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Structure Theorems for Certain Vector Valued Siegel Modular Forms of Degree Two

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Dedicated to my granny

Maria B. Wieber, née Erbacher,

who passed away on April 11th 2003.

ABSTRACT

The aim of this thesis is to develop two structure theorems for vector valued Siegel modular forms with respect to Igusa's subgroup $\Gamma_2\,[2,4]$, the multiplier system induced by the theta constants and the symmetric square of the standard representation $\rho_e \odot \rho_e : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{Sym}^2(\mathbb{C}^2)$.

The thesis rests on the well-known fact that every holomorphic tensor on the upper half-space $\mathbb{H}_2$ that is invariant under $\Gamma_2\,[2,4]$ is associated to a modular form. We define a space of meromorphic tensors that are holomorphic outside a divisor and become holomorphic after a pullback along a certain covering map. Afterwards, we show that modular forms of certain weights correspond to this particular space of tensors on the homogeneous space ${}_{\Gamma_2\,[2,4]}\backslash^{\mathbb{H}_2}$. As an immediate consequence of [Run93], this homogeneous space is a complex manifold and there is a 'nice' embedding into the projective space $\mathbb{P}^3\mathbb{C}$. Therefore, we shall describe modular forms of the aforementioned type and of certain weights as rational tensors with easily handable poles along ten quadrics. It can be shown that these modular forms are linear combinations of Rankin-Cohen brackets of the theta series of the second kind. We extend this result to arbitrary weights by a simple argument from algebraic geometry.

ZUSAMMENFASSUNG

Das Ziel dieser Doktorarbeit ist es zwei Struktur-Theoreme für vektorwertige Siegelsche Modulformen zu Igusas Untergruppe $\Gamma_2\,[2,4]$, dem durch die Theta-Reihen zweiter Art induziertem Multiplikatorsystem und dem symmetrischen Quadrat der Standard-Darstellung $\rho_e \odot \rho_e : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{Sym}^2(\mathbb{C}^2)$ zu entwickeln.

Die Arbeit beruht auf der wohlbekannten Tatsache, dass jeder holomorphe Tensor auf der oberen Halbebene $\mathbb{H}_2$, der unter $\Gamma_2\,[2,4]$ invariant ist, mit einer Modulform identifiziert werden kann. Man definiert einen Raum von meromorphen Tensoren, die holomorph außerhalb eines Divisors sind und deren Pullbacks entlang gewisser Überlagerungen holomorph werden. Die so spezifizierten Tensoren auf dem homogenen Raum ${}_{\Gamma_2\,[2,4]}\backslash^{\mathbb{H}_2}$ entsprechen gerade Modulformen von bestimmten Gewichten. Es folgt sofort aus [Run93], dass dieser homogene Raum eine analytische Mannigfaltigkeit ist und dass es eine Einbettung mit erstrebenswerten Eigenschaften in den projektiven Raum $\mathbb{P}^3\mathbb{C}$ gibt. Deshalb lassen sich die im ersten Abschnitt beschriebenen Modulformen mit passenden Gewichten als rationale Tensoren beschreiben. Diese Tensoren besitzen einfach handhabbare Pole entlang zehn Quadriken. Man kann zeigen, dass diese Modulformen Linearkombinationen von Rankin-Cohen-Klammern der Theta-Reihen zweiter Art sind. Mit einfachen Methoden der algebraischen Geometrie lässt sich dieses Resultat für beliebiges Gewicht verallgemeinern.

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Contents

Nomenclature

Symbols

$M\langle Z\rangle$	action of $\mathrm{Sp}(n,\mathbb{R})$ on $\mathbb{H}_n$, page 50
$A > B$	partial order on the set of matrices, page 49
$\lfloor x\rfloor$	floor function of the real number x, page 67
$B\,[A]$	conjugation with A, i.e. A^tBA, page 49
$[U,f]_p$	the holomorphic germ in p belonging to (U,f), page 34
$\left[\Gamma,\frac{r}{2},v\right]$	vector space of scalar valued modular forms of weight $r/2$ w.r.t. to Γ and v, page 53
$[\Gamma,k]$	vector space of scalar valued modular forms of weight k, page 80
$\left[\Gamma,\frac{r}{2},v\right]_0$	linear subspace of cusp forms in $\left[\Gamma,\frac{r}{2},v\right]$, page 56
$\left[\Gamma,\left(\frac{r}{2},\rho\right),v\right]$	vector space of vector valued modular forms of weight $r/2$ to the representation ρ and the mulitplier system v, page 53
$[\Gamma,(k,\rho)]$	$\left[\Gamma,\left(\frac{r}{2},\rho\right),v\right]$ with trivial v and $r=2k$, page 53
$\{f,g\}$	Rankin-Cohen bracket of the modular forms f and g, page 72
$\{f,g,h\}$	Rankin-Cohen 3-bracket of the modular forms f, g and h, page 76
(f)	principal divisor of a non-zero meromorphic function, page 46
Λ^kV	vector space of alternating tensors on V, page 28
$v_1\wedge\cdots\wedge v_k$	alternating tensor of the vectors $v_1,\ldots,v_k$, page 28
$\bigwedge dZ^{ij}$	lexicographic ordered wedge product over all dZ^{ij}on $\mathbb{H}_n$, page 53
V^*	dual space to a vector space V, page 26
ϕ^*	pullback along the function $\phi: M\to N$, $\Omega^{\otimes q}(M)\to\Omega^{\otimes q}(N)$, page 38
$M_1\otimes_R\cdots\otimes_R M_k$	tensor product of the modules $M_1,\ldots,M_k$ over R, page 26

$\omega \otimes \eta$	tensor product of two covariant tensors ω and η, page 37
$v_1 \otimes_R \cdots \otimes_R v_k$	tensor product of the vectors $v_1, \ldots, v_k$, page 26
$V_1 \otimes_K \cdots \otimes_K V_k$	tensor product of the vector spaces $V_1, \ldots, V_k$, page 27
$V^{\otimes n}$	n-th tensorial power of the K-vector space V, page 28
A^t	transposed matrix of A, page 30
A^{-t}	transposed inverse matrix of A, i.e. $A^{-t} = (A^{-1})^t$, page 30

Letters

$\mathbb{A}_\alpha$	α-th affine space, considered as an open of $\mathbb{P}^n\mathbb{C}$, page 33
$A(\Gamma_2\,[2,4], v_f)$	graded algebra of scalar modular forms with respect to $\Gamma_2\,[2,4]$ and the multiplier system v_f, page 56
$A^{int}(\Gamma_2\,[2,4])$	graded algebra of scalar modular forms with respect to $\Gamma_2\,[2,4]$ and the trivial multiplier system, page 57
$\mathrm{Bihol}(U)$	group of biholomorphic functions on U, page 31
Γ	generic group that is acting totally discontinuously as biholomorphic maps on $\mathcal{D} \subset \mathbb{C}^n$, page 57
$\overline{\Gamma}$	quotient of a group and its action's kernel on a domain, page 57
Γ_n	symplectic group for the integers, i.e. $\mathrm{Sp}(n, \mathbb{Z})$, page 50
$\Gamma_n\,[q, 2q]$	Igusa's group, page 51
$\Gamma_n\,[q]$	principal congruence subgroup, page 51
$\mathbb{C}_d[X^1, \ldots, X^n]$	vector space of polynomials homogeneous of degree d, page 25
$\operatorname{codim} Y$	codimension of an analytic variety Y in a complex manifold M, i.e. $\operatorname{codim} Y = \dim M - \dim Y$, page 43
Δ_n	submanifold of diagonal matrices in $\mathbb{H}_n$, page 49
$\mathcal{D}$	generic domain, in particular $\mathcal{D} \subset \mathbb{C}^n$, page 57
$\mathcal{D}_0$	generic domain without Γ's harmful points, i.e. $\mathcal{D} \setminus \mathrm{Harm}\,(\Gamma)$, page 58
df	1-form associated to the function f, page 36
$d\mathfrak{z}^i$	canonical 1-form on $\mathbb{A}_0$ induced by the coordinate $\mathfrak{z}^i$, page 40
Df	derivative of the function f, page 31

$DM\langle Z_0\rangle$	derivative of the function $Z \mapsto M\langle Z\rangle$ at the point Z_0, page 51
dZ^{ij}	canonical 1-form on $\mathbb{H}_n$, page 53
$\mathbb{E}^n$	unit polycylinder or polydisc in $\mathbb{C}^n$, page 44
$\operatorname{End}(V)$	algebra of endomorphisms of the vector space V, page 30
$\mathbb{F}_2$	field of order 2, page 55
$f_a(Z)$	theta series of the second kind, page 55
$\mathrm{GL}(V)$	general linear group of the vector space V, page 30
$\mathrm{GL}(n,\mathbb{C})$	general linear group of the vector space $\mathbb{C}^n$, i.e. $\mathrm{GL}(\mathbb{C}^n)$, page 30
$\mathrm{GL}(n,R)$	general linear group of a commutative unital ring R, page 50
${}_G\backslash^S$	orbit space of G's action on S, page 29
G_x	stabilizer subgroup, page 28
Gx	orbit of the point x under the group G, page 29
$\operatorname{Harm}(\Gamma)$	set of non-harmless points of a group Γ, page 58
$\mathbb{H}_n$	Siegel upper half-space, page 49
$\operatorname{Im} Z$	imaginary part of the complex matrix Z, page 49
I_n	n-dimensional identity matrix, page 50
$\mathcal{J}$	factor of automorphy $\mathcal{J} : \mathrm{Sp}(n,\mathbb{R}) \times \mathbb{H}_n \longrightarrow \mathrm{GL}(n,\mathbb{C})$, page 50
$\operatorname{Jac}(f,z_0)$	Jacobian matrix of the function f at z_0, page 35
J_n	almost complex structure on $\mathrm{M}(2n,\mathbb{R})$, page 50
$\mathfrak{m}$	characteristic of the theta series $\vartheta\,[\mathfrak{m}]$, page 55
$M\langle Z\rangle$	action of $\mathrm{Sp}(n,\mathbb{R})$ on $\mathbb{H}_n$, page 50
M^+	$\mathbb{C}[f_0,\ldots,f_3]$-module of Rankin-Cohen brackets $\{f_i,f_j\}$, page 73
M^-	$\mathbb{C}[f_0,\ldots,f_3]$-module of Rankin-Cohen 3-brackets $\{f_i,f_j,f_k\}$, page 77
$\mathcal{M}^+(\Gamma_2\,[2,4])$	$\mathbb{C}[f_0,\ldots,f_3]$-module of vector valued modular forms with respect to $\Gamma_2\,[2,4]$, the multiplier system v_f and $\rho_e\odot\rho_e$, page 73
$\mathcal{M}^-(\Gamma_2\,[2,4])$	$\mathbb{C}[f_0,\ldots,f_3]$-module of vector valued modular forms w. resp. to $\Gamma_2\,[2,4]$, the mult. system v_f, the character v_f^2, and $\rho_e\odot\rho_e$, page 77
$\mathrm{M}(n,R)$	R-module of $n\times n$ matrices with values in R , page 49

$\operatorname{ord}(f, Y, p)$	order of the singularity of f along Y at p, page 46
$\mathcal{O}(U, \mathbb{C}^m)$	vector space of holomorphic functions on U and values in $\mathbb{C}^m$, page 31
$\mathcal{O}(U)$	algebra of complex valued holomorphic functions on U, page 31
$\mathcal{O}(M, N)$	set of holomorphic functions between the manifolds M and N, page 33
$\mathcal{O}_{M,p}$	stalk of holomorphic functions at $p \in M$, page 34
$\Omega^{\otimes q}(M)$	vector space of holomorphic tensors, page 37
$\Lambda^p\Omega(M)$	vector space of alternating holomorphic tensors, page 37
$(\Lambda^n\Omega)^{\otimes k}(M)$	vector space of multi-canonical holomorphic tensors, page 37
$(\Omega^{\otimes q}(M))^S$	vector space of S-invariant holomorphic tensors, page 38
$\Omega^{\otimes k}(M, D)$	vector space of covering holomorphic tensors, page 47
$(\Lambda^n\Omega)^{\otimes k}(M, D)$	vector space of covering holomorphic tensors of type $(\Lambda^n\Omega)^{\otimes k}$, page 48
ω	generic meromorphic or holomorphic tensor, page 37
$P(f)$	pole locus of a meromorphic function f, page 40
$\mathbb{P}^n\mathbb{C}$	n-dimensional projective space, page 33
p_n^k	k-th n-dimensional standard element, page 44
$\mathcal{R}$	ramification divisor, page 59
$\operatorname{Ram}(p)$	ramification locus of a covering map p, page 44
Y_{reg}	regular locus of an analytic subvariety Y, page 42
ρ_e	standard representation of $\mathrm{GL}(n, \mathbb{C})$, $A \mapsto A$, page 30
ρ_e^*	dual representation of the std. rep. of $\mathrm{GL}(n, \mathbb{C})$, $A \mapsto A^{-t}$, page 30
$\rho_e \odot \rho_e$	symmetric square of the std. rep., $A \mapsto \{X \mapsto AXA^t\}$, page 30
$\rho_e^* \odot \rho_e^*$	symmetric square of the dual rep., $A \mapsto \{X \mapsto A^{-t}XA^{-1}\}$, page 30
$\mathrm{S}(\Gamma)$	set of fixed points of Γ, page 58
$\mathfrak{S}_n$	group of permutations on $\{1, \ldots, n\}$, page 27
$\operatorname{sgn}(\sigma)$	signature of the permutation σ, page 28
Y_{sing}	singular locus of an analytic subvariety Y, page 42

$\mathrm{SL}(n, R)$	special linear group of a commutative unital ring R, page 50
$\mathrm{Sp}(n, R)$	symplectic group of a commutative unital ring R, page 50
$\operatorname{supp} D$	support of the divisor D, page 43
$\mathrm{Sym}^2(\mathbb{C}^n)$	vector space of symmetric $n \times n$ matrices , page 49
A^t	transposed matrix of A, page 30
A^{-t}	transposed inverse matrix of A, i.e. $A^{-t} = (A^{-1})^t$, page 30
$\vartheta\,[\mathfrak{m}]$	theta series of the first kind, page 55
T_pM	tangent space of M at p, page 34
T_p^*M	cotangent space of M at p, the dual space of T_pM, page 35
v_Γ	multiplier system for a congruence subgroup Γ, page 52
v_f	multiplier system v_f induced by the theta series f_a, n.b. $v_f^4 = 1$, page 56
$\mathfrak{w}^j$	j-th affine coordinate on $\mathbb{A}_1$, a canonical open of $\mathbb{P}^n\mathbb{C}$, page 33
χ_5	Igusa's cusp form of weight 5 and multiple of 10 theta series, page 56
$Z(f)$	zero locus of the function f, page 25
$Z((f_i)_{i\in I})$	common zero locus of the functions $(f_i)_{i\in I}$, page 25
$\mathfrak{z}^i$	i-th affine coordinate on $\mathbb{A}_0$, a canonical open of $\mathbb{P}^n\mathbb{C}$, page 33
$d\mathfrak{z}^i$	canonical 1-form on $\mathbb{A}_0$ induced by the coordinate $\mathfrak{z}^i$, page 40

1 Introduction

A vector valued modular form with respect to a subgroup $\Gamma \subset \mathrm{Sp}(n, \mathbb{Z})$ of finite index is a holomorphic function $f : \mathbb{H}_n \to V$ that transforms under Γ as follows :

$$f(M\langle Z\rangle) = j(M, Z)f(Z).$$

Here j denotes a factor of automorphy on a finite dimensional vector space V, i.e. a map $j : \Gamma \times \mathbb{H}_n \to \mathrm{GL}(V)$ that is holomorphic in the second variable and satisfies the cocycle relation $j(MN, Z) = j(M, N\langle Z\rangle)j(N, Z)$.
The characters of Γ are easy examples of factors of automorphy. Furthermore, a given polynomial representation $\rho : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ induces the factor of automorphy $\rho((CZ + D))$. For integral r, the functions $\sqrt{det(CZ + D)}^{\,r}$ satisfy the cocyle relation up to ± 1. In order to compensate this error, they are multiplied with multiplier systems $v(M)$ of weight $r/2$.

In [Tsu83], Tsushima calculated the dimension of vector spaces of vector valued cusp forms by means of the Riemann-Roch-Hirzebruch-Theorem. In [Sat86], Satoh combined this result with the decomposition of the vector space of vector valued modular forms into the subspace of cusp forms and the subsapce of Eisenstein series, cf. [Ara83]. He obtained a structure theorem for certain vector spaces of vector valued modular forms with respect to the full modular group $\mathrm{Sp}(2, \mathbb{Z})$. We should also mention [Ibu12] and [Aok12] which used the same strategy.

In this thesis, we use a geometric method to get similar results for Igusa's group $\Gamma_2[2, 4]$ (cf. definition 4.8 on page 51) instead of the full modular group. We benefit from the fact that the Satake compactification of $\Gamma_2[2,4] \backslash \mathbb{H}_2$ is simply the 3 dimensional projective space $\mathbb{P}^3\mathbb{C}$. This is a consequence of a couple of basic results by Igusa [Igu64a, Going-down process, p.397] that was proven by Runge [Run93].

We shall investigate modular forms being linked to the symmetric square of the standard representation ρ_e[1] , i. e.

$$\rho_e \odot \rho_e : \mathrm{GL}(2, \mathbb{C}) \longrightarrow \mathrm{Aut}\left(\mathrm{Sym}^2(\mathbb{C}^2)\right).$$

[1] We follow the notations of [Ste64, p.12] and [Olv99, p.64].

Here $\mathrm{Sym}^2(\mathbb{C}^2)$ denotes the symmetric square of $\mathbb{C}^2$, which can be identified with the space of symmetric 2×2 matrices. Then, the representation is given by

$$A \longmapsto \{X \mapsto AXA^t\}.$$

In particular, we shall study the spaces of modular forms with respect to representations of the type

$$\det{}^k \otimes (\rho_e \odot \rho_e), \qquad k \in \mathbb{Z}.$$

For these vector spaces we shall give generators and compute the dimensions. This can be found either on the following pages or in theorem 5.23 on page 79.

Our approach relies on the fact that in the cases where $k = 3r$ and $k = 3r+1$, these forms can be identified with $\Gamma_2\,[2,4]$- invariant holomorphic tensors of the form

(1.1)
$$\big(f_0(Z)\cdot dZ^0 + f_1(Z)\cdot dZ^1 + f_2(Z)\cdot dZ^2\big) \otimes \big(dZ^0 \wedge dZ^1 \wedge dZ^2\big)^{\otimes r}$$

and

(1.2)
$$\big(g_0(Z)\cdot dZ^1\wedge dZ^2 + g_1(Z)\cdot dZ^0\wedge dZ^2 + g_2(Z)\cdot dZ^0\wedge dZ^1\big) \otimes \big(dZ^0 \wedge dZ^1 \wedge dZ^2\big)^{\otimes r},$$

respectively. Here the points in $\mathbb{H}_2$ are of the form

$$Z = \begin{pmatrix} Z_0 & Z_1 \\ Z_1 & Z_2 \end{pmatrix}.$$

The crucial fact is that the map

$$\mathbb{H}_2 \longrightarrow {}_{\Gamma_2\,[2,4]}\backslash^{\mathbb{H}_2} \hookrightarrow \mathbb{P}^3\mathbb{C}$$

branches over 10 explicitly given quadrics in $\mathbb{P}^3\mathbb{C}$. This implies that $\Gamma_2\,[2,4]$-invariant tensors on $\mathbb{H}_2$ correspond to rational tensors on $\mathbb{P}^3\mathbb{C}$ which may have poles of certain types along these 10 quadrics.

