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K-Theory of Intersection Spaces

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Abstract. The construction of an intersection space in [Ban10a] assigns to certain pseudomanifolds a topological space, called intersection space. This intersection space depends on a perversity and the reduced homology with rational coefficients of the intersection space satisfies Poincaré duality across complementary perversities. Therefore, by modifications on a spatial level, this construction restores Poincaré duality for stratified pseudomanifolds. We extend Poincaré duality for certain intersection spaces as given in [Ban10a] and [Gai12] to a broader class of intersection spaces coming from two-strata pseudomanifolds whose link bundles allow a fiberwise truncation. Further properties of this class of intersection spaces are discussed, including the existence of cap products and a calculation of the signature. In [Ada74], Poincaré duality for manifolds is generalized to any homology theory given by a CW-spectrum. We combine these two approaches and show Poincaré duality in complex K-theory for intersection spaces coming from a suitable class of pseudomanifolds, including the class of two strata pseudomanifold mentioned above. Finally, for pseudomanifolds with only isolated singularities, an approach is given, where the spatial homology truncation is performed with respect to any homology theory given by a connective ring spectrum. The objects constructed are not CW-complexes, but CW-spectra. Their rational homology equals intersection homology.

Zusammenfassung. In [Ban10a] wird für eine gewisse Klasse von stratifizierten Pseudomannigfaltigkeiten ein sogenannter Schnittraum konstruiert, der von einer gegebenen Perversität abhängt. Dieser Schnittraum ist ein topologischer Raum, dessen reduzierte Homologie mit rationalen Koeffizienten Poincaré-Dualität erfüllt, wobei die beiden vorkommenden Schnitträume komplementäre Perversitäten besitzen. Durch Modifikationen auf einem räumlichen Level stellt die Schnittraum-Konstruktion daher Poincaré-Dualität für stratifizierte Pseudomannigfaltigkeiten wieder her. Wir erweitern Poincaré-Dualität für Schnitträume, wie sie in [Ban10a] und [Gai12] gegeben ist, zu Poincaré-Dualität für eine größere Klasse von Schnitträumen, die aus gewissen 2-Strata-Pseudomannigfaltigkeiten mit einem Link-Bündel, das eine faserweise Abschneidung zulässt, konstruiert werden. Für diese Art von Schnitträumen werden weitere Eigenschaften diskutiert, darunter die Existenz von Cap-Produkten, sowie eine Berechnung der Signatur. In [Ada74] wird die klassische Poincaré-Dualität für Mannigfaltigkeiten zu einer Dualitätsaussage in einer beliebigen Homologietheorie, die durch ein CW-Spektrum gegeben ist, verallgemeinert. Wir verbinden diese beiden Ansätze und zeigen Poincaré-Dualität in komplexer K-Theorie für eine geeignete Klasse von Schnitträumen, die die oben erwähnte Klasse von Schnitträumen beinhaltet. Schließlich stellen wir einen Ansatz vor, der es für Pseudomannigfaltigkeiten mit isolierten Singularitäten erlaubt, den Prozess des räumlichen Homologie-Abschneidens bezüglich einer beliebigen Homologietheorie, die durch ein konnektives Ring-Spektrum gegeben ist, durchzuführen. Die so konstruierten Objekte sind keine CW-Komplexe mehr, sondern CW-Spektra. Wir zeigen, dass ihre rationale Homologie gerade mit der Schnitthomologie übereinstimmt.

Preface

This thesis deals with the construction of invariants for stratified pseudomanifolds. These are spaces that are not manifolds but admit a stratification, such that each stratum is a manifold. In [GM80] and [GM83], Goresky and MacPherson constructed intersection homology. This topological invariant assigns to a stratified pseudomanifold X a chain complex $IC^{\overline{p}}_{\bullet}(X)$, such that for X closed and oriented, the homology $H_i(IC^{\overline{p}}_{\bullet}(X)) =: IH^{\overline{p}}_i(X)$ satisfies Poincaré duality across complementary perversities. In [GM84] (p.222, Problem 1), Goresky and MacPherson pose the problem, if there is a K-theory or bordism version of intersection homology.

Among other reasons, this led Banagl to an alternative construction, where he assigns to certain stratified pseudomanifolds X a topological space $I^{\overline{p}}X$, called intersection space ([Ban10a], Chapter 2). The idea behind this construction is to get a theory of Poincaré duality for stratified pseudomanifolds that is implemented on a spatial level. The reduced homology $\tilde{H}_i(I^{\overline{p}}X;\mathbb{Q})$ then satisfies Poincaré duality across complementary perversities. One advantage of the intersection space construction is that unlike intersection homology as constructed in [GM80] and [GM83], the intersection space now allows to apply functors that do not factor through chain complexes, such as topological K-theory. The groups $IH^{\overline{p}}_i(X)$ and $\tilde{H}_i(I^{\overline{p}}X;\mathbb{Q})$ are not isomorphic but share certain symmetries. In particular, they can be used for some problems related to the existence of massless D-branes in type II string theory. While intersection homology accounts for the massless D-branes in type IIA string theory, the homology of intersection spaces does the same in type IIB string theory. The two theories form a mirror pair for singular Calabi-Yau conifolds. More generally, the fact that intersection homology is stable under small resolutions (see [GM83]) is "mirrored" in the theory of intersection spaces by the result that the homology of the intersection space of a complex projective hypersurface with one isolated singularity is, roughly speaking, invariant under nearby smooth deformations (see [BM12]). To construct an intersection space, one assigns to a CW-complex L a CW-complex $L_{<k}$ together with a map $f : L_{<k} \to L$ such that on homology $f_* : H_i(L_{<k}) \to H_i(L)$ is an isomorphism for $i < k$ and $H_i(L_{<k}) = 0$ for $i \geq k$. This process is referred to as *spatial homology truncation*. The homology truncation is given by the topological realization of a base change of the k-th cellular chain group and is always possible if L is simply connected. Let M denote the regular part of X, then for isolated singularities the intersection space is given by $cone(L_{<k} \to M)$ and denoted by $I^{\overline{p}}X$ where L is the disjoint union of the links of the isolated singularities. The truncation value k is given by $k = n - 1 - \overline{p}(n)$, where $n = \dim X$. It is not possible to assign to every arbitrary pseudomanifold an intersection space. Examples of constructions of intersection spaces with singularities other than isolated singularities can be found in [Ban10a], [Ban12] and [Gai12].

For manifolds, Adams ([Ada74]) generalized classical Poincaré duality to an isomorphism $E^{d-r}(M) \to E_r(M)$, where E is a CW-spectrum, and M a closed, E-oriented manifold. This isomorphism is given by capping with an E-fundamental class $[M]_E \in E_d(M)$.

Let M be the manifold-with-boundary obtained by cutting off a small open neighborhood of the singular set of a two-strata pseudomanifold X^n. In many cases ∂M looks like a fiber bundle over the singular set Σ^{n-c} with fiber the link L^{c-1}. In certain situations such a fiber bundle can be truncated fiberwise to a bundle $ft_{<k}(\partial M)$. This new bundle

has fibers $L_{<k}$ which are spatial homology truncations of L. Examples of fiber bundles that can be truncated fiberwise are trivial bundles, **ICW**-bundles and $(\overline{p}, \overline{q})$-allowable bundles (as constructed in [Gai12]) or, more generally, bundles where the action of the structure group on the fiber can be restricted to a subspace $L_{<k} \subset L$. Poincaré duality for intersection spaces of two-strata pseudomanifolds has been shown in special cases, for example for trivial bundles in [Ban10a] or in [Gai12] for the bundles constructed in *loc. cit.* In this thesis, we generalize these results and show Poincaré duality for a broader class of intersection spaces coming from two-strata pseudomanifolds and including the following case.

Theorem. *Let X be a compact, oriented pseudomanifold with two strata and a link bundle $(L^{c-1}, \partial M, p, \Sigma^{n-c})$ that admits a fiberwise truncation in two complementary degrees $k = c - 1 - \overline{p}(c)$ and $c - k$ (in the sense of Definition 1.1.3). Assume that the homology truncation $L_{<k}$ is a subspace of L. Then there is a rational Poincaré duality isomorphism*

$$\tilde{H}^{n-r}(I^{\overline{p}}X; \mathbb{Q}) \to \tilde{H}_r(I^{\overline{q}}X; \mathbb{Q}).$$

The proof hereof uses a Mayer-Vietoris argument. Locally, the cohomology of the truncated bundle is calculated by a sheaf theoretic description. The sheaf theoretic construction follows [Gai12], where a special case of the above theorem is proven. As a byproduct it is shown that the map

$$H_r(ft_{<k}(\partial M); \mathbb{Q}) \to H_r(\partial M; \mathbb{Q})$$

is always injective for all $r \in \mathbb{N}$ for compact two-strata spaces that allow a fiberwise truncation at a value k. It turns out, that this injectivity condition is crucial for the generalization of Poincaré duality of intersection spaces to other homology theories. Now assume the injectivity of the map $H_r(ft_{<k}(\partial M); \mathbb{Q}) \to H_r(\partial M; \mathbb{Q})$ and furthermore, that the link bundle can be truncated in all degrees greater than some fixed value, and in all degrees lower than some other fixed value, both values depending on the perversity. We then have the situation that there may be degrees in the middle in which the bundle can not be truncated. Then we also get an isomorphism

$$\tilde{H}^{n-r}(I^{\overline{p}}X; \mathbb{Q}) \to \tilde{H}_r(I^{\overline{q}}X; \mathbb{Q}).$$

That proof does not use sheaf theory at all and relies on the analysis of the Serre spectral sequence associated to the fiber bundle ∂M. The above injectivity condition can be checked by hand in many applications. Moreover, for such pseudomanifolds that satisfy generalized Poincaré duality, the duality isomorphism can be rephrased as capping with the fundamental class $[M, \partial M]$ in a cap product of the form

$$\tilde{H}^i(I^{\overline{p}}X; \mathbb{Q}) \otimes H_j(M, \partial M; \mathbb{Q}) \to \tilde{H}_{j-i}(I^{\overline{q}}X; \mathbb{Q}).$$

Motivated by the above results on two-strata pseudomanifolds, a class Ξ of tuples of spaces $(X, B(\overline{p}))$ with a map $i : B(\overline{p}) \to \partial M$ such that the induced map $i_* : H_*(B(\overline{p})) \to H_*(\partial M)$ on homology is injective, is defined. Here M is still the regular part of a pseudomanifold X and $B(\overline{p})$ is a CW-complex depending on a perversity $\overline{p}$. For $(X, B(\overline{p})) \in \Xi$, set

$I^{B(\overline{p})}X := cone(B(\overline{p}) \to M)$. Moreover, a subclass Θ of Ξ is defined as all objects with the homology of $B(\overline{p})$, ∂M and the homology of the pair being torsion-free. Besides pairs $(ft_{<k}(\partial M), X)$ coming from the class of two-strata pseudomanifolds described above, the class Ξ includes also stratified pseudomanifolds that do not necessarily have two strata but still allow the construction of an intersection space together with a suitable truncation of the boundary of the regular part (for example as constructed in [Ban12]). We prove the following signature formula.

Theorem. *Let X be a compact, oriented pseudomanifold and $(X, B(\overline{p})) \in \Xi$ of dimension $4n$. Assume $B(\overline{p}) = B(\overline{q})$, where $\overline{p}$ and $\overline{q}$ are complementary perversities, and denote $I^{B(\overline{p})}X = I^{B(\overline{q})}X = IX$. Assume, that there is a Poincaré duality isomorphism*

$$\tilde{H}^{4n-r}(IX;\mathbb{Q}) \to \tilde{H}_r(IX;\mathbb{Q})$$

for all $r \in \mathbb{N}$. Then

$$\sigma(M, \partial M) = \sigma(IX).$$

The signature $\sigma(IX)$ is defined as the signature of the bilinear form of the middle homology of IX. This generalizes the calculation of $\sigma(IX)$ in the case of isolated singularities ([Ban10a], Theorem 2.28).
We can now determine which conditions a stratified pseudomanifold X has to satisfy such that an isomorphism

$$\tilde{K}^{n-r}(I^{\overline{p}}X;\mathbb{Z}) \cong \tilde{K}_r(I^{\overline{q}}X;\mathbb{Z})$$

exists. Here, K denotes complex K-theory. The statement remains true for other homology theories with torsion-free coefficient group and a multiplicative character map. This gives a possible answer to the problem posed in [GM84]: For a certain class of pseudomanifolds, a Poincaré duality isomorphism of the associated intersection spaces across complementary perversities in K-theory exists. The main result is the following.

Theorem. *Let X^n be a compact, K-oriented pseudomanifold with $(X, B(\overline{p}))$ and $(X, B(\overline{q}))$ in Θ for complementary perversities $\overline{p}$ and $\overline{q}$. Assume that there is a Poincaré duality isomorphism in ordinary homology*

$$\tilde{H}^{n-r}(I^{B(\overline{p})}X;\mathbb{Q}) \to \tilde{H}_r(I^{B(\overline{q})}X;\mathbb{Q}),$$

Then there is an isomorphism

$$\tilde{K}^{n-r}(I^{B(\overline{p})}X;\mathbb{Z}) \to \tilde{K}_r(I^{B(\overline{q})}X;\mathbb{Z}).$$

Corollary. *Let X^n be a compact, K-oriented 2-strata pseudomanifold with a singular set Σ^{c-1} and a link bundle ∂M that admits a fiberwise truncation in two complementary degrees k and $c-k$ where $\overline{p}(c) = c-1-k$ and $\overline{q}(c) = k-1$. Assume that $L_{<k} \subset L$ and that $H_*(\partial M;\mathbb{Z})$, $H_*(ft_{<k}(\partial M);\mathbb{Z})$ and $H_*(\partial M, ft_{<k}(\partial M);\mathbb{Z})$ are torsion-free. Then there is an integral Poincaré duality isomorphism in K-Theory*

$$\tilde{K}^{n-r}(I^{\overline{p}}X;\mathbb{Z}) \to \tilde{K}_r(I^{\overline{q}}X;\mathbb{Z}).$$

Finally, we deal with the question if the whole process of spatial truncation can be generalized to a spatial truncation with respect to another homology theory than ordinary homology. For CW-complexes, it turns out that this question is hard to attack. Therefore, CW-spectra instead of CW-complexes are taken as the objects to which truncation is applied. For pseudomanifolds with only isolated singularities, objects $XI^{\overline{p}}_E$ in the homotopy category of CW-spectra are defined. The construction of $XI^{\overline{p}}_E$ involves a truncation of the link with respect to the homology theory E, where the spectrum associated to E is a connective ring spectrum. If the truncation is performed with respect to the ordinary Eilenberg-MacLane spectrum, the reduced homology of these spectra turns out to be intersection homology, and not intersection space homology.

Theorem. *Let X be a compact, oriented, PL-stratified pseudomanifold with only isolated singularities. Let E be a connective ring CW-spectrum. Let F be a ring spectrum. Then for all $r \in \mathbb{N}$*

1. *a)* $\tilde{F}^{n-r+1}(XI^{\overline{p}}_H) \cong \tilde{F}_{r+1}(XI^{\overline{q}}_H)$,
   b) $\tilde{H}^{n-r+1}(XI^{\overline{p}}_E) \cong \tilde{H}_{r+1}(XI^{\overline{q}}_E)$ *and*
2. $\tilde{F}_{r+1}(XI^{\overline{p}}_H) \cong IF^{\overline{p}}_r(X;\mathbb{Q})$.

where the generalized intersection homology group $IF^{\overline{p}}_k(X;\mathbb{Q})$ is defined as in [Ban10b].

This thesis is divided into three chapters. The first chapter deals with Poincaré duality isomorphisms of two-strata pseudomanifolds in ordinary homology with rational coefficients. The results of this chapter are used in Chapter 2 but may also be of independent interest. The duality isomorphisms for different types of two-strata pseudomanifolds are proven in Section 1.2. In Section 1.3 a class of pseudomanifolds Ξ is defined that includes the above discussed two-strata pseudomanifolds and with which we will work in the following. The rest of this chapter deals with further properties of pseudomanifolds in Ξ, such as the existence of cap products and a calculation of the signature.
Chapter 2 determines which conditions a stratified pseudomanifold X has to satisfy such that a duality isomorphism of intersection spaces in complex K-theory is possible. We briefly discuss other generalized homology theories in Section 2.3.
Finally, in Chapter 3 we give an approach how to generalize the whole truncation process to a spatial truncation with respect to another homology theory than ordinary homology. This is done for pseudomanfiolds with only isolated singularities. For the convenience of the reader, some results from the literature that are frequently quoted, are collected in the appendix.
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1 Intersection Spaces of Two-strata Stratified Pseudomanifolds

In this chapter we define a suitable class of stratified pseudomanifolds to work with in the following chapter. The focus lies on certain two-strata pseudomanifolds with a link bundle that admits a fiberwise truncation. Poicaré duality and further properties of this class of pseudomanifolds, such as independence of choices made in the construction of the intersection space, the existence of a cap product and a signature formula are discussed. The results of this chapter may be of independent interest from the other chapters.

1.1 Fiberwise Truncation

The guiding idea in the construction of the intersection spaces that have been constructed so far, is, to cut off a neighborhood of the singular part to obtain a compact manifold with boundary M, then truncate the links in the boundary of M for a given perversity $\overline{p}$ (let us call the resulting space $B(\overline{p})$ for the moment) and build the mapping cone of the map $B(\overline{p}) \to \partial M \hookrightarrow M$. In this chapter, we focus on stratified pseudomanifolds with two strata. In this setting it is therefore natural to consider fiberwise truncated fiber bundles together with a fiber bundle map to the fiber bundle ∂M. By a *stratified pseudomanifold*, we mean a topological stratified pseudomanifold as defined in [Ban07], Definition 4.1.1 or equivalently a topological pseudomanifold as defined in [KW06], Definition 4.1.2, with a fixed stratification. In the subsequent discussion of two-strata stratified pseudomanifolds, we additionally assume that the stratified pseudomanifold has a link bundle. This is for example always the case if the pseudomanifold has a Thom-Mather stratification.

Definition 1.1.1 (Definition 1.4 in [Ban10a]). $L_{<k}$ is called a *homology truncation in degree* k of a CW-complex L, if there exists a map

$$i : L_{<k} \to L$$

that induces on homology an isomorphism $H_r(L_{<k}) \cong H_r(L)$ for $r < k$ and $H_r(L_{<k}) = 0$ for $r \geq k$.

Remark 1.1.2. *The construction of $L_{<k}$ is not canonical and involves some choices (see [Ban10a], Section 1.1). We will treat questions concerning the dependence on these choices in Section 1.4.*

In [Ban10a] (Proposition 1.6) it is proven, that for every simply connected finite CW-complex L and for all $k \in \mathbb{N}$, there exists a finite CW-complex $L_{<k}$ which is a homology truncation of L. In many cases, a homology truncation also exists, if L is not simply connected (for example $L = T^2$).

Definition 1.1.3. Let ξ be a fiber bundle with fiber a CW-complex L that allows a homology truncation $L_{<k}$ and base space B. A *fiberwise truncation at k of ξ* is a fiber bundle $ft_{<k}(\xi)$ over B with fiber $L_{<k}$ and a commutative diagram

$$\begin{tikzcd} L_{<k} \arrow[r] \arrow[d, "i|"] & ft_{<k}(\xi) \arrow[r] \arrow[d, "i"] & B \arrow[d, equal] \\ L \arrow[r] & \xi \arrow[r] & B \end{tikzcd}$$

such that the map $i| : L_{<k} \to L$ induces an isomorphism $(i|)_* : H_r(L_{<k}) \to H_r(L)$ for $r < k$.

Definition 1.1.4. Let X be a compact, oriented n-dimensional 2-strata pseudomanifold that has a link bundle $\partial M \to \Sigma^{c-1}$ that admits a fiberwise truncation at some value k. Then define the *intersection space of X with perversity $\overline{p}$* as

$$I^{\overline{p}}X := cone(ft_{<k}(\partial M) \to \partial M \to M)$$

where $k = c - 1 - \overline{p}(c)$.

Example 1.1.5. *The following fiber bundles admit a fiberwise truncation:*

1. *Trivial bundles with a simply connected CW-complex as fiber.*

   The fiberwise truncation is given by $ft_{<k}(\xi) = L_{<k} \times \Sigma$.

2. *Interleaf fiber bundles over a sphere (see Definition 2.1.2 in [Gai12]).*

   *A fiber bundle is called an interleaf fiber bundle over S^m, if the fiber $L \in Ob\boldsymbol{ICW}$ consisting of finitely many cells, the base space is a sphere S^m with $m \geq 2$ and the structure group is a subgroup of the group of cellular homeomorphisms with cellular inverses of L. Here, **ICW** is the full subcategory of the category of CW-complexes with objects simply connected CW-complexes with finitely generated even-dimensional homology and vanishing odd-dimensional homology for any coefficient group. See Section 1.9 in [Ban10a].*

   The fiberwise truncated bundle is constructed in [Gai12], Section 2.1 to 2.3.

3. *$(\overline{p}, \overline{q})$-admissible bundles (see Definition 3.1.4 of [Gai12]).*

   A $(\overline{p}, \overline{q})$-admissible fiber bundle is a fiber bundle with fiber L^n a closed, oriented manifold with finite CW-structure, the base space a closed, oriented topological manifold and the structure group a subgroup of the group of cellular homeomorphisms with cellular inverses of L. Furthermore the cellular boundary map

   $$d_k : H_k(L^k, L^{k-1}) \to H_{k-1}(L^{k-1}, L^{k-2})$$

   vanishes for $k = n - \overline{p}(n-1)$ and $k = 1 + \overline{p}(n+1)$. A fiberwise truncated bundle is defined in Definition 3.4.1 of [Gai12].

4. *More general, bundles where the action of the structure group on the fiber can be restricted to a subspace of the fiber which is a homology truncation. This class of*

examples includes $(\overline{p}, \overline{q})$-admissible bundles. We will give other examples later in this chapter.

5. *Bundles with structure group G and a fiber that allows a G-equivariant Moore decomposition. The construction of a homology truncation involves a Moore space approximation of a simply connected CW-complex L. Given a Moore space decomposition $(L_{\leq r})_{r=1,\ldots}$ the space $L_{\leq n+1}$ is constructed as the homotopy cofiber of a map $k_n : M(H_{n+1}L, n) \to L_{\leq n}$ where $M(H_{n+1}L, n)$ is a Moore space and k_n a k-invariant. Assume now that $L_{\leq n}$ is equipped with a group action of a group G and that $M(H_{n+1}L, n)$ is a G-equivariant Moore space. This yields a decomposition of L in homology truncations $L_{\leq n}$ such that the group action of G can be restricted to all $L_{\leq n}$. When working with fiber bundles, let G be the structure group of the bundle. One then automatically obtains a fiberwise truncation of the fiber bundle in question. The question when G-equivariant Moore spaces exist is treated in [Smi83], where an obstruction theory for the existence of G-equivariant Moore spaces is developed.*

The aim of this section is to develop a machinery that allows in the next section to prove that certain compact, oriented n-dimensional 2-strata pseudomanifolds that have a link bundle $\partial M \to \Sigma^{c-1}$ that admits a fiberwise truncation at some values k and $c-k$, allow a Poincaré duality isomorphism

$$\tilde{H}^{n-r}(I^{\overline{q}}X; \mathbb{Q}) \to \tilde{H}_r(I^{\overline{p}}X; \mathbb{Q})$$

(here $\overline{p}$ and $\overline{q}$ are complementary perversities and $k = c-1-\overline{p}(c)$ so that $c-k = c-1-\overline{q}(c)$). The link bundle $\partial M \to \Sigma$ is not required to allow a truncation in every degree $k \in \mathbb{N}$. The sheaf-theoretic machinery of this section 1.1 stems from [Gai12] (Chapter 3), where certain link bundles that can be trivialized over two open subsets are discussed, whereas here we cover more general link bundles. In particular, the following Lemmas 1.1.7, 1.1.12, 1.1.13 and 1.1.15 are direct generalizations and a re-organization of the material in *loc. cit.* The idea of the following construction is to take a fiber bundle ξ with fiber F that allows a fiberwise truncation. Then restrict ξ to a subbundle ξ_U with base space $U \subset \Sigma$ and truncate it to the fiberwise truncated bundle $ft_{<k}(\xi_U)$. We express the group $H^r_c(ft_{<k}(\xi_U); \mathbb{Q})$ as $\mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_U)$ for some sheaf complex $\mathbf{W}^\bullet_U$ which has the property that the stalk of the sheaf $\mathbf{H}^r(\mathbf{W}^\bullet_U)|_b$ equals the cohomology of the fiber $H^r(F_{<k}; \mathbb{Q})$ in every point $b \in \Sigma$. Equivalently, we express $H^r_c(\xi_U; \mathbb{Q})$ as $\mathcal{H}^r(\Sigma, \mathbf{V}^\bullet_U)$ for some sheaf complex $\mathbf{V}^\bullet_U$ which has the property that the stalk of the sheaf $\mathbf{H}^r(\mathbf{V}^\bullet_U)|_b$ equals $H^r(F; \mathbb{Q})$. This allows us to show in Lemma 1.1.7, that the sheaf complexes $\mathbf{W}^\bullet_U$ and $\tau_{<k}\mathbf{V}^\bullet_U$ are quasi-isomorphic. In Lemma 1.1.15 we use this fact to construct a right inverse to the map $i^*_U : H^r_c(\xi_U; \mathbb{Q}) \to H^r_c(ft_{<k}(\xi_U); \mathbb{Q})$ for all $r \in \mathbb{N}$, where i^*_U is induced by the map $i_U : ft_{<k}(\xi_U) \to \xi_U$. For the rest of this section, we work in the following setting. Let ξ be a compact Hausdorff space and the total space of a fiber bundle

$$\begin{tikzcd} F^{c-1} \arrow[r] & \xi \arrow[d, "p"] \\ & \Sigma^{n-c} \end{tikzcd}$$

with fiber F a compact CW-complex of dimension $c-1$. Assume that ξ admits a fiberwise truncation at some value k. Assume further that the base space Σ is a $(n-c)$-dimensional compact, oriented manifold that is a finite CW-complex. Later we want to apply the results of this section to the fiber bundles $\xi = \partial M$ (in Section 1.2.1) and $\xi = ft/k(\partial M)$ (in Section 1.2.2), where $ft/k(\partial M)$ is a fiber bundle with fiber L/k that is homotopy equivalent to the k-skeleton of L and whose k-th homology has a basis of cells. Let $U \subset \Sigma$ be an open subset. Then the fiber bundle ξ over Σ can be restricted to a fiber bundle ξ_U over U by restricting the transition functions to U. In the same way, we get a fiberwise truncation $ft_{<k}(\xi_U)$ given that the fiberwise truncation $ft_{<k}(\xi)$ exists. Given open subsets $V \subset U \subset \Sigma$, denote the inclusions of ξ_V and ξ_U by

$$\begin{tikzcd} \xi_V \arrow[r, "j_V"] \arrow[d, "j_{VU}"'] & \xi. \\ \xi_U \arrow[ur, "j_U"'] & \end{tikzcd}$$

If there is no confusion possible, we also denote by $j_{VU} : ft_{<k}(\xi_V) \to ft_{<k}(\xi_U)$ the map of truncated bundles. For any open subset $U \subset \Sigma$ denote

$$i_U : ft_{<k}(\xi_U) \to \xi_U.$$

Obviously the following diagram commutes.

$$\begin{tikzcd} ft_{<k}(\xi_V) \arrow[r, "i_V"] \arrow[d, "j_{VU}"'] & \xi_V \arrow[d, "j_{VU}"] \\ ft_{<k}(\xi_U) \arrow[r, "i_U"] & \xi_U \end{tikzcd}$$

Lemma 1.1.6. *For every open subset $U \subset \Sigma$, the map $i_U : ft_{<k}(\xi_U) \to \xi_U$ is a proper and closed map.*

Proof. Let $K \in \xi_U$ be a compact set. As ξ is a fiber bundle, every point in Σ has an open neighborhood U_x such that $\xi|_{U_x}$ is a trivial bundle. $\Sigma = \bigcup_x U_x$ is an open cover of the base space. As the base space Σ is a manifold, we can find compact sets $V_x \subset U_x$ for every open set U_x. Then the compact sets V_x cover the base space and over each V_x, the bundle is trivial. As Σ is compact, a finite number $V_1, ..., V_n$ already cover Σ (take an open set in every set V_i and apply compactness to this open covering). Set $W_i = K \cap (V_i \times F)$ for every $1 \le i \le n$. Clearly $W_i \subset \xi_U$. The preimage $(i_U)^{-1}(W_i)$ is given by $(i_U)^{-1}(K) \cap V_i \times F_{<k}$. As ξ_U is Hausdorff, K is closed and therefore $(i_U)^{-1}(K)$ is also closed. As $F_{<k}$ is a finite CW-complex, it is compact. This shows that $V_i \times F_{<k}$ is compact and hence $(i_U)^{-1}(W_i)$ is compact as it is the intersection of a closed subset with a compact set. Thus

$$(i_U)^{-1}(K) = \bigcup_{i=1}^{n} (i_U)^{-1}(W_i)$$

is a compact set. This shows that i_U is proper. Now U is an open subset of the manifold Σ, therefore Hausdorff and second countable. In particular U is first countable. F is a compact CW-complex. Therefore F is Hausdorff (see [Hat02] Proposition A.3) and first countable (see [FP90] Proposition 1.5.17). Therefore $U \times F$ is Hausdorff and first countable from where follows that ξ_U is Hausdorff and first countable. [Pal70] then shows that the proper map i_U is closed. $\square$

For any topological space Z, let $\underline{\mathbb{Q}}^\bullet_Z$ be the constant sheaf on Z. Then $j_{U!}\underline{\mathbb{Q}}_{\xi_U}$ and $j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}_{\xi_U}$ are sheafs on ξ. Note that $j_{U!}$ is just extension by zero. If $F_{<k} \subset F$, then i_U is an inclusion. As i_U is proper, $i_{U!} = i_{U*}$ is also just extension by zero. Define sheaf complexes, which are given by the above sheafs concentrated in degree 0, and denote them by $j_{U!}\underline{\mathbb{Q}}^\bullet_{\xi_U}$ and $j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_{\xi_U}$. Then set

$$\mathbf{V}^\bullet_U = Rp_*(j_{U!}\underline{\mathbb{Q}}^\bullet_{\xi_U})$$

and

$$\mathbf{W}^\bullet_U = Rp_*(j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_{\xi_U}).$$

These are objects in the derived category of sheaf complexes over Σ. There is a canonical map of sheaf complexes

$$\alpha_U : \mathbf{V}^\bullet_U \to \mathbf{W}^\bullet_U$$

induced by the adjunction morphism $\underline{\mathbb{Q}}_{\xi_U} \to i_{U*}i_U^*\underline{\mathbb{Q}}_{\xi_U}$. For a sheaf complex $\underline{A}^\bullet$, define the truncated sheaf complexes

$$\tau_{<k}(\underline{A}^\bullet) := \ldots \to \underline{A}^{k-2} \to \underline{A}^{k-1} \to \ker d^k \to 0 \to \ldots .$$

We perform this truncation to obtain $\tau_{<k}\mathbf{V}^\bullet_U$ and $\tau_{<k}\mathbf{W}^\bullet_U$. The map of sheaf complexes α_U induces a map

$$\alpha_U : \tau_{<k}\mathbf{V}^\bullet_U \to \tau_{<k}\mathbf{W}^\bullet_U$$

on the truncated sheaf complexes. Furthermore there is the canonical inclusion of sheaf complexes

$$\rho_U : \tau_{<k}\mathbf{W}^\bullet_U \to \mathbf{W}^\bullet_U.$$

Lemma 1.1.7. *Assume that $F_{<k}$ is a subspace of F. The morphisms of sheaf complexes*

$$\alpha_U : \tau_{<k}\,\boldsymbol{V}^\bullet_U \to \tau_{<k}\,\boldsymbol{W}^\bullet_U$$

and

$$\rho_U : \tau_{<k}\,\boldsymbol{W}^\bullet_U \to \boldsymbol{W}^\bullet_U$$

are quasi-isomorphisms for every open subset $U \subset \Sigma$.

Proof. It holds by definition that

$$\mathbf{H}^r(Rp_*(j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_\xi)) = Rp^r_*(j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_\xi).$$

The map $p : \xi \to \Sigma$ is proper, since ξ is compact. Moreover ξ is paracompact, as it is a manifold and Σ is locally compact as it is compact. We can then apply the proper base change theorem ([Har08], Theorem 4.4.17) and obtain

$$\begin{aligned}
\mathbf{H}^r(\mathbf{W}_U^\bullet)|_b &= \mathbf{H}^r(Rp_*(j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}_\xi^\bullet))|_b \\
&= Rp_*^r(j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}_\xi^\bullet)|_b \\
&\cong Rp_*^r(j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}_\xi)|_b \\
&= H^r(p^{-1}(b); j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}_\xi|_{p^{-1}(b)}) \\
&\cong H^r(p^{-1}(b); i_{U*}i_U^*\underline{\mathbb{Q}}_\xi|_{p^{-1}(b)}) \\
&= H^r(F; i_{U*}|_{p^{-1}(b)}i_U^*|_{p^{-1}(b)}\underline{\mathbb{Q}}_F) \\
&\cong H^r(F_{<k}; i_U^*|_{p^{-1}(b)}\underline{\mathbb{Q}}_F) \\
&\cong H^r(F_{<k}; \underline{\mathbb{Q}}_{F_{<k}}) \cong H^r(F_{<k}; \mathbb{Q}).
\end{aligned}$$

where the map

$$i_U^\dagger : H^r(F; i_{U*}|_{p^{-1}(b)}i_U^*|_{p^{-1}(b)}\underline{\mathbb{Q}}_F) \to H^r(F_{<k}; i_U^*|_{p^{-1}(b)}\underline{\mathbb{Q}}_F)$$

is induced by the cohomomorphism $i_U^\dagger|_{p^{-1}(b)}$ and therefore is natural. It remains to show that this map is an isomorphism. Again by the proper base change theorem, we can formulate conditions for this map to be an isomorphism in terms of vanishing conditions of the group $\tilde{H}^j(i_U^{-1}|_{p^{-1}(b)}(y); \mathbb{Q})$. To be precise, Lemma 1.1.6 and the following Lemma 1.1.8 applied to Corollary 4.4.19 of [Har08] show that $i_U^\dagger$ is an isomorphism. Equivalently $\mathbf{H}^r(\mathbf{V}_U^\bullet)|_b \cong H^r(F; \mathbb{Q})$. Naturality of the above equations leads to a commutative diagram

$$\begin{tikzcd}
\mathbf{H}^r(\mathbf{W}_U^\bullet)|_b \arrow[r, "\cong"] \arrow[d] & H^r(F_{<k}; \mathbb{Q}) \arrow[d] \\
\mathbf{H}^r(\mathbf{V}_U^\bullet)|_b \arrow[r, "\cong"] & H^r(F; \mathbb{Q}).
\end{tikzcd}$$

This shows the claim. □

Lemma 1.1.8. *Let $i_U|_{p^{-1}(b)}(y) =: f : F_{<k} \to F$ be the homology truncation of F. Assume that $F_{<k}$ is a subspace of F. Then*

$$H^r(f^{-1}(y); \mathbb{Q}) = 0$$

for all $r > 0$ and all $y \in F$.

Proof. $f^{-1}(y)$ is either y or the empty set and the claim follows. □

Lemma 1.1.9. *ξ_U and $ft_{<k}(\xi_U)$ are locally contractible. ξ_U is locally closed in ξ.*

Proof. F and $F_{<k}$ are locally contractible, as they are CW-complexes (see for example [Hat02], Proposition A.4). U is an open submanifold of Σ and therefore locally contractible. This implies that the fiber bundles ξ_U and $ft_{<k}(\xi_U)$ are locally contractible. As an open subset of ξ, ξ_U is locally closed in ξ. □

Lemma 1.1.10. *The families of compact subsets of ξ_U and $ft_{<k}(\xi_U)$ are paracompactifying for every U.*

Proof. See [Bre97], p. 22 and note that ξ_U and $ft_{<k}(\xi_U)$ are locally compact. □

For every ξ_U, write c_U for the family of compact subsets of ξ_U and $c_U^{<k}$ for the family of compact subsets of $ft_{<k}(\xi_U)$.

Lemma 1.1.11. *Assume that $F_{<k}$ is a subspace of F. The map*

$$H^r_{c_U}(\xi_U, i_{U*}i_U^*\underline{\mathbb{Q}}_{\xi_U}) \to H^r_{c_\xi^{<}|ft_{<k}(\xi_U)}(ft_{<k}(\xi_U), i_U^*\underline{\mathbb{Q}}_{\xi_U})$$

induced by the cohomomorphism $i_U^\dagger$ is an isomorphism.

