathbb{Z}_p) \\
&= M^\vee.
\end{aligned}$$

□

After this technical preparations we are able to prove the following

Theorem 1.6.6. *Let G be a pro-p Poincaré group with $d := \mathrm{cd}(G)+1 < \infty$ and such that $\Lambda = \Lambda(G)$ is Noetherian. Then, for any $M \in \Lambda$-mod,*

$$\mathrm{E}^i(M) \cong \mathrm{Hom}_\Lambda(\mathrm{H}^{d-i}_{\mathfrak{M}}(M), \mathrm{H}^d_{\mathfrak{M}}(\Lambda)) \cong \mathrm{H}^{d-i}_{\mathfrak{M}}(M)^\vee =: T^i(M).$$

Proof. Consider the right exact contravariant additive functor $T^0(-) = \mathrm{H}^d_{\mathfrak{M}}(M)^\vee$ on Λ-mod (note that $\mathrm{H}^i_{\mathfrak{M}}(M) = 0$ for all $i > d$ as Λ has global dimension d). By [53, Thm. 3.36 and Remarks] there is a natural equivalence of functors

$$T^0(-) \cong \mathrm{Hom}_\Lambda(-, T^0(\Lambda)) \cong \mathrm{Hom}_\Lambda(-, \Lambda)$$

on Λ-mod . Therefore, it suffices to show that the functors $T^i(-)$ are the left derived functors of $T^0(-)$. But $\{T^i(-)\}_{i\geq 0}$ is a universal δ-functor because they are effaceable by projectives in Λ-mod (Since T^0 is additive, it is sufficient to verify that $\mathrm{H}^i_{\mathfrak{M}}(\Lambda) = 0$ for all $i < d$, which is done by lemma 1.6.5 (iii)). □

1.7 Auslander-Buchsbaum equality

In this section we assume the same conditions on Λ as in the previous one and, under this conditions, we are going to prove the Auslander-Buchsbaum equality

$$\mathrm{pd}(M) + \mathrm{depth}(M) = \mathrm{depth}(\Lambda)$$

for all $M \in \Lambda$-mod. In the theory of commutative local rings this equality can be proved using *regular sequences*. Since this concept is lacking in the non-commutative theory, we will have to replace it by homological methods, i.e. we will work in derived categories. Our proof is analogous to Jørgensen's proof of the Auslander-Buchsbaum equality in the case of (non-commutative) graded algebras over a field (cf. [35]).

First, we recall the definitions of total Hom and total tensor product. Let $X, Y \in \mathbf{K}(\Lambda\text{-Mod})$ and define

$$(\mathrm{Hom}_\Lambda(X,Y))^n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_\Lambda(X^i, Y^{i+n}),\ d^n = \prod_i (d_X^{i-1} + (-1)^{n+1} d_Y^{i+n})$$

and

$$(X \otimes_\Lambda Y)^n = \bigoplus_{i+j=n} X^i \otimes_\Lambda Y^j,\ d^n = \bigoplus_{i+j=n} (d_X^i \otimes 1 + (-1)^n \otimes d_Y^j).$$

They become bifunctors

$$\mathrm{Hom}_\Lambda(-,-) : \mathbf{K}(\Lambda\text{-Mod})^{op} \times \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\mathbb{Z}_p\text{-Mod}),$$
$$- \otimes_\Lambda - : \mathbf{K}(\text{Mod-}\Lambda) \times \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\mathbb{Z}_p\text{-Mod}),$$

where we denote by Mod-Λ the category of right Λ-modules. Note that the latter category is equivalent to Λ-Mod due to the involution on the group algebra Λ. Moreover, if Y is a complex of bi-modules, then the values of $\mathrm{Hom}_\Lambda(-, Y)$ are in $\mathbf{K}(\text{Mod-}\Lambda)$, if X is a complex of bi-modules, then $X \otimes_\Lambda -$ has values in $\mathbf{K}(\Lambda\text{-Mod})$.

Since Λ-Mod has enough projectives, the derived functors exist (cf. [28, Chap. I, Theorem 5.1] or [61, Thm 10.5.6]):

$$\mathbf{R}\mathrm{Hom}_\Lambda(-,-) : \mathbf{D}^-(\Lambda\text{-Mod})^{op} \times \mathbf{D}(\Lambda\text{-Mod}) \to \mathbf{D}(\mathbb{Z}_p\text{-Mod}),$$

respectively

$$\mathbf{R}\mathrm{Hom}_\Lambda(-,-) : \mathbf{D}^-(\Lambda\text{-Mod})^{op} \times \mathbf{D}(\Lambda\text{-Mod-}\Lambda) \to \mathbf{D}(\text{Mod-}\Lambda)$$

and

$$- \otimes_\Lambda^{\mathbf{L}} - : \mathbf{D}(\text{Mod-}\Lambda) \times \mathbf{D}^-(\Lambda\text{-Mod}) \to \mathbf{D}(\mathbb{Z}_p\text{-Mod}),$$

respectively

$$- \otimes_\Lambda^{\mathbf{L}} - : \mathbf{D}(\Lambda\text{-Mod-}\Lambda) \times \mathbf{D}^-(\Lambda\text{-Mod}) \to \mathbf{D}(\Lambda\text{-Mod}).$$

$\mathbf{R}\mathrm{Hom}$, respectively $\otimes^{\mathbf{L}}$, is computed via a projective resolution in the first, respectively second variable.

Proposition 1.7.1. *Let $Y \in \mathbf{D}^b(\Lambda\text{-}Mod\text{-}\Lambda)$, $Z \in \mathbf{D}^b(\Lambda\text{-}Mod)$ and let $X \in \mathbf{D}^b(\Lambda\text{-}Mod)$ be a bounded complex which is quasi-isomorphic to a bounded complex consisting of finitely generated free Λ-modules. Then*

$$\mathbf{R}Hom_\Lambda(X, Y \otimes^{\mathbf{L}}_\Lambda Z) \cong \mathbf{R}Hom_\Lambda(X, Y) \otimes^{\mathbf{L}}_\Lambda Z.$$

Proof. (See [35, Proposition 2.1] for the case of graded algebras over a field.)

Replacing X with a quasi-isomorphic complex $L \in \mathbf{D}^b(\Lambda\text{-Mod})$ consisting of finitely generated free Λ-modules and replacing Z with a quasi-isomorphic complex $F \in \mathbf{D}^-(\Lambda\text{-Mod})$ consisting of projectives, we see that we have to prove

$$\mathrm{Hom}_\Lambda(L, Y \otimes_\Lambda F) = \mathrm{Hom}_\Lambda(L, Y) \otimes_\Lambda F.$$

But due to the boundedness condition and the fact that L consists of finitely generated free modules, the nth module on either side becomes

$$\bigoplus_{i,j} \mathrm{Hom}_\Lambda(L^i, Y^j) \otimes_\Lambda F^{n+i-j}$$

while the differentials on each summand $\mathrm{Hom}_\Lambda(L^i, Y^j) \otimes_\Lambda F^{n+i-j}$ are given by

$$\begin{aligned} d_L^{i-1} \otimes 1 + d_Y^j \otimes (-1)^{j-i-1} + (-1)^n \otimes d_F^{n+i-j}, &\quad \text{respectively} \\ d_L^{i-1} \otimes 1 + d_Y^j \otimes (-1)^n + (-1)^{i+1} \otimes d_F^{n+i-j}. & \end{aligned}$$

We will construct an isomorphism between the two complexes: If the minimal non-zero module of each of the complexes is $\mathrm{Hom}(L^{i_0}, Y^{j_0}) \otimes \Lambda F^{n_0+i_0-j_0}$, then the multiplication by suitable signs on the summands associated to the triple of indices $(a, b, c) = (i, j, n+i-j)$ defines an isomorphism of complexes. For example, we can choose these signs by the following rules, which determine them uniquely:

(i) $sign((i_0, j_0, n_0 + i_0 - j_0)) = 1$,

(ii) $sign((a + 1, b, c)) = sign(a, b, c)$,

(iii) $sign((a, b + 1, c)) = (-1)^c sign((a, b, c))$,

(iv) $sign((, a, b, c + 1)) = (-1)^{c+b+1} sign((a, b, c))$.

$\square$

In the proof of the next theorem we use the notation

$$\sigma_{\geq n}(Y) := \cdots \longrightarrow 0 \longrightarrow Y^n/\mathrm{im}(Y^{n-1}) \longrightarrow Y^{n+1} \longrightarrow Y^{n+2} \longrightarrow \cdots$$

for the truncation of a complex Y at the degree n.

Theorem 1.7.2. *(Auslander-Buchsbaum equality) For any $M \epsilon$ Λ-mod, it holds*

$$\mathrm{pd}_\Lambda(M) + \mathrm{depth}_\Lambda(M) = \mathrm{depth}_\Lambda(\Lambda).$$

Proof. (See [35, Thm 3.2] for the case of graded algebras over a field.)

Regard k, M, Λ as complexes concentrated in degree zero. Then the invariants in question are related to each other by the following isomorphism

$$\mathbf{R}\mathrm{Hom}_\Lambda(k, M) \cong \mathbf{R}\mathrm{Hom}_\Lambda(k, \Lambda \otimes^{\mathbf{L}}_\Lambda M) \cong \mathbf{R}\mathrm{Hom}_\Lambda(k, \Lambda) \otimes^{\mathbf{L}}_\Lambda M,$$

where we use proposition 1.7.1. Choosing a minimal free resolution L of M and noting that the truncation

$$T = \sigma_{\geq d}(\mathbf{R}\mathrm{Hom}_\Lambda(k, \Lambda))$$

is quasi-isomorphic to $\mathbf{R}\mathrm{Hom}_\Lambda(k, \Lambda)$, we can replace the right term by $T \otimes_\Lambda L$.

The lowest non-zero module in T is T^d with $d = \mathrm{depth}(\Lambda)$ while the lowest non-zero module in L is $L^{-\mathrm{pd}(M)}$ according to corollary 1.1.7 . So the lowest non-zero module in $T \otimes_\Lambda L$ becomes $(T \otimes_\Lambda L)^{d-\mathrm{pd}(M)} = T^d \otimes_\Lambda L^{-\mathrm{pd}(M)}$. Obviously, $\mathrm{depth}(M) \geq d - \mathrm{pd}(M)$, because $\mathrm{depth}(M) = \min\{i \mid \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_\Lambda(k, M) \neq 0\}$. So we need to see that $\mathrm{H}^{d-\mathrm{pd}(M)}(T \otimes_\Lambda L)$ is nonzero.

However, $k \cong \mathrm{Ext}^d_\Lambda(k, \Lambda) = \ker(d^d_T) \subseteq T^d$ and the "beginning" of the complex $T \otimes_\Lambda L$ looks like

$$0 \longrightarrow T^d \otimes_\Lambda L^{-\mathrm{pd}(M)} \longrightarrow T^d \otimes_\Lambda L^{-\mathrm{pd}(M)+1} \oplus T^{d+1} \otimes_\Lambda L^{-\mathrm{pd}(M)} \longrightarrow \cdots.$$

Now it holds that

$$0 \neq \ker(d^d_T) \otimes_\Lambda L^{-\mathrm{pd}(M)} \subseteq \ker(d^{d-\mathrm{pd}(M)}_{T\otimes L}) = \mathrm{H}^{d-\mathrm{pd}(M)}(T \otimes_\Lambda L).$$

Indeed, for $t \otimes l \;\epsilon\; \ker(d^d_T) \otimes_\Lambda L^{-\mathrm{pd}(M)}$, we have

$$d^{d-\mathrm{pd}(M)}_{T\otimes L}(t \otimes l) = d^d_T(t) \otimes l + (-1)^{d-\mathrm{pd}(M)} t \otimes d_L^{-\mathrm{pd}(M)}(l).$$

The first summand is zero because $t \;\epsilon\; \ker(d^d_T)$ while, due to the minimality of L (cf. proposition 1.1.6 (ii)), the second one lies in $\ker(d^d_T) \otimes_\Lambda \mathfrak{M} L^{d-\mathrm{pd}(M)+1} \cong \Lambda/\mathfrak{M} \otimes_\Lambda \mathfrak{M} L^{d-\mathrm{pd}(M)+1} = 0$. □

Corollary 1.7.3. *If M is a finitely generated Λ-module, then*

$$\mathrm{pd}(M) = \max\{i \mid \mathrm{E}^i(M) \neq 0\}.$$

Proof. Using lemma 1.6.5 (ii) and local duality, we get

$$\begin{aligned} \mathrm{pd}(M) &= d - \mathrm{depth}(M) \\ &= d - \min\{i \mid \mathrm{H}^i_{\mathfrak{M}}(M) \neq 0\} \\ &= \max\{i \mid \mathrm{E}^i(M) \neq 0\}. \end{aligned}$$

□

Remark 1.7.4. The statement of the last corollary holds over an arbitrary Noetherian ring for a finitely generated modules M with *finite* projective dimension $\mathrm{pd}_\Lambda M$ and can be proven directly in the following way. Consider a projective resolution of minimal length

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Then the $(n-1)$th syzygy $K = \ker(d_{n-2})$ has projective dimension $\mathrm{pd}_\Lambda K = 1$, i.e. $\mathrm{D}K \simeq \mathrm{E}^1(K) \cong \mathrm{E}^n(M)$. Hence, $\mathrm{E}^n(M)$ cannot vanish because otherwise K would be projective.

1.8 Modules associated with group presentations

Let $\mathcal{C}$ be a class of finite groups closed under taking subgroups, homomorphic images and group extensions. Given an exact sequence of pro-$\mathcal{C}$-groups

$$1 \to \mathcal{H} \to \mathcal{G} \to G \to 1,$$

where $\mathcal{G}$ is assumed to be finitely generated, we choose a presentation $\mathcal{F} \twoheadrightarrow \mathcal{G}$ of $\mathcal{G}$ by a free pro-$\mathcal{C}$-group $\mathcal{F}_d$ of rank d and we associate the following commutative diagram to it:

$$\begin{tikzcd}
 & 1 & 1 & & \\
1 \arrow[r] & \mathcal{H} \arrow[r] \arrow[u] & \mathcal{G} \arrow[r] \arrow[u] & G \arrow[r] & 1 \\
1 \arrow[r] & R \arrow[r] \arrow[u] & \mathcal{F}_d \arrow[r] \arrow[u] & G \arrow[r] \arrow[u, equal] & 1 \\
 & N \arrow[r, equal] \arrow[u] & N \arrow[u] & & \\
 & 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd}$$

Here, R and N are defined by the exactness of the corresponding sequences. In general, the *p-relation module $N^{ab}(p)$ of $\mathcal{G}$ with respect to the chosen free presentation* (and similarly $R^{ab}(p)$ with respect to G instead of $\mathcal{G}$) fits into the following exact sequence, which is called Fox-Lyndon resolution associated with the above free representation of $\mathcal{G}$:

$$0 \longrightarrow N^{ab}(p) \longrightarrow \Lambda(\mathcal{G})^d \longrightarrow \Lambda(\mathcal{G}) \longrightarrow \mathbb{Z}_p \longrightarrow 0. \tag{1.8.1}$$

Hence, if $\mathrm{cd}_p(\mathcal{G}) \leq 2$, then $N^{ab}(p)$ is a projective $\Lambda(\mathcal{G})$-module.

Furthermore, the augmentation ideal $I_{\mathcal{F}_d}$, i.e. the kernel of $\Lambda(\mathcal{F}_d) \to \mathbb{Z}_p$, is a free $\Lambda(\mathcal{F}_d)$-modules of rank d:

$$I_{\mathcal{F}_d} \cong \Lambda(\mathcal{F}_d)^d$$

(for a proof of these facts, see [45] Chap V.6).

Let A be a p-divisible p-torsion abelian group of finite $\mathbb{Z}_p$-corank r with a continuous action of $\mathcal{G}$.

Definition 1.8.1. For a finitely generated $\Lambda = \Lambda(\mathcal{G})$-module M we define the finitely generated Λ-module

$$M[A] := M \otimes_{\mathbb{Z}_p} A^\vee = \mathrm{Hom}_{cont.,\mathbb{Z}_p}(M, A)^\vee$$

with diagonal $\mathcal{G}$-action.We shall also write $M(\rho)$ for this r-dimensional twist where $\rho : G \to Gl_r(\mathbb{Z}_p)$ denotes the operation of G on $A^\vee$.

Note that the functor $-[A]$ is exact.

Lemma 1.8.2. [6] *The module $\Lambda[A]$ is a free Λ-module of rank r.*

Proof. Fix an isomorphism of abelian groups $\phi : A^\vee \cong {\mathbb{Z}_p}^r$ and, for pairs (U, m) consisting of an $m \in \mathbb{N}$ and an open normal subgroup $U \trianglelefteq \mathcal{G}$ such that U acts trivially on $A^\vee/p^m$, consider the well-known isomorphism of Λ-modules

$$\mathbb{Z}_p[\mathcal{G}/U] \otimes_{\mathbb{Z}_p} (A^\vee)/p^m \cong \mathbb{Z}_p[\mathcal{G}/U] \otimes_{\mathbb{Z}_p} {\mathbb{Z}_p}^r/p^m,$$

which sends $gU \otimes (a + p^m A^\vee)$ to $gU \otimes (\phi(g^{-1}a) + p^m {\mathbb{Z}_p}^r)$. It is easily seen that this isomorphisms form a compatible system, i.e.

$$\begin{aligned}
\Lambda \otimes_{\mathbb{Z}_p} A^\vee &= \Lambda \widehat{\otimes}_{\mathbb{Z}_p} A^\vee \\
&= \varprojlim_{(U,m)} \mathbb{Z}_p[\mathcal{G}/U] \otimes_{\mathbb{Z}_p} (A^\vee)/p^m \\
&= \varprojlim_{(U,m)} \mathbb{Z}_p[\mathcal{G}/U] \otimes_{\mathbb{Z}_p} {\mathbb{Z}_p}^r/p^m \\
&= \varprojlim_{U,m} \mathbb{Z}_p/p^m[\mathcal{G}/U]^r \\
&= \Lambda^r
\end{aligned}$$

∎

[6] We thank Alexander Schmidt for drawing our attention to the fact that $\Lambda[A]$ should not only be projective but even free.

Proposition 1.8.3. *For every $i \geq 0$*

$$\mathrm{E}^i(M(\rho)) \cong \mathrm{E}^i(M)(\rho^d),$$

where ρ^d is the contragredient representation, i.e. $\rho^d(g) = \rho(g^{-1})^t$ is the transpose matrix of $\rho(g^{-1})$.

Proof. By homological algebra (and using a free presentation of M) it suffices to prove the case $i = 0$ for free modules. Finally, we only have to check the commutativity of the following diagram which is associated to an arbitrary homomorphism $\phi : \Lambda \to \Lambda$

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(\Lambda(\rho),\Lambda) \arrow[r, "\phi(\rho)^*"] \arrow[d] & \mathrm{Hom}_\Lambda(\Lambda(\rho),\Lambda) \arrow[d] \\
\Lambda^r \arrow[d] & \Lambda^r \arrow[d] \\
\mathrm{Hom}_\Lambda(\Lambda,\Lambda)(\rho) \arrow[r, "\phi^*(\rho^d)"] & \mathrm{Hom}_\Lambda(\Lambda,\Lambda)(\rho).
\end{tikzcd}$$

First note that via the identification $\Lambda^r \overset{\psi_\rho}{=\!=} \Lambda(\rho)$ the matrix representing $\phi(\rho)$ is $A := \sum a_g g\rho(g^{-1})$, where we assume for simplicity that $\phi(1) =: a = \sum a_g g \in \mathbb{Z}_p[G]$. We denote by ι both, the involution $\Lambda \to \Lambda,\ g \mapsto\ g^{-1}$ (also extended to matrices with coefficients in Λ) and the isomorphism of left Λ-modules $\Lambda \to \mathrm{Hom}_\Lambda(\Lambda,\Lambda), g \mapsto (1 \mapsto g^{-1})$. Then its easy to see that the following two diagrams commute

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(\Lambda(\rho),\Lambda) \arrow[r, "\phi(\rho)^*"] \arrow[d, "(\psi_\rho)^*"] & \mathrm{Hom}_\Lambda(\Lambda(\rho),\Lambda) \arrow[d, "(\psi_\rho)^*"] \\
\mathrm{Hom}_\Lambda(\Lambda^r,\Lambda) \arrow[d, "i^r"] & \mathrm{Hom}_\Lambda(\Lambda^r,\Lambda) \arrow[d, "i^r"] \\
\Lambda^r \arrow[r, "\iota(A^t)"] & \Lambda^r,
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda^r \arrow[r, "B"] \arrow[d, "\psi_{\rho^d}"] & \Lambda^r \arrow[d, "\psi_{\rho^d}"] \\
\Lambda(\rho) \arrow[r, "\iota(a)(\rho^d)"] \arrow[d, "i(\rho^d)"] & \Lambda(\rho) \arrow[d, "i(\rho^d)"] \\
\mathrm{Hom}_\Lambda(\Lambda,\Lambda)(\rho) \arrow[r, "\phi^*(\rho^d)"] & \mathrm{Hom}_\Lambda(\Lambda,\Lambda)(\rho),
\end{tikzcd}$$

where $B = \sum a_g g^{-1} \rho^d(g)$, because $\iota(a) = \sum a_g g^{-1}$. We are done if we can verify $B = \iota(A^t)$. But

$$\begin{aligned} \iota(A^t) &= \sum a_g g^{-1} \rho(g^{-1})^t \\ &= \sum a_g g^{-1} \rho^d(g) = B. \end{aligned}$$

□

With the notation

$$\begin{aligned} X &:= X_{\mathcal{H},A} := \mathrm{H}^1(\mathcal{H}, A)^\vee \\ Y &:= Y_{\mathcal{H},A} := (I_{\mathcal{G}}[A])_{\mathcal{H}} \\ J &:= J_{\mathcal{H},A} := \ker(\Lambda(\mathcal{G})[A]_{\mathcal{H}} \to (A^\vee)_{\mathcal{H}}), \end{aligned}$$

we get the following proposition, which generalizes results of Jannsen [33, 4.3] and Nguyen-Quang-Do [47, 1.7], who considered the case $A = \mathbb{Q}_p/\mathbb{Z}_p$ and Ochi[49, lemma 3.5], who extended the result to general p-adic representations:

Proposition 1.8.4. *We have a commutative and exact diagram*

$$\begin{tikzcd}
 & & & 0 & 0 & \\
 & & & J \arrow[u] & J \arrow[u] & \\
0 \arrow[r] & \mathrm{H}^2(\mathcal{H}, A)^\vee \arrow[r] & (N^{ab}(p)[A])_{\mathcal{H}} \arrow[r] & \Lambda(G)^{dr} \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{H}^2(\mathcal{H}, A)^\vee \arrow[r] \arrow[u, equal] & (\mathrm{H}^1(N, A)^{\mathcal{H}})^\vee \arrow[r] \arrow[u, equal] & \mathrm{H}^1(R, A)^\vee \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & 0. \\
 & & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Furthermore, if $\mathrm{cd}_p(\mathcal{G}) \leq 2$, *then* $N^{ab}(p)[A]$ *is a projective* $\Lambda(\mathcal{G})$*-module and* $(N^{ab}(p)[A])_{\mathcal{H}}$ *a projective* $\Lambda(G)$*-module.*

Proof. The proof in the pro-p-case is given by Ochi [49, lemma 3.5]. In the general case, just replace his lemma 3.7 by lemma 1.8.2. □

The next remark shows that the generalization of the diagram with non-trivial coefficients A bears only substantial new information if A is not trivial as a $\mathcal{H}$-module.