We shall elaborate on this result in the subsequent lines. The map $\mathbb{H}_2 \longrightarrow \mathbb{P}^3\mathbb{C}$ is given by the four theta constants of the second kind $f_0, \dots, f_3$. These are Siegel modular forms of weight $1/2$ with respect to a common multiplier system v_f, cf. theorem 4.24 on page 56.

It follows from Runge's results that

$$A(\Gamma_2\,[2,4], v_f) = \bigoplus_{r\in\mathbb{N}} [\Gamma_2\,[2,4], r/2, v_f^r]$$

is the ring of all modular forms of transformation type

$$f(M\langle Z\rangle) = v_f^r(M) \cdot \sqrt{\det(CZ+D)}^{\;r} \cdot f(Z).$$

This and the following theorem can be found on page 56. The ring of all modular forms of integral weight with respect to the trivial multiplier system is

$$\bigoplus_{r\in\mathbb{N}} [\Gamma_2\,[2,4], r] = \left(\bigoplus_{d\geq 0} \mathbb{C}_{4d}\,[f_0, f_1, f_2, f_3]\right) \bigoplus \left(\bigoplus_{d\geq 0} \mathbb{C}_{4d}\,[f_0, f_1, f_2, f_3]\right) \cdot \chi_5.$$

Here χ_5 denotes Igusa's modular form of weight 5 with respect to the full modular group. Furthermore, we use the notation $\mathbb{C}_d[f_0, \ldots, f_3]$ for the space of homogeneous polynomials of degree d.

The simplest case are tensors of the form $f(Z) \cdot \left(dZ^0 \wedge dZ^1 \wedge dZ^2\right)^{\otimes r}$. They belong to complex valued modular forms transforming as follows

$$f(M\langle Z\rangle) = \det(CZ+D)^{3r} \cdot f(Z), \qquad M \in \Gamma_2\,[2,4].$$

Returning to the vector valued case, we start with fixing some notation. Here and subsequently, $\mathcal{M}_r^+$ stands for the vector space of modular forms of transformation type

$$f(M\langle Z\rangle) = v_f^r(M) \cdot \sqrt{\det(CZ+D)}^{\;r} \cdot (CZ+D) f(Z) (CZ+D)^t, \quad M \in \Gamma_2\,[2,4].$$

It is also possible to twist this vector space by the character v_f^2. The space $\mathcal{M}_r^-$ consists of the modular forms satisfying

$$f(M\langle Z\rangle) = v_f^2(M) \cdot v_f^r(M) \cdot \sqrt{\det(CZ+D)}^{\;r} \cdot (CZ+D) f(Z) (CZ+D)^t$$

for all $M \in \Gamma_2\,[2,4]$.

We shall study the graded $A(\Gamma_2\,[2,4], v_f)$-modules

$$\mathcal{M}^+(\Gamma_2\,[2,4]) := \bigoplus_{r\in\mathbb{Z}} \mathcal{M}_r^+ \qquad \text{and} \qquad \mathcal{M}^-(\Gamma_2\,[2,4]) := \bigoplus_{r\in\mathbb{Z}} \mathcal{M}_r^-.$$

The module $\mathcal{M}^+$ contains the so called Rankin-Cohen brackets. These are constructed by means of scalar valued modular forms f, g and derivatives, i.e.

$$\{f, g\} = f \cdot Dg - g \cdot Df.$$

There is a similar construction $\{f, g, h\}$ which defines an element in $\mathcal{M}^-$, cf. definition 5.16 on page 75.

The main results of this thesis (theorems 5.15 and 5.22 on pages 75 and 79, respectively) are

$$\boxed{\begin{gathered}\mathcal{M}^+(\Gamma_2\,[2,4]) = \sum_{0 \le i < j \le 3} (\mathbb{C}[f_0, \dots, f_3])\,\{f_i, f_j\} \\ \text{and} \\ \mathcal{M}^-(\Gamma_2\,[2,4]) = \sum_{0 \le i_1 < i_2 < i_3 \le 3} (\mathbb{C}[f_0, \dots, f_3])\,\{f_{i_1}, f_{i_2}, f_{i_3}\}\,.\end{gathered}}$$

For any given degree r, we shall exhibit explicit bases of $\mathcal{M}_r^+$ and $\mathcal{M}_r^-$. Consequently, we obtain the Hilbert functions

$$\dim \mathcal{M}_r^+ = 3 \cdot \binom{r+1}{3} + 2 \cdot \binom{r}{2} + \binom{r-1}{1}$$

and

$$\dim \mathcal{M}_r^- = 3 \cdot \binom{r-2}{3} + \binom{r-3}{2}.$$

The case of modular forms with trivial multiplier system is of special interest. For integral r, we denote by $\mathcal{M}_r(\Gamma_2\,[2,4])$ the space of all modular forms satisfying

$$f(M\langle Z\rangle) = \det{(CZ+D)}^r \cdot (CZ+D) f(Z) (CZ+D)^t$$

for all $M \in \Gamma_2\,[2,4]$. In what follows, $\mathcal{M}(\Gamma_2\,[2,4])$ stands for the direct sum of all vector spaces $\mathcal{M}_r(\Gamma_2\,[2,4])$.

For $\mathcal{M}(\Gamma_2\,[2,4])$ we obtain

$$\mathcal{M}(\Gamma_2\,[2,4]) = \sum_{0\le i<j\le 3} (\mathbb{C}_{2+4\mathbb{Z}}[f_0,\dots,f_3])\,\{f_i,f_j\} \oplus \sum_{0\le i_1<i_2<i_3\le 3} (\mathbb{C}_{1+4\mathbb{Z}}[f_0,\dots,f_3])\,\{f_{i_1},f_{i_2},f_{i_3}\}$$

and

$$\dim\left(\mathcal{M}_k(\Gamma_2\,[2,4])\right) = \begin{cases} 3\cdot\binom{2k+1}{3} + 2\cdot\binom{2k}{2} + \binom{2k-1}{1}, & \text{if } k \text{ is even,} \\ 3\cdot\binom{2k-2}{3} + \binom{2k-3}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

Here $\mathbb{C}_{a+4\mathbb{Z}}[f_0,\dots,f_3]$ denotes the direct sum of the vector spaces $\mathbb{C}_d[f_0,\dots,f_3]$ where $d \equiv a \mod 4$.

Note that the module $\mathcal{M}(\Gamma_2\,[2,4])$ contains the $\Gamma_2\,[2,4]$-invariant holomorphic tensors shown in eqs. (1.1) and (1.2). Using them, we shall give a brief overview of the structure theorems' proofs. As shown in theorem 4.35, these tensors correspond to rational tensors on $\mathbb{P}^n\mathbb{C}$ having poles of certain types along 10 quadrics, that are given in theorem 4.34 on page 60. Heuristically speaking, these tensors become holomorphic after pulling them back along 2-coverings that are ramified over the quadrics. We refer to definition 3.53 on page 47 for the exact wording. If the parameter $r \in \mathbb{N}$ of eqs. (1.1) and (1.2) is even, then we can work out this condition explicitly, cf. corollaries 5.6 and 5.8 on pages 68 and 71, respectively. This shows the equalities $M_{6r}^+ = \mathcal{M}_{6r}^+$ and $M_{6r}^- = \mathcal{M}_{6r}^-$ for even r, cf. theorems 5.14 and 5.21. The case of arbitrary $\mathcal{M}_s^+$ and $\mathcal{M}_s^-$ can be reduced to the above ones by multiplying with monomials in the f_a, cf. theorems 5.15 and 5.22. This reduction uses the very simple structure of the ring of modular forms, i.e. $A(\Gamma_2\,[2,4],v_f) = \mathbb{C}[f_0,\dots,f_3]$.

We shall close the introduction with an outline of the chapters contents. In Chapter 2 we recapitalize the basic notions of the used algebraic tools, e.g. tensor products and (irreducible) group representations. Chapter 3 is dedicated to the analytic preliminaries such as complex manifolds[2], holomorphic functions, meromorphic tensors, analytic varieties and covering maps. In particular, in section 3.12 we formalize the space of meromorphic tensors with special poles as used above. We aim to present classical results on modular forms and symplectic groups in Chapter 4. In particular, section 4.3 covers Siegel modular forms and their link to holomorphic tensors. Furthermore in section 4.4, we briefly sketch

[2]Here, some parallels to [Wie10] may occur.

the theory of theta series of the first and especially of the second kind. Their structure theorems 4.24 and 4.25 are of particular interest for this thesis. Eventually, section 4.5 discusses the homogeneous space $\Gamma_2\,[2,4] \setminus \mathbb{H}_2$, its structure as a complex manifold, and the used ramification divisor. In the fifth and final chapter all the aforementioned results are combined in order to prove the thesis' main theorems.

2 Algebraic preliminaries

2.1 Polynomials

Definition 2.1 (Zero locus of a function)
We denote by $Z(f)$ the **zero locus** of $f : X \to \mathbb{C}^n$. For a family of functions $(f_i)_{i\in I}$ $Z((f_i)_{i\in I})$ denotes the common zero locus of them, i.e. $\bigcap_{i\in I} Z(f_i)$.

We recall Hilbert's Nullstellensatz.

Theorem 2.2 (Hilbert's Nullstellensatz)
Let $P_1, \dots, P_k \in \mathbb{C}[X^1, \dots, X^n]$. If $\mathfrak{a} = \mathfrak{a}(Z(P_1, \dots, P_k))$ is the ideal of polynomials vanishing on $Z(P_1, \dots, P_k)$, i.e. $\mathfrak{a} = \left\{P \in \mathbb{C}[X^1, \dots, X^n] : P(z) = 0 \ \forall z \in Z(P_1, \dots, P_k)\right\}$, then $\mathfrak{a} = rad(P_1, \dots, P_k) := \left\{P \in \mathbb{C}[X^1, \dots, X^n] : \exists \ m > 0 : P^m \in (P_1, \dots, P_k)\right\}$.

An immediate consequence is following result.

Corollary 2.3
If the zero locus of a polynomial Q is a subset of the zero locus of another polynomial P, then Q divides a power of P.

Definition 2.4 (Square-free element)
In an integral domain an element x is **square-free** if it cannot be divided by the square of a non-unit.

Since the polynomial ring $\mathbb{C}[X^1, \dots, X^n]$ is a unique factorization domain(UFD), we can formulate corollary 2.3 for square-free polynomials.

Corollary 2.5
Given a square-free polynomial Q whose zero set is contained in another's zero set $Z(P)$, then Q divides P.

We recall that we denote by $\mathbb{C}_d[X^1, \dots, X^n]$ the space of polynomials homogeneous of degree d.

Lemma 2.6
For positive d, $\mathbb{C}_d[X^1, \ldots, X^n]$ is a vector space of dimension $\frac{(d+n-1)!}{d!(n-1)!}$.

If the product of two non-zero polynomials P and Q is homogeneous, then so are P and Q. We deduce :

Lemma 2.7
The factorization of a homogeneous polynomial Q consists of homogeneous polynomials.

2.2 Tensor products

We recall some basics about tensor products of modules over commutative and unitary rings.

Definition 2.8 (Tensor product of modules)
The **tensor product** $(M_1 \otimes_R \cdots \otimes_R M_k, \mathrm{ten})$ of the modules $M_1, \ldots, M_k$ over R is a pair consisting of a module $M_1 \otimes_R \cdots \otimes_R M_k$ and a multilinear map

$$\mathrm{ten} : M_1 \times \cdots \times M_k \longrightarrow M_1 \otimes_R \cdots \otimes_R M_k$$

with the following universal property : For each *multilinear* map $M_1 \times \cdots \times M_k \longrightarrow N$, there exists exactly one *linear* map $M_1 \otimes_R \cdots \otimes_R M_k \longrightarrow N$ such that the following diagram commutes :

$$\begin{tikzcd}
 & M_1 \otimes_R \cdots \otimes_R M_k \arrow[d, dashed] \\
M_1 \times \cdots \times M_k \arrow[ur, "\mathrm{ten}"] \arrow[r] & N.
\end{tikzcd}$$

The tensor product is uniquely determined up to isomorphy.

We denote by $v_1 \otimes \cdots \otimes v_k$ the image of $(v_1, \ldots, v_k) \in M_1 \times \cdots \times M_k$ under the map ten.

Definition 2.9 (Dual space and basis)
Let V be a finite dimensional vector space over a field K. Then its **dual space** is $V^* = \mathrm{Hom}_K(V, K)$.

Given a basis $e_1, \dots, e_n$, we define its **dual basis** $e^{1*}, \dots, e^{n*}$ by $e^{i*}(e_j) = \delta^i_j \ \forall i, j$.

In the case of finite dimensional vector spaces (over fields), the tensor product can be constructed as follows :

Remark 2.10 (Tensor product of vector spaces)
Let $V_1, \dots, V_k$ be finite dimensional vector spaces over a field K. There is an isomorphism of $V_1 \otimes_K \cdots \otimes_K V_k$ into the space of multilinear forms on $V_1^* \times \cdots \times V_k^*$,

$$V_1 \otimes_K \cdots \otimes_K V_k \longrightarrow \mathrm{Mult}\,(V_1^* \times \cdots \times V_k^*, K)$$

which sends the element $v_1 \otimes \cdots \otimes v_k$ to the multilinear form

$$(\phi_1, \dots, \phi_k) \longmapsto \prod_{i=1}^{k} \phi_i(v_i).$$

We recall the basic properties of tensor products deduced from the universal property.

1. There is a natural isomorphism between $(M_1 \otimes_R \cdots \otimes_R M_l) \otimes_R (M_{l+1} \otimes_R \cdots \otimes_R M_k)$ and $M_1 \otimes_R \cdots \otimes_R M_k$.

2. The modules $M \otimes_R N$ and $N \otimes_R M$ are naturally isomorphic.

3. A tuple of linear maps $\Psi_i : M_i \to N_i$ induces the linear map $\Psi = \Psi_1 \otimes \ldots \otimes \Psi_k \ : M_1 \otimes_R \cdots \otimes_R M_k \to N_1 \otimes_R \cdots \otimes_R N_k$.

It is also possible to characterize the tensor product via its basis.

Lemma 2.11
Given vector spaces $V_1, \dots, V_k$ with basis $\left\{e^1_{i_1}\right\}_{1 \le i_1 \le \dim V_1}$, $\dots$, and $\left\{e^k_{i_k}\right\}_{1 \le i_k \le \dim V_k}$, respectively, then the tensor product $V_1 \otimes_K \cdots \otimes_K V_k$ has got the basis

$$\left\{e^1_{i_1} \otimes \cdots \otimes e^k_{i_k}\right\}_{\substack{1 \le i_j \le \dim V_j \\ 1 \le j \le k}}.$$

The group of permutations on $\{1, \dots, n\}$ denoted by $\mathfrak{S}_n$ acts on the space

$$V^{\otimes n} := \underbrace{V \otimes_K \ldots \otimes_K V}_{n-\text{times}}$$

by means of

$$v_1 \otimes \ldots \otimes v_n \longmapsto v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}.$$

We denote by T^σ the image of a tensor T under this map. We call a tensor T **alternating** if it holds

$$T^\sigma = \operatorname{sgn}(\sigma)\, T$$

for all permutations σ in $\mathfrak{S}_n$ and all vectors in V.

Remark 2.12 (The vector space of alternating tensors $\Lambda^n V$)
The set of alternating tensors $\Lambda^n V$ is a vector space giving rise to a vector space epimorphism

$$\begin{array}{rccl} alt: & V^{\otimes n} & \twoheadrightarrow & \Lambda^n V, \\ & T & \longmapsto & |\mathfrak{S}_n|^{-1} \cdot \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, T^\sigma, \end{array}$$

that coincides with the identity map on $\Lambda^n V$.

Notation 2.13
We use the notation

$$v_1 \wedge \cdots \wedge v_k = alt\,(v_1 \otimes \ldots \otimes v_k)\,.$$

For a given basis $e_1, \ldots, e_n$ and a set of indices $J = \{j_1, \ldots, j_k\}$, where $1 \le j_1 < \cdots < j_k \le n$, we write

$$e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}.$$

Lemma 2.14
$\Lambda^k V$ has got the basis $\{e_{j_1} \wedge \cdots \wedge e_{j_k}\}_{1 \le j_1 < \cdots < j_k \le n}$ and consequently dimension $\binom{n}{k}$.

Finally, we recall that a linear map $\Psi : V \longrightarrow W$ induces a linear map

$$\Lambda^k V \longrightarrow \Lambda^k W, \qquad v_1 \wedge \cdots \wedge v_k \longmapsto \Psi(v_1) \wedge \cdots \wedge \Psi(v_k).$$

2.3 Group actions

A **group action** of a group G on a set S is a group homomorphism $\rho : G \longrightarrow Aut(S)$ of G into the group of automorphisms of S, i.e. bijective self-maps.

We normally denote $\rho(g)(x)$ by gx or $g(x)$.

Definition 2.15 (Stabilizer subgroup G_x)
For a given point $x \in S$ and a group G acting on S the group $G_x := \{g \in G : g(x) = x\}$ is called **stabilizer subgroup** G_x of x.

The **orbit** Gx of x under G is the set of all gx, $g \in G$. The **orbit space** ${}_G\backslash^S$ is the collection of all orbits.

Definition 2.16 (Free group action)
If each stabilizer subgroup G_x only consists of the identity element in G, then the group action is **free**.

From now on, we assume that $X = S$ is a topological space and that all maps $\rho(g)$ are homeomorphisms. Then, the orbit space carries the quotient topology. Thus, the natural projection $G \longrightarrow {}_G\backslash^X$ is both continuous and open.

Definition 2.17 (Totally discontinuous group action)
A group G acts **totally discontinuously** on a locally compact Hausdorff space if for any two compact subsets K_1 and K_2 the set $\{g \in G : g(K_1) \cap K_2 \neq \emptyset\}$ is finite.

Totally discontinuous group actions have pleasant properties as stated in the following lemma.

Lemma 2.18
Let G be a group which acts totally discontinuously on X. Then

1. *G_p is finite for every p in X;*

2. *the orbit space ${}_G\backslash^X$ is Hausdorff;*

3. *for every p in X there exists a neighbourhood U such that*

   - *$\{g \in G : g(U) \cap U \neq \emptyset\}$ equals G_p;*

   - *the natural map ${}_{G_p}\backslash^U \to {}_G\backslash^X$ is an open embedding;*

4. *given such a neighbourhood U of p, then it holds $G_q \subset G_p$ for all q in U.*

If in addition the group action is free, then the natural projection is a homeomorphism.

