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Pro-reductive groups attached to irreducible representations of the General Linear Supergroup

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Abstract

We study tensor product decompositions of representations of the General Linear Supergroup $Gl(m|n)$. We show that the quotient of $Rep(Gl(m|n), \epsilon)$ by the tensor ideal of negligible representations is the representation category of a pro-reductive supergroup G^{red}. In the $Gl(m|1)$-case we show $G^{red} = Gl(m-1) \times Gl(1) \times Gl(1)$. In the general case we study the image of the canonical tensor functor F_{mn} from Deligne's interpolating category $Rep(Gl_{m-n})$ to $Rep(Gl(m|n), \epsilon)$. We determine the image of indecomposable elements under F_{mn}. This implies tensor product decompositions between projective modules and between certain irreducible modules, including all irreducible representations in the $Gl(m|1)$-case. Using techniques from Deligne's category we derive a closed formula for the tensor product of two maximally atypical irreducible $Gl(2|2)$-representations. We study cohomological tensor functors $DS : Rep(Gl(m|m), \epsilon) \to Rep(Gl(m-1|m-1))$ and describe the image of an irreducible element under DS. At the end we explain how these results can be used to determine the pro-reductive group $G_L \hookrightarrow Gl(m|m)^{red}$ corresponding to the subcategory $Rep(G_L, \epsilon)$ generated by the image of an irreducible element L in $Rep(Gl(m|m)^{red}, \epsilon)$.

Zusammenfassung

Wir untersuchen die Zerlegung von Tensorprodukten von Darstellungen der Allgemeinen Linearen Supergruppe. Wir zeigen, dass der Quotient von $Rep(Gl(m|n), \epsilon)$ nach dem Tensorideal der vernachlässigbaren Morphismen die Darstellungskategorie einer proreduktiven Supergruppe G^{red} ist. Im $Gl(m|1)$-case zeigen wir $G^{red} = Gl(m-1) \times Gl(1) \times Gl(1)$. Im allgemeinen Fall untersuchen wir das Bild des kanonischen Tensorfunktors F_{mn} von Delignes interpolierender Kategorie $Rep(Gl_{m-n})$ in $Rep(Gl(m|n), \epsilon)$. Wir bestimmen das Bild eines indekomposiblen Elements unter F_{mn}. Dies impliziert explizite Tensorproduktzerlegungen für Tensorprodukte zwischen projektiven Moduln und zwischen bestimmten irreduziblen Darstellungen, wobei alle irreduziblen $Gl(m|1)$-Darstellungen so erhalten werden. Mittels der Techniken in Deligne's Kategorie leiten wir eine geschlossene Formel für das Tensorprodukt zweier irreduzibler maximal atypischer $Gl(2|2)$-Darstellungen her. Im Anschluß beschreiben wir kohomologische Tensorfunktoren $DS : Rep(Gl(m|m), \epsilon) \to Rep(Gl(m-1|m-1))$ und das Bild eines irreduziblen Moduls unter DS. Wir beschreiben dann, wie diese Resultate dazu benutzt werden können, um die proreduktive Gruppe $G_L \hookrightarrow Gl(m|m)^{red}$ zu bestimmen, die zu der Unterkategorie $Rep(G_L, \epsilon)$, die von dem Bild eines irreduziblen Elements L in $Rep(G^{red}, \epsilon)$ erzeugt wird, zu bestimmen.

Contents

Introduction

Representations of the General Linear Supergroup

The theory of the finite-dimensional representations of the algebraic group $Gl(n)$ over an algebraically closed field of characteristic 0 is well understood since the work of Schur and Weyl. The category of representations is semisimple, formulas for the character and the dimension of an irreducible representations are known for almost 90 years and the decomposition of the tensor product between two irreducible representation representations can be reduced to the analogous problem for the symmetric group S_n where an explicite algorithm - the Littlewood-Richardson rule - is known since the 1930's (though proven only 1974).

The situation changes drastically if one passes from $Gl(n)$ to the algebraic supergroup $Gl(m,n)$. A $\mathbb{Z}_2$-graded algebra (a superalgebra) is said to be commutative (or super-commutative) if $ab = (-1)^{p(a)p(b)}ba$ for all homogenuous elements $a, b \in A = A_0 \oplus A_1$. As classical groups are usually defined as representable functors, $Gl(m|n)$ can be seen as a functor $salg_k \to grp$ from the category of commutative superalgebras to groups, sending a commutative superalgebra $A = A_0 \oplus A_1$ to $G(A)$, the group of all invertible $(m+n) \times (m+n)$-matrices

$$\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} |A \in M_{m,m}(A_0), B \in M_{m,n}(A_1), C \in M_{n,m}(A_1), D \in M_{n,n}(A_0)\}.$$

We say G has a representation on a $\mathbb{Z}_2$-graded vector space V if there is a natural transformation $\rho : G \to End(V)$ where $End(V) : salg_k \to grp$ is the functor $End(V)(A) = (A \otimes End(V))_0$. We only consider finite-dimensional representations. As representations of algebraic groups are often studied through the representations of their Lie algebras, we will sometimes consider equivalently representations of integrable weight of the underlying Lie superalgebra $gl(m|n)$.

The theory of supergroups or superalgebras has a number of different ancestors. From the mathematical side they were studied first in the early 60's by Gerstenhaber [Ger63], [Ger74] and Milnor and Moore [MM65] in connection with cohomology theories. In physics they occured first in the earlies studies of supersymmetry such as in [Miy68] or [WZ74a], [WZ74b]. Especially in physics they have become an indispensable building block in supersymmetry and supergravity. Again from a mathematical perspective the theory of supergroups and their representations is nowadays studied not only for its intrinsic interest but also for its connections to other fields of mathematics.

- It is expected that representations of supergroups yield new knot invariants and that categorifactions of representations give ways to compute existing categorifications such as knot Floer homology [GPM10], [GKPM11], [Sar13].

- Character formulas for the affine Lie superalgebra $sl(m|n)$ have connections to generalized theta series and mock modular forms [KW94], [BF13], [BO09].
- They are related to categories of perverse sheaves on abelian varieties [Wei06], [KW11]
- They appear in the theory of mixed motives as super tannakian categories.
- Representations of the General Linear Supergroup describe categories of perverse sheaves on Grassmannians.

Due to the deceptively similar definition of $Gl(m|n)$ in comparison to the classical $Gl(n)$-case, one could expect just a $\mathbb{Z}_2$-graded version of the classical theory. This is indeed misleading since hardly any result from the classical world holds for the supergroup $Gl(m|n)$. The category of its representations is not semisimple, rendering many common techniques from Lie theory useless. The classification of the indecomposable modules is an undecidable problem for $m, n \geq 2$ [Ger98], [BS10b]. Even for irreducible modules the analogs of the classical questions - character formulas, formulas for the (super)-dimension, tensor product decomposition - remained open for a long time since the 70's, when their systematic study was pioneered by Kac [Kac78] [Kac77] and Djokovic-Hochschild [DH76]. In recent years a number of very important results have been achieved.

- Serganova [Ser98] (using cohomology of bundles on flag varieties) and Brundan [Bru03] (using quantum group techniques) gave an algorithm to find the character of an irreducible representation. This generalized earlier results in the $Gl(m|1)$-case by Bernstein and Leites.
- Kac and Wakimoto [KW94] conjectured a criterion for an irreducible representation V to have non-vanishing superdimension $dimV_0 - dimV_1$. This became known as the Kac-Wakimoto conjecture. It has been proven independently by Serganova [Ser10] and Weissauer [Wei10b]. Weissauer also gave a closed formula for the superdimension in terms of certain rooted trees.
- The Ext^1 between two simple modules were determined in a series of articles by Brundan and Stroppel [BS08], [BS10a], [BS08], [BS10b]: The dimension of Ext^1 can be described by certain Kazhdan-Lusztig polynomials associated to Grassmannians and admits a combinatorial description.

In particular the latter work has linked the representation theory of $G = Gl(m|n)$ to a large number of other fields in representation theory and geometry. It is based on the construction of a family of diagram algebras baptised as Khovanov algebras. These Khovanov algebras control a number of different categories:

$$\begin{tikzcd}
\text{representations of G} & & \text{parabolic category } O \\
& \text{Khovanov algebras} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{array}{c}\text{Perverse sheaves}\\ \text{on Grassmannians}\end{array} & \begin{array}{c}\text{Geometry of}\\ \text{Springer fibres}\end{array} & \begin{array}{c}\text{Modules over the}\\ \text{walled Brauer algebras}\end{array}
\end{tikzcd}$$

Despite all these achievements a notoriously persistent problem is the decomposition of tensor products of irreducible representations. The problem is wide open and only very special cases have been solved. The cases of $Gl(1|1)$ and $Sl(2|1)$ have been settled in [GQS07]. The work of Brundan and Stroppel [BS10b] gives a rule to decompose the tensor product of an irreducible representation with the standard representation. As the list of the known examples tells us the problem has hardly seen any progress and structural insight is poor. The situation is similar for the other series of simple supergroups, such as the orthosymplectic supergroups $SpO(m,n)$ which generalize the classical B, C and D series of symplectic and orthogonal groups.

The universal semisimple quotient

In this thesis we study the problem of decomposing tensor products from a different point of view. We study either nice tensor functors from a (well-understood) tensor category $C \to Rep(Gl(m|n))$ into $Rep(Gl(m|n))$ or nice tensor functors $Rep(Gl(m|n)) \to C$ into some (well-understood) tensor category C. Here a necessary condition on a tensor functor to be nice is that it maps an irreducible representation to an isotypic one. Indications for the existence of such tensor functors date long back: In the $Gl(m|1)$-case [VdJHKTM90] [VdJ91] van-der-Jeugt obtained a character formula for the simple modules. The knowledge of such a character formula can be used to derive formulas for the superdimension of these modules. It turns out that the superdimension of every such module $L(\lambda)$ equals up to a sign the dimension of an associated simple $Gl(m-1)$-module $L(\lambda')$ where λ' can be explicitely computed from λ. These results were later generalized by [KW94] to other representations of some other simple supergroups. Since a tensor functor preserves the categorial dimensions, these results could be seen as incarnations of a hidden tensor functor as above. In the case of $Sl(2g|2)$ an algebraic-geometric construction of Weissauer suggests the existence of a tensor functor $Rep(Sl(2g|2)) \to Rep(Sl(2g-2)) \otimes svec$.

A first construction of such tensor functors has been given independently by Duflo-Serganova [DS05], Serganova [Ser10] and Weissauer [Wei10b]. In this case we have a tensor functor $Rep(Gl(m|n)) \to Rep(Gl(m-n)) \otimes svec$ (for $m \geq n$) which maps an irreducible representation to an isotypic representation or zero. Here the notion of atypicality plays a role: Generically an irreducible module is a projective object in the category. For some degenerate weights - called atypical - this is no longer true; and this weights can be seen as responsible for the failure of semisimplicity. The atypical weights can be again distinguished by their degree of degeneracy by a number between 1 and n (for instance by measuring the rate of growth of a projective resolution). The Kac-Wakimoto conjecture asserts that the superdimension of an irreducible representation is non-zero if and only if the atypicality is maximal. Both approaches prove this conjecture. However the fiber of an element in $Rep(Gl(m|n))$ under this functor is so huge that these results do not shed light on the tensor structure. Nonetheless these results are an important guideline for the construction of tensor functors into classical representation categories: Since a tensor functor will preserve the internal dimensions and the dimension of an object in a semisimple tensor category is always non-zero, such a tensor functor can be supported only by the "'maximal atypical part"' $\mathcal{A}_n$ of the category and can only be expected to give information about this part.

The main player in this thesis is a construction which has been studied by Jannsen [Jan92] and Andre-Kahn [KA02]. If $\mathbb{A}$ is a k-linear traced tensor category there exists a universal semisimple quotient $\mathbb{A}/\mathcal{N}$ such that the quotient functor ω : $\mathbb{A} \to \mathbb{A}/\mathcal{N}$ is a tensor functor. This quotient is obtained by dividing through the tensor ideal of negligible morphisms $\mathcal{N}$. On objects the functor will kill precisely the indecomposable modules X with $dim_{\mathbb{A}}(X) = 0$ ("'negligible objects"'). This can be seen as an abstract version of results of Jannsen on numerical motives. In [Jan92] Jannsen showed that the category of motives (defined via algebraic correspondences modulo an adequate equivalence relation) is semisimple if and only if the relation is the numerical equivalence. The following theorem generalizes Jannsen's results.

0.1 Theorem. *[KA02] Let $\mathbb{A}$ a k-linear traced tensor category. Then the ideal $\mathcal{N}$ of negligible morphisms is the only proper tensor ideal such that the quotient $\mathbb{A}/\mathcal{N}$ is semisimple.*

Dividing by the ideal of negligible morphism is the abstract counterpart of taking numerical equivalence. We apply this construction to the category of representations of an affine supergroup scheme $\mathbb{A} = Rep(G, \epsilon)$. Regarding this group we have the following fundamental observation which follows easily from a deep characterization of super-tannakian categories by Deligne [Del02].

0.2 Theorem. *If $\mathbb{A} = Rep(G,\epsilon)$ or a pseudoabelian full tensor subcategory of $Rep(G,\epsilon)$, the quotient $\mathbb{A}/\mathcal{N}$ is again of the form $Rep(G^{red},\epsilon)$ for some pro-reductive supergroup G^{red}.*

The main question of this thesis is to study this pro-reductive supergroup in the case of $\mathbb{A} = Rep(Gl(m|n),\epsilon)$.

If $\mathbb{A} = Rep(G)$ for an affine k-group G, the group G^{red}, the *proreductive envelope*, has been thoroughly studied by Andre and Kahn. The classical case is very different from the super-case; in particular no object is killed under the functor $\mathbb{A} \to \mathbb{A}/\mathcal{N}$. The methods and proofs of [KA02] do not apply any more to the super-setting.

In chapter 2 we determine G^{red} in the special case of $Gl(m|1)$, $Sl(m|1)$ and $OSp(2|2n)$ (the latter only conditional). We show

0.3 Theorem. *In the singly atypical case we have*

$$G^{red} = \begin{cases} Gl(m-1) \times Gl(1) \times Gl(1) & G = Gl(m,1) \\ Sl(m-1) \times Gl(1) \times Gl(1) & G = Sl(m,1),\ m \geq 3 \\ Sp(2n-2) \times Gl(1) \times Gl(1) & G = OSp(2|2n) \end{cases}$$

These results follow from estimates on the transcendence degree of the Grothendieck rings, the classification of the indecomposable modules and use a construction of Weissauer [Wei10a] as a black box. None of these results is available in the general case. Since we can't even classify the indecomposable modules (of non-vanishing superdimension) in the general $Gl(m,n)$-case ($m \geq n \geq 2$) the determination of $\mathbb{A}/\mathcal{N}$ is out of reach. We should therefore consider the weaker questions:

What is the pro-reductive supergroup generated by the images of the irreducible modules in $\mathbb{A}/\mathcal{N}$? What is the reductive supergroup associated to the image of an irreducible module in $\mathbb{A}/\mathcal{N}$?

In the later chapters we will restrict ourselves to the case $Gl(n|n)$. Even this reduced questions are very difficult to attack. In general the functor $\mathbb{A} \to \mathbb{A}/\mathcal{N}$ is not well-behaved: It is not exact, hence does not send tensor generators to tensor generators and does not induce a morphism on the underlying Grothendieck groups. It is also not functorial with respect to embeddings of subgroups. Therefore it is unclear how to approach the problem. On the other hand we have no chance to compute any tensor products directly. As usual a construction of Deligne saves the day.

Deligne's interpolating categories

In $Rep(Gl(m|n))$ the decomposition of the tensor product of two irreducible modules is known for a very small class of representations, the so-called covariant modules. Let us sketch the construction. For simplicity we work with $Sl(n)$ and $Sl(m|n)$. If $\mathbb{A}$ is a pseudoabelian k-linear tensor category and $V \in \mathbb{A}$, the symmetric group S_r acts on $V^{\otimes r}$, $r \in \mathbb{N}$. The irreducible representations of S_r are parametrized by partitions λ of r. Denote by V_λ a corresponding irreducible module in the isomorphism class associated to λ. For a partition λ of r define the Schur functor S_λ via $S_\lambda(X) := Hom_{S_r}(V_\lambda, V^{\otimes n})$. Then one has an isomorphism

$$\bigoplus V_\lambda \otimes S_\lambda(V) \to V^{\otimes r}$$

where the sum runs over the partitions of r. For these special elements the following decomposition formula holds [Del02], prop 1.6:

$$S_\mu(V) \otimes S_\nu(V) = \bigoplus c^\lambda_{\mu,\nu} S_\lambda(V)$$

where the sum runs over the partitions of $r = |\mu| + |\nu|$. The coefficients $c^\lambda_{\mu,\nu}$ are known as Littlewood-Richardson coefficients and dozens of different algorithms are known for their computation (though not a closed formula). It is however in general very difficult to determine $S_\lambda(V)$ in practice.

If $\mathbb{A}$ is the category of finite-dimensional representations of $Sl(n)$ and V the standard representation of dimension n, the work of Schur tells us that $S_\lambda(V)$ is nothing but the irreducible representation $L(\lambda')$ with highest weight $\lambda' = (\lambda_1, \ldots, \lambda_n)$. In particular every irreducible representation of $Sl(n)$ is of the form $S_\lambda(V)$ for some partition λ. Hence the formula above solves the problem of decomposing tensor products in the classical case. If $\mathbb{A}$ is on the other hand the category of representations of $Sl(m|n)$ and V the standard representation, the representation $V^{\otimes r}$ is again completely reducible for every r. The irreducible representations obtained in this way - the covariant representations - can be parametrized by certain partitions, so called (m,n)-hook partitions, and their highest weights can be explicitely determined [Ser85] [BR87]. It turns out that these modules form only a very small subset of the irreducible $Sl(m|n)$-modules. The classical approach is therefore of very limited use. This suggests the following idea: Instead of considering the space of covariant tensors $V^{\otimes r}$ one should look at the larger space of mixed tensors $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$, $r, s \in \mathbb{N}$. However the space of mixed tensors is no longer fully reducible. Accordingly the tensor product decomposition of two mixed tensors is not understood. This problem can be solved using a construction of Deligne.

In [Del07] Deligne constructs for any $\delta \in k$ (say $\delta = \pi$) a tensor category $Rep(Gl_\delta)$ which interpolates the classical representation categories $Rep(Gl(n))$

in the sense that for $\delta = n \in \mathbb{N}$ we have an equivalence of tensor categories $Rep(Gl_{\delta=n})/\mathcal{N} \to Rep(Gl(n))$. These interpolating categories possess a canonical element of dimension δ which we call the standard representation st. Deligne's family of tensor categories are the universal tensor categories on an object of dimension δ in the sense of the following universal property.

0.4 Theorem. *[Del07] Let $\mathbb{A}$ be a k-linear tensor category such that $End(\mathbb{1}) = k$. The functor $F \mapsto F(st)$ is an equivalence $Hom^{\otimes}_{\mathbb{A}}(Rep(Gl_\delta), \mathbb{A})$ of the tensor functors of $Rep(Gl_\delta) \to \mathbb{A}$ with the category of objects in $\mathbb{A}$ which are dualisable of dimension δ and their isomorphisms.*

In layman's terms: For any dualisable object $X \in \mathbb{A}$ of dimension δ in a tensor category there exists a unique tensor functor $F : Rep(Gl_\delta) \to \mathbb{A}, st \mapsto X$.

In particular for $d = m-n \in \mathbb{N}_{\geq 0}$ we have two natural tensor functors starting from the Deligne category $Rep(Gl_d)$: One into $Rep(Gl(m-n))$ (determined by the choice of the standard representation of $Gl(m-n)$), the other one into $Rep(Gl(m|n))$ (determined by the choice of the standard representation of $Gl(m|n)$). This opens up a new approach to study the tensor product decomposition in $Rep(Gl(m,n))$: We should understand the tensor product decomposition in Deligne's category. If we are then able to understand the functor $F_{mn} : Rep(Gl_{m-n}) \to Rep(Gl(m|n))$, $st \mapsto st$, we will be able to decompose tensor products in its image. The tensor product decomposition in Deligne's category has been determined by Comes and Wilson [CW11]. They also determine the kernel of the functor F_{mn} and show that its image is precisely the space of mixed tensors T: The full subcategory of $Rep(Gl(m|n))$ of objects which are direct summands in a tensor product $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$ for some $r, s \in \mathbb{N}$. However Comes and Wilson are not able describe the image $F_{mn}(X)$ of an individual element X.

The space of mixed tensors has also been studied by Brundan and Stroppel [BS11]. More generally, in a series of articles [BS08], [BS10a], [BS08], [BS10b] Brundan and Stroppel initiated the study of so-called Khovanov-algebras, a class of diagram algebras arising from Khovanov's categorification of the Jones polynomial. For a particular special case of these Khovanov-algebras, the algebras $K(m,n)$, the main theorem of [BS10b] states an equivalenve of categories

$$E : Rep(Gl(m,n), \epsilon) \to K(m|n) - mod$$

between the categories of finite-dimensional modules of $K(m,n)$ and $Gl(m|n)$. Although the tensor structure is not visible any more on the left-hand side, this result had a fundamental impact on the theory as it allows to analyse many problems from a combinatorial point of view. It turns out that the space of mixed tensors can also be described by the work of Brundan and Stroppel. In both approaches the indecomposable mixed tensors are described by certain pairs $\lambda = (\lambda^L, \lambda^R)$ of partitions, so-called (m,n)-cross bipartitions. The advantage of Brundan and Stroppels results in our case is that it permits to analyse the Loewy structures of the mixed tensors and gives conditions on their highest weights. This not-so-dearly-loved analysis of the complex combinatorics of the Khovanov algebras forms the part of **chapter 3 and chapter 4**. In this chapters we define two invariants $def(\lambda)$ and $k(\lambda)$ of a bipartition. Let us quote two results from chapter 3 and chapter 4.

0.5 Theorem. *A mixed tensor is irreducible iff $def(\lambda) = 0$. A mixed tensor is projective iff $k(\lambda) = n$. Every projective module is a mixed tensor. We have an explicit bijection θ_n between the bipartitions with $k(\lambda) = n$ and the projective covers of irreducible modules. Similarly we have an explicit bijection θ_0 between the bipartitions with $def(\lambda) = 0$ and the irreducible mixed tensors.*

This solves the problem of decomposing tensor products of projective representations of $Gl(m|n)$ since the tensor product decomposition in Deligne's category is known. We call an irreducible module a Kostant module if its weight diagram in the sense of Brundan and Stroppel has a particular simple form (see chapter 2).

0.6 Theorem. *Every irreducible mixed tensor is a Kostant module. Conversely, for $m > n$, every maximally atypical Kostant module is a Berezin twist of an irreducible mixed tensor.*

Again this result gives the tensor product decomposition for the maximally atypical Kostant modules. The result on the maximally atypical modules imply also the tensor equivalence

$$T/\mathcal{N} \simeq Rep(Gl(m-n)).$$

We turn back to our original problem of determining the pro-reductive supergroup associated to the category generated by the images of the irreducible representations in $Rep(Gl(n|n))/\mathcal{N}$. However no maximally atypical irreducible representation of $\mathcal{R}_n = Rep(Gl(n|n), \epsilon)$ is in the image of $F_{nn} : Rep(Gl_0) \to Rep(Gl(n|n), \epsilon)$. To circumvent this problem we proceed as follows: In **chapter 5** we study the class of the "'smallest"' maximally atypical tensors (the ones of minimal Loewy length) which we call the symmetric powers $\mathbb{A}_{S^i}, i \in \mathbb{N}$. We derive a closed formula for their tensor products $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$. As all mixed tensors these have a simple socle which we denote by S^{i-1}. One might hope to infer back from the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$-tensor product to the $S^{i-1} \otimes S^{j-1}$-tensor product. We put this question aside for the moment. Even if this would be possible, the irreducible modules S^i form only a very restricted subset of the irreducible maximally atypical representations; and any other approach to compute tensor products of more general classes of irreducible modules seems out of reach.

Cohomological tensor functors

As a remedy we develop a way to reduce the determination of the pro-reductive supergroup to the the lower rank cases for $k < n$. This will ultimately allow us to make use of the machinery of mixed tensors in the $Gl(2|2)$-case. For any x in the nilpotent cone

$$X = \{x \in \mathfrak{g}_1 \mid [x,x] = 0\}$$

(where $\mathfrak{g}_1$ is the odd part of the Lie Superalgebra $gl(m|n)$) multiplication by x on a module M defines a complex

$$\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots$$

Taking the cohomology of this complex gives a functor

$$DS : \mathcal{R}_n \to Rep(Gl(n-k|n-k))$$

where $k \in \{1, \ldots, n\}$ is the so-called rank of x . The crucial and surprising point is the following theorem.

0.7 Theorem. *[Ser10] For all $x \in X$ the functor DS is a tensor functor.*

This theorem will enable us to transfer information about tensor products in the $\mathcal{R}_{n-1}$-case to the $\mathcal{R}_n$-case once we are able to compute $DS(M)$ for an indecomposable module M. From now on we fix a certain x of rank 1 and denote by $DS : \mathcal{R}_n \to Rep(Gl(n-1, n-1))$ the associated tensor functor. In that case we review in **chapter 7** the Main theorem of [HW13] which gives an explicit formula for $DS(L) = \bigoplus L_i$ for any irreducible module L as a sum of maximally atypical irreducible modules in $\mathcal{R}_{n-1} \oplus \mathcal{R}_{n-1}[1]$. The proof is a long and complicated induction to reduce the statement to the case of mixed tensors for which the situation is well-understood. We make another reduction and study only the full subcategory $\mathcal{R}I_n$ of objects which appear as summands in an iterated tensor product of irreducible elements whose superdimension is non-negative. It is easy to see that DS sends negligible modules to negligible modules. Hence it induces a tensor functor $\mathcal{R}I_n/\mathcal{N} \to \mathcal{R}I_{n-1}/\mathcal{N}$. If H_n and H_{n-1} are the corresponding proreductive groups, we get an injective homomorphism $f : H_{n-1} \to H_n$. The rule for the splitting of $DS(L)$ can be seen as the branching rule with respect to the embedding $H_{n-1} \to H_n$.

This result shows the importance of the $Gl(2|2)$-case: The Tannaka group $\mathcal{R}I_n/\mathcal{N}$ will be the start for an inductive procedure to determine $\mathcal{R}I_n$ from the next lower case $\mathcal{R}I_{n-1}$. To determine the Tannaka group in the $Gl(2|2)$-case, the tensor products of the maximally atypical irreducible modules have to be computed. This is done in **chapter 7**. Every such module is up to a determinant twist of the form S^i. We first determine the K_0-decomposition of $S^i \otimes S^j$ recursively from the K_0-decomposition of the $A_{S^i} \otimes \mathbb{A}_{S^j}$. We then use duality arguments and the functor $DS : \mathcal{R}_2 \to R_1$ to determine the corresponding indecomposable summands and their Loewy structure.

0.8 Theorem. *(see chapter 7 for a precise statement) Up to negligible modules we have $S^i \otimes S^i = Ber^{i-1} \otimes M$ and $S^i \otimes S^j = M'$ for $i > j$ for indecomposable modules M and M'.*

The following theorem is a consequence:

0.9 Theorem. *[HW13] The Tannaka group generated by S^1 is $Sl(2)$ and the Tannaka group generated by S^i, $i \geq 2$ is $Gl(2)$. The proreductive group $H_2 \subset \prod_{\nu=0}^{\infty} Gl(2)$ is the subgroup defined by all elements $g = \prod_{\nu=0}^{\infty} g_\nu$ in the product H_2 with the property $det(g_\nu) = det(g_1)^\nu$.*

In [HW13] the case $Gl(3|3)$ is also established. As of present the proof for the general case $n \geq 4$ is not complete.
This results can be seen as a (weak) version of a tensor product rule for maximally atypical irreducible modules since the well-known decomposition rules of the classical groups tell us the tensor product decomposition up to modules of superdimension zero. Consider as an example the representation $X := \Pi[2, 1, 0]$ of $Gl(3|3)$. It can be shown that $H_X = Sp(6)$. Hence

$$X \otimes X = I_1 \oplus I_2 \oplus I_3 \ mod \ \mathcal{N}$$

with the indecomposable representations corresponding to the irreducible $Sp(6)$ representations $L(2, 0, 0)$, $L(1, 1, 0)$ and $L(0, 0, 0)$. Now consider the tensor product $I_1 \otimes I_1$. Since I_1 corresponds to the $Sp(6)$-representation with highest weight $(2, 0, 0)$, its tensor product will have the same decomposition as the tensor product of $S^2(st) \otimes S^2(st)$ in $Rep(Sp(6))$ up to negligible modules.

Note. Several small parts of this thesis are used in the forthcoming preprint [HW13]: Parts of 1.4, parts of 3.5, parts of 6.1, the section 6.2, section 6.3 and the section 7.1. These sections have been written by myself.

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Notation

$\mathbb{A}_{S^i}$	the generalized symmetric power $R(i;1^i)$ in $\mathcal{R}_n$
$\mathbb{A}_{\Lambda^i}$	the generalized alternating power $R(1^i;i)$ in $\mathcal{R}_n$
$c^{\nu}_{\lambda\mu}$	the Littlewood-Richardson coefficient to the partitions λ, μ and ν
$D_{\lambda\mu}$	the linking number between the cap diagrams $\overline{\lambda}$ and $\overline{\mu}$
$def(\lambda)$	the defect of the bipartition λ
DS	the tensor functor $DS : \mathcal{R}_{mn} \to R_{m-1,n-1}$ for a fixed $x \in X$ of rank 1
F_{mn}	the canonical tensor functor $Rep(Gl_{m-n}) \to Rep(Gl(m\|n), \epsilon)$
F_{m-n}	the canonical tensor functor $Rep(Gl_{m-n}) \to Rep(Gl(m-n))$
$G^t_{\Delta\Gamma}$	a special projective functor $K(m\|n) - mod \to K(m\|n) - mod$
k	algebraically closed field of characteristic 0
$K(\lambda)$	the Kac-module
$k(\lambda)$	the invariant $def(\lambda) + rk(\lambda)$ for the bipartition λ
$K(m\|n)$	a certain Khovanov algebra
$L(\lambda)$	the irreducible module of highest weight λ
λ	either a highest weight, a partition or a bipartition
$\underline{\lambda}$	the cup diagram of the weight λ
$\overline{\lambda}$	the cap diagram of the weight λ
(λ)	indecomposable element in Deligne's category corresponding to λ
$\{\lambda\}$	$S_\lambda(st)$
$\lambda \vdash (r,s)$	$\sum \lambda^L_i = r$ and $\sum \lambda^R_i = s$
Λ^x	the set of $(m\|n)$-cross bipartitions for some fixed m, n
$lift_\delta(.)$	the lift $R_\delta \to R_t$ where t is an indeterminate
M_x	the image of M under Serganova's tensor functor
$\mathcal{N}$	the tensor ideal of negligible morphisms
$P(\lambda)$	the projective cover of $L(\lambda)$
$Proj$	the tensor ideal of projective objects
$p(\lambda)$	the parity $p(\lambda) \in \mathbb{Z}_2$ of the weight λ
$Rep(Gl_\delta)$	Deligne's interpolating category attached to the element $\delta \in k$
$\mathcal{R}_{mn}$	$Rep(Gl(m\|n), \epsilon)$
$\mathcal{R}_n$	$Rep(Gl(n\|n), \epsilon)$
$rk(\lambda)$	the rank of the bipartition λ
$S_\lambda(.)$	the Schur functor associated to the partition λ
$svec$	the category of finite-dimensional super vector spaces over k
θ	the map $\theta : \Lambda^x \to X^+$
T	largest vertex in a bipartition labelled by $x, \circ$ or part of a cap
X	the nilpotent cone $X = \{x \in \mathfrak{g}_1 \mid [x,x] = 0\}$
X^+	the set of integral dominant weights for $Gl(m\|n)$

1. Preliminaries

1.1. Algebraic Supergroups

All the definitions and theorems can be found in the literature, for instance [Var04], [Wei09], [Mas11], [Del02], [CCF11] and notably [Wes] from which most of the defintions have been taken. We assume throughout the thesis that k is an algebraically closed field of characteristic 0.

1.1.1. Super Linear Algebra

A super vector space is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ over k. Elements in V_0 respectively V_1 are called even respectively odd. An element is homogenuous if it is either even or odd. For a homogenuous element v write $p(v)$ for the parity defined by

$$p(v) = \begin{cases} 1 & v \in V_0 \\ -1 & v \in V_1 \end{cases} .$$

We denote by $Hom(V, W)$ the set of k-linear parity-preserving morphism between two super vector spaces V and W. The direct sum has a natural grading via $(V \oplus W)_i = V_i \oplus W_i,\ i \in \mathbb{Z}_2$. The category *svec* of finite-dimensional super vector spaces is an abelian category. It carries also the structur of a symmetric tensor category. The tensor product of two super vector spaces has a natural $\mathbb{Z}_2$-grading via

$$(V \otimes W)_i = \bigoplus_{j+k=i} V_j \otimes W_k,\ i \in \mathbb{Z}_2.$$

The dual V^* has a natural $\mathbb{Z}_2$-grading given by: $w \in (V^*)_i \Leftrightarrow p(w(v)) = p(v) + i = 0$. The inner Hom $\underline{Hom}(V, W)$ is by definition the set of all k-linear maps $V \to W$ of two $\mathbb{Z}_2$-graded vector spaces. This object is $\mathbb{Z}_2$-graded, with the parity-preserving morphisms forming the even part. For the inner Hom functor $(V, W) \mapsto \underline{Hom}(V, W)$ we have the isomorphism $\underline{Hom}(V, W) \simeq V^* \otimes W$. The parity shift functor $\Pi : svec \to svec$ is defined by $(\Pi V)_0 = V_1$, $(\Pi V)_1 = V_0$ and on morphisms $f : V \to W$ via $\Pi f : v \mapsto f(v)$ where v is viewed as an element of ΠV and $f(v)$ as an element of ΠW. We sometimes write $V[1]$ instead of ΠV.

With $k^{p|q}$ we denote the super vector space with $V_0 = k^p,\ V_1 = k^q$. Any morphism $f : k^{p|q} \to k^{r|s}$ can be written as a block matrix

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},\ A \in M_{r,p}(k), D \in M_{s,q}(k).$$

An element of $\underline{Hom}(V,W)_1$ can be represented by a block matrix

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \ B \in M_{r,q}(k), \ C \in M_{s,p}(k).$$

Write $Mat_{p,q}(k)$ for the matrices representing elements of $\underline{Hom}(k^{p|q}, k^{p|q})$. It is naturally $\mathbb{Z}_2$-graded with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}_1 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

with A, B, C, D as above. The supertrace $Mat_{p|q}(k) \to k$ is defined by

$$str \begin{pmatrix} A & B \\ C & D \end{pmatrix} = tr(A) - tr(D).$$

The supertranspose of $X \in Mat_{p|q}(k)$ is defined by

$$X^{ST} = \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix} \text{ for } X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The categorial dimension in the tensor category of super vector spaces is given by the superdimension $sdim(V) = dimV_0 - dimV_1$. With the tensor product, the inner Hom and the dual as above, the category *svec* is a symmetric rigid tensor category.

1.1.2. Superrings and modules

A superring is a $\mathbb{Z}_2$-graded ring $A = A_0 \oplus A_1$ such that the product map $A \times A \to A$ satisfies $A_i A_j \subset A_{i+j}$. A morphism of superrings is a $\mathbb{Z}_2$-grading preserving morphism of rings. Elements of A_0 are called even, elements of A_1 odd. An element that is either even or odd is called homogenuous. The parity function p can be explained for homogenuous elements in the same way as for super vector spaces. A superring is commutative if $ab = (-1)^{p(a)p(b)} ba = 0$ for all $a, b \in A_0 \cup A_1$. We say an abelian group is $\mathbb{Z}_2$-graded if G is a direct sum $G = G_0 \oplus G_1$. Let A be a superring and M a $\mathbb{Z}_2$-graded abelian group $M = M_0 \oplus M_1$. Then M is an A-left module if M is a left A-module in the usual sense and the structure morphism $l : A \times M \to M$ satisfies $A_i \times M_j \to M_{i+j}$. A morphism of A-modules is a parity preserving map that commutes with the action of A.

Let A be a superalgebra over k and let $A^{p|q} = (\bigoplus_{i=1}^p A) \oplus (\bigoplus_{j=1}^q \Pi A)$. Consider maps $A^{p|q} \to A^{r|s}$ that preserve sums and commute with the right action of A: $\varphi(ma) = \varphi(m)a$ for all $m \in A^{p|q}$. The set of all such maps is denoted

$\underline{Hom}_A(A^{p|q}, A^{r|s})$. An element is even if it preserves the $\mathbb{Z}_2$-grading and odd if it reverses it. This makes $\underline{Hom}_A(A^{p|q}, A^{r|s})$ into a $\mathbb{Z}_2$-graded abelian group. Denote by $Mat_{p|q}(A)$ the set of $(p+q) \times (p+q)$-matrices with entries in A. As in the classical case we have a one-to-one correspondence between the elements in $\underline{Hom}_A(A^{p|q}, A^{p|q})$ and $Mat_{p|q}(A)$. A matrix is even respectively odd if it so as an element in $\underline{Hom}_A(A^{p|q}, A^{p|q})$. $Mat_{p|q}(A)$ is an A-module by defininng for $a \in A$ $(F_{ij})a = ((F \circ a)_{ij})$ where we view a as the map $m \mapsto am$. Via the multiplicication of matrices $Mat_{p|q}(A)$ is an associative, unital non-commutative superalgebra.

1.1.3. Algebraic supergroups

A super coalgebra is a super vector space over k together with morphisms of super vector spaces $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ satisfying $id \otimes \epsilon \circ \Delta = \epsilon \otimes id \circ \Delta = id$ where we identify $k \otimes C \simeq C \otimes k \simeq C$. We always assume further $\Delta \otimes id \circ \Delta = id \otimes \Delta \circ \Delta$. A super bialgebra is an associative superalgebra over k that is at the same time a super coalgebra such that the comultiplication $\Delta : B \to B \otimes B$ and the counit $\epsilon : B \to k$ are superalgebra morphisms, ie. $\Delta(xy) = \Delta(x)\Delta(y)$ and $\epsilon(xy) = \epsilon(x)\epsilon(y)$. A super Hopf algebra is a super bialgebra H together with an even linear map $S : H \to H$, called the antipode, such that for all $x \in H$ we have $x'S(x'') = S(x')x'' = \epsilon(x)$. A morphism of super Hopf algebras is a morphism of superalgebras $f : H \to H$ that is also a morphism of super coalgebras and that satisfies $S' \circ f = f \circ S$.

A functor $G : salg \to sets$ is a group functor if it factors over the category of groups. For any superalgebra A we have a functor $F_A : salg \to sets$ given by $F_A(B) = Hom_{salg}(A, B)$. We say a functor is representable if there is an object A such that there is a natural isomorphism $F(B) \simeq F_A(B)$ for all B. A group functor G is called an affine supergroup G if G is a representable group functor and if the representing superalgabra is finitely generated. Let $G : salg \to sets$ be a representable group functor and suppose that $k[G]$ represents G. Then $k[G]$ is a commutative super Hopf algebra. Conversely if A is a commutative super Hopf algebra, the functor $B \mapsto Hom_{salg}(A, B)$ defines a representable group functor. The most important example for us is the group functor $Gl(p|q)$. Define the functor $Gl(p|q)$ from the category of superrings to the category of groups by sending A to $Mat_{p|q}(A)$ and the morphism $f : A \to B$ to the map that sends a matrix $(X_{ij}) \in Gl(p|q)(A)$ to the matrix $(f(X_{ij})) \in Gl(p|q)(B)$. For an invertible even supermatrix X we define the superdeterminant $BerX$ (the Berezinian) by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{det(A - BD^{-1}C)}{det(D)}.$$

For any two $X, Y \in Gl(p|q)(A)$ we have $Ber(XY) = Ber(X)Ber(Y)$. For the representing super Hopf algebra see [Fio11] or [Wes], 8.5. This group functor is called the General Linear supergroup. For the $(p+2q)\times(p+2q)$ matrix Ω defined by

$$\Omega = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & J_q \end{pmatrix}, \quad J_q = \begin{pmatrix} 0 & -\mathbb{1}_q \\ \mathbb{1}_q & 0 \end{pmatrix}$$

we have the group functor $OSp(p|2q)$ defined by

$$OSp(p|2q)(A) = \{A \in Gl(p|q)(A) \mid X^{ST}\Omega X = \Omega\}$$

which is a group subfunctor of $Gl(p|q)$. For a morphism $f : A \to B$ of superalgebras define a morphism of groups $OSp(p|2q)(A) \to OSp(p|2q)(B)$ by applying f to each maxtrix entry. For the representability see [Wes]. This group functor is called the Orthosymplectic Supergroup.

Define the group functor $Gl_V : salg \to sets$ to be the functor that assigns to each commutative superalgebra A the even invertible elements of $End_A(V \otimes A)$. For finite-dimensional V Gl_V is an affine supergroup. Let G be a group functor and $V \in svec$ a finite-dimensional super vector space. A linear representation of G in V is a morphism of group functors $G \to Gl_V$. If G has a linear representation on V, call V a G-module. The action $Gl_V(A)$ on $(V\otimes A)_0$ is a natural transformation $Gl_V \times V \to V$. Hence one can equivalently define a linear representation of G in V to be a natural transformation $G\times V \to V$ that factorises over $Gl_V \times V \to V$. Let V, W be G-modules. A super vector space morphism $f : V \to W$ is a morphism of G-modules if for all commutative superalgebras the diagram

$$\begin{tikzcd} G(A)\times V(A) \arrow[r] \arrow[d, "id\times f_A"] & V(A) \arrow[d, "f_A"] \\ G(A)\times W(A) \arrow[r] & W(A) \end{tikzcd}$$

commutes where f_A is the induced morphism $f \otimes id_A : (V \otimes A)_0 \to (W \otimes A)_0$. As in the classical case one can define the notion of a comodule over a super Hopf algebra and there is a one-to-one corespondence between G-modules and left $k[G]$-comodules.

Let G be a supergroup scheme and let ϵ be an element of $G(k)$ of order dividing 2 such that the automorphism $int(\epsilon)$ of G is the parity automorphism defined by $x \mapsto (-1)^{p(x)}x$ for homogenuous x. Then let $Rep(G,\epsilon)$ be the category of (finite-dimensional) representations $V = (V,\rho)$ such that $\rho(\epsilon)$ is the parity automorphism of V. If G is an affine group scheme, ϵ is central. In this case the category $Rep(G,\epsilon)$ identifies itself with $Rep(G)$ with a new commutativity constraint: For every representation (V,ρ) of G the involution $\rho(\epsilon)$ defines a $\mathbb{Z}_2$-graduation on V and the commutativity isomorphism of the tensor product is given by the Koszul rule. If ϵ is trivial, one recovers $Rep(G)$. These two examples are important since they allow us to see the classical representation categories as a special case of super-representations. For the supergroup $Gl(p|q)$ and non-trivial ϵ we put $Rep(Gl(p|q), \epsilon) = \mathcal{R}_{pq}$. For the whole category $Rep(Gl(p|q))$ we also write R_{pq}.

1.2. Categorial properties

1.2.1. Super-Tannakian categories

1.1 Definition. *A k-linear, abelian, rigid tensor category $\mathbb{A}$ with $k \simeq End(\mathbb{1})$ and with a k-linear exact tensor functor $\rho : \mathbb{A} \to svec$ (super fibre functor) is called a super-tannakian category.*

Recall that rigid means that every object X is dualisable in the sense that there exists an element $X^\vee$ (the dual) with morphisms $\delta : \mathbb{1} \to X \otimes X^\vee$ and $ev : X^\vee \otimes X \to \mathbb{1}$ such that the composed morphisms induced from δ and ev

$$X \to X \otimes X^\vee \otimes X \to X$$
$$X^\vee \to X^\vee \otimes X \otimes X^\vee \to X^\vee$$

are the identity. If ρ would be a functor into vec we would recover the notion of a tannakian category. It is well-known that such a category is tensor-equivalent to the category of representations of a groupscheme.

1.2 Theorem. *[Del02] Every super tannakian category $\mathbb{A}$ is tensor equivalent to the category $\mathbb{A} \simeq Rep(G, \epsilon)$ of representations of a super group G.*

Schur functors Recall [FH91] that the isomorphism classes of irreducible modules in $Rep(S_n)$ are classified by the partitions of n. For every partition λ of n there is an idempotent $c_\lambda \in k[S_n]$ called the Young symmetrizer of λ. If S_n acts on an object $A \in \mathbb{A}$, we have an algebra map $k[S_n] \to End(A)$. By abuse of notation we will not distinguish between the element of $k[S_n]$ and the induced endomorphism of the representation. The object $c_\lambda(A)$ is a direct summand of A.

1.3 Definition. *For every partition λ of n we define*

$$S_\lambda(X) := c_\lambda(X^{\otimes n}).$$

The assignment $S_\lambda(X)$ is called the Schur functor of λ. We put $Sym^n(X) = S_{(n)}(X)$ (the n-th symmetric power) and $\Lambda^n(X) = S_{(1,\ldots,1)}(X)$ (the n-th alternating power).

An object X of $\mathbb{A}$ is called Schur-finite if there exists an integer n and a partition λ of n such that $S_\lambda(X) = 0$. Consider the tensor category *svec*. Every object is isomorphic to $k^{p|q}$ and $S_\lambda(k^{p|q}) = 0$ iff the partition λ contains the rectangle with $p+1$ rows and $q+1$ columns. In particular for every object of $Rep(G, \epsilon)$ there exists a λ such that $S_\lambda(X) = 0$.