Proof. This is a consequence of Theorem II.11.1 of [Bre97]. We check the assumptions of this theorem. By Lemma 1.1.8, the set $f^{-1}(y)$ is either empty or consists of a single point for all $y \in F$. Now for an element $x \in \xi_U$, a neighborhood of x has the form $V \times F$ for some open set $V \subset \Sigma$. Write $x = (v, y)$. The preimage of x is then given by $i_U^{-1}(x) = (v, f^{-1}(y))$. Therefore, the set $i_U^{-1}(x)$ also is either empty or consists of one single point. This shows $H^r(i_U^{-1}(x); \mathbb{Q}) = 0$ for all $r > 0$ and all $x \in \xi_U$. Furthermore, by Lemma 1.1.6, i_U is a closed map. Finally $i_U^{-1}(x)$ is taut in ξ_U (compare Definition 10.5 in [Bre97]). □

Lemma 1.1.12. *Assume that $F_{<k}$ is a subspace of F. For every open subset $U \subset \Sigma$ and every $k \in \mathbb{N}$ such that a fiberwise truncation $ft_{<k}(\xi_U)$ exists, there is an isomorphism*

$$\phi_{U*} : \mathcal{H}^r(\Sigma, \boldsymbol{W}_U^\bullet) \to H^r_{c_U^{<k}}(ft_{<k}(\xi_U); \mathbb{Q})$$

(in particular for $k = n - c + 1$ there is an isomorphism $\phi_{U*} : \mathcal{H}^r(\Sigma, \boldsymbol{V}_U^\bullet) \to H^r_{c_U}(\xi_U; \mathbb{Q})$), such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}^r(\Sigma, \boldsymbol{V}_U^\bullet) \arrow[r, "\phi_{U*}", "\cong"'] \arrow[d, "\alpha_{U*}"'] & H^r_{c_U}(\xi_U; \mathbb{Q}) \arrow[d, "i_U^*"] \\
\mathcal{H}^r(\Sigma, \boldsymbol{W}_U^\bullet) \arrow[r, "\phi_{U*}", "\cong"'] & H^r_{c_U^{<k}}(ft_{<k}(\xi_U); \mathbb{Q}).
\end{tikzcd}$$

Proof. The diagram

$$\begin{tikzcd}
\mathcal{H}^r(\Sigma, Rp_*(j_!\underline{\mathbb{Q}}^\bullet_{\xi_U})) \arrow[r, "\cong"] \arrow[d, "\alpha_{U*}"'] & \mathcal{H}^r(\xi, j_!\underline{\mathbb{Q}}^\bullet_{\xi_U}) \arrow[d, "\alpha_{U*}"] \\
\mathcal{H}^r(\Sigma, Rp_*(j_!i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_{\xi_U})) \arrow[r, "\cong"] & \mathcal{H}^r(\xi, j_!i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_{\xi_U})
\end{tikzcd}$$

commutes. This follows from [Dim08], Corollary 2.3.4 (note that $j_!i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_{\xi_U}$ and $j_!\underline{\mathbb{Q}}^\bullet_{\xi_U}$ are by construction bounded below sheaf complexes). Now $\mathcal{H}^r(\xi, j_!\underline{\mathbb{Q}}^\bullet_{\xi_U}) = H^r(\xi, j_!\underline{\mathbb{Q}}_{\xi_U})$. As ξ is compact, we can pass to sheaf cohomology with compact support and have $H^r(\xi, j_!\underline{\mathbb{Q}}_{\xi_U}) = H^r_{c_\xi}(\xi, j_!\underline{\mathbb{Q}}_{\xi_U})$. By [Bre97] II.10.1, there is a natural isomorphism ϕ_U such that the following diagram commutes

$$\begin{tikzcd}
H^r_{c_\xi}(\xi, j_!\underline{\mathbb{Q}}_{\xi_U}) \arrow[d, "\alpha_{U*}"'] & H^r_{c_U}(\xi_U, \underline{\mathbb{Q}}_{\xi_U}) \arrow[l, "\phi_U"'] \arrow[d, "\alpha_{U*}"] \\
H^r_{c_\xi}(\xi, j_! i_{U*} i_U^* \underline{\mathbb{Q}}_{\xi_U}) & H^r_{c_U}(\xi_U, i_{U*} i_U^* \underline{\mathbb{Q}}_{\xi_U}) \arrow[l, "\phi_U"']
\end{tikzcd}$$

if the family c_ξ is paracompactifying and if ξ_U is locally closed. Both is the case by Lemma 1.1.9 and Lemma 1.1.10.

$$\begin{tikzcd}
H^r_{c_U}(\xi_U, \underline{\mathbb{Q}}_{\xi_U}) \arrow[r, equal] \arrow[d, "\alpha_{U*}"'] & H^r_{c_\xi|\xi_U}(\xi_U, \underline{\mathbb{Q}}_{\xi_U}) \arrow[d, "i_U^*"] \\
H^r_{c_U}(\xi_U, i_{U*} i_U^* \underline{\mathbb{Q}}_{\xi_U}) \arrow[r, "i_U^\dagger"] & H^r_{c_\xi^<|ft_{<k}(\xi_U)}(ft_{<k}(\xi_U), i_U^* \underline{\mathbb{Q}}_{\xi_U})
\end{tikzcd}$$

commutes and $i_U^\dagger$ is an isomorphism by Lemma 1.1.11. Finally consider the diagram

$$\begin{tikzcd}
H^r_{c_\xi|\xi_U}(\xi_U, \underline{\mathbb{Q}}_{\xi_U}) \arrow[r] \arrow[d, "i_U^*"'] & H^r_{c_U}(\xi_U; \mathbb{Q}) \arrow[d, "i_U^*"] \\
H^r_{c_\xi^<|ft_{<k}(\xi_U)}(ft_{<k}(\xi_U), \underline{\mathbb{Q}}_{ft_{<k}(\xi_U)}) \arrow[r] & H^r_{c_U^{<k}}(ft_{<k}(\xi_U); \mathbb{Q}).
\end{tikzcd}$$

For this last diagram, we use Theorem [Bre97] III.1.1., which allows us to identify sheaf cohomology with ordinary cohomology. The families of support c_U and $c_U^{<k}$ are paracompactifying by Lemma 1.1.10. Furthermore ξ_U and $ft_{<k}(\xi_U)$ are homologically locally connected as they are locally contractible (Lemma 1.1.9). Therefore the assumptions of Theorem [Bre97] III.1.1. are satisfied. Composition of the above diagrams yields a diagram

$$\begin{tikzcd}
\mathcal{H}^r(\Sigma, \mathbf{V}_U^\bullet) \arrow[r, "\cong"] \arrow[d, "\alpha_U"'] & H^r_{c_U}(\xi_U; \mathbb{Q}) \arrow[d, "i_U^*"] \\
\mathcal{H}^r(\Sigma, \mathbf{W}_U^\bullet) \arrow[r, "\cong"] & H^r_{c_U^{<k}}(ft_{<k}(\xi_U); \mathbb{Q}).
\end{tikzcd}$$

The maps ϕ_{U*} are defined as the composition of the horizontal maps in the above diagrams. $\square$

For open subsets $V \subset U \subset \Sigma$, the inclusion

$$j_{VU} : ft_{<k}(\xi_V) \to ft_{<k}(\xi_U)$$

induces the following maps

$$\begin{aligned}
i_V^* \underline{\mathbb{Q}}_{\xi_V} &\to i_U^* \underline{\mathbb{Q}}_{\xi_U} \\
i_{V*} i_V^* \underline{\mathbb{Q}}_{\xi_V} &\to i_{U*} i_U^* \underline{\mathbb{Q}}_{\xi_U} \\
j_{V!} i_{V*} i_V^* \underline{\mathbb{Q}}_{\xi_V} &\to j_{U!} i_{U*} i_U^* \underline{\mathbb{Q}}_{\xi_U}.
\end{aligned}$$

The induced maps on hypercohomology and sheaf cohomology will be denoted by j_{VU*}.

Lemma 1.1.13. *Assume that $F_{<k}$ is a subspace of F. The morphism*

$$\mathcal{H}^r(\Sigma,\, \boldsymbol{W}^\bullet_U) \to H^r_{c_U^{<k}}(ft_{<k}(\xi_U);\mathbb{Q})$$

which is constructed in Lemma 1.1.12 for every open subset $U \subset \Sigma$ has the property that for open subsets $V \subset U \subset \Sigma$ the following diagram commutes.

$$\begin{tikzcd}
\mathcal{H}^r(\Sigma,\, \boldsymbol{W}^\bullet_V) \arrow[r, "\cong"] \arrow[d, "j_{VU*}"'] & H^r_{c_V^{<k}}(ft_{<k}(\xi_V);\mathbb{Q}) \arrow[d, "j_{VU*}"] \\
\mathcal{H}^r(\Sigma,\, \boldsymbol{W}^\bullet_U) \arrow[r, "\cong"] & H^r_{c_U^{<k}}(ft_{<k}(\xi_U);\mathbb{Q}).
\end{tikzcd}$$

Proof. The following diagram commutes by the properties of the pushforward of sheaves (see for example [Dim08], Corollary 2.3.4).

$$\begin{tikzcd}
\mathcal{H}^r(\Sigma, Rp_*(j_{V!}i_{V*}i_V^*\underline{\mathbb{Q}}^\bullet_{\xi_V})) \arrow[r, "\cong"] \arrow[d, "j_{VU*}"'] & \mathcal{H}^r(\xi, j_{V!}i_{V*}i_V^*\underline{\mathbb{Q}}^\bullet_{\xi_V}) \arrow[d, "j_{VU*}"] \\
\mathcal{H}^r(\Sigma, Rp_*(j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_{\xi_U})) \arrow[r, "\cong"] & \mathcal{H}^r(\xi, j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}^\bullet_{\xi_U}).
\end{tikzcd}$$

As ξ is compact, we can equally regard the hypercohomology groups as having compact support. Commutativity of the next diagram follows from [Bre97], Theorem II.10.1. The assumptions of that theorem are satisfied by Lemma 1.1.9 and Lemma 1.1.10.

$$\begin{tikzcd}
H^r_{c_\xi}(\xi, j_{V!}i_{V*}i_V^*\underline{\mathbb{Q}}_{\xi_V}) \arrow[d, "j_{VU*}"'] & H^r_{c_V}(\xi_V, i_{V*}i_V^*\underline{\mathbb{Q}}_{\xi_V}) \arrow[l, "\cong"'] \arrow[d, "j_{VU*}"] \\
H^r_{c_\xi}(\xi, j_{U!}i_{U*}i_U^*\underline{\mathbb{Q}}_{\xi_U}) & H^r_{c_U}(\xi_U, i_{U*}i_U^*\underline{\mathbb{Q}}_{\xi_U}). \arrow[l, "\cong"']
\end{tikzcd}$$

Moreover, the diagram

$$\begin{tikzcd}
H^r_{c_V}(\xi_V, i_{V*}i_V^*\underline{\mathbb{Q}}_{\xi_V}) \arrow[r, "i_V^\dagger"] \arrow[d] & H^r_{c_\xi^{<}|ft_{<k}(\xi_V)}(ft_{<k}(\xi_V), i_V^*\underline{\mathbb{Q}}_{\xi_V}) \arrow[d] \\
H^r_{c_U}(\xi_U, i_{U*}i_U^*\underline{\mathbb{Q}}_{\xi_U}) \arrow[r, "i_U^\dagger"] & H^r_{c_\xi^{<}|ft_{<k}(\xi_U)}(ft_{<k}(\xi_U), i_U^*\underline{\mathbb{Q}}_{\xi_U})
\end{tikzcd}$$

commutes, as $i_U^\dagger$ is a natural transformation of functors. Moreover, $i_V^\dagger$ and $i_U^\dagger$ are isomorphisms by Lemma 1.1.11. Finally

$$\begin{tikzcd}
H^r_{c_\xi^{\leq}|ft_{<k}(\xi_V)}(ft_{<k}(\xi_V), \underline{\mathbb{Q}}_{ft_{<k}(\xi_V)}) \arrow[r, "\cong"] \arrow[d] & H^r_{c_V^{<k}}(ft_{<k}(\xi_V);\mathbb{Q}) \arrow[d, "j_{VU*}"] \\
H^r_{c_\xi^{\leq}|ft_{<k}(\xi_U)}(ft_{<k}(\xi_U), \underline{\mathbb{Q}}_{ft_{<k}(\xi_U)}) \arrow[r, "\cong"] & H^r_{c_U^{<k}}(ft_{<k}(\xi_U);\mathbb{Q})
\end{tikzcd}$$

commutes by Theorem [Bre97] III.1.1 (as in the proof of Lemma 1.1.12 the assumptions of that theorem are satisfied by Lemma 1.1.9 and Lemma 1.1.10). Composition yields

$$\begin{tikzcd}
\mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_V) \arrow[r, "\cong"] \arrow[d, "j_{VU*}"'] & H^r_{c_V^{<k}}(ft_{<k}(\xi_V);\mathbb{Q}) \arrow[d, "j_{VU*}"] \\
\mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_U) \arrow[r, "\cong"] & H^r_{c_U^{<k}}(ft_{<k}(\xi_U);\mathbb{Q}).
\end{tikzcd}$$

The vertical maps are exactly the maps constructed in Lemma 1.1.12. □

Corollary 1.1.14. *In particular Lemma 1.1.13 applies for $k > n - c$, so that we get a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}^r(\Sigma, \boldsymbol{V}^\bullet_V) \arrow[r, "\cong"] \arrow[d, "j_{VU*}"'] & H^r_{c_V}(\xi_V;\mathbb{Q}) \arrow[d, "j_{VU*}"] \\
\mathcal{H}^r(\Sigma, \boldsymbol{V}^\bullet_U) \arrow[r, "\cong"] & H^r_{c_U}(\xi_U;\mathbb{Q}).
\end{tikzcd}$$

From now on, we suppress the subscript in the supports and denote the cohomology group with compact support $H^r_{c_U}(\xi_U;\mathbb{Q})$ simply by $H^r_c(\xi_U;\mathbb{Q})$ and similar for the other occurring cohomology groups with compact supports.

Lemma 1.1.15. *Assume that $F_{<k}$ is a subspace of F. For every open subset $U \subset \Sigma$, there exists a right inverse*

$$t^*_U : H^r_c(ft_{<k}(\xi_U);\mathbb{Q}) \to H^r_c(\xi_U;\mathbb{Q})$$

of i^*_U for all $r \in \mathbb{N}$, such that for every $V \subset U \subset \Sigma$ the following diagram commutes

$$\begin{tikzcd}
H^r_c(ft_{<k}(\xi_V);\mathbb{Q}) \arrow[r, "t^*_V"] \arrow[d, "j_{VU*}"'] & H^r_c(\xi_V;\mathbb{Q}) \arrow[d, "j_{VU*}"] \\
H^r_c(ft_{<k}(\xi_U);\mathbb{Q}) \arrow[r, "t^*_U"] & H^r_c(\xi_U;\mathbb{Q}).
\end{tikzcd}$$

Proof. For every $U \subset \Sigma$, the canonical adjunction morphism α_U and the inclusion of the truncated chain complex ρ_U induce a commutative diagram

$$
\begin{tikzcd}
\mathcal{H}^r(\Sigma, \tau_{<k}\mathbf{V}^\bullet_V) \arrow[rrr, "\alpha_{V*}", "\cong"'] \arrow[dr, "\rho_{V*}"] \arrow[ddd, "j_{VU*}"'] & & & \mathcal{H}^r(\Sigma, \tau_{<k}\mathbf{W}^\bullet_V) \arrow[dl, "\rho_{V*}"', "\cong"] \arrow[ddd, "j_{VU*}"] \\
& \mathcal{H}^r(\Sigma, \mathbf{V}^\bullet_V) \arrow[r, "\alpha_{V*}"] \arrow[d, "j_{VU*}"'] & \mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_V) \arrow[d, "j_{VU*}"] & \\
& \mathcal{H}^r(\Sigma, \mathbf{V}^\bullet_U) \arrow[r, "\alpha_{U*}"] & \mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_U) & \\
\mathcal{H}^r(\Sigma, \tau_{<k}\mathbf{V}^\bullet_U) \arrow[rrr, "\alpha_{U*}", "\cong"'] \arrow[ur, "\rho_{U*}"] & & & \mathcal{H}^r(\Sigma, \tau_{<k}\mathbf{W}^\bullet_U). \arrow[ul, "\rho_{U*}", "\cong"']
\end{tikzcd}
$$

Note that the maps $\rho_{V*} : \mathcal{H}^r(\Sigma, \tau_{<k}\mathbf{W}^\bullet_V) \to \mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_V)$ and $\rho_{U*} : \mathcal{H}^r(\Sigma, \tau_{<k}\mathbf{W}^\bullet_U) \to \mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_U)$ are isomorphisms by Lemma 1.1.7. For every $U \subset \Sigma$, construct a right inverse $\beta_U : \mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_U) \to \mathcal{H}^r(\Sigma, \mathbf{V}^\bullet_U)$ to α_U as the composition

$$\beta_{U*} := \rho_{U*} \circ \alpha_{U*}^{-1} \circ \rho_{U*}^{-1}$$

Putting together Lemma 1.1.12 and Lemma 1.1.13, we get the commutative diagram

$$
\begin{tikzcd}
H^r_c(\xi_V; \mathbb{Q}) \arrow[rrr, "i_V^*"] \arrow[ddd, "j_{VU*}"'] & & & H^r_c(ft_{<k}(\xi_V); \mathbb{Q}) \arrow[ddd, "j_{VU*}"] \\
& \mathcal{H}^r(\Sigma, \mathbf{V}^\bullet_V) \arrow[r, "\alpha_{V*}"] \arrow[d, "j_{VU*}"'] \arrow[ul, "\phi_{V*}", "\cong"'] & \mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_V) \arrow[d, "j_{VU*}"] \arrow[ur, "\phi_{V*}"', "\cong"] & \\
& \mathcal{H}^r(\Sigma, \mathbf{V}^\bullet_U) \arrow[r, "\alpha_{U*}"] \arrow[dl, "\phi_{U*}"', "\cong"] & \mathcal{H}^r(\Sigma, \mathbf{W}^\bullet_U) \arrow[dr, "\phi_{U*}", "\cong"'] & \\
H^r_c(\xi_U; \mathbb{Q}) \arrow[rrr, "i_U^*"] & & & H^r_c(ft_{<k}(\xi_U); \mathbb{Q}).
\end{tikzcd}
$$

This finally defines a right-inverse $t_U^* : H^r_c(ft_{<k}(\xi_U); \mathbb{Q}) \to H^r_c(\xi_U; \mathbb{Q})$ of i_U^* that is natural with respect to the restriction map j_{VU*} given by

$$t_U^* = \phi_{U*} \circ \beta_{U*} \circ \phi_{U*}^{-1}$$

$\square$

Now apply the preceeding lemmas to the fiber bundle $\xi = \partial M$ with fiber the link L. Note that ∂M is compact and Hausdorff.

Lemma 1.1.16. *Let $L^{c-1} \to \partial M \to \Sigma^{n-c}$ be a fiber bundle that admits a fiberwise truncation at some value $k \in \mathbb{N}$ and ∂M a closed manifold. Assume that $L_{<k}$ is a subspace of L. Let $U \in \Sigma$ be an open subset. Then a cap product*

$$\tilde{\cap} : H_c^{i-r}(ft_{<k}(\partial M_U); \mathbb{Q}) \otimes H_i(\partial M_U; \mathbb{Q}) \to H_r(\partial M_U, ft_{<k}(\partial M_U); \mathbb{Q})$$

exists such that

$$\begin{tikzcd}
H_c^{i-r}(ft_{<k}(\partial M_U);\mathbb{Q}) \otimes H_i(\partial M_U;\mathbb{Q}) \arrow[r, "-\tilde{\cap}-"] & H_r(\partial M_U, ft_{<k}(\partial M_U);\mathbb{Q}) \\
H_c^{i-r}(\partial M_U;\mathbb{Q}) \otimes H_i(\partial M_U;\mathbb{Q}) \arrow[r, "-\cap-"] \arrow[u] & H_r(\partial M_U);\mathbb{Q}) \arrow[u]
\end{tikzcd}$$

commutes and

$$H_r(ft_{<k}(\partial M);\mathbb{Q}) \hookrightarrow H_r(\partial M;\mathbb{Q})$$

is injective.

Proof. The construction of t_U^* allows the definition of a cap product

$$\begin{tikzcd}
H_c^{i-r}(ft_{<k}(\partial M_U);\mathbb{Q}) \otimes H_i(\partial M_U;\mathbb{Q}) \arrow[r, "\tilde{\cap}"] & H_r(\partial M_U, ft_{<c-k}(\partial M_U);\mathbb{Q}) \\
H_c^{i-r}(\partial M_U;\mathbb{Q}) \otimes H_i(\partial M_U;\mathbb{Q}) \arrow[r, "\cap"] \arrow[u, "i^*\otimes id"] & H_r(\partial M_U;\mathbb{Q}), \arrow[u, "\pi_*"']
\end{tikzcd}$$

where π_* is the obvious map appearing in the long exact sequence of this pair, by setting

$$\partial M \tilde{\cap} x = \pi_*(t_U^*(\partial M) \cap x)$$

for $\partial M \in H_c^{i-r}(ft_{<k}(\partial M_U);\mathbb{Q})$ and $x \in H_r(\partial M_U;\mathbb{Q})$. The right inverse $t_{\partial M}^*$ yields a short exact sequence

$$0 \to H_c^r(\partial M, ft_{<k}(\partial M);\mathbb{Q}) \to H_c^r(\partial M;\mathbb{Q}) \xrightarrow{i^*} H_c^r(ft_{<k}(\partial M);\mathbb{Q}) \to 0$$

for all $r \in \mathbb{N}$. Since ∂M and $ft_{<k}(\partial M)$ are both compact, in particular

$$\dim H^r(\partial M;\mathbb{Q}) = \dim H^r(ft_{<k}(\partial M);\mathbb{Q}) + \dim H^r(\partial M, ft_{<k}(\partial M);\mathbb{Q}).$$

Since for any $\mathbb{Q}$-Vectorspace V we have $\dim \mathrm{Hom}_{\mathbb{Q}}(V;\mathbb{Q}) = \dim V$ it follows by the universal coefficient theorem that

$$\dim H_r(\partial M;\mathbb{Q}) = \dim H_r(ft_{<k}(\partial M);\mathbb{Q}) + \dim H_r(\partial M, ft_{<k}(\partial M);\mathbb{Q}).$$

Exactness of the long exact sequence of the pair $(\partial M, ft_{<k}(\partial M))$ then implies

$$0 \to H_r(ft_{<k}(\partial M);\mathbb{Q}) \xrightarrow{i_*} H_r(\partial M;\mathbb{Q}) \to H_r(\partial M, ft_{<k}(\partial M);\mathbb{Q}) \to 0$$

for all $r \in \mathbb{N}$ by a comparison of the dimensions. □

1.2 Duality Isomorphisms

1.2.1 Duality for Fiberwise Subspace Homology Truncations

The main theorem of this section uses a Mayer-Vietoris argument to patch together the statements of Section 1.1 to a duality isomorphism of intersection spaces.

Theorem 1.2.1. *Let X be a compact, oriented pseudomanifold with two strata and a link bundle $(L^{c-1}, \partial M, p, \Sigma^{n-c})$ that admits a fiberwise truncation in two complementary degrees $k = c - 1 - \bar{p}(c)$ and $c - k$ (in the sense of Definition 1.1.3). Assume that the homology truncation $L_{<k}$ is a subspace of L. Then there is a rational Poincaré duality isomorphism*

$$\tilde{H}^{n-r}(I^{\bar{p}}X;\mathbb{Q}) \to \tilde{H}_r(I^{\bar{q}}X;\mathbb{Q}).$$

Proof. In this proof, all homology and cohomology groups denote homology and cohomology with rational coefficients. The inclusion of the open subset $V \subset U \subset \Sigma$ induces a map $j_{VU*} : H_c^{n-r}(\partial M_V) \to H_c^{n-r}(\partial M_U)$ such that the following diagram commutes (see [May99], p.159)

$$\begin{tikzcd}
H_c^{n-r}(\partial M_V) \arrow[r, "D_V"] \arrow[d, "j_{VU*}"'] & H_r(\partial M_V) \arrow[d, "j_{VU*}"] \\
H_c^{n-r}(\partial M_U) \arrow[r, "D_U"] & H_r(\partial M_U),
\end{tikzcd}$$

where D_V and D_U are the Poincaré duality isomorphisms. Capping with the fundamental class in the cap product of Lemma 1.1.16 yields for every open subset $V \in \Sigma$ a map $D_V^{<}$. It is the composition in the following diagram

$$\begin{tikzcd}
H_c^{n-r}(ft_{<k}(\partial M_V)) \arrow[r, "D_V^{<}"] \arrow[d, "t_V^*"'] & H_r(\partial M_V, ft_{<c-k}(\partial M_V)) \\
H_c^{n-r}(\partial M_V) \arrow[r, "D_V"] & H_r(\partial M_V). \arrow[u, "\pi_*"']
\end{tikzcd}$$

For a trivial bundle $ft_{<k}(\partial M_V) = L_{<k} \times V$, the map D_V is given by $- \cap [L \times V]$ and is an isomorphism and for $V = \Sigma$, the map D_V is just $- \cap [\partial M]$. Finally

$$\begin{tikzcd}
H_c^{n-r}(ft_{<k}(\partial M_V)) \arrow[r, "t_V^*"] \arrow[d, "j_{VU*}"'] & H_c^{n-r}(\partial M_V) \arrow[d, "j_{VU*}"] \\
H_c^{n-r}(ft_{<k}(\partial M_U)) \arrow[r, "t_U^*"] & H_c^{n-r}(\partial M_U)
\end{tikzcd}$$

commutes by construction of t_V^* and

$$\begin{tikzcd}
H_r(\partial M_V) \arrow[r, "\pi_*"] \arrow[d, "j_{VU*}"'] & H_r(\partial M_V, ft_{<c-k}(\partial M_V)) \arrow[d, "j_{VU*}"] \\
H_r(\partial M_U) \arrow[r, "\pi_*"] & H_r(\partial M_U, ft_{<c-k}(\partial M_U))
\end{tikzcd}$$

clearly commutes. Putting these diagrams together to a cube of diagrams, we see that the following diagram commutes.

$$\begin{tikzcd} H_c^{n-r}(ft_{<k}(\partial M_V)) \arrow[r, "D_V^<"] \arrow[d, "j_{VU*}"'] & H_r(\partial M_V, ft_{<c-k}(\partial M_V)) \arrow[d, "j_{VU*}"] \\ H_c^{n-r}(ft_{<k}(\partial M_U)) \arrow[r, "D_U^<"] & H_r(\partial M_U, ft_{<c-k}(\partial M_U)) \end{tikzcd} \tag{1.1}$$

as

$$\begin{aligned} j_{VU*} \circ D_V^< &= j_{VU*} \circ \pi_* \circ D_V \circ t_V^* \\ &= \pi_* \circ j_{VU*} \circ D_V \circ t_V^* \\ &= \pi_* \circ D_U \circ j_{VU*} \circ t_V^* \\ &= \pi_* \circ D_U \circ t_U^* \circ j_{VU*} \\ &= D_U^< \circ j_{VU*}. \end{aligned}$$

To show that $D_V^<$ is an isomorphism, we use induction. Let $\bigcup_{i\in I} U_i$ be an open cover of Σ such that over each U_i the fiber bundle ∂M is trivial. As Σ is compact, we can choose the index set I to be finite and write $\Sigma = \bigcup_{i=1}^N U_i$. Set

$$U^n = \bigcup_{i=1}^n U_i.$$

Without loss of generality, we can assume that for each $1 \leq n \leq N$, the intersection $U^n \cap U_{n+1}$ is non-empty. We know that $D_U^<$ is an isomorphism for a trivial bundle ∂M_U. This is basically an application of the Künneth theorem for cohomology with compact support. The analog case for ordinary cohomology is shown in [Ban10a], Proposition 2.44. The induction basis therefore holds for $U = U_1$. By induction hypothesis, let U^n be a subset such that $D_{U^n}^<$ is an isomorphism. Furthermore $D_{U_{n+1}}^<$ and $D_{U^n \cap U_{n+1}}^<$ are isomorphism, as both $\partial M_{U_{n+1}}$ and $\partial M_{U^n \cap U_{n+1}}$ are trivial bundles. It is clear, that $ft_{<k}(\partial M_{U^n}) \cap ft_{<k}(\partial M_{U_{n+1}}) = ft_{<k}(\partial M_{U^n \cap U_{n+1}})$ and we get the following diagram. For better readability, we abbreviate in this following diagram the map $D_{U^n}^< \oplus D_{U_{n+1}}^<$ by D^U and the fiber bundle ∂M by ξ.

$$\begin{array}{ccc}
H_c^{n-r}(ft_{<k}(\xi_{U^n \cap U_{n+1}})) & \xrightarrow[D^{<}_{U^n \cap U_{n+1}}]{} & H_{r-1}(\xi_{U^n \cap U_{n+1}}, ft_{<c-k}(\xi_{U^n \cap U_{n+1}})) \\
\downarrow & & \downarrow \\
H_c^{n-r}(ft_{<k}(\xi_{U^n})) \oplus H_c^{n-r}(ft_{<k}(\xi_{U_{n+1}})) & \xrightarrow[D^U]{} & H_{r-1}(\xi_{U^n}, ft_{<c-k}(\xi_{U^n})) \oplus H_{r-1}(\xi_{U_{n+1}}, ft_{<c-k}(\xi_{U_{n+1}})) \\
\downarrow & & \downarrow \\
H_c^{n-r}(ft_{<k}(\xi_{U^{n+1}})) & \xrightarrow[D^{<}_{U^{n+1}}]{} & H_{r-1}(\xi_{U^{n+1}}, ft_{<c-k}(\xi_{U^{n+1}})) \\
\downarrow & & \downarrow \\
H_c^{n-r+1}(ft_{<k}(\xi_{U^n \cap U_{n+1}})) & \xrightarrow[D^{<}_{U^n \cap U_{n+1}}]{} & H_{r-2}(\xi_{U^n \cap U_{n+1}}, ft_{<c-k}(\xi_{U^n \cap U_{n+1}})) \\
\downarrow & & \downarrow \\
H_c^{n-r+1}(ft_{<k}(\xi_{U^n})) \oplus H_c^{n-r}(ft_{<k}(\xi_{U_{n+1}})) & \xrightarrow[D^U]{} & H_{r-2}(\xi_{U^n}, ft_{<c-k}(\xi_{U^n})) \oplus H_{r-2}(\xi_{U_{n+1}}, ft_{<c-k}(\xi_{U_{n+1}})).
\end{array} \tag{1.2}$$

The columns are given by the Mayer-Vietoris sequence. As the duality map D_U commutes (for every open subset $U \in \Sigma$) with the maps of the Mayer Vietoris sequence (which is verified in any proof of ordinary Poincaré duality), we can show in analogy to Diagram (1.1) that all squares in the above Diagram (1.2) commute. The five lemma then shows that $D^{<}_{U^n \cup U_{n+1}} = D^{<}_{U^{n+1}}$ is an isomorphism. This is the induction step. We obtain, that $D_{U^N} = D_\Sigma$ is an isomorphism and thus Poincaré duality holds for $\partial M_\Sigma = \partial M$. As above, composition with the Poincaré duality isomorphism on M yields the commutative diagram (the right part commutes by construction of $D^{<}_{U^N}$).

$$\begin{array}{ccccc}
H^{n-r}(M) & \longrightarrow & H^{n-r}(\partial M) & \longrightarrow & H^{n-r}(ft_{<k}(\partial M)) \\
\downarrow{\scriptstyle -\cap[M]} & & \downarrow{\scriptstyle -\cap[\partial M]} & & \downarrow{\scriptstyle D^{<}_{\Sigma}} \\
H_r(M, \partial M) & \longrightarrow & H_{r-1}(\partial M) & \longrightarrow & H_{r-1}(\partial M, ft_{<c-k}(\partial M)).
\end{array}$$

In this diagram, we switched from cohomology with compact support to ordinary cohomology which is possible, as all occurring spaces in this diagram are compact. Lemma 2.46 of [Ban10a] and the five lemma applied to the diagram

$$\begin{tikzcd}
H^{n-r-1}(M) \arrow[r, "-\cap[M,\partial M]", "\cong"'] \arrow[d] & H_{r+1}(M,\partial M) \arrow[d] \\
H^{n-r-1}(ft_{<k}(\partial M)) \arrow[r, "D_\Sigma", "\cong"'] \arrow[d] & H_r(\partial M, ft_{<c-k}(\partial M)) \arrow[d] \\
H^{n-r}(M, ft_{<k}(\partial M)) \arrow[d] & H_r(M, ft_{<c-k}(\partial M)) \arrow[d] \\
H^{n-r}(M) \arrow[r, "-\cap[M,\partial M]", "\cong"'] \arrow[d] & H_r(M,\partial M) \arrow[d] \\
H^{n-r}(ft_{<k}(\partial M)) \arrow[r, "D_\Sigma", "\cong"'] & H_{r-1}(\partial M, ft_{<c-k}(\partial M))
\end{tikzcd}$$

show Poincaré duality

$$\tilde{H}^{n-r}(I^{\bar{p}}X;\mathbb{Q}) \to \tilde{H}_r(I^{\bar{q}}X;\mathbb{Q}).$$

□

1.2.2 Duality for Fiberwise Truncations in the Top Degree

A k-truncation structure is a quadrupel $(L, L/n, h, L_{<k})$ (see [Ban10a], Definition 1.4), where L is a simply connected CW-complex, L/k is a CW-complex with $(L/k)^{k-1} = L^{k-1}$ and such that the group of k-cycles of L/k has a basis of cells. The map $h : L/k \to L^k$ is a cellular homotopy equivalence rel $(k-1)$-skeleton. $L_{<k}$ is a subcomplex of L/k and a homology truncation of L/k in the sense of Definition 1.1.1. We have the following composition of maps

$$L_{<k} \hookrightarrow L/k \xrightarrow{h} L^k \hookrightarrow L.$$

When we consider a fiberwise truncated fiber bundle ζ, it is natural to ask, that the fiber map $ft_{<k}(\zeta) \to \zeta$ factors through this composition.

Definition 1.2.2. Let ζ be a fiber bundle with fiber a CW-complex L and base space Σ. Let L have a k-truncation structure of the form $(L, L/n, h, L_{<k})$. A *strong fiberwise truncation at k* of ζ is a composition of fiber maps between fiber bundles over Σ of the form

$$\begin{tikzcd}
L_{<k} \arrow[r] \arrow[d, "i^{<}|"] & ft_{<k}(\zeta) \arrow[r] \arrow[d, "i^{<}"] & \Sigma \arrow[d, equal] \\
L/k \arrow[r] \arrow[d, "h|"] & ft/k(\zeta) \arrow[r] \arrow[d, "h"] & \Sigma \arrow[d, equal] \\
L^k \arrow[r] \arrow[d, "i^k|"] & ft^k(\zeta) \arrow[r] \arrow[d, "i^k"] & \Sigma \arrow[d, equal] \\
L \arrow[r] & \zeta \arrow[r] & \Sigma.
\end{tikzcd}$$

Remark 1.2.3. *From Lemma 1.1.15 follows that for a compact fiber bundle ζ the map $i^* : H^r(\zeta;\mathbb{Q}) \to H^r(ft_{<k}(\zeta);\mathbb{Q})$ is surjective. The surjectivity of the map i^*, however, does in general not factor through the composition*

$$ft_{<k}(\zeta) \to ft/k(\zeta) \to ft^k(\zeta) \to \zeta$$

This means, that in the following composition in general not all maps are surjective, although the composition may be surjective.

$$i^* : H^r(\zeta) \to H^r(ft^k(\zeta)) \to H^r(ft/k(\zeta)) \to H^r(ft_{<k}(\zeta))$$

(For simplicity we choose a compact fiber bundle, so that we do not have to take care of compact supports. The argument, however, works for non-compact fiber bundles as well). To see the claim, let ζ be the trivial fiber bundle $\Sigma \times S^n$ where the base space Σ is a closed manifold and the fiber is S^n. Let the fiber S^n have the following CW-structure. Take two distinct points and attach two one cells to them, so that each one cell connects the two 0-cells. The resulting space is a S^1. Then attach two 2-cells as the northern and southern hemisphere to the 1-skeleton. Inductively this yields a CW-structure with 2 cells in every dimension.

$$S^n = e^0_1 + e^0_2 + e^1_1 + e^1_2 + \ldots + e^n_1 + e^n_2$$

For $0 < r < n$, the homology $H^r(S^n) = 0$. Thus $H^r(L_{<k}) = H^r(L) = 0$. On the other hand, if $(S^n)^r$ denotes the r-skeleton of S^n, then $H^r((S^n)^r) = H^r(S^r) = \mathbb{Z}$. By the Künneth theorem, $H^r(\zeta;\mathbb{Q}) = H^r(ft_{<k}(\zeta);\mathbb{Q}) \cong H^r(\Sigma;\mathbb{Q})$ and $H^r(ft^k(\zeta);\mathbb{Q}) \cong H^r(\Sigma;\mathbb{Q}) \oplus H^0(\Sigma;\mathbb{Q}) = H^r(\Sigma;\mathbb{Q}) \oplus \mathbb{Q}$. Thus the map

$$H^r(\zeta;\mathbb{Q}) \to H^r(ft_{<k}(\zeta);\mathbb{Q})$$

is surjective, but the map

$$H^r(\zeta;\mathbb{Q}) \to H^r(ft^k(\zeta);\mathbb{Q})$$

is not. This suggests that in general it is not possible to obtain for a fiber bundle ζ a right-inverse $t^* : H^r_c(ft_{<k}(\zeta);\mathbb{Q}) \to H^r_c(\zeta;\mathbb{Q})$ of the map $i^* : H^r_c(\zeta;\mathbb{Q}) \to H^r_c(ft_{<k}(\zeta);\mathbb{Q})$ by constructing right-inverses for every step of the decomposition $H^r_c(\zeta) \to H^r_c(ft^k(\zeta)) \to H^r_c(ft/k(\zeta)) \to H^r_c(ft_{<k}(\zeta))$ and then putting these right-inverses together. This is why we focus in the following on the special case $\zeta^k = \zeta$.

Theorem 1.2.4. *Let X be an oriented, compact stratified pseudomanifold with two strata and a link bundle $(L^{c-1}, \partial M, p, \Sigma^{n-c})$ that allows a strong fiberwise truncation at the value $k = c-1 = \dim L$ and a fiberwise truncation at the value 1 (in other words, a section). Then there is a rational Poincaré duality isomorphism*

$$\tilde{H}^{n-r}(I^{\overline{p}}X;\mathbb{Q}) \to \tilde{H}_r(I^{\overline{q}}X;\mathbb{Q}).$$

where $\overline{p}(c) = 0$ and $\overline{q}(c) = c-2$.