2.4 Group representations

Definition 2.19 (Group representation)
A **representation** of a group G on a vector space V is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V).$$

A map φ from $\mathrm{GL}(n,\mathbb{C})$ to a finite dimensional vector space V is called **polynomial** if for a given basis (e_i) the associated coordinate functions φ^i are polynomial. Clearly, this definition is independent of the chosen basis.

Definition 2.20 (Rational representation of $\mathrm{GL}(n,\mathbb{C})$)
A representation ρ of $\mathrm{GL}(n,\mathbb{C})$ on a vector space V is **rational** if there is a natural number k for which $\det A^k \cdot \rho(A)$ is a polynomial map to $\mathrm{End}\,(V)$ or $\mathrm{GL}(V)$, respectively.

Since $\mathbb{C}\left[X^1,\ldots,X^{n^2}\right]$ is a UFD, there is a minimal *integer* for which $\det A^k \cdot \rho(A)$ is still polynomial. Its additive inverse is referred to as the **weight** of ρ.

We call a representation on V **irreducible** if the only G -invariant subspaces are $\{0\}$ and V.

Definition 2.21 (Reduced group representation)
A rational representation ρ is said to be **reduced** if it is irreducible and has got zero weight.

The following examples of (reduced) representations of $\mathrm{GL}(n,\mathbb{C})$ are important for the theory of Siegel modular forms.

1. We denote by $\rho_e(A) = A$ the **standard representation**, that is of course a *reduced* representation. Its **dual representation** is $\rho_e^*(A) = A^{-t}$. For $n = 2$ the representations $\det \otimes \rho_e^*$ and ρ_e are isomorphic. This follows from the formula
$$JAJ^{-1} = \det(A)A^{-t}, \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2. The representation
$$\begin{array}{rccl} \rho_e \odot \rho_e : & \mathrm{GL}(n,\mathbb{C}) & \longrightarrow & \mathrm{Aut}\left(\mathrm{Sym}^2(\mathbb{C}^n)\right), \\ & A & \longmapsto & \left\{X \mapsto AXA^t\right\}, \end{array}$$
is an other example of a *reduced* representation.

3. An other *reduced* representation is
$$\begin{array}{rccl} \det^2 \rho_e^* \odot \rho_e^* : & \mathrm{GL}(n,\mathbb{C}) & \longrightarrow & \mathrm{Aut}\left(\mathrm{Sym}^2(\mathbb{C}^n)\right)), \\ & A & \longmapsto & \left\{X \mapsto \det^2(A)A^{-t}XA^{-1}\right\}. \end{array}$$

3 Analytic preliminaries

We recall the definition of a complex differentiable function.

Definition 3.1 (Complex differentiable function)
Let U be an open subset of $\mathbb{C}^n$. A function $f : U \to \mathbb{C}^m$ is **complex differentiable** if there is an associated function $Df : U \to L(\mathbb{C}^n, \mathbb{C}^m)$ satisfying

$$f(z) = f(a) + Df(a)\,(z-a) + o(\|z-a\|)$$

in every $a \in U$.

We denote the vector space of complex differentiable functions by $\mathcal{O}(U, \mathbb{C}^m)$. Complex differentiable functions are usually called **holomorphic**.

It is well known that each holomorphic function can be locally expressed as a power series, i.e.

$$f(z) = \sum_{\nu} \alpha_\nu (z-a)^\nu = \sum_{\nu^1,\dots,\nu^n} \alpha_{\nu^1,\dots,\nu^n} \cdot (z^1 - a^1)^{\nu^1} \cdots (z^n - a^n)^{\nu^n}.$$

Definition 3.2 (Biholomorphic function)
We call a bijective holomorphic function $f : U \longrightarrow V$ with a holomorphic inverse

$$f^{-1} : V \longrightarrow U$$

biholomorphic.

A simple lemma shows that a bijective holomorphic function is automatically biholomorphic. We denote by $\mathrm{Bihol}(U)$ the **group of** all **biholomorphic functions** $f : U \to U$.

3.1 Complex manifolds

The aim of this section is to introduce the concept of a complex manifold.

Definition 3.3 (Holomorphic atlas)
A **holomorphic atlas** on a topological space X consists of an open cover $(U_i)_{i\in I}$ of X and a family of homeomorphisms $\phi_i : U_i \to V_i \overset{\text{open}}{\subset} \mathbb{C}^{n_i}$ such that the **transition functions** $\phi_i \circ \phi_j^{-1}$ are biholomorphic. We shall refer to ϕ_i as a **chart**.

Here $\phi_i \circ \phi_j^{-1}$ denotes the homeomorphism with obvious restriction of codomain and domain:

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j).$$

Definition 3.4 (Equivalent atlases)
Two holomorphic atlases $(U_i, \phi_i)_{i\in I}$ and $(\Omega_j, \psi_j)_{j\in J}$ are **equivalent** if their union is still a holomorphic atlas.

Definition 3.5 (Holomorphic structure)
A **holomorphic structure** on a Hausdorff space M is an equivalence class of holomorphic atlases.

Every holomorphic atlas is contained in an equivalence class and induces a holomorphic structure in this way.

Definition 3.6 (Complex manifold)
A Hausdorff space with a holomorphic structure on it is called a **complex manifold**.

Example 3.7

1. Any open subset U of $\mathbb{C}^n$ is a complex manifold because $id : U = M \longrightarrow U \subset \mathbb{C}^n$ gives rise to a holomorphic structure.

2. Every open subset Ω of a complex manifold M is a complex manifold. Indeed, we just have to restrict a holomorphic atlas of M to Ω.

3. We dedicate the whole section 3.2 to an important example, the projective space.

Definition 3.8 (Charts of complex manifolds)
A **chart of a complex manifold** is a map that is a chart in one of the holomorphic structure's atlases.

Given a point p on a manifold M, any chart around p maps into the same $\mathbb{C}^n$. We define the **dimension** of M in p to be this natural number n. This is obviously a locally constant function and hence constant on connected manifolds. In this case it is said to be the **dimension of the manifold**, say M, denoted by $\dim M$.

Definition 3.9 (Holomorphic functions between manifolds)
A continuous function f between two complex manifolds M and N is called **holomorphic**, if for any pair of charts $z : U \to z(U)$ and $w : V \to w(V)$ of M and N, respectively,

satisfying $f(U) \subset V$ the function
$$w \circ f \circ z^{-1}$$
is holomorphic. We denote the set of all these functions by $\mathcal{O}(M, N)$.

3.2 The n-dimensional projective space

The topic of this section is the n-dimensional projective space.

Definition 3.10 (The n-dimensional projective space $\mathbb{P}^n\mathbb{C}$)
We define the **n-dimensional projective space** $\mathbb{P}^n\mathbb{C}$ to be the collection of lines through the origin in $\mathbb{C}^{n+1}$. Each of these lines can be viewed as an equivalence class on $\mathbb{C}^{n+1} \setminus \{0\}$ for the relation

$$z \sim w \iff \exists\, \lambda \in \mathbb{C}^* : z = \lambda w.$$

The equivalence class of z is denoted by $[z]$.

We shall equip the n-dimensional projective space $\mathbb{P}^n\mathbb{C}$ with the structure of a complex manifold of dimension n in the subsequent lines.

The natural projection

$$\begin{array}{rccc} \pi : & \mathbb{C}^{n+1} \setminus \{0\} & \longrightarrow & \mathbb{P}^n\mathbb{C}, \\ & z = \left(z^0, \dots, z^n\right) & \longmapsto & [z] = \left[z^0, \dots, z^n\right], \end{array}$$

induces the quotient topology on $\mathbb{P}^n\mathbb{C}$. The so constructed topological space $\mathbb{P}^n\mathbb{C}$ is compact and Hausdorff.
There are homeomorphisms between the α-th **affine space** $\mathbb{A}_\alpha = \{[z] : z^\alpha \neq 0\}$ and $\mathbb{C}^n$ given by

$$\begin{array}{rccc} \phi_\alpha : & \mathbb{A}_\alpha & \longrightarrow & \mathbb{C}^n, \\ & \left[z^0, \dots, z^n\right] & \longmapsto & \left(\frac{z^0}{z^\alpha}, \dots, \frac{z^{\alpha-1}}{z^\alpha}, \frac{z^{\alpha+1}}{z^\alpha}, \dots \frac{z^n}{z^\alpha}\right). \end{array}$$

These maps are homeomorphisms which define a holomorphic structure on $\mathbb{P}^n\mathbb{C}$. Later, we shall use these charts only in the cases where $\alpha = 0$ or $\alpha = 1$. We use the subsequent notations

$$\phi_0(\left[z^0, \dots, z^n\right]) = (\mathfrak{z}^1, \dots \mathfrak{z}^n)$$

and

$$\phi_1(\left[z^0, \dots, z^n\right]) = (\mathfrak{w}^0, \mathfrak{w}^2, \dots \mathfrak{w}^n).$$

Then, we can transition between coordinates on $\mathbb{A}_1$ and $\mathbb{A}_0$ by

$$(\mathfrak{w}^0, \mathfrak{w}^2, \dots \mathfrak{w}^n) = \left(\frac{1}{\mathfrak{z}^1}, \frac{\mathfrak{z}^2}{\mathfrak{z}^1}, \dots, \frac{\mathfrak{z}^n}{\mathfrak{z}^1}\right).$$

3.3 Stalks of holomorphic functions

Let M be a complex manifold and p a point in M. We consider pairs (U, f) where U is an open neighbourhood of p and f lies in $\mathcal{O}(U)$. Two pairs are called equivalent if there exists a neighbourhood W satisfying $p \in W \subset U \cap V$ such that f and g coincide on W. Such an equivalence class is called a **(holomorphic) germ** at p. The germ at p belonging to (U, f) is denoted by $[U, f]_p$. This germ can be evaluated at p by $f(p)$. We write $\mathcal{O}_{M,p}$ for the collection of all germs in p and call it the **stalk of holomorphic functions** at p.

If U is an open neighbourhood of p, we can identify $\mathcal{O}_{U,p}$ with $\mathcal{O}_{M,p}$. On $\mathcal{O}_{M,p}$ there is a natural algebra structure such that for each open neighbourhood U of p the map

$$\mathcal{O}(U) \longrightarrow \mathcal{O}_{M,p}, \quad f \longmapsto [f]_{M,p},$$

is an algebra homomorphism.
In the case where $M = \mathbb{C}^n$, the algebra of germs can be identified with the algebra of convergent power series around p.

3.4 Tangent spaces

Definition 3.11 (Tangent vector)
A **tangent vector** or derivation at a point p is a $\mathbb{C}$-linear map $v : \mathcal{O}_{M,p} \to \mathbb{C}$ also satisfying Leibniz's law $v(fg) = v(f)g(p) + v(g)f(p)$.

Clearly, the collection of tangent vectors at a point p has got a vector space structure. The so obtained space is called the **holomorphic tangent space** T_pM.

Simple examples of tangent vectors in $T_p\mathbb{C}^n$ are the partial differential operators $\frac{\partial\,\cdot}{\partial z^i}\big|_p$, where i ranges over all integers form 1 to n. Applying these differential operators to a constant function or to the polynomial $(z^i - p^i)(z^j - p^j)$ returns 0. This implies the following lemma.

Lemma 3.12
The partial differential operators $\left\{ \frac{\partial\,\cdot}{\partial z^i}\Big|_p \right\}_{1\le i\le n}$ *form a basis of* $T_p\mathbb{C}^n$*.*

Using this lemma we can identify the tangent space $T_p\mathbb{C}^n$ of every single p with $\mathbb{C}^n$.

The assignment $(M,p) \longmapsto T_pM$ is functorial. Indeed, given a holomorphic map

$$\phi : M \longrightarrow N$$

and a point p in M, there is a natural map

$$\phi^* : \mathcal{O}_{V,\phi(p)} \longrightarrow \mathcal{O}_{U,p}, \quad f \longmapsto f \circ \phi.$$

This map induces a natural linear map

$$\phi_* : T_pM \longrightarrow T_qN, \quad v \longmapsto v \circ \phi^*,$$

called the **pushforward map along** ϕ.

Pushing forward is functorial in the sense that for any two holomorphic functions $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ it holds $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. In particular, it is an isomorphism if ϕ is biholomorphic.

Let T_pM be the tangent space of a point p on a manifold $M = M^n$. Then, there is a canonical isomorphism Φ_* between T_pM and $T_{z(p)}\mathbb{C}^n$, where z is a chart around p.

Lemma 3.13
Let $\phi : U \to V$ *be a holomorphic function between open subsets of* $\mathbb{C}^n$ *and* $\mathbb{C}^m$*, respectively. We identify the tangent space of* U *in a point* p *with* $\mathbb{C}^n$ *by means of the basis, i.e.*

$$T_pU = \mathbb{C} \cdot \frac{\partial \cdot}{\partial z^1} \oplus \cdots \oplus \mathbb{C} \cdot \frac{\partial \cdot}{\partial z^n},$$

cf. lemma 3.12. The same is true mutatis mutandis in $\mathbb{C}^m$*. Then, the matrix associated with the map*

$$\phi_* : T_pM \to T_{\phi(p)}N$$

is the Jacobian matrix $\mathrm{Jac}(\phi, p)$*.*

3.5 Cotangent spaces

Definition 3.14 (Cotangent space)
The dual space of T_pM is called the **holomorphic cotangent space** and denoted by $(T_pM)^* = T_p^*M$.
We call the elements of the cotangent space **co-vector**s, **1-form**s or **covariant vector**s.

Let f be a holomorphic function on some open neighbourhood of $p \in M$. We associate to f the 1-form

$$df_p : T_pM \to \mathbb{C}, \quad v \mapsto v(f).$$

In particular, for $M \overset{\text{open}}{\subset} \mathbb{C}^n$ and

$$z^i : M \longrightarrow \mathbb{C}, \quad z \longmapsto z^i,$$

we stick to the notation dz^i_p. These 1-forms generate the cotangent space. Indeed, suppose $f : M \to \mathbb{C}$ is a holomorphic function we show

$$df_p = \sum_i \left.\frac{\partial f}{\partial z^i}\right|_p \cdot dz^i_p. \tag{3.1}$$

A holomorphic function $\phi : M \longrightarrow N$ induces a linear map called the **pullback**

$$\phi^* : T^*_{\phi(p)}N \longrightarrow T^*_pM, \quad \omega \longmapsto \omega \circ \phi_*.$$

For any two holomorphic functions $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ it holds

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

The cotangent space T_pM^* can be identified with $\mathbb{C}^n$ using the basis $\left\{dz^i_p\right\}_{1\le i\le n}$.

Lemma 3.15
Let $\phi : U \to V$ be a holomorphic function between open subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. We identify the cotangent space of U in a point p with $\mathbb{C}^n$ by means of the basis, i.e.

$$T^*_pU = \mathbb{C} \cdot dz^1_p \oplus \cdots \oplus \mathbb{C} \cdot dz^n_p,$$

cf. eq. (3.1) *on this page. The same is true with necessary modifications in $\mathbb{C}^m$. Then the matrix associated with the map*

$$\phi^* : T^*_pM \to T^*_{\phi(p)}N$$

is the transposed Jacobian matrix $\mathrm{Jac}(\phi, p)^t$.

3.6 Holomorphic tensors

In this section, we shall describe (holomorphic) tensors rather by their local properties. Therefore, we observe the spaces

$$T_p^*M^{\otimes q} = \underbrace{T_p^*M \otimes \ldots \otimes T_p^*M}_{q\text{-times}} \ .$$

Definition 3.16 (Covariant tensor)
A **covariant holomorphic tensor** ω on a manifold M is a map that assigns to each point p an element

$$\omega_p \in T_p^*M^{\otimes q}$$

which depends holomorphically on p. It is clear what holomorphic dependence means : locally, the manifold can be identified with $\mathbb{C}^n$ and consequently $T_p^*M^{\otimes q}$ with $\left(T_p^*\mathbb{C}^n\right)^{\otimes q}$.

We denote the vector space of all holomorphic tensors by

$$\Omega^{\otimes q}(M).$$

For an open subset $U \subset \mathbb{C}^n$, we define

$$dz^i = \left(dz_p^i\right)_{p\in U} .$$

Each holomorphic tensor can be written as

$$\omega = \sum_\nu \omega_\nu \ dz^{\nu_1} \otimes \cdots \otimes dz^{\nu_q}, \quad \nu = (\nu_1, \ldots, \nu_q),$$

where each ω_ν is simply a holomorphic function.

It is clear how to define the **tensor product** $\omega \otimes \eta$ of two tensors ω and η. It is also clear what is meant by an **alternating holomorphic tensor**. Their space is denoted by $\Lambda^p\Omega(M)$. For $n = \dim M$, tensors in the space

$$(\Lambda^n\Omega)^{\otimes k}(M)$$

are called **multi canonical**. Later, we shall study tensors of the type

$$(\Lambda^p\Omega) \otimes (\Lambda^n\Omega)^{\otimes k}.$$

By definition, they are locally of the form

$$\sum_{i_1 \ldots i_p} \omega_{i_1 \ldots i_p} \ dz^{i_1} \wedge \cdots \wedge dz^{i_p} \otimes (dz^1 \wedge \cdots \wedge dz^n)^{\otimes k}.$$

Let $\phi : M \to N$ be a holomorphic function between complex manifolds. Since the tensor product is functorial, the pullback of 1-forms can be generalized in an obvious way to a **pullback of covariant tensors**, i.e.

$$\phi^* : \Omega^{\otimes q}(M) \longrightarrow \Omega^{\otimes q}(N).$$

The above introduced types are preserved by pullbacks.

Definition 3.17 (Invariant tensors)
A covariant tensor field $\omega \in \Omega^{\otimes q}(M)$ is called **invariant** under a subset S of $\mathcal{O}(M, M)$ if it holds $\phi^*\omega = \omega$ for all ϕ in S. The collection of these covariant tensorfields is $\left(\Omega^{\otimes q}(M)\right)^S$.

3.7 Holomorphic functions

We sum up the basic properties on holomorphic functions.

Theorem 3.18 (Implicit function theorem)
Let U be an open subset of $\mathbb{C}^n$ and $f \in \mathcal{O}(U, \mathbb{C}^m)$ with $m \leq n$. Suppose that for a given root $z_0 \in Z(f)$ the leading principal minor $(\mathrm{Jac}(f, z_0)^i_j)_{1\leq i,j\leq m}$ is invertible.
Then, there are open subsets U_1 and U_2 of $\mathbb{C}^{n-m}$ and $\mathbb{C}^m$, respectively, such that z_0 lies in $V := U_1 \times U_2$ and $f(z)$ vanishes on V iff $(z^1, \dots, z^m)$ equals $g(z^{m+1}, \dots, z^n)$.

Theorem 3.19 (Identity theorem)
Two holomorphic functions from a connected complex manifold M are equal if they coincide on a nonvoid open subset of M .

We denote by $\mathbb{E}$ and $\mathbb{E}^*$ the standard and the punctured unit disc in $\mathbb{C}$, respectively.