1.4 Theorem. *[Del02] If $\mathbb{A}$ is an abelian k-linear rigid tensor category with $End(\mathbb{1}) \simeq k$ such that every object is Schur finite, then $\mathbb{A}$ is a super tannakian category. In other words, $\mathbb{A}$ is tensor equivalent to $Rep(G, \epsilon)$ for some supergroup scheme G.*

In the classical case (the tannakian case) one has to replace the statement *every object is Schur finite* by the condition *for every object there is an integer n with* $\Lambda^n(X) = 0$. In this case $\mathbb{A} \simeq Rep(G)$ for a groupscheme G.

1.2.2. Nice categories and quivers

Following [Ger98] we call a small abelian k-linear category nice if morphism spaces are finite-dimensional, every object has a finite composition series and the category has enough projectives. An example is given by the category $\mathcal{R}_{mn}$.

1.5 Lemma. *[Ger98], lemma 1.1.1. If $\mathbb{A}$ is a nice category then*

1. *The endomorphism ring of any indecomposable object is a local ring.*
2. *Every object can be written as a direct sum of indecomposable objects.*
3. *Every indecomposable object has a unique simple quotient.*
4. *Every module has a unique projective cover*
5. *For any module M, the number of indecomposable projective modules with $Hom(P, M) \neq 0$ is finite.*

In particular we get a bijection between the simple objects and the indecomposable projective modules. The simple object correesponding to $\lambda \in X^+$ is denoted $L(\lambda)$. We will describe a nice category by its Ext-quiver following the description in [Ger98].

A quiver is a directed graph. If Q is a quiver with vertex set X^+, define a category kQ as follows: The objects are the vertices of Q. If $\lambda, \mu \in X^+$, the morphism space $Hom_{kQ}(\lambda, \mu)$ is the space of all formal linear combinations of paths between λ and μ. Two morphisms are composed by linearly extending concatenations of paths. A representation V of a quiver is given by an X^+-graded vector space $\bigoplus_{\lambda \in X^+} V_\lambda$ together with linear maps $\phi : V_\lambda \to V_\mu$ for every arrow $\lambda \to \mu$ in Q. This defines a family of maps $Hom_{kQ}(\lambda, \mu) \to Hom_k(V_\lambda, V_\mu)$ which is compatible with composition. A morphism of representations is a morphism of X^+-graded vector spaces which commutes with the action of all arrows in Q. The abelian category of representations of Q is denotes $Q - mod$.

A system of linear representations on Q is a map R which assigns a subspace $R(\lambda, \mu)$ in every $Hom(\lambda, \mu)$ such that for all $\lambda, \mu \in X^+$

$$R(\nu, \mu) \circ Hom_{kQ}(\lambda, \nu) \subset R(\lambda, \mu)$$
$$Hom_{kQ}(\nu, \mu) \circ R(\lambda, \nu) \subset R(\lambda, \mu).$$

Given a system of linear relations on R the category of representations of Q with relations R denoted $kQ/R - mod$ is defined as the full subcategory of $kQ - mod$ such that for all vertices λ, μ the image of $R(\lambda, \mu)$ in $Hom_k(V_\lambda, \mu)$ is zero.

If $\mathbb{A}$ is a nice category we can associate to it its Ext-quiver. The vertex set X^+ is given by the set of isomorphism classes of simple modules and the number of arrows from λ to μ is given by $Ext^1_{\mathbb{A}}(L(\lambda), L(\mu))$.

1.6 Theorem. *[Ger98], Thm 1.4.1. Let $\mathbb{A}$ be a nice category and Q its Ext-quiver. Then there exists (an explicitely given) set of relations R on Q such that we have an equivalence of categories*

$$e : \mathbb{A} \to Q/R - mod$$

such that $e(M) = \bigoplus_{\lambda \in X^+} Hom_{\mathbb{A}}(P(\lambda), M)$ as graded vector spaces.

1.7 Lemma. *[ASS06] Let M be an indecomposable representation of a finite quiver Q which has no cyclic path. Then the number of composition factors of type $L(\mu)$ in M are given by $dim M_\mu$.*

A nice category is said to have wild type if it contains the category of finite-dimensional representations of the free algebra on two generators. The classification of the indecomposable modules of this algebra is not a solvable problem [Ben98], 4.4. If $\mathbb{A}$ has only finitely many isomorphism classes it is said to have finite type and it is said to be of tame type otherwise. Consider the quiver with

$n+1$ vertices $\mu_1, \dots, \mu_n, \lambda$ with one arrow from each μ_1 to λ (picture for $n = 6$)

$$\begin{tikzcd}
\bullet[\mu_6] \arrow[dr] & & \bullet[\mu_1] \arrow[dl] \\
\bullet[\mu_5] \arrow[r] & \bullet[\lambda] & \bullet[\mu_2] \arrow[l] \\
\bullet[\mu_4] \arrow[ur] & & \bullet[\mu_3] \arrow[ul]
\end{tikzcd}$$

This quiver is known as the n-subspace quiver. The category of its representations is of wild type if $n \geq 5$ [Ben98].

A block Γ of X^+ (or $\mathbb{A}$) is a connected component of the Ext-quiver. Let A_Γ be the full subcategory of objects of $\mathbb{A}$ such that all composition factors are in Γ. This gives a decomposition $\mathbb{A} = \bigoplus_\Gamma \mathbb{A}_\Gamma$ of full abelian subcategories. Every indecomposable module lies in a unique A_Γ and all its simple submodules belong to Γ.

Loewy structures We assume now that $\mathbb{A}$ is super tannakian. We recall some definitions from [Hum08]. A semisimple filtration of a module M is a filtration by submodules

$$0 = M^n \subset \ldots \subset M^1 \subset M^0 = m$$

such that each quotient M^i/M^{i+1} is semisimple and non-zero. A composition series of M is a semisimple filtration of maximal length, ie. each quotient is irreducible. The number of composition factors of M is called the length of M. A Loewy series or Loewy filtration is a semisimple filtration of minimal length. Given a Loewy series the semisimple quotients are called the Loewy layers. The number of these layers is the Loewy length. In general a module can have many different Loewy series.

The radical $radM$ is the smallest proper submodule of M such that M/N is semisimple. Define the radical series as follows: Put $Rad^0M = M$, $Rad^1M = RadM$ and put inductively $Rad^kM = Rad(Rad^{k-1}M)$. The radical seris is given by

$$0 \subset Rad^rM \subset \ldots \subset Rad^1M \subset Rad^0M = M$$

where r is the least integer for which $Rad^rM = 0$. Write $Rad_kM = Rad^kM/Rad^{k+1}M$ for the k-th radical layer. We call top $topM$ the quotient $M/radM$, the largest semisimple quotient of M.

The socle $socM$ of M is the largest semisimple submodule of M. Define the socle series as follows: Put $Soc^0M = 0$, $Soc^1M = socM$ and let Sok^kM be the unique submodule of M satisfying $Soc(M/Soc^{k-1}M) = Soc^kM/Soc^{k-1}M$. The socle series is given by

$$0 = Soc^0M \subset Soc^1M \subset \ldots \subset Soc^sM = M$$

where s is the least integer for which $Soc^sM = M$. Write $Soc_kM = soc^kM/Soc^{k-1}M$ for the k-th socle layer.

Both series are Loewy series and hence $r = s$. Given any Loewy series M^i of M, the M^i sit between the radical and the socle series

$$Rad^{r-k}M \subset M^{r-k} \subset Soc^kM.$$

We say M is rigid if the socle series and the radical series coincide. In this case M has a unique Loewy series.

1.3. Lie Superalgebras

We follow [CW12], [Wes] and [Kac78]. A Lie superalgebra is a super vector space $\mathfrak{g}$ with an operation $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that preserves the $\mathbb{Z}_2$-grading and satisfies i) $[x, y] + (-1)^{p(x)p(y)}[y, x] = 0$ and ii)

$$(-1)^{p(x)p(z)}[[x, y]z] + (-1)^{p(y)p(x)}[[y, z]x] + (-1)^{p(y)p(z)}[[z, x]y] = 0.$$

We will always assume that $\mathfrak{g}$ is finite-dimensional. A morphism of Lie superalgebras is a super vector space morphism that preserves the Super Lie bracket. The even part $\mathfrak{g}_0$ is an ordinary Lie algebra and the odd part $\mathfrak{g}_1$ is a module under $\mathfrak{g}_0$. The main example is the super vector space $Mat_{p|q}(k)$. It gets the structure of a Lie superalgebra with the super commutator $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$. It will be denoted $gl(p|q)$. If V is a super vector space, write gl_V for the Lie superalgebra of all linear maps $V \to V$ equipped with the super commutator. The kernel of the supertrace map $Mat_{p|q}(k) \to k$ is denoted $sl(p|q)$. Consider the $(p + 2q) \times (p + 2q)$ matrix Ω defined by

$$\Omega = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & J_q \end{pmatrix}, \; J_q = \begin{pmatrix} 0 & -\mathbb{1}_q \\ \mathbb{1}_q & 0 \end{pmatrix}.$$

Then $osp(p|2q)$ is the super vector space of $(p + 2q) \times (p + 2q)$-matrices satisfying $X^{ST}\Omega + \Omega X = 0$ with the super commutator. It is called the orthosymplectic Lie superalgebra.

A Lie superalgebra is simple if it has no non-trivial ideal (in the $\mathbb{Z}_2$-graded sense). The simple finite-dimensional Lie superalgebras have been classified by

Kac [Kac77]. We assume from now on that $\mathfrak{g}$ is basic classical in the sense of Kac' classification, in particular $\mathfrak{g}$ is simple and $\mathfrak{g}_0$ reductive. The Lie superalgras $sl(m,n)$ for $m \neq n$, $sl(m,m)/ < \mathbb{1}_m >$ and the orthosymplectic Lie superalgebras fulfill this condition. We also count $gl(m,n)$ as being basic classical (although it is not simple).

A representation of a Lie superalgebra $\mathfrak{g}$ is a Lie superalgebra morphism

$$\rho : \mathfrak{g} \to gl(V)$$

for some super vector space V. In this case V will be called a $\mathfrak{g}$-module. Alternatively to our definition of a representation of $Gl(m|n)$ we could have equivalently defined that a representation of G is the same as a finite-dimensional representation ρ of $gl(m|n)$ such that its restriction to $\mathfrak{g}_0$ comes from an algebraic representation of $Gl(m) \times Gl(n)$. In other words studying representations of $Gl(m|n)$ means studying integrable modules over $gl(m|n)$.

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is by definition a Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{h}$ be a Cartan subalgebra. For $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h,g] = \alpha(h)g \ \forall h \in \mathfrak{h}\}$$

be the root space. The root system for $\mathfrak{g}$ is given by

$$\Phi = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}.$$

We define the even and the odd roots as

$$\begin{aligned} \Phi_0 &= \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq 0\} \\ \Phi_1 &= \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq 0\}. \end{aligned}$$

Let Φ be a root system and E a real vector space spanned by Φ. A positive system Φ^+ is a subset of Φ consisting of those roots $\alpha \in \Phi$ which satisfy $\alpha > 0$ for some total ordering of E. The elements in Φ^+ are called positive roots. Likewise we dispose over the set of negative roots Φ^-. Putting $\Phi_i^+ = \Phi^+ \cap \Phi_i$ for $i \in \mathbb{Z}_2$ gives $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$. We define

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$

One obtains a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The solvable subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called the Borel subalgebra corresponding to Φ^+. One has $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$ with $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{g}_i$ for $i \in \mathbb{Z}_2$. As in the classical case we could conversely start with a Borel subalgebra which would determine a subset of positive roots.

Let $\mathfrak{h}$ be the standard Cartan subalgebra and Φ the root system. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra of $\mathfrak{g}$ and let Φ^+ be the associated system of positive roots. If V is a finite-dimensional irreducible representation, V contains a one-dimensional $\mathfrak{b}$-module, which is of the form kv_λ for $\lambda \in \mathfrak{h}^* = (\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$. Then V has a weight space decomposition

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$$

with the μ-weight space

$$V_\mu = \{v \in V \mid hv = \mu(h)v \; \forall v \in \mathfrak{h}\}.$$

The weight space V_μ is 0 unless $\lambda - \mu$ is a $\mathbb{Z}_+$-linear combination of positive roots. The weight λ is called the $\mathfrak{b}$-highest weight and the vector v_λ s called a $\mathfrak{b}$-highest weight vector for $V =: L(\lambda)$. We often fix for $\mathfrak{b}$ the group of upper triangular matrices.

1.3.1. The $gl(m, n)$-case

Let $\mathfrak{h}$ be the Cartan algebra of diagonal matrices, $\mathfrak{n}^+$ respectively $\mathfrak{n}^-$ the subalgebra of strictly upper respectively lower triangular matrices. This gives the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The even subalgebra $\mathfrak{g}_0$ has the decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ with $\mathfrak{n}_0^\pm = \mathfrak{g}_0 \oplus \mathfrak{n}^\pm$. Put $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{n}_0^+$. Let ϵ_i, δ_j be the usual choices of the basis function of the dual of the Cartan subalgebra of diagonal matrices in $gl(m, n)$. With this choice of Borel the positive even roots are

$$\Phi_0^+ = \{\epsilon_i - \epsilon_k \mid 1 \leq i < k \leq m\} \cup \{\delta_j - \delta_l \mid 1 \leq j < l \leq n\}$$

and the positive odd roots are

$$\Phi_1^+ = \{\epsilon_i - \delta_j \mid 1 \leq i \leq m \; and \; 1 \leq j \; \leq n\}.$$

The Lie superalgebra admits also a $\mathbb{Z}$-gradation $\mathfrak{g} = \mathfrak{g}_{-\bar{1}} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ where $\mathfrak{g}_{-\bar{1}}$ respectively $\mathfrak{g}_{\bar{1}}$ is spanned by the elementary matrices E_{ij} with $i > 0 > j$ respectively $i < 0 < j$ and $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$.

For $\lambda \in \mathfrak{h}^*$ let $L_0(\lambda)$ be the simple $\mathfrak{g}_0$-module of highest weight $\lambda \in \mathfrak{h}^*$ relative to $\mathfrak{b}_0$. The $\mathfrak{g}_0$-module $L_0(\lambda)$ can be extended trivially to $\mathfrak{g}_0 \oplus \mathfrak{g}_{\bar{1}}$. The Kac-module is by definition

$$K(\lambda) = Ind_{\mathfrak{g}_0 \oplus \mathfrak{g}_{\bar{1}}}^{\mathfrak{g}} L_0(\lambda)$$

and the AntiKac-module is given by

$$K'(\lambda) = Ind^{\mathfrak{g}}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}} L_0(\lambda).$$

1.8 Lemma. *$K(\lambda)$ has irreducible top and socle. The top is given by $L(\lambda)$. $K(\lambda)$ is finite-dimensional iff $L_0(\lambda)$ is finite-dimensional iff $L(\lambda)$ is finite-dimensional.*

In particular the simple $\mathfrak{g}$-modules are up to a parity shift parametrised by the same set of highest weights as the simple $\mathfrak{g}_0$-modules. Hence the (integral dominant) highest weights X^+ of $gl(m|n)$ are given by

$$\lambda = \sum_{i=1}^{m} \lambda_i \epsilon_i + \sum_{j=m+1}^{m+n} \lambda_j \delta_j = (\lambda_1, \ldots, \lambda_m | \lambda_{m+1}, \ldots, \lambda_{m+n}).$$

Here $\lambda_1 \geq \ldots \geq \lambda_m$ and $\lambda_{m+1} \geq \ldots \geq \lambda_{m+n}$ are integers and every $\lambda \in \mathbb{Z}^{m+n}$ with these properties parametrises a highest weight of an irreducible $\mathfrak{g}$-module. This set of highest weights is called X^+, so that the irreducible modules in $Rep(\mathfrak{g})$ are given by the

$$\{L(\lambda), \Pi L(\lambda) \mid \lambda \in X^+\}.$$

where Π denotes the parity shift.

We say that a module is a Kac-object if it has a filtration whose subquotients are Kac-modules. The full subcategory of these modules is denoted C^+. Similarly we have the category C^- of objects which have a filtration by AntiKac-modules. By [Ger98] $C^+ \cap C^- = Proj$. Both C^+ and C^- are tensor ideals in $\mathcal{R}$.

Atypicality. If $K(\lambda)$ is irreducible the weight λ is called typical. If not, λ is called atypical. $K(\lambda)$ is irreducible if and only if $K(\lambda)$ is projective as a $\mathfrak{g}$-module. Generically a weight is typical. The atypical weights form a thin subset of all weights in X^+. Define a bilinear form on $\mathfrak{h}^*$ by $(\epsilon_i, \epsilon_k) = \delta_{ik}$ and $(\delta_j, \delta_l) = -\delta_{jl}$ and $(\epsilon_i, \delta_j) = 0$ for $i, k = 1, \ldots, m$ and $j, l = 1, \ldots, n$. Define furthermore ρ_0 to be the half sum of positive even roots and ρ_1 the half sum of positive odd roots and $\rho = \rho_0 - \rho_1$. Then

$$\rho_0 = \frac{1}{2} \sum_{i=1}^{m} (m - 2i + 1)\epsilon_i + \frac{1}{2} \sum_{j=1}^{n} (n - 2j + 1)\delta_j$$

$$\rho_1 = \frac{n}{2} \sum_{i=1}^{m} \epsilon_i - \frac{m}{2} \sum_{j=1}^{n} \delta_j.$$

Then the degree of atypicality $at(\lambda) = at(L(\lambda))$ of λ is the number of odd positive roots $\beta \in \Phi_1^+$ for which

$$(\lambda + \rho, \beta) = 0$$

holds. The atypicality of λ is zero iff the weight is typical. If $at(\lambda) = 1$, the weight is singly atypical. The atypicality of a weight is a number between 0 and $min(m, n)$. If the atypicality is maximal, we say the weight is maximally atypical. Examples are the trivial module $\mathbb{1}$ and the standard representation st of highest weight $\lambda = (1, \ldots, 0|0, \ldots, 0)$ for $m \neq n$. Another example is the Berezin determinant (given by the Berezin determinant of the supergroup $Gl(m|n)$)

$$B = Ber = L(1, \ldots, 1 \mid -1, \ldots, 1)$$

of dimension 1. The degree of atypicality is a block-invariant. Hence we can define the degree of atypicality of an arbitrary indecomposable module to be the degree of atypicality of its composition factors. The full subcategory of modules of atypicality i is denoted $\mathcal{A}_i$. Note that the irreducible atypical modules are responsible for the failure of semisimplicity since the typical ones are projective elements. If we are in the $sl(m|n)$-case all results are true; we just have to identify two irreducible modules whose highest weights differ only from a weight of the form $(k, k, \ldots, k| - k, \ldots, -k)$ for $k \in \mathbb{Z}$.

The *-duality Recall that the supertranspose x^{ST} of a graded endomorphism $x \in End(k^{m|n})$ is given by

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto x^{ST} = \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix}.$$

If one identifies $gl(m|n)$ with $End(k^{m|n})$ then $\tau(x) = -x^{ST}$ defines an automorphism of $\mathfrak{g}$ such that $\tau(\mathfrak{g}_{(i)}) = \mathfrak{g}_{(-i)}$ for $i = -1, 0, 1$. If $M \in Rep(Gl(m|n))$ and x homogenuous in $\mathfrak{g}$ the Tannaka dual $M^\vee = (V^\vee, \rho^\vee)$ of $M = (V, \rho)$ is the representation $x \mapsto -\rho(x)^{ST}$ on V with the supertranspose ρ^{ST} on $End(V)$. The *-dual of M is given by $M^* = (V^\vee, \rho^\vee \circ \tau)$ where $\tau(x) = -x^{ST}$. If $M \in \mathcal{R}_{mn}$, then $M^{vee}, M^* \in \mathcal{R}_{mn}$. The duality * is the identity for irreducible and projective M.

1.3.2. The $osp(2|2n)$-case

The situation is completely analogous for the Lie superalgebra $osp(2|2n)$. This is no longer true for the other orthosymplectic Lie superalgebras. The decisive point here is that $osp(2|2n)$ still admits $\mathbb{Z}$-gradation as above wheras this is false in general. The $\mathfrak{g}_0$-part is given by $\mathfrak{g}_0 = so(2) \oplus sp(2n-2)$. As a Cartan subalgebra we pick the space of diagonal matrices. The roots are expressed in terms of the usual linear forms $\epsilon,\ \delta_1, \ldots, \delta_{n-1}$. The positive even roots are

$$\Phi_0^+ = \{\delta_i - \delta_j\ (i < j),\ \delta_i + \delta_j\}$$

and the positive odd roots

$$\Phi_1^+ = \{\epsilon \pm \delta_j\}.$$

Highest weights of $osp(2|2n)$ are written as

$$\lambda = \lambda_0 \epsilon + \sum \lambda_j \epsilon_j = (\lambda_0 | \lambda_1, \ldots, \lambda_n)$$

with integer coefficients satisfying $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ for all $i \in \{1, \ldots, n-1\}$ and basis functions ϵ, ϵ_j as above. As above the irreducible modules in $\mathbb{A} = Rep(\mathfrak{g})$ are the

$$\{L(\lambda), \Pi L(\lambda) \mid \lambda \in X^+\}.$$

The results about Kac-modules and atypicality carry over. Here the atypicality is either zero (the typical case) or one (the singly atypical case). In this respect the Lie superalgebra $osp(2|2n)$ behaves as the Lie superalgebra $gl(m|1)$ respectively $sl(m|1)$.

1.4. The results of Brundan and Stroppel

The most important work regarding representations of the General Linear Supergroup is a series of articles by Brundan and Stroppel [BS08], [BS10a], [BS08], [BS10b], [BS11]. In these articles they develop a general theory of Khovanov-algebras, so-named because they generalize certain diagrammatically defined algebras H_n^n used by Khovanov in his categorification of the Jones polynomial [Kho00]. We treat the existence and definition of these algebras as a blackbox and refer to the very elaborate and lengthy constructions to the articles above. For our purpose only a particular choice of these Khovanov-algebras, denoted $K(m|n)$, is relevant. These algebras are naturally graded. For $K(m|n)$ we have a set of weights or weight diagrams which parametrise the irreducible modules (up to a grading shift). This set of weights is again denoted X^+. For each weight $\lambda \in X^+$ we have the irreducible module $\mathcal{L}(\lambda)$, an indecomposable projective module $\mathcal{P}(\lambda)$ with top $\mathcal{L}(\lambda)$ and the standard or cell module $\mathcal{V}(\lambda)$. If we forget the grading structure on the $K(m|n)$-modules, the the main result of [BS10b] is:

1.9 Theorem. *There is an equivalence of categories E from $\mathcal{R}_{mn}$ to the category of finite-dimensional left $K(m|n)$-modules such that $EL(\lambda) = \mathcal{L}(\lambda)$, $EP(\lambda) = \mathcal{P}(\lambda)$ and $EK(\lambda) = \mathcal{V}(\lambda)$ for $\lambda \in X^+$.*

More precisely $K(m|n)$ is isomorphic to the locally finite endomorphism algebra $End_G^{fin}(P)^{op}$ of a canonical minimal projective generator $P \simeq \bigoplus_{\lambda \in X^+} P(\lambda)$ for $\mathcal{R}_{mn}$. In particular E is a Morita equivalence. Hence E will preserve the Loewy structure of indecomposable modules. This will enable us to study questions regarding extensions or Loewy structures in the category of Khovanov modules. Note however that this is not an equivelence of tensor categories. In fact, all irreducible $K(m|n)$-modules are one-dimensional.

The description of the irreducible $K(m|n)$-modules and their extensions via weight diagrams suggests describing the irreducible modules in $\mathcal{R}_{mn}$ in the same language. This has been proven to be very fruitful, giving a clear picture to phenomena whose structure was not visible in the highest weight language. Examples include the description of blocks, a description of the Ext^1 between irreducible modules and a formula for the superdimension of an irreducible module.

Weight diagrams. To each highest weight $\lambda \in X^+$ we associate, following [BS10b], two subsets of cardinality n of the numberline $\mathbb{Z}$

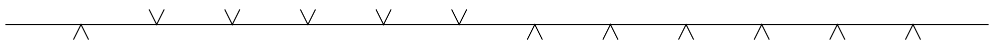
$$\begin{aligned} I_\times(\lambda) &= \{\lambda_1, \lambda_2 - 1, ..., \lambda_n - n + 1\} \\ I_\circ(\lambda) &= \{1 - m - \lambda_{m+1}, 2 - m - \lambda_{m+2}, ..., n - m - \lambda_{m+n}\}. \end{aligned}$$

The integers in $I_\times(\lambda) \cap I_\circ(\lambda)$ are labeled by $\vee$, the remaining ones in $I_\times(\lambda)$ resp. $I_\circ(\lambda)$ are labeled by $\times$ resp. $\circ$. All other integers are labeled by a $\wedge$. This labeling of the numberline $\mathbb{Z}$ uniquely characterizes the weight λ. If the label $\vee$ occurs r times in the labeling, then r is called the degree of atypicality of λ. Notice that $0 \leq r \leq n$, and λ is called maximal atypical if $r = n$. This notion of atypicality agrees with our original definition.

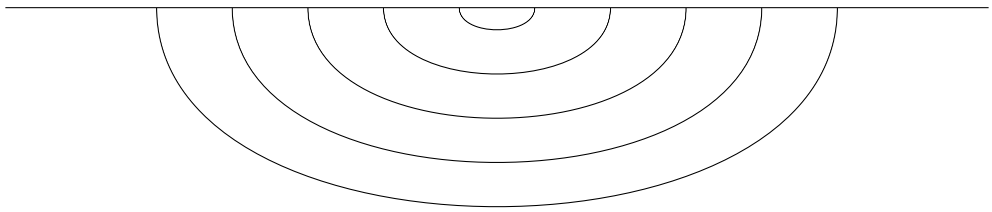
Blocks. Two irreducible representations $L(\lambda)$ and $L(\mu)$ are in the same block if and only if the weights λ and μ define labelings with the same position of the labels $\times$ and $\circ$. The degree of atypicality is a block invariant, and the blocks Λ of atypicality r are in 1-1 correspondence with pairs of disjoint subsets of $\mathbb{Z}$ of cardinality $m - r$ resp. $n - r$.

Bruhat order The Bruhat order $\geq$ is the partial order on the set of weight diagrams generated by the operation of swapping a $\vee$ and a $\wedge$, so that getting bigger in the Bruhat order means moving $\vee$'s to the right.

Cups and Caps. To each such weight diagram we associate its cup diagram as in [BS08]. Here a cup is a lower semi-circle joining two vertices. To construct the cup diagram go from left to right throught the weight diagram until one finds a pair of vertices $\vee \quad \wedge$ such that there only x's, $\circ$'s or vertices which are already joined by cups between them. Then join $\vee\ \wedge$ by a cap. This procedure will result in a diagram with r cups. Now remove all the labels of the vertices and draw rays down to infinity at all vertices which are not part of a cup. If we draw the picture of a cup diagram we will not draw the rays. As an example consider the trivial weight $(0,\ldots,0|0,\ldots,0)$ in $Gl(n|n)$. Its weight diagram is given by

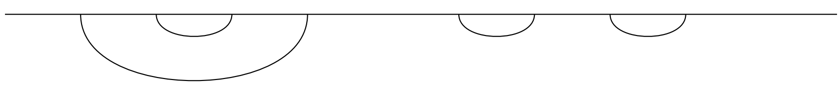


with n $\vee$'s at the vertices $-n+1,\ldots,0$. Its cup diagram is given by

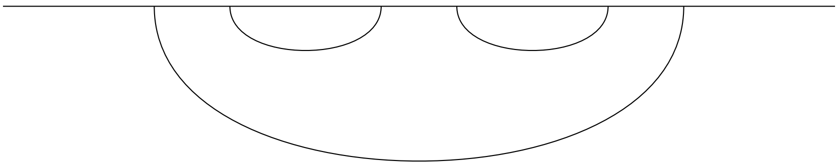


Analogously we define a cap to be a lower semi-circle joining two vertices. The cap diagram is build in the same way as the cup diagram. It is obtained from the latter by reflecting along the numberline. As with the cup diagram we will not draw the rays in pictures.

Sectors and segments. For the purpose of this paragraph we will assume to be in a maximal atypical block, i.e. weight diagrams do not have vertices labelled by either $\times$ or $\circ$. Of the r cups of a cup diagram some may be nested. If we remove all inner parts of the nested cups we obtain a cup diagram defined by the (remaining) outer cups. We enumerate these cups from left to right. The starting points of the j-th lower cups is denoted a_j, its endpoint is denoted b_j. Then there is a label $\vee$ at the position a_j and a label $\wedge$ at position b_j. The interval $[a_j, b_j]$ of $\mathbb{Z}$ will be called the j-th sector of the cup diagram. Adjacent sectors, i.e with $b_j = a_{j+1} - 1$ will be grouped together into segments. The segments again define intervals in the numberline. Let s_j be the starting point of the j-th segment and t_j the endpoint of the j-th segment. Between any two segments there is a distance at least ≥ 1. In the following case the weight diagram has 2 segments and 3 sectors

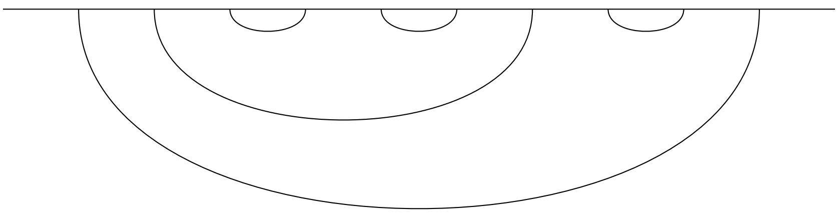


whereas the following weight diagram has 1 segment and 1 sector.



Removing the outer circle would result in a cap diagram with two sectors and two segments.

Plots. Note that the segment and sector structure of a weight diagram is completely encoded by the positions of the $\vee$'s. Hence any finite subset of $\mathbb{Z}$ defines a unique weight diagram in a given block. We call this finite subset λ the plot associated to λ. Sometimes we view a plot also as a function $\lambda : \mathbb{Z} \to \{\vee, \wedge\}$ with finite fibre $\lambda^{-1}(\vee)$. Assume now that we are in the maximally atypical $m = n$-case, ie. our block does not have vertices labelled by $\times$ or $\circ$. Given a cup diagram with r sectors, the plot is the union of the r plots associated to each of the sectors. We call these subsets the prime factors of λ and write $\lambda = \prod_{i=1}^{r} \lambda_i$. For such a prime factor we write (I, K) where K is the finite subset of $\vee$'s in the prime factor (called also its support) and I is the subset of cardinality $2\sharp K$ of vertices in the sector defined by the prime plot. For instance $\{-4, -3, -2, 0, 3\}$ is a prime plot with sector $\{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ as can be seen from its cup diagram



2. The universal semisimple quotient

We define the universal semisimple quotient of a super tannakian category. We prove that it is again a super tannakian category. We determine the pro-reductive cover in the cases $Gl(m|1)$, $Sl(m|1)$ and give a conditional proof in the $OSp(2|2n)$-case.

2.1. The universal semisimple quotient

An additive category $\mathbb{A}$ is a Krull-Schmidt category if every object has a decomposition in a finite direct sum of elements with local endomorphism rings. An ideal in a k-linear category is for any two objects X, Y the specification of a k-submodule $T(X,Y)$ of $Hom_A(X,Y)$, such that for all pairs of morphisms $f \in Hom_(X, X')$, $g \in Hom(Y,Y')$ the inclusion $gT(X',Y)f \ \subseteq T(X,Y')$ holds. Let T be an ideal in $\mathbb{A}$. $\mathbb{A}/T$ is the category with the same objects as A and with $Hom_{\mathbb{A}/T}(X,Y) = Hom_{\mathbb{A}}(X,Y)/T(X,Y)$. It is again a Krull-Schmidt category [Liu09], [KZ08]. Let $\mathbb{A}$ be abelian and every object has finite length. Let X be an indecomposable element and ϕ an endomorphism. By Fitting's lemma ϕ is either invertible or nilpotent. An element X is indecomposable if and only if its endomorphism ring is a local ring.

Assume in the following that $\mathbb{A}$ is a super tannakian category or a pseudoabelian full tensor subcategory. Then all the above conditions hold.

An ideal in a tensor category is a tensor ideal if it is stable under $\mathbb{1}_C \otimes -$ und $- \otimes \mathbb{1}$ for all $C \in \mathbb{A}$ ist. The ideal is then stable under tensor products from left or right with arbitrary morphisms. Let Tr be the trace. For any two objects A, B we define $\mathcal{N}(A,B) \subset Hom(A,B)$ by

$$\mathcal{N}(A,B) = \{f \in Hom(A,B) \mid \forall g \in Hom(B,A),\ Tr(g \circ f) = 0\}.$$

The collection of all $\mathcal{N}(A,B)$ defines a tensor ideal $\mathcal{N}$ of $\mathcal{A}$ [KA02]. We recall [KA02], 8.2.2a):

2.1 Theorem. *(i) $\mathcal{N}$ is the largest proper tensor ideal of $\mathbb{A}$.*
(ii) The only proper tensor ideal $\mathcal{I}$ of $\mathbb{A}$ such that the quotient $\mathbb{A}/\mathcal{I}$ is semisimple, is $\mathcal{I} = \mathcal{N}$.

The quotient $\mathbb{A}/\mathcal{N}$ will be called the universal semisimple quotient of $\mathbb{A}$. We have the following fundamental lemma:

2.2 Proposition. *The quotient $\mathbb{A}/\mathcal{N}$ is again a super tannakian category. If $\mathbb{A}' \subset \mathbb{A}$ is a pseudoabelian full tensor subcategory, the quotient $\mathbb{A}'/(\mathcal{N} \cap \mathbb{A}')$ is a super tannakian category.*

Proof: The quotient of a k-linear rigid tensor category by a tensor ideal is again a k-linear rigid tensor category. Since $\mathcal{N}$ is a tensor ideal the quotient functor $\omega : \mathbb{A} \to \mathbb{A}/\mathcal{N}$ is a tensor functor. The quotient category is semisimple by construction. Since Hom-spaces are finite-dimensional one has idempotent lifting, hence $\mathbb{A}/\mathcal{N}$ is pseudoabelian. A k-linear semisimple pseudoabelian category is abelian by [KA02]. By [Del02] an abelian tensor category is super tannakian if and only if for every object A there exists a Schur functor S_μ with $S_\mu(A) = 0$. Since $\omega(S_\mu(A)) = S_\mu(\omega(A))$ any object in $\mathbb{A}/\mathcal{N}$ is also annulated by a Schur functor. □

The category $\mathbb{A}/\mathcal{N}$ has the following universal property:

2.3 Proposition. *Let $\omega : \mathbb{A} \to C$ be a full tensor functor into a semisimple tensor category C. Then ω factorises over the quotient $\mathbb{A}/\mathcal{N}$.*

Proof: Since C is semisimple there are no negligible morphisms. However the image of a negligible morphism is negligible, since a tensor functor commutes with traces. Hence the image of a negligible morphism under ρ is zero, hence the functor factorizes. □

For lack of references we assemble a few elementary lemmas about this quotient.

2.4 Lemma. *An object X of $\mathbb{A}$ maps to zero in $\mathbb{A}/\mathcal{N}$ if and only if $id_x = 1_X$ belongs to $\mathcal{N}(X, X)$.*

Proof: Let $1_X \in N(X, X)$. By definition: $\forall g \in End(X) : tr(1_X \circ g) = tr(g) = 0$. Let $f \in Hom(X, Y)$. For all $g \in Hom(Y, X)$ $tr(f \circ g) = 0$ since $f \circ g \in End(X)$. Hence $f \in N(X, Y)$ and $Hom_{\mathbb{A}/\mathcal{N}}(X, Y) = 0$. Similarly for $f \in Hom(Y, X)$. □

The collection of these elements - called negligible objects - is denoted by N.

2.5 Lemma. *An indecomposable object is in N if and only if $sdim(X) = 0$.*

Proof: If $X \in N$ we have $\forall g \in End(X) : tr(g) = 0$, in particular for $g = 1_X$. Let $sdim(X) = 0$. We have to show: $1_X \in N(X, X)$, ie. $tr(g) = 0 \; \forall g \in End(X)$. Since X is indecomposable g is either nilpotent or an isomorphism. If g is nilpotent $tr(g) = 0$. Let g be an isomorphism. Since X is indecomposable g has a unique eigenvalue λ and $tr(g) = \lambda sdim(X)$, hence $tr(g) = 0$. □

The following lemma is contained as a remark in [Bru00]. It is a well-known property of quotients. For completeness sake we give a proof.

2.6 Lemma. *The functor $\mathbb{A} \to \mathbb{A}/\mathcal{N}$ induces a bijection between the isomorphism classes of indecomposable elements not in N and the isomorphism classes of irreducible elements in $\mathbb{A}/\mathcal{N}$.*

Before the proof we recall that the trace of a nilpotent endomorphism is zero [KA02] or [Bru00], 1.4.3.

Proof: Let X be indecomposable, $X \notin N$. Since $\mathbb{A}$ and $\mathbb{A}/\mathcal{N}$ are abelian and every object has finite lenght, an object X is indecomposable iff $End(X)$ is a local ring. We have $End_{\mathbb{A}/\mathcal{N}}(X) = End_{\mathbb{A}}(X)/\mathcal{N}(X)$. Since the quotient of a local ring by a (two-sided) ideal is again local, the image of X in $\mathbb{A}/\mathcal{N}$ is indecomposable, hence irreducible. We show: If $M \not\cong N$ in $\mathbb{A}$ we have $Hom_{\mathbb{A}/\mathcal{N}}(M,N) = 0$. Let $f \in Hom_{\mathbb{A}}(M,N)$. Its image is zero in $Hom_{\mathbb{A}/\mathcal{N}}(M,N) = Hom_{\mathbb{A}}(M,N)/\mathcal{N}(M,N)$ iff $tr(fg) = 0 \ \forall g \in Hom_{\mathbb{A}}(N,M)$. Since M is indecomposable any endomorphism is invertible or nilpotent. The endomorphism fg is not bijective, hence nilpotent, hence $tr(fg) = 0 \ \forall g \in Hom(N,M)$, hence $Hom_{\mathbb{A}}(M,N) = 0$. Evidently any irreducible element in $\mathbb{A}/\mathcal{N}$ comes from an indecomposable element in $\mathbb{A}$. $\square$

Let I be an ideal in $\mathbb{A}$. For $X = \bigoplus X_i$ and $Y = \bigoplus Y_j$ we have canonically $I(X,Y) = \bigoplus_{ij} I(X_i, Y_j)$ by [KA02]. Let $X = \bigoplus X_i$ with $X_i \in N$ for all i, ie. $\mathcal{N}(X_i, Y) = Hom(X_i, Y)$ and $\mathcal{N}(Y, X_i) = Hom(Y, X_i)$ for all $Y \in \mathbb{A}$. It follows $\mathcal{N}(X,X) = Hom(X,X)$, hence $X \in N$. If reciprocally $X \in N$ and $X = \bigoplus X_i$, we have $X_i \in N$.

2.7 Corollary. *(i) N is closed under direct sums and direct summands. (ii) If $X \in N$ and $Y \in \mathbb{A}$, we have $X \otimes Y$ in N and each indecomposable summand of $X \otimes Y$ has superdimension* 0. *(iii) Let $X \notin N$ and let $X = \bigoplus X_i$ be its decomposition into indecomposable elements. Then $Hom_{\mathbb{A}/\mathcal{N}}(X,X) = \bigoplus_{i,\ sdim(X_i) \neq 0} k$. (iv) N is neither closed unter submodules nor quotients.*

2.2. The pro-reductive envelope

Since the quotient $\mathbb{A}/\mathcal{N}$ is again a super-tannakian category, this defines a reductive super group scheme G^{red} with $\mathbb{A}/\mathcal{N} \simeq Rep(G^{red}, \epsilon)$ with $\epsilon : \mu_2 \to G$ such that the operation of μ_2 gives the $\mathbb{Z}_2$-graduation of the representations, which we call the pro-reductive envelope of G (following [KA02]). It might be very difficult to determine G^{red} in practice.

2.8 Theorem. *In the singly atypical type I cases we have:* [1]

$$G^{red} = \begin{cases} Gl(m-1)\times Gl(1)\times Gl(1) & G = Gl(m|1) \\ Sl(m-1)\times Gl(1)\times Gl(1) & G = Sl(m|1),\ m\geq 3 \\ Gl(1)\times Gl(1) & G = Sl(2|1), \\ Sp(2n-2)\times Gl(1)\times Gl(1) & G = OSp(2|2n) \end{cases}$$

The proof will occupy the rest of this chapter.

If G is actually an algebraic group, the pro-reductive envelope has been extensively studied by Andre and Kahn. Their proofs do not apply to the supergroup case. In the tannakian case $\mathcal{N} = R$ is equal to the radical ideal. In particular no indecomposable elements maps to zero. In the tannakian case the group G embeds into the group G^{red}. Even in the tannakian case the pro-reductive cover will not be of finite type in general. More precisely we have:

2.9 Theorem. *[KA02], theorem C.5 The proreductive envelope of an affine k-group G is of finite type over k if and only if G is of finite type over k and the prounipotent radical of G is of dimension ≤ 1.*

Consider two examples. If $G = \mathbb{G}_a$, then $G^{red} = Sl(2)$. If $G = \mathbb{G}_a \times \mathbb{G}_a$, then G^{red} is no longer of finite type. In fact, the determination of $G \hookrightarrow G^{red}$ is unsolvable since it would include a classification of the indecomposable representations of G which is a wild problem [KA02], 19.7. This shows that the quotient G^{red} can be very complicated. The situation is even more difficult in the supergroup case since a lot of elements get killed by the quotient functor. For $Gl(m|n)$ the following holds:

2.10 Theorem. *Assume $m,n \geq 2$. Then $Gl(m|n)^{red}$ is not of finite type.*

Proof: This will follow from the description of the Tannaka group generated by the irreducible elements in section 7. □

The statement also follows from the following lemma. This lemma should of course also hold for $m,n \geq 2$, but would require a more difficult argument. Let Q denote the Ext-quiver of $\mathcal{R}_{mn}$. Then there exists a system of relations R on Q such that $\mathcal{R}_{mn} \simeq kQ/R-mod$.

2.11 Lemma. *Assume $m,n \geq 3$. Then the problem of classifying indecomposable*

[1] the proof in the $osp(2|2n)$-case is conditional

representations of non-vanishing superdimension is wild.

Proof: We show that that the classication is wild for every maximally atypical block for $n \geq 3$. Any such block is equivalent to the maximal atypical block Γ of $Gl(n,n)$ [BS10b]. Hence we show that the problem is wild in Γ. By [BS10a], Cor. 5.15 for any two irreducible modules $L(\lambda), L(\mu) \in \mathcal{R}_n$

$$\dim(Ext^1_{\mathcal{R}_n}(L(\lambda), L(\mu))) = p^{(1)}_{\lambda,\mu} + p^{(1)}_{\mu,\lambda}$$

for the Kazhdan-Lusztig polynomials

$$p_{\lambda,\mu}(q) = \sum_{i \geq 0} p^{(i)}_{\lambda,\mu} q^i.$$

By [MS11], lemma 6.10 and [BS10a], lemma 5.2 $p^{(1)}_{\lambda,\mu} \neq 0$ if and only if μ is obtained from λ by interchanging the labels at the ends of one of the cups in the cup diagram of λ. For any $[\lambda] \in \Gamma$ with $\lambda_i > \lambda_{i+} + 1$ the cup diagram looks like

The combinatorial rule from above shows that for every irreducible module $[\lambda]$ away from the diagonal $dimExt^1([\lambda],[\mu_i]) = dimExt^1([\mu_i],[\lambda]) = 1$ for exactly $2n$ different modules μ_i and $dimExt^1([\lambda], \nu) = 0$ for any $\nu \neq \mu_i$. In particular for any vertex away from the diagonal consider the subquiver with vertices $[\lambda], [\mu_1], \ldots, [\mu_{2n}]$ with arrows corresponding to $dimExt^1([\mu_i],[\lambda]) = 1$ and no arrows from $[\lambda]$ to any $[\mu_i]$ (so that $[\lambda]$ becomes a sink) (picture for $n = 3$):

$$\begin{tikzcd}
\bullet[\mu_6] \arrow[dr] & & \bullet[\mu_1] \arrow[dl] \\
\bullet[\mu_5] \arrow[r] & \bullet[\lambda] & \bullet[\mu_2] \arrow[l] \\
\bullet[\mu_4] \arrow[ur] & & \bullet[\mu_3] \arrow[ul]
\end{tikzcd}$$

Since this subquiver has no path of length > 1, it embeds fully into $k(Q)/R$. The classifaction of indecomposable representations of the r-subspace quiver is wild for $r \geq 5$. The superdimension formula of [Wei10b] shows that the superdimension is constant of alternating sign away from the diagonal: if $[\lambda]$ has superdimension d, the $[\mu_i]$ have superdimension $-d$. Hence an indecomposable representation of this subquiver will give an indecomposable representation in Γ of non-vanishing superdimension if and only if

$$dimV_{[\lambda]} \neq \sum_{i=1}^{2n} dimV_{[\mu_i]} \quad (*).$$

We are done when we have shown that the classification of indecomposable representations with $(*)$ is wild. Fix the vertex $[\mu_{2n}]$ and consider an indecomposable representation of the $(2n-1)$-subspace quiver by specifying a vector space for the vertices $[\lambda], [\mu_1], \ldots, [\mu_{2n-1}]$ with injections $V_{[\mu_i]} \to V_{[\lambda]}$. If $dimV_{[\lambda]} \neq \sum_{i=1}^{2n-1} dimV_{[\mu_i]}$ we put $V_{[\mu_{2n}]} = 0$. If $dimV_{[\lambda]} = \sum_{i=1}^{2n} dimV_{[\mu_i]}$ we put $V_{[\mu_{2n}} = k$ and choose some injection of k into $V_{[\lambda]}$. This defines a bijection between the isomorphism classes of indecomposable representations of the $(2n-1)$-subspace quiver with a subset of the indecomposable representations of the $2n$-subspace quiver satisfying $(*)$. □

More generally it seems likely that the following holds:

2.12 Conjecture. *Let G be a super group scheme. Then G^{red} is of finite or tame type iff $Rep(G)$ is of finite or tame type.*

In particular if $Rep(G)$ is of wild type, the problem of classifying indecomposable modules of non-vanishing superdimension should be wild too. Therefore we should not try to determine G^{red} in this case, but ask the following weaker questions: Given any object $X \in Rep(G)$ consider its image in $\mathbb{A}/\mathcal{N}$. The tensor category generated by it will always be a semisimple algebraic tensor category (since $\mathbb{A}/\mathcal{N}$ is semisimple). The semisimple algebraic tensor categories in characteristic zero were classified in [Wei09]:

2.13 Theorem. *Any reductive supergroup G over an algebraically closed field k of characteristic zero is isomorphic to a semidirect product $G' \lhd H$ of a reductive algebraic k-group H with a product $G' = \prod_{r \geq 1} Spo(1, 2r)^{n_r}$ of simple supergroups of BC-type, where the semidirect product is defined by an abstract group homomorphism*

$$p : \pi_0(H) \to \prod_{r \geq 1} S_{n_r} \ .$$

Question What is the tensor category generated by the irreducible elements in $\mathbb{A}/\mathcal{N}$? What is the tensor category generated by all irreducible elements in $\mathbb{A}/\mathcal{N}$? And what is the associated pro-reductive supergroup scheme? Note that this supergroup scheme need not be of finite type: this is already the case for the group $Gl(2|2)$ (see chapter 7).

