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Christian Johannes Enz

aus Heilbronn

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DER INVERSE SKALARE KRÜMMUNGSFLUSS IN ARW-RÄUMEN

Betreuer
Prof. Dr. Claus Gerhardt

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THE INVERSE SCALAR CURVATURE FLOW IN ARW SPACES

Abstract. We consider the inverse scalar curvature flow (ISCF)

$$\dot{x} = -F^{-1}\nu \tag{0.1}$$

in spacetimes N with a special future singularity and some additional structural conditions. We prove the existence of the ISCF for all times, and prove convergence results for the leaves of the flow. Finally, we show that the properly rescaled flow in N has a natural smooth extension across the singularity into a mirrored spacetime $\hat{N}$. With respect to that diffeomorphism we speak of a transition from big crunch to big bang.

zusammenfassung. Wir betrachten den inversen skalaren Krümmungsfluss (ISCF)

$$\dot{x} = -F^{-1}\nu \tag{0.2}$$

in einer Raumzeit N, welche eine spezielle Zukunftssingularität besitzt und einige weitere Struktureigenschaften erfüllt. Wir zeigen die Existenz des ISCFs für alle Zeiten und beweisen Konvergenzresultate für die Blätter des Flusses. Nach geeigneter Reskalierung besitzt der Fluss in N eine natürliche Fortsetzung über die Singularität hinweg in eine gespiegelte Raumzeit $\hat{N}$. Bezüglich dieses Diffeomorphismusses sprechen wir von einem Übergang von Big Crunch nach Big Bang.

Contents

0. Introduction

We start this section with a short introduction into globally hyperbolic spacetimes. The following definition, assumption, and facts stated in the remarks below can be found in [11, Chapter 7.1].

0.1. **Definition.** A globally hyerbolic spacetime N, $\dim N = n+1$, is said to be a asymptotically Robertson-Walker (ARW) with respect to the future, if a future end of N, N_+ can be written as a product $N_+ = I \times \mathcal{S}_0$, where $I = [a, b)$, $\mathcal{S}_0$ is a Riemannian space, and there exists a future directed time function $\tau = x^0$ such that the metric in N_+ can be written as

$$d\breve{s}^2 = e^{2\tilde{\psi}}\{-(dx^0)^2 + \sigma_{ij}(x_0, x)dx^i dx^j\}, \tag{0.1}$$

where $\mathcal{S}_0$ corresponds to $x^0 = a$, $\tilde{\psi}$ is of the form

$$\tilde{\psi}(x_0, x) = f(x_0) + \psi(x^0, x), \tag{0.2}$$

and we assume that there exists a positive constant c_0 and a smooth Riemannian metric $\bar{\sigma}_{ij}$ on $\mathcal{S}_0$ such that

$$\lim_{\tau \to b} e^{\psi} = c_0 \tag{0.3}$$

$$\lim_{\tau \to b} \sigma_{ij}(\tau, x) = \bar{\sigma}_{ij}(x) \tag{0.4}$$

$$\lim_{\tau \to b} f(\tau) = -\infty. \tag{0.5}$$

0.2. **Remark.**

(i) W.l.o.g. we will assume that $c_0 = 1$.

(ii) The first two limits have to be understood to be uniformly in all derivatives of arbitrary order with respect to space and time.

(iii) As a consequence of (ii) N is close to the Robertson-Walker metric

$$d\tilde{s}^2 = e^{2f}\{-(dx^0)^2 + \bar{\sigma}_{ij}(x_0, x)dx^i dx^j\}, \tag{0.6}$$

which means that all derivatives of arbitrary order with respect to space and time of the conformal metric $e^{-2f}\breve{g}_{\alpha\beta}$ converges uniformly to the corresponding derivatives of the limit metric

$$d\hat{s}^2 = -(dx^0)^2 + \bar{\sigma}_{ij}(x)dx^i dx^j. \tag{0.7}$$

(iv) In our setting Robertson-Walker-metric does not mean necessarily that the spacelike metric $(\bar{\sigma}_{ij})$ is a metric of constant curvature.

Proving our main result, we have to have to impose some additional structural conditions on f.

0.3. **Assumption.**

(i)

$$-f^{'} > 0. \tag{0.8}$$

(ii) There exists $\omega \in \mathbb{R}$, $n + w - 2 > 0$ such that

$$\lim_{\tau \to b} |f^{'}|^2 e^{(n+w-2)f} = m > 0. \tag{0.9}$$

(iii) If we set $\tilde{\gamma} = \frac{1}{2}(n + w - 2)$, then there exists the limit

$$\lim_{\tau \to b} (f^{''} + \tilde{\gamma}|f^{'}|^2). \tag{0.10}$$

(iv)

$$|D_\tau^m (f^{''} + \tilde{\gamma}|f^{'}|^2)| \le c_m |f^{'}|^m \quad \forall m \ge 1 \tag{0.11}$$

(v)

$$|D_\tau^m f| \le c_m |f^{'}|^m \quad \forall m \ge 1 \tag{0.12}$$

(vi) Proving the C^3-regularity result of the transition flow, cf. Theorem 13.1, we have further to impose that the following limit exists

$$\lim_{\tau \to 0} (f^{''} + \tilde{\gamma}|f^{'}|^2)^{'} \tau. \tag{0.13}$$

0.4. **Remark.**

(i) From Corollary 1.4 we infer that the range of τ is finite, so that we will assume w.l.o.g. that $I = [a, 0)$.

(ii) If $\mathcal{S}_0$ is compact, then we call N a normalized ARW spacetime, if

$$\int_{\mathcal{S}_0} \sqrt{\det \bar{\sigma}_{ij}} = |S^n|. \tag{0.14}$$

(iii) In the following, $\mathcal{S}_0$ is assumed to be compact. W.l.o.g. we will assume that N is a normalized ARW spacetime.

We consider the scalar curvature function $F = \sigma_2$ and assume $M_0 \subset N^+$ to be a spacelike F-admissable hypersurface. W.l.o.g. we assume in this paper that F is normalized such that $F(1, \dots, 1) = n$. Then, we look at the inverse scalar curvature flow (ISCF) given by the evolution problem

$$\begin{aligned} \dot{x} &= -\frac{1}{F}\nu, \\ x(0) &= x_0, \end{aligned} \tag{0.15}$$

where x_0 is the embedding of an initial hypersurface M_0 and ν denotes the past directed normal. Then, we can express the flow hypersurfaces $M(t)$ as graphs over $\mathcal{S}_0$

$$M(t) = \operatorname{graph} u(t, \cdot) \tag{0.16}$$

and the main result can be formulated as:

0.5. **Theorem.** *Let N and F satisfy the above assumptions and let $M_0 \subset N^+$ be a smooth, closed, spacelike and F-admissable hypersurface, where we assume that*

$$N^+ = [-\epsilon, 0) \times \mathcal{S}_0, \tag{0.17}$$

with

$$\epsilon = \epsilon((N, \breve{g}_{\alpha\beta}), \sup_{M_0} \tilde{v}) > 0 \tag{0.18}$$

small. Then, there holds:

(i) *The inverse scalar curvature flow with initial hypersurface M_0 exists for all times.*

(ii) *Set $\tilde{u} = ue^{\gamma t}$, where $\gamma = \frac{1}{n}\tilde{\gamma}$, then there exist positive constants c_1, c_2 such that*

$$-c_1 \le \tilde{u} \le -c_2 < 0, \tag{0.19}$$

and $\tilde{u}$ converges in $C^\infty(\mathcal{S}_0)$ to a smooth function, if t goes to infinity.

(iii) *Let (g_{ij}) the induced metric of the leaves $M(t)$, then the rescaled metric*

$$e^{\frac{2}{n}t} g_{ij} \tag{0.20}$$

converges in $C^\infty(\mathcal{S}_0)$ to

$$(\tilde{\gamma} m)^{\frac{1}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}} \bar{\sigma}_{ij}. \tag{0.21}$$

(iv) *The leaves $M(t)$ get more umbilical, if t tends to infinity. Namely, there holds*

$$F^{-1} |h_i^j - \frac{1}{n} H \delta_i^j| \le c e^{-2\gamma t}. \tag{0.22}$$

In case $n + \omega - 4 > 0$, we got even a better estimate

$$|h_i^j - \frac{1}{n} H \delta_i^j| \le c e^{-\frac{1}{2n}(n+\omega-4)t}. \tag{0.23}$$

In [9] resp. [19] the theorem was proved for $F = H$ resp. F to be a curvature function of class (K^*). In this paper we go along the lines of both papers as far as possible, with the exception of Section 5 and Section 6, where we use [4].
$F = \sigma_2$ is not of class (K^*) but satisfies at least the so-called (K^*)-condition, cf. (2.20), so that almost all proofs in [19] hold in our setting either. Using the special structure of F_i , cf. (2.18), some proofs are even easier. Since the flow hypersurfaces of the (ISCF) are not convex, we have to impose that ϵ in (0.18) depends from $\sup_{M_0} \tilde{v}$, to ensure that the Riemannian curvature tensor fulfills the important condition (7.40).

1. Notation, definitions and preliminary results

In this section we want to introduce some general notation and basic facts concerning with spacelike hypersurfaces in Lorentzian manifolds. We follow the notation introduced in [8]. Confer further [10, Section 12] for a more detailed treatment.

Let $N = N^{n+1}$ be a Lorentzian space and $M = M^n$ be an embedded spacelike hypersurface

$$x : M \hookrightarrow N \tag{1.1}$$

with differentiable timelike normal ν.

Geometric quantities in M will be denoted by using greek indices which range from 0 to n, i.e. we will write (g_{ij}) resp. (R_{ijkl}) for the metric resp. the Riemannian curvature tensor of M. Geometric quantities in N will be denoted by using latin indices which range from 0 to $n+1$. If there is any ambiguity between quantities of M and N possible, we will use an overbar for the corresponding quantities of N, i.e. we have (ν^α) for the normal of M, and $(\bar{g}_{\alpha\beta})$ resp. $(\bar{R}_{\alpha\beta\gamma\delta})$ for the metric resp. Riemannian curvature tensor of N. If nothing else is stated, the summation convention is used. Local coordinates in M resp. N will be denoted by (ξ^i) resp. (x^α).

Partial differentiation will be marked by a comma, while covariant differentiation will be marked by a semicolon. In the later case we will commonly leave the semicolon out if no misunderstandings are possible. Thus, to give examples, let $f : N \to \mathbb{R}$ be a function, then we would usually write (f_α) for the gradient and $(f_{\alpha\beta})$ for the Hessian of f, but for the covariant derivatives of the Ricci tensor $(R_{\alpha\beta})$ we would write $R_{\alpha\beta;\gamma}$, $R_{\alpha\beta;\gamma\delta}$, etc. We further introduce the indication $R_{\alpha\beta;i} = R_{\alpha\beta;\gamma} x_i^\gamma$ with analogue generalizations to other quantities of N.

Now, we state four fundamental equations which describe the geometry of an embedded hypersurface. In local coordinates we have the Gauß formula

$$x_{ij}^\alpha = h_{ij}\nu^\alpha, \tag{1.2}$$

the Weingarten equation

$$\nu_i^\alpha = h_i^k x_k^\alpha, \tag{1.3}$$

the Codazzi equation

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta, \tag{1.4}$$

and finally the the Gauß equation

$$R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta. \tag{1.5}$$

The Gauß formula can be understood as an implicit definition of the second fundamental form (h_{ij}) of a hypersurface, where the sign of (h_{ij}) depends from the direction of the normal ν. In this paper we will always choose the past directed normal. We emphasize that x^α_{ij}, ν^α_i, etc. are full tensors, i.e. we have

$$x^\alpha_{ij} = x^\alpha_{,ij} - \Gamma^k_{ij} x^\alpha_k + \bar\Gamma^\alpha_{\beta\gamma} x^\beta_i x^\gamma_j, \tag{1.6}$$

where Γ^k_{ij} resp. $\bar\Gamma^\alpha_{\beta\gamma}$ represent the Christoffel symbols of M resp. N.

In our setting N is a topological product $\mathbb{R} \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian space, and there exists a Gaussian coordinate system

$$ds^2 = e^{2\tilde\psi}\{-(dx^0)^2 + \sigma_{ij}(x^0,x)dx^i dx^j\}, \tag{1.7}$$

see [1, Lemma 2.2], and [2, Theorem 1.1] for a proof. Here, σ_{ij} is a Riemannian metric, $\tilde\psi$ a function on N, the (x^i) are local coordinates for $\mathcal{S}_0$, and x^0 is the future oriented time coordinate, which means that x^0 increases on future directed curves.

Next, we will cite a result which shows that in our situation spacelike hypersurfaces M can be written as a graph over $\mathcal{S}_0$, cf. [18, Lemma 3.1] for a proof.

