Risk Management in Banking

Credit Risk Management
and
Bank Closure Policies

Inauguraldissertation
zur Erlangung der Würde eines Doktors
der Wirtschaftswissenschaften
der Wirtschaftswissenschaftlichen Fakultät
der Universität Heidelberg

vorgelegt von
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aus Calw

Heidelberg, Juli 2001

Vorwort

Die vorliegende Arbeit entstand während meiner Tätigkeit als wissenschaftlicher Mitarbeiter am Lehrstuhl für Wirtschaftspolitik der Wirtschaftswissenschaftlichen Fakultät der Universität Heidelberg.

Mein besonderer Dank gilt Herrn Prof. Dr. Hans Gersbach, der mich während meiner gesamten Promotion in vorbildlicher Weise betreut hat. Er hat sich stets Zeit für meine Fragen genommen, mir in vielen Diskussionen wertvolle Denkanstöße gegeben und erheblich zur Vertiefung meiner ökonomischen Kenntnisse beigetragen. Herrn Prof. Dr. Till Requate danke ich für die Bereitschaft, das Koreferat zu übernehmen und Prof. Dr. Lutz Duembgen dafür, dass er stets geduldig bereit war, die Lücken in meiner statistischen Allgemeinbildung zu schließen. Meinen Kollegen am Lehrstuhl danke ich für das angenehme Arbeitsklima, die gute Zusammenarbeit und die wertvollen Hinweise, die sich in der Diskussion über Fragestellungen der Arbeit ergeben haben. Besonderer Dank gilt Hans-Jörg Beilharz, Thomas Fischer, Volker Hahn, Katja Hoffmann und Lars Siemers dafür, dass sie bereit waren große Teile der Arbeit zu lesen. Ihre zahlreichen Verbesserungsvorschläge sind in die Arbeit eingeflossen.

Schließlich möchte ich meinen Eltern dafür danken, dass sie mir den Weg zum Studium und zur Promotion durch ihre Unterstützung geebnet haben.

Ulrich Erlenmaier

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Chapter 1

Introduction

The last three decades have seen a substantial increase in the risks of the banking business. According to HELLWIG (1995, 1997), this trend was mainly driven by two factors. First, intensified competition in banking and finance. This in turn was set off by the global deregulation of the industry and by new communication and information technologies. Deregulation included the abolition of deposit rate regulations, and removals of capital controls and other impediments to international finance, while the new technologies led to a reduction of barriers to competition based on spatial distance and (or) national borders. As a consequence, competition among banks intensified and nonbank intermediaries entered the market (e.g. money market funds). Second, fluctuations in nominal interest rates have become much more pronounced and - due to the abolition of the Bretton Woods System - exchange rate risk began to play a major role. In summary, reduced intermediation margins and higher volatility of important aggregates has left the banking system more exposed to macroeconomic shocks.

As a reaction to this increase in riskiness and competitiveness, banks have started to view risk management as a key issue of their business, and regulators have become increasingly concerned about the risk allocation brought about by the banking system. An important focus was (and still is) the development of quantitative risk management systems in order to bring more objectivity and accuracy to the risk assessments that form the basis on which decisions are made (see e.g. THE ECONOMIST (1993)). Methods for the measurement and management of *market* risk (e.g. exchange rate risk, interest rate risk or equity risk) have evolved quite rapidly since the early eighties.[1] Moreover, during the nineties, quite a few market risk management models have been

[1] A leading example is RiskMetrics, a market risk management framework developed by JP Morgan (see JP MORGAN (1995)).

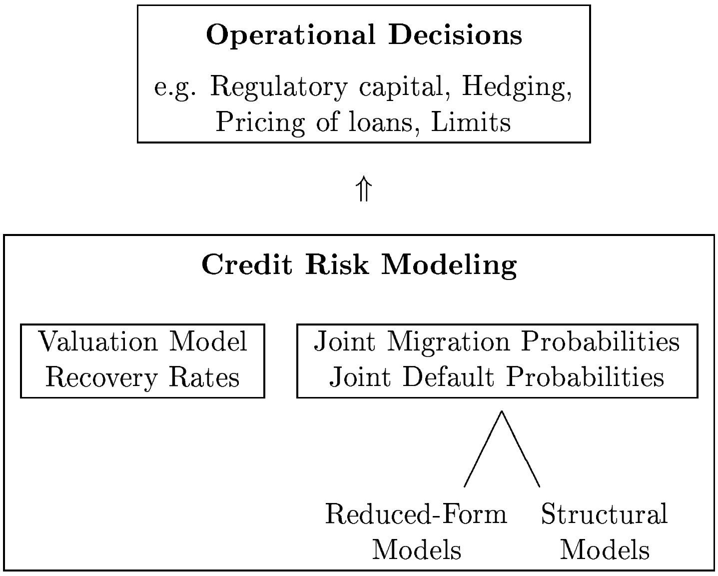

Figure 1.1: Credit Risk Management

accepted by banking regulators as methods of determining an institution's capital adequacy. More recently, banks have also started to address an even more important source of their risk exposure, *credit* risk. The first generation of industry-sponsored credit risk models came to the market in the early nineties. Although they cannot yet be used directly to determine the regulatory credit risk capital of a bank, some of their features have already found their way into the proposals for the New Basel Capital Accord, which is expected to be implemented in the next years.[2]

However, risk measurement and management methods are still at an early stage and quite far from providing exact pictures of a bank's actual risk exposure. This is particularly true for credit risk models, which have been developed and applied only recently. Regulators and academics alike have pointed out that the existing methodologies have to be improved before they can be used to determine a bank's regulatory capital.

But even if risk measurement and controlling (*ex-ante* risk management) becomes more and more sophisticated, the management of bank failures or even system-wide crises (*ex-post* risk management) will remain an important measure in dealing with the increased riskiness in banking. A case in point is the reoccurrence of system-wide banking crises during the eighties and nineties (for example the American savings and loans crisis, the Scandinavian banking crisis or, more recently, the crisis in Asia).

This thesis intends to contribute to both areas ex-ante and ex-post risk management by

[2]See Basel Committee on Banking Supervision (2001).

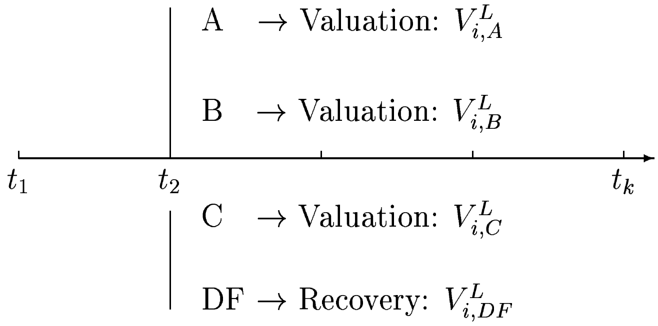

Figure 1.2: Valuation of a loan for different rating scenarios in $t = t_2$.

presenting three theoretically oriented papers, one dealing with optimal bank closure policies in the event of system-wide banking crises, and the other two with ex-ante credit risk management.

1.1 Credit Risk Management

In chapters 2 and 3 we present the two papers on credit risk management. To put the contributions of these chapters into perspective, we first give a brief overview over the most important credit risk management issues. After that we present the papers and show how they are related.

Credit Risk Management Issues

Figure 1.1 summarizes the major issues of credit risk management. While the upper part of figure 1.1 is readily understood, the part on credit risk modeling requires some further detailing.

To assess its credit risk exposure, a bank typically analyzes the possible realizations of the value of its loan portfolio at some point in the future, say in $t = t_2$. The credit risk model specifies the *probability distribution* of these possible realizations. This is done in two steps in the currently proposed models. First, as illustrated in figure 1.2, the $t = t_2$ value of each loan i ($i = 1, .., n$) in the portfolio is determined, given that the firm that has obtained the loan (firm i) is in a certain rating class in t_2. Figure 1.2 contains a stylized rating system consisting of three rating classes A, B and C, and the default case DF. Consequently, $V^L_{i,A}$ denotes the $t = t_2$ value of loan i if firm i has an A-rating in t_2, $V^L_{i,B}$ is the corresponding loan value if the firm has a B-rating and so on.[3] In addition, the amount $V^L_{i,\mathrm{DF}}$ that can be recovered if firm i defaults in t_2 has to be specified. Typically, $V^L_{i,\mathrm{DF}}$ is expressed as a fraction (*recovery rate*) of the overall loan value. Finally, note that the time index t_1 denotes the point in time at which the $t = t_2$ portfolio distribution is determined, and t_2, .., t_k are the points at which interest and (or) principal payments of loan i are due.

In a second step, the joint probabilities that the firms in the portfolio will belong to certain rating classes in t_2 have to be determined, i.e. probabilities of the type

$$\mathrm{Prob}\Big\{\text{Firm } i \text{ has rating } \zeta_i \text{ in } t_2,\ \mathrm{i} = 1, .., n\Big\}.$$

These probabilities are termed *joint migration probabilities.*[4] In the most simple setting, a bank will only consider two possible states for a loan in $t = t_2$, default or non-default. In this case, joint migration modeling is reduced to the modeling of *joint default probabilities.* Both chapters 2 and 3 will deal primarily with joint default probabilities. However, most results generalize to joint migration probabilities in a quite straightforward manner.

Chapters 2 and 3 - Overview

In chapter 2, which is based on ERLENMAIER (2001), we review the models of joint defaults of the major currently available industry-sponsored credit risk frameworks. The main aspects of chapter 2 are the following:

- A detailed description of the different models within a unified framework.
- An overview over of the most important modeling drawbacks.
- Suggestions for the improvement of the existing methodologies.
- The formulation of a research agenda for the development of next-generation models.

[3] The ratings may be taken from public rating agencies such as Moody's or Standard & Poor's or from the proprietary rating system of the particular bank or the company that offers the credit risk model employed by the bank.

[4] The term „migration" refers to the fact that the above probabilities specify the likelihoods of firms migrating from their current ($t = t_1$) ratings to some other ratings (or to default) in $t = t_2$.

While a general survey on credit risk models has recently been provided by CROUHY, GALAI, AND MARK (2000), the overview in chapter 2 differs from the paper of CROUHY ET AL. in three respects. First, by focusing on joint default modeling, our description is more detailed and presents some important modeling features which are not described in CROUHY ET AL. Second, our survey makes the comparison of the models easier and more transparent by presenting the different methodologies in a unified notational framework. Finally, while reviewing the literature, we found two interesting new versions of current credit risk models which - in our view - represent important alternatives to the initial proposals and are therefore discussed as well.

After describing the different joint default models, we identify the most important modeling problems and make suggestions on how the models' performance could be improved. We then set out to assess which of the currently proposed methods for joint default modeling is the most promising basis for next-generation models. Based on this assessment we propose a research agenda for the development of such models.

In the second paper - presented in chapter 3 and partly based on ERLENMAIER AND GERSBACH (2001a) - we derive a theoretical result on the relationship between univariate default probabilities and loan correlations (which are termed *default correlations* in the literature since the probabilities of joint defaults are the major building block of loan correlations).[5] We find that loans with higher default probabilities will not only have higher variances (variance effect) but also higher correlations with other loans (correlation effect). Using numerical examples, we demonstrate that - due to these effects - a portfolio's standard deviation can increase substantially when loan default probabilities rise. These results have some important implications for banks and regulators.

First, loan prices should not only account for higher *expected* losses of loans with higher default probabilities, but should also reflect their higher contributions to a bank's economic capital. Second, macroeconomic shocks (such as business cycle downturns) that increase average default probabilities will not only increase expected losses of banks (which is widely recognized) but also portfolio standard deviations. This observation is indispensable in gaining a more complete picture of the consequences of macroeconomic shocks for the banking system. It will also be important for banks attempting to hedge against fluctuations in required economic capital caused by macroeconomic risk. Finally, our results have consequences for credit risk modeling. They emphasize the necessity of adapting the calibration of credit risk models to the business cycle, an

[5]Referring to figure 1.2, the correlation between the loans to firm i and firm j would be $\mathrm{Corr}(V_i^L, V_j^L)$ where V_i^L denotes the random variable that can take the values $V_{i,A}^L, ..., V_{i,\mathrm{DF}}^L$ in $t = t_2$.

argument that has up to now only been put forward on the basis of varying univariate default probabilities. Moreover, a major class of current credit risk models employs the distribution of firm default rates - i.e. empirically determined firm default frequencies (on an industry, country or rating class level) - for calibration. In chapter 3 we find that default-rate standard deviations will vary through the business cycle for the same reason as portfolio standard deviations do. We therefore suggest that these standard deviations should be estimated conditional on the business cycle when used to parametrize credit risk models, in particular because the outputs of these models have proved to be quite sensitive with respect to the default-rate standard deviations (see e.g. Gordy (2000)).

Relation between Chapters 2 and 3

Concluding the credit risk management introductory section, we illustrate the link between chapters 2 and 3. To do so, we examine the different approaches to joint default modeling a little more closely. Table 1.1 gives an overview and integrates the contributions of both chapters. There are currently two major approaches to joint default modeling, the structural and the reduced-form approach.[6] Both approaches can be described as modeling conditional default probabilities for each firm, given the realization of some systematic variables. Joint default probabilities are then derived by integrating over the distribution of these variables. However, the approaches differ with respect to the way the conditional default probabilities are determined, and with respect to the systematic variables employed. While structural models build on the option-pricing approach as developed in Merton (1974) and assume that a firm goes bankrupt if the value of its assets falls below a certain threshold (which depends on the firm's liability structure), reduced-form models work with general heuristics predicting how default probabilities change due to the changes in the systematic factors. The systematic variables employed are aggregate asset returns and default rates.

In chapter 2 we argue that the structural approach is more suitable for describing joint default behavior. To analyze the relationship between default probabilities and default correlations in chapter 3, we therefore rely on a structural model. Two types of such models have been discussed in the credit risk context. They differ with respect to the modeling of conditional default probabilities. The first type of models calculates the probability of the firm value falling below its default threshold at the *end* of the time

[6]Note that both approaches can be generalized to joint migration modeling.

<table>
<tr><th></th><th>Reduced-Form Models</th><th colspan="2">Structural Models</th></tr>
<tr><td>Cond. Default Probabilities</td><td>Based on general heuristics</td><td>Fixed Default Horizon (FDH)</td><td>Absorbing Barrier</td></tr>
<tr><td>Syst. Variables</td><td>Default rates</td><td colspan="2">Aggregate asset returns</td></tr>
<tr><td>Chapter 2</td><td colspan="3">- Description and assessment of existing models
- Next-generation models *</td></tr>
<tr><td rowspan="3">Chapter 3</td><td></td><td colspan="2">Relationship between default probabilities and default correlations</td></tr>
<tr><td>Implications for:
default rate std. dev.</td><td colspan="2"></td></tr>
<tr><td colspan="3">Implications for:
the adaptation of joint default models to the business cycle, the consequences of macroeconomic shocks for the banking system, the pricing of loans, hedging loan portfolios</td></tr>
</table>

Table 1.1: Joint Default Modeling and the contributions of this thesis.

horizon for which the default probabilities are to be determined.[7] We call this type *Fixed Default Horizon* model (FDH). The second type measures the probability that the firm's asset value falls below the threshold at *any* point in the time horizon. We call such models *Absorbing Barrier* models since the default threshold is assumed to be an absorbing barrier to the firm's asset value process.

We emphasize in chapter 2 that Absorbing Barrier models are a more realistic description of actual default behavior and should therefore be preferred in credit risk modeling. However, when analyzing the relationship between default probabilities and default correlations in chapter 3, we employ a FDH model for tractability reasons, and suggest that the robustness of our results should be assessed with respect to Absorbing Barrier models in future research.

[7]Note that this horizon would be equal to $t = t_2$ in the example given in figure 1.2.

1.2 Bank Closure Policies

The third paper - presented in chapter 4 and partly based on ERLENMAIER AND GERSBACH (2001b) - analyzes bank closure policies in the event of system-wide banking crises. Typically, a large number of banks is at the brink of bankruptcy in such crises, which tends to make the bailout of all banks prohibitively costly. By closing some of them, the share of funds available for the remaining ones will increase since there are fewer banks competing for new deposits. We call this effect the *funds concentration effect*. Moreover, the surviving banks can take over investment projects from closed banks, enabling them to offer higher deposit rates and to attract more depositors. Hence, if enough banks are closed, the others will be able to survive even without any government subsidies: banks are bailed out by closing others (discriminatory bailout). Such an interpretation could be given to the closure policies recently applied during the crisis in Asia and the Swiss regional banking crisis (see RADELET AND SACHS (1998, 1999) and STAUB (1998)).

In chapter 4 we analyze such policies theoretically. While the existing theoretical literature[8] has employed single-bank frameworks where systemic consequences are accounted for only by *exogenous* factors, we take a general-equilibrium view of bank closure policies that allows us to take into account the effect of closing some banks on the financial viability of the others. The contributions of chapter 4 are the following:

- The provision of a simple general-equilibrium framework and a corresponding equilibrium concept.
- A clarification of the major conceptual issues involved.
- The introduction and foundation of the funds concentration effect.
- A comparison of the relative merits of different potential bailout schemes.

To concentrate on the general-equilibrium effect of bank closures, we start our analysis with a situation in which the banks' insolvency is assumed to result solely from an exogenous macroeconomic shock. Since the realization of this shock is not under the control of the banks' managers, it would be most natural to decide randomly about which banks to close (RB) in order to put the funds concentration effect to work.

[8] An overview can be found in BHATTACHARYA, BOOT, AND THAKOR (1998). More recent contributions are REPULLO (1999), GOODHART AND HUANG (1999), CORDELLA AND YEYATI (1999) and FREIXAS (1999).

However, since the closure policies feed back into the banks' strategic behavior, we also consider two other bailout schemes in which the bailout decision depends on the size of a bank with respect to deposits: bailout of big banks (BB) and bailout of small banks (BS).[9]

The most important conceptual issue emerging from the analysis of these closure policies is that - given the banks' exposure to the macroeconomic shock and the regulator's discriminatory closure policy - deposits are risky. Therefore, when deciding about which bank to choose, depositors cannot only rely on the deposit rates offered but also have to assess the *expected* returns paid on the deposits of the respective bank. The actual expected returns paid will depend on the depositors' savings decisions since they determine bank sizes and, hence, bailout probabilities. It is therefore not a priori clear whether return expectations that are correct in equilibrium (consistent assessments) actually exist and whether these assessments are unique. We find that consistent assessments always exist under RB and BB but not necessarily under BS. Moreover, while assessments are unique under RB, this is not necessarily the case under BB.

The comparison of the relative merits of the different bailout schemes focuses on the following issues: stability (with respect to existence and uniqueness of consistent assessments), expected returns, and credibility of regulatory actions. We identify BB as the preferred bailout scheme if depositors can agree on the consistent assessment that guarantees maximum returns on deposits. BS leads to stability problems and may support low-return equilibria, which can both be avoided under BB. Moreover, BB dominates RB with respect to expected returns and credibility.

The intuition for the dominance of BB with respect to expected returns is as follows. Under RB, bailout probabilities are the same for all banks. Hence, depositors will always choose to deposit with the bank that offers the highest deposit rates. This in turn implies that deposit rates will be bidden up until expected profits are zero. Under BB, in contrast, banks stop the bidding when expected returns for depositors are at their maximum, since - due to a self-fulfilling-prophecy effect - depositors will not switch to a bank that offers slightly higher deposit rates. In constellations where expected returns are decreasing in offered deposit rates, BB will therefore implement higher expected equilibrium returns than RB. We show that expected returns can indeed decrease under discriminatory bailout if deposit rates are raised. In this case

[9]In our model, the size of a bank depends only on the attractiveness of the deposit rates it has offered. However, in a more realistic setting, banks can differ in size for many other reasons. The appropriate interpretation of BB and BS will then not refer to the size of the bank with respect to deposits but to the size of the *growth rate* of the banks' deposits.

the decrease in bailout probabilities overcompensates the increase in deposit rates.

However, our overall conclusions have to be modified when the consequences of the bailout policies for the bank's risk-taking incentives are important. If banks can decide about the riskiness of their projects after they have received deposits, BB will have the drawback of providing risk-taking incentives for big banks, since these banks can anticipate to be bailed out with high probability. RB, in contrast, will provide less incentives for risk taking, since banks are uncertain about the regulator's bailout decision. This parallels the „*constructive ambiguity*" approach to bank closures as analyzed by Freixas (1999) in a single-bank model.

In our general-equilibrium framework, however, bailout probabilities have to be chosen in a way ensuring that under *all* realizations of the stochastic decision process, the banks that have not been closed will be able to survive. This makes the design of such a policy more demanding. The simple version of RB we propose in chapter 4 requires that - in out-of-equilibrium strategies - the regulator must commit to bail out significantly less deposits than would be possible and optimal. This undermines the credibility of the regulatory policy. We indicate, however, that more sophisticated RB-type policies may be found that circumvent these drawbacks. The comprehensive construction of such a policy seems, however, to be far from straightforward and is left to future research.

In summary, our findings suggest that closure rules in severe crises should be a mixture of BB and RB. Whether the actual policy will more closely resemble the former or the latter will depend on whether return considerations or excessive risk-taking considerations are more important.

Chapter 2

Models of Joint Default in Credit Risk Management

Abstract. In this chapter we review the models of joint defaults of the current major industry-sponsored credit risk frameworks. Recognizing the need for further improvements of these models, we address the following issues. First, we identify the most important modeling drawbacks that could be fixed on a short-term basis. Second, we analyze which of the proposed models is the conceptually most promising basis for next-generation models. Concluding that the KMV methodology is the most suitable to go forward, we set out a research agenda aiming at further improvements and at extending the KMV model to non-quoted firms.

Keywords: Credit portfolio management, Credit risk models, Joint defaults.

JEL: G11, G21, G28.

2.1 Introduction

The last years have seen a rapid growth of interest in credit risk modeling from banking regulators, practitioners and academic researchers. Since the first generation of models have been developed, big banks have used these models for risk management purposes. Moreover, regulators have started to explore the potential of the models to determine regulatory capital (BASEL COMMITTEE ON BANKING SUPERVISION 1999).

The models' accuracy is the major factor in determining the success of both types of applications. It has therefore become an important point of focus in academic research on credit risk modeling. The findings up to now can be summarized as follows. First, many empirical and theoretical objections regarding the modeling assumptions and the models' calibration have been raised.[1] Second, it has been pointed out that the models' outputs are very sensitive with respect to parameter estimates and differ quite significantly across models (see e.g. GORDY (2000)). Finally, a first backtesting study (NICKELL, PARRAUDIN, AND VAROTTO 1999) for two of the currently available models found that - for portfolios of Eurobonds - the models yielded far more exceptions than they would if they were accurately measuring risk.

The one conclusion emerging from this academic discussion is that current credit risk models cannot yet be used to determine regulatory capital and that new, more sophisticated models have to be developed to measure credit risk more accurately. Moreover, it has been argued that once sufficiently elaborated models are available, extensive backtesting will be necessary before these models can be used to determine regulatory capital.

Some general suggestions for important roads of improvements have been made. Most prominently it has been argued that credit risk models should employ stochastic interest rates instead of deterministic ones (see e.g. CROUHY, GALAI, AND MARK (2000)). It has also been put forward that they should take into account that default probabilities and rating transition probabilities vary through the business cycle or between obligors belonging to different industries (see e.g. NICKELL, PARRAUDIN, AND VAROTTO (2000)). However, to our knowledge there are no contributions that take a broader view, aimed at the description of a detailed research agenda for the development of next-generation models from today's existing methodologies.

This chapter intends to contribute to the closure of this gap by focusing on one particular area of credit risk modeling, *joint default probabilities.* In doing so, we will also

[1] For an overview see JACKSON AND PERRAUDIN (2000).

present a broad picture of modeling drawbacks and suggestions for the improvement of existing methodologies in this area. We will analyze the four major currently available industry-sponsored credit risk models: CreditMetrics (by JP Morgan), Portfolio Manager (by KMV), CreditRisk$^+$ (by Credit Suisse Financial Products) and CreditPortfolioView (by McKinsey). In the following we will also refer to these models as CM (CreditMetrics), KMV, CR (CreditRisk$^+$) and McK (McKinsey).[2]

We start our analysis by identifying the most important modeling problems with respect to joint defaults. While repeating some of the arguments that have already been made, we will be able to point to quite a few drawbacks that have not been discussed yet. We then make suggestions on how the models' performance could be improved.

Reviewing the identified modeling problems and the improvements that could be achieved, we finally set out to assess which of the currently proposed approaches to joint-default modeling is the *conceptually* most promising basis for next-generation models.[3] We argue that a "mixed" model blending modeling features of KMV and of a model recently employed in the literature (Nickell, Parraudin, and Varotto 2000) is the most suitable to go forward. Moreover, we try to set out a research agenda aiming at further improvements and at extending the scope of KMV-type model to non-quoted firms.

In doing so we stress that - besides backtesting *complete* models - it is important to assess (theoretically and empirically) the adequateness of the model's different building blocks with respect to modeling assumptions and parameter estimation. Once sufficiently well-performing blocks have been developed, backtesting of complete models can determine how wide remaining error margins are. This method has not only the advantage that bad performance can be tracked more specifically to single model components but also makes it possible to combine successful parts of different models.

Finally, while we advocate to use the mixed KMV model as a starting point for next-generation models, we recognize that *each* of the currently proposed models will be applied in banking practice in the near future. We think that the suggestions made for CM, CR and McK can help to fix some important drawbacks on a short-term view.

This chapter is organized as follows. In section 2.2 we present a detailed description

[2] A comprehensive description of all of these models can be found in Crouhy, Galai, and Mark (2000). More specific references are JP Morgan (1997) (for CM), Credit Suisse (1997) (for CR) and McKinsey (1998) (for McK). To our knowledge, the best documentation available for the KMV model is the paper of Crouhy, Galai, and Mark (2000). Some interesting information can also be found on the KMV homepage (http://www.kmv.com).

[3] "Non-conceptual" issues have also been put forward in the discussion of the models' relative merits, in particular computational simplicity (see e.g. Crouhy, Galai, and Mark (2000)). Such issues will not be taken up here.

of the currently proposed joint-default models. In section 2.3 we discuss the major drawbacks of the models, argue which conceptual issues should be clarified, and suggest how the models' performance could be improved. The suggestions contain short-term fixes as well as a longer-term research agenda. In section 2.4 we present our conclusions. A summary of the major insights from section 2.3 builds the basis for the identification of the most promising joint-default approach for next-generation models.

2.2 Presentation of the Models

Despite the fact that a comprehensive survey for all of the credit risk models discussed in this paper has recently been provided by Crouhy, Galai, and Mark (2000), we will present a detailed description of each proposed joint-default methodology in this section. This is done for the following reasons. First, since Crouhy, Galai, and Mark (2000) capture all aspects of credit risk modeling, the description of the joint-default models is not as detailed as necessary. In particular, the presentation of the multi-year horizon (for CR and McK), the construction of the KMV country and industry indices, and KMV's estimation of the relative size of the systematic risk component are important modeling features that are not described in Crouhy, Galai, and Mark (2000). Second, focusing on joint-default models allows us to present all models within a unified framework which makes similarities and differences more transparent and provides a clear background for the analysis. Finally, while reviewing the literature, we found two interesting new versions of current credit risk models, one for CR (Gordy 2000), which hereafter will be referred to as CR-GO, and one for KMV (Nickell, Parraudin, and Varotto 1999), hereafter referred to as NPV. Both were not presented in their own rights but as part of studies attempting to compare model outputs and backtest currently proposed models. Nevertheless, we think that these versions represent important alternatives to the initial proposals, and we will therefore discuss them as well.

To focus on joint-default modeling, we employ the most simple credit risk management framework. We consider a bank that holds a loan portfolio and does risk management in $t = 0$. The loans are due in $t = T$ and we assume that the bank's risk management horizon is identical with the date at which the loans mature, i.e. the bank is interested in the distribution of the value of its portfolio at $t = T$. For each loan, two states are possible in $t = T$. Either the firm has not defaulted and the loan's principal and interest are paid back; or the firm has defaulted and the bank receives nothing.[4] Denoting the sum of principal and interest due from firm i by L_i and the default event indicator variable by $B_{i,T}$, we can describe the $t = T$ portfolio value by

$$\sum_{i=1}^{n} L_i B_{i,T}.$$

In general, $B_{i,t}$ is an indicator variable which is zero if firm i has not yet defaulted until time t and which is equal to 1 otherwise ($i = 1, .., n$). Moreover, we use $\mathbf{B}_t$ to denote the vector $(B_{i,t})_{i=1}^{n}$ of default indicators at time t. Obviously, the distribution of the above portfolio value is completely described by the distribution of the vector $\mathbf{B}_T$.

We follow the literature and divide the credit risk models into two classes, labeled "structural models" (CM, KMV and NPV) and "reduced-form models" (CR, CR-GO and McK) respectively. This terminology can be found for example in JARROW AND TURNBULL (2000). It refers to the fact that structural models build on a microeconomic description of firm defaults while reduced-form models employ general heuristics predicting how default probabilities change due to changes in systematic factors such as the business cycle. Since the reduced-form models anualize the credit risk horizon, we will measure time in years and will - for simplicity of representation - assume that $T \in \{1, 2, 3, ...\}$. Negative time indices will be employed to describe historical observations that are used to calibrate the models. T_h denotes the length of the time span from which these observations are taken. Figure 2.1 illustrates the time horizon.

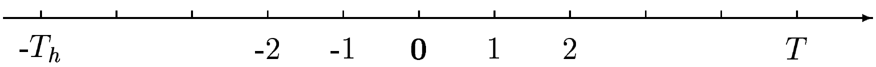

Figure 2.1: Time horizon.

Note that all models derive joint default probabilities by defining a vector $\mathbf{S}$ representing the systematic risk in the economy. Default events are assumed to be independent given the realization of $\mathbf{S}$. The probability of a certain realization $\mathbf{b} = (b_1, ..., b_n)$ of the vector $\mathbf{B}_T$ can then be derived by integrating conditional default probabilities $p_{i,T}(\mathbf{S})$ over the distribution of $\mathbf{S}$:[5]

$$\mathbb{P}\{\mathbf{B}_T = \mathbf{b}\} = \mathbb{E}\prod_{i=1}^{n}\Big\{1\{b_i = 1\}p_{i,T}(\mathbf{S}) + 1\{b_i = 0\}\Big(1 - p_{i,T}(\mathbf{S})\Big)\Big\}.$$

[4] It is easy to integrate fixed recovery rates.

[5] Note how the indicator function $1\{\cdot\}$ is defined. $1\{A\}$ is equal to 1 if statement A holds and equal to 0 if statement A does not hold.

2.2.1 Reduced-Form Models

For both reduced-form models, the vector $\mathbf{S}$ consists of T systematic vectors: $\mathbf{S} = (\mathbf{S}_1, .., \mathbf{S}_T)'$.[6] The entries $\mathbf{S}_1, ..., \mathbf{S}_T$ can be thought of as realizations of a systematic vector at different points in time. At each point $t-1$, the probability that a firm that has survived until $t-1$ will also survive until t is analyzed $(t = 1, .., T)$. Default events are assumed to be independent given $\mathbf{S}_t$ and the conditional one-year default probabilities

$$p_i(\mathbf{S}_t) := \mathbb{P}\Big\{B_{i,t} = 1 \mid B_{i,t-1} = 0, \mathbf{S}_t\Big\}$$

are assumed to be stable over time. The multi-year conditional default probabilities $p_{i,T}(\mathbf{S})$ are then specified in the following way:[7]

$$p_{i,T}(\mathbf{S}) := 1 - \prod_{t=1}^{T}\Big(1 - p_i(\mathbf{S}_t)\Big). \tag{2.1}$$

In the following sections we discuss how the conditional one-year default probabilities $p_i(\cdot)$ and the distribution of the vector $\mathbf{S}$ are determined for the different models. The unconditional one-year default probability of firm i will be denoted by $\bar{p}_i$ $(i = 1, .., n)$. It is important to note that the term unconditional refers to the fact that the default probabilities are integrated over the distribution of the complete time series $(\mathbf{S}_t)_{t=-\infty}^{+\infty}$ of the systematic factors.

2.2.1.1 CreditRisk$^+$ (CR)

To develop the reasoning behind the CR model, we start with the most simple version of CR. CR does not model the distribution of the systematic factors $\mathbf{S}$ *directly* but only

[6] We will use the symbol "$'$" to describe a transposed vector or matrix.

[7] In the appendix, we show that this specification rests on the implicit assumption that

$$\mathbb{P}\{B_{i,t} = b_i \,|\, \mathbf{S}_t, ..., \mathbf{S}_T\} = \mathbb{P}\{B_{i,t} = b_i \,|\, \mathbf{S}_t\} \quad (t = 1, ..., T-1,\ i = 1, ..., n).$$

We will comment on this assumption in section 2.3.1.1 in the paragraph dedicated to multi-year default probabilities (page 38).

the distribution of the resulting portfolio default rate, denoted by μ_t^{P}:[8]

$$\mu_t^{\mathrm{P}} := \mu^{\mathrm{P}}(\mathbf{S}_t) := \frac{1}{n}\sum_{i=1}^{n} p_i(\mathbf{S}_t). \tag{2.2}$$

It is assumed that the series μ_t^{P} is gamma-distributed and independent, identically distributed (iid) over time.[9] Given the realization of the default rate μ_t^{P}, the default events of all obligors are assumed to be independent. The conditional default probability for each firm i is specified by

$$p_i(\mu_t^{\mathrm{P}}) = \bar{p}_i \frac{\mu_t^{\mathrm{P}}}{\bar{\mu}^{\mathrm{P}}}, \tag{2.3}$$

where $\bar{\mu}^{\mathrm{P}}$ is the mean of μ_t^{P}. Note that this definition of conditional default probabilities guarantees (a) that $\mu_t^{\mathrm{P}} = (1/n)\sum_{i=1}^{n} p_i(\mu_t^{\mathrm{P}})$ for all realizations of μ_t^{P} and (b) that $\bar{p}_i = \mathbb{E}\, p_i(\mu_t^{\mathrm{P}})$. Moreover, note that by equations (2.2) and (2.3) the mean $\bar{\mu}^{\mathrm{P}}$ and the standard deviation σ^{P} of μ_t^{P} are given by

$$\bar{\mu}^{\mathrm{P}} = \frac{1}{n}\sum_{i=1}^{n} \bar{p}_i \tag{2.4}$$

$$\sigma^{\mathrm{P}} = \frac{1}{n}\sum_{i=1}^{n} \sigma_i. \tag{2.5}$$

σ_i is the standard deviation of the random variable $p_i(\mu_t^{\mathrm{P}})$, i.e.

$$\sigma_i = \sqrt{\mathrm{Var}\Big(p_i(\mu_t^{\mathrm{P}})\Big)} = \sqrt{\mathrm{Var}\Big(p_i(\mathbf{S}_t)\Big)}.$$

To determine mean and standard deviation of μ_t^{P} from equations (2.4) and (2.5), CR suggests to estimate the parameters $\bar{p}_i$ and σ_i $(i = 1, ..., n)$ from rating class default-rate series by mapping each obligor i to a rating class $\zeta(i)$.[10] The parameter estimates have to be provided by the user. An estimation technique is outlined in section 2.2.1.4. We illustrate the model setup as produced up to now with the following example.

[8] To be more precise, $\mu_t^{\mathrm{P}}(\mathbf{S}_t)$ is the *expected* portfolio default rate conditioned on $\mathbf{S}_t$. However, in this paper we will - for the sake of brevity and to keep track with the terminology used in the CR manual - use the loose vocabulary "portfolio default rate".

[9] The only motivation given in the CR manual for choosing the Gamma distribution, is that it can be paramatrized solely by its mean and standard deviation.

[10] Rating class default rate data can be taken from rating agencies such as Moody's or Standard&Poor's.

Example 1:

Suppose that the conditional default probabilities are given by $p_i(\mathbf{S}_t) = c_i G(\mathbf{S}_t)$ where $G(\cdot)$ is an arbitrary function mapping into the interval $[0, \min_{i=1,..,n}\{1/c_i\}]$. Then

$$\mu_t^{\mathrm{P}} = \frac{G(\mathbf{S}_t)}{n}\sum_{i=1}^{n} c_i,$$
$$\bar{\mu}^{\mathrm{P}} = \frac{\mathbb{E}\, G(\mathbf{S}_t)}{n}\sum_{i=1}^{n} c_i$$

and $\bar{p}_i = c_i \,\mathbb{E}\, G(\mathbf{S}_t)$. Hence, formula (2.3) indeed specifies conditional default probabilities correctly.

To account for diversification effects, the framework outlined above can be extended by dividing the obligors among different subsets $\mathcal{S}_1, .., \mathcal{S}_m$ where each subset is a collection of obligors under the common influence of a systematic factor. These subsets are called "sectors" in the CR manual but do not necessarily represent *industry* sectors. An other example might be the division of a portfolio according to the country of domicile of each obligor. The default rates

$$\mu_{j,t}^{P} := \frac{1}{n}\sum_{i=1}^{n} 1\{i \in \mathcal{S}_j\} p_i(\mathbf{S}_t) \tag{2.6}$$

of each sector are assumed to be gamma distributed, independent of each other and iid in time. Moreover, defaults are assumed to be independent given the realization of the vector $\boldsymbol{\mu}_t^{\mathbf{P}}$ of all sector default rates. Conditional default probabilities are specified by

$$p_i(\boldsymbol{\mu}_t^{\mathbf{P}}) := \bar{p}_i \sum_{j=1}^{m} 1\{i \in \mathcal{S}_j\} \frac{\mu_{j,t}^{P}}{\bar{\mu}_j^{\mathrm{P}}}, \tag{2.7}$$

where $\bar{\mu}_j^{\mathrm{P}}$ is the mean of $\mu_{j,t}^{P}$. This implies that the mean $\bar{\mu}_j^{\mathrm{P}}$ and the standard deviation σ_j^{P} of the default rate in sector j are given by

$$\bar{\mu}_j^{\mathrm{P}} = \frac{1}{n}\sum_{i=1}^{n} 1\{i \in \mathcal{S}_j\}\bar{p}_i \tag{2.8}$$
$$\sigma_j^{\mathrm{P}} = \frac{1}{n}\sum_{i=1}^{n} 1\{i \in \mathcal{S}_j\}\sigma_i. \tag{2.9}$$

The most general version of CR allows for the mapping of each obligor i into more than one sector. This is achieved by assigning obligor-specific weights $(\theta_{i1}, ..., \theta_{im})$ to each sector that are required to sum to 1 $(\sum_{j=1}^{m} \theta_{ij} = 1)$. The weights are specified by the user. By formal analogy (i.e. by substituting $1\{i \in \mathcal{S}_j\}$ with θ_{ij}), the formulas (2.7) - (2.9) are generalized to

$$p_i(\boldsymbol{\mu}_t^{\mathrm{P}}) = \bar{p}_i \sum_{j=1}^{m} \theta_{ij} \frac{\mu_{j,t}^{P}}{\bar{\mu}_j^{\mathrm{P}}} \tag{2.10}$$

$$\bar{\mu}_j^{\mathrm{P}} = \frac{1}{n} \sum_{i=1}^{n} \theta_{ij} \bar{p}_i \tag{2.11}$$

$$\sigma_j^{\mathrm{P}} = \frac{1}{n} \sum_{i=1}^{n} \theta_{ij} \sigma_i. \tag{2.12}$$

It should be noted that the CR manual gives no definition of the variables $\mu_{j,t}^{P}$ $(j = 1, .., m)$ in this general version of the model. Consequently, it is also unclear what formulas (2.11) and (2.12) actually mean.[11]

Finally, note that CR models idiosyncratic risk by employing an additional sector (called idiosyncratic or specific sector). A model with $m-1$ systematic risk factors and obligor-specific idiosyncratic risk can be parametrized as a m-sector model in which one of the sectors (without loss of generality sector 1) represents the idiosyncratic risk components. The default rate associated with this sector is assumed to have zero volatility and thus $\mu_1^P / \bar{\mu}_1^{\mathrm{P}} = 1$.[12] θ_{i1} is supposed to describe the size of the idiosyncratic component of default risk.

In the sequel we will use the following terminology to refer to the different CR versions: *single-sector version* (to denote the simple version which is built on only one default rate), *multi-sector version without weights* (to refer to the version with m sectors where each obligor can only be mapped to one sector) and *weighted multi-sector version* (to refer to the most general specification where each obligor can be mapped to all sectors by assigning obligor-specific weights to each sector).

2.2.1.2 CreditRisk$^+$: The Modified Version of Gordy (CR-GO)

Gordy (2000) compares the model outputs of CR and CM for a variety of hypothetical portfolios. In doing so he presents a more abstract form of the weighted two-sector

[11] We discuss this problem in more detail in section 2.3.1.1.

[12] See the CR manual (Credit Suisse 1997), section A 12.3. Because the specific sector represents diversifiable risk, it is assumed to contribute no volatility to a well diversified portfolio.

version of the CR model (which will hereafter be referred to as CR-GO). He abstracts from the modeling of portfolio default rates and, hence, replaces the default rate $\mu^P_{2,t}$ of the second (systematic-risk-) sector by a general systematic factor $\mathbf{S}_t$. $\mathbf{S}_t$ is a scalar, and the series $(\mathbf{S}_t)^T_{t=-T_h}$ is assumed to be gamma distributed and iid in time. By convention, the first sector represents the idiosyncratic risk (i.e. $\mu^P_{1,t}/\bar{\mu}^{\mathrm{P}}_{1,t} = 1$), and the weight on this sector is given by $\theta_{i1} = 1 - \theta_{i2}$. $\theta_i := \theta_{i2}$ is called the weight on the systematic factor and conditional default probabilities are described in analogy to equation (2.10):

$$p_i(\mathbf{S}_t) = \bar{p}_i\Big((1-\theta_i) + \theta_i \frac{S_t}{\mathbb{E}\, S_t}\Big). \tag{2.13}$$

Gordy then maps obligors to rating classes ($i \to \zeta(i)$). Within each rating grade ζ, obligors are assumed to be *statistically identical,* i.e. they have the same unconditional default probability $\bar{p}_\zeta$ and the same weight θ_ζ on the systematic factor. To calibrate the model, Gordy estimates the unconditional default probabilities $\bar{p}_\zeta$ and the variances of default probabilities $\sigma^2_\zeta := \mathrm{Var}\Big(p_\zeta(\mathbf{S}_t)\Big)$ from rating class default-rate series. The estimation method is outlined in section 2.2.1.4. The parameters θ_ζ are then obtained by inserting the estimates into the formula for default-probability variances that follows from equation (2.13):

$$\sigma^2_\zeta = \bar{p}^2_\zeta \theta^2_\zeta \,\mathrm{Var}(\mathbf{S}_t).$$

However, these estimates are not sufficient to fully specify the model, since $\mathrm{Var}(\mathbf{S}_t)$ is still undetermined. Gordy argues that there is no obvious information to estimate this parameter, and (for illustration purposes) considers three different values for $\sqrt{\mathrm{Var}(\mathbf{S}_t)}$, namely 1, 1.5 and 4.0.

2.2.1.3 CreditPortfolioView (McK)

McK models the distribution of certain industry or country (one-year) default rates $(\hat{\mu}^{\mathrm{C}}_{1,t}, ..., \hat{\mu}^{\mathrm{C}}_{m,t})$ and then assumes that default events are independent given the realization of the vector $\hat{\boldsymbol{\mu}}^{\mathbf{C}}_t := (\hat{\mu}^{\mathrm{C}}_{j,t})^m_{j=1}$ of these default rates. The default-rate model is constructed using macroeconomic variables $\mathbf{S}_t := (S_{1,t}, ..., S_{k,t})$. For each default rate $\hat{\mu}^{\mathrm{C}}_{j,t}$ employed, a macro index $\bar{S}_{j,t}$ is constructed as a linear combination of the macroeconomic variables:

$$\bar{S}_{j,t} = \sum_{l=1}^{k} \tau_{jl} S_{l,t},$$

where the weights τ_{ij} on each of the macroeconomic variables are fitted to the respective default-rate series using a logit model:

$$\hat{\mu}^{\mathrm{C}}_{j,t} = \frac{1}{1+\exp(\bar{S}_{j,t}+\bar{\nu}_{j,t})}.$$

The variables $(\bar{\nu}_{1,t}, ..., \bar{\nu}_{m,t})$ are assumed to be normally distributed zero-mean error terms that are independent in time, while the macroeconomic variables are assumed to follow an AR(2) process with jointly normally distributed innovations $(\nu_{j,t})_{j=1}^{k}$. Hence, by estimating the parameters of the AR process and the covariance matrix

$$\begin{pmatrix} \Sigma_{\bar{\nu}} & \Sigma_{\bar{\nu},\nu} \\ \Sigma_{\nu,\bar{\nu}} & \Sigma_{\nu} \end{pmatrix}$$

of innovation and error terms,[13] the joint distribution of the vectors $\hat{\boldsymbol{\mu}}^{\mathbf{C}}_1, ..., \hat{\boldsymbol{\mu}}^{\mathbf{C}}_T$ is specified conditional on the realization $\mathbf{S}_0$ of the macroeconomic variables in $t = 0$.

To derive the conditional default probabilities $p_i(\hat{\boldsymbol{\mu}}^{\mathbf{C}}_t)$, the model draws on the intuition that with default rates higher than average, default probabilities should also be higher than unconditional default probabilities while the contrary should be true if default rates are lower than average. The formal specification used in McK is

$$p_i(\hat{\boldsymbol{\mu}}^{\mathbf{C}}_t) = \begin{cases} \Big(1-\exp(-\kappa_i)\Big)(1-\bar{p}_i)+\bar{p}_i & \text{if } \kappa_i = \kappa_i(\hat{\boldsymbol{\mu}}^{\mathbf{C}}_t) \geq 0 \\ \exp(\kappa_i)\bar{p}_i & \text{if } \kappa_i = \kappa_i(\hat{\boldsymbol{\mu}}^{\mathbf{C}}_t) < 0, \end{cases}$$

where

$$\kappa_i(\hat{\boldsymbol{\mu}}^{\mathbf{C}}_t) := \Big[\sum_{j=1}^{m} \theta_{ij} \frac{\hat{\mu}^{\mathrm{C}}_{j,t}}{\mathbb{E}\,\hat{\mu}^{\mathrm{C}}_{j,t})}\Big] - 1.$$

θ_{ij} describes the sensitivity of firm i with respect to the default rate $\hat{\mu}^{\mathrm{C}}_{j,t}$. In the current version of McK, conditional default probabilities of a firm i can only depend on the default rate of one industry/country $j = j(i)$ (hence $\theta_{ij} = 0$ for all $j \neq j(i)$) and the intensity of exposure to the respective default rate is assumed to be identical for each

[13] $\Sigma_{\bar{\nu}}$ is the covariance matrix of the error terms $\bar{\nu}_i$, Σ_{ν} is the covariance matrix of the innovations ν_i, and $\Sigma_{\bar{\nu},\nu}$ is the matrix describing the covariances between error terms and innovations.

firm ($\theta_{ij(i)} = \theta$ for all i). Hence, the expression for κ_i simplifies to

$$\kappa_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}) = \theta \frac{\hat{\mu}_{j(i)}^{\mathrm{C}}}{\mathbb{E}\,\hat{\mu}_{j(i)}^{\mathrm{C}}} - 1,$$

implying that if two firms i and $\tilde{i}$ are mapped to the same default rate ($j(i) = j(\tilde{i}) = j$), then the defaults of those firms are assumed to be independent given $\hat{\mu}_{j,t}^{\mathrm{C}}$. To calibrate the model, McK estimates the AR process that describes the stochastic of the macroeconomic variables, and the logit model that links those variables to firm default rates. Moreover, as for CR, unconditional default probabilities are determined using average rating class default frequencies. The sensitivity parameter θ has to be set by the user. It is suggested to estimate θ from default data published by rating agencies.[14]

2.2.1.4 Estimating Mean and Variance of Default Probabilities

In this section we present a method proposed by GORDY (2000), to estimate both mean and variance of annual default probabilities from default rates. It can be used to calibrate both CR versions we have presented. To describe this method, we first introduce some notation. We denote the number of corporate obligors in the rating class ζ in $t-1$ by $\hat{n}_{\zeta,t}$ and the number of obligors in ζ who have defaulted until t by $\hat{d}_{\zeta,t}$. Gordy makes the following assumptions:

1. There is a vector of systematic factors driving default probabilities: for each year $t-1$, obligor defaults until t are independent given the realization $\mathbf{S}_t$ of this systematic vector in t.
2. Conditional one-year default probabilities are identical for all obligors in one rating grade ζ and are stationary over time. They are denoted by $p_\zeta(\mathbf{S}_t)$.
3. The systematic factor series $(\mathbf{S}_t)_{t=-T_h}^{T}$ and the series $(\hat{n}_{\zeta,t})_{t=-T_h}^{-T}$ are iid and independent of each other.
4. The idiosyncratic components of the default risk (which are modeled implicitly) are serially independent.