Remark 1.8.5. Assume A is trivial as $\mathcal{H}$-module. Then the above diagram can be easily obtained by twisting Jannsen's original diagram (i.e. with coefficients $\mathbb{Q}_p/\mathbb{Z}_p$): diagram$(A)$ = diagram$(\mathbb{Q}_p/\mathbb{Z}_p)[A]$. Also the higher Iwasawa adjoints of the occurring modules can be calculated via proposition 1.8.3:

$$\begin{array}{rcl} \mathrm{E}^i(X_{\mathcal{H},A}) & \cong & \mathrm{E}^i(X_{\mathcal{H},\mathbb{Q}_p/\mathbb{Z}_p})(\rho^d) \\ \mathrm{E}^i(Y_{\mathcal{H},A}) & \cong & \mathrm{E}^i(Y_{\mathcal{H},\mathbb{Q}_p/\mathbb{Z}_p})(\rho^d) \\ & \cdots & \end{array}$$

As a consequence of the diagram we obtain the following theorem which is one of our main results in this purely group theoretic setting. The restriction to p-adic Lie groups without p-torsion is necessary in order to apply the dimension theory developed before.

Theorem 1.8.6. *Let* $\mathrm{cd}_p(\mathcal{G}) \leq 2$ *and* G *a* p*-adic Lie group of dimension* h *without* p*-torsion. If the "weak Leopoldt conjecture holds for* A *and* $\mathcal{H}$*", i.e. if* $\mathrm{H}^2(\mathcal{H}, A) = 0$*, then neither* Y *nor* X *have non-zero pseudo-null submodules:* $\mathrm{T}_{h-1}(X) = \mathrm{T}_{h-1}(Y) = 0$.

Proof. Apply proposition 1.5.13 to Y, which has $\mathrm{pd}(Y) \leq 1$ according to the above diagram, and note that $\mathrm{T}_{h-1}(X) \subseteq \mathrm{T}_{h-1}(Y)$ by proposition 1.5.2. □

Let $Z = Z_{\mathcal{H},A} := (D_2^{(p)}(\mathcal{G}, A)^{\mathcal{H}})^\vee$ where

$$D_2^{(p)}(\mathcal{G}, A) = \varinjlim_{U \subseteq_o \mathcal{G}, n} (H^2(U, {}_{p^n}A))^\vee$$

and the direct limit is taken with respect to the p-power map and the dual of the corestriction. Then there is a description of the $\Lambda(G)$-module Y as follows:

Proposition 1.8.7. *Assume that* $\mathrm{cd}_p(\mathcal{G}) = 2$ *and that* $N^{ab}(p)$ *is a finitely generated* $\Lambda(\mathcal{G})$*-module. Then*

$$Y \simeq DZ \quad \text{and} \quad \mathrm{E}^0(Z) \cong \mathrm{H}^2(\mathcal{H}, A)^\vee,$$

thus Y *is determined by* Z *up to projective summands. Suppose, in addition, that* $\mathrm{H}^2(\mathcal{H}, A) = 0$*. Then*

$$\mathrm{E}^1(Y) \cong Z.$$

For a proof of the proposition see [45] 5.6.8 and [49] thm 3.13.

Chapter 2

Local Iwasawa modules

2.1 The general case

In this section we study the structure of Iwasawa modules arising from "p-adic representations" $\mathcal{G} \to \mathrm{Aut}(A)$, where $\mathcal{G} = G_k$ is the absolute Galois group of a finite extension k of $\mathbb{Q}_\ell$ and A is a p-divisible p-torsion abelian group of finite $\mathbb{Z}_p$-corank r. Having fixed a p-adic Lie extension k_∞ of k with Galois group G, we write $\mathcal{H} = G(\overline{k}/k_\infty) \subseteq \mathcal{G}$ where $\overline{k}$ denotes the algebraic closure of k. We are going to apply the general results of section 1.8 to the module

$$X_A := X_{\mathcal{H},A} = \mathrm{H}^1(\mathcal{H}, A)^\vee = \mathrm{H}^1(k_\infty, A)^\vee,$$

i.e. we will determine the $\Lambda(G)$-modules occurring in the canonical exact sequence

$$0 \longrightarrow \mathrm{E}^1\mathrm{D}(X_A) \longrightarrow X_A \longrightarrow \mathrm{E}^0\mathrm{E}^0(X_A) \longrightarrow \mathrm{E}^2\mathrm{D}(X_A) \longrightarrow 0.$$

Partially, the results extend analogous ones obtained by Y. Ochi [49] who restricted himself to pro-p-extensions. But the proofs are almost the same. Since we have fixed $\mathcal{H}$, we shall omit it in the notation and write Y_A, Z_A, etc.

Lemma 2.1.1. (i) *If k is a finite extension of $\mathbb{Q}_\ell$ and k_∞ is a Galois extension of k, then $Z = A^*(k_\infty)^\vee$, where $A^* = (T_pA)^\vee(1)$ by definition,*

(ii) $\mathrm{E}^1\mathrm{D}(X_A) \cong \mathrm{E}^1(A^*(k_\infty)^\vee)$,

(iii) $\mathrm{E}^2\mathrm{D}(X_A) \subseteq \mathrm{E}^2\mathrm{D}(Y_A) \cong \mathrm{E}^2(A^*(k_\infty)^\vee)$,

(iv) *If $\mathrm{cd}_p(G) \leq 1$ or $\mathrm{cd}_p(G) = 2$ and $A(k_\infty)^\vee$ is $\mathbb{Z}_p$-torsion-free, then $\mathrm{D}X_A \simeq \mathrm{E}^1(X_A)$.*

Proof. (i) is just local Tate duality while (ii) is a consequence of (i):

$$\mathrm{E}^1\mathrm{D}(X_A) \cong \mathrm{E}^1\mathrm{D}(Y_A) \cong \mathrm{E}^1(Z_A) \cong \mathrm{E}^1(A^*(k_\infty)^\vee)$$

(Note that the first isomorphism holds because J_A is torsion-free as $\Lambda(U)$-module for a suitable open pro-p-subgroup $U \subseteq G$, such that $\Lambda(U)$ is integral). By the same reason and using the snake lemma, one sees that $\mathrm{E}^2\mathrm{D}(X_A) \subseteq \mathrm{E}^2\mathrm{D}(Y_A)$. To prove (iv) just note that in these cases $\mathrm{pd} X_A \leq 1$ by the diagram 1.8.4, the defining sequence of J_A, corollary 1.7.3 and 1.4.8. $\square$

For a finitely generated abelian p-primary group A we denote by A_{div} the quotient of A by its maximal p-divisible subgroup. The next theorem generalizes results of Greenberg [22] and Ochi [49]:

Theorem 2.1.2. *Let $n = [k : \mathbb{Q}_\ell]$, $\ell = p$, be the finite degree of k over $\mathbb{Q}_p$ and k_∞ a Galois extension of k with Galois group $G \cong \Gamma \rtimes_\omega \Delta$, where $\Gamma \cong \mathbb{Z}_p$ and Δ is a finite group of order t prime to p, which acts on Γ via the character $\omega : \Delta \to \mathbb{Z}_p^*$. If $\chi = \omega^{-1}$ denotes the inverse of the character which determines the action on the p-dualizing module of G, the canonical sequence becomes*

$$0 \longrightarrow T_pA^*(k_\infty)(\chi) \longrightarrow X_A \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is a projective $\Lambda(G)$-module of $\mathrm{rk}_{\Lambda(\Gamma)}P = rnt$ and M is determined by the exact sequence

$$0 \longrightarrow M \longrightarrow A^*(k_\infty)_{div}(\chi) \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}(A(k_\infty)^\vee) \quad .$$

Furthermore,

(i) *if $A^*(k_\infty)$ is finite, then $T_pA^*(k_\infty)(\chi) = 0$. If, in addition, $A(k_\infty)^\vee$ is $\mathbb{Z}_p$-free, then $M \cong A^*(k_\infty)$.*

(ii) *if $A^*(k_\infty)^\vee$ is $\mathbb{Z}_p$-free, then $X_A \cong P \oplus T_pA^*(k_\infty)(\chi)$. In particular, X_A is projective, if $A^*(k_\infty) = 0$.*

Proof. First note that according to lemma 2.1.1 and 1.4.8

$$\begin{aligned}
\mathrm{E}^1\mathrm{D}(X_A) &\cong \mathrm{E}^1(A^*(k_\infty)^\vee)\\
&\cong \mathrm{E}^1(A^*(k_\infty)^\vee/\mathrm{tor}_{\mathbb{Z}_p})\\
&\cong (A^*(k_\infty)^\vee \otimes \mathbb{Q}_p/\mathbb{Z}_p(\chi^{-1}))^\vee\\
&\cong T_pA^*(k_\infty)(\chi).
\end{aligned}$$

To determine $E^2\mathrm{D}(X_A) \cong \mathrm{E}^2\mathrm{E}^1(X_A)$ we use the short exact sequences

$$0 \longrightarrow X_A \longrightarrow Y_A \longrightarrow J_A \longrightarrow 0,$$
$$0 \longrightarrow J_A \longrightarrow \Lambda(G)^d \longrightarrow A(k_\infty)^\vee \longrightarrow 0,$$

i.e. $\mathrm{E}^1(J_A) \cong \mathrm{E}^2(A(k_\infty)^\vee) \cong A(k_\infty)_{div}(\chi)$ by lemma 1.4.8 and

$$A(k_\infty)_{div}(\chi) \longrightarrow \mathrm{E}^1(Y_A) \longrightarrow \mathrm{E}^1(X_A) \longrightarrow 0$$

is exact. Forming the long exact Ext-sequence and applying lemmas 2.1.1 and 1.4.8 again, gives the desired result. □

Let us now consider the case $\ell \neq p$:

Theorem 2.1.3. *In the situation of the last theorem but with $\ell \neq p$ there is an isomorphism*

$$X_A \cong T_pA^*(k_\infty)(\chi).$$

Proof. In [49], prop. 3.12, it was calculated that the $\Lambda(\Gamma)$-rank of X_A equals the $\Lambda(\Gamma)$-corank of $\mathrm{H}^2(k_\infty, A)$, but the latter module vanishes because the order of G is divisible by p^∞ (cf. [45] 7.1.8). □

Theorem 2.1.4. *Let $n = [k : \mathbb{Q}_p]$ be the finite degree of k over $\mathbb{Q}_p$ and k_∞ a p-adic Lie extension of k such that its Galois group G has cohomological dimension $\mathrm{cd}_p(G) = 2$. Let $\Gamma \subseteq G$ be an arbitrary open uniform pro-p-subgroup, i.e. $\Lambda(\Gamma)$ is integral, and let t be the index $(G : \Gamma)$. If χ denotes the inverse of the character which determines the action of G on the p-dualizing module, then the canonical sequence becomes*

$$0 \longrightarrow X_A \longrightarrow R \longrightarrow \mathrm{E}^2\mathrm{D}(X_A) \longrightarrow 0,$$

where R is a reflexive $\Lambda(G)$-module with $\mathrm{rk}_{\Lambda(\Gamma)}R = rnt$. If, in addition, $A(k_\infty)^\vee$ is $\mathbb{Z}_p$-free, then $\mathrm{E}^2\mathrm{D}(X_A)$ is determined by the exact sequence

$$0 \longrightarrow \mathrm{E}^2\mathrm{D}(X_A) \longrightarrow T_pA^*(k_\infty)(\chi) \longrightarrow \mathrm{Hom}(T_pA(k_\infty), \mathbb{Z}_p)\ .$$

Proof. Using again the lemmas 2.1.1 and 1.4.8, the proof is completely analogous to that in the one-dimensional case 2.1.2. □

Note that in the case $p \neq l$ and $\mathrm{cd}_p(G) \geq 2$ we have $\mathcal{H} = 0$, i.e. $X_A = 0$, because the Galois group $G_k(p) \cong \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$ of the maximal p-extension of any local field over $\mathbb{Q}_\ell$ does not have any non-trivial quotient G which satisfies these conditions.

Theorem 2.1.5. *Let* $n = [k : \mathbb{Q}_p]$ *be the finite degree of* k *over* $\mathbb{Q}_p$ *and* k_∞ *a p-adic Lie extension of* k *such that its Galois group* G *has cohomological dimension* $\mathrm{cd}_p(G) \geq 3$. *Let* $\Gamma \subseteq G$ *be an arbitrary open uniform pro-p-subgroup, i.e.* $\Lambda(\Gamma)$ *is integral, and let* t *be the index* $(G : U)$. *Then*

$$X_A \cong \mathrm{E}^0\mathrm{E}^0 X_A$$

is a reflexive $\Lambda(G)$*-module with* $\mathrm{rk}_{\Lambda(\Gamma)} R = rnt$.

Proof. This follows from the lemmas 2.1.1 and 1.4.8 as above. □

At the end of this part we want to restate the results concerning the ranks of the considered modules where we follow Y. Ochi's [49, thm 3.3]. The same results were obtained independently by S. Howson [29, 6.1].

Proposition 2.1.6. *(Howson, Ochi) Let* k *be a finite extension of* $\mathbb{Q}_\ell$ *and* k_∞ *be a pro-p Lie extension of* k *with Galois group* $G = G(k_\infty/k)$. *As before* r *denotes the* $\mathbb{Z}_p$*-rank of* $\mathrm{rank}(A^\vee)$. *Assume that* $\Lambda = \Lambda(G)$ *is integral. Then*

$$\mathrm{rk}_\Lambda \mathrm{H}^1(k_\infty, A)^\vee = \begin{cases} r[k : \mathbb{Q}_p] & \text{if } \ell = p \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Noting the vanishing of $\mathrm{H}^2(k_\infty, A)$ and that $N^{ab}(p) \cong \Lambda(\mathcal{G})$ for $d = [k : \mathbb{Q}_p] + 2$ (conferring [33] thm. 5.1 c)), the result follows from the diagram 1.8.4 and the above remarks with respect to the case $\ell \neq p$. □

2.2 The case $A = \mathbb{Q}_p/\mathbb{Z}_p$

2.2.1 Local units

If we specialize to the important case $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial Galois action, we are able to determine the module structure more exactly using local class field theory: $X := X_{\mathbb{Q}_p/\mathbb{Z}_p} \cong \mathcal{H}^{ab}(p)$[1] is the Galois group of the maximal abelian p-extension of k_∞, which is canonically isomorphic to the inverse limit

$$X \cong \mathbb{A}(k_\infty) := \varprojlim_{k'} \mathbb{A}(k')$$

[1] This notation refers to the diagram of section 1.8 where we represent the absolut local Galois group $\mathcal{G}$ of k by a free profinite group of rank $d = [k : \mathbb{Q}_\ell] + 2$ according to [45] theorem 7.4.1.

of the p-completions $\mathbb{A}(k')$ of the multiplicative groups of finite subextensions k' of k in k_∞ :

$$\mathbb{A}(k') = \varprojlim_m (k')^*/(k')^{*p^m},$$

where the limit is taken via the norm maps. Since the Galois module structure of $\mathbb{A}(k')$ is well known if tensored with $\mathbb{Q}_p$, we get

Theorem 2.2.1. *Let $n = [k : \mathbb{Q}_\ell]$, $\ell = p$, be the finite degree of k over $\mathbb{Q}_p$ and k_∞ a Galois extension of k with Galois group $G \cong \Gamma \rtimes_\omega \Delta$, where $\Gamma \cong \mathbb{Z}_p$ and Δ is a finitely generated profinite group of order prime to p, which acts on Γ via the character $\omega : \Delta \to \mathbb{Z}_p^*$. We write k_0 for the fixed field of Γ and denote by $\chi = \omega^{-1}$ the inverse of the character which determines the action on the p-dualizing module of G.*

(i) *If $\mu_{p^\infty} \subseteq k_\infty$, i.e. k_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k_0 and $G = \Gamma \times \Delta$, then it holds*

$$\mathbb{A}(k_\infty) \cong \Lambda^n \oplus \mathbb{Z}_p(1).$$

(ii) *Let $\mu(k_\infty)(p)$ be finite. Then there is an exact sequence of Λ-modules*

$$0 \longrightarrow \mathbb{A}(k_\infty) \oplus I_G \longrightarrow \Lambda^{n+1} \longrightarrow \mu(k_\infty)(p)(\chi) \longrightarrow 0.$$

For any representation

$$1 \longrightarrow K \longrightarrow \mathcal{F}_{d'} \longrightarrow G \longrightarrow 1$$

by a free profinite group $\mathcal{F}_{d'}$ on $d' \leq n+1$ generators, there exists an exact sequence

$$0 \longrightarrow \mathbb{A}(k_\infty) \longrightarrow \Lambda^{n-d'+1} \oplus K^{ab}(p) \longrightarrow \mu(k_\infty)(p)(\chi) \longrightarrow 0.$$

Remark 2.2.2. (i) The existence of a representations in (ii) is always guaranteed by [31] theorem 4.3. Indeed, one can choose $d' = 2$.
(ii) Using the Krull-Schmidt theorem and Maschke's theorem, it is easily proved (see the proof below) that

$$\begin{aligned} \mathrm{E}^0(I_G)(\omega) \oplus I_G &\cong \mathbb{Z}_p[\![G]\!]^2 \\ \bigoplus_{i=1}^{m-1} I_G(\omega^i) \oplus I_G &\cong \mathbb{Z}_p[\![G]\!]^m, \end{aligned}$$

where m denotes the order of ω. Hence, from the isomorphism $K^{ab}(p) \oplus I_G \cong \mathbb{Z}_p[\![G]\!]^d$ according to the Lyndon sequence (1.8.1), we get isomorphisms (for $m \leq d$)

$$\begin{aligned} K^{ab}(p) &\cong \mathbb{Z}_p[\![G]\!]^{d-2} \oplus \mathrm{E}^0(I_G)(\omega) \\ &\cong \mathbb{Z}_p[\![G]\!]^{d-m} \oplus \bigoplus_{i=1}^{m-1} I_G(\omega^i). \end{aligned}$$

In particular, if ω is an involution and $d = 2$, then $K^{ab}(p) \cong \mathrm{E}^0(I_G)(\omega) \cong I_G(\omega)$ holds.