1.1. **Proposition.** *Let N be globally hyperbolic, $\mathcal{S}_0 \subset N$ a compact, connected Cauchy hypersurface, and $M \subset N$ a compact, connected spacelike hypersurface of class $C^m, m \ge 1$. Then, $M = \operatorname{graph} u_{|\mathcal{S}_0}$ with $u \in C^m(\mathcal{S}_0)$.*

Thus, let $M = \operatorname{graph} u_{|\mathcal{S}_0}$ be a spacelike hypersurface.

$$M = \{\, (x^0, x) \colon x^0 = u(x),\, x \in \mathcal{S}_0 \,\}, \tag{1.8}$$

then the induced metric (g_{ij}) is given by

$$g_{ij} = e^{2\tilde\psi}\{-u_i u_j + \sigma_{ij}\}, \tag{1.9}$$

while its inverse $(g^{ij}) = (g_{ij})^{-1}$ has the form

$$g^{ij} = e^{-2\tilde\psi}\{\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\}, \tag{1.10}$$

where we used the following notation

$$\begin{aligned} (\sigma^{ij}) &= (\sigma_{ij})^{-1} \\ u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j = 1 - |Du|^2. \end{aligned} \tag{1.11}$$

We emphasize that a hypersurface M is spacelike iff $v^2 > 0$.

The covariant resp. contravariant versions of the past directed normal vector of a graph have the form

$$(\nu_\alpha) = \tilde v e^{\tilde\psi}(1, -u_i), \tag{1.12}$$

resp.

$$(\nu^\alpha) = -\tilde{v}e^{-\tilde{\psi}}(1, u^i), \tag{1.13}$$

where we used the notation $\tilde{v} = v^{-1}$.

Thus, considering the component $\alpha = 0$ in the Gauß formula we obtain

$$e^{-\tilde{\psi}}\tilde{v}h_{ij} = -u_{;ij} - \bar{\Gamma}^0_{00}u_iu_j - \bar{\Gamma}^0_{0j}u_i - \bar{\Gamma}^0_{0i}u_j - \bar{\Gamma}^0_{ij}, \tag{1.14}$$

where all derivatives are taken with respect to g_{ij}. For the second fundamental form $\bar{h}_{ij}$ of the slices $\{x^0 = \text{const.}\}$ we obtain

$$\begin{aligned} -\bar{\Gamma}^0_{ij} &= e^{-\tilde{\psi}}\bar{h}_{ij}, \\ &= -\tfrac{1}{2}\dot{\sigma}_{ij} - \dot{\tilde{\psi}}\sigma_{ij}, \end{aligned} \tag{1.15}$$

Here, differentiation with respect to the time coordinate x^0 is marked with a dot.

For later purpose we define a Riemannian reference metric $\overset{+}{g}_{\alpha\beta}$, which is for a given Lorentzian metric $g_{\alpha\beta}$, cf. (1.7), defined by

$$\overset{+}{g}_{\alpha\beta}\, dx^\alpha dx^\beta = e^{2\tilde{\psi}}\{{dx^0}^2 + \sigma_{ij}dx^i dx^j\}. \tag{1.16}$$

For the corresponding norm of a vectorfield η we will write

$$|||\eta||| = (\overset{+}{g}_{\alpha\beta}\, \eta^\alpha\eta^\beta)^{1/2}, \tag{1.17}$$

and the corresponding induced metric will be denoted by $\overset{+}{g}_{ij}$.

An easy calculation leads to the following result, which will be used later, cf. [8, Lemma 2.7] for a proof.

1.2. **Lemma.** *Let $M = \operatorname{graph} u$ be a spacelike hypersurface in N, $p \in M$, and $\xi \in T_p(M)$ a unit vector, then*

$$|||x^\beta_i\xi^i||| \le c(1 + |u_i\xi^i|) \le c\tilde{v}. \tag{1.18}$$

Finally, we draw a few immediate conclusions from our assumptions on f. The proofs can be found in [11, Section 7.3].

1.3. **Lemma.** *Let $f \in C^2([a, b))$ satisfies the conditions*

$$\lim_{\tau\to b} f(\tau) = -\infty \tag{1.19}$$

and

$$\lim_{\tau\to b} |f'|^2 e^{2\tilde{\gamma}f} = m, \tag{1.20}$$

where $\tilde{\gamma}$ and m are positive, then b is finite.

1.4. **Corollary.** *We may and shall therefore assume that* $b = 0$*, i.e., the time interval* I *is given by* $I = [a, 0)$*.*

1.5. **Remark.** A simple application of L'Hospital's rule yields

$$\lim_{\tau \to 0} \frac{e^{\tilde{\gamma}\varphi}}{\tau} = -\tilde{\gamma}\sqrt{m}. \tag{1.21}$$

1.6. **Lemma.** *There holds*

$$f' e^{\tilde{\gamma}\varphi} + \sqrt{m} \sim c\tau^2, \tag{1.22}$$

where c *is a constant and where the relation*

$$\varphi \sim c\tau^2 \tag{1.23}$$

means

$$\lim_{\tau \to 0} \frac{\varphi(t)}{\tau^2} = c. \tag{1.24}$$

1.7. **Lemma.** *The asymptotic relation*

$$\tilde{\gamma} f' \tau - 1 \sim c\tau^2 \tag{1.25}$$

is valid.

2. Curvature functions

In this section we firstly give a short introduction to general curvature functions. We follow the description in [8]. A more detailed treatment can be found in [11, Section 2.1]. Let $\Gamma \in \mathbb{R}^n$ be an open, convex and symmetric cone which contains the positive cone $\Gamma_+ = \{(\kappa_i) : \kappa_i > 0\}$ and $f \in C^{2,\alpha}(\Gamma)$ a positive, symmetric and strictly monotone function.
Let $\mathcal{S}_\Gamma$ be the space of symmetric matrices the eigenvalues of which belong to Γ. Then, we define define a function F on $\mathcal{S}_\Gamma$ by setting

$$F(h^i_j) := f(\kappa), \tag{2.1}$$

where the κ_i, $1 \le i \le n$, are the eigenvalues of $(h^i_j) \in \Gamma_+$. F is well defined and evidently continuous and even of class $C^{m,\alpha}$ if f has this property, cf. [11, 2.1.10 Theorem].

In our setting F resp. f will be evaluated at the mixed tensor $h^i_j = g^{ik}h_{kj}$ resp. the principal curvatures (κ_i) of a spacelike hypersurface M. This approach presumes that M is admissible, i.e. the eigenvalues of h^i_j belong in every point $x \in M$ to the cone Γ.

It is also possible to consider F depending on the covariant tensors (h_{ij}) and (g_{ij})

$$F(h^i_j) = F(h_{ij}, g_{ij}). \tag{2.2}$$

In the rest of the paper we will always write F independently of the considered argument, but we use different notations for the derivatives of F We define

$$F^{ij} := \frac{\partial F}{\partial h_{ij}}, \tag{2.3}$$

$$F^j_i := \frac{\partial F}{\partial h^i_j}, \tag{2.4}$$

and

$$F_i := \frac{\partial f}{\partial \kappa_i}. \tag{2.5}$$

Then, F^{ij} ia a contravariant tensor of order two, while F^j_i is a mixed tensor. Choosing a special coordinate system such that h_{ij} is diagonal, then, F^{ij} is diagonal either and we have

$$F^{ii} = \frac{\partial f}{\partial \kappa_i}. \tag{2.6}$$

cf. [11, 2.1.9 Lemma]. Differentiating F covariantly will be marked with a comma

$$F_{,k} := F_{ij} h^{ij}{}_{;k}. \tag{2.7}$$

For the second derivative of F we write

$$F^{ij,kl} := \frac{\partial^2 F}{\partial h_{ij}\, \partial h_{kl}}, \tag{2.8}$$

and there holds

$$F^{ij,kl}\eta_{ij}\eta_{kl} = \sum_{i,j} \frac{\partial^2 F}{\partial k_i\, \partial k_j}\eta_{ii}\eta_{jj} + \sum_{i\neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j}(\eta_{ij})^2 \le 0 \quad \forall \eta \in \mathcal{S}, \tag{2.9}$$

where $\mathcal{S}$ is the space of all symmetric matrices. If $F = F(\kappa_i)$ is concave, the second summand is non-positive and hence $F = F(h_{ij})$ is concave either, cf. [11, 2.1.23 Proposition].

We state now a well-known and very useful property of general curvature functions, cf. [11, 2.2.19 Lemma].

2.1. **Lemma.** *Let $F \in C^2(\Gamma) \cap C^0(\bar{\Gamma})$ be a strictly monotone, concave curvature function, positively homogeneous of degree 1, then*

$$\sum_{i=1}^n F_i(\kappa) \ge F(1, \ldots, 1), \tag{2.10}$$

where the convex cone Γ is supposed to contain Γ_+.

Now, we turn to more special curvature functions and define as very important examples the elementary symmetric polynomials.

2.2. **Definition.** For $1 \leq k \leq n$ define

$$H_k(\kappa_i) := \sum_{i_1 < \dots < i_k} \kappa_{i_1} \dots \kappa_{i_k}. \tag{2.11}$$

H_k is said to be the elementary symmetric polynomial of order k. The H_k are defined in $\mathbb{R}^n$, however, they are in general not strictly monotone in all of $\mathbb{R}^n$. Therefore, we define

2.3. **Definition.** For fixed $1 \leq k \leq n$ let Γ_k be the connected component of

$$\{(\kappa_i) \in \mathbb{R}^n : H_k(\kappa_i) > 0\} \tag{2.12}$$

containing the positive cone.

2.4. **Lemma.** *The Γ_k are convex cones and an equivalent characterization of the Γ_k is given by*

$$\Gamma_k = \{(\kappa_i) \in \mathbb{R}^n : H_1(\kappa_i) > 0, H_2(\kappa_i) > 0, \dots, H_k(\kappa_i) > 0\}. \tag{2.13}$$

Proof. A proof can be found in [16, Section 2]. □

2.5. **Lemma.** *The H_k are strictly monotone in Γ_k and the k-th roots*

$$\sigma_k = H_k^{\frac{1}{k}} \tag{2.14}$$

are also concave.

Proof. Confer [16, Lemma 2.4] for the monotony and [20] for the concavity. □

In this paper we consider the scalar curvature function $F = \sigma_2$. As shown above, $F = \sigma_2$, defined in Γ_2 , is strictly monotone, concave and positively homogeneous of degree one. We list now some more properties of Γ_2 and $F = \sigma_2$. We emphasize that σ_2 is not normalized in the following lemma in opposite to the assumptions in this paper.

2.6. **Lemma.** *For* $(\kappa_i) \in \Gamma_2$ *and* $F = \sigma_2$ *there holds*

$$H > 0, \tag{2.15}$$

$$|A|^2 \leq H^2, \tag{2.16}$$

$$F \leq \frac{1}{\sqrt{2}} H, \tag{2.17}$$

$$F_i = \frac{1}{F}(H - \kappa_i) > 0, \tag{2.18}$$

$$F_i \geq \frac{F}{H}, \tag{2.19}$$

and,

$$\sum_{i=1}^n F_i \kappa_i^2 \geq \frac{1}{n} F H. \tag{2.20}$$

Proof. We start with

$$H\kappa_i \leq \frac{1}{2} H^2 + \frac{1}{2}|A|^2, \tag{2.21}$$

which is obviously valid and equivalent to (2.19). The proof of (2.20) uses (2.19) and

$$\frac{1}{n} H^2 \leq |A|^2. \tag{2.22}$$

□

3. The evolution problem

We consider the inverse scalar curvature flow

$$\begin{aligned} \dot{x} &= -\frac{1}{F}\nu, \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $F = \sigma_2$, ν is the past directed normal, $x(t)$ is an embedding and x_0 is an embedding of an admissible initial hypersurface M_0.
This is a parabolic problem, so that short time existence, and hence existence on a maximal time interval $[0, T^*)$, $0 < T^* \leq \infty$, is guaranteed.
In the following three chapters we will prove uniform a priori estimates in C^2, so that uniform estimates in $C^{2,\alpha}$ and therefore long-time existence will follow automatically.
Before we can prove the a priori estimates, we need the evolution equations for some important geometric quantities. The proofs can be found in [11, Section 2.3, Section 2.4].

3.1. **Lemma.** *The metric, the normal vector and the second fundamental form of $M(t)$ satisfy the evolution equations*

$$\dot g_{ij} = -\frac{2}{F} h_{ij}, \tag{3.2}$$

$$\dot\nu = g^{ij}\frac{1}{F^2}F_i x_j, \tag{3.3}$$

$$\dot h_i^j = (-\frac{1}{F})_i^j + \frac{1}{F}h_i^k h_k^j + \frac{1}{F}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj}, \tag{3.4}$$

$$\dot h_{ij} = (-\frac{1}{F})_{ij} - \frac{1}{F}h_i^k h_{kj} + \frac{1}{F}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta. \tag{3.5}$$

3.2. **Lemma.** *The Term $\frac{1}{F}$ satisfies the equation*

$$\left(\frac{1}{F}\right)' - \frac{1}{F^2}F^{ij}\left(\frac{1}{F}\right)_{ij} = -\frac{1}{F^3}F^{ij}h_{ik}h_j^k - \frac{1}{F^3}F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta. \tag{3.6}$$

3.3. **Lemma.** *The Term $\tilde v$ satisfies the evolution equation*

$$\begin{aligned}\dot{\tilde v} - \frac{1}{F^2}F^{ij}\tilde v_{ij} = &- \frac{1}{F^2}F^{ij}h_{ik}h_j^k\tilde v - \frac{2}{F}\eta_{\alpha\beta}\nu^\alpha\nu^\beta\\ &- \frac{2}{F^2}F^{ij}h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} - \frac{1}{F^2}F^{ij}\eta_{\alpha\beta\gamma}x_i^\beta x_j^\gamma \nu^\alpha\\ &- \frac{1}{F^2}F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl},\end{aligned} \tag{3.7}$$

where η is the covariant vector field $(\eta_\alpha) = e^{\tilde\psi}(-1, 0, \dots, 0)$.