Theorem 3.20 (Laurent series)
A holomorphic function on $\mathbb{E}^{n-1} \times \mathbb{E}^*$ can be expanded in a Laurent series, i.e.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z^1, \dots, z^{n-1}) \cdot (z^n)^k.$$

Each a_k is uniquely determined and belongs to $\mathcal{O}(\mathbb{E}^{n-1})$.

Theorem 3.21
Let p be a point on a complex manifold M, then $\mathcal{O}_{M,p}$ is a unique factorization domain.

Proof
A proof can be found in [Huy05, Prop 1.1.15, p.14]. □

We formulate Rückert's Nullstellensatz in the ring of power series $\mathbb{C}\{z^1, \dots z^n\}$; it can be identified with the ring of germs at any point p.

Theorem 3.22 (Rückert's Nullstellensatz)
Suppose $P, P_1, \dots, P_k$ are convergent power series in $\mathbb{C}\{z^1, \dots z^n\}$. Take U to be a neighbourhood of 0 such that $P, P_1, \dots, P_k$ converge on U.
If $P_1(z) = \dots = P_k(z) = 0$ implies $P(z) = 0$ for z in U, then P lies in the radical of $(P_1, \dots, P_k)$.

Corollary 3.23
Let f and g be two holomorphic functions on a complex manifold M such that $Z(f) \subset Z(g)$. Then, for any given point p in M there is a natural number k such that $[M, f]_p$ divides $\left([M, g]_p\right)^k$.

Theorem 3.24
If two germs $[f]_{\mathcal{O}_{U,p}}$ and $[g]_{\mathcal{O}_{U,p}}$ are coprime in $\mathcal{O}_{U,p}$, then they are also coprime in $\mathcal{O}_{U,q}$ for all q in a small neighbourhood of p.

Theorem 3.25 (Riemann extension theorem)
Suppose f is a non-zero holomorphic function on a domain $\mathcal{D}$. And let

$$g : \mathcal{D} \setminus Z(f) \to \mathbb{C}$$

be another holomorphic function. Then, g is holomorphic on the whole of $\mathcal{D}$ if it is locally bounded around $Z(f)$.

3.8 Meromorphic functions and tensors

In this section, we follow [Fre11, p. 414].

Definition 3.26 (Meromorphicity of holomorphic functions)
Let f be a holomorphic function on an open and dense subset $\mathcal{D}$ of a manifold M. The function f is **meromorphic** on M if for any $p \in M$ there are two holomorphic functions g and h on an open neighbourhood $U \subset M$ of p satisfying $f(z) = \frac{g(z)}{h(z)}$ for every z in U such that $h(z) \neq 0$.

We call two pairs $(\mathcal{D}, f)$ and $(\mathcal{D}', f')$ with the above property equivalent if f and f' coincide on $\mathcal{D} \cap \mathcal{D}'$. In the following, a **meromorphic function** on M denotes such an equivalence class.

The set of meromorphic functions on a manifold M is a ring. If M is connected, then this ring is a field.

Let f be a meromorphic function on M then there are an open cover of $M = \bigcup_{i\in I} U_i$ and holomorphic functions $g_i, h_i : U_i \to \mathbb{C}$ such that $f|_{U_i} = g_i/h_i$. We may assume that the germs of g_i and h_i are coprime in each stalk $\mathcal{O}_{M,p}$, where $p \in U_i$ due to theorem 3.24 on the previous page. Under this assumption, we call such a family $(U_i, g_i, h_i)_{i\in I}$ a **defining datum for a meromorphic function** f.

Two defining data $(U_i, g_i, h_i)_{i\in I}$ and $(U'_j, g'_j, h'_j)_{j\in J}$ define the same meromorphic function iff for each pair (i,j) there exists a holomorphic function $\phi_{ij} \in \mathcal{O}(U_i \cap U'_j)^*$ such that

$$g_i = \phi_{ij} g'_j \quad \text{and} \quad h_i = \phi_{ij} h'_j \quad \text{on } U_i \cap U'_j.$$

Hence, we make the following definition.

Definition 3.27 (Zero and pole locus of a meromorphic function)
The **zero locus** $Z(f)$ of a meromorphic function f given by a datum $(U_i, g_i, h_i)_{i\in I}$ is the union $\bigcup_{i\in I} Z(g_i)$. Similarly, the **pole locus** $P(f)$ is $\bigcup_{i\in I} Z(h_i)$.

Now, we extend the definitions from above to arbitrary tensors.

Definition 3.28 (Meromorphicity of holomorphic tensors)
Let ω be a holomorphic tensor on an open and dense subset $\mathcal{D}$ of a manifold M. The tensor ω is **meromorphic** on M if for any $p \in M$ there are an connected open neighbourhood U and a non-zero holomorphic function $h : U \to \mathbb{C}$ such that $h \cdot \omega$ is holomorphic on U.

As in the case of meromorphic functions, a **meromorphic tensor** on M is an equivalence class of pairs $(\mathcal{D}, \omega)$. For a complex manifold M the set of meromorphic tensors is a module over the ring of meromorphic functions. The tensor product of two meromorphic tensors is again meromorphic.

The tensor $d\mathfrak{z}^i$ is an example of a meromorphic tensor on $\mathbb{P}^n\mathbb{C}$.

By a theorem of Hurwitz (a special case of Chow's Corollary, cf. [GH78, p. 168]), each meromorphic tensor on $\mathbb{P}^n\mathbb{C}$ is rational, i.e. it is of the form

$$\sum_{i_1\dots i_k} \omega_{i_1\dots i_k} d\mathfrak{z}^{i_1} \otimes \dots \otimes d\mathfrak{z}^{i_k}$$

with $\omega_{i_1 \dots i_k}$ rational (a quotient of two polynomials).

It is also possible to pullback meromorphic tensors.

Remark 3.29 (Pullback of covariant meromorphic tensor fields)
Let ω be a covariant meromorphic tensor field on a manifold N that is holomorphic on $\mathcal{D}$. Then, it can be transported to another manifold M by a holomorphic function $\phi : M \to N$ if $\phi^{-1}(\mathcal{D})$ becomes a dense subset in M.

Theorem 3.30
Any holomorphic rational function $f = \frac{P}{Q}$ is polynomial.

3.9 Complex submanifolds and analytic subvarieties

Definition 3.31 (Complex submanifold)
A subset N of a complex manifold M is called a **complex submanifold** if for every point $p \in N$ there exist natural numbers k and n with $k \leq n$ and a chart $\phi : U \to V \subset \mathbb{C}^n$ of M around p, such that

$$N \cap U \cong \phi(U) \cap \left\{ z \in \mathbb{C}^n : z^j = 0,\ k+1 \leq j \leq n \right\}.$$

Each complex submanifold possesses the structure of a complex manifold.

We can generalize the concept of a submanifold.

Definition 3.32 (Analytic subvariety)
Suppose Y is a subset of a complex manifold M. If for every point $p \in Y$ there is a neighbourhood U and finitely many holomorphic functions in $\mathcal{O}(U)$ satisfying

$$U \cap Y = \bigcap_{1 \leq i \leq m_p} Z\left(f_i\right),$$

then Y is an **analytic subvariety**.

The functions $f_1, \dots, f_{m_p}$ are called **local defining functions** for Y.

The union and intersection of the closed subvarieties Y_1 and Y_2 are again analytic subvarieties.

Definition 3.33 (Regular and singular points of an analytic subvariety)
A point of an analytic subvariety $Y \subset M$ is a **regular point** if there is an open neighbourhood $U \subset M$ such that $Y \cap U$ is a complex submanifold of U. A point that is not regular is **singular**.

We denote the regular and singular points of an analytic subvariety by Y_{reg} and Y_{sing}, respectively.

Definition 3.34 (Irreducible analytic subvariety)
An analytic subvariety Y is **irreducible** if there are no analytic subvarieties Y_1 and Y_2 such that

$$Y_i \overset{closed}{\subsetneq} Y \quad \text{and} \quad Y = Y_1 \cup Y_2.$$

Irreducible polynomials produce irreducible varieties in the following manner.

Theorem 3.35
Let P be an irreducible polynomial then its zero set $Z(P)$ is an irreducible analytic variety.

We need the three subsequent deep theorems. Their proofs can be found in [GR65] on pages 116 and 141, respectively.

Theorem 3.36
The regular locus Y_{reg} is an open dense subset of Y and Y_{sing} is an analytic subvariety.

Theorem 3.37
If Y is irreducible, then Y_{reg} is connected and vice versa.

Theorem 3.38
Let Y be an analytic subvariety. Then, the closures of Y_{reg}'s connected components are irreducible analytic subvarieties.

The above mentioned irreducible subvarieties are called the **irreducible components** of Y. It is also possible to characterize the components as maximal closed subvarieties of Y.

The preimage of an analytic subvariety $Y \subset N$ under a holomorphic function $f : M \to N$ is an analytic subvariety in M.

Definition 3.39 (Dimension of irreducible analytic varieties)
The **dimension of an irreducible analytic variety** is the dimension of its regular locus.

An **analytic subvariety**'s **dimension** is the supremum of its irreducible components' dimensions. An analytic subvariety is **pure dimensional** if all irreducible components have got the same dimension.

Let Y and M be an analytic variety and a complex manifold, respectively, of pure dimensions. If Y is a subvariety of M, then its **codimension** is the natural number

$\operatorname{codim} Y = \dim M - \dim Y$.

Definition 3.40 (Hypersurface)
A **hypersurface** is an analytic subvariety of codimension 1.

Definition 3.41 (Negligible set)
We call a analytic subvariety $A \subset M$ of a connected complex manifold **analytically negligible** if all irreducible components have codimension greater or equal 2.

Given a chain $Y_1 \subsetneq Y_2 \subsetneq M$ of irreducible subvarieties of a connected manifold M, then Y_1 is negligible. Hence, the common zero set of two holomorphic functions $f, g : M \to \mathbb{C}$ is negligible iff the germs of f and g are coprime at each point $p \in M$. A consequence is the subsequent lemma.

Lemma 3.42
Suppose Q is an irreducible polynomial, then the analytic variety $Z\left(Q, \frac{\partial Q}{\partial z^i}\right)$ is negligible.

Proof
Since $\frac{\partial Q}{\partial z^i}$ has a smaller degree than Q, the derivative $\frac{\partial Q}{\partial z^i}$ cannot divide Q. □

Theorem 3.43 (Levi's extension theorem)
Given a negligible set A, any meromorphic function $f : M \setminus A \to \mathbb{C}$ extends to a meromorphic function on M.

Consequently, we can also continue meromorphic tensors over a negligible set.

Definition 3.44 (Weil-divisor)
A **Weil-divisor** D on a connected complex manifold M is a mapping from the collection of irreducible hypersurfaces into the integers. Furthermore, we require it to be locally finite, i.e. every point has got an open neighbourhood U such that only finitely many hypersurfaces with $D(Y) \neq 0$ intersect U. $D(Y)$ is called **multiplicity** of Y. Sometimes, the divisor is denoted by the formal sum

$$\sum_Y D(Y) \cdot Y.$$

The **support** of D is the analytic subvariety

$$\operatorname{supp} D = \bigcup_{D(Y) \neq 0} Y.$$

Therefore, we can define the singular locus of $\operatorname{supp} D$ and denote it by D_{sing}.

3.10 Covering maps and spaces

We recall the notion of a (ramified) covering of topological spaces.

Definition 3.45 (Finite covering map)
We call a map $p : Y \to X$ between two topological spaces **finite covering map** if it is open, continuous, proper, surjective, and finite, i.e. $p^{-1}(x)$ is finite in Y for every x in X.

A point, say $y \in Y$, is a **ramification point** if p is not a homeomorphism around y. We denote by $\mathrm{Ram}\,(p)$ the collection of p 's ramification points in Y. We say that p is **ramified along** $\mathrm{Ram}\,(p)$. We also say that p is **ramified over** $p(\mathrm{Ram}\,(p))$. If the ramification locus is void, then p is called **unramified**. The induced map

$$p : Y \setminus p^{-1}(p(\mathrm{Ram}\,(p))) \longrightarrow X \setminus p(\mathrm{Ram}\,(p))$$

is an unramified covering.

We shall mainly use covering maps that are holomorphic functions between manifolds.

Definition 3.46 (Simple covering)
We call a holomorphic covering map $f : M \to N$ between two n dimensional complex manifolds **simple covering map** if $f\,(\mathrm{Ram}\,(f))$ is a smooth hypersurface.

We give an easy example :

Definition 3.47 (Standard element p_n^k)
We call the map

$$\begin{array}{rccc} p_n^k : & \mathbb{E}^n & \longrightarrow & \mathbb{E}^n, \\ & (z^1, \dots, z^n) & \longmapsto & (z^1, \dots, z^{n-1}, (z^n)^k), \end{array}$$

the **k-th n-dimensional standard element**, where $n, k > 0$.

The standard elements are obviously simple coverings.

Theorem 3.48
Let $f : M \to N$ be a simple covering. Then for each point p in M there are an open neighbourhood U and biholomorphic functions $U \to \mathbb{E}^n$ and $f(U) \to \mathbb{E}^n$ such that the subsequent diagram commutes for a suitable k

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{E}^n \\ \Big\downarrow & & \Big\downarrow p_n^k \\ f(U) & \longrightarrow & \mathbb{E}^n. \end{array}$$

Proof
The proof is deduced from the topological classification of unramified covering maps of $\mathbb{E}^{n-1} \times \mathbb{E}^*$. These coverings correspond to the standard coverings. □

Let R be a closed submanifold of codimension 1 of N. Then, for each point $p \in R$ there is a chart $V \to \mathbb{E}^n$ that sends R to $z^n = 0$. Hence, there is a simple covering $U \to V$ that ramifies over $R \cap V$.

Suppose now that N is an open subset of $\mathbb{C}^n$ and R is given as the zero set of a holomorphic function $Q : N \to \mathbb{C}$. If the assumptions of the implicit function theorem are satisfied, then there is a simple covering that ramifies over $Q = 0$. This simple covering can be given explicitly as follows :

Remark 3.49
Let V be an open subset of $\mathbb{C}^n$ and p an arbitrary point in V. Let $Q : V \to \mathbb{C}$ be a holomorphic function such that the assumptions of the implicit function theorem apply, i.e.

$$Q(z) = 0 \iff z^n = \phi(z^1, \dots z^{n-1}).$$

Then, the map

$$(z^1, \dots z^n) \longmapsto (z^1, \dots, z^{n-1}, (z^n)^k + \phi(z^1, \dots z^{n-1}))$$

defines a simple covering $U_0 \to V_0$ of some open subset $U_0 \subset \mathbb{C}^n$ and an open neighbourhood $p \in V_0 \subset V$.

3.11 Orders of singularities

Let $f : U \to \mathbb{C}$ be a non-vanishing holomorphic function on an open and connected neighbourhood of 0 in $\mathbb{C}^n$. We want to define the **order of** f **along** $Y = \{z^n = 0\}$ **at** 0. For this, we consider a power series expansion of f

$$\sum_{k \geq 0} a_k(z^1, \dots, z^{n-1}) \cdot (z^n)^k.$$

Then, the **order** is the smallest k such that a_k is non-vanishing.

Given a non-vanishing holomorphic function on a connected complex manifold M and a smooth hypersurface Y including p, then the order $\operatorname{ord}(f, Y, p)$ can be defined via charts. For meromorphic functions this can be generalized. Let f be a non-zero meromorphic function on M that can be written as g/h around p. Then, the order is defined as

$$\operatorname{ord}(f, Y, p) := \operatorname{ord}(g, Y, p) - \operatorname{ord}(h, Y, p).$$

This definition is clearly independent of the chosen holomorphic functions g and h.

The function $\operatorname{ord}(f, Y, \cdot)$ is locally constant, in particular, it is constant on connected hypersurfaces.

Now, let X be an irreducible, closed, analytic hypersurface then X_{reg} is a connected smooth hypersurface. Therefore, it is possible to define the **order along** X by

$$\operatorname{ord}(f, X) := \operatorname{ord}(f, X_{reg}).$$

It holds

$$\operatorname{ord}(f + g, Y) \geq \min\{\operatorname{ord}(f, Y), \operatorname{ord}(g, Y)\}$$

and

$$\operatorname{ord}(f \cdot g, Y) = \operatorname{ord}(f, Y) + \operatorname{ord}(g, Y)$$

for all meromorphic functions f and g satisfying $f + g \neq 0$.

The order $\operatorname{ord}(f, X)$ is positive iff X is contained in the zero locus $Z(f)$, cf. definition 3.27 on page 40. A negative order is related to the pole locus $P(f)$ in the same way.

This allows us to associate a divisor to a meromorphic function.

Definition 3.50 (Principal divisor (f))
Given a non-zero meromorphic function f, then its **principal divisor** is defined by

$$(f) := \sum_Y \operatorname{ord}(f, Y) \cdot Y.$$

It is clear that this sum is locally finite.

We recall that the zero locus of an irreducible polynomial Q is an irreducible hypersurface.

Lemma 3.51
The multiplicity of an irreducible polynomial Q along its zero set is one.

Lemma 3.52
Let $p = p_n^k : \mathbb{E}^n \to \mathbb{E}^n$ be the standard element and $f : \mathbb{E}^n \to \mathbb{C}$ a non-zero meromorphic function. Then, the orders vary directly, i.e.

$$\operatorname{ord}\left(f \circ p, \{z^n = 0\}\right) = k \cdot \operatorname{ord}\left(f, \{z^n = 0\}\right).$$

3.12 Covering holomorphic tensors

Definition 3.53 (Covering holomorphic tensors)
Let D be an effective divisor on a n-dimensional complex manifold M, we define $\Omega^{\otimes k}(M, D)$ as the space of tensors $\omega \in \Omega^{\otimes k}(M \setminus \operatorname{supp} D)$ with the supplementary property :

> Let Y be an irreducible component of D and q a point in Y that is a regular point in $\operatorname{supp} D$. Then, there exist an open neighbourhood of $q \in V$ and a simple covering $p : U \to V$ with the subsequent properties :
>
> 1. p is ramified over $Y \cap V$;
>
> 2. p is equivalent to the standard element p_n^k, where $k = D(Y) + 1$ in the terms of theorem 3.48 on page 44;
>
> 3. ω's pullback $p^*\omega$ is holomorphically extendable to the whole of U.