Proof of the Theorem. In this proof all homology and cohomology are understood to be with rational coefficients. By assumption $k = c-1$. Then $L = L^{c-1}$ so that $i^{c-1} = id$. Let U be an open subset of Σ. Define

$$\gamma_U : ft/(c-1)(\zeta_U) \to ft/(c-1)(\partial M)$$

and

$$i_U^{<} : ft_{<c-1}(\zeta_U) \to ft/(c-1)(\zeta_U).$$

Both maps are inclusions. Set

$$\mathbf{X}_U^\bullet := Rp_*(\gamma_{U!} i_{U*}^{<} i_U^{<*} \underline{\mathbb{Q}}^\bullet_{ft/(c-1)(\zeta_U)})$$

and

$$\mathbf{Y}_U^\bullet := Rp_*(\gamma_{U!} \underline{\mathbb{Q}}^\bullet_{ft/(c-1)(\zeta_U)}).$$

$L_{<c-1}$ is a subspace and a homology truncation of $L/(c-1)$. $ft/(c-1)(\partial M)$ is compact as the base space and the fiber both are compact. Furthermore it is Hausdorff, as it is a CW-complex. Set $\xi_U = ft/(c-1)(\zeta_U)$. We can then apply Lemma 1.1.12, Lemma 1.1.13 and Lemma 1.1.15 to the map of fiber bundles $ft_{<c-1}(\zeta_U) = ft_{<c-1}(\xi_U) \to \xi_U = ft/(c-1)(\zeta_U)$. This shows that there exists a right-inverse

$$t_U^{<*} : H_c^r(ft_{<c-1}(\zeta_U)) \to H_c^r(ft/(c-1)(\zeta_U))$$

of $i_U^{<*}$ for all $r \in \mathbb{N}$, such that for every $V \subset U \subset \Sigma$ the following diagram commutes

$$\begin{array}{ccc}
H_c^r(ft_{<c-1}(\zeta_V)) & \xrightarrow{t_V^{<*}} & H_c^r(ft/(c-1)(\zeta_V)) \\
\downarrow{\scriptstyle j_{VU*}} & & \downarrow{\scriptstyle j_{VU*}} \\
H_c^r(ft_{<c-1}(\zeta_U)) & \xrightarrow{t_U^{<*}} & H_c^r(ft/(c-1)(\zeta_U)).
\end{array}$$

The map $h| : L/(c-1) \to L^{c-1}$ is a homotopy equivalence and both spaces are compact. The Künneth theorem for cohomology with compact support (see [Bre97], Theorem II.15.2, noting that L^{c-1} and V are locally compact Hausdorff and identifying sheaf cohomology with singular cohomology by [Bre97], Theorem III.1.1) shows that there is the following natural isomorphism

$$\begin{aligned} H_c^r(L^{c-1} \times V) &\cong \bigoplus_{i+j=r} H_c^i(L^{c-1}) \otimes H_c^j(V) \\ &= \bigoplus_{i+j=r} H^i(L^{c-1}) \otimes H_c^j(V) \\ &\cong \bigoplus_{i+j=r} H^i(L/(c-1)) \otimes H_c^j(V) \\ &= \bigoplus_{i+j=r} H_c^i(L/(c-1)) \otimes H_c^j(V) \cong H_c^r(L/(c-1) \times V). \end{aligned}$$

Naturality of the above isomorphism implies that for an inclusion of subsets $V \hookrightarrow U$, there is a commutative diagram

$$\begin{tikzcd} H_c^r(L^{c-1} \times V) \arrow[r, "\cong"] \arrow[d] & H_c^r(L/(c-1) \times V) \arrow[d] \\ H_c^r(L^{c-1} \times U) \arrow[r, "\cong"] & H_c^r(L/(c-1) \times U) \end{tikzcd}$$

For every open subset V, such that ∂M is trivial over V, this calculation yields a right-inverse

$$t_V^* : H_c^r(ft_{<c-1}(\zeta_V)) \to H_c^r(ft/(c-1)(\zeta_V)) \to H_c^r(ft^{c-1}(\zeta_V)) = H_c^r(\zeta_V) \tag{1.3}$$

such that for open subsets $V \hookrightarrow U$, where ∂M is trivial over U and V, the following diagram commutes

$$\begin{tikzcd} H_c^r(ft_{<c-1}(\zeta_V)) \arrow[r, "t_V^*"] \arrow[d, "j_{VU}*"'] & H_c^r(\zeta_V) \arrow[d, "j_{VU}*"'] \\ H_c^r(ft_{<c-1}(\zeta_U)) \arrow[r, "t_U^*"] & H_c^r(\zeta_U) \end{tikzcd}$$

Let $\bigcup_{i\in I} U_i$ be an open cover of Σ such that over each U_i the fiber bundle ∂M is trivial. As Σ is compact, we can choose the index set I to be finite and write $\Sigma = \bigcup_{i=1}^N U_i$. Set

$$U^n = \bigcup_{i=1}^n U_i.$$

Then both $\zeta_{U_{n+1}}$ and $\zeta_{U^n \cap U_{n+1}}$ are trivial bundles. We want to proceed with an induction on n. Suppose that $h^* : H_c^r(ft^{c-1}(\zeta_{U^i})) \to H_c^r(ft/(c-1)(\zeta_{U^n}))$ is an isomorphism for n. The induction basis is given by the above calculation for a trivial bundle. The Mayer-Vietoris sequence

$$\begin{tikzcd}
H_c^r(ft^{c-1}(\zeta_{U^n\cap U_{n+1}})) \arrow[r] \arrow[d] & H_c^r(ft/(c-1)(\zeta_{U^n\cap U_{n+1}})) \arrow[d] \\
H_c^r(ft^{c-1}(\zeta_{U^n}))\oplus H_c^r(ft^{c-1}(\zeta_{U_{n+1}})) \arrow[r] \arrow[d] & H_c^r(ft/(c-1)(\zeta_{U^n}))\oplus H_c^r(ft/(c-1)(\zeta_{U_{n+1}})) \arrow[d] \\
H_c^r(ft^{c-1}(\zeta_{U^{n+1}})) \arrow[r] \arrow[d] & H_c^r(ft/(c-1)(\zeta_{U^{n+1}})) \arrow[d] \\
H_c^{r+1}(ft^{c-1}(\zeta_{U^n\cap U_{n+1}})) \arrow[r] \arrow[d] & H_c^{r+1}(ft/(c-1)(\zeta_{U^n\cap U_{n+1}})) \arrow[d] \\
H_c^{r+1}(ft^{c-1}(\zeta_{U^n}))\oplus H_c^r(ft^{c-1}(\zeta_{U_{n+1}})) \arrow[r] & H_c^{r+1}(ft/(c-1)(\zeta_{U^n}))\oplus H_c^{r+1}(ft/(c-1)(\zeta_{U_{n+1}})).
\end{tikzcd} \tag{1.4}$$

then shows that $h^*: H_c^r(ft^{c-1}(\zeta_{U^{n+1}})) \to H_c^r(ft/(c-1)(\zeta_{U^{n+1}}))$ is an isomorphism. This is the induction step and shows that

$$h^*: H_c^r(ft^{c-1}(\zeta_{U^m})) \to H_c^r(ft/(c-1)(\zeta_{U^m}))$$

is an isomorphism for all $1 \le m \le N$. Composing this isomorphism with the right-inverse $t_{U^m}^{<*}$ yields a right-inverse

$$t_{U^m}^*: H_c^r(ft_{<c-1}(\zeta_{U^m})) \to H_c^r(ft/(c-1)(\zeta_{U^m})) \to H_c^r(ft^{c-1}(\zeta_{U^m})) = H_c^r(\zeta_{U^m}) \tag{1.5}$$

to the map $(i_{U^m}^{c-1}\circ h\circ i_{U^m}^{<})^*: H_c^r(\zeta_{U^m}) \to H_c^r(ft_{<c-1}(\zeta_{U^m}))$ for all $1\le m\le N$.

Naturality of the Künneth-isomorphism and of the sheaf theoretic calculation of Lemma 1.1.12 to Lemma 1.1.15 yields the following commutative diagram for open subsets $U^n \subset U^m \subset \Sigma$.

$$\begin{tikzcd}
H_c^r(ft_{<c-1}(\zeta_{U^n})) \arrow[r, "t_{U^n}^*"] \arrow[d, "j_{U^nU^m*}"'] & H_c^r(\zeta_{U^n}) \arrow[d, "j_{U^nU^m*}"] \\
H_c^r(ft_{<c-1}(\zeta_{U^m})) \arrow[r, "t_{U^m}^*"] & H_c^r(\zeta_{U^m})
\end{tikzcd}$$

We want to make another induction on n. Using the right-inverses from the equations (1.3) and (1.5), we can construct maps $D_{U_{n+1}}^<$, $D_{U^n}^<$, $D_{U^n\cap U_{n+1}}^<$ and $D_{U^{n+1}}^<$ such that the following diagram commutes (where D^U is an abbreviation for $D_{U^n}^<\oplus D_{U_{n+1}}^<$). The commutativity of the following diagram is explained in the proof of Theorem 1.2.1. Note that we have shown the existence of a natural right inverse t^* for all open sets $V\subset\Sigma$ over which the fiber bundle is trivial and for all sets of the form $U^n\subset\Sigma$. Suppose that $D_{U^n}^<$ is an isomorphism. Clearly $D_{U^n\cap U_{n+1}}^<$ and $D_{U^{n+1}}^<$ are isomorphisms as $\zeta_{U^n\cap U_{n+1}}$ and $\zeta_{U^{n+1}}$ are trivial bundles.

$$
\begin{tikzcd}
H_c^{n-r}(ft_{<c-1}(\zeta_{U^n\cap U_{n+1}})) \arrow[r, "D^{<}_{U^n\cap U_{n+1}}"'] \arrow[d] & H_{r-1}(\zeta_{U^n\cap U_{n+1}}, ft_{<1}(\zeta_{U^n\cap U_{n+1}})) \arrow[d] \\
H_c^{n-r}(ft_{<c-1}(\zeta_{U^n})) \oplus H_c^{n-r}(ft_{<c-1}(\zeta_{U_{n+1}})) \arrow[r, "D^U"'] \arrow[d] & H_{r-1}(\zeta_{U^n}, ft_{<1}(\zeta_{U^n})) \oplus H_{r-1}(\zeta_{U_{n+1}}, ft_{<1}(\zeta_{U_{n+1}})) \arrow[d] \\
H_c^{n-r}(ft_{<c-1}(\zeta_{U^{n+1}})) \arrow[r, "D^{<}_{U^{n+1}}"'] \arrow[d] & H_{r-1}(\zeta_{U^{n+1}}, ft_{<1}(\zeta_{U^{n+1}})) \arrow[d] \\
H_c^{n-r+1}(ft_{<c-1}(\zeta_{U^n\cap U_{n+1}})) \arrow[r, "D^{<}_{U^n\cap U_{n+1}}"'] \arrow[d] & H_{r-2}(\zeta_{U^n\cap U_{n+1}}, ft_{<1}(\zeta_{U^n\cap U_{n+1}})) \arrow[d] \\
H_c^{n-r+1}(ft_{<c-1}(\zeta_{U^n})) \oplus H_c^{n-r}(ft_{<c-1}(\zeta_{U_{n+1}})) \arrow[r, "D^U"'] & H_{r-2}(\zeta_{U^n}, ft_{<1}(\zeta_{U^n})) \oplus H_{r-2}(\zeta_{U_{n+1}}, ft_{<1}(\zeta_{U_{n+1}})).
\end{tikzcd}
\tag{1.6}
$$

The five lemma then shows that $D^{<}_{U^{n+1}}$ is an isomorphism. This is the induction step. We finally get an isomorphism

$$D^{<}_{\Sigma} : H_c^{n-r}(ft_{<c-1}(\partial M)) \to H_{r-1}(\partial M, ft_{<1}(\partial M)).$$

Then proceed as in Theorem 1.2.1 to get a Poincaré duality isomorphism

$$\tilde{H}^{n-r}(I^{\overline{p}}X) \to \tilde{H}_r(I^{\overline{q}}X).$$

□

1.2.3 Duality for Fiber Bundles That Can be Truncated Fiberwise in the Highest and Lowest Degrees

Recall the definition of a system of local coefficients. In [McC01] (Definition 5.19), a system of local coefficients $\mathscr{G}$ on a space B is defined as a collection of groups $\{G_b; b \in B\}$ together with a collection of homomorphisms $h[\lambda] : G_{b_1} \to G_{b_0}$, one homomorphism for each homotopy class of paths from b_0 to b_1. For a fiber bundle with base Σ and fiber N, the action of $\pi_1(\Sigma)$ on $H_*(N)$ defines a group bundle $\mathscr{H}_*(N;\mathbb{Q})$ on Σ by

$$\mathscr{H}_*(N;\mathbb{Q}) = \{H_*(\pi^{-1}(b)) | b \in \Sigma\}$$

together with the collection of isomorphisms

$$\{h_1[\lambda] : H_*(\pi^{-1}(b_2)) \to H_*(\pi^{-1}(b_1)) | \lambda \in \Omega(\Sigma, b_1, b_2)\},$$

where the index λ ranges over all homotopy classes of paths from b_1 to b_2 in Σ.

For the proof of the following theorem we need to introduce some notation and use the following two Lemmas. Assume that a fiberwise truncation

$$\begin{tikzcd} L_{<k} \arrow[r] \arrow[d, "i|"'] & ft_{<k}(\partial M) \arrow[r] \arrow[d, "i"] & \Sigma \arrow[d, equal] \\ L \arrow[r] & \partial M \arrow[r] & \Sigma \end{tikzcd}$$

exists. As i is a map of fiber bundles it induces a map of spectral sequences on the corresponding Serre spectral sequences. Denote the homological Serre spectral sequences of the fibration $L_{<k} \to ft_{<k}(\partial M) \to \Sigma$ by $\tilde{E}$ and the homological Serre spectral sequences of the fibration $L \to \partial M \to \Sigma$ by E. Let

$$i^r_{p,q} : \tilde{E}^r_{p,q} \to E^r_{p,q}$$

be the induced map on the E^r page. We set $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ and $\tilde{d}_r : \tilde{E}^r_{p,q} \to \tilde{E}^r_{p-r,q+r-1}$ for the differentials in the spectral sequence. Then naturality of the Serre spectral sequence for maps of fibrations implies $i^r_{p,q}\tilde{d}_r = i^r_{p,q}d_r$ for all p and q.

Lemma 1.2.5. *For all $r \in \mathbb{N}$, $i^r_{p,q}$ is an isomorphism for $q < k$ and $\tilde{E}^r_{p,q} = 0$ for $q \geq k$.*

Proof. The map of fiber bundles $ft_{<k}(\partial M) \to \partial M$ induces a morphism of group bundles (see [McC01], p.165, Example (g))

$$\begin{tikzcd} H_q(L_{<k}) = \mathscr{H}_q(L_{<k})_{b_1} \arrow[r, "h_1[\lambda]"] \arrow[d, "i|_*"'] & \mathscr{H}_q(L_{<k})_{b_0} = H_q(L_{<k}) \arrow[d, "i|_*"] \\ H_q(L) = \mathscr{H}_q(L)_{b_1} \arrow[r, "h_2[\lambda]"] & \mathscr{H}_q(L)_{b_0} = H_q(L), \end{tikzcd}$$

which is an isomorphism of group bundles for $q < k$. The identifications $H_p(\Sigma; \mathscr{H}_q(L_{<k})) \cong \tilde{E}^2_{p,q}$ and $H_p(\Sigma; \mathscr{H}_q(L)) \cong E^2_{p,q}$ then show that for $r = 2$ the statement of the lemma holds. We want to conclude by induction on r. The basis is given for $r = 2$. Assume that the claim holds for r. We distinguish three cases

(a) $q + r - 1 < k$. Then in the diagram

$$\begin{tikzcd} \tilde{E}^r_{p-r,q+r-1} \arrow[rr, "i^r_{p-r,q+r-1}"] & & E^r_{p-r,q+r-1} & \\ & \tilde{E}^r_{p,q} \arrow[ul, "\tilde{d}_r"] \arrow[rr, "i^r_{p,q}"] & & E^r_{p,q} \arrow[ul, "d_r"'] \end{tikzcd}$$

the maps $i^r_{p,q}$ and $i^r_{p-r,q+r-1}$ are isomorphisms by induction hypothesis. Therefore $\tilde{E}^{r+1}_{p,q} \cong E^{r+1}_{p,q}$ given by $i^{r+1}_{p,q}$ due to the naturality of the spectral sequence.

(b) $q \geq k$. Then $\tilde{E}^{r+1}_{p,q} = 0$ by induction hypothesis.

(c) $k+1-r \leq q < k$. Then the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r, equal] & \tilde{E}^r_{p-r,q+r-1} \arrow[rr, "i^r_{p-r,q+r-1}"] & & E^r_{p-r,q+r-1} & \\ & & \tilde{E}^r_{p,q} \arrow[ul, "\tilde{d}_r"] \arrow[rr, "i^r_{p,q}"] & & E^r_{p,q}. \arrow[ul, "d_r"'] \end{tikzcd}$$

As $i^r_{p,q}$ is an isomorphism by induction hypothesis, it follows, that $d_r = 0$ and therefore $\tilde{E}^{r+1}_{p,q} \cong E^{r+1}_{p,q}$.

This shows the claim. □

Now turn to the cohomological spectral sequence of the fiber bundle $L \to \partial M \to \Sigma$. Here the induced map is denoted $i^{p,q}_r : E^{p,q}_r \to \tilde{E}^{p,q}_r$.

Lemma 1.2.6. *For all $r \in \mathbb{N}$, $i^{p,q}_r$ is an isomorphism for $q < k$ and $\tilde{E}^{p,q}_r = 0$ for $q \geq k$.*

Proof. The same induction argument as for homology (with the differentials going in the opposite direction) shows the claim. □

Theorem 1.2.7. *Let X be a compact, oriented pseudomanifold with two strata and a link bundle $(L^{c-1}, \partial M, p, \Sigma^{n-c})$ that admits a fiberwise truncation in every degree $l \geq c-1-\overline{q}(c)$ and $l \leq c-1-\overline{p}(c)$ (in the sense of Definition 1.1.3) for a pair of complementary perversities $\overline{p}$ and $\overline{q}$. Assume that the map $i_* : H_r(ft_{<k}(\partial M)) \to H_r(\partial M)$, induced by the fiber bundle map given by the fiberwise truncation, is injective for all r. Then there is a rational Poincaré duality isomorphism*

$$\tilde{H}^{n-r}(I^{\overline{p}}X; \mathbb{Q}) \to \tilde{H}_r(I^{\overline{q}}X; \mathbb{Q}).$$

Remark 1.2.8. *If $L_{<k}$ and $L_{<c-k}$ are subspaces of L, then the injectivity of i_* follows from Lemma 1.1.16. In many other cases, where the homology truncations are not subspaces, this condition is still true and can be checked by hand. For example this is true for trivial bundles or* ***ICW****-bundles over a sphere as defined in [Gai12].*

Proof. In this proof all homology and cohomology groups denote homology and cohomology with rational coefficients. Fix $k = c-1-\overline{p}(c)$. By assumption, for every $l \geq c-k = 1+\overline{p}(c) = c-1-\overline{q}(c)$, a fiberwise truncation $ft_{<l}(\partial M)$ is possible. As $i_* : H_r(ft_{<k}(\partial M)) \to H_r(\partial M)$ is injective by assumption, capping with the fundamental class $[\partial M]$ leads to the following commutative diagram with exact lines.

$$\begin{tikzcd} H^{n-1-r}(\partial M) \arrow[r] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-1-r}(ft_{<k}\partial M) \arrow[r] \arrow[d, "-\tilde{\cap}[\partial M]"] & 0 \\ H_r(\partial M) \arrow[r] & H_r(\partial M, ft_{<c-k}\partial M) \arrow[r] & 0. \end{tikzcd}$$

Therefore capping with the fundamental class

$$-\tilde{\cap}[\partial M] : H^{n-1-r}(ft_{<k}\partial M) \to H_r(\partial M, ft_{<c-k}\partial M)$$

is a surjective map. The strategy is to calculate the dimensions of the $\mathbb{Q}$-vector spaces $H^{n-1-r}(ft_{<k}\partial M)$ and $H_r(\partial M, ft_{<c-k}\partial M)$. We want to show that they equal and hence that the above map is an isomorphism.

Apply Lemma 1.2.5 to the truncation $ft_{<k}(\partial M)$. The spectral sequences E (corresponding to the fibration ∂M) and $\tilde{E}$ (corresponding to the fibration $ft_{<k}(\partial M)$) collapse, so that $\tilde{E}^{\infty}_{p,q} \cong E^{\infty}_{p,q}$ for $q < k$ and $\tilde{E}^{\infty}_{p,q} = 0$ for $q \geq k$. As all occurring terms are $\mathbb{Q}$-vector spaces, the recovery problems can be solved (compare the discussion in Chapter 2.2) and it follows that

$$H_r(ft_{<k}\partial M) \cong \bigoplus_{p+q=r} \tilde{E}^{\infty}_{p,q} \cong \bigoplus_{p+q=r, q<k} E^{\infty}_{p,q}.$$

For every $q \geq c-1-\overline{q}(c)$ we can choose by assumption the fiberwise truncation $ft_{<q+1}(\partial M)$. Denote the corresponding spectral sequence again by $\tilde{E}$. Lemma 1.2.5 applied to this truncation shows that $i^2_{p,q}$ is an isomorphism and that $\tilde{E}^2_{p-2,q+1} = H_{p-2}(\Sigma; \mathscr{H}_{q+1}(L_{<q+1}))$ equals zero. The commutative diagram

$$\begin{tikzcd}
\tilde{E}^2_{p-2,q+1} \arrow[rr, "i^2_{p-2,q+1}"] & & E^2_{p-2,q+1} & \\
& \tilde{E}^2_{p,q} \arrow[ul, "\tilde{d}_2"] \arrow[rr, "i^2_{p,q}"] & & E^2_{p,q}. \arrow[ul, "d_2"']
\end{tikzcd}$$

then implies, that the differential $d_2 : E^2_{p,q} \to E^2_{p-2,q+1}$ vanishes.

Inductively this shows that for every $q \geq c-1-\overline{q}(c)$ all higher differentials d_r also vanish and consequently that the spectral sequence has terms $E^{\infty}_{p,q} \cong E^2_{p,q}$ for all $q \geq c-1-\overline{q}(c) = c-k$. Since all terms are finitely generated $\mathbb{Q}$-vector spaces we get

$$\begin{aligned}
H_r(\partial M) &\cong \bigoplus_{p+q=r} E^{\infty}_{p,q} \cong \bigoplus_{p+q=r, q<c-k} E^{\infty}_{p,q} \oplus \bigoplus_{p+q=r, q\geq c-k} E^2_{p,q} \\
&\cong H_r(ft_{<c-k}\partial M) \oplus \bigoplus_{p+q=r, q\geq c-k} H_p(\Sigma; \mathscr{H}_q(L)).
\end{aligned}$$

An equivalent statement holds for cohomology. Lemma 1.2.6 shows that

$$\bigoplus_{\substack{p+q=n-r-1\\ q<k}} E^{p,q}_{\infty} \cong H^{n-r-1}(ft_{<k}(\partial M)).$$

Furthermore, the condition, that the fiber bundle ∂M can be truncated for all $l \leq c-1-\overline{p}(c) = k$ shows by an analogous argument as for homology that $E^{p,q}_{\infty} \cong E^{p,q}_2$ for $q < k$. We therefore have

$$\begin{aligned} H^{n-r-1}(\partial M) &\cong \bigoplus_{p+q=r} E_\infty^{p,q} \cong \bigoplus_{\substack{p+q=n-r-1\\ q<k}} E_2^{p,q} \oplus \bigoplus_{\substack{p+q=n-r-1\\ q\geq k}} E_\infty^{p,q} \\ &\cong \bigoplus_{\substack{p+q=n-r-1\\ q<k}} H^p(\Sigma;\mathscr{H}^q(L)) \oplus \bigoplus_{\substack{p+q=n-r-1\\ q\geq k}} E_\infty^{p,q} \end{aligned}$$

and

$$\bigoplus_{\substack{p+q=n-r-1\\ q<k}} H^p(\Sigma;\mathscr{H}^q(L)) \cong H^{n-r-1}(ft_{<k}(\partial M)).$$

By assumption, $H_r(ft_{<k}(\partial M)) \to H_r(\partial M)$ is injective, from where follows

$$H_r(\partial M) = H_r(ft_{<c-k}(\partial M)) \oplus H_r(\partial M, ft_{<c-k}(\partial M))$$

and

$$H^{n-r-1}(\partial M) = H^{n-r-1}(ft_{<k}(\partial M)) \oplus H^{n-r-1}(\partial M, ft_{<k}(\partial M)).$$

(the last step involves the universal coefficient theorem, noting that we work with rational coefficients). We can therefore identify

$$\bigoplus_{p+q=r, q\geq c-k} H_p(\Sigma;\mathscr{H}_q(L)) \cong H_r(\partial M, ft_{<c-k}(\partial M)).$$

As explained, the action of $\pi_1(\Sigma)$ on $H_*(L)$ defines a group bundle $\mathscr{H}_*(L;\mathbb{Q})$ on Σ. Equivalently, the action of $\pi_1(\Sigma)$ on $H^{c-1-*}(L)$ induces a group bundle $\mathscr{H}^{c-1-*}(L;\mathbb{Q})$ (with a collection of homomorphisms $h_2[\lambda]$). Poincaré duality induces a commutative diagram

$$\begin{tikzcd} H^{c-1-*}(L) \cong (\mathscr{H}^{c-1-*}(L;\mathbb{Q}))_{b_1} \arrow[r, "h_2[\lambda]"] \arrow[d, "-\cap[L]"'] & (\mathscr{H}^{c-1-*}(L;\mathbb{Q}))_{b_0} \cong H^{c-1-*}(L) \arrow[d, "-\cap[L]"] \\ H_*(L) \cong (\mathscr{H}_*(L;\mathbb{Q}))_{b_1} \arrow[r, "h_1[\lambda]"] & (\mathscr{H}_*(L;\mathbb{Q}))_{b_0} \cong H_*(L). \end{tikzcd}$$

Therefore Poincaré duality on L induces an isomorphism of the group bundles $\mathscr{H}_*(L;\mathbb{Q})$ and $\mathscr{H}^{c-1-*}(L;\mathbb{Q})$ (compare the definition of an isomorphism of group bundles as given in [McC01], Section 5.3). For local coefficient system given by a group bundle $\mathscr{G}$, Poincaré duality holds, that is for the closed, connected, oriented manifold Σ^{n-c}

$$H^{n-c-p}(\Sigma;\mathscr{G}) \cong H_p(\Sigma;\mathscr{G}).$$

This is shown for example in [CH96], Theorem 1.1. The definition for a system of local coefficients given in [CH96] coincides with the definition we cited from [McC01]. This equivalence is shown for example in [Hat02], Proposition 3.H.4. It follows that

$$\begin{aligned}
H^{n-r-1}(ft_{<k}\partial M) &\cong \bigoplus_{\substack{p+q=n-r-1\\ q<k}} H^p(\Sigma;\mathscr{H}^q(L)) \cong \bigoplus_{\substack{p+q=n-r-1\\ q<k}} H^p(\Sigma;\mathscr{H}_{c-1-q}(L)) \\
&\cong \bigoplus_{\substack{p+q=n-r-1\\ q<k}} H_{n-c-p}(\Sigma;\mathscr{H}_{c-1-q}(L)) \cong \bigoplus_{\substack{p+q=r\\ q\geq c-k}} H_p(\Sigma;\mathscr{H}_q(L)) \\
&\cong H_r(\partial M, ft_{<c-k}(\partial M)).
\end{aligned}$$

This shows that

$$\dim H^{n-r-1}(ft_{<k}\partial M) = \dim H_r(\partial M, ft_{<c-k}\partial M)$$

and thus $-\tilde{\cap}[\partial M]$ in the following diagram

$$\begin{tikzcd}
H^{n-1-r}(\partial M) \arrow[r] \arrow[d, "-\cap[\partial M]"'] & H^{n-1-r}(ft_{<k}\partial M) \arrow[r] \arrow[d, "-\tilde{\cap}[\partial M]"] & 0 \\
H_r(\partial M) \arrow[r] & H_r(\partial M, ft_{<c-k}\partial M) \arrow[r] & 0
\end{tikzcd}$$

is a surjection of $\mathbb{Q}$-vector spaces of equal dimension, hence an isomorphism. Compose with the Poincaré duality isomorphism on M to the commutative diagram

$$\begin{tikzcd}
H^{n-r-1}(M) \arrow[r] \arrow[d, "-\cap[M]"'] & H^{n-1-r}(\partial M) \arrow[r] \arrow[d, "-\cap[\partial M]"'] & H^{n-1-r}(ft_{<k}\partial M) \arrow[d, "-\tilde{\cap}[\partial M]"] \\
H_{r+1}(M,\partial M) \arrow[r] & H_r(\partial M) \arrow[r] & H_r(\partial M, ft_{<c-k}\partial M).
\end{tikzcd}$$

Lemma 2.46 of [Ban10a] and the five lemma applied to the diagram

$$\begin{tikzcd}
H^{n-r-1}(M) \arrow[r, "-\cap[M,\partial M]", "\cong"'] \arrow[d] & H_{r+1}(M,\partial M) \arrow[d] \\
H^{n-r-1}(ft_{<k}(\partial M)) \arrow[r, "-\tilde{\cap}[\partial M]", "\cong"'] \arrow[d] & H_r(\partial M, ft_{<c-k}(\partial M)) \arrow[d] \\
H^{n-r}(M, ft_{<k}(\partial M)) \arrow[d] & H_r(M, ft_{<c-k}(\partial M)) \arrow[d] \\
H^{n-r}(M) \arrow[r, "-\cap[M,\partial M]", "\cong"'] \arrow[d] & H_r(M,\partial M) \arrow[d] \\
H^{n-r}(ft_{<k}(\partial M)) \arrow[r, "-\tilde{\cap}[\partial M]", "\cong"'] & H_{r-1}(\partial M, ft_{<c-k}(\partial M))
\end{tikzcd}$$

show Poincaré duality

$$\tilde{H}^{n-r}(I^{\overline{p}}X) \to \tilde{H}_r(I^{\overline{q}}X).$$

□

Corollary 1.2.9. *Theorem 1.2.7 in particular proves Poincaré duality for stratified pseudomanifolds with **ICW**-bundles over a sphere independently of the proof given in [Gai12].*

Proof. For the application of Theorem 1.2.7 note that $\pi_1(\Sigma) = 0$ and a fiberwise truncation can be constructed in every degree. □

1.2.4 Summary

We summarize the preceeding results. Let us formulate the following two properties.

(INJ) $H_r(ft_{<k}(\partial M);\mathbb{Q}) \to H_r(\partial M;\mathbb{Q})$ is injective for all $r \in \mathbb{N}$.

(PD) There is a Poincarè duality isomorphism $\tilde{H}^{n-r}(I^{\overline{p}}X;\mathbb{Q}) \to \tilde{H}_r(I^{\overline{q}}X;\mathbb{Q})$ for all $r \in \mathbb{N}$.

In the following situations, we have now proven duality statements.

1. If the link bundle allows a fiberwise truncation and $L_{<k} \subset L$, then (PD) and (INJ) hold. Examples are $(\overline{p}, \overline{q})$-admissible bundles or bundles where the homology truncation is an G-equivariant Moore-approximation.
2. If $k = c - 1$ and the link bundle allows a strong fiberwise truncation, then (PD) and (INJ) hold.
3. If the link bundle allows a fiberwise truncation in all degrees $l \geq c - 1 - \overline{q}(c)$ and $l \leq c - 1 - \overline{p}(c)$ and if (INJ) holds, then (PD) holds. Examples are trivial bundles and **ICW**-bundles over a sphere.

1.2.5 Examples

We give some examples. In the first one, we give a bundle, that allows a truncation in two complementary degrees, but not in all degrees and has an infinite structure group. As expected, the truncated bundles share a duality across the complementary values. Note that in this example, the total space of the fiber bundle is not a manifold. However, the construction of the duality isomorphism D_Σ in the proof of Theorem 1.2.1 only requires that the total space is compact.

Example 1.2.10. *Let $c = (\cos\gamma, \sin\gamma) \in S^1$ and let S^1 act on T^2 by*

$$\begin{aligned} \phi_c &: ((1+\cos\theta)\cos\phi, (1+\cos\theta)\sin\phi, \sin\theta) \\ &\mapsto ((1+\cos(\theta+\gamma))\cos\phi, (1+\cos(\theta+\gamma))\sin\phi, \sin(\theta+\gamma)) \end{aligned}$$

(Rotation by γ along the meridian). Now collapse one meridian of the torus to a point and denote this point by x_0. The result is a pinched torus, which we denote by X. The above group action on the torus induces a group action on the pinched torus with one single fixed point, namely x_0. Set $M = X \vee_{x_0} S^3$ and let $\Phi_c : M \to M$ be the action defined by

$$\Phi_c : x \mapsto \begin{cases} \phi_c(x) & x \in X \\ x & x \in S^3. \end{cases}$$

Let M_Φ be the mapping torus of the group action $\Phi_c : M \to M$ with a fixed element of $c = (\cos\gamma, \sin\gamma) \in S^1$ where γ is not an integer multiple of 2π. Then M_Φ is a fiber bundle over S^1 with fiber M and infinite structure group. A homology truncation of the fiber M is given by

$M_{<3}$	$M_{<2}$	$M_{<1}$
X	S^1	x_0

where the S^1 is the longitude in the pinched torus. The Wang sequence calculates the homology of M_Φ. The induced map $\Phi_{c*} : H_*(M) \to H_*(M)$ is the identity. Therefore the homology is given by

$$H_r(M_\Phi) \cong \begin{cases} \mathbb{Z} & r = 0, 4 \\ \mathbb{Z} \oplus \mathbb{Z} & r = 1, 2, 3. \end{cases}$$

The group action Φ_c on M restricts to a group action on X. Therefore we can form the bundle $ft_{<3}M_\Phi$. Furthermore the restriction to the fixed point x_0 forms a bundle $ft_{<1}M_\Phi$. However, we can not restrict the action to any other homology truncation of M_Φ. In particular the group action Φ_c does not restrict to the 2-truncation $M_{<2}$. We have

$$H_r(ft_{<1}M_\Phi) \cong \begin{cases} \mathbb{Z} & r = 0, 1 \\ 0 & \textit{else} \end{cases}$$

and

$$H_r(ft_{<3}M_\Phi) \cong \begin{cases} \mathbb{Z} & r = 0, 3 \\ \mathbb{Z} \oplus \mathbb{Z} & r = 1, 2 \\ 0 & \textit{else.} \end{cases}$$

Therefore

$$H_r(M_\Phi, ft_{<1}(M_\Phi)) \cong H^{4-r}(ft_{<3}(M_\Phi))$$

and

$$H_r(M_\Phi, ft_{<3}(M_\Phi)) \cong H^{4-r}(ft_{<1}(M_\Phi)).$$

Example 1.2.11. *Let $p : S^3 \to S^2$ be the Hopf fibration. Define*

$$\tilde{p} : S^3 \times S^2 \to S^2$$

by $\tilde{p}(x, y) = p(x)$. Then $\tilde{p}$ is a fiber bundle (whose total space we will denote by ξ) with fiber $S^1 \times S^2$. We can form the bundle $ft_{<2}(\xi)$ which is just the Hopf fibration and have a map of fibrations $ft_{<2}(\xi) \to \xi$ given by

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "i|"] & S^3 \arrow[r, "p"] \arrow[d, "i"] & S^2 \arrow[d, equal] \\ S^1 \times S^2 \arrow[r] & S^3 \times S^2 \arrow[r, "\tilde{p}"] & S^2. \end{tikzcd}$$

Comparison of the homology shows

$$\dim H_r(S^3 \times S^2; \mathbb{Q}) \neq \dim \bigoplus_{i+j=r} H_i(S^2, H_j(S^1 \times S^2; \mathbb{Q}))$$

for example for $r = 1$ and therefore the Serre spectral sequence of the fibration ξ can not collapse at E_2. This in turn shows, that we can not truncate the bundle ξ in all degrees. Indeed, in degrees 1 and 3 a truncation is not possible, as this would imply the existence of a section of the Hopf bundle. On the other hand, as predicted by Theorem 1.2.1, there is an isomorphism

$$H^{5-r}(ft_{<2}(\xi); \mathbb{Q}) \to H_r(\xi, ft_{<2}(\xi); \mathbb{Q})$$

for all r ($\xi = S^3 \times S^2$ and $ft_{<2}(\xi) = S^3$).

1.3 A Good Class of Pseudomanifolds

In Section 1.1, we have seen, that certain two-strata pseudomanifolds with a link bundle that admits a fiberwise truncation have the property that $H_r(ft_{<k}(\partial M); \mathbb{Q}) \to H_r(\partial M; \mathbb{Q})$ is injective and that there is a Poincaré duality isomorphism between the intersection spaces of complementary perversity. In this section, we want to define a wider class of pseudomanifolds, for which these properties still hold. We do this, as many of the subsequent results can be formulated in more generality than only for two-strata spaces. Let X be a stratified pseudomanifold and let as usual denote M the manifold-with-boundary obtained by cutting off an open neighborhood of the singular set of X.

Definition 1.3.1. Let $\overline{p}$ be a perversity. Define Ξ as the class of pairs $(X, B(\overline{p}))$, where X is a compact, stratified pseudomanifold and $B(\overline{p})$ is a space together with a map $i : B(\overline{p}) \to \partial M$ such that the induced map on homology $i_* : H_r(B(\overline{p}); \mathbb{Q}) \to H_r(\partial M; \mathbb{Q})$ is injective in every degree $j \in \mathbb{N}$.

Denote

$$I^{B(\overline{p})}X := cone(B(\overline{p}) \to \partial M \hookrightarrow M).$$

In the following Lemma (2.-6.) $I^{B(\overline{p})}X$ is the intersection space as defined in [Ban10a], [Gai12] and [Ban12].

Lemma 1.3.2. *The following pairs are in Ξ and $I^{B(\overline{p})}X$ possesses Poincaré duality across complementary perversities.*

1. *Any compact, oriented stratified pseudomanifold when $B(\overline{p}) = pt$ and $B(\overline{q}) = \partial M$ for complementary perversities $\overline{p}$ and $\overline{q}$.*

2. *Compact, oriented pseudomanifolds with two strata and a trivial link bundle, with a simply connected link. $B(\overline{p})$ is a fiberwise truncation of the link bundle.*

3. *Compact, oriented pseudomanifolds with two strata and an interleaf link bundle over a sphere. $B(\overline{p})$ is a fiberwise truncation of the link bundle.*

4. *Compact, oriented pseudomanifolds with two strata and a $(\overline{p}, \overline{q})$-admissible link bundle that has a base space which is the union of two open subsets, such that over each of these subsets, the bundle is trivial. $B(\overline{p})$ is a fiberwise truncation of the link bundle.*

5. *Even-dimensional, compact, oriented pseudomanifolds with a stratification $X_n \supset X_1 \supset X_0$ with $X_1 \cong S^1$ and X_0 a point, such that the links of the strata are simply connected and X satisfies the strong Witt condition. $B(\overline{p}) = |H\Gamma^{\overline{p}}|$ as constructed in [Ban12].*

Remark 1.3.3. *Lemma 1.3.2 shows, that for all stratified pseudomanifolds for which intersection spaces are constructed so far (see [Ban10a],[Ban12] and [Gai12]), a space $B(\overline{p})$ can be found, such that $(X, B(\overline{p})) \in \Xi$ and $I^{B(\overline{p})}X = I^{\overline{p}}X$ is the intersection space.*

Proof of the Lemma. Follow the numbering in the Proposition.

1. This is just ordinary Poincaré duality for compact manifolds with boundary.

2. See [Ban10a], Section 2.9.

3. by Theorem 1.2.7

4. by Theorem 1.2.1

5. Let $e : |H\Gamma^{\overline{p}}| \to \partial M$ and let all homology groups be with rational coefficients. Proposition 6.1 of [Ban12] shows, that there is an isomorphism ϕ such that

$$\begin{tikzcd} H^{n-r}(\partial M) \arrow[r, "e^*"] \arrow[d, "-\cap[\partial M]"'] & H^{n-r}|H\Gamma^{\overline{p}}| \arrow[d, "\phi"] \\ H_{r-1}(\partial M) \arrow[r] & H_{r-1}(e) \end{tikzcd}$$

commutes. By the usual five lemma argument, this diagram can be extended to the following commutative diagram

$$\begin{tikzcd} H^{n-r}(\partial M) \arrow[r, "e^*"] \arrow[d, "-\cap[\partial M]"'] & H^{n-r}|H\Gamma^{\overline{p}}| \arrow[r, "\partial^*"] \arrow[d, "\phi"] & H^{n-r+1}(e) \arrow[d, "\cong"] \\ H_{r-1}(\partial M) \arrow[r] & H_{r-1}(e) \arrow[r, "\partial_*"] & H_{r-2}|H\Gamma^{\overline{p}}|. \end{tikzcd}$$

Proposition 6.1 of [Ban12] further shows that the maps $H_{r-1}(\partial M) \to H_{r-1}(e)$ and e^* are isomorphisms in degree $r > k$ whereas $H^{n-r}|H\Gamma^{\overline{p}}| = H_{r-1}(e) = 0$ in degree

$r \leq k$. In all cases this shows that $\partial^* = \partial_* = 0$ and therefore the map

$$H_r|H\Gamma^{\overline{p}}| \xrightarrow{e_*} H_r(\partial M)$$

is injective for all r.