3.4. **Lemma.** *The mixed tensor h_i^j satisfies the parabolic equation*

$$\begin{aligned}&\dot h_i^j - \frac{1}{F^2}F^{kl}h_{i;kl}^j\\ &= -\frac{1}{F^2}F^{kl}h_{rk}h_l^r h_i^j + \frac{2}{F}h_i^k h_k^j + \frac{1}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j\\ &\quad + \frac{2}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta}x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h_l^m g^{rj} + \frac{1}{F^2}F^{kl,rs}h_{kl;i}h_{rs;}{}^j\\ &\quad - \frac{1}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} - \frac{1}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^{mj}\\ &\quad - \frac{1}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_i^j + \frac{2}{F}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} - \frac{2}{F^3}F_i F^j\\ &\quad - \frac{1}{F^2}F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}\}.\end{aligned} \tag{3.8}$$

3.5. **Lemma.** *Let $M(t) = \operatorname{graph} u(t)$ be the flow hypersurfaces, then we have*

(3.9)
$$\dot u - \frac{1}{F^2}F^{ij}u_{ij} = \frac{2}{F}e^{-\tilde\psi}\tilde v + \frac{1}{F^2}\bar\Gamma^0_{00}F^{ij}u_iu_j + \frac{2}{F^2}F^{ij}\bar\Gamma^0_{0i}u_j + \frac{1}{F^2}F^{ij}\bar\Gamma^0_{ij}.$$

4. C^0-estimates

In this chapter we show on the one hand that the flow stays in a precompact region of N for finite time, but on the other hand that the flow leaves any precompact region of N if the flow exists for all time. The following results are from [19], where ideas from [11, Section 6] were used.

4.1. **Lemma.** *Let $M_\tau = \{x^0 = \tau\}$ denote the coordinate slices. Then, there exists τ_0 such that M_τ is convex for all $\tau \ge \tau_0$.*

Proof. The second fundamental form $\bar h_{ij}$ of the M_τ is given by

$$\bar h^i_j = -e^{-\tilde\psi}(\tfrac12\sigma^{ik}\dot\sigma_{kj} + \dot{\tilde\psi}\delta^i_j), \tag{4.1}$$

cf. (1.15).
From the properties (0.3), (0.4), (0.5), and the uniform convergence of the metric, cf. (0.6) et seq., we deduce that

$$\lim_{\tau\to0} -e^{-\tilde\psi} = -\infty \quad\wedge\quad \lim_{\tau\to0}\frac12\sigma^{ik}\dot\sigma_{kj} = 0 \quad\wedge\quad \lim_{\tau\to0}\dot{\tilde\psi}\delta^i_j = -\infty, \tag{4.2}$$

where we also used for the last convergence that $f' \longrightarrow -\infty$ for $t \longrightarrow 0$, in view of (0.8) and (0.9). Using these relations we get the claim.

□

In order to prove the main result of this chapter we have to show the existence of a special time function.

4.2. **Lemma.** *There exists a time function $\tilde x^0 = \tilde x^0(x^0)$ and τ_1 such that for all $\tau \ge \tau_1$ holds*

$$e^{\tilde{\tilde\psi}}F_{|_{M_\tau}} \ge 1, \tag{4.3}$$

where $e^{\tilde{\tilde\psi}}$ is the conformal factor of the metric in N with respect to coordinates $(\tilde x^0, x^i)$, i.e.

$$d\breve s = e^{2\tilde{\tilde\psi}}\{-(d\tilde x^0)^2 + \tilde\sigma_{ij}(\tilde x^0, x)dx^idx^j\}. \tag{4.4}$$

The time function $\tilde x^0$ is strictly increasing, and we have

$$\tilde x^0(\{\tau_1 \le x^0 < 0\}) = [0,\infty). \tag{4.5}$$

Proof. Using the arguments of the previous lemma we conclude that there exists τ_1 such that

$$e^{\tilde{\psi}}F|_{M_\tau} = F(e^{\tilde{\psi}}\bar{h}^i_j) = F(-\frac{1}{2}\sigma^{ik}\dot{\sigma}_{kj} - \dot{\tilde{\psi}}\delta^i_j) \geq -\delta_0 f^{'} \quad \forall \tau \geq \tau_1, \tag{4.6}$$

where $\delta_0 > 0$ is a constant. With these relations, we infer that there exists τ_1 such that M_τ is convex for all $\tau \geq \tau_1$. Choosing τ_1 if necessary large, we conclude

$$e^{\tilde{\psi}}F_{|_{M_\tau}} = e^{\tilde{\psi}}F(\bar{h}^i_j) = F(-\frac{1}{2}\sigma^{ik}\dot{\sigma}_{kj} - \dot{\tilde{\psi}}\delta^i_j) \geq -\delta_0 f^{'} \quad \forall \tau \geq \tau_1, \tag{4.7}$$

where $\delta_0 > 0$ is a constant. Now we define a function φ and a new time function $\tilde{x}^0$ by

$$\varphi(\tau) := -\delta_0 f^{'} > 0, \tag{4.8}$$

and

$$\tilde{x}^0(\tau) := \int_{\tau_0}^{\tau} \varphi(s) ds. \tag{4.9}$$

Using (0.5) we receive

$$\tilde{x}^0(\tau) = -\delta_0(f(\tau) - f(\tau_0)) \quad \longrightarrow \quad \infty, \qquad \tau \longrightarrow 0. \tag{4.10}$$

The relation of the conformal factors is given by

$$e^{2\tilde{\tilde{\psi}}} = e^{2\tilde{\psi}} \frac{\partial x^0}{\partial \tilde{x}^0} \frac{\partial x^0}{\partial \tilde{x}^0} = e^{2\tilde{\psi}} \varphi^{-2}. \tag{4.11}$$

Thus, we get

$$e^{\tilde{\tilde{\psi}}}F_{|_{M_\tau}} = e^{\tilde{\psi}}F_{|_{M_\tau}}\varphi^{-1} \geq 1, \tag{4.12}$$

due to (4.8). □

4.3. **Lemma.**

(i) *For any finite* $0 < T \leq T^*$ *the flow stays in a precompact set* Ω_T *for* $0 \leq t < T$.

(ii) *The flow runs into the future singularity if it exists for all time, i.e. with the respect to the above chosen coordinates* $(\tilde{x}^0, x^i)$ *we have*

$$\lim_{t\to\infty} \inf_{S_0} u(t, \cdot) = \infty. \tag{4.13}$$

Proof. We choose coordinates $(\tilde{x}^0, x^i)$, where $(\tilde{x}^0)$ is the time function the existence of which was shown in the previous lemma. Now, let $M(t) = \operatorname{graph} u(t, \cdot)$ be the flow hypersurfaces and define

$$\varphi(t) = \sup_{S_0} u(t, \cdot). \tag{4.14}$$

It is a well-known fact that $\varphi(t)$ is Lipschitz continuous and for a.e. $0 \le t < T$ differentiable by Rademacher's theorem, cf. [11, 6.3.2 Lemma],

$$\dot\varphi(t) = \frac{\partial}{\partial t} u(t, x_t), \tag{4.15}$$

where x_t is the point where the supremum is attained, i.e.

$$\sup_{S_0} u(t,\cdot) = u(t, x_t). \tag{4.16}$$

Applying the maximum principle we deduce that in x_t holds

$$\bar h_{ij} \le h_{ij}, \tag{4.17}$$

and thus we have in view of the monotony of F in x_t

$$F_{|_{M_\tau}} \le F_{|_M}. \tag{4.18}$$

Now, we look at the component $\alpha = 0$ of the flow equation, cf. (3.1),

$$\dot u = \frac{\tilde v}{e^{\tilde\psi} F_{|_M}}, \tag{4.19}$$

here, $\dot u$ is a a total derivative, i.e.

$$\dot u = \frac{\partial u}{\partial t} + u_i \dot x^i, \tag{4.20}$$

and we get for the partial derivative the relation

$$\frac{\partial u}{\partial t} = \frac{v}{e^{\tilde\psi} F_{|_M}} \tag{4.21}$$

Inserting (4.12) and (4.18) we get in x_t

$$\frac{\partial u}{\partial t} \le 1, \tag{4.22}$$

and using (4.15) we deduce further

$$\varphi \le \varphi(0) + t \qquad \forall 0 \le t \le T^*, \tag{4.23}$$

and hence the claim of (i). The claim of (ii) can be proven similarly by defining φ by

$$\varphi(t) = \inf_{S_0} u(t,\cdot). \tag{4.24}$$

□

5. C^1-ESTIMATES

In this chapter we want to prove the C^1-estimates for the (ISCF). With the exception of some smaller modifications we are able to apply the arguments of the corresponding result for the (SCF) in [4]. But, we want to emphasize that the main achievement for the C^1-estimates was done by Gerhardt in [8, Proposition 4.8]. For the convenience of the reader we will present the modified proof of [8, Proposition 4.8] without mentioning modifications explicitly. We start with four lemmas, cf. [8, Section 4] for a proof.

5.1. **Lemma.** *The composite function*

$$\varphi = e^{\mu e^{\lambda u}}, \tag{5.1}$$

where μ, λ are constants, satisfies the equation

$$\begin{aligned} \dot\varphi - \frac{1}{F^2}F^{ij}\varphi_{ij} =& \frac{2}{F}e^{-\tilde\psi}\tilde v \mu\lambda e^{\lambda u}\varphi + \frac{1}{F^2}F^{ij}u_iu_j\,\bar\Gamma^0_{00}\,\mu\lambda\, e^{\lambda u}\varphi \\ &+ 2\frac{1}{F^2}F^{ij}u_i\bar\Gamma^0_{0j}\,\mu\lambda e^{\lambda u}\varphi + \frac{1}{F^2}F^{ij}\bar\Gamma^0_{ij}\,\mu\lambda e^{\lambda u}\varphi \\ &- [1+\mu e^{\lambda u}]\frac{1}{F^2}F^{ij}u_iu_j\,\mu\lambda^2 e^{\lambda u}\varphi. \end{aligned} \tag{5.2}$$

5.2. **Lemma.** *Let $\Omega \subset N$ be precompact. As long as the flow stays in Ω there exists a constant $c = c(\Omega)$ such that for any positive function $0 < \epsilon = \epsilon(x)$ on $\mathcal{S}_0$ and any hypersurface $M(t)$ of the flow we have*

$$|||\nu||| \le c\tilde v, \tag{5.3}$$

$$g^{ij} \le c\tilde v^2 \sigma^{ij}, \tag{5.4}$$

$$F^{ij} \le F^{kl}g_{kl}g^{ij}, \tag{5.5}$$

$$|F^{ij}h^k_j x^\alpha_i x^\beta_k \eta_{\alpha\beta}| \le \frac{\epsilon}{2}F^{ij}h^k_i h_{kj}\tilde v + \frac{c}{2\epsilon}F^{ij}g_{ij}\tilde v^3, \tag{5.6}$$

$$|F^{ij}\eta_{\alpha\beta\gamma}x^\beta_i x^\gamma_j \nu^\alpha| \le c\tilde v^3 F^{ij}g_{ij}, \tag{5.7}$$

$$|F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_k x^\delta_j \eta_\epsilon x^\epsilon_l g^{kl}| \le c\tilde v^3 F^{ij}g_{ij}. \tag{5.8}$$

5.3. **Lemma.** *Let $\Omega \subset N$ be precompact and $M \subset \Omega$ be a graph over $\mathcal{S}_0$, $M = \operatorname{graph} u$ and $\epsilon = \epsilon(x)$ a function defined on $\mathcal{S}_0$, $0 < \epsilon < \frac{1}{2}$. Let φ be defined through*

$$\varphi = e^{\mu e^{\lambda u}}, \tag{5.9}$$

where $0 < \mu$ and $\lambda < 0$. Then there exists $c = c(\Omega)$ such that

$$\begin{aligned} 2|F^{ij}\tilde{v}_i\varphi_j| \le{}& cF^{ij}g_{ij}\tilde{v}^3|\lambda|\mu e^{\lambda u}\varphi + (1-2\epsilon)F^{ij}h_i^k h_{kj}\tilde{v}\varphi \\ &+ \frac{1}{1-2\epsilon}F^{ij}u_i u_j \mu^2\lambda^2 e^{2\lambda u}\tilde{v}\varphi. \end{aligned} \tag{5.10}$$

5.4. **Lemma.** *Let $\Omega \subset N$ be precompact. As long as the flow stays in Ω there exists a constant $c = c(\Omega)$ such that for any positive function $0 < \epsilon = \epsilon(x) < 1$ on $\mathcal{S}_0$ and any hypersurface $M(t)$ of the flow the term $\tilde{v}$ satisfies an evolution inequality of the form*

$$\dot{\tilde{v}} - \frac{1}{F^2}F^{ij}\tilde{v}_{ij} \le -(1-\epsilon)\frac{1}{F^2}F^{ij}h_i^k h_{kj}\tilde{v} + \frac{c}{\epsilon}\frac{1}{F^2}F^{ij}g_{ij}\tilde{v}^3. \tag{5.11}$$

Proof. To estimate the last three terms in the evolution equation of $\tilde{v}$, cf. (3.7), we apply (5.6), (5.7) and (5.8). The second term in (3.7) which is associated with F^{-1} is estimated by

$$\begin{aligned} -\frac{2}{F}\eta_{\alpha\beta}\nu^\alpha\nu^\beta &= -\frac{2}{F^2}F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &\le \frac{\epsilon}{2}\frac{1}{F^2}F^{ij}h_i^k h_{kj}\tilde{v} + 8\epsilon^{-1}\frac{1}{F^2}F^{ij}g_{ij}\tilde{v}^{-1}(\eta_{\alpha\beta}\nu^\alpha\nu^\beta)^2 \\ &\le \frac{\epsilon}{2}\frac{1}{F^2}F^{ij}h_i^k h_{kj}\tilde{v} + \tilde{c}\epsilon^{-1}\frac{1}{F^2}F^{ij}g_{ij}\tilde{v}^3. \end{aligned} \tag{5.12}$$

□

Now, we are able to prove the C^1-estimates.

