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Construction of Harmonic Maass Forms in Small Weight

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To Gina

"A friend is one soul abiding in two bodies."

-Aristotle

Abstract

This thesis deals with various problems regarding automorphic forms of small weight. We study the continuation of Poincaré and Eisenstein series, as well as more abstract construction principles for harmonic Maass forms. Further, we show by using Riemann-Roch that the automorphic forms we constructed provide us with a basis for the weakly harmonic Maass forms. The difficulties encountered are solved by the introduction of a new type of vector space for square integrable automorphic forms, the Petersson-Sobolev spaces, which are defined in analogy to functional analytic Sobolev spaces. We show that these spaces provide us with a notion of invertibility of the Laplace operator as well as regularity theorems for the Petersson-Sobolev spaces, which are similar to the Sobolev imbedding theorems of functional analysis. The properties of the Petersson-Sobolev spaces then provide us with the tools required to solve the problems studied.

Diese Arbeit behandelt verschiedene Probleme in der Theorie automorpher Formen, unter anderen betrachten wir die Fortsetzung von Poincaré- und Eisensteinreihen und weitere Konstruktionsmethoden harmonischer Maassformen. Mit Hilfe von Riemann-Roch zeigen wir, dass die so konstruierten automorphen Formen ein Erzeugendensystem des Raumes der schwach harmonischen Maassformen bilden. Hierbei ergeben sich einige Schwierigkeiten die wir durch Einführung einer neuen Art von Vektorräumen für quadratintegrierbare automorphe Formen lösen, der sogenannten Petersson-Sobolev Räume, in Analogie zur Definition der funktionalanalytischen Sobolevräume. Wir zeigen dann, dass diese Räume es uns erlauben, den Laplaceoperator zu invertieren und uns zusätzlich mit Regularitätsaussagen, ähnlich der Sobolevschen Einbettungssätze, ausstatten die wir dann als Werkzeuge nutzen um die analysierten Probleme zu lösen.

Introduction

In this thesis, we study the vector space of automorphic forms $\mathcal{A}_{k,L}(\Gamma)$ with respect to some congruence subgroup $\Gamma \subseteq \mathrm{Mp}_2(\mathbb{Z})$ for intergal or half integral weight k and a lattice L. Then, if we denote the dual lattice by L', the discriminant group L'/L is finite. For us, an element of $\mathcal{A}_{k,L}$ is a smooth function

$$f : \mathbb{H} \to \mathbb{C}[L'/L],$$

that for all $M \in \Gamma$ is invariant under the Petersson slash operator $|_{k,L} M$. There are two important operators defined on spaces of automorphic forms, the Laplace operator

$$\Delta_k = -4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar\tau} - 2ik\frac{\partial}{\partial\bar\tau},$$

which is an endomorphism of $\mathcal{A}_{k,L}$ and the ξ-operator, given by

$$\xi_k = -2iy^k\overline{\frac{\partial}{\partial\bar\tau}}.$$

Let L^- denote the lattice L with the conjugated Weil representation on $\mathbb{C}[L'/L]$. Then the operator ξ_k maps $\mathcal{A}_{k,L}$ to $\mathcal{A}_{2-k,L^-}$ and satisfies

$$\Delta_k = \xi_{2-k}\xi_k.$$

To each of these operators a sub vector space of $\mathcal{A}_{k,L}$ is associated. The first vector space is the space of weakly harmonic Maass forms $H_{k,L}$, which are all elements of $\mathcal{A}_{k,L}$ that are of at most exponential growth at each cusp and are annihilated by Δ_k. The second vector space is the space of weakly holomorphic modular forms $M^!_{k,L}$, i.e. those elements of $H_{k,L}$ that are holomorphic as a function on $\mathbb{H}$. Equivalently they are the elements of $H_{k,L}$ that are annihilated by ξ_k. From these definitions it follows that the operator ξ_k maps $H_{k,L}$ to M_{2-k,L^-}, the proof of which, however, involves some calculations.

These definitions give rise to several questions. The first one that arises is the one of examples of weakly holomorphic modular forms, or, alternatively, weakly harmonic Maass forms. If the weight is sufficiently big, namely $k > 2$, examples of weakly holomorphic modular forms are readily constructed. If one considers the stabilizer subgroup Γ_∞ of Γ that stabilizes the cusp $i\infty$, one knows that it is of the form

$$\Gamma_\infty = \left\{ \begin{pmatrix} \pm1 & Nx \\ 0 & \pm1 \end{pmatrix} \mid x \in \mathbb{Z} \right\}$$

for a suitable $N \in \mathbb{N}$. If $m \in \frac{1}{N}\mathbb{Z}$ and $\mathfrak{e}_\gamma$ is a basis vector of $\mathbb{C}[L'/L]$, one can give standard examples of weakly holomorphic modular forms, the so called Poincaré series

$$P_k(m,\tau,\gamma) = \sum_{M \in \Gamma_\infty \backslash \Gamma} (e^{2\pi i m\tau}\mathfrak{e}_\gamma) \mid_k M,$$

which converge absolutely for $k > 2$ and as such, are holomorphic. One can use a trick and define Poincaré series that analytically depend on some parameter $s \in \mathbb{C}$ via

$$P_k(m, \tau, s, \gamma) = \sum_{M \in \Gamma_\infty \backslash \Gamma} (y^s e^{2\pi i m \tau} \mathfrak{e}_\gamma) \mid_k M.$$