Proof. Let us first consider the case that Δ is a finite group, which grants that $\Lambda(G)$ is Noetherian. Then the statements are consequences of theorem 2.1.2 once having determined the structure of $P = \mathrm{E}^0\mathrm{E}^0 X$. We will apply the Krull-Schmidt theorem 1.3.1 and we first observe that for any open normal subgroup $U \trianglelefteq \Gamma$ and $\bar{G} := G/U$ it holds: $X_U \otimes \mathbb{Q}_p \cong P_U \otimes \mathbb{Q}_p$ and, if k' denotes the fixed field of U, there are exact sequences of $\bar{G}$-modules

$$0 \longrightarrow U^{ab}(p) \longrightarrow (I_G)_U \longrightarrow \mathbb{Z}_p[\bar{G}] \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$
$$0 \longrightarrow X_U \longrightarrow \bar{G}^{ab}_{k'}(p) \longrightarrow U^{ab}(p) \longrightarrow 0.$$

Hence, by Maschke's theorem and using $\bar{G}^{ab}_{k'}(p) \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[\bar{G}]^n \oplus \mathbb{Q}_p$ (cf. [45] 7.4.3), we get

$$P_U \otimes \mathbb{Q}_p \oplus (I_G)_U \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[\bar{G}]^{n+1},$$

i.e.

$$P \oplus I_G \cong \Lambda^{n+1}.$$

Now, taking U-coinvariants of the augmentation sequence

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}_p[\![G]\!] \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

and tensoring with $\mathbb{Q}_p(\omega^i)$ gives

$$\mathbb{Q}_p[\bar{G}] \oplus \mathbb{Q}_p(\omega^{i+1}) \cong (I_G(\omega^i))_U \otimes \mathbb{Q}_p \oplus \mathbb{Q}_p(\omega^i).$$

For (i) just note that I_G is projective and ω trivial because Δ acts trivially on Γ, hence: $I_G \cong \mathbb{Z}_p[\![G]\!]$. The first sequence in (ii) is immediate while the second one results from the isomorphism $K^{ab}(p) \oplus I_G \cong \mathbb{Z}_p[\![G]\!]^d$ according to the Lyndon sequence (1.8.1).
Now let us assume that Δ is infinite. If $\Delta' \subseteq \Delta$ is an open subgroup then the functor obtained by taking Δ'-coinvariants is exact because $\mathrm{H}_1(\Delta', M) = 0$ for any Λ-module M. Since the automorphism group is virtually pro-p, there

is an open normal subgroup Δ_0 of Δ which acts trivially on Γ, in particular any open normal subgroup Δ' of Δ which is contained in Δ_0 is normal with in G. Now a free presentation of G

$$1 \longrightarrow K \longrightarrow \mathcal{F}_{d'} \longrightarrow G \longrightarrow 1$$

induces a free presentation of $G' := G/\Delta'$

$$1 \longrightarrow K_{\Delta'} \longrightarrow \mathcal{F}_{d'} \longrightarrow G/\Delta' \longrightarrow 1.$$

Using the Lyndon sequence, it is easy to verify that $(I_G)_{\Delta'} \cong I_{G/\Delta'}$ and $K^{ab}(p)_{\Delta'} \cong K^{ab}_{\Delta'}(p)$. Now the strategy is as follows. Take a $\Lambda(G)$-module M and show that for any Δ' as above its Δ'-coinvariants are isomorphic to certain finitely generated $\Lambda(G')$-modules of the same type, e.g. $\mathbb{A}(k') \oplus I_{G'}$, where k' is the fixed field of k_∞ by Δ'. Then it follows easily (using a compactness argument to grant the existence of a compatible system of isomorphisms) that $M \cong \mathbb{A}(k_\infty) \oplus I_G$. As an example we prove the first statement in (ii): choose a surjection $\Lambda(G)^{n+1} \twoheadrightarrow \mu(k_\infty)(p)(\chi)$ and define M to be the kernel of it. Taking Δ'-coinvariants and comparing it with the result for k', i.e. for (finite) Δ/Δ', we obtain an isomorphism $M_{\Delta'} \cong \mathbb{A}(k') \oplus I_{G'}$ by Schanuel's lemma (see [33, 1.3] for a generalized version). The other statements follow by similar arguments.

The second isomorphism of the remark can be deduced by summing up $(I_G(\omega^i))_U \otimes \mathbb{Q}_p$ for $0 \leq i \leq m$. For the first one, use that due to the projectivity of I_G

$$\begin{aligned}
\mathrm{E}^0(I_G)_U \otimes \mathbb{Q}_p &\cong \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(I_G, \mathbb{Z}_p[\![G]\!])_U \otimes \mathbb{Q}_p \\
&\cong \mathrm{Hom}_{\mathbb{Z}_p[\bar{G}]}((I_G)_U, \mathbb{Z}_p[\bar{G}]) \otimes \mathbb{Q}_p \\
&\cong \mathrm{Hom}_{\mathbb{Q}_p}((I_G)_U, \mathbb{Q}_p) \otimes \mathbb{Q}_p
\end{aligned}$$

holds. □

Theorem 2.2.3. *In the situation of the last theorem but with $\ell \neq p$ there is an isomorphism*

$$X \cong \begin{cases} \mathbb{Z}_p(1)(\chi) & \text{if } \mu_p \subseteq k_0 \\ 0 & \text{otherwise} \end{cases}.$$

The next theorem generalizes results of Wintenberger [64] who restricts himself to the case in which G is abelian. It applies for example to $\Gamma \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$.

Theorem 2.2.4. *Let* $n = [k : \mathbb{Q}_p]$ *be the finite degree of* k *over* $\mathbb{Q}_p$ *and* k_∞ *an Galois extension of* k *with Galois group* $G \cong \Gamma \rtimes_\rho \Delta$, *where* Γ *is a pro-p Lie group of dimension* 2 *and* Δ *is a profinite group of order prime to* p, *which acts on* Γ *via* $\rho : \Delta \to Aut(\Gamma)$. *Let* k_0 *be the fixed field of* Γ *and let* $\chi = det\rho^{-1}$ *denote the inverse of the character which determines the action on the p-dualizing module of* G.

(i) *If* $\mu(k_0)(p) = 1$, *then* $X \oplus \Lambda \cong R^{ab}(p)$. *If* ρ *is trivial, then* $X \cong \Lambda^n$.

(ii) *If* $\mu_{p^\infty} \subseteq k_\infty$ *and* G *is without p-torsion and such that its dualizing module is not isomorphic to* μ_{p^∞}, *then there is an exact sequence of* Λ*-modules*

$$0 \longrightarrow X \oplus \Lambda \longrightarrow R^{ab}(p) \longrightarrow \mathbb{Z}_p(1)(\chi) \longrightarrow 0.$$

If ρ *is trivial, then*

$$0 \longrightarrow X \longrightarrow \Lambda^n \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0$$

is exact.

(iii) *If* $\mu(k_\infty)(p)$ *and* Δ *are finite, then* $X \cong \mathrm{E}^0\mathrm{E}^0(X)$ *is reflexive, i.e. there is an exact sequence*

$$0 \longrightarrow X \longrightarrow R^{ab}(p) \longrightarrow \Lambda \longrightarrow \mu(k_\infty)(p).$$

If, in addition, $\mu(k)(p) = 1$, *but* $\mu(k_\infty)(p) \neq 1$ *and* $\chi^{-1} \neq \chi_{cycl}$, *then the right map is also surjective (in particular* X *is not free in this case).*

Remark 2.2.5. For extensions $k_\infty|k$ of the type $G \cong \Gamma \times \Delta$ with $\Gamma \cong \mathbb{Z}_p^s$, $s \geq 3$ and finite Δ, we can consider the relative situation

$$0 \longrightarrow X(k_\infty)_{\Gamma'} \longrightarrow X(K_\infty) \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$

where Γ' is direct factor of Γ isomorphic to $\mathbb{Z}_p$, i.e. $\Gamma \cong \Gamma' \times \mathbb{Z}_p^{s-1}$, and K_∞ is the fixed field of k_∞ with respect to Γ'. By induction and applying Diekert's theorem ([45]) one reobtains at once Wintenberger's results (but now more generally with not necessarily abelian Δ): For any irreducible character $\chi \neq 1, \chi_{cycl}$ the component $X(k_\infty)^{e_\chi}$ is a free $\Lambda(G)^{e_\chi}$-module of rank n

$$X(k_\infty)^{e_\chi} \cong (\Lambda(G)^{e_\chi})^n.$$

But since we already know that $\mathrm{pd}_\Lambda X = s - 2$ for $s \geq 3$, X can not be projective in this case, i.e. $X(k_\infty)^{e_\chi}$ or $X(k_\infty)^{e_{\chi_{cycl}}}$ is definitely not of this type.

We will prove the theorem only for finite Δ because the general case follows similarly as in theorem 2.2.1. Just note that also in this case the automorphism group of Γ is virtually pro-p (see [16, 5.6]).
But before giving the proof we need some preparation:

Lemma 2.2.6. *Let $G = \Gamma \times \Delta$ be the product of a pro-p Lie group Γ with $\mathrm{cd}_p(\Gamma) = 2$ and a finite group Δ of order prime to p. Then*

$$R^{ab}(p) \cong \Lambda^{n+1}.$$

Proof. Let $U_n := p^n\Gamma \trianglelefteq G$. By the Lyndon sequence and using proposition 1.3.4, we calculate the Euler characteristic $h_{U_n}(R^{ab}(p)) = h_{U_n}(\mathbb{Z}_p) + h_{U_n}(\Lambda^{n+1}) = h_{U_n}(\Lambda^{n+1})$. The result follows. $\square$

Lemma 2.2.7. *If in the situation of the theorem $\mu(k_\infty)(p)$ is infinite, then both $\mathrm{E}^0(X)$ and $\mathrm{E}^0\mathrm{E}^0(X)$ are projective.*

Proof. Since $\mathrm{E}^0(-)$ preserves projectives and $\mathrm{E}^0\mathrm{E}^0\mathrm{E}^0(X) \cong \mathrm{E}^0(X)$ by 1.5.14, it is sufficient to prove the statement for $\mathrm{E}^0\mathrm{E}^0(X)$. But according to proposition 1.5.16 the latter module is the 2-syzygy of $\mathrm{E}^3\mathrm{E}^1(X)$. We claim that

$$Y \simeq X \oplus \Lambda,$$

i.e. that $\mathrm{E}^3\mathrm{E}^1(X) \cong \mathrm{E}^3\mathrm{E}^1(Y) \cong \mathrm{E}^3(\mu(k_\infty)(p)^\vee) = 0$, which implies the lemma. Indeed, due to Poincaré-duality

$$\mathrm{H}^2(G, \mu(k_\infty)(p))^\vee \cong \mathrm{Hom}_G(\mu(k_\infty)(p), D_2^{(p)}) = 0,$$

if $D_2^{(p)} \neq \mu_{p^\infty}$. Hence, $Y \simeq X \oplus \Lambda$ by the second description of 4.5 b) in [33][2]. $\square$

Proof (of the theorem). Let $U_m = p^m\Gamma \trianglelefteq G$ and denote the fixed field of U_m by k_m. From the exact sequence

$$1 \longrightarrow G_{k_\infty} \longrightarrow G_{k_m} \longrightarrow U_m \longrightarrow 1$$

we obtain the associated homological Hochschild-Serre sequence

$$0 = \mathrm{H}_2(k_m, \mathbb{Z}_p) \longrightarrow \mathrm{H}_2(U_m, \mathbb{Z}_p) \longrightarrow X_{U_m} \longrightarrow G_{k_m}^{ab}(p) \longrightarrow \mathrm{H}_1(U_m, \mathbb{Z}_p) \longrightarrow 0.$$

[2]For $\Gamma = \mathbb{Z}_p^2$ this statement was proved by Jannsen ([33] 5.2 c): Though there the claimed isomorphism $R^{ab}(p) \cong \Lambda^{d-1}$ is only correct if ρ is trivial, the arguments (which we restated above) still prove $X \oplus \Lambda \simeq Y$.

After tensoring with $\mathbb{Q}_p$, it follows that

$$X_{U_m} \otimes \mathbb{Q}_p \oplus \mathrm{H}_1(U_m, \mathbb{Z}_p) \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[\bar{G}]^n \oplus \mathbb{Q}_p \oplus \mathrm{H}_2(U_m, \mathbb{Z}_p) \otimes \mathbb{Q}_p,$$

where we used Maschke's theorem and $\bar{G}^{ab}_{k_m}(p) \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[\bar{G}]^n \oplus \mathbb{Q}_p$ (cf. [45] 7.4.3). On the other hand, the Euler characteristic of the projective module $R^{ab}(p)$ can be calculated by means of the Lyndon sequence:

$$\begin{aligned}
[R^{ab}(p)_{U_m} \otimes \mathbb{Q}_p] &= h_{U_m}(R^{ab}(p)) \\
&= h_{U_m}(\mathbb{Z}_p) + h_{U_m}(\Lambda^{n+1}) \\
&= [\mathbb{Q}_p] - [\mathrm{H}_1(U_m, \mathbb{Z}_p) \otimes \mathbb{Q}_p] + [\mathrm{H}_2(U_m, \mathbb{Z}_p) \otimes \mathbb{Q}_p] \\
&\quad + [\mathbb{Q}_p[\bar{G}]^{n+1}]
\end{aligned}$$

and hence $X_{U_m} \otimes \mathbb{Q}_p \oplus \mathbb{Q}_p[\bar{G}] \cong R^{ab}(p)_{U_m} \otimes \mathbb{Q}_p$.

Assume that $\mu(k_0)(p) = 1$, i.e. $\mathrm{tor}_{\mathbb{Z}_p}\mathbb{A}(k_0) = 1$ and X_{U_0} is $\mathbb{Z}_p$-free. Therefore, since t is prime to p, it follows that X_{U_0} is $\mathbb{Z}_p[\Delta]$-projective. If ρ is trivial, we conclude, by the calculation above under consideration of $h_{U_m}(\mathbb{Z}_p) = 0$ (by lemma 1.3.4) and using the Krull-Schmidt theorem, that $X_{U_0} \cong \mathbb{Z}_p[\Delta]^n$. Applying lemma 1.3.3, gives the desired result in this case. Anyway, these arguments show that X is projective also in the case with non-trivial ρ, i.e. we obtain $X \oplus \Lambda \cong R^{ab}(p)$ in the general case.

In order to prove (ii), we apply theorem 2.1.4: Since $X \oplus \Lambda \simeq Y$ in this case (see the proof of lemma lemma 2.2.7), we obtain

$$\begin{aligned}
\mathrm{E}^2 D(X) &\cong \mathrm{E}^2 D(Y) \\
&\cong \mathrm{E}^2(\mathbb{Z}_p(-1)) \\
&\cong \mathbb{Z}_p(1)(\chi),
\end{aligned}$$

where we applied the lemmas 2.1.1 and 1.4.8. Note that $\chi^{-1}(x) = \det(Adx) = \det\rho(x) : G \to \Delta \stackrel{\det\rho}{\to} \mathbb{Z}_p^*$ (cf. [38] V 2.5.8.1). We still have to determine the module $P = \mathrm{E}^0\mathrm{E}^0(X)$, which is projective according to lemma 2.2.7: it is easily seen that $P_{U_m} \otimes \mathbb{Q}_p \cong X_{U_m} \otimes \mathbb{Q}_p$, i.e. $P \oplus \Lambda \cong R^{ab}(p)$, by the above calculations. If ρ is trivial, lemma 2.2.6 gives the desired result.

The first statement of (iii) is just theorem 2.1.4 and lemma 2.1.1. By proposition 1.5.16, we obtain an exact sequence

$$0 \longrightarrow X \longrightarrow P \longrightarrow \Lambda^s \longrightarrow \mu(k_\infty)(p)$$

for some s. Splitting up the sequence, taking the long exact $\mathrm{H}_i(U_m, -)$-sequences and using the above calculations, one immediately sees that $P_{U_m} \otimes$

$\mathbb{Q}_p \cong R^{ab}(p)_{U_m} \otimes \mathbb{Q}_p \oplus \mathbb{Q}_p[\bar{G}]^{s-1}$, i.e. $P \cong R^{ab}(p) \oplus \Lambda^{s-1}$. After possibly changing the basis of Λ^d and using the Krull-Schmidt theorem 1.3.1, one easily sees that we can rid off the summand Λ^{s-1}.
In order to prove the last statement, we assume that $\chi^{-1} \neq \chi_{cycl}$ and consider the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{E}^1(X)^\vee \arrow[r] & \mathrm{E}^1(Y)^\vee \arrow[r] \arrow[d, equal] & \mathrm{E}^1(I)^\vee \arrow[d, equal] \\
& & \mu(k_\infty)(p) & \mathbb{Q}_p/\mathbb{Z}_p(\chi^{-1}).
\end{tikzcd}$$

The decomposition of the sequence with respect to the irreducible $\mathbb{Q}_p$-characters of Δ gives $(\mathrm{E}^1(X)^\vee)^{\chi_{cycl}} = \mu(k_\infty)(p)^{\chi_{cycl}} = \mu(k_\infty)(p)$. □

2.2.2 Principal units

When $l = p$, we are also interested in the Λ-structure of the inverse limit of the principal units

$$\mathbb{U}^1(k_\infty) := \varprojlim_{k'} \mathbb{U}^1(k'),$$

where k' runs through all finite subextensions of $k_\infty|k$ and the limit is taken with respect to the norm maps.

Proposition 2.2.8. *Let k be a finite extension of $\mathbb{Q}_p$ and k_∞ a Galois extension of k.*

(i) *If k_∞ contains the maximal unramified p-extension of k, i.e. if p^∞ divides the degree of the residue field extension associated with $k_\infty|k$, then*

$$\mathbb{U}^1(k_\infty) \cong \mathbb{A}(k_\infty).$$

(ii) *In the other case there is the following exact sequence*

$$0 \longrightarrow \mathbb{U}^1(k_\infty) \longrightarrow \mathbb{A}(k_\infty) \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

Proof. For finite extensions $K'|K|k$ of k with associated residue field extensions $\lambda'|\lambda|\kappa$ consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{U}^1(K')/p^m \arrow[r] \arrow[d, "N_{K'/K}"] & \mathbb{A}(K') \arrow[r] \arrow[d, "N_{K'/K}"] & \mathbb{Z}_p/p^m \arrow[r] \arrow[d, "{[\lambda':\lambda]}"] & 0 \\
0 \arrow[r] & \mathbb{U}^1(K)/p^m \arrow[r] & \mathbb{A}(K) \arrow[r] & \mathbb{Z}_p/p^m \arrow[r] & 0.
\end{tikzcd}$$

While in case (i) the inverse limit $\varprojlim_{K,m} \mathbb{Z}_p/p^m$ vanishes, because for any m and any K there is an extension K' such that $p^m | [\lambda' : \lambda]$, in the second case it is isomorphic to $\mathbb{Z}_p$. $\square$

Theorem 2.2.9. *Assume in the situation of theorem 2.2.4 that k_∞ contains μ_{p^∞} but not the maximal unramified p-extension of k. Then there exists an exact sequence*

$$0 \longrightarrow \mathbb{U}^1(k_\infty) \oplus \Lambda \longrightarrow R^{ab}(p) \longrightarrow M \longrightarrow 0.$$

In particular, if ρ is trivial, there exists an exact sequence

$$0 \longrightarrow \mathbb{U}^1(k_\infty) \longrightarrow \Lambda^n \longrightarrow M \longrightarrow 0,$$

where M fits into the exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow M \longrightarrow \mathbb{Z}_p(1)(\chi) \longrightarrow 0.$$

Proof. Evaluating the long exact E^i-sequence associated with the exact sequence from the proposition above and noting that $\mathrm{pd}_\Lambda \mathbb{U}^1(k_\infty) \leq 1$ due to $\mathrm{pd}_\Lambda \mathbb{A}(k_\infty) \leq 1$ and $\mathrm{pd}_\Lambda \mathbb{Z}_p = 2$, one obtains that

(i) $\mathrm{E}^0(\mathbb{U}^1(k_\infty)) \cong \mathrm{E}^0(X)$,

(ii) $\mathrm{E}^1\mathrm{D}(\mathbb{U}^1(k_\infty)) = 0$ and an exact sequence

(iii) $0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathrm{E}^2\mathrm{D}(\mathbb{U}^1(k_\infty)) \longrightarrow \mathbb{Z}_p(1)(\chi) \longrightarrow 0.$

Here we used that $\mathrm{E}^2\mathrm{E}^2(\mathbb{Z}_p) \cong \mathbb{Z}_p$, because $\mathbb{Z}_p$ is a Cohen-Macaulay module of dimension 2. The result follows from the canonical sequence. $\square$

Remark 2.2.10. In the situation of theorem 2.2.1 with trivial action of Δ the structure of the principal units is described in [45] as follows:

(i) If $\mu_{p^\infty} \subseteq k_\infty$, then
$$\mathbb{U}^1(k_\infty) \cong \Lambda^n \oplus \mathbb{Z}_p(1).$$

(ii) If $\mu(k_\infty)(p)$ is finite, then there is an exact sequence
$$0 \longrightarrow \mathbb{U}^1(k_\infty) \longrightarrow \Lambda^n \longrightarrow \mu(k_\infty)(p).$$

(iii) If $k_\infty | k$ is unramified, then
$$\mathbb{U}^1(k_\infty) \cong \mathbb{A}(k_\infty).$$

But the proof of [45] works also if ω is not trivial.

2.3 The local CM-case

As a consequence of theorem 2.2.4 we can also determine the structure of $X_A = \mathrm{H}^1(k_\infty, A)^\vee$ in the trivializing case, i.e. $k(A) \subseteq k_\infty$:

Theorem 2.3.1. *Let $n = [k : \mathbb{Q}_p]$ be the finite degree of k over $\mathbb{Q}_p$ and k_∞ a Galois extension of k with Galois group $G \cong \Gamma \rtimes_\rho \Delta$, where Γ is a pro-p Lie group of dimension 2 and Δ is a finite group of order t prime to p, which acts on Γ via $\rho : \Delta \to Aut(\Gamma)$. Let k_0 be the fixed field of Γ and let $\chi = det\rho^{-1}$ denote the inverse of the character which determines the action on the p-dualizing module of G. For any A with $\mathrm{rk}_{\mathbb{Z}_p} A^\vee = r$ such that $k(A) \subseteq k_\infty$ the following is true.*

(i) *If $\mu(k_0)(p) = 1$, then $X_A \oplus \Lambda^r \cong R^{ab}(p)[A]$, in particular, if ρ is trivial: $X_A \cong \Lambda^{nr}$.*

(ii) *If $\mu_{p^\infty} \subseteq k_\infty$ and G is p-torsion-free and its dualizing module is not isomorphic to μ_{p^∞}, then there is an exact sequence of Λ-modules*

$$0 \longrightarrow X_A \oplus \Lambda^r \longrightarrow R^{ab}(p)[A] \longrightarrow A^\vee(1)(\chi) \longrightarrow 0.$$

In particular, if ρ is trivial, then

$$0 \longrightarrow X_A \longrightarrow \Lambda^{nr} \longrightarrow A^\vee(1) \longrightarrow 0$$

is exact.