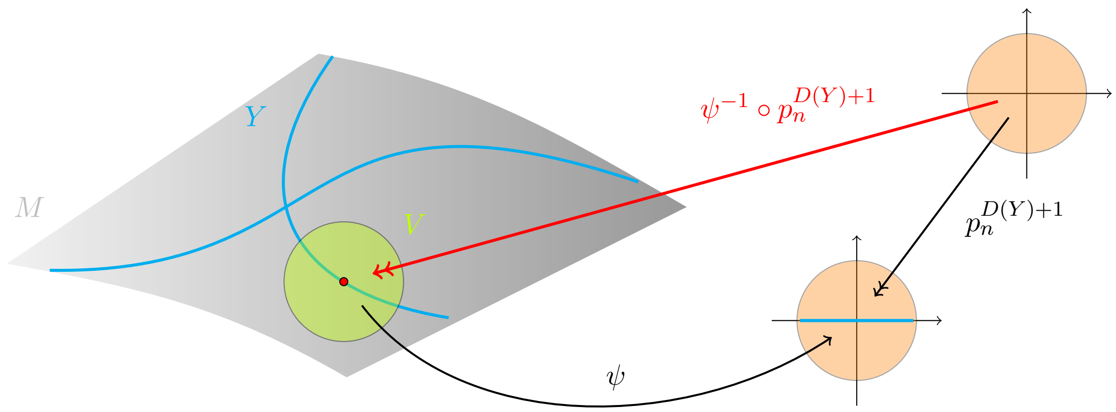

Figure 3.1: Illustration of definition 3.53

Of course, the third condition in the previous defintion is independent of the chosen p. It is not hard to show that these tensors are meromorphic on M. Later, we shall use tensors of the following type (the notation should be self explicatory)

$$\left(\Lambda^p\Omega \otimes (\Lambda^n\Omega)^{\otimes k}\right)(M, D), \quad \text{where } n = \dim M.$$

4 Modular forms

This chapter is a generalisation of the theory of modular functions of degree 1, which can be found in such books as [FB09], to the ones with higher degrees as introduced in [Fre91] and [Fre11].

4.1 Symplectic groups

Definition 4.1 (Partial order on symmetric matrices)
We say a real symmetric matrix A is greater (or equal) than the symmetric matrix B if $A - B$ is positive (semi-)definite, and denote this fact by $A \geq B$ or $A > B$, respectively.

We denote the vector space of **symmetric** $n \times n$ **matrices** by

$$\mathrm{Sym}^2(\mathbb{C}^n) := \left\{ Z \in M(n, \mathbb{C}) : Z = Z^t \right\}.$$

$\mathrm{Sym}^2(\mathbb{C}^n)$ is naturally isomorphic to $\mathbb{C}^{\frac{n(n+1)}{2}}$ and hence a complex manifold.

Definition 4.2 (Siegel upper half-space)
The **Siegel upper half-space** is the set $\mathbb{H}_n := \left\{ Z \in \mathrm{Sym}^2(\mathbb{C}^n) : \mathrm{Im}\, Z > 0 \right\}$.

The upper half-space $\mathbb{H}_n$ is an open and convex complex manifold. Since $\mathbb{H}_n$ is simply connected, every non-vanishing holomorphic function on $\mathbb{H}_n$ has got a holomorphic square root.

We denote by Δ_n the **set of diagonal matrices** in $\mathbb{H}_n$.It can be seen easily that Δ_n is a closed submanifold of $\mathbb{H}_n$. Hence, in the important case where $n = 2$, Δ_n is a hypersurface.

Given matrices B in $R^{n \times n}$ and A in $R^{n \times m}$, we use the standard notation $B[A]$ for $A^t B A$.

Definition 4.3 (Symplectic group)
Given the matrix

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

in the linear group of a commutative unital ring $\mathrm{GL}(2n, R)$, then the **symplectic group** is the subgroup

$$\mathrm{Sp}(n, R) := \{M \in \mathrm{SL}(2n, R) : J_n [M] = J_n\}.$$

Furthermore, we shorten $\mathrm{Sp}(n, \mathbb{Z})$ to Γ_n.

For $n = 1$, the symplectic group $\mathrm{Sp}(1, R)$ coincides with $\mathrm{SL}(2, R)$.

Theorem 4.4 (Generators for $\mathrm{Sp}(n, R)$)
The symplectic group $\mathrm{Sp}(n, R)$ *for an Euclidean domain is generated by* J_n *and matrices of the form* $\begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}$ *with symmetric matrices* $S \in \mathrm{M}(n, R)$.

Proof
For a proof we refer to [Fre83, A5.4 Satz, p.326]. □

Throughout this thesis, we denote the generic element of $\mathrm{Sp}(n, R)$ by $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Definition 4.5
By $\mathcal{J}$ we refer to the map

$$\mathcal{J} : \mathrm{Sp}(n, \mathbb{R}) \times \mathbb{H}_n \longrightarrow \mathrm{M}(n, \mathbb{C}), \quad (M, Z) \longmapsto CZ + D.$$

Lemma 4.6
With the help of $\mathcal{J}$ *we can prove the following facts :*

1. $\mathcal{J}$ *maps into* $\mathrm{GL}(n, \mathbb{C})$*;*
2. $M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$ *lies in* $\mathbb{H}_n$ *for* Z *in* $\mathbb{H}_n$ *and* M *in* $\mathrm{Sp}(n, \mathbb{R})$*;*
3. *it holds* $M\langle Z\rangle = Z$ *for all* Z *in* $\mathbb{H}_n$ *iff* $M = \pm I_{2n} \in \mathrm{Sp}(n, \mathbb{R})$*;*
4. $M\langle N\langle Z\rangle\rangle = (M \cdot N)\langle Z\rangle$ *for* M *and* N *in* $\mathrm{Sp}(n, \mathbb{R})$*;*
5. *the induced action of* $\mathrm{Sp}(n, \mathbb{Z})$ *is totally discontinuous;*

6. *$M\langle Z\rangle$ is holomorphic with derivative* $DM\langle Z_0\rangle(W) = (CZ_0+D)^{-t}W(CZ_0+D)^{-1}$ *and Jacobian determinant* $\det(CZ_0+D)^{-(n+1)}$.

Definition 4.7 (Principal congruence subgroup)
The kernel of the natural group homomorphism $\mathrm{Sp}(n,\mathbb{Z}) \to \mathrm{Sp}(n, \mathbb{Z}/q\mathbb{Z})$ is the **principal congruence subgroup** $\Gamma_n[q]$.

We call a subgroup $\Gamma \subset Sp(n,\mathbb{R})$ a **congruence subgroup** if it contains a principal congruence subgroup $\Gamma_n[q]$ as subgroup of finite index.

Let A be a square matrix then $(A)_0$ denotes the diagonal vector $(a_{11},\dots,a_{nn})$.

Definition 4.8 (Igusa's group)
The **Igusa's group** of level q is defined to be

$$\Gamma_n[q,2q] := \left\{M \in \Gamma_n[q] : \left(AB^t\right)_0 \equiv \left(CD^t\right)_0 \equiv 0 \mod 2q\right\}.$$

It can be shown that Igusa's group is actually a group. Suppose that q is even then $\Gamma_n[q,2q]$ is normal in the full modular group.

Lemma 4.9
The non-trivial subgroups of $\left(\Gamma_2[2,4]/\pm I_4\right)$ of finite order are all of order 2. Their generators are conjugated in $\Gamma_2/\pm I_4$ to the image of the matrix

$$\begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & 1 & \\ & & & -1 \end{pmatrix}.$$

Proof
It is clear that squaring an arbitrary matrix M in $\Gamma_2[2,4] < \Gamma[2]$ gives a matrix in $\Gamma[4]$. It follows from rather basic algebraic facts that this group acts without fixed points. Therefore, M is of order 2 and [Run93]'s Lemma 5.3 on page 23 completes the proof. □

4.2 Factors of automorphy and multiplier systems of half integral weight

Now, we shall state a definition of the factor of automorphy sufficient for this thesis.

Definition 4.10 (Factor of automorphy)
For a subgroup Γ of $\mathrm{Sp}(n, \mathbb{R})$ and a finite dimensional $\mathbb{C}$-vector space V a **factor of automorphy** is a map

$$j : \Gamma \times \mathbb{H}_n \to \mathrm{GL}(V)$$

that is holomorphic in the second variable and satisfies the **cocycle relation**

$$j(MN, Z) = j(M, N\langle Z\rangle) \circ j(N, Z).$$

The first example of a factor of automorphy is of the form $j(M, Z) = \chi(M)$, where χ is a character on Γ.
Another example is the function $\mathcal{J} : \mathrm{Sp}(n, \mathbb{R}) \times \mathbb{H}_n \longrightarrow \mathrm{GL}(n, \mathbb{C})$ as defined in definition 4.5 on page 50.

In the following we chose once and forall a holomorphic square root and denote it by

$$\sqrt{det(\mathcal{J}(M, Z))} = \sqrt{det(CZ + D)}.$$

Definition 4.11 (Multiplier system)
Let Γ be a congruence subgroup in $\mathrm{Sp}(n, \mathbb{R})$. A map $v_\Gamma = v : \Gamma \longrightarrow \mathbb{C}^*$ is called a multiplier system of weight $r/2$ if the function

$$v(M)\sqrt{\det CZ + D}^{\,r}$$

is a factor of automorphy.

It is worth mentioning that multiplier systems of integral weight are characters.

4.3 Modular forms of half integral weight

Let Γ be a congruence subgroup in $\mathrm{Sp}(n, \mathbb{R})$, v a multiplier system of weight $r/2$ and ρ be a rational representation of $\mathrm{GL}(n, \mathbb{C})$ on a vector space V. Then,

$$v(M)\sqrt{\det CZ + D}^{\,r}\rho(CZ + D)$$

is a factor of automorphy.
By a **vector valued modular form** with respect to this factor of automorphy we mean a holomorphic function $f : \mathbb{H}_n \to V$ which transforms as

$$f(M\langle Z\rangle) = v(M)\sqrt{\det CZ + D}^{\,r}\rho(CZ + D)f(Z)$$

under Γ.

In the case where $n = 1$, the usual condition at the cusps has to be added. For $n \geq 2$, the Koecher principle ensures this, cf. [Fre83, Hilfssatz 4.11, p.175].

The vector space of modular forms is denoted by $\left[\Gamma, \left(\frac{r}{2}, \rho\right), v\right]$. If v is trivial and $r = 2k$ is even, then we shorten this to $[\Gamma, (k, \rho)]$.

The pairs $(\frac{r}{2}, \rho)$ and $(\frac{r-2k}{2}, \det^k \cdot \rho)$ define the same factor of automorphy. Hence, for an irreducible representation ρ we always may assume that ρ is reduced, i.e. it is polynomial and does not vanish on $\{\det(A) = 0\}$. After this normalization we call $r/2$ the **weight** of a vector valued modular form.

Lemma 4.12
Vector valued modular forms of negative weight are identically zero.

The proof of the lemma is a modification of [Fre83, Satz 3.13, I.3. Modulformen n-ten Grades, p.48].

Definition 4.13 (Graded algebra of modular forms)
We can construct from a multiplier system of weight $1/2$ its graded algebra of modular forms of half integral weight $A(\Gamma, v) := \bigoplus_{r \in \mathbb{Z}} \left[\Gamma, \frac{r}{2}, v^r\right]$.

Example 4.14
The vector space $[\Gamma, ((n+1)k, \rho_e \odot \rho_e)]$ consists of all the holomorphic functions

$$f : \mathbb{H}_n \longrightarrow \mathrm{Sym}^2(\mathbb{C}^n)$$

satisfying

$$f(M\langle Z\rangle) = \det(CZ + D)^{(n+1)k} \cdot (CZ + D) f(Z) (CZ + D)^t \qquad \forall\ M \in \Gamma.$$

We want to identify vector valued modular forms with tensors on $\mathbb{H}_n$; therefore, we fix some notation.

We consider the wedge product of all 1-forms dZ^{ij} in lexicographic order by

$$\bigwedge dZ^{ij} = \bigwedge_{1 \leq i < j \leq n} dZ^{ij}.$$

Lemma 4.15
There is an isomorphism between the vector space $[\Gamma, ((n+1)k, \rho_e \odot \rho_e)]$ *of vector valued modular forms transforming as*

$$f(M\langle Z\rangle) = \det(CZ + D)^{(n+1)k} \cdot (CZ + D) f(Z) (CZ + D)^t$$

and the subspace of Γ*-invariant tensors in* $\left(\Omega \otimes_{\mathcal{O}} \left(\Lambda^{\frac{n(n+1)}{2}}\Omega\right)^{\otimes k}\right)(\mathbb{H}_n)$*, i.e.*

$$\begin{array}{rccl} \Phi : & \left(\left(\Omega \otimes_{\mathcal{O}} \left(\Lambda^{\frac{n(n+1)}{2}}\Omega\right)^{\otimes k}\right)(\mathbb{H}_n)\right)^{\Gamma} & \longrightarrow & [\Gamma, ((n+1)k, \rho_e \odot \rho_e)]\,, \\ & \omega = \sum_{i<j} f_{ij}\ dZ^{ij} \otimes \left(\bigwedge dZ^{ij}\right)^{\otimes k} & \longmapsto & (f_{ij})_{1 \le i,j \le n}\,. \end{array}$$

Proof
This follows directly from lemma 4.6 on page 50. □

We shall also consider Γ-invariant tensors in $\left(\Lambda^{(\frac{n(n+1)}{2}-1)}\Omega \otimes_{\mathcal{O}} \left(\Lambda^{\frac{n(n+1)}{2}}\Omega\right)^{\otimes k}\right)(\mathbb{H}_n)$. They can be identified with vector valued modular forms transforming as

$$f(M\langle Z\rangle) = \det(CZ+D)^{(n+1)(k+1)}(CZ+D)^{-t} f(Z)(CZ+D)^{-1}$$

under Γ, cf. [Fre83, 4.6_1 Folgerung, p.172].

We have already seen on page 30 that in the case where $n = 2$, the reduced representation $\det^2(A)A^{-t}XA^{-1}$ is isomorphic to $\rho_e \odot \rho_e$. Therefore, we obtain the subsequent result.

Theorem 4.16
The vector space $[\Gamma, (3k+1, \rho_e \odot \rho_e)]$ *is isomorphic to the subspace of* Γ*-invariant tensors in* $\left(\Lambda^2\Omega \otimes_{\mathcal{O}} (\Lambda^3\Omega)^{\otimes k}\right)(\mathbb{H}_2)$*. The tensor*

$$\left(f_0\ dZ^1 \wedge dZ^2 + f_1\ dZ^0 \wedge dZ^2 + f_2\ dZ^0 \wedge dZ^1\right) \otimes \left(\bigwedge dZ^{ij}\right)^{\otimes k}$$

is mapped to the vector valued modular form

$$\begin{pmatrix} f_2 & -f_1 \\ -f_1 & f_0 \end{pmatrix}.$$

4.4 Theta series

Definition 4.17 (Even vector)
We call a vector $\mathfrak{m} = (\mathfrak{m}^1, \mathfrak{m}^2) \in \mathbb{Z}^{2n}$ **even** if the Euclidean scalar product of $\mathfrak{m}^1$ and $\mathfrak{m}^2$ $\left\langle \mathfrak{m}^1, \mathfrak{m}^2 \right\rangle \equiv 0 \mod 2$.

Definition 4.18 (Theta series)
On $\mathrm{Sym}^2(\mathbb{C}^n)$ we define different kinds of theta series :

of the first kind

$$\vartheta\left[\mathfrak{m}\right](Z) := \vartheta\begin{bmatrix}\mathfrak{m}^1\\ \mathfrak{m}^2\end{bmatrix}(Z) := \sum_{g\in\mathbb{Z}^n} exp\left(\pi i\left(Z\left[g+\frac{\mathfrak{m}^1}{2}\right]+\left(g+\frac{\mathfrak{m}^1}{2}\right)^t\mathfrak{m}^2\right)\right)$$

for the **characteristic** $\mathfrak{m} = (\mathfrak{m}^1, \mathfrak{m}^2) \in \{0,1\}^{2n} \subset \mathbb{Z}^{2n}$;

of the second kind

$$f_a(Z) := \vartheta\begin{bmatrix}a\\ 0\end{bmatrix}(2Z) = \sum_{g\in\mathbb{Z}^n} exp\left(2\pi i Z\left[g+\frac{a}{2}\right]\right), \quad \text{for } a \in \{0,1\}^n.$$

If desired we may observe the above parameter a as an element of $(\mathbb{F}_2)^n$.
We shall multiply the different theta series of the first kind with even characteristic

$$\theta(Z) := \prod_{\mathfrak{m}\in\{0,1\}^{2n} even} \vartheta\left[\mathfrak{m}\right](Z).$$

The theta series of the first and second kind are holomorphic functions on $\mathbb{H}_n$. We state another of their properties that can be found on page 233 of [Igu64b].

Theorem 4.19
The theta series of the first and second kind are related in the following manner

$$\vartheta^2\begin{bmatrix}\mathfrak{m}^1\\ \mathfrak{m}^2\end{bmatrix}(Z) = \sum_{a\in(\mathbb{F}_2)^n} (-1)^{\langle a,\mathfrak{m}^2\rangle} f_{a+\mathfrak{m}^1}(Z)\cdot f_a(Z).$$

Corollary 4.20
The function θ^2 can be expressed in terms of f_a, i.e.

$$\theta^2 = \prod_{\mathfrak{m}\in\{0,1\}^{2n} even} \vartheta^2\left[\mathfrak{m}\right] = \prod_{\mathfrak{m}\in\{0,1\}^{2n} even}\left(\sum_{a\in(\mathbb{F}_2)^n} (-1)^{\langle a,\mathfrak{m}^2\rangle} f_{a+\mathfrak{m}^1}(Z)\cdot f_a(Z)\right).$$

Runge deduces on page 59 of [Run93] the following corollary.

Corollary 4.21
There is not a single Z in $\mathbb{H}_2$ which is a common root of all four thetas, i.e. $f_a(Z) = 0$.

On $\mathbb{H}_2$ the modular form θ looks like :

$$\begin{aligned}\chi_5(Z) := \theta(Z) = \vartheta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(Z)\,\cdot\,\vartheta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}(Z)\,\cdot\,\vartheta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}(Z)\,\cdot\,\vartheta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}(Z)\,\cdot\\ \vartheta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}(Z)\,\cdot\,\vartheta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}(Z)\,\cdot\,\vartheta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(Z)\,\cdot\,\vartheta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}(Z)\,\cdot\\ \vartheta\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}(Z)\,\cdot\,\vartheta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(Z).\end{aligned}$$

Theorem 4.22 (Igusa)
χ_5 is a cusp form of weight 5 to a multiplier system v_{χ_5}, i.e. $\chi_5 \in [\mathrm{Sp}(2,\mathbb{Z}), 5, v_{\chi_5}]_0$.

PROOF
We refer to [Maa64] and in particular to page 135 for the explicit characterisation of v_{χ_5} by

$$v_{\chi_5}(J_2) = 1, \qquad v_{\chi_5}\begin{pmatrix}I_2 & S\\ 0 & I_2\end{pmatrix} = \exp(\pi i(s_0+s_1+s_2)) \text{ with } S = \begin{pmatrix}s_0 & s_1\\ s_1 & s_2\end{pmatrix}$$

$$\text{and} \quad v_{\chi_5}\begin{pmatrix}U & 0\\ 0 & U^{-t}\end{pmatrix} = \exp(\pi i(u_{11}u_{22} + (1+u_{11}+u_{22})(1+u_{12}+u_{21})))$$

$$\text{with } U = \begin{pmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}). \qquad \square$$

A fundamental result on the zero locus of χ_5 is the subsequent one, cf. Satz 2 in [Fre65].

Theorem 4.23
The zero locus of the above χ_5 on $\mathbb{H}_2$ is $\bigcup_{M\in\Gamma_2} M\langle\Delta_2\rangle$.