These series converge absolutely as long as the real part σ of s satisfies $2\sigma + k > 2$. If $k > 2$, the holomorphic Poincaré series are obtained by evaluating the series at $s = 0$. For $k \leq 2$, the question arises whether these series have a continuation to $2\sigma + k \leq 2$ and hence can be evaluated in $s = 0$. We will show that analytic continuation to the set $2\sigma + k > 1$ is indeed possible, as long as $k \neq 1$. The problem that arises in the case $k = 1$ is that the continuation involves inverting the Laplace operator Δ_k, which we will see is only possible if $k \neq 1$. We will further see that there is a way to extend the definition to the set $\{s \in \mathbb{C} \mid 2\sigma + k \notin 1 - 2\mathbb{N}_0\}$, however, the extension to this set is not analytic as the set is not connected.

After finishing this thesis, the author was kindly informed by Kathrin Maurischat that similar methods are used in the as of yet unpublished paper [7] to study the continuation of scalar valued Poincaré series of weight $k = 2$ for $m > 0$.

One requires however a different approach to continue Eisenstein series, i.e. Poincaré series for $m = 0$. The question of their continuation was first answered positively in [8, 12]. In this thesis we use an innate connection between the Fourier coefficients of Eisenstein series and the constant coefficients of the series for $m \neq 0$ to prove this result.

Another question one can ask is whether the map ξ_k is surjective. If $2 - k > 2$, it is well known that the Poincaré series $P_{2-k}(m, \tau, \gamma)$ generate a basis of the weakly holomorphic modular forms and in a recent preprint of Andersen, Bringmann and Rolen titled "Images of Maass-Poincaré Series in the Lower Half Plane", preimages of $P_{2-k}(m, \tau, \gamma)$ under ξ_k were constructed for unimodular lattices by continuing Fourier coefficients to the lower half plane, hence proving the surjectivity in that case. In a similar manner to Poincaré series, one can construct Maass forms for weights $k > 2$. The challenges of construction using these methods thus only arise for small weights, i.e. $k \in \{0, 1/2, 3/2, 2\}$, hence the title of this thesis. Note that regardless of this fact, our methods of construction work for all weights $k \neq 1$.

The most general result pertaining to the surjectivity of ξ_k was already derived in the paper "On two geometric theta lifts" by Bruinier and Funke (see [4]), where they prove the surjectivity of ξ_k for arbitrary half integral weight and all lattices. The method employed in this paper relies on the theory of logarithmic sheaves on analytic curves.

Encouraged by their result, it is the aim of this thesis to provide a proof of the surjectivity of ξ_k by means of explicitly constructing a generating system for the weakly holomorphic modular forms. We will see that not only we can construct such a generating system, but also can extend it to a generating system of the weakly harmonic Maass forms for arbitrary lattices and all half integral weights $k \neq 1$.

The method of construction primarily relies on a theorem by Roelcke, as found in [9, 10]. In the two papers, Roelcke shows that, when one considers unimodular lattices (and so $\mathbb{C}[L'/L] = \mathbb{C}$) and the space of square integrable automorphic forms of weight k with respect to the Petersson scalar product

$$\langle f, g\rangle_k = \int_{\mathcal{F}} f\overline{g}\, y^k \, \frac{dx\, dy}{y^2},$$

the operator $\Delta_k - \lambda$ is, essentially, invertible if $\lambda \notin [(k-1)^2/4, \infty)$. Note that one needs to take some more care regarding the notion of invertibility, which will be discussed in detail in this thesis. We generalize this result to arbitrary integral lattices, as long as $k \neq 1$. This then allows us to invert Δ_k. The problem that arises in weight one is that Δ_0 is not invertible as 0 is in its continuous spectrum, as already indicated above. Hence, we only consider weights $k \neq 1$ in this thesis.

Another problem one encounters when inverting Δ_k is that one needs to estimate the cuspidal growth of $\Delta_k^{-1} f$, for a suitable square integrable automorphic form f of weight k, as we defined Maass forms to be of at most exponential growth.