5.5. **Proposition.** *Let $\Omega \subset N$ be precompact. Then, as long as the flow stays in Ω the term $\tilde{v}$ remains uniformly bounded*

$$\tilde{v} \le c = c(\Omega, \sup_{M_0}\tilde{v}). \tag{5.13}$$

Proof. We show that the function

$$w = \tilde{v}\varphi, \tag{5.14}$$

φ as in (5.1), is uniformly bounded, if we choose

$$0 < \mu < 1 \quad \text{and} \quad \lambda << -1, \tag{5.15}$$

appropriately, and assume furthermore, without loss of generality, that $u \le -1$, for otherwise replace u by $(u-c)$, c large, in the definition of φ. With the help of Lemma 5.1, Lemma 5.3 and Lemma 5.4 we derive from the relation

$$\dot{w} - \frac{1}{F^2}F^{ij}w_{ij} = [\dot{\tilde{v}} - \frac{1}{F^2}F^{ij}\tilde{v}_{ij}]\varphi + [\dot{\varphi} - \frac{1}{F^2}F^{ij}\varphi_{ij}]\tilde{v} - \frac{2}{F^2}F^{ij}\tilde{v}_i\varphi_j \tag{5.16}$$

the parabolic inequality

$$\begin{aligned} \dot{w} - \frac{1}{F^2}F^{ij}w_{ij} \le &-\epsilon\frac{1}{F^2}F^{ij}h^k_ih_{kj}\tilde{v}\varphi + c[\epsilon^{-1} + |\lambda|\mu e^{\lambda u}]\frac{1}{F^2}F^{ij}g_{ij}\tilde{v}^3\varphi \\ &+ [\frac{1}{1-2\epsilon} - 1]\frac{1}{F^2}F^{ij}u_iu_j\mu^2\lambda^2e^{2\lambda u}\tilde{v}\varphi \\ &- \frac{1}{F^2}F^{ij}u_iu_j\mu\lambda^2e^{\lambda u}\tilde{v}\varphi, \end{aligned} \tag{5.17}$$

where we have chosen the same function $\epsilon = \epsilon(x)$ in Lemma 5.3 resp. Lemma 5.4. We claim that w is uniformly bounded provided μ and λ are chosen appropriately. We shall use the maximum principle, therefore let $0 < T < T^*$ and $x_0 = x(t_0, \xi_0)$ be such that

$$\sup_{[0,T]}\sup_{M(t)} w = w(t_0, \xi_0). \tag{5.18}$$

To exploit the good term

$$-\epsilon\frac{1}{F^2}F^{ij}h^k_ih_{kj}\tilde{v}\varphi, \tag{5.19}$$

we use the fact that $Dw(x_0) = 0$, or, equivalently

$$\begin{aligned} -\tilde{v}_i &= \mu\lambda e^{\lambda u}\tilde{v}u_i \\ &= e^{\psi}h^k_iu_k - \eta_{\alpha\beta}\nu^\alpha x^\beta_i, \end{aligned} \tag{5.20}$$

Next, we choose a coordinate system (ξ^i) such that in the critical point

$$g_{ij} = \delta_{ij} \qquad \text{and} \qquad h^k_i = \kappa_i\delta^k_i, \tag{5.21}$$

and the labelling of the principal curvatures corresponds to

$$\kappa_1 \le \kappa_2 \le \cdots \le \kappa_n. \tag{5.22}$$

Then, we deduce from (5.20)

$$e^{\psi}\kappa_iu_i = \mu\lambda e^{\lambda u}\tilde{v}u_i + \eta_{\alpha\beta}\nu^\alpha x^\beta_i. \tag{5.23}$$

Assume that $\tilde{v}(x_0) \ge 2$, and let $i = i_0$ be an index such that

$$|u_{i_0}|^2 \ge \frac{1}{n}\|Du\|^2. \tag{5.24}$$

Setting $(e^i) = \frac{\partial}{\partial \xi^{i_0}}$ and assuming without loss of generality that $0 < u_i e^i$ in x_0, we infer from Lemma 1.2

$$\begin{aligned} e^{\psi}\kappa_{i_0} u_i e^i &= \mu\lambda e^{\lambda u}\tilde{v} u_i e^i + \eta_{\alpha\beta}\nu^\alpha x_i^\beta e^i \\ &\le \mu\lambda e^{\lambda u}\tilde{v} u_i e^i + c\tilde{v}^2, \end{aligned} \tag{5.25}$$

and we deduce further in view of (1.10), (1.11) and (5.24) that

$$\kappa_{i_0} \le [\mu\lambda e^{\lambda u} + c]\tilde{v} e^{-\psi} \le \frac{1}{2}\mu\lambda e^{\lambda u}\tilde{v} e^{-\psi}, \tag{5.26}$$

if $|\lambda|$ is sufficiently large, i.e. κ_{i_0} is negative and of the same order as $\tilde{v}$. Next, let us estimate the crucial term in (5.19). Using the particular coordinate system (5.21), as well as the inequalities (5.22), together with the fact that κ_{i_0} is negative, we conclude

$$-F^{ij} h_i^k h_{kj} \le -\sum_{i=1}^{i_0} F_i^i \kappa_i^2 \le -\sum_{i=1}^{i_0} F_i^i \kappa_{i_0}^2. \tag{5.27}$$

F is concave, and therefore, we have in view of (5.22)

$$F_1^1 \ge F_2^2 \ge \cdots \ge F_n^n, \tag{5.28}$$

cf. [3, Lemma 2]. Hence, we conclude

$$-\sum_{i=1}^{i_0} F_i^i \le -F_1^1 \le -\frac{1}{n}\sum_{i=1}^{n} F_i^i. \tag{5.29}$$

Using (5.26), (5.27) and (5.29) we deduce further

$$-F^{ij} h_i^k h_{kj} \le -cF^{ij} g_{ij}\mu^2\lambda^2 e^{2\lambda u}\tilde{v}^2. \tag{5.30}$$

Inserting this estimate in (5.17), with $\epsilon = e^{-\lambda u}$, we obtain

$$\begin{aligned} 0 \le &-cF^{ij} g_{ij}\mu^2\lambda^2 e^{\lambda u}\tilde{v}^3\varphi + cF^{ij} g_{ij}\mu|\lambda| e^{\lambda u}\tilde{v}^3\varphi \\ &+ \frac{2}{1-2\epsilon} F^{ij} u_i u_j \mu^2\lambda^2 e^{\lambda u}\tilde{v}\varphi - F^{ij} u_i u_j \mu\lambda^2 e^{\lambda u}\tilde{v}\varphi \end{aligned} \tag{5.31}$$

where $|\lambda|$ is chosen so large that

$$e^{-\lambda u} \le \frac{1}{4}. \tag{5.32}$$

Choosing $\mu = \frac{1}{4}$ and $|\lambda|$ sufficient large, we see that the right-hand side of the preceding inequality is negative, contradicting the maximum principle, i.e. the maximum of w cannot occur at a point where $\tilde{v} \ge 2$. Thus, the desired uniform estimate for w and hence $\tilde{v}$ is proved. □

5.6. **Remark.** Notice that the proof of the preceding C^1-estimate is valid for any curvature function F that is monotone, concave and homogeneous of degree 1.

Let us close this section with an interesting observation that is an immediate consequence of the preceding proof, we have especially (5.27) and (5.29) in mind.

5.7. **Lemma.** *Suppose $F = \sigma_2$ is evaluated at a point (κ_i) and assume that κ_{i_0} is a component that is either negative or the smallest component of that particular n-tuple, then*

$$\sum_{i=1}^{n} F_i \kappa_i^2 \geq \frac{1}{n} \sum_{i=1}^{n} F_i \kappa_{i_0}^2. \tag{5.33}$$

6. C^2-estimates

We want to prove that the principal curvatures of the flow hypersurfaces are uniformly bounded. Firstly, we have to show that F is bounded from above. Therefore, we need the following Lemma.

6.1. **Lemma.** *The term $\log F$ satisfies the evolution equation*

$$\begin{aligned} (\log F)' - \frac{1}{F^2} F^{ij} (\log F)_{ij} =& \frac{1}{F^2} F^{ij} h_{ik} h_j^k + \frac{1}{F^2} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\ & - \frac{1}{F^4} F^{ij} F_i F_j. \end{aligned} \tag{6.1}$$

6.2. **Lemma.** *Let $\Omega \subset \mathbb{N}$ be precompact and assume that the flow stays in Ω for $0 \leq t < T^*$, then there exists a constant $c(\Omega)$ such that*

$$F \leq c(\Omega). \tag{6.2}$$

Proof. We define

$$\varphi = \log F \tag{6.3}$$

and set

$$w = \varphi + \lambda \tilde{v}. \tag{6.4}$$

We claim that w is bounded, if λ is chosen appropriately. We shall use the maximum principle, therefore let $0 < T < T^*$, and $x_0 = x(t_0, \xi_0)$ be a point in $M(t_0)$ such that

$$w(t_0, x_0) = \sup_{[0,T]} \sup_{M(t)} w. \tag{6.5}$$

Applying the maximum principle we receive

$$0 \leq (1 - \frac{\lambda}{2}) F^{ij} h_{ik} h_j^k + c(1 + \lambda) F^{ij} g_{ij}, \tag{6.6}$$

in view of Lemma 5.4. Choosing λ larger than 4, we obtain with (2.18) and (2.20)

$$0 \le -\frac{1}{n} F H + c F^{-1} H. \tag{6.7}$$

Thus, we have an a priori estimate for F. □

To prove the following proposition we have to assume, that a strictly convex function χ exists.

6.3. **Remark.** Let χ be the strictly convex function. Its evolution equation is

$$\begin{aligned} \dot{\chi} - \frac{1}{F^2} F^{ij} \chi_{ij} &= -\frac{2}{F^2} F \chi_\alpha \nu^\alpha - \frac{1}{F^2} F^{ij} \chi_{\alpha\beta} x_i^\alpha x_j^\beta \\ &\le -\frac{2}{F} \chi_\alpha \nu^\alpha - c_0 \frac{1}{F^2} F^{ij} g_{ij}, \end{aligned} \tag{6.8}$$

where $c_0 > 0$ is independent of t.

If we are close enough to the future singularity, the existence of a strictly convex function is automatically satisfied.

6.4. **Remark.** Due to [11, Lemma 1.8.3] and the convexity of the flow hypersurfaces, cf. (4.1) and the following lines, the existence of a strictly convex function $\chi \in C^2(\bar{\Omega})$ for a relative compact subset Ω of N is guaranteed, if Ω lies sufficiently far in the future of N, i.e. $|\inf_\Omega x^0| < \epsilon$ with $\epsilon > 0$ chosen sufficiently small.