Later, we shall use the subsequent two theorems.

Theorem 4.24 (Structure theorem for $A(\Gamma_2[2,4], v_f)$)
The functions $f_0,\ldots,f_3$ are modular forms of weight $1/2$ with respect to Igusa's group $\Gamma_2[2,4]$ and a common multiplier system v_f. Nota bene $v_f^4 = 1$.
In particular, the whole ring of modular forms with respect to $\Gamma_2[2,4]$ is generated by the theta constants, i.e.

$$A(\Gamma_2[2,4], v_f) = \bigoplus_{r\in\mathbb{N}} \left[\Gamma_2[2,4], \frac{r}{2}, v_f^r\right] = \mathbb{C}[f_0, f_1, f_2, f_3].$$

The modular form χ_5 has a trivial multiplier system on $\Gamma_2[2,4]$ and hence is not contained in $A(\Gamma_2[2,4], v_f)$. The following variant of the previous theorem is also true.

Theorem 4.25 (Structure theorem for $A^{int}(\Gamma_2\,[2,4])$)
The graded algebra of scalar modular forms with respect to $\Gamma_2\,[2,4]$ *and the trivial multiplier system can be decomposed*

$$\begin{aligned} A^{int}(\Gamma_2\,[2,4]) &= \bigoplus_{r\in\mathbb{N}} [\Gamma_2\,[2,4], r] \\ &= \left(\bigoplus_{d\geq 0} \mathbb{C}_{4d}\,[f_0, f_1, f_2, f_3]\right) \bigoplus \left(\bigoplus_{d\geq 0} \mathbb{C}_{4d}\,[f_0, f_1, f_2, f_3]\right) \cdot \chi_5 \end{aligned}$$

This version can be found in [Run93, Remark 3.15, p.74]. And theorem 4.24 is a rather simple consequence using the methods of Chapter 5, cf. theorem 5.24 on page 80.

4.5 Quotient spaces

We are now interested in the equivalence classes of $\Gamma_2\,[2,4]$'s action on $\mathbb{H}_2$. We state and prove the basic results for an arbitrary group Γ acting totally discontinuously on a domain $\mathcal{D}$ in $\mathbb{C}^n$.

4.5.1 The general case ${}^{\Gamma}\backslash{}^{\mathcal{D}}$

In this subsection, we consider a domain $\mathcal{D} \subset \mathbb{C}^n$ and a group Γ acting totally discontinuously by biholomorphic functions on $\mathcal{D}$

$$\rho : \Gamma \to \mathrm{Bihol}(\mathcal{D})\,.$$

We use the notation

$$\overline{\Gamma} := {}^{\Gamma}/_{\ker(\rho)}.$$

Lemma 4.26
Suppose a group Γ acts totally discontinuously on $\mathcal{D} \subset \mathbb{C}^n$. Then for any point p there is a Γ_p-invariant neighbourhood U and a biholomorphic map $\lambda : U \to V \subset \mathbb{C}^n$, such that $\lambda\Gamma_p\lambda^{-1}$ acts **linear**, *i.e. $\lambda\overline{\Gamma}_p\lambda^{-1}$ is a subgroup of $\mathrm{GL}(n,\mathbb{C})$, say G_p, that induces a biholomorphic map ${}^{\Gamma_p}\backslash{}^{U} \to {}^{G_p}\backslash{}^{V}$.*

Proof
The proof can be found in [Car54, Lemme 1, p. 12-3]. □

We denote by $\mathrm{S}(\Gamma)$ the set of **fixed point**s of Γ, i.e. $\mathrm{S}(\Gamma) = \{p \in \mathcal{D} : \overline{\Gamma}_p \supsetneq \{id\}\}$. This is a closed analytic subvariety.

The group $\overline{\Gamma}$ acts freely on $\mathcal{D} \setminus \mathrm{S}(\Gamma)$. The quotient ${}_{\Gamma}\backslash^{(\mathcal{D} \setminus \mathrm{S}(\Gamma))}$ carries the structure of a complex manifold such that the natural projection is locally biholomorphic.

Definition 4.27 (Harmless point)
We call a point p in $\mathcal{D}$ Γ-**harmless** if it satisfies

1. $\overline{\Gamma}_p$ is cyclic with generator γ_p;

2. the fixed point set of γ_p is a *smooth* hypersurface around p.

By $\mathrm{Harm}(\Gamma)$ we denote the collection of non-harmless or harmful points in $\mathrm{S}(\Gamma)$.

Let p be a harmless point in $\mathcal{D}$ and G_p a linearization of $\overline{\Gamma}_p$ in the sense of lemma 4.26. Let γ_p be the generator of $\overline{\Gamma}_p$we may assume that it is diagonal due to its finite order. Only one diagonal entry may differ from 1 because the fixed point set has got codimension 1. Therefore, γ_p is of the form

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \zeta \end{pmatrix},$$

where ζ is a d-th root of unity.

Using the map

$$(z^1, \ldots, z^n) \longmapsto (z^1, \ldots, (z^n)^d)$$

we can identify ${}_{G}\backslash^{\mathbb{C}^n}$ with $\mathbb{C}^n$.

For the sake of simplicity, we introduce the abbreviation of $\mathcal{D}\backslash\mathrm{Harm}(\Gamma)$ to $\mathcal{D}_0$.

Lemma 4.28
The set $\mathrm{S}(\Gamma) \setminus \mathrm{Harm}(\Gamma)$ *of harmless fixed points is a closed analytic submanifold of* $\mathcal{D}_0$ *of codimension 1. The quotient* ${}_{\Gamma} \backslash {}^{\mathcal{D}_0}$ *is a complex analytic manifold. The projection*

$$\mathcal{D}_0 \longrightarrow {}_{\Gamma} \backslash {}^{\mathcal{D}_0}$$

is locally a simple covering.

Lemma 4.29
A point p *of* $\mathrm{S}(\Gamma)$ *is harmless iff* $\mathrm{S}(\Gamma)$ *is a smooth hypersurface around* p*. As a consequence, the set of harmful points is negligible in* $\mathcal{D}$*.*

In chapter 5 the occurring fixed points will all be harmless so this lemma is redundant. Hence, we shall just sketch the proof.
We make use of the fact that the quotient ${}_{\Gamma}\backslash^{\mathcal{D}}$ carries the structure of a complex normal space in the sense of Serre, cf. [GR84, Chap 1, §1, 5. Complex Spaces, p.8]. The image T of $\mathrm{S}(\Gamma)$ under the natural projection is a closed analytic subvariety of ${}_{\Gamma}\backslash^{\mathcal{D}}$. We may assume that T is smooth and of codimension 1. In this new setting, the natural projection is locally a simple covering.

If the group $\overline{\Gamma}$ acts freely on $\mathcal{D}$, then the Γ-invariant holomorphic tensors on $\mathcal{D}$ can be identified with holomorphic tensors on ${}_{\Gamma}\backslash^{\mathcal{D}}$.
We want to generalize this correspondence to groups with non-free action on $\mathcal{D}$. Therefore, we define the **ramification divisor** $\mathcal{R}$ in the quotient manifold ${}_{\Gamma}\backslash^{\mathcal{D}_0}$. $\mathcal{R}$ is supported by the image of $\mathrm{S}(\Gamma) \setminus \mathrm{Harm}(\Gamma)$. The multiplicity of an irreducible component Y is given as follows. Take p to be in Y and x to be a point in $\mathcal{D}$ that also lies in the preimage of p. Then, we set the multiplicity $\mathcal{R}(Y)$ to be $\#G_x - 1$.

For the subsequent theorem, we refer the reader to the definition of the space $\Omega^{\otimes k}({}_{\Gamma}\backslash^{\mathcal{D}_0}, \mathcal{R})$ on page 47.

Theorem 4.30
There is an isomorphism between the vector space of Γ*-invariant tensors on* $\Omega^{\otimes k}(\mathcal{D})^{\Gamma}$ *and the vector space* $\Omega^{\otimes k}({}_{\Gamma}\backslash^{\mathcal{D}_0}, \mathcal{R})$*, i.e.*

$$\pi^* : \Omega^{\otimes k}({}_{\Gamma}\backslash^{\mathcal{D}_0}, \mathcal{R}) \longrightarrow \Omega^{\otimes k}(\mathcal{D})^{\Gamma}, \quad \eta \longmapsto \pi^*\eta = \omega.$$

Proof
Since the set $\mathrm{Harm}(\Gamma)$ is negligible, the space $\Omega^{\otimes k}(\mathcal{D})^{\Gamma}$ coincides with $\Omega^{\otimes k}(\mathcal{D}_0)^{\Gamma}$. The natural projection $\mathcal{D}_0 \to {}_{\Gamma}\backslash^{\mathcal{D}_0}$ is locally isomorphic to a standard element. The definition of $\Omega^{\otimes k}({}_{\Gamma}\backslash^{\mathcal{D}_0}, \mathcal{R})$ yields the desired result. □

4.5.2 The quotient space ${}_{\Gamma_2[2,4]}\backslash^{\mathbb{H}_2}$

We are now observing the special case of the group $\Gamma_2[2,4]$ acting on $\mathbb{H}_2$. This group acts without harmful points on $\mathbb{H}_2$, as we can deduce from lemma 4.9 on page 51. It also follows that all elements of finite order are conjugated to the matrix

$$\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}.$$

The fixed point set $\mathrm{S}(\Gamma_2[2,4])$ is a disjoint union of closed smooth hypersurfaces, the images of the diagonal Δ_2 under Γ_2.

Lemma 4.31
The ramification divisor $\mathcal{R}$ in ${}_{\Gamma_2[2,4]}\backslash^{\mathbb{H}_2}$ consists of 10 components all with multiplicity 1.

We deduce from theorem 4.23 on page 56 the succeeding lemma.

Lemma 4.32
$\mathrm{S}(\Gamma_2[2,4])$*, the subset of fixed points in $\mathbb{H}_2$, is the zero locus of $\chi_5 = \prod_{\mathfrak{m}\in\{0,1\}^4 even} \vartheta[\mathfrak{m}]$. Moreover, the image of the zero locus of an individual $\vartheta[\mathfrak{m}]$ is one of the above mentioned 10 components of the type $M\langle\Delta_2\rangle$.*

The map

$$F : \mathbb{H}_2 \longrightarrow \mathbb{C}^4 \setminus \{0\}, \quad Z \longmapsto (f_0(Z), f_1(Z), f_2(Z), f_3(Z)),$$

factors through a map

$$\begin{tikzcd}
\mathbb{H}_2 \arrow[r, "F"] \arrow[d, "p"'] & \mathbb{C}^4 \setminus \{0\} \arrow[d, "\pi"] \\
{}_{\Gamma_2[2,4]}\backslash^{\mathbb{H}_2} \arrow[r, dotted, "\phi"] & \mathbb{P}^3\mathbb{C}.
\end{tikzcd}$$

From Runge's results [Run93] one can deduce the following striking theorem.

Theorem 4.33
The map $\phi : {}_{\Gamma_2[2,4]}\backslash^{\mathbb{H}_2} \to \mathbb{P}^3\mathbb{C}$ is a holomorphic embedding and even biholomorphic on its open image. The image's complement is negligible.

Instead of observing divisors, functions, and tensors on ${}_{\Gamma_2[2,4]}\backslash^{\mathbb{H}_2}$ we can study their counterparts on $\mathbb{P}^3\mathbb{C}$. In particular, we can consider R as a divisor on $\mathbb{P}^3\mathbb{C}$. From theorem 4.19 on page 55 we deduce the subsequent theorem.

Theorem 4.34
The ramification divisor $\mathcal{R}$ considered on $\mathbb{P}^3\mathbb{C}$ is the sum of 10 quadrics given by the following 10 polynomials

- $Q_0(z^0, \ldots, z^3) = (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2;$
- $Q_1(z^0, \ldots, z^3) = (z^0)^2 - (z^1)^2 + (z^2)^2 - (z^3)^2;$

- $Q_2(z^0, \dots, z^3) = (z^0)^2 + (z^1)^2 - (z^2)^2 - (z^3)^2$;
- $Q_3(z^0, \dots, z^3) = (z^0)^2 - (z^1)^2 - (z^2)^2 + (z^3)^2$;
- $Q_4(z^0, \dots, z^3) = 2(z^0 z^1 + z^2 z^3)$;
- $Q_6(z^0, \dots, z^3) = 2(z^0 z^1 - z^2 z^3)$;
- $Q_8(z^0, \dots, z^3) = 2(z^0 z^2 + z^1 z^3)$;
- $Q_9(z^0, \dots, z^3) = 2(z^0 z^2 - z^1 z^3)$;
- $Q_{12}(z^0, \dots, z^3) = 2(z^0 z^3 + z^1 z^2)$;
- $Q_{15}(z^0, \dots, z^3) = 2(z^0 z^3 - z^1 z^2)$.

We close this section with its main result.

Theorem 4.35
There is a natural isomorphism

$$\Omega^{\otimes q}(\mathbb{H}_2)^{\Gamma_2[2,4]} \cong \Omega^{\otimes q}(\mathbb{P}^3\mathbb{C}, \mathcal{R}).$$

The tensors on the left hand side can be considered as certain vector valued Siegel modular forms and the ones on the right hand side can be easily described in algebraic terms.

5 Existence results and structure theorems

5.1 Construction of meromorphic tensors with prescribed poles

For the construction of these tensors, we consider a homogeneous polynomial Q of degree d in $n+1$ variables $X^0, \ldots, X^n$. We always assume that Q is square-free and that X^0 does not divide Q. We want to describe meromorphic tensors on $\mathbb{P}^n\mathbb{C}$ that are holomorphic outside Q's zero locus $Z(Q)$. Any such tensor is determined by its restriction to the affine chart $\mathbb{A}_0$. Recall that the projective coordinates of $\mathbb{P}^n\mathbb{C}$ are denoted by $z^0, \ldots, z^n$ and the coordinates on $\mathbb{A}_0$ are

$$(\mathfrak{z}^1, \ldots, \mathfrak{z}^n) := \left(\frac{z^1}{z^0}, \ldots, \frac{z^n}{z^0}\right).$$

Hence, every meromorphic tensor on $\mathbb{P}^n\mathbb{C}$ can be written in the form

$$\omega = \sum_{i_1,\ldots,i_r} \omega_{i_1,\ldots,i_r} d\mathfrak{z}^{i_1} \otimes \ldots \otimes d\mathfrak{z}^{i_r}.$$

The coefficient functions $\omega_{i_1,\ldots,i_r}$ are rational functions in the variables $\mathfrak{z}^1, \ldots, \mathfrak{z}^n$.

By means of homogenization, every rational function $f(\mathfrak{z})$ can be written as the quotient of two homogeneous polynomials $A, B \in \mathbb{C}[X^0, \ldots, X^n]$ of common degree, i.e.

$$f(\mathfrak{z}) = \frac{A(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n)}{B(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n)}.$$

The restriction of the above introduced tensor ω to $\mathbb{A}_0$ is holomorphic outside $\left\{Q(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n) = 0\right\}$ iff the coefficients of ω can be written in the form

$$\omega_{i_1,\ldots,i_r}(\mathfrak{z}^1, \ldots, \mathfrak{z}^n) = \frac{A_{i_1,\ldots,i_r}(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n)}{Q^N(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n)},$$

where N is an appropriate natural number and $A_{i_1,\dots,i_r}$ is a homogeneous polynomial of degree $N \cdot \deg Q$ in $n+1$ variables.

This condition does not imply that ω is holomorphic on $\mathbb{P}^n\mathbb{C} \setminus Z\,(Q)$. But, it is sufficient to consider a second chart $\mathbb{A}_1$ because $\mathbb{A}_0 \cup \mathbb{A}_1$ is complemented by a negligible set. Recall that the coordinates in this affine space are defined as

$$\left(\mathfrak{w}^0, \mathfrak{w}^2 \dots, \mathfrak{w}^n\right) := \left(\frac{z^0}{z^1}, \frac{z^2}{z^1}, \dots, \frac{z^n}{z^1}\right).$$

We change between these two charts by

$$(\mathfrak{z}^1, \dots, \mathfrak{z}^n) = \left(\frac{1}{\mathfrak{w}^0}, \frac{\mathfrak{w}^2}{\mathfrak{w}^0}, \dots, \frac{\mathfrak{w}^n}{\mathfrak{w}^0}\right).$$

For the sake of simplicity we take ω to be of the type $\Omega \otimes_{\mathcal{O}} (\Lambda^n \Omega)^{\otimes k}$.

Lemma 5.1 (Construction of meromorphic 1-forms with prescribed poles) *Let ω be a meromorphic tensor on $\mathbb{P}^n\mathbb{C}$ which is holomorphic outside $Z\,(Q)$, then ω can be written in the form*

$$\omega = \sum_{i=1}^{n} \omega_i \; d\mathfrak{z}^i \otimes \left(d\mathfrak{z}^1 \wedge \dots \wedge d\mathfrak{z}^n\right)^{\otimes k},$$

where

$$\omega_i(\mathfrak{z}^1, \dots, \mathfrak{z}^n) = \frac{A_i(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}{Q^N(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}$$

with the following properties

1. *N is a natural number;*
2. *A_i is a homogeneous polynomial of degree $N \cdot \deg Q$;*
3. *$(X^0)^{k(n+1)+1}$ divides A_i;*
4. *and $(X^0)^{k(n+1)+2}$ divides the sum $\sum_{i=1}^{n} X^i A_i$.*

PROOF
On $\mathbb{A}_1$ the tensor ω is of the form

$$\begin{aligned}\omega = & \sum_{i=2}^{n} \frac{(-1)^k}{(\mathfrak{w}^0)^{k(n+1)+1}}\, \omega_i \left(\frac{1}{\mathfrak{w}^0}, \frac{\mathfrak{w}^2}{\mathfrak{w}^0}, \ldots, \frac{\mathfrak{w}^n}{\mathfrak{w}^0} \right) d\mathfrak{w}^i \otimes \left(d\mathfrak{w}^0 \wedge d\mathfrak{w}^2 \wedge \ldots \wedge d\mathfrak{w}^n \right)^{\otimes k} \\ & - \frac{(-1)^k}{(\mathfrak{w}^0)^{k(n+1)+2}} \left(\omega_1 + \sum_{i=2}^{n} \mathfrak{w}^i \cdot \omega_i \right) d\mathfrak{w}^0 \otimes \left(d\mathfrak{w}^0 \wedge d\mathfrak{w}^2 \wedge \ldots \wedge d\mathfrak{w}^n \right)^{\otimes k}.\end{aligned}$$

The coefficients in the first line are holomorphic outside $\left\{\mathfrak{w}^0 = 0\right\}$ iff $(\mathfrak{w}^0)^{k(n+1)+1}$ divides $A_i(\mathfrak{w}^0, 1, \mathfrak{w}^2, \ldots, \mathfrak{w}^n)$. Since A_i is homogeneous, this means that $(X^0)^{k(n+1)+1}$ divides A_i for all $i \geq 2$. Observing ω_1 or its numerator A_1, respectively, on $\mathbb{A}_2$ gives an analogous result. The same argument shows that the coefficient in the second line is holomorphic outside $\left\{\mathfrak{w}^0 = 0\right\}$ iff condition 4 is satisfied. $\square$

It is convenient to write the fourth condition in the following way. We introduce the matrix

$$T = \begin{pmatrix} -X^1 & -X^2 & \ldots & -X^n \\ 0 & X^0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & X^0 \end{pmatrix}.$$

Remark 5.2
Condition 4 in lemma 5.1 is equivalent to the condition that

$$X^0 \mid T \cdot \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}.$$

There is a generalisation of the above lemma that can be proven analogously. We consider tensors of the type

$$\Lambda^p \Omega \otimes_{\mathcal{O}} (\Lambda^n \Omega)^{\otimes k}.$$

For this, we introduce a handy notation for the canonical basis elements of $\Lambda^p\Omega$. Given a subset $I \subset \{1, \ldots, n\}$, say $I = \{i_1, \ldots, i_p\}$ and $i_1 < \cdots < i_p$, then $d\mathfrak{z}^I$ is short for

$$d\mathfrak{z}^I = d\mathfrak{z}^{i_1} \wedge \ldots \wedge d\mathfrak{z}^{i_p}.$$

Lemma 5.3 (Construction of meromorphic *p*-forms with prescribed poles)
Let ω be a meromorphic tensor on $\mathbb{P}^n\mathbb{C}$ which is holomorphic outside $Z(Q)$, then ω can

be written in the form

$$\omega = \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=p}} \omega_I \; d\mathfrak{z}^I \otimes \left(d\mathfrak{z}^1 \wedge \cdots \wedge d\mathfrak{z}^n\right)^{\otimes k},$$

where

$$\omega_I(\mathfrak{z}^1,\dots,\mathfrak{z}^n) = \frac{A_I(1,\mathfrak{z}^1,\dots,\mathfrak{z}^n)}{Q^N(1,\mathfrak{z}^1,\dots,\mathfrak{z}^n)}$$

with the following properties

1. *N is a natural number;*

2. *A_I is a homogeneous polynomial of degree $N\cdot\deg Q$;*

3. *$(X^0)^{k(n+1)+p}$ divides A_I;*

4. *and it holds for all $J\subset\{2,\dots,n\}$ with $|J|=p-1$*

$$(X^0)^{k(n+1)+p+1} \;\Big|\; \sum_{\substack{1\le j\le n\\ j\notin J}} (-1)^{pos(j,J\cup\{j\})}\cdot X^j \cdot A_{J\cup\{j\}},$$

where $pos(i,I)$ returns i's position in the ordered set I.