The central input of this thesis is the introduction of the Petersson-Sobolev spaces, which can be seen as a generalization of square integrable automorphic forms. They are an inductively defined family of vector spaces $\mathcal{H}^n_{k,L}$ such that $\mathcal{H}^0_{k,L}$ is the space of square integrable automorphic forms and $\mathcal{H}^n_{k,L}$ consists of the elements f of $\mathcal{H}^0_{k,L}$ such that $\xi_k f$ is in $\mathcal{H}^{n-1}_{2-k,L^-}$. Their most important property is that they provide us with bounds for the cuspidal growth of elements therein. The main results can be briefly summarized as follows. First, we show that for any Fourier coefficient

$$f_n(y) := \int_0^N f(x+iy) e^{-2\pi i n x}\, dx$$

of an element f of $\mathcal{H}^1_{k,L}$, one has

$$f_n(y) = O(y^{(3-k)/2})$$

at every cusp. For elements of $\mathcal{H}^2_{k,L}$, we have a stronger result, namely that

$$f(y) = O(y^{(1-k)/2})$$

at every cusp. Further, for suitable square integrable f, the element $\Delta_k^{-1} f$ is in $\mathcal{H}^2_{k,L}$. This provides us with the desired growth estimates at each cusp and allow us to prove that the automorphic forms we construct are indeed weakly holomorphic modular forms, respectively weakly harmonic Maass forms.

Last, we show using Riemann-Roch that the set of weakly holomorphic modular forms we construct are indeed a generating system for the space $M^!_{k,L}$, in the sense that any weakly holomorphic modular form is a finite linear combination of elements in the generating system. We proceed to construct preimages under ξ_{2-k} for those, where again we need our theory of Petersson-Sobolev spaces to prove that the preimages are indeed weakly harmonic Maass forms. This method of construction then proves the surjectivity of ξ_k for all weights $k \neq 1$. As a consequence of this proof, we will also see that we obtain a natural generating system for the weakly harmonic Maass forms.

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1 Maass Forms on Lattices

1.1 Introduction and Notation

1.1.1 Complex Variables and Functions

In the following, we will be considering Maass forms and as such, functions $f_s : \mathbb{M} \times \mathbb{H} \to V$ with $\mathbb{M}$ being some discrete parameter space and V a suitable vector space, that analytically depend on some parameter s. We will denote the argument of f_s in $\mathbb{H}$ by $\tau \in \mathbb{H}$ and decompose it into real and imaginary part, denoted by x and y, i.e.

$$\tau = x + iy.$$

Here, s will be an element of $\mathbb{C}$ such that

$$\sigma := \mathrm{Re}(s) > \text{Const.}.$$

Another thing we will commonly need is a concept of a square root. By the square root $\sqrt{}$, we always mean the principal part, such that for $z \in \mathbb{C} \backslash \{0\}$

$$-\pi/2 < \mathrm{Arg}(\sqrt{z}) \leq \pi/2.$$

Lastly, we abbreviate

$$e(x) := e^{2\pi i x}. \tag{1.1}$$

1.1.2 The Metaplectic Group, Lattices and Weil Representations

The first object we need to define is the metaplectic group.

Definition 1.1. *The metaplectic group* $\mathrm{Mp}_2(\mathbb{Z})$ is the set of pairs (M, ϕ) where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\phi : \mathbb{H} \to \mathbb{C}$ such that ϕ is holomorphic and $\phi^2(\tau) = c\tau + d$. The multiplication is defined via $(M, \phi) \cdot (M', \phi') := (M \cdot M', (\phi \circ M') \cdot \phi')$ where $(\phi \circ M') \cdot \phi'(\tau) = \phi(M'\tau) \cdot \phi'(\tau)$.

Is is easy to see that this is indeed a group; as for (M, ϕ) and (M', ϕ') in $\mathrm{Mp}_2(\mathbb{Z})$ we denote $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and then quickly calculate

$$\begin{aligned} (\phi(M'\tau) \cdot \phi'(\tau))^2 &= (c(\frac{a'\tau + b'}{c'\tau + d'}) + d) \cdot (c'\tau + d') \\ &= (ca' + dc')\tau + (cb' + dd'), \end{aligned}$$

as well as

$$M \cdot M' = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

Remark 1.2. The canonical projection $\pi : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$, defined via

$$\pi((M, \phi)) = M,$$

is a twofold cover as there are precisely two square roots of any function $c\tau + d$.

Lemma 1.3. *The metaplectic group* $\mathrm{Mp}_2(\mathbb{Z})$ *is generated by the elements* $S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right)$ *and* $T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right)$.

Proof. We use the generally known fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $S' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let G be the subgroup of $\mathrm{Mp}_2(\mathbb{Z})$ generated by S and T. Thus $\pi|_G : G \to \mathrm{SL}_2(\mathbb{Z})$ is surjective as $\pi(S) = S'$ and $\pi(T) = T'$. Further, $S^4 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1\right)$ is in G, hence for any $(M, \phi) \in G$, so is $(M, \phi) \cdot (\mathrm{id}, -1) = (M, -\phi)$. Since for any $M \in \mathrm{SL}_2(\mathbb{Z})$, there are exactly two preimages under π, we must have $G = \mathrm{Mp}_2(\mathbb{Z})$. □