□

Definition 1.3.4. Let $\mathbf{C}$ be the category whose objects are pairs of CW-complexes (X, A) with $A \subset X$ and a morphism between (X, A) and (Y, B) is a cellular map $f : X \to Y$ which maps A into B. Let $\mathbf{TfCW}$ denote the full subcategory of $\mathbf{C}$, with objects (X, A) that have the property

$$\operatorname{Tor}_{\mathbb{Z}} H_*(X, A; \mathbb{Z}) = 0.$$

For a pair $(X, \emptyset) \in \operatorname{Ob}\mathbf{TfCW}$, we will simply write $X \in \operatorname{Ob}\mathbf{TfCW}$.

Definition 1.3.5. Let Θ be the subclass of pairs $(X, B(\overline{p})) \in \Xi$ with ∂M, $B(\overline{p})$ and $(\partial M, B(\overline{p}))$ in $\operatorname{Ob}\mathbf{TfCW}$.

Lemma 1.3.6. *The following pseudomanifolds are in Θ:*

1. *Compact, oriented pseudomanifolds with two strata and a trivial link bundle such that the link is simply connected, and $B(\overline{p})$ a fiberwise truncation of the link bundle, with* $\operatorname{Tor}(H_*(L)) = \operatorname{Tor}(H_*(\Sigma)) = 0$*,*

2. *Compact, oriented pseudomanifolds with two strata and an interleaf link bundle over a sphere (without further restrictions), and $B(\overline{p})$ a fiberwise truncation of the link bundle,*

Proof. Numbering as in the Lemma.

1. By construction of the homology truncation and the Künneth formula, the map $B(\overline{p}) = \Sigma \times L_{<k} \to \Sigma \times L = \partial M$ induces an injective map $H_r(B(\overline{p}); \mathbb{Z}) \to H_r(\partial M; \mathbb{Z})$ in every degree $r \in \mathbb{N}$. If $\operatorname{Tor}(H_*(\partial M); \mathbb{Z}) = 0$ then it follows that also $H_*(B(\overline{p}); \mathbb{Z})$ is torsion-free. As $H_r(L, L_{<k}; \mathbb{Z}) \cong H_r(L; \mathbb{Z})$ for $r \geq k$ and $H_r(L, L_{<k}; \mathbb{Z}) = 0$ for $r < k$, it follows again from the Künneth-Theorem, that $H_r(\partial M, B(\overline{p}); \mathbb{Z})$ is torsion free.

2. The link is assumed to be in the interleaf category and the singular stratum is a sphere. Then the homology of L is torsion-free ([Ban10a], Lemma 1.64). In [Gai12] it is shown by an argument involving the collapse of the Serre spectral sequence of the fiber bundle, that

$$H_r(\partial M; \mathbb{Q}) \cong H_r(L; \mathbb{Q}) \oplus H_{r-n+c}(L; \mathbb{Q}).$$

By the same argument, this holds also for integral coefficients. Indeed, if $n - c$ is even, then all differentials d_2 map from or to zero entries. If $n - c$ is odd, then we can choose a truncation $L_{<k}$ for a k that is small enough and employ the naturality of spectral sequences with respect to the map of fibrations

$$\begin{tikzcd} L_{<k} \arrow[r] \arrow[d] & W(ft_{<k}(p)) \arrow[r] \arrow[d] & B \arrow[d] \\ L \arrow[r] & Y \arrow[r] & S^{n-c} \end{tikzcd}$$

to conclude, that the differentials are 0. Here $W(ft_{<k}(p))$ is the associated Hurewicz fibration to the Dold fibration $ft_{<k}(p)$ (for details see the construction in [Gai12]). In particular this shows, that the homology of ∂M is also torsion-free. Again naturality of the Serre-spectral sequence shows, that

$$H_r(ft_{<k}(\partial M);\mathbb{Z}) \cong H_r(L_{<k};\mathbb{Z}) \oplus H_{r-n+c}(L_{<k};\mathbb{Z}).$$

Since $\operatorname{Tor} H_*(L) = 0$ implies $\operatorname{Tor} H_*(L_{<k}) = 0$ it then follows that

$$\operatorname{Tor}(B(\overline{p})) = \operatorname{Tor}(H_*(ft_{<k}(\partial M))) = 0.$$

By a five lemma argument (Lemma 2.6.3 in [Gai12]), it follows that

$$H_r(\partial M, ft_{<c-k}(\partial M)) \cong H_r(L_{\geq c-k}) \oplus H_{r-n+c}(L_{\geq c-k}).$$

This shows the claim.

□

Example 1.3.7. *We want to list some examples of naturally occurring pseudomanifolds which are in* Θ*:*

(a) Topological conifolds as arise in a conifold transition between Calabi-Yau manifolds have links $L = S^2 \times S^3$*.*

(b) Let $(X, 0) \subset (\mathbb{C}^n, 0)$ *be an irreducible, isolated surface singularity and let* $\tilde{X}$ *be a smooth resolution. The link* L *is a compact, oriented 3-dimensional manifold. The homology of the link is torsion-free if* $\det(I(\tilde{X})) = \pm 1$*, where* $I(\tilde{X})$ *is the intersection matrix of the resolution* $\tilde{X}$ *(see [Dim92], § 3).*

(c) All examples of pseudomanifolds with a link in ***ICW*** *as listed in [Ban10a], Example 1.65. This includes for example links that are complex projective spaces or simply connected, closed 4-manifolds.*

Remark 1.3.8. *Let* $X \in \Theta$*. Assume that the cap product of Lemma 1.1.16 can already be constructed with integral coefficients.*

$$\begin{tikzcd} H^{n-1-r}(\partial M;\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "-\cap[\partial M]"] & H^{n-1-r}(ft_{<k}\partial M;\mathbb{Z}) \arrow[r] \arrow[d, "-\tilde{\cap}[\partial M]"] & 0 \\ H_r(\partial M;\mathbb{Z}) \arrow[r, "\pi_*"] & H_r(\partial M, ft_{<c-k}\partial M;\mathbb{Z}) \arrow[r] & 0. \end{tikzcd}$$

Then there is an integral duality isomorphism

$$H_r(I^{\overline{p}}X;\mathbb{Z}) \to H^{n-r}(I^{\overline{q}}X;\mathbb{Z}).$$

This is a consequence of the following Lemma 1.3.10 applied to the diagram

$$\begin{tikzcd} {} \arrow[r] & H^r(M,\partial M;\mathbb{Z}) \arrow[r] \arrow[d, "-\cap[M]"] & H^r(M, ft_{<c-k}\partial M;\mathbb{Z}) \arrow[r] & H^r(\partial M, ft_{<c-k}\partial M;\mathbb{Z}) \arrow[r] \arrow[d, "-\cap[\partial M]"] & {} \\ {} \arrow[r] & H_{n-r}(M;\mathbb{Z}) \arrow[r] & H_{n-r}(M, ft_{<k}\partial M;\mathbb{Z}) \arrow[r] & H_{n-1-r}(ft_{<k}\partial M;\mathbb{Z}) \arrow[r] & {}. \end{tikzcd}$$

Corollary 1.3.9. *In particular Poincaré duality holds yet integrally for a stratified pseudomanifold X with two strata and a link bundle that is an **ICW**-bundle over a sphere, as constructed in [Gai12].*

Lemma 1.3.10 (Modified Five Lemma)**.** *Let $A, B, C, D, E, A', B', C', D', E'$ be finitely generated abelian groups and let D, D' be torsion-free. If the following diagram of exact sequences is commutative up to sign*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

and if the maps $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then there exists an isomorphism

$$\gamma : C \to C'.$$

fitting in the diagram commutatively. The isomorphism γ also exists, if A, B, C, D, E, A', B', C', D' and E' are $\mathbb{Q}$-vector spaces.

Proof. For rational vector spaces, this is Lemma 2.46 of [Ban10a] and the five lemma (where the outer squares commute up to sign). We use the same lines of argument as in Lemma 2.46 of [Ban10a] (for the five lemma compare for example [Hat02]) and just show, that commutativity up to sign and passing from rational vector spaces to $\mathbb{Z}$-modules where D, D' are torsion-free, does not change anything. Since D is torsion-free there is a splitting $s_k : \mathrm{im}(k) \to C$ and we can write $c = j(b) + x$ for $c \in C$ and for some element $b \in B$ and $x \in \mathrm{im}(s_k)$. Define

$$\gamma(c) = \gamma(j(b) + x) := j'\beta(b) + s_{k'}(\delta k(x))$$

where $s_k : \mathrm{im}(k') \to C'$ is a splitting for k'. This is well defined: Let $b' \in B$ such that $j(b) = j(b')$, then there is a $a \in A$ with $i(a) = b - b'$. Thus $j'\beta(b - b') = j'\beta i(a) = \pm j'i'\alpha(a) = 0$. It follows that $j'\beta(b) = j'\beta(b')$. The inner squares commute, since

$$\gamma j(b) = j'\beta(b)$$

and

$$k'\gamma(c) = k'\gamma(j(b) + x) = k'j'\beta(b) + k's_{k'}\gamma k(x) = \gamma k(x).$$

γ is injective: If the left square commutes, then injectivity of γ follows from the 5 lemma. So we assume that $\beta i = -i'\alpha$. Assume $\gamma(c) = 0$ We want to show that $c = 0$. $\delta k(c) = k'\gamma(c) = 0$. Since δ is injective it follows that $k(c) = 0$, thus there is a $b \in B$ with $j(b) = c$. Furthermore $j'\beta(b) = \gamma j(b) = \gamma(c) = 0$ and therefore $\beta(b) = i'(a')$ for some $a' \in A'$. Since α is surjective, there is an element $a \in A$ with $a' = -\alpha(a)$. Then $\beta(i(a) - b) = \beta i(a) - \beta(b) = -i'\alpha(a) - \beta(b) = 0$. β is injective and so $i(a) - b = 0$ from where follows $c = j(b) = ji(a) = 0$.
γ is surjective: Assume $\epsilon l = -l'\delta$, otherwise the five lemma shows the claim. Let $c' \in C'$. δ is surjective, thus $k'(c') = \delta(d)$ for some $d \in D$. ϵ is injective, therefore $\epsilon l(d) = -l'\delta(d) = -l'k'(c') = 0$ implies $l(d) = 0$ from where follows that $d = k(c)$ for a $c \in C$. Now $k'(c' - \gamma(c)) = k'(c') - k'(\gamma(c)) = k'(c) - \delta k(c) = k'(c') - \delta(d) = 0$ and thus $c' - \gamma(c) = j'(b')$ for a $b' \in B'$. Moreover β is surjective, showing $b' = \beta(b)$ for a $b \in B$. Finally $\gamma(c + j(b)) = \gamma(c) + \gamma j(b) = \gamma(c) + j'\beta(b) = \gamma(c) + j'(b') = c'$ □

Remark 1.3.11. *Often, it is not necessary that the homology groups are torsion free in every degree, but sufficient, that they are torsion free in some degrees. The next Corollary shows that in the case of isolated singularities.*

Corollary 1.3.12. *Let X be an n-dimensional, oriented, stratified pseudomanifold with only isolated singularities. Let $H_{k-1}(L)$ be torsion-free where $k = n - 1 - \overline{p}(n)$ and let $\overline{p}$ and $\overline{q}$ be complementary perversities. Then there is an isomorphism*

$$H_r(I^{\overline{p}}X; \mathbb{Z}) \cong H^{n-r}(I^{\overline{q}}X; \mathbb{Z})$$

for every i.

Proof. In all degrees other than k, the homology of the intersection space is given by

$$H_r(I^{\overline{p}}X) \cong \begin{cases} H_r(M, \partial M) & r < k \\ \\ H_r(M) & r > k. \end{cases}$$

Furthermore

$$H^r(L_{<n-k}) \cong \begin{cases} H^r(L) & r < n-k \\ \operatorname{Ext}(H_{k-1}(L), \mathbb{Z}) = 0 & r = n-k \\ 0 & r > n-k \end{cases}$$

and the long exact sequence of the pair $(M, L_{<n-k})$ then shows that

$$H^r(I^{\overline{p}}X) \cong \begin{cases} H^r(M, \partial M) & r < n-k \\ \\ H^r(M) & r > n-k. \end{cases}$$

□

Assume, that $(X, B(\overline{p})), (X, B(\overline{q})) \in \Xi$ (see Definition 1.3.1).

Definition 1.3.13. We call an isomorphism

$$\phi : \tilde{H}^r(I^{B(\overline{p})};\mathbb{Q}) \to \tilde{H}_{n-r}(I^{B(\overline{q})};\mathbb{Q})$$

a *Poincaré duality isomorphism* if the induced map

$$\psi : H^r(\partial M, B(\overline{p});\mathbb{Q}) \to H_{n-r-1}(B(\overline{q});\mathbb{Q})$$

that is given by Lemma 1.3.10 applied to the diagram

$$\begin{tikzcd}
H^r(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap[M,\partial M]"] \arrow[d] & H_{n-r}(M;\mathbb{Q}) \arrow[d] \\
\tilde{H}^r(I^{\overline{p}}X;\mathbb{Q}) \arrow[r, "\phi"] \arrow[d] & \tilde{H}_{n-r}(I^{\overline{q}}X;\mathbb{Q}) \arrow[d] \\
H^r(\partial M, B(\overline{p});\mathbb{Q}) \arrow[d] & H_{n-r-1}(B(\overline{q});\mathbb{Q}) \arrow[d] \\
H^{r+1}(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap[M,\partial M]"] \arrow[d] & H_{n-r-1}(M;\mathbb{Q}) \arrow[d] \\
\tilde{H}^{r+1}(I^{\overline{p}}X;\mathbb{Q}) \arrow[r, "\phi"] & \tilde{H}_{n-r-1}(I^{\overline{q}}X;\mathbb{Q})
\end{tikzcd}$$

fits into the following diagram commutatively.

$$\begin{tikzcd}
H^r(\partial M, B(\overline{p});\mathbb{Q}) \arrow[r] \arrow[d, "\psi"'] & H^r(\partial M;\mathbb{Q}) \arrow[r] \arrow[d, "-\cap[\partial M]"] & H^{r+1}(M,\partial M;\mathbb{Q}) \arrow[d, "-\cap[M,\partial M]"] \\
H_{n-r-1}(B(\overline{q});\mathbb{Q}) \arrow[r] & H_{n-r-1}(\partial M;\mathbb{Q}) \arrow[r] & H_{n-r-1}(M;\mathbb{Q}).
\end{tikzcd}$$

Example 1.3.14. *If X is a 2-strata pseudomanifold with a link bundle that admits a fiberwise truncation in two complementary degrees, then for certain pseudomanifolds we constructed in Section 1.2 a Poincaré duality isomorphism*

$$\tilde{H}^{n-r}(I^{\overline{q}}X;\mathbb{Q}) \to \tilde{H}_r(I^{\overline{p}}X;\mathbb{Q}).$$

1.4 Choices in the Construction of the Intersection Space

We recall some details of the construction of intersection spaces from [Ban10a], Chapter 1. The category $\mathbf{CW}_{n\supset\partial}$ has as objects pairs (K,Y) where K is a simply connected CW-complex and $Y \subset C_n(K)$ is a subgroup of the n-th cellular chain group of K. A morphism is a cellular map $f : K \to L$ with $f_\#(Y_K) \subset Y_L$, where $f_\#$ is the map induced by f on the cellular chain groups. There is an assignment

$$t_{<n} : \mathbf{CW}_{n\supset\partial} \to \mathbf{HoCW}_{n-1}$$

where $\mathbf{HoCW}_{n-1}$ is the homotopy category of rel $n-1$-skeleton homotopy equivalences between CW-complexes. Set $K_{<n} = t_{<n}(K, Y)$. It is an open question, if the homotopy type of $K_{<n}$ is independent of the choice of Y. Therefore, if $ft_{<k}\partial M$ is a fiberwise truncated link bundle, then it is not clear if the homotopy type of $I^{\overline{p}}X$ is independent of the choice of $Y \subset C_n(K)$. We prove that for links with torsion-free homology, the homotopy type of intersection spaces with a trivial link bundle is independent of Y.
Let X be a stratified pseudomanifold with a trivial link bundle and whose link L is has torsion-free homology. Then there is a canonical way of defining the intersection space. Since L has only integral homology, the theorem about minimal cell structures ([Hat02], Proposition 4C.1), shows that there is a homotopy equivalence

$$f : L \to E(L)$$

where $E(L)$ is a CW-complex with every cell being a generator for the corresponding homology group. By Lemma 1.2 of [Ban10a] this means that $E(L)$ is n-segmented for every $n \in \mathbb{N}$ and thus we can set $E(L)_{<n} = E(L)^{n-1}$ - the $(n-1)$-skeleton of $E(L)$. We can then form the intersection space by setting

$$G : \Sigma \times E(L)_{<n} \hookrightarrow \Sigma \times E(L) \xrightarrow{\overline{f}} \Sigma \times L \hookrightarrow M$$

where $\overline{f}$ is a homotopy inverse of f and set

$$I^{\overline{p}}X := cone(G).$$

Stronger, we obtain the following Proposition that shows that no matter how the truncation of L is done, the resulting intersection space always is homotopy equivalent to the intersection space $cone(G)$ as defined above.

Proposition 1.4.1. *Let X be a compact stratified pseudomanifold with two strata and a trivial link bundle. Let the link L have torsion-free homology. Then the homotopy type of $I^{\overline{p}}X$ is independent of the truncation.*

Proof. The proof is almost identical to the one given in [Ban10a] where this is shown for links in the interleaf category **ICW** (see Example 1.1.5 for a definition). We just check that the same requirements are fulfilled. Note that for links in Ob**ICW**, the truncation is even functorial, which is not true for links that only have torsion-free homology. But like in the interleaf case, the homotopy type of the intersection space is well defined. The goal is to show that $cone(G) \simeq cone(g)$ for any truncation $L_{<k}$ and structure map $g : L_{<k} \to L \hookrightarrow M$.
First step: For $i \geq k$

$$C_i(E(L_{<k})) = H_i(E(L_{<k})) = 0,$$

since all differentials are zero (see construction of minimal cell structures, using the fact that we only have generator cells and no relator cells). This shows, that $E(L_{<n})$ has only cells in dimension $< n$. Cellular approximation yields a cellular map

$$E(L_{<k}) \to L_{<k} \to L \to E(L)$$

that factors through

$$\begin{tikzcd} E(L_{<k}) \arrow[r] \arrow[d] & E(L) \\ E(L)^{k-1} \arrow[ur] & \end{tikzcd},$$

since $E(L_{<k})$ only has cells of dimension $\leq k-1$.
Second step: Now follow the proof of [Ban10a], Theorem 2.26 word by word to obtain a homotopy commutative diagram

$$\begin{tikzcd} L_{<k} \arrow[r] \arrow[d] & M \arrow[d] \\ E(L)^{k-1} \arrow[r] & M \end{tikzcd}$$

and therefore the following diagram also homotopy commutes

$$\begin{tikzcd} \Sigma \times L_{<k} \arrow[r] \arrow[d] & M \arrow[d] \\ \Sigma \times E(L)^{k-1} \arrow[r] & M. \end{tikzcd}$$

Theorem 6.6 of [Hil65] then proofs the claim. □

Corollary 1.4.2. *For stratified pseudomanifolds with a trivial link bundle, $K^*(I^{\overline{p}}X)$ is free of choices made in the construction of $I^{\overline{p}}X$.*

1.5 Cap Products

In this section we want to introduce a cap product of the form

$$\tilde{H}^i(I^{\overline{p}}X;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \to \tilde{H}_{j-i}(I^{\overline{q}}X;\mathbb{Q}).$$

for all pairs $(X, B(\overline{p}))$ and $(X, B(\overline{q}))$ in Ξ. If the associated spaces $I^{B(\overline{p})}X$ satisfy Poincaré duality, then capping with the fundamental class is an isomorphism. However, the construction of this cap product relies on some arbitrary choices and is not completely natural. It is therefore not suitable to give a new way of proving Poincaré duality for a given intersection space, but it rather rephrases the Poincaré duality isomorphism and gives it the form of a cap product. Furthermore, we will treat the question, to what extent this cap product is compatible with the cap product defined in [Ban10a]. We first introduce some notation. The commutative triangle

$$\begin{tikzcd} B(\overline{p})(\partial M) \arrow[r, "s"] \arrow[dr, "f"'] & \partial M \arrow[d, "i"] \\ & M \end{tikzcd} \tag{1.7}$$

induces the following long exact sequences.

$$H_r(B(\overline{p})(\partial M)) \xrightarrow{f_*} H_r(M) \xrightarrow{g_*} \tilde{H}_r(I^{\overline{p}}X) \xrightarrow{\partial_{fg*}} H_{r-1}(B(\overline{p})(\partial M)), \tag{1.8}$$

$$H_r(\partial M) \xrightarrow{i_*} H_r(M) \xrightarrow{j_*} H_r(M, \partial M) \xrightarrow{\partial_{ij*}} H_{r-1}(\partial M), \tag{1.9}$$

$$H_r(\partial M, B(\overline{p})(\partial M)) \xrightarrow{q_*} \tilde{H}_r(I^{\overline{p}}X) \xrightarrow{r_*} H_r(M, \partial M) \xrightarrow{\partial_{qr*}} H_{r-1}(\partial M, B(\overline{p})(\partial M))$$

and

$$H_r(B(\overline{p})(\partial M)) \xrightarrow{s_*} H_r(\partial M) \xrightarrow{t_*} H_r(\partial M, B(\overline{p})(\partial M)) \xrightarrow{\partial_{st*}} H_{r-1}(B(\overline{p})(\partial M))$$

and the corresponding sequences on cohomology. For a pair $(X, B(\overline{p}))$ in Ξ, we can define a cap product

$$\begin{tikzcd}
H^i(\partial M;\mathbb{Q}) \otimes H_{j-i-1}(\partial M;\mathbb{Q}) \arrow[r] \arrow[d, "s^*\otimes id"'] & H_{j-i-1}(\partial M;\mathbb{Q}) \arrow[d, "t_*"] \\
H^i(B(\overline{p})(\partial M);\mathbb{Q}) \otimes H_{j-1}(\partial M;\mathbb{Q}) \arrow[r] & H_{j-i-1}(\partial M, B(\overline{q})(\partial M);\mathbb{Q})
\end{tikzcd}$$

by choosing a splitting of the map surjective map s^*.

1.5.1 Construction of a Cap Product

Proposition 1.5.1. *Let $(X, B(\overline{p}))$ and $(X, B(\overline{q}))$ in Ξ. Then there is a cap product*

$$\tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \otimes H_j(M, \partial M;\mathbb{Q}) \to \tilde{H}_{j-i}(I^{B(\overline{q})}X;\mathbb{Q})$$

such that

$$\begin{tikzcd}
\tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \otimes H_j(M, \partial M;\mathbb{Q}) \arrow[r] \arrow[d, "g^*\otimes id"'] & \tilde{H}_{j-i}(I^{B(\overline{q})}X;\mathbb{Q}) \arrow[d, "r_*"] \\
H^i(M;\mathbb{Q}) \otimes H_j(M, \partial M;\mathbb{Q}) \arrow[r, "-\cap-"] & H_{j-i}(M, \partial M;\mathbb{Q})
\end{tikzcd}$$

commutes.

We first need two Lemmas about the naturality of the cap product. In the first Lemma, the existence of a cap product as constructed in Lemma 1.1.16 is crucial.

Lemma 1.5.2. *The following diagram commutes*

$$\begin{tikzcd}
H^i(M;\mathbb{Q}) \otimes H_j(M, \partial M;\mathbb{Q}) \arrow[r] \arrow[d, "f^*\otimes\partial_{ij*}"'] & H_{j-i}(M, \partial M;\mathbb{Q}) \arrow[d, "\partial_{qr*}"] \\
H^i(B(\overline{p})(\partial M);\mathbb{Q}) \otimes H_{j-1}(\partial M;\mathbb{Q}) \arrow[r] & H_{j-i-1}(\partial M, B(\overline{q})(\partial M);\mathbb{Q}).
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
H^i(M;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r] \arrow[d, "i^*\otimes\partial_{ij*}"'] & H_{j-i}(M,\partial M;\mathbb{Q}) \arrow[d, "\partial_{ij*}"] \\
H^i(\partial M;\mathbb{Q}) \otimes H_{j-i-1}(\partial M;\mathbb{Q}) \arrow[r] & H_{j-i-1}(\partial M;\mathbb{Q})
\end{tikzcd}$$

commutes by [Spa66], Chapter 5, Section 6.

$$\begin{tikzcd}
H^i(\partial M;\mathbb{Q}) \otimes H_{j-i-1}(\partial M;\mathbb{Q}) \arrow[r] \arrow[d, "s^*\otimes id"'] & H_{j-i-1}(\partial M;\mathbb{Q}) \arrow[d, "t_*"] \\
H^i(B(\overline{p})(\partial M);\mathbb{Q}) \otimes H_{j-1}(\partial M;\mathbb{Q}) \arrow[r] & H_{j-i-1}(\partial M, B(\overline{q})(\partial M);\mathbb{Q})
\end{tikzcd}$$

commutes by definition. From the triangle (1.7) it follows that $s^* \circ i^* = f^*$ and $t_* \circ \partial_{ij*} = \partial_{qr*}$. □

Lemma 1.5.3. *The following diagram commutes*

$$\begin{tikzcd}
H^i(M,\partial M) \otimes H_j(M,\partial M) \arrow[r, "-\cap-"] \arrow[d, equal] & H_{j-i}(M) \arrow[d, "j_*"] \\
H^i(M,\partial M) \otimes H_j(M,\partial M) \arrow[r, "-\cap-"] \arrow[d, "j^*\otimes id"'] & H_{j-i}(M,\partial M) \arrow[d, equal] \\
H^i(M) \otimes H_j(M,\partial M) \arrow[r, "-\cap-"] & H_{j-i}(M,\partial M).
\end{tikzcd}$$

Proof. Let X be a topological space and (A_1, A_2) an excisive couple in X (as defined in [Spa66], Chapter 4, Section 6) then there is a cap product (see [Spa66], Chapter 5, Section 6)

$$H^q(X, A_1) \otimes H_n(X, A_1 \cup A_2) \to H_{n-q}(X, A_2).$$

Set (X, A_1, A_2) equals $(M, \partial M, \emptyset)$, $(M, \partial M, \partial M)$ and $(M, \emptyset, \partial M)$. By the naturality property of the cap product (see [Spa66], Chapter 5, Section 6, 16), for triples (X, A_1, A_2) and (Y, B_1, B_2) and a map $f : X \to Y$ that takes A_1 to B_1 and A_2 to B_2, the following formula holds:

$$f_{2*}(f_1^* u \cap z) = u \cap \overline{f}_* z.$$

Here, $f_1 : (X, A_1) \to (Y, B_1)$, $f_2 : (X, A_2) \to (Y, B_2)$ and $\overline{f} : (X, A_1 \cup A_2) \to (Y, B_1 \cup B_2)$ are induced by f. For $\alpha \in H^i(M, \partial M)$, $b \in H_j(M, \partial M)$ and maps

$$f = g = id : M \to M$$

that induce maps

$$\begin{aligned} f_1 = id &:(M, \partial M) \to (M, \partial M) \\ f_2 = j &:(M, \emptyset) \to (M, \partial M) \\ \overline{f} = id &:(M, \partial M) \to (M, \partial M) \end{aligned}$$

and

$$\begin{aligned} g_1 = j &:(M, \emptyset) \to (M, \partial M) \\ g_2 = id &:(M, \partial M) \to (M, \partial M) \\ \overline{g} = id &:(M, \partial M) \to (M, \partial M). \end{aligned}$$

we thus get

$$\begin{aligned} j_*(\alpha \cap b) &= \alpha \cap \beta \\ &= j^* \alpha \cap \beta. \end{aligned}$$

This shows commutativity of

$$\begin{tikzcd} H^i(M, \partial M) \otimes H_j(M, \partial M) \arrow[r, "- \cap -"] \arrow[d, equal] & H_{j-i}(M) \arrow[d, "j_*"] \\ H^i(M, \partial M) \otimes H_j(M, \partial M) \arrow[r, "- \cap -"] \arrow[d, "j^* \otimes id"'] & H_{j-i}(M, \partial M) \arrow[d, equal] \\ H^i(M) \otimes H_j(M, \partial M) \arrow[r, "- \cap -"] & H_{j-i}(M, \partial M). \end{tikzcd}$$

□

Proof of the Proposition. Write

$$\tilde{H}^i(I^{B(\overline{p})}X; \mathbb{Q}) = V \oplus W \tag{1.10}$$

with $V = \ker g^*$ and $W \cong \operatorname{im} g^*$ (see Sequence (1.8)). Denote with V_1 a vector subspace in $H^{i-1}(B(\overline{p})(\partial M); \mathbb{Q})$ which is the image of a splitting of ∂^*_{fg}, so that $\partial^*_{fg}|_{V_1}$ is an isomorphism. By definition of Ξ, the map s^* is surjective. Denote by $V_2 \subset H^{i-1}(\partial M; \mathbb{Q})$ a preimage of V_1 under s^* and set $\partial^*_{ij}(V_2) =: V_3 \subset H^i(M, \partial M; \mathbb{Q})$. We then get

$$\begin{aligned} r^*(V_3) &= (r^* \circ \partial^*_{ij})(V_2) \\ &= (\partial^*_{fg} \circ s^*)(V_2) \\ &= \partial^*_{fg}(V_1) \\ &= V \end{aligned}$$

and therefore the following diagram commutes.

$$\begin{tikzcd}
V_3 \arrow[rrr, "r^*", "\cong"'] \arrow[dr] & & & V \arrow[dl] \\
& H^i(M,\partial M;\mathbb{Q}) \arrow[r, "r^*"] & \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) & \\
& H^{i-1}(\partial M;\mathbb{Q}) \arrow[u, "\partial^*_{ij}"] \arrow[r, "s^*"] & H^{i-1}(B(\overline{p})(\partial M);\mathbb{Q}) \arrow[u, "\partial^*_{fg}"'] & \\
V_2 \arrow[uuu, "\partial^*_{ij}", "\cong"'] \arrow[ur] \arrow[rrr, "\cong", "s^*"'] & & & V_1 \arrow[uuu, "\cong", "\partial^*_{fg}"'] \arrow[ul]
\end{tikzcd}$$

We can therefore choose a splitting

$$s_{r^*} : \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \to H^i(M,\partial M;\mathbb{Q})$$

with $\operatorname{im}(s_{r^*}) = V_3$. Let $x \in W$ and $b \in H_j(M,\partial M;\mathbb{Q})$. We apply Lemma 1.5.2 and obtain

$$\partial_{qr*}(g^*(x) \cap b) = f^* g^*(x) \cap \partial_{ij*}(b) = 0.$$

Thus $g^*(x) \cap b \in \operatorname{im} r_*$. For $\xi \in \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q})$ write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in V$ and $\xi_2 \in W$. Let $\tilde{\xi_1} \in V_3 \subset H^i(M,\partial M;\mathbb{Q})$ be a preimage of ξ_1 under r^*. Then define a cap product

$$\tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \to \tilde{H}_{j-i}(I^{B(\overline{q})}X;\mathbb{Q})$$

by setting

$$\xi \cap b := g_*(\tilde{\xi_1} \cap b) + s_{r*}(g^*(\xi_2) \cap b) \tag{1.11}$$

for a splitting $s_{r*} : H_{j-i}(M,\partial M;\mathbb{Q}) \to \tilde{H}_{j-i}(I^{B(\overline{q})}X;\mathbb{Q})$ of r_*. Finally

$$\begin{aligned} r_*(\xi \cap b) &= r_*(g_*(\tilde{\xi_1} \cap b) + s_{r*}(g^*(\xi_2) \cap b)) \\ &= j_*(\tilde{\xi_1} \cap b) + g^*(\xi_2) \cap b \\ &= j^*(\tilde{\xi_1}) \cap b + g^*(\xi_2) \cap b \\ &= g^*(\xi_1) \cap b + g^*(\xi_2) \cap b \\ &= g^*(\xi) \cap b \end{aligned}$$

using Lemma 1.5.3 and thus

$$\begin{tikzcd} \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r] \arrow[d, "g^*\otimes id"'] & \tilde{H}_{j-i}(I^{B(\overline{q})}X;\mathbb{Q}) \arrow[d, "r_*"] \\ H^i(M;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap-"] & H_{j-i}(M,\partial M;\mathbb{Q}) \end{tikzcd}$$

commutes. □

1.5.2 Capping With the Fundamental Class

Proposition 1.5.4. *Let $(X, B(\overline{p}))$ and $(X, B(\overline{q}))$ in Ξ. Assume that there is a Poincaré duality isomorphism $\tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \to \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q})$ (in the sense of Definition 1.3.13). Then this isomorphism is relized by capping with the fundamental class $[M,\partial M]$. That is*

$$\begin{array}{rcl} \Phi : \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) & \to & \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q}) \\ x & \mapsto & x \cap [M,\partial M] \end{array}$$

is an isomorphism.

Proof. Let $V = \ker g^*$ and $W = \operatorname{im} g^*$ as defined in Equation (1.10). Let $\xi_2 \in W$.

$$\Phi|_W(\xi_2) = s_{r*}(g^*(\xi_2) \cap [M,\partial M]).$$

Now $g^*|_W : W \to \operatorname{im} g^*$ is an isomorphism by definition of W. Moreover

$$- \cap [M,\partial M] : H^i(M) \to H_{n-i}(M,\partial M;\mathbb{Q})$$

is an isomorphism by Poincaré duality, hence injective. Finally

$$s_{r*} : \operatorname{im} r_* \to \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q})$$

is injective by definition. Summing up, the composition $\Phi|_W : W \to \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q})$ is injective and thus

$$\Phi|_W : W \to \operatorname{im} \Phi|_W$$

is an isomorphism. For an element $\xi_1 \in V$, set $\Phi|_V(\xi_1) = g_*(\tilde{\xi_1} \cap [M,\partial M])$. As we assumed a Poincarè duality isomorphism (in the sense of Definition 1.3.13), a commutative diagram

$$
\begin{tikzcd}
H^i(M,\partial M;\mathbb{Q}) \arrow[r, "r^*"] \arrow[d, "-\cap[M,\partial M]"'] & \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \arrow[r] \arrow[d, "D"] & H^i(\partial M, B(\overline{p})(\partial M);\mathbb{Q}) \arrow[d, "-\cap[\partial M]"] \\
H_{n-i}(M;\mathbb{Q}) \arrow[r, "g_*"] & \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q}) \arrow[r] & H_{n-i-1}(B(\overline{q})(\partial M);\mathbb{Q})
\end{tikzcd}
$$

exists. D is the Poincaré duality isomorphism. The commutativity of the above diagram then implies

$$
\begin{aligned}
\Phi|_V(\xi_1) &= g_*(\tilde{\xi_1} \cap [M,\partial M]) \\
&= D(\xi_1)
\end{aligned}
$$

As D is an isomorphism, it follows, that $\Phi|_V : V \to \operatorname{im}\Phi|_V$ is an isomorphism. We want to determine $\operatorname{im}\Phi|_V \cap \operatorname{im}\Phi|_W$. Let $x \in \operatorname{im}\Phi|_V \cap \operatorname{im}\Phi|_W$. Then there are elements $\xi_1 \in V$ and $\xi_2 \in W$ with

$$
\begin{array}{rrl}
 & \Phi|_V(\xi_1) & = \Phi|_W(\xi_2) \\
\Leftrightarrow & g_*(\tilde{\xi_1} \cap [M,\partial M]) & = s_{r*}(g^*(\xi_2) \cap [M,\partial M]) \\
\Rightarrow & r_* g_*(\tilde{\xi_1} \cap [M,\partial M]) & = g^*(\xi_2) \cap [M,\partial M] \\
\Leftrightarrow & j_*(\tilde{\xi_1} \cap [M,\partial M]) & = g^*(\xi_2) \cap [M,\partial M].
\end{array}
$$

The following diagram commutes

$$
\begin{tikzcd}
H^i(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap[M,\partial M]"] & H_{n-i}(M;\mathbb{Q}) \\
H^{i-1}(\partial M;\mathbb{Q}) \arrow[u, "\partial^*_{ij}"] \arrow[r, "-\cap[\partial M]"] & H_{n-i}(\partial M;\mathbb{Q}). \arrow[u, "i_*"']
\end{tikzcd}
$$

The element $\tilde{\xi_1}$ is defined as $\tilde{\xi_1} = \partial^*_{ij}(\eta)$ for some $\eta \in H^{i-1}(\partial M;\mathbb{Q})$. Then

$$
\begin{aligned}
j_*(\tilde{\xi_1} \cap [M,\partial M]) &= j_*(\partial^*_{ij}(\eta) \cap [M,\partial M]) \\
&= j_* \circ i_*(\eta \cap [\partial M]) = 0
\end{aligned}
$$

by exactness of Sequence (1.9). It follows that

$$
g^*(\xi_2) \cap [M,\partial M] = 0
$$

and as $g^*(-) \cap [M,\partial M]|_W$ is an isomorphism, it follows that $\xi_2 = 0$. But then by assumption $\Phi|_V(\xi_1) = \Phi|_W(\xi_2) = 0$ and we conclude that

$$
\operatorname{im}\Phi|_V \cap \operatorname{im}\Phi|_W = \{0\}.
$$

We have $\tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \cong \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q})$. Summing up

$$\begin{aligned}\dim \operatorname{im} \Phi &= \dim \operatorname{im} \Phi|_V + \dim \operatorname{im} \Phi|_W \\ &= \dim V + \dim W \\ &= \dim \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \\ &= \dim \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q}).\end{aligned}$$

Therefore

$$\Phi = \Phi|_V + \Phi|_W : \tilde{H}^i(I^{B(\overline{p})}X;\mathbb{Q}) \to \tilde{H}_{n-i}(I^{B(\overline{q})}X;\mathbb{Q})$$

is a surjective homomorphism of $\mathbb{Q}$-vectorspaces of equal dimension, hence an isomorphism. □

1.5.3 Consistency With Other Definitions of Cap Products

Lemma 1.5.5. *Let X^n have only isolated singularities, where $n = \dim X \equiv 2(4)$, and let $j - 2l \leq \frac{n}{2}$. Then the cap product*

$$- \cap - : \tilde{H}^{2l}(I^{\overline{m}}X;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \to \tilde{H}_{j-2l}(I^{\overline{m}}X;\mathbb{Q})$$

as defined in Equation (1.11) *coincides with the cap product defined in [Ban10a], Proposition 2.31.*

Remark 1.5.6. *For $j - 2l > k = \frac{n}{2}$, however, the cap product defined in this section depends on the choice of the splitting $s_{r*} : H_{j-2l}(M,\partial M;\mathbb{Q}) \to \tilde{H}_{j-2l}(I^{\overline{m}}X;\mathbb{Q})$.*

Proof. Let $n = \dim X \equiv 2(4)$. Then for $k = n - 1 - \overline{m}(n) = \frac{n}{2}$ we have either $2l > k$ or $2l < k$. Let first be $2l > k$. Then $g^* : \tilde{H}^{2l}(I^{\overline{m}}X) \to H^{2l}(M)$ is an isomorphism and in the notation of (1.10) we get $V = 0$ and $W = \tilde{H}^{2l}(I^{\overline{m}}X)$. Moreover $r_* : H^{i-2l}(M,\partial M) \to \tilde{H}_{i-2l}(I^{\overline{m}}X)$ is an isomorphism. The cap product is then defined through the diagram

$$\begin{tikzcd} \tilde{H}^{2l}(I^{\overline{m}}X;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r] \arrow[d, "g^*\otimes id"', "\cong"] & \tilde{H}_{j-2l}(I^{\overline{m}}X;\mathbb{Q}) \arrow[d, "\cong"', "r_*"] \\ H^{2l}(M;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap-"] & H_{j-2l}(M,\partial M;\mathbb{Q}). \end{tikzcd}$$

Expressed as a formula, for $\xi \in \tilde{H}^{2l}(I^{\overline{m}}X;\mathbb{Q})$ and $b \in H_j(M,\partial M;\mathbb{Q})$, the cap product is given by

$$\xi \cap b = (r^*)^{-1}(g^*(\xi) \cap b)$$

and therefore coincides with the cap product of Proposition 2.31 of [Ban10a].
Now if $2l < k$, then $r^* : H^{2l}(M,\partial M) \to \tilde{H}^{2l}(I^{\overline{m}}X)$ is an isomorphisms. The cap product is defined through the diagram

$$\begin{tikzcd}
H^{2l}(M,\partial M;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap-"] \arrow[d, "r^*\otimes id"', "\cong"] & H_{j-2l}(M) \arrow[d, "g_*"] \\
\tilde{H}^{2l}(I^{\overline{m}}X;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap-"] \arrow[d, "g^*\otimes id"'] & \tilde{H}_{j-2l}(I^{\overline{m}}X;\mathbb{Q}) \arrow[d, "r_*"] \\
H^{2l}(M;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap-"] & H_{j-2l}(M,\partial M;\mathbb{Q})
\end{tikzcd}$$

as

$$\xi \cap b := g_*(\tilde{\xi_1} \cap b) + s_{r*}(g^*(\xi_2) \cap b).$$

For V, the definition $g_*(\tilde{\xi_1} \cap b)$ coincides with the definition in Proposition 2.31 of [Ban10a]. For W, Lemma 1.5.3 shows that

$$\begin{tikzcd}
H^{2l}(M,\partial M;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap-"] \arrow[d, "j^*\otimes id"'] & H_{j-2l}(M) \arrow[d, "j_*"] \\
H^{2l}(M;\mathbb{Q}) \otimes H_j(M,\partial M;\mathbb{Q}) \arrow[r, "-\cap-"] & H_{j-2l}(M,\partial M;\mathbb{Q})
\end{tikzcd}$$

commutes. Let $j - 2l \leq k = \frac{n}{2}$, then $r_* : \tilde{H}_{j-2l}(I^{\overline{m}}X;\mathbb{Q}) \to H_{j-2l}(M,\partial M;\mathbb{Q})$ is an isomorphism and for $\xi_2 \in W$ we have

$$\begin{aligned}
j^*(r^*)^{-1}(\xi_2) \cap b &= j_*((r^*)^{-1}(\xi_2) \cap b) \\
\Leftrightarrow g^*(\xi_2) \cap b &= r_* g_*((r^*)^{-1}(\xi_2) \cap b) \\
\Rightarrow (r_*)^{-1}(g^*(\xi_2) \cap b) &= g_*((r^*)^{-1}(\xi_2) \cap b)
\end{aligned}$$

as $j = r \circ g$. It follows that

$$\begin{aligned}
\xi \cap b &:= g_*(\tilde{\xi_1} \cap b) + s_{r*}(g^*(\xi_2) \cap b) \\
&= g_*(\tilde{\xi_1} \cap b) + g_*((r^*)^{-1}(\xi_2) \cap b) \\
&= g_*((r^*)^{-1}(\xi_1 + \xi_2) \cap b) \\
&= g_*((r^*)^{-1}(\xi) \cap b)
\end{aligned}$$

and therefore the cap products coincide.