2.3. Irreducible elements and the toy example $Sl(2|1)$

We describe the irreducible elements in $\mathbb{A}/\mathcal{N}$. Then we determine G^{red} in the $Sl(2|1)$-case.

Assume from now on that we are in the $Gl(m|1)$-case. Recall that Kac-modules have a simple socle. The highest weight of the socle is denoted by $T^-\lambda$. The highest weight of the socle of the AntiKac-module $K'(\lambda)$ is denoted by $T^+\lambda$. If λ is atypical $K(\lambda)$ is an extension of $L(\lambda)$ by $L(T^-\lambda)$:

$$0 \longrightarrow L(T^-\lambda) \longrightarrow K(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

Similarly we have the exact sequence

$$0 \longrightarrow L(T^+\lambda) \longrightarrow K'(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

By [Ger98] this sequences are up to equivalence all non-trivial extensions between simple modules: $Ext^1_{\mathbb{A}}(L(\lambda), L(\mu)) = \mathbb{C}\ for\ \mu \in \{T^+\lambda, T^-\lambda\}$ and zero else.
In the case of $g = sl(m,1)$ or $gl(m,1)$ or $osp(2,2n)$ an irreducible element is mapped to zero iff it is typical or, equivalently, projective.

Indecomposable elements. We recall the results about the classification of indecomposable modules in the singly atypical case obtained by [Su00] and [Ger98]. We parametrise an atypical block as in [Ger98] by $\mathbb{Z}$ and denote the corresponding weight with $a \in \mathbb{Z}$. The blocks of $osp(2|2n)$ are equivalent to those of $sl(m|1)$.

By Germoni the indecomposable modules are either the Kac objects C^+ or the AntiKac objects C^- or V sits in an exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow L \longrightarrow 0$$

with $U \in C^+$ and L irreducible, or Q sits in an exact sequence

$$0 \longrightarrow L' \longrightarrow U \longrightarrow Q \longrightarrow 0$$

with $U \in C^+$ and L' irreducible. In down to earth terms: Fix an arbitrary $a \in \mathbb{Z}$ (that is, an arbitrary weight a in the block, or its corresponding simple module $L(a)$). From a we can either go a finite number of steps to the left to a point

$b \leq a$ using the extensions described by Kac-modules or a finite number of steps to the right to a point $b \geq a$ using the extensions described by Anti-Kac-modules. To any such intervall $[a,b]$ or $[b,a]$ corresponds a unique indecomposable module with composition factors $L(a),\ L(a-1),\ldots,L(b)$ (ie. $L(\lambda),\ L(T^{-}\lambda),\ldots,L(T^{-l}\lambda)$ where $l = |b-a|$) in the case of $a \geq b$ resp with composition factors $L(a),\ L(a+1),\ldots,L(b)$ ((ie. $L(\lambda),\ L(T^{+}\lambda),\ldots,L(T^{+l}\lambda)$). For $b = a$ one obtains the simple modules $L(a)$ and for $b = a-1$ resp. $b = a+1$ the Kac resp. Anti-Kac-modules. These two families of indecomposable modules are called ZigZag resp Anti-ZigZag-modules and denoted by $Z^l(a)$ resp. $\bar{Z}^l(a)$ where l is the number of composition factors. They form a complete system of representatives of the isomorphism classes of non-projective indecomposable modules in $\mathbb{A}$.

Superdimension. The superdimension of any Kac resp. Anti-Kac-module is zero for any Typ I Lie superalgebra (easily seen be the explicite description of $K(\lambda)$ as $K(\lambda) = U(\mathfrak{g}_{-1}) \otimes L(\lambda)$ with $U(\mathfrak{g}_{-1}) = \bigoplus_{k=0}^{dim(\mathfrak{g}_{-1})} \Lambda^k(\mathfrak{g}_{-1})$). The superdimension is clearly additive in short exact sequences, hence the short exact sequence

$$0 \longrightarrow L(T^{-}\lambda) \longrightarrow K(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

yields $sdimT^{-}L(\lambda) = -sdimL(\lambda)$ for any atypical λ and likewise $sdimT^{+}L(\lambda) = -sdimL(\lambda)$.

2.14 Corollary. *The superdimension of the indecomposable modules $Z^l(a)$ is given by*

$$sdimZ^l(a) = \sum_{i=1}^{l}(-1)^i sdimL(a) = \begin{cases} sdimL(a) & l\ odd \\ 0 & l\ even. \end{cases}$$

The same for $sdim\bar{Z}^l(a)$.

The only other remaining indecomposable modules are the projective covers of the atypical simple modules $P(\lambda)$. Since $\mathcal{N}$ is the biggest proper tensor ideal of $\mathbb{A}$ any projective module is killed under the quotient functor $\omega : \mathbb{A} \to \mathbb{A}/\mathcal{N}$.

2.15 Corollary. *The irreducible objects in $\mathbb{A}/\mathcal{N}$ are up to isomorphism given by the*

$$\{Z^l(\lambda), \bar{Z}^l(\lambda) \mid \lambda\ atypical,\ l\ odd\}.$$

2.3.1. The toy example $sl(2|1)$

Consider at first the general case of $g = sl(m|n)$, $m > n$. The supergroup attached to $\mathbb{A}/\mathcal{N}$ will contain a group of type A_{m-n-1} as a factor: The standard representation st of $sl(m|n)$ is irreducible, atypical and its symmetric and exterior square are irreducible. The tensor functor ω commutes with Λ^2 and Sym^2, hence the image $\omega(st)$ is an irreducible representation such that $\Lambda^2(st)$ and $Sym^2(st)$ are irreducible. By the classification of [KW09] the tensor category generated by $\omega(st)$ in $\mathbb{A}/\mathcal{N}$ contains a group of type A_{m-n} since the rang is also preserved under ω. However a tensor generator of $\mathbb{A}$ need not be mapped to a tensor generator of $\mathbb{A}/\mathcal{N}$ since the functor does not preserve injections or surjections. For the same reason ω does not induce a morphism of Grothendieck groups which might allow us to control the size of $\mathbb{A}/\mathcal{N}$.

In the case of $sl(m|1)$ we will determine the quotient by a direct inspection of the tensor product rules up to elements in N. Consider first the case of $sl(2|1)$. The tensor products of the indecomposable modules have been computed by [GQS07] and can be used to compute the formulas in the quotient.

2.16 Lemma. *The tensor product of the irreducible modules in $\mathbb{A}/\mathcal{N}$ is given by the following rules:*

- $Z^{2p_1+1}(j_1) \otimes Z^{2p_2+1}(j_2) = Z^{2(p_1+p_2)+1}(j_1+j_2)$
- $\bar{Z}^{2p_1+1}(j_1) \otimes \bar{Z}^{2p_2+1}(j_2) = \bar{Z}^{2(p_1+p_2)+1}(j_1+j_2)$
- $Z^{2p_1+1}(j_1) \otimes \bar{Z}^{2p_2+1}(j_2) = \bar{Z}^{2(p_2-p_1)+1}(j_1+j_2-p_1)$ *for* $p_1 \leq p_2$
- $Z^{2p_1+1}(j_1) \otimes \bar{Z}^{2p_2+1}(j_2) = Z^{2(p_1-p_2)+1}(j_1+j_2-p_2)$ *for* $p_2 \leq p_1$

Proof. This is an inspection using [GQS07]. The tensor products between ZigZag-modules are given by Proposition 4 in loc.cit. as a direct sum of a $\mathcal{T}$-part and a Θ-part. Since ω preserves direct sums we omit any projective module in the formulas. By definiton $\mathcal{T}(,)$ consists of a direct sum of typical modules (formula (44)). For the contributions of the Θ-part see p.836 in loc.cit. Note that Θ maps projective modules to projective modules. The $gl(1|1)$-formulas yield then the above result. □

We use the following reparametrization: We put

$$\bar{Z}^{2p+1}(j) := (-p,-j),$$
$$Z^{2p+1}(j) := (p, p-j).$$

The irreducible elements in $\mathbb{A}/\mathcal{N}$ are then parametrized by $\mathbb{Z} \times \mathbb{Z}$. The rules for the tensor products read now:

$$\begin{aligned}(p_1, p_1 - j_1) \otimes (p_2, p_2 - j_2) &= (p_1 + p_2, p_1 + p_2 - (j_1 + j_2))\\ (-p_1, -j_1) \otimes (-p_2, -j_2) &= (-(p_1 + p_2), -(j_1 + j_2))\\ (p_1, p_1 - j_1) \otimes (-p_2, -j_2) &= (p_1 - p_2, p_1 - j_1 - j_2)\\ (p_1, p_1 - j_1)) \otimes (-p_2, -j_2) &= (p_1 - p_2, p_1 - j_1 - j_2).\end{aligned}$$

Note that this is exactly the tensor product for the group $Gl(1) \times Gl(1)$.

2.17 Corollary. $G^{red} = Gl(1) \times Gl(1)$

Proof: We have to define a functor $\rho : \mathbb{A}/\mathcal{N} \to Rep(Gl(1) \times Gl(1))$ which is an equivalence of tensor categories. Use the parametrisation above of the irreducible elements in $\mathbb{A}/\mathcal{N}$ by $\mathbb{Z} \times \mathbb{Z}$. Define ρ on objects by mapping the irreducible element corresponding to $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ to the irreducible representation $t^a \otimes t^b$ of $Gl(1) \times Gl(1)$. Note that Hom-spaces are either zero or one-dimensional by Schur's lemma. The results on tensor products show that this is a tensor functor. $\square$

2.4. Tensor products in the $Gl(m|1)$-case

Motivated by the reparametrisation in the last section we change our notation. We denote by $R(a, \ldots, b)$ the indecomposable module corresponding to the exact sequence

$$0 \to K \to R[a, \ldots, b] \to L(b) \to 0$$

where K is a Kac-object with composition factors $L(a), \ldots, L(b-1)$. We call R a roof module. Similarly we denote by $B[a, \ldots, b]$ the indecomposable module corresponding to

$$0 \to L(a) \to B[a, \ldots, b] \to K' \to 0$$

where K' is a Kac-object with composition factors $L(a+1), \ldots, L(b)$. We call B a bottom module.

The homotopy category. Here we use the results of [Wei10a] to compute the tensor product of indecomposible elements up to superdimension zero. In [Wei10a] Weissauer constructs a k-linear pseudoabelian tensor category HoT and a tensor functor $Rep(Gl(m|n), \epsilon) \to HoT$ such that the image of an irreducible object is irreducible. Let us assume that $n = 1$. Any object of $Rep(Gl(m|1))$ is a direct sum of indecomposable modules. Those in C^+ become zero in HoT, those in C^- not. The remaining ones get identified in HoT to the image of a simple module.

More precisely the obvious morphisms $R[a,\ldots,b] \to L(b)$ and $L(a) \to B[a,\ldots,b]$ are isomorphisms in HoT. The indecomposable objects in HoT are therefore up to isomorphism the images of the simple modules $L(a)$ or the AntiKac objects $K^+[a,\ldots,a+2i+1]$ with an even number of composition factors.

The quotient $HoT/\mathcal{N}$ is a semisimple k-linear rigid tensor category with $End(\mathbb{1}) = k$. Since the superdimension of the K_+ is zero they get killed. The irreducible elements are hence parametrised by the atypical weights. The main result of loc.cit in the singly atypical case is the equivalence of tensor categories

$$HoT/\mathcal{N} \simeq Rep(Gl(m-1) \times Gl(1)) \otimes svec.$$

The homotopy category HoT is tensor equivalent to $\overline{\mathcal{R}}/ < C^+ >$, the quotient of the stable category $\overline{\mathcal{R}}$ by the thick tensor ideal C^+ of Kac objects. Similarly we could consider $HoT^- = \overline{\mathcal{R}}/ < C^- >$. Then $HoT/\mathcal{N} \simeq HoT^-/\mathcal{N}$. However the identification of the indecompable roof and bottom module with an irreducible object changes: For the roof and bottom modules we have also exact sequences

$$0 \to \tilde{K} \to R[a,\ldots,b] \to L(a) \to 0$$

where K is an AntiKac-object and the exact sequence

$$0 \to L(a) \to B[\alpha,\ldots,a] \to \tilde{K}' \to 0$$

where K' is an AntiKac-object. This gives us the identifications $R[a,\ldots,b] \to L(a)$ and $L(b) \to B[a,\ldots,b]$ in HoT^-. We write $\omega : \mathcal{R} \to HoT/\mathcal{N}$ and $\omega^- : \mathcal{R} \to HoT^-/\mathcal{N}$ for the two tensor functors so obtained. We write $c^c_{a,b}$ for the coefficients in a $Gl(n-1)$ tensor product $L(a) \otimes L(b) = \bigoplus c^c_{a,b} L(c)$.

2.18 Lemma. *Denote the image of $L(a)$ under ω by $L(\lambda_a)$.*

$$B(a_1,\ldots,a_s) \otimes B(c_1,\ldots,c_t) = \bigoplus_\nu c^\nu_{\lambda_{a_1},\lambda_{c_1}} B(\nu,\ldots,\nu+\Delta),\ \Delta = (s-1)+(t-1)$$

$$R(a_1,\ldots,a_s) \otimes R(c_1,\ldots,c_t) = \bigoplus_\nu c^\nu_{\lambda_{a_1},\lambda_{c_1}} R(\nu,\ldots,\nu-\Delta),\ \Delta = (s-1)+(t-1)$$

$$B(a_1,\ldots,a_s) \otimes R(c_1,\ldots,c_t) =$$

$$\bigoplus_\nu c^\nu_{\lambda_{a_1},\lambda_{c_1}} \begin{cases} B(\nu,\ldots,\nu+\Delta),\ \Delta = (s-1)-(t-1),\ s > t \\ R(\nu-\Delta,\ldots,\nu),\ \Delta = (s-1)-(t-1),\ s < t \\ L(\nu) \qquad\qquad s = t \end{cases}$$

Proof: Consider the tensor product of two indecomposable modules of non-vanishing superdimension, say, for simplicity $B(a,\ldots,\alpha) \otimes B(c,\ldots,\gamma)$. Under ω they map to two irreducible representations of $gl(m-1) \times gl(1)$, say, $L(a) = L(\lambda_a) \times t^a$ and $L(c) = L(\lambda_c) \times t^c$. Their tensor product is given by the Littlewood-Richardson-rule

$$L(a) \otimes L(c) = \bigoplus_\nu c^\nu_{\lambda_a,\lambda_c} L(\nu) \boxtimes (t^a \otimes t^c).$$

The fiber of an element $L(a) \times t^a$ consists of a) the irreducible representation $L(a)$, b) the roof module $R(\alpha,\ldots,a)$ for any $\alpha < a$ and c) the bottom modules $B(a,\ldots,b)$ for any $b > a$. Under the tensor functor ω^- the two indecomposable elements map again to two irreducible $gl(m-1) \times gl(1)$-representations. Since ω^- respects Schur functors as well, it will map an irreducible module to the same irreducible $gl(m-1) \times gl(1)$-module as ω, and on the indecomposable modules R and B the two functors differ by a $gl(1)$-twist: $L(\alpha) = L(\lambda_\alpha) \times t^\alpha$ and $L(\gamma) = L(\lambda_\gamma) \times t^\gamma$. Their tensor product is given by the Littlewood-Richardson-rule

$$L(\alpha) \otimes L(\gamma) = \bigoplus_\nu c^\nu_{\lambda_\alpha,\lambda_\gamma} L(\nu) \boxtimes (t^\alpha \otimes t^\gamma).$$

Introduce $\Delta = \alpha - a + \beta - b$. The tensor product in the two quotients differs then by a twist with t^Δ. Taking fibres we just have to look for common elements in the fibre of ω and ω^-: The possible bottoms which may appear in the tensor product are of the form $B(a,\ldots,a_i)$ for varying a_i (contribution from ω) and of the form $R(b_i,\ldots,a+\Delta)$ for varying b_i (contribution from ω^-). A possible equality happens only in the case $b_i = a,\ a_i = a + \Delta$. Roofs can never appear for in the tensor product: The possible bottoms are of the form $R(c_i,\ldots,a)$ (varying c_i) (from ω)) and $R(a+\Delta,\ldots,d_i)$ (varying d_i) (from ω^-). Since $\Delta > 0$ these can never be equal. The $\Delta = 0$ case is easy, in that case both tensor products in the two quotients are equal, and hence neither roofs nor bottoms may appear in the tensor product. The same argument works in the case of the tensor product of two roof modules with a negative Δ. In the case of a tensor product of a roof with a bottom module $R(a,\ldots,b) \otimes B(c,\ldots,d)$ and Δ one has roofs in the tensor product for $\Delta < 0$, Bottoms for $\Delta > 0$ and an irreducible element for $\Delta = 0$. $\square$

Remark Note that this is analogously to the results in the $sl(2|1)$-case: Consider eg. $B(a_1,\ldots,a_s) \otimes B(c_1,\ldots,c_t)$, $\Delta = (s-1)+(t-1)$ and define p_1 and p_2 via $s = 2p_1+1,\ t = 2p_2+1$. Then $\Delta + 1 = (s-1)+(t-1)+1 = 2(p_1+p_2)+1$. In the old notation

$$Z^{2p_1+1}(a) \otimes Z^{2p_2+1}(b) = \bigoplus_\nu c^\nu_{\lambda_a,\lambda_b} Z^{2(p_1+p_2)+1}(\nu).$$

2.5. The pro-reductive envelope of $Gl(m,1)$

In this section we prove the following theorem.

2.19 Theorem. *The quotient $\mathbb{A}/\mathcal{N}$ is equivalent as a tensor category to*

$$\mathbb{A}/N \simeq Rep(Gl(m-1) \times Gl(1) \times Gl(1).$$

Let us describe the image of an atypical representation. Any tensor functor commutes with Schur functors. The atypical covariant representations (i.e. the atypical representations obtained by applying a Schur functor S_λ to the standard representaion) are of the form $L(\lambda_1, \ldots, \lambda_{m-1}, 0|0)$ [BR87]. Hence the irreducible representation $L(\lambda_1, \ldots, \lambda_{m-1}, 0|0))$ maps to the irreducible representation $L_0(\lambda_1, \ldots, \lambda_{m-1})$ of the tensor subcategory generated by the standard representation in $\mathbb{A}/\mathcal{N}$.

If $L(\lambda)$ is atypical there exists an integer $i \in \{1, \ldots, m\}$ such that $\lambda_{m+1} + \lambda_i = i - m$. Tensoring with the Berezinian $Ber = L(1, \ldots, 1|-1)$ shifts any representation of highest weight $\lambda = (\lambda_1, \ldots, \lambda_m| - \lambda_m)$ to a covariant representation:

$$L(\lambda) \simeq Ber^{\lambda_m} \otimes L(\tilde{\lambda})$$

for $\tilde{\lambda} = (\lambda_1 - \lambda_m, \ldots, \lambda_{m-1} - \lambda_m, 0|0) = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{m-1}, 0|0)$. Hence

$$\omega(L(\lambda)) = \omega(Ber^{\lambda_m}) \otimes S_{\tilde{\lambda}}(X).$$

However $S_{\tilde{\lambda}}(X) \simeq det(X)^{-\lambda_m} \otimes S_\lambda(X)$, so

$$\omega(L(\lambda)) = \omega(Ber^{\lambda_m}) \otimes det(X)^{-\lambda_m} \otimes S_{(\lambda_1,\ldots,\lambda_{m-1})}(X).$$

Let $\Pi := Ber^{-1} \otimes \Lambda^{m-1}(X)$. Then Π equals the irreducible socle $L(0, \ldots, 0, -1|-1)$ of the Kac module $K(\mathbb{1})$ by [SZ07]. The irreducible representation Π becomes an invertible object in the homotpy category. In fact $\Pi \simeq \mathbb{1}[1]$. Since $\mathbb{1}[1] \otimes \mathbb{1}[m] \simeq \mathbb{1}[n+m]$, Π generates a tensor subcategory isomorphic to $Rep(Gl(1), \epsilon)$ for nontrivial $\epsilon : \mathbb{Z}/2\mathbb{Z} \to Gl(1)$ in the homotopy category.

Fix an atypical block and identify the weights with $\mathbb{Z}$ as in [Ger98]. Assume that λ is atypical for some fixed $i \neq m: \ \lambda_{m+1} + \lambda_i = i - m$. Tensoring with Π shifts (after projection to the block) an atypical representation by 1, so

$$\Pi^{\lambda_i + 1 - \lambda_m} \otimes L(\lambda) = L(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \ldots, \lambda_{m-1}, \lambda_i + \lambda_{m+1} + m - i + 1).$$

This corresponds to moving $\vee$ to the left of all x. The resulting irreducible representation is covariant. The factor $(\Pi \otimes Ber/det(X))^{\lambda_{m-1}}$ is trivial in HoT. Hence the irreducible representation $L(\lambda_1, \ldots, \lambda_m|\lambda_{m+1})$ is given in HoT by

$$\Pi^{\lambda_i} \otimes S_{(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1}-1,\ldots,\lambda_m-1)}(X) =: \Pi^{\lambda_i} \otimes S_{\lambda'}(X).$$

We want to show $\mathbb{A}/\mathcal{N} \simeq Rep(Gl(m-1) \times Gl(1) \times Gl(1)$. We define a functor $\rho : \mathbb{A}/\mathcal{N} \to Rep(Gl(m-1) \times Gl(1) \times Gl(1)$. The discussion above forces the following Ansatz:

$$\rho(L(\lambda)) = \mathbb{1} \otimes t^{\lambda_i} \otimes S_{\lambda'}(X)$$

for $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \ldots, \lambda_m - 1)$. To each weight a in our block are attached the indecomposable modules

$$R(a, \ldots, a-s),\ B(a, \ldots, a+r) \text{ for some } r, s \geq 0.$$

For these we make the Ansatz

$$\rho(R(a, \ldots, a-s)) = t^{-s} \otimes t^{a_i} \otimes S_{a'}(X)$$
$$\rho(B(a, \ldots, a+r)) = t^{-r} \otimes t^{a_i} \otimes S_{a'}(X)$$

Note that the Hom-spaces between the irreducible elements are either zero or one-dimensional since Schur's lemma holds in any semisimple tensor category, hence the functor is trivially defined for the morphisms. Our results on the tensor product decomposition above and on the image in $\mathbb{A}/NN$ show that ρ is a tensor functor. It is clearly one-to-one on objects and fully faithful. This proves the theorem.

The special linear case Let us discuss the case $Sl(m|1)$, $m \geq 3$. In that case Ber is trivial and $\Pi = \Lambda^{m-1}(X)$. Hence

$$L(\lambda_1, \ldots, \lambda_m | -\lambda_m) \simeq L(\lambda_1 - \lambda_m, \ldots, \lambda_{m-1} - \lambda_m, 0|0) \simeq L(\tilde{\lambda}) \simeq S_{\tilde{\lambda}}(X),$$

hence $\omega(L(\lambda)) \simeq S_{\tilde{\lambda}}(X)$. So we obtain that the irreducible elements of $Gl(m|1)$ obtained that way are the $S_\lambda(X)$ for some partition λ.

2.20 Corollary. *$G^{red} = Sl(m-1) \times Gl(1) \times Gl(1)$ for $m \geq 3$.*

Remark Assume $m = 2$: In this case Π equals the standard representation and does not generate an additional $Gl(1)$.

2.6. The homotopy category in the $OSp(2|2n)$-case

Now we want to repeat the steps in the calculation of $\mathbb{A}/\mathcal{N}$ in the $Gl(m|1)$-case for the $OSp(2|2n)$-case. Hence we will first determine $HoT/\mathcal{N}$ and then use this quotient to calculate the tensor products of certain indecomposable modules. Since the blocks of $Rep(OSp(2|2n))$ are equivalent to those of $Rep(Gl(m|1))$ a number of results concerning the block structure or Ext^1-arguments in the $Gl(m|1)$-case hold immediately in the $OSp(2|2n)$-case. We will just refer to [Wei10a] in this case.

We have two tensor functors from HoT into semisimple tensor categories: i) the quotient $HoT/\mathcal{N}$ and ii) as in [Wei10a] one can localize HoT by the class of the so-called isogenies Σ. In the category $HoT[\Sigma^{-1}] =: \tilde{T}$ any two simple atypical modules which lie in the same block in $Rep(\mathfrak{g})$ are identified. Any atypical block contributes therefore a simple module in $\tilde{T}$; and any simple module in $\tilde{T}$ is obtained in this way. The localization yields a second tensor functor $HoT \to \tilde{T}$. The composition of this tensor functor with the one to the homotopy category defines a tensor functor

$$\omega' : R \to HoT \to \tilde{T}.$$

In the following we will study the category $\tilde{T}$ in the case of $\mathfrak{g} = osp(2|2n)$. For the case $gl(m|1)$ see [Wei10a].

Consider the localization of the homotopy category $\omega : Rep(OSp(2|2n)) \to \tilde{T}$. It is a tensor functor which maps exact sequences to distinguished triangles. Hence it induces a homomorphism of Grothendieck rings $K_0(\omega) : K_0(Rep(OSp(2|2n)) \to K_0(\tilde{T})$. By [Wei10a] $\tilde{T}$ is an abelian semisimple k-linear category. The category is super-tannakian and hence by Deligne equivalent to the semisimple representation category of a supergroup scheme $\tilde{G}$ over k. It is however not clear that $\tilde{T}$ is algebraic, i.e. has a tensor generator, and is hence equivalent to the representation category of an algebraic supergroup. Clearly the tensor subcategory of any (self-dual) object X in $\tilde{T}$ is algebraic and equivalent to the representation category of a supergroup.

Let st be the standard representation of $osp(2|2n)$ on $k^{2|2n}$ and $X = \Pi(st)$. By [LS02] the tensor module $\bigotimes^f(st)$ splits for $n \geq f$ as

$$\bigotimes^f(st) = \bigotimes \sum_{j=0}^{[f/2]} \sum_{\lambda \vdash f-2j} |ud_\lambda| L(\lambda)$$

where $\lambda \vdash n$ means partitions of length n. Here $L(\lambda)$ is the simple module associated to the partition λ and ud_λ is the number of up-down Tableaux of length f and shape λ. For details see [LS02].

2.21 Lemma. *$Sym^2(X) = S(1,1,0,\ldots,0)$ and $\Lambda^2(X) = \mathbb{C} \oplus S(2,0,...0)$.*

Proof: By [CWZ07] there exists a surjective map $\Delta : S^k(st) \to S^{k-2}(st)$ with kernel $S(\lambda')$ where $\lambda' = (k,0,\ldots,0)$. Since $Sym^0(st) = k$, $S^2(st)$ has the composition factors k and $S(k,0,\ldots,0)$. For $f=2$ the result of [LS02] yields

$$st \otimes st = \bigotimes_{\lambda \vdash 2} |ud_\lambda| L(\lambda) \oplus \bigoplus_{\lambda \vdash 0} |ud_\lambda| L(\lambda).$$

The last sum gives the trivial module. In the first sum only the two following partitions may occur: $(2,0)$ and $(1,1)$. In both cases $ud_{\lambda=1}$, hence $st \otimes st = S(2,0,\ldots,0) \oplus S(1,1,0,\ldots,0) \oplus k$. When passing from st to X we get $Sym^2(\Pi(st)) = \Lambda^2(st)$. Since $st \otimes st = Sym^2(st) \oplus \Lambda^2(st)$ the result follows. □

Consider now $\omega(X)$. Note first that $L(1,1,0,\ldots,0)$ and $L(2,0,...0)$ are both atypical, hence map to an irreducible module in $A[\Sigma^{-1}]$. Since Schur functors commute with ω we have

$$\omega(Sym^2(X)) = Sym^2(\omega(X)) = irreducible$$

since ω maps irreducible to irreducible objects. Further

$$\begin{aligned}\Lambda^2(\omega(X)) &= \omega(\Lambda^2(X)) = \omega(k \oplus L(2,0,\ldots,0)) = \omega(k) \oplus \omega(L(2,0,\ldots,0) \\ &= k \oplus irreducible\end{aligned}$$

since ω preserves direct sums.

2.22 Corollary. *The tensor subcategory generated by X is equivalent to $Rep(Sp(n-2))$.*

Proof: Since the action of st on $k^{2|2n}$ is faithful, $< X >_\otimes$ is connected. Since $< X >_\otimes$ is semisimple, the classification of [Wei09] yields that it is equivalent to the representation category of a product $H \times \prod_r Spo(1,2r)^{n_r}$ where H is a reductive group. The classification of [KW09] yields that $< X >_\otimes$ is either $Rep(Sp(2n-2))$ where $X = st$ or $Rep(osp(1,2r)$ with $X = st$. However the last case is excluded since $dim(\omega(X)) = sdim(X) = 2n-2 \neq 1-2r$. □

We obtain an epimorphism $\rho : \tilde{G} \to Sp(2n-2)$ of supergroups. The homomorphism splits and hence

$$\tilde{G} \simeq ker(\rho) \times Sp(2n-2).$$

We need to determine $ker(\rho)$.

2.23 Lemma. *$ker(\rho)^\circ$ is a torus of rank 1 or zero.*

Proof: We will argue by the transcendence degree of the Grothendieck ring as in [Wei10b]. Put $T' = Rep(OSp(2|2n))$. By [SV07] the Grothendieck group of

$osp(2|2n)$ is given by

$$K_0(Rep(\mathfrak{g})) = \{f \in \mathbb{Z}[u, v_1, \ldots, v_n]^{S_n} \mid u\frac{\partial f}{\partial u} + v_j\frac{\partial f}{\partial v_j} \in (u - v_j),\ j = 1, \ldots, n\}$$

with the principal ideal $(u - v_j)$. This is a ring of transcendence degree $n + 1$. ω induces a homomorphism of Grothendieck rings $K_0(\omega) : K_0(T) \to K_0(A[\Sigma^{-1}])$. The exterior power $\Lambda^m(st)$ of the standard representation of $sp(m)$ vanishes. Since $\omega(\Lambda^{2n-2}(X)) = \Lambda^{2n-2}(\omega(X)) = \Lambda^{2n-2}(st) = 0$ $\Lambda^{2n-2}(st)$ gets killed in $\tilde{T}$, $K_0(\tilde{T})$ is a quotient of $K_0(T)/\Lambda^{2n-2}(X)K_0(T)$. Recall that $K_0(sp(2n-2))$ has transcendence degree $n - 1$. Since the transcendence degree is $td(K_0(T)) = n + 1$, we have $td(K_0(\tilde{T}) \leq n$. Now

$$n \geq td(K_0(\tilde{T})) = td(K_0(sp(2n - 2)) + td(K_0(ker(\rho)) = n - 1 + td(K_0(ker(\rho)),$$

hence $td(K_0(ker(\rho)) \leq 1$. Since $ker(\rho)$ is semisimple and algebraic, its connected component is a torus of rank 1 or zero. $\square$

2.7. The pro-reductive envelope of $osp(2|2n)$-case

We form the quotient of HoT by the ideal of negligible morphisms $\mathcal{N}$. This is a semisimple super-tannakian category and $\omega^s : HoT \to HoT/\mathcal{N}$ is a tensor functor by the former sections and [Wei10a]. The associated supergroup-scheme is called G^{AK}. Although the tensor functor $\omega^s : HoT \to HoT/\mathcal{N}$ does not induce a morphism of Grothendieck groups since it is not exact, one can show that the vanishing of $\Lambda^{2n-2}(X)$ in $HoT/\mathcal{N}$ leads to the estimate $td(K_0(HoT/\mathcal{N})) \leq n$ [Wei10a]. By the same argument as before the image of X in $HoT/\mathcal{N}$ creates a tensor subcategory equivalent to $Rep(Sp(2n - 2))$ with trivial twisting factor. However there is additionally the tensor subcategory $< \mathbb{1}[1] >_{\otimes}$ isomorphic to the tensor category $Rep(\mu_0, Gl(1)) = svec_k^{\mathbb{Z}}$ for nontrivial $\mu_0 : \mathbb{Z}/2\mathbb{Z} \to Gl(1)$. By the discussion in [Wei10a] this implies that the transcendence degree of $K_0(HoT/\mathcal{N})$ is strictly bigger than the one of $K_0(\tilde{T})$.

2.24 Corollary. *$td(K_0(HoT/\mathcal{N}) = n$ and $td(K_0(\tilde{T})) = n - 1$.*

2.25 Corollary. *$(G^{AK})^\circ = Sp(2n - 2) \times Gl(1)$ and $(\tilde{G})^\circ = Sp(2n - 2)$.*

We need to determine the group π of connected components. At the moment we assume the following: The group of connected components is abelian. Then the following lemma shows that $\pi = \mathbb{Z}/2\mathbb{Z}$. The following results depend on this assumption, hence the result for $OSp(2|2n)$ is conditional.

2.26 Lemma. *$\tilde{T}$ contains up to isomorphism only two irreducible modules with dimension ± 1.*

Proof: By [Wei10a] any simple object in $\tilde{T}$ is the image of a simple atypical object in T. Assume that $L(\lambda)$ is atypical with respect to $\epsilon \pm \delta_k$. By [KW94] and [VdJ91] the superdimension of any such module $L(\lambda)$ is up to a sign given by the dimension of the simple $sp(2n-2)$-module $L(\lambda')$ with highest weight $\lambda' = \sum_{i \neq k} \lambda_i \delta_i + (\delta_1 + \ldots \delta_{k-1})$. The only simple $sp(2n-2)$-module with dimension 1 is the trivial module. Any integral dominant λ such that $\lambda' = 0$ is of the form $\lambda = \lambda_0 \epsilon + \lambda_1 \delta_1$. It fulfilles one of the atypicality conditions $\lambda_0 + \lambda_1 = 0$ (atypical with respect to $\epsilon - \delta_1$) or $\lambda_0 - \lambda_1 - 2n + 2 = 0$ (atypical with respect to $\epsilon + \delta_1$). By [SZ07]

$$Ext^1(k, L(\lambda)) = \begin{cases} k & \lambda = -\alpha_{min}, 2\delta_1 \\ 0 & else. \end{cases}$$

Here δ_1 is the half sum of the positive odd roots, hence $\delta_1 = \frac{1}{2}\sum_{\beta \in \Delta_1^+} \beta = (n-1)\epsilon = (2n, 0, \ldots, 0)$; and $-\alpha_{min} = (-1, 1, 0, \ldots, 0)$. By [VdJ91] if λ is singly atypical with respect to $\beta \in \Phi^+$ and $\lambda - \beta \in X^+$ then $T^- \lambda = \lambda - \beta$. Hence for $L(-\alpha_{min}) = L(-1, 1, 0, \ldots, 0)$ the successive modules $T^{-k} L(\lambda)$ are the $L(\lambda_0 - k, \lambda_1 + k, 0, \ldots, 0)$. Similarly for $L(2\delta_1) = L(2n, 0, \ldots, 0)$ the modules $T^{+k} L(2\delta_1)$ (using $T^+ T^- = T^- T^+ = id$) are of the form $L(2n + k, k, 0, \ldots, 0)$. Hence any atypical module with highest weight $\lambda = \lambda_0 \epsilon + \lambda_1 \delta_1$ is of the form $T^{-k_1} L(\alpha_{min})$ or $T^+ k_2 L(2\delta_1)$ for suitable k_1, k_2, hence is in the block of $\mathbb{1}$. The parity shift $\mathbb{1}[1]$ is of superdimension -1. Since the tensor functor ω preserves the categorial rang the claim follows $\square$

2.27 Corollary. *We have $\pi(\tilde{G}) = \mathbb{Z}/2\mathbb{Z}$, hence $\tilde{G} = Sp(2n-2) \times \mathbb{Z}/2\mathbb{Z}$.*

2.28 Corollary. *G^{AK} is connected, hence $G^{AK} = Sp(2n-2) \times Gl(1)$.*

Proof: Ths follows from the exact sequence [Wei10a]

$$0 \to \mathbb{Z}/2Z \to \pi_0(\tilde{G}) \to \pi_0(G^{AK}) \to 0.$$

$\square$

We can now proceed as in the $Gl(m|1)$-case. Analogously to this case we have two tensor functors ω, ω^- for the two homotopy categories obtained by dividing the stable category by either the tensor ideal C^+ or the tensor ideal C^-. The computation of the tensor products between the $R(a, \ldots, a-s)$ and $B(a, \ldots, a+r)$ is a formal consequencem, replacing $Gl(m-1)$ by $Sp(2n-2)$. We claim:

2.29 Theorem. *$G^{red} \simeq Sp(2n-2) \times Gl(1) \times Gl(1)$.*

Proof: Since $HoT/\mathcal{N}$ equals $Rep(Sp(2n-2) \times Gl(1))$ any irreducible object $L(\lambda)$ is of the form $t^a \otimes L_0(\lambda')$ for some $a \in \mathbb{Z}$ and $Sp(2n-2)$ weight λ'. We make the ansatz

$$\rho(L(\lambda)) = \mathbb{1} \otimes t^a \otimes L_0(\lambda')$$

to define $\rho : A/\mathcal{N} \to Rep(Sp(2n-2) \times Gl(1) \times Gl(1))$. For $R(a, \ldots, a-s)$ and $B(a, \ldots, a+r)$ we proceed as in the $Gl(m|1)$-case. This defines a tensor functor which is an equivalence of categories. □

Remark: Contrary to the $Gl(m|1)$-case we do not state the image of an irreducible representation explicitely. The reason is that we do not know how Schur functors behave for $OSp(2|2n)$. Van-der-Jeugt showed that the superdimension of an irreducible $L(\lambda)$ is given $(-1)^{\lambda_i} dim L_0(\lambda')$ for the explicite λ' from above. Hence one should obtain $\omega(L(\lambda)) = \mathbb{1} \otimes t^{\lambda_i} \otimes L_0(\lambda')$ for λ atypical with respect to i.

3. Deligne's interpolating categories and mixed tensors

We introduce Deligne's interpolating categories and explain how to decompose tensor products in them. Then we describe the image of of a canonical functor from Deligne's category for the parameter $\delta \in \mathbb{N}$ into $Rep(Gl(m|n))$, $m - n = \delta$. As a result we get rules to decompose the tensor product of two projective modules and a description of the duals.

3.1. Bipartitions and indecomposable modules

For every $\delta \in k$ we dispose over Deligne's interpolating category [Del07] [CW11] denoted $Rep(Gl_\delta)$. This is a k-linear abelian rigid tensor category. By construction it contains an object of dimension δ, called the standard representation. Given any k-linear pseudoabelian tensor category C with unit object and a tensor functor

$$F : Rep(Gl_t) \to C$$

the functor $F \to F(st)$ is an equivalence between the category of $\otimes$-functors of $Rep(Gl_t)$ to C with the category of δ-dimensional dualisable objects $X \in C$ and their isomorphisms. In particular, given a dualizable object X of dimension δ in a k-linear pseudoabelian tensor category, a unique tensor functor $F_X : Rep(Gl_t) \to C$ exists mapping st to X.

Let $\lambda = (\lambda^L, \lambda^R)$ a bipartition (a pair of partitions). Call $|\lambda| = (|\lambda^L|, |\lambda^R|)$ (where $|\lambda^L| = \sum \lambda_i^L$) the size of the bipartition (notation $\lambda \vdash |\lambda|$) and $l(\lambda) = l(\lambda^L) + l(\lambda^R)$ the length of λ. We denote by P the set of all partitions, by Λ the set of all bipartitions.

To each bipartition is attached an indecomposable element $R(\lambda)$ in $Rep(Gl_\delta)$. By [CW11] the assignement $\lambda \to R(\lambda)$ defines a bijection between the set of bipartitions of arbitrary size and the set of isomorphism classes of nonzero indecomposable objects in $Rep(GL_\delta)$. We often write (λ) instead of $R(\lambda)$. By the universal property of Deligne's category there exists for $\delta = d \in \mathbb{N}$ a full tensor functor

$$F_d : Rep(GL_d) \to Rep(GL(d)).$$

The correspondence between bipartitions and highest weights: Given a bipartition $\lambda = (\lambda^L, \lambda^R)$ of length d, $\lambda^L = (\lambda_1^L, \ldots, \lambda_s^L, 0, \ldots)$, $\lambda_s^L > 0$, $\lambda^R = (\lambda_1^R, \ldots, \lambda_t^L, 0, \ldots)$, $\lambda_t^R > 0$, put $wt(\lambda) = \lambda_1^L \epsilon_1 + \ldots + \lambda_s^L \epsilon_s - \lambda_t^R \epsilon_{d+1-t} - \ldots - \lambda_1^R \epsilon_d$. This defines the irreducible $GL(d)$-module $L(wt(\lambda))$ with highest weight $wt(\lambda)$. By [CW11]

$$F_d(R(\lambda)) = \begin{cases} L(wt(\lambda)) & l(\lambda) \leq d \\ 0 & l(\lambda) > d. \end{cases}$$

This defines a bijection between bipartitions of length $\leq d$ with highest weights of $GL(d)$. Similarly we dispose over a tensor functor $F_{mn} : Rep(Gl_d) \to \mathcal{R}_{mn}$ for $d = m - n$ given by standard representation of superdimension $m - n$.

3.1 Theorem. *[CW11] The image of F_{mn} is the space of mixed tensors, the full subcategory of objects which appear as a direct summand in a decomposition of*

$$T(r, s) := V^{\otimes r} \otimes (V^\vee)^{\otimes s}$$

for some $r, s \in \mathbb{N}$. The functor F_{mn} is full. If $\lambda \neq \mu$, we have $F_{mn}(R(\lambda)) \neq F_{mn}(R(\mu))$.

A bipartition is said to be (m, n)-cross if there exists some $1 \leq i \leq m + 1$ with $\lambda_i^L + \lambda_{m+2-i}^R < n+1$. For a graphical interpretation see [CW11], section 8, or [BS11], section 8. The set of (m, n)-cross bipartitions is denoted Λ_{mn}^x or simply Λ^x. By [CW11] the modules $R(\lambda) := F_{mn}(L(\lambda))$ are $\neq 0$ iff λ is an (m, n)-cross bipartition. Up to isomorphism the indecomposable nonzero summands of $V^{\otimes r} \otimes W^{\otimes s}$ are the modules [BS11], Thm 8.19,

$$\{R(\lambda) \mid \lambda \in \dot{\Lambda}_{r,s} \ (m, n) - cross\}$$

where

$$\Lambda_{r,s} := \{\lambda \in \Lambda^x \mid |\lambda^L| = r - t,\ |\lambda^R| = s - t \ for\ 0 \leq t \leq min(r,s)\}$$

$$\dot{\Lambda}_{r,s} := \begin{cases} \Lambda_{r,s} & if\ \delta \neq 0,\ or\ r \neq s\ or\ r = s = 0 \\ \Lambda_{r,s} \setminus (0,0) & if\ \delta = 0\ and\ r = s > 0. \end{cases}$$

For any bipartition define the two sets λ

$$I_\wedge(\lambda) := \{\lambda_1^L, \lambda_2^L - 1, \lambda_3^L - 2, \ldots\}$$
$$I_\vee(\lambda) := \{1 - \delta - \lambda_1^R, 2 - \delta - \lambda_2^R, \ldots\}.$$

Here we use the convention that a partition is always continued by an infinite number of zeros. To these two sets one can attach a weight diagram in the sense of [BS08] as follows: Label the integer vertices i on the numberline by the symbols $\wedge, \vee, \circ, \times$ according to the rule

$$\begin{cases} \circ & if\ i\ \notin I_\wedge \cup I_\vee, \\ \wedge & if\ i \in I_\wedge,\ i \notin I_\vee, \\ \vee & if\ i \in I_\vee,\ i \notin I_\wedge, \\ \times & if\ i \in I_\wedge \cap I_\vee. \end{cases}$$

To any such weight diagram one attaches a cap-diagram as in [BS08]. For integers $i < j$ one says that (i,j) is a $\vee\wedge$-pair if they are joined by a cap. For $\lambda, \mu \in \Lambda$ one says that μ is linked to λ if there exists an integer $k \geq 0$ and bipartitions $\nu^{(n)}$ for $0 \leq n \leq k$ such that $\nu^{(0)} = \lambda, \nu^{(k)} = \mu$ and the weight diagramm of $\nu^{(n)}$ is obtained from the one of $\nu^{(n-1)}$ by swapping the labels of some pair $\vee\wedge$-pair. Then set

$$D_{\lambda,\mu} = \begin{cases} 1 & \mu \text{ is linked to } \lambda \\ 0 & \text{otherwise.} \end{cases}$$

One has $D_{\lambda,\lambda} = 1$ for all λ. Further $D_{\lambda,\mu} = 0$ unless $\mu = \lambda$ or $|\mu| = (|\lambda^L| - i, |\lambda^R| - i)$ for some $i > 0$. Let t be an indeterminate and R_δ respective R_t the Grothendieck rings of $Rep(GL_\delta)$ over k respective of $Rep(GL_t))$ over $k(t)$. Now define $lift_\delta$: $R_\delta \to R_t$ be the $\mathbb{Z}$-linear map defined by

$$lift_\delta(\lambda) = \sum_\mu D_{\lambda,\mu}\mu.$$

By [CW11], Thm. 6.2.3, $lift_\delta$ is a ring isomorphism for every $\delta \in K$.