Under these assumptions, the observed rating class default frequencies $\hat{\mu}_{\zeta,t}^{R} := \hat{d}_{\zeta,t}/\hat{n}_{\zeta,t}$ are iid in time. By dropping the time index, it can be shown that the mean $\bar{p}_\zeta$ and the

[14] See MCKINSEY (1998), p. 56. Unfortunately, an estimation procedure is not described.

variance σ_ζ^2 of $p_\zeta(\mathbf{S}_t)$ are given by $\bar{p}_\zeta = \mathbb{E}\,\hat{\mu}_\zeta^R$ and

$$\sigma_\zeta^2 = \frac{\mathrm{Var}(\hat{\mu}_\zeta^R) - \mathbb{E}[1/\hat{n}_\zeta]\bar{p}_\zeta(1-\bar{p}_\zeta)}{1-\mathbb{E}[1/\hat{n}_\zeta]} \tag{2.14}$$

respectively.[15] Hence, estimates for $\bar{p}_\zeta$ and σ_ζ^2 can be derived by estimating $\mathbb{E}\,\hat{\mu}_\zeta^R$, $\mathrm{Var}(\hat{\mu}_\zeta^R)$, and $\mathbb{E}[1/\hat{n}_\zeta]$ from the series $(\hat{\mu}_{\zeta,t}^R)_{t=-1}^{-T_h}$ and $(\hat{n}_{\zeta,t})_{t=-1}^{-T_h}$ and by inserting these results in equation (2.14).[16]

2.2.2 Structural Approach

The structural approach is employed by CM, KMV and NPV. Under this approach, the specification of joint defaults is derived from an option-pricing model as developed in Merton (1974). A firm goes bankrupt if the value of its assets falls below a certain threshold that depends on the firm's liability structure. Models of this type consist of two building blocks: (a) assumptions about the joint dynamics of the firms' asset values and (b) the firms' liability structures. We will start with the description of the former and denote firm i's asset value in t by $V_{i,t}$. The process V_i is assumed to follow the stochastic differential equation

$$dV_{i,t} = \mu_i^v V_{i,t}dt + \sigma_i^v V_{i,t}dW_{i,t} \quad (i = 1,..,n) \tag{2.15}$$

where the vector process $(W_1,..,W_n)$ is a multidimensional standard Brownian motion.[17] The correlation structure of this vector process is described using aggregate asset value processes V_j^S $(j = 1,..,k)$ that represent the systematic risk in the economy. In analogy to equation (2.15), those processes are assumed to follow

$$dV_{j,t}^S = \mu_j^S V_{j,t}^S dt + \sigma_j^S V_{j,t}^S dW_{j,t}^S \quad (j = 1,..,k). \tag{2.16}$$

[15]The formula for $\bar{p}_\zeta$ is obvious, and the formula for σ_ζ^2 is derived in Gordy (2000).

[16]Since both series are iid, standard techniques can be used for estimation. If $(X_i)_{i=1}^n$ is an iid series, then the standard estimates for mean and variance are $\bar{X} := \frac{1}{n}\sum_{i=1}^n X_i$ and

$$\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$$

respectively.

[17]The term "standard" refers to the fact that the means and the standard deviations of the instantaneous returns of the processes $W_1,..,W_n$ are assumed to be equal to 0 and 1 respectively.

The vector process $(W_1^S, .., W_k^S)$ is assumed to be a multidimensional standard Brownian motion with covariance matrix Σ^S. The firms' asset value processes are linked to the systematic processes by the joint log returns

$$
\begin{aligned}
Z_{i,t}(\Delta t) &:= \log(V_{i,t}/V_{i,t-\Delta t}) \quad (i = 1, .., n) && (2.17)\\
Z_{j,t}^S(\Delta t) &:= \log(V_{j,t}^S/V_{j,t-\Delta t}^S) \quad (j = 1, ..., k) && (2.18)
\end{aligned}
$$

which are normally distributed.[18] More precisely, it is assumed that a firm's returns can be described as a linear combination of the systematic returns and an idiosyncratic component:

$$
Z_{i,t}(\Delta t) = \sum_{j=1}^{k} \theta_{ij} Z_{j,t}^S(\Delta t) + \epsilon_{i,t}(\Delta t) \quad (i = 1, .., n), \tag{2.19}
$$

where the random variables $\epsilon_{1,t}(\Delta t), .., \epsilon_{n,t}(\Delta t)$ are normally distributed for fixed Δt, mutually independent and independent of the systematic processes $V_1^S, .., V_k^S$.

To determine joint default probabilities, the firms' liability structures have to be specified and parameters have to be estimated. We first present the approaches of CM and KMV, and then the alternative approach suggested by Nickell, Parraudin, and Varotto (1999).

2.2.2.1 CreditMetrics and KMV

Both CM and KMV assume that default occurs if a firm's asset value drops below a certain level at the *end of the risk management horizon,* i.e in $t = T$.[19] In this case the default event can be equivalently expressed as the event that the standard normally distributed variable

$$
Z_i := \frac{Z_{i,T}(T) - \mathbb{E}\, Z_{i,T}(T)}{\sqrt{T}\,\sigma_i^v}
$$

hits some low level z_i (called default point). To derive the distribution of the vector $\mathbf{B}$ of default indicators, it is therefore sufficient to specify the default points and the joint

[18] Δt is the time interval on which the returns are reported.

[19] Note that - in our description of the KMV model - we primarily refer to the version described in the paper of Crouhy, Galai, and Mark (2000). To our knowledge, this paper contains the most precise publicly available documentation of the KMV model. We do not know whether KMV has also implemented other versions of its model.

distribution of the normalized returns Z_i $(i = 1,..,n)$. Default points are determined by $z_i := \Phi^{-1}(\bar{p}_{i,T})$, where $\Phi^{-1}(\cdot)$ is the inverse standard-normal distribution function and $\bar{p}_{i,T}$ is the default probability of firm i conditional on the information available in $t = 0$ but averaged over all realizations of Z_i $(i = 1, ..., n)$.

The joint distribution of standardized returns is determined in the following way. First, for each firm i, weights $\tilde{\theta}_{i1}, ..., \tilde{\theta}_{ik}$ are chosen that determine the extent to which firm i is exposed to the systematic processes $V_1^S, ..., V_k^S$. These weights are required to sum to 1 $(\sum_{j=1}^k \tilde{\theta}_{ij} = 1)$. We denote the matrix of all weights by $\tilde{\Theta} := (\tilde{\theta}_{ij})_{1 \le i \le n; 1 \le j \le k}$. Second, to allow for the ratio of systematic return risk to overall return risk to take all values between 0 and 1, the parameters θ_{ij} are supposed to be multiples of $\tilde{\theta}_{ij}$: $\theta_{ij} = \theta_i \tilde{\theta}_{ij}$. Instead of determining the parameters θ_i directly, the variables Z_i are written as

$$Z_i = \lambda_i^S \frac{(\tilde{\Theta}\mathbf{S})_i}{\sqrt{(\tilde{\Theta}\Sigma^S\tilde{\Theta}^T)_i}} + \epsilon_i, \tag{2.20}$$

where

$$\lambda_i^S := \frac{\theta_i \sqrt{(\tilde{\Theta}\Sigma^S\tilde{\Theta}^T)_i}}{\sigma^v}.$$

$\mathbf{S}$ is a random vector distributed according to $\mathcal{N}(0, \Sigma^S)$ and the variables ϵ_i are mutually independent, normally distributed, independent of $\mathbf{S}$, and have zero mean and variance $1 - (\lambda_i^S)^2$. Obviously, the parameter λ_i^S represents the ratio of systematic to overall return risk for firm i (in terms of standard deviation). Moreover, from equation (2.20) we can conclude that conditional default probabilities are given by

$$p_{i,T}(\mathbf{S}) = \Phi\Big\{\frac{z_i - \lambda_i^S (\tilde{\Theta}\mathbf{S})_i / \sqrt{(\tilde{\Theta}\Sigma^S\tilde{\Theta}^T)_i}}{\sqrt{1 - (\lambda_i^S)^2}}\Big\}.$$

Hence, the distribution of the vector $\mathbf{B}$ is completely specified by choosing the unconditional default probabilities $\bar{p}_i$, the matrices Σ^S and $\tilde{\Theta}$, and the systematic risk ratios λ_i^S $(i = 1,..,n)$. CM and KMV differ in the way these parameters are determined. Before we describe the different approaches, we note that given the modeling assumptions, the parameters Σ^S and λ_i^S can be estimated from time series data. To see this, define

$$\mathbf{Z}_t(\Delta t) := \Big(Z_{i,t}(\Delta t)\Big)_{i=1}^n, \quad \mathbf{Z}_t^S(\Delta t) := \Big(Z_{i,t}^S(\Delta t)\Big)_{i=1}^n \tag{2.21}$$

and note that the following lemma holds.

Lemma 2.1

Suppose that the set of assumptions presented up to now holds. Then the vector series $\mathbf{Z} := \Big(\mathbf{Z}_{l\cdot\Delta t}(\Delta t)\Big)_{l=-1}^{T_h/\Delta t}$ *and* $\mathbf{Z}^S := \Big(\mathbf{Z}^S_{l\cdot\Delta t}(\Delta t)\Big)_{l=-1}^{T_h/\Delta t}$ *are both iid. Moreover,*

$$\mathrm{Cov}\Big(\mathbf{Z}_t^S(\Delta t)\Big) = \Delta t \Sigma^S, \tag{2.22}$$

and for $Z_{t,i}$*, the ratio of systematic to overall risk is constant over time and equal to* λ_i^S $(i = 1, .., n)$.

Lemma 2.1 implies that if the vector series $\mathbf{Z}$ and $\mathbf{Z}^\mathrm{S}$ are available, Σ^S and λ_i^S can be estimated from these series. Standard techniques can be used for the estimation of $\mathrm{Cov}\Big(\mathbf{Z}_t^\mathrm{S}(\Delta t)\Big)$[20] which, by formula (2.22), also yield an estimate for Σ^S. An estimation technique for the parameters λ_i^S is provided by KMV and is described in section 2.2.2.3.

2.2.2.2 CreditMetrics: Rating Class Default Frequencies and Equity Index Returns

In CM unconditional default probabilities $\bar{p}_i$ are determined by mapping each obligor i into a rating class $\zeta(i)$ and by setting $\bar{p}_i$ to the average historical $T-$year default frequency in the rating class $\zeta(i)$. The underlying rating system can be Moody's, Standard&Poor's, or the internal rating system of the bank.

Concerning the correlation model, CM interprets the indices $V_1^S, ..., V_k^S$ as describing aggregate asset values of certain industries in specific countries. For example V_1^S may describe the assets in the German banking industry, V_2^S the assets in the German insurance, V_3^S the United States chemical industry and so on. The covariance matrix Σ^S of aggregate industry asset returns is approximated by the corresponding matrix for *equity* index returns, which in turn is estimated from widely available equity index time series. The ratio λ_i^S of systematic to total return risk and the weights $\tilde{\theta}_{i1}, ..., \tilde{\theta}_{ik}$ of each firm i on the different industries have to be specified on a judgmental basis by the user. We use the following example for illustration:

[20] If $(\mathbf{X}_l)_{l=1}^n$ is an iid series of random vectors then the standard estimate for $\mathrm{Cov}(\mathbf{X}_{1i}, \mathbf{X}_{1j})$ is given by

$$\frac{1}{n-1}\sum_{l=1}^{n}(\mathbf{X}_{li} - \bar{\mathbf{X}}_i)(\mathbf{X}_{lj} - \bar{\mathbf{X}}_j)$$

where $\bar{\mathbf{X}}_i := \frac{1}{n}\sum_{l=1}^n \mathbf{X}_{li}$.

	Database	Portfolio
Default Probabilities	Calculate DD Map firms to rating classes Determine EDFs	Calculate DD Map firms to rating classes $\bar{p}_{i,T} := \text{EDF}_{\zeta(i),T}$
Correlation Model	Choose $\tilde{\Theta}$ Construct indices V_i^S Estimate Σ^S	Choose $\tilde{\Theta}$ Estimate λ_i^S
Input	Asset value series, parameters μ^v and σ^v	

Table 2.1: Calibration of the KMV model.

Example 2:

Suppose that firm i participates in the German banking (30%) and the German insurance industry (70%). Assume that V_1^S represents the German banking industry while V_2^S is the index for the German insurance industry. The ratio of systematic risk to total risk is assessed to be 0.9. In this case we have $\lambda_i^S = 0.9$, $\tilde{\theta}_{i1} = 0.3$ and $\tilde{\theta}_{i2} = 0.7$; all other coefficients $\tilde{\theta}_{ij}$ $(j > 2)$ are set to zero.

The approaches of KMV and NPV avoid the approximation of asset returns with equity returns by deriving asset values from observable variables (equity and liability data) using option-pricing models similar to that of Merton (1974).

2.2.2.3 KMV: Distance to Default, Country- and Industry Indices

Table 2.1 gives an overview over the steps involved in the calibration of the KMV model.[21] Before we start to describe the calibration steps associated with the respective entries of the table, we note that KMV uses data of two firm pools to calibrate the model. The first pool consists of the firms $i = 1, .., n$ in the actual portfolio and the second consists of firms from a large KMV database. The database, which includes firms that have actually defaulted, is used to empirically determine default frequencies and to construct the aggregate asset value indices $V_1^S, ..., V_k^S$. A major input into all calibration steps are the historical asset value series of the firms in both pools (reported on a certain time grid $t = -T_h, -T_h + \Delta t, ..., 0$) and the parameters μ^v and σ^v of the firms' asset value processes. To derive both asset values and parameter estimates from

[21]The terms DD and EDF will be explained during the description of the calibration steps on pages 28 and 29 respectively.

observable variables, KMV employs an option pricing model which is outlined in the next paragraph.

Firm Model As already described in equation (2.15), a firm's asset value process is assumed to follow

$$dV_t = \mu^v V_t dt + \sigma^v V_t dW_t$$

where W is a standard Brownian motion. Furthermore, it is assumed that the capital structure of a firm is composed of equity, short-term debt D_t (which is considered to be equivalent to cash), and of long-term debt, which is assumed to be a perpetuity paying an average coupon c. Under these assumptions, V_t can be determined as function of the value X_t of the firm's equity, the leverage ratio $K_t := X_t/D_t$, and the instantaneous risk-free interest rate r by solving the corresponding option pricing model. This function is denoted by H_{KMV}:

$$V_t = H_{\mathrm{KMV}}(\sigma^v;\, X_t, K_t, c, r). \tag{2.23}$$

To calibrate equation (2.23) for σ^v, KMV uses an iterative technique.[22] Once σ^v is determined the asset value series can be derived from equation (2.23) and the parameter μ^v can be estimated from the obtained series.

Distance to Default Like CM, KMV determines univariate default probabilities by mapping firms into different rating classes and by calculating the average default frequencies in these classes. In contrast to CM, however, KMV constructs its own rating classes, and default frequencies are derived using the database firms. To map firms into different rating classes, a cardinal measure for the respective firm's default probability is calculated. This measure is called "distance to default" (DD).

If T-year default frequencies shall be determined, then the DD of a firm at a point t in time is given by

$$\mathrm{DD} = \mathrm{DD}_{T,t} := \frac{\log(V_t/v_{T,t}) + \left(\mu^v - 0.5(\sigma^v)^2\right)T}{\sigma^v\sqrt{T}}.$$

[22] To our knowledge, this technique is not public. But most likely it proceeds by choosing a starting value σ_0^v for σ^v and calculating the asset value series derived from σ_0^v. The estimate for σ^v that is derived from the obtained asset value series would then provide the next value for the iteration.

Note that given the firms' asset value processes follow a geometric Brownian motion (as described in the previous paragraph) and given a firm defaults within a T- year horizon if and only if its asset value falls below a certain threshold $v_{T,t}$ at the end of the horizon T, all firms with the same $\mathrm{DD}_{\mathrm{T},t}$ would have the same T-year default probability in t, namely $\Phi(-\mathrm{DD}_{T,t})$.[23]

The critical threshold $v_{T,t}$ is set at the par value of liabilities in t including short-term debt (STD) to be serviced over the time horizon, plus half the long-term debt (LTD): $v_{T,t} := \mathrm{STD}_{T,t} + (1/2)\mathrm{LTD}_{T,t}$.[24]

Univariate Default Probabilities To determine T-year default frequencies, the DD measure is calculated for a time horizon of T years for each firm in the database at each point t on the time grid. Firms with a similar $\mathrm{DD}_{T,t}$ are mapped into one rating class ζ. We denote the set that contains all firms in the data base which are mapped to the rating class ζ in t if a T-year horizon is considered by $\mathcal{C}_{\zeta,T,t}$. Then the average T-year default frequency $\mathrm{EDF}_{\zeta,T}$[25] in each rating class ζ and for each time horizon T is calculated:

$$\mathrm{EDF}_{\zeta,T} := \frac{\sum_{t=-T_h}^{-T} \#(\text{Firms defaulted in } \mathcal{C}_{\zeta,T,t} \text{ within } T \text{ years})}{\sum_{t=-T_h}^{-T} \#\mathcal{C}_{\zeta,T,t}}.$$

To determine the unconditional T-year default probability $\bar{p}_{i,T}$ of firm i in the portfolio, the measure $\mathrm{DD}_{T,0}$ is calculated implying the mapping of the firm into a class $\zeta(i)$. Finally, $\bar{p}_{i,T}$ is set to the default frequency in class $\zeta(i)$: $\bar{p}_{i,T} = \mathrm{EDF}_{\zeta(i),T}$.

Correlation Model To specify the correlation model, KMV interprets the processes $V_1^S, .., V_k^S$ as describing aggregate asset values in certain industries and countries (industry and country indices). The processes $(V_1^S, .., V_{k_I}^S)$ may represent industry indices while the remaining processes $(V_{k_I+1}^S, .., V_{k_I+k_C}^S)$ represent the country indices $(k_I + k_C = k)$. The weights $\tilde{\Theta}$ for the firms in both pools are determined using accounting data (sales and assets) as is illustrated in the following example.

[23]This follows directly from the fact that the event $V_T \leq v_{T,t}$ can be equivalently written as $\log(V_T/V_t) \leq \log(v_{T,t}/V_t)$ and that the variable $\log(V_T/V_t)$ is normally distributed with mean $\Big(\mu^v - 0.5(\sigma^v)^2\Big)T$ and standard deviation $\sigma^v\sqrt{T}$. See Crouhy, Galai, and Mark (2000), p. 75.

[24]Note that the formula for $v_{T,t}$ is an ad-hoc specification. Using a sample of several hundred companies, KMV has observed that firms default when their asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt.

[25]The term EDF is used by KMV and stands for Expected Default Frequency.

Example 2 (continued):

As in example 2 we assume that firm i participates in the German banking and the German insurance industry. Let V_1^S be the industry index for banking, V_2^S the index for insurance, and $V_{k_I+1}^S$ the country index for Germany. Furthermore assume that the following data have been extracted from a database providing financial information about firms:[26]

Business line	Assets (%)	Sales (%)
Banking	35	45
Insurance	65	55
Total	100	100

In this case we would set $\tilde{\theta}_{i1} = (0.35 + 0.45)/2 = 0.4$, $\tilde{\theta}_{i2} = (0.65 + 0.55)/2 = 0.6$. and $\tilde{\theta}_{k_I+1} = 1$.

To derive the correlation structure Σ^S between the industry and country indices, KMV constructs the returns on these indices from the asset returns of the database firms. For each point on the time grid, a (general least square) *cross-section* regression on the database asset returns is estimated:

$$\mathbf{Z}_t(\Delta t) = \boldsymbol{\alpha} + \tilde{\Theta}\boldsymbol{\beta}_t + \boldsymbol{\epsilon}_t. \tag{2.24}$$

Note that, in order to avoid overstretching the notation, we have used the variables $\mathbf{Z}_t(\Delta t)$ and $\tilde{\Theta}$ to denote the vector of the *database* returns and the matrix of weights for the *database* firms respectively, despite the fact that elsewhere they are used to describe the *portfolio* firms.[27] Hence, if N denotes the number of firms in the database, then $\mathbf{Z}_t$ is an N-dimensional return vector and the vectors $\boldsymbol{\alpha}$ of constants and $\boldsymbol{\epsilon}_t$ of error terms also have the dimension N. Moreover, $\tilde{\Theta}$ is a $N \times k$ matrix and the vector $\boldsymbol{\beta}_t$ of regression coefficients has dimension k.

The estimates $\hat{\boldsymbol{\beta}}_t$ obtained for the regression coefficients are interpreted as index returns: $\mathbf{Z}_t^{\mathrm{S}}(\Delta t) := \hat{\boldsymbol{\beta}}_t$. The matrix Σ^S is estimated from the obtained vector *time series* $\left(\mathbf{Z}_{l\cdot\Delta t}^{\mathrm{S}}(\Delta t)\right)_{l=-1}^{T_h/\Delta t}$ as described in the conclusions from lemma 2.1. For explanatory purposes, industry and country indices are further decomposed into independent factors as illustrated in figure 2.2.[28]

[26] An example for such a database would be Compustat.

[27] The definitions of the database variables are completely analogous to the definitions of the respective portfolio variables. See equations (2.17) and (2.21) for $\mathbf{Z}_t$ and page 25 for $\tilde{\Theta}$.

[28] Figure 2.2 is quoted from CROUHY, GALAI, AND MARK (2000).

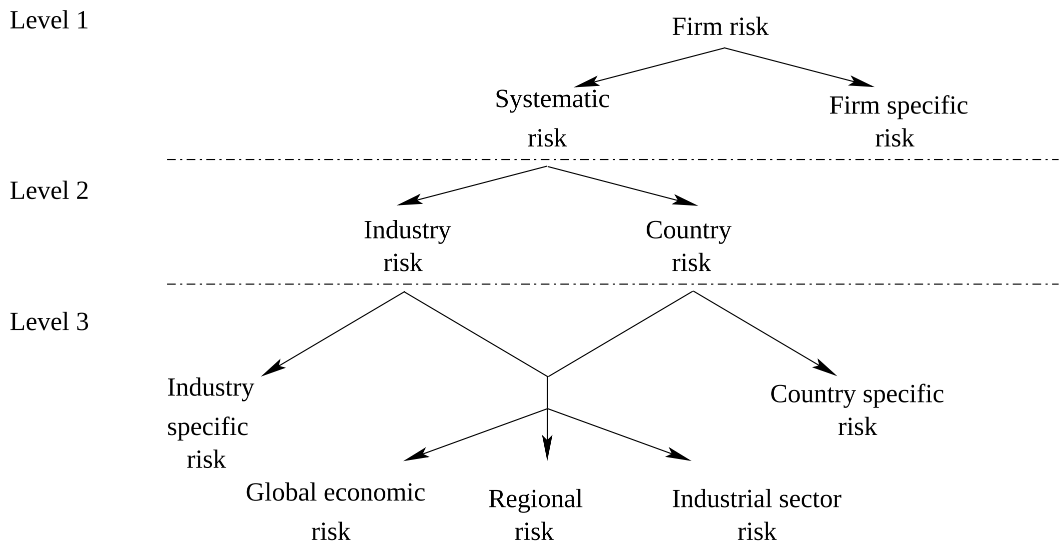

Figure 2.2: Risk decomposition in the KMV model (source: KMV Corporation).

Finally, the systematic risk ratios λ_i^S have to be determined for the portfolio firms $i = 1, .., n$. They are estimated from a *time series* regression of firm returns on index returns:

$$Z_{i,l\cdot\Delta t}(\Delta t) = \alpha_i + \beta_i \cdot (\tilde{\Theta}\mathbf{Z}^{\mathrm{S}}_{l\cdot\Delta t})_i \;\; (l = -1, ..., -T_h/\Delta t).$$

λ_i^S is set to the square root of the regression's R^2.

2.2.2.4 Nickell et. al.: Market Portfolio

The NPV model differs from the KMV model in the following points:

1. It employs only one systematic process ($k = 1$) which is interpreted as the asset value of the market portfolio.[29] Moreover, the returns on this systematic process are not constructed explicitly. Only the parameters μ_1^S and σ_1^S of this process are estimated.

2. It uses a slightly different liability structure to specify the firm model.

3. Firm default is assumed to occur if the firm's asset value falls below some low level at *any* point t in the risk management horizon T. Consequently, default probabilities are calculated as *absorbing-barrier* probabilities.

[29] The term "market" refers to national or international equity markets.

Firm Model Each firm is assumed to have an earning flow $\delta \cdot (V_t - D_t)$ where δ is the dividend payout rate and D_t denotes the firm's liabilities, assumed to follow $dD_t = \mu^d D_t dt$. Under the assumption that a firm is declared bankrupt when its *asset*-to-liability ratio $K_t := V_t/D_t$ first hits some low level k, the observable *equity*-to-liability ratio X_t/D_t can be expressed as (non-linear) function of K_t by solving the corresponding option-pricing model. We denote this function by H_{NPV}:

$$\frac{X_t}{D_t} = H_{\text{NPV}}(K_t;\, r, \delta, \sigma^v, \mu^d). \tag{2.25}$$

r is the instantaneous risk-free interest rate.

Parameter Estimation The parameters μ_i^v, σ_i^v, δ_i, μ_i^d and θ_{i1} for each firm i in the portfolio and the parameters μ_1^S, σ_1^S of the systematic process are determined by a Maximum-Likelihood estimation of the complete model described by equations (2.15), (2.16), (2.19), and the specifications of equation (2.25) for each firm i in the portfolio:[30]

$$\frac{X_{i,t}}{D_{i,t}} = H_{\text{NPV}}(K_{i,t};\, r, \delta_i, \sigma_i^v, \mu_i^d) \qquad (i = 1, .., n). \tag{2.26}$$

Joint Default Probabilities For given parameter estimates, NPV derive a formula for the default probability of firm i conditional on the realization of the systematic return variable $Z_T^S(\Delta t)$.[31] In contrast to KMV and CM, NPV calculates an absorbing-barrier probability, i.e. the probability that the asset-to-liability ratio K_t falls below the default threshold k at any point t in the risk management horizon T. To stay consistent with our notation, we normalize $Z_T^S(\Delta t)$ with respect to mean and variance and denote the normalized variable by $\mathbf{S}$. Using the formulas in NPV, conditional default probabilities can be expressed as a function of $\mathbf{S}$: $p_{i,T} = p_{i,T}(\mathbf{S})$. The distribution of the default indicator vector $\mathbf{B}_t$ can then be derived by integrating $p_{i,T}(\mathbf{S})$ over the distribution of $\mathbf{S}$.

Finally note that, to ensure better fit to empirical data, observed default frequencies could also be used for calibration (as in the KMV model). In this case the default barriers k_i would be chosen in a way ensuring that default probabilities are equal to empirical default frequencies, instead of being explicitly derived from the theoretical

[30] To account for the non-linearity of equations (2.26) in K_t, NPV include a Jacobian term in the likelihood.

[31] Since there is only one systematic process, the subscript j can be dropped.

model. We will discuss this more closely in section 2.3.2.3 where we present our suggestions for further improvements of the structural approach.

2.3 Assessment and Suggestions

In this section we try to identify the major problems of the models, argue which conceptual issues should be clarified, and make suggestions on how the models' performance could be improved. In sections 2.3.1 we discuss the reduced-form models while the structural models are analyzed in section 2.3.2. For both approaches we perform the assessment in two steps. In the first step we look at the model setup and in the second at the calibration procedure.

2.3.1 Reduced-Form Models

2.3.1.1 Model setup

We begin the analysis of the model setup by assessing how joint *one-year* default probabilities are determined. After that we examine how multi-year default probabilities are derived from the one-year ones.

One-Year Default Probabilities The major problem of all reduced-form models is that there is no adequate theoretical or empirical foundation of either the assumptions about the distribution of the vectors $\mathbf{S}_t$, $\boldsymbol{\mu}_t^{\mathbf{P}}$ and $\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}$ describing the systematic risk in the portfolio, or of the respective conditional default probabilities' functional form. Moreover, the assumption of conditional independence given the realization of the systematic factors can also be problematic. Table 2.2 gives an overview.[32]

	CR	**McK**
$\mathcal{L}(\mathbf{S}_t)$, $\mathcal{L}(\boldsymbol{\mu}_t^{\mathbf{P}})$, $\mathcal{L}(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}})$	No foundation	Monotonic in macro variables
$p_i(\mathbf{S}_t)$, $p_i(\boldsymbol{\mu}_t^{\mathbf{P}})$, $p_i(\hat{\boldsymbol{\mu}}_t^{\mathbf{C}})$	Monotonic in systematic factors	
Cond. Independence	Problematic for multi-factor systematic risk	

Table 2.2: Assessment of the reduced-form models' setup.

[32]Note that $\mathcal{L}(X)$ denotes the law (or distribution) of a random variable X.

Starting with the first row of the table we note that while the CR-type models give no foundation for the employed distributions at all,[33] McK employs time series of macroeconomic indices for which the assumption of normally distributed innovation can be tested and might be adequate. However, except for a general intuition that country/industry default rates should be increasing functions of those indices, it cannot be justified why default rates should be constructed via a *logit* transformation of these indices. Nonetheless, this transformation, together with the indices' distribution, determines the distribution of $\hat{\boldsymbol{\mu}}_t^{\mathbf{C}}$. Turning to the second row of table 2.2, we observe that while all models build on the reasoning that default probabilities should be increasing in the systematic factors, none of the models delivers a foundation for the specific functional form employed. It is, however, worth noting that in the special case where conditional default probabilities are identical for all obligors ($p_i(\mathbf{S}_t) = p(\mathbf{S}_t)$ for all i), they are specified correctly in the single sector-version of CR, since $\mu_t^{\mathrm{P}}(\mathbf{S}_t) = p(\mathbf{S}_t)$. In this case it does not make sense to introduce an additional idiosyncratic sector (as proposed by CR), because the diversification effect that stems from the presence of idiosyncratic risk is already accounted for by the fact that default events are independent given $\mathbf{S}_t$. Concerning CR-GO, it should be noted that the assumption of statistical homogeneity within rating classes contradicts the intuitive argument that a firms' exposure to macroeconomic risk should depend on the type of business it is in rather than on its default probability.

We finally turn to the last row of table 2.2. We ask what happens if the distribution of the systematic factors and the conditional default probabilities are specified correctly (i.e. if the concerns raised in the first two rows of the table are not justified), and want to analyze in which cases the assumption of conditional independence given the realization of the systematic factors may be problematic. For the sake of clarity, we will consider a portfolio where all obligors are mapped to the same default rate (sector). Hence, for CR, defaults are assumed to be independent given $\mu_t^{\mathrm{P}} = (1/n)\sum_{i=1}^{n} p_i(\mathbf{S}_t)$. The corresponding statistic for McK can, by the law of the large numbers, be approximated by $\hat{\mu}_t^C \approx (1/N)\sum_{i=1}^{N} p_i(\mathbf{S}_t)$ where N is the number of firms in the industry/country from which the default rate is taken. Assuming - for the sake of simplicity - that the portfolio is representative for the corresponding industry/country (with respect to conditional default probabilities), we have $\hat{\mu}_t^C \approx \mu_t^{\mathrm{P}}$ and, hence, conditioning on μ_t^{P} and $\hat{\mu}_t^C$ will have the same effect. Finally, since we analyze one-year default probabilities in this paragraph, we will consider a risk management horizon of one year ($T = 1$).

[33] Except for the gamma distribution being a distribution that can be completely specified by choosing its mean and variance.

We start the analysis with the case where the vector $\mathbf{S}_1$ is a scalar (i.e. the dimension of $\mathbf{S}_1$ is 1). By our assumptions, CR-GO will be correct in that case and we obtain that this can also be said for CR and McK under the plausible assumption that the risk factor can be described in a way such that $\mu^{\mathrm{P}}(\cdot)$ is strictly de- or increasing in $\mathbf{S}_1$.

Lemma 2.2

Suppose that the distribution of μ_1^P and the conditional default probabilities $p_i(\cdot)$ are specified correctly, and that the vector $\mathbf{S}_1$ of systematic risk is a scalar. Then the one-year portfolio distribution produced by CR and McK is correct if $\mu^P(\cdot)$ is strictly de- or increasing in $\mathbf{S}_1$.

Now consider the case where the dimension of $\mathbf{S}_1$ is higher than 1. In this case it is unclear whether it is possible to construct a one-dimensional statistic of the systematic vector $\mathbf{S}_1$ that contains all the information relevant for the portfolio distribution (we call such a statistic *sufficient*). Before we present two cases where μ_1^{P} is indeed a sufficient statistic, we present a more precise definition of what sufficiency refers to in our context.

Definition 2.1

A statistic μ of the systematic variable $\mathbf{S}_1$ is called sufficient if the portfolio distributions derived under the assumption of conditional independence given $\mathbf{S}_1$ and under the assumption of independence given μ are the same.

Lemma 2.3

In the following cases μ_1^P is a sufficient statistic:

(i) If all loans in the portfolio are homogeneous with respect to size.

(ii) If all obligors are homogeneous with respect to their exposure to the systematic vector, i.e. if $p_i(\mathbf{S}_t) = c_i G(\mathbf{S}_t)$ with some constants $c_1, ..., c_n$ and an arbitrary function G mapping into the interval $[0, \min_{i=1,..,n}\{1/c_i\}]$.

In general, however, it will not be possible to construct sufficient statistics. In this case the assumption of conditional independence will understate portfolio risk. This is illustrated in the next example.

Example 3:

We consider a portfolio of 4 loans with face values $L_i = i$ $(i = 1, ..4)$. Unconditional default probabilities are assumed to be identical for all obligors ($\bar{p}_i =$

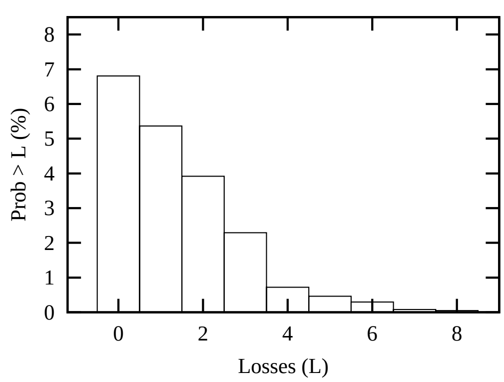

Figure 2.3: Correct specification.

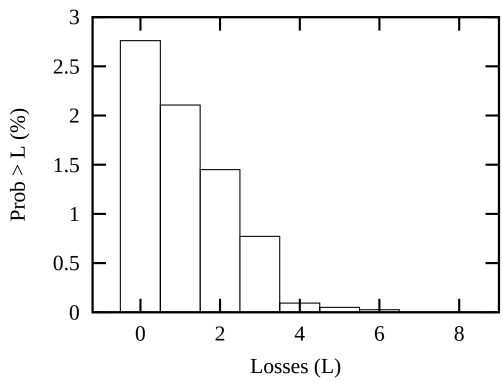

Figure 2.4: Assuming conditional independence given μ^{P}.

$\bar{p} = 0.2\%$) and the systematic vector $\mathbf{S}_1$ has dimension two: $\mathbf{S}_1 = (S_{1,1}, S_{2,1})$. Furthermore, we assume that obligors can be divided into two groups, A (which consists of obligors 1 and 2) and B (obligors 3 and 4) so that conditional default probabilities are identical within groups:

$$p_g(\mathbf{S}_1) = \Phi\Big(z - \theta_{g1}S_{1,t} - \theta_{g2}S_{2,t}\Big), \quad g = A, B.$$

Note that the specification of $p_g(\cdot)$ is taken from the two-factor version of KMV. We choose $\theta_{A1} = \theta_{B2} = 0.5$ and $\theta_{A2} = \theta_{B1} = 1$ and $\mathbf{S}_1$ to have independent components with mean zero and variance 1. Figures 2.3 and 2.4 show the probability of one-year portfolio losses exceeding the quantiles 0 to 8 for the correct specification and the specification assuming conditional independence given μ_1^{P}, respectively. Note that the scaling of both graphs is different, which illustrates the misspecification caused by assuming conditional independence for our example.[34]

For McK, homogeneity with respect to conditional default probabilities might be realistic if industry default rates are used but it should be problematic for country default rates. Concerning CR, it is generally unrealistic that the conditions of lemma 2.3 hold for the entire portfolio and, hence, portfolio risk will be underestimated. Moreover, note that the one-sector version, which has been used for this discussion is the most prudential one, producing the highest quantile for a given confidence level.[35] Therefore, the underestimation of portfolio risk will be even worse if the multi-sector version is used instead of the single-sector one.

[34]The formula for the conditional default probabilities is derived in the appendix.
[35]See Credit Suisse (1997), p. 23.

We conclude the discussion of the one-year framework by summarizing the conceptual critique of CR that has not already been presented. CR employs the economic intuition that the correlation between the defaults of single obligors can be accounted for by modeling the variation of the portfolio default rate due to changes in the macroeconomic environment. We have already argued that, using this intuition, it is inconsistent to introduce a second type of idiosyncratic risk on the sector level, since the diversification effect that stems from the presence of idiosyncratic risk is already accounted for by assuming that default events are independent given $\mathbf{S}_t$. Moreover, only the multi-sector version of the model can account for the fact that default probabilities of obligors depend on the systematic factors in different forms. The multi-sector version without weights is consistent with the default-rate intuition but would imply that default events of obligors in different sectors are independent of each other, which is not plausible. The multi-sector version with weights is constructed only by *formal* analogy to the non-weighted version: the expressions $1\{i \in \mathcal{S}_j\}$ are replaced by θ_{ij} in formulas (2.7) - (2.9). However, the CR manual does not mention how the variables $\mu^P_{j,t}$ $(j = 1, .., m)$ should be interpreted in this weighted multi-sector version.

It would be most natural to interpret the variables $\mu^P_{j,t}$ as weighted default rates by applying the same formal analogy as used to derive the formulas (2.7) - (2.9):[36]

$$\mu^P_{j,t} := \frac{1}{n}\sum_{i=1}^{n}\theta_{ij}p_i(\mathbf{S}_t). \tag{2.27}$$

This would also be consistent with formula (2.8) for the mean of $\mu^P_{j,t}$. However, it would not be consistent with formula (2.9) for the standard deviation of $\mu^P_{j,t}$. Moreover, inserting equation (2.10), which specifies the conditional default probabilities $p_i(\cdot)$, into the definition of $\mu^P_{j,t}$ given in (2.27) would imply that the following condition would have to hold if the model should be consistent:

$$\mu^P_{j,t} = \frac{1}{n}\sum_{i=1}^{n}\bar{p}_i\theta_{ij}\sum_{l=1}^{s}\theta_{il}\frac{\mu^P_{l,t}}{\bar{\mu}^{\mathrm{P}}_l} \qquad (j = 1, .., m).$$

This condition lacks a sensible economic interpretation. We are also not aware of any other plausible interpretation for the variables $\mu^P_{j,t}$ that is consistent with the formulas for conditional default probabilities and with the formulas for mean and standard deviation employed by CR (formulas 2.10 - 2.12). In summary, the economic interpretation

[36] I.e. by replacing $1\{i \in \mathcal{S}_j\}$ with θ_{ij} in the definition of $\mu^P_{j,t}$ for the non-weighted multi-sector version (see equation 2.6).

of the variables μ_j^P, which drive the joint-default behavior, is unclear in the weighted multi-sector version of CR. Consequently, the formulas for the mean and the standard deviation of these variables are somewhat arbitrary. This is in particular problematic since these formulas are a key ingredient to the calibration procedure.

Multi-Year Default Probabilities Recall that formula (2.1) provides the link between the one-year and the multi-year default probabilities for all reduced-form models. In the appendix we show that formula (2.1) rests on the assumption that

$$\mathbb{P}\{B_{i,t} = b_i \mid \mathbf{S}_t, ..., \mathbf{S}_T\} = \mathbb{P}\{B_{i,t} = b_i \mid \mathbf{S}_t\} \quad (t = 1, ..., T-1,\ i = 1, ..., n).$$

Hence, it is assumed that future realizations $\{\mathbf{S}_{t+1}, ..., \mathbf{S}_T\}$ of the systematic factors will not affect default probabilities in t, and only $\mathbf{S}_t$ is decisive. However, this is in sharp contrast to the basic idea of the structural approach to firm defaults. Since a firm's value reflects the discounted sum of all future cash flows, knowing that the macroeconomic environment will be bad in the future (e.g.) will affect the asset value of the firm today and hence will also affect the probability that equity holders choose to default on the obligations of the firm. Therefore, by projecting on the σ-algebra generated by the random vector $\mathbf{S}_t$ rather than on that generated by the vectors $\{\mathbf{S}_t, \mathbf{S}_{t+1}, ..., \mathbf{S}_T\}$, the volatility of $\mathbf{B}_T$ and therefore portfolio risk is underestimated.

Note also that while the implicit assumption expressed by equation (2.1) is consistent within the CR framework (since the systematic factors are assumed to be independent), this is not the case for McK (since McK explicitly stresses the importance of recognizing the autocorrelation of default-rate time series, which allows them to model changes of default probabilities over the business cycle). Of course this observation should not lead to the conclusion that CR dominates McK in this respect. Rather it should be stressed that the autocorrelation of default frequency series is an empirically well documented phenomenon,[37] which should be taken into account by CR and by McK.

2.3.1.2 Calibration

A major point of critique with respect to the calibration of all CR-type models is the following. While the model setup is built on the intuition that default probabilities change due to changes in systematic factors (of which the business cycle is the most

[37] See e.g. BÄR (2000).

important example), they do not condition parameter estimates on business cycle information. This will not only have the (rather obvious) effect that default probabilities will be underestimated in recessions. It will also - as we show in chapter 3 - have the effect that the variances $\mathrm{Var}(\hat{\mu}_\zeta^R)$ of rating-class default frequencies and, hence, the variances of default probabilities will be underestimated. Of course the opposite will be true during expansion states of the business cycle. Moreover, the rating class default-frequency series $(\hat{\mu}_{\zeta,t}^R)_{t=1}^{-T_h}$ is very likely serially dependent.[38] This is inconsistent with the assumptions posed by GORDY (2000) to set up an estimation technique for default-probability variances (see section 2.2.1.4).

Addressing the problems mentioned above is important, since the portfolio quantiles produced by CR-type models tend to be very sensitive with respect to changes in default probability variances as has been demonstrated for example by GORDY (2000). Additionally, to make CR-GO applicable, a method of determining the variance of the systematic factor $\mathbf{S}_t$ has to be developed.

Concerning McK, it remains unclear which statistical framework should be used for the estimation of the sensitivity parameter θ and whether this parameter is stable over the business cycle.

2.3.1.3 Suggestions

For all CR-type models, the most important issue is the adaptation of default probability moments to the business cycle. Recently, some authors have proposed statistical frameworks to predict default rates with macroeconomic variables.[39] They suggested that this might be a method of adjusting unconditional default probabilities in credit risk models. However, it is much more unclear how to adjust default-probability variances empirically. A starting point could be the framework of GORDY (2000), which has been presented in section 2.2.1.4. It could be extended by additionally modeling changes in conditional default probabilities $p_i(\mathbf{S}_t)$ over time while keeping the assumption that the systematic vector $\mathbf{S}_t$ is iid in time. A reasoning for this kind of modeling is that, as in the structural approach, $\mathbf{S}_t$ could represent returns on aggregate asset value indices. This would make the iid assumption for $\mathbf{S}_t$ look like a reasonable approximation. Explicitly constructing such indices - as has been done e.g. in ANDERSON AND SUNDARESAN (2000) - would also make it possible to determine the lacking estimate for $\mathrm{Var}(\mathbf{S}_t)$ in the CR-GO model.

[38] Default frequencies in t should be correlated with the respective frequencies in $t-1$ due to the common dependency on the business cycle. See for example BÄR (2000).

[39] Albeit only on the level of *country* default rates: see e.g. BÄR (2000).

Concerning CR, we think that a clarification of the theoretical model is necessary. Either the factors that drive joint defaults are not interpreted as portfolio default rates; then the model can be reduced to the CR-GO version. Or the portfolio default rate is still used as economic intuition; then it is necessary to construct a consistent model that (a) allows for a different dependence of obligors on systematic factors and (b) still has interpretable components. In particular, the role of the idiosyncratic sector should be clarified. We have demonstrated that by employing such a sector in the way it is done in the current CR framework, portfolio risk will be underestimated.[40]

Finally, the underestimation of portfolio risk arising in the context of multi-year default probabilities could be avoided for all reduced-form models if multi-year default probabilities were derived directly from multi-year default rates.

2.3.2 Structural Models

2.3.2.1 Model Setup

The structural approach rests on the following two building blocks.

1. The *joint* distribution of the obligors' asset value processes.
2. The obligors' capital structure.
3. The default event definition employed to determine joint default probabilities.

We first turn to the theoretical justification of the distributional assumptions concerning the firms' asset value processes. If (unanticipated) asset returns are independent in time, identically distributed over each time interval with the same length and have finite variance, then - by the central limit theorem - the asset value processes will follow a multidimensional geometric Brownian motion.[41] Serial independence in time can be derived from the efficient market hypothesis (see FAMA (1970)). Identical distribution over time intervals with the same length builds on the idea that the nature of unexpected asset returns does not change over time.

Can this model be backed by empirical results? When trying to test hypotheses about the distribution of asset returns, one faces the problem that asset market values cannot be observed directly, since not all of a firm's obligations are traded in the market.

[40] See the discussion in section 2.3.1.1.
[41] See e.g. COX AND MILLER (1990).

As a first approximation, one can investigate *equity* returns of firms with a very low probability of default. In this case, equity and asset returns should perform sufficiently similar. In general, studies on equity returns tend to find that - while not being perfectly accurate - the normal distribution seems to be a first approximation. However, there are important deviations of empirical return distributions from the normal family (in particular skewness and fat tails). A next natural step towards a better fit to empirical data is to drop the (rather artificial) assumption of finite variances in the reasoning given above. This enlarges the family of potential distributions for returns to the (parametric) class of stable-law distributions. Stable-law distributions can account for fat tails and skewness.[42] The second important empirical objection against a Brownian motion model of equity returns is time-varying volatility.[43]

These results on equity returns give first hints on whether the Brownian motion model might be a good description for a firm's asset value process. However, for a final assessment it is important to test this assumption more directly using asset-value series derived from stock prices and debt structures. In doing so, the two building blocks of the structural approach, distributional and capital structure assumptions, become interdependent. To our best knowledge, the only empirical evidence on the distribution of model-implied asset value returns has been provided by KMV. According to their studies, actual data conform quite well to this hypothesis for the univariate case,[44] but nothing is said about the accuracy of the multivariate distributional assumptions. Of course it also lies in the commercial interest of KMV to provide evidence in favor of their own approach; therefore these statements should be treated with some caution.

Concerning the second building block (capital structure assumptions), it should be noted that - compared to arrangements in the practice - KMV and NPV assume a quite simplistic capital structure. We will comment on this later when presenting our suggestions for the structural approach in section 2.3.2.3.

Finally, a major advantage of the NPV model compared to KMV and CM is that default is modeled as an absorbing barrier that can be reached at any time within the risk management horizon when equity holders choose to exercise their default option. KMV and CM on the other hand only allow for the default of a firm at the end of the risk management horizon which, of course, is unrealistic. On the other hand, the KMV technique of using theoretically derived univariate default probabilities not

[42] See Fama (1970) for a discussion of the literature and Rachev, Schwartz, and Khindanova (2000) for a more recent contribution.

[43] See again Rachev, Schwartz, and Khindanova (2000).

[44] See Crouhy, Galai, and Mark (2000), p. 87.

directly, but only as an intermediate measure for group-building, seems superior to the sole reliance on theoretical results. It is, however, completely unclear why the theoretical default probabilities are not employed *directly* as group-building indices. Using a different scaling, such as KMV's DD measure (which is the quantile of the default probability under the standard normal distribution), implies that the variation of default probabilities will be much higher in some classes than in others. This, of course, is unfavorable.

2.3.2.2 Calibration

The CM approach uses rating-class default frequencies to determine univariate default probabilities and equity-index return correlations as proxies for asset index return correlations. The former is problematic since it does not take into account that default probabilities vary through the business cycle. The latter might be an appropriate approximation for highly rated firms, for which equity and asset values should perform sufficiently similar. However, it will be problematic for firms with a substantial probability of default. Additionally, equity volatility and therefore equity index volatility is relatively unstable over time,[45] which is not reflected in CM's estimation procedure. Finally, it is also unsatisfactory that the relative size λ_i^S of a firm's systematic return component is chosen by rules of thumb and is not based on a quantitative analysis.

The approach of KMV and NPV avoids these problems by relying on asset value data. However, the proposed methods are not practicable for non-quoted firms.[46] Comparing the methods of determining asset correlations of JPM and KMV on the one hand and of NPV on the other hand, it should be noted that empirical studies on the correlation structure of *equity* returns come to the very robust result that traditional-industry-index models are dominated by market-index models. Provided that these results can be generalized for asset returns, the NPV approach is superior to the one employed by KMV. In this context it should also be noted that it is straightforward to make the current NPV model suitable for the management of international portfolios with obligors placed in different markets (e.g. US and Europe): the single-index model can easily be generalized to a multi-index one. Finally, recent research seems also to suggest that fundamental models, relating equity returns to macroeconomic variables,

[45] See e.g. Crouhy, Galai, and Mark (2000), p. 88.

[46] Note that KMV has extended its model to non-quoted firms. However, there is, to our knowledge, no publicly available documentation of this extension.

might outperform market-index models;[47] this points to a potential source of further improvements.

2.3.2.3 Suggestions

Our suggestions for the structural approach can be summarized as follows. First, an appropriate mix of the KMV and the NPV firm model can be seen as a reasonable starting point for producing asset value data and modeling joint defaults. Using this model, the most important assumptions about the distribution of the firms' asset value processes should be tested empirically, i.e. multivariate normality of asset returns and the stability of the parameters μ^v and σ^v over time.[48] Second, to fully specify the mixed structural model, a method for determining asset-return correlations has to be chosen. Using the asset-value data obtained from the model, empirical research should be conducted to assess which of the discussed approaches is the most promising (market-index models, industry-index models or fundamental models). Third, we suggest how to extend the KMV model to non-quoted firms. We will discuss each of these points in turn.