(iii) *If $\mu(k_\infty)(p)$ is finite, then $X_A \cong \mathrm{E}^0\mathrm{E}^0(X_A)$ is reflexive, i.e. there is an exact sequence*

$$0 \longrightarrow X_A \longrightarrow R^{ab}(p)[A] \longrightarrow \Lambda^r \longrightarrow \mu(k_\infty)(p)[A].$$

If, in addition, $\mu(k)(p) = 1$, but $\mu(k_\infty)(p) \neq 1$ and $\chi^{-1} \neq \chi_{cycl}$, then the right map is also surjective (in particular, X_A is not free in this case).

Proof. In this case the subgroups $\mathcal{H}, R$ and N act trivially on $A = A(k_\infty)$, i.e. $X_A \cong X[A]$. □

This result applies to the following situation: Let K be a imaginary quadratic number field, F a finite, abelian extension of K and E an elliptic curve defined over F with complex multiplication (CM) by the ring of integers $\mathcal{O}_K$ of K such that $F(E_{tor})$ is an abelian extension of K. Assume that the rational prime p splits in K, i.e. $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, $\mathfrak{p} \neq \bar{\mathfrak{p}}$, and that E

has good reduction at all places lying over p. Set $G = G(F(E(p))/F)_{\mathfrak{P}}$ the decomposition group at some $\mathfrak{P}|\mathfrak{p}$. According to [15, 1.9], the prime $\mathfrak{P}$ ramifies totally in $F(E(\mathfrak{p}))|F$ and decomposes only finitely (and is unramified) in $F(E(\bar{\mathfrak{p}}))|F$. Therefore the decomposition group G is an open subgroup of $G(F(E(p))/F)$, i.e. of type $\mathbb{Z}_p^2 \times \Delta$ where Δ is a finite abelian group. Thus we obtain an exact sequence

$$0 \longrightarrow \mathrm{H}^1(F(E(p))_{\mathfrak{P}}, E(p))^\vee \longrightarrow \Lambda(G)^{2n} \longrightarrow T_pE \longrightarrow 0,$$

where $n = [F_{\mathfrak{p}} : \mathbb{Q}_p]$. By the same argument, but now using theorem 2.2.1(ii), there exists an exact sequence

$$0 \longrightarrow \mathrm{H}^1(F(E(\mathfrak{p})_{\mathfrak{P}}, E(\mathfrak{p}))^\vee \longrightarrow \Lambda(G')^n \longrightarrow \mu(F(E(\mathfrak{p})_{\mathfrak{P}})[E(\mathfrak{p})] \longrightarrow 0,$$

where $G' = G(F(E(\mathfrak{p}))/F)_{\mathfrak{P}}$, and a similar one for $\bar{\mathfrak{p}}$.

Chapter 3

Global Iwasawa modules

Let k_∞ be a p-adic Lie extension of the number field k contained in k_S with Galois group G and let A be a p-divisible p-torsion abelian group with $\mathbb{Z}_p$-corank r and on which $G_S(k) = G(k_S/k)$ acts continuously where S is a finite set of places of k containing all places S_p over p and all infinite places S_∞ (and by definition all places at which A is ramified). Here k_S denotes the maximal S-ramified extension of k, i.e. the maximal extension of k which is unramified outside S. In order to derive information about the $\Lambda = \Lambda(G)$-modules $\mathrm{H}^i(G(k_S/k_\infty), A)$ we would like to apply the diagram 1.8 to the group $\mathcal{G} = \mathcal{G}_S := G(k_S/k)$. On the other hand we have to guarantee that $\mathcal{G}$ is finitely generated as a profinite group which, unfortunately, is not known for the group $\mathcal{G}_S$. But using a theorem of Neumann, i.e. the inflation maps are isomorphisms

$$\mathrm{H}^i(G(\Omega/k_\infty), A) \cong \mathrm{H}^i(G_S(k_\infty), A), \qquad i \geq 0,$$

for any (p, S)-closed extension Ω of k (i.e. Ω is a S-ramified extension of k which does not possess any non-trivial S-ramified p-extension) and for any p-torsion $G(\Omega/k_\infty)$-module A, we are free to replace $G_S(k)$ for example by the Galois group $\mathcal{G} := G(\Omega/k)$ where Ω is the maximal S-ramified p-extension of $k'(A)$ and k' is a Galois subextension of k_∞/k such that $G(k_\infty/k')$ is an open (normal) pro-p-group. Regarding this technical detail, we assume in what follows that k_∞ is contained in such a (p, S)-closed field Ω. Then, since $\mathcal{G}$ has an open pro-p Sylow group, it is finitely generated and has $\mathrm{cd}_p(\mathcal{G}) \leq 2$ for odd p. Note that $Y_{S,A} := Y_{G(\Omega/k_\infty),A}$ and $X_{S,A} := X_{G(\Omega/k_\infty),A}$ do not depend on the choice of Ω. The next lemma shows among other things that the corresponding module Z only depends on k_∞, A and S. Recall that $T_pA = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A)$ denotes the "Tate module" of A. We shall write $\mathrm{H}^*_{cts}(G_S(k), T_pA) \cong \varprojlim_n \mathrm{H}^*(G_S(k), {}_{p^n}A)$ for the continuous cochain cohomology groups (see [45, II.§3.]).

Lemma 3.0.2. *Let k, k_∞ and A be as above. Then*

$$Z_{S,A} := Z_{G(\Omega/k_\infty),A} \cong \varprojlim_{k\subseteq k'\subseteq k_\infty} H^2_{cts}(G_S(k'), T_pA).$$

Proof. Note that, by the theorem of Neumann, $D_2^{(p)}(\mathcal{G}, A) \cong D_2^{(p)}(\mathcal{G}_S, A)$, which can be determined via the exact sequence of Tate-Poitou (see [33, 5.4 d] for the case $A = \mathbb{Q}_p/\mathbb{Z}_p$). □

One of the main theorems of this thesis is the following

Theorem 3.0.3. *Let G a p-adic Lie group without p-torsion. If the "weak Leopoldt conjecture holds for A and k_∞", i.e. $\mathrm{H}^2(G_S(k_\infty), A) = 0$, then neither $Y_{S,A}$ nor $X_{S,A} \cong \mathrm{H}^1(G_S(k_\infty), A)^\vee$ have non-zero pseudo-null submodules.*

Proof. The conditions of theorem 1.8.6 are fulfilled. □

Remark 3.0.4. The weak Leopoldt conjecture for A and k_∞ holds for example if $k(A)$ and the cyclotomic $\mathbb{Z}_p$-extension of k are contained in k_∞. Indeed, it is a result of Iwasawa that the weak Leopoldt conjecture (for $A = \mathbb{Q}_p/\mathbb{Z}_p$) holds for the cyclotomic $\mathbb{Z}_p$-extension of any number field (see [45, 10.3.25] for a cohomological proof). The claim follows by expressing $\mathrm{H}^2(G_S(k_\infty), A)$ (considered as abelian group) as direct limit $\varinjlim_{k'} \mathrm{H}^2(G_S(k'_{cyc}), \mathbb{Q}_p/\mathbb{Z}_p)^r$, where k' runs through the finite extensions of k in k_∞.

Furthermore, Λ-rank of $X_{S,A}$ can be determined, using the diagram 1.8:

Theorem 3.0.5. *(Ochi) Let $k_\infty|k$ be a p-adic pro-p extension. Assume that $k(A)|k$ is a pro-p-extension and that Λ is an integral domain. Then*

$$\mathrm{rk}_\Lambda \mathrm{H}^1(G_S(k_\infty), A)^\vee - \mathrm{rk}_\Lambda \mathrm{H}^2(G_S(k_\infty), A)^\vee = r_2(k)r$$

Here $r_2(k)$ denotes as usual the number of complex places of k.

Proof. We give a slightly different proof than Y. Ochi: using the diagram 1.8, we calculate

$$\begin{aligned}
\mathrm{rk}_\Lambda X - \mathrm{rk}_\Lambda \mathrm{H}^2(G_S(k_\infty), A)^\vee &= \mathrm{rk}_\Lambda Y - \mathrm{rk}_\Lambda J - \mathrm{rk}_\Lambda \mathrm{H}^2(G_S(k_\infty), A)^\vee \\
&= rd - \mathrm{rk}_\Lambda (N^{ab}(p)[A])_{\mathcal{H}} - r \\
&= rr_2(k),
\end{aligned}$$

where we used Jannsen's determination of the Λ-structure of $N^{ab}(p)_{\mathcal{H}}$, see 3.1.3. Note that under our assumptions $\mathcal{G}$ is a pro-p-group, thus $N^{ab}(p)$ is a free $\Lambda(\mathcal{G})$-module and hence $\mathrm{rk}_\Lambda(G)(N^{ab}(p)[A])_{\mathcal{H}} = \mathrm{rk}_\Lambda(G)(N^{ab}(p))_{\mathcal{H}}[A] = r\mathrm{rk}_\Lambda(G)N^{ab}(p)$. □

3.1 The multiplicative group $\mathbb{G}_m$

3.1.1 The maximal abelian p-extension of k_∞ unramified outside S

We still consider p-adic Lie extensions $k_\infty|k$ with Galois group $G = G(k_\infty/k)$ such that k_∞ is contained in the maximal S-ramified extension k_S of k. Here, as before, S denotes a finite set of places of k containing all places S_p over p and all infinite places S_∞. For $K|k$ finite let $S_f(K)$ be the set of finite primes in K lying above S. In this paragraph we specialize to the case $A = \mathbb{Q}_p/\mathbb{Z}_p$ and we shall write X_S for the $\Lambda = \Lambda(G)$-module $X_{S,\mathbb{Q}_p/\mathbb{Z}_p}$

$$X_S := X_{S,\mathbb{Q}_p/\mathbb{Z}_p} = \mathrm{H}^1(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong G(k_S/k_\infty)^{ab}(p),$$

and respectively for Y_S and Z_S.

In this case our main theorem 3.0.3 is a generalization of the theorems of Greenberg [21] and Nguyen-Quang-Do [47], who considered the case $G \cong \mathbb{Z}_p^d$. Indeed, it confirms Greenberg's suggestion that an analogous statement also should hold for p-adic Lie extensions.

Theorem 3.1.1. *Let G be a p-adic Lie group without p-torsion. If the "weak Leopoldt conjecture holds for k_∞", i.e. $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, then $X_S \cong G_S(k_\infty)^{ab}(p)$ has no non-zero pseudo-null Λ-submodule.*

Remark 3.1.2. Recall that the weak Leopoldt conjecture for k_∞ holds if the cyclotomic $\mathbb{Z}_p$-extension of k is contained in k_∞.

We will also consider the Λ-modules

$$\begin{aligned} X_{nr} &= G(L/k_\infty), \\ X_{cs}^S &= G(L'/k_\infty), \end{aligned}$$

where L is the maximal abelian unramified pro-p-extension of k_∞ and L' is the maximal subextension in which every prime above S is completely decomposed.

For an arbitrary number field K, we denote the ring of integers (resp. S-integers) by $\mathcal{O}_K$ (resp. $\mathcal{O}_{K,S}$) and its units by $E(K) := \mathcal{O}_K^\times$ (resp. $E_S(K) := \mathcal{O}_{K,S}^\times$). Then we define

$$\begin{aligned} \mathbb{E} :&= \varprojlim_{k'} (\mathcal{O}_{k'}^\times \otimes \mathbb{Z}_p), \\ \mathbb{E}_S :&= \varprojlim_{k'} (\mathcal{O}_{k',S}^\times \otimes \mathbb{Z}_p), \end{aligned}$$

where the limit is taken with respect to the norm maps. This should not be confused with the discrete module of units (resp. S-units) $E(k_\infty) = \varinjlim_{k'} E(k')$ (resp. $E_S(k_\infty) = \varinjlim_{k'} E_S(k')$).

Finally, we write for the local-global modules

$$\begin{aligned}\mathbb{A}_S &= \bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu} \mathbb{A}_\nu,\\ \mathbb{U}_S &= \bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu} \mathbb{U}_\nu,\end{aligned}$$

where $\mathbb{A}_\nu = \mathbb{A}(k_{\infty,\nu})$ (resp. $\mathbb{U}_\nu = \mathbb{U}^1(k_{\infty,\nu})$) are the local modules introduced in chapter 2.
The above modules are connected via global class field theory and the Tate-Poitou sequence as follows

Proposition 3.1.3. *(Jannsen) There are the following exact and commutative diagrams of Λ-modules:*

(i)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\vee \arrow[r] \arrow[d, equal] & \mathbb{E} \arrow[r] \arrow[d, hook] & \mathbb{U}_S \arrow[r] \arrow[d, hook] & X_S \arrow[r] \arrow[d, equal] & X_{nr} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\vee \arrow[r] & \mathbb{E}_S \arrow[r] & \mathbb{A}_S \arrow[r] & X_S \arrow[r] & X_{cs}^S \arrow[r] & 0
\end{tikzcd}$$

(ii)

$$0 \longrightarrow \mathbb{E} \longrightarrow \mathbb{E}_S \longrightarrow \bigoplus\nolimits_{S_{un}(k)} \mathrm{Ind}_G^{G_\nu} \mathbb{Z}_p \longrightarrow X_{nr} \longrightarrow X_{cs}^S \longrightarrow 0,$$

where $S_{un} := \{\nu \in S(k) \mid p^\infty \nmid f_\nu\}$ and $f_\nu = f(k_{\infty,\nu}/k_\nu)$ denotes the degree of the extension of the corresponding residue class fields.

(iii)

$$0 \longrightarrow X_{cs}^S \longrightarrow Z_{S,\mathbb{Q}_p/\mathbb{Z}_p(1)} \longrightarrow \bigoplus\nolimits_{S_f(k)} \mathrm{Ind}_G^{G_\nu} \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$

and, if $\mu_{p^\infty} \subseteq k_\infty$,

$$0 \longrightarrow X_{cs}(-1) \longrightarrow Z_S \longrightarrow \bigoplus\nolimits_{S_f(k)} \mathrm{Ind}_G^{G_\nu} \mathbb{Z}_p(-1) \longrightarrow \mathbb{Z}_p(-1) \longrightarrow 0.$$

In particular, $X_{cs}^S = X_{cs} := X_{cs}^{S_p}$ is independent of S in this case.

(iv) *$N^{ab}(p)$ is a finitely generated, projective $\Lambda(G(k_{\infty S}(p)/k))$-module and, if the free presentation of $\mathcal{G} = G(k_{\infty S}(p)/k)$ (cf. section 1.8) is chosen such that $d \geq r_1' + r_2 + 1$, then*

$$N^{ab}(p)_{G_S(k_\infty)(p)} \cong \bigoplus_{S_\infty'} \mathrm{Ind}_G^{G_\nu} \mathbb{Z}_p \oplus \Lambda(G)^{d-r_2-r_1'-1},$$

where S_∞' is the set of real places of k which ramify (i.e. become complex) in k_∞, r_1' is the cardinality of S_∞', and r_2 is the number of complex places of k.

Proof. The assertions (i) and (iii) are obtained by taking inverse limits of the Tate-Poitou sequence (see [33, Thm 5.4]) and recalling lemma 3.0.2 while (ii) follows from (i) by the snake lemma and prop. 2.2.8. □

From these diagrams and the fact that Λ is Noetherian it follows that the modules X_{nr}, X_{cs}^S are finitely generated. Furthermore, S. Howson [29, 7.14-7.16] and independently Y. Ochi [49, 4.10] proved that X_{nr} and X_{cs}^S are Λ-torsion. Actually, this result was first proved by M. Harris [25, thm 3.3] but, as S. Howson remarked, his proof is incomplete because it relies on the false "strong Nakayama" lemma (loc. cit. lem 1.9), see the discussion in [2]. However, in a recent correction M. Harris [27] has given a new proof of the result. In the case $G \cong \mathbb{Z}_p^d$, this result is originally proved by Greenberg [20]. Here, we present a slight modification of Y. Ochi's proof:

Corollary 3.1.4. (i) *If $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p(1)) = 0$ (e.g. if $\dim(G_\nu) \geq 1$ for all $\nu \in S_f$), then X_{cs}^S is a Λ-torsion module.*

(ii) *If $\dim(G_\nu) \geq 1$ for all $\nu \in S_f$, then X_{nr} is a Λ-torsion module.*

For example, the conditions of the corollary are satisfied if k_∞ contains the cyclotomic $\mathbb{Z}_p$-extension.

Proof. (cf. Ochi)[1] The first statement follows from 1.8.7 while the second one is a consequence of the first one and the above proposition (To calculate the (co)dimension of $\mathrm{Ind}_G^{G_\nu}\mathbb{Z}_p$ use 1.5.17 and 1.4.8). Note that the condition "$\dim(G_\nu) \geq 1$ for all $\nu \in S_f$" implies, using Tate-Poitou duality,

$$\begin{aligned} \mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p(1)) &= Ш^2(G_S(k_\infty), \mu_{p^\infty}) \\ &= \varinjlim_{k',n} Ш^1(G_S(k'), \mathbb{Z}/p^n)^\vee \end{aligned}$$

[1] Y. Ochi's setting is restricted to the situation $\mu_p \subseteq k_\infty$ (otherwise μ_{p^∞} is not a $G_S(k_\infty)(p)$-module), but this problem can be avoided by the arguments given at the beginning of this chapter.

$$\begin{aligned} &= \varinjlim_{k'} Cl_S(k') \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \\ &= 0 \end{aligned}$$

because $Cl_S(k')$ is finite. □

Theorem 3.1.5. *If* $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, $\mu_{p^\infty} \subseteq k_\infty$, *and* $\dim(G_\nu) \geq 2$ *for all* $\nu \in S_f$, *then*

$$X_{nr}(-1) \sim X^S_{cs}(-1) \sim \mathrm{E}^1(Y_S) \sim \mathrm{E}^1(\mathrm{tor}_\Lambda Y_S) \cong \mathrm{E}^1(\mathrm{tor}_\Lambda X_S).$$

If, in addition, $G \cong \mathbb{Z}_p^r$, $r \geq 2$, *then even the following holds:*

$$X_{nr}(-1) \sim X^S_{cs}(-1) \sim (\mathrm{tor}_\Lambda X_S)^\circ,$$

where $^\circ$ *means that* G *operates via the involution* $g \mapsto g^{-1}$.

Proof. The first two pseudo-isomorphisms follow again from proposition 3.1.3 using 1.5.17, 1.4.8 and 1.8.7. The third one is just prop. 1.5.9. Note that there is even an isomorphism $\mathrm{tor}_\Lambda Y_S \cong \mathrm{tor}_\Lambda X_S$ because the augmentation ideal I_G is torsion-free. In the commutative case it is well known that a Λ-torsion module M is pseudo-isomorphic to $\mathrm{E}^1(M)$ (see [52, prop. I.2.2.8] for $r = 2$, but for $r > 2$ the same proof holds). □

The following consequence generalizes a result of McCallum [43, thm 8] who considered the $\mathbb{Z}_p^r$-case:

Corollary 3.1.6. *With the assumptions of the theorem the following holds.*

(i) *There is a pseudo-isomorphism*

$$\mathrm{tor}_\Lambda X_S \sim \mathrm{E}^1(X^S_{cs}(-1)).$$

(ii) *If* $\dim(G) \geq 3$, *then there is an isomorphism*

$$\mathrm{tor}_\Lambda X_S \cong \mathrm{E}^1(X^S_{cs}(-1)).$$

Proof. The cokernel $K := coker(X^S_{cs}(-1) \hookrightarrow Z_S \cong \mathrm{E}^1(Y_S)$ is pseudo-null, i.e. $\mathrm{E}^1(K) = 0$. If $\dim(G) \geq 3$, then $\mathrm{E}^2(K) = 0$, too, as can be calculated using 1.2.3. Now, the long exact E-sequence gives the result observing $\mathrm{E}^1\mathrm{E}^1(Y_S) \cong \mathrm{E}^1\mathrm{D}Y_S \cong \mathrm{tor}_\Lambda Y_S \cong \mathrm{tor}_\Lambda X_S$. □

Remark 3.1.7. The condition "$\dim(G_\nu) \geq 2$ for all $\nu \in S_f$," is known to hold in "most" extensions arising from geometry, see remark 3.2.12 and (the proof of) corollary 3.2.5 below. Other important cases are the following ones:
(a) Let k_∞ be the maximal multiple $\mathbb{Z}_p$-extension $\tilde{k}$ of k, i.e. the composite of all $\mathbb{Z}_p$-extensions of k, and assume that $\mu_p \subseteq k$ or
(b) let k_∞ be a multiple $\mathbb{Z}_p$-extension with $G \cong \mathbb{Z}_p^r$, $r \geq 2$, and assume that there is only one prime of k lying over p.
Then, as has been observed independently by T. Nguyen-Quang-Do [48, thm 3.2] and McCallum [43, proof of thm 7], the condition holds for $S = S_p \cup S_\infty$. Indeed, since $\mathbb{Q}(\mu_p)$ has only one prime dividing p, it suffices to consider the second case. But then all inertia groups T_ν, $\nu \in S_p$, are conjugate, thus they are all equal and hence an open subgroup of G due to the finiteness of the ideal class group.

With respect to the composite $\tilde{k}$ of all $\mathbb{Z}_p$-extensions of k there is the following outstanding

Conjecture 3.1.8. *(R. Greenberg) For any number field k, the $\Lambda(G(\tilde{k}/k))$-module X_{nr} is pseudo-null.*

Recently, W. McCallum [43] proved this conjecture for the base field $k = \mathbb{Q}(\mu_p)$ under some mild assumptions. For a list of other cases in which this conjecture is known to hold, see [48, rem 4.6]. Assume that $\mu_p \subseteq k$ and that the condition "$\dim(G_\nu) \geq 2$, for all $\nu \in S_f$," holds. Then, by the above theorem and theorem 3.1.1, Greenberg's conjecture is equivalent to the statement that X_S is Λ-torsion-free, compare with [48, 4.4] and [43, Cor 13].