Tensor products. We recall the results of Comes and Wilson about the decomposition of tensor products of the indecomposable modules $R(\lambda)$ in Deligne's category. To get the tensor product in $\mathcal{R}_{mn}$ the tensor product is computed in $Rep(GL_d)$ and then pushed to $Rep(GL(m,n))$ by means of the tensor functor $F_{m,n}$. By [CW11], Thm 7.1.1, the following decomposition holds for arbitrary bipartitions in R_t:

$$\lambda\mu = \sum_{v\in\Lambda} \Gamma^{\nu}_{\lambda,\mu}\nu$$

with the numbers

$$\Gamma^{\nu}_{\lambda,\mu} = \sum_{\alpha,\beta,\eta,\theta\in P} \Big(\sum_{\kappa\in P} c^{\lambda^L}_{\kappa\alpha} c^{\mu^R}_{\kappa\beta}\Big) \Big(\sum_{\gamma\in P} c^{\lambda^R}_{\gamma\eta} c^{\mu^L}_{\gamma\theta}\Big)\ c^{\nu^L}_{\alpha\theta} c^{\nu^R}_{\beta\eta},$$

see [CW11], Thm 5.1.2. In particular if $\lambda \vdash (r,s)$, $\mu \vdash (r',s')$, then $\Gamma^{\nu}_{\lambda\mu} = 0$ unless $|\nu| \leq (r+r', s+s')$. As a special case we obtain

$$(\lambda^L;0)\ (0;\mu^R) = \sum_{\nu}\sum_{\kappa\in P} c^{\lambda^L}_{\kappa\nu^L} c^{\mu^R}_{\kappa\nu^R}\nu$$

in R_t. So to decompose tensor products in R_δ apply the following three steps: Determine the image of the lift $lift_\delta(\lambda\mu)$ in R_t, uses the above formula and then take $lift_\delta^{-1}$.

3.2. The modules $R(\lambda)$

The interpretation of the mixed tensors as elements in the image of the tensor functor F_{mn} gives a tensor product decomposition for these modules. However it does not say anything about the structure of these modules and hence does not give any way to identify the image $F_{mn}(R(\lambda))$. This is possible by a different approach by Brundan and Stroppel [BS11]. There the mixed tensors are interpreted as the images of certain Khovanov-modules under the equivalence of categories E^{-1} : $K(m|n)-mod \to \mathcal{R}_{mn}$. We urge the reader to consult [BS08], [BS10a] for extensive discussions of the following definitions.

Some terminology of Brundan and Stroppel Let α, β be weight diagrams for $K(m|n)$. Let $\alpha \sim \beta$ mean that β can be obtained from α by permuting $\vee$s and $\wedge$s. The equivalence classes of this relation are called blocks. Given $\lambda, \mu \sim \alpha$ one can label the cup diagram λ resp. the cap diagram μ with α to obtain $\underline{\lambda}\alpha$ resp. $\alpha\bar{\mu}$. These diagrams are by definition consistently oriented iff each cup resp cap has exactly one $\vee$ and one $\wedge$ and all the rays labelled $\wedge$ are to the left of all rays labelled $\vee$. Set $\lambda \subset \alpha$ iff $\lambda \sim \alpha$ and $\underline{\lambda}\alpha$ is consistently oriented.

A **crossingless matching** is a diagram obtained by drawing a cap diagram underneath a cup diagram and then joining rays according to some order-preserving bijection between the vertices. Given blocks Δ, Γ a $\Delta\Gamma$-matching is a crossingless matching t such that the free vertices (not part of cups, caps or lines) at the bottom are exactly at the position as the vertices labelled $\circ$ or $\times$ in Δ; and similarly for the top with Γ. Given a $\Delta\Gamma$-matching t and $\alpha \in \Delta$ and $\beta \in \Gamma$, one can label the bottom line with α and the upper line with β to obtain $\alpha t \beta$. $\alpha t \beta$ is consistently oriented if each cup resp cap has exactly one $\vee$ and one $\wedge$ and the endpoints of each line segment are labelled by the same symbol. Notation: $\alpha \rightarrow^t \beta$.

For a crossingless $\Delta\Gamma$-matching t and $\lambda \in \Delta,\ \mu \in \Gamma$, label the bottom and the upper line as usual. The **lower reduction** $red(\underline{\lambda} t)$ is the cup diagram obtained from $\underline{\lambda} t$ by removing the bottom number line and all connected components that do not extend up to the top number line. The upper reduction $red(t\bar{\mu})$ is the cap diagram obtained from $t\bar{\mu}$ by removing the top line.

If $M = \bigoplus_{j \in \mathbb{Z}} M_j$ is a graded $K(m|n)$-module, write $M < j >$ for the same module with new grading $M < j >_i := M_{i-j}$. The modules $\{L(\lambda) < j > \mid \lambda \in X^+,\ j \in \mathbb{Z}\}$ give a complete set of isomorphism classes of irreducible graded $K(m|n)$-modules. The Grothendieck group is the free $\mathbb{Z}$-module with basis the $L(\lambda) < j >$. Viewing it instead as a $\mathbb{Z}[q, q^{-1}]$-module so that $q^j[M] := [M < j >]$ $K_0(Rep(K(m|n))$ becomes the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{L(\lambda) \mid \lambda \in X^+\}$.

Then for any $\Delta\Gamma$-matching t we have the so-called special projective functors $G^t_{\Delta\Gamma}$ in the category of graded $K(m|n)$-modules [BS10a].
The mixed tensors $R(\lambda)$ are special cases of the projective functors of the theorem. Given a bipartition λ we denote by the defect of λ $def(\lambda)$ the number of caps in the cap diagram and by rank of λ $rk(\lambda) = min(\#x, \#\circ)$. For $\delta \geq 0$ one has $rk(\lambda) = \#\circ$'s. Then set

$$k(\lambda) := def(\lambda) + rk(\lambda).$$

Denote by η the weight diagram

$$\wedge \quad \wedge \quad \wedge \quad \times \quad \times \quad \times \quad \times \quad \vee \quad \vee \quad \vee \quad \vee$$

where the rightmost x is at position zero, and there are $\delta = m - n$ crosses. Let α be the weight diagram obtained from η by switching the rightmost $k(\lambda)$ $\wedge$'s with the leftmost $k(\lambda)$ $\vee$'s.

Let t be the crossingless matching between $\bar{\lambda}$ and $\underline{\alpha}$ obtained as follows: Draw the cap diagram $\overline{\lambda}$ underneath the cup diagram $\underline{\alpha}$ and then join the rays in the unique way such that rays coming from a vertex $a \in \mathbb{Z}$ get joined with rays coming from the vertex a except for a finite number of vertices. Now replace α with the weight diagram ζ (the weight diagram of the trivial representation)

$$\wedge \quad \vee \quad \vee \quad \vee \quad \times \quad \times \quad \times \quad \wedge \quad \wedge \quad \wedge \quad \wedge$$

Here the rightmost x is at position zero, and there are $\delta = m - n$ crosses as well as n $\vee$'s to the left of the crosses. Then adjust the labbels of λ that are at the bottoms of line segments to obtain $\lambda^\dagger$ such that $\lambda^\dagger t \zeta$ is consistently oriented.

Let Γ be the block of ζ, Δ be the block containing $\lambda^\dagger$ and set

$$R(\lambda) = G^t_{\Delta\Gamma} L(\zeta).$$

We transport $R(\lambda)$ by the equivalence of categories $E : \mathcal{R}_{mn} \to K(m,n) - mod$ to a g-supermodule. By Morita equivalence the Loewy layers are preserved. We denote by $\lambda^\dagger$ the highest weight of the irreducible socle of $R(\lambda)$. This defines a map $\theta : \Lambda^x \to X^+$, $\lambda \mapsto \lambda^\dagger$.

3.3. Irreducible modules and projective covers

In order to make use of the results of Comes and Wilson the image of F_{mn} has to be described more explictely. There is a priori hardly any connection between the natural labelling with highest weights and the labelling by bipartitions. The only transparent examples are the co- and contravariant modules. Bipartitions in the image are neither bounded by degree nor length. For instance what indecomposable module of $Gl(3,2)$ is described by the bipartition $((1^{10^{10}}); (38))$? Among other results we describe the $R(\lambda)$ which are irreducible and those which are the projective cover of some atypical representation. Additionally we show that certain classes of modules are in the image.

3.2 Theorem. *[BS11], Thm 3.4. and [BS10a], Thm 4.11: (i) Given a $\Delta\Gamma$-matching t as above. Then $G^t_{\Delta\Gamma} L(\mu)$ is an indecomposable module with irreducible head and socle which differ only by a grading shift. (ii) In the graded Grothendieck group*

$$[G^t_{\Delta\Gamma} L(\mu)] = \sum_\gamma (q + q^{-1})^{n_\gamma} [L(\gamma)]$$

where n_γ denotes the number of lower circles in $\underline{\gamma}t$ and the sum is over all $\gamma \in \Lambda$ such that a) $\underline{\mu}$ is the lower reduction of $\underline{\gamma}t$ and b) the rays of each lower line in $\underline{\gamma}\overline{\gamma}t$ are oriented so that exactly one is $\vee$ and one is $\wedge$. (iii) If we forget the grading then

$$[G^t_{\Delta\Gamma} L(\mu)] = \sum_{\lambda \subset \alpha \to^t \mu,\ red(\underline{\lambda}t) = \underline{\mu}} [L(\lambda)].$$

The information about the *graded composition multiplicities* is much finer than the mere information about the composition factors since it gives rise to a grading filtration with semisimple quotients.

3.3 Corollary. *$R(\lambda)$ has Loewy length $2def(\lambda) + 1$. It is rigid.*

Proof: Let $R(j)$ be the submodule of $R(\lambda)$ spanned by all graded pieces of degree $\geq j$. Then

$$R(\lambda) = R(-def(\lambda)) \supset R(-def(\lambda) + 1) \supset \ldots \supset R(def(\lambda))$$

with successive semisimple quotients $R(j)/R(j+1)$ of degree j. By [BS10b] every block of $\mathcal{R}_n$ is Koszul. We already know that the top and socle are simple. Since Koszul algebras are quadratic, the following proposition finishes the proof. □

3.4 Proposition. *[BGS96], prop. 2.4.1. Let A be a graded ring such that i) A_0 is semisimple, ii) A is generated by A_1 over A_0. Let M be a graded A-module of finite length. If $soc(M)$ (resp. $top(M)$ is simple, the socle (resp. the radical) filtration on M coincides with the grading filtration (up to a shift).*

3.5 Corollary. *Every indecomposable module in $\mathcal{R}$ with irreducible top and socle is rigid.*

3.6 Corollary. *$R(\lambda)$ is irreducible iff $def(\lambda) = 0$.*

Projective covers. Recall that the indecomposable projective modules in $Rep(Gl(m, n))$ are precisely the irreducible typical modules by [Kac78] and the projective covers of the irreducible atypical modules.

3.7 Lemma. *Every indecomposable projective module appears as some $R(\lambda)$.*

Proof: The module $st \otimes st^\vee$ is a tensor generator of $\mathcal{R}_{mn}$ [Wei10a]. Hence every module M appears as a subquotient of some direct sum of $T(r, s)$. If M is indecomposable projective the surjection will split, hence M appears as some direct summand. □

Since every atypical weight appears in the socle and top of its projective cover we obtain also

3.8 Corollary. *The map $\theta : \Lambda \to X^+$ is surjective.*

Now that we know that every projective cover appears as some $R(\lambda)$, we characterize the projective covers in this part.

3.9 Lemma. *The crosses and circles of the bipartition λ are at the same places as the crosses and circles of the highest weight $\lambda^\dagger$. In particular $at(R(\lambda)) = n - rk(\lambda)$.*

Proof: This is clear since only the labels of λ which bear a $\vee$ or a $\wedge$ are changed when applying θ. $\square$

Consider a block of atypicality k, i.e there are $m-k$ crosses and $n-k$ circles. Since $k(\lambda) \leq min(m,n)$ there can be at most k cups in $\underline{\lambda}$. In that case we say that λ has maximal defect (i.e. when $k(\lambda) = n$).

3.10 Theorem. *The projective covers of the irreducible modules are precisely the modules $R(\lambda)$ where λ has maximal defect m. In this case $R(\lambda) = P(\lambda^\dagger)$.*

Proof: We remark the following reduction: For every indecomposable module M with $head(M) = L(\lambda^\dagger)$ there exists a surjection $P(\lambda^\dagger) \to M$ by [Zou96], lemma 3.4. If M has the same composition factors as $P(\lambda)$, this surjection has to have trivial kernel and gives an isomorphism. By [BS10b] the following formulas hold in the Grothendieck group:

$$[P(\lambda^\dagger)] = \sum_{\mu \supset \lambda^\dagger} [K(\mu)] \qquad K(\mu) = \sum_{\rho \subset \mu} [L(\rho)].$$

On the other hand

$$R(\lambda) = G^t_{\Delta\Gamma} L(\zeta) = \sum_{\mu \subset \alpha \to^t \zeta, red(\underline{\mu}t) = \underline{\zeta}} L(\mu).$$

We will show that the second formula is equal to the first one. Since ζ and t are fixed, the condition $\alpha \to^t \zeta$ and $\alpha \sim \zeta$ implies that $\alpha(i)$ is fixed up to the choice of the position of $\vee$ and $\wedge$ in each cup: All other coordinates are determined by the condition that the endpoints of line segments of t must be labelled by the same symbol (and implies that α has m cups and no free $\vee$'s). Hence any such α differs from $\lambda^\dagger$ only by the the position of $\vee$ and $\wedge$ in each cup. The set of α so obtained is precisely the set of α with $\alpha \supset \lambda^\dagger$: the condition that there cannot be free $\vee$ to the left of free $\wedge$s forces all m $\vee$s to be bound in cups. Hence

$$R(\lambda) = G^t_{\Delta\Gamma} L(\zeta) = \sum_{\mu \subset \alpha \supset \lambda^\dagger, red(\underline{\mu} t) = \underline{\zeta}} L(\mu).$$

The condition $red(\underline{\mu} t) = \underline{\zeta}$ is however automatic: Since ζ is completely nested and t is fixed we only have to show that no further cup can arise in ζ. In that case there would exist two line segments starting from $\zeta(i_1)$, $\zeta(i_2)$ (both marked with a $\wedge$) mapping to $\alpha(j_1)$ resp. $\alpha(j_2)$ such that $\alpha(j_1) = \vee$ and $\alpha(j_2) = \wedge$ and such that α_{j_1} is connected to α_{j_2} by a chain of cups or caps. This is not possible because α does not have free $\vee$'s. Hence we know $R(\lambda) = P(\lambda^\dagger)$ for $k(\lambda) = n$. For the converse we show that any module with $k(\lambda) < n$ is not the projective cover of $\lambda^\dagger$. For maximal defect the condition $\alpha \leftrightarrow^t \zeta$ is equivalent to $\alpha \supset \lambda^\dagger$. Generally $\alpha \leftrightarrow^t \zeta$ fixes α up to the positions of $\vee$ and $\wedge$ in the cups. For $k(\lambda) = n - r$ cups one has r cups less which eliminates r choices. Hence the condition $\alpha \leftrightarrow^t \zeta$ is stricter than the condition $\alpha \supset \lambda^\dagger$. Hence the composition factors of $R(\lambda)$ are just a proper subset of the ones of $P(\lambda^\dagger)$. □

Example: The module $R((3,2,1),(3,2,1))$ is the projective cover $P([2,1,0])$ in $\mathcal{R}_{33}$.

3.4. Injectivity of θ

As noted by [BS11] the map $\theta : \Lambda^x \to X^+$ is in general not injective. For instance every projective cover appears. It is not even injective if one fixes the defect and the rank of the bipartition as sample calculations in the the maximal atypical $Gl(3,3)$ and $Gl(4,4)$-case show. If one wants to use the tensor product formula of Comes and Wilson one has to be able to go back from $\lambda^\dagger$ to λ.

3.11 Lemma. *θ is injective for minimal and maximal defect (i.e. $def(\lambda) = 0$ and in the case $k(\lambda) = n$).*

Proof: Consider the maximal defect case first. The position of the crosses and circles is fixed by θ. Since all $\vee$ of $\lambda^\dagger$ are bound in cups their position is unchanged under θ. These three labels determine the weight $\lambda^\dagger$ uniquely. Since every module of maximal defect is a projective cover, the module is determined by its socle.

Now assume $def(\lambda) = 0$. Assume the claim would be wrong. Then there would exist two bipartitions (λ, ρ), (λ', ρ') with same image under θ. However since both modules are irreducible this would imply that they are equal. This is in contradiction to $R(\lambda) = R(\mu)$ iff $\lambda = \mu$. $\square$

Since θ is injective for minimal and maximal defect we can describe its inverse $\theta^{-1} : X^+ \to \Lambda$ in these fixed situations. Here and in the following we use implicitely the following obvious lemma:

3.12 Lemma. *The labelled matching $\lambda t \alpha$ is consistently oriented.*

Proof: We have to show that line segments of t starting with a $\vee$ connect with line segments labelled by a $\vee$ and likewise for the $\wedge$'s. After removing all crosses, circles and cups α and λ look both like

$$\wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \vee \quad \vee \quad \vee \quad \vee \quad \vee \quad \vee$$

We will choose specific points T^+, T^- such that the matching is the identity for labels $\geq$ resp. $\leq T^+$ resp T^-. Then we just have to count the numbers of $\wedge$'s and $\vee$'s occuring in α and λ. If the numbers agree we are done. We choose the minimal positions T^+, T^- from which on t is the identity. We put

$$T^+_\lambda = max(l(\lambda^R) + 1 - (m-n), \lambda^L_1 + 1)$$
$$T^+ = max(l(\lambda^R) + 1 - (m-n), \lambda^L_1 + 1, k(\lambda) + 1).$$

Then T^+_λ is the label left to the first position coming from $\lambda^R_{l(\lambda)}$ or the position of the rightmost x. Similarly put

$$T^-_\lambda = min(-l(\lambda^L), -(m-n) - \lambda^R_1)$$
$$T^- = min(-l(\lambda^L), -(m-n) - \lambda^R_1, -(m-n) - k(\lambda)).$$

We want to count the $\vee$ and $\wedge$ between T^+ and T^-. For α we count the $\wedge$'s $> T^-_\lambda$ and $\leq -(m-n) - k(\lambda)$. There are

$$(-1)(T^-_\lambda - (-(m-n) - k(\lambda))) = -T^-_\lambda (m-n) - k(\lambda).$$

The number of $\vee \geq k(\lambda) + 1$ and $< T^+_\lambda$ is

$$T^+_\lambda - k(\lambda) - 1.$$

Now we do the same count for λ: Between T^-_λ and T^+_λ $2k(\lambda) + (m-n)$ positions are bound, that is part of a cup or a x or a $\circ$. The number of $\wedge$'s used in cups and x's is $k(\lambda) + (m-n) := b$. We count the total number of all $\wedge$'s and substract

b. The $\wedge$-contributions come from the $l(\lambda^l)$ nontrivial λ_i^L and from the $\lambda_i^L = 0$-contributions in (T^-, T^+). This gives an additional $-T_\lambda^- - l(\lambda^L)$. So the total number of $\wedge$'s is

$$l(\lambda^L) + (-T_\lambda^- - l(\lambda^L)) = -T_\lambda^-.$$

Substracting b we obtain the number of free $\wedge$'s:

$$-T_\lambda - k(\lambda) - (m-n).$$

That agrees with the count for α. Then the number of $\vee$'s is also the same. □

As a consequence any $\alpha(i) \neq \zeta(i)$ will result in a switch of a label in λ when passing from $\lambda \to \lambda^\dagger$. This results in the following simplified algorithm for $\lambda \mapsto \lambda^\dagger$:

An algorithm: α differs from ζ in the following way: To the left of the $m-n$ crosses we have $n-k(\lambda)$ different labels. To the right infinitely many. Define M = maximal coordinate of a x or $\circ$ or part of a cup in λ. The matching t will be the identity (meaning t connects the ith position of α with the ith position of λ) from positions greater or equal to

$$T = max(k(\lambda)+1, M+1).$$

Since $\alpha(i) \neq \zeta(i)$ for all $i \geq T$, all labels in λ at positions greater or to T will be switched. Now define

$$X = \begin{cases} 0 & M+1 \leq k(\lambda)+1 \\ M-k(\lambda) & else. \end{cases}$$

A free position is one which does not have a cross, or a circle or is not part of a cup.

3.13 Corollary. *The weight diagram of $\lambda^\dagger$ is obtained from the weight diagram of λ by switching all labels at vertices $> T$ and switching the first $X+n-k(\lambda)$ free positions $< T$.*

Example: The typical case: Say the x are at position $v_1 > v_2 > \ldots > v_m$ and the circles at position $w_1 > \ldots > w_n$. Then

$$\lambda_1^\dagger = v_1, \lambda_2^\dagger = v_2+1, \ldots, \lambda_m^\dagger = v_m+m-1, \lambda_{m+1}^\dagger = w_n+m-1, \ldots, \lambda_{m+n}^\dagger = w_1+m-n.$$

The inverse $\lambda^\dagger \mapsto \lambda$: Given any typical weight $\lambda^\dagger$. If $T = n+1$, all the free entries up to n are labelled with $\wedge$'s and the remaining ones to the right with $\vee$'s. Otherwise there will be $\vee$'s in the $T-n-1$ free positions to the left of the rightmost cross or circle. After that (to the left) there will be $\wedge$'s. This describes θ and θ^{-1} for $\lambda^\dagger$ typical.

3.4.1. The map θ in the typical case

It is an ill-posed problem to express the map θ and θ^{-1} in terms of the coordinates of the bipartition. In the simplest case when $\lambda^\dagger$ is typical we give an explicit expression for the two maps.

3.14 Definition. *(compare with [MVdJ04], [MVdJ06]) A bipartition (μ,ν) is in the subset $\Lambda^{st} \subset \Lambda^x$ if there exist J and L such that*

$$J = min\{j \mid \mu^*_{j+1} + \nu^*_{n-j+1} \leq m\} \text{ with } 0 \leq J \leq n$$
$$L = min\{l \mid \mu_{m-l+1} + \nu^*_{l+1} \leq n\} \text{ with } 0 \leq L \leq m$$

In that case put $I = m - L$ and $K = n - J$.

We associate to such a bipartition the highest weight $\tilde{\theta}(\mu,\nu) = \lambda_{\mu,\nu}$

$$\lambda_{\mu,\nu} = (\mu_1, \ldots, \mu_I, n - \nu_l, \ldots, n - v_1 \mid \mu^*_1 - m, \ldots, \mu^*_J - m, -\nu^*_K, \ldots, -\nu^*_1).$$

Conversely we associate to any typical weight $\lambda^\dagger$ following [Moe06], lemma 3.15, the following bipartition (μ,ν): let JL be the set of all (j,l) with $0 \leq j \leq n$ and $0 \leq l \leq m$ such that

$$\begin{aligned} j = 0 \quad & \quad or\ \lambda^\dagger_{m+j} > -l \\ and\ j = n \quad & \quad or\ \lambda^\dagger_{m+j+1} \leq -l, \\ and\ l = m \quad & \quad or\ \lambda^\dagger_{m-l} \geq j \\ and\ l = 0 \quad & \quad or\ \lambda^\dagger_{m-l+1} < j. \end{aligned}$$

Let then $J = min\{j \mid (j,l) \in JL\}$ and $L = min\{l \mid (J,l) \in JL\}$. The associated bipartition is given via

$$\mu_i = \lambda^\dagger_i \ for\ i = 1, \ldots, I = m - L \ and\ \mu_i \leq J \ for\ i > J$$
$$\mu^*_j = m + \lambda^\dagger_{m+j} \ for\ j = 1, 2, \ldots, J \ and\ \mu^*_j \leq I \ for\ j > J$$
$$\nu^*_k = -\lambda^\dagger_{m+n-k+1} \ for\ k = 1, 2, \ldots, K = n - J \ and\ \nu^*_k \leq L \ for\ k > K$$
$$\nu_l = n - \lambda^\dagger_{m-l+1} \ for\ l = 1, \ldots, L \ and\ \nu_l \leq K \ for\ l > L$$

3.15 Lemma. *Let λ be such that $R(\lambda) = L(\lambda^\dagger)$ is typical. Then $\lambda^\dagger = \lambda_{\lambda^L,\lambda^R}$ and the inverse $\theta^{-1}(\lambda^\dagger)$ is given by the prescription above.*

Proof: The set Λ^{st} is a subset of Λ^x. Hence both $\lambda^\dagger$ and $\lambda_{\mu,\nu}$ are defined on Λ^{st}. Every typical weight in X^+ is in the image of $\tilde{\theta}$ by [Moe06], lemma 3.15.

The character of $L(\lambda_{\mu,\nu})$ is computed in [MVdJ06] and is given by the supersymmetric Schur function $s_{\mu,\nu}$. Similarly the character of $R(\lambda) = L(\lambda^\dagger)$ is computed in [CW11]. The two characters are equal. Since the character determines the irreducible representation the result follows. $\square$

Remark: A warning: Note that the condition $gl(m|m)$-standard of loc.cit is not equivalent to the condition (m,m)-cross. Furthermore the map of loc.cit which associates to any bipartition the weight $\lambda_{\mu,\nu}$ does in general not agree with $\lambda^\dagger$.

3.4.2. Kostant weights

A weight μ is called a Kostant weight, if the cup diagram of $L(\mu)$ is completely nested. In other words if its weight diagram is $\wedge\vee\wedge\vee$-avoiding in the sense that there are no vertices $i < j < k < l$ labelled in this order by $\wedge\vee\wedge\vee$.

3.16 Lemma. *Every irreducible mixed tensor is a Kostant module.*

Proof: This follows from the simplified algorithm since the weight diagram of a bipartition with $d(\lambda) = 0$ looks like

$$\wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \vee \quad \vee \quad \vee \quad \vee \quad \vee$$

after removing the crosses and circles. Applying θ means specifying a vertex, say V, and switching all free labels at vertices $\geq V$. This will not create any neighbouring vertices labelled $\vee\wedge\vee\wedge$. $\square$

3.17 Corollary. *If $L(\mu)$ is an irreducible mixed the tensor then:*

1. *The Kazhdan-Lusztig polynomials are multiplicity free: $p_{\lambda,\mu}(q) = q^{l(\lambda,\mu)}$ for all $\lambda \leq \mu$.*
2. *$\sum_{i\geq 0} dim Ext^i(V(\lambda), L(\mu)) \leq 1$ for all $\lambda \in X^+$.*
3. *L possesses a resolution by multiplicity free direct sums of Kac modules (BGG-resolution).*

Proof: This are properties of Kostant weights [BS10a], lemma 7.2 and theorem 7.3

Remark: In fact an irreducible module has a BGG-resolution if and only if it is Kostant [BS10a], thm 7.3. It has been known that the covariant modules possess a BGG-resolution [CKL08].

3.4.3. Tensor products of projective modules.

We obtain now an algorithm to decompose tensor products of projective modules. Note that $Proj$ is a tensor ideal, ie. the tensor product of a projective module with any other module will split in a direct sum of irreducible typical representations and projective covers of atypical modules

$$P \otimes M = \bigoplus P_i \oplus \bigoplus L(\lambda).$$

Since we have shown that a) every projective module is in the image of F_{mn} and b) given an explicit bijection θ between the projective modules and bipartitions with $k(\lambda) = n$, the tensor product formula in the Deligne category gives us an explicit algorithm for the decomposition.

3.5. Tannaka duals

We also obtain an explicit description of the Tannaka dual of any irreducible module. This is non-trivial since a closed expression for the highest weight of the dual module is not known. Brundan [Bru03] gave a complicated algorithm using certain operators on crystal graphs. In [BS10a] an algorithm on the cup diagram $\underline{\lambda}$ is given to determine the highest weight of the dual module.

Any irreducible module occurs as socle and head in its projective cover.

3.18 Lemma. *We have* $P(\lambda^\dagger)^\vee = P((\lambda^\dagger)^\vee)$

Proof: Clear. □

On the other hand $P(\lambda^\dagger)^\vee = R(\lambda^L, \lambda^R)^\vee = R(\lambda^R, \lambda^L) = P((\lambda^\dagger)^\vee)$. So to compute the Tannaka dual of an irreducible module, take its highest weight and associate to it the unique (m,n)-cross bipartition (λ^L, λ^R) of maximal defect as given above (labelling the projective cover of the irreducible module), switch it to $\tilde{\lambda} = (\lambda^R, \lambda^L)$ and then compute $\tilde{\lambda}^\dagger$. Then

$$L(\lambda^\dagger)^\vee = L(\tilde{\lambda}^\dagger).$$

In two instances we use this algorithm to describe the dual weight more explicitely: For contravariant modules and for maximally atypical modules in the $m = n$-cae.

3.5.1. Duals in the maximally atypical $m = n$-case

Let λ be a maximal atypical weight, and $[\lambda] = [\lambda_1, \ldots, \lambda_n]$ the associated irreducible representation. Note that $(Ber^k \otimes [\lambda])^\vee = Ber^{-k} \otimes [\lambda]^\vee$. To compute the Tannaka duals, by a twist with $Ber^{-\lambda_1}$, we therefore may assume $\lambda_1 = 0$. Furthermore recall that λ uniquely corresponds to a plot, also denoted λ. Let $\lambda(s) = \prod_i \lambda_i(s)$ be its prime factorization. For each prime factor $\lambda_i(s) = (I, K)$ with segment I and support K we define $\lambda_i^c(s) := (I, K^c)$, where $K^c = I - K$ denotes the complement of K in I. Then put

$$\lambda^c(s) := \prod_i \lambda_i^c(s) \ .$$

3.19 Proposition. *The Tannaka dual representation $\lambda^\vee$ of a maximal atypical representation λ is given by the plot*

$$\lambda^\vee(s) = \lambda^c(1-s).$$

We have to compute the socle of $P(\lambda)^\vee = R(\lambda^L, \lambda^R)^\vee = R(\lambda^R, \lambda^L)$ for the unique bipartition $(\lambda^L, \lambda^R) = \theta^{-1}([\lambda])$ with $k(\lambda^L, \lambda^R) = n$. In the case where $\lambda_1 = 0$ the weight diagram of (λ^L, λ^R) is obtained from the weight diagram of λ as follows: Put $\vee$'s at the same vertices as in λ and put $\vee$'s at all vertices $\geq n$. The remaining vertices are labelled by $\wedge$. Since the position of the $\vee$'s is given by the set

$$I_\vee(\lambda^L, \lambda^R) = \{1 - \lambda_1^R, 2 - \lambda_2^R, \ldots\}$$

this shows

$$\lambda^R = (n - \lambda_n, n - \lambda_{n-1}, \ldots, n - \lambda_2, n).$$

We will see in chapter 4 that a mixed tensor is maximally atypical if and only if $(\lambda^L)^* = \lambda^R$, so we simply write $R(\lambda^L, \lambda^R) = R(\lambda^L)$ in this case. We say that a symbol $\vee$ or $\wedge$ is bound in a cup if the underlying vertex is the starting or ending point of a cup.

Proof: For $\lambda = [0, \lambda_2, \ldots, \lambda_n]$ the set

$$I_x(\lambda) = \{0, \lambda_2 - 1, \ldots, \lambda_n - n + 1\}.$$

defines the left starting points of the sectors of the weight λ. Let (λ^L, λ^R) be such that $P(\lambda) = R(\lambda^L, \lambda^R)$. Since dualising means interchanging λ^L and λ^R we have to compute the socle of $R(\lambda^R, \lambda^L)$ for λ^R as above. For this specific λ^R we get

$$I_\wedge = \{n - \lambda_n, n - \lambda_{n-1} - 1, \ldots, n - n + 1, -n, -n-1, \ldots\}$$

and $I_\vee = \mathbb{Z} \setminus I_\wedge$. Exactly n of the vertices in $I_\vee$ will be bound in cups. The largest vertex labelled by $\vee$ is at position $n - \lambda_n$. We go from the right to the left starting from $n - \lambda_n$ to determine the n labels $\vee$'s bound in cups. If

$$\lambda_1^L = \ldots = \lambda_{s_1}^L > \lambda_{s_1+1}^L = \ \ldots \ = \lambda_{s_2}^L > \lambda_{s_2+1}^L = \ldots$$

put $\delta_1 = s_1$ and $\delta_i = s_i - s_{i-1}$ and $\Delta_i = \lambda^L_{s_i} - \lambda^L_{s_i+1}$:

$$\ldots \overbrace{\vee \ldots \vee}^{\Delta_3} \overbrace{\wedge \ldots \wedge}^{\delta_3} \overbrace{\vee \ldots \vee}^{\Delta_2} \overbrace{\wedge \ldots \wedge}^{\delta_2} \overbrace{\vee \ldots \vee}^{\Delta_1} \overbrace{\wedge \ldots \wedge}^{\delta_1}.$$

In the weight diagram we have δ_1 labels $\wedge$ at the positions

$$\{n - \lambda_n, n - \lambda_n - 1, \ldots, n - \lambda_n - \delta_1 + 1\}$$

followed by Δ_1 labels $\vee$ to the left. These labels $\vee$ will form $min(\Delta_1, \delta_1)$ cups with the $\wedge$'s to the right. Hence we get $min(\Delta_1, \delta_1)$ bound $\vee$'s at the positions $n - \lambda_n - s_1, n - \lambda_n - s_1 - 1, \ldots, n - \lambda_n - s_1 - min(\delta_1, \Delta_1) + 1$ and so on. Now consider the weight diagram of λ. The n labels $\vee$ are at the positions $\lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1$. These are bound in n cups. These $\vee$'s will be transformed into $\wedge$'s by the change $(I, K) \to (I, I - K)$. Going from left to the right throught the weight diagram we have s_1 $\vee$'s to the left, then Δ_1 $\wedge$'s, then δ_2 $\vee$'s, then Δ_2 $\wedge$'s and so on:

$$\overbrace{\vee \ldots \vee}^{\delta_1} \overbrace{\wedge \ldots \wedge}^{\Delta_1} \overbrace{\vee \ldots \vee}^{\delta_2} \overbrace{\wedge \ldots \wedge}^{\Delta_2} \overbrace{\vee \ldots \vee}^{\delta_3} \overbrace{\wedge \ldots \wedge}^{\Delta_3} \ldots.$$

Hence the rule for determining the n lables $\wedge$ bound in cups is exactly the same in reverse order as the rule for the n labels $\vee$ bound in cups in the weight diagram of λ^L. Hence (if $\tilde{\lambda}$ denotes the weight defined by the plot $\lambda^\vee(s)$) after the reflection $s \mapsto 1 - s$ we get $\tilde{\lambda} = (\lambda)^\vee + Ber^k$ for some $k \in \mathbb{Z}$. We have to show $k = 0$. Now the leftmost $\vee$ in the weight diagram of λ is at position $\lambda_n - n + 1$ and the leftmost $\wedge$ in a cup is at position $\lambda_n - n + 1 + s_1$. It will give the rightmost $\vee$ after the reflection $s \mapsto 1 - s$. It maps under the reflection to $n - \lambda_n - s_1$. This is also the position of the rightmost $\vee$ in the weight diagram of $\lambda^\vee$, hence $k = 0$. $\square$

3.20 Corollary. $L(\lambda) = L(0, \lambda_2, \ldots, \lambda_n) \neq \mathbb{1}$ *is never selfdual.*

Proof: $L(\lambda)$ is selfdual if and only if

$$(\lambda^L)^* = \lambda^L \text{ for } \lambda^L = (n - \lambda_n, n - \lambda_{n-1}, \ldots, n - \lambda_2, n).$$

This follows from the proof since a maximally atypical $R(\lambda)$ is self-dual if and only if $\lambda = \lambda^*$. However the partition is evidently never self-conjugate. $\square$

Example 1. Suppose $\lambda = [0, \lambda_2, \ldots, \lambda_n]$ holds with $0 > \lambda_2$ and $\lambda_i > \lambda_{i+1}$ for $2 \leq i \leq n - 1$. Then $\lambda^\vee = [n - \lambda_n - 1, n - \lambda_{n-1} - 1, \ldots, n - \lambda_2 - 1, n - 1]$.

3.21 Lemma. *For maximal atypical irreducible representations* $L = [\lambda_1, ..., \lambda_n]$ *such that* $\lambda_n = 0$ *the following assertions are equivalent*

1. *$L^\vee \cong [\rho_1, ..., \rho_n]$ such that $\rho_n \geq 0$.*

2. *L is basic, i.e. $\lambda_1 \geq \ldots \lambda_n \geq 0$ and $\lambda_i \leq n - i$.*

3. *$\lambda_1 \leq n-1$ and $L^\vee \cong [\lambda_1^*, ..., \lambda_n^*]$ holds for the transposed partition $\lambda^* = (\lambda_1^*, ..., \lambda_n^*)$ of the partition $\lambda = (\lambda_1, ..., \lambda_n)$.*

Proof: i) implies ii): If $\rho_n = 0$ the leftmost $\vee$ in the weight diagram of $[\rho]$ is at position $-n+1$. Then the smallest $\wedge$ bound in a cup is at a position ≤ 1 and $\geq 1-n$. After the change $(I,K) \to (I, I-K)$ and the reflection $s \mapsto 1-s$ this means that the rightmost $\vee$ in $[\rho]^\vee$ is at position $\leq n-1$ and ≥ 0 which is equivalent to $0 \leq \lambda_1 \leq n-1$. Likewise the i-th leftmost $\wedge$ bound in a cup is at a position $\geq -n+i+1$ and $\leq n$. It will give the i-th largest $\vee$ in the weight diagram of $[\lambda]$. After the change $(I,K) \mapsto (I, I-K)$ and the reflection the i-th largest $\vee$ is at a position $\leq n-2i+1$ which is equivalent to $\lambda_i \leq n-i$. ii) implies i): If λ is basic the largest $\vee$ is at position $\leq n-1$, hence the largest $\wedge$ bound in a cup is at position $\leq n$. It gives the smallest $\vee$ of $[\lambda]^\vee$. Hence the smallest $\vee$ of $[\lambda]^\vee$ is at a position $\geq 1-n$ which is equivalent to $\lambda_n^\vee \geq 0$.
ii) implies iii): If λ is basic, the $2n$ vertices in cups form the intervall $J := [-n+1, n]$ of length $2n$. If $J_\vee$ is the subset of vertices labelled by $\vee$, the subset $J \setminus J_\vee$ is the subset of vertices labelled by $\wedge$. The intervall J is preserved by the reflection $s \mapsto 1-s$. If λ is basic, so is λ^*. As in the proof of 3.19 we use the following notation: If

$$\lambda_1 = \ldots = \lambda_{s_1} > \lambda_{s_1+1} = \ \ldots \ = \lambda_{s_2} > \lambda_{s_2+1} = \ldots = \lambda_{s_r} > \lambda_{s_r+1} = 0$$

put $\delta_1 = s_1$ and $\delta_i = s_i - s_{i-1}$ and $\Delta_i = \lambda_{s_i} - \lambda_{s_i+1}$: Likewise for λ^* with δ_i^* and Δ_i^*. Then

$$\delta_i = \Delta_{i-r}^*, \ \Delta_i = \delta_{i-r}^*.$$

Then the weight diagram of $[\lambda^*]$ looks, starting from n and going to the left

$$\ldots \overbrace{\vee \ldots \vee}^{\delta_3^*} \overbrace{\wedge \ldots \wedge}^{\Delta_2^*} \overbrace{\vee \ldots \vee}^{\delta_2^*} \overbrace{\wedge \ldots \wedge}^{\Delta_1^*} \overbrace{\vee \ldots \vee}^{\delta_1^*} \wedge \ldots \wedge$$

and the weight diagram of $[\lambda]$ looks, starting from $-n+1$ and going to the right like

$$\vee \ldots \vee \overbrace{\wedge \ldots \wedge}^{\Delta_r = \delta_1^*} \overbrace{\vee \ldots \vee}^{\delta_r = \Delta_1^*} \overbrace{\wedge \ldots \wedge}^{\Delta_{r-1} = \delta_2^*} \overbrace{\vee \ldots \vee}^{\delta_{r-1} = \Delta_2^*} \ldots .$$

We can argue now exactly as in the proof of 3.19. The two weight diagrams are mirror images of each other and the rule for the $\vee$'s in cups in one is the same as the rule for the $\wedge$'s in the cups of the other. Hence after the change $(I,K) \mapsto (I, I-K)$ and the reflection $s \mapsto 1-s$ the two weight diagrams agree.
iii) implies i): trivial. $\square$

Example 3. Duals in the $\mathcal{R}_3$-case. If $a > b > 0$, then $[a,b,0]^\vee = [2, 2-b, 2-a] = Ber^{2-a}[a, a-b, 0]$. If $a \geq 1$ then $[a,a,0]^\vee = [2, 1-a, 1-a] = Ber^{1-a}[a+1, 0, 0] = Ber^{1-a}S^{a+1}$.

Example 4 Duals in the $\mathcal{R}_4$-case: Assume $[\lambda] = [0, \lambda_2, \lambda_3, \lambda_4]$. In the table we use the following convention: If we write $[0, \lambda_2, \lambda_3, \lambda_4]$ we mean $0 > \lambda_2 > \lambda_3 > \lambda_4$. If we had say $\lambda_2 = \lambda_3$ we would replace λ_3 by λ_2, ie. write $[0, \lambda_2, \lambda_2, \lambda_4]$.

Case	$[\lambda]$	extra condition	$[\lambda]^\vee$
1	$[0, \lambda_2, \lambda_3, \lambda_4]$		$[3-\lambda_4, 3-\lambda_3, 3-\lambda_2, 3]$
2	$[0, \lambda_2, \lambda_3, \lambda_3]$	$\lambda_2 > \lambda_3 + 1$	$[2-\lambda_3, 2-\lambda_3, 3-\lambda_2, 3]$
3	$[0, \lambda_2, \lambda_3, \lambda_3]$	$\lambda_2 = \lambda_3 + 1,\ 0 > \lambda_2 + 1$	$[2-\lambda_3, 2-\lambda_2, 3-\lambda_2, 3]$
4	$[0, -1, -2, -2]$		$[4, 3, 2, 2]$
5	$[0, \lambda_2, \lambda_2, \lambda_2]$	$0 > \lambda_2 + 2$	$[1-\lambda_2, 1-\lambda_2, 1-\lambda_2, 3]$
6	$[0, -2, -2, -2]$		$[3, 3, 2, 2]$
7	$[0, -1, -1, -1]$		$[2, 1, 1, 1]$
8	$[0, \lambda_2, \lambda_2, \lambda_4]$	$0 > \lambda_2 + 1$	$[3-\lambda_4, 2-\lambda_2, 2-\lambda_2, 3]$
9	$[0, -1, -1, \lambda_3]$		$[3-\lambda_3, 3, 2, 2]$
10	$[0, 0, \lambda_3, \lambda_4]$		$[3-\lambda_4, 3-\lambda_3, 2, 2]$
11	$[0, 0, \lambda_3, \lambda_3]$	$\lambda_3 + 2 < 0$	$[2-\lambda_3, 2-\lambda_3, 2, 2]$
12	$[0, 0, -1, -1]$		$[3, 1, 1, 1]$
13	$[0, 0, 0, \lambda_4]$		$[3-\lambda_4, 1, 1, 1]$

Weakly selfdual representations. A representation M is called weakly selfdual, if $M^\vee = Ber^k \otimes M$ holds for some $k \in \mathbb{Z}$. Any Berezin twist of a self-dual module is weakly self-dual, but the converse is true if and only if $k \in 2\mathbb{Z}$.

Example. Weakly selfdual irreducible modules in the $\mathcal{R}_4$-case: Going through the list above we see that in case 2, 3, 5, 6, 9, 10, 12 and 13 the module is not weakly self-dual. In case 1 the module is selfdual iff $[\lambda] = [0, \lambda_2, \lambda_3, \lambda_2 + \lambda_3]$ with twist factor $Ber^{-\lambda_2 - \lambda_3 + 3}$. The module $[0, -1, -2, -2]$ is weakly selfdual with twist factor Ber^4. The module $[0, -1, -1, -1]$ is weakly self-dual with twist factor Ber^2. The module $[0, \lambda_2, \lambda_2, \lambda_4]$ is weakly selfdual if and only if $\lambda_4 = 1 + 2\lambda_2$, in which case the twist factor is $Ber^{-2\lambda_2 + 2}$. The module $[0, 0, \lambda_3, \lambda_3]$ is always weakly selfdual with twist $Ber^{-\lambda_3 + 2}$.