For $p=n$ conditions 3 and 4 are merged to

3'. *$(X^0)^{k(n+1)+n+1}$ divides $A_{\{1,\dots,n\}}$.*

Again, we reformulate condition 4. For this, we consider the $\Lambda^p\mathbb{C}^n$-valued polynomial (A_I) where I runs through all subsets of $\{1,\dots,n\}$ of order p.

Remark 5.4
Condition 4 in lemma 5.1 is equivalent to the condition that

$$\left(X^0\right)^p \;|\; (\Lambda^p T)\cdot(A_I).$$

5.2 Existence results for covering holomorphic tensors

As in the previous section, we consider a square-free polynomial Q and a divisor with support in $\{Q=0\}$

$$\mathcal{R} = \sum a_{\mathfrak{m}} Z\left(Q_{\mathfrak{m}}\right), \quad a_{\mathfrak{m}} \ge 0.$$

We want to describe the spaces

$$\left(\Omega \otimes_{\mathcal{O}} (\Lambda^n \Omega)^{\otimes k}\right)(\mathbb{P}^n\mathbb{C}, \mathcal{R}).$$

Since each element of this space, say ω, is holomorphic outside $\{Q = 0\}$ it is of the form

$$\omega_i(\mathfrak{z}^1, \ldots, \mathfrak{z}^n) = \frac{A_i(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n)}{Q^N(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n)} \tag{5.1}$$

with the following properties

1. N is a natural number;
2. A_i is a homogeneous polynomial of degree $D \cdot \deg Q$;
3. $(X^0)^{k(n+1)+1}$ divides A_i;
4. and $(X^0)^{k(n+1)+2}$ divides the sum $\sum_{i=1}^n X^i A_i$,

cf. lemma 5.1 on page 64.

Theorem 5.5
We introduce the numbers [1]

$$d := \max_{1 \leq \mathfrak{m} \leq M} \left\lfloor \frac{a_\mathfrak{m}}{a_\mathfrak{m} + 1} k \right\rfloor$$

and

$$D := \max_{1 \leq \mathfrak{m} \leq M} \left\lfloor \frac{a_\mathfrak{m}}{a_\mathfrak{m} + 1} (k+1) \right\rfloor .$$

Let ω be a tensor as in eq. (5.1) where we choose N minimal. We can state necessary and sufficient conditions in terms of d and D for ω to lie in $\left(\Omega \otimes_{\mathcal{O}} (\Lambda^n \Omega)^{\otimes k}\right)(\mathbb{P}^n\mathbb{C}, \mathcal{R})$:

$$\omega \in \left(\Omega \otimes_{\mathcal{O}} (\Lambda^n \Omega)^{\otimes k}\right)(\mathbb{P}^n\mathbb{C}, \mathcal{R}) \quad \Longrightarrow \quad N \leq D,$$

and

$$N \leq d \quad \Longrightarrow \quad \omega \in \left(\Omega \otimes_{\mathcal{O}} (\Lambda^n \Omega)^{\otimes k}\right)(\mathbb{P}^n\mathbb{C}, \mathcal{R}).$$

[1] in the following formula $\lfloor x \rfloor$ denotes the floor function $\max\{m \in \mathbb{Z} \mid m \leq x\}$.

Corollary 5.6
If it holds for all $\mathfrak{m}$

$$a_{\mathfrak{m}} + 1 \mid k,$$

then $d{=}D$*. Hence, the space* $\left(\Omega \otimes_{\mathcal{O}} (\Lambda^n\Omega)^{\otimes k}\right)(\mathbb{P}^n\mathbb{C}, \mathcal{R})$ *is completely determined.*

Proof (Theorem)
Let x be a smooth point of Q's zero locus and $Q_{\mathfrak{m}}$ its prime factor that vanishes at x. We have to consider an open neighbourhood $x \in V \subset \mathbb{P}^n\mathbb{C}$ and a simple covering $U \to V$ which ramifies over $Q_{\mathfrak{m}} = 0$. Since z^0 does not divide Q, we may assume that V is a subset of $\mathbb{A}_0$. Without loss of generality, we may assume that the n-th partial derivative of $Q_{\mathfrak{m}}$ is non-zero in V. Now, we can use the construction from remark 3.49 on page 45. The covering p is of the form

$$(\mathfrak{z}^1, \dots, \mathfrak{z}^n) \longmapsto (\mathfrak{z}^1, \dots, \mathfrak{z}^{n-1}, (\mathfrak{z}^n)^{a_{\mathfrak{m}}+1} + \varphi(\mathfrak{z}^1, \dots, \mathfrak{z}^{n-1})),$$

where φ is implicitly defined by

$$Q_{\mathfrak{m}}(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n) = 0 \quad \Longleftrightarrow \quad \mathfrak{z}^n = \varphi(\mathfrak{z}^1, \dots, \mathfrak{z}^{n-1}).$$

We have to observe the pullback $p^*\omega$ of the tensor

$$\omega = \sum_{i+1}^{n} \omega_i \; d\mathfrak{z}^i \otimes \left(d\mathfrak{z}^1 \wedge \cdots \wedge d\mathfrak{z}^n\right)^{\otimes k},$$

where

$$\omega_i(\mathfrak{z}^1, \dots, \mathfrak{z}^n) = \frac{A_i(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}{Q^N(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}$$

with the properties formulated in lemma 5.1. We have

$$p^*\omega = (p^*\omega)_j d\mathfrak{z}^j \otimes \left(\bigwedge_{m=1}^{n} d\mathfrak{z}^m\right)^{\otimes k}$$

with coefficient functions

$$(p^*\omega)_n = \left((a_{\mathfrak{m}} + 1)\,(\mathfrak{z}^n)^{a_{\mathfrak{m}}}\right)^{k+1} \omega_n \circ p \qquad \text{and}$$
$$(p^*\omega)_j = \left((a_{\mathfrak{m}} + 1)\,(\mathfrak{z}^n)^{a_{\mathfrak{m}}}\right)^{k} \left(\omega_j \circ p + \omega_n \circ p \cdot \frac{\partial\varphi}{\partial\mathfrak{z}^j}\right) \qquad \text{for } j \neq n.$$

We have to check whether the pullback $p^*\omega$ is holomorphic on U. This means that the holomorphic functions $(p^*\omega)_j$ must have non-negative orders along the line $\mathfrak{z}^n = 0$, i.e.

$$\operatorname{ord}\left((p^*\omega)_j, Z\left(\mathfrak{z}^n\right)\right) \geq 0, \quad 1 \leq j \leq n.$$

In the case where $j = n$, this is equivalent to

$$\operatorname{ord}\left(\omega_n \circ p, Z\left(\mathfrak{z}^n\right)\right) \geq -a_{\mathfrak{m}}(k+1).$$

Claim 1

The condition on all $j \in \{1, \ldots n\}$

$$\operatorname{ord}\left(\omega_j \circ p, Z\left(\mathfrak{z}^n\right)\right) \geq -a_{\mathfrak{m}}(k+1)$$

is necessary for $p^*\omega$ to be holomorphic.

Proof

Assuming that

$$\operatorname{ord}\left(\omega_j \circ p, Z\left(\mathfrak{z}^n\right)\right) < -a_{\mathfrak{m}}(k+1)$$

yields

$$\operatorname{ord}\left(\omega_j \circ p + \omega_n \circ p \cdot \frac{\partial \varphi}{\partial \mathfrak{z}^j}, Z\left(\mathfrak{z}^n\right)\right) = \operatorname{ord}\left(\omega_j \circ p, Z\left(\mathfrak{z}^n\right)\right) < -a_{\mathfrak{m}}(k+1)$$

because we have already verified the claim for $j = n$. □

In the same way we derive sufficient conditions for $p^*\omega$ to be holomorphic.

Condition 2

The condition on all $j \in \{1, \ldots n\}$

$$\operatorname{ord}\left(\omega_j \circ p, Z\left(\mathfrak{z}^n\right)\right) \geq -a_{\mathfrak{m}} \cdot k$$

is sufficient for $p^*\omega$ to be holomorphic.

Now, we want to consider the coefficient functions ω_j instead of the functions $\omega_j \circ p$. The zero set $Z\left(\mathfrak{z}^n\right)$ in U corresponds to the zero divisor $(Q_{\mathfrak{m}})$ of the polynomial $Q_{\mathfrak{m}}(1, \mathfrak{z}^1, \ldots, \mathfrak{z}^n)$ on V. We have

$$\operatorname{ord}\left(\omega_j \circ p, Z\left(\mathfrak{z}^n\right)\right) = (a_{\mathfrak{m}} + 1) \cdot \operatorname{ord}\left(\omega_j, (Q_{\mathfrak{m}})\right).$$

So, we obtain the necessary condition

$$\operatorname{ord}\left(\omega_j, (Q_{\mathfrak{m}})\right) \geq -\left\lfloor \frac{a_{\mathfrak{m}}}{a_{\mathfrak{m}}+1}(k+1) \right\rfloor, \quad 1 \leq j \leq n,$$

and the sufficient condition

$$\operatorname{ord}\left(\omega_j, (Q_{\mathfrak{m}})\right) \geq -\left\lfloor \frac{a_{\mathfrak{m}}}{a_{\mathfrak{m}}+1} k \right\rfloor, \quad 1 \leq j \leq n,$$

for ω to be in the space $\left(\Omega \otimes_{\mathcal{O}} \left(\Lambda^n \Omega\right)^{\otimes k}\right)(\mathbb{P}^n\mathbb{C}, \mathcal{R})$. This completes the proof of the theorem. □

The same method as above also works for tensors of the type $\left(\Lambda^p\Omega \otimes_{\mathcal{O}} (\Lambda^n\Omega)^{\otimes k}\right)(\mathbb{P}^n\mathbb{C}, \mathcal{R})$. Such a tensor ω can be written in the form

$$\omega = \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=p}} \omega_I \; d\mathfrak{z}^I \otimes \left(d\mathfrak{z}^1 \wedge \cdots \wedge d\mathfrak{z}^n\right)^{\otimes k}, \tag{5.2}$$

where

$$\omega_I(\mathfrak{z}^1, \dots, \mathfrak{z}^n) = \frac{A_I(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}{Q^N(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}$$

with the following properties

1. N is a natural number;
2. A_I is a homogeneous polynomial of degree $N \cdot \deg Q$;
3. $(X^0)^{k(n+1)+p}$ divides A_I;
4. and it holds for all $J \subset \{2, \dots, n\}$ with $|J| = p - 1$

   $$(X^0)^{k(n+1)+p+1} \;\Big|\; \sum_{\substack{1\le j\le n\\ j\notin J}} (-1)^{pos(j,J\cup\{j\})} \cdot X^j \cdot A_{J\cup\{j\}},$$

   where $pos(i, I)$ returns i's position in the ordered set I.

For $p = n$ conditions 3 and 4 are merged to

3'. $(X^0)^{k(n+1)+n+1}$ divides $A_{\{1,\dots,n\}}$,

cf. lemma 5.3 on page 65.

We recall the constants from theorem 5.5 on page 67 :

$$d = \max_{1\le\mathfrak{m}\le M} \left\lfloor \frac{a_{\mathfrak{m}}}{a_{\mathfrak{m}}+1} k \right\rfloor$$

and

$$D = \max_{1\le\mathfrak{m}\le M} \left\lfloor \frac{a_{\mathfrak{m}}}{a_{\mathfrak{m}}+1} (k+1) \right\rfloor.$$

Theorem 5.7
Let ω be a tensor as in eq. (5.2), where we choose N minimal. A necessary condition for ω to lie in $\Lambda^p\Omega \otimes_{\mathcal{O}} (\Lambda^n\Omega)^{\otimes k}(\mathbb{P}^n\mathbb{C}, \mathcal{R})$ is $N \le D$. If p equals n, this is also sufficient. Otherwise, it suffices to set $N \le d$.

Corollary 5.8
If it holds for all $\mathfrak{m}$

$$a_{\mathfrak{m}} + 1 \mid k,$$

then d=D. As a consequence, the space $\Lambda^p\Omega \otimes_{\mathcal{O}} (\Lambda^n\Omega)^{\otimes k}(\mathbb{P}^n\mathbb{C}, \mathcal{R})$ *is completely determined.*

5.3 A structure theorem for vector valued modular forms with respect to the multiplier system $\mathrm{v_f}$

We consider the polynomial $Q = \prod_{\mathfrak{m}} Q_{\mathfrak{m}}$ of degree 20 that gives the divisor of χ_5 in $\mathbb{P}^3\mathbb{C}$, cf. theorem 4.34 on page 60. We recall that

$$\left[\Gamma_2\,[2,4], \left(\frac{12k}{2}, \rho_e \odot \rho_e\right), v_f^{12k}\right] \cong \left(\Omega \otimes_{\mathcal{O}} \left(\Lambda^3\Omega\right)^{\otimes 2k}\right)(\mathbb{P}^3\mathbb{C}, (Q))$$

according to lemma 4.15 on page 53.

Hence, we can reformulate the results of the previous sections in terms of modular forms.

Theorem 5.9
If we denote by f_a *the theta constants of the second kind, then the determinant*

$$\begin{vmatrix} \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^0} & \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^1} & \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^2} \\ \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^0} & \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^1} & \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^2} \\ \frac{\partial\left(\frac{f_3}{f_0}\right)}{\partial Z^0} & \frac{\partial\left(\frac{f_3}{f_0}\right)}{\partial Z^1} & \frac{\partial\left(\frac{f_3}{f_0}\right)}{\partial Z^2} \end{vmatrix}$$

equals

$$c_5 \cdot \frac{\chi_5}{(f_0)^4}$$

with a constant c_5 *in* $\mathbb{C}$.

PROOF
A proof can be found on pages 15 and 16 of [FSM10]. □

We shall present an easy and well-known lemma.

Lemma 5.10 (Rankin-Cohen bracket)
For f and g in $\left[\Gamma_2\,[2,4], \frac{1}{2}, v_f\right]$ the ***Rankin-Cohen bracket***

$$\{f,g\} := f\cdot Dg - g\cdot Df = \begin{pmatrix} \left(f\frac{\partial g}{\partial Z^0} - g\frac{\partial f}{\partial Z^0}\right) & \left(f\frac{\partial g}{\partial Z^1} - g\frac{\partial f}{\partial Z^1}\right) \\ \left(f\frac{\partial g}{\partial Z^1} - g\frac{\partial f}{\partial Z^1}\right) & \left(f\frac{\partial g}{\partial Z^2} - g\frac{\partial f}{\partial Z^2}\right) \end{pmatrix} = f^2\cdot D\left(\frac{g}{f}\right)$$

lies in $\left[\Gamma_2\,[2,4], (1, \rho_e\odot\rho_e), v_f^2\right]$.

The next theorem will rely heavily on the just defined Rankin-Cohen brackets.

Theorem 5.11
Every modular form $f \in [\Gamma_2\,[2,4], (6s, \rho_e\odot\rho_e)]$ is of the form

$$f = \sum_{1\leq i\leq 3} P_i(f_0,\dots f_3)\,\{f_0, f_i\}\,\frac{1}{f_0},$$

where all P_i are homogeneous polynomials of degree $12s-1$. Conversely, such a sum lies in $[\Gamma_2\,[2,4], (6s, \rho_e\odot\rho_e)]$ iff it is holomorphic which means

$$f_0 \mid \sum_{1\leq i\leq 3} f_i\cdot P_i.$$

Proof
A modular form $f \in [\Gamma_2\,[2,4], (6s, \rho_e\odot\rho_e)]$ can be considered as tensor on $\mathbb{P}^3\mathbb{C}$ of the form

$$\sum_{i=1}^{3} R_i\; d\left(\frac{f_i}{f_0}\right) \otimes \left(d\left(\frac{f_1}{f_0}\right)\wedge\cdots\wedge d\left(\frac{f_3}{f_0}\right)\right)^{\otimes 2s}.$$

In theorem 5.5 on page 67, we have seen that R_i is of the form

$$R_i = \frac{P_i\cdot f_0^{8s+1}}{\chi_5^{2s}}$$

with P_i a polynomial of degree $12s-1$ and

$$f_0 \mid \sum_{1\leq i\leq 3} f_i\cdot P_i.$$

We want to observe this tensor on the upper half plane. The functional determinant of $d\left(\frac{f_1}{f_0}\right)\wedge\cdots\wedge d\left(\frac{f_3}{f_0}\right)$ is χ_5/f_0^4, cf. theorem 5.9 on the previous page. Hence, the modular form is of the desired type. □

We could have formulated the above theorem replacing f_0 by any other f_a.