6.5. **Proposition.** *Let $\Omega \subset \mathbb{N}$ be precompact and assume that the flow stays in Ω for $0 \le t < T^*$, then there exists a constant $c(\Omega)$ such that*

$$\kappa_i \le c(\Omega), \quad 1 \le i \le n. \tag{6.9}$$

Proof. Let ζ and w be respectively defined by

$$\zeta = \sup\{\, h_{ij}\eta^i\eta^j : \|\eta\| = 1 \,\}, \tag{6.10}$$

$$w = \log \zeta + \lambda \chi, \tag{6.11}$$

where $\lambda > 0$ is supposed to be large. We claim that w is bounded, if λ is chosen sufficiently large.
Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \le T$, be a point in $M(t_0)$ such that

$$\sup_{M_0} w < \sup\{\, \sup_{M(t)} w : 0 < t \le T \,\} = w(x_0). \tag{6.12}$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \zeta = h^n_n. \tag{6.13}$$

Let $\tilde\eta = (\tilde\eta^i)$ be the contravariant vector field defined by

$$\tilde\eta = (0, \dots, 0, 1), \tag{6.14}$$

and set

$$\tilde\zeta = \frac{h_{ij}\tilde\eta^i\tilde\eta^j}{g_{ij}\tilde\eta^i\tilde\eta^j}. \tag{6.15}$$

$\tilde\zeta$ is well defined in neighbourhood of (t_0, ξ_0).
Now, define $\tilde w$ by replacing ζ by $\tilde\zeta$ in (6.11); then, $\tilde w$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$\dot{\tilde\zeta} = \dot h^n_n, \tag{6.16}$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde\zeta$ satisfies the same differential equation (3.8) as h^n_n. For the sake of greater clarity, let us therefore treat h^n_n like a scalar and pretend that w is defined by

$$w = \log h^n_n + \lambda\chi. \tag{6.17}$$

We assume that the section curvatures are labelled according to (5.22).
At (t_0, ξ_0) we have $\dot w \ge 0$, and, in view of the maximum principle, we deduce from (2.9), (3.8), (5.28) and (6.8)

$$\begin{aligned} 0 \le &-\tfrac12 F^{ij}h_{ki}h^k_j + 2F\kappa_n + cF^{ij}g_{ij} + (1+\lambda)cF - \lambda c_0 F^{ij}g_{ij} \\ &+ F^{ij}(\log h^n_n)_i(\log h^n_n)_j + \frac{2}{\kappa_n - \kappa_1}\sum_{i=1}^n (F_n - F_i)(h_{ni;}{}^n)^2(h^n_n)^{-1}, \end{aligned} \tag{6.18}$$

where we have estimated bounded terms by a constant c, and assumed that h^n_n and λ are larger than 1. We distinguish two cases

Case 1. Suppose that

$$|\kappa_1| \ge \epsilon_1\kappa_n, \tag{6.19}$$

where $\epsilon_1 > 0$ is small. Then, we infer from Lemma 5.7

$$F^{ij}h_{ki}h^k_j \ge \tfrac1n F^{ij}g_{ij}\epsilon_1^2\kappa_n^2, \tag{6.20}$$

and

$$F^{ij}g_{ij} \ge F(1, \dots, 1), \tag{6.21}$$

for a proof see [11, Lemma 2.2.19].
Since $Dw = 0$,

$$D\log h^n_n = -\lambda D\chi, \tag{6.22}$$

hence

$$F^{ij}(\log h^n_n)_i(\log h^n_n)_j \le \lambda^2 F^{ij}\chi_i\chi_j. \tag{6.23}$$

Hence, we conclude that κ_n is a priori bounded in this case.

Case 2. Suppose that

$$\kappa_1 \geq -\epsilon_1 \kappa_n, \tag{6.24}$$

then the last term in inequality (6.18) is estimated from above by

$$\begin{aligned}
&\frac{2}{1+\epsilon_1}\sum_{i=1}^n (F_n - F_i)(h_{ni;}{}^n)^2 (h_n^n)^{-2} \\
&\leq \frac{2}{1+2\epsilon_1}\sum_{i=1}^n (F_n - F_i)(h_{nn;}{}^i)^2 (h_n^n)^{-2} \\
&\quad + c(\epsilon_1)\sum_{i=1}^n (F_i - F_n)\kappa_n^{-2},
\end{aligned} \tag{6.25}$$

where we used the Codazzi equation. The last sum can be easily balanced. The terms in (6.18) containing the derivative of h_n^n can therefore be estimated from above by

$$\begin{aligned}
&-\frac{1-2\epsilon_1}{1+2\epsilon_1}\sum_{i=1}^n F_i (h_{nn;}{}^i)^2 (h_n^n)^{-2} \\
&+\frac{2}{1+2\epsilon_1} F_n \sum_{i=1}^n (h_{nn;}{}^i)^2 (h_n^n)^{-2} \\
&\leq 2F_n \sum_{i=1}^n (h_{nn;}{}^i)^2 (h_n^n)^{-2} \\
&= 2\lambda^2 F_n \|D\chi\|^2.
\end{aligned} \tag{6.26}$$

Hence, we infer

$$\begin{aligned}
0 \leq &-\tfrac{1}{2} F_n \kappa_n^2 + 2F\kappa_n + cF^{ij} g_{ij} \\
&+ \lambda^2 c F_n + (1+\lambda) cF - \lambda c_0 F^{ij} g_{ij}.
\end{aligned} \tag{6.27}$$

From (2.18) and Lemma 6.2 we deduce

$$F^{ij} g_{ij} \geq c\kappa_n, \tag{6.28}$$

with the boundness of F , we obtain an a priori estimate

$$\kappa_n \leq \text{const}, \tag{6.29}$$

if λ is chosen large enough. Notice that ϵ_1 is only subject to the requirement $0 < \epsilon_1 < \frac{1}{2}$. □

With the help of (2.15) we conclude further that there exists a positive constant $c(\Omega)$ such that

$$|\kappa_i| \leq c(\Omega), \quad 1 \leq i \leq n. \tag{6.30}$$

In the next chapter we will show independently of the following results, that F is bounded from below

$$F \geq \inf_{M_0} F > 0, \tag{6.31}$$

cf. Corollary 7.9, as long as the flow exists. Combining (6.31) with (6.2) we deduce that there are positive constants $c_1 = c_1(\Omega)$ and $c_2 = c_2(\Omega)$ such that

$$0 < c_1 \le F \le c_2 \tag{6.32}$$

as long as the flow stays in a relative compact subset Ω of N. We now look at the scalar version of the flow as in (5.19)

$$\frac{\partial u}{\partial t} = e^{-\tilde{\psi}} v \frac{1}{F}, \tag{6.33}$$

defined in the cylinder

$$Q_{T^*} = [0, T^*) \times \mathcal{S}_0 \tag{6.34}$$

with initial value $u(0) \in C^\infty(\mathcal{S}_0)$. We deduced hat for $T^* < \infty$ the flow stays in a compact subset Ω of N and proved uniform C^2-estimates for u. In view of (6.30) and (6.31) we know that the principal curvatures of the flow stay in a compact subset of Γ_2. Hence, the differential operator on the right-hand side of (6.33) is uniformly elliptic in u independent of t. Thus, we can apply the $C^{2,\alpha}$-estimates of Krylov and Safonov and conclude that a maximal T^* cannot be finite.

7. Results in the conformal space

Proving the convergence results for the ISCF, we shall for technical reasons consider the flow hypersurfaces to be embedded in $(N, \bar{g})$, where $\bar{g}$ stands for the conformal metric

$$d\bar{s}^2 = -(dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j. \tag{7.1}$$

We will write h_{ij}, g_{ij}, ν, etc. for geometric quantities of hypersurfaces in $(N, \bar{g})$ and $\breve{h}_{ij}, \breve{g}_{ij}, \breve{\nu}$, etc. for geometric quantities of hypersurfaces in $(N, \breve{g})$, i.e. standard notation now apply to the case when N is equipped with the the metric in (7.1). We have

$$\breve{g}_{\alpha\beta} = e^{2\tilde{\psi}} \bar{g}_{\alpha\beta} \tag{7.2}$$

and the second fundamental forms $\breve{h}_i^j$ and h_i^j are related by

$$e^{\tilde{\psi}} \breve{h}_i^j = h_i^j + \tilde{\psi}_\alpha \nu^\alpha \delta_i^j, \tag{7.3}$$

cf. [11, 1.1.11 Proposition]. In accordance with the introduced notation we define $\breve{F}$ by

$$\breve{F} = F(\breve{h}_i^j), \tag{7.4}$$

and a new function $\check{F}$ by setting

$$\check{F}(h_i^j) = e^{\check{\psi}} F(\breve{h}_i^j) = F(h_i^j - \tilde{v} f' \delta_i^j + \psi_\alpha \nu^\alpha) = F(\check{h}_i^j), \tag{7.5}$$

where $\check{h}_i^j$ stands for

$$\check{h}_i^j = e^{\tilde{\psi}} \breve{h}_i^j = h_i^j - \tilde{v} f' \delta_i^j + \psi_\alpha \nu^\alpha \delta_i^j. \tag{7.6}$$

However, we will from now on write F instead of $\check{F}$, so that the evolution equation can then be written as

$$\dot{x} = -\frac{1}{F}\nu, \tag{7.7}$$

since

$$\check{\nu} = e^{-\tilde{\psi}}\nu. \tag{7.8}$$

The flow exists for all time and is smooth, cf. our results in the last chapters. We want to emphasize that the argument of F is now $\check{h}_i^j$. These notations introduced above will be used until the end of this paper.

For further reference we state now some evolution equations, cf. [11, Section 2.3, Section 2.4] for the proofs.

7.1. **Lemma.** *Consider the flow in (7.7), then the metric, the normal vector and the second fundamental form of $M(t)$ satisfy the evolution equations*

$$\dot{g}_{ij} = -\frac{2}{F}h_{ij}, \tag{7.9}$$

$$\dot{\nu} = g^{ij}\frac{1}{F^2}F_i x_j, \tag{7.10}$$

$$\dot{h}_i^j = (-\frac{1}{F})_i^j + \frac{1}{F}h_i^k h_k^j + \frac{1}{F}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj}, \tag{7.11}$$

$$\dot{h}_{ij} = (-\frac{1}{F})_{ij} - \frac{1}{F}h_i^k h_{kj} + \frac{1}{F}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta. \tag{7.12}$$

Looking at the component $\alpha = 0$ in (7.7), we infer that the total and partial derivate of u satisfy the equations

$$\dot{u} = \frac{\tilde{v}}{F}, \tag{7.13}$$

$$\frac{\partial u}{\partial t} = \frac{v}{F}. \tag{7.14}$$

7.2. **Lemma.** *The evolution equation of u has the form*

$$\dot{u} - \frac{1}{F^2}F^{ij}u_{ij} = \frac{2}{F}\tilde{v} + \frac{1}{F^2}\tilde{v}^2 f' F^{ij}g_{ij} - \frac{1}{F^2}\tilde{v}\psi_\alpha \nu^\alpha F^{ij}g_{ij} - \frac{1}{F^2}F^{ij}\bar{h}_{ij}. \tag{7.15}$$

Proof. From the component $\alpha = 0$ of the Gauß formula we obtain

$$u_{ij} = -\tilde{v}h_{ij} + \bar{h}_{ij} \tag{7.16}$$

and from (7.6) we deduce

$$-F^{ij}h_{ij} = -F - \tilde{v}f'F^{ij}g_{ij} + \psi_\alpha\nu^\alpha F^{ij}g_{ij}, \tag{7.17}$$

where we used the homogeneity of F. Combining these two identities and the identity in (7.13) lead to the claim. □

7.3. **Lemma.** *The Term $\tilde{v}$ satisfy the evolution equation*

$$\begin{aligned}\dot{\tilde{v}} - \frac{1}{F^2}F^{ij}\tilde{v}_{ij} = &- \frac{1}{F^2}F^{ij}h_{ik}h_j^k\tilde{v} + \frac{1}{F^2}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ &- \frac{1}{F^2}F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{F}\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &- \frac{1}{F^2}(F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma + 2F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k) \\ &- \frac{1}{F^2}(-\tilde{v}f^{''}\|Du\|^2F^{ij}g_{ij} - \tilde{v}_k u^k f^{'}F^{ij}g_{ij}) \\ &+ \psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k F^{ij}g_{ij} + \psi_\alpha x_l^\alpha h_k^l u^k F^{ij}g_{ij},\end{aligned} \tag{7.18}$$

where $\eta = (\eta_\alpha) = (-1, 0, ..., 0)$ is a covariant unit vector field.

Proof. There holds $\tilde{v} = \eta_\alpha\nu^\alpha$. Differentiating $\tilde{v}$ we deduce

$$\tilde{v}_i = \eta_{\alpha\beta}x_i^\beta\nu^\alpha + \eta_\alpha\nu_i^\alpha, \tag{7.19}$$

$$\tilde{v}_{ij} = \eta_{\alpha\beta\gamma}x_i^\beta x_j^\gamma\nu^\alpha + \eta_{\alpha\beta}\nu_j^\alpha x_i^\beta + \eta_{\alpha\beta}\nu^\alpha\nu^\beta h_{ij} + \eta_\alpha\nu_{ij}^\alpha, \tag{7.20}$$

$$\dot{\tilde{v}} = \eta_{\alpha\beta}\nu^\alpha\dot{x}^\beta + \eta_\alpha\dot{\nu}^\beta. \tag{7.21}$$

Inserting the evolution equation of ν, cf. (7.10), in (7.21), we receive

$$\dot{\tilde{v}} = -\frac{1}{F}\eta_{\alpha\beta}\nu^\alpha\nu^\beta + \frac{1}{F^2}F_k\eta_\alpha x_k^\alpha. \tag{7.22}$$

From the the definition of F we get

$$\begin{aligned}F_k = &F^{ij}h_{ij;k} - \tilde{v}_k f^{'}F^{ij}g_{ij} - \tilde{v}f^{''}u_kF^{ij}g_{ij} \\ &+ \psi_{\alpha\beta}\nu^\alpha x_k^\beta F^{ij}g_{ij} + \psi_\alpha x_r^\alpha h_k^r F^{ij}g_{ij}.\end{aligned} \tag{7.23}$$

From these relations the claim follows with the help of the Weingarten equation, the Codazzi equation and the Gauß formula. □

From now on to the end of this paper we will basically follow the descriptions in [11, Section 7] and especially [19]. Now, we present some results, namely, Lemma 7.4, Lemma 7.5, Lemma 7.6, Lemma 7.7, Lemma 7.8, and Corollary 7.9, which can be found in [19, Section 4, Section 5]. For the convenience of the reader we will present some of the proofs from [19], where the proof of Lemma 7.8 had to be modified to our situation.