We list the weakly self-dual modules of type $[0, \lambda_2, \lambda_3, \lambda_4]$. Any other weakly selfdual module is a Berezin twist of one of these. We use the same convention as for the table above.

Case	$[\lambda]$	Berezin twist
1	$[0, \lambda_2, \lambda_3, \lambda_2 + \lambda_3]$	$Ber^{-\lambda_3-\lambda_2+3}$
2	$[0, \lambda_2, \lambda_2, 1 + 2\lambda_2]$	$Ber^{-2\lambda_2+2}$
3	$[0, 0, \lambda_3, \lambda_3]$	$Ber^{-\lambda_3+2}$
4	$[0, -1, -2, -2]$	Ber^4
5	$[0, -1, -1, -1]$	Ber^2

None of these modules is selfdual. If $[\lambda]$ is selfdual, write $[\lambda] = Ber^{\lambda_1}[0, \lambda_2-\lambda_1, \lambda_3-\lambda_1, \lambda_4 - \lambda_1]$. For selfduality the Berezin twist of $[0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1]$ has to be $Ber^{2\lambda_1}$. Case 4 and 5 give the selfdual modules $[1, 0, 0, 0]$, $[2, 1, 0, 0]$. In case 3 we get a selfdual module only if $\lambda_3 = 2\tilde{\lambda}_3$ is even (< 0). The corresponding selfdual module is $[-\tilde{\lambda}_3 + 1, -\tilde{\lambda}_3 + 1, \tilde{\lambda}_3 + 1, \tilde{\lambda}_3 + 1]$ (for $\tilde{\lambda}_3 = -1$ we obtain $[2, 2, 0, 0]$). In case 3 we obtain the selfdual modules $[-\lambda_2 + 1, 1, 1, 2 - \lambda_2]$ for $\lambda_2 \leq -2$ (eg for $\lambda_2 = -2$ we get $[3, 1, 1, 0]$). In case 1 $-\lambda_3 + \lambda_2 + 3$ must be even to get a selfdual module. Hence either i) λ_2 odd and λ_3 even or ii) λ_2 even and λ_3 odd. In the first case write $\lambda_2 = 2\tilde{\lambda}_2 + 1$ and $\lambda_3 = \tilde{\lambda}_3$. By a Berezin twist with $-\tilde{\lambda}_3 - \tilde{\lambda}_2 + 1$ we obtain the selfdual module $[-\tilde{\lambda}_3 - \tilde{\lambda}_2 + 1, \tilde{\lambda}_2 - \tilde{\lambda} - 3 + 2, \tilde{\lambda}_3 - \tilde{\lambda}_2 + 1, \tilde{\lambda}_2 + \tilde{\lambda}_3 + 2]$ for two negative numbers $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ (possibly equal) (example: $\tilde{\lambda}_2 = \tilde{\lambda}_3 = -1$, then we get $[3, 2, 1, 0]$). In the second case we substitute likewise and get the selfdual modules $[-\tilde{\lambda}_3 - \tilde{\lambda}_2 + 1, \tilde{\lambda}_2 - \tilde{\lambda} - 3 + 1, \tilde{\lambda}_3 - \tilde{\lambda}_2 + 2, \tilde{\lambda}_2 + \tilde{\lambda}_3 + 2]$ for two negative numbers $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ with $\tilde{\lambda}_2 > \tilde{\lambda}_3$ (example: $\tilde{\lambda}_2 = -1$ and $\tilde{\lambda}_3 = -2$ lead to the selfdual weight $[4, 2, 1, -1]$).

3.5.2. Contravariant modules for $m = n$

The contravariant modules are the modules in the decomposition $T(0, r) = (V^*)^{\otimes r}$. Hence they are the duals of the covariant modules $\{\lambda\}$. Recall that the highest weight of $\{\lambda\} =: L(\mu)$ is obtained as follows: Put $\mu_i = \lambda_i$ for $i = 1, \ldots, m$ and $\mu_{m+i} = max(0, \lambda_i^* - m)$ for $i = 1, \ldots, m$. Put further $(\lambda_1, \ldots, \lambda_r)^v = (-\lambda_r, \ldots, -\lambda_1)$. Recall further that for $\lambda \in H(m, m)$ we have $\lambda^* \in H(m, m)$. We give a closed formula for the highest weight of a contravariant module.

3.22 Theorem. *$\{\lambda\}^\vee$ has highest weight μ^v where μ is the highest weight of $\{\lambda^*\}$.*

Proof: Assume $at\{\lambda\} = m - k$. We determine $\lambda^\dagger$ of $R(0, \lambda) = L(\lambda^\dagger)$. We have

$$I_\wedge = \{0, -1, -2, \ldots\}$$
$$I_\vee = \{1 - \lambda_1, 2 - \lambda_2, \ldots, m - \lambda_m, m + 1 - \lambda_{m+1}, \ldots\}.$$

The crosses are at the positions $1-\lambda_1, \ldots, k-\lambda_k$ with the property $\lambda_i - i \geq 0$. The remaining $k + 1 - \lambda_{k+1}, \ldots, \ldots$ are positive and will give $\vee$. We have $\alpha(i) \neq \zeta(i)$

for $i \geq k+1$ and for $i \in [-k-1, -m]$. Indices in λ which are connected to this elements via t get switched. To obtain $\lambda^\dagger$ from λ take the first $l(\lambda)+m-k$ free positions (no crosses or circles) $\leq l(\lambda)+k$ and switch the symbols. In $(0, l(\lambda)+k]$ there are $l(\lambda)$ switches since we have k circles. Hence there are $m-k$ switches in $[-(m-k-1), 0]$. All free $\lambda(i)$ for $i > 0$ are $\vee$s which get turned into $\wedge$s. The first $m-k$ free positions ≤ 0 are switched: All this yield $\vee$s in $\lambda^\dagger$.
Now we compute the other side: The highest weight of $\{\lambda\}$ is given by $\mu_i = \lambda_i$ for $i = 1, \ldots, m$ and $\mu_{m+i} = max(0, \lambda_i^* - m)$ for $i = 1, \ldots, m$. For the transposed partition $\lambda_i^* = \sharp\lambda_i :\ \lambda_i \geq i$. Hence the highest weight of λ^* is given by $\mu_i = \lambda_i^* = \sharp\lambda_i :\ \lambda_i \geq i$ for $i = 1, \ldots, m$ and $\mu_{m+i} = max(0, \lambda_i - m)$ for $i = 1, \ldots, m$. Applying $()^v$ yields the conjectured highest weight of $\{\lambda\}^\vee$

$$\mu = (-max(0, \lambda_m - m), \ldots, -max(0, \lambda_1 - m) | -(\sharp\lambda_i : \lambda_i \geq m, \ldots, -(\sharp\lambda_i :\ \lambda_i \geq 1).$$

Therefore

$$I_x = \{-max(0, \lambda_m - m), -max(0, \lambda_{m-1} - m) - 1, \ldots, -max(0, \lambda_1 - m) - m + 1\}$$
$$I_0 = \{1 - m + (\sharp\lambda_i :\ \lambda_i \geq m), \ldots, \sharp\lambda_i :\ \lambda_i \geq 1).$$

Assume that there are d λ_i with $\lambda_i \geq m$. Then one has crosses at the position $1 - \lambda_1, \ldots, d - \lambda_d$. For the remaining λ_i one has $max(0, \lambda_i - m) = 0$, hence

$$I_x = \{0, -1, \ldots, -(m-d) + 1, d - \lambda_d, \ldots, 1 - \lambda_1\}.$$

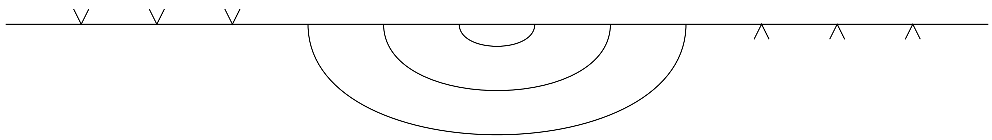
This agrees with the position of the $\vee$ and $\times$ in $\lambda^\dagger$.
Hence the weights will be equal if the circles are at the same vertices. In the I_0-picture the circles are at the k positions $\sharp(\lambda_i : \lambda_i \geq 1) = l(\lambda)$, $-1 + \sharp(\lambda_i : \lambda_i \geq 2), \ldots, -k + 1 + \sharp(\lambda_i : \lambda_i \geq k)$. In the $\lambda^\dagger$-picture the circles occupy the first k free positions ≥ 1, that is they occur when one has $\lambda_i > \lambda_{i+1} \geq 1$ for $i \geq k$. For $\lambda_i - \lambda_{i+1} = r$ one has r circles after the $\wedge$ at position $i - \lambda_i > 0$. Now use a counting argument just as in the proof of 3.19. $\square$

3.6. The constituent of highest weight

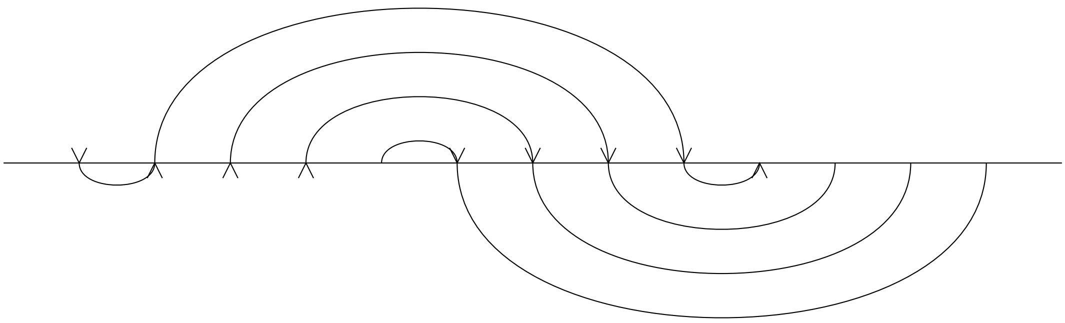
We have seen that the irreducible modules in T are the ones with $def(\lambda) = 0$. We describe the constituent of highest weight of $R(\lambda)$ for $def(\lambda) > 0$. The constituents of $R(\lambda)$ are given by $[R(\lambda)] = [G^t_{\Delta\Gamma} L(\zeta)] = \sum_{\mu \subset \alpha \to^t \zeta,\ red(\underline{\mu} t) = \underline{\zeta}} [L(\mu)]$. The condition $\mu \subset \alpha$ implies $\alpha \geq \mu$ in the Bruhat order, hence the constituent of highest weight must be among the $\alpha \to^t \zeta$. We define A_λ by taking the weight diagram of $\lambda^\dagger$ and by labelling all caps in the matching t by $\wedge\vee$. This is the maximal element in the Bruhat order among all the possible α. It will give the constituent of highest weight if A_λ satisfies the condition $red(\underline{A_\lambda} t) = \underline{\zeta}$.

3.23 Lemma. *A_λ is the constituent of highest weight of $R(\lambda)$. It occurs with multiplicity 1 in the middle Loewy layer.*

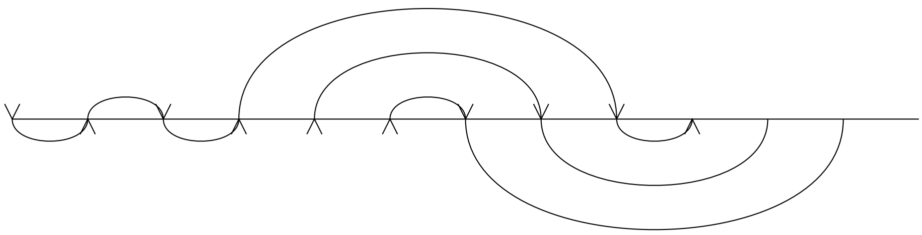
Proof: If $k(\lambda) = n$ the assertion is clear (see the section on the projective covers). So assume $k(\lambda) < n$. The cup diagram of α is completely nested with $k(\lambda)$ cups with the innermost cup at position $(-1, 0)$. After the change from α to ζ the upper line in the matching t looks like



with $n - k(\lambda)$ free $\vee$ to the left of the nested cups and $n - k(\lambda)$ free $\wedge$'s to the right of the nested cups. We call the ones to the left $k_1^-, k_2^-, \ldots, k_{n-k(\lambda)}^-$, the ones to the right $k_1^+, k_2^+, \ldots, k_{n-k(\lambda)}^+$ We have $red(\underline{A_\lambda}t) = \underline{\zeta}$ iff k_1^- will be connected with k_1^+ via t when performing the lower reduction, k_2^- with k_2^+ and so forth. Under t k_1^- is connected to a position in $\underline{A_\lambda}$ which we call again k_1^-, k_2^- to a position which we call k_2^- etc. Since t is oriented the $-$-positions are labelled by a $\vee$, the $+$-positions by a $\wedge$. Assume first that $k(\lambda) = n - 1$. If $k_1^- = k_1^+ - 1$ then we are done. If not, we look at the cup diagram in the intervall $I = [k_1^- + 1, k_1^+ - 1]$. By construction of t there are no free $\vee$ or $\wedge$ in I. We may ignore x and $\circ$'s and assume that the cup diagram consists of one segment and r different sectors $C_1, \ldots, C_r$. If $r = 1$ the cup diagram is completely nested and we get



The situation generalizes immediately if the cup diagram is a union of $r > 1$ sectors, eg



Hence the assertion is true for $k(\lambda) = n-1$. In case $k(\lambda) < n-1$ we may connect k_1^- to k_1^+ as above. We may then remove the part of the cup diagram connected to k_1^- and k_1^+ and obtain a diagram with one $k_i^{\pm}$ less. We can then connect k_2^- to k_2^+ as above and iterate this procedure to finish the proof. $\square$

3.24 Theorem. *Given two direct sums $\bigoplus P_i$, $\bigoplus Q_j$ of projective modules, these two sums are equal iff they are equal in K_0.*

Proof: It suffices to test this for a single block Γ. Any indecomposable projective module is a projective cover $P(w)$ of an irreducible module $L(w) \in \Gamma$. Given two projective covers $P(w) = R(\lambda)$, $P(\tilde{w}) = R(\tilde{\lambda})$, $w \neq \tilde{w}$ in a fixed block, the matching $t_1 = t(\lambda)$ and $t_2 = t(\tilde{\lambda})$ differ. Otherwise $\lambda^{\dagger} = \tilde{\lambda}^{\dagger}$. Since the defect and the position of the $x, \circ$ are fixed this means that $\underline{\lambda} \neq \underline{\tilde{\lambda}}$. Hence $A_{\lambda} \neq A_{\tilde{\lambda}}$. Conversely $R(\lambda)$ of maximal defect is uniquely determined by giving A_{λ} and the block. Hence it is equivalent to give the direct sum $\bigoplus_{i=1}^n P_i$ and the set $\{A_i\}$. Hence $\bigoplus P_i = \bigoplus Q_j$ iff $\{A_i\} = \{A_j\}$. We are done if we can determine the set $\{A_i\}$ uniquely from the decomposition $[\bigoplus P_i]$ in K_0. We will give an algorithm to do so. The block will be represented by the numberline with k $\vee$'s (with variable position) and $m-k$ x and $n-k$ $\circ$ (with fixed position). Let P be the set of composition factors of $\bigoplus P_i$. It may be identified with the set of the corresponding weight diagrams. We go from the right to the left through these diagrams. Let i_1 be the rightmost position with a $\vee$ in P. We restrict to the subset P_{i_1} of P of diagrams with a $\vee$ at position i_1. From i_1 we move to the left. Let i_2 be the next position with a $\vee$ among the diagrams in P_{i_1}. Let $P_{i_1 i_2}$ the set of weight diagrams with a $\vee$ at position i_1 and i_2. Iterating this procedure we obtain $P_{i_1 i_2 \ldots i_k}$. This set consists of the weight diagram of a unique weight, possibly with multiplicity ≥ 1 (since x, $\circ$ and $\vee$'s are fixed). We claim that this weight is of the form A_i for some P_i. This is clear: The weight determines a composition factor of some $P(a)$. If $L(...) \neq A_a$, then $A_a > L(...)$ in contradiction to the construction above.

The factor A_i determines the corresponding projective module P_i. We remove all the composition factors of the copies of P_i from P. Now we apply the same algorithm again to the set $P \setminus r[P_i]$ to obtain again a weight of the form A_l with corresponding projective module P_l. We remove its composition factors etc until there are no weights left in P. Hence we have constructed all the weights A_i from the K_0-decomposition. □

3.7. Serganova's functor and base change

Let M be any $\mathfrak{g}$-module. For any $\xi \in X$ there exists $g \in Gl(m) \times Gl(n)$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \ldots, \alpha_k$ such that $Ad_g(\xi) = \xi_1 + \ldots + \xi_k$ with $\xi_i \in \mathfrak{g}_{\alpha_i}$. The number k is called the rank of ξ [Ser10]. For any $x \in X = \{x \in \mathfrak{g}_1 \mid [x,x] = 0\}$ of $rk(x) = k$ we dispose over the (cohomological) tensor functor - called fibre functor - $M \to M_x$ from $Rep(gl(m,n)) \to Rep(gl(m-k, n-k))$. We quote [Ser10], thm 2.1, cor 2.2

3.25 Theorem. *If $at(M) < rk(M)$, then $DS(M) = 0$. If $at(M) = rk(x)$, then $DS(M)$ is a typical module. If $rk(x) = r$, then $at(M_x) = at(M) - r$.*

From now on we will study Serganova's tensor functor for special x. We denote by

$$x_r = \begin{pmatrix} 0 & \epsilon_r \\ 0 & 0 \end{pmatrix}, \quad \epsilon_r = \begin{pmatrix} 1 & 0 & \ldots & & & \\ 0 & 1 & 0 & \ldots & & \\ \ldots & & \ldots & & & \\ 0 & \ldots & & 0 & & \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with r 1's on the diagonal.

3.26 Lemma. *DS_{x_r} maps st to the standard representation of $Gl(m-r, n-r)$.*

Proof: Let $*$ denote an arbitrary element. Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be some element of $k^{m,n}$. Then

$$xv = \begin{pmatrix} 0 & \epsilon_r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \epsilon_r y \\ 0 \end{pmatrix} = \big((1,1,\ldots,1,0,\ldots,0\big)$$

with r 1's. So

$$ker(x \cdot st) = \{\big((x,0,\ldots,0,*,\ldots,*\big) \mid x \in k^m, \ (n-r) \ 0's\}$$

and

$$image(x \cdot st) = \{\big((*, \ldots, *, 0, \ldots, 0\big) \mid r \;\; *' \; s\}.$$

So

$$\frac{ker(x \cdot st)}{image(x \cdot st)} = \{v \in k^{m,n} \mid v = \begin{pmatrix} 0 \\ v_1 \\ 0 \\ w_1 \end{pmatrix},\ v_1 \in k^{m-r}, w_1 \in k^{n-r} \ arbitr.\}$$

3.27 Proposition. *Under* DS_{x_r}

$$R(\lambda^L, \lambda^R) \mapsto \begin{cases} 0 & (\lambda^L, \lambda^R) \ not \ (m,n) - cross \\ R(\lambda^L, \lambda^R) & else \end{cases}$$

In the case $r = 1$ *this specialises to*

$$R(\lambda^L, \lambda^R) \mapsto \begin{cases} 0 & R(\lambda^L, \lambda^R) \ projective \\ R(\lambda^L, \lambda^R) & else \end{cases}.$$

Proof: This follows from the diagram

$$\begin{tikzcd} Rep(Gl_{m-n}) \arrow[d, "F_{m,n}"] \arrow[dr, "F_{m-r,n-r}"] & \\ T_{m,n} \arrow[r, "DS_{x_r}"] & T_{m-r,n-r} \end{tikzcd}.$$

Since DS_{x_r} maps the standard representation to the standard representation the universal property of Deligne's category implies that the diagram is commutative. In the case $r = 1$ the kernel of DSx_1 consists of the $(m-1, n-1)$-cross bipartitions which are not (m,n)-cross. By [BS11] (λ^L, λ^R) is (m,n)-cross iff $k(\lambda^L, \lambda^R) \leq k(\lambda)$. Similarly (λ^L, λ^R) is $(m-1, n-1)$-cross iff $k(\lambda^L, \lambda^R) \leq n-1$. Hence $R(\lambda^L, \lambda^R) \in ker(DS_{x_1})$ iff $k(\lambda^L, \lambda^R) = n$ which is equivalent to $R(\lambda^L, \lambda^R)$ projective. $\square$

Remark: This is a special case of a more general result. We quote from [BKN09a], page 16: If M is $*$-invariant, then M is projective iff $M_x = 0$ for some x of rank 1. For general M the following holds: M is projective iff there exist two elements $x \in g_1$, $y \in g_{-1}$ such that $M_x = M_y = 0$, see loc.cit.

Example: If $M := R(n-1, n-2, \ldots)$ in $Gl(n,n)$ then the socle is $L[n-2, n-3, \ldots, 1, 0, 0]$. We obtain $M_x = P[n-2, n-3, \ldots, 1, 0]$ in $Rep(Gl(n-1, n-1)$ for x of rank 1.

3.28 Lemma. *Let $y \in X$ of rank r such that DS_y maps the standard representation to the standard representation. Then $DS_y = DS_{x_r}$ when restricted to T.*

Proof: This follows from the diagram above and the universal property of Deligne's category. □

3.29 Lemma. *If $R(\lambda^L, \lambda^R)$ is irreducible so is $DS_{x_r}(R(\lambda^L, \lambda^R)$.*

Proof: $R(\lambda^L, \lambda^R)$ is irreducible iff $def(\lambda^L, \lambda^R) = 0$. The defect of a bipartition only depends on the difference $m - n = (m-r) - (n-r)$.

3.8. Elementary properties of the $R(\lambda)$

Given two (m,n)-Hook partitions λ^L, λ^R we form the bipartition (λ^L, λ^R). It is in general not (m,n)-cross. We will assume this from now on.

3.30 Lemma. *Given two (m,n)-Hook partitions λ^L, λ^R such that (λ^L, λ^R) is (m,n)-cross. Then $\{\lambda^L\} \otimes \{\lambda^R\}^\vee$ contains $R(\lambda^L, \lambda^R)$ as a direct summand. In the decomposition*

$$\{\lambda^L\} \otimes \{\lambda^R\}^\vee = R(\lambda^L, \lambda^R) \oplus \bigoplus R(\mu^j)$$

all μ^j satisfy $(\mu^j)^L_i \leq \lambda^L_i$ and $(\mu^j)R_i \leq \lambda^R_i$ for all i and $deg(\mu^j) < deg(\lambda^L, \lambda^R)$.

Proof: Recall that in R_t

$$(\lambda^L, 0) \otimes (0, \lambda^R) = \sum_\nu \sum_{\kappa \in P} c^{(\lambda^L)}_{\kappa,\nu^L} c^{(\lambda^R)}_{\kappa,\nu^R} \; \nu.$$

Putting $\kappa = 0$ yields $\nu^L = \lambda^L,\ \nu^R = \lambda^R$. Hence

$$(\lambda^L, 0) \otimes (0, \lambda^R) = (\lambda^L, \lambda^R) + \sum_\nu \sum_{\kappa \in P, \kappa \neq 0} c^{(\lambda^L)}_{\kappa,\nu^L} c^{(\lambda^R)}_{\kappa,\nu^R} \; \nu.$$

All other bipartitions $\nu = (\nu^L, \nu^R)$ will have degree stricty lower than (λ^L, λ^R) and length $\geq$ than (λ^L, λ^R) (well-known properties of LR-coefficients). By Comes-Wilson $lift_d(\lambda) = \lambda + \ldots$ where the other bipartitions are obtained by swapping successively $\vee\wedge$-pairs, i.e. decreasing the coefficients of the bipartition. Since (λ^L, λ^R) is the largest bipartition, $R(\lambda^L, \lambda^R)$ will occur with multiplicity one in the decomposition. □

For any two partitions λ^L, λ^R such that the pair (λ^L, λ^R) is (m,n)-cross we define

$$\mathbb{A}_{\lambda^L,\lambda^R} := \{\lambda^L\} \otimes \{\lambda^R\}^\vee.$$

3.31 Proposition. *$R(\lambda^L, \lambda^R)$ is $*$-invariant*

Proof: Clearly $\mathbb{A}_{\lambda^L,\lambda^R}$ is $*$-invariant since irreducible modules are $*$-invariant. In the decomposition

$$\mathbb{A}_{\lambda^L,\lambda^R} = R(\lambda^L, \lambda^R) \oplus \bigoplus_i R(\mu_i)$$

$R(\lambda^L, \lambda^R)$ occurs as a direct summand with multiplicity 1; and $deg(\lambda^L, \lambda^R) > deg(\mu_i)$. Assume $R(\lambda^L, \lambda^R)$ would not be $*$-invariant. Then there exists a μ_i occuring with multiplicity 1 in the decomposition with $R(\lambda^L, \lambda^R)^* = R(\mu_i)$. Write $\mu_i = (\mu_i^L, \mu_i^R)$. As for (λ^L, λ^R) $R(\mu_i)$ occurs with multiplicity 1 in the decomposition of the $*$-invariant

$$\mathbb{A}_{\mu_i^L,\mu_i^R} = R(\mu_i) \oplus \bigoplus_j R(\nu_j)$$

with degree strictler larger then the other bipartitions ν_j. Hence there exists a ν_j with $R(\mu_i)^* = R(\nu_j)$. Since $*^2 = id$ this forces $\nu_j = (\lambda^L, \lambda^R)$. However $deg(\lambda^L, \lambda^R) > deg(\mu_i) > deg(\nu_j)$. □

Hence by the lemma the $R(\lambda)$ are the modules *with largest bipartition* in the decomposition $\{\lambda^L\} \otimes \{\lambda^R\}^\vee = R(\lambda^L, \lambda^R) \oplus \bigoplus R(\mu^j)$. This does not give a new interpretation of the modules $R(\lambda)$ at all. We would like to have an intrinsic interpretation in $Rep(Gl(m,n))$ not using either the Khovanov algebra nor Deligne's category. Can $R(\lambda)$ be characterised as a certain direct summand in the decomposition $\{\lambda^L\} \otimes \{\lambda^R\}^\vee$?

3.32 Lemma. *Assume $\mu \leq \lambda$. Then $k(\lambda) \geq k(\mu)$.*

Proof: By [BS11] λ is (m,n)-cross iff $k(\lambda) \leq n$. Choose $\tilde{n}$ minimal such that λ is $(m,\tilde{n})$-cross. Then $k(\lambda) = \tilde{n}$. Since $\mu \leq \lambda$ μ is $(m,\tilde{n})$-cross, hence $k(\mu) \leq \tilde{n} = k(\lambda)$. qed

3.33 Lemma. *If $R(\lambda)$ is maximally atypical then $def(\lambda) \geq def(\mu_j)$ $\forall j$. If $R(\lambda)$ is maximally atypical and irreducible then $\{\lambda^L\} \otimes \{\lambda^R\}^\vee$ is completely reducible and splits into maximally atypical irreducible summands.*

Proof: $R(\lambda)$ is maximally atypical iff $rk(\lambda) = 0$. Hence $k(\lambda) \geq k(mu_j)$ implies the first statement. If $R(\lambda)$ is additionally irreducible, then $def(\mu_j) = 0$ $\forall j$. □

Remark: We will see that for $m > n$ all maximally atypical modules are irreducible. For $m = n$ the second case cannot happen.

3.8.1. Tensor ideals

3.34 Lemma. *In the tensor product*

$$R(\lambda) \otimes R(\mu) = \sum_i \kappa_{\lambda\mu}^{\nu_i} R(\nu_i)$$

all ν_i satisfy

$$k(\nu_i) \geq max(k(\lambda), k(\mu)).$$

Proof: Let $n' = max(k(\lambda), k(\mu))$. Apply $DS_{n-n'} : T_{mn} \to T_{m'n'}$. Wlog $n' = k(\lambda)$. Then $R(\lambda)$ is projective in $T_{m'n'}$. The projective modules form a tensor ideal, hence $R(\lambda) \otimes R(\mu)$ decomposes in $T_{m'n'}$ into indecomposable projective modules. Sicne the tensor product comes from the Deligne category

$$\begin{tikzcd} & Rep(Gl_{m-n}) \arrow[dl, "F_{mn}"'] \arrow[dr, "F_{m'n'}"] & \\ T_{mn} \arrow[rr, "DS_{n-n'}"] & & T_{m'n'} \end{tikzcd}$$

we have in T_{mn}

$$\sum_i \kappa_{\lambda\mu}^{\nu_i} R(\nu_i) \oplus ker(DS_{n-n'})$$

with $k(\nu_i) \geq n'$ $\forall i$. Further $ker(DS_{n-n'})$ are the mixed tensors $R(\gamma)$ with $n' < k(\gamma) \leq n$. $\square$

Example: Any irreducible summand in $R(\lambda) \otimes R(\mu)$ has atypicality $\leq n - n'$.

We denote by T^i_{mn} the subset of mixed tensors with $k(\lambda) \geq i$.

3.35 Corollary. *The T^i are tensor ideals in T_{mn} for $m > n$ and tensor ideals in $T_{mm} \cup \mathbf{1}$. We have strict inclusion*

$$T^0 \supsetneq T^1 \supsetneq \ldots \supsetneq T^n$$

with $T^0 = T$ and $T^n = Proj$.

By [Ser10] any two irreducible object of atypicality k generate the same tensor ideal in $\mathcal{R}_{mn}$. Therefore write I_k for the tensor ideal generated by any irreducible object of atypicality k. Clearly $I_0 = Proj$ and $I_n = T_n$ since it contains the identity. This gives the following filtration of $\mathcal{R}$

$$Proj = I_0 \subsetneq I_1 \subsetneq \ldots I_{n-1} \subsetneq I_n = T_n$$

with strict inclusions by [Ser10] and [Kuj11].

3.36 Lemma. *$I_k|_T = T^{n-k}$ for $m > n$ for all $k = 0, \ldots, n$. For $m = n$ $I_k|_T = T^{n-k}$ for all $k < n$.*

Proof: For any atypicality k there exists an irreducible mixed tensor with that atypicality (except for $m = n$ and $k = n$), hence $I_k|_T \subset T^{n-k}$. Conversely let $R \in T^{n-k}$. It occurs as a direct summand in $R(\lambda^L, 0) \otimes R(0, \lambda^R)$. Then $max(k(\lambda^L, 0), k(0, \lambda^R)) \leq n - k$, hence $rk(\lambda^L, 0), rk(0, \lambda^R) \leq n - k$, hence $at(R(\lambda^L, 0), at(R(0, \lambda^R)) \geq k$, hence $R \in I_l$ for any $l \geq k$. $\square$

3.37 Lemma. *For $m > n$ $I_{n-1}|_T = \mathcal{N}|_T$. For $m = n$ $\mathcal{N}|_T = T$.*

Proof: Clearly $T_1 \subset \mathcal{N}|_T$. Ket $m > n$. If $R \in \mathcal{N}|_T$, then $k(\lambda) \geq 1$. Indeed $k(\lambda) = 0$ implies $R(\lambda)$ is maximally atypical irreducible, hence $sdim R(\lambda) \neq 0$. $\square$

4. Maximally atypical modules in the space of mixed tensors

We have seen that every irreducible typical module occurs as some $R(\lambda)$. In this section we characterize among other things the irreducible maximally atypical modules in the image of F_{mn}. We give some applications regarding tensor products.

4.1. Multiplicities and tensor quotients

For $d = m - n > 0$ we have the two tensor functors

$$\begin{tikzcd} & Rep(Gl_{m-n}) \arrow[dl, "F_{m,n}"'] \arrow[dr, "F_{m-n}"] & \\ \mathcal{R}_{mn} & & Rep(Gl(m-n)) \end{tikzcd}$$

given by mapping the standard representation to the two standard representations. We also dispose over Weissauer's tensor functor: By [Wei10b] there exists a purely transcendental field extension K/k of transcendence degree n and a K-linear exact tensor functor

$$\rho : \mathcal{R}_{mn} \otimes_k K \to Rep(Gl(m-n)) \otimes svec_K.$$

By [Wei10b] each simple maximal atypical object $L(\mu)$ maps to the isotypic representation $m(\mu)\rho(V)[p(\mu)]$ where $m(\mu)$ is a positive integer, V is the ground state (see chapter 4.5) of the block of μ and $p(\mu)$ is the parity of μ. After a suitable specialisation of ρ we may assume that ρ is defined over k and maps the standard to the standard representation. Hence we ge the commutative diagramm of tensor functors (due to Deligne's universal property)

$$\begin{tikzcd}
 & Rep(Gl_{m-n}) \arrow[dl, "F_{m,n}"'] \arrow[dd, "F_{m-n}\otimes svec"] \\
Rep(GL(m,n)) \arrow[dr, "\rho"] & \\
 & Rep(GL(m-n)) \otimes svec.
\end{tikzcd}$$

Here the functor $F_{m-n} \otimes svec$ maps $R(\lambda)$ to the even representation

$$L(wt(\lambda)) \in Rep(Gl(m-n)) \subset Rep(Gl(m-n)) \otimes svec.$$

4.1 Lemma. *Let $m > n$ and $d = m - n$. Then $R(\lambda)$ has superdimension $\neq 0$ iff $l(\lambda) \leq d$.*

Proof: This follows from the commutative diagram above. Use the bijection between the highest weights of $GL(d)$ and bipartitions of length $\leq d$ to choose for any (m,n)-cross bipartition λ the irreducible highest weight module $L(wt(\lambda))$. By the commutativity the indecomposable module $R(\lambda)$ has to map to $L(wt(\lambda))$. Its superdimension is the dimension of $L(wt(\lambda))$. $\square$

Assume $m > n$. Let

$$T = \bigoplus_{r,s} T(r,s)$$

be the full subcategory of $F(m,n)$ whose objects are direct sums of $R(\lambda)$ for all bipartitions λ. It is a pseudoabelian tensor subcategory of $\mathcal{R}_{mn}$. It is closed under duals ($T(r,s)^\vee = T(s,r)$) and contains the identity. The functor of Weissauer

$$\rho : \mathcal{R}_{mn} \to Rep(Gl(m-n)) \otimes svec$$

can be restricted to T.

4.2 Theorem. *The functor $\rho_T : T \to Rep(Gl(m-n)) \otimes svec$ factorises over $T/\mathcal{N}$ and defines an equivalence of categories*

$$T/\mathcal{N} \simeq Rep(Gl(m-n)).$$

It maps the element $R(\lambda)$ to the irreducible element $L(wt(\lambda))$.

Proof: The functor will factorize if ρ_T is full 2.3. This follows from the commutative diagram since an indecomposable module maps to an irreducible module. $R(\lambda) \mapsto L(wt(\lambda))$ is forced by the commutativity of the diagram. By the bijection between highest weights of $Gl(m-n)$ and bipartitions of lenght $\leq m-n$ the functor is one-to-one on objects. Fully faithful follows from Schur's lemma in the semisimple tensor category $T/\mathcal{N}$. $\square$

Remark: Pulling back to T gives the tensor product of the modules in T up to superdimension zero. We will see that the modules $R(\lambda)$ are essentially the maximally atypical irreducible modules with completely nested cup diagram. We will give a second proof of the above theorem in the next section which does not use Weissauer's functor.

4.1.1. An alternative approach

Assume $m > n$. All bipartitions are (m,n)-cross. We provide an alternative proof that $\rho : T/\mathcal{N} \simeq Rep(Gl(m-n))$ which does not use the existence of a tensor functor $Rep(Gl(m,n)) \to Rep(Gl(m-n)) \otimes svec$.

4.3 Proposition. *Let λ be a bipartition of length $\leq m-n$. Then the weight diagram of λ does not contain any $\vee\wedge$-pair (i.e. has no caps).*

Proof: Let k be the length of $\lambda^L = (a_1, \ldots, a_k)$, hence length of $\lambda^R \leq m-n-k$. We use the notation $\lambda^R = (b_1, b_2, \ldots)$. Define the sets

$$\begin{aligned} I_\wedge &= I_\wedge^{\leq k} \cup I_\wedge^{>k} = \{a_1, \ldots, a_k - k + 1\} \cup I_\wedge^{>k} \\ I_\vee &= I_\vee^{\leq m-n-k} \cup I_\vee^{>m-n-k} \\ &= \{1-m-n-b_1, \ldots, m-n-k-(m-n)-b_{m-n-k}\} \cup I_\vee^{>k}. \end{aligned}$$

We have

$$I_\wedge^{>k} \cap I_\vee^{>m-n-k} = \emptyset.$$

More precisely:

$$I_\wedge^{>k} = [-k, -\infty), \quad I_\vee^{>m-n-k} = [-k+1, \infty).$$

Hence crosses can only appear by the intersections

$$I_1 = I_\wedge^{\leq k} \cap I_\vee^{\leq m-n-k},\ I_2 = I_\wedge^{>k} \cap I_\vee^{\leq m-n-k},\ I_3 = I_\vee^{>m-n-k} \cap I_\wedge^{\leq k}.$$

Note that

$$I_1 \cup I_2 \cup I_3 \subseteq (I_\wedge^{\leq k} \cup I_\vee^{\leq m-n-k}).$$

However any λ has at least $m-n$ crosses. Since $|I_\wedge^{\leq k} \cup I_\vee^{\leq m-n-k}| = m-n$ we obtain that the crosses are at the positions

$$I_\wedge^{\leq k} \cup I_\vee^{\leq m-n-k}.$$

This implies $def(\lambda) = 0$: Since $I_\vee^{>m-n-k} > I_\wedge^{>k}$ a $\vee\wedge$-pair is not possible. $\square$

4.4 Corollary. *$lift_d(\lambda) = \lambda$ for all d.*

4.5 Corollary. *Let λ, ν be bipartitions of lenght $\leq m-n$. Then their tensor product is given by the Littlewood-Richardson rule for $Gl(m-n)$ up to superdimension 0. More precisely*

$$R(\lambda) \otimes R(\mu) = \bigoplus_{\nu,\ l(\nu) \leq m-n} c^{wt(\nu)}_{wt(\lambda),wt(\mu)} R(\nu) \ mod\ \mathcal{N}.$$

Proof: (cf the proof of 7.1.1 in [CW11]) Let $\nu_1, \ldots \nu_k$ bipartition such that

$$\lambda\mu = \nu_1 + \ldots \nu_k$$

in R_t. Since $lift(\lambda) = \lambda,\ lift(\mu) = \mu$ we may assume mod $\mathcal{N}$ that all ν_i have length $\leq m-n = d$. So d fulfills $d \geq l(\nu_i)$ for all i and $lift_d$ fixes $\lambda, \mu, \nu_1, \ldots \nu_k$. Hence $\lambda\mu = \nu_1 + \ldots \nu_k$ holds in R_d as well. Using the tensor functor $F_d : Rep(Gl_d) \to Rep(Gl(d))$ which maps λ to $L(wt(\lambda))$ we obtain

$$\begin{aligned} L(wt(\lambda)) \otimes L(wt(\mu)) &= L(wt(\nu_1)) \oplus \ldots \oplus L(wt(\nu_k)) \\ &= \bigoplus_{\nu, l(\nu) \leq m-n} c^{wt(\nu)}_{wt(\lambda),wt(\mu)} R(\nu) \end{aligned}$$

by the Littlewood-Richardson rule in $Rep(Gl(d))$. Taking the preimage one obtains modulo $\mathcal{N}$ the result. $\square$

Remark: The proof show more: For bipartitions of lenght $\leq k$ and without cups - up to contributions of higher length one gets a $Gl(k)$-tensor product. However some contributions might not be (m,n)-cross.

4.6 Corollary. *Let λ and μ be such that $l(\lambda) + l(\mu) \leq m - n$. Then $R(\lambda) \otimes R(\mu)$ splits completely into irreducible maximally atypical modules. The decomposition rule is given by the Littlewood-Richardson rule for $Gl(m-n)$.*

Example: Consider the irreducible representation $\Lambda^{m-n}(X) = R(1^{m-n}; 0)$ and tensor products $R(1^{m-n}; 0) \otimes R(\lambda)$ for $l(\lambda) \leq m - n$. The weight of (1^{m-n}) for $Gl(m-n)$ is $(1, \ldots, 1)$, so $\Lambda^{m-n}(X) \otimes L(\lambda) = L(\lambda_1 + 1, \ldots, \lambda_{m-n} + 1)$ in $Rep(Gl(m-n))$. If $R(\lambda) = R(a_1, \ldots, a_k; b_{k+1}, \ldots, b_{k+r})$ for $k + r \leq m - n$, then tensoring with Λ^{m-n} gives

$$R(a_1 + 1, \ldots, a_k + 1, 1^{(m-n)-(k+r)} \; ; \; b_{k+1} - 1, \ldots, b_{m-n} - 1).$$

Recall that any $R(\lambda)$ with $def(\lambda) = 0$ is irreducible. Hence

4.7 Corollary. *Let $m > n$ and $l(\lambda) \leq m - n$. Then $R(\lambda)$ is irreducible.*

It is now easy to recover the theorem from the previous section. Since

$$F_{mn} : Rep(Gl_{m-n}) \to \mathcal{R}_{mn}$$

has its image in T we can consider the diagram

$$\begin{tikzcd}
& Rep(Gl_{m-n}) \arrow[dl, "F_{m,n}"'] \arrow[dd] \\
T \arrow[d] & \\
T/\mathcal{N} \arrow[r, dotted] & Rep(Gl(m-n)).
\end{tikzcd}$$

Using the bijection between the irreducible elements $R(\lambda)$ and the irreducible elements in $Rep(GL(m-n))$, we define the lower horizontal functor by putting $R(\lambda) \mapsto L(wt(\lambda))$ on objects. Since both categories are semisimple tensor categories, Schur's lemma holds and the functor sends the morphism $id : R(\lambda) \to R(\lambda)$ to $id : L(wt(\lambda)) \to L(wt(\lambda))$. The results on the tensor products show that this defines a tensor functor. It is clearly fully faithful.

4.2. Maximally atypical irreducible modules

In two cases the highest weight $\lambda^\dagger$ is trivial to determine, namely in the case of covariant and contravariant modules. The covariant modules are the irreducible modules in the decomposition $T(r, 0) = \bigoplus_{\lambda \in H(m,n), |\lambda| \leq r} \{\lambda\}$, where λ

runs through the set of partitions which obey the (m,n)-hook condition and $\{\lambda\} = S_\lambda(st)$. The contravariant modules are the duals of the covariant ones: $T(0,s) = \bigoplus_{\rho \in H(m,n), |\rho| \leq s} \{\rho\}^\vee$. The highest weight of $\{\lambda\} =: L(\mu)$ is given as follows: Put $\mu_i = \lambda_i$ for $i = 1, \ldots, m$ and $\mu_{m+i} = max(0, \lambda_i^* - m)$ for $i = 1, \ldots, m$, cf [Ser85], [BR87].

We will show in this section:

4.8 Proposition. *Let $m > n$. Every maximally atypical Kostant module is a Berezin twist of an irreducible mixed tensor.*

We will show this by analysing the effect of θ on a bipartition of length $\leq m-n$. Assume $m > n$, $l(\lambda) \leq m-n$. Assume that

$$\lambda = ((a_1, \ldots, a_k), (b_{k+1}, \ldots, b_{m-n}))$$

and assume additionally a_1 and b_{k+1} to be greater zero (otherwise we have covariant or contravariant modules). Recall that the crosses are at the positions

$$I_\wedge^{\leq k} \cup I_\vee^{\leq m-n-k}.$$

We have

$$I_\wedge = \{a_1, a_2 - 1, \ldots, a_k - (k-1), -k, -(k+1), \ldots\}$$
$$I_\vee = \{1 - (m-n) - b_{k+1}, \ldots, (m-n) - k - (m-n) - b_{m-n}, -k+1, +k+2, \ldots\}.$$

The $m-n$ crosses are at the positions

$$a_1, a_2 - 1, \ldots, a_k - (k-1), 1 - (m-n) - b_{k+1}, \ldots, -k - b_{m-n}.$$

Since $a_1, \ldots, a_k, b_{k+1}, \ldots, b_{m-n}$ are arbitrary, the position of the crosses is arbitrary. Note that the crosses coming from the a_i are to the right of the b_i-crosses: $a_k - (k-1) > -k - b_{m-n}$. The position of the $\vee$'s: We have $a_1 = \lambda_1^L$, hence there are $a_1 + n$ switches in the free positions left from the cross at a_1. To know the position of the $\vee$'s, the change from the $\wedge$ to the $\vee$'s has to be known: In fact $I_\wedge^{>k} = [-k, -\infty)$, $I_\vee^{>m-n-k} = [-k+1, \infty)$, hence the free positions $\leq -k$ have $\wedge$'s, the free ones $\geq -k+1$ have $\vee$'s. In the free slots $\geq -k+1$ λ has $\vee$s. These get turned into $\wedge$s. This are precisely a_1 free slots since there are k-crosses between a_1 and $-k$. The next n free slots $\leq -k$ contain $\wedge$s. These get turned into $\vee$s. After that all free slots contain a $\wedge$. The cup diagram is completely nested: All the $\vee$ are in the first n free slots to the left of $-k$.

4.9 Corollary. *The irreducible maximally atypical modules in T (and are not co- or contravariant) are precisely the highest weights with weight diagram as follows: The weight diagram has $m-n$-crosses, n $\vee$s and there exists some negative number $-k \in \{-1, \ldots, -(m-n-1)\}$ such that all the $\vee$ are in the free vertices $\leq -k$ and k crosses lie to the right of $-k$.*

4.10 Corollary. *In $Sl(m|n)$ $L(\lambda) \in im(F_{m,n}) \Leftrightarrow L(\lambda)$ is a Kostant module. In the $Gl(m|n)$ case it is correct up to a twist with a suitable power of the Berezinian.*

In particular the tensor product of two such modules can be decomposed explicitely.