Firm Model When deciding about the appropriate option-pricing model to be employed to produce asset value data, one can choose among quite a variety of models. Both models proposed in the credit risk context (KMV and NPV) assume a very simple capital structure, which might be regarded as unrealistic for many firms. More sophisticated modifications of the original Merton model have been discussed in the literature (see BOHN (1999b) for an overview). However, as Bohn notes:[49]

> "The cost of these modifications is tractability. The more realistic the model becomes, the more complex is the resulting valuation equation. In some of the more extreme cases we must rely on numerical solutions which can be unintutitve and computationally expensive. Even in the cases where we can find closed-form solutions, we may lose clarity regarding the factors driving the value. More often than not, however, we end up with equations characterized by numerous parameters that are difficult to estimate. Finding the

[47] For an overview of the empirical evidence on correlation models for equity returns see ELTON AND GRUBER (1995).

[48] Remember that μ^v and σ^v denote the mean and the standard deviation of instantaneous asset returns respectively.

[49] See BOHN (1999b), p. 20.

> appropriate balance between realism and tractability requires assumptions and approximations. Empirical research can illuminate the aspects of these models that can be simplified or even ignored."

Our own assessment is very much along these lines. We therefore advocate to start with firm models that are as simple as possible. Empirical support for such a position has been provided by (Bohn 1999a) who showed that credit spread data can be fitted reasonably well with a model even simpler than the original Merton framework.

Both models discussed here (KMV and NPV) are simple enough to provide a reasonable starting point. The NPV approach seems to be more accurate with respect to the theoretical derivation of default probabilities (since it can account for default prior to the risk management horizon). On the other hand, the KMV technique of using theoretically derived univariate default probabilities not directly, but only as an index for group-building, seems superior to the sole reliance on theoretical results. However, group-building should rely *directly* on theoretical default probabilities and not on a derived measure such as KMV's "distance to default".

We therefore suggest to use the theoretical NPV default probabilities as index for group-building. Group default frequencies can then provide the estimates for the univariate default probabilities of the group members. Finally, *joint* default probabilities can again be calculated from the NPV formula, using exogenous default barriers k_i derived from the empirically determined univariate default probabilities. [50]

Using this "mixed" firm model, the most important underlying distributional assumptions about the firms' asset value processes should be tested empirically: multivariate normally distributed returns and the stability of the parameters μ^v and σ^v over time. Concerning the latter assumption, it should be noted that the well documented phenomenon of time-varying *equity*-return volatility can be explained by a firm model with stable *asset* return distributions over time. As has been pointed out by Bensoussan, Crouhy, and Galai (1994), stable asset return volatility would imply that equity-return volatility fluctuates with the firm's default probability, since the elasticity of the equity value with respect to the underlying asset value changes with a firm's leverage. Empirically investigating asset value models as proposed above could therefore also provide important insights for equity research.

[50] If the KMV version is used nevertheless, then at least the sensitivity of joint default probabilities should be assessed with respect to the simplifying assumption that default can only occur at the end of the risk management horizon.

Correlation Model To assess which of the proposed methods of determining asset return correlations works best, the empirical research on equity-return correlations should be reproduced for asset returns. The evidence of equity research suggests that it should be sensible to start with simple index models that explain correlations solely by the co-movements of the firms' asset values with the market. While performing robustly better than traditional-industry-index models for equity returns, such models would also have the charm of simplicity. However, it should be checked whether fundamental models might not further improve the correlation models' performance.

Finally, it should also be stressed that in risk management correlation models are intimately related to the topic of hedging undiversifiable risk. If the correlation models are constructed around publicly available and well documented indices, then a bank can hedge its exposure to these indices. Fundamental models rely on such indices per definition (e.g. interest rates, economic growth rates) and the method of KMV explicitly constructs such indices (which could be published). The approach of NPV would have to be refined in so far that an index for the asset value of the complete market would have to be constructed. This should, however, be achieved rather easily by aggregating microeconomic firm data.[51]

Extension to Non-Quoted Firms While CM can deal with non-quoted firms, we have already seen (a) that the determination of univariate default probabilities by CM is problematic, (b) that equity index returns might be too crude an approximation for asset returns and (c) that the grouping of firms according to traditional industries might not be optimal. If the KMV and NPV approach is extended to non-quoted firms, two issues have to be taken up. First, the determination of univariate default probabilities and second the derivation of asset return correlations. For both issues, group-building could be the method of choice. If groups of firms can be identified that are sufficiently homogeneous with respect to default probabilities and asset returns, and if the groups contain quoted as well as non-quoted firms, then the correlation and default probability results obtained for the former can be used as a proxy for the latter.[52]

For univariate default probabilities, rating classes may be a first obvious group choice which later on might be refined by industry- or country- specific rating classes.[53] Con-

[51] For examples of how to construct such aggregates see Anderson and Sundaresan (2000).

[52] Of course it is necessary that "group membership" is identifiable for quoted and non-quoted firms.

[53] Note that the crucial difference to the JPM method is that - by determining average rating class default probabilities for quoted firms from a KMV- or NPV-type firm model - estimates for univariate default probabilities will reflect business cycle information contained in the stock prices.

cerning asset-return correlations, remember that group building via traditional industries proved to be not particularly successful for equity-return correlations. It should therefore be thought of trying to construct pseudo industries as it is done in equity-return research. Started by FARELL (1974), who extracted four types of pseudo industries (growth stocks, cyclical stocks, stable stocks, and oil stocks), pseudo-industry models are applied in more sophisticated ways in today's investment banking practice.

To assess the relative performance of the different group-building techniques, estimates (of default probabilities and return correlations respectively) for quoted firms can be calculated once using the procedure that is only viable for quoted firms and once using group building. It can then be evaluated which group-building technique produces results that fit best to the estimates obtained under the quoted-firm procedure.

2.4 Conclusions

In this chapter we have discussed the joint-default models of the four major, currently available credit risk frameworks: CreditMetrics (CM), Portfolio Manager (by KMV), CreditRisk$^+$ (CR) and CreditPortfolioView (by McKinsey, McK). Moreover, we have also included two new versions of current joint-default models, one for CR, presented by GORDY (2000) (CR-GO), and one for KMV, presented by NICKELL, PARRAUDIN, AND VAROTTO (1999) (NPV). Following the literature we have divided the six models into two classes, labeled "structural models" (CM, KMV and NPV) and "reduced-form models" (CR, CR-GO and McK) respectively.

We have attempted to identify the most severe drawbacks of the proposed models and suggested measures for improving them. These measures contained short-term fixes as well as a long-term research agenda. In sections 2.4.1 and 2.4.2 we summarize our major insights with respect to reduced-form models and structural models respectively. Drawing on these results, we finally compare the structural and the reduced-form approach to joint-default modeling in section 2.4.3. We conclude that the mixed structural model proposed in section 2.3.2.3 is the conceptually most convincing basis for second-generation models.

2.4.1 Reduced-Form Models

Our findings with respect to the reduced-form models are as follows. First, we have argued that the arbitrary nature of the assumptions about the systematic risk factors' distribution and about the functional form of conditional default probabilities is the most important drawback of the reduced-form approach to joint-default modeling.

Second, the assumption of conditional independence given portfolio default rates (CR) or industry- (country-) default rates (McK) is adequate as long as all systematic risk can be represented by a *one*-dimensional random variable. This could be appropriate for portfolios where all obligors are placed in one single (equity) market (e.g. the US market).[54] However, for multi-market portfolios (e.g. US and Europe), more factors should be necessary. We have argued that if this is the case, McK should still perform quite well while CR will underestimate portfolio risk.

Third, we have demonstrated that all reduced-form models underestimate portfolio risk for multi-year risk management horizons; when calculating multi-year default probabilities from the one-year formulas, they neglect the influence of future realization of the systematic variables on today's conditional default probabilities. We have suggested that this could be avoided by directly modeling multi-year default rates.

Concerning the reduced-form models' calibration, we have argued that it is important for the CR-type models to adapt the estimations of mean and variance of default probabilities to the business cycle. We have outlined that this may be achieved by using returns on aggregate asset indices as systematic factors and by additionally modeling changes in conditional default probabilities over the business cycle. This would also allow to complete the calibration of CR-GO through a direct estimate of the systematic factor's variance. For McK, the estimation of the parameter that measures the sensitivity of default probabilities with respect to default-rate realizations seems to be the most problematic part of the model's calibration. It has to be assessed empirically whether a robust estimation technique can be found and whether this parameter is stable over the business cycle.

Finally, concerning CR, we think that a clarification of the theoretical model is necessary. Either the reduced version CR-GO is used or a consistent interpretation of the weighted multi-sector version should be developed. In particular, it has to be clarified why risk-reducing diversification effects enter the CR model via *two* channels. First, via the fact that default events are assumed to be independent given the systematic risk variable. Second, via an additional idiosyncratic sector. We have argued that the first channel is sufficient and that adding the second one will lead to an underestimation of portfolio risk.

[54] Research on equity-return correlations seems to suggest that a market portfolio index is a quite good measure for the systematic-risk exposure of firms in a common market (see ELTON AND GRUBER (1995)).

2.4.2 Structural Models

The second part of this chapter was concerned with structural models. We have pointed out that JPM has the important disadvantages that (a) univariate default probabilities are not sensitive to changes in the macroeconomic environment and (b) that equity index returns might be too crude an approximation for asset returns. Our analysis has therefore focused on the models of KMV and NPV, which avoid these problems. These models rely on asset value data which are produced by employing option-pricing. While recognizing that both models might be quite simplistic with respect to the specification of the capital structure, we have argued that it makes sense to start with such simple models.

Comparing both firm models, we have identified the most problematic assumption in the KMV model, namely that (as in CM) firms are assumed to default only at the end of the risk management horizon. NPV uses the more realistic approach of modeling default as an absorbing barrier of the firm's asset value process that can be reached at any time within the risk management horizon. We have therefore proposed that a "mixed" firm model employing the theoretical setup of NPV, and the group-building technique of KMV (to empirically determine univariate default probabilities) would be the best starting point for next-generation structural joint-default models. Using the asset value data produced by this mixed model, the following issues should be addressed empirically.

First, the adequacy of the assumptions about the distribution of asset returns should be assessed (multivariate normal distribution and stability of mean and standard deviation in time). Second, to fully specify the mixed structural model, a method of determining asset return correlations has to be chosen (market-index models, industry-index models or fundamental models). Using the asset value data obtained from the model, the best-performing method should be identified empirically.

Finally, we have suggested a method of extending the scope of structural models to non-quoted firms, which does not share the major deficits of the CM approach. Univariate default probabilities and asset return correlations could be determined by building groups of quoted and non-quoted firms. If group members are sufficiently homogeneous, results obtained for the quoted firms can then be used as a proxy for non-quoted ones. A case in point would be group-building via rating classes (for default probabilities) and empirically determined pseudo-industries (for asset correlations).

2.4.3 Comparison

Using the above observations and suggestions we finally want to argue why - in our view - the structural approach to the modeling of joint defaults should be favored. As has already been pointed out, the major drawback of the reduced-form models is the arbitrary nature of the distributional assumptions involved. When backtesting these models or comparing them with other models, it is completely unclear why the actually proposed distribution should be used and not *any* other distribution that does not violate the few qualitative restrictions that can be made. This question is especially hard to answer when recognizing that quantiles belonging to extreme probability levels are the most important output of credit risk models; these quantiles will very likely depend on the higher moments of the default rates' distribution[55] and on the specific functional form chosen for conditional default probabilities. Both cannot be estimated from empirical data.

Moreover, even if a potential default-rate distribution is determined, it will be difficult to test its accuracy empirically since default data are recorded on an annual rather than on a monthly or even shorter-term basis (as it is the case for stock returns). Aggravating this problem, there is strong theoretical and empirical evidence that default-rate distributions vary significantly over the business cycle.

In contrast, the structural approach can use the Brownian motion model for the asset-value process and a sufficiently simple capital structure as a starting point to test distributional assumptions and model performance. Data availability is much better and the case for the distributional stability of unexpected asset returns over the business cycle is much stronger than for default probabilities and default-rate distributions.

If the starting point proves to be too simplistic, it would be possible to move to more sophisticated models that are already available in the theoretical literature. Of course it could well turn out that reality is too complex for sufficiently simple firm models to be a sensible first-order approximation. But we think that the successful application of very simple structural models for default prediction, estimation of univariate default probabilities, and in the area of bond pricing is a quite encouraging sign for structural models to become a practicable and reasonably accurate approach to the management of default risk.[56]

[55] Recall that default rates are the systematic factors used by CR and McK.

[56] KMV has published several studies on the performance of its EDF measure with respect to default prediction and default probability estimation (for further assessments see also CROUHY, GALAI, AND MARK (2000), pp. 92-93). For the application to bond pricing see BOHN (1999a).

Do these conclusions stand up when extending the focus from the narrow view on joint-default modeling to the complete credit risk management framework where loans may not mature at the end of the risk horizon T and thus have to be repriced in T? In this general setting the case for structural models seems to be even stronger. Pricing loans is a natural application of the option-pricing models on which the structural approach is based. Indeed, these models have been developed to price bonds and were then extended to the field of credit portfolio management. The same cannot be said for the reduced-form models we have discussed. While CR does not allow for repricing of loans at all, McK can deal with repricing but employs an ad-hoc specification for pricing risky future cash flows that uses average credit spread data. KMV, in contrast, builds on a full-fledged risk-neutral-pricing framework.

It should, however, be noted that a new reduced-form model with a state-of-the-art contingent claims pricing framework and abstract systematic risk factors (instead of default rates) has been proposed by JARROW AND TURNBULL (2000). While this model does not share the methodological problems of McK and CR, it does share the major drawback of reduced-form models, namely the arbitrary nature of distributional assumptions. Moreover, the model is calibrated using price data for publicly traded bonds. This raises the question of how the bond prices entering the calibration have been determined in the first place, or more generally, how the different securities issued by a firm should be priced.

While a financial analyst can be expected to assess expected future cash flows and derive a firm's asset value by risk-adjusted discounting, she will need the guidance of a theoretical model to split the asset value into the values of a firm's different contingent securities (such as bonds and equity). But the generally accepted and also most convincing approach to do this is the option-pricing framework as employed by the structural approach.

In summary, the structural approach - compared to the reduced-form approach - presents a unified, consistent and more complete framework for the pricing and management of portfolios of corporate securities. These properties will prove to be particularly important for the development of future models that should allow to manage *all* assets of a bank within a single risk management framework.

2.A Appendix

Proof of lemma 2.2.

If $\mathbf{S}_1$ is a scalar and $\mu^{\mathrm{P}}(\cdot)$ is strictly de- or increasing in $\mathbf{S}_1$, then the event $\{\mathbf{S}_1 = s\}$ can be equivalently described by $\{\mu_1^{\mathrm{P}} = \mu^{\mathrm{P}}(s)\}$. Hence, denoting the density of $\mathbf{S}_1$ and μ_1^{P} by f_S and $f_{\mu^{\mathrm{P}}}$ respectively, we obtain:

$$\begin{aligned}
\mathbb{P}\{\mathbf{B}_1 = \mathbf{b}\} &= \int \prod_{i=1}^{n} \mathbb{P}\{B_{i,1} = b_i \,|\, \mathbf{S}_1 = s\} f_S(s)\, ds \\
&= \int \prod_{i=1}^{n} \mathbb{P}\{B_{i,1} = b_i \,|\, \mu_1^{\mathrm{P}} = \mu^{\mathrm{P}}(s)\} f_S(s)\, ds \\
&= \int \prod_{i=1}^{n} \mathbb{P}\{B_{i,1} = b_i \,|\, \mu_1^{\mathrm{P}} = r\} f_{\mu^{\mathrm{P}}}(r)\, dr.
\end{aligned}$$

$\square$

Proof of lemma 2.3.

(i) If loans are homogeneous with respect to size (i.e. $L_i = L$ for $i = 1, ..., n$), then

$$\begin{aligned}
\mathbb{P}\{\sum_{i=1}^{n} L_i B_{i,1} = q\} &= \mathbb{E}\,\mathbb{P}\Big\{\sum_{i=1}^{n} B_{i,1} = q/L \,|\, \mathbf{S}_1\Big\} \\
&= \mathbb{E}\,\mathbb{P}\Big\{X_{\mu^{\mathrm{P}}(\mathbf{S}_1)} = q/L\Big\}
\end{aligned}$$

where $X_{\mu^{\mathrm{P}}}$ is a random variable with $\mathcal{L}(X_{\mu^{\mathrm{P}}}) = \mathcal{L}(\sum_{i=1}^{n} X_i)$ and $X_1, .., X_n$ are independent Bernoulli variables with $\sum_{i=1}^{n} \mathbb{P}\{X_i = 0\} = \mu^{\mathrm{P}}$. The same formula is derived under the assumption of conditional independence given μ_1^{P}.

(ii) If conditional default probabilities are homogeneous, then we obtain

$$\mathbb{P}\{\mathbf{B}_1 = \mathbf{b}\} = \mathbb{E}\, G(\mathbf{S}_1)^n \prod_{i=1}^{n} c_i.$$

Moreover, $\mu_1^{\mathrm{P}} = G(\mathbf{S}_1)(\sum_{i=1}^{n} c_i)$, implying that conditional default probabilities are given by $p_i(\mu_1^{\mathrm{P}}) = c_i \mu_1^{\mathrm{P}} / (\sum_{i=1}^{n} c_i)$. This in turn shows that joint probabilities are calculated correctly under the assumption of conditional independence given μ_1^{P}.

$\square$

Derivation of conditional default probabilities for example 3

To derive the conditional default probabilities, we assume that, for arbitrary s_1, $\mu^{\mathrm{P}}(s_1,\cdot)$ is a strictly decreasing, continuous function of the second systematic factor $S_{2,t}$ and that $\mu^{\mathrm{P}}(s_1,\cdot)$ takes all values between 0 and 1 if s_2 varies between $-\infty$ and $+\infty$.[57] This implies that if $\mu_t^{\mathrm{P}} = r$, then for an arbitrary realization s_1 of $S_{1,t}$ there is an unique value $S_2(s_1,r)$ so that $\mu^{\mathrm{P}}\Big(s_1, S_2(s_1,r)\Big) = r$ (Mean value theorem). Therefore

$$p_i(\mu_t^{\mathrm{P}}) = \frac{\int_{-\infty}^{+\infty} p_i\Big(s_1, S_2(s_1,\mu_t^{\mathrm{P}})\Big)\, f\Big(s_1, S_2(s_1,\mu_t^{\mathrm{P}})\Big)\, ds_1}{\int_{-\infty}^{+\infty} f\Big(s_1, S_2(s_1,\mu_t^{\mathrm{P}})\Big)\, ds_1}.$$

Multi-year default probabilities (reduced-form models)

Without loss of generality we assume that $T = 2$ and that $B_{i,2} = 0$ for $i = 1, .., l$ and $B_{i,2} = 1$ for $i = l+1, .., n$. In this case we obtain:

$$\begin{aligned}
\mathbb{P}\Big\{\mathbf{B}_2 = \mathbf{b}_2\Big\} &= \mathbb{E}\,\mathbb{P}\Big\{\mathbf{B}_2 = \mathbf{b}_2 \,|\, \mathbf{S}_2, \mathbf{B}_1\Big\} \\
&= \mathbb{E}\Bigg[\prod_{i=1}^{l} 1\{B_{i,1} = 0\}[1 - p_i(\mathbf{S}_2)] \\
&\qquad \cdot \prod_{i=l+1}^{n} \Big\{1\{B_{i,1} = 0\}p_i(\mathbf{S}_2) + 1\{B_{i,1} = 1\}\Big\}\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{i=1}^{l} \mathbb{P}\{B_{i,1} = 0|\mathbf{S}_1, \mathbf{S}_2\}[1 - p_i(\mathbf{S}_2)] \\
&\qquad \cdot \prod_{i=l+1}^{n} \Big\{\mathbb{P}\{B_{i,1} = 0|\mathbf{S}_1, \mathbf{S}_2\}p_i(\mathbf{S}_2) + \mathbb{P}\{B_{i,1} = 1|\mathbf{S}_1, \mathbf{S}_2\}\Big\}\Bigg].
\end{aligned}$$

Note that the last equation can be derived by applying the conditional expectation operator $\mathbb{E}[\cdot|\mathbf{S}_1, \mathbf{S}_2]$. On the other hand, using formula (2.1) for $T = 2$, we find that

$$\begin{aligned}
\mathbb{P}\Big\{\mathbf{B}_2 = \mathbf{b}_2\Big\} &= \mathbb{E}\,\mathbb{P}\Big\{\mathbf{B}_2 = \mathbf{b}_2 \,|\, \mathbf{S}_1, \mathbf{S}_2\Big\} \\
&= \mathbb{E}\Bigg[\prod_{i=1}^{l} [1 - p_i(\mathbf{S}_1)] \cdot [(1 - p_i(\mathbf{S}_2)] \\
&\qquad \cdot \prod_{i=l+1}^{n} \Big\{[1 - p_i(\mathbf{S}_1)]p_i(\mathbf{S}_2) + p_i(\mathbf{S}_1)\Big\}\Bigg].
\end{aligned}$$

[57] Note that this condition is fullfilled for the specifications of $p_i(\cdot)$ given in example 3.

Comparing expressions leads to the statement that, by using formula (2.1) to determine multi-year default probabilities from the one-year framework, one assumes that $\mathbb{P}\{B_{i,1} = b_i \mid \mathbf{S}_1, \mathbf{S}_2\} = \mathbb{P}\{B_{i,1} = b_i \mid \mathbf{S}_1\}$ $(i = 1, .., n)$.

Chapter 3

Default Probabilities and Default Correlations

Abstract. Starting from the Merton framework for firm defaults, we provide the analytics and robustness of the relationship between default probabilities and default correlations. We then derive the implications of these results for the impact of macroeconomic shocks on credit portfolios, for the pricing of loans, and for the design of credit risk models.

Keywords: Credit portfolio management, Default correlations, Pricing of loans, Macroeconomic risk, Credit risk models.

JEL: G11, G12, G21, G31

3.1 Introduction

During the last two decades portfolio considerations have become a central issue in credit risk management. A crucial ingredient for any portfolio consideration in the credit risk context are the correlations of loan returns. They are termed default correlations in the literature since the probabilities of joint defaults are the major building blocks of loan correlations. In this chapter we examine the relationship between (univariate) default probabilities and default correlations. The analysis is motivated by two questions.

The first question concerns the pricing of loans with different default probabilities. In current practice, loan prices usually merely reflect the impact of higher default probabilities on expected returns. If, however, loans with a higher default probability also contribute more to the portfolio standard deviation (as we will show to be the case), then the marginal increase of economic capital when adding such a loan to the portfolio will be higher than for loans with lower default probabilities. Moreover, if these differences are substantial (as we will also show), then it is important for loan prices to reflect these differences. By the variance-covariance formula, the contribution of a loan to a portfolio's standard deviation consists of its own standard deviation and of its correlations with other loan returns, i.e. its default correlations. In the simplest setting, the standard deviation of a loan is a multiple of $\sqrt{p(1-p)}$ where p is the firm's default probability.[1] Therefore it is easy to see that the standard deviation will increase in p. This is called the *variance effect* of an increase in default probabilities. However, it is not clear how default correlations will react to changes in p. This effect is called the *correlation effect*.

Second, it has long been recognized that default probabilities change with the state of the economy and that credit risk models should take this into account, since higher default probabilities imply higher expected losses.[2] However, the way in which these changes in default probabilities affect the second important building block of the loan portfolio distribution - the portfolio standard deviation - has only recently been addressed. Using a simulation approach, GERSBACH AND LIPPONER (2000) have demonstrated that adverse macroeconomic shocks - by increasing default probabilities - can raise default correlations. They show that this effect may account for more than 50% of the increase in the credit risk caused by the shock. In this chapter we will provide

[1] In such a setting, the return on the loan can be described as multiple of a Bernoulli variable that is either equal to 1 (default) with probability p or equal to 0 (no default) with probability $(1-p)$.

[2] See e.g. WILSON (1998) or, more recently CROUHY, GALAI, AND MARK (2000).

an analytic foundation for the connection between negative macroeconomic shocks and loan default correlations.

Following the structural approach to credit risk, which has already been described in chapter 2, we construct our model along the same lines as MERTON (1974). The returns on a firm's assets are assumed to be normally distributed and loans are modeled as a claim on the value of the firm. This value is measured by the price at which the firm's total liabilities can be purchased; it is thus equal to the value of the stock and the value of the debt. Default on loans occurs if the market value of the firm falls below a certain threshold which depends on the firm's liability structure. In a sufficiently simple framework, the joint default behavior of two firms can therefore be described by two indicator variables $1\{Z_1 \leq z_1\}$ and $1\{Z_2 \leq z_2\}$.[3] Z_1 and Z_2 are two normalized, correlated, jointly normally distributed random variables that describe the firm's standardized returns. If, for example, the standardized returns of the first firm fall below the threshold z_1, the firm will default. We call z_1 and z_2 the respective default points of the firms. An increase in default probabilities will shift default points to the right. We examine how such a shift changes the default correlation of the two firms.

Our findings are as follows. Default correlations increase under a homogeneous shift to the right (i.e. both default points increase by the same amount). The same is true if the shift is more pronounced for the firm with the lower likelihood of default. Default correlations may only decline if the downward shift for the lower rated firm is significantly higher than that for the higher rated firm.

Furthermore, at a structural level the correlation effect is made up of two interwined effects. First, when default points move to the right, the skewness of each of the indicator variables will be reduced.[4] As a consequence, they will reveal more information about the correlated underlying firm returns. Default correlations rise and move closer towards return correlations. We call this phenomenon the *skewness effect* (SE). Second, the distance $d := z_2 - z_1$ between default points may change. Default correlations should decrease in d since it will become harder to infer the state (default/no default) of one firm when observing the state of the other firm. We call this effect the *distance-*

[3]Note how the indicator function $1\{\cdot\}$ is defined. $1\{A\}$ is equal to 1 if statement A holds and equal to 0 if statement A does not hold.

[4]The term "skewness" in the context of Bernoulli variables refers to the fact that the probability of one outcome is higher than the probability of the other one. In our case both indicator variables would be unskewed for $z_1 = z_2 = 0$. Moreover, note that default probabilities of loans are usually lower than 50%, implying that z_1 and z_2 are smaller than zero and that the binary default variables are skewed towards zero. A shift to the right will therefore reduce skewness.

of-default-points effect (DDE). However, changing the distance between default points necessarily changes the distance from zero for at least one default point. Hence DDE cannot be completely separated from SE. If the default point of the firm with the lower probability of default increases more than that of the other firm, skewness increases and d decreases, which implies that both effects work in the same direction and default correlations increase. In the opposite case, both effects work in different directions and it will depend on the parameters (the location of z_1 and z_2 and the asset correlation) whether the default correlations increase or decrease.

With respect to the two questions posed at the beginning of this introduction, our results suggest that loan prices should reflect the higher contributions to economic capital of loans with a higher default probability. Moreover, we indicate that portfolio standard deviation and hence economic capital can increase significantly under negative macroeconomic shocks. While for the pricing of loans both effects (SE and DDE) are relevant, we argue that, for the impact of macroeconomic shocks on credit portfolios, the distance-of-default-points effect will tend to cancel out while the skewness effect remains. We will also discuss the consequences of these results for the adaptation of credit risk models to the business cycle.

Finally, our results remain robust under various generalizations of our original model. First, we allow for endogenous recovery rates where the severity of the default determines the value of firm assets that can be recovered. Second, by considering loan maturities that are longer than the risk management horizon, we address scenarios where changes in the portfolio value stem from rating migrations rather than from firm defaults. We show that the qualitative nature of our results is robust with respect to such scenarios. Finally, we demonstrate that any alternative distribution for asset returns yields the same results as long as a monotonic transformation into a bivariate normal distribution exists.

This chapter is organized as follows. In the next section we introduce the model and present our analytic results. Moreover, we discuss and illustrate the consequences of these results for credit risk management. In section 3.3 we investigate the robustness of our results with respect to crucial assumptions. Section 3.4 presents our conclusions.

3.2 Analytic Results and Applications

In this section we analyze the relationship between default probabilities and default correlations, and discuss the consequences of this analysis for the impact of macroeconomic shocks on portfolio standard deviation and for the pricing of loans. In section 3.2.1 we present the model, and in section 3.2.2 we analyze the relationship between default probabilities and default correlations qualitatively, deriving our main analytic results. The applications of these results are discussed in section 3.2.3.

3.2.1 The Model

As a starting point, we employ the risk-of-ruin or option-pricing model developed in WILCOX (1973), MERTON (1974) and SCOTT (1981). The probability of a firm going bankrupt depends on both the market value of the firm's assets relative to its outside debt and on the volatility of the market value of the assets.

Using t as time index, we consider a bank holding a credit portfolio consisting of loans to two firms (1 and 2) and undertaking risk management in $t = t_1$. The loans are due in $t = t_2$, and we assume that the bank's risk management horizon is identical with the date at which the loans mature, i.e. the bank is interested in the distribution of the $t = t_2$ value of its portfolio.[5] We denote the two firms' asset values in t by $V_{1,t}$ and $V_{2,t}$ respectively, and assume that the debt obligations of both firms are due in $t = t_2$ (we denote the sum of these obligations by v_1 and v_2 respectively). According to the option-pricing model, default of firm i in $t = t_2$ occurs if $V_{i,t_2} \leq v_i$. We assume that in this case the firm will repay an exogenously determined fraction of the loan's principal (recovery rate),[6] while in the other case the complete amount is repaid. Therefore, the stochastics of the portfolio payoff in $t = t_2$ can be characterized by the joint distribution of the binomial random variables $1\{V_{1,t_2} \leq v_1\}$ and $1\{V_{2,t_2} \leq v_2\}$. In the standard framework of the option pricing approach, this distribution is characterized via the distribution of the continuously compounded rates of asset returns $Z_{i,t_2} := \log(V_{i,t_2}/V_{i,t_1})$. The vector (Z_{1,t_2}, Z_{2,t_2}) is assumed to be independent of (V_{1,t_1}, V_{2,t_1}) and bivariate normally distributed with correlation coefficient $\rho > 0$.[7] Note that the event $V_{i,t_2} \leq v_i$ can be equally well described as

$$Z_{i,t_2} \leq \log(v_i) - \log(V_{i,t_1}). \tag{3.1}$$

Moreover, from a $t = t_1$ perspective the vector (V_{1,t_1}, V_{2,t_1}) is fixed and the joint distribution of (Z_{1,t_2}, Z_{2,t_2}) does not depend on the realization of this vector. Hence we can

[5] In section 3.3.2 we consider the case where the loans mature after the risk management horizon.

[6] We consider the case of endogenously determined recovery rates in section 3.3.1.

[7] Note that this scenario is usually derived from an extension of the MERTON (1974) framework. Asset values in time are described by a two dimensional geometric Wiener process. This model is described in more detail in section 3.3.2.

normalize equation (3.1) with respect to mean and variance of Z_{i,t_2}. We conclude that it is sufficient to analyze the joint distribution of the Bernoulli variables $1\{Z_1 \leq z_1\}$ and $1\{Z_2 \leq z_2\}$ where (Z_1, Z_2) are standardized, bivariate normally distributed random variables with correlation ρ and

$$z_i := \frac{\log(v_i) - \log(V_{i,t_1}) - \mathbb{E}\, Z_{i,t_2}}{\sqrt{\mathrm{Var}(Z_{i,t_2})}} \quad (i = 1, 2). \tag{3.2}$$

Throughout this chapter we will assume that $z_1 \leq z_2$. The correlation between the two Bernoulli variables is termed default correlation and is denoted by $\rho^{\mathrm{def}} = \rho^{\mathrm{def}}(z_1, z_2, \rho)$. Note that since default probabilities are given by $p_i = \Phi(z_i)$,[8] the relationship between default points (z_1, z_2) and ρ^{def} monotonically translates into a respective relationship between default probabilities (p_1, p_2) and ρ^{def}. We will use either of these representations as convenient.

3.2.2 Analytic Results

In this section we describe the relationship between default probabilities and default correlations in qualitative terms. If default probabilities change, default points will also change accordingly. Such a shift in default points has two consequences that prove to be important in understanding the relationship between default probabilities and default correlations. First, the distance of the default points from zero will change. Second, the distance $d = z_2 - z_1$ between default points may change. In order to isolate these two effects, we first consider an increase of z_1 and z_2 with the distance between the default points remaining constant. We denote the partial derivatives of ρ^{def} with respect to z_1 and z_2 by ρ_1^{def} and ρ_2^{def} respectively and obtain the following result.

Proposition 3.1
Consider a homogeneous move of both default points to the right (i.e. a move where the distance between default points remains constant).

(i) If $p_1, p_2 < 50\%$, then default correlations ***increase*** *($\rho_1^{\mathrm{def}} + \rho_2^{\mathrm{def}} > 0$);*

(ii) If $p_1, p_2 > 50\%$, then default correlations ***decrease*** *($\rho_1^{\mathrm{def}} + \rho_2^{\mathrm{def}} < 0$).*

As for all other propositions, the proof of proposition 3.1 is given in appendix 3.A. The reasoning behind proposition 3.1 runs as follows: If $z_i < 0$ $(i = 1, 2)$, then a

[8] $\Phi(\cdot)$ denotes the cumulative standard normal density function.

homogeneous shift of z_1 and z_2 to the right will reduce the skewness of the binary variables $1\{Z_1 \leq z_1\}$ and $1\{Z_2 \leq z_2\}$. The less skewed these binary variables are, the more information they reveal about the correlated underlying variables Z_1 and Z_2. Accordingly, default correlations increase towards the higher correlations of returns. If $z_1, z_2 > 0$, skewness increases when default points shift to the right and hence default correlations decrease. We call this phenomenon the skewness effect (SE). In practical applications we are mainly interested in the case $z_1, z_2 < 0$ since default probabilities higher than 50% are not relevant. Nevertheless, the result for $z_1, z_2 > 0$ confirms the reasoning we propose. For the rest of the analysis we will focus on the case $z_1, z_2 < 0$.[9]

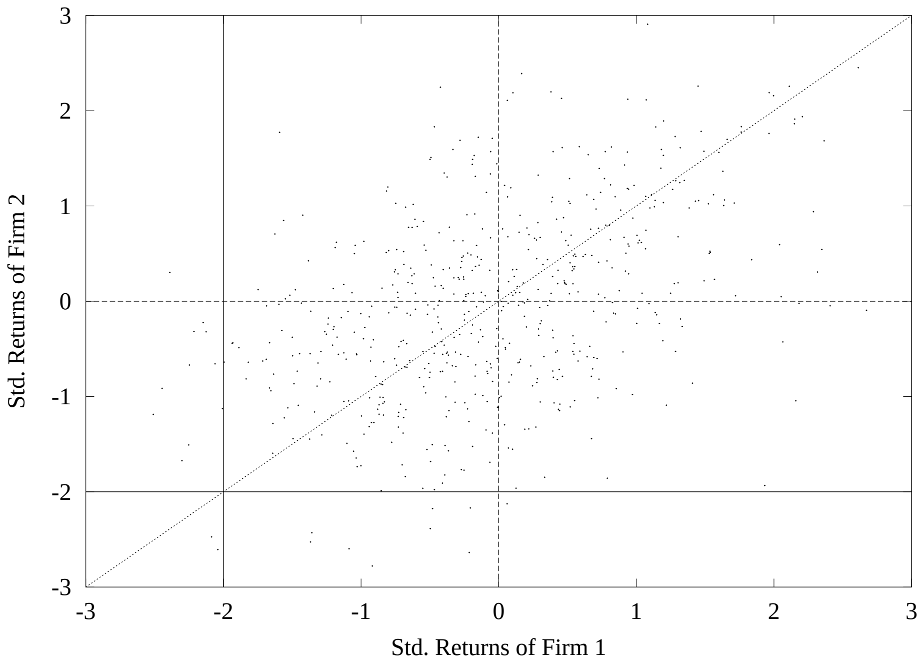

Figure 3.1: Scatter plot of two standardized, bivariate normally distributed random variables with correlation $\rho = 50\%$.

Why does a reduction of skewness increase default correlations? Figure 3.1 shows the scatter plot of two normally distributed correlated random variables describing realizations of the pair (Z_1, Z_2). Note that the corresponding scatter plot for the derived binary variables $1\{Z_i \leq z_i\}$ $(i = 1, 2)$ would depict only four points ($(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$). The frequency with which each of these four possible realizations occurs

[9] Note that results for $z_1, z_2 > 0$ and for negative correlations are also available in most cases. But since these parameter constellations are not relevant in practice we will not discuss them.

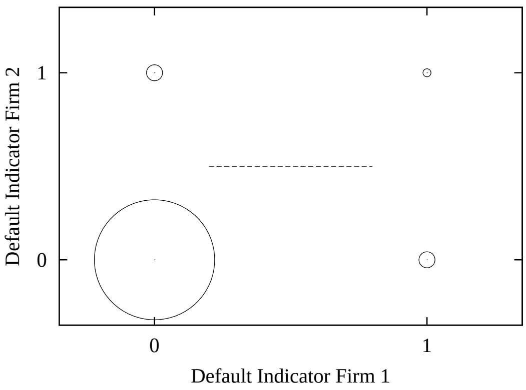

Figure 3.2: Stylized "Scatter Plot" of the indicator variables: asymmetric (skewed) case. Almost no upward sloping tendency in the data.

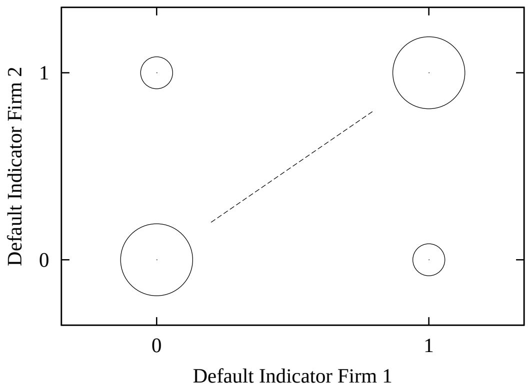

Figure 3.3: Stylized "Scatter Plot" of the indicator variables: symmetric (unskewed) case. Strong upward sloping tendency in the data.

can be inferred from figure 3.1 by counting the number of points in the respective quadrants of the two "coordinate systems" inserted in the figure. The system depicted with solid lines illustrates the case $z_1 = z_2 = -2$. In this case the distribution of the indicator variables is strongly asymmetric: nearly all data points lie in $(0,0)$ (both firms survive) while there are only very few points in $(1,1)$ (both firms default).

What happens to the frequencies when we move the origin of the coordinate system along the (broken) 45^o line from $(-2,-2)$ to $(0,0)$ (the origin of the system with the broken lines)? The major effect is that point mass is shifted from $(0,0)$ to $(1,1)$ which reduces the asymmetry of the distribution. As a consequence, the upward sloping tendency in the data increases and this is a manifestation of a higher correlation coefficient ρ^{def}. This is illustrated in figures 3.2 and 3.3. The size of the circles around the four possible realizations is used to illustrate the number of observations (big circle - many observation, small circle - few observation).

Finally, note that the skewness effect consists of two counteracting effects concerning the information revealed about $1\{Z_2 \leq z_2\}$ when observing $1\{Z_1 \leq z_1\}$ (or vice versa). The information content of the event $1\{Z_1 \leq z_1\} = 1$ decreases when default points increase jointly, while the information content of the event $1\{Z_1 \leq z_1\} = 0$ increases. This is because the information about the underlying return realizations decreases (increases), which in turn impacts on the information revealed about the other binary variable. If firm 1 defaults, then one can infer that $Z_1 \in (-\infty, z_1)$. This interval increases with z_1 decreasing the information available about Z_1 from the default event. This in turn implies that less information about Z_2 and, hence, about $1\{Z_2 \leq z_2\}$ is obtained. The opposite is true for the non-default event. If firm 1 does not default, then $Z_1 \in (z_1, \infty)$, an interval decreasing in size if z_1 increases.

We illustrate this point for fully correlated firm returns ($\rho = 100\%$). Conditional on the default of firm 2, the probability that firm 1 will also default is $\Phi(z_1)/\Phi(z_2)$, and the ratio of conditional and unconditional default probability for firm 1 is therefore given by $1/\Phi(z_2)$. Hence, this ratio (and therefore the information content of firm 2's default) decreases if z_1 and z_2 increase. The contrary is true for the information content of the event where firm 1 has not defaulted. Conditional on the information that firm 1 has survived, the probability that firm 2 has also survived is $[1 - \Phi(z_2)]/[1 - \Phi(z_1)]$. Hence the ratio of the conditional and unconditional probability that firm 2 will survive is given by $1/[1 - \Phi(z_1)]$ and increases if skewness is reduced. As can be seen from the arguments in the previous paragraphs, the effect of non-default event information increasing dominates the effect of default event information decreasing.

We now turn to the second consequence of a change in default probabilities, namely that the distance d between default points can change. The reasoning based on information revelation about return realizations implies that default correlations should decrease in d. We call this effect the distance-of-default-points effect (DDE). Unfortunately, DDE cannot be completely separated from SE since changing the distance between default points necessarily changes the distance from zero for at least one default point. Suppose, for example, that one default point is fixed while the other one moves to the right. If the smaller of the two points, z_1, moves, then both effects should work in the same direction. Skewness is reduced and the distance between default points decreases, which should increase default correlations. If, however, z_2 moves, then the two effects work in opposite directions and it is no longer clear which one dominates the other. Proposition 3.2 shows that whether ρ^{def} decreases or increases depends on the default point ratio $\lambda := z_1/z_2$ and on the asset-return correlation ρ. To prepare for the formulation of proposition 3.2, we define

$$\rho_+(\lambda) := (25/32)\left\{\lambda - \sqrt{\lambda^2 + 24/25}\right\}.$$

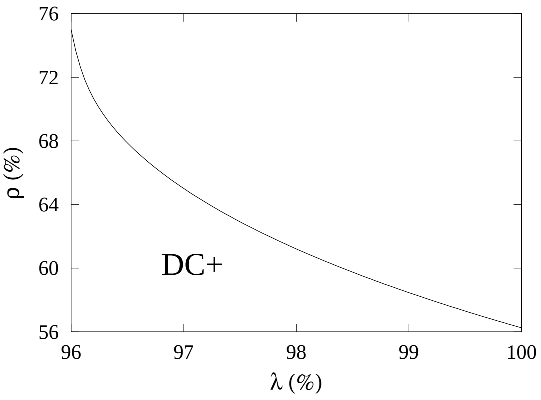

Figure 3.4: The set DC+ contains all points below the solid line and visualizes all combinations (λ, ρ) that fulfill the inequality $\rho < \rho_+(\lambda)$.

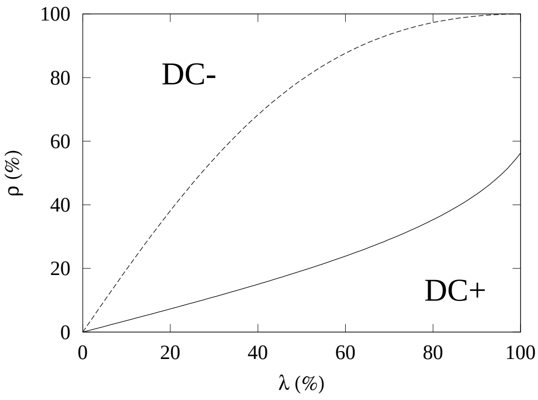

Figure 3.5: The set DC+ contains all points below the solid line and visualizes all combinations (λ, ρ) that fulfill the inequality $\rho < \rho_+(1/\lambda)$. The set DC- contains all points above the broken line and visualizes all combinations (λ, ρ) that fulfill the inequality $\rho > 2\lambda/(1+\lambda^2)$.

Proposition 3.2
Suppose that $p_1, p_2 < 50\%$ and consider a move of only one default point to the right.

*(i) If z_1 moves, then default correlations **increase** ($\rho_1^{\text{def}} > 0$) if $\rho < \rho_+(\lambda)$. Note that this inequality is fulfilled if $\lambda \leq 96\%$ or if $\rho \leq 56\%$.*

(ii) If z_2 moves, then default correlations

- ***increase** ($\rho_2^{\text{def}} > 0$) if $\rho < \rho_+(1/\lambda)$*
- ***decrease** ($\rho_2^{\text{def}} < 0$) if $\rho > 2\lambda/(1+\lambda^2)$.*

Figures 3.4 and 3.5 visualize the parameter sets for which, according to our theoretical results, default correlations will increase (DC+) or decrease (DC-). Figure 3.4 depicts the case where z_1 moves. In this case, default correlations increase if (λ, ρ) lies in DC+. The intuition developed for SE and DDE would imply the more general statement that $\rho_1^{\text{def}} > 0$ for all $0 < \lambda < 1$ and $0 < \rho \leq 1$. While a formal proof is not in reach yet, our simulation exercises have confirmed this conjecture. Figure 3.5, on the other hand, depicts the case where z_2 moves. In this case, default correlations increase if (λ, ρ) lies in DC+ and decrease if (λ, ρ) lies in DC-.

Figures 3.6 and 3.7 illustrate the two effects identified in propositions 3.1 and 3.2. Figure 3.6 demonstrates the skewness effect for the symmetric case where both firms

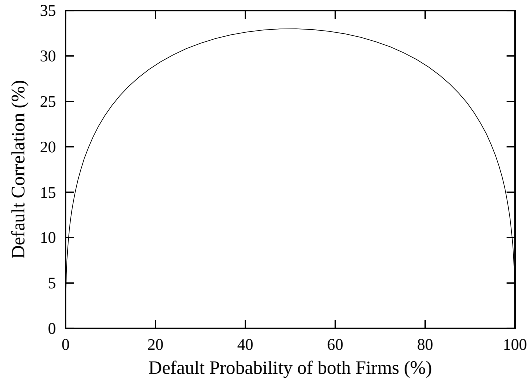

Figure 3.6: The skewness effect. The figure depicts the default correlation of two firms with the same default probability p when p ranges from 0 to 100% ($\rho = 50\%$).

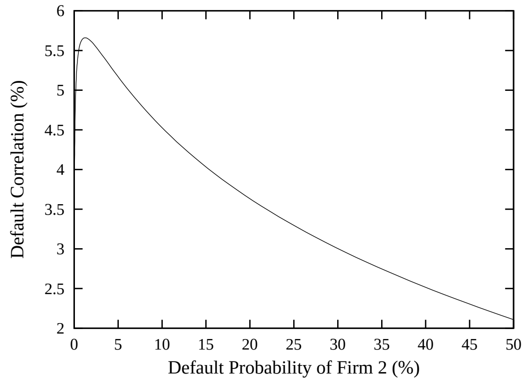

Figure 3.7: The distance-of-default-points effect. The figure depicts the default correlation between firm 1 ($p_1 = 0.05\%$) and firm 2 (p_2 ranging from 0 to 50%) ($\rho = 50\%$).

have the same default probability. Default correlations are maximum when the binary variables are unskewed, i.e. when default probabilities are equal to 50%. Figure 3.7 visualizes the distance-of-default-points effect for the case where firm 1 has a default probability of 0.05% and the default probability of firm 2 varies from 0.05% to 50%. In this case default correlations first increase (high λ) and then decrease (low λ).

Finally, figure 3.8 summarizes the major insights from propositions 3.1 and 3.2 and our conjecture. Default correlations increase if default points move to the right and if the move of the default point associated with the lower default probability is more pronounced. An analogous result holds if default points move to the left. In the question-mark ranges both default points move to the left or to the right but it depends on the parameter vector (λ, ρ) whether default correlations will increase or decrease.

The next proposition completes our picture of ρ^{def} by exploring the boundary cases $z_1 = z_2 = 0$ and $z_1, z_2 \to -\infty$.

Proposition 3.3

(i) ρ^{def} *has a local maximum in* $z_1 = z_2 = 0$ *and*

$$\rho^{def}(0, 0, \rho) = \frac{2}{\pi} \arcsin(\rho).$$

(ii) If $\rho < 1$, *then* $\lim_{z \to -\infty} \rho^{def}(z, z, \rho) = 0$. *Moreover,* $\rho^{def}(z, z, 1) = 1$ *for all* z.

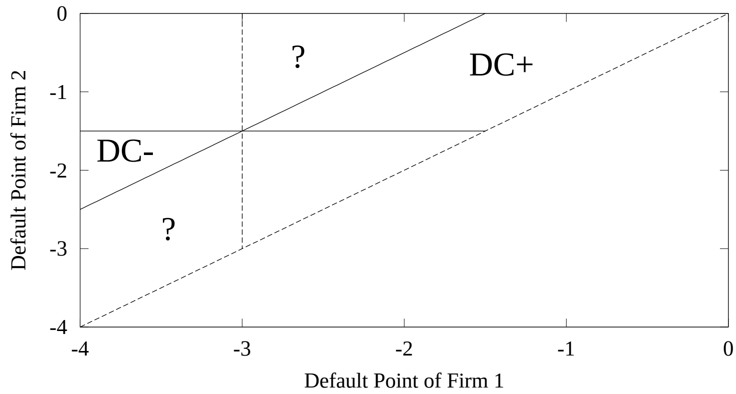

Figure 3.8: For a given combination of default points $(-3, -1.5)$, the figure depicts the default-point range in which default correlations are higher than $\rho^{\text{def}}(-3, -1.5)$ (i.e. DC+) and the range in which they are lower (DC-).