Next, consider an integral lattice L of signature (b_+, b_-) and scalar product $(\cdot, \cdot)$. For us, an integral lattice L is a $\mathbb{Z}$-submodule of rank n in $\mathbb{R}^n$ such that $\langle \mathbb{R}L \rangle = \mathbb{R}^n$. The scalar product on L is the one induced by the euclidean scalar product on $\mathbb{R}^n$. By denoting its dual lattice by L', it is well known that L'/L is a finite abelian group. We can next consider the associated monoid ring $\mathbb{C}[L/L']$, whose base elements we denote by $\mathfrak{e}_\gamma$, $\gamma \in L'/L$, and define a representation of $\mathrm{Mp}_2(\mathbb{Z})$ thereon. As $\mathrm{Mp}_2(\mathbb{Z})$ is generated by S and T, it is sufficient to define it on those two elements.

Definition 1.4. *The Weil representation of* $\mathrm{Mp}_2(\mathbb{Z})$. Let $\mathfrak{e}_\gamma \in \mathbb{C}[L'/L]$. We define

$$\rho_L(S)\mathfrak{e}_\gamma := \frac{\sqrt{i}^{b_+ - b_-}}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e(-(\gamma, \delta))\mathfrak{e}_\delta,$$

$$\rho_L(T)\mathfrak{e}_\gamma := e(\frac{1}{2}(\gamma, \gamma))\mathfrak{e}_\gamma$$

and extend ρ_L to a representation of the full metaplectic group using these relations. The extension is outlined, for example, in [14, Lemma 1.3]. We use this opportunity to point out the $e(\cdot)$ notation as defined in (1.1).

If L is unimodular, and hence $\mathbb{C}[L'/L] = \mathbb{C}$, the representation on elements of

$$\Gamma(4) := \ker(\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}))$$

is given via

$$\rho_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} \left(\frac{c}{d}\right) \epsilon_d^{-1} & \text{if } 2k \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol (as defined in [13]) and

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4, \\ i & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

We also expand our previous shorthand

$$e_\gamma(x) \coloneqq e(x) \cdot \mathfrak{e}_\gamma,$$

where $\mathfrak{e}_\gamma \in \mathbb{C}[L'/L]$ is one of the generators. Lastly, we consider the conjugated representation, i.e. conjugation of the matrix entries, with respect to L as the one given by $\overline{\rho_L}$. We denote this representation by L^-.

1.2 Maass Forms

When considering Maass forms for lattices apart from the trivial lattice 0, it is important to note note that there are no non zero Maass forms unless $2k = b^+ - b^- \bmod 2$. This statement will only be marginally relevant later, specifically for deriving (2.4) by using the explicit representations. To see that there are no Maass forms if $2k \neq b^+ - b^- \bmod 2$, let us first give the following

Definition 1.5. *The Petersson slash operator.* Let f be some function that takes its values on the upper half plane $\mathbb{H}$. Let M be an element of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$. Then we define the Petersson slash operator $|_{k,L}\ M$ by

$$(f\ |_{k,L}\ M)(\tau) \coloneqq \frac{1}{\sqrt{c\tau + d}^{2k}} \rho_L(M)^{-1} f(M\tau).$$

Further, we need the notion of a congruence subgroup, which is given by

Definition 1.6. *Congruence subgroups of* $\mathrm{Mp}_2(\mathbb{Z})$. Let Γ be a subgroup of $\mathrm{Mp}_2(\mathbb{Z})$. We say Γ is a congruence subgroup if it contains contains the preimage $\pi^{-1}(\Gamma')$ for some congruence subgroup Γ' of $\mathrm{SL}_2(\mathbb{Z})$ under the canonical projection $\pi : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$.

Definition 1.7. *An automorphic form of half integer weight k w.r.t. a lattice L and some congruence subgroup* $\Gamma \subseteq \mathrm{Mp}_2(\mathbb{Z})$ is a smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ such that for $M = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi \right) \in \Gamma$, we have

$$(f\ |_{k,L}\ M)(\tau) \quad = \quad f(\tau).$$

We denote by $\mathcal{A}_{k,L}(\Gamma)$ the space of all automorphic forms of weight k for L and Γ. Note that our definition of automorphic form slightly deviates from the convention as we require only invariance under the group action and no restriction on growth in the cusps. If L is unimodular, we further allow dropping the subscript L and just write $\mathcal{A}_k(\Gamma)$ instead. Now, to prove the claim that there are no non vanishing automorphic forms if $2k \neq b^+ - b^- \bmod 2$, consider the element $S^4 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1 \right)$. It is contained in any congruence subgroup, and by definition, we have for any base vector $\mathfrak{e}_\gamma$ of $\mathbb{C}[L'/L]$

$$\rho_L(S)^2 \mathfrak{e}_\gamma \quad = \quad \frac{i^{b_+ - b_-}}{|L'/L|} \sum_{\delta, \delta'} e(-(\gamma, \delta) - (\delta, \delta')) \mathfrak{e}_{\delta'}$$

$$= \frac{i^{b_+-b_-}}{|L'/L|}\sum_{\delta,\delta'} e(-(\delta,\delta'))\mathfrak{e}_{\delta'-\gamma}.$$