7.4. **Lemma.** *The following estimates are valid*

$$|\eta_{\alpha\beta}\nu^\alpha\nu^\beta| \le c\tilde{v}^2 |||\eta_{\alpha\beta}|||, \tag{7.24}$$

$$|F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma| \le c\tilde{v}^3 |||\eta_{\alpha\beta\gamma}||| F^{ij} g_{ij}, \tag{7.25}$$

$$|\eta_{\alpha\beta}\nu^\alpha x_k^\beta u^k| \le c|||\eta_{\alpha\beta}|||\tilde{v}^3. \tag{7.26}$$

For any $\epsilon > 0$ we have

$$|F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k| \le c\epsilon\tilde{v}F^{ij}h_{kj}h_i^k |||\eta_{\alpha\beta}||| + c_\epsilon \tilde{v}^3 F^{ij}g_{ij}|||\eta_{\alpha\beta}|||. \tag{7.27}$$

$$|F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l| \le c\tilde{v}^3 F^{ij}g_{ij}. \tag{7.28}$$

In points where $\tilde{v}_i = 0$, there holds

$$|\psi_\alpha x_k^\alpha h_i^k u^i| \le c|||D\psi|||\tilde{v}^3. \tag{7.29}$$

Considering the flow hypersurfaces embedded in $(N, \bar{g}_{\alpha\beta})$, we can easier take advantage of the asymptotical behaviour of some quantities. This turns out to be very useful in the next chapters, but firstly, we are able to prove an uniform estimate for $\tilde{v}$.

7.5. **Lemma.** *$\tilde{v}$ is uniformly bounded on the maximal existence interval $[0, T^*)$, i.e. there holds*

$$\sup_{[0,T^*)} \tilde{v} \le c = c(\sup_{M_0} \tilde{v}, (N, \breve{g}_{\alpha\beta})). \tag{7.30}$$

Proof. For $0 < T < T^*$ assume that there are $0 < t_0 \le T$ and $x_0 \in \mathcal{S}_0$ such that

$$\sup_{[0,T^*]} \sup_{M(t)} \tilde{v} = \tilde{v}(t_0, x_0) \ge 2. \tag{7.31}$$

Applying the maximum principle we shall deduce that either $\tilde{v} \le 2$ or that t_0 is a priori bounded. From Lemma 7.3, Lemma 7.4 and the monotony of F we deduce

$$\begin{aligned} 0 \le &- F^{ij}h_{kj}h_i^k\tilde{v} + F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l - F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &- F\eta_{\alpha\beta}\nu^\alpha\nu^\beta - F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma + 2F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k \\ &+ \tilde{v}f^{''}\|Du\|^2 F^{ij}g_{ij} + \tilde{v}_k u^k f^{'} F^{ij}g_{ij} - \psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k F^{ij}g_{ij} \\ &- \psi_\alpha x_l^\alpha h_k^l u^k F^{ij}g_{ij} \\ \le &- \frac{1}{2}F^{ij}h_{kj}h_i^k\tilde{v} + c\tilde{v}^3(|f^{'}| + 1)F^{ij}g_{ij} + \tilde{v}f^{''}\|Du\|^2 F^{ij}g_{ij}. \end{aligned} \tag{7.32}$$

If $\tilde{v} \ge 2$, we have

$$\|Du\| \ge \frac{1}{4}\tilde{v}^2 \tag{7.33}$$

and if t_0 would be very large, then $f^{''}$ would be very large. Since, the term containing $f^{''}$ dominates the other terms, cf. (0.10), we would have a contradiction. Hence, t_0 and consequently $\tilde{v}$ are a priori bounded. □

Proving the exponential decay for $\|Du\|$ and $\|A\|$ in the next chapters, we need a decay property for certain tensors.

7.6. **Lemma.** *(i) Let $\varphi \in C^\infty([a,0))$, $a<0$, and assume*

$$\lim_{\tau\to 0} \varphi^{(k)}(\tau) = 0 \quad \forall k \in \mathbb{N}, \tag{7.34}$$

then for every $k\in\mathbb{N}$ there exists a $c_k>0$ such that

$$|\varphi(\tau)| \le c_k |\tau|^k. \tag{7.35}$$

(ii) Let T be a tensor such that for all $k\in\mathbb{N}$

$$|||D^k T(x^0,x)||| \longrightarrow 0 \quad \text{as} \quad x^0 \longrightarrow 0 \quad \text{uniformly in } x \tag{7.36}$$

then

$$\forall_{k\in\mathbb{N}} \quad \exists_{c_k>0} \quad \forall_{x\in S_0} \quad |||T(x^0,x)||| \le c_k |x^0|^k \tag{7.37}$$

(iii) For $T=(\eta_{\alpha\beta})$ the relation (7.37) is true, analogously for $|||\eta_{\alpha\beta\gamma}|||$, $|||D\psi|||$, $|||\bar{R}_{\alpha\beta\gamma\delta}\eta^\alpha|||$, or more generally for any tensor that would vanish identically, if it would have been formed with respect to the product metric

$$-(dx^0)^2 + \bar{\sigma}_{ij}dx^i dx^j. \tag{7.38}$$

In the next Lemma we will state how the the Riemannian curvature tensors of the metric $\breve{g}_{\alpha\beta}$ and its conformal counterpart $\bar{g}_{\alpha\beta}$ are related.

7.7. **Lemma.** *The Riemannian curvature tensors of the metrics $\breve{g}_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ are related by*

$$\begin{aligned} e^{-2\tilde{\psi}}\breve{R}_{\alpha\beta\gamma\delta} =& \bar{R}_{\alpha\beta\gamma\delta} - \bar{g}_{\alpha\gamma}\tilde{\psi}_{\beta\delta} - \bar{g}_{\beta\delta}\tilde{\psi}_{\alpha\gamma} + \bar{g}_{\alpha\delta}\tilde{\psi}_{\beta\gamma} + \bar{g}_{\beta\gamma}\tilde{\psi}_{\alpha\delta} \\ &+ \bar{g}_{\alpha\gamma}\tilde{\psi}_\beta\tilde{\psi}_\delta + \bar{g}_{\beta\delta}\tilde{\psi}_\alpha\tilde{\psi}_\gamma - \bar{g}_{\alpha\delta}\tilde{\psi}_\beta\tilde{\psi}_\gamma + \bar{g}_{\beta\gamma}\tilde{\psi}_\alpha\tilde{\psi}_\delta \\ &+ \{\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}\}\|D\tilde{\psi}\|^2, \end{aligned} \tag{7.39}$$

where the norms and the covariant derivatives on the right-hand side are with respect to $\bar{g}_{\alpha\beta}$.

Now, we are able to prove an exponentiell growth of the Riemannian curvature tensor.

7.8. **Lemma.** *There exists a constant $c>0$ such that for the leaves of the ISCF in $N^+=[-\epsilon,0)\times\mathcal{S}_0$ the following estimate holds*

$$\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\breve{\nu}^\alpha x_i^\beta \breve{\nu}^\gamma x_j^\delta \ge c|f^{'}|^2 e^{-2\tilde{\psi}}, \tag{7.40}$$

provided $\epsilon=\epsilon((N,\breve{g}_{\alpha\beta}),\sup_{M_0}\tilde{v})$ is chosen sufficiently small.

Proof. Since F^j_i is positive homogeneous of degree 0, we have

$$F^i_j = \breve{F}^i_j \tag{7.41}$$

and hence

$$F^{ij} = e^{2\tilde\psi}\breve{F}^{ij} \tag{7.42}$$

We have due to Lemma 7.7

$$\begin{aligned} e^{2\tilde\psi}\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\breve{\nu}^\alpha x^\beta_i\breve{\nu}^\gamma x^\delta_j =& F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i\nu^\gamma x^\delta_j + F^{ij}x^\beta_i x^\delta_j\tilde\psi_{\beta\delta} \\ & - F^{ij}g_{ij}\tilde\psi_{\alpha\gamma}\nu^\alpha\nu^\gamma - F^{ij}x^\beta_i x^\delta_j\tilde\psi_\beta\tilde\psi_\delta \\ & + F^{ij}g_{ij}\tilde\psi_\alpha\tilde\psi_\gamma\nu^\alpha\nu^\gamma + F^{ij}g_{ij}\|D\tilde\psi\|^2. \end{aligned} \tag{7.43}$$

Using (5.5) we estimate the summands in (7.43)

$$|F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i\nu^\gamma x^\delta_j| \le c\tilde{v}^4F^{ij}g_{ij}, \tag{7.44}$$

$$F^{ij}x^\beta_i x^\delta_j\tilde\psi_{\beta\delta} = F^{ij}u_iu_jf^{''} + F^{ij}x^\beta_i x^\delta_j\psi_{\beta\delta} \ge F^{ij}u_iu_jf^{''} - c\tilde{v}^2F^{ij}g_{ij}, \tag{7.45}$$

$$-F^{ij}g_{ij}\tilde\psi_{\alpha\gamma}\nu^\alpha\nu^\gamma \ge -\tilde{v}^2F^{ij}g_{ij}f^{''} - c\tilde{v}^2F^{ij}g_{ij}, \tag{7.46}$$

(7.47)
$$\begin{aligned} -F^{ij}x^\beta_i x^\delta_j\tilde\psi_\beta\tilde\psi_\delta =& -F^{ij}u_iu_j(\psi_0+f^{'})^2 - F^{ij}\psi_i\psi_j - 2F^{ij}u_j\psi_i(\psi_0+f^{'}) \\ \ge& -F^{ij}u_iu_j(\psi_0+f^{'})^2 - c|f^{'}|\tilde{v}^2F^{ij}g_{ij}, \end{aligned}$$

$$F^{ij}g_{ij}\tilde\psi_\alpha\tilde\psi_\gamma\nu^\alpha\nu^\gamma \ge \tilde{v}^2(\psi_0+f^{'})^2F^{ij}g_{ij} - c\tilde{v}|f^{'}|F^{ij}g_{ij}, \tag{7.48}$$

$$\begin{aligned} F^{ij}g_{ij}\|D\tilde\psi\|^2 =& -(f^{'}+\psi_0)^2F^{ij}g_{ij} + \sigma^{ij}\psi_i\psi_jF^{ij}g_{ij} \\ \ge& -(f^{'}+\psi_0)^2F^{ij}g_{ij} - cF^{ij}g_{ij}. \end{aligned} \tag{7.49}$$

Using the estimate $u_iu_j \le (\tilde{v}^2-1)g_{ij}$ we conclude

(7.50)
$$\begin{aligned} e^{2\tilde\psi}\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\breve{\nu}^\alpha x^\beta_i\breve{\nu}^\gamma x^\delta_j \ge& -c\tilde{v}^4F^{ij}g_{ij} + f^{''}F^{ij}(u_iu_j - \tilde{v}^2g_{ij}) \\ & - c\tilde{v}^2|f^{'}|F^{ij}g_{ij} + (\psi_0+f^{'})^2F^{ij}(\tilde{v}^2g_{ij} - u_iu_j - g_{ij}) \\ \ge& -c\tilde{v}^4F^{ij}g_{ij} - f^{''}F^{ij}g_{ij} - c|f^{'}|\tilde{v}^2F^{ij}g_{ij}, \end{aligned}$$

and hence the claim if ϵ is chosen sufficiently small. Here, we used (0.10) and the uniform boundedness of $\tilde{v}$, cf. Lemma 7.5. □

Finally, we will prove that $\breve{F}$ is bounded from below, which is necessary for the C^2 estimates in the previous section.

7.9. **Corollary.** *We assume that* (7.40) *is valid for the leaves of the ISCF. Then we have*

$$\breve{F} \ge \inf_{M_0}\breve{F}, \tag{7.51}$$

as long as the flow exists. If the flow exists for all times, then $\breve{F}$ *increases exponentially fast, namely, we have*

$$\breve{F} \geq c_0 e^{(\gamma + \frac{1}{n})t}, \tag{7.52}$$

where $c_0 = c(M_0) > 0$.

Proof. We define

$$\varphi(t) = \inf_{M(t)} \breve{F} \tag{7.53}$$

and infer from Lemma 3.2

$$\begin{aligned} \frac{d}{dt}\breve{F} - \frac{1}{\breve{F}^2}\breve{F}^{ij}\breve{F}_{ij} =& \frac{1}{\breve{F}}\breve{F}^{ij}\breve{h}_{ik}\breve{h}_j^k + \frac{1}{\breve{F}}\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\ &- \frac{2}{\breve{F}^3}\breve{F}^{ij}\breve{F}_i\breve{F}_j, \end{aligned} \tag{7.54}$$

hence, using Lemma 7.8 we deduce

$$\dot{\varphi}(t) \geq \tilde{c}\frac{|f'|^2}{\breve{F}}e^{-2f}, \tag{7.55}$$

especially $\dot{\varphi}(t) \geq 0$ for a.e. $0 < t < T^*$. If the flow exists for all times, we know already that the flow runs into the future singularity, i.e.

$$\lim_{t\to\infty} \inf_{\mathcal{S}_0} u(t,\cdot) = 0. \tag{7.56}$$

Using Theorem 8.2, which is proven independently of all previous results, (0.9) and (1.25) we infer

$$\frac{d}{dt}(\varphi^2) \geq c e^{2(\gamma + \frac{1}{n})t} \tag{7.57}$$

for a.e. $t > 0$ and a positive constant $c > 0$. Integration yields the claim. □

8. Decay of the C^0-norm

We want to prove the optimal decay of u. We show that there are positive constants c_1, c_2 such that

$$-c_1 \leq \tilde{u} \leq -c_2 \quad \forall t \in \mathbb{R}_+, \tag{8.1}$$

where $\tilde{u}$ is defined by

$$\tilde{u} = u e^{\gamma t}. \tag{8.2}$$

and u is the solution of the scalar version of the ISCF, that means, u is the solution of (7.13). The next three results and the corresponding proofs can be found in [19, Section 6].