The intuition of the SE and DDE effects developed above suggests that $(0, 0)$ is also a global maximum, which is confirmed by our simulation results. Hence, $(2/\pi) \arcsin(\rho)$ can be used as an upper boundary for default correlations. We conclude this section by stating a result expressing how default correlation changes due to shifts in default points depend on the return correlation ρ.

Proposition 3.4

Suppose that $z_1, z_2 < 0$. Then there are real numbers $\bar{\rho}_L = \bar{\rho}_L(z_1, z_2)$ and $\bar{\rho}_H = \bar{\rho}_H(z_1, z_2)$ with $\bar{\rho}_L < \bar{\rho}_H$ such that the following statements hold:

(i) $(\rho_1^{\text{def}} + \rho_2^{\text{def}})$ *is increasing in ρ for $\rho < \bar{\rho}_H$ and decreasing in ρ for $\rho > \bar{\rho}_H$.*

(ii) ρ_2^{def} *is increasing in ρ for $\rho < \bar{\rho}_L$ and decreasing in ρ for $\rho > \bar{\rho}_L$.*

Moreover, if $10^{-6} \leq p_1, p_2 \leq 0.46$, then $\bar{\rho}_H \in [0.53, 0.89]$.

3.2.3 Applications

After stating our analytic results, we will now discuss their potential applications.

3.2.3.1 Default Correlations and Macroeconomic Shocks

In this section we explore how macroeconomic shocks impact on credit portfolios. In particular, we address the following questions:

1. Can default correlations of two firms decrease after a negative macroeconomic shock?

2. Which correlation effect (SE or DD) is most relevant at the portfolio level?

3. What can be said about the size of the effects?

We start with the first question. Suppose that a macroeconomic shock scales down the $t = t_1$ asset value of firm i by the factor Δ_i ($V_{i,t_1} \to \Delta_i V_{i,t_1}$). Then, according to equation (3.2), z_i increases by

$$\delta_i := \frac{-\log(\Delta_i)}{\sqrt{\mathrm{Var}(Z_{i,t_2})}}, \tag{3.3}$$

i.e. $z_i \to z_i + \delta_i$.[10] Hence, such a macroeconomic shock will move both default points to the right, but generally by different magnitudes. We can describe this shift by a move of δ_1 units in direction $(1, \delta)$ where $\delta := \delta_2/\delta_1$. Note that according to propositions 3.1 and 3.2 and our conjecture, the default correlation between firm 1 and 2 can only decrease if $\delta > 1$. Moreover, we can describe the marginal change of default correlations by the derivative in the direction of $(1, \delta)$, i.e. by

$$\rho_1^{\mathrm{def}} + \delta\rho_2^{\mathrm{def}} = (\rho_1^{\mathrm{def}} + \rho_2^{\mathrm{def}}) + (\delta - 1)\rho_2^{\mathrm{def}}. \tag{3.4}$$

Since $\rho_1^{\mathrm{def}} + \rho_2^{\mathrm{def}} > 0$, we obtain that $\rho_1^{\mathrm{def}} + \delta\rho_2^{\mathrm{def}}$ can only be negative if $\rho_2^{\mathrm{def}} < 0$. Note that for arbitrary asset correlations ρ we can find a pair (z_1, z_2) for which $\rho_2^{\mathrm{def}} < 0$ by simply choosing z_1 close enough to zero (see figure 3.5). However, if we restrict our analysis to default points associated with default probabilities lower than, say, 10 percent and higher than 10^{-4} percent we obtain that $\lambda \geq 27\%$ and can infer from figure 3.5 that ρ_2^{def} can only be negative if $\rho \geq 9.9\%$. Moreover, our simulation results suggest that for fixed δ and fixed default points z_1 and z_2, there is a critical value $\bar{\rho} = \bar{\rho}(\delta, z_1, z_2)$ so that $\rho_1^{\mathrm{def}} + \delta\rho_2^{\mathrm{def}}$ is increasing in ρ for $\rho < \bar{\rho}$ and decreasing in ρ for $\rho > \bar{\rho}$.[11] Hence, default correlations will only decrease if ρ exceeds a certain threshold $\tilde{\rho}(\delta, z_1, z_2)$.

Summing up, we have found that default correlations will only increase if both variables δ and ρ are high. In order to assess which combinations of δ and ρ can occur, it is

[10] Note that $\delta_i > 0$ since $\Delta_i < 1$.

[11] Note that for $\rho < \bar{\rho}_L$ and $\rho > \bar{\rho}_H$ this observation is backed by proposition 3.4. For $\bar{\rho}_L < \rho < \bar{\rho}_H$, we only know that $(\rho_1^{\mathrm{def}} + \rho_2^{\mathrm{def}})$ increases in ρ while $(\delta - 1)\rho_2^{\mathrm{def}}$ decreases.

important to keep in mind that the firms' exposure to the macroeconomic shock in $t = t_1$ might explain quite a large part of the correlation between returns in $t = t_2$. For greater concreteness, we add a third point in time, t_0, define the asset returns from t_0 to t_1 by $Z_{i,t_1} := \log(V_{i,t_1}/V_{i,t_0})$, and assume that the firms' normalized returns depend linearly on a macroeconomic factor Z^S and on idiosyncratic components ϵ_i:

$$\frac{Z_{i,t} - \mathbb{E}\, Z_{i,t}}{\sqrt{\mathrm{Var}(Z_{i,t})}} = \theta_i Z_t^S + \epsilon_{i,t} \;\; (i = 1, 2;\; t = t_1, t_2). \tag{3.5}$$

The random variables $(\epsilon_{i,t}, Z_i^S)_{i=1,2;\,t=t_1,t_2}$ are assumed to be mutually stochastically independent, and θ_1 and θ_2 are positive real numbers. Inserting equation (3.5) in equation (3.2) and using the fact that $\log(V_{i,t_1}) = \log(V_{i,t_0}) + Z_{i,t_1}$, we finally obtain that the default points z_1 and z_2 can be written as

$$z_i = \bar{z}_i - \theta_i Z_{t_1}^S \;\; (i = 1, 2).$$

$\bar{z}_1$ and $\bar{z}_2$ are constants that depend on (V_{1,t_0}, V_{2,t_0}) , (v_1, v_2), $(\epsilon_{1,t_1}, \epsilon_{2,t_1})$, and on the means and variances of Z_{1,t_2} and Z_{2,t_2} respectively.[12] By normalizing the variance of $Z_{t_2}^S$ to 1 we obtain that $\rho = \theta_1\theta_2$, $\delta = \theta_2/\theta_1$ and that $0 \le \theta_i \le 1$ $(i = 1, 2)$. This in turn implies that $\delta \le 1/\rho$.

What can we learn from this exercise? First, since the macroeconomic factor responsible for the scale-down in asset values will also explain a certain fraction of the asset-return correlation ρ,[13] there is a structural relationship between ρ and δ. Second, for fixed ρ this relationship limits the possible values for δ from above, since ρ determines a minimal *joint* exposure to the macroeconomic factor. Hence, an increase in ρ will be associated with a decrease of the upper boundary for δ. Whether a macroeconomic shock can decrease default correlations at all will therefore depend on how strongly the upper bound for δ is depressed when ρ rises, i.e. on the extent to which the asset correlations are explained by the macroeconomic factor that has triggered the shock. In our example, where the macroeconomic factor fully explains the correlation, simulation results suggest that default correlations always increase if the default probabilities p_1 and p_2 lie between 10^{-4} and 10 percent.

[12] Here we have assumed that $\mathrm{Var}(Z_{i,t_1}) = \mathrm{Var}(Z_{i,t_2})$ $(i = 1, 2)$. In the Merton framework, which is more precisely described in section 3.3.2, this is equivalent to the assumption that $(t_2 - t_1) = (t_1 - t_0)$. Hence we assume that the risk management horizon and the period during which the macroeconomic shock is analyzed have the same length.

[13] In our example the macroeconomic factor fully explains ρ.

We now turn to the portfolio level. In an average portfolio, the DD effect will tend to cancel out, since there should be as many pairs of loans where d decreases as pairs where d increases. As a benchmark, consider a portfolio where for each loan in the portfolio with default points (z_1, z_2) and exposure direction $(1, \delta)$ there is another loan with (z_1, z_2) and the opposite exposure direction $(\delta, 1)$. If we add correlation effects pairwise across the portfolio, the marginal effect of each pair is given by

$$(\rho_1^{\text{def}} + \delta\rho_2^{\text{def}}) + \delta\rho_1^{\text{def}} + \rho_2^{\text{def}} = (1+\delta)(\rho_1^{\text{def}} + \rho_2^{\text{def}}) > 0.$$

On a more practical level, suppose that loans are subdivided into classes according to their default probability (for example by rating classes). Now calculate the δ of each class by summing all single exposures of loans in that class. If the values for δ in all classes are about the same, then DDE should approximately cancel out and the whole effect of the shock on portfolio correlations can be described by SE. If, however, class exposures differ significantly, then DDE might modify or intensify the rise of default correlations caused by SE. However, even if DDE is significant at portfolio level and thus modifies SE, then the reasoning above suggests that the overall effect should still be a rise in default correlations. Moreover, in terms of the whole economy, the argument that DDE cancels out could be made even stronger, since if credit portfolios of all banks are considered, the δ for each loan class is averaged over a much higher number of firms. This in turn should imply that the values of δ are much more similar among classes. Therefore, as we turn to the third question posed at the beginning of this section, we will concentrate on SE.

In order to give an impression of the size of the effects at work, we have calculated default correlations for different loan types and different macroeconomic scenarios. The default probabilities associated with these scenarios are shown in table 3.1.

	AV	A	B	C
Initial		**.05**	**.20**	**7.12**
Shock 1	-15%	.12	.43	11.20
Shock 2	-30%	.31	1.01	17.96
Shock 3	-50%	1.35	3.56	34.52

Table 3.1: Default probabilities in percent for different loan types and different shock scenarios.

	A			B			C		
A	**.11**	**.98**	**4.21**	**.18**	**1.38**	**5.18**	**.53**	**2.35**	**5.02**
	1.89	1.58	1.36	1.80	1.53	1.33	1.54	1.39	1.28
	3.75	2.62	1.91	3.38	2.44	1.84	2.43	1.98	1.67
	10.11	5.46	3.17	8.28	4.76	2.93	4.48	3.18	2.43
B				**.30**	**2.03**	**6.89**	**.92**	**3.92**	**8.45**
				1.72	1.49	1.31	1.47	1.34	1.24
				3.04	2.27	1.75	2.18	1.81	1.56
				6.77	4.13	2.66	3.65	2.70	2.10
C							**3.10**	**11.30**	**22.65**
							1.26	1.19	1.13
							1.56	1.39	1.27
							1.95	1.65	1.43

Table 3.2: Default correlations (bold) in percent and scaling factors by which default correlations increase under the shock scenarios. The scaling factor describes the multiplier by which default correlations change compared to the initial scenario. In each cell of the matrix, the columns correspond to the different asset correlations chosen. Asset correlations ρ increase from left to right (ρ = 10, 30, and 50 percent). The rows correspond to the different shock scenarios. The intensity of the shock increases from the top to the bottom row.

The first scenario (initial scenario) serves as a benchmark and the three others describe deviations from the initial scenario induced by macroeconomic shocks of different intensities. We have chosen three different loan types (labeled A, B and C) which have different initial default probabilities. The initial default probabilities for each loan type have been chosen as average one-year default rates of the high, medium and low rating segments of Moody's.[14] In the three shock scenarios, asset values will be reduced by 15, 30 and 50 percent (i.e. $\Delta = 0.85$, 0.7 and 0.5 respectively).

In order to calculate the resulting changes in default points we finally need to fix the standard deviation of returns (see equation (3.3)). We have chosen an average value 0.64 for yearly returns from BERNDT (1991). We have then calculated default correlations in the initial scenario and derived the factors by which these correlations increase under the three shock scenarios. All calculations have been made for all

[14] More precisely, we have divided all firms that were rated by Moody's in 2000 into three equal-sized groups. The highly rated segment (labeled A) includes firms with ratings from Aaa to A3, the medium-rated segment (labeled B) includes firms with ratings from Baaa1 to Ba2 and the low-rated segment (labeled C) includes firms with ratings from Baa3 to C. See MOODY'S (2000).

possible combinations of the three loan types and for different asset correlations ($\rho =$ 10%, $\rho = 30\%$ and $\rho = 50\%$). Table 3.2 shows the results of these calculations.

In each cell of the matrix, initial default correlations are bold and are followed by the scaling factors by which default correlations increase under the different shock scenarios. The different columns correspond to the different asset correlations chosen. For example, the third line in each cell shows the factors by which default correlations increase under the shock scenario 2 for the different values of ρ. The second entry in this line presents the factor for $\rho = 50\%$. The reported results suggest that the changes in default correlations may be substantial. Moreover, default correlations rise most strongly relative to their pre-shock value if return correlations and initial default probabilities are low. The effect becomes less pronounced if those parameters increase.

At this point we have concluded that at portfolio level, the increase in variances caused by higher default probabilities after a macroeconomic shock will be reinforced by an increase in default correlations. We now give an impression of the size of these effects for an average loan portfolio. Table 3.3 shows the characteristics of the portfolios used in order to illustrate our theoretical results. Note that the principal is chosen such that all loans have an expected repayment of 1.

Principal	Recovery rate	ρ	Number of firms
$1/(1-p_i)$	0	50%	300

Table 3.3: Characteristica of the portfolios used for illustration.

We have considered three different types of such portfolios: portfolios only consisting of type A loans (labeled AAA), portfolios only consisting of type C loans (labeled CCC), and portfolios with 100 loans of each type (labeled ABC). Table 3.4 illustrates how the standard deviations (Std) and the economic capital (EC) of an ABC portfolio increase under the different shock scenarios we have considered. For example, if asset values decrease by 15%, then portfolio standard deviation will increase by 33%. In order to identify the size of the correlation effect, we have calculated the increase in portfolio standard deviation that is achieved when only the increase in loan variances is considered and default correlations are fixed to their pre-shock levels. These results are reported in column (Std-Corr).

A more exhaustive study of the relevance of the correlation effect for standard deviation and economic capital of credit portfolios under macroeconomic shocks has been

AV	Std-Corr	Std	EC
-15%	24	33	24
-30%	55	79	51
-50%	105	167	100

Table 3.4: Percentage increase in standard deviation (Std) and economic capital (EC) after macroeconomic shocks for an ABC portfolio. The column (Std-Corr) gives the percentage increase in portfolio standard deviation when the increase in default correlations is not taken into account.

provided by GERSBACH AND LIPPONER (2000) and LIPPONER (2000). Using a simulation approach, they isolate the correlation effect of a macroeconomic shock. After the shock has occurred, they scale down return correlations until default correlations are at the same level as before the shock. Comparing standard deviation and economic capital obtained with the correct and the scaled-down return correlations enables them to measure the size of the correlation effect relative to the other effects. They show that the correlation effect may account for more than 50% of the increase in credit risk induced by the macroeconomic shock.

An important further implication of our results is that the standard deviation of default rates will vary throughout the business cycle, as is the case with the standard deviation of credit portfolios. This observation is important since - as has been described in chapter 2 - the currently proposed reduced-form credit-risk models use default rate distributions as input for the value-at-risk analysis of credit portfolios. In table 3.5 we illustrate how default-rate standard deviations vary throughout the business cycle. Considering an industry with an ABC firm portfolio, we have calculated the percentage increase of the standard deviation when the economy moves from the expansion to the recession state of the business cycle.[15] Three different expansion/recession scenarios - where the asset values increase (decrease) by 10, 20 and 30 percent compared to the average case - have been evaluated. Our results suggest that - when using default rate distributions as an input for credit risk models - the standard deviations of default rates should be adapted to the business cycle. For example the simulation results in GORDY (2000) show that the percentile values calculated by reduced-form models are very sensitive to changes in default-rate standard deviation. For the portfolios considered by Gordy, an increase of variances by 100% increases percentiles by two to three times.

[15] Employing an ABC portfolio, we have in particular assumed that the number of firms in the industry is 300. However, the figures in table 3.5 do not change substantially for a larger number of firms.

AV	Std-Corr	Std
±10%	34	47
±20%	78	114
±30%	135	214

Table 3.5: Percentage increase of default-rate standard deviation when the economy moves from the expansion to the recession state of the business cycle. We have assumed that the firm portfolio of the sector or industry under consideration is of type ABC.

3.2.3.2 Pricing of Loans

The price of a loan should reflect the costs of the additional amount of capital that has to be held against the credit portfolio when the specific loan is added to the portfolio. In current practice, interest rates on loans in most cases merely reflect the impact of higher default probabilities on *expected* returns. Our results suggest that the impact on portfolio standard deviation should also be taken into account. We believe that the following two observations are especially important in this respect. First, as can be seen from table 3.2, the contribution of a loan to the standard deviation of the credit portfolio varies with the composition of the portfolio. For example, the default correlation of a C loan with loans in an AAA portfolio is 5.02%, while default correlations will rise to 22.65% if the considered portfolio is of type CCC (distance-of-default-points effect). Second, the variance effect (loans with higher default probability have a higher standard deviation than loans with lower default probability) is reinforced by the correlation effect (they also have a higher correlation with other loans).

Table 3.6 illustrates that it is important to recognize that a loan's contribution to the portfolio standard deviation varies strongly with its default probability. We compare standard deviation and economic capital of an AAA and a CCC portfolio by calculating the ratio by which both of these measures increase when moving from the analysis of a CCC portfolio to the respective values for an AAA portfolio. For example, the CCC standard deviation is 25.28 times higher than the AAA standard deviation. As before, we have calculated the increase in portfolio standard deviation when the increase of default correlations is not taken into account in column (Std-Corr). Note that all of these ratios also reflect differences in the principals of the loans in both portfolios (see table 3.3). Since we are mainly interested in the variance and correlation effect, the second row of the table calculates the respective ratios when the principals of the loans in both portfolios are normalized to 1.

Principals	Std-Corr	Std	EC
Heterogeneous	12.10	27.20	57.13
Homogeneous	11.24	25.28	53.1

Table 3.6: Ratios by which standard deviation (Std) and economic capital (EC) increase from AAA to CCC portfolios. The first column (Std-Corr) displays the ratio for portfolio standard deviation when the increase of default correlations is not taken into account. The second row shows the ratios in the case where the loans' principals are normalized to 1.

3.3 Robustness

In the preceding section we established the theoretical foundation of the correlation effect and illustrated its importance for credit portfolio management. In this section we explore the robustness of our results with respect to the assumptions made during the specification of the model. In section 3.3.1 we consider endogenous recovery rates, in section 3.3.2 we examine the case where the risk management horizon is not identical with the maturity of the loans, and in section 3.3.3 we discuss how alternative distributional assumptions would affect our results.

3.3.1 Endogenous Recovery Rates

In this section we uphold the assumption of bivariate normally distributed returns but consider endogenous recovery rates. To account for endogenous recovery rates, we assume that a certain fraction β of the asset value V_2 can be recovered in the event of default. Recall that $Z_{i,t_2} = \log(V_{i,t_2}/V_{i,t_1})$ and hence $V_{i,t_2} = V_{i,t_1}\exp(Z_{i,t_2})$. Normalizing the loan repayment in the case of non-default to 1, the payoff of a portfolio of two loans is given by $2 - I_1 - I_2$ where

$$I_i := 1\{Z_{i,t_2} \leq z_i\}\Big(1 - \beta V_{i,t_1}\exp(Z_{i,t_2})\Big) \quad (i = 1, 2).$$

We standardize V_{1,t_1} to 1 and adjust β and V_{2,t_1} such that a certain fixed fraction $\tilde{\beta}$ of the loan is recovered for $Z_{1,t_2} = z_1$ and $Z_{2,t_2} = z_2$ respectively. Hence,

$$\beta = \tilde{\beta} \exp(-z_1) \quad \text{and} \quad V_{2,t_1} = \tilde{\beta} \exp(-z_2)/\beta.$$

Finally, by normalizing the mean of (Z_{1,t_2}, Z_{2,t_2}) to zero and by assuming, for simplicity, that Z_{1,t_2} and Z_{2,t_2} have the same variance σ^2, we can calculate the building blocks of the correlation between loan repayments. Variances can be derived from[16]

$$\begin{aligned}
\mathbb{E}\, I_i^2 &= \int_{-\infty}^{z_i/\sigma} \Big(1 - \tilde{\beta} \exp(\sigma\zeta - z_i)\Big)^2 \varphi(\zeta)\; d\zeta \\
\mathbb{E}\, I_i &= \int_{-\infty}^{z_i/\sigma} \Big(1 - \tilde{\beta} \exp(\sigma\zeta - z_i)\Big) \varphi(\zeta)\; d\zeta,
\end{aligned}$$

and the covariance from the formulas for $\mathbb{E}\, I_i$ $(i = 1, 2)$ and from

$$\begin{aligned}
\mathbb{E}\, I_1 I_2 &= \int_{-\infty}^{z_1/\sigma} \int_{-\infty}^{z_2/\sigma} \Big(1 - \tilde{\beta} \exp(\sigma\zeta_1 - z_1)\Big) \Big(1 - \tilde{\beta} \exp(\sigma\zeta_2 - z_2)\Big) \varphi_\rho(\zeta_1, \zeta_2, \rho)\; d\zeta_1\, d\zeta_2 \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{z_1/\sigma} \Big(1 - \tilde{\beta} \exp(\sigma\zeta - z_1)\Big) \exp\Big\{-\frac{\zeta^2}{2(1-\rho^2)}\Big\} g(\zeta)\, d\zeta
\end{aligned}$$

where

$$\varphi_\rho(\zeta_1, \zeta_2, \rho) := \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big\{-\frac{\zeta_1^2 - 2\rho\zeta_1\zeta_2 + \zeta_2^2}{2(1-\rho^2)}\Big\}$$

and

$$g(z) := \int_{-\infty}^{z_2/\sigma} \Big(1 - \tilde{\beta} \exp(\sigma\zeta - z_2)\Big) \exp\Big\{-\frac{\zeta^2 - 2\rho z \zeta}{2(1-\rho^2)}\Big\}\; d\zeta.$$

For the robustness analysis we have relied on numerical results for three reasons. First, the analytic tractability of the problem seems questionable. Second, there is a quite clear-cut intuition of how default-correlation changes for the endogenous case relate to such changes in the exogenous case. If recovery rates are endogenous, loan repayments in the event of default provide full information about realized returns. This implies that default correlations in general should be higher than in the exogenous case, where the default event only reveals that the firm's standardized returns are below the default point. Moreover, with higher default probabilities, the event of full information

[16] $\varphi(\cdot)$ denotes the density function of the standard normal distribution.

revelation is more likely which increases the information about joint returns available from loan repayments. This adds to the skewness effect and default correlations should therefore increase even more if recovery rates are endogenous. Third, the log-normal specification of asset values implies that the amount that can be recovered in the event of default decreases exponentially when returns decrease (see e.g. the equations for $\mathbb{E}\, I_i^2$ or $\mathbb{E}\, I_1 I_2$). This suggests that the difference between the endogenous and the exogenous case should decrease rapidly when returns decrease from default points, which in turn implies that the correlation measures should not differ very strongly.

We have calculated default correlations for a wide variety of parameter constellations. The results obtained confirm the intuition outlined above. Table 3.7 documents the case $\sigma = 1$ and $\tilde{\beta} = 0.8$.

	A			B			C		
A	**.10**	**1.03**	**4.80**	**.18**	**1.46**	**5.91**	**.54**	**2.61**	**6.20**
	1.89	1.58	1.35	1.81	1.54	1.33	1.56	1.41	1.29
	3.78	2.62	1.90	3.42	2.45	1.83	2.51	2.03	1.70
	10.36	5.50	3.14	8.54	4.83	2.91	4.84	3.40	2.56
B				**.30**	**2.14**	**7.81**	**.93**	**4.29**	**10.11**
				1.73	1.49	1.30	1.49	1.35	1.25
				3.09	2.28	1.75	2.26	1.86	1.58
				7.03	4.21	2.66	3.97	2.88	2.21
C							**3.20**	**12.32**	**25.83**
							1.28	1.20	1.14
							1.64	1.45	1.29
							2.20	1.81	1.52

Table 3.7: Default correlations for endogenous recovery rates ($\sigma = 1$ and $\tilde{\beta} = 0.8$). The table can be read in the same way as table 3.2.

3.3.2 Rating Migration

In the last section we suggested that endogenous recovery rates do not affect our qualitative results. We now return to the assumption of exogenous recovery rates in examining the robustness of our results with respect to the relationship between risk management horizon and loan maturity. In this section we investigate how our results are affected when loans do not mature at the end of the risk management horizon. In order to conduct this analysis we add an additional point in time, t_3. We assume that the bank does risk management in $t = t_1$, uses the risk management horizon $t = t_2$ and holds zero-coupon loans maturing in $t = t_3$. As in the previous section, the loans default if the firms' asset values fall below the respective default points, i.e. if $V_{t_3,i} \leq v_i$ $(i = 1, 2)$. Note that in such a setting the loans have to be reevaluated in $t = t_2$. In order to derive a complete valuation framework, we make the following assumptions:

1. The asset values of both firms follow a two-dimensional geometric Brownian motion, i.e.:

$$\begin{pmatrix} dV_{1,t}/V_{1,t} \\ dV_{2,t}/V_{2,t} \end{pmatrix} = dW_t.$$

   $W := (W_t)_{t\in[0,\infty)}$ is a two-dimensional Brownian motion with mean vector $\mu = (\mu_1, \mu_2)^T$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}.$$

   μ_i, σ_i and ρ describe the mean, variance and correlation of the two firms' instantaneous assets returns respectively $(i = 1, 2)$.

2. The instantaneous risk-free interest rate is r, i.e. a dollar invested from time t to time s in risk-free securities yields $e^{r(s-t)}$ dollars.

When pricing both loans in $t = t_2$, we work with the standard risk-neutral or martingale probability measure.[17] We use p^*_{it} to denote the risk-neutral conditional probability that firm i defaults in $t = t_3$, given all the information available in t. Since under the risk-neutral probability measure, the expected return on all securities is the risk-free rate r, the risk-neutral default probabilities are given by the probabilities of the corresponding risk-neutral asset value processes $V^*_i := (V^*_{i,t})_{t\in[0,\infty)}$ falling below the firms' respective default points v_1 and v_2 in $t = t_3$.[18]

[17] For a description of the martingale-measure approach to the pricing of securities see e.g. Jarrow and Turnbull (1996), ch. 5 and 6. Note that this approach is the state of the art for repricing loans before maturity. It does, however, not take into account problems arising from the fact that - due to asymmetric information - the bank might not be able to sell its loans to the market in $t = t_2$ at the martingale measure price. To our knowledge there is currently no risk management approach that combines risk neutral pricing and asymmetric information among market participants to reprice loans.

[18] See Crouhy, Galai, and Mark (2000), p.100.

Note that V_i^* is defined in the same way as the asset value process V_i except that the return rates μ_i are replaced by the riskless return r. Hence, the log increments

$$\Big(\log(V_{i,s}^*) - \log(V_{i,t}^*)\Big)_{i=1,2} \quad (s > t)$$

of the risk neutral processes are distributed jointly normally with means $(s-t)(r-\sigma_i^2/2)$, variances $(s-t)\sigma_i^2$ and correlation ρ, and are independent of $V_{i,t}^*$.[19] For $t < s = t_3$ we therefore obtain that

$$\begin{aligned}
p_{i,t}^* &= \mathbb{P}\Big\{\frac{\log(V_{i,t_3}^*) - \log(V_{i,t}^*) - (r-\sigma_i/2)(t_3-t)}{\sigma_i\sqrt{t_3-t}} \le \frac{\log(v_i) - \log(V_{i,t}^*) - (r-\sigma_i/2)(t_3-t)}{\sigma_i\sqrt{t_3-t}}\Big\} \\
&= \Phi\Big\{\frac{\log(v_i) - \log(V_{i,t_1}^*) - (r-\sigma_i/2)(t_3-t_1)}{\sigma_i\sqrt{t_3-t}} - \frac{\log(V_{i,t}^*) - \log(V_{i,t_1}^*) - (r-\sigma_i/2)(t-t_1)}{\sigma_i\sqrt{t_3-t}}\Big\} \\
&= \Phi\Big\{\sqrt{\frac{t_3-t_1}{t_3-t}}z_i^* - \sqrt{\frac{t-t_1}{t_3-t}}\,\frac{\log(V_{i,t}^*) - \log(V_{i,t_1}^*) - (r-\sigma_i/2)(t-t_1)}{\sigma_i\sqrt{t-t_1}}\Big\}
\end{aligned}$$

where

$$z_i^* := \frac{\log(v_i) - \log(V_{i,t_1}^*) - (r-\sigma_i/2)(t_3-t_1)}{\sigma_i\sqrt{t_3-t_1}} \quad (i = 1, 2).$$

For $t = t_1$ we obtain $p_{i,t_1}^* = \Phi(z_i^*)$, which allows us to fix the $t = t_1$ risk-neutral default probabilities by choosing appropriate values for z_i^*. Once these values are fixed, the $t = t_2$ risk neutral default probabilities are given by

$$p_{i,t_2}^* = \Phi\left(\sqrt{\tau+1}\,z_i^* - \sqrt{\tau}\,Z_i\right)$$

where

$$Z_i := \frac{\log(V_{i,t_2}^*) - \log(V_{i,t_1}^*) - (r-\sigma_i/2)(t_2-t_1)}{\sigma_i\sqrt{t_2-t_1}} \quad (i = 1, 2)$$

and

$$\tau := \frac{t_2-t_1}{t_3-t_2}.$$

[19] See e.g. Jarrow and Turnbull (1996). Of course the same is true for the asset value processes themselves. The only difference is that in this case the means for the log increments are given by $(s-t)(\mu_i - \sigma_i^2/2)$.

Note that τ describes the relative sizes of the risk management period and the loan duration after the risk management horizon. Using the $t = t_2$ risk-neutral default probabilities, we can describe the t_2 value V^L_{i,t_2} of loan i in the following way:

$$V^L_{i,t_2} = L_i e^{-r(t_3-t_2)}\Big[1 - p^*_{i,t_2}(1-\beta)\Big].$$

L_i is the principal that has to be paid back in t_3 and β is the (exogenous) recovery rate. Note that from a $t = t_1$ perspective, only the probabilities p^*_{1,t_2} and p^*_{2,t_2} are random, implying that

$$\begin{aligned}\rho^{\text{mig}} &:= \operatorname{Corr}(V^L_{1,t_2}, V^L_{2,t_2}) \\ &= \operatorname{Corr}(p^*_{1,t_2}, p^*_{2,t_2}) \\ &= \operatorname{Corr}\Big(H_{z_1,\tau}(Z_1)\,,\, H_{z_2,\tau}(Z_2)\Big).\end{aligned}$$

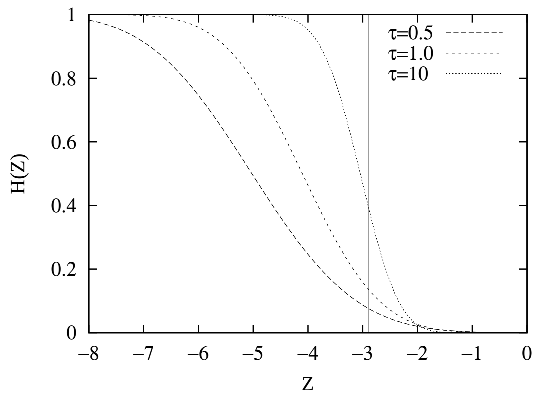

Figure 3.9: The function $H_{-2.9,\tau}(\cdot)$ for different values of τ.

$H_{z,\tau}$ is defined by

$$H_{z,\tau}(Z) := \Phi(\sqrt{\tau+1}\,z - \sqrt{\tau}\,Z)$$

and (Z_1, Z_2) is a standard bivariate normally distributed random vector with correlation ρ. Comparing ρ^{mig} with ρ^{def}, we observe that the functions $1\{\cdot \leq z_i\}$ are replaced by $H_{z_i,\tau}(\cdot)$ $(i = 1, 2)$. Figure 3.9 depicts the indicator function and the function H for $z_1 = z_2 = -2.9$ (which corresponds to a risk-neutral default probability of 0.2%, i.e. a B loan) and for different values of τ. It can be seen that H is a "continuous version" of the indicator function and that H converges towards the indicator form for $\tau \to \infty$. The intuition about the relationship between default probabilities and default correlations that led to the analytic results in section 3.2.2 should therefore still apply qualitatively. In order to confirm this reasoning we have calculated ρ^{mig} for different values of τ. The results for $\tau = 1$ are displayed in table 3.8.[20]

Note that the results do not differ substantially from the ones obtained in section 3.2.3. Moreover, the differences should be most pronounced for low τ, since $H_{z_i,\tau}(\cdot)$ will differ the stronger from the indicator functions the lower the values for τ are. But even for $\tau = 0.05$ (which implies that loan duration is 20 times longer than the length of the

[20]Note that the labeling of the different loan types in this section refers to the risk neutral default probabilities (e.g. an A loan has an initial risk neutral default probability of 0.05%).

	A			B			C		
A	**.11**	**1.05**	**4.81**	**.18**	**1.48**	**5.91**	**.55**	**2.59**	**6.00**
	1.90	1.58	1.35	1.81	1.54	1.33	1.56	1.40	1.28
	3.79	2.62	1.89	3.42	2.45	1.82	2.48	1.99	1.67
	10.37	5.48	3.11	8.51	4.80	2.88	4.68	3.26	2.42
B				**.31**	**2.18**	**7.80**	**.96**	**4.29**	**9.83**
				1.73	1.49	1.30	1.48	1.34	1.24
				3.09	2.28	1.74	2.23	1.83	1.55
				6.99	4.17	2.63	3.82	2.77	2.10
C							**3.28**	**12.32**	**25.29**
							1.27	1.19	1.13
							1.60	1.42	1.27
							2.06	1.71	1.45

Table 3.8: Loan correlations ρ^{mig} for $\tau = 1$. The table can be read in the same way as table 3.2.

risk management period)[21] no significant deviations from the values in table 3.8 result.

3.3.3 Distributional Assumptions

In the last two sections we derived the stability of our results with respect to recovery rates and the relationship between risk management horizon and loan duration. We now return to the assumptions that loans mature at the end of the risk management horizon and that recovery rates are exogenous, and examine the robustness of our results with respect to the assumptions about the distribution of asset returns.

We will show that the crucial point of our assumptions is in fact not that returns are bivariate normally distributed but that an arbitrary monotonic transformation of the asset value process has this property. Let us describe the general framework we have in mind in more detail. Suppose that there is a monotonic transformation $\mathcal{T}$ of the asset values V_{1,t_2} and V_{2,t_2} which may depend on some aspects $\mathcal{H}_{i,t_1}$ of the asset value history up to time t_1,[22] so that the transformed variables $Z_{i,t_2} := \mathcal{T}(V_{i,t_2}, \mathcal{H}_{i,t_1})$ are bivariate normally distributed. The transformation has to fulfill the following two conditions:

[21] The typical length of a risk management period in the banking industry is one year (see e.g. JP Morgan (1997)).

[22] More technically, $\mathcal{H}_{i,t_1}$ is assumed to be an arbitrary sub σ - algebra of $\sigma\Big((V_{i,s})_{s\leq t_1}\Big)$.

1. Z_{1,t_2} and Z_{2,t_2} are independent of $\mathcal{H}_{1,t_1}$ and $\mathcal{H}_{2,t_1}$;

2. $\mathcal{T}(\cdot, \mathcal{H}_{i,t_1})$ is strictly increasing for all realizations of $\mathcal{H}_{i,t_1}$.

In this case the random variable $1\{V_{i,t_2} \leq v_i\}$ that describes the default behavior of firm i can be equivalently written as $1\{Z_{i,t_2} \leq z_i\}$ where $z_i = z_i(v_i, \mathcal{H}_{i,t_1})$ is increasing in v_i. We call the transformation $\mathcal{T}$ the "correlation model". Note that the model we have formulated in section 3.2.1 is a special case of this general framework. The transformation used in section 3.2.1 was

$$\mathcal{T}(V_{i,t_2}, V_{i,t_1}) := \log(V_{t_2}/V_{t_1}),$$

and hence $\mathcal{H}_{i,t_1}$ is the σ - algebra generated by $V_{i,t_1} (i = 1, 2)$.

As can be seen from the preceding arguments, the robustness of our analysis with respect to the underlying distributional assumptions boils down to the robustness of the correlation model chosen and has nothing to do with assumptions about the univariate distribution of returns. If the correlation model is a good approximation for the correlation structure, then our whole analysis applies, since arbitrary univariate distributions are supported by our model.[23]

Moreover, we conjecture that the intuition developed in section 3.2.2 about the relationship between default points and default correlations should apply more generally for linear correlation models, i.e. for models where asset-return correlations are derived from a common linear dependency on some independent factors.[24] For example, it has long been argued that stable-law distributions might provide a better description of actual returns than the normal distribution.[25] Moreover, since the sum of independent stable-law-distributed random variables also follows a stable law law, a straightforward specification of a linear correlation model is possible. It might therefore be insightful to test our conjecture for such random variables.

The most important empirical objection against the linear correlation model for asset returns is that correlations of large negative returns seem to be much greater than

[23]Note that, given a continuously distributed random variable R with distribution function F and given any other distribution function G, we can find a monotonic transformation of R which is distributed according to G. This transformation $\mathcal{T}$ is given by $\mathcal{T} = G^{-1} \circ F$.

[24]Note that multivariate normally distributed random variable are a special case of linear correlation models.

[25]See FAMA (1970) for a discussion of the literature or RACHEV, SCHWARTZ, AND KHINDANOVA (2000) for a more recent contribution. Stable-law distributions can capture two important empirical deviation from normality (thick tails and excess kurtosis).

expected under bivariate normality.[26] Moreover, there are quite a few theoretical arguments of why this may be the case, ranging from contagion from some markets to others, joint credit constraints, to changes in market structures and practices.[27] In appendix 3.B we explore the consequences of this (potential) deviation from a linear correlations model for our results. We find that higher return correlations for low returns will tend to moderate the skewness effect or even reverse it. Whether this mechanism has a significant impact on default correlations will have to be assessed by empirical research.

3.4 Conclusions

In this chapter we have established the structural relationship between default probabilities and default correlations. Loans with higher default probabilities will not only have higher variances (variance effect) but also higher correlations with other loans (correlation effect). Hence, the variance effect (which is an obvious consequence of higher default probabilities) is reinforced by the correlation effect. We have seen that due to these effects, portfolio standard deviation can increase substantially with higher loan default probabilities. These results have important implications for banks and regulators.

First, when determining relative prices of loans with high and loans with low default probability, banks should take into account the differences in the contribution to the overall standard deviation of their portfolio and hence to the economic capital needed to be held against the credit portfolio.

Second, during economic downturns default probabilities will increase. This will not only increase expected losses but also the standard deviation of loan portfolios. Hence the increase of required economic capital during downturns will stem from at least these two sources. We feel that this observation is important for regulators in seeking

[26] See Longin and Solnik (1999). Note that Longin and Solnik (1999) investigate *equity* returns. Of course it is not clear whether these results can be extended to asset returns. However, since asset returns are not directly observable, the results for equity returns can be seen as a first indication that similar results may be obtained for asset returns, especially since for highly rated firms, asset and equity returns should exhibit similar patterns if the value of debt does not vary too much.

[27] Recent discussions of possible routes of contagion include Drazen (1998), Eichengreen, Rose, and Wyplosz (1996), and Gerlach and Smets (1995). The CGFS report (Committee on the Global Financial System 1999) on the events following the Russian default in August 1998 presents a narrative account of how the effects of shocks were reinforced and spread to other markets by market practices. Concerning the literature on credit constraints see Holmström and Tirole (1997) and references therein.

to gain a more complete picture of the consequences of macroeconomic shocks for the banking system. It will also be important for banks which attempt to hedge against fluctuations in required economic capital caused by macroeconomic risk.

Third, consider the consequences for current credit risk models.[28] Academics and regulators have pointed out that credit risk models should take into account the fact that default probabilities increase during economic downturns, since the banks' expected losses will increase. Our results emphasize this point, showing that not only expected losses but also default correlations and, accordingly, portfolio standard deviation will increase. Moreover, the correlation effect has different consequences for structural models on the one hand and reduced-form models on the other. The structural approach to credit events is used by Credit Portfolio Manager from KMV and (indirectly) by Credit Metrics (CM) from JP Morgan. This approach essentially employs the framework outlined in section 3.2.1 to derive joint default probabilities. While KMV uses a firm's stock market value and its debt structure to derive default points (a method sensitive to changes in asset values), CM uses historical rating class default frequencies to calibrate default points. Our results emphasize that adjusting the CM default points to the business cycle is important. Once these default points have been appropriately adjusted to changed default probabilities, CM will also take the variance and correlation effect into account.

Reduced-form models, on the other hand, are used by Credit Risk$^+$ (CR) from the Credit Suisse Group and Credit Portfolio View from McKinsey (McK). A major building block of these models is the distribution of default rates. While Bär (2000) has demonstrated that default rates can be predicted quite well using macroeconomic variables, thereby presenting a method of how to adjust default rate means to the current economic environment, it is much less clear how to adjust the standard deviation of default rates. Our results suggest that default-rate standard deviation will vary significantly throughout the business cycle and that reduced-form models should reflect such changes. How to estimate default rate variances conditional on the business cycle therefore emerges as an important empirical issue that still awaits a satisfactory answer.[29]

[28] A survey of these models with respect to joint default probabilities has been given in chapter 2. Other aspects of the models are reviewed in Crouhy, Galai, and Mark (2000).

[29] Note that formally, McK does not need default rate distributions as input. These distributions are derived from the distributions of macroeconomic variables that are supposed to explain default rate changes. However, in order to justify that the implied default rate variances not only reflect the variances of the underlying macroeconomic variables, conditional default rate variances need to be predicted accurately. But this again rises the question of how to estimate these conditional variances.

We have derived our results for a fairly simple joint default model. However, we have shown that they remain robust under endogenous recovery rates, loan reevaluation at the end of the credit risk horizon and under alternative distributional assumptions for asset returns (as long as a monotonic transformation into a bivariate normal distribution exists). To gain a more complete picture, two issues have to be taken up in future research. First, rather than assuming that default can only occur at the date of the loan's maturity, it should be more realistic to model default as an absorbing barrier to a firm's asset value process, which can be reached at *any* point before the loan's maturity.[30] The robustness of our results in such a modeling framework should be investigated.

An equally important issue is whether the linear correlation model used by KMV and CM is appropriate. We have indicated that for correlation models where the return correlation is higher for low returns than suggested by the linear correlation model, the relationship between default probabilities and default correlations might even be reversed. These results suggest that a detailed empirical assessment may become necessary of whether other correlation models should enter credit risk frameworks of the structural type. In the interim, simple regime-switching correlation models could be used for stress testing.[31]

Finally, while the impact of the macroeconomic environment on expected default rates (and hence on average default probabilities) has been studied quite carefully, changes in default correlations are much more difficult to handle empirically. We have highlighted the difficulties of both modeling approaches in assessing the changes of default correlations due to macroeconomic shocks. When determining the bank capital needed to hold against a credit portfolio, it might therefore be useful to increase the default correlations calculated by the models by a "security factor". The expected losses produced should be more stable across time.

[30] This issue has been discussed in more detail in chapter 2.

[31] See e.g. ANG AND BEKAERT (1999).

3.A Proofs

In this part of the appendix we prove our main results concerning the relationship between default probabilities and default correlations. First of all note that

$$\rho^{\mathrm{def}} = \rho^{\mathrm{def}}(z_1, z_2, \rho) = \frac{\mathrm{Cov}\Big(1\{Z_1 \le z_1\}\,,\, 1\{Z_2 \le z_2\}\Big)}{\sqrt{\mathrm{Var}(1\{Z_1 \le z_1\})\,\mathrm{Var}(1\{Z_2 \le z_2\})}} =: \frac{D}{N}.$$

In section 3.A.1 we derive some useful expressions for D and N and for their first and second derivatives. This will be helpful in proving our main results in section 3.A.2. The following notation will be used throughout the appendix: Φ^ρ for the distribution function of (Z_1, Z_2), $\varphi(\cdot)$ and $\Phi(\cdot)$ for the one-dimensional standard normal density and distribution function respectively. Moreover, throughout the appendix we will use subscripts to indicate partial derivatives. D_1 for example will denote the partial derivative of D with respect to z_1 and D_{12} is a short form for $\partial^2 D/\partial z_1 \partial z_2$. Finally note that z_1 always indicates the default point associated with the lower default probability $(z_1 \le z_2)$.

3.A.1 The Building Blocks D and N

Since $D = \Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2)$, we will first derive a formula for $\Phi^\rho(z_1, z_2)$. Note that (Z_1, Z_2) has the same distribution as $(Z_1, \tilde{Z}_2)$ where $\tilde{Z}_2 = \rho Z_1 + \epsilon$, and ϵ is distributed according to $\mathcal{N}\Big(0, 1-\rho^2\Big)$ and independent of Z_1.[32] Hence,

$$\Phi^\rho(z_1, z_2) = \int_{-\infty}^{z_1} \Phi\Big(\frac{z_2 - \rho\zeta}{\sqrt{1-\rho^2}}\Big)\varphi(\zeta)\; d\zeta. \tag{3.6}$$

Note that by the theorem of Lebesgue, formula (3.6) also applies for the case $\rho = \pm 1$. In this case the limit of the right-hand side of the equation is taken. In the next sections we will take first and second derivatives of the right-hand side of equation (3.6). Again by the theorem of Lebesgue, the respective derivatives for $\rho = \pm 1$ can be obtained by taking the limit of the formulas derived for $|\rho| < 1$.

[32] $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2.

3.A.1.1 The Functions D and N

From equation (3.6) we obtain

$$D = \left[\int_{-\infty}^{z_1} \Phi\Big(\frac{z_2 - \rho\zeta}{\sqrt{1-\rho^2}}\Big)\varphi(\zeta)\ d\zeta\right] - \Phi(z_1)\Phi(z_2). \tag{3.7}$$

The function N can be represented as

$$N = \sqrt{\Phi(z_1)\Big(1 - \Phi(z_1)\Big)}\sqrt{\Phi(z_2)\Big(1 - \Phi(z_2)\Big)} \tag{3.8}$$

and[33]

$$D\,|_{z_1=z_2=0} \ = \frac{1}{2\pi}\arcsin(\rho) \text{ and } N\,|_{z_1=z_2=0} \ = \frac{1}{4}.$$

3.A.1.2 The First Derivatives of D and N

$$D_1 = \varphi(z_1)\Big[\Phi\Big(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\Big) - \Phi(z_2)\Big] \tag{3.9}$$

$$N_1 = \frac{1}{2}\Big[\Phi(z_2)\Big(1 - \Phi(z_2)\Big)\Big]^{1/2}\Big[\Phi(z_1)\Big(1 - \Phi(z_1)\Big)\Big]^{-1/2}\varphi(z_1)\Big(1 - 2\Phi(z_1)\Big). \tag{3.10}$$

Moreover,

$$D_1\,|_{z_1=z_2=0} \ = 0 \text{ and } N_1\,|_{z_1=z_2=0} \ = 0.$$

3.A.1.3 The Second Derivatives of D and N

$$D_{11} = -z_1\varphi(z_1)\Big[\Phi\Big(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\Big) - \Phi(z_2)\Big] - \varphi(z_1)\frac{\rho}{\sqrt{1-\rho^2}}\varphi\Big(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\Big)$$

[33] The proof for the arcsin representation of D is given in Gersbach and Lipponer (2000).

$$D_{12} = \varphi(z_1)\left[\frac{1}{\sqrt{1-\rho^2}}\varphi\Big(\frac{z_2-\rho z_1}{\sqrt{1-\rho^2}}\Big) - \varphi(z_2)\right]$$

$$\begin{aligned} N_{11} &= \frac{1}{2}\Big[\Phi(z_2)\Big(1-\Phi(z_2)\Big)\Big]^{1/2}\Bigg\{-\frac{1}{2}\Big[\Phi(z_1)\Big(1-\Phi(z_1)\Big)\Big]^{-3/2}\Big[\varphi(z_1)\Big(1-2\Phi(z_1)\Big)\Big]^2 \\ &\quad +\Big[\Phi(z_1)\Big(1-\Phi(z_1)\Big)\Big]^{-1/2}\Big[-z_1\varphi(z_1)\Big(1-2\Phi(z_1)\Big)-2\varphi(z_1)^2\Big]\Bigg\} \end{aligned}$$

$$N_{12} = \frac{1}{4}\Big[\Phi(z_2)\Big(1-\Phi(z_2)\Big)\Big]^{-1/2}\varphi(z_2)\Big(1-2\Phi(z_2)\Big)\Big[\Phi(z_1)\Big(1-\Phi(z_1)\Big)\Big]^{-1/2}\varphi(z_1)\Big(1-2\Phi(z_1)\Big).$$

Moreover,

$$D_{11}\,|_{z_1=z_2=0} \;= -\frac{\rho}{2\pi\sqrt{1-\rho^2}} \text{ and } D_{12}\,|_{z_1=z_2=0} \;= \frac{1}{2\pi}\Big(\frac{1}{\sqrt{1-\rho^2}}-1\Big),$$

$$N_{11}\,|_{z_1=z_2=0} \;= -\varphi(0)^2 = -\frac{1}{2\pi} \text{ and } N_{12}\,|_{z_1=z_2=0} \;= 0.$$

3.A.1.4 Derivatives of Fractions

The following general formulas provide the link between the derivatives of D and N and those of ρ^{def}. We define

$$F := F(z_1, z_2) = \frac{D(z_1,z_2)}{N(z_1,z_2)}.$$

and observe that

$$F_1 = \frac{D_1 N - D N_1}{N^2},$$

$$\begin{aligned} F_{11} &= \frac{D_{11}N - D_1 N_1}{N^2} - \frac{(D_1 N_1 + D N_{11})N^2 - D N_1 2 N N_1}{N^4} && (3.11)\\ &= \frac{D_{11}N - D N_{11} - 2D_1 N_1}{N^2} + \frac{2DN_1^2}{N^3} && (3.12) \end{aligned}$$

and

$$
\begin{aligned}
F_{12} &= \frac{D_{12}N - D_1N_2}{N^2} - \frac{(D_2N_1 + DN_{12})N^2 - DN_1 2NN_2}{N^4} && (3.13)\\
&= \frac{D_{12}N - DN_{12} - D_1N_2 - D_2N_1}{N^2} + \frac{2DN_1N_2}{N^3}. && (3.14)
\end{aligned}
$$

3.A.2 Proofs

In section 3.A.2.1 we present the proofs of propositions 3.1, 3.2 and 3.4, all of which cover the case $z_1, z_2 < 0$. In section 3.A.2.2 we prove proposition 3.3, which is concerned with the boundary cases $z_1 = z_2 = 0$ and $z_1 = z_2 \to \infty$.