$\square$

The cases $n \equiv k(4)$ with $k \neq 2$ are similar.

1.6 The Signature of Intersection Spaces

Let M be a compact manifold with boundary of dimension $4n$. One can define a non-degenerate intersection form on

$$H^{2n}(M,\partial M)/\operatorname{Ker}(H^{2n}(M,\partial M)\to H^{2n}(M))\cong \operatorname{im}(H^{2n}(M,\partial M;\mathbb{Q})\to H^{2n}(M;\mathbb{Q}))$$

This is done for example in [AS68], Section 7, where the signature of a manifold with boundary is defined as the signature of the following non-degenerate symmetric bilinear form

$$B: j^*H^{2n}(M,\partial M;\mathbb{Q})\otimes j^*H^{2n}(M,\partial M;\mathbb{Q})\to\mathbb{Q}$$

where

$$B(j^*(\xi),j^*(\eta)) :=< ([\xi]\cup[\eta]),[M] >= \epsilon_*(([\xi]\cup[\eta])\cap[M]).$$

and denoted by $\sigma(M,\partial M)$. (Equivalently, the signature of a manifold with boundary $(M^{4n},\partial M)$ can be defined as the signature of the non degenerate pairing on the respective homology groups).

Theorem 1.6.1. *Let X be a compact, oriented pseudomanifold and $(X,B(\overline{p}))\in\Xi$ of dimension $4n$. Assume $B(\overline{p})=B(\overline{q})$, where $\overline{p}$ and $\overline{q}$ are complementary perversities, and denote $I^{B(\overline{p})}X=I^{B(\overline{q})}X=IX$. Assume, that there is a Poincaré duality isomorphism*

$$\tilde{H}^{4n-r}(IX;\mathbb{Q})\to\tilde{H}_r(IX;\mathbb{Q})$$

for all $r\in\mathbb{N}$ in the sense of Definition 1.3.13. Then

$$\sigma(M,\partial M)=\sigma(IX)$$

The signature of the intersection space $\sigma(IX)$ is defined as the signature of the bilinear form on the middle homology. The form is non-degenerate by generalized Poincaré Duality.

Corollary 1.6.2. *If X^{4n} is a compact, oriented, stratified pseudomanifold with two strata with the properties (INJ) and (PD) and if the codimension of the singular set is even, then*

$$\sigma(M,\partial M)=\sigma(IX)$$

Proof of the Theorem. The proof is subdivided in four parts. First we decompose the group $\tilde{H}_{2n}(IX)$ in suitabel subspaces. Then we show that the intersection form of IX restricted to one of these subspaces equals the intersection form of the manifold-with-boundary $(M,\partial M)$. Third, it is shown that the rest of the intersection form of IX does not contribute to the signature of IX and finally we conclude by the application of Novikov additivity. In this proof, all homology and cohomology groups are understood to be with rational coefficients.
Step 1: A suitable decomposition of $\tilde{H}_{2n}(IX)$
The various long exact sequences of the commutative triangle

$$\begin{tikzcd} B(\overline{p}) \arrow[r, "s"] \arrow[dr, "f"'] & \partial M \arrow[d, "i"] \\ & M \end{tikzcd}$$

form the following commutative diagrams

$$\begin{tikzcd} & H_{2n}(M,\partial M) \arrow[r, equal] & H_{2n}(M,\partial M) & \\ H_{2n}(B(\overline{p})) \arrow[r, "f_*"] \arrow[d, equal] & H_{2n}(M) \arrow[r, "g_*"] \arrow[u, "j_*"] & \tilde{H}_{2n}(IX) \arrow[r, "\partial_{fg*}"] \arrow[u, "r_*"'] & H_{2n-1}(B(\overline{p})) \arrow[d, equal] \\ H_{2n}(B(\overline{p})) \arrow[r, "s_*"] & H_{2n}(\partial M) \arrow[r, "t_*"] \arrow[u, "i_*"] & H_{2n}(\partial M, B(\overline{p})) \arrow[r, "\partial_{st*}"] \arrow[u, "q_*"'] & H_{2n-1}(B(\overline{p})) \\ & H_{2n+1}(M,\partial M) \arrow[r, equal] \arrow[u, "\partial_{ij*}"] & H_{2n+1}(M,\partial M) \arrow[u, "\partial_{qr*}"'] & \end{tikzcd} \tag{1.12}$$

and

$$\begin{tikzcd} & H^{2n}(M) \arrow[r, equal] & H^{2n}(M) & \\ H^{2n-1}(\partial M, B(\overline{q})(\partial M)) \arrow[r, "q^*"] \arrow[d, equal] & H^{2n}(M,\partial M) \arrow[r, "r^*"] \arrow[u, "j^*"] & \tilde{H}^{2n}(IX) \arrow[r, "q^*"] \arrow[u, "g^*"'] & H^{2n}(\partial M, B(\overline{q})(\partial M)) \arrow[d, equal] \\ H^{2n-1}(\partial M, B(\overline{q})(\partial M)) \arrow[r, "t^*"] & H^{2n-1}(\partial M) \arrow[r, "s^*"] \arrow[u, "\partial^*_{ij}"] & H^{2n-1}(B(\overline{q})) \arrow[r, "\partial^*_{st}"] \arrow[u, "\partial^*_{fg}"'] & H^{2n}(\partial M, B(\overline{q})(\partial M)) \\ & H^{2n-1}(M) \arrow[r, equal] \arrow[u, "i^*"] & H^{2n-1}(M) \arrow[u, "f^*"'] & \end{tikzcd} \tag{1.13}$$

All maps are the usual induced maps of long exact sequences of pairs and triples. Each term of the homological diagram is connected to the term at the same entry in the cohomological diagram by a duality isomorphism and the duality isomorphisms are natural with respect to the maps in the two diagrams as we assumed a Poincaré duality isomorphism in the sense of Lemma 1.3.13. Explicitly we will need the following isomorphisms

$$d_M : H_{4n-i}(M) \to H^i(M,\partial M)$$

and

$$d'_M : H_{4n-i}(M,\partial M) \to H^i(M).$$

Both isomorphisms are inverses of the duality isomorphism $- \cap [M,\partial M]$.

$$d_{\partial M} : H_{4n-i-1}(\partial M) \to H^i(\partial M)$$

is inverse to the duality isomorphism $- \cap [\partial M]$. Finally

$$d_{IX} : \tilde{H}_{4n-i}(IX) \to \tilde{H}^i(IX)$$

is the duality isomorphism of the intersection space. As $(X, B(\overline{p})) \in \Xi$, the boundary map $\partial_{st*} : H_i(\partial M, B(\overline{p})) \to H_{i-1}(B(\overline{p}))$ equals zero for all i. Equivalently the corresponding coboundary map $\partial_{st}^* : H^{i-1}(B(\overline{p})) \to H^i(\partial M, B(\overline{p}))$ equals zero for all i. We want to describe the intersection form of $\tilde{H}_{2n}(IX)$. Let $\{e_1, ..., e_r\}$ be a basis of $\operatorname{im} j_*$ and let $\overline{e}_i \in H_{2n}(M)$ be elements such that $j_*(\overline{e}_i) = e_i$. Set

$$Q := \{q \in H_{2n}(M, \partial M) | d_M(\overline{e}_i)(q) = 0 \text{ for all } i\}$$

Then it is shown in [Ban10a], Theorem 2.28, that

$$H_{2n}(M, \partial M) = \operatorname{im}(j_*) \oplus Q$$

There is a subspace $Y \subset \tilde{H}_{2n}(IX)$ such that $\tilde{H}_{2n}(IX) = \operatorname{Ker}(r_*) \oplus Y$. Write $\operatorname{Ker}(r_*) = V_3$. The restriction

$$r_* : Y \to \operatorname{im}(r_*) \subset H_{2n}(M, \partial M)$$

is an isomorphism. From above we take the decomposition $H_{2n}(M, \partial M) = \operatorname{im}(j_*) \oplus Q$. Define

$$\tilde{V}_1 = \operatorname{im}(r_*) \cap Q$$

and

$$\tilde{V}_2 = \operatorname{im}(r_*) \cap \operatorname{im}(j_*).$$

Set $r_*^{-1}(\tilde{V}_1) =: V_1$ and $r_*^{-1}(\tilde{V}_2) =: V_2$ so that

$$\tilde{H}_{2n}(IX) = V_1 \oplus V_2 \oplus V_3.$$

Let $\{a_1, ..., a_l\}$ be a basis of V_3. Since $\partial_{st*} = 0$ and using $g_* i_* = q_* t_*$, there are linearly independent elements $\{x_1, .., x_l\} \in H_{2n}(\partial M)$ with $g_* i_*(x_i) = a_i$ for all $0 \leq i \leq l$. Then $\{i_*(x_1), ..., i_*(x_l)\}$ are linearly independent in $H_{2n}(M)$. Set

$$y^i := d_M(i_*(x_i)) \in H^{2n}(M, \partial M)$$

and let $y_i \in H_{2n}(M, \partial M)$ be the dual elements of the elements y^i. In Lemma 1.6.3 below, we show that the elements $\{r_*^{-1}(y_1), ..., r_*^{-1}(y_l)\}$ form a basis of V_1.

Step 2: The intersection form on V_2

The signature of the manifold-with-boundary M is the signature of the following intersection form.

$$B(j^*(\xi), j^*(\eta)) :=< ([\xi] \cup [\eta]), [M] >= \epsilon_*(([\xi] \cup [\eta]) \cap [M]).$$

Let $[\xi], [\eta] \in H^{2n}(M, \partial M)/\operatorname{Ker}(H^{2n}(M, \partial M) \to H^{2n}(M))$, where ξ and η are representatives of the equivalence classes. The following diagram commutes, where j is the quotient map.

$$\begin{tikzcd} H^{4n-r}(M,\partial M) \arrow[r, "j^*"] \arrow[d, "-\cap[M]"'] & H^{4n-r}(M) \arrow[d, "-\cap[M]"] \\ H_r(M) \arrow[r, "j_*"] & H_r(M,\partial M) \end{tikzcd} \tag{1.14}$$

Let now be $[M] \in H_n(M, \partial M)$ the orientation class. Furthermore, define

$$\begin{aligned} x' &= \xi \cap [M] \\ y' &= \eta \cap [M] \end{aligned}$$

Set $x := g_*(x')$ and $y := g_*(y')$ and note that $r_*(x) = r_* g_*(x') = j_*(x')$. Lemma 1.5.3 shows, that the following diagram commutes

$$\begin{tikzcd} H^{2n}(M,\partial M) \otimes H_{2n}(M,\partial M) \arrow[r, "\cap"] \arrow[d] & H_0(M) \arrow[d, "j_*"] \\ H^{2n}(M,\partial M) \otimes H_{2n}(M,\partial M) \arrow[r, "\cap"] & H_0(M,\partial M) \end{tikzcd} \tag{1.15}$$

j_* is the map induced by the quotient map on chain level and by abuse of notation, we denote both relative cap products simply by $- \cap -$. For the relative cap product (as defined in [Spa66] Chapter 5, Section 6) we have the following property: Let $\alpha \in H^p(X, A)$, $\beta \in H^q(X, A)$ and $x \in H_m(X, A)$, then in $H_{m-p-q}(X)$ the following formula holds:

$$\alpha \cap (\beta \cap x) = (\alpha \cup \beta) \cap x \tag{1.16}$$

It follows that

$$\begin{aligned} \epsilon_*((\xi \cup \eta) \cap [M]) &= \epsilon_*(\xi \cap (\eta \cap [M])) && \text{by (1.16)} \\ &= \epsilon_*(\xi \cap j_*(\eta \cap [M])) && \text{by (1.15)} \\ &= \xi(j_*(\eta \cap [M]) && \text{by definition} \\ &= (d_M(x'))(j_*(y')) && \text{by definition} \\ &= (d_M(j_*(x')))(y') && \text{by (1.14)} \\ &= (d_M(r_* g_*(x')))(y') && \text{by (1.12)} \\ &= g^*(d_{IX}(g_*(x'))(y') && \text{by (1.12) and (1.13)} \\ &= d_{IX}(g_*(x'))(g_*(y')) && \\ &= d_{IX}(x)(y) && \text{by definition} \end{aligned}$$

We check that this is well defined on the equivalence classes. Let ξ_1 and ξ_2 be such that $\xi_1 - \xi_2 \in \ker(j^*)$. Then $x'_1 - x'_2 \in \ker(j_*)$ by Diagram (1.14). Thus

$$\begin{aligned} d_{IX}(x_1)(y) - d_{IX}(x_2)(y) &= (d_M(j_*(x_1')))(y') - (d_M(j_*(x_2')))(y') \\ &= (d_M(j_*(x_1' - x_2')))(y') \\ &= (d_M(0))(y') = 0 \end{aligned}$$

The argument is symmetric in ξ and η and therefore shows, that the above calculation is well defined on equivalence classes and thus

$$\epsilon_*(([\xi] \cup [\eta]) \cap [M]) = d_{IX}(x)(y)$$

Choose a basis $\{x^1, \dots, x^j\}$ of $H^{2n}(M, \partial M)/\operatorname{Ker}(H^{2n}(M, \partial M) \to H^{2n}(M))$. With respect to this basis, the bilinear form B has a matrix representation, say

$$A = (a_{kl})_{1 \leq k,l \leq j}$$

Then $(x^i \cap [M], \dots, x^i \cap [M])$ is a basis of $H_{2n}(M)/\operatorname{Ker}(H_{2n}(M) \to H_{2n}(M, \partial M))$. Using the isomorphism

$$j_* : H_{2n}(M)/\ker(H_{2n}(M) \to H_{2n}(M, \partial M)) \to \operatorname{im}(H_{2n}(M) \to H_{2n}(M, \partial M)) \tag{1.17}$$

we deduce that $(j_*(x^i \cap [M]), \dots, j_*(x^i \cap [M]])$ is a basis of $\operatorname{im}(H_{2n}(M) \to H_{2n}(M, \partial M)))$. By construction $r_*(V_2) = \operatorname{im}(j_*)$. The isomorphisms (1.17) and $r_*|_{V_2} : V_2 \to \operatorname{im}(j_*)$ show that the elements $\{r_*^{-1} j_*(x^1 \cap [M]), \dots, r_*^{-1} j_*(x^j \cap [M]])\}$ form a basis of V_2 and with respect to that basis the intersection form d_{IX} restricted to V_2 has the matrix representation

$$A = (a_{kl})_{1 \leq k,l \leq j}$$

This shows, that the intersection form

$$\Phi_{IX}(v \otimes w) := d_{IX}(v)(w)$$

restricted to V_2 equals the bilinear form B.

Step 3: We now calculate the complete intersection form of $H_{2n}(IX)$. The following calculations were already used by the author in [Spi10] (unpublished) to show a similar result in the special case of intersection spaces coming from a two-strata pseudomanifold with a trivial link bundle.

(a) The pairing of elements in V_3. Let $x, y \in V_3$. Then there are elements $\xi, \eta \in H_{2n}(\partial M)$ with $q_* t_*(\xi) = x$ and $q_* t_*(\eta) = y$.

$$\begin{aligned} d_{IX}(x)(y) &= d_{IX}(q_* t_*(\xi))(q_* t_*(\eta)) \\ &= d_{IX}(g_* i_*(\xi))(q_* t_*(\eta)) \\ &= (r^* d_M(i_*(\xi)))(q_* t_*(\eta)) \\ &= d_M(i_*(\xi))(r_* q_* t_*(\eta)) = 0 \end{aligned}$$

(b) The pairing of elements in V_1 and V_2. Let $x \in V_1$ and $y \in V_2$. Then $x = r_*^{-1}(x')$ and $y = g_*(y')$.

$$\begin{aligned} d_{IX}(x)(y) &= d_{IX}(r_*^{-1}(x'))(g_*(y')) \\ &= d'_M(x')(y') = d_M(y')(x') \end{aligned}$$

Now $x' \in Q$ and $y' = \Sigma\lambda_k\bar{e}_k$. Therefore

$$d_M(y')(x') = \Sigma\lambda_k d_M(\bar{e}_k)(x') = 0$$

(c) The pairing of elements in V_1 and V_3. By Lemma 1.6.3, we can check this on the basis elements a_i and $r_*^{-1}(y_i)$.

$$\begin{aligned} d_{IX}(a_i)(r_*^{-1}(y_j)) &= d_{IX}(g_*i_*(x_i))(r_*^{-1}(y_j)) \\ &= d_M(i_*(x_i))(y_j) \\ &= y^i(y_j) = \delta_{ij} \end{aligned}$$

(d) The pairing of elements in V_2 and V_3. Let $x \in V_2$ and $y \in V_3$. Then $x = g_*(\xi)$ by definition of V_2. Write $y = q_*t_*(\eta)$. Then

$$\begin{aligned} d_{IX}(x)(y) &= d_{IX}(g_*(\xi))(q_*t_*(\eta)) \\ &= d_{IX}(g_*(\xi))(g_*i_*(\eta)) \\ &= d'_M(r_*g_*(\xi))(i_*(\eta)) \\ &= d'_M(j_*(\xi))(i_*(\eta)) \\ &= d_M(\xi)(j_*i_*(\eta)) = 0 \end{aligned}$$

Choose the following basis for $H_{2n}(IX)$:

$$\{r_*^{-1}j_*(x^1 \cap [M]), \dots, r_*^{-1}j_*(x^j \cap [M]), r_*^{-1}(y_1), \dots, r_*^{-1}(y_l), a_1, \dots, a_l\}.$$

The matrix representation of the intersection form with respect to that basis is

$$\begin{pmatrix} A & 0 & 0 \\ 0 & * & I_l \\ 0 & I_l & 0 \end{pmatrix}$$

where the block matrix $(*)$ is symmetric.

Step 4:

Now $\begin{pmatrix} * & I_l \\ I_l & 0 \end{pmatrix}$ is a split inner product space (see [MH73], Lemma 6.3). Thus

$$\sigma(M, \partial M) = \sigma(IX)|_{V_2} = \sigma(IX)$$

□

Lemma 1.6.3. *The elements* $\{r_*^{-1}(y_1), ..., r_*^{-1}(y_l)\}$ *form a basis of* V_1.

Proof. We first show, that $r_*^{-1}(y_i) \in V_1$.

$$d_{IX}(\partial_{qr*}(y_i))(b) = f^* d'_M(y_i)(b) = d'_M(y_i)(f_*(b))$$

for all $b \in H_{2n}(B(\overline{q}))$. As $f_*(b) \in \mathrm{Ker}(g_*)$, it follows that $f_*(b) \notin \langle i_*(x_1), ..., i_*(x_l)\rangle$. Therefore $d'_M(y_i)(f_*(b)) = d_M(f_*(b))(y_i) = 0$ by definition of y_i. We conclude that $\partial_{qr*}(y_i) = 0$ for all i and thus $y_i \in \mathrm{im}(r_*)$. Finally assume $y_i \in \mathrm{im}(j_*)$, then $y_i = j_*(\tilde{y}_i)$ for some element $\tilde{y}_i$.

$$\begin{aligned} 1 &= y^i(y_i) \\ &= d_M(i_*(x_i)(j_*(\tilde{y}_i)) \\ &= d'_M(j_* i_*(x_i)(\tilde{y}_i) = 0 \end{aligned}$$

This shows $r_*^{-1}(y_i) \in V_1$ and the elements $\{r_*^{-1}(y_i) | 1 \leq i \leq l\}$ are linearly independent. By a rank argument we now show, that the elements $r_*^{-1}(y_i)$ form a basis of V_1. Assume there is an element $\tilde{y} \in V_1$ linearly independent of all $r_*^{-1}(y_i)$ for $1 \leq i \leq l$. Let $z \in H_{2n}(\partial M)$ be the dual element of $d_{\partial M}(\partial_{ij*} r_*(\tilde{y}))$. Then

$$\begin{aligned} d_{IX}(\tilde{y})(g_* i_*(z)) &= d'_M(r_* \tilde{y})(i_*(z)) \\ &= d_{\partial M}(\partial_{ij*} r_*(\tilde{y}))(z) = 1 \end{aligned}$$

So $d_{IX}(\tilde{y})$ is the dual element of $g_* i_*(z)$. Since $d_{IX}(r_*^{-1}(y_i))$ is the dual element of a_i for all i (see (c) in the proof of Theorem 1.6.1), it follows by assumption that $d_{IX}(\tilde{y})$ is linearly independent of the elements a_i. But this is a contradiction as we chose the elements a_i to be a basis of V_3. □

2 Duality in Generalized Homology Theories

The aim of this chapter is to find duality statements for intersection spaces in generalized homology theories. Let E denote a ring spectrum. Adams shows in [Ada74] (III, Corollary 10.13) that for an E-oriented, closed manifold M exists a Poincaré-duality isomorphism

$$E^{d-r}(M) \to E_r(M)$$

given by capping with an E-fundamental class $[M]_E \in E_d(M)$. As discussed in Chapter 1, for a certain class of pseudomanifolds, an assignment $X \rightsquigarrow I^{\bar{p}}X$ can be constructed. $I^{\bar{p}}X$ is called the intersection space of X. There is a Poincaré-duality isomorphism

$$\tilde{H}^{n-r}(I^{\bar{p}}X;\mathbb{Q}) \to \tilde{H}_r(I^{\bar{q}}X;\mathbb{Q}).$$

In this chapter, we want to combine these two approaches. More precisely, we want to answer the question, which assumptions on the spectrum E and the pseudomanifold X we have to make, that an isomorphism of the form

$$\tilde{E}^{n-r}(I^{\bar{p}}X;\mathbb{Z}) \to \tilde{E}_r(I^{\bar{q}}X;\mathbb{Z})$$

is possible. We will see, that this is possible for all stratified pseudomanifolds in Θ (see Definition 1.3.5) and all commutative ring spectra E with torsion-free coefficient ring and a suitable character map $ch : E \to H(\pi_*E \otimes \mathbb{Q})$. The focus lies on $E = KU$, the complex K-theory spectrum. If not otherwise mentioned, all homology groups and cohomology groups in this chapter which are written without specific coefficients are understood to be with integral coefficients.

2.1 Preliminaries

In this section, we want to provide some tools and show some properties that are used in the calculations that follow. Throughout this chapter, we work in the homotopy category of spectra as defined by Adams ([Ada74], Part 3). Consequently, π_* denotes stable homotopy and $[-,-]_*$ denotes stable homotopy classes of maps of spectra. The generalized homology theories are denoted consistently with the spectra defining them. In particular, H is the Eilenberg Mac-Lane spectrum, KU denotes the complex K-theory spectrum (we will, however, write simply K^* for the induced cohomology theory), KO the real K-theory spectrum, S the sphere spectrum and so on.

Definition 2.1.1. Let E and X be CW-spectra. $E_r(X) := \pi_r(E \wedge X)$ are called the *E-homology groups* of X and $E^r(X) := [X, E]_r$ the *E-cohomology groups*.

For a graded abelian group G_* we define the Eilenberg-MacLane spectrum with coefficients in G as $\Pi_n H(G_n, n)$ and denote it by HG. This means that $HG_r(X) = \bigoplus_n H_{r-n}(X; G_n)$

and $HG^r(X) = \Pi_n H^{r+n}(X; G_n)$. In particular, we use the notation $H(\pi_* KU \otimes \mathbb{Q})_r(X)$ or $H_r(X; \pi_* KU \otimes \mathbb{Q})$.

2.1.1 Coefficients

The duality isomorphism of intersection spaces in Theorem 1.2.1 (or also the duality isomorphism in [Ban10a]) has rational coefficients. It is well known that when tensored with the rationals, every generalized homology theory decomposes into a product of ordinary homology theories. This can be shown by the following argument: Let E be a CW-spectrum. Let $\alpha_i : S^n \to E$ represent generators of the the group $\pi_n E \otimes \mathbb{Q}$. Then the map of spectra

$$f : S(\pi_* E \otimes \mathbb{Q}) \to E$$

which is given in degree n by

$$f_n := \bigvee \alpha_i : \bigvee_i S^n \to E$$

induces an isomorphism on the stable homotopy groups

$$f_* : \pi_* S(\pi_* E \otimes \mathbb{Q}) \to \pi_* E \otimes \mathbb{Q}.$$

The Whitehead theorem for CW-spectra (Corollary 3.5 in [Ada74] Part 3, §3) then shows, that f induces an equivalence of spectra

$$f : S(\pi_* E \otimes \mathbb{Q}) \to E \wedge S\mathbb{Q}.$$

The rational sphere spectrum $S\mathbb{Q}$ is isomorphic to the rational Eilenberg-MacLane spectrum $H\mathbb{Q}$ (see for example [Ada74], Part 3, §6,). Therefore we obtain an equivalence of spectra

$$f : H(\pi_* E \otimes \mathbb{Q}) \to E \wedge S\mathbb{Q}.$$

It is thus not very interesting to look for duality statements in generalized homology theories when taking rational coefficients. We therefore want to consider integral coefficients only. As we have seen in Section 1.3, also in ordinary homology, the duality statement of [Ban10a] does not hold integrally for all intersection spaces.

2.1.2 Where to choose splittings

Let X be a stratified pseudomanifold with only isolated singularities. The link L is the disjoint union of the links of the singularities and the manifold-with-boundary M is obtained by cutting off a small open conical neighborhood of the singularities in X. Following the construction of intersection spaces in [Ban10a], one obstruction for an integral duality isomorphism of intersection spaces in any generalized homology theory is the existence of splittings

$$s : E_{r-1}(L_{<k}) \to E_r(M, L_{<k})$$

and

$$t : E^{n-r}(L, L_{<n-k}) \to E^{n-r}(M, L_{<n-k})$$

in the following diagram

$$\begin{tikzcd}
E_r(L_{<k}) \arrow[r] \arrow[d] & E_r(M) \arrow[r] \arrow[d] & E_r(M, L_{<k}) \arrow[r] \arrow[d, dotted] & E_{r-1}(L_{<k}) \arrow[r] \arrow[d] \arrow[l, bend right, "s"'] & E_{r-1}(M) \arrow[d] \\
E^{n-r-1}(L, L_{<n-k}) \arrow[r] & E^{n-r}(M, L) \arrow[r] & E^{n-r}(M, L_{<n-k}) \arrow[r] & E^{n-r}(L, L_{<n-k}) \arrow[r] \arrow[l, bend left, "t"] & E^{n-r+1}(M, L)
\end{tikzcd} \tag{2.1}$$

or alternatively the existence of splittings

$$q : E_r(M, L) \to E_r(M, L_{<k})$$

and

$$r : E^{n-r}(M) \to E^{n-r}(M, L_{<n-k})$$

in the following diagram

$$\begin{tikzcd}
E_{r+1}(M, L) \arrow[r] \arrow[d] & E_r(L, L_{<k}) \arrow[r] \arrow[d] & E_r(M, L_{<k}) \arrow[r] \arrow[d, dotted] & E_r(M, L) \arrow[r] \arrow[d] \arrow[l, bend right, "q"'] & E_{r-1}(L, L_{<n-k}) \arrow[d] \\
E^{n-r-1}(M) \arrow[r] & E^{n-r-1}(L_{<n-k}) \arrow[r] & E^{n-r}(M, L_{<n-k}) \arrow[r] & E^{n-r}(M) \arrow[r] \arrow[l, bend left, "r"] & E^{n-r}(L_{<n-k}).
\end{tikzcd} \tag{2.2}$$

Note that long exact sequences of pairs and triples exist in any generalized homology theory (see [Ada74], Part 3, §6).

We first want to give an argument that it is better to work with the splittings in Diagram (2.1), as in this case we need less assumptions than working with Diagram (2.2). The splittings q and r in diagram (2.2) exist, if both $E_*(M, \partial M)$ and $E^*(M)$ are torsion-free. For the homology theories that we are mostly concerned with here, such as complex K-theory or ordinary homology, there are universal coefficient theorems (see for example [And] or [Ada69]) that show that then $E^*(M, \partial M)$ and $E_*(M)$ are torsion-free, too. On the other hand, for the existence of the splittings s and t, it suffices to assume only that the homology of L is torsion-free and we have no assumptions on the homology or cohomology of the manifold M. Moreover we will see that we need the assumption that the homology of L is torsion-free in any case to show isomorphisms of the form $E_r(L, L_{<k}) \to E^{n-r-1}(L_{<n-k})$ or $E^{n-r}(L, L_{<n-k}) \to E_{r-1}(L_{<k})$. We therefore use diagram (2.1) to construct a duality isomorphism, as in that case we need less assumptions.

Example 2.1.2. *One very easy example of a manifold whose homology has torsion subgroups but whose boundary has torsion-free homology is a closed manifold M^n whose* *homology has torsion subgroups. Drill out an n-ball. The resulting space is a manifold with boundary S^n. The homology groups of M and $(M, \partial M)$ have torsion but $H_*(\partial M) = H_*(S^n)$ is torsion-free. A very easy example of a pseudomanifold whose regular part M has this property is the following. Let M^n be a closed manifold whose homology has torsion subgroups. Drill out two n-balls. This results in a manifold-with-boundary with two boundary components S^n. Set $X = M \cup_{\partial M} cone(\partial M)$. This is a pseudomanifold (moreover it is not a manifold) whose regular part M has homology with torsion subgroups but the homology of the link is torsion-free.*

2.1.3 Torsion-freeness

In Section 1.3, we defined the class of pseudomanifolds Θ. We can compare the torsion-freeness conditions we impose on the groups $H_*(\partial M)$ for pseudomanifolds in Θ to get a duality isomorphism of intersection spaces (compare Remark 1.3.8) with the conditions that allow a torsion pairing in intersection homology, using the results of [GS83] (Theorem 4.4) or an integral duality isomorphism in intersection homology, using [GS83] (Theorem 7.1). In that paper, Goresky and Siegel show, that a pseudomanifold X which is locally $\overline{p}$-torsion-free has a non-degenerate torsion pairing

$$T_r^{\overline{p}}(X) \times T_{n-r-1}^{\overline{q}}(X) \to \mathbb{Q}/\mathbb{Z}$$

between the torsion subgroups $T_r^{\overline{p}}(X)$ of the intersection homology $IH_r^{\overline{p}}(X)$ of X. Here, a pseudomanifold is called *locally $\overline{p}$-torsion-free* if

$$T_{c-2-\overline{p}(c)}^{\overline{p}}(L) = 0$$

for every link of a stratum of X with codimension c. If X is a pseudomanifold with isolated singularities, then the only relevant value is $c = n$. Assume as above that $H_{k-1}(L)$ is torsion-free, then

$$IH_{c-2-\overline{p}(c)}^{\overline{p}}(L) = IH_{n-1-\overline{p}(n)-1}^{\overline{p}}(L) = IH_{k-1}^{\overline{p}}(L).$$

Since $H_{k-1}(L)$ is torsion-free, so is $IH_{k-1}^{\overline{p}}(L)$ (L is a manifold here) and thus the assumption $\operatorname{Tor}(H_{k-1}(L)) = 0$ implies both the existence of integral Poincaré duality for the intersection space and the existence of a non-degenerate torsion pairing in the intersection homology of X.

Let us now assume that $n = \dim X$ is even and again that $H_{k-1}(L)$ is torsion-free. Since n is even, we can denote by IX the intersection space with respect to the (upper or lower) middle perversity (denoted by $\overline{m}$ and $\overline{n}$). The condition that $H_{k-1}(L)$ is torsion-free, implies, that the torsion subgroup $T_{n/2-1}^{\overline{p}}(L) = 0$. Theorem 7.1 in [GS83] states that under that condition the following sequence is split exact:

$$0 \to \operatorname{Hom}(T_{r-1}^{\overline{m}}(X); \mathbb{Q}/\mathbb{Z}) \to IH_{n-r}^{\overline{m}}(X; \mathbb{Z}) \to \operatorname{Hom}(IH_r^{\overline{m}}(X; \mathbb{Z}); \mathbb{Z}) \to 0.$$

On the other hand, the condition that $H_{k-1}(L)$ is torsion-free, implies a duality isomorphism

$$\tilde{H}_{n-r}(IX) \to \tilde{H}^r(IX).$$

Applying this duality to the universal coefficient theorem gives a spilt exact sequence

$$0 \to \operatorname{Ext}(\tilde{H}_{r-1}(IX);\mathbb{Z}) \to \tilde{H}_{n-r}(IX) \to \operatorname{Hom}(\tilde{H}_r(IX);\mathbb{Z}) \to 0.$$

We want to compare the terms $\operatorname{Ext}(\tilde{H}_{r-1}(IX);\mathbb{Z})$ and $\operatorname{Hom}(T_{r-1}(X);\mathbb{Q}/\mathbb{Z})$. Every finitely generated abelian group G can be written as the direct sum of a free and a torsion part. Say $G \cong F \oplus T$. Now

(a) $\operatorname{Ext}(F;\mathbb{Z}) = 0 = \operatorname{Hom}(\operatorname{Tor}(F);\mathbb{Q}/\mathbb{Z})$

(b) Furthermore, we have isomorphisms

$$\operatorname{Hom}(\mathbb{Z}/p\mathbb{Z};\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

and

$$\operatorname{Ext}(\mathbb{Z}/p\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$

We can thus identify

$$\operatorname{Hom}(\operatorname{Tor} G;\mathbb{Q}/\mathbb{Z}) \cong \operatorname{Tor} G \cong \operatorname{Ext}(G;\mathbb{Z})$$

and get analogous conditions for integral duality statements in intersection homology and in the homology of the intersection space.

2.1.4 Naturality of the Cap Product

The following definitions and the notation are taken from [Ada74], Part 3, §9. Let X, Y be CW-spectra and let S be the sphere spectrum. Let E be a ring spectrum. This means that we have maps

$$\mu : E \wedge E \to E$$

and

$$\eta : S \to E$$

of degree zero. μ is required to be associative and η serves as the neutral element. Examples of ring spectra are HR for a ring R and KU. The product map μ determines the various products on the induced homology and cohomology groups. For example, the cap product is defined as follows.

Definition 2.1.3 ([Ada74], Part 3, §9)**.** Let X and Y be CW-complexes. Let

$$\triangle : X \to X \times X$$

be the diagonal map and

$$E^p(X) \otimes E_q(X \times Y) \xrightarrow{\backslash} (E \wedge E)_{q-p}(Y)$$

be the slant product, which is defined by

$$S \xrightarrow{v} E \wedge X \wedge Y \longrightarrow X \wedge E \wedge Y \xrightarrow{u\wedge 1\wedge 1} E \wedge E \wedge Y$$

Here $u \in E^p(X)$ and $v \in E_q(X \times Y)$. The cap product

$$\cap : E^p(X) \otimes E_q(X) \to E_{q-p}(X)$$

is then defined by

$$\cap : E^p(X) \otimes E_q(X) \overset{\backslash(id\otimes\triangle_*)}{\rightarrow} (E \wedge E)_{q-p}(X) \overset{\mu_*}{\rightarrow} E_{q-p}(X)$$

where the last map is induced by the ring operation μ.