Next, note that

$$\sum_{\delta} e(-(\delta,\delta')) = \begin{cases} |L'/L| & \text{if } \delta' = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and thus

$$\rho_L(S)^2\mathfrak{e}_\gamma = i^{b_+-b_-}\,\mathfrak{e}_{-\gamma}$$

and

$$\rho_L(S)^4{=}(-1)^{b_+-b_-}.$$

Thus for any $f \in \mathcal{A}_{k,L}(\Gamma)$

$$\begin{aligned} f &= f|_k\, S^4 \\ &= (-1)^{2k+b_+-b_-} f, \end{aligned}$$

i.e. $f = 0$ unless $2k + b_+ - b_- \equiv 0 \bmod 2$.

There are two important operators defined on automorphic forms, the Maass lowering operator (see [4],[5])

$$\begin{aligned} l_k : \mathcal{A}_{k,L}(\Gamma) &\to \mathcal{A}_{k-2,L}(\Gamma) \\ l_k f &:= -2iy^2\frac{\partial f}{\partial\bar\tau} \end{aligned}$$

where

$$\begin{aligned} \frac{\partial}{\partial\tau} &:= \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \\ \frac{\partial}{\partial\bar\tau} &:= \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}), \end{aligned}$$

as well as the conjugation

$$\begin{aligned} *_k : \mathcal{A}_{k,L}(\Gamma) &\to \mathcal{A}_{-k,L^-}(\Gamma) \\ *_k f &:= y^k\bar f. \end{aligned}$$

We will see that all these are well defined. These further define the Maass raising operator

$$\begin{aligned} r_k : \mathcal{A}_{k,L}(\Gamma) &\to \mathcal{A}_{k+2,L}(\Gamma) \\ r_k &:= *_{-k-2}l_{-k}*_k\,. \end{aligned}$$

To give a coordinate representation, note that

$$r_k f = y^{-k-2}\overline{(-2iy^2\frac{\partial}{\partial\bar\tau}(y^k\bar f))}$$

$$
\begin{aligned}
&= 2iy^{-k}\frac{\partial}{\partial\tau}(y^k f)\\
&= 2i\frac{\partial f}{\partial\tau} + \frac{k}{y}f
\end{aligned}
$$

Next we define ξ-operator

$$
\begin{aligned}
\xi_k : \mathcal{A}_{k,L}(\Gamma) &\to \mathcal{A}_{2-k,L^-}(\Gamma)\\
\xi_k &:= *_{k-2} l_k,
\end{aligned}
$$

for which we have the coordinate representation

$$
\xi_k f = 2iy^k \frac{\partial \bar{f}}{\partial\tau}.
$$

The last operator we need is the Laplace operator

$$
\begin{aligned}
\Delta_k : \mathcal{A}_{k,L}(\Gamma) &\to \mathcal{A}_{k,L}(\Gamma)\\
\Delta_k &:= -\xi_{2-k}\xi_k\\
&= -r_{k-2}l_k\\
&= -l_{k+2}r_k - k. \qquad (1.2)
\end{aligned}
$$

To prove the last line and derive a coordinate representation, observe that

$$
\begin{aligned}
\Delta_k f &= -r_{k-2}l_k f\\
&= -4y^{-k+2}\frac{\partial}{\partial\tau}(y^k\frac{\partial f}{\partial\bar\tau})\\
&= -4y^{2-k}\frac{\partial}{\partial\tau}(y^k\frac{\partial f}{\partial\bar\tau})\\
&= -4y^2\frac{\partial^2 f}{\partial\bar\tau\partial\tau} + 2ik\frac{\partial f}{\partial\bar\tau}\\
&= 2iy^2\frac{\partial}{\partial\bar\tau}(2i\frac{\partial f}{\partial\tau} + \frac{k}{y}f) - kf\\
&= -l_{k+2}r_k f - kf.
\end{aligned}
$$

The well-definedness of all of these operations follows from the commutation relations for arbitrary $M \in \mathrm{Mp}_2(\mathbb{Z})$

$$
\begin{aligned}
(l_k\cdot)|_{k-2,L}\, M &= l_k((\cdot)|_{k,L}\, M),\\
(*_k\cdot)|_{-k,L^-}\, M &= *_{-k}((\cdot)|_{k,L}\, M).
\end{aligned}
$$