We introduce the $\mathbb{C}[f_0, \ldots, f_3]$-module

$$\mathcal{M}^+(\Gamma_2\,[2,4]) := \bigoplus_{r\in\mathbb{Z}} \Big[\Gamma_2\,[2,4], \Big(\frac{r}{2}, \rho_e \odot \rho_e\Big), v_f^r\Big].$$

One of its $\mathbb{C}[f_0, \ldots, f_3]$-submodules is

$$M^+ := \sum_{0\le i<j\le 3} \mathbb{C}[f_0, \ldots, f_3]\,\{f_i, f_j\}.$$

Theorem 5.12 (Bracket relations)
The Rankin-Cohen brackets are related in the following manner :

$$\begin{aligned}
R_1 &: f_1\{f_0,f_2\} = f_2\{f_0,f_1\} + f_0\{f_1,f_2\} \;\; ; \\
R_2 &: f_1\{f_0,f_3\} = f_3\{f_0,f_1\} + f_0\{f_1,f_3\} \;\; ; \\
R_3 &: f_2\{f_0,f_3\} = f_3\{f_0,f_2\} + f_0\{f_2,f_3\} \;\; ; \\
R_4 &: f_2\{f_1,f_3\} = f_3\{f_1,f_2\} + f_1\{f_2,f_3\} \;\; .
\end{aligned}$$

These are defining relations of the module M^+. Therefore, the Hilbert function is

$$\dim M_r^+ = 3\cdot\binom{r+1}{3} + 2\cdot\binom{r}{2} + \binom{r-1}{1}.$$

Proof
Firstly, the four relations are simply consequences of the product rule. Secondly, we mention that the three brackets $\{f_0, f_j\}$ are $\mathbb{C}[f_0, \ldots, f_3]$- linearily independent, i.e. the equality

$$\sum_{j=1}^{3} P_j\,\{f_0, f_j\} = 0 \tag{5.3}$$

implies that the polynomials P_j are all zero.

Thirdly, we consider an arbitrary relation

$$R: \quad \sum_{0\le i<j\le 3} P_{ij}\,\{f_i, f_j\} = 0.$$

It is equivalent to the subsequent three relations

$$f_0P_{0j} + \sum_{i=1}^{j-1} f_iP_{ij} - \sum_{i=j+1}^{3} f_iP_{ji} = 0, \quad \forall\, j\in\{1,\ldots,3\}$$

due to the given relations $R_1, \ldots, R_4$ and eq. (5.3).

Applying $R_1, \ldots, R_4$, the relation R can be transformed to a form where P_{ij} is a polynomial in the variables $f_0, \ldots, f_j$. In this normal form we see that each P_{ij} is zero. Indeed, for $j = 1$ we obtain

$$f_0 P_{01} - \sum_{i>1} f_i P_{1i} = 0.$$

Setting the variables f_2, f_3 zero yields

$$P_{01} = P_{01}(f_1) = 0.$$

Specialising now f_3 gives $P_{12} = P_{12}(f_1, f_2) = 0$ and hence $P_{13} = 0$.
The just proven equality $P_{12} = 0$ simplifies the above relation for $j = 2$ to

$$f_0 P_{02} - f_3 P_{23} = 0.$$

A similiar line of argument shows

$$P_{02} = P_{23} = 0.$$

Setting $j = 3$ gives the remaining coefficients. □

The module $\mathcal{M}^+$ can be considered as a submodule of the free module

$$\sum_a \mathbb{C}[f_0, \ldots, f_3] \cdot Df_a.$$

In this setting, an element of $\mathcal{M}^+$, say $\sum_a P_a \cdot Df_a$, can be characterized by a single $\mathbb{C}[f_0, \ldots, f_3]$- linear equation :

$$\sum_a f_a P_a = 0.$$

This can be shown by decomposing each P_a into $Q_a + f_3 R_a$, where any Q_a is independent of f_3. Then, the first three Q_a satisfy $\sum_{a=0}^{2} f_a Q_a = 0$ simplifying the problem.

A short MAGMA[BCP97] code or the above appealing observation, that is due to Dr. Uwe Weselmann, both imply the subsequent theorem.

Theorem 5.13
We have

$$\bigcap_{i=0}^{3} \frac{f_0 \cdots f_3}{f_i} M^+ = f_0 \cdots f_3 \cdot M^+.$$

An immediate consequence is the succeeding theorem.

Theorem 5.14
Every modular form $f \in [\Gamma_2\,[2,4],(6s,\rho_e\odot\rho_e)]$ *is of the form*

$$f = \sum_{0\le i<j\le 3} P_{ij}(f_0,\dots f_3)\,\{f_i,f_j\}\,,$$

where any P_{ij} *is a homogeneous polynomials of degree* $12s-1$.

We want to generalize this now to arbitrary weights.

Theorem 5.15 (Structure theorem)
We have

$$\mathcal{M}^+(\Gamma_2\,[2,4]) = \bigoplus_{r\in\mathbb{Z}} \left[\Gamma_2\,[2,4], \left(\frac{r}{2},\rho_e\odot\rho_e\right), v_f^r\right] = \sum_{0\le i<j\le 3} \mathbb{C}[f_0,\dots,f_3]\,\{f_i,f_j\}\,.$$

Proof
Let f be a modular form of weight $r/2$. If r is a multiply of 12, then we apply theorem 5.14. Therefore, we may assume that $f_i\cdot f$ lies in the right hand side. Now, we can apply theorem 5.13. □

5.4 A structure theorem for vector valued modular forms twisted with the character $\mathrm{v_f}^2$

We consider the character v_f^2 on $\Gamma_2\,[2,4]$ and the twisted version of $\mathcal{M}^+$:

$$\mathcal{M}^-(\Gamma_2\,[2,4]) := \bigoplus_{r\in\mathbb{Z}} \left[\Gamma_2\,[2,4], \left(\frac{r}{2},\rho_e\odot\rho_e\right), v_f^r\cdot v_f^2\right].$$

We shall present the analogous proclaims to the ones in section 5.3. Interestingly, most of the proofs stay (almost) unchanged. The twist by v_f^2 is important if one is interested in modular forms of odd weight and trivial multiplier system.

Definition 5.16 (Rankin-Cohen 3-bracket)
For f, g and h in $\mathcal{O}(\mathbb{H}_2)$ and $0\le j_1<j_2\le 2$ we define

$$\{f,g,h\}_{(j_1,j_2)} := \begin{vmatrix} \frac{\partial f}{\partial Z^{j_1}} & \frac{\partial f}{\partial Z^{j_2}} & f \\ \frac{\partial g}{\partial Z^{j_1}} & \frac{\partial g}{\partial Z^{j_2}} & g \\ \frac{\partial h}{\partial Z^{j_1}} & \frac{\partial h}{\partial Z^{j_2}} & h \end{vmatrix}.$$

Then the **Rankin-Cohen 3-bracket** of f, g and h is

$$\{f,g,h\} := \begin{pmatrix} \{f,g,h\}_{(1,2)} & \{f,g,h\}_{(0,2)} \\ \{f,g,h\}_{(0,2)} & \{f,g,h\}_{(0,1)} \end{pmatrix}.$$

Theorem 5.17
Let f,g,h be elements of $[\Gamma_2\,[2,4],\frac{1}{2},v_f]$ then

$$\{f,g,h\} \in \left[\Gamma_2\,[2,4],\left(\frac{5}{2},\rho_e\odot\rho_e\right),v_f^5\cdot v_f^2\right].$$

Proof
The proof can be given by a direct computation which may be somewhat tedious. But, we shall argue differently : We may assume that

$$f = f_0, \quad g = f_1 \quad \text{and} \quad h = f_2.$$

We consider the Γ-invariant tensor

$$d\left(\frac{f_1}{f_0}\right)\wedge d\left(\frac{f_2}{f_0}\right) = \omega_0\cdot dZ^1\wedge dZ^2 + \omega_2\cdot dZ^0\wedge dZ^1 + \omega_1\cdot dZ^0\wedge dZ^2,$$

where

$$\omega_0 = \begin{vmatrix} \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^1} & \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^2} \\ \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^1} & \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^2} \end{vmatrix}, \quad \omega_1 = \begin{vmatrix} \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^0} & \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^2} \\ \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^0} & \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^2} \end{vmatrix} \quad \text{and} \quad \omega_2 = \begin{vmatrix} \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^0} & \frac{\partial\left(\frac{f_1}{f_0}\right)}{\partial Z^1} \\ \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^0} & \frac{\partial\left(\frac{f_2}{f_0}\right)}{\partial Z^1} \end{vmatrix}.$$

We conclude from theorem 4.16 on page 54 that

$$f_0^3\cdot\begin{pmatrix} \omega_2 & -\omega_1 \\ -\omega_1 & \omega_0 \end{pmatrix} \in \left[\Gamma_2\,[2,4],\left(\frac{5}{2},\rho_e\odot\rho_e\right),v_f^5\cdot v_f^2\right].$$

Calculating the determinants completes the proof. □

Theorem 5.18
Every modular form $f\in[\Gamma_2\,[2,4],(6s+1,\rho_e\odot\rho_e)]$ is of the form

$$f = \frac{1}{f_0}\left(P_3\{f_0,f_1,f_2\} + P_2\{f_0,f_1,f_3\} + P_1\{f_0,f_2,f_3\}\right),$$

where all P_i are homogeneous polynomials of degree $12s-3$ in the variables $f_0,\dots,f_3$. Conversely, such a sum lies in $[\Gamma_2\,[2,4],(6s,\rho_e\odot\rho_e)]$ iff it is holomorphic which means

$$f_0 \quad | \quad f_1\cdot P_3 - f_3\cdot P_1$$

and

$$f_0 \quad | \quad f_1\cdot P_2 + f_2\cdot P_1.$$

Proof
The same line of argument as in theorem 5.11 on page 72 yields the desired result. □

It is again possible to replace f_0 by any other f_a in the theorem's formulation.

We recall the definition of the $\mathbb{C}[f_0,\dots,f_3]$-module

$$\mathcal{M}^-(\Gamma_2\,[2,4]) := \bigoplus_{r\in\mathbb{Z}}\left[\Gamma_2\,[2,4],\left(\frac{r}{2},\rho_e\odot\rho_e\right),v_f^r\cdot v_f^2\right].$$

An important $\mathbb{C}[f_0,\dots,f_3]$-submodule is

$$M^- := \sum_{0\le i<j<k\le 3}\mathbb{C}[f_0,\dots,f_3]\,\{f_i,f_j,f_k\}\,.$$

In contrast to theorem 5.12 on page 73, the Rankin-Cohen 3-brackets just satisfy a single relation.

Lemma 5.19 (3-bracket relation)
The Rankin-Cohen 3-brackets are related in the following manner

$$R_5: -f_0\,\{f_1,f_2,f_3\} + f_1\,\{f_0,f_2,f_3\} - f_2\,\{f_0,f_1,f_3\} + f_3\,\{f_0,f_1,f_2\} = 0.$$

This is a defining relation of the module M^-. Hence, the Hilbert function is

$$\dim M_r^- = 3\cdot\binom{r-2}{3}+\binom{r-3}{2}.$$

Proof

i. Consider the matrix

$$A=\begin{pmatrix} f_0 & f_1 & f_2 & f_3\\ \frac{\partial f_0}{\partial Z^0} & \frac{\partial f_1}{\partial Z^0} & \frac{\partial f_2}{\partial Z^0} & \frac{\partial f_3}{\partial Z^0}\\ \frac{\partial f_0}{\partial Z^1} & \frac{\partial f_1}{\partial Z^1} & \frac{\partial f_2}{\partial Z^1} & \frac{\partial f_3}{\partial Z^1}\\ \frac{\partial f_0}{\partial Z^2} & \frac{\partial f_1}{\partial Z^2} & \frac{\partial f_2}{\partial Z^2} & \frac{\partial f_3}{\partial Z^2}\end{pmatrix}$$

and its adjoint matrix

$$\mathrm{Adj}(A) = \begin{pmatrix} \frac{\partial f_{\{1,2,3\}}}{\partial Z^{\{0,1,2\}}} & -\{f_1,f_2,f_3\}_{(1,2)} & +\{f_1,f_2,f_3\}_{(0,2)} & -\{f_1,f_2,f_3\}_{(0,1)} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{\partial f_{\{0,1,2\}}}{\partial Z^{\{0,1,2\}}} & +\{f_0,f_1,f_2\}_{(1,2)} & -\{f_0,f_1,f_2\}_{(0,2)} & +\{f_0,f_1,f_2\}_{(0,1)} \end{pmatrix}.$$

The product $A \cdot \mathrm{Adj}(A)$ is just $c_5 \cdot \chi_5 \cdot I_4$; this yields the desired relation.

ii. First, we mention that the three brackets $\{f_0, f_i, f_j\}$ are $\mathbb{C}[f_0,\dots,f_3]$- linearly independent. Second, we take R to be an arbitrary relation, i.e.

$$R: \quad \sum_{0\le i<j<k\le 3} P_{ijk}\,\{f_i,f_j,f_k\} = 0.$$

Indexing P_{ijk} by the missing index P_l, e.g. P_{123} by P_0, R is equivalent to the subsequent relations

$$f_0P_i + (-1)^i f_iP_0 = 0, \quad \forall\, i \in \{1,\dots,3\}$$

due to the given relation R_5 and the aforementioned linear independence.
Third, applying R_5 the relation R can be transformed to a form where P_0 is a polynomial in the variables $f_1,\dots,f_3$. In this normal form we see that all P_{ijk} are zero. Indeed, setting $f_0 = 0$ yields $P_0 = 0$. This completes the proof as the remaining brackets are linearly independent. □

The module M^- can be considered as a submodule of the free module

$$\sum_{0\le a<b\le 3} \mathbb{C}[f_0,\dots,f_3]\cdot Df_a \wedge Df_b.$$

In this setting, an element of $\mathcal{M}^-$, say $\sum_{0\le a<b\le 3} P_{ab}\cdot Df_a\wedge Df_b$, can be characterized by four $\mathbb{C}[f_0,\dots,f_3]$- linear equations :

$$\begin{aligned} f_1P_{01} + f_2P_{02} + f_3P_{03} &= 0; \\ -f_0P_{01} + f_2P_{12} + f_3P_{13} &= 0; \\ -f_0P_{02} - f_2P_{12} + f_3P_{23} &= 0; \\ -f_0P_{03} - f_1P_{13} - f_2P_{23} &= 0. \end{aligned}$$

A short MAGMA[BCP97] code or the previous appealing observation, that is also due to Dr. Uwe Weselmann, both imply the subsequent theorem.

Theorem 5.20
We have

$$\bigcap_{i=0}^{3} \frac{f_0 \cdots f_3}{f_i} M^- = f_0 \cdots f_3 \cdot M^- .$$

Similarly as in theorem 5.14 on page 75, we deduce easily the following theorem.

Theorem 5.21
Every modular form $f \in [\Gamma_2\,[2,4],(6s+1,\rho_e \odot \rho_e)]$ *is of the form*

$$f = \sum_{0 \le i < j < k \le 3} P_{ijk}(f_0, \ldots f_3)\,\{f_i, f_j, f_k\}\,,$$

where every P_{ijk} *is a homogeneous polynomials of degree* $12s-3$.

Again, generalising this to arbitrary weights is desirable and attainable.

Theorem 5.22 (Structure theorem)
We have

$$\mathcal{M}^-(\Gamma_2\,[2,4]) = \bigoplus_{r\in\mathbb{Z}} \left[\Gamma_2\,[2,4], \left(\frac{r}{2}, \rho_e \odot \rho_e\right), v_f^r \cdot v_f^2\right] = \sum_{0 \le i < j < k \le 3} (\mathbb{C}[f_0, \ldots, f_3])\,\{f_i, f_j, f_k\}\,.$$

Proof
The proof is analogous to the one of theorem 5.15 on page 75. □

5.5 A structure theorem for vector valued modular forms with respect to the trivial multiplier system

We can extract from the modules $\mathcal{M}^+(\Gamma_2\,[2,4])$ and $\mathcal{M}^-(\Gamma_2\,[2,4])$ the modular forms with trivial multiplier system.

Theorem 5.23 (Structure theorem for $\mathcal{M}(\Gamma_2\,[2,4])$)
We have

$$\bigoplus_{k\in\mathbb{Z}} [\Gamma_2\,[2,4],(k,\rho_e \odot \rho_e)] =$$
$$\sum_{0 \le i < j \le 3} (\mathbb{C}_{2+4\mathbb{Z}}[f_0, \ldots, f_3])\,\{f_i, f_j\} \oplus \sum_{0 \le i_1 < i_2 < i_3 \le 3} (\mathbb{C}_{1+4\mathbb{Z}}[f_0, \ldots, f_3])\,\{f_{i_1}, f_{i_2}, f_{i_3}\}$$

and

$$\dim\left[\Gamma_2\,[2,4],(k,\rho_e\odot\rho_e)\right]=\begin{cases}3\cdot\binom{2k+1}{3}+2\cdot\binom{2k}{2}+\binom{2k-1}{1}, & \textit{if } k \textit{ is even},\\ 3\cdot\binom{2k-2}{3}+\binom{2k-3}{2}, & \textit{if } k \textit{ is odd}.\end{cases}$$

5.6 A structure theorem for scalar valued modular forms

So far, we have treated tensors of the type

$$\Lambda^p\Omega\otimes_{\mathcal{O}}(\Lambda^3\Omega)^{\otimes k},$$

where $p=1,2$. It is worthwhile to mention that our method is also successful for $p=3$. In this case we get the structure theorems for the ring of scalar valued modular forms. Recall that our method relied on the injectivity and ramification behaviour of the map

$$[f_0,\dots,f_3]:{}_{\Gamma_2\,[2,4]}\backslash^{\mathbb{H}_2}\longrightarrow\mathbb{P}^3\mathbb{C}.$$

More precisely, we obtain the following theorem which is essentially due to Runge.

Theorem 5.24 (Structure theorem)
We have

$$\bigoplus_{r\in\mathbb{Z}}\left[\Gamma_2\,[2,4],\frac{r}{2},v_f^r\right]=\mathbb{C}[f_0,\dots,f_3].$$

Twisting with v_f^2 yields

$$\bigoplus_{r\in\mathbb{Z}}\left[\Gamma_2\,[2,4],\frac{r}{2},v_f^r\cdot v_f^2\right]=\chi_5\cdot\mathbb{C}[f_0,\dots,f_3].$$

As a consequence,

$$\bigoplus_{r\in\mathbb{Z}}\left[\Gamma_2\,[2,4],r\right]=\mathbb{C}_{4\mathbb{Z}}[f_0,\dots,f_3]\oplus\mathbb{C}_{4\mathbb{Z}}[f_0,\dots,f_3]\cdot\chi_5.$$

We omit the details of the proof, but stress that we start with tensorial weights, i.e. $k\in 3\mathbb{Z}$, as in the case of vector valued modular forms. Afterwards, we extend the results to all weights again by intersecting appropriate modules. It is quite interesting that Igusa's modular form χ_5 comes up automatically in our approach. This happens while studying the holomorphicity of tensors by means of the ramification behaviour of $[f_0,\dots,f_3]$.

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