Lemma 2.1.4. *The slant product* $\backslash$ *is natural with respect to maps between spectra.*

Proof. This follows from the definition of the slant product. Let $e, f : E \to E'$ and $u \in E^p(X)$, $v \in E_q(X \times Y)$. Then the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, "v"] \arrow[d] & E \wedge X \wedge Y \arrow[r] \arrow[d, "f\wedge 1\wedge 1"'] & X \wedge E \wedge Y \arrow[r, "u\wedge 1\wedge 1"] \arrow[d, "1\wedge f\wedge 1"'] & E \wedge E \wedge Y \arrow[d, "e\wedge f\wedge 1"] \\ S \arrow[r, "f_*(v)"] & E' \wedge X \wedge Y \arrow[r] & X \wedge E' \wedge Y \arrow[r, "e_*(u)"] & E' \wedge E' \wedge Y. \end{tikzcd}$$

□

Now assume, that f is a map of ring spectra. In particular, we want the following diagram to commute

$$\begin{tikzcd} E \wedge E \arrow[r, "\mu"] \arrow[d, "f\wedge f"'] & E \arrow[d, "f"] \\ E' \wedge E' \arrow[r, "\mu"] & E'. \end{tikzcd}$$

Lemma 2.1.5. *The cap product*

$$E^p(X) \otimes E_q(X) \to E_{q-p}(X)$$

is natural with respect to maps of ring spectra.

Proof. Lemma 2.1.4 shows that

$$\begin{tikzcd} E^p(X) \otimes E_q(X) \arrow[r, "\backslash(id\otimes\triangle_*)"] \arrow[d, "f\otimes f"'] & (E \wedge E)_{q-p}(X) \arrow[r, "\nu_*"] \arrow[d, "(f\wedge f)_*"] & E_{q-p}(X) \arrow[d, "f_*"] \\ E'^p(X) \otimes E'_q(X) \arrow[r, "\backslash(id\otimes\triangle_*)"] & (E' \wedge E')_{q-p}(X) \arrow[r, "\nu_*"] & E'_{q-p}(X) \end{tikzcd}$$

commutes. □

We want to point out two examples of maps of ring spectra that we will need later. First, let R and S be rings and $f : R \to S$ a ring homomorphism. Then f induces a map of ring spectra

$$f : HR \to HS.$$

Second, the chern character $ch : KU \to H(\pi_* KU \otimes \mathbb{Q})$ is a map of ring spectra (compare for example [Smi73b]).

Remark 2.1.6. *This can also be seen as a consequence of the more classical fact that the chern character preserves the multiplicative structure of the cohomology theories K and $H(\pi_* KU \otimes \mathbb{Q})$ (see for example [Dol72] or [AH60]).*
To see this, write $G := \pi_* KU \otimes \mathbb{Q}$. To preserve the multiplicative structure means for the chern character that the following diagram is commutative for all X and Y

$$\begin{tikzcd} K^p(X) \times K^q(Y) \arrow[r, "\times"] \arrow[d, "ch \times ch"'] & K^{p+q}(X \times Y) \arrow[d, "ch"] \\ H^p(X;G) \times H^q(Y;G) \arrow[r, "\times"] & H^{p+q}(X \times Y; G). \end{tikzcd}$$

For CW-spectra E, X and Y, the product $\times$ is given by

$$E^p(X) \times E^q(Y) \xrightarrow{\overline{\times}} (E \wedge E)^{p+q}(X \wedge Y) \xrightarrow{\mu_*} E^{p+q}(X \wedge Y).$$

$\overline{\times}$ is natural with respect to any map between CW-spectra by construction. Thus the chern character commutes with the product $\times$ if and only if it commutes with the induced map μ_*.
Specialize to $X = Y = KU$ and $p + q = 0$, this leads to the following commutative diagram

$$\begin{tikzcd} (KU \wedge KU)^0(KU \wedge KU) \arrow[r, "\mu_*"] \arrow[d, "ch \wedge ch"'] & KU^0(KU \wedge KU) \arrow[d, "ch"] \\ (HG \wedge HG)^0(KU \wedge KU) \arrow[r, "\mu_*"] & HG(KU). \end{tikzcd}$$

Apply this to the map $id \in [KU \wedge KU, KU \wedge KU] = (KU \wedge KU)^0(KU \wedge KU)$. Then the maps

$$KU \wedge KU \xrightarrow{id} KU \wedge KU \xrightarrow{\mu} KU \xrightarrow{ch} HG$$

and

$$KU \wedge KU \xrightarrow{id} KU \wedge KU \xrightarrow{ch \wedge ch} HG \wedge HG \xrightarrow{\mu} HG$$

equal. In other words, the following diagram commutes

$$\begin{tikzcd} KU \wedge KU \arrow[r, "\mu"] \arrow[d, "ch \wedge ch"'] & KU \arrow[d, "ch"] \\ HG \wedge HG \arrow[r, "\mu"] & HG. \end{tikzcd}$$

2.1.5 The Atiyah-Hirzebruch spectral sequence

There is a spectral sequence F with

$$E^2_{p,q}(X) \cong H_p(X;\pi_q E) \Rightarrow E^{\infty}_{p+q}(X)$$

and the same for cohomology

$$E_2^{p,q}(X) \cong H^p(X;\pi_{-q}E) \Rightarrow E_{\infty}^{p+q}(X).$$

For a construction see for example [Ada74], Part3, §7. We will constantly make use of the following fact, which can be found in [Arl92] (and is already remarked in [Dol66], Bemerkung 14.18):
In an Atiyah-Hirzebruch spectral sequence with X a bounded below spectrum or a CW-complex, the image of all differentials d_r with $r \geq 2$ are torsion subgroups.
In particular this implies, that if E has torsion-free coefficient group $\pi_*(E)$ and X torsion-free ordinary homology, then all differentials d_r with $r \geq 2$ vanish.

2.1.6 Recovery Problems

We will frequently make use of the Atiyah-Hirzebruch spectral sequence, applied to the truncated link. Its E_2-terms are isomorphic to the ordinary homology groups with coefficients π_*E and its limit is isomorphic to the E-homology groups. These are the groups we are interested in, so we have to consider recovery problems in order to obtain the E-homology groups from the spectral sequence.
Let X be a bounded below spectrum or a CW-complex with torsion-free homology and let E be spectrum with π_*E torsion-free. Then the spectral sequence collapses at E_2 (see the remark above). Let F^pE^r be a filtration of $E^r(X)$. By construction, there is a short exact sequence

$$0 \to F^{p+1}E^r \to F^pE^r \to E_{\infty}^{p,r-p} \to 0.$$

Splitting this sequence for every p means to solve the recovery problem. As the spectral sequence collapses at the 2 table,

$$E_{\infty}^{p,r-p} \cong E_2^{p,r-p}.$$

But

$$E_2^{p,r-p} \cong H^p(X;E^{r-p}(pt))$$

is torsion-free by assumption and the universal coefficient theorem. Therefore we can always choose splittings and obtain

$$F^pE^r \cong F^{p+1}E^r \oplus E_{\infty}^{p,r-p}.$$

This solves the extension problem and we denote the resulting isomorphism

$$E^r(X) \cong F^0E^r \cong F^1E^r \oplus E_{\infty}^{0,r} \cong \ldots \cong \bigoplus_{i=0}^{r} E_{\infty}^{i,r-i}$$

by

$$d_\infty : \bigoplus_{i=0}^{r} E_\infty^{i,r-i} \to E^r(X).$$

Exactly the same holds for homology. Here, let $F^p E_r$ be a filtration of $E_r(X)$. The short exact sequence then has the form

$$0 \to F^p E_r \to F^{p-1} E_r \to E^\infty_{p,r-p} \to 0.$$

Again, the spectral sequence collapses at the 2 table and thus

$$E^\infty_{p,r-p} \cong E^2_{p,r-p}.$$

But

$$E^2_{p,r-p} \cong H_p(X; E_{r-p}(pt))$$

is torsion-free by assumption. Therefore we can always choose splittings and obtain

$$F^{p-1} E_r \cong F^p E_r \oplus E^\infty_{p,r-p}.$$

This solves the extension problem in the case of homology as well and again we denote the resulting isomorphism by d_∞.

$$E_r(X) \cong F^n E_r \cong F^{n-1} E_r \oplus E^\infty_{0,r} \cong \ldots \cong \bigoplus_{i=0}^{r} E^\infty_{i,r-i}.$$

2.1.7 Orientation

There are various notions of orientation of a manifold, among them orientation given by a fundamental class or in the smooth case by an orientation of the tangent bundle. It is well known that these definitions coincide for orientation in the classical sense. For an E-orientation, where E is a ring spectrum, basically the same holds. We want to work with the following definition of orientation.
The rings $\tilde{E}^*(S^n)$ and $\tilde{E}_*(S^n)$ are π_*E-modules via the maps $\pi_*E = \tilde{E}_*(S^0) \to \tilde{E}_*(S^n)$ and $\pi_{-*}E = \tilde{E}^*(S^0) \to \tilde{E}^*(S^n)$. Following [Ada74], Part 3, §10, we call an element $\Phi \in \tilde{E}_*(S^n)$ a generator, if $< \Phi >$ is a π_*E- basis of $\tilde{E}_*(S^n)$. Equivalently for cohomology.

Definition 2.1.7. Let E be a ring spectrum. An n-dimensional manifold M is called *E-orientable* if an element $\omega \in E^n(M \times M, M \times M - \triangle)$ exists, that restricts for every $p \in M$ with $i_p : p \hookrightarrow M$ to $i_p^*(\omega) = \pm\gamma^n \in E^n(M \times p, M \times p - p \times p) \cong \tilde{E}^n(S^n)$ where $\gamma^n \in \tilde{E}^n(S^n)$ is a generator.

Remark 2.1.8. *This definition of orientation is slightly stronger than the one Adams gives in [Ada74], Part 3, §10, and implies the latter. It ensures that the orientation class lies in the top (co-)homology.*

In the following we want the manifolds $M, \partial M$ and Σ to be E-oriented. Therefore we need an appropriate definition of orientation of a stratified pseudomanifold that fulfills this requirement.

Definition 2.1.9. A stratified pseudomanifold X is called *E-orientable* if every stratum $X_{n-k} - X_{n-k-1}$ is an E-orientable manifold.

Lemma 2.1.10. *Let M be a smooth compact manifold. An E-orientation of the tangent bundle and the existence of an E-fundamental class is equivalent to an E-orientation of M in the sense of Definition 2.1.7.*

Proof. The proof is as in the case of ordinary homology (see for example in [DK01], Chapter 10.7). The exponential map $exp : T_pM \to M$ is a diffeomorphism in a neighborhood $T_pM \supset W \to U \subset M$. Thus exp induces an isomorphism on cohomology (the first and third isomorphism is given by excision):

$$\begin{aligned} E^n(T_pM, T_pM - 0) &\cong E^n(W, W - 0) \\ &\stackrel{exp}{\to} E^n(U, U - p) \\ &\cong E^n(M, M - p). \end{aligned}$$

This implies that a local orientation on the tangent bundle is equivalent to a local orientation given by a cohomological fundamental class. The existence of a cohomological fundamental class in turn is equivalent to the existence of a (homological) fundamental class by the formula

$$< [M]^*, [M] >= 1.$$

Note that the Kronecker product exists in any generalized homology theory E (see [Ada74], part 3, §9). Thus the notions of orientation coincide. Finally $E^n(M, M - p) \cong E^n(M \times p, M \times p - p \times p)$ and thus a local orientation given by a cohomological fundamental class is equivalent to a local orientation in the sense of Definition 2.1.7. □

Lemma 2.1.11. *An orientation on X induces orientations on Σ, M and ∂M.*

Proof. Let $p \in M$. The orientation on X induces by Definition 2.1.9 an orientation on $X - \Sigma$, given by an element $\omega \in E^n((X - \Sigma) \times (X - \Sigma), (X - \Sigma) \times (X - \Sigma) - \triangle)$. Now for $p \in X - \Sigma$, the element

$$i^*(\omega) \in E^n((X - \Sigma) \times p, (X - \Sigma) \times p - p \times p) \cong E^n(M \times p, M \times p - p \times p)$$

is a generator. Thus the restriction to $\omega|_M$ is an orientation for M. Again by definition, the orientation on X induces an orientation on the singular stratum Σ.

The induced orientation on ∂M can be derived as in [May99], Chapter 21, Section 4, substituting E_* for ordinary homology. Let $U \ni x$ be a coordinate chart of ∂M, $V = \partial U$ and $y \in \mathring{U}$, the interior of U. We work with a fundamental class as definition for an orientation (compare Lemma 2.1.10).

$$\begin{aligned}
E_n(\mathring{M}, \mathring{M} - \mathring{U}) &\cong E_n(\mathring{M}, \mathring{M} - y) \\
&\cong E_n(M, M - \mathring{U}) && \text{homotopy equivalence} \\
&\cong E_{n-1}(M - \mathring{U}, M - U) && \text{connecting homomorphism} \\
&\cong E_{n-1}(M - \mathring{U}, (M - \mathring{U}) - x) && \text{homotopy equivalence} \\
&\cong E_{n-1}(\partial M, \partial M - x) && \text{excision} \\
&\cong E_{n-1}(\partial M, \partial M - V)
\end{aligned}$$

Here the connecting homomorphism of the triple $(M, M - \mathring{U}, M - U)$ is an isomorphism since M and $M - U$ are homotopy equivalent. □

Remark 2.1.12. *Given a homological fundamental class $[M] \in E_d(M)$ of a closed manifold M and an orientation ω of M in the sense of Definition 2.1.7, both elements correspond in the following way*

$$\begin{gathered}
E^r(M) \to E_{d-r}(M) \\
x \mapsto x \cap [M] \\
\omega / y \leftarrow\!\shortmid y.
\end{gathered}$$

See [Ada74], Part 3, Section 10. / is the slant product defined in [Ada74], Part 3, Section 9.

2.2 Complex K-theory

The goal of this section is to determine, under which assumptions on the stratified pseudomanifold X a duality isomorphism

$$\tilde{K}^r(I^{\overline{p}}X) \to \tilde{K}_{n-r}(I^{\overline{q}}X)$$

can exist. We defined in Definition (2.1.1) for any CW-complex X,

$$K_r(X) := \pi_r(KU \wedge X)$$

and

$$K^r(X) := [KU \wedge X]_r.$$

In many cases it is, however, difficult to compute the actual groups by these definitions. Better tools are the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q}(X) \cong H_p(X; \pi_q E) \Rightarrow E_{p+q}(X)$$

(and its cohomological equivalent) or the chern character map

$$ch : K^*(X) \to H^*(X; \pi_* K \otimes \mathbb{Q}).$$

When we tensor with $\mathbb{Q}$, then ch is an isomorphism and the spectral sequence collapses at E_2. We thus have two different isomorphisms

$$ch : K^r(X) \otimes \mathbb{Q} \to H^r(X; \pi_* K \otimes \mathbb{Q})$$

and

$$d_\infty : H^r(X; \pi_* K \otimes \mathbb{Q}) \to K^r(X) \otimes \mathbb{Q}.$$

We will describe and constantly make use of the properties of these isomorphisms. The goal of this section is the following Lemma.

Lemma 2.2.1. *Let N be a closed, oriented manifold of dimension n. Then the diagram*

$$\begin{tikzcd}
K^r(N) \arrow[r, "\cap_K [N]_K"] \arrow[d, "ch"'] & K_{n-r}(N) \arrow[d, "ch"] \\
H^r(N; \pi_* KU \otimes \mathbb{Q}) \arrow[r, "\cap[N]"] & H_{n-r}(N; \pi_* KU \otimes \mathbb{Q})
\end{tikzcd}$$

commutes for all $r \in \mathbb{N}$.

Proof. Lemma 2.1.5 showed that

$$\begin{tikzcd}
K^r(N) \otimes K_n(N) \arrow[r, "\cap_K"] \arrow[d, "ch \otimes ch"'] & K_{n-r}(N) \arrow[d, "ch"] \\
H(\pi_* KU \otimes \mathbb{Q})^r(N) \otimes H(\pi_* KU \otimes \mathbb{Q})_n(N) \arrow[r, "\cap"] & H(\pi_r KU \otimes \mathbb{Q})_{n-r}(N)
\end{tikzcd}$$

commutes. The claim follows as a consequence of the following Lemmas 2.2.2 and 2.2.3. □

Lemma 2.2.2. *Let $[N]_K$ be the K-orientation class of a K-orientable manifold N. Then*

$$ch([N]_K) =: [N]_{H(\pi_* KU \otimes \mathbb{Q})}$$

is an $H(\pi_* KU \otimes \mathbb{Q})$ orientation class of N.

Proof.

$$ch : \tilde{K}_n(S^n) \to \tilde{H}(\pi_* KU \otimes \mathbb{Q})_n(S^n) \tag{2.3}$$

maps a generator (a base as a $\pi_* KU$-module) to a generator (a base as a $\pi_* H(KU \otimes \mathbb{Q})$-module). Furthermore ch is natural in the sense that the following diagram commutes for all i:

$$\begin{tikzcd}
K_r(N) \arrow[r, "i_*"] \arrow[d, "ch"'] & K_r(N, N - x) \arrow[d, "ch"] \\
H(\pi_* E \otimes \mathbb{Q})_r(N) \arrow[r, "i_*"] & H(\pi_* E \otimes \mathbb{Q})_r(N, N - x).
\end{tikzcd}$$

Let $\gamma \in K_n(N, N - x) \cong \tilde{K}_n(S^n)$ be a generator. Then

$$i_*(ch([N]_K)) = ch(i_*([N]_K)) = ch(\gamma)$$

is a generator of $\tilde{H}(\pi_* KU \otimes \mathbb{Q})_n(S^n)$. Thus the restriction $i_*(ch([N]_K))$ of $ch([N]_K) \in H(\pi_* E \otimes \mathbb{Q})_n(N)$ is a generator of $H(\pi_* E \otimes \mathbb{Q})_n(N, N - x)$. Therefore $ch([N]_K)$ is an orientation class of N. □

Lemma 2.2.3. *Let G be a torsion-free group and HG the Eilenberg-MacLane spectrum of the group G. Capping with the fundamental class of HG factors as capping with the ordinary fundamental class tensored with the identity on G.*

Proof. If G is a torsion-free group, then Proposition 6.7 of [Ada74] states that for any ring spectrum E,

$$E_*(X) \otimes G \cong (EG)_*(X).$$

In particular, the orientation class $[N]_{HG} \in HG_n(N)$ is given as the image of the ordinary orientation class by the canonical map $[N]_H \in H_n(N) \to H_n(N) \otimes G \cong HG_n(N)$. The cap product factors through this identification.
Indeed, there is a map of spectra $H \to HG$. Thus the following diagram commutes (see Lemma 2.1.5)

$$\begin{tikzcd} H^p(N) \otimes H_q(N) \arrow[r, "\cap_H"] \arrow[d] & H_{q-p}(N) \arrow[d] \\ HG^p(N) \otimes HG_q(N) \arrow[r, "\cap_{HG}"] & HG_{q-p}(N) \end{tikzcd}$$

and therefore the diagram also commutes when tensoring the first line with G

$$\begin{tikzcd} H^p(N) \otimes H_q(N) \otimes G \arrow[r, "\cap_H \otimes id"] \arrow[d] & H_{q-p}(N) \otimes G \arrow[d] \\ HG^p(N) \otimes HG_q(N) \arrow[r, "\cap_{HG}"] & HG_{q-p}(N). \end{tikzcd}$$

Finally,

$$\begin{tikzcd} H^p(N) \otimes G \arrow[r, "(\cap [N]_H) \otimes id"] \arrow[d, "\cong"] & H_{q-p}(N) \otimes G \arrow[d, "\cong"] \\ HG^p(N) \arrow[r, "\cap [N]_{HG}"'] & HG_{q-p}(N) \end{tikzcd}$$

commutes.

□

For a cap product of the form $HG^r(N) \overset{\cap_{HG}[N]_{HG}}{\to} HG_{n-r}(N)$, we will therefore simply write $H^r(N; G) \overset{\cap[N]}{\to} H_{n-r}(N; G)$.

2.2.1 K-Orientation

Atiyah, Bott and Shapiro ([ABS64], Theorem 12.3) showed that a necessary and sufficient condition for KO-orientability of a vector bundle is the existence of a *spin*-structure. In the complex case, the existence of a *spin*^{c}-structure is necessary and sufficient. K(KO)-

orientability of smooth manifolds can thus be verified by the existence of a *spin*^{c}(*spin*) structure on the tangent bundle.
Note that the notion of orientability in [ABS64] coincides with the notion of orientability that we give. In [ABS64], orientation for a n-dimensional vector bundle $\xi \to B$ and a generalized homology theory F is defined by a Thom class

$$\mu_\xi \in \tilde{F}^n(T(\xi))$$

that restricts to a generator in $\tilde{F}^n(T(\xi|_{\{x\}})) \cong \tilde{F}^n(S^n)$ under the inclusion map $\{x\} \to B$. Let $S(\xi)$ denote the sphere bundle, then homotopy equivalence and excision show that

$$\tilde{F}^n(T(\xi)) \cong F^n(S(\xi), B) \cong F^n(S(\xi), S(\xi)_0) \cong F^n(\xi, \xi_0)$$

(compare [May99], Chapter 23, Section 5). This implies orientation of a vector bundle as Adams defines it. Furthermore, a smooth manifold is F-orientable in the sense of [ABS64] if and only if it is orientable in the sense of Definition 2.1.7.
It is a well known result that the obstruction of a vector bundle to allow a spin structure can be described in terms of characteristic classes. In particular, an oriented vector bundle E admits a *spin*^{c} structure if and only if the third integral Stiefel-Whitney class $W_3(E)$ equals 0.

Example 2.2.4.

(a) Compact, orientable manifolds of dimension 3 or less are spin whereas compact, oriented, smooth manifolds of dimension 4 or less are spin^c *(see for example [KR85]).*

(b) Almost complex manifolds are spin^c *(see for example [GGK02], Appendix D).*

2.2.2 Motivation

As we have explained in Section 2.1, rationally, the K-theory of a space is completely determined by its ordinary homology. Thus the interesting part of K-theory are the torsion subgroups. Indeed, the torsion subgroups of the K-groups of a space are in general not given by its integral homology.

Example 2.2.5. *An easy class of examples where the torsion of the K-groups is not given by the torsion of the cohomology groups are real projective spaces. In [Ada62] (Theorem 7.3) a formula for the complex K-cohomology of real projective spaces can be found. In particular: Let n be an odd number.*

$$K^r(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{2^k} & r = 0 \\ \mathbb{Z} & r = 1 \end{cases}$$

where $k = \frac{n-1}{2}$ *and*

$$H^r(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & r = 0 \\ \mathbb{Z}_2 & r \ \textit{even}, 0 < r < n \\ \mathbb{Z} & r = n \\ 0 & \textit{else.} \end{cases}$$

If $n \geq 5$, then the torsion subgroups differ. For $n \geq 7$ even the order of the torsion subgroups of K-homology and ordinary homology are different. Note that this also implies, that not all differentials in the Atiyah-Hirzebruch spectral sequence are 0.

To motivate our approach, we first try to construct a duality isomorphism in the case of isolated singularities, and point out where the difficulties lie. We begin with the Atiyah-Hirzebruch spectral sequence of the link L

$$E^2_{p,q}(L) \cong H_p(L; \pi_q KU) \Rightarrow K_{p+q}(L)$$

and the same for cohomology

$$E_2^{p,q}(L) \cong H^p(L; \pi_{-q} KU) \Rightarrow K^{p+q}(L).$$

As an illustration, we draw the E_2 page of this spectral sequence for the K-cohomology explicitly. It has non-zero entries only in the first and fourth quadrant since $H^r(L; G) = 0$ for $r < 0$ and any coefficient group G. Furthermore $K^r(pt) = \mathbb{Z}$ if r is even and $K^r(pt) = 0$ if r is odd. The page then reads as follows:

$$\begin{array}{ccccc}
 & & \vdots & & \\
q = 2 & H^0(L;\mathbb{Z}) & H^1(L;\mathbb{Z}) & H^2(L;\mathbb{Z}) & \\
q = 1 & 0 & 0 & 0 & \\
q = 0 & H^0(L;\mathbb{Z}) & H^1(L;\mathbb{Z}) & H^2(L;\mathbb{Z}) & \cdots \\
q = -1 & 0 & 0 & 0 & \\
 & & \vdots & & \\
 & p = 0 & p = 1 & p = 2 & .
\end{array}$$

If the link L has torsion-free ordinary homology, then, as remarked in Section 2.1.5, the spectral sequence collapses at E_2. In particular this shows, that then the K-cohomology of L is also torsion-free. This is consistent with the following criterion, proven by Atiyah and Hirzebruch ([AH60], Section 2.4) by more elementary means.

Lemma 2.2.6. *Let $H_*(X)$ be torsion-free. Then the above spectral sequence for cohomology collapses at the E_2 page. In particular $K^*(X)$ is torsion-free.*

No matter if we take the proof by [Arl92] or the modify the one in [AH60], the analogous statement for homology holds as well.

Remark 2.2.7. *Of course for real K-theory this statement is false. The above arguments both fail, since the coefficient group of real K-theory already has torsion.*

In a first step, we recall the ordinary homology and cohomology groups of the truncated link $L_{<k}$. The truncation structure $L_{<k}$ is designed to give the following isomorphism in ordinary homology:

$$H_r(L_{<k}) \cong \begin{cases} H_r(L) & r < k \\ 0 & r \geq k \end{cases}$$

and applying the universal coefficient theorem (see [Ban10a] Remark 1.42)

$$H^r(L_{<k}) \cong \begin{cases} H^r(L) & r < k \\ \operatorname{Ext}(H_{k-1}(L); \mathbb{Z}) & r = k \\ 0 & r > k. \end{cases}$$

Thus if the homology of L is torsion-free, the Ext-term vanishes and therefore both the homology and the cohomology of $L_{<k}$ are torsion-free, too. Applying the Atiyah-Hirzebruch spectral sequence to the truncated link $L_{<k}$ we get, that $K^*(L_{<k})$ and $K_*(L_{<k})$ are torsion-free.

The next task is to construct a duality isomorphism between the groups $K^*(L_{<k})$ and $K_{n-*}(L, L_{<n-k})$. As a motivation for the following construction, let us first take a naive construction of this isomorphism and then explain why this doesn't work. Since we know that the Atiyah-Hirzebruch spectral sequence collapses, the easiest thing to do, was to define such an isomorphism by using the maps d_∞.

$$D : K^r(L) \xrightarrow{d_\infty^{-1}} H^r(L; \pi_{-*}KU) \xrightarrow{-\cap[L]} H_{n-r}(L; \pi_* KU) \xrightarrow{d_\infty} K_{n-r}(L)$$

and

$$\begin{aligned} D| : K^r(L_{<k}) &\xrightarrow{d_\infty^{-1}} H^r(L_{<k}; \pi_{-*}KU) \xrightarrow{-\cap[L]} H_{n-r}(L, L_{<n-k}; \pi_* KU) \\ &\xrightarrow{d_\infty} K_{n-r}(L, L_{<n-k}). \end{aligned}$$

The middle isomorphism is given in every component by the (ordinary) cap product and by the cap product constructed in [Ban10a], Proposition 2.43 (In fact, there, this cap product is defined only for homology with rational coefficients, but it is easy to see and straightforward to prove that exactly the same construction works for integral coefficients as well, when all occurring terms are torsion-free). Then the following diagram commutes by definition

$$\begin{tikzcd}
H^r(L;\pi_{-*}KU) \arrow[rrr, "\cap[L]"] \arrow[dr, "d_\infty"] \arrow[ddd] & & & H_{n-r}(L;\pi_*KU) \arrow[dl, "d_\infty"'] \arrow[ddd] \\
& K^r(L) \arrow[r, "D"] \arrow[d] & K_{n-r}(L) \arrow[d] & \\
& K^r(L_{<k}) \arrow[r, "D|"] & K_{n-r}(L, L_{<k}) & \\
H^r(L_{<k};\pi_{-*}KU) \arrow[rrr, "\cap[L]"] \arrow[ur, "d_\infty"] & & & H_{n-r}(L, L_{<k};\pi_*KU). \arrow[ul, "d_\infty"']
\end{tikzcd}$$

The problem now is, that it is not clear whether D equals the K-theory cap product with the K-fundamental class of L. More precisely, although we know that the chern character behaves well with respect to the cap product (see Lemma 2.2.1) and thus

$$\begin{tikzcd}
K^r(L)\otimes\mathbb{Q} \arrow[r, "ch\circ - \cap[L]\circ ch^{-1}"] \arrow[d, equal] & K_{n-r}(L)\otimes\mathbb{Q} \arrow[d, equal] \\
K^r(L)\otimes\mathbb{Q} \arrow[r, "\cap_K([L]_K)"] & K_{n-r}(L)\otimes\mathbb{Q}
\end{tikzcd}$$

does commute, we do not know if the following diagram commutes

$$\begin{tikzcd}
K^r(L) \arrow[r, "D"] \arrow[d, equal] & K_{n-r}(L) \arrow[d, equal] \\
K^r(L) \arrow[r, "\cap_K([L]_K)"] & K_{n-r}(L)
\end{tikzcd}$$

If d_∞ and ch were simply inverse to each other, then this diagram would commute. But this is certainly not true, first because d_∞ is not natural but involves arbitrary choices. More important, the chern character map does in general not factor through ordinary homology with integral coefficients. In other words

$$\begin{tikzcd}
K^r(L) \arrow[rr, "d_\infty^{-1}"] \arrow[dr, "ch"'] & & H^r(L;\pi_{-*}KU) \arrow[dl, "\otimes\mathbb{Q}"] \\
& H^r(L;\pi_{-*}KU\otimes\mathbb{Q}) &
\end{tikzcd}$$

is in general not commutative. Thus we can't use the map D to construct a duality isomorphism that is natural with respect to the K-cap product.

Example 2.2.8. *This example is taken from [Hat], Chapter 4: Let $L \in K(\mathbb{C}P^n)$ be the canonical line bundle. Then $ch(L) = 1 + c + c^2/2 + ... + c^n/n!$ where $c = c_1(L)$ is the first chern class of L. Thus c is a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$ and c^k is a generator of $H^{2k}(\mathbb{C}P^n;\mathbb{Z})$. This shows, that ch does not factor through homology with integral coeffi-*

cients, even though $H^*(\mathbb{C}P^n)$ is torsion-free.

We thus have to replace the isomorphism D by some map that commutes with the cap product in K-theory. In the following, we construct such a map.

2.2.3 Duality

For the rest of this section let us assume, that the pairs $(X, B(\overline{p}))$ and $(X, B(\overline{p}))$ are both in Θ (see Definition 1.3.5) and that there is a Poincaré duality isomorphism (see Definition 1.3.13)

$$D : \tilde{H}^r(I^{B(\overline{p})};\mathbb{Q}) \to \tilde{H}_{n-r}(I^{B(\overline{q})};\mathbb{Q}).$$

Let us denote the induced isomorphism

$$E : H^r(\partial M, B(\overline{p});\mathbb{Q}) \to H_{n-r-1}(B(\overline{q});\mathbb{Q}).$$

By Lemma 1.3.10 applied to the long exact sequence of the pair $(\partial M, B(\overline{p}))$ we obtain equivalently an isomorphism

$$F : H^r(B(\overline{p});\mathbb{Q}) \to H_{n-r-1}(\partial M, B(\overline{q});\mathbb{Q})$$

that fits commutatively into the following diagram

$$\begin{tikzcd}
H^r(M;\mathbb{Q}) \arrow[r] \arrow[d, "-\cap[M,\partial M]"'] & H^r(\partial M;\mathbb{Q}) \arrow[r] \arrow[d, "-\cap[\partial M]"] & H^r(B(\overline{p});\mathbb{Q}) \arrow[d, "F"] \\
H_{n-r}(M.\partial M;\mathbb{Q}) \arrow[r] & H_{n-r-1}(\partial M;\mathbb{Q}) \arrow[r] & H_{n-r-1}(\partial M, B(\overline{q});\mathbb{Q}).
\end{tikzcd}$$

Together this shows the existence of the following commutative diagram for all $r \in \mathbb{N}$

$$\begin{tikzcd}
{} \arrow[r] & H^r(\partial M, B(\overline{p});\mathbb{Q}) \arrow[r, "j^*"] \arrow[d, "E"'] & H^r(\partial M;\mathbb{Q}) \arrow[r, "i^*"] \arrow[d, "-\cap[\partial M]"] & H^r(B(\overline{p});\mathbb{Q}) \arrow[r] \arrow[d, "F"] & {} \\
{} \arrow[r] & H_{n-r-1}(B(\overline{q});\mathbb{Q}) \arrow[r, "i_*"] & H_{n-r-1}(\partial M;\mathbb{Q}) \arrow[r, "j_*"] & H_{n-r-1}(\partial M, B(\overline{q});\mathbb{Q}) \arrow[r] & \cdot
\end{tikzcd}$$

Now i_* is injective as we assumed $(X, B(\overline{p})), (X, B(\overline{q})) \in \Xi$, so the boundary map equals zero. The commutativity then implies, that the coboundary map also equals zero, so that we obtain

$$\begin{tikzcd}
0 \arrow[r] & H^r(\partial M, B(\overline{p});\mathbb{Q}) \arrow[r, "j^*"] \arrow[d, "E"'] & H^r(\partial M;\mathbb{Q}) \arrow[r, "i^*"] \arrow[d, "-\cap[\partial M]"] & H^r(B(\overline{p});\mathbb{Q}) \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & H_{n-r-1}(B(\overline{q});\mathbb{Q}) \arrow[r, "i_*"] & H_{n-r-1}(\partial M;\mathbb{Q}) \arrow[r, "j_*"] & H_{n-r-1}(\partial M, B(\overline{q});\mathbb{Q}) \arrow[r] & 0.
\end{tikzcd} \tag{2.4}$$

Lemma 2.2.9. *The induced chern character map on homology*

$$ch : K_*(X) \to H_*(X, \pi_* KU \otimes \mathbb{Q})$$

is an isomorphism when tensored with the rationals and injective when $K_*(X)$ is torsion-free.