Proof. Let $f : \mathbb{H} \to \mathbb{C}[L'/L]$ be a smooth function. Let $M = \left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \pm 1\right) \in \mathrm{Mp}_2(\mathbb{Z})$. By using the chain rule we obtain

$$
l_k(f|_{k,L}\, M)(\tau) = -2iy^2\frac{\partial}{\partial\bar\tau}\left(\frac{1}{\sqrt{c\tau+d}^{2k}}\rho_L(M)^{-1}f(M\tau)\right)
$$

$$
\begin{aligned}
&= \frac{-2iy^2}{\sqrt{c\tau+d}^{2k}}\rho_L(M)^{-1}\frac{1}{(c\bar{\tau}+d)^2}\frac{\partial f}{\partial\bar{\tau}}(M\tau)\\
&= \frac{-2iy^2}{|c\tau+d|^4}\rho_L(M)^{-1}\frac{1}{\sqrt{c\tau+d}^{2(k-2)}}\frac{\partial f}{\partial\bar{\tau}}(M\tau)\\
&= \left\{\left(-2iy^2\frac{\partial f}{\partial\bar{\tau}}\right)|_{k-2,L}\,M\right\}(\tau)\\
&= ((l_k f)|_{k-2,L}\,M)(\tau).
\end{aligned}
$$

Now, let $f \in \mathcal{A}_{k,L}(\Gamma)$. Let $M' \in \Gamma$. Then, by definition, $f|_k\, M' = f$ and

$$
\begin{aligned}
(l_k f)|_{k-2,L}\,M' &= l_k(f|_{k,L}\,M')\\
&= l_k f,
\end{aligned}
$$

hence $l_k f \in \mathcal{A}_{k-2,L}(\Gamma)$.

As to $*_k$, let again f be smooth (but not necessarily invariant under $|_{k,L}$) and $M = \left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \pm 1\right) \in \mathrm{Mp}_2(\mathbb{Z})$ as above. Then

$$
\begin{aligned}
(*_k(f|_{k,L}\,M))(\tau) &= y^k\overline{(f|_{k,L}\,M)}(\tau)\\
&= y^k\frac{1}{\sqrt{c\bar{\tau}+d}^{2k}}\overline{\rho_L(M)^{-1}}\bar{f}(M\tau)\\
&= \frac{1}{\sqrt{c\tau+d}^{-2k}}\rho_{L^-}(M)^{-1}\frac{y^k}{|c\tau+d|^{2k}}\bar{f}(M\tau)\\
&= ((y^k\bar{f})|_{-k,L^-}\,M)(\tau)\\
&= ((*_k f)|_{-k,L^-}\,M)(\tau),
\end{aligned}
$$

and, for $f \in \mathcal{A}_{k,L}(\Gamma)$ and $M' \in \Gamma$,

$$
\begin{aligned}
(*_k f)|_{-k,L^-}\,M' &= *_k(f|_{k,L}\,M')\\
&= *_k f,
\end{aligned}
$$

hence $f \in \mathcal{A}_{-k,L^-}(\Gamma)$. □

Note that in the progress of this thesis, when applying $|_{k,L}$ it will always be to elements of a certain $\mathcal{A}_{k,L}$, which is known beforehand. Thus, we can drop the subscript L of $|_{k,L}$ and simply write $|_k$ instead for easier reading. The next important space are the harmonic Maass forms, which we define via

Definition 1.8. *The harmonic Maass forms $H_{k,L}(\Gamma)$ are the subspace of $f \in \mathcal{A}_{k,L}(\Gamma)$ that, for arbitrary $M \in \mathrm{Mp}_2(\mathbb{Z})$, satisfy*

$$|(f|_k\,M)(\tau)| \leq O(e^{Cy}),$$

as well as

$$\Delta_k f = 0.$$

Harmonic Maass forms have been discussed to great extent in the literature. At this point, we would like to give a review of some structural results, all of which are found in [4]. Note that by definition of congruence subgroups, there is some N such that, for

$$\tilde{\Gamma}(N) := \ker(\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})),$$

$\tilde{\Gamma}(N) \leq \Gamma$. Thus, $T^N = \left(\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, 1 \right) \in \Gamma$. And therefore, for any $f \in H_{k,L}(\Gamma)$,

$$\begin{aligned} f(\tau + N) &= f|_k T^N \\ &= f, \end{aligned}$$

i.e. f has period N. Since f is smooth, it has an absolutely converging Fourier expansion

$$f(\tau) = \sum_{n \in \frac{1}{N}\mathbb{Z}} c(n,y) e(nx).$$

Note that in contrast to, e.g. [4], we do not expand our constants $c(n,y)$ with respect to the basis $\{\mathfrak{e}_h\}$ of $\mathbb{C}[L'/L]$. Instead, we just take them as constant vectors in $\mathbb{C}[L'/L]$. Further

$$c(n,y) = \frac{1}{N} \int_0^N f(x+iy) e(-nx)\, dx,$$

where $n \in \frac{1}{N}\mathbb{Z}$. Applying Δ_k yields

$$\begin{aligned} -\Delta_k c(n,y)e(nx) &= y^2 \frac{\partial^2 c(n,y)}{\partial y^2} + ky \frac{\partial c(n,y)}{\partial y} \\ &+ (-(2\pi y n)^2 + 2\pi k y n) c(n,y) \\ &= \frac{1}{N} \int_0^N (-\Delta_k f(x+iy)) e(-nx)\, dx \\ &= 0. \end{aligned}$$