Explicite Berezin twist. The explicite desription of $\lambda \to \lambda^\dagger$ tells how to twist with the Berezin to obtain an irreducible module in T. Given a maximally atypical irreducible module with completely nested cup diagram. Let $\vee_m$ be the rightmost $\vee$. Count the crosses with labels to the right of $\vee_m$. Name that number k. Then move $\vee_m$ with a Berezin twist to the position $-k$. An inspection of the algorithm above shows that this is an irreducible module in T.

Example: Consider the highest weight $\lambda = (12, 12, 10, 10, 10, 10, 0|-11, -11, -12)$ of $Gl(7, 3)$ It is maximally atypical with rightmost $\vee$ at position 8 and two crosses at position 11 and 12 to the right. Hence twist $L(\mu)$ with Ber^{-10} to move $\vee$ to position -2 and obtain $\tilde{\mu} = (2, 2, 0, 0, 0, 0, -10| - 1, -1, -2)$. The two crosses are now at position 1 and 2 and give $a_1 = 2,\ a_2 = 2$. The two other crosses are at the positions -3 and -16 giving $b_1 = 13$ and $b_2 = 1$, hence

$$Ber^{-10} \otimes L(\mu) = R(2, 2; 13, 1).$$

For two weights $\lambda = (\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n})$ and $\mu = (\mu_1, \ldots, \mu_m \mid \mu_{m+1}, \ldots, \mu_{m+n})$ say that $\lambda \succeq \mu$ if there exists $i \in \{1, \ldots, m\}$ with the property $\lambda_j = \mu_j \forall j < i$ and $\lambda_i > \mu_i$.

4.11 Lemma. *Let $R(\lambda)$ be maximally atypical irreducible. Then $R(\lambda) = L(\lambda^\dagger)$ with $L(\lambda^\dagger) \succ L(\mu_j^\dagger)$ for all j.*

Proof: Define $I_x^{max}(\lambda)$ = largest label with an x or $\vee$. We claim $I_x^{max}(\lambda) \geq I_x^{max}(\mu_j)$ for all j. The position of the crosses is given by the elements in $I_\wedge^{\leq k} \cup I_\vee^{\leq m-n-k}$. Since $\lambda_i^L \geq \mu_{j,i}^L$ $I_\wedge^{\leq k}(\mu_j) \leq I_\wedge^{\leq k}(\lambda)$. There are k crosses to the right of $-k$ (meaning for k_λ and k_{μ_j}). Hence for the first k_{μ_j} $\lambda_i \geq \mu_{j,i}$ for all $i \in \{1, \ldots, k_{\mu_j}\}$. This holds in fact for the first k_λ-coordinates: There are k_λ crosses at positions $> -k_\lambda$ k_{μ_j} crosses at positions $> -k_{\mu_j}$. The next $k_\lambda - k_{\mu_j}$ positions with crosses or

$\vee$'s in μ_j are then at the positions $-k_{\mu_j}, -k_{\mu_j} - 1, \ldots, -k_\lambda + 1$. Since there exists at least one i with $\lambda_i^L > \mu_{j,i}$ the claim follows. $\square$

So the maximally atypical $R(\lambda)$ for $m > n$ of $sdim \neq 0$ could be characterized as follows: Take all the tensor products of two (m,n)-Hook partitions λ^L, λ^R such that (λ^L, λ^R) is (m,n)-cross. Then the $R(\lambda)$ are the indecomposable modules in the decomposition $\{\lambda^L\} \otimes \{\lambda^R\}^\vee$ which satisfy $R(\lambda) = L(\lambda^\dagger) \succ L(\mu_j^\dagger)$ for all j.

4.3. Multiplicity 1

Given two λ, μ Kostant weights we shift both into T

$$L(\lambda) \otimes Ber^{\lambda'} = L(\tilde{\lambda}) \in T$$
$$L(\mu) \otimes Ber^{\mu'} = L(\tilde{\mu}) \in T$$

where $\lambda',\ \mu'$ only depend on the position of the unique segment. Therefore

$$\begin{aligned} L(\lambda) \otimes L(\mu) &= (L(\tilde{\lambda}) \otimes L(\tilde{\mu}))\ \otimes\ (Ber^{\lambda'} \otimes Ber^{\mu'}) \\ &= \bigoplus_\nu c^\nu_{\tilde{\lambda},\tilde{\mu}} L(\nu)) \otimes Ber^{\lambda'+\mu'} \end{aligned}$$

for certain coefficients $c^\nu_{\tilde{\lambda},\tilde{\mu}}$ which can be calculated explicitely from [CW11]. In the case $Gl(m|1)$ and $Sl(m|1)$ every weight is a Kostant weight. Since Ber is trivial in the Sl-case we obtain:

4.12 Proposition. *Up to a twist of a suitable power of Ber every atypical irreducible module of $Gl(m|1)$ is in T. Every irreducible atypical module of $Sl(m|1)$ is in T.*

Example: Let us study the $Gl(2|1)$-case. Since $l(\lambda) \leq 1$, the irreducible atypical mixed tensors are the covariant and contravariant tensors. The irreducible tensor products with highest weight $(\lambda_1, \lambda_2|\lambda_3)$ is atypical iff either $\lambda_2 = -\lambda_3$ or $\lambda_3 = -\lambda_1 - 1$. The covariant module $R(a;0)$ has highest weight $(a,0|0)$ and the contravariant module $R(0;b)$ has highest weight $(0, -b+1| -1)$. The modules with highest weights $(\lambda_1, \lambda_2| -\lambda_2)$ are Berezin twists of covariant modules and the modules with highest weights $(\lambda_1, \lambda_2| -\lambda_1 - 1)$ are Berezin twists of contravariant modules.

By [Ger98] and Su the indecomposable modules are the (Anti-)ZigZag-modules and the projective hulls of the irreducible atypical representations.

4.13 Corollary. *$Gl(m|1)$-case: If $l(\lambda) \leq m-1$, then $R(\lambda^\dagger$ is irreducible singly atypical. If $l(\lambda) > m-1$ and $def(\lambda) = 0$ then $R(\lambda) = L(\lambda^\dagger)$ is typical. If λ is any bipartition with $def(\lambda) = 1$ then $R(\lambda) = P(\lambda^\dagger)$.*

4.14 Corollary. *In the decomposition $L(\lambda) \otimes L(\mu)$ between two irreducible $Gl(m,1)$-modules no ZigZag module $Z^l(a)$ with $l \geq 2$ appears.*

Example: The conditions $\vee \leq 0$ and $\lambda^\dagger$ atypical imply $1 - m - \lambda_{m+1} \leq 0$ (atypicality condition), hence $1 - m \leq \lambda_{m+1}$. Additionally $\vee$ cannot be too much to the left: The leftmost possible position of $\vee$ is at $-(m-1)$. Hence $1 - m + (m-1) \geq \lambda_{m+1}$, hence $0 \geq \lambda_{m+1}$. Hence a necessary condition for $\lambda^\dagger$ to be in T is $1 - m \leq \lambda^\dagger \leq 0$.

Tensor products. Any irreducible $Gl(m,1)$-module is up to an explicit Berezin-Twist in T. So the tensor product formula in Deligne's category and the description of the image of $F_{m,1}$ solves the problem of decomposing any two irreducible $Gl(m,1)$-representations.

Example 1: We compute the tensor product between two irreducible atpyical $Gl(4,1)$-modules, namely $L(2,0,0,0|0) \otimes L(1,0,0,0|-1)$. Applying θ^{-1} we see that the corresponding bipartitions are $(2;0)$ and $(1;1)$. Since $def = 0$ we only have to compute $(2;0) \otimes (1;1)$ in R_3. By [CW11], p.35 we have

$$(2;0) \otimes (1;1) = ((2,1);1) + (3;1) + (1^2;0) + (2;0)$$

for $\delta = 3$ in R_δ. Each of these bipartitions describes an irreducible singly atypical module. We have

$$\theta(3;1) = (3,0,0,0|-1), \ \theta((2,1);1) = (2,1,0,0|-1).$$

Hence

$$\begin{aligned} L(2,0,0,0|0) \otimes L(1,0,0,0|-1) = & L(1,1,0,0|0) \oplus L(2,0,0,0|0) \oplus L(3,0,0,0|-1) \\ & \oplus L(2,1,0,0|-1) \end{aligned}$$

in $Rep(Gl(4,1))$.

Example 2: One could hope that the tensor product of two atypical irreducible modules splits into a sum of irreducible atypical and typical modules. This is wrong: Take $Gl(4,1)$, $\lambda^L = (3,2,1)$, $\lambda^R = (1,1)$. Then $R(3,2,1;1,1)$ is projective.

ZigZag modules of length greater than 1 never occur in the image of F_{m1}. However the tensor product between an indecomposable projective module with a ZigZag-module is easily reduced to the known cases by the following well-known fact:

4.15 Proposition. *Let P be projective and M any module. Then $P \otimes M = \bigoplus_i P \otimes M_i$ where the sum runs over the composition factors M_i of M.*

Proof: Use induction on the length of M. If M is of length n consider an sequence

$$0 \longrightarrow M_i \longrightarrow M \longrightarrow M' \longrightarrow 0$$

with $length(M') = n-1$. Tensoring with P and using that $Proj$ is a tensor ideal we see that the sequence splits. $\square$

4.16 Lemma. *Let P be an indecomposable projective $Gl(m,1)$-Module. Then*

$$P \otimes Z^r(a) = \bigoplus_{a_i} P \otimes L(a_i)$$
$$P \otimes \overline{Z}^r(a) = \bigoplus_{a_i} P \otimes L(a_i)$$

where the sums run over the composition factors $L(a_i)$ of $Z^r(a)$ respectively $\overline{Z}^r(a)$.

All in all the only remaining unknown tensor products in the $Gl(m,1)$-case are the tensor products $Z^r(a) \otimes Z^s(b)$ and vice versa for the Anti-ZigZag-modules.

4.4. Maximally atypical $R(\lambda)$ for $m = n$

We have seen that all maximally atypical modules of superdimension $\neq 0$ are irreducible and that for $m = n$ no maximally atypical irreducible modules are in T. In this section we characterise the maximally atypical modules for $m = n$.
Assume from now on that $\lambda^\dagger$ is in the maximal atypical block, i.e. the weight diagram has no x, no $\circ$ and exactly m $\vee$.

4.17 Lemma. *If $R(\lambda, \rho)$ is maximally atypical, then $\rho = \lambda^*$. Conversely $R(\lambda^L, (\lambda^L)^*)$ is maximally atypical.*

Proof: We start with the following observation: Since there are no $\circ$:

$$I_\vee \cup I_\wedge = \mathbb{Z}.$$

Since there are no x

$$I_\vee \cap I_\wedge = \emptyset.$$

Hence ρ and λ determine each other uniquely. The biggest $\wedge$ is at position λ_1. We use the same same notation as in 3.19 and recall $\delta_i = \Delta^*_{r-i}$ and $\Delta_i = \delta^*_{r-i}$. Note further that the leftmost $\vee$ is at the vertex

$$\lambda^L_1 - \sum \delta_i - \sum \Delta_i + 1 = \lambda^L_1 - l(\lambda^L) - \lambda^L_1 + 1 = 1 - (\lambda^L)^*_1. \quad \square$$

4.18 Corollary. *$T(r,s)$ contains a maximally atypical summand only for $r = s$.*

Proof: By [BS11] and the characterisation of maximally atypical R

$$pr_\Gamma T(r,s) = \bigoplus R(\lambda, \lambda^*)$$

where $|\lambda| = r - t,\ |\lambda^*| = s - t$. Since $|\lambda| = |\lambda^*|$ this can only happen for $r = s$. □

Notation: From now on we always write $R(\lambda)$ where λ is a partition such that (λ, λ^*) is (m,m)-cross.

4.19 Lemma. *Assume $l(\lambda) \leq m$ and $def(\lambda) = m$. Then $\lambda^\dagger = [\lambda]^0$.*

Proof: This is easily seen using the algorithm of determining $[\lambda]^0$ given in [BS10a], page 36. and the fact that the positions of all $\vee$'s is determined due to maximal defect. □

Examples: $\lambda = (3,2,1)$, then $[\lambda]^0 = [2,1]$; $\lambda = (3,3,1)$, then $[\lambda]^0 = [1,1]$.

Remark. Let λ be any partition and let β the intersection of the Young diagram with the box of lenght m and width m with upper left corner at position $(0,0)$. Then the Young diagram has the following shape

$$\lambda = \begin{pmatrix} \beta & \alpha \\ \gamma & \end{pmatrix}.$$

Hence if $l(\lambda) \leq m$ then $\gamma = 0$. Consider the following weight $A_\lambda := [\alpha + \beta + (\gamma^*)^v]$. For $l(\lambda) \leq m$ this is nothing but the weight $[\lambda]$. $\mathbb{A}_\lambda = \{\lambda\} \otimes \{\lambda^*\}^\vee$ always contains the maximal atypical constituent $[\lambda] = [\alpha + \beta + (\gamma^*)^\nu]$ as heighest weight representation with multiplicity 1 [Wei10c]. Since the restriction of $\mathbb{A}_\lambda$ to the maximal atypical block decomposes as

$$pr_\Gamma \mathbb{A}_\lambda = R(\lambda) \oplus \bigoplus R(\lambda^i)$$

for partitions λ^i with $\lambda_j \geq \lambda^i_j$ for all i,j, it seems likely to conjecture that the unique constituent of highest weight in $R(\lambda)$ is given by $A_\lambda := [\alpha + \beta + (\gamma^*)^v]$. This is wrong, as the following example shows: Take $Gl(4,4)$ and choose $\lambda = (3^4, 1^2)$. Then $\lambda^\dagger = [1,1,1,0]$, $A_\lambda = [3,3,1,1]$ but $[\alpha + \beta + (\gamma^*)^v] = [3,3,3,1]$. In particular $R(\lambda)$ cannot be characterised as the constituent of highest weight in the $\mathbb{A}_\lambda$-decomposition.

4.4.1. The involution I

Recall that the Tannaka dual of an indecomposable element in T is given by $R(\lambda^L, \lambda^R)^\vee = R(\lambda^R, \lambda^L)$. Similarly we define

$$IR(\lambda^L, \lambda^R) := R(\lambda^{R^*}, \lambda^{L^*}).$$

4.20 Lemma. *This is a well-defined operation on T for $m = n$ (ie. $(\lambda^{R^*}, \lambda^{L^*})$ is again (m,m)-cross). I is an involution and commutes with Tannaka duality. I is the identity iff $R(\lambda)$ is maximally atypical.*

Proof: Let $i \in 1, \ldots, m$ have the property $\lambda^L_{i+1} + \lambda^R_{m-i+1} \leq m$, so $\lambda^L_{i+1} \leq k$ and $\lambda^R_{m-i+1} \leq m - k$ for some k. Then $(\lambda^L_{k+1})^* \leq i$ and $(\lambda^R_{m-k+1})^* \leq m - i$, hence $(\lambda^L_{k+1})^* + (\lambda^R_{m-k+1})^* \leq m$. The other statements are clear. $\square$

Remark: For $m > n$ the bipartition $((\lambda^R)^*, (\lambda^L)^*)$ may fail to be (m,n)-cross. Take for instance $Gl(7|3)$, $\lambda^L = (10, 10, 3, 3, 3, 3, 3, 3)$ and $\lambda^R = (10, 10, 10, 10)$. Then (λ^L, λ^R) is $(7,3)$-cross, but $I\lambda$ is not.

4.21 Lemma. *I preserves dimensions.*

Proof: Since the dimension is preserved under dualising $(\lambda^L, \lambda^R) \mapsto (\lambda^R, \lambda^L)$, we only have to take care of $(\lambda^L, \lambda^R) \mapsto (\lambda^{L*}, \lambda^{R*})$. By [CW11], (43)

$$dim R(\lambda) = \sum_{\mu \subset \lambda} D_{\lambda,\mu} d_\mu$$

where d_μ is obtained from the composite supersymmetric Schur polynomial $s_\mu(\underline{x}, \underline{y})$, $\underline{x} = (x_1, \ldots, x_m)$, $\underline{y} = (y_1, \ldots, y_m)$ by setting $x_i = 1 = y_i \ \forall i = 1, \ldots, m$. By [Moe06], (2.39) $s_\mu(x|y) = s_{\mu^*}(y|x)$, hence $d_\mu = d_{\mu^*}$. Let $\lambda \vdash (r,s)$. Then $\lambda^* \vdash (r,s)$. By [CDV11] the number $D_{\lambda\mu}$ is the decomposition number $[\Delta_{r,s}(\lambda) : L_{r,s}(\mu)]$ where $\Delta_{r,s}$ is a cell module for the walled Brauer algebra B_{rs}. It is clear that $D_{\lambda\mu} = D_{\lambda^*,\mu^*}$, hence if $\sum_{\mu \subset \lambda} D_{\lambda\mu} d_\mu = d_{\mu_1} + \ldots + d_{\mu_r}$, then $\sum_{\mu' \subset \lambda^*} D_{\lambda^* \mu'} d_{\mu'} = d_{\mu_1^*} + \ldots d_{\mu_r^*}$.

Example: $I([i; 1^j]) = [j; 1^i]$, hence $\lambda^\dagger = [i, 0|0, -j]$ and $I\lambda^\dagger = [j, 0|0, -i]$.

In the typical case the interpretation is as follows: The irreducible module $L(\lambda_1, \ldots, \lambda_m | \lambda_{m+1}, \ldots, \lambda_{2m})$ is induced from the irreducible $Gl(m) \times Gl(m)$-module $L(\lambda_1, \ldots, \lambda_m) \otimes L(\lambda_{m+1}, \ldots, \lambda_{2m})$. The dual of the irreducible $Gl(m)$-representation $L(\lambda_1, \ldots, \lambda_m)$ is given by $L(-\lambda_m, \ldots, -\lambda_1)$. Hence $IR(\lambda)$ is just obtained by taking the $Gl(m) \times Gl(m)$-dual and then inducing.

4.5. Lower Atypicality

The favourable properties of mixed tensors (for instance the behaviour under DS or the tensor product rule) motivate the question which irreducible modules lie in T. The results of the maximally atypical case raise the question whether every Kostant module is a Berezin-twist of an irreducible mixed tensor. The answer is negative. A counterexample will be given below in the $Gl(2|2)$-case. We content ourselves to prove two existence results about irreducible mixed tensors which are not maximally atypical.

4.5.1. Ground states

We show that in the $\mathcal{R}_n$-case every ground state of a block of atypicality $< n$ is a Berezin-twist of an irreducible mixed tensor.

4.22 Proposition. *Let*

$$L = L(\lambda^\dagger) = L(\lambda_1, \ldots, \lambda_{n-i}, 0, \ldots, 0|0, \ldots, 0, \lambda_{n+i+1}, \ldots, \lambda_{2n})$$

be any irreducible i-fold atypical representation. Then L is a mixed tensor $L = R(\lambda)$ for a unique bipartition of defect 0 and $rk = n - i$. Under DS

$$DS(L) = R_{n-1}(\lambda) = L(\bar{\lambda}^\dagger)$$

where $\bar{\lambda}^\dagger$ is obtained from $\lambda^\dagger$ by removing the two innermost zeros corresponding to $\lambda^\dagger_n$ and $\lambda^\dagger_{n+1}$.

Proof: We apply θ^{-1} to λ. It transforms the weight diagram of λ into some other weight diagram which might not be the weight diagram of a bipartition. However if the resulting weight diagram is the weight diagram of an (n, n)-cross bipartition of defect 0, then $\theta(\lambda) = \lambda^\dagger$ and $def(\lambda) = L(\lambda^\dagger)$. For $\lambda^\dagger$

$$I_x = \{\lambda_1, \lambda_2 - 1, \ldots, \lambda_{n-i} - (n-i) + 1, -n + i, \ldots, -n + 1\}$$
$$I_\circ = \{1 - n, 2 - n, \ldots, i - n, i + 1 - n - \lambda_{n+1+i}, \ldots, -\lambda_{2n}\}.$$

Then $I_x \cap I_\circ = \{-n + 1, \ldots, -n + 1\}$ (since the atypicality is i) and the $n - i$ crosses are at the positions $\lambda_1, \lambda_2 - 1, \ldots, \lambda_{n-i} - (n-i) + 1$ and the $n - i$ circles at the positions $i + 1 - n - \lambda_{n+1+i}, \ldots, -\lambda_{2n}$. Define M, T, X as above. Note that $k(\lambda^\dagger) = n - i$. We distinguish two cases, either $T = n - i + 1$ or $t = M + 1$. Assume first $M + 1 \ \leq n - i + 1$. Switch all free labels at positions $\geq T$ and the first $n - (n - i) = i$ free labels at positions $< T$. By assumption the $2n - 2i$ crosses and circles lie at positions $> i - n$ and $< n - i + 1$. However there are exactly $2n - 2i$ such positions. Hence the switches at positions $< T$ turn exactly the $n \vee$

at positions $i-n, \ldots, 1-n$ into $\wedge$'s. In the second case $T = M+1 > n-i+1$ switch the first $M+n-2(n-i)$ free labels at positions $< T$. There are exactly $M+n-i$ positions between M and $i-n$, $M-n+2i$ switches and $2n-2i$ crosses and circles between $i-n$ and T. This results in $M-n+i$ free positions between $i-n$ and T. The remaining i switches switch the i $\vee$'s. Hence in both cases θ^{-1} transforms the weight diagram into a weight diagram where the rightmost $\wedge$ is at position $i-n$ and the leftmost $\vee$ is at the first free position $> i-n$ and all labels at positions $\geq T$ are given by $\vee$'s. This is the weigth diagram of a bipartition of defect 0 and rank $n-i$. Indeed the labelling defines the two sets $I_\wedge$ and $I_\vee$ and this defines a two tuples $\lambda^L = (\lambda_1^L, \lambda_2^L, \ldots)$ and $\lambda^R = (\lambda_1^R, \lambda_2^R, \ldots)$. The positioning of the $\wedge$'s implies that $\lambda_{n-i+1}^L = 0$ and the positioning of the $\vee$'s implies $\lambda_t^R = 0$. Clearly $\lambda_1^L = \lambda_1 > 0$ and $\lambda_1^R \geq 0$. Hence the pair $\lambda := (\lambda^L, \lambda^R)$ is a bipartition (of defect 0 and rank $n-i$) and $\theta(\lambda) = \lambda^\dagger$. It remains to compute the highest weight of $R_{n-1}(\lambda)$. The two sets $I_\vee$ and $I_\wedge$ and accordingly the weight diagram of λ do not depend on n. Neither do t, m, s and the switches at positions $\geq t$. To get $\lambda^\dagger$ in $\mathcal{R}_n$ from λ we switch the first $s+n-(n-i)$ free labels $< t$. To get $\lambda^\dagger$ in $\mathcal{R}_{n-1}$ from λ we switch the first $s+(n-1)-(n-i)$ free labels $< t$. This results in removing the leftmost $\vee$ at position $1-n$. $\square$

Following Weissauer we call a groundstate of a block of atypicality i an irreducible representation of this block such that its weight is i) a Kostant weight and ii) all $\vee$'s are to the left of all x's and all $\circ$'s. Such a weight is of the form

$$\lambda = (\lambda_1, \ldots, \lambda_{n-i}, \lambda_n, \ldots, \lambda_n \mid -\lambda_n, \ldots, -\lambda_n, \lambda_{n+i+1}, \ldots, \lambda_{2n})$$

with $\lambda_n \leq min(\lambda_{n-i}, -\lambda_{n+1+i})$.

4.23 Corollary. *Every Groundstate of a block of atypicality $i, i < n$, is a Berezin-twist of an irreducible mixed tensor.*

4.24 Corollary. *Denote by $\bar{\lambda}$ the weight in $\mathcal{R}_{n-1}$ obtained from λ by removing the innermost $(\ldots, \lambda_n \mid \lambda_{n+1}, \ldots)$. Then $DS(L(\lambda)) = L(\bar{\lambda})[\lambda_{n+1}]$ for a groundstate of a block.*

4.5.2. Twisted symmetric powers

We classify the $m-1$-times irreducible atypical modules appearing as a direct summand in $T \subset \mathcal{R}_m$.

Since θ preserves the x and $\circ$ positions, $I_\vee \cap I_\wedge = \{point\}$ and $I_\vee \cup I_\wedge = \mathbb{Z} \setminus \{point\}$. Further $I_\wedge < I_\vee$ with the exception of a single point. We determine the possible λ^L. Every jump $\lambda_i^L > \lambda_{i+1}^L$ in λ^L will give a gap in the numberline. Exactly one gap (one $\circ$) has to appear. Since $def(\lambda) = 0$ no $\vee$ may fill the resulting gaps in the numberline. Hence there can be either at most one jump of size 1 in λ^L, leading to $\lambda^L = (1^i)$ for some $i \geq 0$, or the λ_1^L position is given by a cross, leading to $\lambda^L = (i, 0, \ldots)$.

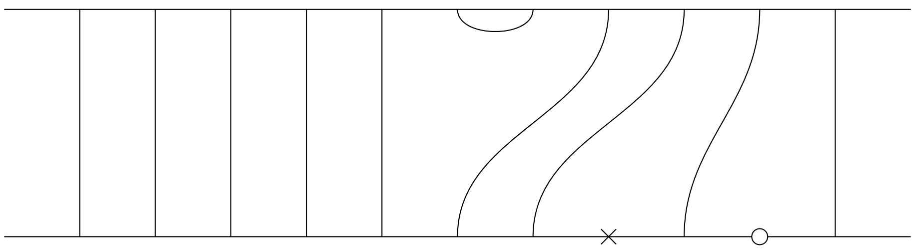
4.25 Lemma. *The $(m-1)$-times atypical irreducible modules are precisely the*

- $[(i), (1^j)]$, $i \geq 0,\ j \neq i,\ (i,j) \neq (0,0)$ *or the duals*
- $[(1^j), (i)]$ *with the same conditions.*

We call the $(i; 1^j)$ **twisted symmetric powers**. We compute their highest weights. Clearly

$$I_\wedge = \{i, -1, -2, \ldots\};\ I_\vee = \{0, 1, 2, \ldots, j_1, j+1, j+2, \ldots\}.$$

Since $k(\lambda) = 1$ the resulting matching looks like



with the x at position i and the $\circ$ at position j. We obtain $\lambda^\dagger$ by switching all free positions $\geq -m+1$. The first $m-1$ are $\wedge$'s and get turned to $\vee$'s. Hence

$$I_x = \{-m+1, \ldots, -1, i\},\ I_\circ = \{-m+1, \ldots, -1, j\}.$$

This implies $\lambda^\dagger = (i, 0, \ldots, 0|0, \ldots, 0, -j)$, hence

$$R(\lambda) = L(\lambda^\dagger) = L(i, 0, \ldots, 0|0, \ldots, 0, -j).$$

Duals. Now we compute the highest weights of the duals. Here

$$I_\wedge = \{1, 0, -1, \ldots, -j+2, -j, -j-1, \ldots\};\ I_\vee = \{1-i, 2, 3, 4, \ldots\}.$$

All labels at positions ≥ 2 will be switched as well as the first $m-1$ free positions < 2. We have to make a case distinction depending on whether x or $\circ$ lies at one of the first $m-1$ positions ≥ 1.

Case a): The first $m-1$ positions ≥ 1 are only $\wedge$'s. Then $I_x = \{1, 0, \ldots, -m + 3, 1-i\}$ and $I_\circ = \{1, 0, \ldots, -m+3, 1-j\}$, hence

$$\lambda^\dagger = (1, 1, \ldots, 1, m-i| - m + j, -1, \ldots, -1).$$

This case occurs for $i > m-2, \ j > m-2$.

Case b): Both x and $\circ$ lie in that intervall. Then $I_x = \{1, 0, \ldots, -j + 2, -j, \ldots, -m+1\}$ and $I_\circ = \{1, 0, \ldots, 2-i, -i, \ldots, -m+1\}$, hence

$$\lambda^\dagger = (1, 1, \ldots, 1, 0, \ldots, 0|0, 0, \ldots, 0, -1, \ldots, -1)$$

where the gaps occur from positions j to $j+1$ and from $2m-i-1$ to $2m-i$.

Case c): x is in the intervall, $\circ$ not. Then

$$\lambda^\dagger = (1, 1, \ldots, 1| - m + j, 0, \ldots, 0, -1, -1, \ldots, -1).$$

Case d): $\circ$ in the intervall, x not. Then

$$\lambda^\dagger = (1, \ldots, 1, m-i| - 1, \ldots, -1).$$

4.5.3. The $Gl(2,2)$-case

We specialise to $Gl(2,2)$. We will see that most singly atypical irreducible $Gl(2,2)$-modules are Berezin-twists of a twisted symmetric power.

A stupid calculation shows that the atypicality condition for $\lambda = (\lambda_1, \lambda_2|\lambda_3, \lambda_4)$ are: 1) $\lambda_1 + \lambda_3 + 1 = 0$, 2) $\lambda_1 + \lambda_4 = 0$, 3) $\lambda_2 + \lambda_3 = 0$, 4) $\lambda_2 + \lambda_4 - 1 = 0$. So a singly atypical weight is of one of the following four types:

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2| - \lambda_1 - 1, \lambda_4) \\ \lambda &= (\lambda_1, \lambda_2|\lambda_3, -\lambda_1), \\ \lambda &= (\lambda_1, \lambda_2| - \lambda_2, \lambda_4) \\ \lambda &= (\lambda_1, \lambda_2|\lambda_3, -\lambda_2 + 1) \end{aligned}$$

Evidently any λ with $\lambda_1 - \lambda_2 \neq \lambda_3 - \lambda_4$ is a Berezin twist of a twisted symmetric power. However there are singly atypical modules for which this does not hold. For λ of type 2 or 3 that cannot happen. For λ of type 1, the condition $\lambda_1 - \lambda_2 \neq \lambda_3 - \lambda_4$ forces $\lambda_1 = \lambda_2$, hence the only such examples are the $\lambda = (\lambda_1, \lambda_1| - \lambda_1 - 1, -\lambda_1 - 1)$. For case 4 we obtain the λ of type $(\lambda_1, \lambda_2|\lambda_1 - 2\lambda_2 + 1, -\lambda_2 + 1)$.

The duals of the $(i,0|0,-j)$: If there is a $\wedge$ at position 1, then

$$\lambda^{\dagger} = (1, 2-i| -2+j, -1).$$

This is always the case if $i, j \neq 0$. For $j = 0, i > 1$ $\lambda^{\dagger} = (0, 2-i| -1, -1)$, for $i = 1$ $\lambda^{\dagger} = (0, 0| -1, 1)$. For $i = 0,\ j > 1$ $\lambda^{\dagger} = (1, 1|j-2, 0)$; for $j = 1$ $\lambda^{\dagger} = (1, 0|0, 0)$. If $L(\lambda)$ is a module of type 1, then it is a Berezin-twist of the dual module with $j = 0, i = 2$. If $L(\lambda)$ is of type 4 it is in general not a Berezin-twist of a twisted symmetric power or their duals.

4.26 Corollary. *Not every Kostant module is a Berezin-twist of an irreducible mixed tensor.*

4.27 Lemma. *The bipartitions which label $(m-2)$fold atypical irreducible representations in T are the following: First series:*

$$\textit{Any}\ \ [(2^r, 1^s), (a, b)]\ \ \textit{with}\ \ a \neq -1 + r,\ b \neq 1 + r + s.$$

and their duals $[(a, b); (2^r, 1^s)]$. Second series:

$$\lambda = [(a, 1^s); (c, 1^t)]\ \textit{with}\ a > 1,\ c \geq 1, c\ neq 2, t > 0, t \neq a - 1.$$

and their duals.

Proof: For a bipartition which will yield an $(m-2)$-fold atypical weight $\lambda^{\dagger}$ we must have

$$I_{\wedge} \cap I_{\vee} = \{p_1, p_2\},\ I_{\vee} \cup I_{\wedge} = \mathbb{Z} \setminus \{q_1, q_2\}.$$

Further $def(\lambda) = 0$. Any gap $\lambda_i^L - \lambda_{i+1}^L > 0$ can't be filled with $\vee$ because a cap would result unless at least one of the values $\lambda_i^L - (i-1)$ or $\lambda_{i+1}^L - i$ is occupied by a $\times$. This gives the following possibilities for λ^L: $(2^r, 1^s)$, $(a^r, 1^s), a > 2$, $(a, 0), a \geq 2$, (2^r), $1^s, s \geq 2$, (a, b). $\square$

4.6. Appendix: The orthosymplectic case

In this section we study a toy model: We divide the space of mixed tensors of an orthosymplectic Lie superalgebra by the ideal $\mathcal{N}$. Recall that for $m > n$ we have $T/\mathcal{N} \simeq Rep(Gl(m-n))$.

Here we prove the analogous result in the orthosymplectic case. As for $Gl(n)$ there exists an interpolating category $Rep(O_t)$, $t \in k$ with a standard representation st. Following Deligne [Del07] we define for $t = n \in \mathbb{Z}$ the following triples (G, ϵ, X) where G is a supergroup, ϵ an element of order 2 such that $int(\epsilon)$ induces on $\mathcal{O}(G)$ its grading modulo 2 and $X \in Rep(G, \epsilon)$:

- $n \geq 0$: $(O(n), id, st)$
- $n = -2m \leq 0$: $(Sp(2m), -1, st$ seen as odd $)$
- $n = 1 - 2m \leq 0$: $(OSp(1, 2m), diag(1, -1, \ldots, -1), st)$

By the universal property [Del07], prop 9.4 the assignment $st \mapsto X$ defines a tensor functor $Rep(O_t) \to Rep(G, \epsilon)$.

4.28 Theorem. *[Del07], thm 9.6 The functor $st \mapsto X$ of $Rep(O_t) \to Rep(G, \epsilon)$ defines an equivalence of $\otimes$-categories*

$$Rep(O_t)/\mathcal{N} \to Rep(G, \epsilon).$$

By the universal property we also have a tensor functor $Rep(O_t) \to Rep(OSp(n, m))$ for $t = n - m$.

4.29 Proposition. *For $t = n - m$ we have a commutative diagram of tensor functors*

$$\begin{tikzcd}
& Rep(O_t) \arrow[dl] \arrow[dd, "\mathcal{N}"] \\
Rep(OSp(n,m)) \arrow[dr, "S_x"] & \\
& Rep(G, \epsilon) \otimes svec.
\end{tikzcd}$$

Proof: We construct S_x. Take

$$x = \begin{pmatrix} (0 & diag(1, \ldots, 1) \\ 0 & 0 \end{pmatrix}.$$

Then $x \in \mathfrak{g}_1$ and $[x, x] = 0$. An easy computation shows $rk(x) = def(\mathfrak{g})$. For any such x the formalism of [Ser10] gives a tensor functor $M \mapsto M_x$ from $Rep(OSp(m, n)) \to Rep(G, \epsilon)$. A second calculation shows that it maps the standard representation to the standard representation. Hence $st \mapsto st$ on both sides of the diagram. By the universal property a tensor functor from Deligne's category is already determined by the image of the standard representation. $\square$

Let T denote the image of $F_{m,n} : Rep(0_t) \to Rep(OSp(m, n))$. Instead of the above diagram we consider the commutative diagram

$$\begin{tikzcd} & Rep(O_t) \arrow[dl] \arrow[dd, "\mathcal{N}"] \\ T \arrow[dr, "S_x"'] & \\ & Rep(G,\epsilon). \end{tikzcd}$$

4.30 Theorem. *We have $T/\mathcal{N} \simeq Rep(G,\epsilon)$*

Proof: The functor $T \to Rep(G,\epsilon)$ factorises over $T/\mathcal{N}$. The equivalence $Rep(O_t)/\mathcal{N} \simeq Rep(G,\epsilon)$ gives us a bijection between the irreducible elements of $Rep(G,\epsilon)$ and the indecomposable modules X in $Rep(O_t)$ with $id_X \notin \mathcal{N}$. Any X in $Rep(O_t)$ with $id_X \in \mathcal{N}$ maps to zero in $T/\mathcal{N}$. Note that the image of an indecomposable element of $Rep(O_t)$ in $T/\mathcal{N}$ is indecomposable by [CW11], lemma 2.7.4 since F_{mn} is full. This shows that the functor $T/\mathcal{N} \to Rep(G,\epsilon)$ is one-to-one on objects. Fully faithfullness follows trivially from Schur's lemma. $\square$

Similarly to the $Gl(m,n)$-case the maximally atypical modules of non-vanishing superdimension in T are those which are parametrized by partitions of length $\leq t$.

5. Symmetric powers and their tensor products

We study a remarkable class of indecomposable mixed tensors living in the maximal atypical block for any $n \geq 1$ of Loewy length 3. They are the smallest indecomposable modules in T with these properties. We then compute their tensor products. This will be crucial for the evaluation of the tensor products between the irreducible maximally atypical modules $S^i := [i, 0, \ldots, 0]$.

5.1. The symmetric and alternating powers

We define

$$\mathbb{A}_{S^i} := R(i; 1^i) = R(i) \text{ and } \mathbb{A}_{\Lambda^i} := (\mathbb{A}_{S^i})^\vee = R(1^i; i) = R(1^i).$$

5.1 Lemma. *If $def(\lambda) = 1$, then $R(\lambda) = \mathbb{A}_{S^i}$ or $\mathbb{A}_{\Lambda^i}$ for some $i > 0$.*

Proof: For $def(\lambda) = 1$ there can be at most one jump $\lambda_j > \lambda_{j+1}$ in the bipartition, hence $\lambda = (a, 0, \ldots)$ or $\lambda = (\underbrace{b, b, \ldots, b}_{n}, 0, \ldots)$ for $n > 1$. For $b > 1$ two $\vee$ will occur, hence $def(\lambda) > 1$. $\square$

We want to compute $((i); 0) \otimes (0; 1^i)$ in R_t, hence the sum $\sum_\nu \sum_{\kappa \in P} c^{(i)}_{\kappa,\nu^L} c^{(1^i)}_{\kappa,\nu^R}$, hence we search the pairs (κ, ν), (κ, ν^*) in λ^{-1} resp $(\lambda^*)^{-1}$. The Pieri rules tells one that the only such pairs are the pairs

$$((0), (i)) \longleftrightarrow ((0), (1^i)) \text{ and } ((1), (i-1)) \longleftrightarrow ((1), (1^{i-1})).$$

Hence

$$(i; 0) \otimes (0; 1^i) = (i) + (i-1).$$

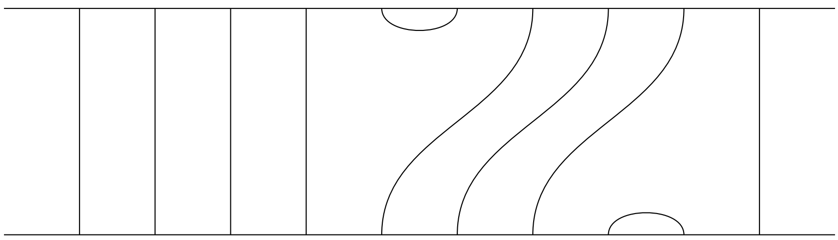
in R_t. Now clearly $lift(i) = (i) + (i-1)$, hence

5.2 Lemma. $\mathbb{A}_{S^i} = \{(i)\} \otimes \{(1^i)\}^\vee$. *Dito for* $\mathbb{A}_{\Lambda^j}$.

5.3 Lemma. *The Loewy structure of the* $\mathbb{A}_{S^i}$ *is given by*

$$\begin{aligned} \mathbb{A}_{S^1} &= (\mathbb{1}, S^1, \mathbb{1}) \\ \mathbb{A}_{S^i} &= (S^{i-1}, S^i \oplus S^{i-2}, S^{i-1}) \quad 1 < i \neq n \\ \mathbb{A}_{S^n} &= (S^{n-1}, S^n \oplus S^{n-2} \oplus B^{-1}, S^{n-1}). \end{aligned}$$

Proof: We sketch the computation for $\mathbb{A}_{S^i}$, $1 < i < m$. The module in the socle can be computed by applying θ. The matching t looks schematically like (picture for $i = 4$)



with the upper cup at the vertices $(0, 1)$ and the lower one at the vertices $(i-1, i)$. To determine the remaining composition factors we search the μ with $\mu \subset \alpha \to^t \zeta$, $red(\underline{\mu}t) = \underline{\zeta}$. Since t and ζ are fixed and the matching has to be consistently oriented this determines α up to the position at the unique cup in t at position $(i-1, i)$. Now consider μ where μ is obtained from $\lambda^\dagger = S^{i-1}$ by moving the $\vee$ at position $i-1$ to position $i-2$. This gives a cup at position $(i-2, i-1)$. The lower reduction property is satisfied and gives the weight S^{i-2}. No other $\mu \subset \lambda^\dagger$ fulfill the summation conditions. The second possible case for α (switching the $\wedge$ with the $\vee$ in the rightmost cup, hence moving $\vee$ one to the right) gives the module $[S^i] = [i, 0, \ldots, 0]$. As in the case of $\alpha = \lambda^\dagger$ a second $\mu \subset [S^i]$ may be otained by moving the rightmost $\vee$ one to the left. The corresponding module is $[S^{i-1}]$ and gives the second copy of $[S^{i-1}]$. One can check that no other weight diagrams fulfills the summation conditions. The Loewy layers can be determined from the number of lower circles in $red(\underline{\mu}t) = \underline{\mathbb{1}}$. The remaining cases can be treated in the same way. $\square$

This recovers an older result of Weissauer [Wei10c] who had determined the Loewy structures of the $\mathbb{A}_{S^i}$ by considering the restriction to the even part G_0.

Application: Typical $\otimes S^i$ As an application we sketch an explicite recursive algorithm to compute the tensor product $L(v) \otimes S^i$ where $L(v)$ is any typical module in the $m = n$-case. The tensor product $L(v) \otimes \mathbb{A}$ is known since both modules are in the image of F_{mm}. Since $L(v)$ is projective and $\mathbb{A} = \begin{pmatrix} k \\ S^1 \\ k \end{pmatrix}$ it splits into $2L(v) \oplus L(v) \otimes S^1$. Removing the two $L(v)$ we obtain $L(v) \otimes S^1$. Similarly $L(v) \otimes \mathbb{A}_{S^1} = L(\nu) \otimes S^2 \oplus 2L(\nu) \otimes S^1 \oplus L(v)$ which gives a formula for $L(v) \otimes S^2$. Iterating this procedure gives the decomposition of $L(v) \otimes S^i$ for any i. In particular it gives an

5.4 Corollary. *Explicite algorithm to decompose $L(v) \otimes L[a,b]$ where $L(v)$ is any typical $Gl(2,2)$-module and $L[a,b]$ is any maximall atypical weight of $Gl(2,2)$.*

5.1.1. Alternating and symmetric powers in other blocks

By Serganova every block of atypicality k in $Rep(Gl(m,n))$ is equivalent to the maximal atypical block in $Rep(Gl(k,k))$. Since the $\mathbb{A}_{S^i}$ exist in the maximal atypical block of any $Gl(m,m)$ every atypical block of $Rep(Gl(m,n))$ contains such that a family of modules (by abuse of notation again denoted by $\mathbb{A}_{S^i}$). The case of a singly atypical block is not very interesting: In that case the $\mathbb{A}_{S^i}$ are just the projective covers of the atypical irreducible modules. For $k \geq 2$ the $\mathbb{A}_{S^i}$ are new. They are also interesting since the length of the modules in the other known families of indecomposable modules - the projective covers and the Kac modules - grows very fast when enlarging m.

A first try to mimick the construction in the $m > n$-case fails: Consider the tensor product $\{(i)\} \otimes \{(1^i)\}^\vee = R(i;0) \otimes R(0;1^i)$. In R_t

$$(i;0) \otimes (0;1^i) = (i;1^i) \oplus (i-1, 1^{i-1}).$$

We have

$$\begin{aligned} &I_\wedge = \{i, -1, -2, -3, \ldots\} \\ &I_\vee = \{-(m-n), 1-(m-n), \ldots, (i-1)-(m-n), (i+1)-(m-n), \ldots\}. \end{aligned}$$

Hence

$$lift(i;1^i) = (i;1^i) \quad (m \neq n).$$

A cup only arises for $m = n$. Accordingly the tensor product splits into a sum of two irreducible modules

$$\{(i)\} \otimes \{(1^i)\}^\vee = (i;1^i) \oplus (i-1, 1^{i-1}).$$

Since the $\mathbb{A}_{S^i}$ do not appear in T we neither have an interpretation as Khovanov modules nor as modules in the Deligne category. In order to do so one has to know the analogs of the S^i in some random atypical block under the block equivalence. This is possible via Serganova's algorithm to associate to a k-fold atypical weight in *Rep*$(Gl(m,n))$ the corresponding k-fold atypical weight in *Rep*$(Gl(k,k))$.

Extensions panachees. We recall from [Gro72], vol.1, IX.9.3 and [Dro09], ch.4 the notion of extension panachee: Given $x \in Ext^1(P,R)$, $y \in Ext^1(R,Q)$, their Yoneda product is an element of $Ext^2(P,Q)$. It is zero iff there exists a module M with a filtration

$$M = X_0 \supset X_1 \supset X_2 \supset X_3 = \{0\}$$

with quotients $X_0/X_1 \simeq P$, $X^1/X^2 \simeq R$, $X^2 = Q$ with x corresponding to X_0/X_2 and y to X_1/X_3. We apply this for $i \neq m$ to

$$\mathbb{A}_{S^i} = \begin{pmatrix} S^{i-1} \\ S^i \oplus S^{i-2} \\ S^{i-1} \end{pmatrix}$$

We put $P = S^{i-1} = Q$ and $R = S^i \oplus S^{i-2}$. Clearly $\mathbb{A}_{S^i}$ is an extension panachee; so the Yoneda product $x \bullet y$ equals zero for $x \in Ext^1(P,R)$, $y \in Ext^1(R,Q)$. This gives a way to define $\mathbb{A}_{S^i}$ in any block of atypicality r by saying it is the extension panachee corresponding to the vanishing Yoneda product of x and y.