Proof. The chern character is induced by a map of spectra

$$ch : KU \to H(\pi_* KU \otimes \mathbb{Q})$$

ch induces an isomorphism on the coefficient groups

$$ch_* : \pi_*(KU\mathbb{Q}) \to \pi_*(H(\pi_* KU \otimes \mathbb{Q}))$$

since the following diagram commutes, and the bottom row is an isomorphism

$$\begin{tikzcd}
\pi_*(KU\mathbb{Q}) \arrow[r, "ch"] \arrow[d, "\cong"'] & \pi_*(H(\pi_* KU \otimes \mathbb{Q})) \arrow[d, "\cong"] \\
(KU\mathbb{Q})^{-*}(pt) \arrow[r, "ch"] \arrow[d, "\cong"'] & H(\pi_* KU \otimes \mathbb{Q})^{-*}(pt) \arrow[d, "\cong"] \\
K^{-*}(pt) \otimes \mathbb{Q} \arrow[r, "ch"] & H^{-*}(pt; \pi_* KU \otimes \mathbb{Q}).
\end{tikzcd}$$

This uses the generalized universal coefficient theorem Proposition 6.6 of [Ada74], Part 3. The Whitehead-type theorem [Ada74], Part 3, §3, Corollary 3.5 then shows, that

$$ch : KU\mathbb{Q} \to H(\pi_* KU \otimes \mathbb{Q})$$

is an equivalence of spectra in the homotopy category as defined in [Ada74]. In particular, this shows, that the induced map on homology is also an isomorphism

$$ch : K_*(X) \otimes \mathbb{Q} \to H_*(X, \pi_* KU \otimes \mathbb{Q}).$$

Finally,

$$\begin{tikzcd}
KU \arrow[r, "ch"] \arrow[d] & H(\pi_* KU \otimes \mathbb{Q}) \arrow[d, "="] \\
KU\mathbb{Q} \arrow[r, "ch"] & H(\pi_* KU \otimes \mathbb{Q})
\end{tikzcd}$$

commutes. Thus the map

$$ch : KU \to H(\pi_* KU \otimes \mathbb{Q})$$

factors as

$$K_*(X) \to KU\mathbb{Q}_*(X) \to H(\pi_* KU \otimes \mathbb{Q})$$

which shows that

$$ch : K_*(X) \to H_*(X, \pi_* KU \otimes \mathbb{Q})$$

is injective if $K_*(X)$ is torsion-free. $\square$

Lemma 2.2.10. *The pairs* $(\partial M, (X, B(\overline{p})))$ *and* $(\partial M, (X, B(\overline{q})))$ *induce the following short exact sequences for all* $r \in \mathbb{N}$*.*

$$0 \longrightarrow K^r(\partial M, B(\overline{p})) \xrightarrow{g^*} K^r(\partial M) \xrightarrow{f^*} K^r(B(\overline{p})) \longrightarrow 0$$

$$0 \longrightarrow K_r(B(\overline{q})) \xrightarrow{f_*} K_r(\partial M) \xrightarrow{g_*} K_r(\partial M, B(\overline{q})) \longrightarrow 0.$$

Proof. The Atiyah-Hirzebruch spectral sequence shows, that $K_r(\partial M)$ and $K^r(\partial M)$ are torsion-free. Therefore the chern character maps

$$ch : K_r(\partial M) \to H_r(\partial M; \pi_* KU \otimes \mathbb{Q})$$

and

$$ch : K^r(\partial M) \to H^r(\partial M; \pi_{-*} KU \otimes \mathbb{Q})$$

are injective (see Lemma 2.2.9). Since $H_*(B(\overline{p})) \to H_*(\partial M)$ is injective, $H_*(B(\overline{p}))$ is torsion-free and

$$ch : K_r(B(\overline{p})) \to H_r(B(\overline{p}); \pi_* KU \otimes \mathbb{Q})$$

and

$$ch : K^r(B(\overline{p})) \to H^r(B(\overline{p}); \pi_{-*} KU \otimes \mathbb{Q})$$

are also injective. We obtain the following commutative diagram

$$\begin{tikzcd}
K_r(\partial M) \arrow[r, hook, "ch"] & H_r(\partial M; \pi_* KU \otimes \mathbb{Q}) \\
K_r(B(\overline{p})) \arrow[u, "f_*"] \arrow[r, hook, "ch"] & H_r(B(\overline{p})); \pi_* KU \otimes \mathbb{Q}) \arrow[u, "i_*"']
\end{tikzcd}$$

where f_* and i_* are both induced by the inclusion $B(\overline{p}) \to \partial M$. Now i_* is injective. Therefore, for all indices $r \in \mathbb{Z}$, f_* is injective, too. The long exact sequence of the pair $(\partial M, B(\overline{p}))$ then shows that the map

$$K_r(\partial M) \to K_r(\partial M, B(\overline{p}))$$

is surjective for all $r \in \mathbb{Z}$. An equivalent statement holds for cohomology. Here we get a commutative diagram

$$\begin{tikzcd}
K^r(\partial M) \arrow[r, hook, "ch"] & H^r(\partial M; \pi_{-*} KU \otimes \mathbb{Q}) \\
K^r(\partial M, B(\overline{p})) \arrow[u, "g^*"] \arrow[r, hook, "ch"] & H^r(\partial M, B(\overline{p})); \pi_{-*} KU \otimes \mathbb{Q}) \arrow[u, "j^*"']
\end{tikzcd}$$

where g^* and j^* are induced by the quotient map of the pair $(\partial M, B(\overline{p}))$. j^* is injective by Diagram (2.4). The commutativity of the above diagram then shows that g^* is also injective for all indices $r \in \mathbb{N}$. This shows that the map

$$K^r(\partial M) \to K^r(B(\overline{p}))$$

is surjective for all $r \in \mathbb{N}$. Thus

$$0 \longrightarrow K^r(\partial M, B(\overline{p})) \xrightarrow{g^*} K^r(\partial M) \xrightarrow{f^*} K^r(B(\overline{p})) \longrightarrow 0$$

$$0 \longrightarrow K_r(B(\overline{q})) \xrightarrow{f_*} K_r(\partial M) \xrightarrow{g_*} K_r(\partial M, B(\overline{q})) \longrightarrow 0.$$

□

The application of the chern character to Lemma 2.2.10 leads to the diagrams

$$\begin{tikzcd} 0 \arrow[r] & K_r(B(\overline{p})) \otimes \mathbb{Q} \arrow[r, "f_* \otimes id"] \arrow[d, "ch"'] & K_r(\partial M) \otimes \mathbb{Q} \arrow[r, "g_* \otimes id"] \arrow[d, "ch"] & K_r(\partial M, B(\overline{p})) \otimes \mathbb{Q} \arrow[r] \arrow[d, "ch"] & 0 \\ 0 \arrow[r] & H_r(B(\overline{p}); G) \arrow[r, "i_*"] & H_r(\partial M; G) \arrow[r, "j_*"] & H_r(\partial M, B(\overline{p}); G) \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

$$\begin{tikzcd} 0 \arrow[r] & K^r(\partial M, B(\overline{p})) \otimes \mathbb{Q} \arrow[r, "g^* \otimes id"] \arrow[d, "ch"'] & K^r(\partial M) \otimes \mathbb{Q} \arrow[r, "f^* \otimes id"] \arrow[d, "ch"] & K^r(B(\overline{p})) \otimes \mathbb{Q} \arrow[r] \arrow[d, "ch"] & 0 \\ 0 \arrow[r] & H^r(\partial M, B(\overline{p}); G) \arrow[r, "j^*"] & H^r(\partial M; G) \arrow[r, "i^*"] & H^r(B(\overline{p}); G) \arrow[r] & 0. \end{tikzcd} \tag{2.6}$$

We collect the results. Lemma 2.2.1 shows that the following diagram commutes.

$$\begin{tikzcd} K^r(\partial M) \otimes \mathbb{Q} \arrow[r, "\cap_K [\partial M]_K \otimes id"] \arrow[d, "ch"'] & K_{n-r-1}(\partial M) \otimes \mathbb{Q} \arrow[d, "ch"] \\ H^r(\partial M; G) \arrow[r, "\cap[\partial M]"] & H_{n-r-1}(\partial M; G). \end{tikzcd} \tag{2.7}$$

Diagram (2.4) is a consequence of the assumption, that the duality isomorphism of the intersection spaces commutes with the ordinary Poincaré duality map of the manifold M. Finally, Diagram (2.5) and Diagram (2.6) use that the chern character is a natural transformation. Define $E_K = ch^{-1} \circ E \circ ch$ and $F_K = ch^{-1} \circ F \circ ch$. Then the commutativity of the Diagrams (2.4) to (2.7) show, that for an element $x \otimes y \in K^r(\partial M, B(\overline{p})) \otimes \mathbb{Q}$,

$$\begin{aligned} ch(g^*(x) \cap_K [\partial M]_K \otimes y) &= (ch(g^*(x) \otimes y)) \cap [\partial M] \\ &= (j^*(ch(x \otimes y))) \cap [\partial M] \\ &= i_*(E(ch(x \otimes y))) \\ &= i_*(ch(E_K(x) \otimes y)) \\ &= ch(f_*(E_K(x)) \otimes y) \end{aligned}$$

Using that $ch : K_{n-r-1}(\partial M) \otimes \mathbb{Q} \to H_{n-r-1}(\partial M; G)$ is an isomorphism and doing a similar calculation to show that for $x \otimes y \in K^i(\partial M) \otimes \mathbb{Q}$ the equation $(F_K(f^*(x) \otimes y)) = g_*(x \cap_K [\partial M]_K \otimes y)$ holds, leads to the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & K^r(\partial M, B(\overline{p})) \otimes \mathbb{Q} \arrow[r, "g^* \otimes id"] \arrow[d, "E_K"'] & K^r(\partial M) \otimes \mathbb{Q} \arrow[r, "f^* \otimes id"] \arrow[d, "- \cap_K [\partial M]_K \otimes id"] & K^r(B(\overline{p})) \otimes \mathbb{Q} \arrow[r] \arrow[d, "F_K"] & 0 \\
0 \arrow[r] & K_{n-r-1}(B(\overline{q})) \otimes \mathbb{Q} \arrow[r, "f_* \otimes id"] & K_{n-r-1}(\partial M) \otimes \mathbb{Q} \arrow[r, "g_* \otimes id"] & K_{n-r-1}(\partial M, B(\overline{q})) \otimes \mathbb{Q} \arrow[r] & 0
\end{tikzcd} \tag{2.8}$$

We want this factorization to hold also with coefficients $\mathbb{Z}$. To show this, set

$$\begin{aligned}
V_r &= \operatorname{im} f_* \\
W_r &= g_*^{-1}(K_r(\partial M, B(\overline{q}))) \\
V^r &= (f^*)^{-1}(K^r(B(\overline{p}))) \\
W^r &= \operatorname{im} g^*.
\end{aligned}$$

Then

$$K_r(\partial M) = V_r \oplus W_r$$

and

$$K^r(\partial M) = V^r \oplus W^r.$$

Lemma 2.2.11. *The duality map* $K^r(\partial M) \overset{\cap_K [\partial M]_K}{\to} K_{n-r-1}(\partial M)$ *factors as*

$$\begin{tikzcd}
K^r(\partial M) \arrow[d, "\cap_K [\partial M]_K"'] & = & V^r \arrow[drr] & \oplus & W^r \arrow[dll] \\
K_{n-r-1}(\partial M) & = & V_{n-r-1} & \oplus & W_{n-r-1}.
\end{tikzcd}$$

Proof. Diagram (2.8) shows that tensored with the rationals the duality map factors as

$$\begin{tikzcd}
K^r(\partial M) \otimes \mathbb{Q} \arrow[d, "\cap_K [\partial M]_K \otimes id"'] & \cong & V^r \otimes \mathbb{Q} \arrow[drr] & \oplus & W^r \otimes \mathbb{Q} \arrow[dll] \\
K_{n-r-1}(\partial M) \otimes \mathbb{Q} & \cong & V_{n-r-1} \otimes \mathbb{Q} & \oplus & W_{n-r-1} \otimes \mathbb{Q}.
\end{tikzcd} \tag{2.9}$$

Let ϕ for now denote the duality isomorphism

$$\phi := - \cap_K [\partial M]_K : K^r(\partial M) \to K_{n-r-1}(\partial M).$$

We restrict ϕ to the subspaces V^r and W^r of $K^r(\partial M)$. Since ϕ is an isomorphism, we know that

$$\operatorname{im} \phi|_{V^r} \oplus \operatorname{im} \phi|_{W^r} = K_{n-r-1}(\partial M) \cong V_{n-r-1} \oplus W_{n-r-1}.$$

Assume that $\operatorname{im} \phi|_{V^r} \nsubseteq W_{n-r-1}$. Then

$$\operatorname{im}(\phi \otimes id|_{V^r \otimes \mathbb{Q}}) \nsubseteq W_{n-r-1} \otimes \mathbb{Q}.$$

But this is a contradiction to (2.9). Therefore

$$\operatorname{im} \phi|_{V^r} \subseteq W_{n-r-1}.$$

Exactly the same argument shows that

$$\operatorname{im} \phi|_{W^r} \subseteq V_{n-r-1}.$$

Because $K^r(\partial M) \cong V^r \oplus W^r$ and

$$\operatorname{im} \phi = V_{n-r-1} \oplus W_{n-r-1}.$$

it then follows that

$$\operatorname{im} \phi|_{V^r} = W_{n-r-1}$$

and

$$\operatorname{im} \phi|_{W^r} = V_{n-r-1}$$

and thus

$$\begin{tikzcd}
K^r(\partial M) \arrow[d, "\cap_K [\partial M]_K"'] & \cong & V^r \arrow[drr] & \oplus & W^r \arrow[dll] \\
K_{n-r-1}(\partial M) & \cong & V_{n-r-1} & \oplus & W_{n-r-1}.
\end{tikzcd}$$

□

Lemma 2.2.12. *The following diagram is commutative up to sign.*

$$\begin{tikzcd}
K^r(\partial M, B(\overline{p})) \arrow[r, hook] \arrow[d, "\cap_K [\partial M]_K|"'] & K^r(\partial M) \arrow[r] \arrow[d, "\cap_K [\partial M]_K"] & K^{r+1}(M, \partial M) \arrow[d, "\cap_K [M, \partial M]_K"] \\
K_{n-r-1}(B(\overline{q})) \arrow[r, hook] & K_{n-r-1}(\partial M) \arrow[r] & K_{n-r-1}(M).
\end{tikzcd}$$

The composition of the horizontal maps

$$K_{n-r-1}(B(\overline{q})) \to K_{n-r-1}(M)$$

and

$$K^{r-1}(\partial M, B(\overline{p})) \to K^r(M, \partial M)$$

are the induced maps of the long exact sequence of the triple $(M, \partial M, B(\overline{p}))$ and of the pair $(M, B(\overline{q}))$.

Proof. Commutativity of the left square follows by Lemma 2.2.11. The right square commutes up to sign by [Ada74], part III, §10. Observe, that the map $B(\overline{q}) \to \partial M \hookrightarrow M$ is by definition the map that induces

$$K_{n-r}(B(\overline{q})) \to K_{n-r}(M)$$

in the long exact sequence. Finally, the connecting homomorphism

$$\partial^* : K^{r-1}(\partial M, B(\overline{p})) \to K^r(M, \partial M)$$

of the long exact sequence of the triple factors as

$$K^{r-1}(\partial M, B(\overline{p})) \xrightarrow{i^*} K^{r-1}(\partial M) \xrightarrow{\partial^*} K^r(M, \partial M).$$

This is a direct consequence of the commutativity of

$$\begin{tikzcd} B(\overline{p}) \arrow[r] \arrow[d] & M \\ \partial M \arrow[ur] & \end{tikzcd} .$$

□

Theorem 2.2.13. *Let X^n be a compact, K-oriented pseudomanifold with $(X, B(\overline{p}))$ and $(X, B(\overline{q}))$ in Θ for complementary perversities $\overline{p}$ and $\overline{q}$. Assume that there is a Poincaré duality isomorphism in ordinary homology*

$$\tilde{H}^{n-r}(I^{B(\overline{p})}X; \mathbb{Q}) \to \tilde{H}_r(I^{B(\overline{q})}X; \mathbb{Q}).$$

Then there is an isomorphism

$$\tilde{K}^{n-r}(I^{B(\overline{p})}X; \mathbb{Z}) \to \tilde{K}_r(I^{B(\overline{q})}X; \mathbb{Z}).$$

Proof of the Theorem. Lemma 2.2.12 shows, that the outer squares in the following diagram commute at least up to sign.

$$\begin{tikzcd} & K^{n-r-1}(\partial M, B(\overline{p})) \arrow[r, "E_K"] \arrow[d] & K_r(B(\overline{q})) \arrow[d] & \\ & K^{n-r}(M, \partial M) \arrow[r, "-\cap_K [M,\partial M]_K"] \arrow[d] & K_r(M) \arrow[d] & \\ \tilde{K}^{n-r}(I^{\overline{p}}X; \mathbb{Z}) \arrow[r, equal] & K^{n-r}(M, B(\overline{p})) \arrow[r, dotted] \arrow[d] & K_r(M, B(\overline{q})) \arrow[r, equal] \arrow[d] & \tilde{K}_r(I^{\overline{q}}X; \mathbb{Z}) \\ & K^{n-r}(\partial M, B(\overline{p})) \arrow[r, "E_K"] \arrow[d] & K_{r-1}(B(\overline{q})) \arrow[d] & \\ & K^{n-r}(\partial M, B(\overline{p})) \arrow[r, "-\cap_K [M,\partial M]_K"] & K_{r-1}(M) & \end{tikzcd} .$$

Lemma 1.3.10 then shows that there is an isomorphism

$$\tilde{K}^{n-r}(I^{\overline{p}}X;\mathbb{Z}) \cong K^{n-r}(M, B(\overline{p});\mathbb{Z}) \to K_r(M, B(\overline{q});\mathbb{Z}) \cong \tilde{K}_r(I^{\overline{q}}X;\mathbb{Z}).$$

□

2.2.4 Examples

Example 2.2.14. *Drill out a small 3-ball out of $\mathbb{R}P^3$ and call the resulting space Y. Y is a manifold with boundary S^2. Then take*

$$X = (Y \times S^1) \cup cone(\partial Y \times S^1).$$

X is a pseudomanifold with one isolated singularity. The link of X is $L = S^2 \times S^1$, therefore $(X, L_{<2}) \in \Theta$. X is K-orientable, since M and L are compact, oriented manifolds of dimension ≤ 4 that can be given a smooth structure. The manifold obtained by cutting off a conical neighborhood of the singular stratum is $M = Y \times S^1$ which has K-(co)homology with torsion subgroups. We show that the duality theorem holds integrally and not only rationally.
Recall the integral homology and cohomology of $\mathbb{R}P^3$:

$$H_r(\mathbb{R}P^3) = \begin{cases} \mathbb{Z} & r=0 \\ \mathbb{Z}_2 & r=1 \\ 0 & r=2 \\ \mathbb{Z} & r=3 \end{cases} \qquad H^r(\mathbb{R}P^3) = \begin{cases} \mathbb{Z} & r=0 \\ 0 & r=1 \\ \mathbb{Z}_2 & r=2 \\ \mathbb{Z} & r=3. \end{cases}$$

The long exact sequence of the pair (Y, S^2) in reduced homology is

$$\ldots \to \tilde{H}_r(S^2) \to \tilde{H}_r(Y) \to H_r(Y, S^2) \to \tilde{H}_{i-1}(S^2) \to \ldots . \tag{2.10}$$

Now

$$H_r(Y, S^2) \cong \tilde{H}_r(Y/S^2) \cong \tilde{H}_r(\mathbb{R}P^3).$$

Therefore $H_3(Y, S^2) \cong \mathbb{Z}$ and the connecting homomorphism maps the orientation class $[Y, \partial Y]$ to $[S^2] \in H_2(S^2)$. Sequence (2.10) then shows, that the homology of Y is

$$H_r(Y) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \\ 0 \\ 0. \end{cases}$$

The same argument for cohomology (or Poincaré duality for Y) shows

$$H^r(Y) = \begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Z}_2 \\ 0. \end{cases}$$

In the Atiyah-Hirzebruch spectral sequence both for homology and cohomology, the differentials d_2 are all zero, as they all point from or to an entry that is zero. Thus

$$\begin{array}{ll} K^0(Y) = \mathbb{Z} \oplus \mathbb{Z}_2 & K^1(Y) = 0 \\ K_0(Y) = \mathbb{Z} & K_1(Y) = \mathbb{Z}_2. \end{array}$$

Let $\overline{m}$ be the middle perversity. Note that the codimension of the singular set is 4, thus the lower and upper middle perversity agree on this value and we write IX for both $I^{\overline{m}}X$ and $I^{\overline{n}}X$. Then $k = 4-1-\overline{p}(4) = 2$ and $L_{<2} = S^1 \times pt$. Then $K^*(L, L_{<2}) \cong K^*(S^2) \cong K^*(pt)$. The long exact sequences of the pair $(M, L_{<2})$ and $(M, L, L_{<2})$ are in this case:

$$\begin{tikzcd} {} \arrow[r] & K_r(Y \times S^1) \arrow[r] \arrow[d] & K_r(Y \times S^1, pt \times S^1) \arrow[r] \arrow[d, dotted] & K_{r-1}(S^1) \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & K^{n-r}(Y \times S^1, S^2 \times S^1) \arrow[r] & K^{n-r}(Y \times S^1, pt \times S^1) \arrow[r] & K^{n-r}(S^2 \times S^1, S^1) \arrow[r] & \cdot \end{tikzcd}$$

Then

$$\tilde{K}^*(IX) \cong K^*(Y \times S^1, pt \times S^1) \cong \tilde{K}^*(Y) \cong \begin{cases} \mathbb{Z}_2 & * = 0 \\ 0 & * = 1 \end{cases}$$

and

$$\tilde{K}_*(IX) \cong K_*(Y \times S^1, pt \times S^1) \cong \tilde{K}_*(Y) \cong \begin{cases} 0 & * = 0 \\ \mathbb{Z}_2 & * = 1. \end{cases}$$

Note that $\tilde{K}_{4-*}(IX) \cong \tilde{K}_*(IX)$ by Bott periodicity. Thus, there is an isomorphism

$$\tilde{K}^*(IX) \cong \tilde{K}_{4-*}(IX)$$

that contains information about torsion subgroups. Of course this is consistent with the duality isomorphism in ordinary homology, since

$$\tilde{H}_*(IX; \mathbb{Q}) \cong \tilde{H}_*(Y; \mathbb{Q}) = 0 = \tilde{H}^*(IX; \mathbb{Q}).$$

The following example illustrates, that in general it is not possible to drop the assumption that the ordinary homology of the link is torsion-free.

Example 2.2.15. *Let $X = \Sigma\mathbb{R}P^3$, the suspension of the real projective space. X is a 4-dimensional pseudomanifold and we endow it with the canonical stratification. The cutoff value for the middle perversities is $k = 4-1-\overline{p}(4) = 2$. The regular part is $M = I \times \mathbb{R}P^3$ and the link is $L = \mathbb{R}P^3 \dot\cup \mathbb{R}P^3$. The homology of the link is $H_*(L) = H_*(\mathbb{R}P^3) \oplus$*

$H_*(\mathbb{R}P^3) \cong \mathbb{Z}^4 \oplus (\mathbb{Z}/2)^2$ (as graded groups). In particular the homology of the link is not torsion-free. A 2-truncation of $\mathbb{R}P^3$ is given by $\mathbb{R}P^2$, so that $L_{<2} = \mathbb{R}P^2 \dot\cup \mathbb{R}P^2$. Then $(X, L_{<2}) \in \Xi$ and we calculate the homology of the intersection space $IX = I^{\bar{n}}X = I^{\bar{m}}X$.

$$H_1(IX) \cong H_1(M, \partial M) \cong H_1(X) \oplus \mathbb{Z} = H_1(\Sigma\mathbb{R}P^3) \oplus \mathbb{Z} \cong H_0(\mathbb{R}P^3) \cong \mathbb{Z}.$$

$H_2(IX)$ can be calculated by the long exact sequence of the pair $(M, L_{<2})$.

$$\underbrace{H_2(I \times \mathbb{R}P^3)}_{=0} \to H_2(IX) \xrightarrow{\partial} \underbrace{H_1(\mathbb{R}P^2 \dot\cup \mathbb{R}P^2)}_{\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2} \xrightarrow{i} \underbrace{H_1(I \times \mathbb{R}P^3)}_{\cong \mathbb{Z}/2}.$$

The map i is given by $i(a,b) = a+b$. We then deduce that $H_2(IX) \cong \mathbb{Z}/2$ and the map ∂ is given by $\partial(x) = (x,x)$.

$$H_3(IX) \cong H_3(M) \cong H_3(\mathbb{R}P^3) \cong \mathbb{Z}$$

and

$$H_4(IX) \cong H_4(M) \cong H_4(\mathbb{R}P^3) = 0.$$

To sum up,

$$\tilde{H}_r(IX) \cong \begin{cases} 0 & r = 0 \\ \mathbb{Z} & r = 1 \\ \mathbb{Z}/2 & r = 2 \\ \mathbb{Z} & r = 3 \\ 0 & r = 4. \end{cases}$$

The universal coefficient theorem shows that

$$\tilde{H}^r(IX) \cong \begin{cases} 0 \\ \mathbb{Z} \\ \mathrm{Hom}(\mathbb{Z}/2; \mathbb{Z}) \\ \mathrm{Hom}(\mathbb{Z}; \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}/2; \mathbb{Z}) \\ 0 \end{cases} \cong \begin{cases} 0 & r = 0 \\ \mathbb{Z} & r = 1 \\ 0 & r = 2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & r = 3 \\ 0 & r = 4. \end{cases}$$

In the cohomological Atiyah-Hirzebruch sequence, the entries in the 0 and 2- column survive to the E_∞ table, since all differentials in question point from or to 0. Therefore

$$K^0(IX) \cong \mathbb{Z}$$

and

$$\tilde{K}^0(IX) = 0.$$

Using that all differentials in the homological Atiyah-Hirzebruch spectral sequence are torsion, we see, that the homological sequence collapses at E_2 and therefore

$$K_0(IX) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

and

$$\tilde{K}_0(IX) \cong \mathbb{Z}/2.$$

This shows that

$$\tilde{K}^0(IX) = 0 \neq \mathbb{Z}/2 \cong \tilde{K}_0(IX) \cong \tilde{K}_4(IX),$$

the last step by Bott periodicity. Rationally the duality isomorphism holds, as expected.

2.3 Other Homology Theories

Basically, it is possible to apply the same approach also to other ring spectra with torsion-free coefficient group and a multiplicative character map.

Example 2.3.1. *Multiplicative generalized homology theories with torsion-free coefficients are for example*

1. *complex bordism,*
2. *Brown-Peterson cohomology,*
3. *connective complex K-theory.*

In the following we want to discuss, where the difficulties lie, when the coefficient ring of a spectrum has torsion, and discuss the case of complex bordism in more detail.

2.3.1 Real K-theory

There is little hope for the whole construction to work in real K-theory as well. The reason is, that the coefficient group of KO is not torsion-free. This assumption entered in the calculation of $K^*(I^{\overline{p}}X)$ in various ways. First, it was crucial to determine the groups $K^*(L_{<k})$. For KO the associated spectral sequence does not collapse. Second, for the construction of the duality isomorphism

$$K^r(L_{<k}) \to K_{n-r}(L, L_{<n-k}),$$

we needed torsion-freeness of $\pi_* KU$. Third the construction of the duality isomorphism

$$\tilde{K}_r(I^{\overline{p}}X; \mathbb{Z}) \to \tilde{K}^{n-r}(I^{\overline{q}}X; \mathbb{Z})$$

requires the existence of splittings which can not be guaranteed if the occurring groups have torsion subgroups. The assumption of torsion-freeness of a group $KO^*(X)$ of some space X is - other than for KU - extremely restrictive.

2.3.2 Complex Bordism

In this section we want to focus on complex bordism, and point out some specific properties such as orientability with respect to complex bordism and its relation to complex K-theory. There is a map of ring spectra

$$\mu : MU \to KU$$

(see [Smi73a], Section 1). The composition with the Chern character is the map of ring spectra

$$th : MU \to H(\pi_* KU \otimes \mathbb{Q}).$$

The induced map on homology is the Todd character. We want to use μ and th instead of the Chern character ch to obtain duality isomorphisms in complex bordism. However, the induced map μ_* on homology is not injective (compare [Smi73b], Section 2). In the case of complex K-theory, the chern character kept all information, when working with CW-complexes with torsion-free homology. In this case, however, we lose a lot of information when applying the map μ. Nevertheless, μ preserves sufficient structure to construct a duality isomorphism for complex bordism, as we will show. By Diagram (2.4), there is a decomposition

$$H_r(\partial M; \mathbb{Q}) \cong H_r(B(\overline{p}); \mathbb{Q}) \oplus H_r(\partial M, B(\overline{p}); \mathbb{Q})$$

and

$$H^r(\partial M; \mathbb{Q}) \cong H^r(B(\overline{p}); \mathbb{Q}) \oplus H^r(\partial M, B(\overline{p}); \mathbb{Q}).$$

If we assume, that the integral homology and cohomology of the above terms is torsion-free, this decomposition also holds integrally and

$$H_r(\partial M; \mathbb{Z}) \cong H_r(B(\overline{p}); \mathbb{Z}) \oplus H_r(\partial M, B(\overline{p}); \mathbb{Z})$$

and

$$H^r(\partial M; \mathbb{Z}) \cong H^r(B(\overline{p}); \mathbb{Z}) \oplus H^r(\partial M, B(\overline{p}); \mathbb{Z}).$$

As ∂M has torsion-free homology and $\mathrm{Tor}(\pi_* MU) = 0$, the Atiyah-Hirzebruch spectral sequence collapses and we get

$$MU_*(\partial M) \cong MU_*(B(\overline{p})) \oplus MU_*(\partial M, B(\overline{p}))$$

and

$$MU^*(\partial M) \cong MU^*(B(\overline{p})) \oplus MU^*(\partial M, B(\overline{p}))$$

(but the isomorphisms are not canonical and it is not clear if they are natural). All of the above groups are torsion-free, finitely generated abelian groups. Inserting this in the long exact sequence of the pair $(\partial M, B(\overline{p}))$, we get by a comparison of the ranks

$$\begin{tikzcd}
0 \arrow[r] & MU_*(B(\overline{p})) \arrow[r, "i_*"] \arrow[d, "\mu_*"'] & MU_*(\partial M) \arrow[r, "j_*"] \arrow[d, "\mu_*"] & MU_*(\partial M, B(\overline{p})) \arrow[r] \arrow[d, "\mu_*"] & 0 \\
0 \arrow[r] & K_*(B(\overline{p})) \arrow[r, "f_*"] & K_*(\partial M) \arrow[r, "g_*"] & K_*(\partial M, B(\overline{p})) \arrow[r] & 0
\end{tikzcd} \tag{2.11}$$

and

$$\begin{tikzcd}
0 \arrow[r] & MU^*(\partial M, B(\overline{p})) \arrow[r, "j^*"] \arrow[d, "\mu_*"'] & MU^*(\partial M) \arrow[r, "i^*"] \arrow[d, "\mu_*"] & MU^*(B(\overline{p})) \arrow[r] \arrow[d, "\mu_*"] & 0 \\
0 \arrow[r] & K^*(\partial M, B(\overline{p})) \arrow[r, "g^*"] & K^*(\partial M) \arrow[r, "f^*"] & K^*(B(\overline{p})) \arrow[r] & 0.
\end{tikzcd} \tag{2.12}$$

Indeed,

$$MU_*(\partial M) \cong \operatorname{Ker} j_* \oplus \operatorname{im} j_* = \operatorname{im} i_* \oplus \operatorname{im} j_*$$

so

$$\begin{aligned}
\operatorname{rank} MU_*(L) &= \operatorname{rank} \operatorname{im} i_* + \operatorname{rank} \operatorname{im} j_* \\
&\leq \operatorname{rank} MU_*(B(\overline{p})) + \operatorname{rank} MU_*(\partial M, B(\overline{p})) = \operatorname{rank} MU_*(L).
\end{aligned}$$

Therefore $\operatorname{im} i_* \cong MU_*(B(\overline{p}))$ and $\operatorname{im} j_* \cong \operatorname{im} MU_*(\partial M, B(\overline{p}))$. Equivalently for cohomology. Furthermore

$$\begin{tikzcd}
MU^r(\partial M) \arrow[r, "-\cap_{MU}[\partial M]_{MU}"] \arrow[d, "\mu_*"'] & MU_{n-r-1}(\partial M) \arrow[d, "\mu_*"] \\
K^r(\partial M) \arrow[r, "-\cap_K[\partial M]_K"] & K_{n-r-1}(\partial M)
\end{tikzcd}$$

commutes as μ is a map of ring spectra. As in the case of complex K-theory, this leads to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & MU^r(\partial M, B(\overline{p})) \arrow[r, "j^*"] \arrow[d, "D_{MU}"'] & MU^r(\partial M) \arrow[r, "i^*"] \arrow[d, "-\cap_{MU}[\partial M]_{MU}"] & MU^r(B(\overline{p})) \arrow[r] \arrow[d, "E_{MU}"] & 0 \\
0 \arrow[r] & MU_{n-r}(B(\overline{q})) \arrow[r, "i_*"] & MU_{n-r}(\partial M) \arrow[r, "j_*"] & MU_{n-r}(\partial M, B(\overline{q})) \arrow[r] & 0
\end{tikzcd}$$

as the following lemma shows. Define

$$\begin{aligned}
V_r^{MU} &= \operatorname{im} i_* \\
W_r^{MU} &= j_*^{-1}(MU_r(\partial M, B(\overline{q}))) \\
V_{MU}^r &= (i^*)^{-1}(MU^r(B(\overline{p}))) \\
W_{MU}^r &= \operatorname{im} j^*.
\end{aligned}$$

Lemma 2.3.2. *The duality map* $MU^r(\partial M) \overset{\cap_{MU}[\partial M]_{MU}}{\to} MU_{n-r-1}(\partial M)$ *factors as*

$$\begin{tikzcd}
MU^r(\partial M) \arrow[d, "\cap_{MU}[\partial M]_{MU}"'] & = & V_{MU}^r \arrow[drr] & \oplus & W_{MU}^r \arrow[dll] \\
MU_{n-r-1}(\partial M) & = & V_{n-r-1}^{MU} & \oplus & W_{n-r-1}^{MU}.
\end{tikzcd}$$

Proof. Use the notation of Lemma 2.2.11. Assume $V^r_{MU} \cap_{MU} [\partial M]_{MU} \not\subseteq W^{MU}_{n-r-1}$. Then there is an $x \in V^r_{MU}$ with $x \cap_{MU} [\partial M]_{MU} \in V^{MU}_{n-r-1}$. Diagrams (2.11) and (2.12) show that

$$\begin{aligned}
\mu(V^r_{MU}) &= V^r \\
\mu(W^r_{MU}) &= W^r \\
\mu(V^{MU}_{n-r-1}) &= V_{n-r-1} \\
\mu(W^{MU}_{n-r-1}) &= W_{n-r-1}.
\end{aligned}$$

It follows that $\mu x \cap_{MU} [\partial M]_{MU} \in V_{n-r-1}$ and therefore $V^r \cap_{KU} [\partial M]_{KU} \not\subseteq W_{n-r-1}$. But this is a contradiction to Lemma 2.2.11. $\square$

Therefore the map $- \cap_{MU} [\partial M]_{MU}$ can be restricted to

$$\begin{tikzcd}
MU^{r-1}(\partial M, B(\overline{p})) \arrow[r] \arrow[d, "-\cap_{MU}[\partial M]_{MU}|"'] & MU^{r-1}(\partial M) \arrow[d, "-\cap_{MU}[\partial M]_{MU}"] \\
MU_{n-r}(B(\overline{p})) \arrow[r] & MU_{n-r}(\partial M).
\end{tikzcd}$$

We can then proceed as in the case of complex K-theory and obtain a duality isomorphism.

Theorem 2.3.3. *Let X^n be a compact, MU-oriented pseudomanifold with $(X, B(\overline{p}))$ and $(X, B(\overline{q}))$ in Θ for complementary perversities $\overline{p}$ and $\overline{q}$. Assume that there is a Poincaré duality isomorphism in ordinary homology*

$$\tilde{H}^{n-r}(I^{B(\overline{p})}X; \mathbb{Q}) \to \tilde{H}_r(I^{B(\overline{q})}X; \mathbb{Q}).$$

Then there is an isomorphism

$$D_{MU} : \tilde{MU}^{n-r}(I^{\overline{p}}X; \mathbb{Z}) \to \tilde{MU}_r(I^{\overline{q}}X; \mathbb{Z}).$$

MU-Orientation For example, a complex vector bundle is MU-orientable. Thus, if M has a tangent bundle that is a complex vector bundle, then M is MU-orientable.

Example 2.3.4. *Let X be a stratified pseudomanifold with two strata and a trivial link bundle, such that the manifold M obtained by cutting off small conical neighborhoods of the singular set is a complex manifold. Then X is MU-orientable.*

The Relation to complex K-theory Here we want to relate the duality isomorphism in complex bordism to the duality isomorphism in complex K-theory.

Lemma 2.3.5. *The map of ring spectra*

$$\mu : MU \to KU$$

induces the following commutative diagram

$$\begin{tikzcd}
\tilde{MU}^{n-r}(I^{\overline{q}}X) \arrow[r, "\mu_*"] \arrow[d, "D_{MU}"'] & \tilde{K}^{n-r}(I^{\overline{q}}X) \arrow[r, "ch_*"] \arrow[d, "D_K"] & \tilde{H}^{n-r}(I^{\overline{q}}X;G) \arrow[d, "D"] \\
\tilde{MU}_r(I^{\overline{p}}X) \arrow[r, "\mu_*"] & \tilde{K}_r(I^{\overline{p}}X) \arrow[r, "ch_*"] & \tilde{H}_r(I^{\overline{p}}X;G)
\end{tikzcd}$$

where

$$D_K : \tilde{K}^{n-r}(I^{\overline{p}}X;\mathbb{Z}) \to \tilde{K}_r(I^{\overline{q}}X;\mathbb{Z})$$

and

$$D : \tilde{H}^{n-r}(I^{\overline{p}}X;\mathbb{Z}) \to \tilde{H}_r(I^{\overline{q}}X;\mathbb{Z})$$

are the generalized Poincaré duality isomorphisms.

Proof. As μ is a map of ring spectra, it induces the following commutative diagram

$$\begin{tikzcd}
MU^r(X) \arrow[r] \arrow[d, "\mu_*"'] & MU^r(Y) \arrow[r] \arrow[d, "\mu_*"] & MU^r(X,Y) \arrow[d, "\mu_*"] \\
K^r(X) \arrow[r] & K^r(Y) \arrow[r] & K^r(X,Y).
\end{tikzcd} \tag{2.13}$$

The induced map μ_* on homology and cohomology is natural with respect to the generalized Poincaré duality maps of the manifolds M and ∂M. The commutativity of Diagram (2.13) and of the following diagram

$$\begin{tikzcd}
MU^{n-r}(M,\partial M) \arrow[rr, "\mu_*"] \arrow[dr, "D_{MU}"'] \arrow[dd] & & K^{n-r}(M,\partial M) \arrow[dr, "D_K"] \arrow[dd] & \\
& MU_r(M) \arrow[rr, "\mu_*"] \arrow[dd] & & K_r(M) \arrow[dd] \\
\tilde{MU}^{n-r}(I^{\overline{q}}X) \arrow[rr, "\mu_*"] \arrow[dr, "D_{MU}"'] \arrow[dd] & & \tilde{K}^{n-r}(I^{\overline{q}}X) \arrow[dr, "D_K"] \arrow[dd] & \\
& \tilde{MU}_r(I^{\overline{p}}X) \arrow[rr, "\mu_*"] \arrow[dd] & & \tilde{K}_r(I^{\overline{p}}X) \arrow[dd] \\
MU^{n-r}(\partial M, B(\overline{p})) \arrow[rr, "\mu_*"] \arrow[dr, "D_{MU}"'] & & K^{n-r}(\partial M, B(\overline{p})) \arrow[dr, "D_K"] & \\
& MU_{i-1}(B(\overline{p})) \arrow[rr, "\mu_*"] & & K_{i-1}(B(\overline{p}))
\end{tikzcd}$$

then shows that the left part of the following diagram commutes.

$$\begin{tikzcd}
\tilde{MU}^{n-r}(I^{\overline{q}}X) \arrow[r, "\mu_*"] \arrow[d, "D_{MU}"'] & \tilde{K}^{n-r}(I^{\overline{q}}X) \arrow[r, "ch_*"] \arrow[d, "D_K"] & \tilde{H}^{n-r}(I^{\overline{q}}X;G) \arrow[d, "D"] \\
\tilde{MU}_r(I^{\overline{p}}X) \arrow[r, "\mu_*"] & \tilde{K}_r(I^{\overline{p}}X) \arrow[r, "ch_*"] & \tilde{H}_r(I^{\overline{p}}X;G).
\end{tikzcd}$$

Commutativity of the right part follows analogously.

□

3 Truncation in Generalized Homology Theories

Up to now, we have looked at generalized homology groups of intersection spaces $I^{\bar{p}}X$. It is natural to ask, if we can perform the whole truncation process with respect to a generalized homology theory. More precisely, for a given space X and a spectrum E, is there a space $X^E_{<n}$ and a map $f : X^E_{<n} \to X$ so that f induces isomorphisms $E_r(X^E_{<n}) \cong E_r(X)$ for $r < n$ and $E_r(X^E_{<n}) = 0$ else?

3.1 Construction

We suggest an answer to this question by constructing CW-spectra, not CW-spaces, for which a notion of truncation with respect to a connective ring spectrum E can be defined. The aim of this chapter is to construct such a truncation assignment with respect to a connective ring spectrum E and show Poincaré Duality. To this end, we construct a spectrum $XI^{\bar{p}}_E$ for a given stratified pseudomanifold with isolated singularities X, such that the homology of $XI^{\bar{p}}_E$ can be seen as intersection homology with rational coefficients with respect to the spectrum E. We then take the homology of this spectrum $XI^{\bar{p}}_E$. This construction assigns to a pseudomanifold with isolated singularities a $\mathbb{Q}$-vector spaces. We show that the graded $\mathbb{Q}$-vector spaces $E_{*+1}(XI^{\bar{p}}_H)$ and $H_{*+1}(XI^{\bar{p}}_E)$ both possess Poincaré duality and are compatible with the rational generalized intersection homology as defined in [Ban10b]. We use the following notation.

$\mathscr{S}$	The category of spectra.
$h\mathscr{S}$	The homotopy category of spectra.
$\overline{h}\mathscr{S}$	The stable homotopy category of spectra, which is obtained from $h\mathscr{S}$ by formally inverting the weak equivalences.
$hCW\mathscr{S}$	The homotopy category of CW-spectra (for example as [Ada74] defines it).

Recall the following fact.

Theorem 3.1.1 (Approximation by CW-spectra)**.** *There is an equivalence of categories* $\overline{h}\mathscr{S} \cong hCW\mathscr{S}$.