If $n = 0$, this implies that

$$c(0,y) = c_0^+ + c_0^- y^{1-k}$$

for some constants $c_0^\pm \in \mathbb{C}[L'/L]$. If $n \neq 0$, we set $w = 2\pi n y$ and set $c(n,y) = b(n,w)$. Inserting this into above equation yields

$$0 = \left(\frac{\partial^2 b(n,w)}{\partial w^2} + \frac{k}{w} \frac{\partial b(n,w)}{\partial w} - b(n,w) + \frac{k}{w} b(n,w) \right).$$

One solution to this is e^{-w}. The other solution is given by the Whittaker function

$$H(w) := M_{\mathrm{sgn}(n)k/2,(k-1)/2}(|w|),$$

which is in great detail discussed in [1]. It contains the estimates

$$H(w) \approx \begin{cases} |2w|^{-k} e^{-|w|} & \text{as } w \to -\infty, \\ (-2w)^{-k} e^{|w|} & \text{as } w \to +\infty. \end{cases}$$

[4] also gives the integral representation

$$H(w) = e^{w} \int\limits_{-2w}^{\infty} e^{-t} t^{-k}\, dt, \tag{1.3}$$

where one needs to take appropriate values of w and k for convergence and eventually continue analytically in k and w. Hence we can uniquely decompose

$$c(n, y) = c_n^{+} e^{-2\pi n y} + c_n^{-} H(2\pi n y).$$

Now, note that, by definition of harmonic Maass forms,

$$\begin{aligned} |c(n,y)| &= \left| \frac{1}{N} \int\limits_{0}^{N} f(x+iy) e(-nx)\, dx \right| \\ &\leq e^{Cy} \end{aligned}$$

for some constant $C > 0$. By above estimates, we further have

$$c(n, y) \approx \begin{cases} c_n^{+} e^{-2\pi n y} & \text{for } n < 0 \text{ as } y \to \infty, \\ (-4\pi n y)^{-k} c_n^{-} e^{2\pi n y} & \text{for } n > 0 \text{ as } y \to \infty. \end{cases}$$

Thus, $c_n^{+} = 0$ for $n \ll \infty$ and $c_n^{-} = 0$ for $n \gg -\infty$. We can then define

$$\begin{aligned} f^{+}(\tau) &:= \sum_{n \in \frac{1}{N}\mathbb{Z}} c_n^{+} e(n\tau), \\ f^{-}(\tau) &:= c_0^{-} y^{1-k} + \sum_{n \in \frac{1}{N}\mathbb{Z}\setminus\{0\}} c_n^{-} H(2\pi n y) e(nx), \end{aligned}$$

which implies

$$f = f^{+} + f^{-}. \tag{1.4}$$

As each sum has only finitely many positive resp. negative coefficients, they both converge absolutely. We say f^{+} is the holomorphic part of f.

A second subspace of importance are

Definition 1.9. *The weakly holomorphic Maass forms $M^{!}_{k,L}(\Gamma)$ are the subspace of $f \in H_{k,L}(\Gamma)$ such that*

$$l_k f = 0.$$

Note that this is equivalent to saying that f is holomorphic on $\mathbb{H}$ with poles of finite order at the cusps.

Lemma 1.10. *The restriction of* ξ_k *as an operator*

$$\xi_k : H_{k,L}(\Gamma) \to M^!_{2-k,L^-}(\Gamma) \tag{1.5}$$

is well defined.

Proof. We only need to consider $k \neq 1$ as the weight $k = 1$ is not studied in this thesis. Let $f \in H_{k,L}(\Gamma)$. We can then decompose

$$f = f^+ + f^-$$

as in (1.4). Then, by holomorphy of f^+ and absolute convergence of the series of f^-,

$$\begin{aligned}\xi_k f &= \xi_k f^- \\ &= 2(1-k)c_o^- \\ &+ y^k \sum_{n \in \frac{1}{N}\mathbb{Z}\setminus\{0\}} c_n^- (\frac{\partial}{\partial y} + 2\pi n) H(2\pi n y) e(-nx).\end{aligned}$$

Now, by the integral representation (1.3), it becomes obvious that

$$\frac{\partial H}{\partial w} + H = (-2w)^{-k} e(-w),$$

and thus

$$\xi_k f = 2(1-k)c_o^- + 2 \sum_{n \in \frac{1}{N}\mathbb{Z}\setminus\{0\}} c_n^- (-4\pi n)^{1-k} e(-n\tau).$$

Since the Fourier series has only finitely many non vanishing coefficients for $n > 0$, it is of at most polynomial growth in $i\infty$. Since for any harmonic Maass form f, for any $M \in \mathrm{Mp}_2(\mathbb{Z})$, $f|_k M$ is a harmonic Maass form as well, f must be of at most exponential growth at the other cusps too. □

As part of this thesis, we will prove surjectivity of ξ_k by constructing a suitable basis. For an alternative proof of this, see e.g. [4], where the mentioned result is derived using complex analysis of fiber bundles.