3.A.2.1 Interior Points ($z_1, z_2 < 0$)

We first of all calculate the derivative of ρ^{def} with respect to z_1 and z_2.

Lemma 3.1
If $|\rho| < 1$, *then*

$$
\rho_1^{def} = \frac{\varphi(z_1)}{N}\left\{\left(\Phi\Big(\frac{z_2 - \rho z_1,}{\sqrt{1-\rho^2}}\Big) - \Phi(z_2)\right) - \psi(z_1)\left(\Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2)\right)\right\} \quad (3.15)
$$

$$
\rho_2^{def} = \frac{\varphi(z_2)}{N}\left\{\left(\Phi\Big(\frac{z_1 - \rho z_2,}{\sqrt{1-\rho^2}}\Big) - \Phi(z_1)\right) - \psi(z_2)\left(\Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2)\right)\right\} \quad (3.16)
$$

where

$$
\psi(z) := \frac{1 - 2\Phi(z)}{2\Phi(z)\Big(1 - \Phi(z)\Big)}.
$$

For $\rho = 1$ *we obtain*

$$
\begin{aligned}
\rho_1^{def} &= \frac{\varphi(z_1)\Big(1 - \Phi(z_2)\Big)}{2N\Big(1 - \Phi(z_1)\Big)}\\
\rho_2^{def} &= -\frac{\varphi(z_2)\Phi(z_1)}{2N\Phi(z_2)}.
\end{aligned}
$$

Remark 3.1
The formulas for $\rho = 1$ can be derived as limits of the formulas (3.15) and (3.16) for $\rho \to 1$.

Corollary 3.1
If

$$\rho > \frac{2\lambda}{1+\lambda^2},$$

then $\rho_2^{def} < 0$.[34]

Proof of corollary 3.1.

Use equation (3.16), and note that $\Phi^\rho(z_1, z_2) > \Phi(z_1)\Phi(z_2)$[35] and that $\psi(z) > 0$ if $z < 0$. Hence, $\rho_2^{\text{def}} < 0$ if

$$\frac{z_1 - \rho z_2}{\sqrt{1-\rho^2}} < z_1,$$

which is equivalent to $(1 - \rho\lambda)^2 > (1 - \rho^2)$.[36]

$\square$

Proof of lemma 3.1.

First of all note that by the theorem of Lebesgue the formulas for $\rho = 1$ can be derived as limits of the respective formulas for $|\rho| < 1$ and that by symmetry $\rho_2^{\text{def}}(z_1, z_2, \rho) = \rho_1^{\text{def}}(z_2, z_1, \rho)$. It is therefore sufficient to derive the formula for ρ_1^{def} if $|\rho| < 1$. But this formula follows directly from

$$\rho_1^{\text{def}} = \frac{D_1}{N} - \frac{DN_1}{N^2},$$

equations (3.8) - (3.10) and the fact that $D = \Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2)$.

$\square$

[34]Remember that $\lambda = z_2/z_1$.

[35]This inequality follows from the more general inequality, $\mathbb{E}\Big(f(Z)g(Z)\Big) \geq \mathbb{E}\Big(f(X)\Big)\,\mathbb{E}\Big(g(Z)\Big)$, which applies for arbitrary real-valued random variables Z and monotonically increasing functions f and g (see e.g. Hardy, Littlewood, and Pólya (1991)). In our case we can choose $f(z) := 1\{z \leq z_1\}$ and $g(z) := \Phi\Big((z_2 - \rho z)/(\sqrt{1-\rho^2})\Big)$. Note also that in this case the strict inequality applies, which can be seen by examining the proof in Hardy, Littlewood, and Pólya (1991). We thank Lutz Duembgen for this suggestion.

[36]Note that $z_1 < 0$.

To derive our main results, we draw on the fact that ρ_1^{def} and ρ_2^{def} are increasing (decreasing) functions of ρ in certain areas. Therefore, we now examine how ρ_1^{def} and ρ_2^{def} depend on ρ. The derivatives of ρ_1^{def} and ρ_2^{def} with respect to ρ will be denoted by $\rho_{1,\rho}^{\text{def}}$ and $\rho_{2,\rho}^{\text{def}}$ respectively.

Lemma 3.2

$$\begin{aligned}\rho_{1,\rho}^{def} &= \xi\Big\{\frac{\rho z_2 - z_1}{1-\rho^2} - \psi(z_1)\varphi(z_1)\Big\}\\ \rho_{2,\rho}^{def} &= \xi\Big\{\frac{\rho z_1 - z_2}{1-\rho^2} - \psi(z_2)\varphi(z_2)\Big\}\end{aligned}$$

where

$$\xi := \frac{\varphi(z_1)\varphi\Big((z_2-\rho z_1)/\sqrt{1-\rho^2}\Big)}{N\sqrt{1-\rho^2}}.$$

Hence

$$\rho_{1,\rho}^{def} + \rho_{2,\rho}^{def} = \xi\Big\{\frac{-(z_1+z_2)}{1+\rho} - \psi(z_1)\varphi(z_1) - \psi(z_2)\varphi(z_2)\Big\}.$$

Proof.

First we calculate the derivative of Φ^ρ with respect to ρ. In order to do so we use that

$$\frac{\partial}{\partial\rho}\Big(\frac{z_2-\rho\zeta}{\sqrt{1-\rho^2}}\Big) = \frac{\rho z_2 - \zeta}{(1-\rho^2)^{3/2}}. \tag{3.17}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial\rho}\Phi^\rho(z_1,z_2) &= \int_{-\infty}^{z_1}\varphi(\zeta_1)\frac{\partial}{\partial\rho}\int_{-\infty}^{\frac{z_2-\rho\zeta_1}{\sqrt{1-\rho^2}}}\varphi(\zeta_2)\,d\zeta_1 d\zeta_2\\ &= \frac{1}{(1-\rho^2)^{3/2}}\int_{-\infty}^{z_1}(\rho z_2-\zeta_1)\varphi(\zeta_1)\varphi\Big(\frac{z_2-\rho\zeta_1}{\sqrt{1-\rho^2}}\Big)\,d\zeta_1.\end{aligned}$$

But since $\sqrt{2\pi}\varphi(\zeta) = e^{-\zeta^2/2}$, we obtain

$$2\pi\varphi(\zeta)\varphi\Big(\frac{z_2-\rho\zeta}{\sqrt{1-\rho^2}}\Big) = \exp\Big\{-\frac{\zeta^2(1-\rho^2)+z_2^2-2z_2\rho\zeta+\rho^2\zeta^2}{2(1-\rho^2)}\Big\} =: \exp(T).$$

Moreover, T can be written as

$$\begin{aligned} T &= -\frac{\zeta^2 + z_2^2 - 2z_2\rho\zeta}{2(1-\rho^2)} \\ &= -\frac{(\zeta - z_2\rho)^2 + z_2^2(1-\rho^2)}{2(1-\rho^2)} \\ &= -\frac{(\zeta - z_2\rho)^2}{2(1-\rho^2)} - \frac{z_2^2}{2}. \end{aligned}$$

Hence

$$\varphi(\zeta)\varphi\Big(\frac{z_2 - \rho\zeta}{\sqrt{1-\rho^2}}\Big) = \varphi\Big(\frac{\zeta - z_2\rho}{\sqrt{1-\rho^2}}\Big)\varphi(z_2) \tag{3.18}$$

and therefore

$$\frac{\partial}{\partial\rho}\Phi^\rho = \frac{\varphi(z_2)}{(1-\rho^2)^{3/2}}\int_{-\infty}^{z_1}(\rho z_2 - \zeta)\varphi\Big(\frac{\zeta - z_2\rho}{\sqrt{1-\rho^2}}\Big)\,d\zeta.$$

But since the integrand can be integrated analytically to

$$(1-\rho^2)\varphi\Big(\frac{\zeta - z_2\rho}{\sqrt{1-\rho^2}}\Big),$$

we obtain

$$\frac{\partial}{\partial\rho}\Phi^\rho = \frac{\varphi(z_2)}{\sqrt{1-\rho^2}}\varphi\Big(\frac{z_1 - \rho z_2}{\sqrt{1-\rho^2}}\Big).$$

We are now able to calculate the derivative of ρ_1^{def} with respect to ρ. Recall that

$$\rho_1^{\text{def}} = \frac{\varphi(z_1)}{N}\Bigg\{\Bigg(\Phi\Big(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\Big) - \Phi(z_2)\Bigg) - \psi(z_1)\Bigg(\Phi^\rho(z_1, z_2) - \Phi(z_1)\Phi(z_2)\Bigg)\Bigg\}.$$

Therefore, using equations (3.17) and (3.18), we find:

$$\begin{aligned} \rho_{1,\rho}^{\text{def}} &= \frac{\varphi(z_1)}{N\sqrt{1-\rho^2}}\Bigg\{\frac{\rho z_2 - z_1}{1-\rho^2}\varphi\Big(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\Big) - \psi(z_1)\varphi(z_2)\varphi\Big(\frac{z_1 - \rho z_2}{\sqrt{1-\rho^2}}\Big)\Bigg\} \\ &= \frac{\varphi(z_1)\varphi\Big((z_2 - \rho z_1)/\sqrt{1-\rho^2}\Big)}{N\sqrt{1-\rho^2}}\Big\{\frac{\rho z_2 - z_1}{1-\rho^2} - \psi(z_1)\varphi(z_1)\Big\}. \end{aligned}$$

Moreover, since $\rho_2^{\text{def}}(z_1, z_2, \rho) = \rho_2^{\text{def}}(z_2, z_1, \rho)$, a formula for $\rho_{2,\rho}^{\text{def}}$ is obtained by interchanging the roles of z_1 and z_2 in the formula for $\rho_{1,\rho}^{\text{def}}$. The expression for $\rho_{2,\rho}^{\text{def}}$ given in the lemma can then be derived by using that - according to equation (3.18) - the following statement holds:

$$\varphi(z_2)\varphi\Big(\frac{z_1 - \rho z_2}{\sqrt{1-\rho^2}}\Big) = \varphi(z_1)\varphi\Big(\frac{z_2 - \rho z_1}{\sqrt{1-\rho^2}}\Big).$$

□

Lemma 3.2 provides the key to our main results. Note that according to lemma 3.2 the following holds:

$$\rho_{1,\rho}^{\text{def}} = \xi H\Big(g_\lambda(\rho), z_1\Big) \tag{3.19}$$

$$\rho_{2,\rho}^{\text{def}} = \xi H\Big(g_{1/\lambda}(\rho), z_2\Big) \tag{3.20}$$

$$\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}} = \xi\Big[H\Big(\frac{1}{1+\rho}, z_1\Big) + H\Big(\frac{1}{1+\rho}, z_2\Big)\Big] \tag{3.21}$$

where

$$H(\alpha, z) := -\alpha z - \psi(z)\varphi(z) = -\alpha z - \frac{\Big(1 - 2\Phi(z)\Big)\varphi(z)}{2\Phi(z)\Big(1-\Phi(z)\Big)},$$

$$g_\lambda(\rho) := \frac{1-\rho\lambda}{1-\rho^2}.$$

Since $\xi > 0$, the signs of $\rho_{1,\rho}^{\text{def}}$, $\rho_{2,\rho}^{\text{def}}$ and $\rho_{1,\rho}^{\text{def}} + \rho_{2,\rho}^{\text{def}}$ are equal to the signs of the right-hand sides of equations (3.19), (3.20) and (3.21) respectively. To prepare for the proofs, we will therefore - in the next two lemmata - derive some properties of the functions H and g.

Lemma 3.3
Suppose that $z < 0$. Then the following statements hold.

(i) *If $\alpha > 0.5$, then $H(\alpha, z) > 0$ for all*

$$z < -\sqrt{\frac{4\alpha - 1}{2\alpha - 1}} =: \bar{z} = \bar{z}(\alpha).$$

(ii) *$H(\alpha, z) > 0$ for all $\alpha \geq 0.65$.*

(iii) *If $10^{-6} \leq \Phi(z) \leq 0.46$, then $H(\alpha, z) < 0$ for all $\alpha \leq 0.52$.*

Proof.

(i) First we observe that

$$1 - \frac{\Phi(z)}{1-\Phi(z)}$$

is strictly decreasing in z and converges to 1 for $z \to -\infty$. Hence

$$H(\alpha, z) \geq -\alpha z - \frac{\varphi(z)}{2\Phi(z)} =: h_1(z).$$

Note that the anti-derivative of h_1 is given by

$$\begin{aligned} \int h_1 &= -\frac{1}{2}\Big[\alpha z^2 + \log\Big(\Phi(z)\Big)\Big] \\ &= \frac{1}{2}\log\Big(\frac{e^{-\alpha z^2}}{\Phi(z)}\Big). \end{aligned}$$

We will show that $\tilde{h}_1(z) := e^{-\alpha z^2}/\Phi(z)$ is strictly increasing in z for $z < \bar{z}$, which implies that $\int h_1$ is strictly increasing in z; this in turn leads to $h_1(z) > 0$ for $z < \bar{z}$. To show that $\tilde{h}_1(\cdot)$ is strictly increasing in z, note that

$$\begin{aligned} \tilde{h}_1'(z) &= \frac{-e^{-\alpha z^2}\Big[2\alpha z \Phi(z) + \varphi(z)\Big]}{\Phi(z)^2} \\ &=: -\frac{e^{-\alpha z^2}}{\Phi(z)^2}\tilde{h}_2(z). \end{aligned}$$

Moreover,

$$\tilde{h}_2'(z) = (2\alpha - 1)z\varphi(z) + 2\alpha\Phi(z)$$

and $\tilde{h}_2''(z) = \varphi(z)[(4\alpha-1)-(2\alpha-1)z^2]$. Hence, if $\alpha > 0.5$, then $\tilde{h}_2''(z) < 0$ if $z < \bar{z}$ and therefore $\tilde{h}_2'(z) < \lim_{z\to-\infty}\tilde{h}_2'(z) = 0$. Therefore, $\tilde{h}_2(z) < \lim_{z\to-\infty}\tilde{h}_2(z) = 0$, implying that $\tilde{h}_1'(z) > 0$ for $z < \bar{z}$.

(ii) Note that H is increasing in α and that it is therefore sufficient to show that $H(0.65, z) > 0$ for all $z < 0$. Moreover, according to (i), $H(0.65, z) > 0$ for $z < \bar{z}(0.65)$. But since $\bar{z}(0.65) \geq -2.31$, we only need to consider the interval $[-2.31, 0)$. The proof will proceed in two steps. For $z \in [-2.31, 0.1]$ we have relied on numerical methods. These methods use an approximation of the second term of H, which is no longer feasible when z approaches zero. The area $[0.1, 0)$ is therefore treated in the second step.

For the numerical analysis we have chosen a grid of 40,000 equidistant points and have evaluated the function

$$\psi(z)\varphi(z) = \frac{\Big(1-2\Phi(z)\Big)\varphi(z)}{2\Phi(z)\Big(1-\Phi(z)\Big)}.$$

The values on the grid have been approximated by standard numerical integration (for $\Phi(z)$) and Taylor series expansion (for $\varphi(z)$). The values between grid points have been approximated from above by the mean value theorem of differential calculus, using that

$$|\frac{d}{dz}\Big(\psi(z)\varphi(z)\Big)| \le \frac{\varphi(z_2)}{2\Phi(z_1)}\Big\{\frac{|z_1|+2\varphi(z_2)}{1-\Phi(z_2)} + \frac{\varphi(z_2)}{\Phi(z_1)}\Big\}$$

for $z \in [z_1, z_2]$.[37]

Now we turn to the second step of the proof, where we show that $H(.65, z) > 0$ for $z \in [-0.1, 0)$. We use the fact that $\tilde{h}_3(z) := \varphi(z)/\Phi(z)$ is strictly decreasing in z for $z < 0$.[38] This implies that if $z \ge z_1$ then

$$H(z) \ge -\alpha z - \frac{c}{2}\Big[1 - \frac{\Phi(z)}{1-\Phi(z)}\Big] =: h_2(z)$$

where $c := \varphi(z_1)/\Phi(z_1)$. On the other hand,

$$\begin{aligned} h_2'(z) &= -\alpha + \frac{c}{2\Big(1-\Phi(z)\Big)^2}\Big[\varphi(z)\Big(1-\Phi(z)\Big) + \varphi(z)\Phi(z)\Big] \\ &= -\alpha + \frac{c\varphi(z)}{2\Big(1-\Phi(z)\Big)^2} \end{aligned}$$

[37]We denote the nominator of $2\psi(z)\varphi(z)$ by f_1 and the denominator by f_2. Note that $f_2' = f_1$ and that we can therefore write the derivative of $2\psi(z)\varphi(z)$ as $(f_1'/f_2) - (f_1/f_2)^2$. But $f_2(z) \ge \Phi(z_1)\Big(1-\Phi(z_2)\Big)$ and

$$f_1'(z) = -z\Big(1-2\Phi(z)\Big)\varphi(z) - 2\varphi(z)^2.$$

Finally, $f_1/f_2 \le \varphi(z_2)/\Phi(z_1)$.

[38]Note that

$$\tilde{h}_3'(z) = -\frac{\varphi(z)}{\Phi(z)^2}\Big[z\Phi(z) + \varphi(z)\Big] =: -\frac{\varphi(z)}{\Phi(z)^2}\tilde{h}_4(z).$$

$\tilde{h}_4(z)$ is strictly increasing in z since $\tilde{h}_4'(z) = z\varphi(z) + \Phi(z) - z\varphi(z) = \Phi(z) > 0$. Therefore $\tilde{h}_4(z) > \lim_{z\to-\infty}\tilde{h}_4(z) = 0$ and hence $\tilde{h}_3'(z) < 0$.

and hence we obtain that $h_2'(z) \leq -\alpha + cd$ for $z \leq z_2$ where

$$d := \frac{\varphi(z_2)}{2\Big(1 - \Phi(z_2)\Big)^2}.$$

For $z_1 = -0.1$ and $z_2 = 0$ we have $c \leq 0.74$ and $d \leq 0.8$. Hence $h_2'(z) < 0$ if $\alpha \geq 0.65$ and therefore $h_2(z) > h_2(0) = 0$.

(iii) Since H is increasing in α, it is sufficient to show that $H(0.52, z) < 0$ for all $\Phi^{-1}(10^{-6}) \leq z \leq \Phi^{-1}(0.46)$. This result has been derived by the same numerical method as used in the proof of (ii).

□

Lemma 3.4

(i) *If $\lambda > 1$, then $g_\lambda(\cdot)$ is strictly decreasing in ρ.*

(ii) *If $0 < \lambda \leq 1$, then $g_\lambda(\cdot)$ is strictly decreasing for $0 < \rho < \rho_{min}$ and strictly increasing for $\rho_{min} < \rho < 1$ where*

$$\rho_{min} := \frac{1 - \sqrt{1 - \lambda^2}}{\lambda}.$$

(iii) *The equation $g_\lambda(\rho) = c$ has the following solutions:*

$$\rho_{1/2}(c) = \frac{1}{2c}\Big\{\lambda \mp \sqrt{\lambda^2 - 4c(1-c)}\Big\}.$$

(iv) *For all $\lambda > 0$ we have $g_\lambda(\rho) \geq .64$ for $0 < \rho < \rho_1(.64)$. Moreover,*

$$\rho_1(.64) = \frac{25}{32}\Big\{\lambda - \sqrt{\lambda^2 + 24/25}\Big\} =: \rho_+(\lambda).$$

Proof.

Note that

$$g_\lambda'(\rho) = -\frac{\lambda\rho^2 - 2\rho + \lambda}{(1-\rho^2)^2}.$$

All results can be obtained by straightforward calculations.

□

Proof of proposition 3.1.

First of all note that by symmetry the statement for $z_1, z_2 > 0$ can be deduced from the statement for $z_1, z_2 < 0$. If $z_1, z_2 > 0$, we can use that

$$1\{Z_i \leq z_i\} = 1 - 1\{Z_i > z_i\} \quad (i = 1, 2)$$

and that

$$\mathrm{Corr}\Big([1 - 1\{Z_1 > z_1\}]\,,\,[1 - 1\{Z_2 > z_2\}]\Big) = \mathrm{Corr}\Big(1\{Z_1 > z_1\}\,,\,1\{Z_2 > z_2\}\Big).$$

But by the symmetry of the bivariate normal distribution we have[39]

$$\Big(1\{Z_1 > z_1\}\,,\,1\{Z_2 > z_2\}\Big) \stackrel{\mathcal{D}}{=} \Big(1\{Z_1 \leq -z_1\}\,,\,1\{Z_2 \leq -z_2\}\Big)$$

and thus it is sufficient to consider the case $z_1, z_2 < 0$ for the rest of the proof.

Recall from lemma 3.2 that

$$\frac{\rho_{1,\rho}^{\mathrm{def}} + \rho_{2,\rho}^{\mathrm{def}}}{\xi} = -\frac{z_1 + z_2}{1 + \rho} - \psi(z_1)\varphi(z_1) - \psi(z_2)\varphi(z_2) =: h_1(\rho),$$

which implies that h_1 determines the sign of $\rho_{1,\rho}^{\mathrm{def}} + \rho_{2,\rho}^{\mathrm{def}}$. First of all note that h_1 is strictly decreasing in ρ. Hence, h_1 is either always higher or always lower than zero or there is a $\rho_0 = \rho_0(z_1, z_2)$ such that h_1 is positive for $\rho < \rho_0$ and negative for $\rho > \rho_0$. Concluding that the same statement applies for $\rho_{1,\rho}^{\mathrm{def}} + \rho_{2,\rho}^{\mathrm{def}}$, we obtain that

$$\rho_1^{\mathrm{def}}(\rho) + \rho_2^{\mathrm{def}}(\rho) > \min\Big\{[\rho_1^{\mathrm{def}}(0) + \rho_2^{\mathrm{def}}(0)], [\rho_1^{\mathrm{def}}(1) + \rho_2^{\mathrm{def}}(1)]\Big\}$$

for $0 < \rho < 1$. But $\rho_1^{\mathrm{def}}(0) = \rho_2^{\mathrm{def}}(0) = 0$ and it remains to show that $\rho_1^{\mathrm{def}}(1) + \rho_2^{\mathrm{def}}(1) > 0$.

Recall from lemma 3.1 that for $\rho = 1$

$$\begin{aligned} \rho_1^{\mathrm{def}} &= \frac{\varphi(z_1)\Big(1 - \Phi(z_2)\Big)}{2N\Big(1 - \Phi(z_1)\Big)} \\ \rho_2^{\mathrm{def}} &= -\frac{\varphi(z_2)\Phi(z_1)}{2N\Phi(z_2)}. \end{aligned}$$

[39] By $X \stackrel{\mathcal{D}}{=} Y$ we mean that X and Y have the same distribution.

Hence:

$$\begin{aligned}[\rho_1^{\text{def}} + \rho_2^{\text{def}}]\Big\{2N\Big(1-\Phi(z_1)\Big)\Phi(z_2)\Big\} &= \varphi(z_1)\Phi(z_2)\Big(1-\Phi(z_2)\Big) - \varphi(z_2)\Phi(z_1)\Big(1-\Phi(z_1)\Big) \\ &=: h_2(z_1)\end{aligned}$$

and

$$h_2'(z_1) = \varphi(z_1)\Big\{-z_1\Phi(z_2)\Big(1-\Phi(z_2)\Big) - \varphi(z_2)\Big(1-2\Phi(z_1)\Big)\Big\}.$$

The following three statements imply that $h_2(z_1) > 0$ for all $-\infty < z_1 < z_2$, which in turn proves that $\rho_1^{\text{def}} + \rho_2^{\text{def}} > 0$.

1. There is a $\bar{z}$ $(-\infty \le \bar{z} \le z_2)$ such that $h_2'(z_1) > 0$ for $z_1 < \bar{z}$ and $h_2'(z_1) < 0$ for $z_1 > \bar{z}$.
2. $\lim_{z_1 \to -\infty} h_2(z_1) = 0$.
3. $h_2(z_2) = 0$.

Statements 2 and 3 are obvious. To prove statement 1 it is sufficient to show that

$$h_3(z_1) := -z_1\Phi(z_2)\Big(1-\Phi(z_2)\Big) - \varphi(z_2)\Big(1-2\Phi(z_1)\Big)$$

has the proposed property. But

$$h_3'(z_1) := -\Phi(z_2)\Big(1-\Phi(z_2)\Big) + 2\varphi(z_2)\varphi(z_1)$$

and statement 1 follows from

1. $h_3(z_2) < 0$ and
2. there is a $\tilde{z}$ $(-\infty < \tilde{z} \le z_2)$ so that $h_3'(z_1) < 0$ for all $z_1 < \tilde{z}$ and $h_3'(z_1) > 0$ for all $z_1 > \tilde{z}$.

Point 2 is obvious and 1 is equivalent to

$$-\frac{z_2}{2} - \frac{\varphi(z_2)\Big(1-2\Phi(z_2)\Big)}{2\Phi(z_2)\Big(1-\Phi(z_2)\Big)} = H(\frac{1}{2}, z_2) \overset{!}{<} 0,$$

which is true according to lemma 3.3.

□

Proof of proposition 3.2.

The second part of statement (ii) has already been derived as corollary to lemma 3.1. Concerning the other statements, recall that, according to lemma 3.2, $\rho^{\text{def}}_{1,\rho}(\rho) > 0$ as long as $H\Big(g_\lambda(\rho), z_1\Big) > 0$, and that $\rho^{\text{def}}_{2,\rho}(\rho) > 0$ as long as $H\Big(g_{1/\lambda}(\rho), z_2\Big) > 0$. By lemma 3.3 this is the case if $g_\lambda(\rho) \geq .64$ and $g_{1/\lambda}(\rho) \geq .64$ respectively. But according to lemma 3.4, the latter conditions are fulfilled for all $\rho < \rho_+(\lambda)$ and $\rho < \rho_+(1/\lambda)$ respectively. For ρ in these areas we therefore obtain that $\rho^{\text{def}}_1(\rho) > \rho^{\text{def}}_1(0) = 0$.

□

Proof of proposition 3.4.

(i) Recall that the sign of $\rho^{\text{def}}_{1,\rho} + \rho^{\text{def}}_{2,\rho}$ is the same as the sign of

$$H(\frac{1}{1+\rho}, z_1) + H(\frac{1}{1+\rho}, z_2).$$

Obviously, $1/(1+\rho)$ is decreasing in ρ which implies the existence of $\bar{\rho}_H$. Moreover, if both summands are positive, which is the case for $0.64 \cdot (1+\rho) < 1$, $\rho^{\text{def}}_{1,\rho} + \rho^{\text{def}}_{2,\rho}$ will also be positive. The opposite is true if they are negative, which is the case for $0.52 \cdot (1+\rho) > 1$. These conditions translate into $\rho < 0.54$ and $\rho > 0.89$ respectively.

(ii) By lemma 3.2, the sign of $\rho^{\text{def}}_{2,\rho}$ is the same as that of $H\Big(g_{1/\lambda}(\rho), z_2\Big)$. But from lemma 3.4 we know that $g_{1/\lambda}(\cdot)$ is strictly decreasing for $\lambda \leq 1$, which implies the existence of $\bar{\rho}_L$. Moreover, since $g_1(\rho) = 1/(1+\rho)$ and $g_{1/\lambda}(\rho) < g_1(\rho)$ for $\lambda < 1$, we obtain that $\bar{\rho}_L < \bar{\rho}_H$.

□

3.A.2.2 Boundary Cases

In this section we prove proposition 3.3. To derive statement (ii), note that the numerator D and the denominator N of $\rho^{\text{def}}(z, z)$ converge to zero if z approaches $-\infty$.[40]

[40]See equations (3.7) and (3.8).

We therefore derive the limit of $\rho^{\mathrm{def}}(z,z)$ by differentiating $D(z,z)$ and $N(z,z)$ respectively. Using equations (3.9) and (3.10), and the fact that $dD(z,z)/dz = 2D_1(z,z)$ and $dN(z,z)/dz = 2N_1(z,z)$, we obtain for $|\rho| < 1$:

$$\lim_{z\to-\infty} \rho^{\mathrm{def}}(z,z) = \lim_{z\to-\infty} \frac{D_1(z,z)}{N_1(z,z)} = \lim_{z\to-\infty} \frac{2\Phi\left(\frac{z-\rho z}{\sqrt{1-\rho^2}}\right) - 2\Phi(z)}{1-2\Phi(z)} = 0.$$

The proof of the arcsin representation of ρ^{def} is given in Gersbach and Lipponer (2000). Finally, we show that ρ^{def} has a local maximum in $(0,0)$. Obviously, the first-order conditions for a local extremum are fulfilled.[41] The second-order conditions can be derived from the following properties of the Hess matrix of ρ^{def}.

Lemma 3.5
Suppose that $|\rho| < 1$.

(i) *The determinant $D(\rho) := \det J(\rho^{\mathrm{def}})|_{z_1=z_2=0}$ of the Hess matrix $J(\rho^{\mathrm{def}})$ in $z_1 = z_2 = 0$ is given by*

$$D(\rho) = \frac{1}{64\pi^2 N^4}\left\{\left(\frac{\rho}{\sqrt{1-\rho^2}} - \frac{2}{\pi}\arcsin(\rho)\right)^2 - \left(\frac{1}{\sqrt{1-\rho^2}} - 1\right)^2\right\}.$$

Moreover, $D(\rho) > 0$ for $-1 < \rho < 0$ and $0 < \rho < 1$ and $D(\rho) = 0$ for $\rho = 0$.

(ii) *The top left entry of $J(\rho^{\mathrm{def}})$ in $z_1 = z_2 = 0$ is given by*

$$J_{11}(\rho) := \rho^{\mathrm{def}}_{11}|_{z_1=z_2=0} = \frac{1}{8\pi N^2}\left\{\frac{2}{\pi}\arcsin(\rho) - \frac{\rho}{\sqrt{1-\rho^2}}\right\}.$$

Moreover, $J_{11}(\rho) > 0$ for $-1 < \rho < 0$, $J_{11}(\rho) < 0$ for $0 < \rho < 1$ and $J_{11}(\rho) = 0$ for $\rho = 0$.

(iii) *$J(\rho^{\mathrm{def}})$ is negative definite if $\rho > 0$ and positive definite if $\rho < 0$.*

Proof.

Recall that

$$D\,|_{z_1=z_2=0} = \frac{1}{2\pi}\arcsin(\rho) \text{ and } N\,|_{z_1=z_2=0} = \frac{1}{4}$$

[41] See section 3.A.1.2.

$$D_1\,|_{z_1=z_2=0} = 0 \text{ and } N_1\,|_{z_1=z_2=0} = 0$$

$$D_{11}\,|_{z_1=z_2=0} = -\frac{\rho}{2\pi\sqrt{1-\rho^2}} \text{ and } D_{12}\,|_{z_1=z_2=0} = \frac{1}{2\pi}\Big(\frac{1}{\sqrt{1-\rho^2}}-1\Big)$$

$$N_{11}\,|_{z_1=z_2=0} = -\frac{1}{2\pi} \text{ and } N_{12}\,|_{z_1=z_2=0} = 0.$$

Using equations (3.12) and (3.14) we therefore obtain:

$$J_{11}(\rho) = \rho_{11}^{\text{def}}\,|_{z_1=z_2=0} = \frac{D_{11}N - DN_{11}}{N^2}|_{z_1=z_2=0} = \frac{1}{8\pi N^2}\Big\{\frac{2}{\pi}\arcsin(\rho) - \frac{\rho}{\sqrt{1-\rho^2}}\Big\}$$

and

$$\rho_{12}^{\text{def}}\,|_{z_1=z_2=0} = \frac{D_{12}N}{N^2}|_{z_1=z_2=0} = \frac{1}{8\pi N^2}\Big(\frac{1}{\sqrt{1-\rho^2}}-1\Big).$$

Hence

$$\begin{aligned}
\det J(\rho^{\text{def}})\,|_{z_1=z_2=0} &= \frac{1}{N^4}\Big[(D_{11}N - DN_{11})^2 - (D_{12}N)^2\Big]|_{z_1=z_2=0} \\
&= \frac{1}{N^4}\Bigg\{\Big(\frac{1}{4\pi^2}\arcsin(\rho) - \frac{\rho}{8\pi\sqrt{1-\rho^2}}\Big)^2 - \frac{1}{64\pi^2}\Big(\frac{1}{\sqrt{1-\rho^2}}-1\Big)^2\Bigg\} \\
&= \frac{1}{64\pi^2 N^4}\Bigg\{\Big(\frac{\rho}{\sqrt{1-\rho^2}} - \frac{2}{\pi}\arcsin(\rho)\Big)^2 - \Big(\frac{1}{\sqrt{1-\rho^2}}-1\Big)^2\Bigg\}.
\end{aligned}$$

We now need to verify the properties of $D(\cdot)$ and $J_{11}(\cdot)$ as stated under (i) and (ii). To prove statement (i) we define $F(\rho) := \frac{2}{\pi}\arcsin(\rho)$ and observe that by the symmetry of $D(\cdot)$ it is sufficient to show that

$$\tilde{D}(\rho) := \Big(\frac{\rho}{\sqrt{1-\rho^2}} - F(\rho)\Big)^2 - \Big(\frac{1}{\sqrt{1-\rho^2}}-1\Big)^2$$

is positive for $0 < \rho < 1$. We write $\tilde{D} = (A+E)^2 - A^2 = 2AE + E^2$ where

$$A(\rho) := \frac{1}{\sqrt{1-\rho^2}} - 1$$

and

$$\begin{aligned} E(\rho) &:= \Big[\frac{\rho}{\sqrt{1-\rho^2}} - F(\rho)\Big] - \Big[\frac{1}{\sqrt{1-\rho^2}} - 1\Big] \\ &= \Big(1 - F(\rho)\Big) - \sqrt{\frac{1-\rho}{1+\rho}}. \end{aligned}$$

Since $A(\rho) > 0$, it is sufficient to show that $E(\rho) > 0$ for $0 < \rho < 1$. But this follows from

$$\begin{aligned} E'(\rho) &= -\frac{2}{\pi}\frac{1}{\sqrt{1-\rho^2}} + \frac{1}{2}\sqrt{\frac{1+\rho}{1-\rho}}\frac{2}{(1+\rho)^2} \\ &= \frac{1}{\sqrt{1-\rho^2}}\Big[\frac{1}{1+\rho} - \frac{2}{\pi}\Big], \end{aligned}$$

which implies that $E(\rho) > \min\{E(0), E(1)\} = 0$.

Finally we have to show that $J_{11}(\rho) > 0$ for $-1 < \rho < 0$ and $J_{11}(\rho) < 0$ for $0 < \rho < 1$. Since $J_{11}(\cdot)$ is anti-symmetric, it is sufficient to prove that

$$\tilde{F}(\rho) := F(\rho) - \frac{\rho}{\sqrt{1-\rho^2}}$$

is negative for $\rho > 0$. Observing that

$$\begin{aligned} \tilde{F}'(\rho) &= \frac{2}{\pi}\frac{1}{\sqrt{1-\rho^2}} - \frac{1}{(1-\rho^2)^{3/2}} \\ &= \frac{1}{(1-\rho^2)^{3/2}}\Big[\frac{2}{\pi}(1-\rho^2) - 1\Big] \\ &< 0, \end{aligned}$$

we find that $\tilde{F}(\rho) < \tilde{F}(0) = 0$ for $\rho > 0$.

$\square$

3.B Deviations from the Linear Correlation Model

In this appendix we explore the consequences for our results when return correlations are much higher for low returns than suggested under the linear correlation model. To do so, we consider an extreme example of two return variables that are perfectly correlated for low realizations and uncorrelated for high realizations. More precisely, let Z_1 and $\tilde{Z}_1$ be stochastically independent random variables with arbitrary distribution functions $F^1_{(\cdot)}$ and $\tilde{F}^1_{(\cdot)}$ respectively and let $c > 0$ and d be real numbers. We define Z_2 by

$$Z_2 := \begin{cases} cZ_1 + d & \text{if } Z_1 \leq z^* \\ \tilde{Z}_1 & \text{else,} \end{cases}$$

and denote the resulting distribution functions of the vector (Z_1, Z_2) by $F_{(\cdot,\cdot)}$. First we observe that the correlation $\rho := \operatorname{Corr}(Z_1, Z_2)$ between Z_1 and Z_2 is usually positive.

Lemma 3.6
Suppose that (Z_1, Z_2) is distributed according to F and that $\mathbb{E}\,\tilde{Z}_1 > \mathbb{E}\,Z_2$. Then $\rho > 0$.

Proof.

Note that

$$\begin{aligned} \operatorname{Cov}(Z_1, Z_2) &= \mathbb{E}\Big[(Z_1 - \mathbb{E}\,Z_1)(Z_2 - \mathbb{E}\,Z_2)\Big] \\ &= F^1_{z^*}\,\mathbb{E}\Big[(Z_1 - \mathbb{E}\,Z_1)(cZ_1 + d - \mathbb{E}\,Z_2) \,|\, Z_1 \leq z^*\Big] \\ &\quad +(1 - F^1_{z^*})\Big(\mathbb{E}[Z_1|Z_1 > z^*] - \mathbb{E}\,Z_1\Big)\Big(\mathbb{E}\,\tilde{Z}_1 - \mathbb{E}\,Z_2\Big). \end{aligned}$$

Since

$$\mathbb{E}\,Z_2 = F^1_{z^*}\Big(c\,\mathbb{E}[Z_1|\,Z_1 \leq z^*] + d\Big) + (1 - F^1_{z^*})\,\mathbb{E}\,\tilde{Z}_1,$$

we obtain that

$$cZ_1 + d - \mathbb{E}\,Z_2 = c\Big(Z_1 - \mathbb{E}[Z_1|Z_1 \leq z^*]\Big) + (1 - F^1_{z^*})\Big(c\,\mathbb{E}[Z_1|\,Z_1 \leq z^*] + d - \mathbb{E}\,\tilde{Z}_1\Big).$$

Therefore, using that

$$\mathbb{E}\,\tilde{Z}_1 - \mathbb{E}\,Z_2 = F^1_{z^*}\Big(\mathbb{E}\,\tilde{Z}_1 - c\,\mathbb{E}[Z_1|Z_1 \leq z^*] - d\Big),$$

we obtain

$$\begin{aligned}\operatorname{Cov}(Z_1, Z_2) &= F^1_{z^*} c\,\mathbb{E}\Big[\Big(Z_1 - \mathbb{E}[Z_1|Z_1 \le z^*]\Big)^2 \,|\, Z_1 \le z^*\Big] \\ &\quad + (1 - F^1_{z^*})\Big(\mathbb{E}[Z_1|Z_1 > z^*] - \mathbb{E}[Z_1|Z_1 \le z^*]\Big)\Big(\mathbb{E}\,\tilde{Z}_1 - \mathbb{E}\,Z_2\Big)\end{aligned}$$

which is positive by our assumption that $\mathbb{E}\,\tilde{Z}_1 > \mathbb{E}\,Z_2$.

□

The next proposition shows that for default points larger than

$$\bar{z} := \max\{cz^* + d\,,\, z^*\},$$

the relationship between default probabilities and default correlations, obtained for jointly normally distributed returns, is reversed if the returns are distributed according to F.

Proposition 3.5
Suppose that (Z_1, Z_2) is distributed according to F and that $\bar{z} \le z_1, z_2 < 0$. Then ρ^{def} is decreasing in z_1 and z_2. Moreover, ρ^{def} is strictly *decreasing if $F^1_{(\cdot)}$ is a strictly increasing function.*

Proof.

Note that for $z_1, z_2 > \bar{z}$ we obtain

$$\begin{aligned}F_{z_1,z_2} &= F^1_{z^*} + (1 - F^1_{z^*})\tilde{F}^1_{z_2}\frac{F^1_{z_1} - F^1_{z^*}}{1 - F^1_{z^*}} \\ &= F^1_{z^*}(1 - \tilde{F}^1_{z_2}) + \tilde{F}^1_{z_2} F^1_{z_1}.\end{aligned}$$

Therefore

$$\begin{aligned}L(z_1) := \sqrt{F_{\infty,z_2}(1 - F_{\infty,z_2})}\,\rho^{\text{def}}(z_1, z_2) &= \frac{F_{z_1,z_2} - F^1_{z_1} F_{\infty,z_2}}{\sqrt{F^1_{z_1}(1 - F^1_{z_1})}} \\ &= \frac{F^1_{z^*}(1 - \tilde{F}^1_{z_2}) + F^1_{z_1}(\tilde{F}^1_{z_2} - F_{\infty,z_2})}{\sqrt{F^1_{z_1}(1 - F^1_{z_1})}}.\end{aligned}$$

But $F_{\infty,z_2} = F^1_{z^*} + (1 - F^1_{z^*})\tilde{F}^1_{z_2}$ and hence

$$L(z_1) = F^1_{z^*}(1 - \tilde{F}^1_{z_2})\frac{1 - F^1_{z_1}}{\sqrt{F^1_{z_1}(1 - F^1_{z_1})}}.$$

Obviously, $L(\cdot)$ is decreasing in z_1 since the numerator is decreasing and the denominator is increasing in z_1 for $z_1 < 0$. Interchanging the roles of z_1 and z_2, one can derive the same statement for z_2.

□

To provide the intuition for this result, we first recall the effects identified for the jointly normal case. When moving default points to the right in this case, the information content of the event $1\{Z_1 \leq z_1\} = 1$ with respect to the underlying return variable Z_1 decreases. This implies that less information about Z_2 and, hence, about $1\{Z_2 \leq z_2\}$ is obtained when observing that $1\{Z_1 \leq z_1\} = 1$. The opposite is true for the non-default event: the information content of this event increases. We have shown that the latter effect dominates the former because of the skewness effect.[42]

In contrary, for pairs of return variables distributed according to F, moving default points to the right has only the effect of decreasing the information content of $1\{Z_1 \leq z_1\} = 1$ with respect to the other indicator variable while the information revealed from the event $1\{Z_1 \leq z_1\} = 0$ does not change. Note that the event $1\{Z_1 \leq z_1\} = 0$ does contain more information about the underlying return variable Z_1 when default points have moved to the right. However, this will not reveal any additional information about the other return variable Z_2, since for $Z_1, Z_2 > \bar{z}$ the variables Z_1 and Z_2 are uncorrelated.

Summing up, we have identified an effect that indicates in which direction the relationship between default probabilities and default correlations will change when return correlations are higher for low returns than suggested by linear correlation models. Higher return correlations for low returns will tend to moderate the skewness effect or even reverse it.

[42] See section 3.2.2.

Chapter 4

Discriminatory Bailout

Abstract. In the presence of macroeconomic shocks severe enough to threaten the liquidity or solvency of the banking system, the regulator can rely on the funds concentration effect to save long-term investment projects. Some banks are forced into bankruptcy with the result that other banks obtain more new funds and remain solvent. We investigate two different implementations of the funds concentration effect and the corresponding discriminatory bailout scheme: "random bailout" and "bailout the big ones". While the latter can be problematic in terms of stability, it is superior to the former in terms of welfare and credibility. A third bailout scheme, "bailout the small ones", would lead to severe stability problems and may support low-return equilibria.

Keywords: Financial intermediation, Macroeconomic risk, Banking regulation, Discriminatory bailout, Funds concentration, Aggregate liquidity, Consistent expectations.

JEL: D84, E44, G28

4.1 Introduction

The frequency and virulence of financial crises has led to serious rethinking concerning the appropriate form of government intervention in financial markets. A major issue is whether such crises can or should be avoided or whether a workout approach is superior to prevention. In particular, it is unclear how financial intermediaries should be regulated when they are subject to large macroeconomic shocks, as has been the case in the recent crisis in Asia.[1]

While in the period before 1970 less intensive competition in banking in connection with interest rate ceilings created oligopoly profits which acted as a buffer against macroeconomic shocks, the present regulatory frameworks are focussed on the prevention of banking crises through cash-asset reserves and risk-sensitive capital requirements. If a banking crisis nevertheless occurs, a variety of approaches are applied. In the most common case of explicit or implicit deposit insurance, the taxpayers' money is used to bail out banks. In some cases, banking crises have been dealt with by closing some banks or by takeovers, which smacks of a discriminatory approach to bailout.[2]

Since the prevention of crises via restricted competition or ex-post bailout with taxpayers' money has costs of its own, and because equity will not always be sufficient to buffer severe macroeconomic shocks,[3] we will focus in this chapter on the possibility of discriminatory bailout.

Under discriminatory bailout, the regulator forces only one or a small number of banks into bankruptcy while the remaining banks are allowed to continue with their operations although all banks may be identical with respect to their balance sheet. The rationale for discriminatory bailout can best be understood in an overlapping generation framework where banks invest short-term deposits in long-term productive investment. During the fruition time of the long-term investments, banks need to refinance themselves by taking the deposits from new generations of savers in order to pay back deposits from the old generation.

Suppose that, during the fruition time, new information reveals that the real return

[1] See e.g. HELLWIG (1998) and BHATTACHARYA, BOOT, AND THAKOR (1998).

[2] This has happened e.g. during the crisis in Asia and the Swiss regional bank crisis (see RADELET AND SACHS (1998, 1999) and STAUB (1998)).

[3] For example HELLWIG (1995) notes (p. 723): "Given the difficulties of recapitalization after a spell of bad luck - and given the possibility of repeated bad spells - it is not clear what one means in asking a bank to follow a strategy of having more equity as a buffer. More equity at the beginning - certainly! But thereafter?" Moreover, GERSBACH (2001) shows that requiring large equity buffers for banks reduces equity in firms, thereby increasing credit rationing which has negative macroeconomic consequences.

on long-term investment is low. This shock will not allow all banks to refinance long-term productive investment by taking new deposits, since they cannot credibly promise interest rates that are sufficiently high to attract enough deposits from the new generation. If there is no coordination mechanism that allows depositors to concentrate savings on a fraction of banks, regulatory intervention is desirable.

In order to save both, banks and long-term investment projects, the regulator can rely on the following general-equilibrium effect, which we call *funds concentration effect*. By forcing some illiquid banks into bankruptcy, the share of funds available for the remaining banks will increase, since there are fewer banks competing for new deposits. Moreover, the surviving banks can buy investment projects from bankrupt banks at liquidation value, thus enabling them to credibly offer higher deposit rates to the second generation. The bailout policy of the regulator is discriminatory.

To concentrate on the funds concentration effect of bank closures, we start our analysis with a situation where the banks' insolvency is assumed to result solely from an exogenous macroeconomic shock. As the realization of this shock is not under the control of the banks' managers, it would be most natural to decide randomly about which banks to close (RB). However, closure policies feed back into the banks' strategic behavior and we therefore also consider two other bailout schemes, namely bailout of big banks (BB) and bailout of small banks (BS).

We compare the discriminatory bailout approach with scenarios where banking crises are prevented completely and with the no-regulation case. Moreover, the different implications of the discriminatory bailout schemes with respect to stability, welfare and credibility are analyzed. We identify BB as the preferred bailout scheme if depositors can coordinate on maximum-return assessments. BS raises severe stability problems and may support low-return equilibria, which both can be avoided under BB. Moreover, BB dominates RB with respect to welfare and credibility of regulatory actions.

Finally, recognizing that the welfare implication of this chapter can only be a first step towards a more complete assessment of the pros and cons of discriminatory bailout, we want to stress that an important aspect of this chapter is the provision of a simple analytical framework and a clarification of the major conceptual issues involved. Given the possibility of a macroeconomic shock and discriminatory bailout, deposits are risky. If an individual bank raises deposit rates, it will affect its own bailout probability as well as that of all other banks since the refinancing needs rise accordingly. Therefore, the expected returns for depositors of all banks are influenced by the decision of an individual bank. Moreover, the distribution of deposits among banks will affect expected returns on deposits as well, since some banks have higher refinancing needs under asymmetric distributions than others. These banks might have to offer higher second-period deposit rates than under symmetric distributions, forcing the other banks to offer higher rates as well in order to obtain any savings at all. As expected deposit returns at all banks are affected by individual bank decisions and by depositors' savings decisions, it is not a priori clear whether consistent assessments of depositors' expected returns actually exist. We establish a general existence result for consistent return assessments and also identify the constellations in which such assessments may not exist.