Sketch of Proof. For a proof see for example Theorem 1.5 of [EKMM95]. CW-approximation assigns to a spectrum E a CW spectrum ΓE and a weak equivalence $\gamma : \Gamma E \to E$. On the homotopy category $h\mathscr{S}$, Γ is a functor such that γ is natural. This induces an equivalence of categories $\overline{h}\mathscr{S} \cong hCW\mathscr{S}$. $\square$

Let $S\mathbb{Q}$ denote the Moore spectrum for $\mathbb{Q}$ and $H\mathbb{Q}$ the Eilenberg-MacLane spectrum for $\mathbb{Q}$. There is a truncation functor

$$t_{<k} : h\mathscr{S} \to h\mathscr{S}$$

that gives rise to a Postnikov-decomposition of spectra. It assigns to a connective ring spectrum E a truncated spectrum $t_{<k}(E)$, together with a natural map

$$E \to t_{<k}(E)$$

that induces

$$\pi_r(t_{<k}(E)) \cong \begin{cases} \pi_r(E) & r < k \\ 0 & \text{else.} \end{cases}$$

In every component n, $t_{<k}(E_n)$ is the usual Postnikov section functor for spaces (for a discussion of the Postnikov tower of connective ring spectra see [DS06]). Obviously $t_{<k}$ induces a functor on the stable homotopy category of spectra

$$t_{<k} : \overline{h}\mathscr{S} \to \overline{h}\mathscr{S}.$$

We define the following objects in $\mathrm{Ob}hCW\mathscr{S}$.

$$M_E := E \wedge \Sigma^\infty M \wedge H\mathbb{Q}$$

and

$$L_E := E \wedge \Sigma^\infty L \wedge H\mathbb{Q}.$$

The smash product is the one of the category $hCW\mathscr{S}$. Furthermore, using the equivalence of categories $\overline{h}\mathscr{S} \cong hCW\mathscr{S}$ we can define a spectrum

$$L_{k,E} := t_{<k} L_E$$

in $hCW\mathscr{S}$ and a morphism

$$f_k : L_E \to L_{k,E}$$

with $f_k \in \mathrm{Mor}hCW\mathscr{S}$, and the smash product $\wedge$ is the smash product in $hCW\mathscr{S}$ as defined in [Ada74]. We form the mapping cone spectrum $cone(f_k)$ (The construction of the mapping cone commutes with the suspension, so that there is no problem in taking the mapping cone of a map of spectra. Compare also [Ada74], Part 3, Section 6). The stable homotopy groups of $cone(f_k)$ can be calculated from the long excact sequence of the pair $(L_E, L_{k,E})$ as

$$\pi_r(cone(f_k)) \cong \begin{cases} 0 & r \leq n \\ \pi_{r-1}(L_E) & r > n. \end{cases}$$

The map $j : L \to M$ induces a map $\Sigma^\infty(j) : \Sigma^\infty L \to \Sigma^\infty M$ and finally a map $j_E : L_E \to M_E$. Then the following triangle commutes

$$\begin{tikzcd} cone(f_k) \arrow[r, "q_E"] \arrow[dr, "g_k"'] & S \wedge L_E \arrow[d, "id \wedge j_E"] \\ & S \wedge M_E. \end{tikzcd} \tag{3.1}$$

As for spaces, there is a map

$$g_k : cone(f_k) \xrightarrow{q_E} S \wedge L_E \xrightarrow{id \wedge j_E} S \wedge M_E$$

(Compare [Ada74], Part 3, Section 6, Page 209).

Definition 3.1.2. Let X^n be a compact, oriented, stratified pseudomanifold with only isolated singularities and $k = n - 1 - \overline{p}(n)$. Let E be a connective ring spectrum. The *rational intersection homology spectrum of X with respect to E* is an object in $hCW\mathscr{S}$ defined as

$$XI_E^{\overline{p}} := cone(g_k).$$

Remark 3.1.3. *As every spectrum E is stable homotopy equivalent to a product of Eilenberg-MacLane spectra when tensored with the rationals, it suffices to assume ordinary orientability of a manifold M in order to get $E\mathbb{Q}$ orientability of M.*

Compare also Lemma 2.2.3. We will constantly make use of the fact that there is an equivalence of spectra between $S\mathbb{Q}$ and $H\mathbb{Q}$ in $hCW\mathscr{S}$ (see for example [Ada74], Part 3, Section 6). We first need the following Lemma.

Lemma 3.1.4. *Let E be a connective ring spectrum. Then $XI_E^{\overline{p}}$ is a bounded below spectrum.*

Proof. The calculation

$$\pi_{r+1}(cone(f_{n-k})) \cong \begin{cases} 0 & r < n-k \\ \pi_r(L_E) \cong E_r(L;\mathbb{Q}) & r \geq n-k \end{cases} \tag{3.2}$$

shows that $cone(f_k)$ is a connective spectrum, as E is connective. As $\pi_r(S \wedge M_E) \cong E_{r-1}(M;\mathbb{Q})$, exactness of the long exact sequence of the pair $(S \wedge M_E, cone(f_k))$ then shows, that $XI_E^{\overline{p}}$ is a bounded below spectrum. $\square$

Lemma 3.1.5. *For any homology theory F and connective ring spectrum E, we have*

$$F_r(XI_E^{\overline{p}}) \cong \bigoplus_{i+j=r} H_i(XI_E^{\overline{p}}, \pi_j(F) \otimes \mathbb{Q}).$$

Proof. By Lemma 3.1.4, the spectrum $XI_E^{\overline{p}}$ is bounded below. Therefore the Atiyah-Hirzebruch spectral sequence of $XI_E^{\overline{p}}$ converges. As the differentials are torsion (see [Arl92]) and the coefficients are rational vector spaces, it follows that the spectral sequence collapses at the E_2-page. As the E_2-terms are rational vector spaces, the recovery problem can be solved. This calculates $F\mathbb{Q}_r(XI^{\overline{p}}_E) \cong F_r(XI^{\overline{p}}_E) \otimes \mathbb{Q}$ (using the universal coefficient theorem Proposition 6.6 of [Ada74], Part III). Now $F_r(XI^{\overline{p}}_E)$ is already a rational vector space, as

$$\begin{aligned}\tilde{F}_r(XI^{\overline{p}}_E) &= \tilde{F}_r(cone(g_k))\\ &\cong F_{r-1}(L_E, M_E)\\ &\cong F_{r-1}(E \wedge \Sigma^\infty L, E \wedge \Sigma^\infty M) \otimes \mathbb{Q}.\end{aligned}$$

Then the claim follows. □

3.2 Duality

The main theorem of this section is

Theorem 3.2.1. *Let X be a compact, oriented, PL-stratified pseudomanifold with only isolated singularities. Let E be a connective ring CW-spectrum and let $XI^{\overline{p}}_E$ be the object in $hCW\mathscr{S}$ defined in Definition 3.1.2. Let F be a ring spectrum. Then for all $r \in \mathbb{N}$*

1. *a) $\tilde{F}^{n-r+1}(XI^{\overline{p}}_H) \cong \tilde{F}_{r+1}(XI^{\overline{q}}_H)$,*
   b) $\tilde{H}^{n-r+1}(XI^{\overline{p}}_E) \cong \tilde{H}_{r+1}(XI^{\overline{q}}_E)$ and
2. *$\tilde{F}_{r+1}(XI^{\overline{p}}_H) \cong IF^{\overline{p}}_r(X;\mathbb{Q})$.*

where the generalized intersection homology group $IF^{\overline{p}}_k(X;\mathbb{Q})$ is defined as in [Ban10b].

Corollary 3.2.2. *In particular $\tilde{H}_{r+1}(XI^{\overline{p}}_H) \cong IH^{\overline{p}}_r(X;\mathbb{Q})$.*

Proof. 1. (a) follows from point 2 and [Ban10a]. We proof (b):

Let us first introduce some notation. The cofiber sequence of the triple $(S \wedge M_E, S \wedge M_L, cone(f_k))$ induces the long exact sequence

$$\ldots \to \pi_r(S \wedge L_E, cone(f_k)) \overset{a_{E*}}{\to} \pi_r(S \wedge M_E, cone(f_k)) \overset{b_{E*}}{\to} \pi_r(S \wedge M_E, S \wedge L_E) \overset{\delta_{E*}}{\to} \ldots.$$

Similarly, there is the following long exact sequence

$$\ldots \to \pi_r(S \wedge L_E) \overset{q_{E*}}{\to} \pi_r(S \wedge M_E) \overset{id \wedge j_{E*}}{\to} \pi_r(cone(f_k)) \overset{\partial_{E*}}{\to} \ldots.$$

The occurring terms are all rational vector spaces, since

$$
\begin{aligned}
\pi_r(S \wedge L_E) &\cong \pi_{r+1}(L_E) \\
&\cong \pi_{r+1}((E \wedge \Sigma^\infty L)\mathbb{Q}) \\
&\cong \pi_{r+1}(E \wedge \Sigma^\infty L) \otimes \mathbb{Q} \oplus Tor_1(\pi_r(E \wedge \Sigma^\infty L); \mathbb{Q}) \\
&\cong \pi_{r+1}(E \wedge \Sigma^\infty L) \otimes \mathbb{Q} \cong E_{r+1}(L) \otimes \mathbb{Q}
\end{aligned}
$$

using the universal coefficient theorem [Ada74], Part 3, Theorem 6.6 (for the other terms in the above sequences similar). For a $\mathbb{Q}$-vector space V, let V^* denote the dual space of V. We want to construct a commutative diagram

$$
\begin{tikzcd}
(\pi_{n-r}(S \wedge L_E, cone(f_k)))^* \arrow[r, "\delta_E^*"] \arrow[d, "\cong"'] & (\pi_{n-r+1}(S \wedge M_E, S \wedge L_E))^* \arrow[d, "\cong"] \\
\pi_{r+1}(cone(f_{n-k})) \arrow[r, "g_{n-k*}"] & \pi_{r+1}(S \wedge M_E).
\end{tikzcd}
\tag{3.3}
$$

As

$$
\begin{aligned}
\pi_{n-r}(S \wedge L_E, cone(f_k)) &\cong \begin{cases} \pi_{n-r}(S \wedge L_E) & n-r \leq k \\ 0 & n-r > k \end{cases} \\
&= \begin{cases} \pi_{n-r}(S \wedge L_E) & r \geq n-k \\ 0 & r < n-k \end{cases},
\end{aligned}
\tag{3.4}
$$

Diagram (3.3) commutes trivially for $r < n-k$. Let now be $r \geq n-k$. In the following diagrams, all horizontal maps come from long exact sequences of pairs and triples. The diagram

$$
\begin{tikzcd}
(\pi_{n-r}(S \wedge L_E, cone(f_k)))^* \arrow[r, "\delta_E^*"] \arrow[d, "\cong"'] & (\pi_{n-r+1}(S \wedge M_E, S \wedge L_E))^* \arrow[d, equal] \\
(\pi_{n-r}(S \wedge L_E))^* \arrow[r, "\partial_E^*"] & (\pi_{n-r+1}(S \wedge M_E, S \wedge L_E))^*
\end{tikzcd}
\tag{3.5}
$$

commutes for $r \geq n-k$ as Diagram (3.1) induces a commutative diagram

$$
\begin{tikzcd}
\pi_{n-r+1}(S \wedge M_E, S \wedge L_E) \arrow[d, "\partial_{E*}"'] \arrow[dr, "\delta_{E*}"] & \\
\pi_{n-r}(S \wedge L_E) \arrow[r] & \pi_{n-r}(S \wedge L_E, cone f_k).
\end{tikzcd}
$$

The diagram

$$\begin{tikzcd}
(\pi_{n-r}(S \wedge L_E))^* \arrow[r] \arrow[d, "S(L_E)^*"', "\cong"] & (\pi_{n-r+1}(S \wedge M_E, S \wedge L_E))^* \arrow[d, "\cong"', "S(M_E,L_E)^*"] \\
(\pi_{n-r-1}(L_E))^* \arrow[r] & (\pi_{n-r}(M_E, L_E))^*
\end{tikzcd} \tag{3.6}$$

commutes, where $S(X) : \pi_r(X) \to \pi_{r+1}(SX)$ is induced by shifting the index of the spectra by 1 (equivalently for pairs of spectra).

$$\begin{tikzcd}
(\pi_{n-r-1}(L_E))^* \arrow[r] \arrow[d, equal] & (\pi_{n-r}(M_E, L_E))^* \arrow[d, equal] \\
(E\mathbb{Q}_{n-r-1}(L))^* \arrow[r] & (E\mathbb{Q}_{n-r}(M, L))^*
\end{tikzcd} \tag{3.7}$$

commutes by definition of E-homology. Furthermore

$$\begin{tikzcd}
(E\mathbb{Q}_{n-r-1}(L))^* \arrow[r] \arrow[d, "\cong"'] & (E\mathbb{Q}_{n-r}(M, L))^* \arrow[d, "\cong"] \\
E\mathbb{Q}^{n-r-1}(L) \arrow[r] & E\mathbb{Q}^{n-r}(M, L)
\end{tikzcd} \tag{3.8}$$

commutes as $E\mathbb{Q}$ is equivalent to a product of Eilenberg-Mac-Lane spectra in the homotopy category of spectra. The universal coefficient theorem then gives a natural isomorphism (it is natural as $\pi_*E \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space)

$$\begin{aligned}
(E\mathbb{Q}_{n-r-1}(L))^* &= \operatorname{Hom}(E\mathbb{Q}_{n-r-1}(L); \mathbb{Q}) \\
&\cong \operatorname{Hom}(H_{n-r-1}(L; \pi_*E \otimes \mathbb{Q}); \mathbb{Q}) \\
&\cong H^{n-r-1}(L; \pi_*E \otimes \mathbb{Q}) \\
&\cong E\mathbb{Q}^{n-r-1}(L).
\end{aligned}$$

The Diagram

$$\begin{tikzcd}
E\mathbb{Q}^{n-r-1}(L) \arrow[r] \arrow[d, "\cong"'] & E\mathbb{Q}^{n-r}(M, L) \arrow[d, "\cong"] \\
E\mathbb{Q}_r(L) \arrow[r, "j_*"] & E\mathbb{Q}_r(M)
\end{tikzcd} \tag{3.9}$$

commutes by Poincaré duality (L and M are $E\mathbb{Q}$ oriented, compare Remark 3.1.3). Then

$$\begin{tikzcd}
E\mathbb{Q}_r(L) \arrow[r, "j_*"] \arrow[d, "\cong"'] & E\mathbb{Q}_r(M) \arrow[d, "\cong"] \\
\pi_r(L_E) \arrow[r, "j_*"] & \pi_r(M_E)
\end{tikzcd} \tag{3.10}$$

commutes by the same line of arguments. Finally

$$\begin{tikzcd}
\pi_{r+1}(S \wedge L_E) \arrow[r, "id \wedge j_*"] \arrow[d, "\cong"'] & \pi_{r+1}(S \wedge M_E) \arrow[d, "\cong"] \\
\pi_{r+1}(cone(f_{n-k})) \arrow[r, "g_{n-k*}"] & \pi_{r+1}(S \wedge M_E)
\end{tikzcd} \tag{3.11}$$

commutes by definition of g_{n-k*} and the vertical maps are isomorphisms as $r \geq n-k$. Composing the diagrams (3.5) to (3.11), we get the desired commutative diagram for all values $r \in \mathbb{N}$.

$$\begin{tikzcd}
(\pi_{n-r}(S \wedge L_E, cone(f_k)))^* \arrow[r, "\delta_E^*"] \arrow[d, "\cong"'] & (\pi_{n-r+1}(S \wedge M_E, S \wedge L_E))^* \arrow[d, "\cong"] \\
\pi_{r+1}(cone(f_{n-k})) \arrow[r, "g_{n-k*}"] & \pi_{r+1}(S \wedge M_E).
\end{tikzcd}$$

Then apply Lemma 1.3.10 to the diagram

$$\begin{tikzcd}
(\pi_{n-r}(S \wedge L_E), cone(f_k))^* \arrow[r, "\cong"] \arrow[d, "\delta_E^*"'] & \pi_{r+1}(cone(f_{n-k})) \arrow[d, "g_{n-k*}"] \\
(\pi_{n-r+1}(S \wedge M_E), S \wedge L_E)^* \arrow[r, "\cong"] \arrow[d, "b_E^*"'] & \pi_{r+1}(S \wedge M_E) \arrow[d] \\
(\pi_{n-r+1}(S \wedge M_E, cone(f_k))^* \arrow[d, "a_E^*"'] & \pi_{r+1}(S \wedge M_E), cone(f_{n-k})) \arrow[d] \\
(\pi_{n-r+1}(S \wedge L_E), cone(f_k))^* \arrow[r, "\cong"] \arrow[d, "\delta_E^*"'] & \pi_r(cone(f_{n-k})) \arrow[d, "g_{n-k*}"] \\
(\pi_{n-r+2}(S \wedge M_E), S \wedge L_E)^* \arrow[r, "\cong"] & \pi_r(S \wedge M_E).
\end{tikzcd}$$

Using $\pi_* \otimes \mathbb{Q} \cong H_* \otimes \mathbb{Q}$, it follows that

$$
\begin{aligned}
H^{n-r-1}(S \wedge M_E, cone(f_k)) &\cong H^{n-r-1}(S \wedge M_E, cone(f_k)) \otimes \mathbb{Q} \\
&\cong (H_{n-r-1}(S \wedge M_E, cone(f_k)) \otimes \mathbb{Q})^* \\
&\cong (\pi_{n-r-1}(S \wedge M_E, cone(f_k)) \otimes \mathbb{Q})^* \\
&\cong (\pi_{n-r-1}(S \wedge M_E, cone(f_k))^* \\
&\cong \pi_{r+1}(S \wedge M_E, cone(f_{n-k})) \\
&\cong H_{r+1}(S \wedge M_E, cone(f_{n-k})).
\end{aligned}
$$

This implies the isomorphism

$$\tilde{H}^{n-r+1}(XI_E^{\overline{p}}) \cong \tilde{H}_{r+1}(XI_E^{\overline{q}}).$$

2. In [Ban10b] the generalized intersection homology groups of a pseudomanifold X is defined as $(F\mathbb{Q})_r^{H\mathbb{Q}}(\Phi IC_\bullet^{\overline{p}}(X;\mathbb{Q})) \cong F_r(\Phi IC_\bullet^{\overline{p}}(X;\mathbb{Q}))$. In Section 10 of *loc. cit.* the generalized rational intersection homology of a distinguished neighborhood is calculated. We use this calculation to first show, that our claim holds locally.

$$
\begin{aligned}
IF_{r+1}^{\overline{p}}(c^\circ L;\mathbb{Q}) &= (t_{\leq r+1+\overline{p}(n)-n}F)_r(L;\mathbb{Q}) \\
&= \bigoplus_{\substack{p+q=r \\ p\geq n-\overline{p}(n)-1}} H_p(L;\pi_q F\otimes\mathbb{Q}) \\
&= \bigoplus_{\substack{p+q=r \\ p\geq n-\overline{p}(n)-1}} \pi_p(H\wedge\Sigma^\infty L\wedge S\mathbb{Q})\otimes\pi_q F\otimes\mathbb{Q} \\
&= \bigoplus_{p+q=r} \pi_{p+1}(cone(f_k))\otimes\pi_q F\otimes\mathbb{Q} \\
&= \bigoplus_{p+q=r+1} H_p(cone(f_k);\pi_q F\otimes\mathbb{Q}) \\
&= F_{r+1}(cone(f_k));\mathbb{Q}).
\end{aligned}
$$

Here the Atiyah-Hirzebruch spectral sequence is used. Equation (3.2) shows that $cone(f_k)$ is a connective spectrum, so that the Atiyah-Hirzebruch spectral sequence of $cone(f_k)$ converges. Furtermore, all entries of the E_2-table of the spectral sequence are rational vector spaces, therefore, the Atiyah-Hirzebruch spectral sequence collapses at the E_2-table and we don't have any recovery problems. We now could use this local calculation and patch it together to a global statement. However, for technical reasons, it is better to first perform the truncation for $F = H$ and then to pass to any spectrum F by using the Atiyah-Hirzebruch spectral sequence.

Let $IC_\bullet^{\overline{p}}$ denote the simplicial intersection chain complex with closed support. The restriction map induces the following short exact sequence

$$0 \to IC^{\overline{p}}_{\bullet}(X - c^{\circ}L; \mathbb{Q}) \to IC^{\overline{p}}_{\bullet}(X; \mathbb{Q}) \to IC^{\overline{p}}_{\bullet}(c^{\circ}L; \mathbb{Q}) \to 0$$

that induces a long exact sequence

$$\ldots \to IF^{\overline{p}}_{r}(X - c^{\circ}L; \mathbb{Q}) \to IF^{\overline{p}}_{r}(X; \mathbb{Q}) \to IF^{\overline{p}}_{r}(c^{\circ}L; \mathbb{Q}) \to \ldots . \tag{3.12}$$

By definition of M,

$$IC^{\overline{p}}_{\bullet}(X - c^{\circ}L) \cong C_{\bullet}(M)$$

($C_{\bullet}(M)$ is the simplicial chain complex with closed support here). We want to construct isomorphisms α and β, such that the following diagram commutes.

$$\begin{tikzcd}
H_r(\Phi(IC^{\overline{p}}_{\bullet}(c^{\circ}L)); \mathbb{Q}) \arrow[r, "\iota"] \arrow[d, "\alpha"', "\cong"] & H_{r-1}(\Phi(IC^{\overline{p}}_{\bullet}(X - c^{\circ}L)); \mathbb{Q}) \arrow[d, "\cong"', "\beta"] \\
H_r(cone(f_k)) \arrow[r] & H_r(S \wedge M_H).
\end{tikzcd} \tag{3.13}$$

The map

$$\iota : H_r(\Phi(IC^{\overline{p}}_{\bullet}(c^{\circ}L)); \mathbb{Q}) \to H_{r-1}(\Phi(IC^{\overline{p}}_{\bullet}(X - c^{\circ}L)); \mathbb{Q})$$

is defined as follows. The boundary map of the long exact sequence of the pair $(X - c^{\circ}L, c^{\circ}L)$ in intersection homology

$$IH_r(c^{\circ}L) \to IH_{r-1}(X - c^{\circ}L)$$

induces by definition a map

$$\partial_* : H_r(IC^{\overline{p}}_{\bullet}(c^{\circ}L)) \to H_{r-1}(IC^{\overline{p}}_{\bullet}(X - c^{\circ}L)) = H_{r-1}(C_{\bullet}(M)).$$

The naturality of the functor Φ, then induces

$$\pi_r(\Phi(IC^{\overline{p}}_{\bullet}(c^{\circ}L)); \mathbb{Q}) \to \pi_{r-1}(\Phi(IC^{\overline{p}}_{\bullet}(X - c^{\circ}L)); \mathbb{Q})$$

and thus in turn

$$\iota : H_r(\Phi(IC^{\overline{p}}_{\bullet}(c^{\circ}L)); \mathbb{Q}) \to H_{r-1}(\Phi(IC^{\overline{p}}_{\bullet}(X - c^{\circ}L)); \mathbb{Q})$$

by the natural equivalence of $\pi_* \otimes \mathbb{Q} \cong H_* \otimes \mathbb{Q}$. Let $k = n - 1 - \overline{p}(n)$. The intersection homology of a cone can be calculated as follows (see for example [Ban07], Example 4.1.15).

$$IH^{\overline{p}}_{r}(c^{\circ}L) \cong \begin{cases} H_{r-1}(L) & r \geq k \\ 0 & r < k. \end{cases}$$

If $r < k$, then Diagram (3.13) clearly commutes, as $IH_r^{\overline{p}}(c^\circ L) = H_r(\Phi(IC_\bullet^{\overline{p}}(c^\circ L)); \mathbb{Q})$ and therefore $H_r(\Phi(IC_\bullet^{\overline{p}}(c^\circ L)); \mathbb{Q}) = 0$ in that case. Let now be $r \geq k$. By the above discussion, the following diagram commutes

$$\begin{tikzcd} H_r(\Phi(IC_\bullet^{\overline{p}}(c^\circ L)); \mathbb{Q}) \arrow[r, "\iota"] \arrow[d] & H_{r-1}(\Phi(C_\bullet(X - c^\circ L)); \mathbb{Q}) \arrow[d] \\ H_r((IC_\bullet^{\overline{p}}(c^\circ L)); \mathbb{Q}) \arrow[r, "\partial"] & H_{r-1}(C_\bullet(X - c^\circ L); \mathbb{Q}). \end{tikzcd}$$

Furthermore, in the diagram

$$\begin{tikzcd} H_r((IC_\bullet^{\overline{p}}(c^\circ L)); \mathbb{Q}) \arrow[r, "\delta"] & H_{r-1}(C_\bullet(X - c^\circ L); \mathbb{Q}) \arrow[d, "="] \\ H_{r-1}((C_\bullet(L)); \mathbb{Q}) \arrow[u, "\alpha_*"] \arrow[r, "incl_*"] & H_{r-1}(C_\bullet(M); \mathbb{Q}) \end{tikzcd}$$

the upper map is the boundary operator δ from the long exact sequence (3.12). The map $\alpha : H_{r-1}(C_\bullet(L)) \to H_r(IC_\bullet^{\overline{p}}(c^\circ L))$ is induced by taking the cone on a chain and is an isomorphism on homology, as $r \geq k$. Thus the diagram commutes.

In the following diagrams, all horizontal maps are induced by the inclusion $L \to M$. The following diagram commutes by definition of homology

$$\begin{tikzcd} H_{r-1}((C_\bullet(L)); \mathbb{Q}) \arrow[r] \arrow[d, "\cong"'] & H_{r-1}(C_\bullet(M); \mathbb{Q}) \arrow[d, "\cong"] \\ H_{r-1}(L; \mathbb{Q}) \arrow[r] & H_{r-1}(M; \mathbb{Q}). \end{tikzcd}$$

The next diagram commutes by definition of homology defined by spectra

$$\begin{tikzcd} H_{r-1}(L; \mathbb{Q}) \arrow[r] \arrow[d, "\cong"'] & H_{r-1}(M; \mathbb{Q}) \arrow[d, "\cong"] \\ \pi_{r-1}(H \wedge \Sigma^\infty L \wedge S\mathbb{Q}) \arrow[r] & \pi_{r-1}(H \wedge \Sigma^\infty M \wedge S\mathbb{Q}). \end{tikzcd}$$

The commutativity of the next diagram is a consequence of Diagram (3.1)

$$\begin{tikzcd} \pi_{r-1}(H \wedge \Sigma^\infty L \wedge H\mathbb{Q}) \arrow[r] \arrow[d, "\cong"'] & \pi_{r-1}(H \wedge \Sigma^\infty M \wedge H\mathbb{Q}) \arrow[d, "\cong"] \\ \pi_r(cone(f_k)) \arrow[r] & \pi_r(S \wedge M_H) \end{tikzcd}$$

and finally the following diagram commutes as $\pi_* \otimes \mathbb{Q} \cong H \otimes \mathbb{Q}$ and this isomorphism is natural.

$$\begin{tikzcd}
\pi_r(cone(f_k)) \arrow[r] \arrow[d, "\cong"'] & \pi_r(S \wedge M_H) \arrow[d, "\cong"] \\
H_r(cone(f_k)) \arrow[r] & H_r(S \wedge M_H).
\end{tikzcd}$$

Since

$$XI^{\overline{p}}_H := cone(cone(f_k) \to S \wedge M_H),$$

we conclude from Lemma 1.3.10, applied to

$$\begin{tikzcd}
H_{r+1}(\Phi(IC^{\overline{p}}_\bullet(c^\circ L));\mathbb{Q}) \arrow[r, "\alpha", "\cong"'] \arrow[d, "\iota"'] & H_{r+1}(cone(f_k)) \arrow[d] \\
H_r(\Phi(C_\bullet(X - c^\circ L));\mathbb{Q}) \arrow[r, "\beta", "\cong"'] \arrow[d] & H_{r+1}(S \wedge M_H) \arrow[d] \\
H_r(\Phi(C_\bullet(X));\mathbb{Q}) \arrow[d] & \tilde{H}_{r+1}(XI^{\overline{p}}_H) \arrow[d] \\
H_r(\Phi(IC^{\overline{p}}_\bullet(c^\circ L));\mathbb{Q}) \arrow[r, "\alpha", "\cong"'] \arrow[d, "\iota"'] & H_r(cone(f_k)) \arrow[d] \\
H_{r-1}(\Phi(C_\bullet(X - c^\circ L));\mathbb{Q}) \arrow[r, "\beta", "\cong"'] & H_r(S \wedge M_H)
\end{tikzcd}$$

that

$$\tilde{H}_{r+1}(XI^{\overline{p}}_H) \cong H_r(\Phi(IC^{\overline{p}}_\bullet(X));\mathbb{Q}) = IH^{\overline{p}}_r(X;\mathbb{Q}).$$

The last equation is by Proposition 3.1 of [Ban10b]. By Lemma 3.1.5

$$\begin{aligned}
F_{r+1}(XI^{\overline{p}}_H) &\cong \bigoplus_{p+q=r+1} H_p(XI^{\overline{p}}_H;\pi_q(F)\otimes\mathbb{Q}) \\
&\cong \bigoplus_{p+q=r+1} \tilde{H}_p(XI^{\overline{p}}_H;\pi_q(F)\otimes\mathbb{Q}) \oplus H_p(pt;\pi_q(F)\otimes\mathbb{Q}) \\
&\cong \bigoplus_{p+q=r+1} H_{p-1}(\Phi(IC^{\overline{p}}_\bullet(X));\pi_q(F)\otimes\mathbb{Q}) \oplus H_p(pt;\pi_q(F)\otimes\mathbb{Q}) \\
&\cong \bigoplus_{p+q=r} H_p(\Phi(IC^{\overline{p}}_\bullet(X));\pi_q(F)\otimes\mathbb{Q}) \oplus \bigoplus_{p+q=r+1} H_p(pt;\pi_q(F)\otimes\mathbb{Q}) \\
&\cong F_r(\Phi(IC^{\overline{p}}_\bullet(X));\mathbb{Q}) \oplus F\mathbb{Q}_{r+1}(pt)\otimes\mathbb{Q} = IF^{\overline{p}}_r(X;\mathbb{Q}) \oplus F_{r+1}(pt)\otimes\mathbb{Q}.
\end{aligned}$$

Thus, there is an isomorphism

$$\tilde{F}_{r+1}(XI_H^{\overline{p}}) \cong IF_r^{\overline{p}}(X;\mathbb{Q})$$

□

Corollary 3.2.3. *Let X be a compact, oriented pseudomanifold with only isolated singularities of dimension n. Let E be a connective ring CW-spectrum. Then for all $r \in \mathbb{N}$*

$$\tilde{K}^{n-r+1}(XI_E^{\overline{p}}) \cong \tilde{K}_{r+1}(XI_E^{\overline{q}}).$$

Proof. Let n be even. Use the isomorphism

$$\tilde{H}^{n-r+1}(XI_E^{\overline{p}}) \cong \tilde{H}_{r+1}(XI_E^{\overline{q}})$$

and the property that $\pi_{-q}(KU) = \pi_q(KU)$ and $\pi_{n-r+1}(KU) = \pi_{r+1}(KU)$.

$$\begin{aligned}
K^{n-r+1}(XI_E^{\overline{p}}) &\cong \bigoplus_{p+q=n-r+1} H^p(XI_E^{\overline{p}};\pi_q(KU)\otimes\mathbb{Q}) \\
&\cong \bigoplus_{p+q=n-r+1} \tilde{H}^p(XI_E^{\overline{p}};\pi_q(KU)\otimes\mathbb{Q}) \oplus H^p(pt;\pi_q(KU)\otimes\mathbb{Q}) \\
&\cong (\bigoplus_{p+q=n-r+1} \tilde{H}_{n-p+2}(XI_E^{\overline{q}};\pi_q(KU)\otimes\mathbb{Q})) \oplus \pi_{n-r+1}(KU)\otimes\mathbb{Q} \\
&\cong (\bigoplus_{p+q=r+1} \tilde{H}_p(XI_E^{\overline{q}};\pi_{-q}(KU)\otimes\mathbb{Q})) \oplus \pi_{n-r+1}(KU)\otimes\mathbb{Q} \\
&\cong (\bigoplus_{p+q=r+1} \tilde{H}_p(XI_E^{\overline{q}};\pi_q(KU)\otimes\mathbb{Q})) \oplus \pi_{r+1}(KU)\otimes\mathbb{Q} \\
&\cong \bigoplus_{p+q=r+1} \tilde{H}_p(XI_E^{\overline{q}};\pi_q(KU)\otimes\mathbb{Q}) \oplus H_p(pt;\pi_q(KU)\otimes\mathbb{Q}) \\
&\cong \bigoplus_{p+q=r+1} H_p(XI_E^{\overline{q}};\pi_q(KU)\otimes\mathbb{Q}) \cong K_{r+1}(XI_E^{\overline{q}}).
\end{aligned}$$

As n is even $K^{n-r+1}(pt) \cong K_{r+1}(pt)$ and this implies an isomorphism

$$\tilde{K}^{n-r+1}(XI_E^{\overline{p}}) \cong \tilde{K}_{r+1}(XI_E^{\overline{q}}).$$

Let now n be odd and r odd. Then $\pi_{n-r+1}(KU) = 0$.

$$
\begin{aligned}
K^{n-r+1}(XI_E^{\overline{p}}) &\cong \bigoplus_{p+q=n-r+1} H^p(XI_E^{\overline{p}}; \pi_q(KU)\otimes\mathbb{Q}) \\
&\cong \bigoplus_{p+q=n-r+1} \tilde{H}^p(XI_E^{\overline{p}}; \pi_q(KU)\otimes\mathbb{Q}) \oplus H^p(pt; \pi_q(KU)\otimes\mathbb{Q}) \\
&\cong \Big(\bigoplus_{p+q=n-r+1} \tilde{H}_{n-p+2}(XI_E^{\overline{q}}; \pi_q(KU)\otimes\mathbb{Q})\Big) \oplus \pi_{n-r+1}(KU)\otimes\mathbb{Q} \\
&\cong \bigoplus_{p+q=r+1} \tilde{H}_p(XI_E^{\overline{q}}; \pi_{-q}(KU)\otimes\mathbb{Q}) \\
&\cong \bigoplus_{p+q=r+1} \tilde{H}_p(XI_E^{\overline{q}}; \pi_q(KU)\otimes\mathbb{Q}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
K^{n-r+1}(XI_E^{\overline{p}}) \oplus K_{r+1}(pt;\mathbb{Q}) &\cong K^{n-r+1}(XI_E^{\overline{p}}) \oplus \pi_{r+1}(KU)\otimes\mathbb{Q} \\
&\cong \bigoplus_{p+q=r+1} H_p(XI_E^{\overline{q}}; \pi_q(KU)\otimes\mathbb{Q}) \\
&\cong K_{r+1}(XI_E^{\overline{q}})
\end{aligned}
$$

and this implies an isomorphism

$$\tilde{K}^{n-r+1}(XI_E^{\overline{p}}) \cong \tilde{K}_{r+1}(XI_E^{\overline{q}}).$$

Let finally be n odd and r even. Then $\pi_{r+1}(KU) = 0$ and a similar argument shows

$$
\begin{aligned}
K^{n-r+1}(XI_E^{\overline{p}}) &\cong K_{r+1}(XI_E^{\overline{q}}) \oplus \pi_{n-r+1}(KU)\otimes\mathbb{Q} \\
&\cong K_{r+1}(XI_E^{\overline{q}}) \oplus K^{n-r+1}(pt;\mathbb{Q})
\end{aligned}
$$

from where follows

$$\tilde{K}^{n-r+1}(XI_E^{\overline{p}}) \cong \tilde{K}_{r+1}(XI_E^{\overline{q}})$$

in that case, too.

□

A Appendix

For the convenience of the reader, we collect here some results that are cited throughout this thesis.

A.1 List of Selected Facts

Theorem A.1.1 (Theorem 4.4.17 of [Har08]). *Assume that X is paracompact, that Y is locally compact and Hausdorff and that*

$$f : X \to Y$$

is a proper map. Then for any sheaf $\mathcal{F}$ on X and $y \in Y$ we have

$$R^q f_*(\mathcal{F})_y = H^q(f^{-1}(y), i_y^*(\mathcal{F}))$$

where $i_y : f^{-1}(y) \hookrightarrow X$.

Theorem A.1.2 (Theorem II.11.1 of [Bre97]). *Let $f : X \to Y$ be a closed map, $\mathcal{A}$ a sheaf on X and Φ a family of supports on Y. Suppose that $H^p(f^{-1}(y); \mathcal{A}) = 0$ for all $p > 0$ and all $y \in Y$, and that each $f^{-1}(y)$ is taut in X. Then the natural map*

$$f^\dagger : H^*_\Phi(Y, f_*\mathcal{A}) \to H^*_{f^{-1}\Phi}(X; \mathcal{A})$$

induced by the f-cohomorphism $f : f_*\mathcal{A} \rightsquigarrow \mathcal{A}$, is an isomorphism.

Theorem A.1.3 (Theorem III.1.1 of [Bre97]). *There exists a natural multiplicative transformation of functors*

$$H^*_\Phi(X; \mathcal{A}) \to_\triangle H^*_\Phi(X; \mathcal{A})$$

which is an isomorphism if X is homologically locally connected and Φ is paracompactifying. In that case and if $\mathcal{A}$ is locally constant, the groups ${}_\triangle H^*_\Phi(X; \mathcal{A})$ are the classical singular cohomology groups.

Theorem A.1.4 (Theorem II.10.1 of [Bre97]). *Suppose that Φ is a paracompactifying family of supports on X and that $i : A \subset X$ is locally closed. Then there is a natural isomorphism*

$$H^*_\Phi(X; i_!\mathcal{A}) \cong H^*_{\Phi|A}(A; \mathcal{A})$$

Theorem A.1.5 (Corollary 2.3.4 of [Dim08]). *For any continous mapping $f : X \to Y$ and any sheaf complex $\mathcal{F}^\bullet \in D^+(X)$ there is a functorial isomorphism*

$$\mathcal{H}^\bullet(X, \mathcal{F}^\bullet) = \mathcal{H}^\bullet(Y, Rf_*\mathcal{F}^\bullet)$$

Notation

$L_{<k}$	k-th homology truncation of the CW-complex L. (Definition 1.1.1).
$ft_{<k}(\xi)$	k-th fiberwise truncation of the fiber bundle ξ. (Definition 1.1.3).
$\tau_{<k}(C^\bullet)$	k-th (good) truncation of the cochain complex $C^\bullet$.
$\mathbf{H}^r(\mathbf{A}^\bullet)$	r-th cohomology sheaf of the complex of sheaves $\mathbf{A}^\bullet$.
$\mathcal{H}^r(X, \mathbf{A}^\bullet)$	r-th hypercohomology group of the space X with coefficients in the complex of sheaves $\mathbf{A}^\bullet$.
$\mathscr{H}_*(N; \mathbb{Q})$	Group bundle defined by the action of the base space of a fiber bundle on the fiber N with coefficients in $\mathbb{Q}$.
$\Sigma^\infty X$	Suspension spectrum of the space X.
$t_{<k}(E)$	k-th Postnikov section of the spectrum E.

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