5.2. The tensor product $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$

We derive a closed formula for the tensor product $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$.

5.2.1. The $Gl(1,1)$-case

5.5 Corollary. *The atypical $Gl(1,1)$-modules in T are the $\mathbb{A}_{S^i}$ and their duals $\mathbb{A}_{\Lambda^j}$. They are the projective covers $\mathbb{A}_{S^i} = P[i-1]$ and $\mathbb{A}_{\Lambda^j} = P[-j+1]$.*

Proof: This follows since the defect of $(i, 0, \ldots)$ and $(1^i, 0, \ldots)$ is maximal for $Gl(1|1)$. □

5.6 Corollary. *In $Gl(1|1)$*

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} =& \mathbb{A}_{S^{|-i+j|+2}} \oplus 2\mathbb{A}_{S^{|-i+j|+1}} \oplus \mathbb{A}_{S^{|-i+j|}} \\ \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \end{aligned}$$

Proof: This is just rewriting the known formula ($a, b \in \mathbb{Z}$)

$$P(a) \otimes P(b) = P(a+b+1) \oplus 2P(a+b) \oplus P(a+b-1)$$

from [GQS07].

Let us assume from now on $m, n \geq 2$.

5.7 Lemma. *After projection to the maximal atypical block ($n \geq 2$)*

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} =& \mathbb{A}_{S^{|-i+j|+2}} \oplus 2\mathbb{A}_{S^{|-i+j|+1}} \oplus \mathbb{A}_{S^{|-i+j|}} \oplus R_1 \\ \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \oplus R_2 \end{aligned}$$

where R_1 and R_2 are direct sums of modules which do not contain any $\mathbb{A}_{S^i}$ or $\mathbb{A}_{\Lambda^j}$.

Proof: This follows from the $Gl(1|1)$-case and the identification between the projective covers and the symmetric and alternating powers. In $Gl(1|1)$ [GQS07]

$$P(a) \otimes P(b) = P(a+b-1) \oplus 2P(a+b) \oplus P(a+b+1).$$

Hence this formula holds for the corresponding $\mathbb{A}_{S^i}$ respectively $\mathbb{A}_{\Lambda^j}$. It then holds in $Rep(Gl_0)$ and hence in any $Rep(Gl(m|m))$ up to contributions which lie in the kernel $F_{mm} : Rep(Gl_0) \to Rep(Gl(m|m))$ and which are not $(1,1)$-cross. □

We carry out the tensor product decomposition in $Rep(Gl_0)$. Recall that this consists of three steps: i) take the lift $R_0 \to R_t$; ii) decompose the lift in R_t according to Comes-Wilson, iii) take $lift^{-1}$. From the resulting sum in $Rep(Gl_0)$ we remove the terms in $ker(F_n)$ and get the result in $\mathcal{R}_n$.

Lifts: Clearly $lift(i) = (i) + (i-1)$, $lift(1^i) = (1^i) + (1^{i-1})$. In order to compute the tensor product $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ we have to compute the tensor product $(i) \otimes (j) \oplus (i) \otimes (j-1) \oplus (i-1) \otimes (j) \oplus (i-1) \otimes (j-1)$ in R_t.

We derive first a closed formula for $(i) \otimes (j)$ in R_t, i.e. $((i,0,\ldots),(1^i)) \otimes (j,0,\ldots),(1^j)$. Without saying we often restrict to the maximal atypical case where $\nu^L = (\nu^R)^*$ and omit the other factors.

- **The contribution** $\sum_{\gamma \in P} c^{\nu^L}_{\alpha,\theta} c^{\nu^R}_{\beta,\eta}$: Here $\lambda^R = (1^i)$ and $\mu^L = (j,0,\ldots)$. iow the Pieri rule gives $(\mu^L)^{-1} = (0,j),(1,j-1),\ldots,(j-1,1),(j,0)$ and $(\lambda^R)^{-1} = (0,1^i),(1,1^{i-1}),\ldots,(1^i,0)$. In the sum over all bipartitions ν we consider only those with $\nu^L = (\nu^R)^*$. This condition permits only the pairs $(0,i) \leftrightarrow (0,1^j)$ and $(1,i-1) \leftrightarrow (1,1^{j-1})$ (to have same γ).
- **The contribution** $\sum_{\kappa \in P} c^{\lambda^L}_{\kappa,\alpha} c^{\mu^R}_{\kappa,\beta}$: Here $\mu^R = (1^j)$, $\lambda^L = (i)$. As in the previous case this gives only the possibilities $c^i_{0,i} c^{1^j}_{0,1^j}$ and $c^i_{1,i-1} c^{1^j}_{1,1^{j-1}}$.

Hence the sum
$$\sum_{\alpha,\beta,\eta,\theta} (\sum_{\kappa \in P} c^{\lambda^L}_{\kappa,\alpha} c^{\mu^R}_{\kappa,\beta})(\sum_{\gamma \in P} c^{\nu^L}_{\alpha,\theta} c^{\nu^R}_{\beta,\eta})$$
collapses to
$$(c^i_{0,i} c^{1^j}_{0,1^j} \; + \; c^i_{1,i-1} c^{1^j}_{1,1^{j-1}}) \; (c^{1^i}_{0,1^i} c^j_{0,j} + c^{1^i}_{1,1^{i-1}} c^j_{1,j-1}).$$
This corresponds to the choices

- (A) $\alpha = i, \; \beta = 1^j$
- (B) $\alpha = i-1, \; \beta = 1^{j-1}$
- (C) $\eta = 1^i, \; \theta = j$
- (D) $\eta = 1^{i-1}, \; \theta = j-1$.

Only for these choices AC, AD, BC, BD can there be a non-vanishing contribution $c^{\nu^L}_{\alpha,\theta} c^{\nu^R}_{\beta,\eta}$. We assume always $\nu^L = (\nu^R)^*$.

- The AC-case: $c^{\nu^L}_{i,j} c^{\nu^R}_{1^j,1^i}(\nu^L,\nu^R)$. By the Pieri rule ν^L can be any of $(i+j)$, $(i+j-1,1)$, $(i+j-2,2),\ldots$ and ν^R any of (1^{i+j}), $(2,1^{i+j-2},\ldots,(i,|i-j|)$. Hence the following bipartitions ν appear with multiplicity 1:
$$(i+j),(i+j-1,1),\ldots,((max(i,j),min(i,j)).$$

- The AD-case: $c^{\nu^L}_{i,j-1}c^{\nu^R}_{1^j,1^{i-1}}$. Restricting to $\nu^L = (\nu^R)^*$ we obtain
$$\nu \in \{(i+j-1),(i+j-2,1),\dots,((max(i,j),min(i,j)-1))\}.$$
- The BC-case: $c^{\nu^L}_{i-1,j}c^{\nu^R}_{1^{j-1},1^i}$. Here ν is any of
$$\nu \in \{((i+j-1),(i+j-2,1),\dots,((max(i,j),min(i,j)-1))\}.$$
- The BD-case: $c^{\nu^L}_{i-1,j-1}c^{\nu^R}_{1^{j-1},1^{i-1}}$. Here
$$\nu \in \{((i+j-2),(i+j-3,1),\dots,(max(i-1,j-1),min(i-1,j-1))\}$$

Summary: Hence

$$\begin{aligned}
&(i)\otimes(j) = \\
&(i+j)+(i+j-1,1)+\dots+((max(i,j),min(i,j)) \\
&\ +(i+j-1)+(i+j-2,1)+\dots+((max(i,j),min(i,j)-1)) \\
&+(i+j-1)+(i+j-2,1)+\dots+((max(i,j),min(i,j)-1)) \\
&\ +((i+j-2)+(i+j-3,1)+\dots+(max(i-1,j-1),min(i-1,j-1)).
\end{aligned}$$

We want to compute $R((i))\otimes R((j))$. We know $lift(i) = (i)+(i-1)$. This gives in R_t $((i)+(i-1))\cdot((j)+(j-1)) = (i)(j)+(i)(j-1)+(i-1)(j)+(i-1)(j-1)$.

The special case $j=1, i>1$: Then $(j-1)=0$. In this case $lift((i)\otimes(1)) = (i)\otimes(1)\oplus(i)\oplus(i-1)\oplus(i-1)\otimes(1)$. In R_t we have
$$(i)\otimes(1) = (i+1)+(i,1)+2(i)+(i-1)$$
so that
$$lift((i)\otimes(1) = (i+1)+(i,1)+4(i)+(i-1,1)+4(i-1)+(i-2).$$
After removing the contributions which will lead to $\mathbb{A}_{S^{i+1}}\oplus 2\mathbb{A}_{S^i}\oplus\mathbb{A}_{S^{i-1}}$ we are left with $(i,1)+(i)+(i-1,1)+(i-1)$. Hence

5.8 Lemma. *For $i\geq 2$*
$$\mathbb{A}_{S^i}\otimes\mathbb{A}_{S^1} = \mathbb{A}_{S^{i+1}}\oplus 2\mathbb{A}_{S^i}\oplus\mathbb{A}_{S^{i-1}}\oplus R(i,1).$$

In the general case we add up the contributions $((i)+(i-1))\cdot((j)+(j-1)) = (i)(j)+(i)(j-1)+(i-1)(j)+(i-1)(j-1)$. All the summands are of the following type: $(a,0)$, $(a,b), a>b>0, (a,a), a>0$. We have
$$\begin{aligned}
lift(a,b) =& (a,b)+(a,b-1)+(a-1,b)+(a-1,b-1) \quad a>b>0 \\
lift(a,a) =& (a,a)+(a,a-1)+(a-1,a-2)+(a-2,a-2).
\end{aligned}$$

After removing the contributions in R_t which will give the $\mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}}$ and applying successively the liftings from above we get the following decompositions. We assume $m = n \geq 2,\ i > j$:
Recall that for $i > 1,\ j = 1$

$$\begin{aligned}\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^1} =& \mathbb{A}_{S^{i+1}} \oplus 2 \cdot \mathbb{A}_{S^i} \oplus \mathbb{A}_{S^{i-1}} \\ & \oplus R(i,1).\end{aligned}$$

For $i > 2, j = 2$ we get

$$\begin{aligned}\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^2} =& \mathbb{A}_{S^{i+2}} \oplus 2 \cdot \mathbb{A}_{S^{i+1}} \oplus \mathbb{A}_{S^i} \\ & \oplus R(i+1,1) \oplus R(i,2) \oplus 2 \cdot R(i,1) \oplus R(i-1,1)\end{aligned}$$

Assume now $i > 2,\ j \geq 2$ und $i \neq j$ (for $i = j$ see below) and $i > j$. Then

$$\begin{aligned}\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\ & \oplus R(i+j-1,1) \\ & \oplus R(i+j-2,2) \oplus 2 \cdot R(i+j-2,1) \\ & \oplus R(i+j-3,3) \oplus 2 \cdot R(i+j-3,2) \oplus R(i+j-3,1) \\ & \oplus R(i+j-4,4) \oplus 2 \cdot R(i+j-4,3) \oplus R(i+j-4,2) \\ & \oplus R(i+j-5,5) \oplus 2 \cdot R(i+j-5,4) \oplus R(i+j-5,3) \\ & \oplus R(i+j-6,6) \oplus \dots \\ & \oplus R(i,j) \oplus 2 \cdot R(i,j-1) \oplus R(i,j-2) \\ & \oplus R(i-1,j-1).\end{aligned}$$

Now assume $i = j$. For $i = j = 2$ we get:

$$\begin{aligned}\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} =& \mathbb{A}_{S^4} \oplus 2 \cdot \mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \\ & \oplus R(3,1) \oplus R(2,2) \oplus 2 \cdot R(2,1).\end{aligned}$$

Now $i = j > 2$. Then

$$\begin{aligned}\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\ & \oplus R(i+j-1,1) \\ & \oplus R(i+j-2,2) \oplus 2 \cdot R(i+j-2,1) \\ & \oplus R(i+j-3,3) \oplus 2 \cdot R(i+j-3,2) \oplus R(i+j-3,1) \\ & \oplus R(i+j-4,4) \oplus 2 \cdot R(i+j-4,3) \oplus R(i+j-4,2) \\ & \oplus R(i+j-5,5) \oplus 2 \cdot R(i+j-5,4) \oplus R(i+j-5,3) \\ & \oplus R(i+j-6,6) \oplus \dots \\ & \oplus R(i,j) \oplus 2 \cdot R(i,j-1) \oplus R(i,j-2).\end{aligned}$$

.
We get the same result as for $i \neq j$ with omitting the last factor $\oplus R(i+j-min(i,j)-1, min(i,j)-1)$.

Example We then obtain the following formula

$$\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} = \mathbb{A}_{S^4} \oplus 2\mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \oplus R(3,1) \oplus R(2,2) \oplus 2 * R(2,1)$$
$$\mathbb{A}_{S^3} \otimes \mathbb{A}_{S^2} = \mathbb{A}_{S^5} \oplus 2\mathbb{A}_{S^4} \oplus \mathbb{A}_{S^3} \oplus R(4,1) \oplus R(3,2) \oplus 2 * R(3,1) \oplus R(2,1).$$

Highest weights The highest weights appearing in the socle and head of these indecomposable modules are $(4,1)$ with highest weight $[3,0,\ldots,0]$, $(3,2)$ with highest weight $[2,1,0,\ldots,0]]$, $(3,1)$ with highest weight $[2,0,\ldots,0]]$, $(2,2)$ with highest weight $[0,0,\ldots,0]$ and $(2,1)$ with highest weight $[1,0,\ldots,0]$ where one should insert the appropriate number of zeros depending on m.

5.3. The tensor products $\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j}$

We derive a closed formula for the tensor product $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$. We have

$$lift((i) \otimes (1^j)) = (i) \otimes (1^j) \oplus (i-1) \otimes (1^j) \oplus (i) \otimes (1^{j-1}) \oplus (i-1) \otimes (1^{j-1}).$$

in the Grothendieck ring R_t.

We may assume that $j > 1$ since $\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^1} = \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^1}$. We may also assume that $i \geq j$ since $(\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j})^\vee = \mathbb{A}_{\Lambda^i} \otimes \mathbb{A}_{S^j}$.

We compute $(i) \otimes (1^j)$ in R_t. Recall the classical Pieri rule $(i) \otimes (1^j) = (i+1, 1^{j-1}) \oplus (i, 1^j)$.

- $\sum_{\gamma \in P} c^{\nu^L}_{\alpha,\theta} c^{\nu^R}_{\beta,\eta}$: We evaluate this for $\lambda^R = (1^i)$, $\mu^L = (1^j)$. $(\lambda^R)^{-1} = (0, 1^i)$, $(1, 1^{i-1}), \ldots, (1^i, 0)$ and $(\mu^L) - 1 = (0, 1^j)$, $(1, 1^{j-1}), \ldots, (1^j, 0)$. Pairs with the same γ are

$$(0, 1^i) \leftrightarrow (0, 1^j),$$
$$(1, 1^{i-1}) \leftrightarrow (1, 1^{j-1}),$$
$$\ldots,$$
$$(1^{min(i,j)}, 1^{i-|i-j|}) \leftrightarrow (1^{min(i,j)}, 1^{j-|i-j|}).$$

- $\sum_{\kappa\in P} c^{\lambda^L}_{\kappa,\alpha} c^{\mu^R}_{\kappa,\beta}$: Here $\mu^R = (j),\ \lambda^L = (i)$. Here the permitted pairs are the

$$
\begin{gathered}
(0,i) \leftrightarrow (0,j),\\
(1,i-1) \leftrightarrow (1,j-1),\\
\dots,\\
(min(i,j),(i-|i-j|) \leftrightarrow (min(i,j),(j-|i-j|).
\end{gathered}
$$

The big sum collapses to

$$
\begin{aligned}
&(c^i_{0,i}c^j_{0,j} + \dots + c^i_{min(i,j),i-|i-j|}c^j_{min(i,j),j-|i-j|})\\
&(c^{1^i}_{0,1^i}c^{1^j}_{0,1^j} + \dots + c^{1^i}_{min(i,j),1^{i-|i-j|}}c^{1^j}_{min(i,j),1^{j-|i-j|}})
\end{aligned}
$$

We multiply these sums and identify $c^{1^i}_{0,1^i} = c^{1^j}_{0,1^i}$ etc and ignore squares since all the coefficients $c^{\dots}_{\dots}$ are 1 anyway.

We have to evaluate $\sum_\nu \sum_{\alpha,\beta,\eta,\theta} c^{\nu^L}_{\alpha,\theta} c^{\nu^R}_{\beta,\eta}\nu$. The following values for these for α,β,η,θ give non-vanishing coefficients (let $t = min(i,j)$):

- a) $\alpha = i,\ \beta = j$
- b) $\alpha = i-1,\ \beta = j-1$.
- ...
- t) $\alpha = i-t,\ \beta = j-t$

and

- a)' $\eta = 1^i,\ \theta = 1^j$
- b)' $\eta = 1^{i-1},\ \theta = 1^{j-1}$
- ...
- t) $\eta = 1^{i-t},\ \theta = 1^{j-t}$.

This gives $(t+1)^2$ non-vanishing products, namely aa', ab', ..., at, ba', bb', ..., tt. iow we use $(i) \otimes (1^j) = (i+1, 1^{j-1}) \oplus (i, 1^j)$ in order so see which ones will give maximally atypical ν.

iow $\Gamma^{\nu}_{\lambda,\mu} = \sum_{\alpha,\beta,\theta,\eta} \dots = 0$ unless the indices form one of the admissible tuples aa', ab', $\dots, at', ba', bb', \dots, tt'$. A bipartion ν will appear iff there exists an admissible tuple such that $c^{\nu^L}_{\alpha,\theta} c^{\nu^R}_{\beta\eta} \neq 0$. The classical formula $(i)\otimes(1^j) = (i+1, 1^{j-1}) \oplus (i, 1^j)$ tells us that such a ν is necessarily of the form

$$\nu = [(n, 1^{\tilde{n}}), (\tilde{n}+1, 1^{n-1})]$$

for $n, \tilde{n}$ in a suitable range. We have

$$(\nu^L)^{-1} = (n, 1^{\tilde{n}}) \ \ resp \ (n-1, 1^{\tilde{n}+1})$$
$$(\nu^L)^{-1} = (\tilde{n}+1, 1^{n-1}) \ \ resp \ (\tilde{n}, 1^{n})$$

Hence a given ν can be realised in maximally 4 different ways: Through either one of

- i) $\alpha = n, \theta = 1^{\tilde{n}}, \beta = \tilde{n}+1, \eta = 1^{n-1}$
- ii) $\alpha = n, \theta = 1^{\tilde{n}}, \beta = \tilde{n}, \eta = 1^{n}$
- iii) $\alpha = n-1, \theta = 1^{\tilde{n}+1}, \beta = \tilde{n}+1, \eta = 1^{n-1}$
- iv) $\alpha = n-1, \theta = 1^{\tilde{n}+1}, \beta = \tilde{n}, \eta = 1^{n}$

Define $a(\nu) =$ the number of cases i) - iv) which are fulfilled. Then

$$\lambda\mu = \bigoplus \Gamma^{\nu}_{\lambda,\mu}\nu = \bigoplus_{n\in\{i-t,\dots,i\},\tilde{n}\in\{i-t-1,\dots,j\}} a(\nu)(n, 1^{\tilde{n}}).$$

We carry out the summation $\sum_{l=0}^{t}\sum_{k'=0}^{t} lk'$.

We first treat the partial sum $aa' + ab' + \dots at'$. In that case only aa' and ab' give a contribution. aa' yields $(i+1, 1^{j-1})$ and $(i, 1^j)$ and ab' yields $(i, 1^{j-1})$.

Now consider a generic summand lk', $i \neq a, t$. The corresponding product of the Littlewood-Richardson coefficients is

$$c^{\nu^L}_{i-l,1^{j-k}} c^{\nu^R}_{j-k,1^{i-l}}.$$

The possible ν^L are of the form

$$\nu^L_1 = (i-l+1, 1^{j-k-1}), \ \ \nu^L_2 = (i-l, 1^{j-k})$$

and the possible ν^R are of the form

$$\nu^R_1 = (j-k+1, 1^{i-l-1}), \ \ \nu^R_2 = (j-k, 1^{i-l}).$$

We only consider ν with $\nu^R = (\nu^L)^*$. We have

$$(\nu_1^L)^* = (j-k, 1^{i-l}).$$

This is equal to one of the two ν^R for $k = l$ in which case we get (ν_1^L) and (ν_2^L) as a contribution. The pair lk will not give any contribution for $k \notin \{l-1, l, l+1\}$. For $l = k+1$ we get the contribution (ν_1^L) and fpr $l = k-1$ we get the contribution (ν_2^L).

The sum $ta' + \ldots + tt'$ gives the contribution

$$\begin{cases} (i-j+1) \oplus (i-j) & i > j \\ (1^{j-i+1}) \oplus (1^{j-i}) & j > i \\ (1) \oplus (0) & i = j \end{cases}$$

Hence we obtain the following **closed formula**:

$$\begin{aligned} (i) \otimes (1^j) &= (i+1, 1^{j-1}) \oplus (i, 1^j) \oplus (i, 1^{j-1}) \\ &\oplus \bigoplus_{l=1}^{t-1} [\ (i-l, 1^{j-l-1}) \oplus (i-l, 1^{j-l}) \oplus (i-l+1, 1^{j-l-1}) \oplus (i-l+1, 1^{j-l})\ ] \\ &\oplus \begin{cases} (i-j+1) \oplus (i-j) & i > j \\ (1^{j-i+1}) \oplus (1^{j-i}) & j > i \\ (1) \oplus (0) & i = j \end{cases} \end{aligned}$$

We apply this formula to the four summands of $lift((i) \otimes (1^j))$, $(i) \otimes (1^j)$, $(i-1) \otimes (1^j)$, $(i) \otimes (1^{j-1})$, $(i-1) \otimes (1^{j-1})$. The contributions in the total sum are either of the form (i) or (1^j) or $(i, 1^j)$. We have

$$lift(i, 1^j) = (i, 1^j) \oplus (i-1, 1^j) \oplus (i, 1^{j-1}) \oplus (i-1, 1^{j-1}).$$

From the $Gl(1,1)$-case we know that the contribution of the alternating and symmetric powers will be given by $(i > j)$

$$\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} = \mathbb{A}_{S^{|-i+j|+2}} \oplus 2\mathbb{A}_{S^{|-i+j|+1}} \oplus \mathbb{A}_{S^{|-i+j|}} \oplus R$$

and by

$$\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^i} = \mathbb{A}_{S^{+2}} \oplus 2\mathbb{A} \oplus \mathbb{A}_{\Lambda^2} \oplus R$$

for $i = j$ for some R-term which does not involve any alternating or symmetric powers. Removing all the corresponding bipartitions from the total sum and working downwards as in the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$-case we obtain the **final result**:

For $i = j = 2$ we obtain:

$$\mathbb{A}_{S^2} \otimes \mathbb{A}_{\Lambda^2} = \mathbb{A}_{S+2} \oplus 2\mathbb{A} \oplus \mathbb{A}_{\Lambda^2} \oplus R(3,1) \oplus R(2,1^2) \oplus 2R(2,1)$$

and for $i > j = 2$ we obtain

$$\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^2} = \mathbb{A}_{S^i} \oplus 2\mathbb{A}_{S^{i-1}} \oplus \mathbb{A}_{S^{i-2}} \oplus R(i+1,1) \oplus R(i,1^2) \oplus 2R(i,1) \oplus R(i-1,1)$$

The **general formula** is for $i > j > 2$ as follows:

$$\begin{aligned}
\mathbb{A}_{S^i} \otimes \mathbb{A}_{\Lambda^j} =& \mathbb{A}_{S^{|-i+j|+2}} \oplus 2\mathbb{A}_{S^{|-i+j|+1}} \oplus \mathbb{A}_{S^{|-i+j|}} \\
& \oplus R(i+j-(j-1), 1^{j-1}) \\
& \oplus R(i+j-j, 1^j) \oplus 2\dot{R}(i, 1^{j-1}) \oplus R(i, 1^{j-2}) \\
& \ldots \\
& \oplus R(i+j-k, 1^k) \oplus 2 \cdot R(i+j-k, 1^{k-1}) \oplus R(i+j-k, 1^{k-2}) \\
& \oplus \ldots \\
& \oplus R(i-j+2, 1^2) \oplus 2 \cdot R(i-j+2, 1) \\
& \oplus R(i-j+1, 1).
\end{aligned}$$

For $i = j > 2$ one has to remove the last term $R(i-j+1,1)$.

We list some sample computations:

$$\begin{aligned}
\mathbb{A}_{S^3} \otimes \mathbb{A}_{\Lambda^2} &= \mathbb{A}_{S^3} \oplus 2\mathbb{A}_{S^2} \oplus \mathbb{A}_{S^1} \oplus R(4,1) \oplus R(3,1^2) \oplus 2R(3,1) \oplus R(2,1) \\
\mathbb{A}_{S^3} \otimes \mathbb{A}_{\Lambda^3} &= \mathbb{A}_{S^2} \oplus 2\mathbb{A} \oplus \mathbb{A}_{\Lambda^2} \oplus R(4,1^2) \oplus R(3,1^3) \oplus 2R(3,1^2) \oplus R(3,1) \\
&\oplus R(2,1^2) \oplus 2R(2,1) \\
\mathbb{A}_{S^9} \otimes \mathbb{A}_{\Lambda^5} &= \mathbb{A}_{S^6} \oplus 2\mathbb{A}_{S^5} \oplus \mathbb{A}_{S^4} \oplus R(10,1^4) \oplus R(9,1^5) \oplus 2R(9,1^4) \\
&\oplus R(9,1^3) \oplus R(8,1^4) \oplus 2R(8,1^3) \oplus R(8,1^2) \oplus R(7,1^3) \oplus R(7,1^2) \\
&\oplus R(7,1) \oplus R(6,1^2) \oplus 2R(6,1) \oplus R(5,1).
\end{aligned}$$

5.4. The Example $Gl(2,2)$

As an example let us compute the $Gl(2|2)$-case explicitely. Let λ be a bipartition.

- If $rk(\lambda) = 2$, $R(\lambda)$ is irreducible typical; and every typical module occurs as some $R(\lambda)$

- If $rk(\lambda) = 1$, the defect of λ is either 0 or 1. If it is zero, the irreducible modules are the twisted symmetric powers and their duals studied in the last chapter. If the defect is 1, $R(\lambda) = P(\lambda^\dagger)$.
- If $rk(\lambda) = 0$, $R(\lambda)$ is maximally atypical. We focus on this case for now. The defect is either 1 or 2, leading either to the alternating and symmetric powers or to the projectice covers $P([a, b])$.

We will not make use of the characterization of the projective covers and calculate everything explicitely.

The condition $(2, 2)$-cross means: There exists $i \in \{1, 2, 3\}$ such that $\lambda_i + \lambda^*_{m+2-i} \leq 2$, i.e. one of the following three conditions should hold:

$$\begin{aligned} i = 1 :& \lambda_1 + \lambda_3^* \leq 2 \\ i = 2 :& \lambda_2 + \lambda_2^* \leq 2 \\ i = 3 :& \lambda_3 + \lambda_1^* \leq 2. \end{aligned}$$

This gives the following possibilities for partitions $\neq 0$:

$$\begin{aligned} i = 1 :& \lambda = (1, *, \ldots),\ \lambda = (2, *, \ldots) \\ i = 2 :& \lambda = (*, 0, \ldots),\ \lambda = (*, 1, *, \ldots) \\ i = 3 :& \lambda = (*, 0, \ldots),\ \lambda = (*, *, 0, \ldots) \end{aligned}$$

where $*$ is any number ≥ 0 such that λ is a partition.

Case 1: $\lambda = (a, 0, \ldots),\ a > 0$. In that case $X = a,\ k(\lambda) = 1$. One obtains

$$\lambda^\dagger = [a - 1, 0].$$

Case 2: $\lambda = (a, b, 0, \ldots)$, $b \neq 0$. In that case $k(\lambda) = 2$ unless $a = b = 1$ where $k(\lambda) = 1$. For $a > 1$ and $a > b$ we obtain

$$\lambda^\dagger = [a - 1, b - 1].$$

In the second case we obtain

$$\lambda^\dagger = [1, 0].$$

For $a = b$, $a,\ b \geq 2$

$$\lambda^\dagger = [a - 2, a - 2].$$

Case 3: $\lambda = (1, \underbrace{1, \ldots, 1}_{a}, 0, \ldots)$, $a > 0$. For $a > 1$ $k(\lambda) = 1,\ X = 1 - a$ and

$$\lambda^\dagger = [1, 2 - a].$$

For $a = 1$

$$\lambda^\dagger = [1, 0].$$

Case 4: $\lambda = (2, \underbrace{2, \ldots, 2}_{a}, \underbrace{1, \ldots, 1}_{b}, 0, \ldots)$ and $a > 0,\ b > 0$. Then $k(\lambda) = 2,\ X = 2 - a$ and

$$\lambda^\dagger = [1 - a, 1 - a - b].$$

The case $a = 0$ has already been treated. For $b = 0$ (and $a > 0$)

$$\lambda^\dagger = [1 - a, 1 - a].$$

Case 5: $\lambda = (a, 1, \underbrace{1, \ldots, 1}_{b}, 0, \ldots)$. Assume $a > 1,\ b > 0$. Then $k(\lambda) = 2,\ X = a$ and

$$\lambda^\dagger = [a - 1, -b].$$

We see by direct inspection:

- Every maximally atypical weight appears as a $\lambda^\dagger$.
- For the following highest weights there exist precisely two bipartitions in the fiber:
  - highest weights of the type $[1, -a], a \geq 0$,
  - highest weights of the type $[a - 1, 0], a \geq 2$.

5.9 Theorem. *($Gl(2,2)$-case) a) Assume that $\lambda^\dagger \neq \mathbb{1}$. The weight $\lambda^\dagger$ determines $R(\lambda)$ uniquely up to projective covers. b) The only non-projective modules among the $R(\lambda)$ are the symmetric and alternating powers. c) Considering only non-projective and indecomposable modules we have $\mathbb{A}_\lambda \simeq R(\lambda)$. d) $R(\lambda)$ is projective iff $def(\lambda) = 2$ (i.e. maximal).*

Remark: For $m \geq 3$ the analogs of statements a,b,c) are wrong.

Duals We compute the duals of the irreducible modules in the maximal atypical block following the recipe from section 2. Since every such module is a Berezin-twist of one of the S^i we may restrict to these case. The projective cover of $S^i = [i, 0]$ is the module $R(i+1, 1)$. The conjugate partition of $(i+1, 1)$ is the partition $(2, 1^i)$. Hence the dual of the projective cover $P[i, 0]$ is the module $R(2, 1^i)$. Via the list above the irreducible module in the socle has weight $[1, 1-i]$, hence

$$(S^i)^\vee = [1, 1 - i],$$

ie. $S^i = Ber^{i-1}(S^i)^\vee$. In particular the representations $Ber^{-l}S^{2l+1}$ are selfdual.

6. Cohomological tensor functors

In the last two chapters we sketch an approach to determine the reductive group $G_L \hookrightarrow G^{red}$ attached to an maximally atypical irreducible representation in $\mathcal{R}_n$. In chapter 6 we review shortly the main result about the image of an irreducible representation under DS We also determine the kernel of DS and show, how the main theorem implies a formula for the modified superdimension.

6.1. The Duflo-Serganova functor

We have already studied the Duflo-Serganova functor

$$DS : \mathcal{R}_{mn} \to R_{m-r,n-r}$$

where r is the rank of the fixed element $x \in X$. We fix an element x of $rk(x) = 1$ such that DS maps the standard to the standard representation. Define the functor $H^+(V)$ via

$$H^+ : \mathcal{R}_n \to R_{n-1} \to^{pr} \mathcal{R}_{n-1}$$

where $pr : R_{n-1} \to \mathcal{R}_n$ is the projection onto the even part of $R_n = \mathcal{R}_n \oplus \mathcal{R}_n[1]$. Then we define H^- by the decompositon $DS(V) = H^+(V) \oplus \Pi H^-(V)$. For H^+ and H^- the long exact sequence defines an exact hexagon in $\mathcal{R}_{n-1}$

$$\begin{tikzcd}
 & H^+(A) \arrow[r] & H^+(B) \arrow[dr] & \\
H^-(C) \arrow[ur] & & & H^+(C) \arrow[dl] \\
 & H^-(B) \arrow[ul] & H^-(A). \arrow[l] &
\end{tikzcd}$$

We refer to the sections 6-11 in [HW13] where refined versions of these cohomology functors are studied.

6.1 Theorem. *Suppose $L(\lambda) \in \mathcal{R}_n$ is irreducible and atypical so that λ corresponds to a cup diagram*

$$\bigcup_{j=1}^{r} [a_j, b_j]$$

with r sectors $[a_j, b_j]$ for $j = 1, \ldots, r$. Then

$$DS(L(\lambda)) \simeq \bigoplus_{i=1}^{r} L(\lambda_i)[n_i]$$

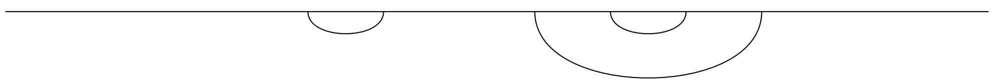
with shift n_i as in [HW13]. The irreducible representation $L(\lambda_i)$ is uniquely defined by the property that its cup diagram is

$$[a_i + 1, b_i - 1] \cup \bigcup_{j=1, j \neq i}^{r} [a_j, b_j].$$

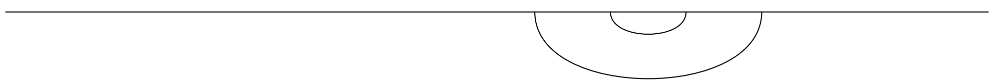
In other words: To each fixed sector of λ corresponds a summand in $DS(L(\lambda))$. This summand has the same sector structure as λ except that we we remove the outer cup of the given sector.

Example: If L is a ground state of a block of atypicality $< n$, L is a mixed tensor. By the permanence properties of the mixed tensors under DS, $DS(L(\lambda)) = L(\bar{\lambda})[\lambda_{n+1}]$ (see chapter 4). This example is very important since it will give us the induction start in the proof of the theorem.

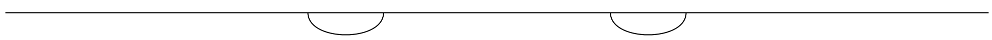
Example. Let $L(\lambda) = [2, 2, 0]$ in $\mathcal{R}_3$. Then λ has the following cup diagram



with the two sectors $[-2, -1]$ and $[1, 4]$. Hence $DS([2, 2, 0]$ will split into two maximally atypical irreducible modules in R, namely in the module with cup diagram



and the module with cup diagram



The corresponding irreducible modules are $[2, 2]$ and $[2, -1]$ with signs 1 and -1. The parity shift is then given by $n_1 = 0$ and $n_1 = 1$, hence

$$DS(L(2, 2, 0)) = L(2, 2) \oplus L(2, -1)[1].$$

6.2. The kernel of DS

Support varieties. We review some results from [BKN10], [BKN09b] and [BKN09a] on support varieties and the connection to DS. Let $\langle \xi \rangle$ be the Lie superalgebra generated by an odd vector ξ with $[\xi, \xi] = 0$. Then $\langle \xi \rangle$ is an abelian Lie superalgebra. It has two indecomposable modules: The trivial module and its projective cover $U(\langle \xi \rangle)$. For ξ in the cone $X = \{\xi \in \mathfrak{g}_{\bar{1}} \mid [\xi, \xi] = 0\}$ the condition $M_\xi \neq 0$ is equivalent to the condition that M is not projective as a $U(\langle \xi \rangle)$-module [BKN09a], 3.6.1. By [BKN09a] $V_{\mathfrak{g}_{(\pm 1)}}(M)$ is canonically isomorphic to the rank variety

$$V^{rank}_{\mathfrak{g}_{(\pm 1)}}(M) = \{\xi \in g_{(\pm 1)} \mid M \text{ not projective as a } U(\langle x \rangle) - \text{module}\} \cup \{0\}.$$

By [BKN10], prop 6.3.1 it satisfies

$$V^{rank}_{\mathfrak{g}_{(\pm 1)}}(M \otimes N) = V^{rank}_{\mathfrak{g}_{(\pm 1)}}(M) \cap V^{rank}_{\mathfrak{g}_{(\pm 1)}}(N).$$

We have further the associated variety of Duflo and Serganova

$$\begin{aligned} X_M &= \{\xi \in X \mid M_\xi \neq 0\} \\ &= \{\xi \in X \mid M \text{ is not projective as a } U(<\xi>) - \text{module}\} \cup \{0\}. \end{aligned}$$

By the rank variety description of $V_{\mathfrak{g}_{(\pm 1)}}(M)$ then

$$V_{\mathfrak{g}_{(-1)}}(M) \cup V_{\mathfrak{g}_{(1)}}(M) \subseteq X_M \quad , \quad V_{\mathfrak{g}_{(\pm 1)}}(M) = X_M \cap \mathfrak{g}_{(\pm 1)} \ .$$

Kac and anti-Kac objects. We denote by C^+ the tensor ideal of modules with a filtration by Kac modules (Kac objects) and by C^- the tensor ideal of modules with a filtration by anti-Kac modules (anti-Kac objects). We quote [BKN09a], thm 3.3.1, thm 3.3.2

$$\begin{aligned} M \in C^+ &\Leftrightarrow V_{\mathfrak{g}_{(1)}}(M) = 0 \\ M \in C^- &\Leftrightarrow V_{\mathfrak{g}_{(-1)}}(M) = 0. \end{aligned}$$

Hence a module M is projective if and only if $V_{\mathfrak{g}_{(1)}}(M) = V_{\mathfrak{g}_{(-1)}}(M) = 0$.

Vanishing criterion. For any $\xi \in X$ there exists $g \in Gl(n) \times Gl(n)$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \ldots, \alpha_k$ such that $Ad_g(\xi) = \xi_1 + \ldots + \xi_k$ with $\xi_i \in \mathfrak{g}_{\alpha_i}$. The number k is called the rank of ξ [Ser10]. By a minimal orbit for the adjoint action of $Gl(n) \times Gl(n)$ on $g_{(\pm 1)}$ we mean a minimal non-zero orbit with respect to the partial order given by containment in closures. Let $\{\xi_i \mid i \in I\}$ be a set of orbit representatives for the minimal orbits on $\mathfrak{g}_{(1)}$ and $\{y_i \mid i \in I\}$ one for the minimal orbits on $\mathfrak{g}_{(-1)}$. In both cases one has a single minimal orbit. The n orbits for the action of $Gl(n) \times Gl(n)$ on $g_{(1)}$ are [BKN09a], 3.8.1

$$(\mathfrak{g}_{(1)})_r = \{\xi \in \mathfrak{g}_{(1)} \mid r(\xi) = r\}, \ 0 \leq r \leq n\}$$

with closure $\overline{(\mathfrak{g}_{(1)})_r} = \{\xi \in \mathfrak{g}_{(1)} \mid r(\xi) \leq r\}$, hence $(\mathfrak{g}_{(1)})_r \subset \overline{(\mathfrak{g}_{(1)})_s}$ if and only if $r \leq s$. The unique minimal orbit is then the set $(\mathfrak{g}_{(1)})_1$, which is the orbit of the element x defined earlier. The situation is analogous for $\mathfrak{g}_{(-1)}$. A slight modification of [BKN09a], thm 3.7.1 and its proof gives

6.2 Theorem. *If $\xi \in \mathfrak{g}_{(1)}$ then $C^- \subset ker(DS_\xi)$. If $\xi \in \mathfrak{g}_{(-1)}$ then $C^+ \subset ker(DS_\xi)$. If $rk(\xi) = 1$, then $ker(DS_\xi) = C^-$ for $\xi \in \mathfrak{g}_{(1)}$ and $ker(DS_\xi) = C^+$ for $\xi \in \mathfrak{g}_{(-1)}$. For $\xi = x$ we have $M_x = 0$ iff $M \in C^-$ and $M_{\tau x} = 0$ iff $M \in C^+$.*

Proof. Let $M \in C^-$. Then $V_{\mathfrak{g}_{(1)}}(M) = 0$. Hence

$$\{\xi \in \mathfrak{g}_{(1)} \mid M_\xi \neq 0\} = 0.$$

Similarly for $M \in C^+$. Conversely let $M_\xi = 0$ for $\xi \in \mathfrak{g}_{(1)}$ of rank 1. Since $V_{\mathfrak{g}_{(1)}}(M)$ is a closed $Gl(n) \times Gl(n)$-stable variety it contains a closed orbit. Since the orbits $(\mathfrak{g}_{(1)})_m$ are closed only for $m = 1$, $V_{g_{(1)}}(M) \neq 0$ if and only if it contains $(\mathfrak{g}_{(1)})_1$. But $M_\xi = 0$ implies

$$\xi \notin V^{rank}_{\mathfrak{g}_{(1)}}(M) = V_{\mathfrak{g}_{(1)}}(M),$$

hence $V_{\mathfrak{g}_{(1)}}(M) = 0$, contradiction. Likewise for $\xi \in \mathfrak{g}_{(-1)}$. □

6.3 Corollary. *Let $x \in (\mathfrak{g}_{(1)})_1$. Then:*

1. *M is projective if and only if $M_x = 0$ and $M_{\tau x} = 0$.*
2. *M is projective if and only if $M_x = 0$ and $M_x^* = 0$*
3. *If $M = M^*$, then M is projective if and only if $M_x = 0$.*

Proof. $M_x = 0$ implies $V_{\mathfrak{g}_{(1)}}(M) = 0$ and $M_{\tau(x)} = 0$ implies $V_{\mathfrak{g}_{(-1)}}(M) = 0$, hence i). (ii) and (iii) follow from [BKN09a], 3.4.1

$$V_{\mathfrak{g}_{(\pm 1)}}(M^*) = \tau(V_{\mathfrak{g}_{(\mp 1)}}(M)).$$

□

Example. [BKN09a], 3.8.1 For any M we have $V_{\mathfrak{g}_{(1)}}(M) = \overline{(\mathfrak{g}_{(1)})_r}$ for some r and $V_{\mathfrak{g}_{(-1)}}(M) = \overline{(\mathfrak{g}_{(-1)})_s}$ for some s. If M is either a Kac module, or an anti-Kac module or an irreducible module of atypicality k [BKN09a], 3.8.1 and

$$V_{\mathfrak{g}_{(\pm 1)}}(M) = \overline{(\mathfrak{g}_{(\pm 1)})_k}.$$

6.3. Modified superdimension

The main theorem gives a closed formula for the modified superdimension of an irreducible module in $\mathcal{R}_n$. For the normal superdimension such a formula was found in [Wei10b].

Modified superdimensions and Kac-Wakimoto . We recall some definitions and results from [Kuj11], [GKPM11] and [Ser10]. The following definitions are copied from [Kuj11]. Assume $m \geq n$. Denote by $c_{V,W} : V \otimes W \to W \otimes V$ the usual flip $v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v$. Put $ev'_V = ev_V \circ c_{V,V^\vee}$ and $coev'_V = c_{V,V^\vee} \circ coev_V$. For any pair of objects V, W and an endomorphism $f : V \otimes W \to V \otimes W$ we define

$$tr_L(f) = (ev_V \otimes id_W) \circ (id_{V^\vee} \otimes f) \circ (coev'_V \circ id_w) \in End_T(W)$$
$$tr_R(f) = (id_V \otimes ev'_W) \circ (f \otimes id_{W^\vee}) \circ (id_V \otimes coev_W) \in End_T(V)$$

For an object $J \in \mathcal{T}_n$ let I_J be the tensor ideal of J. A trace on I_J is by definition a family of linear functions

$$t = \{t_V : End_{\mathcal{T}_n}(V) \to k\}$$

where V runs over all objects of I_J such that the following two conditions hold.

1. If $U \in I_J$ and W is an object of $\mathcal{T}_n$, then for any $f \in End_{\mathcal{T}_n}(U \otimes W)$ we have
$$t_{U \otimes W}(f) = t_U(t_R(f)).$$
2. If $U, V \in I$ then for any morphisms $f : V \to U$ and $g : U \to V$ in $\mathcal{T}_n$ we have
$$t_V(g \circ f) = t_U(f \circ g).$$

For V an object of $\mathcal{R}_n$ a linear function $t : End_{\mathcal{T}_n}(V) \to K$ is an ambidextrous trace on V if for all $f \in End_{\mathcal{T}_n}(V \otimes V)$ we have

$$t(t_L(f)) = t(t_R(f)).$$

An object is ambidextrous if it is irreducible and admits a nonzero ambidextrous trace.

6.4 Theorem. *[Kuj11], thm 2.3.1 Let L be irreducible. If I_L admits a trace then the map t_L is an ambidextrous trace on L. Conversely, an ambidextrous trace on L extends uniquely to a trace on I_L. The trace on I_L and the ambidextrous trace on L are unique up to multiplication by an element of k.*

Given a trace on I_J, $\{t_V\}_{V\in I_J}$, define the modified dimension function on objects of I_J as the modified trace of the identity morphism:

$$d_J(V) = t_V(id_V).$$

6.5 Theorem. *[Ser10] Let J be irreducible. Then J is ambidextrous and if L is another simple supermodule with*

$$atyp(L) \leq atyp(J),$$

then L is an object of I_J and

$$atyp(L) = atyp(J) \text{ if and only if } d_J(L) \neq 0.$$

Tensor ideals. Recall from chapter 3 that by [Ser10] any two irreducible object of atypicality k generate the same tensor ideal. Therefore write I_k for the tensor ideal generated by any irreducible object of atypicality k. Clearly $I_0 = Proj$ and $I_n = T_n$ since it contains the identity. This gives the following filtration

$$Proj = I_0 \subsetneq I_1 \subsetneq \ldots I_{n-1} \subsetneq I_n = T_n$$

with strict inclusions by [Ser10] and [Kuj11].