The observation of the previous proof leads also to:

Proposition 3.1.9. *If $\dim(G_\nu) \geq 2$ for all $\nu \in S_f$, then*

$$\mathbb{E} \sim \mathbb{E}_S.$$

We are also interested in the (Pontryagin duals of the) direct limits

$$\begin{aligned} Cl_S(k_\infty)(p) &= \varinjlim_{k'} Cl_S(k')(p), \\ \mathcal{E}_S(k_\infty) &:= (E_S(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p)^\vee, \end{aligned}$$

of the p-part of the ideal class group, resp. of the global (S-)units of finite extensions k' of k inside k_∞.

Proposition 3.1.10. *Let T be a set of places of k such that $S_\infty \subseteq T \subseteq S$. Assume that $\dim(T_\nu) \geq 1$ for all $\nu \in S \setminus T$, where $T_\nu \subseteq G_\nu$ denotes the inertia group of ν.*

(i) *There is an exact sequence of* Λ*-modules*

$$0 \to Cl_S(k_\infty)(p)^\vee \xrightarrow{\psi} Cl_T(k_\infty)(p)^\vee \to \mathcal{E}_S(k_\infty) \xrightarrow{\varphi} \mathcal{E}_T(k_\infty) \to 0.$$

(ii) *Assume that* $S \setminus T = \{\nu\}$*. Then, if* $\dim(G_\nu) \geq 1$ *(resp.* $\dim(G_\nu) \geq 2$*), then* $coker(\psi) \cong ker(\varphi)$ *is* Λ*-torsion (resp. pseudo-null).*

(iii) *If* $\dim(G_\nu) \geq 2$ *for every* $\nu \in S \setminus T$*, then there are canonical pseudo-isomorphisms*

$$Cl_S(k_\infty)(p)^\vee \sim Cl_T(k_\infty)(p)^\vee, \qquad \mathcal{E}_S(k_\infty) \sim \mathcal{E}_T(k_\infty).$$

Proof. Consider the canonical exact diagram for a finite extension k' of k in k_∞

$$E_T(k') \otimes_{\mathbb{Z}} \mathbb{Z}_p \overset{i_{k'}}{\hookrightarrow} E_S(k') \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \bigoplus\nolimits_{(S\setminus T)(k')} \mathbb{Z}_p \to Cl_T(k')(p) \overset{\pi_{k'}}{\twoheadrightarrow} Cl_S(k')(p).$$

Setting $C(k') := coker(i_{k'})$ (resp. $D(k') := ker(\pi_{k'})$), $C_\infty = \varinjlim C(k')$ (resp. $D_\infty = \varinjlim D(k')$) and tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$ we get the following exact sequences

$$0 \to E_T(k') \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to E_S(k') \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to C(k') \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$

$$0 \longrightarrow D(k') \longrightarrow C(k') \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus\nolimits_{(S\setminus T)(k')} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0,$$

$$0 \longrightarrow D(k') \longrightarrow Cl_T(k')(p) \longrightarrow Cl_S(k')(p) \longrightarrow 0.$$

Taking the direct limit over all finite subextensions k', we get an isomorphism $D_\infty \cong C_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p$ because the transition maps for the sum of the $\mathbb{Q}_p/\mathbb{Z}_p$'s is just the multiplication with the ramification index. The first result follows after taking the Pontryagin dual.
Now assume that $S \setminus T$ consists of a single prime and set $\bar{G} := G(k'/k)$ Since then $\bar{G}_\nu = G_\nu G(k_\infty/k')/G(k_\infty/k')$ acts trivial on $\bigoplus_{(S\setminus T)(k')} \mathbb{Z}_p \cong \mathrm{Ind}_{\bar{G}}^{\bar{G}_\nu} \mathbb{Z}_p$ and therefore also on $C(k') \otimes \mathbb{Q}_p/\mathbb{Z}_p$, it follows that G_ν acts trivial on $(C_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. But then any surjection $\Lambda^n \twoheadrightarrow (C_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ factors through $(\mathrm{Ind}_G^{G_\nu} \mathbb{Z}_p)^n$ which is torsion (resp. pseudo-null) if $\dim(G_\nu) \geq 1$ (resp. $\dim(G_\nu) \geq 2$). The last statement is a consequence of the second one. □

The Λ-modules $Cl_S(k_\infty)(p)^\vee$ and $\mathcal{E}_S(k_\infty)$ are related to each other and to X_S via Kummer theory:

Proposition 3.1.11. *Assume that $\mu_{p^\infty} \subseteq k_\infty$. Then the following holds:*

(i) *There are exact sequences of Λ-modules*

$$0 \longrightarrow Cl_S(k_\infty)(p)^\vee \longrightarrow X_S(-1) \longrightarrow \mathcal{E}_S(k_\infty) \longrightarrow 0$$

and, if k_∞ is contained in k_Σ, where $\Sigma = S_p \cup S_\infty$,

$$0 \longrightarrow Cl(k_\infty)(p)^\vee \longrightarrow X_\Sigma(-1) \longrightarrow \mathcal{E}(k_\infty) \longrightarrow 0.$$

(ii) *$Cl_S(k_\infty)(p)^\vee$ is Λ-torsion. If $\dim(G_\nu) \geq 1$ for every $\nu \in S_p$, then $Cl(k_\infty)(p)^\vee$ is Λ-torsion, too. In particular, there are exact sequences*

$$0 \longrightarrow Cl_S(k_\infty)(p)^\vee \longrightarrow \mathrm{tor}_\Lambda X_S(-1) \longrightarrow \mathrm{tor}_\Lambda \mathcal{E}_S(k_\infty) \longrightarrow 0,$$

$$0 \longrightarrow Cl(k_\infty)(p)^\vee \longrightarrow \mathrm{tor}_\Lambda X_\Sigma(-1) \longrightarrow \mathrm{tor}_\Lambda \mathcal{E}(k_\infty) \longrightarrow 0.$$

Proof. The long exact $\mathrm{H}^i(G(k_S/k_\infty), -)$ -sequence of

$$0 \to \mu_{p^n} \to E_S(k_S) \xrightarrow{p^n} E_S(k_S) \to 0$$

induces the short exact sequence

$$0 \to E_S(k_\infty)/p^n \to \mathrm{H}^1(G(k_S/k_\infty), \mu_{p^\infty}) \to {}_{p^n}\mathrm{H}^1(G(k_S/k_\infty), E_S(k_s)) \to 0,$$

i.e. after taking the direct limit with respect to n

$$0 \to E_S(k_\infty) \otimes_\mathbb{Z} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{H}^1(G(k_S/k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)(1) \to Cl_S(k_\infty)(p) \to 0.$$

Taking the dual, we obtain the first statement. A canonical map $Cl(k_\infty(p)^\vee) \to X_S(-1)$ which is compatible with the inclusion $Cl_S(k_\infty)(p)^\vee \to X_S(-1)$ from the first sequence can be defined exactly as in the $\mathbb{Z}_p$-case, see [45, 11.4.2 and errata]. Then the exactness of the second sequence at the first term is obtained from the first one and prop. 3.1.11:

$$Cl(k_\infty)(p)^\vee/Cl_\Sigma(k_\infty)(p)^\vee \subseteq \mathcal{E}_\Sigma \cong X_\Sigma(-1)/Cl_\Sigma(k_\infty)(p)^\vee,$$

i.e. $Cl(k_\infty)(p)^\vee$ can be considered as submodule of $X_\Sigma(1)$ and then its quotient is $\mathcal{E}$.
Comparing the ranks of X_S and $\mathcal{E}_S$ (see 3.1.22) (with respect to an arbitrary open subgroup $H \subseteq G$ such that $\Lambda(H)$ is integral), we conclude that $Cl_S(k_\infty)(p)^\vee$ is Λ-torsion while the analogous result for $Cl(k_\infty)(p)^\vee$ follows from prop. 3.1.11. Now, the last sequences can be derived from the prior ones by rank considerations or by applying the snake lemma to the canonical sequence of homotopy theory (1.2.4). □

In the $\mathbb{Z}_p$-case there exists a remarkable duality between the inverse and direct limit of the (S-) ideal class groups in the $\mathbb{Z}_p$-tower, viz the pseudo-isomorphisms

$$\begin{aligned} X_{nr} &\sim \mathrm{E}^1(Cl(k_\infty)(p)^\vee) \sim (Cl(k_\infty)(p)^\vee)^\circ, \\ X_{cs}^S &\sim \mathrm{E}^1(Cl_S(k_\infty)(p)^\vee) \sim (Cl_S(k_\infty)(p)^\vee)^\circ. \end{aligned}$$

Therefore it seems natural (though maybe very optimistic) to pose the following

Question 3.1.12. *Is it true that for any p-adic Lie extension (at least under the assumption "$\dim(G_\nu) \geq 2$, for all $\nu \in S_f$,") there exist pseudo-isomorphisms*

$$\begin{aligned} X_{nr} &\sim \mathrm{E}^1(Cl(k_\infty)(p)^\vee), \\ X_{cs}^S &\sim \mathrm{E}^1(Cl_S(k_\infty)(p)^\vee) \ ? \end{aligned}$$

In this context we remind the reader also to our question 1.5.10.

Observe, that $X_{nr} \sim X_{cs}^S$ and $Cl_S(k_\infty)(p)^\vee \sim Cl(k_\infty)(p)^\vee$ by 3.1.3, 3.1.10. Hence, it would suffice to prove the existence of one of the pseudo-isomorphisms. By prop. 3.1.11(ii) and theorem 3.1.5 the question would be true if one could show that the Λ-torsion of $\mathcal{E}_S(k_\infty)$ is pseudo-null. But it seems difficult to prove the latter statement directly. In fact, in the case of a multiple $\mathbb{Z}_p$-extension $k_\infty|k$ where $\mu_{p^\infty} \subseteq k_\infty$ and k has only one prime above p , W. McCallum [43, thm 7] answers the above question positively and then derives $\mathrm{tor}_\Lambda \mathcal{E}_S(k_\infty) = 0$ just from the desired pseudo-isomorphism. This is the only case to the knowledge of the author where a positive answer to this question is known. Also J. Nekovar (unpublished) announced a result in the direction of the question.

For the next result, which generalizes theorem 11.3.7 of [45], recall that the μ-invariant of a Λ-module was defined as the $\mathbb{F}_p[\![G]\!]$-rank of $\bigoplus_{i\geq 0} {}_{p^{i+1}}M/{}_{p^i}M$ in case the latter is well-defined, see (1.5.29).

Theorem 3.1.13. *Let $k_\infty|k$ be a p-adic Lie extension such that G is without p-torsion and $\mathbb{F}_p[\![G]\!]$ is an integral domain. Then $\mathcal{G} = G(k_S(p)/k_\infty)$ is a free pro-p-group if and only if $\mu(X_S) = 0$ and the weak Leopoldt conjecture holds:* $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

Proof. Since $\mathcal{G}$ is pro-p it is free if and only if $\mathrm{H}^2(\mathcal{G}, \mathbb{Z}/p) = 0$, i.e. if and only if ${}_p(X_S)$ and $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$ vanish. But, by remark 1.5.30 and since X_S does not contain any pseudo-null submodule, these two conditions are equivalent to the vanishing of $\mu(X_S)$ and the validity of the weak Leopoldt conjecture. $\square$

The next theorem, which generalizes theorem 11.3.8 in [45], shows that the validity of the weak Leopoldt conjecture and the vanishing of the μ-invariant are properties which should be considered simultaneously if one studies the behaviour of X_S under change of the base field.

Theorem 3.1.14. *Let $K|k$ be a finite Galois p-extension inside k_S, $k_\infty|k$ a p-adic pro-p Lie extension such that*

$$(*)\quad G = G(k_\infty/k) \text{ is without } p\text{-torsion and } \mathbb{F}_p[\![G]\!] \text{ is integral.}$$

Set $K_\infty = Kk_\infty$ and $G' = G(K_\infty/K)$. Then G' satisfies the condition $(*)$, too, and the following holds

$$\left\{\begin{array}{c} \mu(X_S(k_\infty/k)) = 0 \text{ and} \\ \mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \mu(X_S(K_\infty/K)) = 0 \text{ and} \\ \mathrm{H}^2(G_S(K_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0 \end{array}\right\}.$$

In particular, if k_∞ contains the cyclotomic $\mathbb{Z}_p$-extension, then

$$\mu(X_S(k_\infty/k)) = 0 \Leftrightarrow \mu(X_S(K_\infty/K)) = 0.$$

Proof. Let $\mathcal{H}' := \mathcal{H} \cap G(k_S(p)/K)$. Then, by theorem 3.1.13, the statements to be compared are equivalent to the freeness of $\mathcal{H}$, resp. $\mathcal{H}'$, thus equivalent to $\mathrm{cd}_p(\mathcal{H}) = 1$, resp. $\mathrm{cd}_p(\mathcal{H}') = 1$. But, since $\mathcal{H}'$ is open in $\mathcal{H}$ and $\mathrm{cd}_p(\mathcal{H}) < \infty$, we have $\mathrm{cd}_p(\mathcal{H}') = \mathrm{cd}_p(\mathcal{H})$ by [45] 3.3.5,(ii). $\square$

The same arguments prove the following

Theorem 3.1.15. *Let $K_\infty|k_\infty|k$ be p-adic pro-p Lie extensions (inside k_S) such that for both $G(K_\infty/K)$ and $G(k_\infty/k)$ the condition $(*)$ of the previous theorem holds. Then*

$$\left\{\begin{array}{c} \mu(X_S(k_\infty/k)) = 0 \text{ and} \\ \mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \mu(X_S(K_\infty/k)) = 0 \text{ and} \\ \mathrm{H}^2(G_S(K_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0 \end{array}\right\}.$$

The next theorem, which generalizes theorem 11.3.5 in [45], describes the "difference" if we vary the finite set of places S defining the module X_S. By $T(K/k) \subseteq G(K/k)$ we shall denote the inertia subgroup for a Galois extension $K|k$ of local fields and, for an arbitrary set of places S of k and a p-adic analytic extension $k_\infty|k$, we write $S^{cd}(k)$ for the subset of finite places which decompose completely in $k_\infty|k$.

Theorem 3.1.16. *Let $S \supseteq T \supseteq S_p \cup S_\infty$ be finite sets of places of k and let $k_\infty|k$ be a p-adic pro-p Lie extension inside k_T with Galois group G. Assume that G does not contain any p-torsion element and that the weak Leopoldt* *conjecture holds for $k_\infty|k$. Then there exists a canonical exact sequence of Λ-modules*

$$0 \longrightarrow \bigoplus_{(S\setminus T)(k)} \mathrm{Ind}_G^{G_\nu} T(k_\nu(p)/k_\nu)_{G_{k_\infty,\nu}} \longrightarrow X_S \longrightarrow X_T \longrightarrow 0$$

and the direct sum on the left is isomorphic to

$$\bigoplus_{\substack{(S\setminus T)(k) \\ p^\infty|f_\nu,\ \mu_p\subseteq k_\nu}} \mathrm{Ind}_G^{G_\nu} \mathbb{Z}_p(1) \oplus \bigoplus_{(S\setminus T)^{cd}(k)} \Lambda/p^{t_\nu},$$

where $p^{t_\nu} = \#\mu(k_\nu)(p)$ and, as before, $f_\nu = f(k_{\infty,\nu}/k_\nu)$ denotes the degree of the extension of the corresponding residue class fields. In particular, there is an exact sequence of Λ-torsion modules

$$0 \longrightarrow \bigoplus_{(S\setminus T)(k)} \mathrm{Ind}_G^{G_\nu} T(k_\nu(p)/k_\nu)_{G_{k_\infty,\nu}} \longrightarrow \mathrm{tor}_\Lambda X_S \longrightarrow \mathrm{tor}_\Lambda X_T \longrightarrow 0.$$

Proof. Since $\mathrm{H}^2(G_T(k_\infty)(p), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, we have an exact sequence

$$0 \longrightarrow G(k_S(p)/k_T(p))^{ab}_{G_T(k_\infty)} \longrightarrow X_S \longrightarrow X_T \longrightarrow 0.$$

Setting $\mathcal{G} = G_T(k)(p)$ and using [45, 10.5.4,10.6.1] as well as lemma 1.3.6, we obtain

$$\begin{aligned} G(k_S(p)/k_T(p))^{ab}_{G_T(k_\infty)} &\cong \Big(\bigoplus_{(S\setminus T)(k)} \mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}_\nu} T(k_T(p)_\nu(p)/k_T(p)_\nu) \Big)_{G_T(k_\infty)} \\ &\cong \bigoplus_{(S\setminus T)(k)} \mathrm{Ind}_G^{G_\nu} T(k_\nu(p)/k_\nu)_{G_{k_\infty,\nu}}. \end{aligned}$$

Observe that, for $\nu \in S \setminus T$,

$$T(k_\nu(p)/k_\nu) \cong \begin{cases} \mathbb{Z}_p(1) & \text{if } \mu_p \subseteq k_\nu, \\ 0 & \text{otherwise.} \end{cases}$$

Since G is without p-torsion and $\nu \in S \setminus T$ is unramified in $k_\infty|k$, there are only two possibilities for G_ν :

$$G_\nu = \begin{cases} 0 & \text{if } \nu \text{ is completely decomposed in } k_\infty|k, \\ \mathbb{Z}_p & \text{if } p^\infty|f_\nu, \end{cases}$$

respectively,

$$G_{k_\infty,\nu}(p) \cong \begin{cases} \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p, & \text{if } \nu \text{ is completely decomposed in } k_\infty|k, \\ \mathbb{Z}_p(1), & \text{if } p^\infty|f_\nu. \end{cases}$$

It follows that

$$G(k_S(p)/k_T(p))^{ab}_{G_T(k_\infty)} \cong \bigoplus_{\substack{(S\backslash T)(k) \\ p^\infty | f_\nu,\ \mu_p \subseteq k_\nu}} \mathrm{Ind}_G^{G_\nu} \mathbb{Z}_p(1) \oplus \bigoplus_{(S\backslash T)^{cd}(k)} \Lambda/p^{t_\nu}.$$

In particular, this module is Λ-torsion and therefore the second statement follows from the first. □

Recalling that μ is additive on short exact sequences of Λ-torsion modules by 1.5.34, we obtain the following

Corollary 3.1.17. *Under the assumptions of the theorem,*

$$\mu(X_S) = \mu(X_T) + \sum_{(S\backslash T)^{cd}(k)} t_\nu,$$

where $p^{t_\nu} = \#\mu(k_\nu)(p)$.

3.1.2 Global units

We still consider p-adic Lie extensions $k_\infty | k$ with Galois group $G = G(k_\infty/k)$.

Recall that we denote the norm compatible S-units of k_∞ by $\mathbb{E}_S := \varprojlim_{k'} (\mathcal{O}^\times_{k',S} \otimes \mathbb{Z}_p)$. Noting that $\mathbb{E}_S \cong \varprojlim_{k'} \mathrm{H}^1(G_S(k'), \mathbb{Z}_p(1))$ by Kummer theory and the finiteness of the S-ideal class group, we are going to derive some relations between $\mathbb{E}_S$ and $\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee$ by interpreting Jannsen's spectral sequence for Iwasawa adjoints with respect to $A = \mathbb{Q}_p/\mathbb{Z}_p(1) = \mu_{p^\infty}(k_S)$. We assume that G does not have any p-torsion, i.e. G is a Poincaré group at p, and we denote the character which gives the operation of G on the dualizing module by χ^{-1}.

Proposition 3.1.18. (i) *If* $\mu_{p^\infty} \subseteq k_\infty$, *then*

(a) *if* $\mathrm{cd}_p(G) = 1$:

$$\begin{aligned} \mathbb{E}_S &\cong \mathbb{Z}_p(1)(\chi) \oplus \mathrm{E}^0(X_S(-1)) \\ \varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)) &\cong \mathrm{E}^1(X_S(-1)), \\ \mathrm{E}^i(X_S(-1)) &= 0 \ \ \textit{for} \ \ i \geq 2. \end{aligned}$$

(b) *if* $\mathrm{cd}_p(G) = 2$: *there is an exact sequence*

$$0 \longrightarrow \mathbb{E}_S \longrightarrow \mathrm{E}^0(X_S(-1)) \longrightarrow \mathbb{Z}_p(1)(\chi) \longrightarrow$$

$$\varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)) \longrightarrow \mathrm{E}^1(X_S(-1)) \longrightarrow 0,$$

and

$$\mathrm{E}^i(X_S(-1)) = 0 \text{ for } i \geq 2.$$

(c) *if* $\mathrm{cd}_p(G) = 3$: *there is an exact sequence*

$$0 \to \varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)) \to \mathrm{E}^1(X_S(-1)) \to \mathbb{Z}_p(1)(\chi) \to 0,$$

and

$$\begin{aligned} \mathbb{E}_S &\cong \mathrm{E}^0(X_S(-1)), \\ \mathrm{E}^i(X_S(-1)) &= 0 \text{ for } i \geq 2. \end{aligned}$$

(d) *if* $\mathrm{cd}_p(G) \geq 4$:

$$\mathbb{E}_S \cong \mathrm{E}^0(X_S(-1)),$$

$$\varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)) \cong \mathrm{E}^1(X_S(-1)),$$

$$\mathrm{E}^i(X_S(-1)) = \begin{cases} \mathbb{Z}_p(1)(\chi) & \text{if } i = \mathrm{cd}_p(G) - 2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i \geq 2.$$

Similar results hold for arbitrary A *with* $k(A) \subseteq k_\infty$ *if* $\mathbb{E}_S$ *is replaced by* $\varprojlim_{k'} \mathrm{H}^1(G_S(k'), T_pA)$, $X_S(-1)$ *by* $X_S[A]$, ...