4.2 Review of the Literature

The role governments should play in managing illiquid banks remains one of the main unresolved issues in banking regulation (see BHATTACHARYA, BOOT, AND THAKOR (1998)). The existing theoretical literature primarily draws on a partial equilibrium point-of-view where systemic consequences are accounted for only by *exogenous* factors. It has been stressed that closure policies have to weigh the costs of bailout (subsidies to uninsured debtholders) with the closure costs (direct bankruptcy costs, externalities). Excessive risk-taking incentives can occur as both costs of bailout and costs of closure. On the one hand, bailout creates moral hazard, as the probability of surviving depends less on the bank's risk choice and more on the regulator's actions. On the other hand, it increases the bank's probability of survival, thus raising the value at stake and, in turn, the bank's incentive to protect it.[4]

Depending on how the different costs are weighed, authors come to different conclusions about the desirability of governmental intervention. While for example HUMPHREY (1986) and SCHWARTZ (1995) advocate a non-interventionist view, the opposite view, namely that in some cases bailing out banks is socially desirable, has been put forward by MISHKIN (1995), SANTOMERO AND HOFFMAN (1998), FREIXAS, PARIGI, AND ROCHET (1998) or CORDELLA AND YEYATI (1999).[5] This chapter gives a new slant to this debate. In our model, closing some banks is necessary so that others can survive without further government intervention. In this sense, putting the funds concentration effect to work is both interventionistic and non-interventionistic.

A further question raised in the literature is how the decision to close a bank should depend on important bank-specific or macroeconomic variables such as the level of unin-

[4]See CORDELLA AND YEYATI (1999) for a formalization of the tradeoffs resulting from these two mutually offsetting effects.

[5]For a comprehensive discussion of this issue see GOODHART (1995).

sured debt on a bank's balance sheet (FREIXAS 1999), the size of a bank (GOODHART AND HUANG 1999) or aggregate investment returns (CORDELLA AND YEYATI 1999). FREIXAS (1999) finds that under optimal policies, banks will be closed either if they have a too low *or* a too high level of uninsured debt on their balance sheet. Whether the former or the latter of these policies should be applied depends on the respective dominance of two counteracting effects: the costs of the subsidies to uninsured debt holders on the one hand and the monitoring incentives for debtholders (which are increasing in the level of uninsured debt) on the other hand. CORDELLA AND YEYATI (1999), investigating how closure policies can minimize the risk-taking incentives of banks, find that banks should be bailed out if aggregate investment returns fall below a certain threshold level. The intuition behind their conclusion is that if aggregate returns are high and a bank fails nevertheless, this will signal excessive risk taking, which is discouraged by threatening closure. Bailing out banks in low states of the variable will increase a bank's charter value and therefore decrease risk-taking incentives.

While the conditionality introduced in CORDELLA AND YEYATI (1999) would have no sensible application in the crises scenarios we are mainly interested in,[6] distinguishing between the relative levels of insured deposits and uninsured debt on a bank's balance sheet would be a further useful step for the analysis of bank closure policies in general-equilibrium frameworks.

Finally, GOODHART AND HUANG (1999) provide a framework that justifies a "bail out the big ones" policy as long as risk-taking incentives are not taken into account. If these incentives are important, the optimal rescuing policy may depend on the size of the bank in a non-monotonic way. While GOODHART AND HUANG (1999) derive their results by comparing the costs of bank failure (contagion) and of bailout (rescuing insolvent banks with the taxpayers' money), we stress the following advantages of BB. First, it helps to avoid low-return equilibria. Second, it is more credible ex-post than RB and - in contrary to BS - guarantees the existence of consistent deposit-return assessments. However, BB is subject to self-fulfilling prophecies and hence return assessments might not be unique. Moreover, it might provide risk-taking incentives for big banks.

Besides the analysis of optimal bank closure policies, an important strand of the literature has investigated the regulator's incentives to apply such rules. BOOT AND THAKOR (1993) examined the regulator's incentives to close banks in a manner that results in socially optimal bank portfolio choices. They find that the regulator's opti-

[6] While the aggregate-investment indicator would surely indicate that all banks should be rescued in such scenarios, it would still be too costly to do so.

mal bank closure policy is less tight than is socially optimal. The analysis has been extended by Acharya and Dreyfus (1989), Fries, Mella-Barral, and Perraudin (1997) and Mailath and Mester (1994). Finally, Repullo (1999) considers government agencies with different objective functions and investigates which of these agencies should make bailout decisions. He finds that central banks should be responsible for dealing with small liquidity shocks, while the deposit insurance agency should deal with large ones. While we do not address institutional design issues - as considered in Repullo (1999) -, incentives for regulators are briefly discussed during the analysis of the bailout schemes' credibility.

On a conceptual level, this chapter is related to the literature in three respects. First, discriminatory bailout can be interpreted as a version of the "*constructive ambiguity*" principle, where regulators have full discretion to let one bank go bankrupt. Two concepts of constructive ambiguity have been discussed in the literature. In Freixas (1999) the central bank deciding which banks are to be rescued follows a mixed strategy. In Goodfriend and Lacker (1999) and Repullo (1999), the bailout policy is not random from the perspective of the central bank but is perceived as such by outsiders that cannot observe the supervisory information that leads to the bailout decision. Our closure policy RB introduces a constructive ambiguity concept similar to Freixas (1999) since the regulator will choose to bail out a bank with a certain probability. The BB and the BS concept are subtle mixtures of predetermined bailout (if banks differ in size) and constructive ambiguity (if banks are equal in size). In contrast to Freixas (1999), who considers a regulator that follows a mixed strategy when deciding about a *single* bank's bailout, we investigate the whole banking system and motivate constructive ambiguity with aggregate solvency concerns. Therefore, bailout probabilities have to be chosen in a way ensuring that under *all* realizations of the stochastic decision process, the banks that have not been closed will be able to survive. This makes the design of such a policy more demanding.

Second, this chapter is related to the literature discussing banking competition when deposits are risky and depositors' assessments about the performance of a bank might become self-fulfilling. In particular, Matutes and Vives (1996) have highlighted the importance of the perceptions of depositors about the probability of success in banking competition. They show that return expectations can become self-fulfilling because of economies of scale or diversification effects. In this chapter the probability of an individual bank failure is determined by the amount and distribution of savings obtained by banks in conjunction with the regulatory bailout approach. This creates existence problems for consistent return assessments by depositors. Moreover, we show that BB (but not RB) can lead to self-fulfilling prophecies. Under BB, a bank that is assessed as offering high returns on deposits attracts a larger share of depositors than competitors and hence increases the likelihood of bailout in the event of a negative macroeconomic shock. This may validate the higher expected returns that depositors have associated with this bank in the first place. As in MATUTES AND VIVES (1996), such self-fulfilling prophecies can create stability problems, which do not occur under RB.

Finally, our analysis bears on the recent tradition of integrating financial intermediation in overlapping generation models. An important strand in this literature (FULGHIERI AND ROVELLI (1998), BHATTACHARYA AND PADILLA (1996) and QI (1994)) has examined the relative merits of intermediaries and financial markets for the possibility of sharing liquidity risk across generations. In this chapter we investigate how macroeconomic shocks can be dealt with in OLG frameworks.

4.3 The Model

The model encompasses two overlapping generations; the first generation lives from $t = 0$ to $t = 1$ and the second from $t = 1$ to $t = 2$. Each generation consists of a continuum of households. There is one single physical good in the economy, which can be used for production and consumption. Moreover, there is a number of banks owned and managed by bankers. Banks gather the households' savings and invest them in a production technology.[7] The key features of the model are the following.

1. Returns on the production technology are subject to macroeconomic risk.
2. Banks offer *uncontingent* deposit contracts to households, thereby exposing themselves to macroeconomic risk.

We first have to justify why some of the macroeconomic risk remains on the balance sheets of the banks. According to HELLWIG (1998), a bank could in principle reduce its exposure to macroeconomic risk traceable to easily observable indicators such as GDP or interest rates (either by offering state contingent deposit contracts or by transferring risk to third parties via hedging contracts). However, banks bear substantial macroeconomic risk in reality. HELLWIG (1998) offers a detailed account of why this is the case. First, available indicators are only an incomplete measure of exposure to aggregate

[7] For simplicity of representation we do not model bank loans to entrepreneurs.

risk. Second, in practice banks do not conclude contingent deposit contracts for the following reasons: the inflexibility of indexed deposit rates as a risk management tool, the existence of transaction costs, and the market-making role of banks. Moreover, the on-demand clause of deposit contracts may invite runs on banks if repayments are made contingent on the realization of macroeconomic variables such as GDP at a certain point in time. Third, hedging counterparties are often banks themselves and hence our analysis can be applied to the counterparty banks. Moreover, banks that shift their risk to third parties are still exposed to credit risk; this risk is likely to be correlated with the macroeconomic risk they want to insure themselves against.[8]

In order to keep the analysis as simple as possible, we do not focus on the moral hazard of banks or risk aversion of households as further possible explanations for aggregate risk exposure of banks. However, our analysis can be applied to the excessive risk-taking problem, which has been identified as one of the major problems of prudential banking regulation (see e.g. DEWATRIPONT AND TIROLE (1994)). If all banks in the industry undertake portfolio choices with a common macroeconomic risk component that cannot be diversified, regulatory intervention can follow a logic similar to the one outlined in this chapter. The additional question emerging in this context is how regulatory bailout schemes affect the banks' risk choices. We will briefly discuss this issue as an extension to our analysis.

Finally, our model allows an alternative interpretation for the banks' exposure to macroeconomic risk. It draws on the uncertainty about the accuracy of the banks' risk management systems rather than on uncontingent deposit contracts. Suppose that banks write contingent contracts that - according to their risk management tools - isolate them from macroeconomic risk. If banks use similar risk management tools, the aggregate uncertainty about future returns can be interpreted as aggregate uncertainty with respect to the accuracy of the contingencies in the deposit contracts: risk management tools may overestimate production returns in one macroeconomic scenario while they underestimate them in an other. This leaves the banking system exposed to systematic risk.[9]

[8] GERSBACH (1998) describes two additional scenarios in which banks do not offer contingent deposit contracts. In the first scenario, the regulator can commit to the failure of insolvent banks. Macroeconomic shocks are then borne by risk-neutral entrepreneurs, as long as their inside funds are a sufficient buffer for these shocks. In the second scenario, banking crises are worked out. Banks offer uncontingent deposit rates that can only be paid back when the state of returns is good. Downturn macroeconomic risk is shifted to future generations.

[9] This view is for example substantiated by SHIN (1999), who suggests that the risk management tools of financial institutions tend to heavily underestimate risk during episodes of market turbulence since they do not take into account the endogeny of future market outcomes (i. e. the fact that outcomes depend on their own actions and that of other market participants).

4.3.1 Technology

We assume that there is a long-term technology that pays a random return of R_2 units of the good in $t = 2$ for each unit invested in $t = 0$. If liquidated in $t = 1$, returns are zero.[10] Production returns in $t = 2$ are subject to aggregate risk. Two different realizations of R_2 are possible. In the first state, occurring with probability p_l, we have low returns: $R_2 = r_{2l}$. In the second state, with probability $p_h := (1-p_l)$, we have high returns $R_2 = r_{2h}$. The realization of the aggregate productivity shock is revealed in $t = 1$ and will be observed by all market participants. We assume (a) constant returns to scale and (b) that investment at arbitrary scale is possible.

4.3.2 Banks

The need for financial intermediation can arise for several reasons (see Bhattacharya and Thakor (1993) for a comprehensive overview). We take this need for granted and do not model it explicitly here. A special feature of our model is that banks finance long-term investments with short-term saving contracts. In contrast to the standard Diamond and Dybvig (1983) framework, there is no risk of consumption timing for the first generation in our model. The individuals of the first generation know that they will never see the fruits of their long-term investments. However, there is an aggregate production risk that makes consumption uncertain in the second period. The economic problem lies in enabling both generations to participate in the benefits of a risky long-term investment though only the second generation will see the returns of the investment.

In $t = 0$ there are n banks, denoted by $B_1, ..., B_n$. They are long-living institutions enabling both generations to participate in long-term investments. Banks offer deposit contracts at deposit rates d_1^i to the first generation and receive an amount D_1^i of deposits ($i = 1, .., n$); all deposits are invested in the production technology. In $t = 1$ banks have to pay back their debt $d_1^i D_1^i$ to first-generation depositors. To obtain new funds, they offer deposit contracts to the second generation at deposit rates d_2^i. After banks have received their second-period deposits, two cases can occur for each individual bank. First, it has raised enough funds from second-generation depositors to pay back its debt; in this case it receives investment returns in $t = 2$ and pays back its second-period depositors. If returns are not sufficient to service all depositors in $t = 2$, investment proceeds are uniformly distributed among depositors. Second,

[10] We will relax this assumption later on (see section 4.6.4.1 and appendix 4.B).

the bank cannot raise enough funds; in this case it has to declare bankruptcy, and the investments are liquidated. First-period depositors of such banks receive only the bank's cash, i.e. the savings of second-period depositors if there are any. Second-period depositors receive nothing.

We complete the description of the banking sector by assuming (a) that banks are owned by risk-neutral bankers[11] who live for three periods and consume in $t = 2$, and (b) that bank managers maximize expected bank profits and hence internalize losses that accrue to depositors in case their claims cannot be fully served.[12]

4.3.3 Households

There are two overlapping generations of consumers (first and second generation), each consisting of a continuum of households living for two periods. They are risk-neutral but want to smooth consumption over time.[13] We denote the individual saving function that describes how much funds household h in generation g ($g = 1, 2$) is willing to deposit with banks by s_{gh}. $s_{gh}(\cdot)$ is assumed to be an increasing function of the *expected* return paid on bank deposits, which we denote by u.

Note that since in both periods some banks might not be able to fully pay back their debts to depositors, both generations of households have to assess the expected returns paid by each bank given first-period deposit rates (for the first generation) and given the first-period allocation and second-period deposit rates (for the second generation). We denote the resulting aggregate saving function for generation g as $S_g(\cdot)$ and assume that S_g is continuous and strictly increasing in u. $S_g(u)$ can be represented as an

[11] Note that for the sake of tractability we have excluded the possibility of issuing equity. We could allow for equity as long as bank reserves cannot buffer losses completely in the event of negative macroeconomic shocks.

[12] To fully close the model, a microeconomic foundation of the bank managers' objective function would have to be derived, and it would have to be specified how bank ownership gets transfered from one generation to the other. We deliberately abstract from the latter issue by considering only one generation of bank managers. Concerning the former issue, it should be noted that the appropriate modeling of firm objectives under imperfect competition is still an open issue (see e.g. Dierker and Grodal (1999)). By employing expected profit maximization, we assume that the usual risk-taking incentive - when bank managers do not take the losses accruing to depositors into account - does not play a major role. For instance, we can assume that the regulatory body imposes some kind of penalty on failing banks or bank managers that induces them to internalize losses (see e.g. Dewatripont and Tirole (1994)). If bank managers only maximized returns for shareholders and sought excessive risks, banks would bid up deposit rates even higher than derived in this chapter, thereby aggravating the refinancing problems of banks in $t = 1$. Consequently, the benefits of regulation, based on funds concentration, would be even more pronounced when risk-taking incentives are present.

[13] The assumption of risk-neutrality is made for convenience and tractability as in Bernanke and Gertler (1988) and Kijotaki and Moore (1997).

integral of the saving density function $s_{gh}(u)$ over an interval on the real line (without loss of generality $[0,1]$), each point on the interval representing one household: $S_g(u) = \int_0^1 s_{gh}(u)\,dh$. We will refer to this representation when using the expressions "*full measure of savings*" and "*zero measure of savings*" later on. A certain bank has obtained the full measure of savings if it has not attracted all depositors but if the integral of the saving density function over all the banks' depositors is equal to the integral over all households ("full-measure bank"). If a bank has attracted some depositors but the integral of the saving density function over the banks' depositors is zero, then we say that the bank has obtained a zero measure of savings ("zero-measure bank"). In the sequel we will use functions of the type $S(u) = au^{\alpha}$ with $a, \alpha \in (0, \infty)$ as an example for the saving functions of both generations.

Finally, note that the saving functions S_g for deposits can be interpreted as a result of a portfolio decision. Deposits may only be one of several saving possibilities[14] that are imperfect substitutes. In this case, the expected-return elasticity of deposits can be quite high.

4.3.4 Example

Throughout this chapter we will use the example presented in table 4.1 to illustrate our results. Note that $\bar{R}_2$ denotes the expected investment return $p_l r_{2l} + p_h r_{2h}$.

$S_1(u) = u$	$p_l = 0.2$	$r_{2l} = 1.03$	$\bar{R}_2 = 1.18$
$S_2(u) = 1.07 \cdot u$	$p_h = 0.8$	$r_{2h} = 1.22$	

Table 4.1: Example A.

4.3.5 Regulatory Policy

We will derive the necessity of regulation precisely in sections 4.4.1 and 4.5.1. For the time being, note that it will result from the following reasoning. In the case of low production returns it might not be possible for all banks to refinance in $t = 1$ since they cannot credibly offer sufficiently high second-period deposit rates. Nevertheless,

[14] The others are not modeled explicitly but enter the model via the specification of the saving functions.

it might be possible for a fraction of the banks to refinance if depositors concentrated their savings on these banks. Without regulation though, depositors have no possibility of coordinating their savings on such a fraction of banks; equilibria in which no bank is able to refinance can therefore not be excluded. We will consider two types of regulatory scenarios designed to avoid these problems. The first one (prudential banking) ensures that the whole banking system is able to refinance in both states of production returns by encouraging banks to offer low deposit rates in the first period. The second one (discriminatory bailout) allows for situations where the banking system is not able to refinance itself. The regulator solves the coordination problem of depositors by closing a fraction of banks in order to make sure that the others can survive. Closing some banks will have two effects: first, it will reduce the amount of second-period deposits needed by the banking system; second, by taking over investment projects of closed banks, surviving banks can offer higher returns on deposits. In this section we describe the different regulatory approaches formally.

4.3.5.1 Bailout Schemes

Suppose that there are $m \leq n$ banks in $t = 1$ that have received deposits. The regulator observes the realization r_2 of the macroeconomic shock, i.e. the future prospects of aggregate production returns. The banking system is able to refinance if and only if

$$S_2(r_2/d_1^{\max}) \geq \sum_{i=1}^{n} d_1^i D_1^i. \tag{4.1}$$

$d_1^{\max}$ is the highest deposit rate that has been offered by a bank in the first period. Note that $r_2/d_1^{\max}$ is the highest return that *all* banks can credibly offer to the second generation in $t = 1$ (because $d_2^i d_1^i$ cannot exceed r_2). If refinancing condition (4.1) holds, then all banks can survive (for example, if a uniform deposit rate of $r_2/d_1^{\max}$ is offered to second-period depositors) and the regulator will not intervene. Consequently all banks will be allowed to compete for second-generation deposits in this case. In the following we will use the matrix $\boldsymbol{\Delta} := (\Delta_i)_{i=1}^n$ with $\Delta_i := (\Delta_{iD}, \Delta_{iI})$ to summarize deposits and investments of the banks after the regulatory decision. Δ_{iD} denotes the obligations to first-period depositors and Δ_{iI} denotes the units of investment projects that a bank holds. Hence, if (4.1) holds, deposits and investments are given by $\Delta_i = (d_1^i D_1^i, D_1^i)$ for $i = 1, .., n$ since the regulator has not stepped in.

If condition (4.1) does not hold, then not all banks will be able to refinance themselves because new funds at the largest credible uniform deposit rate are less than the aggregate obligations of the banking system.[15] In this case the regulator will close a certain number $(m-k)$ of banks and additionally eliminate a fraction $(1-b)$ of the surviving banks' deposits. Depositors whose deposits have been eliminated will loose their claims on the bank. The bailout schemes differ with respect to the manner in which the subset of surviving banks, which we denote by $\mathcal{B}^+$, is determined. Under **random bailout (RB)**, $\mathcal{B}^+$ is chosen by *randomly* drawing k banks (out of the m banks which have received any deposits). Under **prudential banking (PB)**, the regulator also applies RB but additionally imposes a penalty P on all banks that had to be closed. P is assumed to be so high that a bank strategy with a positive probability of leading to P will always be eschewed in favor of any strategy that does not involve the possibility of insolvency, including exiting from the market.[16] While the surviving banks are chosen randomly under RB and PB, banks are ordered with respect to the amount of first-period deposits they have gathered under **bail out the big ones (BB)**:

$$D_1^{\tau(1)} > \cdots > D_1^{\tau(\underline{k})} = \cdots = D_1^{\tau(k)} = \cdots = D_1^{\tau(\bar{k})} > \cdots > D_1^{\tau(m)}.$$

The set $\mathcal{B}^+$ will contain the banks $B_{\tau(1)}, ..., B_{\tau(\underline{k}-1)}$ and another $k-(\underline{k}-1)$ banks which are chosen randomly from the set $\{B_{\tau(\underline{k})}, ..., B_{\tau(\bar{k})}\}$. The scheme **bail out the small ones (BS)** is defined in a symmetric way; the only difference is that the ordering scheme is reversed, i.e. big banks will be closed first.[17] Since both bailout schemes BB and BS work completely symmetrically, we will describe only BB in more detail; also all examples that will be discussed refer to BB.

The investment projects of closed banks are distributed among surviving banks in proportion to the amount of deposits they have gathered. Hence, after regulatory intervention, the balance sheet of a surviving bank i consists of obligations $bd_1^i D_1^i$ to first-period depositors and of $b_I D_1^i$ units of investment projects where[18]

$$b_I = b_I(\mathcal{B}^+) := \frac{\sum_{i=1}^n D_1^i}{\sum_{i\in\mathcal{B}^+} D_1^i}. \tag{4.2}$$

[15] If higher deposit rates would be credibly offered by some banks, then at least the bank that has offered $d_1^{\max}$ in $t=0$ would not be able to receive any deposits.

[16] Note that by offering very unfavorable deposit rates a bank can always ensure that it will never become insolvent, regardless of the behavior of the other banks.

[17] Regarding BB and BS we might also envisage ordering banks with respect to the value of their obligations instead of ordering them with respect to their deposits, as above. Our arguments would not be affected by a switch to outstanding-debt ordering.

[18] Note that we use $\mathcal{B}^+$ not only to denote the set of surviving banks but also to denote the set of indices $i_1, ..., i_k$ that identify the surviving banks. Note also that if all banks in $\mathcal{B}^+$ have only a zero measure of deposits on their balance sheet, then the denominator of equation (4.2) is zero. In this case investment projects are uniformly distributed among all banks in $\mathcal{B}^+$.

Hence

$$\Delta_i := \begin{cases} (bd_1^i D_1^i\,,\, b_I D_1^i) & \text{if i} \in \mathcal{B}^+ \\ (0\,,\,0) & \text{else.} \end{cases}$$

Recall now that regulatory policy must ensure that all remaining banks are able to pay back their first-period depositors with the savings of the second generation. Note that if a bank i receives exactly the amount of second-period deposits that it needs to service its obligations (i.e. $D_2^i = bd_1^i D_1^i$), then it will be able to credibly offer a deposit rate $r_2 b_I/(bd_1^i)$ to its second-period depositors; hence the rate $r_2 b_I/(bd_1^{\max})$ can be offered by *all* surviving banks, and the total amount of second-period savings that can be attracted is at least $S_2\Big(r_2 b_I/(bd_1^{\max})\Big)$. Since b/b_I is equal to the fraction of overall deposits which have not been eliminated (we denote this fraction by q), we conclude that all remaining banks will be able to refinance if

$$S_2\Big(\frac{r_2}{qd_1^{\max}}\Big) \geq qd_1^{\max} \sum_{i=1}^{n} D_1^i.$$

The highest possible fraction $\bar{q}$ of first-period deposits that can be bailed out under the constraint that the surviving banks shall be able to refinance is therefore given as solution of the equation[19]

$$S_2\Big(\frac{r_2}{qd_1^{\max}}\Big) = qd_1^{\max} \sum_{i=1}^{n} D_1^i. \tag{4.3}$$

Note that under BB we have

$$q = \frac{b\sum_{i=1}^{k} D_1^{\tau(i)}}{\sum_{i=1}^{n} D_1^i}. \tag{4.4}$$

Hence, k and b can be chosen to ensure that $q = \bar{q}$. Obviously there is more than one combination of k and b that leads to $q = \bar{q}$. It is therefore important to note that allowing the regulator to additionally eliminate a fraction $(1-b)$ of all the surviving banks' balance sheets only serves technical purposes.[20] In principle we do not allow for the balance sheets of all banks to be scaled down arbitrarily without disruptive consequences for the banks when continuing their operations and thus for the economy.

[19] Note that the left-hand side of the equation is decreasing while the right-hand side is strictly increasing in q which, together with the fact that inequality (4.1) does not hold, implies that there is a unique solution $\bar{q} \in [0,1]$ of equation (4.3).

[20] This assumption allows us to avoid discontinuities (see page 130).

If such an arbitrary scale-down were possible, an alternative implementation of the funds concentration effect would be to scale down the balance sheets of all banks without closing any of them completely. But under severe macroeconomic shocks (in which our major interest lies), the scale-down needed would most likely disrupt the banks' operations and thus shrinking all banks simultaneously is no viable alternative. Therefore, under all bailout schemes we will try to choose b as high as possible. Hence, using equation (4.4) we determine k and b under BB by

$$k = \min\Big\{ l \in \mathbb{N} \mid \sum_{i=1}^{l} D_1^{\tau(i)} \geq \bar{q} \sum_{i=1}^{n} D_1^i \Big\}, \tag{4.5}$$

$$b = \Big(\bar{q} \sum_{i=1}^{n} D_1^i\Big) / \Big(\sum_{i=1}^{k} D_1^{\tau(i)}\Big). \tag{4.6}$$

Contrary to BB, the fraction q of bailed out depositors under RB is in general *not* determined by the choice of k and b, since it is not clear which banks will be chosen to survive. To determine k under RB, we must therefore take into account that the fraction of remaining deposits should not exceed $\bar{q}$, regardless of which banks have been chosen. The worst case that can be thought of in terms of remaining deposits is that - as under BB - the k largest banks have been chosen to survive. Hence, to ensure that the fraction of bailed out deposits is equal to $\bar{q}$ in this case, k and b are determined as under BB, i.e. according to equations (4.5) and (4.6).

4.3.5.2 Bailout Schemes: The Symmetric Case

In this section we illustrate the working of the bailout schemes for an arbitrary symmetric first-period allocation (d_1, D_1) where all banks have offered the same deposit rate d_1 and received the same amount of first-period deposits D_1. In this case all bailout schemes will proceed in the same way. First, $\bar{q}$ is determined as the solution of a simplified version of equation (4.3):

$$qnd_1D_1 = S_2\Big(\frac{r_2}{qd_1}\Big). \tag{4.7}$$

Figure 4.1 illustrates the solution of equation (4.7) for example A, which will be used for all illustrations unless otherwise indicated. Second, to achieve a fraction $\bar{q}$ of bailed out deposits with b as a high as possible, we choose $k = \lceil nq \rceil$ and $b = (nq)/\lceil nq \rceil$.[21] The k banks that will survive are chosen randomly under all schemes since all banks

[21]Note that $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

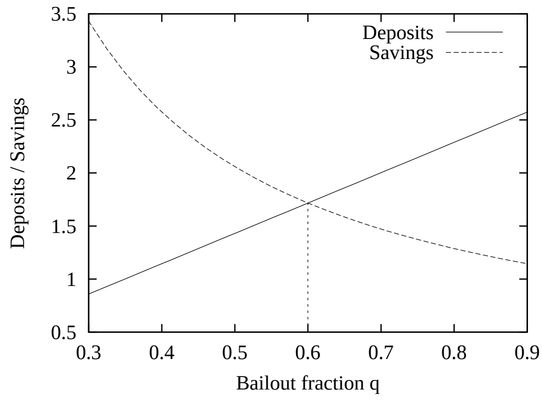

Figure 4.1: Bailed out first-period Deposits qnd_1D_1 and second-period savings $S_2(r_2/qd_1)$ as functions of q ($nD_1 = 2.7$ and $d_1 = 1.05$, example A).

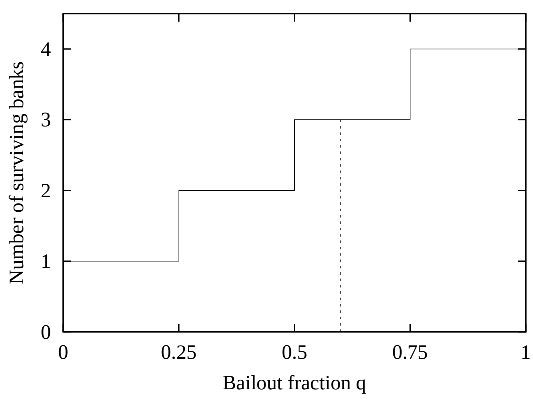

Figure 4.2: Number k of surviving banks as function of q.

have raised the same amount of first-period deposits; hence the bailout probability of each single deposit is equal to the fraction $\bar{q}$ of bailed out deposits.

As an example consider the case $n = 4$. Figure 4.2 depicts the function $q \to \lceil nq \rceil$. If e.g. $\bar{q} = 0.6$, then $\lceil n\bar{q} \rceil = 3$ and one bank will be closed. Moreover, $b = 1.8/3 = 0.6$ and 40% of each surviving bank's deposits are eliminated.

4.3.5.3 Bailout Schemes: The Asymmetric Case

If the first-period deposit distribution is asymmetric (i.e. if not all banks have received the same amount of deposits), the schemes RB and BB will generally produce different regulatory decisions; under BB, always a fraction $\bar{q}$ of depositors is bailed out and the bailout probabilities for deposits depend on the size of the bank at which the deposits are held. Under RB, in contrary, the bailout fraction can be lower than $\bar{q}$ and the bailout probability of each deposit is given by $(k - 1 + b)/m$.

We start, however, with an important case of asymmetric deposit distribution where RB and BB produce the same regulatory decision: the case where one bank has obtained all deposits and the other banks none. In this case we have $k = 1$ and $b = \bar{q}$ implying that the bailout probability of each deposit is equal to the fraction $\bar{q}$ of bailed out deposits.

As an example illustrating the differences between BB and RB, consider a deposit distribution as depicted in figure 4.3 and suppose that as above $\bar{q} = 0.6$. To determine

k, note that the sum of all first-period deposits is 10. Since $5/10 < 0.6$ and $(5+2)/10 > 0.6$, we have $k = 2$ and $b = 10 \cdot 0.6/(5+2) \approx 0.85$. Under BB, the regulator will therefore close bank 4 and either bank 2 or bank 3; the choice between those two banks is performed randomly with each bank having a probability of 0.5 to survive. After that a fraction $(1-b) = 0.15$ of the surviving banks' deposits is eliminated. Hence, the bailout probability for deposits at bank 4 is zero, it is $0.5 \cdot 0.85 = 0.425$ for deposits with banks 2 and 3, and it is given by 0.85 for deposits at banks 1. Under RB, in contrast, the bailout probability is equal to $(2 - 1 + 0.85)/4 = 0.462$ for all deposits. Moreover, the fraction of bailed out deposits is 0.6 if bank 1 and bank 2 have been chosen to survive while it drops to $[0.85 \cdot (2+1)]/10 = 0.255$ if banks 3 and 4 have been chosen.

Finally, consider the case where one bank has obtained the full measure of savings and the other $(n-1)$ have obtained zero measures of savings. As for the case where only one bank has received any deposits, we obtain $k = 1$ and $b = \bar{q}$. But while under BB the $(n-1)$ small banks are closed and the bailout probability for the depositors of the big bank is $\bar{q}$, the bailout probability under RB drops to $\bar{q}/n$. Moreover, with probability $(n-1)/n$, the full measure of deposits is eliminated.

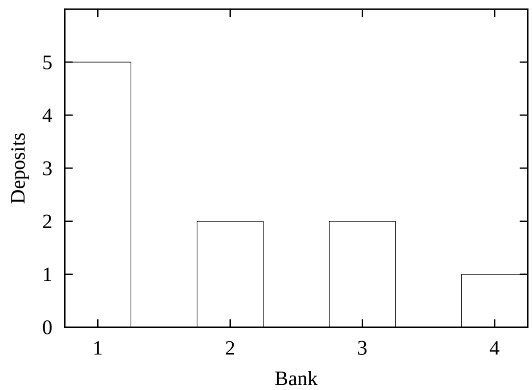

Figure 4.3: Example for a first-period deposit distribution.

4.3.5.4 Bailout the Big Ones: The Case of Zero-Measure Banks

Concluding the description of the bailout schemes, we note that BB will slightly differ from the procedure described above in case that some banks have only gathered a zero measure of deposits in $t = 0$. Note that under BB such banks will always be closed if the refinancing condition (4.1) is not fulfilled. Hence, when determining $\bar{q}$ in such a case, the first-period deposit rates offered by zero-measure banks do not have to be taken into account. In such a situation, the regulator will therefore define $d_1^{\max}$ as the maximum first-period deposit rate offered by positive-measure banks and will close all zero-measure banks that have offered deposit rates higher than $d_1^{\max}$. The BB bailout scheme described in the previous sections will then be applied to the positive-measure banks and to the remaining zero-measure banks.

4.3.6 Summary: Sequence of Events

We now summarize the sequence of events.

1. **Banks offer first-period deposit rates.**
   In $t = 0$ banks simultaneously offer their first period-deposit rates d_1^i $(i = 1, .., n)$. $\mathbf{d_1} = (d_1^i)_{i=1}^n$ denotes the vector of all first-period deposit rates.

2. **Households (first generation) assess expected returns and make their saving decisions.**
   First-generation households make assessments $\mathbf{u_1} = (u_1^i)_{i=1}^n$ about the expected returns that will be paid on deposits by each bank. Based on these assessments, they decide on the amount of savings they want to deposit with each bank. We denote the vector of all first-period deposits by $\mathbf{D_1} = (D_1^i)_{i=1}^n$. Finally, banks invest the deposits obtained in the production technology.

3. **Regulatory policy.**
   The regulator observes the realization of the productivity variable r_2. If the refinancing condition (4.1) is fulfilled, the regulator will not intervene. In this case the deposits and investments of bank i are given by $\Delta_i = (d_1^i D_1^i, D_1^i)$. If condition (4.1) is not fulfilled, then one of the bailout schemes will be applied and some banks will be closed. The set of surviving banks is denoted by $\mathcal{B}^+$. Investment projects of closed banks are distributed among surviving banks in proportion to the amount of first-period deposits they have gathered. Deposits and investments of bank i after regulatory policy are given by $\Delta_i = (0, 0)$ if it has been closed and by $\Delta_i = (bd_1^i D_1^i, b_I D_1^i)$ if it has survived.

4. **Surviving banks offer second-period deposit rates.**
   Surviving banks simultaneously offer their second-period deposit rates $d_2^i(\mathbf{\Delta})$ $(i \in \mathcal{B}^+)$. The vector of all second-period deposit rates is denoted by $\mathbf{d_2} = (d_2^i)_{i \in \mathcal{B}^+}$.

5. **Households (second generation) assess expected returns and make their saving decisions.**
   Second-generation households make assessments $\mathbf{u_2} = (u_2^i)_{i \in \mathcal{B}^+}$ about the expected returns that will be paid on deposits by each bank. Based on this assessment, they decide on the amount of savings they want to deposit with each bank. The vector of all second-period deposits is denoted by $\mathbf{D_2} = (D_2^i)_{i \in \mathcal{B}^+}$.

6. **Surviving banks pay their second-period depositors back.**
   In $t = 2$ surviving banks receive returns from investments and pay their second-period depositors back. Profits are consumed by managers.

We call steps 4 -6 the second-period subgame of the intermediation game.

4.3.7 Equilibrium Concept

In order to derive the subgame-perfect equilibrium of the game described in section 4.3.6, some subtle points have to be taken into account. In particular we need to discuss how the households' return assessments can be derived. Two issues are important in this respect.

First, given an assessment $\mathbf{u_g}$ by households in generation g ($g = 1, 2$) about expected deposit returns, the deposit distribution $\mathbf{D_g} = \mathbf{D_g}(\mathbf{u_g})$ is derived from the households' utility maximization. We use $\mathcal{B}_g^{\max}$ to denote the subset of all banks that are assessed to pay the maximum expected return $u_g^{\max}$ among all banks for generation g. The banks in $\mathcal{B}_g^{\max}$ will receive all the savings of the households: $\sum_{i \in \mathcal{B}_g^{\max}} D_g^i(\mathbf{u_g}) = S(u_g^{\max})$ and $D_g^i(\mathbf{u_g}) = 0$ for all $i \notin \mathcal{B}_g^{\max}$.

Since depositors are indifferent with regard to all banks in $\mathcal{B}_g^{\max}$, it is unclear how deposits are distributed among these banks. We will assume that if two banks are in $\mathcal{B}_g^{\max}$, they will receive the same amount of deposits if all of their characteristics are identical.[22] This means that indifferent depositors will randomize among their preferred banks with equal probability and independently of each other.

Second, the households' assessments have to be consistent. In order to give a precise definition of consistency, we use $\mathbf{U_1}(\mathbf{d_1}, \mathbf{D_1})$ to denote the vector of expected returns on first-period deposits resulting from the allocation $(\mathbf{d_1}, \mathbf{D_1})$ and from regulatory policy. Furthermore, given the matrix $\boldsymbol{\Delta}$ of deposits and investments after regulatory policy and given second-period deposit rates $\mathbf{d_2}$ and deposit distribution $\mathbf{D_2}$, we can define the resulting vector of expected second-period returns as $\mathbf{U_2}\Big(\boldsymbol{\Delta}, \mathbf{d_2}, \mathbf{D_2}\Big)$. We will show in section 4.4.1 that the functions $\mathbf{U_1}(\cdot)$ and $\mathbf{U_2}(\cdot)$ are well defined for all entries i with $i \in \mathcal{B}_g^{\max}$, i.e. for banks that are assessed to pay the maximum expected returns on deposits. However, if $i \notin \mathcal{B}_g^{\max}$ then bank i will receive no deposits and exit the market; it is therefore unclear whether the assessment was correct in the first place. To

[22] In $t = 0$, banks are identical if they have offered the same first-period deposit rates and in $t = 1$ they are identical if their balance sheets are identical and if they have offered the same second-period deposit rate.

deal with this problem, we introduce a so called "*zero-measure test*". We calculate the expected returns for each bank resulting from a deposit distribution $\mathbf{\hat{D}_g(u_g)}$. $\mathbf{\hat{D}_g(u_g)}$ differs from $\mathbf{D_g(u_g)}$ only in one respect, namely that banks $i \notin \mathcal{B}_g^{\max}$ receive a zero measure of savings instead of no savings at all:

$$\hat{D}_g^i(\mathbf{u_g}) := \begin{cases} D_g^i(\mathbf{u_g}) & \text{if i} \in \mathcal{B}_g^{\max} \\ \text{zero measure} & \text{else.} \end{cases}$$

Definition 4.1 (Consistent assessments)
Given first-period deposit rates $\mathbf{d_1}$*, an assessment* $\mathbf{u_1}$ *is consistent if and only if*

$$\mathbf{U_1}\Big(\mathbf{d_1}, \mathbf{\hat{D}_1}(\mathbf{u_1})\Big) = \mathbf{u_1}.$$

Given the matrix $\mathbf{\Delta}$ *of post-regulation deposits and investments, and given second-period deposit rates* $\mathbf{d_2}$*, an assessments* $\mathbf{u_2}$ *is consistent if and only if*

$$\mathbf{U_2}\Big(\mathbf{\Delta}, \mathbf{d_2}, \mathbf{\hat{D}_2}(\mathbf{u_2})\Big) = \mathbf{u_2}.$$

Note that we will only consider different assessments for two banks if they are different with regard to at least one of their characteristics.

Consistent assessments mean that depositors make optimal saving decisions[23] and that expected returns are equal to returns generated when depositors distribute themselves among the preferred banks. Whether or not consistent assessments exist will be discussed at length in the next section. If more than one consistent assessment exists, we apply the Pareto selection criterion and assume that the assessment which generates the highest returns will be realized. We therefore define:

Definition 4.2 (Optimal assessments)
An assessment $\mathbf{u_g}$ $(g = 1, 2)$ *is called optimal if it is consistent and if* u_g^{max} *is at least as high as the maximum expected return resulting from any other consistent assessment.*

We will see that under regulation the best assessment and the corresponding deposit distribution are always unique. We conclude this section by summarizing our equilibrium concept. Note that, since banks are identical ex-ante, we constrain ourselves to the analysis of symmetric equilibria.

[23] I.e. savings decisions that lead to the highest expected returns, given the deposit rates offered by the banks.

Definition 4.3 (Equilibrium concept)
For any given regulatory policy, a symmetric subgame-perfect Bayesian equilibrium is a set consisting of first-period deposit rates $\mathbf{d_1} = (d_1, .., d_1)$*, assessments* $\mathbf{u_1} = (u_1, ..., u_1)$*, a deposit distribution* $\mathbf{D_1} = (D_1, ..., D_1)$*, reaction functions* $\mathbf{d_2} = \mathbf{d_2}(\mathbf{\Delta})$ *that assign a vector of second-period deposit rates* $\mathbf{d_2}$ *to each possible set* $\mathbf{\Delta}$ *of post-regulation deposits and investments, and a second-period deposit distribution* $\mathbf{D_2} = (D_2, .., D_2)$*. This set has to fulfill the following conditions:*

1. *Given* $\mathbf{\Delta}$*, second-period deposit rates* $\mathbf{d_2}(\mathbf{\Delta})$ *constitute an equilibrium in the subgame.*

2. *The second-period subgame equilibrium is symmetric, i.e. banks that are identical in* $t = 1$ *offer the same second-period deposit rate.*

3. *The strategies* $\left(\mathbf{d_1}, \mathbf{d_2}(\cdot)\right)$ *constitute a subgame-perfect Bayesian Nash equilibrium in the entire game.*

4. *Assessments are optimal.*

The equilibrium concept is a subgame-perfect Bayesian Nash equilibrium involving two subtleties. First, individual deposit decisions have no influence on return assessments since the contribution of each single depositor to overall deposits has zero measure. However, the *distribution* of deposits matters. Second, different deposit distributions for the same vector of deposit rates can imply different probabilities for bank defaults, which feeds back into the return assessments. Both subtleties raise considerable problems for the determination of return assessments. These problems will be addressed in the following section.

4.4 Equilibria in the Second Period and Consistent Assessments

In this section we first solve the second-period subgame and then analyze the existence of consistent assessments in the first period. All proofs in this and the next sections are deferred to the appendix.

4.4.1 Equilibria in the Second Period

Recall that a surviving bank i in $t = 1$ has $bd_1^i D_1^i$ first-period deposits and $b_I D_1^i$ units of investment projects. If the refinancing condition (4.1) holds or if no regulation is applied in $t = 1$, then $b = b_I = 1$ and all banks that have received any deposits in $t = 0$ compete for second-period deposits. If, on the other hand, condition (4.1) does not hold and regulation is applied, then $b \leq 1$, $b_I > 1$, and the regulator closes all banks outside of $\mathcal{B}^+$.[24] The surviving banks' profits in both cases are given by[25]

$$\Pi_2^i := \begin{cases} r_2 b_I D_1^i - (1-b) d_1^i D_1^i - d_2^i D_2^i & \text{if } D_2^i \geq b d_1^i D_1^i \\ -d_1^i D_1^i & \text{else.} \end{cases}$$

To analyze the second-period subgame equilibrium we define $\tilde{d}_1^{\max} := \max_{i \in \mathcal{B}^+}\{d_1^i\}$, $\bar{d}_2^* := S_2^{-1}\left(\sum_{i \in \mathcal{B}^+} bd_1^i D_1^i\right)$ and

$$\mathcal{E}_2^* := \left\{ d_2^i = \bar{d}_2^*, \;\; D_2^i = bd_1^i D_1^i \;\;\; (i \in \mathcal{B}^+) \right\}.$$

Note that $\bar{d}_2^*$ is the lowest deposit rate that generates enough second-period deposits for all surviving banks to refinance and that $\mathcal{E}_2^*$ is the (potential) second-period equilibrium where all surviving banks offer $\bar{d}_2^*$.

We start with the analysis of the no-regulation case. This case is only presented to derive the necessity of regulation. We will restrict ourselves in this case to first-period constellations where all banks have offered the same deposit rate d_1 and therefore have received the same amount D_1 of deposits. Note that this implies that $\tilde{d}_1^{\max} = d_1$ and $\bar{d}_2^* = S_2^{-1}(md_1 D_1)$ where m is the number of banks that have received any deposits in $t = 0$.

Proposition 4.1 (No-regulation case)
Suppose that banks have offered the same deposit rate d_1 and therefore have received the same amount D_1 of deposits in $t = 0$. Then the following statements hold:

(i) If $r_2/d_1 \geq \bar{d}_2^*$, then $\mathcal{E}_2^*$ is an equilibrium. Moreover, from the point of view of the banks, $\mathcal{E}_2^*$ Pareto-dominates all other possible equilibria.

[24] Note that if no bank has been closed by the regulator, then $\mathcal{B}^+$ simply denotes the set of all banks that have received any deposits in $t = 0$.

[25] Note again that we assume bank managers to internalize losses that accrue to depositors.

(ii) *If $r_2/d_1 < \bar{d}_2^*$, then there is no equilibrium where all banks can refinance themselves. Moreover, in all symmetric second-period equilibria, where all banks offer the same deposit rate d_2, no bank can refinance itself and we have $D_2^i = 0$ for all $i \in \mathcal{B}^+$.*

Intuitively statement (i) stems from the following reasoning. First, deviations from $\mathcal{E}_2^*$ are not profitable since higher deposit rates increase repayment obligations; deviation to lower deposit rates either leads to the loss of all second-period deposits to the other banks or to the concentration of all savings on the deviating bank, which both takes the deviating bank's profits down to zero. Second, $\mathcal{E}_2^*$ Pareto-dominates all other equilibria, since equilibria with higher deposit rates lead to an increase in repayment obligations and because in equilibria with lower deposit rates no bank will receive any deposits. The mechanism leading to the latter observation is also responsible for the second part of statement (ii) and can be explained as follows.

Assume that the refinancing condition were fulfilled for $\tilde{m}$ banks ($\tilde{m} < m$), i.e. that

$$\frac{r_2}{d_1} \geq S_2^{-1}\Big(\tilde{m} d_1 D_1\Big).$$

If depositors could manage to deposit their savings only with a subset of $\tilde{m}$ banks, these banks would be able to refinance. But since all banks are identical, depositors cannot coordinate to deposit with a particular subset of banks; rather they would randomize independently between banks and, by the law of the large numbers, every bank would receive the same amount of savings, which is not enough to refinance. This in turn implies that none of the banks will receive *any* savings. Discriminatory bailout solves this coordination problem by closing some of the banks so that the remaining ones can raise enough new funds to refinance.

Of course there can be asymmetric constellations where one bank is able to refinance. Imagine the case where there are only two banks and $r_2/d_1 \geq S_2^{-1}(d_1 D_1)$. If one bank offers u_*, defined as the positive solution of $u = r_2 D_1 / S_2(u)$, and the other bank offers a lower deposit rate, the depositors' coordination problem is solved, since they know that the bank that has offered u_* can pay strictly higher returns. Without regulation, however, there is a severe coordination problem, because both of them would like to be the bank that is able to pay depositors back. Therefore, in our analysis of the no-regulation case in section 4.5.1 we will assume that no bank will receive any second-period savings if $r_2/d_1 < \bar{d}_2^*$. On the other hand, if $r_2/d_1 \geq \bar{d}_2^*$, we assume that $\mathcal{E}_2^*$ is played which - according to proposition 4.1 - can be justified by the Pareto selection criterion.

We now turn to the case where regulation (PB, RB, BB or BS) ensures that $r_2/(q\tilde{d}_1^{\max}) \geq \bar{d}_2^*$. The regulation case will be analyzed in the general setting where banks may have offered different deposit rates in $t = 0$. We have already derived that under symmetric first-period allocations, banks will offer second-period deposit rates that are just sufficient to attract enough second-period savings to pay back their obligations to first-period depositors. Deviations to lower deposit rates can be excluded as the bank which has offered lower deposit rates will receive no second-period savings. Under asymmetric first-period constellations, however, deviations to lower deposit rates could be profitable for big banks, since the smaller non-deviating banks cannot cope with all second-period savings alone. This in turn could lead to non-existence of equilibria or to equilibria where not all banks can refinance (despite the fact that all banks would be able to refinance if offered deposit rates were high enough). To avoid these problems, we assume that the regulator imposes a **lower bound on second-period deposit rates** (**LBD**), i.e. she guarantees that no banks offers a deposit rate lower than $\bar{d}_2^*$.[26] This ensures that refinancing of all banks indeed occurs in equilibrium under asymmetric first-period constellations as the next proposition indicates.

Proposition 4.2 (Regulation case)
Suppose that regulation ensures that $r_2/(q\tilde{d}_1^{max}) \geq \bar{d}_2^*$ and that LBD is applied in $t = 1$. Then $\mathcal{E}_2^*$ is a second-period equilibrium. Moreover, from the point of view of the banks, $\mathcal{E}_2^*$ Pareto-dominates all other possible equilibria.

Regulation LBD ensures that banks do not undercut the rate $\bar{d}_2^*$. Moreover, banks that have offered higher deposit rates than $\bar{d}_2^*$ have higher repayment obligations than under $\mathcal{E}_2^*$. This implies that deviations from $\mathcal{E}_2^*$ are not profitable and that all other possible equilibria are Pareto-dominated by $\mathcal{E}_2^*$.