8.1. **Lemma.** *For any* $0 < \lambda < \gamma$, *there exists a constant* $c(\lambda)$ *sucht that*

$$|ue^{\lambda t}| \leq c(\lambda) \quad \forall t \in \mathbb{R}_+. \tag{8.3}$$

With the help of the previous result it is possible to prove the optimal decay.

8.2. **Theorem.** *We define $\tilde{u} = ue^{\gamma t}$. Then, there are positive constants c_1, c_2 such that*

$$-c_1 \le \tilde{u} \le -c_2. \tag{8.4}$$

8.3. **Corollary.** *For any $k \in N^*$ there exists c_k such that*

$$|f^{(k)}| \le c_k e^{k\gamma t}, \tag{8.5}$$

where $f^{(k)}$ is evaluated at u.

Proof. In view of (0.12) there holds

$$|f^{(k)}| \le c_k |f^{'}|^k = c_k |f^{'}|^k u^k \tilde{u}^{-k} e^{k\gamma t}. \tag{8.6}$$

Then use (1.25) and the preceding theorem. □

9. Decay of the C^1-norm

Our final goal is to show that $||D\tilde{u}||$ is uniformly bounded, where we recall that

$$\tilde{u} = ue^{\gamma t}, \tag{9.1}$$

but this estimate has to be deferred to the next section. At the moment we only prove that $||Du||e^{\lambda t}$, with $0 < \lambda < \gamma$, is uniformly bounded.

9.1. **Lemma.** *There holds*

$$|F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l| \le c(|u|^2 + \|Du\|^2)F^{ij}g_{ij}. \tag{9.2}$$

Proof. We start with the identity

$$\begin{aligned} F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l = &- \tilde{v}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\eta^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ &- \tilde{v}F^{ij}\bar{R}_{r\beta\gamma\delta}\check{u}^r x_i^\beta x_l^\gamma x_j^\delta u^l, \end{aligned} \tag{9.3}$$

where $\eta = (\eta_\alpha) = (-1, 0, \dots, 0)$ is a covariant vectorfield. With the help of Lemma 7.5 and Lemma 7.6 we estimate the first term

$$-\tilde{v}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\eta^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \le cu^2 F^{ij}g_{ij} \tag{9.4}$$

and for the second term we receive

$$-\tilde{v}F^{ij}\bar{R}_{r\beta\gamma\delta}\check{u}^r x_i^\beta x_l^\gamma x_j^\delta u^l \le c\|Du\|^2 F^{ij}g_{ij}, \tag{9.5}$$

where we used Lemma 1.2. □

The following two lemmas and the corresponding proofs can be found in [19, Section 7], but we will give exemplarily a proof of the next lemma.

9.2. **Lemma.** *There exists $\epsilon > 0$ and a constant c_ϵ such that*

$$\|Du\| e^{\epsilon t} \le c_\epsilon \tag{9.6}$$

Proof. We employ the relation

$$\tilde{v}^2 = 1 + \|Du\|^2 \tag{9.7}$$

and the fact that $\tilde{v}$ is uniformly bounded to conclude that there exists a positive constant c, such that

$$c\|Du\|^2 \le 2\log\tilde{v} = \log\tilde{v}^2 \le \|Du\|^2, \tag{9.8}$$

i.e. we can equivalently prove that $\log\tilde{v}e^{\epsilon t}$ is uniformly bounded. Let ϵ be small and set

$$\varphi = \log\tilde{v}e^{2\epsilon t}, \tag{9.9}$$

then φ satisfies

$$\dot{\varphi} - \frac{1}{F^2}F^{ij}\varphi_{ij} = \frac{1}{\tilde{v}}(\dot{\tilde{v}} - \frac{1}{F^2}F^{ij}\tilde{v}_{ij})e^{2\epsilon t} + \frac{1}{F^2}\frac{1}{\tilde{v}^2}F^{ij}\tilde{v}_i\tilde{v}_j e^{2\epsilon t} + 2\epsilon\varphi. \tag{9.10}$$

and we get with the evolution equation of $\tilde{v}$, cf. Lemma 7.3,

$$\begin{aligned} F^2e^{-2\epsilon t}(\dot{\varphi} - \frac{1}{F^2}F^{ij}\varphi_{ij}) = &- F^{ij}h_{ki}h^k_j + \frac{1}{\tilde{v}}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_l x^\delta_j u^l \\ &- \frac{1}{\tilde{v}}F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{\tilde{v}}F\eta_{\alpha\beta}\nu^\alpha n^\beta \\ &- \frac{1}{\tilde{v}}F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x^\beta_i x^\gamma_j - \frac{2}{\tilde{v}}F^{ij}\eta_{\alpha\beta}x^\alpha_k x^\beta_i h^k_j \\ &+ f^{''}\|Du\|^2F^{ij}g_{ij} + \frac{1}{\tilde{v}}\tilde{v}_k u^k f^{'}F^{ij}g_{ij} \\ &- \psi_{\alpha\beta}\nu^\alpha x^\beta_k u^k\frac{1}{\tilde{v}}F^{ij}g_{ij} - \frac{1}{\tilde{v}}\psi_\alpha x^\alpha_l h^l_k u^k F^{ij}g_{ij} \\ &+ F^{ij}\tilde{v}_i\tilde{v}_j\frac{1}{\tilde{v}^2} + 2\epsilon F^2\log\tilde{v} \end{aligned} \tag{9.11}$$

Now, let T, $0 < T < \infty$, assume that

$$\sup_{[0,T]}\sup_{M(t)}\varphi = \varphi(t_0, x_0), \tag{9.12}$$

where $0 < t_0 \le T$ large and $x_0 \in \mathcal{S}_0$. Applying the maximum principle, we infer with the help equation (9.11), Lemma 7.4, Lemma 7.5, Lemma 7.6 and Lemma 9.1, after multiplying by $F^2e^{-2\epsilon t}$, that

$$\begin{aligned} 0 \le &- \frac{1}{2}F^{ij}h_{kj}h^k_i + c_\epsilon(u^2 + \|Du\|^2)F^{ij}g_{ij} \\ &+ c\epsilon|f^{'}|^2\|Du\|^2F^{ij}g_{ij} + f^{''}\|Du\|^2F^{ij}g_{ij}, \end{aligned} \tag{9.13}$$

where we estimated the term $|2\epsilon F^2 \log \tilde{v}|$ with the help of (2.17), (2.18), (7.6) and (9.8) in the following way

$$\begin{aligned} |2\epsilon F^2 \log \tilde{v}| &\le \sqrt{2}\epsilon F \check{H} \log \tilde{v} \\ &\le \sqrt{2} n\epsilon F^{ij} \check{h}_{ki} \check{h}^k_j \log \tilde{v} \\ &\le c\epsilon \big(F^{ij} h_{ki} h^k_j + \tilde{v}^2 |f^{'}|^2 F^{ij} g_{ij} + (\psi_\alpha \nu^\alpha)^2 F^{ij} g_{ij}\big) \log \tilde{v} \\ &\le c\epsilon \big(F^{ij} h_{ki} h^k_j + \|Du\|^2 |f^{'}|^2 F^{ij} g_{ij} + |u|^2 F^{ij} g_{ij}\big), \end{aligned} \tag{9.14}$$

and the term $|\frac{1}{\tilde{v}} F \eta_{\alpha\beta} \nu^\alpha n^\beta|$ is estimated by

$$\begin{aligned} |\frac{1}{\tilde{v}} F \eta_{\alpha\beta} \nu^\alpha n^\beta| &\le c|u|^3 F \\ &\le \epsilon |u|^4 F^2 + \epsilon^{-1} \tilde{c} |u|^2 F^{ij} g_{ij}, \end{aligned} \tag{9.15}$$

where we used Lemma 2.1. Using

$$\lim_{t\to\infty} f^{'} u = \frac{1}{\tilde{\gamma}} = \frac{1}{n\gamma}, \tag{9.16}$$

we conclude

$$\begin{aligned} \epsilon |u|^4 F^2 &\le c\epsilon |u|^4 \big(F^{ij} h_{ki} h^k_j + \tilde{v}^2 |f^{'}|^2 F^{ij} g_{ij} + (\psi_\alpha \nu^\alpha)^2 F^{ij} g_{ij}\big) \\ &\le c\epsilon \big(F^{ij} h_{ki} h^k_j + |u|^2 F^{ij} g_{ij}\big). \end{aligned} \tag{9.17}$$

If we now choose $0 < \epsilon < \gamma$ sufficient small and t_0 large enough, then we receive from (9.13), that

$$\|Du\|^2 \le \frac{cu^2}{|f''|}. \tag{9.18}$$

Hence, we have in (t_0, x_0)

$$\varphi = \log \tilde{v} e^{2\epsilon t} \le c\|Du\|^2 e^{2\epsilon t} \le \frac{cu^2}{|f''|} e^{2\epsilon t} \le c. \tag{9.19}$$

□

After having established the exponential decay of $\|Du\|$, we can improve the decay rate.

9.3. **Lemma.** *For any $0 < \lambda < \gamma$ there exists a constant c_λ such that*

$$\|Du\| e^{\lambda t} \le c_\lambda. \tag{9.20}$$

10. Decay of the C^2-Norm

Our final goal in this chapter is to show that $\|A\|e^{\gamma t}$ is uniformly bounded. We start with some preliminary results. Firstly, we prove that F grows exponentially fast.

10.1. **Theorem.** *We have the estimate*

$$F \geq ce^{\gamma t}, \tag{10.1}$$

where $c > 0$ depends on M_0.

Proof. From Corollary 7.9 we deduce

$$F = e^{\tilde{\psi}}\breve{F} \geq ce^{f}\breve{F} \geq ce^{\gamma t} \tag{10.2}$$

Here we used the relation

$$e^{f} \sim ce^{-\frac{1}{n}t}, \tag{10.3}$$

due to (0.9) and

$$|f^{'}u - \frac{1}{\tilde{\gamma}}| \leq cu^2. \tag{10.4}$$

□

Now, we state the evolution equations for h^k_l and F.

10.2. **Lemma.** *The second fundamental h_l^k form satisfies the evolution equation*

(10.5)

$$
\begin{aligned}
\dot h_l^k - \frac{1}{F^2}F^{ij}h^k_{l;ij} =& -\frac{2}{F^3}F^kF_l + \frac{1}{F}h^{kr}h_{rl} + \frac{1}{F}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_l^\beta\nu^\gamma x_r^\delta g^{rk}\\
&-\frac{1}{F^2}F^{ij}h_{aj}h_i^a h_l^k + \frac{1}{F^2}F^{ij}h_{ij}h_{al}h^{ak}\\
&+\frac{2}{F^2}g^{pk}F^{ij}\bar R_{\alpha\beta\gamma\delta}x_r^\alpha x_p^\beta x_i^\gamma x_l^\delta h_j^r - \frac{1}{F^2}F^{ij}\bar R_{\alpha\beta\gamma\delta}x_a^\alpha x_i^\beta x_l^\gamma x_j^\delta h^{ak}\\
&-\frac{1}{F^2}g^{pk}F^{ij}\bar R_{\alpha\beta\gamma\delta}x_r^\alpha x_i^\beta x_p^\gamma x_j^\delta h_l^r - \frac{1}{F^2}F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta h_l^k\\
&+\frac{1}{F^2}g^{pk}F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_p^\beta\nu^\gamma x_l^\delta h_{ij} + \frac{1}{F^2}g^{pk}F^{ij}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x_p^\beta x_i^\gamma x_l^\delta x_j^\epsilon\\
&+\frac{1}{F^2}g^{pk}F^{ij}\bar R_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x_i^\beta x_j^\gamma x_p^\delta x_l^\epsilon + \frac{1}{F^2}g^{pk}F^{ij;rs}\check h_{ij;p}\check h_{rs;l}\\
&+\frac{1}{F^2}F^{ij}g_{ij}(-u_lu^k\tilde v f''' + g^{pk}\psi_{\alpha\beta\gamma}\nu^\alpha x_p^\beta x_l^\gamma + \psi_{\alpha\beta}\nu^\alpha\nu^\beta h_l^k\\
&+g^{pk}\psi_{\alpha\beta}x_r^\alpha x_p^\beta h_l^r + \psi_{\alpha\beta}x_r^\alpha x_l^\beta h^{rk} + \psi_\alpha\nu^\alpha h_{lr}h^{rk} + \psi_\alpha x_r^\alpha h^{rk}{}_{;l}\\
&-g^{pk}f'\eta_{\alpha\beta\gamma}\nu^\alpha x_p^\beta x_l^\gamma - g^{pk}f'\eta_{\alpha\beta}x_r^\alpha x_p^\beta h_l^r - f'\eta_{\alpha\beta}\nu^\alpha\nu^\beta h_l^k\\
&-f'\eta_{\alpha\beta}x_r^\alpha x_l^\beta h^{rk} - f'h_{rl}h^{rk}\tilde v + f'u^rh^k_{l;r} + f'u^rg^{kp}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_p^\beta x_r^\gamma x_l^\delta\\
&-f''(\tilde v^ku_l + \tilde v_lu^k) + f''\tilde v^2h_l^k + f''\tilde v\eta_{\alpha\beta}x_l^\alpha x_r^\beta g^{rk}).
\end{aligned}
$$