(ii) *If* $\mu(k_\infty)(p)$ *is finite, then*

(a) *if* $\mathrm{cd}_p(G) = 1$: *then there is an exact sequence*

$$0 \to \mathbb{E}_S \to \mathrm{E}^0(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) \to \mu(k_\infty)(p)^\vee(\chi) \to \varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)).$$

(a_1) *If in addition* $\mathrm{H}^2(G_S(k_\infty), \mu_{p^\infty}) = 0$, *then the cokernel of the sequence is* $\mathrm{E}^1(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee)$ *and* $\mathrm{E}^i(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) = 0$ *for* $i \geq 2$.

(a$_2$) *If in addition* $\mathrm{H}^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$, *then there is a short exact sequence*

$$0 \to \mathbb{E}_S \to \mathrm{E}^0(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) \to \mu(k_\infty)(p)^\vee(\chi) \to 0.$$

(b) *if* $\mathrm{cd}_p(G) = 2$, *then* $\mathbb{E}_S \cong \mathrm{E}^0(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee)$.

If in addition $\mathrm{H}^2(G_S(k_\infty), \mu_{p^\infty}) = 0$, *then there is an exact sequence*

$$0 \to \varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)) \to \mathrm{E}^1(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) \to \mu(k_\infty)(p)^\vee(\chi) \to 0$$

and

$$\mathrm{E}^i(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) = 0 \text{ for } i \geq 2.$$

(c) *if* $\mathrm{cd}_p(G) \geq 3$, *then* $\mathbb{E}_S \cong \mathrm{E}^0(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee)$.

If in addition $\mathrm{H}^2(G_S(k_\infty), \mu_{p^\infty}) = 0$, *then*

$$\begin{aligned} \mathbb{E}_S &\cong \mathrm{E}^0(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee), \\ \varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)) &\cong \mathrm{E}^1(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee), \end{aligned}$$

$$\mathrm{E}^i(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) = \begin{cases} \mu(k_\infty)(p)(\chi) & \text{if } i = \mathrm{cd}_p(G) - 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i \geq 2.$$

(iii) *If* $\mu(k_\infty)(p) = 0$, *then there is in addition to the results for finite* $\mu(k_\infty)(p)$ *the following exact sequence:*

$$\begin{aligned} 0 \to \mathrm{E}^1(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) \to \varprojlim_{k'} H^2(G_S(k'), \mathbb{Z}_p(1)) \to \\ \mathrm{E}^0(\mathrm{H}^2(G_S(k_\infty), \mu_{p^\infty})^\vee) \to \mathrm{E}^2(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) \to 0, \end{aligned}$$

and

$$\mathrm{E}^i(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee) \cong \mathrm{E}^{i-2}(\mathrm{H}^2(G_S(k_\infty), \mu_{p^\infty})^\vee)).$$

For the proof apply theorem 1.4.5 and its corollary and note the following facts: $\mathrm{H}^1(G_S(k_\infty), A)^\vee \cong X_S[A]$ if $k(A) \subseteq k_\infty$, $\mathrm{H}^2(G_S(k_\infty), A) = 0$ if $\mu_{p^\infty} \subseteq k_\infty$ because the weak Leopoldt conjecture is true for the cyclotomic extension of any number field. Furthermore, we applied several times corollary 1.4.8. Also observe, that the reflexive module $\mathrm{E}^0(X_S(-1))$ is projective in the case

$\mathrm{cd}_p(G) = 1$ regarding the defining sequence of the transpose functor D and using prop. 1.1.2. The last statement of (ii)(a) is proved in [45] 11.3.9.

These results bear a lot of information about the structure of $\mathbb{E}_S$ and $\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee$, e.g. one can derive the projective dimension of the latter module (using corollary 1.7.3) and some information about the dimensions of the modules occurring above, in particular whether a module is torsion(free). Furthermore, we see that $\mathbb{E}_S$ is reflexive for almost all cases with $\mathrm{cd}_p(G) \geq 2$ by proposition 1.5.14.
In order to relate $\mathbb{E}_S$ to the finitely generated Λ-module

$$\mathcal{E}_S(k_\infty) = (E_S(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p)^\vee$$

we need some technical lemmas.

Lemma 3.1.19. (i) *Let* $G = G(k_\infty/k) \cong \mathbb{Z}_p^d$, $d \geq 1$, *and* $G_n := p^n G$.

(a) *If* $\mu_{p^\infty} \subseteq k_\infty$, *then with* $\Gamma = G(k(\mu_{p^\infty})$ *and* $\Gamma_n = p^n\Gamma$ *the following holds*

$$\mathrm{H}^i(G_n, \mu_{p^\infty}) = \mu(k_n)(p)^{\binom{d-1}{i}},$$

where $k_n = k(\mu_{p^\infty})^{\Gamma_n}$.

(b) *If* $\mu(k_\infty)(p)$ *is finite, then for any* n *such that* $\mu(k_\infty)(p)^{G_n} = \mu(k_\infty)(p)$ *it holds*

$$\mathrm{H}^i(G_n, \mu_{p^\infty}) = \mu(k_\infty)(p)^{\binom{d}{i}}.$$

(ii) *Let* G *be a finitely generated pro-p Lie group without p-torsion which fits into a exact sequence*

$$1 \longrightarrow U \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 1,$$

with $\Gamma \cong \mathbb{Z}_p$ *and let* G_n *be an open subgroup. Assume that* $\Gamma_n := \pi(G_n)$ *acts via a splitting trivially on* $U_n = G_n \cap U$. *Then* $\mathrm{H}^2(G_n, \mu(k_\infty)(p))$ *is finite and the following holds*

(a) *If* $\mu_{p^\infty} \subseteq k_\infty$ *and* $\Gamma = G(k(\mu_{p^\infty}))$, *then*

$$\mathrm{H}^1(G_n, \mu(k_\infty)(p)) \cong \mu(k_n)(p)^s \oplus \bigoplus_i \mu_{p^{\nu_i}}(k_n),$$

where $U_n^{ab} \cong \mathbb{Z}_p{}^s \oplus \bigoplus_i \mathbb{Z}_p/p^{\nu_i}$ *with* $U_n = U \cap G_n$.

(b) *If* $\mu(k_\infty)(p)$ *is finite, then for any* n *such that* $\mu(k_\infty)(p)^{G_n} = \mu(k_\infty)(p)$ *there is an exact sequence*

$$0 \to \mu(k_\infty)(p) \to \mathrm{H}^1(G_n, \mu(k_\infty)(p)) \to \mu(k_\infty)(p)^s \oplus \bigoplus_i \mu_{p^{\nu_i}}(k_\infty) \to 0.$$

(c) *If* $\mathrm{cd}_p(G) = 2$, *then*

$$\mathrm{H}^2(G_n, \mu(k_\infty)(p)) \cong \begin{cases} 0 & \textit{if } \mu_{p^\infty} \subseteq k_\infty, \\ \mu(k_\infty)(p) & \textit{otherwise.} \end{cases}$$

Proof. Consider the exact sequence

$$1 \longrightarrow U \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 1,$$

and let $U_n = G_n \cap U$ and $\Gamma_n = \pi(G_n)$. The Hochschild-Serre spectral sequence gives

$$\mathrm{H}^1(\Gamma_n, \mathrm{H}^i(U_n, \mu(k_\infty)(p))) \hookrightarrow \mathrm{H}^{i+1}(G_n, \mu(k_\infty)(p)) \twoheadrightarrow \mathrm{H}^{i+1}(U_n, \mu(k_\infty)(p))^{\Gamma_n}$$

for $i \geq 0$.
Let us first assume that $\mu_{p^\infty} \subseteq k_\infty$: Since U_n acts trivially on μ_{p^∞}, we get

$$\mathrm{H}^i(U_n, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathrm{H}^i(U_n, \mathbb{Q}_p/\mathbb{Z}_p)(1) = (\mathbb{Q}_p/\mathbb{Z}_p)^{\binom{d-1}{i}}.$$

in the abelian case by the Künneth formula. As $\mathbb{Q}_p/\mathbb{Z}_p(1)_{\Gamma_n} = 0$ it follows that $\mathrm{H}^i(G_n, \mu_{p^\infty}) = \mathrm{H}^i(U_n, \mu_{p^\infty})^{\Gamma_n} = \mu(k_n)^{\binom{d-1}{i}}$. In the non-abelian case we calculate

$$\begin{aligned} \mathrm{H}^1(G_n, \mu_{p^\infty}) &= \mathrm{H}^1(U_n, \mathbb{Q}_p/\mathbb{Z}_p)(1)^{\Gamma_n} \\ &= (U_n^{ab})^\vee(1)^{\Gamma_n} \\ &= \mu(k_n)(p)^s \oplus \bigoplus \mu_{p^{\nu_i}}(k_n). \end{aligned}$$

Hence $\mathrm{H}^1(\Gamma_n, \mathrm{H}^1(U_n, \mu(k_\infty)(p)))$ is finite and the finiteness of $\mathrm{H}^2(G_n, \mu_{p^\infty})$ follows because $\mathrm{H}^2(U_n, \mu_{p^\infty})^{\Gamma_n} \cong \mathrm{H}^2(U_n, \mathbb{Q}_p/\mathbb{Z}_p)(1)^{\Gamma_n}$ is also finite ($H^2(U_n, \mathbb{Q}_p/\mathbb{Z}_p)$ is a cofinitely generated abelian group).
Now we consider the case of finite $\mu(k_\infty)(p)$: Here $\mathrm{H}^1(\Gamma_n, \mu(k_\infty)(p)) = \mu(k_\infty)(p)$ and the abelian case follows again using the Künneth formula. In the non-abelian case the finiteness of $\mathrm{H}^2(G_n, \mu_{p^\infty})$ is trivial while $\mathrm{H}^1(U_n, \mu(k_\infty)(p))^{\Gamma_n}$ can be calculated similarly as above. For the last assertion just note that $U_n \cong \mathbb{Z}_p$. □

Lemma 3.1.20. (i) *In the situation of the previous lemma (ii) it holds*

(a) $\varprojlim_{m,n} {}_{p^m}\mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty)) \cong \varprojlim_{m,n} {}_{p^m}\mathrm{H}^1(G_n, E_S(k_\infty)) = 0,$

(b) $\varprojlim_{n} \mathrm{H}^1(G_n, E_S(k_\infty)) \subseteq X_{cs}^S,$

$$\begin{aligned} \text{(c)}\quad \mathrm{E}^0(\mathcal{E}_S(k_\infty)) &\cong \varprojlim_{m,,n}{}_{p^m}(E_S(k_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p)^{G_n} \\ &\cong \varprojlim_{m,n}(E_S(k_\infty)/\mu(k_\infty))^{G_n}/p^m, \end{aligned}$$

(d) $\mathrm{T}_0(\varprojlim_n \mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty))) = \mathrm{T}_0(\mathrm{E}^1((E_S(k_\infty)\otimes_{\mathbb{Z}}\mathbb{Q}_p/\mathbb{Z}_p)^\vee))$,

(e) *that the following sequence is exact:*

$$0 \to \varprojlim_n \mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty)) \to \mathrm{E}^1(\mathcal{E}_S(k_\infty)) \to \varprojlim_{m,n}{}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty)/\mu(k_\infty)) \to 0.$$

(ii) *If, in addition,* $\mathrm{cd}_p(G)\le 2$, *then with* $\kappa = \begin{cases} 1 & \textit{if } \mu(k_\infty)(p) \textit{ is finite,} \\ 0 & \textit{otherwise,e} \end{cases}$ *there are the following exact sequences*

(a) *if* $\mathrm{cd}_p(G) = 2:$

$$0 \to \varprojlim_n \mathrm{H}^1(G_n, E_S(k_\infty)) \to \varprojlim_{m,n} \mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty))/p^m \to \mu(k_\infty)(p)^\kappa \to D \to 0,$$

$$0 \to \varprojlim_{m,n}{}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty)) \to \varprojlim_{m,n}{}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty)/\mu(k_\infty)) \to D \to$$

$$\varprojlim_{m,n}\mathrm{H}^2(G_n, E_S(k_\infty))/p^m \to \varprojlim_{m,n}\mathrm{H}^2(G_n, E_S(k_\infty)/\mu(k_\infty))/p^m \to 0,$$

where D *is some finite module.*

(b) *if* $\mathrm{cd}_p(G) = 1:$

$$0 \to \mathrm{E}^2\mathrm{E}^1(\mathbb{E}_S) \to \mu(k_\infty)(p)^\kappa \to \varprojlim_n \mathrm{H}^1(G_n, E_S(k_\infty)) \to \varprojlim_{m,n}\mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty))/p^m \to 0$$

and

$$\varprojlim_{m,n}{}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty)) \cong \varprojlim_{m,n}{}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty)/\mu(k_\infty)).$$

Proof. If we split the long exact cohomology sequence induced by

$$0 \longrightarrow \mu(k_\infty) \longrightarrow E_S(k_\infty) \longrightarrow E_S(k_\infty)/\mu(k_\infty) \longrightarrow 0,$$

we get the following short exact sequences

$$\begin{aligned} &0 \longrightarrow F_n \longrightarrow \mathrm{H}^1(G_n, \mu(k_\infty)) \longrightarrow A_n \longrightarrow 0, \\ &0 \longrightarrow A_n \longrightarrow \mathrm{H}^1(G_n, E_S(k_\infty)) \longrightarrow B_n \longrightarrow 0, \\ &0 \longrightarrow B_n \longrightarrow \mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty)) \longrightarrow C_n \longrightarrow 0 \end{aligned}$$

and furthermore a map $C_n \hookrightarrow \mathrm{H}^2(G_n, \mu(k_\infty)(p))$. Evaluating the associated long exact sequences of p^m-torsion (snake lemma) and noting the finiteness of A_n and C_n according to the previous lemma, we get

$$\varprojlim_m {}_{p^m}B_n \cong \varprojlim_m {}_{p^m}\mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty)),$$

$$0 \longrightarrow \varprojlim_m {}_{p^m}\mathrm{H}^1(G_n, E_S(k_\infty)) \longrightarrow \varprojlim_m {}_{p^m}B_n \longrightarrow A_n,$$

and therefore

$$0 \to \varprojlim_{m,n} {}_{p^m}\mathrm{H}^1(G_n, E_S(k_\infty)) \to \varprojlim_{m,n} {}_{p^m}\mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty)) \to \varprojlim_n A_n$$

is exact.

But $\varprojlim_n A_n$ is a quotient of

$$\varprojlim_n \mathrm{H}^1(G_n, \mu(k_\infty)(p)) = \begin{cases} \mu(k_\infty)(p) & \text{if } d = 1 \text{ and } \mu(k_\infty)(p) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

(See the previous lemma and note that the transition maps are partially norm maps besides the non-trivial case where they are the natural projections, i.e. identities for n sufficiently big.). Since the middle term is $\mathbb{Z}_p$-torsion free, we get the desired isomorphism, because, by the Hochschild-Serre spectral sequence, it can be seen in any case that the first group is contained in $\varprojlim_{m,n} {}_{p^m}Cl_S(k_n) = 0$. This proves (i)(a) while (b) is again the cited spectral sequence.

The first equality of (i)(c) is just 1.4.7 (iii) because $\mathcal{E}_S(k_\infty)$ has no $\mathbb{Z}_p$-torsion while the second one follows by the exact sequence

$$(E_S(k_\infty)/\mu(k_\infty))^{G_n}/p^m \hookrightarrow {}_{p^m}(E_S(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p)^{G_n} \twoheadrightarrow {}_{p^m}\mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty))$$

and (a). Similar arguments apply for (i)(e), i.e.

$$\mathrm{E}^1(\mathcal{E}_S(k_\infty)) \cong \varprojlim_{m,n} \mathrm{H}^1(G_n, {}_{p^m}(E_S(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p)).$$

The assertion (d) is a direct consequence of (e), because $\varprojlim_{m,n} {}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty)/\mu(k_\infty))$ is $\mathbb{Z}_p$-torsion-free.

Now let us assume that $\mathrm{cd}_p(G) \leq 2$. With the notation as above and recalling that A_n, B_n and C_n are finite, we get exact sequences

$$0 \longrightarrow A_n \longrightarrow \mathrm{H}^1(G_n, E_S(k_\infty)) \longrightarrow B_n \longrightarrow 0,$$

$$0 \longrightarrow B_n \longrightarrow \varprojlim_m \mathrm{H}^1(G_n, E_S(k_\infty)/\mu(k_\infty))/p^m \longrightarrow C_n \longrightarrow 0$$

and

$$0 \longrightarrow C_n \longrightarrow \mathrm{H}^2(G_n, \mu(k_\infty)) \longrightarrow D_n \longrightarrow 0.$$

Passing to the limit gives the first exact sequence in (ii)(a) (Note that the transition maps of the system $\{C_n\}$ are the canonical projections, i.e. identities for n sufficiently large). The second one is proved similarly using

$$D_n \hookrightarrow \mathrm{H}^2(G_n, E_S(k_\infty)) \to \mathrm{H}^2(G_n, E_S(k_\infty)/\mu(k_\infty)) \to \mathrm{H}^3(G_n, \mu(k_\infty)(p)) = 0$$

and $\mathrm{H}^2(G_n, \mu(k_\infty)(p)) \twoheadrightarrow D_n$. The proof of (ii)(b) is completely analogous, just note that $\varprojlim_n F_n \cong \mathrm{E}^2\mathrm{E}^1(\mathbb{E}_S)$ because the latter module is the cokernel of $\mathbb{E}_S \to \mathrm{E}^0\mathrm{E}^0(\mathbb{E}_S) \cong \mathrm{E}^0(\mathcal{E}_S(k_\infty))$. $\square$

Proposition 3.1.21. *There is an exact sequence*

$$0 \longrightarrow \mathbb{Z}_p(1)^\delta \longrightarrow \mathbb{E}_S \longrightarrow \mathrm{E}^0(\mathcal{E}_S(k_\infty)) \longrightarrow C$$

with

$$C = \begin{cases} \mu(k_\infty)(p) & \text{if } d = 1 \text{ and } \mu(k_\infty)(p) \text{ finite} \\ \mathbb{Z}_p(1) & \text{if } d = 2 \text{ and } \mu_{p^\infty} \subseteq k_\infty \\ \text{f.g. } \mathbb{Z}_p\text{-module} & d \geq 3 \text{ and } G \text{ non-abelian} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta = \begin{cases} 1 & \text{if } d = 1, \mu_{p^\infty} \subseteq k_\infty, \\ 0 & \text{otherwise.} \end{cases}$$

If in addition the weak Leopoldt conjecture holds, the right map is onto in the case $d = 1$ and $\mu(k_\infty)(p)$ finite.

Proof. Taking G_n-invariants of the exact sequence

$$0 \longrightarrow \mu(k_\infty)(p) \longrightarrow E_S(k_\infty) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow (E_S(k_\infty)/\mu(k_\infty)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow 0$$

and passing to the inverse limit, we get

$$0 \to \varprojlim_{n} \mu(k_n)(p) \to \mathbb{E}_S \to \varprojlim_{m,n} (E_S(k_\infty)/\mu(k_\infty))^{G_n}/p^m \to \varprojlim_{n} \mathrm{H}^1(G_n, \mu(k_\infty)(p))$$

The result follows except the fact that E^0 maps onto the finite group of roots of unity in the case when $d = 1$. But this is proved in [45] 11.3.9 under the assumption that the weak Leopoldt conjecture holds. □

Corollary 3.1.22. *Let $k_\infty|k$ be a p-adic Lie extension such that G does not have any p-torsion. Then*

$$\mathrm{E}^0(\mathbb{E}_S) \cong \mathrm{E}^0\mathrm{E}^0(\mathcal{E}_S(k_\infty)) \cong \mathrm{E}^0(\mathrm{H}^1(G_S(k_\infty), \mu_{p^\infty})^\vee).$$

In particular, if G is in addition pro-p and $\mathrm{H}^2(G_S(k_\infty), \mu_{p^\infty}) = 0$ (e.g. if $\mu_{p^\infty} \subseteq k_\infty$), then

$$\mathrm{rk}_\Lambda \mathbb{E}_S = \mathrm{rk}_\Lambda \mathcal{E}_S = r_2(k).$$

Now the question arises whether the module $\mathrm{E}^0(\mathbb{E}_S)$ is not only reflexive but also projective. While in the case $\mathrm{cd}_p(G) = 1$ this is always true, in higher dimensions one needs additional conditions. We will only get a satisfying answer in the two dimensional case:

Proposition 3.1.23. *Let $k_\infty|k$ be a p-adic Lie extension such that $\mathrm{cd}_p(G) = 2$ and assume that the weak Leopoldt conjecture holds for k_∞. Then the following is equivalent:*

(i) $\mathrm{E}^0(\mathbb{E}_S)$ *is projective,*

(ii) $\mathrm{T}_0\mathrm{E}^1(\mathcal{E}_S(k_\infty)) = 0$.

Remark 3.1.24. These equivalent statements hold for example, if either $\mu_{p^\infty} \subseteq k_\infty$ or $\mu(k_\infty)(p) = 0$, and $\mathrm{T}_0(X_{cs}^S) = 0$, i.e. if X_{cs}^S does not have any non-zero finite submodule, because then $\mathrm{T}_0\mathrm{E}^1(\mathcal{E}_S(k_\infty)) = 0$ by lemma 3.1.20.