Throughout this chapter we will assume that under regulation, banks will play $\mathcal{E}_2^*$ which can be justified by the Pareto criterion. In this case, second-period deposits just suffice to cover the refinancing needs of the banks and we can describe expected returns on first-period deposits of bank i as $u_1^i = (p_l q_l^i + p_h q_h^i) d_1^i$ $(i = 1, ..., n)$. q_l^i (q_h^i) denotes the bailout probability for bank i in the case of low (high) production returns.

[26] Again we could think that a high enough penalty is imposed in case that banks do not obey.

4.4.2 Consistent Assessments in the First Period

In this section we analyze the existence and uniqueness of consistent assessments in the first period, assuming that one of the regulatory schemes is applied.[27] Consider a situation where there are two groups of banks, $\mathcal{B}_l$ and $\mathcal{B}_h$, that have offered first-period deposit rates d_{1l} and d_{1h} ($d_{1l} \leq d_{1h}$) respectively. Note that the assessments and deposits for all banks in $\mathcal{B}_l$ and for all banks in $\mathcal{B}_h$ must be identical. Note also that this scenario includes two important cases that need to be considered in order to analyze symmetric equilibria of the intermediation game: first, the symmetric *non-deviation case* where all banks offer the same first-period deposit rate $d_{1l} = d_{1h} = d$ and second, the *deviation case* in which one bank deviates to a lower or a higher deposit rate. In the non-deviation case all banks are in one group (without loss of generality in $\mathcal{B}_h$) while in the deviation case either the deviating bank is in $\mathcal{B}_h$ while the the non-deviating banks are in $\mathcal{B}_l$ or vice versa.

We will now examine expected first-period returns on deposits in the non-deviation case and in the deviation case if one group of banks receives all deposits.[28] We denote the deposit rate in the non-deviation case and the deposit rate offered in the group of banks that has received all savings in the deviation case by d. The corresponding return assessment is denoted by u and the bailout probability when productivity is low (high) is denoted by q_l (q_h). Using equation (4.7) we observe that in both cases u can only be consistent if it solves the system $\mathcal{S}(d)$ that consists of the equations

$$u = (p_l q_l + p_h q_h) d \tag{4.8}$$

$$q_l = \min\left\{ \frac{1}{dS_1(u)} S_2\left(\frac{r_{2l}}{q_l d}\right) \, , \; 1 \right\} \tag{4.9}$$

$$q_h = \min\left\{ \frac{1}{dS_1(u)} S_2\left(\frac{r_{2h}}{q_h d}\right) \, , \; 1 \right\} \tag{4.10}$$

and of the constraints $q_l > 0$ and $q_h > 0$. Note that refinancing condition (4.1) will hold in both states of production returns if and only if $dS_1(d) \leq S_2(r_{2l}/d)$. We denote the highest first-period deposit rate at which this is the case by $d_{\mathbf{L}}$. Hence, $d_{\mathbf{L}}$ is the unique solution of the equation

$$S_2^{-1}\Big(dS_1(d)\Big) = \frac{r_{2l}}{d}.$$

[27] The no-regulation case will be summarized in section 4.5.1.

[28] We will see later (in propositions 4.3 and 4.4) that we do not have to consider the case where one bank deviates and both the deviating bank and the non-deviating banks receive deposits.

Moreover, we use $\underline{u} := \min\{u | S(u) \geq 0\}$. The next lemma is crucial for the analysis of consistent first-period assessments.

Lemma 4.1
Suppose that $d > \underline{u}$. Then the system $\mathcal{S} = \mathcal{S}(d)$ has a unique solution which we denote by $(\bar{u}_d, \bar{q}_{l,d}, \bar{q}_{h,d})$. Moreover, this solution has the following properties:

(i) $\bar{u}_{(\cdot)}$, $\bar{q}_{l,(\cdot)}$ *and* $\bar{q}_{h,(\cdot)}$ *are continuous functions of* d.

(ii) $\bar{u}_d = d$ *for* $d \leq d_{\mathbf{L}}$ *and* $\bar{u}_d < d$ *for* $d > d_{\mathbf{L}}$.

(iii) $\bar{q}_{l,d} < 1$ *for* $d > d_{\mathbf{L}}$; $\bar{q}_{l,d}, \bar{q}_{h,d} > 0$ *for all* d.

(iv) $\bar{q}_{l,(\cdot)}$ *is strictly decreasing in* d *for all* $d \in \mathcal{D}_M$ *where*

$$\mathcal{D}_M := \Big\{ d \mid dS_1(\bar{u}_d) \leq S_2(r_{2h}/d) \text{ and } d > d_{\mathbf{L}} \Big\}.$$

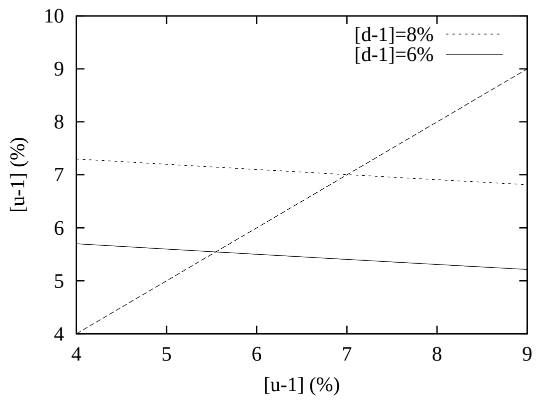

Figure 4.4: The left-hand and the right-hand side of equation (4.8) as functions of u for different values of d (example A) Note that u is represented by the percentage points by which it exceeds 1.

The existence and uniqueness of a solution for $\mathcal{S}$ is derived from a fixed-point argument: the right-hand-side of equation (4.8) is a decreasing function of u (since bailout probabilities q_l and q_h are decreasing in u) while the left-hand-side is strictly increasing. This implies existence and uniqueness of $\bar{u}_d$ because of the continuity of the bailout probabilities as functions of u. Figure 4.4 illustrates the solution of $\mathcal{S}$. Note that u and d are represented by the percentage points by which they exceed 1, i.e. $u = 1.06$ is represented by 6. This scale will be used for u and d in all following illustrations.

Note that at this point our technical device that allows for a fraction $(1 - b)$ of the surviving banks' deposits to be eliminated guarantees the continuity of the bailout probabilities as functions of u and thus the existence of a solution for $\mathcal{S}$. If only entire banks could be closed, the bailout probabilities would not be continuous in u. Discontinuities would appear for all u where a marginal higher value of u requires to close an additional bank: in such points bailout probability would fall by $1/n$. Hence, consistent assessments might not exist for some values of d.

For the description of the deviation case we need to analyze the system $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}(d_{1l}, d_{1h})$ consisting of equations (4.8) - (4.10) where d is replaced by d_{1l} in equation (4.8) and by d_{1h} in equations (4.9) and (4.10).[29]

Lemma 4.2

Suppose that $d_{1l}, d_{1h} > \underline{u}$. Then the system $\tilde{\mathcal{S}}$ has a unique solution which we denote by $(\tilde{u}_{d_{1l},d_{1h}}, \tilde{q}_{l,d_{1l},d_{1h}}, \tilde{q}_{h,d_{1l},d_{1h}})$.

In the next two propositions we characterize consistent and optimal assessments. Note that in the non-deviation case where all banks have offered the same first-period deposit rate d_1, assessments for expected returns of banks are denoted by u_1. In the deviation case there are two groups of banks, $\mathcal{B}_l$ and $\mathcal{B}_h$, that have offered different first-period deposit rates d_{1l} and d_{1h} respectively ($d_{1l} < d_{1h}$). Here we denote the corresponding assessments by u_{1l} and u_{1h} respectively.

Proposition 4.3 (Consistent assessments: Non-deviation case)
If all banks have offered the same first-period deposit rate d_1, then $\mathbf{u_1} = (\bar{u}_{d_1}, ..., \bar{u}_{d_1})$ is the only consistent assessment under all bailout regimes.

Proposition 4.4 (Consistent assessments: Deviation case)
In the deviation case only the following types of assessments can be consistent:

a) $u_{1l} < u_{1h} = \bar{u}_{d_{1h}}$ *b)* $u_{1h} < u_{1l} = \bar{u}_{d_{1l}}$ *c) assessments of the type* $u_{1l} = u_{1h}$.

More specifically, we obtain:

(i) Under the RB or PB bailout scheme, $u_{1l} < u_{1h} = \bar{u}_{d_{1h}}$ is the only *consistent assessment.*

(ii) Under the BB bailout scheme, $u_{1l} < u_{1h} = \bar{u}_{d_{1h}}$ is a consistent assessment and $u_{1h} < u_{1l} = \bar{u}_{d_{1l}}$ is a consistent assessment if and only if

$$\Big(p_l I_{l,d_{1l},d_{1h}} + p_h I_{h,d_{1l},d_{1h}}\Big) d_{1h} < \bar{u}_{d_{1l}}$$

where[30]

$$I_{i,d_{1l},d_{1h}} := 1\Big\{S_2(r_{2i}/d_{1h}) \geq dS_1(\bar{u}_{d_{1l}})\Big\} \quad (i = l, h).$$

Moreover, an assessment $u_{1l} = u_{1h}$ is never optimal.

[29] This will become apparent when going through the proof of proposition 4.4.

[30] Note how the indicator function $1\{\cdot\}$ is defined. $1\{A\}$ is equal to 1 if statement A holds and equal to 0 if statement A does not hold.

(iii) Under the BS bailout scheme, $u_{1l} < u_{1h} = \bar{u}_{d_{1h}}$ is a consistent assessment if and only if $\bar{u}_{d_{1h}} > d_{1l}$; an assessment $u_{1h} < u_{1l}$ can never be consistent. Finally, if $\bar{u}_{d_{1h}} < d_{1l}$ and $p_h d_{1h} < \tilde{u}_{d_{1l},d_{1h}}$, then no consistent assessments exist.

Corollary 4.1
If $\bar{q}_{h,d_{1h}} = 1$ and $d_{1l} > d_{\mathbf{L}}$, then no consistent assessments exist under BS if $d_{1h} - d_{1l}$ is sufficiently small.

Proposition 4.3 follows directly from lemma 4.1 since expected returns for depositors can be expressed by equations (4.8) - (4.10). Moreover, under RB, bailout probabilities for all banks are the same, implying that the banks that have offered the highest deposit rates will always pay the highest returns. Therefore assessments are also unique in the deviation case. Under BB, however, we cannot generally exclude assessments that assign higher expected returns to banks in $\mathcal{B}_l$ despite the fact that those banks have offered lower deposit rates than the banks in $\mathcal{B}_h$. This is due to a *self-fulfilling prophecy* effect caused by BB. Suppose that a bank is assessed to pay higher expected returns than the other banks. This bank will obtain more deposits than the others and hence will be "bigger" in terms of the bailout regime. Under BB it will therefore have a higher bailout probability. This effect can indeed compensate for lower deposit rates. To see why an assessment $u_{1l} = u_{1h}$ cannot be optimal under BB, note that in this case banks in $\mathcal{B}_h$ must be smaller with respect to first-period deposits than banks in $\mathcal{B}_l$; otherwise bailout probability *and* offered deposit rates would be higher for $\mathcal{B}_h$-banks. But this implies that $\mathcal{B}_h$-banks have a lower bailout probability than in the case where they receive all deposits. The formalization of these arguments leads to statement (ii) in proposition 4.4.

Finally, the mechanism leading to consistency problems under BS is the following. If some banks are assessed to pay higher returns than the other ones, the former will attract all deposits. But this implies that these banks are bigger than the banks associated with lower expected returns and will therefore have a lower bailout probability. This could proof the initial assessment to be incorrect even if higher assessed banks have offered higher deposit rates. In analogy to the self-fulfilling prophecy effect under BB, this mechanism could be termed *self-contradicting prophecy:* by assessing a bank to pay high returns, depositors lower its bailout probability and hence its expected returns. To give an impression of the size of the effects, we note that if (for our example A) $d_{1l} = 1.05$ then no consistent assessments exist for $d_{1h} \leq 1.059$, a difference of nearly 1 percentage point. This can prevent banks from biding up deposit rates in low-return equilibria. Suppose, for example, that depositors will not switch to deviating banks in case the deviation deposit rate lies in the non-consistency region. Then, if e.g. all banks have offered 1.05, a deviating bank would have to offer at least 1.059 to attract the other banks' depositors. Such a rate may proof to be too large to be attractive, since a deviating bank would have to trade off the higher amount of deposits with a lower bailout probability (in our case it would decline from 0.97 to 0.95) and higher deposit rates. Because of these problems we will not further analyze the BS bailout scheme and rather add some additional comments when we summarize our results.

Propositions 4.3 allow us to characterize symmetric equilibria under regulation solely in terms of the first-period deposit rate d_1 offered by all banks:

1. Banks offer $\mathbf{d_1} = (d_1, ..., d_1)$ which leads to the assessments $\mathbf{u_1} = (\bar{u}_{d_1}, ..., \bar{u}_{d_1})$ and to the deposit distribution $\mathbf{D_1} = \Big(D_1, ..., D_1\Big)$ where $D_1 = S_1(\bar{u}_{d_1})/n$.

2. The regulator observes the realization r_2 of the aggregate productivity shock and determines $\bar{q}$ as the positive solution of

$$qd_1S_1(\bar{u}_{d_1}) = S_2(\frac{r_2}{qd_1})$$

   if that solution is lower than 1; otherwise $\bar{q}$ is set equal to 1. k and b are determined by $k = \lceil n\bar{q} \rceil$ and $b = (n\bar{q})/\lceil n\bar{q} \rceil$.

3. A set $\mathcal{B}^+$ of k banks is randomly chosen from all n banks; each bank has the same probability of being chosen. Investment projects of closed banks are uniformly distributed among surviving banks. Deposits and investments of surviving banks are given by

$$\Delta = \Big(bd_1D_1\,,\; nD_1/k\Big).$$

4. Banks offer $\mathbf{d_2} = (d_2, ..., d_2)$ where

$$d_2 := S_2^{-1}\Big(\bar{q}S_1(\bar{u}_{d_1})\Big).$$

We will characterize symmetric equilibria by using the short form $\mathcal{E} = (d_1)$.

4.5 Allocations Under Different Regulatory Approaches

Note that from a $t = 0$ perspective expected profits for bank i are given by

$$\Pi_1^i := -\operatorname{Prob}(A_i)(d_1^i D_1^i + P) + \Big(1 - \operatorname{Prob}(A_i)\Big)\, \mathbb{E}\Big[R_2 b_I D_1^i - (1-b) d_1^i D_1^i - d_2^i D_2^i \mid A_i^c\Big].$$

A_i denotes the eventuality of bank i being closed by the regulator or not being able to meet its obligation in $t = 1$ and A_i^c denotes the complement of A_i, i.e. the possibility of bank i living on until $t = 2$.[31] Remember that P is the penalty imposed by the regulator under A_i. While $P > 0$ under PB, we have $P = 0$ under RB and BB. Note again that under all regulatory approaches, banks are assumed to internalize losses that accrue to depositors. The penalty P under PB will be imposed additionally to any other penalties that might be used to force banks to internalize losses.

4.5.1 No Regulation

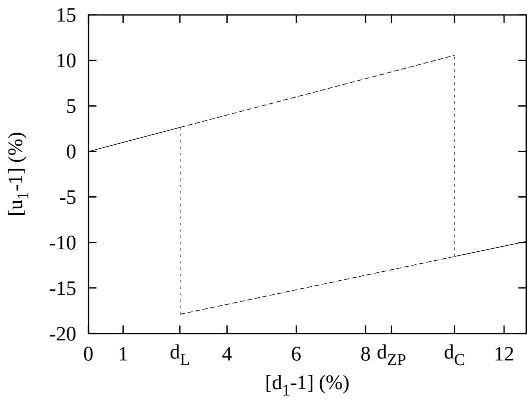

Figure 4.5: Expected returns u_1 for the first generation as function of the first-period deposit rate d_1 (no-regulation case, example A). d_{ZP} stands for $\tilde{d}_{\mathbf{ZP}}$.

In this section we analyze the no-regulation case to motivate the potential benefits of regulation. Consider a symmetric equilibrium where all banks have offered the same deposit rate d_1 in $t = 0$. Note that according to proposition 4.1 only three first-period return assessments are possible, namely d_1, $p_h d_1$ and zero. Obviously, if $d_1 \leq d_{\mathbf{L}}$, then only $u_1 = d_1$ is consistent and banks can refinance in both states of production returns.[32] If $d_1 > d_{\mathbf{L}}$, then $u_1 = d_1$ is no longer consistent since under this assessment banks would go bankrupt for $r_2 = r_{2l}$ (because $\bar{d}_2^* > r_{2l}/d$), which would lead to $u < d_1$. Equilibria where banks are correctly assessed to pay zero returns can also be excluded. Hence, the only other possible assessment is $u_1 = p_h d_1$. Such an assessment will be correct if and only if $r_{2h}/d_1 \geq \tilde{d}_2^*(d_1) > r_{2l}/d_1$ where $\tilde{d}_2^*(d) := S_2^{-1}\Big(d S_1(p_h d)\Big)$. Hence, by defining $d_{\mathbf{C}}$ as the

[31] Note that given A_i bank i cannot pay anything to depositors since the liquidation value of the project is zero.

[32] Note that in this case first-period savings amount to $S_1(d)$ and hence $\bar{d}_2^* = S_2^{-1}\Big(d S_1(d)\Big)$ which by definition of $d_{\mathbf{L}}$ is not higher than r_{2l}/d.

unique solution of $\tilde{d}_2^*(d) = r_{2l}/d$, we have derived that without regulation no consistent assessments exist if $d_{\mathbf{L}} < d_1 \leq d_{\mathbf{C}}$.

Using the parameter values from example A, we illustrate the consistent-assessment problem in figure 4.5: if $d_{\mathbf{L}} < d_1 \leq d_{\mathbf{C}}$, and depositors assume that all banks will survive in both states of production returns, then return assessments are given by the upper (broken) line in figure 4.5. Actual returns paid are represented by the lower (broken) line; if depositors assume that banks can only refinance in the high state, then assessments are given by the lower line and actual returns paid by the higher line. Finally, if $d_1 \leq d_{\mathbf{L}}$ or $d_1 > d_{\mathbf{C}}$, the solid lines represent the respective consistent return assessments for first-generation depositors.

In order to assess the benefits of regulation we also want to compare expected returns for depositors resulting with and without regulation. For our purposes it is sufficient to observe that banks will not bid deposit rates higher than $d_1 = \tilde{d}_{\mathbf{ZP}}$ which is defined as the unique solution of the equation $\tilde{\pi}(d) = 0$. $\tilde{\pi}(d)$ are the banks' profits per deposit in a symmetric equilibrium $d_1 = d$ if $d > d_{\mathbf{C}}$. They can be described by

$$\tilde{\pi}(d) = -p_l d + p_h \left(r_{2h} - d\tilde{d}_2^*(d) \right).$$

Symmetric equilibria with higher deposit rates will not occur since such equilibria would imply negative bank profits (because $\pi(\cdot)$ is strictly increasing in d). Our results are summarized in the following proposition.

Proposition 4.5

Suppose that banks play a symmetric strategy $\mathbf{d_1} = (d_1, ..., d_1)$ in the first period and that there is no regulation. Then the following statements hold:

(i) If $d_{\mathbf{L}} < d_1 \leq d_{\mathbf{C}}$, then no consistent assessments exist.

(ii) For both generations, the highest possible symmetric equilibrium returns are either achieved if $d_1 = d_{\mathbf{L}}$ or $d_1 = \tilde{d}_{\mathbf{ZP}}$. The corresponding unique first-period assessments are $d_{\mathbf{L}}$ and $p_h\tilde{d}_{\mathbf{ZP}}$ respectively.

(iii) If $\tilde{d}_{\mathbf{ZP}} \leq d_{\mathbf{C}}$, then the highest possible symmetric equilibrium returns are achieved for $d_1 = d_{\mathbf{L}}$.

Note that in example A we have $\tilde{d}_{\mathbf{ZP}} = 1.079 < d_{\mathbf{C}} = 1.105$ and hence statement (iii) applies. Proposition 4.5 points to the potential benefits of regulation. Without regulation, the existence of consistent assessments is not guaranteed and it can occur that none of the banks is able to refinance in $t = 1$, implying that intermediation services break down completely for the second generation. In the following, we discuss how regulatory approaches can avoid the breakdown of intermediation. In section 4.5.2, we consider the enforcement of prudential equilibria with $d_1 \leq d_{\mathbf{L}}$, and in section 4.5.3 we analyze the case of discriminatory closure of some banks in order to allow the others to refinance. Both scenarios also help to avoid the problem of nonexistent assessments, as we have already observed in proposition 4.3. In section 4.6 we explicitly compare the no-regulation and the different regulatory approaches with respect to stability and expected returns paid on deposits.

4.5.2 Prudential Banking

In this section we assume that the regulatory regime forces banks to avoid the possibility of default.

Proposition 4.6
The unique symmetric equilibrium under prudential banking is $\mathcal{E}_L$.

Obviously, prudential banking can heavily depress deposit rates and investments if a serious productivity shock can occur. Moreover, in the case $r_{2l} < \underline{u}$, intermediation is impossible. In the next sections we therefore examine work-out type regulatory approaches to banking crises and their implications.

4.5.3 Discriminatory Bailout

In this section we investigate the equilibria that occur under discriminatory bailout. In order to describe the banks' profits under discriminatory regulation schemes, we recall the definition of the set $\mathcal{D}_M$ and additionally introduce the set $\mathcal{D}_H$:

$$\begin{aligned}\mathcal{D}_M &:= \Big\{d \mid dS_1(\bar{u}_d) \leq S_2(r_{2h}/d) \text{ and } d > d_{\mathbf{L}}\Big\} \\ \mathcal{D}_H &:= \Big\{d \mid dS_1(\bar{u}_d) > S_2(r_{2h}/d)\Big\}.\end{aligned}$$

The sets $\mathcal{D}_M$ and $\mathcal{D}_H$ refer to a situation where all banks have symmetrically offered a deposit rate $d_1 = d$ in $t = 0$. If $d \in \mathcal{D}_M$, then all banks can refinance in the good state but not in the bad state of production returns while banks cannot refinance in both states for $d \in \mathcal{D}_H$. Moreover, $d_2^*(d) := S_2^{-1}\Big(dS_1(\bar{u}_d)\Big)$ is the second-period deposit rate that - if symmetrically offered by all n banks in $t = 1$ - generates just enough savings for all banks to refinance.

Now consider a potential symmetric equilibrium $\mathcal{E} = (d_1)$, or a deviation d_1^{dev} from such a symmetric equilibrium where the deviating bank receives all savings. Then the expected profits that a bank makes on each unit of deposits are given by $\pi(d_1)$ and $\pi(d_1^{\text{dev}})$ respectively where π is defined by[33]

$$\pi(d) := \begin{cases} \bar{R}_2 - dd_2^*(d) & \text{if } d \leq d_{\mathbf{L}} \\ -p_l(1-\bar{q}_{l,d})d + p_h\Big(r_{2h} - dd_2^*(d)\Big) & \text{if } d \in \mathcal{D}_M \\ -p_l(1-\bar{q}_{l,d})d - p_h(1-\bar{q}_{h,d})d & \text{if } d \in \mathcal{D}_H. \end{cases}$$

Lemma 4.3
$\pi(\cdot)$ is a continuous function of d.

We note that $\pi(d) > 0$ if $d \leq d_{\mathbf{L}}$ and that $\pi(d) < 0$ if $d \in \mathcal{D}_H$. Hence, by the continuity of $\pi(\cdot)$, there is a first-period deposit rate d with $\pi(d) = 0$. We will work with the following assumption:

Assumption 1 (UZP)
There is a unique first-period deposit rate d where profits are zero ($\pi(d) = 0$). We denote this deposit rate by $d_{\mathbf{ZP}}$ and further assume that $\pi(d) < 0$ for $d > d_{\mathbf{ZP}}$.

Note that assumption UZP is satisfied if $\bar{u}_{(\cdot)}$ is increasing in d for all $d \in \mathcal{D}_M$ which can be verified for our example saving functions.

Lemma 4.4
If $S_i(u) = a_i u^{\alpha_i}$ with $a, \alpha \in (0, \infty)$ $(i = 1, 2)$, then $\bar{u}_{(\cdot)}$ is increasing in d for all $d \in \mathcal{D}_M$.

We can now turn to the analysis of the random bailout regime:

Proposition 4.7
Suppose that UZP holds. Then the unique symmetric equilibrium under random bailout is $\mathcal{E}_{ZP} := (d_{\mathbf{ZP}})$.

$\mathcal{E}_{ZP}$ is the zero-profit equilibrium. Equilibria with higher deposit rates imply negative profits for banks and will thus not be played. Equilibria with lower deposit rates do not exist, since banks will have an incentive to deviate to slightly higher deposit rates thereby collecting all savings. Figures 4.6 and 4.7 show expected returns for the first

[33]Remember that $\bar{R}_2 = p_l r_{2l} + p_h r_{2h}$.

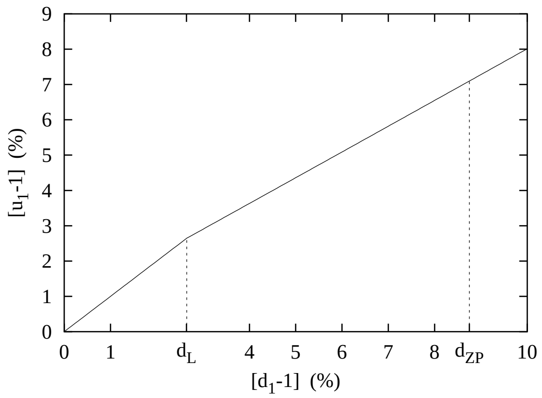

Figure 4.6: Expected returns u_1 for the first generation as function of the first-period deposit rate d_1 (example A).

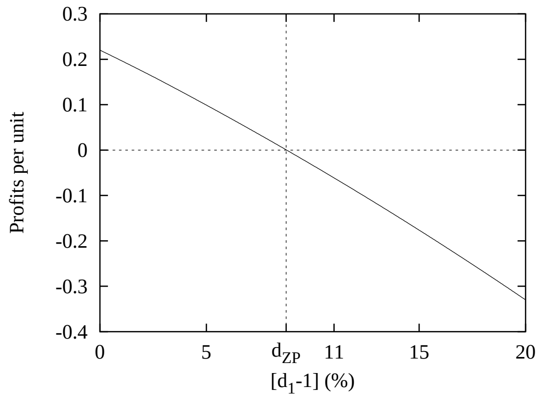

Figure 4.7: Profits per unit of deposits as function of the first-period deposit rate d_1 (example A).

generation and banks' expected profits as functions of the first-period deposit rate d_1 that has been offered.

Having derived this result, we now set out to examine whether and how allocations are affected if the regulator follows BB instead of RB. We have already indicated that BB can lead to self-fulfilling prophecy effects when first-period deposit rates are set asymmetrically. Banks that have offered lower deposit rates can consistently be assessed to pay higher returns than banks that have offered higher deposit rates. To present our results we introduce the following tie-breaking rule:

(TR) *If depositors receive the same expected returns when depositing with non-deviating banks as when depositing with the deviating bank, they choose the non-deviating ones.*

Moreover, we introduce the function

$$\Pi^{\text{dev}}(d) := \max\Big\{\pi(\tilde{d})S(\bar{u}_{\tilde{d}}) \ : \ \bar{u}_{\tilde{d}} > \bar{u}_d\Big\},$$

which describes the maximum profits that can be obtained when deviating from a symmetric equilibrium where all banks have offered a first-period deposit rate d. Finally we distinguish the following cases for the relationship between expected equilibrium returns for the first generation and offered first-period deposit rates:

1. There is a deposit rate $d_{\mathbf{UH}}$ ($d_{\mathbf{L}} \leq d_{\mathbf{UH}}$) such that $\bar{u}_{(\cdot)}$ is strictly increasing in d for $d < d_{\mathbf{UH}}$ and strictly decreasing for $d_{\mathbf{UH}} < d \leq d_{\mathbf{ZP}}$. **(UID)**

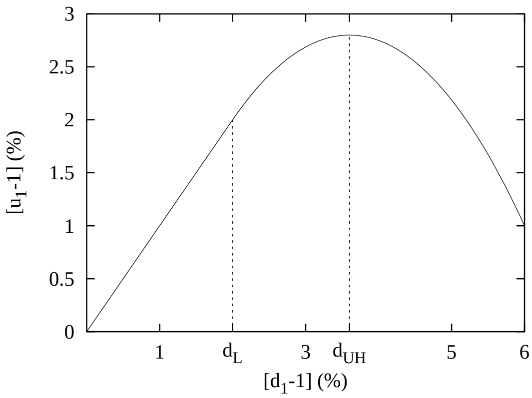

Figure 4.8: The case UID.

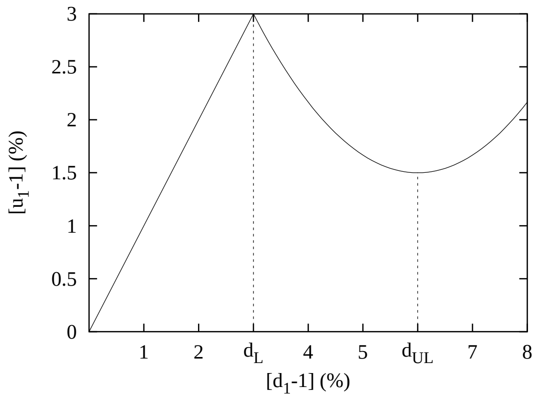

Figure 4.9: The case UDI.

2. There is a deposit rate $d_{\mathbf{UL}}$ ($d_{\mathbf{L}} \leq d_{\mathbf{UL}}$) such that $\bar{u}_{(\cdot)}$ is strictly decreasing in d for $d < d_{\mathbf{UL}}$ and strictly increasing for $d_{\mathbf{UL}} < d \leq d_{\mathbf{ZP}}$. (**UDI**)

These cases are illustrated in figures 4.8 and 4.9. Note that under UID (UDI), both constellations are possible: (a) $d_{\mathbf{UH}} < d_{\mathbf{ZP}}$ ($d_{\mathbf{UL}} < d_{\mathbf{ZP}}$) and (b) $d_{\mathbf{UH}} \geq d_{\mathbf{ZP}}$ ($d_{\mathbf{UL}} \geq d_{\mathbf{ZP}}$).

Proposition 4.8

Suppose that the assumption UZP holds and that TR is applied. Then the following holds under bail out the big ones:

(i) $\mathcal{E} = (d)$ *is an equilibrium for each deposit rate* $d \in \mathcal{U}_{max} := \operatorname{argmax} \bar{u}_{(\cdot)} |_{\pi(d)\geq 0}$.

(ii) Under UID we obtain that $\mathcal{E}_{UH} := \Big(\min\{d_{\mathbf{UH}}, d_{\mathbf{ZP}}\}\Big)$ *is the unique symmetric equilibrium.*

(iii) Under UDI we obtain:

- *If* $d_{\mathbf{L}} > \bar{u}_{d_{\mathbf{ZP}}}$, *then* $\mathcal{E}_L$ *is the unique symmetric equilibrium.*
- *If* $d_{\mathbf{L}} < \bar{u}_{d_{\mathbf{ZP}}}$, *then* $\mathcal{E}_{ZP}$ *is an equilibrium and* $\mathcal{E}_L$ *is an equilibrium if and only if* $\Pi^{dev}(d_{\mathbf{L}}) \leq \pi(d_{\mathbf{L}})S(d_{\mathbf{L}})/n$. *No other equilibria exist.*

The next corollary is an immediate consequence of proposition 4.8. It is concerned with the cases where $\bar{u}_{(\cdot)}$ is strictly increasing (**UI**) or strictly decreasing (**UD**) for $d_{\mathbf{L}} < d \leq d_{\mathbf{ZP}}$.

Corollary 4.2
Suppose assumption UZP holds and that TR is applied. Then under bail out the big ones we obtain:

(i) Under UI, $\mathcal{E}_{ZP}$ is the unique equilibrium.

(ii) Under UD, $\mathcal{E}_L$ is the unique equilibrium.

What is the economic intuition behind the results in proposition 4.8? Let us first turn to statement (i). Under BB, maximum expected return equilibria are supported, since even if banks deviate to higher deposit rates, depositors can consistently assess non-deviating banks as paying higher returns, thereby securing maximum expected returns. This is not possible under RB. Let us now turn to the interesting case of statement (ii) where $d_{\mathbf{UH}} < d_{\mathbf{ZP}}$. Equilibria with higher deposit rates are not possible, because banks would deviate to lower rates, and depositors would switch to the deviating banks since they can guarantee higher expected returns. Again this is made possible by the self-fulfilling prophecy effect of BB. Lower deposit rates are not possible because banks will deviate to higher rates. Statement (iii) can be explained by the same reasoning.

4.6 Comparison

In this section we compare the three regulatory scenarios (prudential banking, random bailout and bail out the big ones) and the no-regulation scenario. Our comparison is concerned with three issues: fragility issues, credibility issues and expected returns. For two points of the analysis we have relied on simulation results: first, for the determination of the shape of $\bar{u}_{(\cdot)}$ as function of d; second, for the comparison of expected returns in the $\mathcal{E}_L$ and the $\mathcal{E}_{ZP}$ equilibrium.

For the time being, we will focus on what we call the "normal case", namely the case where $\bar{u}_{(\cdot)}$ is strictly increasing in d. The label "normal" is justified by the fact that $\bar{u}_{(\cdot)}$ has this property for our family of example saving functions and because $\bar{u}_{(\cdot)}$ has behaved in this way for a wide range of other numerical examples. In section 4.6.4 we will consider scenarios where $\bar{u}_{(\cdot)}$ is not increasing in d.

4.6.1 Fragility Issues

Regulation improves the stability of intermediation. First, the existence of second-period equilibria is guaranteed under regulation. Moreover, these equilibria can be ranked by banks according to the Pareto criterion. Without regulation, neither are guaranteed. Second, under regulation the nonexistence of consistent assessments in the first period, which may occur without regulation (see proposition 4.5), can be avoided.

The crucial question in comparing the stability across regulatory schemes is whether the proposed coordination mechanism for depositors return assessments works if there is more than one consistent assessment. We have assumed that if there is more than one consistent assessment, then depositors choose the assessment that promises the highest expected returns (optimal assessment). If this is the case, all regulatory regimes yield the same stable result, namely unique assessments and a unique equilibrium: $\mathcal{E}_{ZP}$ (under RB and BB) and $\mathcal{E}_L$ (under PB). The result for PB and RB is independent of whether the optimal-assessment criterion holds or not. The stability of the BB regime on the other hand depends upon it heavily: if it does not hold, uniqueness is not guaranteed (see proposition 4.4).

4.6.2 Return Issues

Recall that in the normal case we have to compare the equilibria $\mathcal{E}_L$ (implemented by prudential banking and possibly implemented without regulation), $\tilde{\mathcal{E}}_{ZP}$ (possibly implemented without regulation) and $\mathcal{E}_{ZP}$ (implemented by RB and BB). The expected returns for the first generation (u_1) and for the second generation (u_2) in the different equilibria are presented in table 4.2.

	First Generation	Second Generation
$\mathcal{E}_L$	$d_{\mathbf{L}}$	$d_2^*(d_{\mathbf{L}})$
$\tilde{\mathcal{E}}_{ZP}$	$p_h \tilde{d}_{\mathbf{ZP}}$	$p_l \underline{u} + p_h d_2^*(\tilde{d}_{\mathbf{ZP}})$
$\mathcal{E}_{ZP}$	$p_l \bar{q}_{l,d} d_{\mathbf{ZP}} + p_h d_{\mathbf{ZP}}$	$p_l S_2^{-1}\Big(\bar{q}_{l,d} d_{\mathbf{ZP}} S_1(\bar{u}_{d_{\mathbf{ZP}}})\Big) + p_h d_2^*(d_{\mathbf{ZP}})$

Table 4.2: Expected returns under different equilibria

The most important question is whether regulation can improve expected returns for both generations. We observe that returns $u_i(\mathcal{E}_{ZP})$ in $\mathcal{E}_{ZP}$ are higher for both generations than returns $u_i(\tilde{\mathcal{E}}_{ZP})$ in $\tilde{\mathcal{E}}_{ZP}$ $(i = 1, 2)$. This is stated in the next proposition:

Proposition 4.9
$u_i(\mathcal{E}_{ZP}) > u_i(\tilde{\mathcal{E}}_{ZP})$ for $i = 1, 2$.

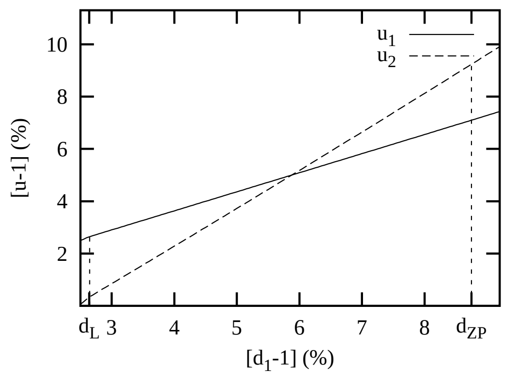

Figure 4.10: Expected returns for both generations as function of the first-period deposit rate d_1 (example A).

Proposition 4.9 implies that regulation can improve welfare. On the other hand, it is not clear whether $\mathcal{E}_{ZP}$ also delivers higher returns than $\mathcal{E}_L$. Obviously, $u_1(\mathcal{E}_{ZP}) > u_1(\mathcal{E}_L)$, but the effect for the second generation is ambiguous, since $\bar{q}_{l,d}d_{\mathbf{ZP}}$ might be smaller than $d_{\mathbf{L}}$ and hence might offset the effect that $d_2^*(d_{\mathbf{L}}) < d_2^*(d_{\mathbf{ZP}})$. However, in all simulation exercises $\mathcal{E}_{ZP}$ also improves returns for the second generation compared to $\mathcal{E}_L$. Hence, in these cases discriminatory bailout improves expected returns for both generations compared to PB and compared to the non-regulation case. As illustration we show in figure 4.10 expected returns for the first and the second generation under discriminatory bailout as function of offered first-period deposit rates (for example A). The returns resulting under $\mathcal{E}_L$ and $\mathcal{E}_{ZP}$ are presented in table 4.3.[34]

	$[d_1 - 1](\%)$	$[u_1 - 1](\%)$	$[u_2 - 1](\%)$	q_l
$\mathcal{E}_L$	2.64	2.64	0.34	1
$\mathcal{E}_{ZP}$	8.75	7.00	9.24	0.92

Table 4.3: First-period deposit rates, expected returns, and fraction of bailed out depositors for example A.

4.6.3 Credibility Issues

The issue of credibility obviously only has a bearing on the three regulatory schemes. The most important difference with respect to the credibility of those schemes is the *out-of- equilibrium* strategy that is required. While the credibility of PB first of all depends on the credibility of the penalties that have to be applied (which will not be

[34]Note that the equilibrium $d_1 = \tilde{d}_{\mathbf{ZP}}$ does not exist in the no-regulation case for example A. Hence the highest possible equilibrium returns are achieved under $\mathcal{E}_L$.

taken up here), the credibility of BB and RB depends on the impact of the respective out-of-equilibrium closure rules.

In section 4.3.5.3 we have already illustrated that while the maximum fraction of depositors is always bailed out under BB, under RB it might be necessary to bail out a significantly lower fraction of first-period deposits than would be possible. This occurs if the deposit distribution is very unequal. The necessity to commit to lower-than-possible bailout fractions might well reduce the credibility of the RB scheme. Agents might expect the regulator to abandon RB and bail out more depositors if an asymmetric deposit distribution occurs. As mentioned above, this kind of credibility problem does not occur under BB.

4.6.4 Extensions

In this section we discuss three important extensions of the current model: positive liquidation values and takeover costs, differences in banks sizes that might stem from other sources than considered in this chapter, and risk-taking incentives.

4.6.4.1 Liquidation Value and Takeover Costs

In this section we will briefly explore cases where $\bar{u}_{(\cdot)}$ is not increasing in d and discuss the consequences for the comparison of the schemes RB and BB. In order to obtain such scenarios we had to relax two implicit assumptions: (a) that the takeover of investment projects from closed banks does not involve any deadweight costs; and (b) that the $t = 1$ liquidation value of investments is zero. In a more general setting we assume that a fraction $(1-\delta)$ of project returns is consumed by the takeover procedure. Hence returns of such projects in the second period are given by δr_2.[35] Moreover, we allow for a positive liquidation value R_1 of investments when liquidated in $t = 1$; the realization of the liquidation value can be either high ($R_1 = r_{1h}$ if $R_2 = r_{2h}$) or low ($R_1 = r_{1l}$ if $R_2 = r_{2l}$). We require that depositors whose claims have been eliminated by the regulator will receive a return equal to the liquidation value of investments.

In appendix 4.B we show that if $r_{1h} \leq \delta d_{\mathbf{L}}$, our whole analysis also applies for this more general case. Moreover, concerning the expected returns achieved under the different regulatory schemes, our numerical exercises have generated the following pattern. First, profits per deposits are always decreasing in d, i.e. UZP is always fulfilled and hence

[35] Note that we have employed the most simple and plausible specification of takeover costs, namely that they are proportional to the *ex-post* size of the project.

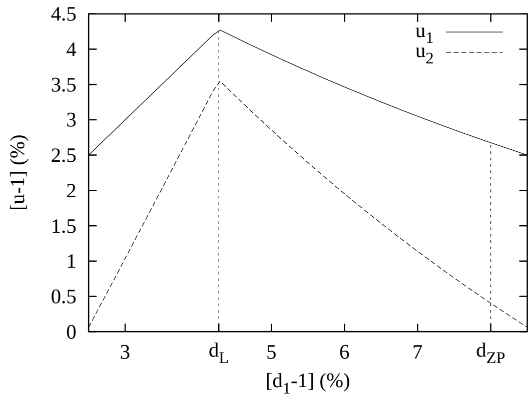

Figure 4.11: Expected returns for both generations as function of the first-period deposit rate d_1 (example B).

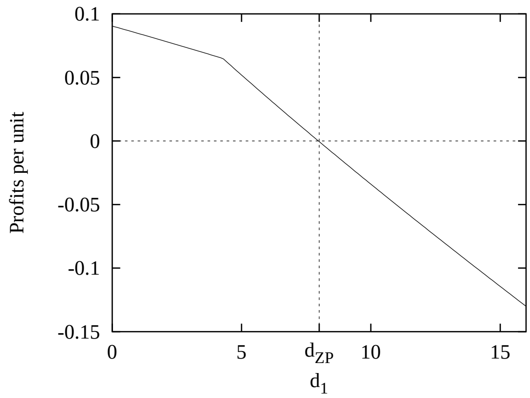

Figure 4.12: Profits per unit of deposits as function of the first-period deposit rate d_1 (example B).

propositions 4.7 and 4.8 apply. Second, in all cases investigated, either UI or UDI applies. Moreover, under UDI we always obtained $d_{\mathbf{L}} > \bar{u}_{d_{\mathbf{ZP}}}$. Hence, for all examples considered, BB implements either the same expected first-period returns as RB or higher returns. The same is true for second-period returns: they qualitatively behave in the same way as expected first-period returns (increase when u_1 increases and decrease when u_2 decreases). Moreover, we observe that scenarios where the normal case UI is not fulfilled only occur if δ is sufficiently small *and* p_l (the probability of low returns) is high enough.[36]

For illustration we use the example that is presented in table 4.4. Figures 4.11 and 4.12 show expected returns for the first and the second generation, and profits per unit of deposits as functions of offered first-period deposit rates under discriminatory bailout. It can be inferred from figure 4.11 that the case UD applies (i.e. $\bar{u}_d$ is decreasing in d if $d \in \mathcal{D}_M$). Hence BB implements the equilibrium $\mathcal{E}_L = (d_{\mathbf{L}})$ and RB implements $\mathcal{E}_{ZP} := (d_{\mathbf{ZP}})$. Table 4.5 shows that expected returns for both generations are higher under BB.

4.6.4.2 Bank Size and Growth Rates of Deposits

In this section we discuss how the interpretation of the proposed bailout policies has to be adapted if the size of a bank depends not only on the attractiveness of the deposit rates it has offered. In a more realistic setting, banks sizes will differ for many other

[36] This result is intuitively convincing. Return losses due to lower deposit rates are decreasing in p_l and increasing in δ.

$S_1(u) = u$	$p_l = 0.8$	$r_{1l} = 0.7$	$r_{2l} = 1.08$	$\bar{R}_2 = 1.14$
$S_2(u) = 1.05 \cdot u$	$p_h = 0.2$	$r_{1h} = 0.7$	$r_{2h} = 1.40$	$\delta = 0.6$

Table 4.4: Example B.

	$[d_1 - 1](\%)$	$[u_1 - 1](\%)$	$[u_2 - 1](\%)$	q_l
$\mathcal{E}_L$	4.28	4.28	3.48	1
$\mathcal{E}_{ZP}$	8.00	2.67	0.4	0.82

Table 4.5: First-period deposit rates, expected returns, and the fraction of bailed out depositors for example B.

reasons.[37] It is therefore important to recall the actual mechanism that characterizes the BB scheme, namely that banks that are assessed to pay higher returns (or, in a more general view, to provide better intermediation services) will also have a higher bailout probability. Conditioning bailout policies on the *growth rates* rather than on the actual size of a bank's deposits should therefore be a more appropriate implementation of the spirit of BB in realistic banking competition environments.

4.6.4.3 Risk-Taking Incentives

The risk-taking incentives generated by bailout policies are an important focus of the existing literature on bank closures.[38] In this section we briefly discuss the consequences of such considerations for the policy suggestions derived by now.

If banks can decide about the riskiness of their projects after they have received deposits, BB will have the drawback of providing risk-taking incentives for big banks since they can anticipate to be bailed out with high probability. RB - in contrast - will provide less incentives for risk taking as banks are more uncertain about the regulator's bailout decision. This foundation of a constructive ambiguity approach to bailout has recently been discussed by FREIXAS (1999). In FREIXAS (1999), constructive ambiguity is achieved by assuming that the regulator follows a mixed strategy when deciding about a *single* bank's bailout. In our general-equilibrium framework, bailout probabil-

[37]See TIROLE (1994) and FREIXAS AND ROCHET (1999) for general industrial-organization and bank-specific reasons, respectively.
[38]See CORDELLA AND YEYATI (1999) and references therein.

ities have to be chosen in a way ensuring that under *all* realizations of the stochastic decision process, the banks that have not been closed will be able to survive. This makes the design of such a policy more demanding. The simple version of RB we have proposed in this chapter requires that - in out-of-equilibrium strategies - the regulator must commit to bail out significantly less deposits than would be possible. This in turn undermines the credibility of the RB policy (see sections 4.3.5.3 and 4.6.3). However, it might be possible to construct RB-type policies which do not share this drawback.

To explore this possibility, recall how bank closures are determined under our RB specification. We require the regulator to randomly draw a subset of k banks from the complete sample with each bank having the same probability to be be drawn. The integer k is chosen in a way that ensures that the surviving banks will always be able to refinance, independently of which subsample of banks has been drawn. Thus the choice of k has to ensure that even if the biggest banks are chosen, the fraction q of bailed out deposits will never exceed $\bar{q}$. In cases where small banks have been drawn, the fraction of bailed out deposits can therefore drop sharply, leading to the credibility problems of RB.

To avoid this problem, the regulator could *sequentially* draw from the sample of all banks. Each bank that is drawn will survive and the regulator continues drawing until the critical fraction $\bar{q}$ is reached. All banks that have not been drawn will be closed and eventually a fraction of the last drawn banks' balance sheet will be eliminated. The most intuitive application of the RB principle to sequential drawing would be to give all banks remaining in the sample the same probability to be drawn in the next step. For example, if there are 10 banks and 2 have already been drawn to survive, the remaining 8 banks will all have a probability of 1/8 to be drawn next.

However, this sequential drawing procedure would lead to a higher bailout probability for big banks than for small banks, as is illustrated in the following example. Suppose that there are only two banks, which have obtained deposits of $D_1^1 = 0.8$ and $D_1^2 = 0.2$ and assume that $\bar{q} = 0.6$. If the regulator draws the big bank first, then the small will be closed and a fraction 0.6/0.8 of the big bank's deposits will be bailed out. If the small bank is drawn first, the maximum fraction $\bar{q}$ will not have been reached yet and a fraction 0.4/0.8 of the big bank's deposits will be bailed out. Hence, the a-priori bailout probability for deposits at the big bank is given by $q_1 = 0.5 \cdot 0.6/0.8 + 0.5 \cdot 0.4/0.8 = 0.625$ while the respective probability for the small bank is $q_2 = 0.5 \cdot 0 + 0.5 = 0.5$. In general, big banks will have a higher bailout probability since, given that small banks have been drawn already, their conditional bailout probability will still be high while the contrary is true for small banks.