The projective case. Define for any typical module the following function

$$d(L(\lambda)) = \prod_{\alpha\in\Delta_0^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)} / \prod_{\alpha\in\Delta_1^+} (\lambda+\rho,\alpha)$$

with the pairing $(,)$ defined in chapter 1. Then $d(L(\lambda)) \neq 0$ for every typical $L(\lambda)$. By [GKPM11], 6.2.2 for typical L

$$d_J(L) = \frac{d(L)}{d(J)}.$$

Since the ideal I_0 is independent of the choice of a particular J and any ambidextrous trace is unique up to a scalar, we normalize and define the modified superdimension on I_0 to be

$$sdim_0(L) := d(L).$$

A formula for the superdimension. Applying DS iteratively k-times to a module of atypicality k we obtain the functor

$$DS^k := DS \circ \ldots \circ DS : \mathcal{T}_n \to T_{n-k}$$

which sends M with $atyp(M) = k$ to a direct sum of typical modules (possibly zero).

We know that an ambidextruous trace exists on I_k and is unique up to a scalar. This applies in particular to I_0 where we have normalized it already. Call this trace t^P. Now we define for $M \in I_k$

$$t_M := t^p_{DS^k(M)} f_{DS^k(M)} : End_{\mathcal{T}_n}(M) \to k$$

where $f_{DS^k(M)}$ is the image of f under the functor DS^k. We claim that this actually defines a nontrivial trace on I_k. Let $M = L$ be irreducible. Then

$$t_L(id_L) := t^p_{DS^k(L)}(id_{DS^k(L)}).$$

Now we compute $DS^k(L)$. By the main theorem the irreducible summands in $DS(L)$ are obtained by removing one of the outer cups of each sector. Applying DS k-times gives then the typical module in T_{n-k} given by the cup diagram of L with all $\vee$'s removed. Applying DS^k to any other irreducible module in the same block will result in the same typical weight. We call this unique irreducible module the core of the block L^{core}. Hence $DS^k(L) = m_1(L) \cdot L^{core} \oplus m_2(L) \cdot L^{core}[1]$ for some $m_1(L), m_2(L)$ which can be calculated explicitely from the structure of the cup diagram of L. However by the sign rule of the main theorem one of $m_1(L)$ and $m_2(L)$ is zero. Hence

$$t^p_{DS^k(L)}(id_{DS^k(L)}) = t^p_{DS^k(L)}(id_{m_1(L)L^{core}}) \neq 0.$$

Therefore this defines a nontrivial trace on I_k.
We define

$$sdim_k(M) := t_M(id_L).$$

Now one should determine $m(L)$. For L maximally atypical $DS^k(L) = V$ for some super vector space of superdimension $(-1)^{p(\lambda)} m(\lambda)$ by Weissauer's superdimension formula where $m(\lambda)$ is some constant explicitely given in terms of the cup diagram of $L = L(\lambda)$. On the other hand $DS^n(L) : \mathcal{T}_n \to T_0 = svec$ gives a super vector space of the same superdimension, so clearly $m(L) = m(\lambda)$. Note that $m(\lambda)$ can be defined equally well for any block of atypicality k (using Weissauer's recursion formula) and this formula ignores any crosses and circles. Likewise the rule for $m(L)$ given by the refined conjecture ignores any crosses and circles as well, hence

$$m(L) = m(\lambda).$$

6.6 Theorem. *For any irreducible module $L(\lambda)$ of atypicality k we have*

$$sdim_k(L(\lambda)) = \pm m(\lambda) \cdot sdim_0 L^{core}$$

for some sign as in [HW13].

7. The Tannaka group of an irreducible representation

The formalism of cohomological tensor functors permits us get information about the $\mathcal{R}_n$-case from the $\mathcal{R}_{n-1}$-case. Hence the $Gl(2|2)$-case plays a special role.

7.1. Tensor product decomposition in the $\mathcal{R}_2$-case

We compute the tensor product of any two maximally irreducible modules in $\mathcal{R}_2$. The summands in $S^i \otimes S^j$ which are not maximally atypical are computed for any $n \geq 2$.

7.1.1. The $\mathcal{R}_2$-case: Setup

Every maximally atypical irreducible representation $L(\lambda) = [\lambda_1, \lambda_2]$ is a Berezin twist of a representation of the form $S^i := [i, 0]$ for $i \in \mathbb{N}$. The Ext-quiver of the maximal atypical block Γ of $\mathcal{R}_2$ can be easily determined from [BS10a]. It has been worked out by [Dro09]. For all irreducible modules in Γ we have $dimExt^1(L(\lambda), L(\mu)) = dimExt^1(L(\mu), L(\lambda)) = 0$ or 1. The Ext-quiver can be picturised as follows where a line segment between two irreducible modules denotes a non-trivial extension class between these two modules and where an irreducible module $[x, y]$ is represented as a point in $\mathbb{Z}^2$.

$$\begin{tikzcd}
 & & & & & \cdots \arrow[d, dash] & \\
 & & & & B^{j+3} \arrow[r, dash] \arrow[d, dash] \arrow[urr, dash] & B^{j+3}S^1 \arrow[r, dash] \arrow[d, dash] \arrow[dll, dash] & \cdots \\
 & & & B^{j+2} \arrow[r, dash] \arrow[d, dash] & B^{j+2}S^1 \arrow[r, dash] \arrow[d, dash] \arrow[dll, dash] & B^{j+2}S^2 \arrow[r, dash] \arrow[d, dash] & \cdots \\
 & & B^{j+1} \arrow[r, dash] \arrow[d, dash] & B^{j+1}S^1 \arrow[r, dash] \arrow[d, dash] \arrow[dll, dash] & B^{j+1}S^2 \arrow[r, dash] \arrow[d, dash] & B^{j+1}S^3 \arrow[r, dash] \arrow[d, dash] & \cdots \\
 & B^j \arrow[r, dash] \arrow[d, dash] & B^jS^1 \arrow[r, dash] \arrow[d, dash] \arrow[dll, dash] & B^jS^2 \arrow[r, dash] \arrow[d, dash] & B^jS^3 \arrow[r, dash] \arrow[d, dash] & B^jS^4 \arrow[r, dash] \arrow[d, dash] & \cdots \\
\cdots \arrow[r, dash] & \cdots \arrow[r, dash] & \cdots \arrow[r, dash] & \cdots \arrow[r, dash] & \cdots \arrow[r, dash] & \cdots \arrow[r, dash] & \cdots
\end{tikzcd}$$

The Loewy structure of the projective covers of a maximally atypical irreducible module can also be computed from [BS10a] or be taken from Drouot: For $[a,b], a = b+k, k \geq 3$ the Loewy structure is

$$\begin{pmatrix} B^{a-k}Sk \\ B^{a-k}S^{k+1} \oplus B^{a-k}S^{k-1} \oplus B^{a-k-1}S^{k+1} \oplus B^{a-k+1}S^{k-1} \\ 2B^{a-k}S^{k} \oplus B^{a-k-1}S^{k+2} \oplus B^{a-k-1}S^{k} \oplus B^{a-k+2}S^{k-3} \\ B^{a-k}S^{k+1} \oplus B^{a-k}S^{k-1} \oplus B^{a-k-1}S^{k+1} \oplus B^{a-k+1}S^{k-1} \\ B^{a-k}S^{k} \end{pmatrix}.$$

For $[a,b], a = b+2$ the Loewy structure is

$$\begin{pmatrix} B^{a-2}S^{2} \\ B^{a-2}S^{3} \oplus B^{a-2}S^{1} \oplus B^{a-3}S^{3} \oplus B^{a-1}S^{1} \\ 2B^{a-2}S^{2} \oplus B^{a-3}S^{4} \oplus B^{a-3}S^{2} \oplus B^{a-1}S^{2} \oplus B^{a-1} \oplus B^{a-2} \\ B^{a-2}S^{3} \oplus B^{a-2}S^{1} \oplus B^{a-3}S^{3} \oplus B^{a-1}S^{1} \\ B^{a-2}S^{2} \end{pmatrix}.$$

For $[a,b], a = b+1$ the Loewy structure is

$$\begin{pmatrix} B^{a-1}S^{1} \\ B^{a-1}S^{2} \oplus B^{a-1} \oplus B^{a-2}S^{2} \oplus B^{a} \oplus B^{a-2} \\ 2B^{a-1}S^{1} \oplus B^{a-2}S^{3} \oplus B^{a-2}S^{1} \oplus B^{a}S^{1} \oplus \\ B^{a-1}S^{2} \oplus B^{a-1} \oplus B^{a-2}S^{2} \oplus B^{a} \oplus B^{a-2} \\ B^{a-1}S^{1} \end{pmatrix}.$$

For $[a,b], a = b$ the Loewy structure is

$$\begin{pmatrix} B^{a} \\ B^{a}S^{1} \oplus B^{a-1}S^{1} \oplus B^{a+1}S^{1} \\ 2B^{a} \oplus B^{a-1} \oplus B^{a-2} \oplus B^{a-1}S^{2} \oplus B^{a}S^{2} \oplus B^{a+1} \oplus B^{a+2} \\ B^{a}S^{1} \oplus B^{a-1}S^{1} \oplus B^{a+1}S^{1} \\ B^{a} \end{pmatrix}.$$

7.1.2. The $\mathcal{R}_2$-case: Mixed tensors

We specialise the general formula for the symmetric powers to the $Gl(2|2)$-case. There all the $R(a,b)$ are projective as in 5.4. In this case the following decompositions hold after projection to the maximal atypical block.

First the result $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ for $i, j \leq 2$: We have

$$\begin{aligned}
\mathbb{A}_{S^1} \otimes \mathbb{A}_{S^1} =& \mathbb{A}_{S^2} \oplus 2\mathbb{A}_{S^1} \oplus \mathbb{A}_{S^2}^{\vee} \\
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^1} =& \mathbb{A}_{S^{i+1}} \oplus 2 \cdot \mathbb{A}_{S^i} \oplus \mathbb{A}_{S^{i-1}} \oplus P[i-1, 0]. \\
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^2} =& \mathbb{A}_{S^{i+2}} \oplus 2 \cdot \mathbb{A}_{S^{i+1}} \oplus \mathbb{A}_{S^i} \\
& \oplus P([i,0]) \oplus P([i-1,1] \oplus 2 \cdot P([i-1,0]) \oplus P([i-2,0])
\end{aligned}$$

where we assumed $i > 1$ respectively $i > 2$.
Assume now $i > 2,\ j \geq 2$ und $i \neq j$ (for $i = j$ see below), wlog $i > j$.

$$\begin{aligned}
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\
& \oplus P[i+j-2, 0]) \\
& \oplus P[i+j-3, 1] \oplus 2 \cdot P[i+j-3, 0] \\
& \oplus P[i+j-4, 2] \oplus 2 \cdot P[i+j-4, 1] \oplus P[i+j-4, 0] \\
& \oplus P[i+j-5, 3] \oplus 2 \cdot P[i+j-5, 2] \oplus P[i+j-5, 1] \\
& \oplus P[i+j-6, 4] \oplus 2 \cdot P[i+j-6, 3] \oplus P[i+j-6, 2] \\
& \oplus P[i+j-7, 5] \oplus \ldots \\
& \oplus P[i-1, j-1] \oplus 2 \cdot P[i-1, j-2] \oplus P[i-1, j-3] \\
& \oplus P[i-2, j-2].
\end{aligned}$$

For $i = j = 2$

$$\mathbb{A}_{S^2} \otimes \mathbb{A}_{S^2} = \mathbb{A}_{S^4} \oplus 2\mathbb{A}_{S^3} \oplus \mathbb{A}_{S^2} \oplus P[2,0] \oplus P[0,0] \oplus 2P[1,0].$$

For $i = j > 2$ we have

$$\begin{aligned}
\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& \mathbb{A}_{S^{i+j}} \oplus 2 \cdot \mathbb{A}_{S^{i+j-1}} \oplus \mathbb{A}_{S^{i+j-2}} \\
& \oplus P[i+j-2, 0]) \\
& \oplus P[i+j-3, 1] \oplus 2 \cdot P[i+j-3, 0] \\
& \oplus P[i+j-4, 2] \oplus 2 \cdot P[i+j-4, 1] \oplus P[i+j-4, 0] \\
& \oplus P[i+j-5, 3] \oplus 2 \cdot P[i+j-5, 2] \oplus P[i+j-5, 1] \\
& \oplus P[i+j-6, 4] \oplus 2 \cdot P[i+j-6, 3] \oplus P[i+j-6, 2] \\
& \oplus P[i+j-7, 5] \oplus \ldots \\
& \oplus P[i-2, i-2] \oplus 2 \cdot P[i-1, i-2] \oplus P[i-1, i-3].
\end{aligned}$$

7.1.3. The $\mathcal{R}_2$-case: K_0-decomposition

The tensor product decomposition of the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ along with the knowledge of the composition factors of the indecomposable summands permits to give recursive formulas for the K_0-decomposition of the tensor products $S^i \otimes S^j$. Due to the asymmetry of the formulas and the asymmetry of the $K_=$-decompositions for $\mathbb{A}_{S^i}$ and $P[a,b]$ for small i and $a-b$ we compute the tensor products for small i and j first. As before the K_0-decomposition is only derived up to contributions which are not in Γ.

The K_0-decomposition $S^1 \otimes S^1$ follows immediately from the $\mathbb{A}_{S^1} \otimes \mathbb{A}_{S^1}$-decomposition since all other factors are known. We get

$$S^1 \otimes S^1 = 2\mathbb{1} \oplus 2S^1 \oplus B \oplus B^{-1} \oplus B^{-1}S^2 \oplus S^2.$$

Similarly one computes

$$S^2 \otimes S^1 = 2S^2 + S^3 + B^{-1}S^3 + S^1 + BS^1$$

and

$$S^2 \otimes S^2 = S^4 + B^{-1}S^4 + 2S^3 + S^2 + BS^2 + 2BS^1 + \mathbf{1} + 2B + B^2.$$

7.1 Lemma. *For $i \geq 1$ we have $P[i,0] = 2\mathbb{A}_{S^{i+1}} + B^{-1}\mathbb{A}_{S^{i+2}} + B\mathbb{A}_{S^i}$.*

Proof. This is just a direct inspection of the Loewy structures above. $\square$

7.2 Lemma. *For all $i > j$ we have in the Grothendieck group*

$$\begin{aligned} S^i \otimes S^j =& 2(S^{i+j-1} + BerS^{i+j-3} + \cdots + Ber^{j-1}S^{i-j+1}) \\ &+ S^{i+j}(1 + Ber^{-1}) + \cdots + Ber^b S^{i-j}(1 + Ber^{-1}) \ . \end{aligned}$$

For $i = j$ we get

$$\begin{aligned} S^i \otimes S^i =& 2(S^{2i-1} + BerS^{2i-3} + \cdots + Ber^{i-1}S^1) \\ &+ S^{2i}(1 + Ber^{-1}) + \cdots + Ber^i(1 + Ber^{-1}) + B^{i-1} + B^{i-2} \ . \end{aligned}$$

Proof. We first consider the cases $S^i \otimes S^1$ and $S^i \otimes S^2$ for $i > 1$ respectively $i > 2$. The case $S^i \otimes S^1$, $i > 1$: For the induction start $i = 2$ see above. Put $C_i = S^i \otimes S^1$ in $K_0(\mathcal{R}_n)$. For $i \geq 4$ we get then the uniform formula $S^i \otimes S^1 + 2S^{i-1} \otimes S^1 + S^{i-2} \otimes S^1 = (S^{i+1} + 2S^i + S^{i-1}) + (S^{i-1} + 2S^{i-2} + S^{i-3}) + (2C_{i-1} + Ber^{-1}S^{i+1} + Ber^{-1}S^{i-1} + BerS^{i-1} + BerS^{i-3})$. Hence using the induction assumption $S^{i-2} \otimes S^1 = 2S^{i-2} + S^{i-1} + Ber^{-1}S^{i-1} + S^{i-3} + BerS^{i-3}$ we get $S^i \otimes S^1 =$

$2S^i + S^{i+1} + S^{i-1} + Ber^{-1}S^{i+1} + BerS^{i-1}$, and this proves the induction step. Likewise for $S^i \otimes S^2$.
Now assume $i > j > 2$. Then for $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ we get the regular formula in $K_0(\mathcal{R}_n)$

$$\begin{aligned}\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& S^i \otimes S^j + 4(S^{i-1} \otimes S^{j-1}) + 2(S^{i-1} \otimes S^j) + \\ & 2(S^{i-1} \otimes S^{j-2}) + 2(S^i \otimes S^{j-1}) + S^i \otimes S^{j-2} + \\ & 2(S^{i-2} \otimes S^{j-1}) + S^{i-2} \otimes S^j + S^{i-2} \otimes S^{j-2}.\end{aligned}$$

All tensor products except $S^i \otimes S^j$ are known by induction. On the other hand this sum equals $\mathbb{A}_{S^{i+j}} + 2\mathbb{A}_{S^{i+j}} + \mathbb{A}_{S^{i+j-2}} + P[i+j-2,0] + 2P[i+j-3,0] + P[i+j-4,0](1+B) + 2BP[i+j-5,0] + BP[i+j-6,0](1+B) + \ldots + 2B^{j-2}P[i-j+1,0] + B^{j-2}P[i-j,0](1+B)$. Plugging in $P[a,0] = 2\mathbb{A}_{S^{a+1}} + B^{-1}\mathbb{A}_{S^{a+2}} + \mathbb{A}_{S^a}$ for all $a \geq 1$ and comparing terms with the same B-power on both sides finishes the proof. The case $i = j$ works exactly the same way. $\square$

7.1.4. The $\mathcal{R}_2$-case: Socle Estimates

We say $w(M) = k$ for a module M, if $M^\vee \cong Ber^{-k}M$. Examples: $w(S^i) = i - 1$ and $w(Ber) = 2$, and therefore

$$w(S^i \otimes S^j) \ = \ i + j - 2 \ .$$

On the other hand for $*$-selfdual modules M we have

$$soc(M) \cong cosoc(M) \ ,$$

since $*$-duality is trivial on semisimple modules. On the other hand $w(M) = k$ implies $soc(M)^\vee \cong Ber^{-k}cosoc(M)$, so that both conditions together imply $w(soc(M)) = k$. Hence being semi-simple, it is a direct sum of modules

$$soc(M) \cong soc'(M) \oplus \bigoplus_{\nu \in \mathbf{Z}} m(\nu) \cdot Ber^\nu S^{k+1-2\nu}$$

with $S^i = 0$ for $i < 0$ and certain multiplicities $m(\nu)$, plus a sum $soc'(M)$ of modules of type

$$\big(Ber^\nu \oplus Ber^{k-\nu-j+1}\big) S^j$$

for certain $\nu \in \mathbf{Z}$ and certain natural numbers j with $k - \nu - j + 1 \neq \nu$.

7.3 Proposition. *For $i > j \geq 2$ we have $soc'(M) = 0$ for $M = S^{i-1} \otimes S^{j-1}$ and*

$$soc(S^{i-1} \otimes S^{j-1}) \hookrightarrow 3 \cdot S^{i+j-3} \oplus 2 \cdot BerS^{i+j-5} \oplus \cdots \oplus 2 \cdot Ber^{j-2}S^{i-j+1} .$$

For $i = j \geq 2$ we have

$$soc(S^{i-1} \otimes S^{i-1}) \hookrightarrow 3 \cdot S^{2i-3} \oplus 2 \cdot BerS^{2i-5} \oplus \cdots \oplus 2 \cdot Ber^{i-2}S^{1} \oplus B^{i-4} .$$

Proof: . Assume $i > j$. Notice $soc(M) \hookrightarrow soc(\mathbf{A}_i \otimes \mathbf{A}_j)$ and by the above formulas the latter is

$$\begin{aligned} S^{i+j-1} \oplus & 3S^{i+j-2} \oplus 3S^{i+j-3} \oplus (Ber \oplus \mathbf{1})S^{i+j-4} \oplus 2BerS^{i+j-5} \\ & \oplus (Ber \oplus \mathbf{1})BerS^{i+j-6} \oplus 2Ber^2S^{i+j-7} \oplus \cdots \\ & \oplus (Ber \oplus \mathbf{1})Ber^{j-2}S^{i-j} \oplus 2Ber^{j-2}S^{i-j+1} . \end{aligned}$$

Since $k = w(M) = (i-1) - 1 + (j-1) - 1 = i + j - 4$, this implies the assertion $soc'(M) = 0$. Indeed the terms $S^{i+j-1} \oplus 3S^{i+j-2}$ and also $N = (Ber \oplus \mathbf{1})Ber^{\nu}S^{i+j-4-2\nu}$ can not contribute to $soc'(M)$, since

$$\begin{aligned} N^{\vee} =& (Ber^{-1} \oplus \mathbf{1})Ber^{-\nu}Ber^{-i-j+3+2\nu}S^{i+j-4-2\nu} \\ =& (Ber^{-1} \oplus \mathbf{1})Ber^{-i-j+3+\nu}S^{i+j-4-2\nu} \end{aligned}$$

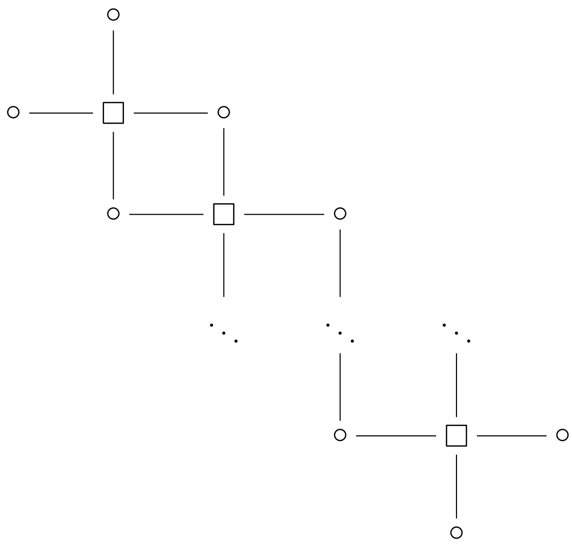
and

$$\begin{aligned} Ber^{-k}N =& Ber^{-k}(Ber \oplus \mathbf{1})Ber^{\nu}S^{i+j-4-2\nu} \\ =& (Ber^2 \oplus Ber)Ber^{-i-j+3+\nu}S^{i+j-4-2\nu} \end{aligned}$$

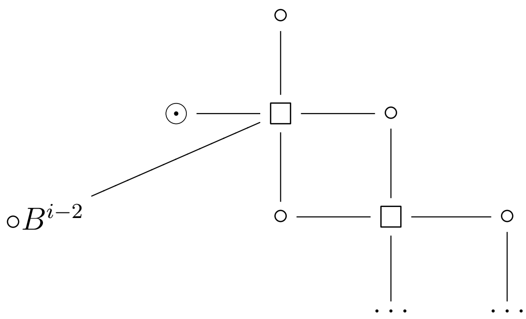
have no common irreducible summand. Hence $soc(M)$ is contained in $3 \cdot S^{i+j-3} \oplus 2 \cdot BerS^{i+j-5} \oplus \cdots \oplus 2 \cdot Ber^{j-2}S^{i-j+1}$. The proof is analogous for $i = j$. $\square$

7.1.5. The $\mathcal{R}_2$-case: Indecomposability

If we display the composition factors of $S^i \otimes S^j$ in the weight lattice of Γ, we get the following picture. Here $\square$ denotes composition factors occuring with multiplicity 2 and the $\circ$ appear with multiplicity 1. The socle is contained in the subset of composition factors denoted by $\square$.



with the two $\circ$ to the upper left at position $B^j S^{i-j}$ and $B^{j-1}S^{i-j}$ and the ones to the lower right at position $B^{-1}S^{i+j}$ and S^{i+j}. The picture in the $i = j$-case is similar



with the composition factor $\odot$ at position B^{i-1} appearing with multiplicity 2 and the additional $\circ$ at position B^{i-2}.

In the $Gl(1,1)$-case $S^i \simeq B^i$ and hence $S^i \otimes S^j = S^{i+j}$. We have already seen $DS(S^i) = S^i + B^{-1}[1-i]$ and $DS(B) = B[-1]$. Hence $DS(S^i \otimes S^j)$ splits into four indecomposable summands each of superdimension 1 or of superdimension -1:

$$\begin{aligned} DS(S^i \otimes S^j) &= (S^i \oplus B^{-1}) \otimes (S^j \oplus B^{-1}[1-j]) \\ &= B^{i+j} \oplus B^{i-1}[1-j] \oplus B^{j-1}[1-i] \oplus B^{-2}[2-i-j]. \end{aligned}$$

Hence $M = S^i \otimes S^j$ splits into at most four indecomposable summands of $sdim \neq 0$.

7.4 Lemma. *Every atypical direct summand is *-invariant.*

Proof: . If I is a direct summand which is not *-invariant, M contains I^* as a direct summand and $[I] = [I^*]$ in $K_0(\mathcal{R}_n)$. However any summand of length > 1 must contain a factor of type $\circ$ which occur in M only with multiplicity 1. Contradiction. $\square$

7.5 Corollary. *The superdimension of any maximally atypical summand is* $\neq 0$.

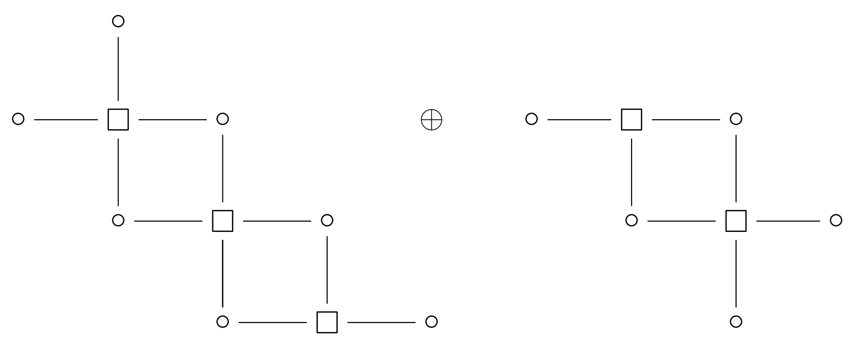
Proof. M does not contain any projectice cover (look at composition factors). If $sdim(I) = 0$, $DS(I) = 0$. However $ker(DS) = AKac$ which are not *-invariant, unless they are projective. $\square$

Assume $i > j$. By $*$-invariance the Loewy length of a direct summand is either 1 or 3. If I is irreducible, then necessarily $I = \square$ for a composition factor of the socle. By socle considerations both $\square$ will split as direct summands. The remaining module has superdimension zero, hence the Loewy length of a direct summand is 3. Fix a composition factor of type $\square$. The multiplicity of $\square$ in the socle cannot be 2. If the multiplicity of $\square$ in the socle is zero, then $\square$ has to be in the middle Loewy layer. But this would force composition factors of type $\circ$ to be in the socle. Contradiction. Hence

7.6 Corollary. *For* $i > j$

$$soc(S^i \otimes S^j) \;=\; S^{i+j-1} \oplus Ber S^{i+j-3} \oplus \cdots \oplus Ber^{j-1} S^{i-j+1}.$$

Assume $i > j$. Then the superdimension of a direct summand is either 2 or 4. Hence M is either indecomposable or splits into two summands $M = I_1 \bigoplus I_2$ of superdimension 2. If M would split, it would split in the following way:



Recall that $d(M) = H^+(M) - H^-(M)$ defines a ring homomorphism $d : K_0(\mathcal{R}_n) \to$

$K_0(\mathcal{R}_{n-1})$ fitting into the commutative diagram

$$\begin{tikzcd} K_0(T_n) \arrow[r] \arrow[d, "DS"'] & K_0(\mathcal{R}_n) \arrow[d, "d"] \\ K_0(T_{n-1}) \arrow[r] & K_0(\mathcal{R}_{n-1}) \end{tikzcd}$$

where the horizontal maps are surjective ring homomorphisms defined by $\Pi \mapsto -1$.

We know

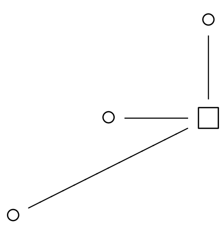
$$d(S^i \otimes S^j) = B^{i+j} + (-1)^{1-j} B^{i-1} + (-1)^{1-i} B^{j-1} + (-1)^{2-i-j} B^{-2}.$$

Since DS maps Anti-Kac modules to zero, $d()$ of any square with edges $B^k S^i, B^{k+1} S^{i-1}, B^{k+1} S^i, B^k S^{i+1}$ is zero. Hence $d(I_2)$ is given by applying d to the hook in the lower right $d(S^{i+j} + S^{i+j-1} + B^{-1} S^{i+j})$ and to $(B^v S^{i+j+1-2v} + B^v S^{i+j-2v})$ from the upper left of I_2. We get $d(I_2) = B^{i+j} + (-1)^{i-j} B^{-2} + (-1)^v B^{i+j+1-v} + (-1)^v B^{i+j-v}$ with the two additional summands $(-1)^v B^{i+j+1-v} + (-1)^v B^{i+j-v}$. Contradiction, hence M is indecomposable.

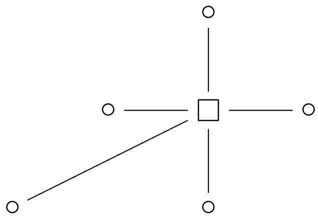
Now assume $i = j$. By the socle estimates for $S^i \otimes S^i$ and $*$-duality either B^{i-1} splits as a direct summand or both B^{i-1} lie in the middle Loewy layer. Note that $Hom(B^{i-1}, S^i \otimes S^i) = Hom(B^{i-1} \otimes (S^i)^\vee, S^i) = End(S^i) = k$, hence the last case cannot happen. Hence B^{i-1} splits as a direct summand. We show that the remaining module M' in $S^i \otimes S^i = B^{i-1} \oplus M'$ is indecomposable. As in the $i > j$-case the Loewy length of any direct summand of M' must be 3. As before we obtain for $i = j$

$$soc(S^i \otimes S^i) \;=\; S^{2i-1} \oplus Ber S^{2i-3} \oplus \cdots \oplus Ber^{i-1} S^1 \oplus B^{i-1}.$$

The remaining part M' can either split into three indecomposable modules of superdimension one each, in a direct sum of two modules of superdimension one respectively two or is indecomposable. One cannot split the upper left $\tilde{I}$



as a direct summand since its superdimension is -1. Similarly one cannot split



as a direct summand since the remaining module would have superdimension zero. Since all composition factors except the B's have superdimension zero, M' can split only into $M' = I_1 \oplus I_2$ with $sdim(I_1) = 1$ and $sdim(I_2) = 2$ with I_2 as above. We argue now as in the $i > j$-case. We have

$$d(M) = B^{2i} + (-1)^{1-i} B^{i-1} + (-1)^{1-i} B^{i-1},$$

but $d(I_2)$ has four summands as in the $i > j$-case. Contradiction, hence M is indecomposable.

7.7 Corollary. *$S^j \otimes S^i \simeq M$ $j \neq i$ where M is indecomposable with Loewy structure*

$$\begin{pmatrix} S^{i+j-1} \oplus Ber S^{i+j-3} \oplus \cdots \oplus Ber^{j-1} S^{i-j+1} \\ S^{i+j}(1 + Ber^{-1}) + \cdots + Ber^j S^{i-j}(1 + Ber^{-1}) \\ S^{i+j-1} \oplus Ber S^{i+j-3} \oplus \cdots \oplus Ber^{j-1} S^{i-j+1} \end{pmatrix}$$

and $S^i \otimes S^i = B^{i-1} \oplus M'$ where M' is indecomposable with Loewy structure

$$\begin{pmatrix} S^{2i-1} \oplus Ber S^{2i-3} \oplus \cdots \oplus Ber^{i-1} S^1 \\ S^{2i}(1 + Ber^{-1}) + \cdots + Ber^i S^0(1 + Ber^{-1}) + B^{i-2} \\ S^{2i-1} \oplus Ber S^{2i-3} \oplus \cdots \oplus Ber^{i-1} S^1 \end{pmatrix}.$$

Remark. A similar result has been obtained in the $psl(2|2)$-case in [GQS05]. However the authors do not give any proofs.

7.1.6. Typical contributions

We compute the remaining contributions to the tensor product $S^i \otimes S^j$ in $\mathcal{R}_n$ for $n \geq 2$.

7.8 Lemma. *$\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ is a direct sum of maximally atypical summands and $(n-2)$-times atypical irreducible representations. Likewise for $\mathbb{A}_{\Lambda^i} \otimes \mathbb{A}_{\Lambda^j}$. The $(n-2)$-times atypical summands are irreducible.*

Proof: . In the decomposition of $lift((i) \otimes (j))$ in R_t the bipartitions which will not contribute to the maximal atypical block are of the form

$$(i+j-k, k); (2^r, 1^{i+j-2r})]$$

for some $k, r \geq 0$ and $k \neq r$ 5. We have

$$\begin{aligned} I_\wedge =& \{i+j-k, k-1, -2, -3, -4, \ldots\} \\ I_\vee =& \{-1, 0, 1, \ldots, r-2, r, r+1, \ldots, i+j-r-1, i+j-r+1, \ldots\} \end{aligned}$$

Since $k \neq r$, neither one of the two conditions $i+j-k = i+j-r$, $k-1 = r-1$ is fulfilled, hence the two sets intersect at two points, hence the weight diagram of any such bipartition has two crosses and two circles. Clearly the weight diagrams do not have any $\vee\wedge$-pair, hence the corresponding modules are irreducible. $\square$

7.9 Lemma. *The direct summands of $S^i \otimes S^j$ which are not maximally atypical are given by the set*

$$R((i+j-k,k);(2^r,1^{i+j-2r})), \;\; k,r = 0,1,\ldots,min(i,j), \;\; k \neq r.$$

All these modules are $(n-2)$-fold atypical irreducible.

Proof: . This is again a recursive determination from the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ tensor products. As before the $S^i \otimes S^1$ and $S^i \otimes S^2$-cases for $i \geq 1$ respectively $i \geq 2$ should be treated separately. For $S^i \otimes S^j$, $i,j \geq 3$ we obtain the regular formulas

$$\begin{aligned} \mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j} =& (S^i + 2S^{i-1} + S^{i-2}) \otimes (S^j + 2S^{j-1} + S^{j-2}) \\ =& S^i \otimes S^j + \text{ lower terms} \end{aligned}$$

where the lower terms are known by induction. In the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ tensor product the $R(,)$'s from above cannot occur (for degree reasons) in any tensor product $\mathbb{A}_{S^p} \otimes \mathbb{A}_{S^q}$ for $p \leq i$, $q \leq j$ where either $p < i$ or $q < j$. Hence they cannot occur in any tensor product decomposition of any $S^p \otimes S^q$ for p,q as above, hence they have to occur in the $S^i \otimes S^j$-decomposition. The number of these modules is $(min(i,j)^2 - min(i,j)$. Substracting the inductively known numbers of not maximally atypical contributions in $S^p \otimes S^q$ in the $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$-tensor product from the number of all such contributions in $\mathbb{A}_{S^i} \otimes \mathbb{A}_{S^j}$ we get $(min(i,j)^2 - min(i,j)$ remaining modules. Hence there are no other summands in $S^i \otimes S^j$. $\square$

7.10 Lemma. *The highest weight of the irreducible module $R((i+j-k,k);(2^r,1^{i+j-2r}))$ is given by*

$$L(i+j-k, k, 0, \ldots, 0|0, \ldots, 0, -r, -i-j+r).$$

Proof: Let M denote the maximal coordinate of a cross or circle in the weight diagram of the bipartition. To obtain the weight diagram of the highest weight we have to switch all labels to the right of this coordinate as well as the first $M-n+2$ labels to its left which are not labelled $\times$ or $\circ$. Since we have four such labels this amounts to switching all the labels at positions ≥ -1 and $< M$ (all of them $\vee$'s) and the $n-2$ $\wedge$'s at positions $-2, \ldots, -n+1$ to $\vee$'s. The crosses are at the positions $i+j-k, k-1$ and the circles at the positions $i+j-r, r-1$. The result follows. $\square$

For a maximally atypical weight $[\lambda_1, \ldots, \lambda_n| - \lambda_n, \ldots, -\lambda_1]$ denote by $L_0(\lambda_1, \ldots, \lambda_n) \boxtimes L_0(-\lambda_n, \ldots, -\lambda_1)$ the underlying irreducible $Gl(n) \times Gl(n)$-module. Denote by π the following map from irreducible $Gl(n) \times Gl(n)$ modules to irreducible $Gl(n,n)$-modules:

$$\pi((L_0(\lambda_1, \ldots, \lambda_n) \boxtimes L_0(\mu_1, \ldots, \mu_n)) \\ = \begin{cases} 0 & L(\lambda_1, \ldots, \lambda_n|\mu_1, \ldots, \mu_n) \in \Gamma \\ L(\lambda_1, \ldots, \lambda_n|\mu_1, \ldots, \mu_n) & else \end{cases} .$$

7.11 Corollary. *The not maximally atypical contributions to $S^i \otimes S^j$ are given by*

$$\pi(\ (L_0(i, 0, \ldots, 0) \boxtimes L_0(0, \ldots, 0, -i)\ ) \otimes (\ L_0(j, 0, \ldots, 0) \boxtimes L_0(0, \ldots, 0, -j)\ ).$$

We will see that the maximally atypical part of $S^i \otimes S^j$ is always given by

$$S^i \otimes S^j = \begin{cases} M_1 \oplus M_2 & i = j \\ M_{ij} & i \neq j \end{cases}$$

for some indecomposable modules M_1, M_2 respectively M_{ij}. Together with the decomposition of the not maximally atypical part above this gives the complete tensor product decomposition of the $\mathrm{S}^i \otimes S^j$ (or their Berezin twists) in $\mathcal{R}_n$ up to projective covers.

Example. The tensor product $S^1 \otimes S^1$ has the $(n-2)$-times atypical contributions $R(2;2) \oplus R(1^2;1^2)$. Both modules are irreducible with highest weight $L(2,0,\ldots,0|0,\ldots,0,-1,-1)$ respectively $L(1,1,0,\ldots,0|0,\ldots,0,-2)$.

7.2. The Tannaka groups

We denote by $\mathcal{R}I_n$ the full subcategory in $R = \mathcal{R}_n \oplus \mathcal{R}_n[1]$ of all objects which appear as summands in iterated tensor products of irreducible representations of superdimension ≥ 0. This includes all irreducible representations of atypicality $< n$ and all maximally atypical representations up to a parity shift. This is a pseudoabelian k-linear rigid tensor category, hence the quotient $\mathcal{R}I_n/\mathcal{N}$ is defined and tensor equivalent to $Rep(H_n, \epsilon)$ for some proreductive supergroup H_n. We prove in [HW13]

7.12 Lemma. *The functor DS induces*

$$DS : \mathcal{R}I_n \to \mathcal{R}I_{n-1}.$$

By definition all the irreducible representations in $\mathcal{R}I_n$ have superdimension ≥ 0. This inherits to all objects in $\mathcal{R}I_n$: Indeed DS preserves the superdimension and by induction we may assume that it holds in $\mathcal{R}_{n-1}$. By a characterization of Tannaka categories by [**?**] an abelian k-linear rigid tensor category such that every object has $sdim \geq 0$ is a Tannakian category. Hence the next lemma.

7.13 Lemma. *$\mathcal{R}_n$ is tensor equivalent to the representation category of a proreducive algebraic group H_n.*

Since all objects in $\mathcal{R}_n$ have superdimension ≥ 0, negligible objects map to negligible objects. More precisely the results on the kernel of DS can be used to show that the only possible maximally atypical modules of superdimension 0 in $\mathcal{R}I_n$ are projective covers. We get

7.14 Lemma. *$DS : \mathcal{R}I_n \to \mathcal{R}I_{n-1}$ induces a k-linear exact tensor functor*

$$\eta : \mathcal{R}_n \to \mathcal{R}_{n-1}.$$

It can be shown that this induces an injective homomorphism of affine k-groups $f : H_{n-1} \to H_n$.

7.15 Proposition. *The functor $\eta : Rep(H_n) \to Rep(H_{n-1})$ can be identified with the restriction functor for f.*

Now consider the case $Gl(2|2)$. Up to a Berezin-twist the irreducible modules are the S^i. Their superdimension is $(-1)^i 2$. Hence the irreducible maximally atypical modules in $\mathcal{R}I_2$ are the $X^i := \Pi^i S^i$ for $i \geq i$ and their Berezin-twists. Note that S^1 is selfdual. For an irreducible element M in $\mathcal{R}_n$ denote by H_X the tensor category generated by the image of M in $\mathcal{R}I_n/\mathcal{N}$. Then we get

$$H_{X^1} = Sl(2) \text{ and } H_{X^i} = Gl(2) \text{ for } i \geq 2$$

For the full Tannaka group H_2 we get that $H_2 \subset \prod_{\nu=0}^{\infty} Gl(2)$ is the subgroup defined by all elements $g = \prod_{\nu=0}^{\infty} g_\nu$ in the product H_2 with the property $det(g_\nu) = det(g_1)^\nu$.

In the $Gl(3|3)$-case put $L_i = \Pi^i[i,0,0]$. These modules have superdimension 3 for $i \geq 2$. If H_{L_i} is the associated Tannaka group, then

$$H_{L_i} = Gl(3).$$

More generally for the modules $L_i = \Pi^i[i,0,\ldots,0]$ one has $H_{L_i} = Gl(n)$ for $i \geq n \geq 4$. If $X = \Pi(Ber^{1-b} \otimes [2b,b,0]$ for $b > 1$, then H_X is either $Sp(6),\ SO(6)$ or $O(6)$. If $X = \Pi^{a+b}[a,b,0],\ a > b > 0,\ a \neq 2b,\ H_X = Gl(6)$.

7.16 Lemma. *([HW13]) The derived connected group $G_3 = (H_3)^\circ_{der}$ of H_3 is*

$$G_3 \simeq \prod_\lambda H_\lambda,$$

where λ runs over all $\lambda = [\lambda_1, \lambda_2, 0]$ with integers λ_1, λ_2 such that $0 \leq 2\lambda_2 \leq \lambda_1$ and

$$H_\lambda = \begin{cases} 1 & \lambda = [0,0,0] \\ Sl(2) & \lambda = [1,0,0] \\ Sl(3) & \lambda = [2+\nu,0,0], \nu \geq 0 \\ Sp(6) & \lambda = [2\lambda_2, \lambda_2, 0], \lambda_2 > 0 \\ Sl(6) & \lambda = [\lambda_1, \lambda], 0 < 2\lambda_2 < \lambda_1. \end{cases}$$

These results can be seen as a rule for the tensor product decomposition up to elements of superdimension 0. Consider for instance the tensor product of two maximally atypical irreducible elements $L(\lambda)$ and $L(\mu)$ in $\mathcal{R}I_n$ which are not Berezin-twists of each other. Then

$$L(\lambda) \otimes L(\mu) = I \ mod \ \mathcal{N}$$

for an indecomposable representation I of superdimension $sdimL(\lambda)sdimL(\mu)$. Indeed $L(\lambda)$ and $L(\mu)$ correspond to the standard representations of their Tannaka groups G_λ and G_μ. Since G_λ and G_μ are disjoint grouos in G_n, tensoring with $L(\lambda)$ and $L(\mu)$ corresponds to taking the external tensor product $st_{G_\lambda} \boxtimes st_{G_\mu}$ of the two groups. If on the other hand we consider the tensor product $L(\lambda) \otimes L(\lambda)$ this corresponds to the tensor product of the standard representation of G_λ with itself.

As an example consider the representation $X := \Pi[2,1,0]$ of $Gl(3|3)$. It can be shown that $H_X = Sp(6)$. Hence

$$X \otimes X = I_1 \oplus I_2 \oplus I_3 \ mod \ \mathcal{N}$$

with the indecomposable representations corresponding to the irreducible $Sp(6)$ representations $L(2,0,0)$, $L(1,1,0)$ and $L(0,0,0)$. Now consider the tensor product $I_1 \otimes I_1$. It decomposes as

$$I_1 \otimes I_1 = \bigoplus_{i=1}^{6} J_i \quad \mathrm{mod}\ \mathcal{N}$$

with the 6 indecomposable representations J_i corresponding to the 6 irreducible $Sp(6)$-representations in the decomposition

$$L(2,0,0) \otimes L(2,0,0) = L(4,0,0) \oplus L(3,1,0) \oplus L(2,2,0) \oplus L(2,0,0) \oplus L(1,1,0) \oplus \mathbb{1}.$$

Although $n = 3$ and the weight $[2,1,0]$ are small, such a result is impossible to achieve by a brute force calculation.

We end with some limitations of the present approach. If M is a maximally atypical elements in $\mathcal{R}I_n$ with $sdim(M) = 0$, then M is projective by the results on the kernel of DS. We have no control about elements which are not maximally atypical. Note that in the special case when X is a Berezin-twist of an S^i all such elements are mixed tensors. It is not known wether such a result should happen in general. We also do not know much about the modules I_i, for instance their Loewy structure or the elements in the socle. Note however that we can detect whether I_i is irreducible or not since an irreducible I_i will correspond to the occurence of the standard representation in the tensor product decomposition.

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