Proof. Since we already know that $\mathrm{pd}(\mathrm{E}^0(\mathbb{E}_S)) \leq 1$, because $\mathrm{E}^0(\mathbb{E}_S)$ is the second syzygy of $\mathrm{D}\mathbb{E}_S$, the projectivity is equivalent to the vanishing of $\mathrm{E}^1\mathrm{E}^0(\mathbb{E}_S)$. Now the equivalence stated above follows from the next lemma. □

Lemma 3.1.25. *In the situation of the proposition it holds*

$$\mathrm{T}_0\mathrm{E}^1(\mathcal{E}_S(k_\infty)) \cong \mathrm{E}^1\mathrm{E}^0(\mathbb{E}_S) \cong \mathrm{E}^3\mathrm{E}^1(\mathbb{E}_S)$$

Proof. Set $M := \mathcal{E}_S(k_\infty)$ and consider the exact sequence

$$0 \longrightarrow M/\mathrm{T}_1(M) \longrightarrow \mathrm{E}^0(\mathbb{E}_S) \longrightarrow \mathrm{E}^2 D(M) \longrightarrow 0.$$

The long exact sequence for E^i gives

$$0 = \mathrm{E}^1\mathrm{E}^2 D(M) \longrightarrow \mathrm{E}^1\mathrm{E}^0(M) \longrightarrow \mathrm{E}^1(M/\mathrm{T}_1(M)) \longrightarrow \mathrm{E}^2\mathrm{E}^2 D(M).$$

On the other hand there is the exact sequence

$$0 = \mathrm{E}^0(\mathrm{T}_1(M)) \longrightarrow \mathrm{E}^1(M/\mathrm{T}_1(M)) \longrightarrow \mathrm{E}^1(M) \longrightarrow \mathrm{E}^1\mathrm{E}^1 D(M).$$

Since $\mathrm{E}^i\mathrm{E}^i D(M)$ is pure of codimension i, the isomorphism follows. But $\mathrm{E}^1\mathrm{E}^0(\mathbb{E}_S) \cong \mathrm{E}^3\mathrm{E}^1(\mathbb{E}_S)$ by the spectral sequence due to Björk, see 1.5.16. □

The proposition above should be compared with the following result which has already been observed by Kay Wingberg (unpublished):

Proposition 3.1.26. *If* $\mathrm{cd}_p(G) = 1$, *then for sufficiently large* n *there is a canonical exact sequence*

$$0 \longrightarrow \mathcal{E}_S(k_\infty)^{G_n} \longrightarrow \mathcal{E}_S(k_\infty) \longrightarrow \mathrm{E}^0(\mathbb{E}_S) \longrightarrow C \longrightarrow 0$$

where $C = \mathrm{E}^2\mathrm{D}(\mathcal{E}_S(k_\infty))$ *is connected with* $\mathrm{E}^2\mathrm{D}(\mathbb{E}_S)$ *by the exact sequence*

$$0 \longrightarrow \mathrm{E}^2\mathrm{D}(\mathbb{E}_S) \longrightarrow \mu(k_\infty)(p)^\kappa \longrightarrow \mathrm{T}_0 X_{cs}^S \longrightarrow C^\vee \longrightarrow 0.$$

Proof. The first sequence is just the canonical sequence 1.2.4 for the module $\mathcal{E}_S(k_\infty)$ while the second one already occurred in lemma 3.1.20 (ii)(b) as we show now: The fact that $\mathrm{T}_0(X_{cs}^S) \cong \varprojlim_n \mathrm{H}^1(G_n, E_S(k_\infty))$ is well known (see for example [45, XI.§3.]). Recall that $\mathrm{E}^2\mathrm{E}^1(\mathcal{E}_S(k_\infty)) \cong \mathrm{T}_0(\mathrm{E}^1(\mathcal{E}_S(k_\infty)))^\vee$ and apply lemma 3.1.20 (i)(d) to recover C. Using 3.1.20, (i)(e) and (ii)(b) we see that $\mathrm{E}^1\mathrm{E}^1(\mathcal{E}_S(k_\infty)) \cong \mathrm{E}^1(\varprojlim_{m,n}{}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty)))$, which we will determine by means of 1.4.7 (iii):

$$M := \varprojlim_{m,n}{}_{p^m}\mathrm{H}^2(G_n, E_S(k_\infty))^\vee \cong \varinjlim_{m,n} \mathcal{E}_S(k_\infty)^{G_n}/p^m = \varinjlim_{m} \mathcal{E}_S(k_\infty)^{G_n}/p^m$$

for n sufficiently large, because $\mathcal{E}_S(k_\infty)$ is a finitely generated Λ-module. Hence

$$\mathrm{E}^1(M) \cong \varprojlim_{m,n} ({}_{p^m}M)_{G_n} = \mathcal{E}_S(k_\infty)^{G_n}.$$

for n large enough. □

Proposition 3.1.27. *Let k_∞/k be a p-adic Lie extension such that $G \cong \Gamma\times\Delta$ where Γ is a pro-p-Lie group of $\mathrm{cd}_p(\Gamma) = 2$, Δ is a finite group of order prime to p. Assume that the weak Leopoldt conjecture holds for k_∞. Then the following is true:*

(i) *There is an exact sequence*

$$0 \longrightarrow \mathrm{E}^0\mathrm{E}^0(\mathbb{E}_S) \longrightarrow \Lambda^{r_2+r_1-r_1'-s} \oplus \bigoplus_{S^{cd}\cup S'_\infty} \mathrm{Ind}_G^{G_\nu}(\mathbb{Z}_p) \longrightarrow$$
$$\Lambda^s \longrightarrow \mathrm{T}_0\mathrm{E}^1(\mathcal{E}_S(k_\infty)) \longrightarrow 0.$$

(ii) *If $\mathrm{E}^0\mathrm{E}^0(\mathbb{E}_S)$ is projective, then*

$$\mathrm{E}^0\mathrm{E}^0(\mathbb{E}_S) \cong \Lambda^{r_2+r_1-r_1'} \oplus \bigoplus_{S^{cd}\cup S'_\infty} \mathrm{Ind}_G^{G_\nu}(\mathbb{Z}_p).$$

Proof. We calculate the Euler characteristic with respect to an arbitrary open normal subgroup $U \trianglelefteq \Gamma$ using lemma 1.3.4, proposition 1.8.4,[33] 5.4 b),

$$\begin{aligned}
h_U(\mathrm{E}^0\mathrm{E}^0(\mathbb{E}_S)) &= h_U(\mathbb{E}_S)\\
&= h_U(\mathbb{A}_S) - h_U(X_S) + h_U(X_{cs}^S)\\
&= h_U(\mathbb{A}_S) - h_U(Y_S) + h_U(I_G)\\
&= h_U(\mathbb{A}_S) - h_U(\Lambda^d) + h_U(N_{\mathcal{H}}^{ab}(p)) + h_U(\Lambda) - h_U(\mathbb{Z}_p)\\
&= h_U(\mathbb{A}_S) - h_U(\Lambda^{r_2+r_1'}) + h_U(\bigoplus_{S'_\infty} \mathrm{Ind}_G^{G_\nu}(\mathbb{Z}_p))\\
&= \sum_S \mathrm{Ind}_G^{G_\nu} h_{U\cap G_\nu}(\mathbb{A}_\nu) - h_U(\Lambda^{r_2+r_1'}) + h_U(\bigoplus_{S'_\infty} \mathrm{Ind}_G^{G_\nu}(\mathbb{Z}_p))\\
&= \sum_{S^{cd}} \mathrm{Ind}_G^{G_\nu} h_U(\mathbb{Z}_p) + h_U(\Lambda^{r_2+r_1-r_1'}) + h_U(\bigoplus_{S'_\infty} \mathrm{Ind}_G^{G_\nu}(\mathbb{Z}_p)).
\end{aligned}$$

Therefore, if $\mathrm{E}^0\mathrm{E}^0(\mathbb{E}_S)$ is projective, it follows that

$$\mathrm{E}^0\mathrm{E}^0(\mathbb{E}_S) \cong \Lambda^{r_2+r_1-r_1'} \oplus \bigoplus_{S^{cd}\cup S'_\infty} \mathrm{Ind}_G^{G_\nu}(\mathbb{Z}_p).$$

This proves (ii) while (i) follows easily applying proposition 1.5.16. $\square$

3.2 Selmer groups of abelian varieties

In this section let k be a number field, $\mathcal{A}$ a g-dimensional abelian variety defined over k and p a fixed rational *odd* prime number. For a non-empty, finite set S of places of k containing the places S_{bad} of bad reduction of $\mathcal{A}$, the places S_p lying over p and the places S_∞ at infinity we write $\mathrm{H}^i(G_S(k), \mathcal{A})$, respectively $\mathrm{H}^i(k_\nu, \mathcal{A})$, for the cohomology groups $\mathrm{H}^i(G_S(k), \mathcal{A}(k_S))$, respectively $\mathrm{H}^i(G_\nu, \mathcal{A}(\bar{k}_\nu))$, where $G_S(k)$ denotes the Galois group of the maximal outside S unramified extension of k, $\bar{k}_\nu$ the algebraic closure of the completion of k at ν and G_ν the corresponding decomposition group. The (p^m)-)Selmer group $\mathrm{Sel}(\mathcal{A}, k, p^m)$ and the Tate-Shafarevich group $\text{Ш}(\mathcal{A}, k, p^m)$[2] fit by definition into the following commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{A}(k)/p^m \arrow[r] \arrow[d, equal] & \mathrm{Sel}(\mathcal{A}, k, p^m) \arrow[r] \arrow[d] & \text{Ш}(\mathcal{A}, k, p^m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}(k)/p^m \arrow[r] & \mathrm{H}^1(G_S(k), {}_{p^m}\mathcal{A}) \arrow[r] \arrow[d] & {}_{p^m}\mathrm{H}^1(G_S(k), \mathcal{A}) \arrow[r] \arrow[d] & 0 \\
 & & \bigoplus_{S(k)} \mathrm{H}^1(k_\nu, \mathcal{A})(p) \arrow[r, equal] & \bigoplus_{S(k)} \mathrm{H}^1(k_\nu, \mathcal{A})(p). &
\end{tikzcd}$$

If k_∞ is an infinite Galois extension of k with Galois group $G = G(k_\infty/k,)$ we get the following commutative exact diagram by passing to the direct limit with respect to m and finite subextensions k' of k_∞/k:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{A}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d, equal] & \mathrm{Sel}(\mathcal{A}, k_\infty, p^\infty) \arrow[r] \arrow[d] & \text{Ш}(\mathcal{A}, k_\infty, p^\infty) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & \mathrm{H}^1(G_S(k_\infty), \mathcal{A}(p)) \arrow[r] \arrow[d] & \mathrm{H}^1(G_S(k_\infty), \mathcal{A})(p) \arrow[r] \arrow[d] & 0 \\
 & & \bigoplus_{S(k)} \mathrm{Coind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \mathcal{A})(p) \arrow[r, equal] & \bigoplus_{S(k)} \mathrm{Coind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \mathcal{A})(p). &
\end{tikzcd}$$

[2] It is not difficult to show that both the Selmer and Tate-Shafarevich group are independent of S.

Note that $\varinjlim_{k'} \bigoplus_{S(k')} \mathrm{H}^1(k'_\nu, \mathcal{A})(p) \cong \bigoplus_{S(k)} \mathrm{Coind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \mathcal{A})(p)$. Alternatively, we can pass to the inverse limits and we will get the following commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\mathcal{A}}_{k_\infty} \arrow[r] \arrow[d, equal] & \widehat{Sel}(k_\infty, \mathcal{A}) \arrow[r] \arrow[d] & \varprojlim_{k',m} Ш(\mathcal{A}, k', p^m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\mathcal{A}}_{k_\infty} \arrow[r] & \varprojlim_{k'} \mathrm{H}^1(G_S(k'), T_p\mathcal{A}) \arrow[r] \arrow[d] & \varprojlim_{k'} T_p\mathrm{H}^1(G_S(k'), \mathcal{A}) \arrow[r] \arrow[d] & 0 \\
 & & \varprojlim_{k'} \bigoplus_{S(k')} T_p\mathrm{H}^1(k'_\nu, \mathcal{A}) \arrow[r, equal] & \varprojlim_{k'} \bigoplus_{S(k')} T_p\mathrm{H}^1(k'_\nu, \mathcal{A}). &
\end{tikzcd}$$

where $\widehat{\mathcal{A}}_{k_\infty} := \varprojlim_{k',m} \mathcal{A}(k')/p^m$ and $\widehat{Sel}(k_\infty, \mathcal{A}) := \varprojlim_{k',m} \mathrm{Sel}(k', \mathcal{A}, p^m)$ (The limits are taken with respect to corestriction maps and multiplication by p).

Henceforth we will drop the p from the notation of the Selmer group: $\mathrm{Sel}(\mathcal{A}, k_\infty) := \mathrm{Sel}(\mathcal{A}, k_\infty, p^\infty)$. Furthermore, we shall use the following notation for the local-global modules

$$\begin{aligned}
\mathbb{U}_{S,\mathcal{A}} &:= \bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \mathcal{A})(p)^\vee, \\
\mathbb{A}_{S,\mathcal{A}} &:= \bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \mathcal{A}(p))^\vee, \\
\mathbb{T}_{S,\mathcal{A}} &:= \bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu} (\mathcal{A}(k_{\infty,\nu}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee.
\end{aligned}$$

As a consequence of the long exact sequence of the Tate-Poitou duality theorem we have the following (compact) analogue of proposition 3.1.3, where we shall write $\mathcal{A}^\vee$ for the dual abelian variety of $\mathcal{A}$ and $Ш^1_S(k_\infty, \mathcal{A}(p))$ for the kernel of the localization map

$$\mathrm{H}^1(G_S(k_\infty), \mathcal{A}(p)) \to \bigoplus_{S(k)} \mathrm{Coind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \mathcal{A}(p)).$$

Proposition 3.2.1. *Let $k_\infty|k$ be a p-adic Lie extension with Galois group G. Then, there are the following exact commutative diagrams of $\Lambda = \Lambda(G)$-modules*

(i)

$$\begin{tikzcd}
0 & 0 & \\
\mathrm{Sel}(\mathcal{A}, k_\infty)^\vee \arrow[u] \arrow[r, two heads] & Ш^1_S(k_\infty, \mathcal{A}(p))^\vee \arrow[u] & \\
\mathrm{H}^1(G_S(k_\infty), \mathcal{A}(p))^\vee \arrow[u] \arrow[r, equal] & \mathrm{H}^1(G_S(k_\infty), \mathcal{A}(p))^\vee \arrow[u] & \\
\mathbb{U}_{S,\mathcal{A}} \arrow[u] \arrow[r, hook] & \mathbb{A}_{S,\mathcal{A}} \arrow[u] \arrow[r, two heads] & \mathbb{T}_{S,\mathcal{A}} \\
\widehat{Sel}(k_\infty, \mathcal{A}^\vee) \arrow[u] \arrow[r, hook] & \varprojlim_{k'} \mathrm{H}^1(G_S(k'), T_p(\mathcal{A}^\vee)) \arrow[u] \arrow[r] & \mathbb{T}_{S,\mathcal{A}}, \arrow[u, equal] \\
\mathrm{H}^2(G_S(k_\infty), \mathcal{A}(p))^\vee \arrow[u] \arrow[r, equal] & \mathrm{H}^2(G_S(k_\infty), \mathcal{A}(p))^\vee \arrow[u] & \\
0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

(ii)

$$0 \longrightarrow \widehat{Sel}(k_\infty, \mathcal{A}^\vee) \longrightarrow \varprojlim_{k'} \mathrm{H}^1(G_S(k'), T_p\mathcal{A}^\vee) \longrightarrow \mathbb{T}_{S,\mathcal{A}} \longrightarrow$$
$$\mathrm{Sel}(\mathcal{A}, k_\infty)^\vee \longrightarrow Ш^1_S(k_\infty, \mathcal{A}(p))^\vee \longrightarrow 0,$$

(iii)

$$0 \to Ш^1_S(k_\infty, \mathcal{A}(p))^\vee \to Z_{S,\mathcal{A}^\vee(p)} \to \bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu}(\mathcal{A}(k_{\infty,\nu})(p))^\vee \to \mathcal{A}(k_\infty)(p)^\vee \to 0.$$

For the proof, just note that by virtue of local Tate duality ([44, Cor.3.4]), the Weil-pairing and 3.0.2,

(i) $\mathrm{H}^1(k_{\infty,\nu}, \mathcal{A})(p)^\vee \cong \widehat{(\mathcal{A}^\vee)}_{\infty,\nu} := \varprojlim_{k',m} \mathcal{A}^\vee(k'_\nu)/p^m,$

(ii) $Z_{S,\mathcal{A}^\vee(p)} \cong \varprojlim_{k'} \mathrm{H}^2(G_S(k'), T_p(\mathcal{A}^\vee))$,

(iii) $(\mathcal{A}(k_{\infty,\nu}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \varprojlim_{k'} T_p\mathrm{H}^1(k'_\nu, \mathcal{A}^\vee)$ and

(iv) $\mathrm{H}^1(k_{\infty,\nu}, \mathcal{A}(p))^\vee \cong \varprojlim_{k'} \mathrm{H}^1(k'_\nu, T_p(\mathcal{A}^\vee))$

hold.

By a well-known theorem of Mattuck, we have an isomorphism

$$\mathcal{A}(k'_\nu) \cong \mathbb{Z}_l^{g[k'_\nu:\mathbb{Q}_l]} \times \text{(a finite group)},$$

for any finite extension k'_ν of $\mathbb{Q}_l$. Recall that g denotes the dimension of the abelian variety $\mathcal{A}$. Clearly

$$\mathcal{A}(k'_\nu) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p = 0$$

for all $l \neq p$ and $\nu \mid l$, i.e.

$$\mathrm{H}^1(k'_\nu, \mathcal{A})(p) \cong \mathrm{H}^1(k'_\nu, \mathcal{A}(p)),$$

respectively

$$\mathrm{H}^1(k'_{\infty,\nu}, \mathcal{A})(p) \cong \mathrm{H}^1(k'_{\infty,\nu}, \mathcal{A}(p)),$$

in this case. On the other hand, Coates and Greenberg proved that for primes $\nu \mid p$ with good reduction

$$\mathrm{H}^1(k_{\infty,\nu}, \mathcal{A})(p) \cong \mathrm{H}^1(k_{\infty,\nu}, \widetilde{\mathcal{A}}(p))$$

holds, if k_∞ is a deeply ramified, where $\widetilde{\mathcal{A}}$ denotes the reduction of $\mathcal{A}$ (see [9, Prop. 4.8]). We recall that an algebraic extension k of $\mathbb{Q}_p$ is called *deeply ramified* if $\mathrm{H}^1(k, \overline{\mathfrak{m}})$ vanishes, where $\overline{\mathfrak{m}}$ is the maximal ideal of the ring of integers of an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$; see [9, p. 143] for equivalent conditions and for the following statement (loc. cit. thm. 2.13): A field k_∞ which is a p-adic Lie extension of a finite extension k of $\mathbb{Q}_p$ is deeply ramified if the inertial subgroup of $G(k_\infty/k)$ is infinite.
For arbitrary reduction at $\nu \mid p$, the same result as above holds, if one replaces $\widetilde{\mathcal{A}}_{p^\infty}$ by the quotient $\mathcal{A}(p)/\mathcal{F}_{\mathcal{A}}(\overline{\mathfrak{m}})(p)$, where $\mathcal{F}_{\mathcal{A}}$ denotes the formal group associated with the Neron model of $\mathcal{A}$ over a possibly finite extension of k_ν, such that the Neron model has semi-stable reduction. Taking these facts into account, we get the following description for $\mathbb{U}_{S,\mathcal{A}}$, where $T(k_{\infty,\nu}/k_\nu)$ denotes the inertia subgroup of G_ν.

Proposition 3.2.2. *(cf.* [50, lemma 5.4]*) Assume that* $\dim(T(k_{\infty,\nu}/k_\nu) \geq 1$ *for all* $\nu \in S_p$*. Then there is an isomorphism of* Λ*-modules*

$$\mathbb{U}_{S,\mathcal{A}} \cong \bigoplus_{S_p(k)} \mathrm{Ind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \widetilde{\mathcal{A}}(p))^\vee \oplus \bigoplus_{S_f \setminus S_p(k)} \mathrm{Ind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \mathcal{A}(p))^\vee.$$

In particular, if $\dim(G_\nu) \geq 2$ *for all* $\nu \in S_f$*, then*

$$\mathbb{U}_{S,\mathcal{A}} \cong \bigoplus_{S_p(k)} \mathrm{Ind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \widetilde{\mathcal{A}}(p))^\vee$$

and $\mathbb{U}_{S,\mathcal{A}}$ *is* Λ*-torsion-free.*

Proof. The first assertion has been explained above while the second statement follows from the local calculations 2.1.4, 2.1.5 with respect to the p-adic representations $A = \widetilde{\mathcal{A}}(p)$ respectively $A = \mathcal{A}(p)$ and the comment before 2.1.5. $\square$

Before going on we would like to recall some well-known facts about abelian varieties:

Remark 3.2.3. (i) $\mathrm{rk}_{\mathbb{Z}_p}(\mathcal{A}(p)^\vee) = 2g$, where g denotes the dimension of $\mathcal{A}$.
(ii) There exists always an isogeny from $\mathcal{A}$ to its dual $\mathcal{A}^\vee$, by which the Weil-pairing induces a non-degenerate skew-symmetric pairing on the Tate-module $T_p\mathcal{A}$ of $\mathcal{A}$. If $\mathcal{A} = E$ is an elliptic curve this isogeny can be chosen as a canonical isomorphism between E and $E^\vee$. Again for an arbitrary abelian variety it follows that $k(\mu_{p^\infty}) \subseteq k(\mathcal{A}(p)) = k(\mathcal{A}^\vee(p))$ (see [55, §0 lem. 7]).