Proof. We start with the equation for $\dot h_l^k$, cf. (7.11), which contains the summand

$$(-\frac{1}{F})_l^k = \frac{1}{F^2}F_l^k - \frac{2}{F^3}F_lF^k. \tag{10.6}$$

Calculating the covariant derivative F_{kl} we get

$$F_{kl} = F^{ij}\check h_{ij;kl} + F^{ij,rs}\check h_{ij;k}\check h_{rs;l}, \tag{10.7}$$

then we use the definition of $\check h_{ij}$, the Codazzi equation, the Ricci identities and the Gauß equation to express $F^{ij}\check h_{ij;kl}$ through $F^{ij}h_{lk,ij}$. □

10.3. **Lemma.** *The term F satisfies the evolution equation*

(10.8)

$$
\begin{aligned}
\dot F - \frac{1}{F^2}F^{ij}F_{ij} =& -\frac{2}{F^3}F^{ij}F_iF_j + \frac{1}{F}F^{ij}h_i^kh_{kj} + \frac{1}{F}F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta\\
&+\frac{1}{F}\eta_{\alpha\beta}\nu^\alpha\nu^\beta f'F^{ij}g_{ij} - \frac{1}{F}\tilde v^2f''F^{ij}g_{ij} + \frac{1}{F^2}f'F_ku^kF^{ij}g_{ij}\\
&-\frac{1}{F}\psi_{\alpha\beta}\nu^\alpha\nu^\beta F^{ij}g_{ij} + \frac{1}{F^2}\psi_\alpha x_k^\alpha F^kF^{ij}g_{ij}.
\end{aligned}
$$

Proof. We have

$$\dot{F} = F^i_j \dot{\check{h}}^j_i. \tag{10.9}$$

Using (7.6) and (7.11) leads to the claim. □

The proofs of the next five results will be omitted and can be found in [19, Chapter 8].

10.4. **Theorem.** *The principal curvatures κ_i of $M(t)$ are uniformly bounded from above during the evolution, e.g. there exists a constant c such that*

$$\kappa_i \le c, \quad 1 \le i \le n. \tag{10.10}$$

Now, we know that the biggest principle curvature is bounded from above. In the next lemma we examine the behaviour of the absolute value of the smallest principle curvature during the evolution.

10.5. **Lemma.** *We have*

$$\sup_{M(t)} \max_i |\kappa_i u| \to 0 \quad t \to \infty, \tag{10.11}$$

for $1 \le i \le n$.

Now, we are able to state a decay of $\|A\|$.

10.6. **Lemma.** *For any $0 < \lambda < \gamma$ there exists a positive constant $c(\lambda)$ such that*

$$\|A\| e^{\gamma t} \le c(\lambda) \quad \forall t \in \mathbb{R}_+. \tag{10.12}$$

In the next two theorems we will state the optimal decay of $\|Du\|$ and $\|A\|$.

10.7. **Theorem.** *Let $\tilde{u} = u e^{\gamma t}$, then there exists a positive constant such that*

$$\|D\tilde{u}\| \le c \quad \forall t \in \mathbb{R}_+ \tag{10.13}$$

10.8. **Theorem.** *The quantity $w = \frac{1}{2}\|A\|^2 e^{2\gamma t}$ is uniformly bounded during the evolution.*

11. Higher order estimates

After having established the exponential decay of $\|A\|$, we want to prove the exponential decay of the higher order derivatives. Therefore, we introduce in the following lemma a new notation, cf. [11, Section 7.6].

11.1. **Definition.** (i) For arbitrary tensors S, T denote by $S \star T$ any linear combination of tensors formed by contracting S over T. The result can be a tensor or a function. Using this notation we do not distinguish between $S \star T$ and $cS \star T$, where c is a constant.
(ii) The symbol A represents the second fundamental form of the hypersurfaces $M(t)$ in N. $\tilde{A} = Ae^{\gamma t}$ is the scaled version and $D^m A$ resp. $D^m \tilde{A}$ represent the covariant derivatives of order m.
(iii) For $m \in N$ denote by $\mathcal{O}_m$ a tensor expression defined on $M(t)$ that satisfies the pointwise estimates

$$\|\mathcal{O}_m\| \le c_m (1 + \|\tilde{A}\|_m)^{p_m}, \tag{11.1}$$

where c_m, p_m are positive constants and

$$\|\tilde{A}\|_m = \sum_{|\alpha| \le m} \|D^\alpha \tilde{A}\|. \tag{11.2}$$

Moreover, the derivative of $\mathcal{O}_m$ is of class $\mathcal{O}_{m+1}$ and can be estimated by

$$\|D\mathcal{O}_m\| \le c_m (1 + \|\tilde{A}\|_m)^{p_m} (1 + \|D^{m+1} \tilde{A}\|) \tag{11.3}$$

with constants c_m, p_m.
(iv) The symbol $\mathcal{O}$ represents a tensor such that $D\mathcal{O}$ is of class $\mathcal{O}_0$.

11.2. **Remark.** We emphasize that

$$D^m \mathcal{O}_0 = \mathcal{O}_m \quad \forall m \in \mathbb{N}. \tag{11.4}$$

11.3. **Lemma.** *We have*

$$D(uf') = e^{-2\gamma t} \mathcal{O}. \tag{11.5}$$

Proof. Differentiation and adding a zero yields

$$D_i(uf') = u_i f' (1 - \tilde{\gamma} f' u) + u u_i (\tilde{\gamma} |f'|^2 + f'') \tag{11.6}$$

and hence the claim in view of (0.10), (0.11) and (0.12). □

Now, we state the main result of this section, a proof of which can be found in [19, Chapter 9].

11.4. **Theorem.** *The quantities $\frac{1}{2}\|D^m \tilde{A}\|^2$ are uniformly bounded during the evolution for all $m \in \mathbb{N}^*$.*

12. Convergence of $\tilde{u}$ and the behaviour of derivatives in t

The proofs of this chapter are identical to those in [19, Section 10] and are omitted. Firstly, we state that $\tilde{u}$ converges when t tends to infinity.

12.1. **Lemma.** *$\tilde{u}$ converges in $C^m(\mathcal{S}_0)$ for any $m \in \mathbb{N}$, if t tends to infinity, and hence $D^m \tilde{A}$ converges.*

The following technical lemmas proof parts of the claim in Theorem 0.5.

12.2. **Lemma.** *Let $(\breve{g}_{ij})$ be the induced metric of the leaves $M(t)$ of the ISCF, then the rescaled metric*

$$e^{\frac{2}{n}t}\breve{g}_{ij} \tag{12.1}$$

converges in $C^\infty(\mathcal{S}_0)$ to

$$(\tilde{\gamma}^2 m)^{\frac{1}{\gamma}}(-\tilde{u})^{\frac{2}{\gamma}}\bar{\sigma}_{ij}, \tag{12.2}$$

where we are slightly ambiguous by using the same symbol to denote $\breve{u}(t,\cdot)$ and $\lim \tilde{u}(t,\cdot)$.

12.3. **Lemma.** *The leaves $M(t)$ of the ISCF get more umbilical, if t tends to infinity, namely*

$$\breve{F}^{-1}|\breve{h}_i^j - \frac{1}{n}\breve{H}\delta_i^j| \le ce^{-2\gamma t}. \tag{12.3}$$

In case $n+\omega-4>0$ we even get a better estimate, namely

$$|\breve{h}_i^j - \frac{1}{n}\breve{H}\delta_i^j| \le ce^{-\frac{1}{2n}(n+w-4)t}. \tag{12.4}$$

13. Transition from big crunch to big bang

In this chapter we want to present the concept of a transition from big crunch to big bang. The following is literally adapted from [11, Section 7.8]. Only some formulas had to be modified.

We shall define a new spacetime $\hat{N}$ by reflection and time reversal such that the ISCF in the old spacetime transforms to the ISCF in the new one. By switching the light cone we obtain a new spacetime $\hat{N}$. The flow equation in N is independent of the time orientation. We extend F, which is defined in the cone $\Gamma_2 \subset \mathbb{R}^n$, to $\Gamma_2 \cup (-\Gamma_2)$ by setting

$$F(\kappa_i) = -F(-\kappa_i) \tag{13.1}$$

for $(\kappa_i) \in -\Gamma_2$. The flow equation can then be written as

$$\dot{x} = -\frac{1}{\breve{F}}\breve{\nu} = -(-\frac{1}{\breve{F}})(-\breve{\nu}) =: -\frac{1}{\hat{F}}\hat{\nu}, \tag{13.2}$$

where the normal vector $\hat\nu = -\breve\nu$ is past directed in $\hat N$ and the curvature $\hat F = -\breve F$ negative. Introducing a new time function $\hat x^0 = -x^0$ and formally new coordinates $(\hat x^\alpha)$ by setting

$$\hat x^0 = -x^0, \quad \hat x^i = x^i, \tag{13.3}$$

we define a spacetime $\hat N$ having the same metric as N – only expressed in the new coordinate system – such that the flow equation has the form

$$\dot{\hat x} = -\hat F^{-1}\hat\nu, \tag{13.4}$$

where $M(t) = \operatorname{graph}\hat u(t)$, $\hat u = -u$, and

$$(\hat\nu^\alpha) = -\tilde v e^{-\tilde\psi}(1, \hat u^i) \tag{13.5}$$

in the new coordinates, since

$$\hat\nu^0 = -\breve\nu^0 \frac{\partial \hat x^0}{\partial x^0} = \breve\nu^0 \tag{13.6}$$

and

$$\hat\nu^i = -\breve\nu^i. \tag{13.7}$$

The singularity in $\hat x^0 = 0$ is now a past singularity, and can be referred to as a big bang singularity. The union $N \cup \hat N$ is a smooth manifold, topologically a product $(-a, a) \times \mathcal S_0$ – we are well aware that formally the singularity $\{0\} \times \mathcal S_0$ is not part of the union; equipped with the respective metrics and time orientations it is a spacetime which has a (metric) singularity in $x^0 = 0$. The time function

$$\hat x^0 = \begin{cases} x^0, & \text{in } N \\ -x^0, & \text{in } \hat N \end{cases} \tag{13.8}$$

is smooth across the singularity and future directed. $N \cup \hat N$ can be regarded as a cyclic universe with a contracting part $N = \{\hat x^0 < 0\}$ and an expanding part $\hat N = \{\hat x^0 > 0\}$ which are joined at the singularity $\{\hat x^0 = 0\}$. We shall show that the ISCF, properly rescaled, defines a natural C^3-diffeomorphism across the singularity and with respect to this diffeomorphism we speak of a transition from big crunch to big bang. The inverse ISCF in N and $\hat N$ can be uniformly expressed in the form

$$\dot{\hat x} = -\hat F^{-1}\hat\nu, \tag{13.9}$$

where (13.9) represents the original flow in N, if $\hat x^0 < 0$, and the flow in (13.4), if $\hat x^0 > 0$. Let us now introduce a new flow parameter

$$s = \begin{cases} -\gamma^{-1} e^{-\gamma t}, & \text{for the flow in } N \\ \gamma^{-1} e^{-\gamma t}, & \text{for the flow in } \hat N \end{cases} \tag{13.10}$$

and define the flow $y = y(s)$ by $y(s) = \hat x(t)$. $y = y(s, \xi)$ is then defined in $[-\gamma^{-1}, \gamma^{-1}] \times \mathcal S_0$, smooth in $\{s \ne 0\}$, and satisfies the evolution equation

$$y' := \frac{d}{ds} y = \begin{cases} -\hat F^{-1}\hat\nu e^{\gamma t}, & s < 0 \\ \hat F^{-1}\hat\nu e^{\gamma t}, & s > 0. \end{cases} \tag{13.11}$$

The flow y is certainly continuous across the singularity, and also future directed, i.e., it runs into the singularity, if $s < 0$, and moves away from it, if $s > 0$.
As in the previous sections we again view the hypersurfaces as embeddings with respect to the ambient metric

$$d\bar{s}^2 = -(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j. \tag{13.12}$$

The flow equation for $s < 0$ can therefore be written as

$$y' = -F^{-1}\nu e^{\gamma t}. \tag{13.13}$$

13.1. **Theorem.** *The flow $y = y(s, \xi)$ is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times \mathcal{S}_0$ and defines a natural diffeomorphism across the singularity. The flow parameter s can be used as a new time function.*

A detailed proof of Theorem 13.1 can be found in [11, Section 7.8] or in [19, Section 11].

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