Note that in our example it would be possible to achieve an equal bailout probability of $\bar{q} = 0.6$ for depositors of both banks. This could be done by altering the probabilities with which the two banks are drawn. Denoting the respective probabilities by μ_i $(i = 1, 2)$ we have $\mu_2 = 1 - \mu_1$ and hence

$$q_1 = \mu_1 \cdot 0.6/0.8 + (1 - \mu_1) \cdot 0.4/0.8 \stackrel{!}{=} 0.6.$$

This implies that $\mu_1 = 0.4$ and $\mu_2 = 0.6$ would lead to homogenous bailout probabilities. The most simple generalization to the case of m banks would be to assign a certain probability weight μ_i to each bank i $(i = 1, .., m)$ and to define the probability of bank i to be drawn, given that banks $B_{i_1}, .., B_{i_k}$ are still in the sample by

$$\frac{\mu_i}{\sum_{j=1}^{k} \mu_{i_j}}.$$

It is, however, unclear whether the probabilities $\mu_1, .., \mu_m$ can be chosen in way that a-priori bailout probabilities are equal for all banks.[39] An even more general probabilistic structure could be formulated by allowing for conditional default probabilities to depend on any piece of information available from draws that have already been conducted (e.g. on the size or number of banks which have already been drawn). Therefore, we conjecture that it should be possible to construct a drawing mechanism that produces uniform bailout probabilities while bailing out the maximum fraction $\bar{q}$ of deposits under all circumstances. However, to construct such a mechanism in general seems to be far from straightforward and is left to future research.

Putting together the observations from this section and those from section 4.6.4.1, a mixture of BB and RB might be sensible. Whether the actual policy will more closely resemble the former or the latter will depend on whether return consideration or excessive risk-taking consideration are more important. However, to make such combinations of BB and RB possible, an algorithm has to be found that - for any given deposit distribution - can deliver drawing mechanisms generating *any* pattern $q_1, .., q_m$ of bailout probabilities.

[39] Note that it is quite complicated to actually calculate these probabilities. Moreover, even once this is done one would very likely end up with a system of n polynomial equations of degree n.

4.7 Conclusion

We have attempted to provide a general-equilibrium analysis of the funds concentration effect and the corresponding regulatory bailout schemes in an overlapping-generations framework. Banks invest in long-term projects that are subject to macroeconomic risk. The projects are financed by short-term deposit contracts, offered to both generations of households. The banks' obligations to the first generation have to be served with the savings of the second generation. Since deposit contracts are not contingent on the realizations of investment returns, banks are exposed to macroeconomic risk: if investment returns are expected to be low, banks might not be able to attract enough second-generation savings to pay back first-generation depositors.

Without regulation, banks might bid up first-period deposit rates so that they are able to refinance if investment returns are high but are not able to refinance if they are low. Hence, all banks may go bankrupt in the state of low production returns since they cannot offer sufficiently high returns to second-period depositors. This has the unfavorable effect that long-term projects are liquidated in the state of low returns. Moreover, for a range of first-period deposit rates no consistent return assessments exist for first-period depositors.

In this chapter we have considered two different regulatory approaches to deal with these problems. First, prudential banking where the regulator ensures that banks bid up first-period deposit rates only to the point where they are still able to refinance themselves in both future states of production returns. Second, discriminatory bailout where in the low state of production returns the regulator closes some banks and distributes the investment projects of these banks among the surviving ones. This increases returns that can be paid by the surviving banks and decreases the amount of second-period savings needed by the banking system to refinance.

Our results are as follows. First, discriminatory bailout improves the stability of intermediation by guaranteeing the existence of consistent assessments for first-period depositors. Moreover, it dominates both prudential banking and the no-regulation case with respect to expected returns paid on deposits. In the case where expected returns $\bar{u}_d$ on first-period deposits are an increasing function of first-period deposit rates d, the two different variants of discriminatory bailout, random bailout (RB) and bail out the big ones (BB) implement the same equilibrium. Non-increasing functional forms of $\bar{u}_{(\cdot)}$ occur if takeover of investment projects involves deadweight costs and if the probability of low returns is high. In this case BB dominates RB with respect to expected returns. The reason for this result is as follows. Under RB, bailout probabilities are the same for all banks. Hence, depositors will always chose to deposit with the bank that offers the highest deposit rates. This in turn implies that deposit rates will be bidden up until expected profits are zero. Under BB, in contrast, banks stop the bidding when expected returns for depositors are at their maximum, since - due to a self-fulfilling-prophecy effect - depositors will not switch to a bank that offers slightly higher deposit rates.

We have also considered a third variant of discriminatory bailout, bail out the small ones (BS), and have identified that under such a regime the existence of consistent assessments cannot be guaranteed when banks deviate from a symmetric equilibrium. This result stems from what we call a self-contradicting-prophecy effect. While under BB, a bank that is assessed to pay higher returns than an other bank will also have a higher bailout probability (which can validate the initial assessment), under BS the same bank will have a lower bailout probability than the bank which has been assessed to pay lower returns (an effect that undermines the initial assessment). Additional to the resulting instability, the non-existence problem may also prevent banks from biding up deposit rates which might support low-return equilibria.

In order to ultimately assess which of the discriminatory bailout schemes, BB or RB, should be preferred, two further issues have to be taken into account. First, to what degree does return maximization help to coordinate depositors' assessments if more than one consistent assessment exists? Under RB, only one consistent assessment exists for each vector of first-period deposit rates while under BB there might be more than one consistent assessment. We have suggested that if more than one consistent assessment exists, depositors will choose the assessment which promises the maximum expected return (optimal assessment). Under this assumption assessments are unique under BB.

Second, to what extent do bailout policies influence the risk-taking incentives of banks? We have argued that BB will have the drawback of providing risk-taking incentives for big banks, because they can anticipate to be bailed out with high probability. RB - in contrast - will provide less incentives for risk taking since banks are uncertain about the regulator's bailout decision. This parallels the „constructive ambiguity" approach to bailout as analyzed by FREIXAS (1999) in a single-bank model. In our general-equilibrium framework, bailout probabilities have to be chosen in a way ensuring that - under all possible realization of the stochastic bailout decision - the banks that have not been closed will be able to survive. This makes the design of such a policy more demanding. The simple version of RB we have proposed in this chapter requires that - in out-of-equilibrium strategies - the regulator must commit to bail out significantly less deposits than would be possible and optimal. This undermines the credibility of the regulatory policy. We have indicated, however, that a RB-type policy may be found which circumvents these drawbacks. But the comprehensive construction of such a policy seems to be far from obvious and is left to future research.

In summary, our findings suggest that closure rules in severe crises should be a mixture of BB and RB. Whether the actual policy will more closely resemble the former or the latter will depend on whether return consideration or stability consideration (risk-taking incentives, unique assessments) are more important.

Future research should integrate risk-taking incentives into the general-equilibrium framework developed in this chapter. In particular, the following two issues have to be addressed. First - as has already been outlined above - the construction of bailout schemes that allow for arbitrary combinations of RB and BB and that do not share the credibility problem of the RB scheme introduced in this chapter. Second, the analysis of closure rules where the bailout decision may not only depend on the size of a bank but also on other bank-specific variables such as the level of uninsured debt on a bank's balance sheet.

Finally, future research should also allow for banks to differ with respect to their positions in the matrix of interbank connections. Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000) have argued that this position can be crucial for the stability of the financial system.

4.A Proofs

In this appendix, we present the proofs of all lemmata and propositions. We start with the proofs for section 4.4.

4.A.1 Proofs for Section 4.4

Proof of proposition 4.1.

(i) The proof will proceed in two steps.

Step 1: $\mathcal{E}_2^*$ is an equilibrium.

First, we note that if $d_2^i = \bar{d}_2^*$ is played $(i \in \mathcal{B}^+)$, the assessment $u_2^i = \bar{d}_2^*$ $(i \in \mathcal{B}^+)$ is optimal and only the deposit distribution $D_2^i = d_1^i D_1^i$ $(i = 1, .., n)$ is consistent with this assessment. Second, we have to show that deviations from $d_2^i = \bar{d}_2^*$ are not profitable. Deviations to higher deposit rates can be excluded since they raise repayment obligations. If a bank deviates to $d_2^{\text{dev}} < \bar{d}_2^*$, it is not possible for all banks to receive a positive measure of deposits.[40] But then either all or none of the non-deviating banks receive a positive measure of deposits.[41] Hence, there can only be one case where the deviating bank receives a positive measure of savings, namely if it can attract the full measure of second-period savings. But in this case the deviation cannot be profitable for the following reasons. Depositors only choose to give resources to the deviating bank if returns are at least as high as returns at the non-deviating banks. But if depositors chose to deposit with the non-deviating banks, returns are given by $\min\{u_*, \bar{d}_2^*\}$, where u_* is the positive solution of

$$u = \frac{(m-1)r_2 D_1}{S_2(u)}. \tag{4.11}$$

If, on the other hand, depositors deposit with the deviating bank, returns cannot be higher than $\min\{u_*^{\text{dev}}, d_2^{\text{dev}}\}$, where u_*^{dev} is the positive solution of equation (4.11) when $m-1$ is replaced by 1. Hence, the inequality $u_*^{\text{dev}} \geq u_*$ can only be fulfilled if $m = 2$ and $d_2^{\text{dev}} \geq u_*$. But in this case the deviating banks' profits cannot be higher than zero.

[40] If that were the case, all banks would have to be assessed paying the *same* return $u_2 \leq d_2^{\text{dev}}$. But then $S_2(u_2) < md_1D_1$, and hence at least one bank cannot refinance. This implies that $u_2 = 0$ and $S_2(u_2) = 0$ in contradiction to the assumption that all banks receive a positive measure of deposits.

[41] Since the non-deviating banks are identical with respect to all their characteristics, they will receive the same amount of deposits.

Step 2: Pareto-dominance.

If $d_2 > \bar{d}_2^*$, then banks obtain lower profits than in $\mathcal{E}_2^*$ because repayment obligations are higher. If $d_2 < \bar{d}_2^*$, then the amount of overall savings is bounded by $S_2(d_2)$ and hence second-period deposits of a single bank are limited by $S_2(d_2)/m$ (since depositors cannot coordinate on a subset of banks). But this implies that no bank can refinance and the only consistent assessment is $u_2 = 0$ for all banks, implying that no bank receives any savings.

(ii)

The first observation is obvious and the second has already been derived under step 2 in the proof of (i).

□

Proof of proposition 4.2.

Step 1: $\mathcal{E}_2^*$ is an equilibrium. We observe that if $d_2^i = \bar{d}_2^*$ is played, the assessment $u_2^i = \bar{d}_2^*$ $(i \in \mathcal{B}^+)$ is optimal and only the deposit distribution $D_2^i = bd_1^i D_1^i$ $(i = 1, .., n)$ is consistent with this assessment. Deviations to higher deposit rates would increase repayment obligations and can therefore not be profitable. Deviations to lower rates are excluded by LBD.

Step 2: Pareto-dominance. Equilibria where some banks have offered lower deposit rates than $\bar{d}_2^*$ are excluded by LBD, and all other equilibria are Pareto-dominated by $\mathcal{E}_2^*$ since repayment obligations are higher than under $\mathcal{E}_2^*$.

□

Proof of lemma 4.1.

Step 1: The system $\mathcal{S}$ has a unique solution.

First, note that if $\bar{u}_d$ is a solution of the system $\mathcal{S}$, then $\bar{u}_d > \underline{u}$. But for all $u > \underline{u}$, equations (4.9) and (4.10) have unique, strictly positive solutions, which we denote by $q_l = q_l(d, u)$ and $q_h = q_h(d, u)$ respectively.[42] Figure 4.13 illustrates the argument by depicting the left-hand and the right-hand side of equation (4.9) for example A and $d = 1.05$. Figure 4.13 also illustrates that the functions $q_l(d, \cdot)$ and $q_h(d, \cdot)$ are

[42] The left-hand sides of the equations are strictly increasing in q_l and q_h respectively and they take all values in $(0, \infty)$; the right-hand sides are decreasing in q_l and q_h respectively.

decreasing in u for fixed d by depicting the solutions of the equation for $u = 1.05$ and $u = 1.07$. Moreover, $q_l(\cdot,\cdot)$ and $q_h(\cdot,\cdot)$ are continuous in d and u since the left and the right-hand side of the equations (4.9) and (4.10) are continuous functions of d and u. Inserting $q_l(d,u)$ and $q_h(d,u)$ in equation (4.8), we obtain an implicit equation for u.

As we saw above, we have to restrict the range of this equation to $u > \underline{u}$. Hence the left-hand side of this equation is strictly increasing in u and takes all values in $\mathbb{R}$ that are strictly higher than $\underline{u}$; the right-hand side is decreasing in u and higher than $\underline{u}$ if u is close enough to $\underline{u}$.[43] Hence, by the mean value theorem, a unique solution $\bar{u}_d$ of this implicit equation exists. This solution is a continuous function of d since the left and the right-hand side of the equation are continuous functions of d. Finally, inserting $\bar{u}_d$ in q_l and q_h we obtain $\bar{q}_{l,d} := q_l(d,\bar{u}_d)$ and $\bar{q}_{h,d} := q_h(d,\bar{u}_d)$.

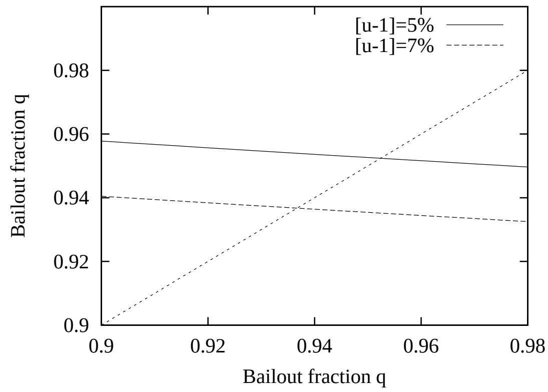

Figure 4.13: The left-hand and the right-hand side of equation (4.9) as functions of q (example A, $d = 1.05$).

Step 2: Proof of statements (i) - (iv)

The continuity of $\bar{u}_{(\cdot)}$ has already been shown in step 1 and the continuity of $\bar{q}_{l,(\cdot)}$ and $\bar{q}_{h,(\cdot)}$ follows directly from the continuity of $\bar{u}_{(\cdot)}$, $q_l(\cdot,\cdot)$ and $q_h(\cdot,\cdot)$. Statements (ii) and (iii) are straightforward.We now need to substantiate (iv), i.e. the monotony of $\bar{q}_{l,(\cdot)}$ for all $d \in \mathcal{D}_M$. Recall that

$$\mathcal{D}_M = \Big\{ d \mid dS_1(\bar{u}_d) \leq S_2(r_{2h}/d) \text{ and } d > d_{\mathbf{L}} \Big\},$$

and note that for $d \in \mathcal{D}_M$ we have

$$\bar{q}_{l,d} = \frac{1}{dS_1(\bar{u}_d)} S_2\Big(\frac{r_{2l}}{\bar{q}_{l,d} d}\Big), \tag{4.12}$$

$\bar{q}_{h,d} = 1$ and

$$\bar{u}_d = (p_l \bar{q}_{l,d} + p_h)d. \tag{4.13}$$

Suppose now that $d < \tilde{d}$. If $\bar{u}_d \leq \bar{u}_{\tilde{d}}$, then we obtain $\bar{q}_{l,d} > \bar{q}_{l,\tilde{d}}$ from equation (4.12). If on the other hand $\bar{u}_d > \bar{u}_{\tilde{d}}$, then $\bar{q}_{l,d} > \bar{q}_{l,\tilde{d}}$ follows from equation (4.13).

□

[43]Note that if $u \to \underline{u}$, the right-hand side approaches d, which is assumed to be higher than $\underline{u}$.

Proof of lemma 4.2.

The proof is along the same lines as the proof for lemma 4.1.

□

Proof of proposition 4.3.

Since banks are identical, assessments and deposit distribution have to be symmetrical: $\mathbf{u_1} = (u, ..., u)$ and $\mathbf{D_1} = (D, .., D)$ where $D = S(u)/n$. Hence the expected return on first-period bank deposits is given by equations (4.8) - (4.10). But we know from lemma 4.1 that in this case $u = \bar{u}_{d_1}$ is the only consistent assessment.

□

Proof of proposition 4.4.

We denote the share of deposits that the banks in $\mathcal{B}_l$ receive by λ_l . First, note that an assessment $u_{1l} < u_{1h}$ always leads to $\lambda_l = 0$ and to a symmetric distribution of all savings among banks in $\mathcal{B}_h$. Hence, by lemma 4.1, this assessment is consistent if and only if $u_{1h} = \bar{u}_{d_{1h}}$ and if the assessment passes the zero measure test: if all banks in $\mathcal{B}_l$ receive a zero measure of deposits, then depositors at $\mathcal{B}_l$-banks must receive lower expected returns than $\bar{u}_{d_{1h}}$. Of course, the same is true for the converse assessment $u_{1h} < u_{1l}$. It leads to $\lambda_l = 1$ and $u_{1l} = \bar{u}_{d_{1l}}$ and is consistent if and only if the return paid by zero-measure $\mathcal{B}_h$-banks is smaller than $\bar{u}_{d_{1l}}$.

The assessment $u_{1l} < u_{1h} = \bar{u}_{d_{1h}}$ is consistent under RB, PB and BB, since under all those bailout schemes the bailout probability for depositors at zero-measure banks is never higher than the bailout probability for deposits at positive-measure banks. Under BS, however, a zero-measure bank is bailed out with probability 1. Hence, an assessment $u_{1h} < u_{1l} = \bar{u}_{d_{1l}}$ will never pass the the zero-measure test, and an assessment $u_{1l} < u_{1h} = \bar{u}_{d_{1h}}$ passes this test if and only if $d_{1l} < \bar{u}_{d_{1h}}$. Furthermore, under PB and RB bailout probabilities are the same for all deposits. Hence, expected returns on deposits of $\mathcal{B}_h$-banks are strictly higher than those on deposits of $\mathcal{B}_l$-banks for any consistent assessment. Finally, under BB, the assessment $u_{1h} < u_{1l} = \bar{u}_{d_{1l}}$ passes the zero-measure test if and only if

$$\Big(p_l I_{l,d_{1l},d_{1h}} + p_h I_{h,d_{1l},d_{1h}}\Big) d_{1h} < \bar{u}_{d_{1l}}.$$

This follows from the fact that under BB zero-measure banks are closed if they have offered higher first-period deposit rates than all positive-measure banks and if the refinancing condition (4.1) for the banking system does not hold. Therefore, $\mathcal{B}_l$-bank deposits will be bailed out with the same probability as in the symmetric case where all banks have offered d_{1l}.[44] Hence, expected returns on $\mathcal{B}_l$-bank deposits are equal to $\bar{u}_{d_{1l}}$. The variables $I_{i,d_{1l},d_{1h}}$ $(i = l, h)$ indicate whether the overall refinancing condition (4.1) holds in the low (high) state of production returns, thereby setting the bailout probability of the zero-measure $\mathcal{B}_h$-banks to 0 or 1.

It remains to show that an assessment $u_{1l} = u_{1h}$ cannot be optimal under BB and that no consistent assessments exist under BS if $\bar{u}_{d_{1h}} < d_{1l}$ and $p_h d_{1h} < \tilde{u}_{d_{1l},d_{1h}}$. Note that, concerning the last point, we have already shown that $u_{1l} < u_{1h}$ and $u_{1h} < u_{1l}$ are not consistent under these circumstances. Hence it remains to analyze assessments of the type $u_{1l} = u_{1h}$ under BB and BS. Note that such assessments can only be consistent if $\lambda_l > n_l/n$ (under BB) and $\lambda_l < n_l/n$ (under BS) respectively where n_l denotes the number of banks in $\mathcal{B}_l$. Hence, the statement that $u := u_{1l} = u_{1h}$ is a consistent assessment is equivalent to the existence of a real number $\lambda_l > n_l/n$ $(\lambda_l < n_l/n)$ with u solving the system $\bar{\mathcal{S}}(\lambda_l)$ described by equations (4.14) - (4.19):

$$
\begin{aligned}
u &= (p_l q_{l,l} + p_h q_{h,l}) d_{1l} && (4.14)\\
u &= (p_l q_{l,h} + p_h q_{h,h}) d_{1h} && (4.15)\\
q_{i,l} &= \begin{cases} 1 & \text{if } q_i \geq \lambda_l \\ q_i/\lambda_l & \text{else} \end{cases} \qquad (i = l, h) && (4.16)\\
q_{i,h} &= \begin{cases} (q_i - \lambda_l)/(1-\lambda_l) & \text{if } q_i \geq \lambda_l \\ 0 & \text{else} \end{cases} \qquad (i = l, h) && (4.17)\\
q_l &= \min\Big\{ \frac{1}{d_{1h} S_1(u)} S_2\Big(\frac{r_{2l}}{q_l d_{1h}}\Big), 1 \Big\} && (4.18)\\
q_h &= \min\Big\{ \frac{1}{d_{1h} S_1(u)} S_2\Big(\frac{r_{2h}}{q_h d_{1h}}\Big), 1 \Big\}. && (4.19)
\end{aligned}
$$

Note that $q_{i,l}$ and $q_{i,h}$ denote the bailout probabilities of banks in $\mathcal{B}_l$ and $\mathcal{B}_h$ respectively ($i = l, h$ denotes the state of production returns). The remaining statements therefore follow from lemma 4.5.

□

[44] See section 4.3.5.4 for the special treatment of zero-measure banks under BB.

Lemma 4.5
Suppose that $\underline{u} < d_{1l} < d_{1h}$.

(i) *If* $\bar{\mathcal{S}}(\lambda_l)$ *has a solution* u *for arbitrary* $\lambda_l \in (0,1]$, *then* $u < \bar{u}_{d_{1h}}$.

(ii) *If* $\bar{u}_{d_{1h}} < d_{1l}$ *and* $p_h d_{1h} < \tilde{u}_{d_{1l},d_{1h}}$, *then* $\bar{\mathcal{S}}(\lambda_l)$ *has no solution for all* $\lambda_l \in [0,1]$.

Proof.

First of all note that a solution of the complete system has to solve the subsystems $\bar{\mathcal{S}}_l$ (consisting of equations 4.14, 4.16, 4.18 and 4.19) and the subsystem $\bar{\mathcal{S}}_h$ (consisting of equations 4.15, 4.17, 4.18 and 4.19).

(i) The proof of part (i) rests on the observation that the sub-system $\bar{\mathcal{S}}_h$ consists of the same equations as the system $\mathcal{S}(d_{1h})$, which has the solution $\bar{u}_{d_{1h}}$. The only difference is that in the system $\mathcal{S}(d_{1h})$, $q_{i,h}$ is replaced by q_i $(i = l, h)$ in equation (4.15). The statement $u < \bar{u}_{d_{1h}}$ therefore follows from the fact that $q_{i,h} \leq q_i$.

To present this argument in a more formal way, we use the index i to indicate both $i = l$ and $i = h$. Note that a solution of $\bar{\mathcal{S}}_h$ can be derived by solving equations (4.18) and (4.19) and inserting the solutions in equation (4.17), which yields $q_{i,h}(u, \lambda_l)$. Since the right-hand side of equation (4.17) is decreasing in λ_l for arbitrary $q_i \leq 1$, we obtain that $q_{i,h}(u, \cdot)$ is decreasing in λ_l. By inserting $q_{i,h}$ in equation (4.15) we can therefore conclude that the solution $u(d_{1h}, \lambda_l)$ of the resulting equation is decreasing in λ_l. Hence $u(d_{1h}, \lambda_l) \leq u(d_{1h}, 0) = \bar{u}_{d_{1h}}$. In order to show that the inequality holds strictly for $\lambda_l > 0$, we assume that $u(d_{1h}, \lambda_l) = \bar{u}_{d_{1h}}$ for $\lambda_l > 0$. By inserting $u = \bar{u}_{d_{1h}}$ in equations (4.18) and (4.19) we can then see that $q_i = \bar{q}_{i,d_{1h}}$ and therefore that $q_{i,h}(\lambda_l) < \bar{q}_{i,d_{1h}}$. But then insertion into equation (4.15) would imply that $u(d_{1h}, \lambda_l) < \bar{u}_{d_{1h}}$, in contradiction to our assumption.

(ii) We denote the solutions of $\bar{\mathcal{S}}_i(\lambda_l)$ by u_i $(i = l, h)$. Note that if $u_h > p_h d_{1h}$, then $q_{l,h} > 0$ implying that $q_{l,l} = q_{h,l} = 1$.[45] Therefore $u_l = d_{1l} > \bar{u}_{d_{1h}}$ which by statement (i) is not possible if $u = u_l$ solves $\bar{\mathcal{S}}$. The case $u \leq p_h d_{1h}$ can be excluded by observing that $u_l \geq \tilde{u}_{d_{1l},d_{1h}}$ which by our assumption leads to $u = u_l > p_h d_{1h}$. To see that $u_l \geq \tilde{u}_{d_{1l},d_{1h}}$, assume to the contrary that $u = u_l < \tilde{u}_{d_{1l},d_{1h}}$. Then, by equations (4.18) and (4.19), we have $q_i > \tilde{q}_{i,d_{1l},d_{1h}}$, implying that $q_{i,l} > \tilde{q}_{i,d_{1l},d_{1h}}$. This in turn leads to $u_l > \tilde{u}_{d_{1l},d_{1h}}$ in contradiction to our assumption.

□

[45] If $q_{l,h} > 0$, then $q_l \geq \lambda_l$ by equation (4.17) and, since $q_l \leq q_h$, we also obtain that $q_h \geq \lambda_l$. Hence $q_{l,l} = q_{h,l} = 1$ by equation (4.16).

Proof of corollary 4.1.

Since $\tilde{q}_{i,d_{1l},d_{1h}} \geq \bar{q}_{i,d_{1h}}$ $(i = l, h)$, we obtain that $\tilde{u}_{d_{1l},d_{1h}} \geq (p_h + p_l \bar{q}_{l,d_{1h}}) d_{1l}$, which implies the statement of the corollary because $\bar{q}_{l,d_{1h}} > 0$.

□

4.A.2 Proofs for Section 4.5

The propositions in section 4.5 are concerned with symmetric equilibria in $t = 0$. Hence we will either have to analyze the case where all banks offer the same first-period deposit rate d_1 or the deviation case where $(n-1)$ banks offer the same first-period deposit rate d_1 and one bank j offers a different rate d_1^{dev}. We will always denote the assessment for the non-deviating banks by u_1 and that for the deviating bank by u_1^{dev}. Resulting first-period deposits are denoted by D_1 and D_1^{dev} respectively.

Proof of proposition 4.5.

It only remains to substantiate (ii) and (iii). Statement (ii) is obvious for first-period returns and follows for second-period returns, because they are given by $S_2^{-1}\Big(dS_1(d)\Big)$ if $d_1 \leq d_{\mathbf{L}}$ and by $p_l \underline{u} + p_h \tilde{d}_2^*(d_1)$ if $d_1 > d_{\mathbf{L}}$. Both expressions are increasing in d_1. Statement (iii) follows immediately from (ii) since if $d_{\mathbf{ZP}} \leq d_{\mathbf{C}}$ only symmetric equilibria with $d_1 \leq d_{\mathbf{L}}$ are possible.

□

Proof of proposition 4.6.

Step 1: $\mathcal{E}_L$ is an equilibrium.

Given $\mathbf{d_1} = (d_{\mathbf{L}}, ..., d_{\mathbf{L}})$, the only consistent assessment is $\mathbf{u_1} = (d_{\mathbf{L}}, ..., d_{\mathbf{L}})$. Hence the equilibrium deposit distribution is $D_1^i = S_1(d_{\mathbf{L}})/n$ $(i = 1, ..., n)$ and each bank's expected profits are given by

$$\Pi_1 = \frac{S_1(d_{\mathbf{L}})}{n} p_h (r_{2h} - r_{2l}) > 0.$$

Consider now a deviation of one bank. Deviation to $d_1^{\text{dev}} < d_1$ leads to $u_1^{\text{dev}} < u_1$ and hence to $D_1^{\text{dev}} = 0$, which cannot be profitable. On the other hand $d_1^{\text{dev}} > d_1$ leads to $u_1 < u_1^{\text{dev}} = \bar{u}_{d_1^{\text{dev}}}$ and hence to $D_1^{\text{dev}} = S_1(\bar{u}_{d_1^{\text{dev}}})$. This implies that the deviating bank cannot refinance in the case of low production returns. By our assumption, the punishment P which is imposed by the regulator in this case would outweigh all possible deviation gains.

Step 2: No other equilibria $\mathcal{E} = (d_1)$ with $d_1 \neq d_{\mathbf{L}}$ exist.

If $d_1 < d_{\mathbf{L}}$, then deviation to a slightly higher deposit rate d_1^{dev} ($d_1 < d_1^{\text{dev}} < d_{\mathbf{L}}$) would lead to $u_1 < u_1^{\text{dev}} = \bar{u}_{d_1^{\text{dev}}} = d_1^{\text{dev}}$. Therefore $D_1^{\text{dev}} = S_1(d_1^{\text{dev}}) < S_1(d_{\mathbf{L}})$ and the deviating bank would be able to refinance in both states of production returns. The collection of all savings outweighs the slightly higher interest payment. If $d_1 > d_{\mathbf{L}}$, the only possible assessment is $\mathbf{u_1} = (\bar{u}_{d_1}, ..., \bar{u}_{d_1})$ where $\bar{u}_{d_1} < d_1$. But this implies a positive probability of being closed and suffering the punishment P, which leads to negative expected profits. Of course this is not possible in equilibrium.

□

Proof of lemma 4.3.

The proof rests on the continuity of the functions $S_i(\cdot)$ $(i = 1, 2)$, $\bar{u}_{(\cdot)}$ and $\bar{q}_{i,(\cdot)}$ $(i = l, h)$. First we prove the continuity for the points $d \in \mathcal{D}_M$. Consider a sequence $(d_n)_{n \in \mathbb{N}}$ with $d_n \to d$ $(n \to \infty)$. If $dS_1(\bar{u}_d) < S_2(r_{2h}/d)$, then $d_n S_1(\bar{u}_{d_n}) < S_2(r_{2h}/d)$ for sufficiently large n. If $dS_1(\bar{u}_d) = S_2(r_{2h}/d)$, we have $\bar{q}_{h,d_n} \to 1$ by equation (4.10) and $d_2^*(d_n) \to r_{2h}/d$ $(n \to \infty)$. Hence, in both cases we obtain $\pi(d_n) \to \pi(d)$.

For points $d \in \mathcal{D}_H$ the proof is completely analogous. Now suppose that $d \leq d_{\mathbf{L}}$. Again, only the case $d = d_{\mathbf{L}}$ is interesting. If $d_n \to d$, then $q_{l,d_n} \to 1$ by equation (4.9) and $d_n d_2^*(d_n) \to d_{\mathbf{L}} d_2^*(d_{\mathbf{L}}) = r_{2l}$. Hence $\pi(d_n) \to p_h(r_{2h} - r_{2l})$.

□

Proof of lemma 4.4.

If $d \in \mathcal{D}_M$, then by equation (4.9) we obtain

$$q_l = \frac{a_2}{d a_1 u^{\alpha_1}} (\frac{r_{2l}}{q_l d})^{\alpha_2}$$

implying that $q_l = c d^{-1} u^{-\tilde{\alpha}_1}$ where

$$c := \left(\frac{a_2 r_{2l}^{\alpha_2}}{a_1} \right)^{1/(1+\alpha_2)}$$

and $\tilde{\alpha}_i := \alpha_i/(1+\alpha_i)$ $(i = 1, 2)$. Inserting q_l in equation (4.8) we find that $\bar{u}_d$ can be described as solution of the equation $F(u, d) = 0$ where

$$F(u,d) := u - p_l c u^{-\tilde{\alpha}_1} - p_h d.$$

But since $F(\bar{u}_d, d) = 0$ we find that $F_u \bar{u}'_d + F_d = 0$ implying that $\bar{u}'_d = -F_d/F_u$.[46] Hence, $\bar{u}'_d > 0$ follows from

$$\begin{aligned} F_d(u,d) &= -p_h \\ F_u(u,d) &= 1 + \tilde{\alpha}_1 p_l u^{-(\tilde{\alpha}_1+1)}. \end{aligned}$$

□

Proof of proposition 4.7.

The proof follows the same arguments as the proof of proposition 4.6:

(1) $\mathcal{E}_{ZP}$ is an equilibrium. Deviation to $d_1^{\text{dev}} < d_{\mathbf{ZP}}$ leads to $u_1^{\text{dev}} < u_1$ and hence to $D_1^{\text{dev}} = 0$, which cannot be profitable. On the other hand, $d_1^{\text{dev}} > d_1$ leads to $u_1 < u_1^{\text{dev}} = \bar{u}_{d_1^{\text{dev}}}$ and hence to $D_1^{\text{dev}} = S(\bar{u}_{d_1^{\text{dev}}})$. But since $\pi(d_1^{\text{dev}}) < 0$, deviation profits are negative.

(2) No other equilibria $\mathcal{E} = (d_1)$ with $d_1 \neq d_{\mathbf{ZP}}$ exist. If $d_1 < d_{\mathbf{ZP}}$, then deviation to a slightly higher deposit rate d_1^{dev} $(d_1 < d_1^{\text{dev}} < d_{\mathbf{ZP}})$ leads to $D_1^{\text{dev}} = S(\bar{u}_{d_1^{\text{dev}}})$. By continuity, losses in profits per unit of deposits can be offset by the collection of all savings if $(d_1^{\text{dev}} - d_1)$ is small enough. If $d_1 > d_{\mathbf{ZP}}$, then $\pi(d_1) < 0$, which cannot be the case in equilibrium.

□

Proof of proposition 4.8.

Considering a deviation $d_1^{\text{dev}} \neq d_1$ from a symmetric equilibrium $\mathbf{d_1} = (d_1, ..., d_1)$, we make the following preliminary remarks:

(1) From proposition 4.4 we know that only the following two assessments and deposit constellations are possible:

$$\begin{array}{llll} \text{(A1)} & u_1^{\text{dev}} < u_1 & D_1^{\text{dev}} = 0 & D_1 = S_1(\bar{u}_{d_1})/n \\ \text{(A2)} & u_1^{\text{dev}} > u_1 & D_1^{\text{dev}} = S_1(\bar{u}_{d_1^{\text{dev}}}) & D_1 = 0. \end{array}$$

[46] F_u and F_d denote the partial derivatives of F with respect to u and d respectively.

A1 is consistent if $d_1^{\text{dev}} < d_1$ and A2 if $d_1^{\text{dev}} > d_1$. Moreover, if $d_{\mathbf{L}} \leq d_1, d_1^{\text{dev}} \leq d_{\mathbf{ZP}}$, then the following additional consistency conditions hold. A1 is consistent if $\bar{u}_{d_1^{\text{dev}}} \leq \bar{u}_{d_1}$ and A2 if $\bar{u}_{d_1^{\text{dev}}} \geq \bar{u}_{d_1}$. This follows directly from proposition 4.3 and the fact that (e.g. for A1)

$$\left(p_l I_{l,d_1,d_1^{\text{dev}}} + p_h I_{h,d_1,d_1^{\text{dev}}}\right) d_1^{\text{dev}} \leq p_h d_1^{\text{dev}} < \bar{u}_{d_1^{\text{dev}}} \leq \bar{u}_{d_1}.$$

(2) We can exclude equilibria $\mathcal{E} = (d_1)$ where $d_1 > d_{\mathbf{ZP}}$ because they are negative expected profits equilibria.

(3) We do not have to consider deviations to $d_1^{\text{dev}} \geq d_{\mathbf{ZP}}$ since they cannot be profitable. They lead to zero profits in the case of A1 and to zero or negative profits in the case of A2.

(4) If $d_1 \in \mathcal{U}_{\max}$, then $d_1 \geq d_{\mathbf{L}}$.

Now we turn to the proof of the proposition.

(i) Suppose that $d_1 \in \mathcal{U}_{\max}$. Deviation to $d_1^{\text{dev}} \neq d_1$ with $d_1^{\text{dev}} \leq d_{\mathbf{ZP}}$ cannot be profitable since by TR depositors would always choose to deposit with the non-deviating banks (A1 is consistent and $\bar{u}_{d_1} \geq \bar{u}_{d_1^{\text{dev}}}$).

(ii) We define $\tilde{d} =: \min\{d_{\mathbf{UH}}, d_{\mathbf{ZP}}\}$. From statement (i) we know that $\mathcal{E}_{UH}$ is a Nash equilibrium because $\mathcal{E}_{UH} = (\tilde{d})$ and $\tilde{d} \in \mathcal{U}_{\max}$. No other equilibrium with $d_1 \neq \tilde{d}$ exists, since for $d_1 < \tilde{d}$ deviation to a slightly higher deposit rate $d_1^{\text{dev}} > d_1$ leads to A2 (because $\bar{u}_{d_1^{\text{dev}}} > \bar{u}_{d_1}$) and hence such a deviation is always profitable if $(d_1^{\text{dev}} - d_1)$ is small enough. The same argument applies for $d_1 > \tilde{d}$ if $\tilde{d} = d_{\mathbf{UH}}$. In this case, deviation to slightly lower deposit rates is profitable.

(iii) Case 1 follows from the same arguments as used under (ii). Consider now case 2. $\mathcal{E}_{ZP}$ is a Nash equilibrium according to statement (i). Now turn to the question whether $\mathcal{E}_L$ is an equilibrium. Obviously, only deviations to $d_1^{\text{dev}} > d_{\mathbf{L}}$ with $\bar{u}_{d_1^{\text{dev}}} > d_{\mathbf{L}}$ can be profitable. Hence deviation is profitable if and only if $\Pi^{\text{dev}}(d_{\mathbf{L}}) - \pi(d_{\mathbf{L}})S_1(d_{\mathbf{L}})/n > 0$. Moreover, no other equilibria $\mathcal{E} = (d_1)$ can exist since for $d_{\mathbf{L}} < d_1 < d_{\mathbf{UL}}$, deviation to slightly lower, and for $d_1 < d_{\mathbf{L}}$ and $d_1 \geq d_{\mathbf{UL}}$ deviation to slightly higher deposit rates is profitable by the same arguments as under (ii).

□

Proof of proposition 4.9.

Suppose that $u_1(\tilde{\mathcal{E}}_{ZP}) \geq u_1(\mathcal{E}_{ZP})$. Then $\tilde{d}_{\mathbf{ZP}} > d_{\mathbf{ZP}}$ (see table 4.2) and hence $\tilde{d}_2^*(\tilde{d}_{\mathbf{ZP}}) > d_2^*(d_{\mathbf{ZP}})$. But since under $\tilde{\mathcal{E}}_{ZP}$ banks' profits per deposit are given by

$$p_h\Big(r_{2h} - \tilde{d}_2^*(\tilde{d}_{\mathbf{ZP}})\Big),$$

this would imply that those profits are smaller than $\pi(d_{\mathbf{ZP}}) = 0$, which is impossible in equilibrium. To prove that $u_2(\tilde{\mathcal{E}}_{ZP}) < u_2(\mathcal{E}_{ZP})$, we draw on the fact that $\tilde{d}_2^*(\tilde{d}_{\mathbf{ZP}}) \leq d_2^*(d_{\mathbf{ZP}})$.[47] This implies that

$$\begin{aligned} u_2(\tilde{\mathcal{E}}_{ZP}) &= p_l\underline{u} + p_h\tilde{d}_2^*(\tilde{d}_{\mathbf{ZP}}) \\ &\leq p_l\underline{u} + p_h d_2^*(d_{\mathbf{ZP}}) \\ &< u_2(\mathcal{E}_{ZP}). \end{aligned}$$

□

[47]Suppose that $\tilde{d}_2^*(\tilde{d}_{\mathbf{ZP}}) > d_2^*(d_{\mathbf{ZP}})$. Then $\tilde{d}_{\mathbf{ZP}} > d_{\mathbf{ZP}}$ which - as above - would imply that profits per deposits are negative under $\tilde{\mathcal{E}}_{ZP}$.

4.B Liquidation Value and Takeover Costs

In this appendix we will relax two implicit assumptions which have been imposed to make the presentation of our results more transparent: (a) the assumption that the takeover of investment projects from closed banks does not involve any deadweight costs, and (b) that the $t = 1$ liquidation value of investments is zero. In the sequel, the general version of the model (with liquidation value and takeover costs) will be called the "*general setting*", compared to the "*specific setting*" where liquidation value and takeover costs are zero. In the general setting we assume that a fraction $(1 - \delta)$ of project returns is consumed by the takeover procedure. $(1 - \delta)$ is assumed to represent all conceivable costs of transferring ownership. Hence, second-period returns of projects that have been taken over from other banks are given by δr_2. Moreover, we allow for a positive liquidation value R_1 of investments when liquidated in $t = 1$; the realization of the liquidation value can be either high ($R_1 = r_{1h}$ if $R_2 = r_{2h}$) or low ($R_1 = r_{1l}$ if $R_2 = r_{2l}$). We require that first-period depositors whose claims have been eliminated by the regulator will receive a return equal to the liquidation value of investments. A surviving bank will therefore take over the same amount of first-period depositors as it takes investment projects, implying that such banks will have $b_I D_1^i$ units of investment projects and repayment obligations of $\left(bd_1^i + (b_I - b)r_1\right)D_1^i$. Moreover, due to takeover costs, the project cash flows in $t = 2$ (if all projects are continued and a fraction q of depositors has been bailed out) are given by $[q + \delta(1 - q)]r_2 b_I D_1^i$.[48] Hence, the maximum return that can be offered to second-period depositors by bank i is given by

$$\frac{[q + \delta(1 - q)]r_2}{qd_1^i + (1 - q)r_1}. \tag{4.20}$$

Consequently, the maximum fraction $\bar{q}$ of bailed out depositors in the case where refinancing condition (4.1) does not hold, is determined as the solution of the equation[49]

$$S_2\left(\frac{[q + \delta(1 - q)]r_2}{qd_1^{\max} + (1 - q)r_1}\right) = \left(qd_1^{\max} + (1 - q)r_1\right)\sum_{i=1}^{n} D_1^i. \tag{4.21}$$

[48] Note that the fraction of own projects in all projects hold by a surviving bank is $1/(n/k) = k/n$, which we approximate by q.

[49] The corresponding equation in the specific setting is equation (4.3). Note that since $k/n \geq q$, banks can *at least* pay a return as given in (4.20). Hence, by bailing out a fraction $\bar{q}$ of depositors, the regulator guarantees that all surviving banks can refinance.

Moreover, in the general setting, the definition of $\bar{d}_2^*$ has to be changed to

$$\bar{d}_2^* := S_2^{-1}\left(\sum_{i\in\mathcal{B}+}[bd_1^i + (1-b)r_1]D_1^i + \sum_{i\notin\mathcal{B}+} r_1 D_1^i\right),$$

and consistent first-period assessments in a symmetric equilibrium are now described by a general version of the system $\mathcal{S}(d)$:[50]

$$\begin{aligned} u &= \Big[p_l q_l + p_h q_h\Big]d + \Big[p_l(1-q_l)r_{1l} + p_h(1-q_h)r_{1h}\Big] \\ q_l &= \min\left\{\frac{1}{d-r_{1l}}\left[\frac{1}{S_1(u)}S_2\left(\frac{[q_l+(1-q_l)\delta]r_{2l}}{q_l d + (1-q_l)r_{1l}}\right) - r_{1l}\right]\ ,\ 1\right\} \\ q_h &= \min\left\{\frac{1}{d-r_{1h}}\left[\frac{1}{S_1(u)}S_2\left(\frac{[q_h+(1-q_h)\delta]r_{2h}}{q_h d + (1-q_h)r_{1h}}\right) - r_{1h}\right]\ ,\ 1\right\} \end{aligned}$$

under the constraints $q_l > 0$ and $q_h > 0$. In remark 4.1 we state the condition under which this generalized version of $\mathcal{S}(d)$ has a unique solution. As in the specific setting, we denote this solution by $\bar{u}_d$ and the corresponding bailout probabilities by $\bar{q}_{l,d}$ and $\bar{q}_{h,d}$, respectively. The profit-per-deposit function π now reads as[51]

$$\pi(d) := \begin{cases} \bar{R}_2 - dd_2^*(d) & \text{if } d \le d_{\mathbf{L}} \\ -p_l(1-\bar{q}_{l,d})(d-r_{1l}) + p_h\Big(r_{2h} - dd_2^*(d)\Big) & \text{if } d \in \mathcal{D}_M \\ -p_l(1-\bar{q}_{l,d})(d-r_{1l}) - p_h(1-\bar{q}_{h,d})(d-r_{1l}) & \text{if } d \in \mathcal{D}_H. \end{cases}$$

Remark 4.1

The results obtained for the specific setting can be adapted to the general setting in the following way:

1. *Proposition 4.2 holds in the general setting. Inequality $\bar{d}_2^* \le r_2/(q\tilde{d}_1^{max})$ is replaced by*

$$\bar{d}_2^* \le \frac{[q+\delta(1-q)]r_2}{q\tilde{d}_1^{max} + (1-q)r_1}.$$

2. *If $r_{1h} \le \delta d_{\mathbf{L}}$, then lemmata 4.1, 4.3 and 4.5 hold in the general setting.*

[50] Note that the formulas for q_l and q_h are derived from equation (4.21).

[51] Note that the definitions of the sets $\mathcal{D}_M$ and $\mathcal{D}_H$ and of the function $d_2^*(\cdot)$ given in section 4.5.3 remain valid (see page 136). The description of $\mathcal{D}_M$, $\mathcal{D}_H$ and $d_2^*(\cdot)$ in the general setting follows therefore directly from the generalization of the function $\bar{u}_{(\cdot)}$.

3. *If $r_{1h} \leq \delta d_{\mathbf{L}}$, then propositions 4.3, 4.4 and 4.6 - 4.8 hold in the general setting. In proposition 4.4, the following adaptations have to be made:*

   (a) *BB: inequality*

   $$(p_l I_{l,d_{1l},d_{1h}} + p_h I_{h,d_{1l},d_{1h}}) d_{1h} < \bar{u}_{d_{1l}}$$

   has to be replaced by

   $$(p_l I_{l,d_{1l},d_{1h}} + p_h I_{h,d_{1l},d_{1h}}) d_{1h} + p_l(1 - I_{l,d_{1l},d_{1h}}) r_{1l} + p_h(1 - I_{h,d_{1l},d_{1h}}) r_{1h} < \bar{u}_{d_{1l}}.$$

   (b) *BS: inequality $p_h d_{1h} < \bar{u}_{d_{1l}}$ has to be replaced by $p_h d_{1h} + (p_l r_{1l} + p_h r_{1h}) < \bar{u}_{d_{1l}}$.*

Proof.

Part 1: Lemmata 4.1, 4.3 and 4.5.

The proofs are along the lines of the proofs for the specific setting. Concerning lemma 4.5, note that in the general setting the system $\bar{\mathcal{S}}(\lambda_l)$ reads as

$$\begin{aligned}
u &= \Big[p_l q_{l,l} + p_h q_{h,l})\Big] d_{1l} + \Big[p_l(1 - q_{l,l}) r_{1l} + p_h(1 - q_{h,l})) r_{1h}\Big] \\
u &= \Big[p_l q_{l,h} + p_h q_{h,h})\Big] d_{1h} + \Big[p_l(1 - q_{l,h}) r_{1l} + p_h(1 - q_{h,h})) r_{1h}\Big] \\
q_{i,l} &= \begin{cases} 1 & \text{if } q_i \geq \lambda_l \\ q_i/\lambda_l & \text{else} \end{cases} \qquad (i = l, h) \\
q_{i,h} &= \begin{cases} (q_i - \lambda_l)/(1 - \lambda_l) & \text{if } q_i \geq \lambda_l \\ 0 & \text{else} \end{cases} \qquad (i = l, h) \\
q_l &= \min\left\{ \frac{1}{d_{1h} - r_{1l}} \left[\frac{1}{S_1(u)} S_2 \left(\frac{[q_l + (1 - q_l)\delta] r_{2l}}{q_l d_{1h} + (1 - q_l) r_{1l}} \right) - r_{1l} \right] \ , \ 1 \right\} \\
q_h &= \min\left\{ \frac{1}{d_{1h} - r_{1h}} \left[\frac{1}{S_1(u)} S_2 \left(\frac{[q_h + (1 - q_h)\delta] r_{2h}}{q_h d_{1h} + (1 - q_h) r_{1h}} \right) - r_{1h} \right] \ , \ 1 \right\}.
\end{aligned}$$

With respect to lemmata 4.1 and 4.5 we just have to verify that

$$L(q) := \frac{\Big(q + (1 - q)\delta\Big) r_2}{qd + (1 - q) r_1}$$

is decreasing in q for the cases $\{r_1 = r_{1l}, r_2 = r_{2l}\}$ and $\{r_1 = r_{1h}, r_2 = r_{2h}\}$ if $d \geq d_{\mathbf{L}}$.

To derive this result we calculate the first derivative of L with respect to q:

$$\begin{aligned}\Big[qd+(1-q)r_1\Big]^2 L'(q)/r_2 &= (1-\delta)\Big[qd+(1-q)r_1\Big]-(d-r_1)\Big[q+(1-q)\delta\Big] \\ &=: H(q)\end{aligned}$$

and

$$\begin{aligned}H(q) &= qd+(1-q)r_1-\delta qd-\delta(1-q)r_{1l}-dq-d(1-q)\delta+r_1q+r_1(1-q)\delta \\ &= r_1\Big[(1-q)+\delta(1-q)+q-(1-q)\delta\Big]-\delta d(q+(1-q)) \\ &= \alpha(r_1-\delta d).\end{aligned}$$

Hence we have $H(q)\leq 0$ for both cases if $r_{1h}\leq \delta d_{\mathbf{L}}$.

Part 2: Propositions 4.3, 4.4 and 4.6 - 4.8.

The proofs for the specific setting can be generalized directly.

$\square$

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