Theorem 3.2.4. *Assume that* $\mathrm{H}^2(G_S(k_\infty), (\mathcal{A}^\vee)(p)) = 0$*. If* $\dim(G_\nu) \geq 2$ *for all* $\nu \in S_f$*, then*

$$Ш^1_S(k_\infty, \mathcal{A}(p))^\vee \sim \mathrm{E}^1(Y_{S,\mathcal{A}^\vee(p)}) \sim \mathrm{E}^1(\mathrm{tor}_\Lambda Y_{S,\mathcal{A}^\vee(p)}) \cong \mathrm{E}^1(\mathrm{tor}_\Lambda X_{S,\mathcal{A}^\vee(p)}).$$

If, in addition, $G \cong \mathbb{Z}_p^r$, $r \geq 2$*, then the following holds:*

$$Ш^1_S(k_\infty, \mathcal{A}(p))^\vee \sim (\mathrm{tor}_\Lambda X_{S,\mathcal{A}^\vee(p)})^\circ,$$

where $^\circ$ *means that the* G *acts via the involution* $g \mapsto g^{-1}$*.*

Proof. The first condition implies $Z_{S,\mathcal{A}^\vee(p)} \cong \mathrm{E}^1(Y_{S,\mathcal{A}^\vee(p)})$ while the other condition grants that $\bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu}(\mathcal{A}(k_{\infty,\nu})(p))^\vee$ is pseudo-null because $\mathcal{A}(k_{\infty,\nu})(p)^\vee$ is a finitely generated (free) $\mathbb{Z}_p$-module. Now everything follows as in 3.1.5 using here prop. 3.2.1. $\square$

Corollary 3.2.5. *Let $\mathcal{A}$ be an abelian variety over k with good* supersingular *reduction, i.e. $\widetilde{\mathcal{A}_{k_\nu}}(p) = 0$, at all places ν dividing p. Set $k_\infty = k(\mathcal{A}(p))$ and assume that $G(k_\infty/k)$ is a pro-p-group without any p-torsion. Then, for $\Sigma_{bad} := S_{bad} \cup S_p \cup S_\infty$ the following holds:*

$$X_{cs}[\mathcal{A}^\vee(p)] \cong \text{Ш}^1_{\Sigma_{bad}}(k_\infty, \mathcal{A}^\vee(p))^\vee \sim \mathrm{E}^1(\mathrm{tor}_\Lambda \mathrm{Sel}(\mathcal{A}, k_\infty)^\vee).$$

If, in addition, $\mathcal{A}$ has CM, then the following holds

$$X_{cs}[\mathcal{A}^\vee(p)] \sim (\mathrm{tor}_\Lambda \mathrm{Sel}(\mathcal{A}, k_\infty)^\vee)^\circ.$$

Therewith, in the case of an elliptic curve with CM, we reobtain a theorem of P. Billot [3, 3.23]. Over a $\mathbb{Z}_p$-extension an analogous statement was proved by K. Wingberg [63, cor. 2.5].

Proof. First note that by the Néron-Ogg-Shafarevich criterion the sets of bad reduction of $\mathcal{A}$ and its dual $\mathcal{A}^\vee$ coincide. Therefore, it suffices to prove that $\dim(G_\nu) \geq 2$ for all $\nu \in S_{bad} \cup S_p$ because then the theorem applies to $\mathcal{A}^\vee$ and proposition 3.2.2 shows that $\mathbb{U}_{S,\mathcal{A}} = 0$, i.e. $X_{S,\mathcal{A}(p)} \cong \mathrm{Sel}(\mathcal{A}, k_\infty)^\vee$.
So, let ν be either in S_p or in S_{bad}. Since $k_\nu(\mathcal{A}(p))$ contains $k_\nu(\mu_{p^\infty})$, we only have to show that $G(k_\nu(\mathcal{A}(p))/k_\nu(\mu_{p^\infty}))$ is not trivial because then it automatically has to be infinite as $G_\nu \subseteq G$ has no finite subgroup by assumption.
If $\nu|p$, by a theorem of Imai[3] [30] $\mathcal{A}(k(\mu_{p^\infty})$ is finite and thus $k_\nu(\mathcal{A}(p)) \neq k_\nu(\mu_{p^\infty})$.
If $\nu \in S_{bad}$, then the Néron-Ogg-Shafarevich criterion implies that $G(k_\nu(\mathcal{A}(p))/k_\nu(\mu_{p^\infty})) = T(k_\nu(\mathcal{A}(p))/k_\nu)$ is non-trivial. □

By remark 3.0.4 and 3.2.3 the conditions of theorem 3.0.3 are fulfilled for the p-torsion points $\mathcal{A}(p)$ and its trivializing extension of k, i.e. the extension which is obtained by adjoining the p-torsion points of $\mathcal{A}$:

Theorem 3.2.6. *Let $k_\infty = k(\mathcal{A}(p))$ and assume that G does not have any p-torsion. Then $\mathrm{H}^1(G_S(k_\infty), \mathcal{A}(p))^\vee$ has no non-zero pseudo-null submodule.*

Recall that G does not have any p-torsion if $p \geq 2\dim(\mathcal{A}) + 2$. Otherwise one only has to replace k by a finite extension inside k_∞.

For the convenience of the reader we recall Ochi's results on the ranks in the situation of abelian varieties. Then, theorem 3.0.5 yields

Proposition 3.2.7. *(Ochi [49, 5.11]) Assume in the above situation that G is pro-p and such that Λ is an integral domain. Then*

$$\mathrm{rk}_\Lambda \mathrm{H}^1(G_S(k_\infty), \mathcal{A}(p))^\vee = g[k : \mathbb{Q}].$$

[3] I owe to John Coates the idea to use Imai's theorem here.

The next result should also be compared to proposition 6.1 of S. Howson's PhD-thesis [29] where she proved (i)(a) by different methods.

Proposition 3.2.8. *(cf. Ochi [49, 5.7]) Let k be a finite extension of $\mathbb{Q}_\ell$, $\mathcal{A}$ an abelian variety over k of dimension g and $k_\infty = k(A(p))$. Assume that $G = \mathrm{Gal}(k_\infty/k)$ is pro-p and that $\mathcal{A}$ has good reduction. Let r denote the $\mathbb{Z}_p$-rank of $\widetilde{\mathcal{A}}(p)^\vee$. Then,*

(i) *if $\ell = p$,*

  (a) $\mathrm{rk}_\Lambda \mathrm{H}^1(k_\infty, \mathcal{A}(p^\infty))^\vee = 2g[k : \mathbb{Q}_p]$,

  (b) $\mathrm{rk}_\Lambda \mathrm{H}^1(k_\infty, \mathcal{A})(p)^\vee = r[k : \mathbb{Q}_p]$,

  (c) $\mathrm{rk}_\Lambda (\mathcal{A}(k_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee = (2g - r)[k : \mathbb{Q}_p]$,

(ii) *if $\ell \neq p$, $\mathrm{H}^1(k_\infty, \mathcal{A}(p^\infty))^\vee$ and $\mathrm{H}^1(k_\infty, \mathcal{A})(p)^\vee$ are Λ-torsion.*

Proof. The first two statements of (i) and (ii) are just 2.1.6 recalling the isomorphism $\mathrm{H}^1(k_{\infty,\nu}, \mathcal{A})(p) \cong \mathrm{H}^1(k_{\infty,\nu}, \widetilde{\mathcal{A}}(p))$ due to Coates-Greenberg. Now, (i)(c) results from Kummer theory. □

For the rank-description of the local-global modules, i.e. those global modules which are "induced from local modules," suppose that $\mathcal{A}$ has good reduction at each $\nu | p$. Let r_ν be the p-rank of the reduction of (a Neron model of) $\mathcal{A}_{k_\nu}$, i.e. the $\mathbb{Z}_p$-rank of $\widetilde{\mathcal{A}_{k_\nu}}(p)^\vee$, where $\nu \in S_p(k)$, and define

$$\begin{aligned} \alpha_\nu(\mathcal{A}/k) &= r_\nu[k_\nu : \mathbb{Q}_p], \\ \alpha_p(\mathcal{A}/k) &= \sum_{\nu \in S_p(k)} \alpha_\nu(\mathcal{A}/k), \end{aligned}$$

$$\begin{aligned} \beta_\nu(\mathcal{A}/k) &= (g - r_\nu)[k_\nu : \mathbb{Q}_p], \\ \beta_p(\mathcal{A}/k) &= \sum_{\nu \in S_p(k)} \beta_\nu(\mathcal{A}/k), \end{aligned}$$

and

$$\begin{aligned} \gamma_\nu(\mathcal{A}/k) &= (2g - r_\nu)[k_\nu : \mathbb{Q}_p] = 2\beta_\nu(\mathcal{A}/k) + \alpha_\nu(\mathcal{A}/k), \\ \gamma_p(\mathcal{A}/k) &= \sum_{\nu \in S_p(k)} \gamma_\nu(\mathcal{A}/k) = 2\beta_p(\mathcal{A}/k) + \alpha_p(\mathcal{A}/k). \end{aligned}$$

Similar results as in the following proposition were also obtained by S. Howson [29, 5.30,6.5-6.9,6.13-6.14,7.3].

Proposition 3.2.9. *(cf. Ochi* [49, 5.12]*) Assume in the above situation that G is pro-p and such that Λ is an integral domain. Then*

(i) $\mathrm{rk}_\Lambda \mathbb{A}_{S,\mathcal{A}} = 2g[k:\mathbb{Q}] = 4r_2(k)g = 2(\alpha_p(\mathcal{A}/k) + \beta_p(\mathcal{A}/k))$.

(ii) $\mathrm{rk}_\Lambda \mathbb{U}_{S,\mathcal{A}} = \alpha_p(\mathcal{A}/k)$.

(iii) $\mathrm{rk}_\Lambda \mathbb{T}_{S,\mathcal{A}} = \gamma_p(\mathcal{A}/k)$.

(iv) $\mathrm{rk}_\Lambda \mathrm{Sel}(\mathcal{A}, k_\infty)^\vee - \mathrm{rk}_\Lambda \widehat{Sel}(k_\infty, \mathcal{A}^\vee) = \beta_p(\mathcal{A}/k)$.

(v) $\mathrm{rk}_\Lambda \varprojlim_{k'} \mathrm{H}^1(G_S(k'), T_p(\mathcal{A}^\vee)) = 2r_2(k)g = \alpha_p(\mathcal{A}/k) + \beta_p(\mathcal{A}/k)$.

(vi) $Z_{S,\mathcal{A}^\vee(p)} \cong \varprojlim_{k'} \mathrm{H}^2(G_S(k'), T_p(\mathcal{A}^\vee))$ *is Λ-torsion.*

If $\dim(G_\nu) \geq 2$ *for all* $\nu \in S_f$*, then* $\mathbb{A}_{S,\mathcal{A}}, \mathbb{U}_{S,\mathcal{A}}$ *and* $\varprojlim_{k'} \mathrm{H}^1(G_S(k'), T_p(\mathcal{A}^\vee))$ *are Λ-torsion-free.*

Furthermore, in the case of elliptic curves S. Howson proved the following result.

Proposition 3.2.10. *(Howson* [29, 6.14-15]*) Let E be an elliptic curve over k without complex multiplication and with good* ordinary *reduction at all places over p. Assume that $G = G(k(E(p)/K)$ is pro-p without any p-torsion. Then*

$$\mathbb{T}_{S,E} \cong \mathbb{A}_{S,\widetilde{E}} \cong \bigoplus_{S_f(k)} \mathrm{Ind}_G^{G_\nu} \varprojlim_{k'} \mathrm{H}^1(k'_\nu, T_p(\widetilde{E}))$$

and these modules are $\Lambda(G)$-torsion-free. Furthermore, there is an isomorphism

$$\mathbb{U}_{S,E} \cong \mathrm{E}^0(\mathbb{T}_{S,E}).$$

3.2.1 Elliptic curves without CM

The following results on the Selmer group of an elliptic curve E over a number field k without complex multiplication (CM) are taken from a joint paper with Y. Ochi [50]. They demonstrate once more the efficiency of the methods developed in this thesis. First we will recall what is conjectured and known about the Λ-rank of the Pontryagin dual of the Selmer group. At the end we give an example where the Pontryagin dual of the Selmer group has a positive μ-invariant.

In this section we assume that $p \geq 5$ and set $k_\infty := k(E(p))$. Recall that under these condition $G = G(k_\infty/k)$ does not have any p-torsion. Furthermore, we assume that E has *good reduction at all primes over p*. By proposition 3.2.7 and 3.2.9 (resp. thm 2.5 in [11], see also [29]) we know that

$$\beta_p(E/k) \leq \mathrm{rk}_\Lambda \mathrm{Sel}(E, k_\infty)^\vee \leq [k : \mathbb{Q}]$$

(here, but only for rank considerations, we assume that G is pro-p such that Λ is integral). Harris conjectured in ([25]) that $\mathrm{Sel}(E, k_\infty)^\vee$ is Λ-torsion if E has good ordinary reduction at all primes above p. Coates and Howson ([11, Conj. 2.4]) generalized his conjecture to a more precise conjecture concerning the Λ-rank of the Selmer group for arbitrary (but still good) reduction types.

Conjecture 3.2.11. *(Coates-Howson)* $\mathrm{rk}_\Lambda \mathrm{Sel}(E, k_\infty)^\vee = \beta_p(E/k)$.

Remark 3.2.12. Assume that G is pro-p. Due to results of Serre [57, appendix to chapter IV] and Serre-Tate [59] we have the following descriptions of G_ν (cf. [11, lemma 5.1]): first, it does not occur that E has additive reduction at some place, because G does not have any finite subgroup by our assumptions. Hence, $\dim(G_\nu) = 2$ for any $\nu \in S_{bad}$. At places $\nu \in S_p$ the elliptic curve E has either supersingular reduction, i.e. $\mathrm{H}^1(k_{\infty,\nu}, \widetilde{E}(p)) = 0$, or $\dim(G_\nu) = 3$. Together with 3.2.2 this means that for $\Sigma_{bad} = S_{bad} \cup S_p \cup S_\infty$ we get an isomorphism

$$\mathbb{U}_{\Sigma_{bad},E} \cong \bigoplus_{S_p^{ord}(k)} \mathrm{Ind}_G^{G_\nu} \mathrm{H}^1(k_{\infty,\nu}, \widetilde{\mathcal{A}}(p))^\vee,$$

where $S_p^{ord}(k) \subseteq S_p(k)$ denotes the subset of places over p at which E has good ordinary reduction. Furthermore, $\mathbb{U}_{\Sigma_{bad},E}$ is reflexive by 2.1.5.

Proposition 3.2.13. *The following assertions are equivalent:*

(i) $\mathrm{rk}_\Lambda \mathrm{Sel}(E, k_\infty)^\vee = \beta_p(E/k)$.

(ii) $\widehat{Sel}(k_\infty, E) = 0$.

(iii) *The sequence*

$$0 \longrightarrow \mathbb{U}_{S,E} \longrightarrow \mathrm{H}^1(G_S(k_\infty), E(p))^\vee \longrightarrow \mathrm{Sel}(E, k_\infty)^\vee \longrightarrow 0.$$

is exact.

Proof. According to proposition 3.2.9 the first assertion is equivalent to the vanishing of $\widehat{Sel}(k_\infty, E)$, because this module is Λ-torsion-free as a submodule of $\mathbb{U}_{\Sigma_{bad},E}$. But the latter module is torsion-free by proposition 3.2.2 and the previous remark. On the other hand, the vanishing of $\widehat{Sel}(k_\infty, E)$ is equivalent to assertion (iii) by proposition 3.2.1. □

Theorem 3.2.14. *Assume that the conjecture of Coates and Howson holds. Then* $\mathrm{Sel}(E, k_\infty)^\vee$ *has no non-zero pseudo-null* Λ*-submodule.*

In particular, the theorem applies if E has supersingular reduction at all places above p, because in this case $\mathbb{U}_{\Sigma_{bad},E} = 0$.

Let us assume that the conjecture 3.2.11 holds, i.e. the exactness of the above short exact sequence (in (iii) of prop. 3.2.13). In order to prove this theorem it is sufficient to show that $\mathrm{E}^i\mathrm{E}^i(\mathrm{Sel}(E, k_\infty)^\vee) = 0$ for all $i \geq 2$ by proposition 1.5.5, (i),(c). Since we can control the projective dimensions of the modules $\mathbb{U}_{S,E}$ and $X_{S,E(p)} = \mathrm{H}^1(G_S(k_\infty), E(p))^\vee$ by the diagram 1.8, it is easy to see that

$$\mathrm{pd}_\Lambda(\mathrm{Sel}(E, k_\infty)^\vee) = \mathrm{pd}_\Lambda(X_{S,E(p)}) = \mathrm{pd}_\Lambda(\mathbb{U}_{S,E}) + 1 = 2,$$

i.e. the only outstanding case to show is $i = 2$. But the vanishing of $\mathrm{E}^2\mathrm{E}^2(\mathrm{Sel}(E, k_\infty)^\vee)$ can be shown by evaluating the long exact E^i-sequence associated with the above short exact sequence using theorem 3.2.6 (for details see [50, section 5]).

Assume now that G is pro-p. Since the cyclotomic $\mathbb{Z}_p$-extension $k_{cycl} = k(\mu_{p^\infty})$ is contained in k_∞ we are able to compare the above Selmer group over k_∞ to the Selmer group $\mathrm{Sel}(E, k_{cycl})$, the Pontryagin dual of which is a finitely generated $\Lambda(\Gamma)$-module. Here we write Γ for the Galois group $G(k_{cycl}/k)$ and set $H := G(k_\infty/k_{cycl})$. An observation of Coates and Howson [11] is that if $\mathrm{Sel}(E, k_{cycl})^\vee$ is finitely generated as $\mathbb{Z}_p$-module (i.e. $\Lambda(\Gamma)$-torsion and with zero μ-invariant), then $\mathrm{Sel}(E, k_\infty)^\vee$ is finitely generated over $\Lambda(H)$. The $\Lambda(H)$-torsion submodule $N := \mathrm{tor}_{\Lambda(H)}\mathrm{Sel}(E, k_\infty)^\vee$ does not change under a finite base change of k inside k_{cycl}. So we may assume for a moment that $G \cong H \times \Gamma$ conferring our observation below remark 1.5.19. But then, due to the fact that the map det induces the cyclotomic character on G ([57, p. I-4]), the lemma 1.5.18 tells us that N is a pseudo-null $\Lambda(G)$-module, i.e. zero by theorem 3.2.14.

Therefore we can answer a question of John Coates positively as follows:

Theorem 3.2.15. *Assume that G is pro-p and that* $\mathrm{Sel}(E, k_{cycl})^\vee$ *is a finitely generated* $\mathbb{Z}_p$*-module. Then* $\mathrm{Sel}(E, k_\infty)^\vee$ *is a finitely generated* $\Lambda(H)$*-module, whose* $\Lambda(H)$*-torsion submodule is zero.*

We conclude this section with an example of an elliptic curve where the Pontryagin dual of its Selmer group has a positive μ-invariant[4] . Let $p = 5$

[4]We would like to thank J. Coates for drawing our attention to the elliptic curve $X_0(11)$ and Greenberg's calculations in [23].

and consider the elliptic curve $E = X_0(11)$ without CM and its 5-Selmer group $\mathrm{Sel}(E, k_\infty)$ over $k_\infty = k(E(p))$, where $k := \mathbb{Q}(\mu_5) = \mathbb{Q}(E[5])$. The last equality holds because $E[5] \cong \mu_5 \times \mathbb{Z}/5$ as a $G_{\mathbb{Q}}$-module (cf. [23, p. 120]). Since $X_0(11)$ is isogenous to $X_1(11)$ it is easy to see that $\mathrm{Sel}(E, k_\infty)^\vee$ is a $\mathbb{Z}_5[\![G]\!]$-torsion module (see [11, §7]). Now it is known that μ_5 lies in the kernel of the reduction map at 5

$$E(5) \to \widetilde{E}(5),$$

see [23, p. 121]. It follows that the $\Lambda(G)$-submodule

$$\mathrm{Hom}(G_S(k_\infty), \mu_5) \subseteq \mathrm{H}^1(G_S(k_\infty), E(5))$$

is contained in $\mathrm{Sel}(E, k_\infty)$ because it satisfies the only local condition, at $\nu = 5$ (cf. 3.2.12). Therefore, it suffices to show that $\mu(X_S/5) > 0$:
Since $\mathrm{rk}_\Lambda X_S = 2$ there exists an exact sequence of Λ-modules

$$0 \longrightarrow \Lambda^2 \longrightarrow X_S \longrightarrow B \longrightarrow 0,$$

where B is a finitely generated Λ-torsion module. This induces an exact sequence

$$0 \longrightarrow {}_pX_S \longrightarrow {}_pB \longrightarrow (\Lambda/p)^2 \longrightarrow X_S/p \longrightarrow B/p \longrightarrow 0.$$

Since the vanishing of $\mu(X_S/p)$ would imply that $\mu(B/p) = \mu({}_pB) = 0$ by the additivity of μ for Λ-torsion-modules and using lemma 1.5.35, it would contradict the fact that $\mu((\Lambda/5)^2) = 2$. Hence

$$\mu(\mathrm{Sel}(E, k_\infty)^\vee) > 0.$$

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