1.3 The Petersson Inner Product

We let $\mathcal{F}$ denote a fundamental domain for Γ. For $f, g \in \mathcal{A}_{k,L}(\Gamma, L)$ we define the scalar product to be

$$\langle f, g\rangle_k := \int_{\mathcal{F}} y^k f \bar{g} \, \frac{dx \cdot dy}{y^2}, \tag{1.6}$$

where $f\bar{g}$ is the scalar product of two vectors in $\mathbb{C}[L'/L]$. This enables us to define the space of square integrable automorphic forms.

Definition 1.11. We denote by $\mathcal{L}^2_{k,L}(\Gamma)$ *the vector space of square integrable automorphic forms*, which is the completion of the pre-Hilbert space $X := \{f \in \mathcal{A}_{k,L}(\Gamma) | \langle f, f\rangle_k < \infty\}$ with respect to the scalar product $\langle , \rangle_k$.

As we will see a later, this space will play a crucial role in continuing the soon to be defined Poincaré series and finding a basis of the harmonic Maass forms.

1.4 Congruence Subgroups, Inclusions and Traces

Let in the following denote B an element of the set $\{\mathcal{L}^2, \mathcal{A}, H, M^!\}$. For two congruence subgroups $\Gamma' \leq \Gamma$ there exists a natural inclusion

$$\begin{array}{rcl} \iota : B_{k,L}(\Gamma) & \hookrightarrow & B_{k,L}(\Gamma') \\ f & \mapsto & f \end{array}$$

and trace, which is defined via a set of representatives $\{\alpha_i\}$ of $\Gamma'\backslash\Gamma$ by

$$\begin{array}{rcl} \pi : B_{k,L}(\Gamma') & \twoheadrightarrow & B_{k,L}(\Gamma) \\ f & \mapsto & \dfrac{1}{[\Gamma' : \Gamma'']} \displaystyle\sum_i f|_k \, \alpha_i. \end{array} \tag{1.7}$$

It is easy to see that

$$\pi \circ \iota = \mathrm{id}_{B_{k,L}(\Gamma)}.$$

We can use this in the following way: when we want to show that a Maass form is well defined, it will suffice to do so for a perhaps smaller subgroup. We define

$$\tilde{\Gamma}(N) := \ker(\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})).$$

Since for any congruence subgroup Γ, by definition, there is an N such that

$$\tilde{\Gamma}(MN) \leq \tilde{\Gamma}(N) \leq \Gamma, \tag{1.8}$$

it will suffice to show our continuation theorems for a certain suitable subgroup $\tilde{\Gamma}(MN)$, which will allow us to assume without loss of generality $8|N$, simplifying our notation.

2 Petersson-Sobolev Spaces

2.1 Introduction and Definition

2.1.1 Definition

In this section, we would like to introduce the Petersson-Sobolev spaces $\mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$. At first, recall the definition of $\mathcal{L}_k^2(\Gamma\backslash\mathbb{H}, L)$, which denotes the space of square integrable, vector valued, modular forms of weight k with respect to the Petersson scalar product as in Definition (1.11).

Definition 2.1. Define $\mathcal{H}_k^0(\Gamma\backslash\mathbb{H}, L) = \mathcal{L}_k^2(\Gamma\backslash\mathbb{H}, L)$. Let $X_{k,n+1}(\Gamma\backslash\mathbb{H}, L) := \{f \in \mathcal{A}_{k,L}(\Gamma\backslash\mathbb{H}) \mid f \in \mathcal{L}_k^2,\ l_k f \in \mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)\}$. Further, define the scalar product $\langle f, g\rangle_{k,l} := \langle f, g\rangle_k + \langle l_k f, l_k g\rangle_{k-2,n-1}$ with $\langle f, g\rangle_{k,0} := \langle f, g\rangle_k$. Clearly, $X_{k,n}(\Gamma\backslash\mathbb{H}, L)$ together with the scalar product $\langle,\rangle_{k,n}$ is a pre-Hilbert space. We denote the completion of the $X_{k,n}(\Gamma\backslash\mathbb{H}, L)$ with respect to the scalar product by $\mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$ and call them *Petersson-Sobolev spaces.* If the lattice L and congruence subgroup Γ are fixed, we allow dropping the indices $\Gamma\backslash\mathbb{H}$ and L and just write $\mathcal{H}_k^n$ instead.

Remark 2.2. Note that this definition is perfectly natural as we could replace l_k with r_k as by (2.3) they would generate equivalent norms on the $X_{k,n}(\Gamma\backslash\mathbb{H}, L)$.

The importance of the Petersson-Sobolev spaces follows from (3.3) as by this all the integrable Poincaré series P_k defined in the next section (see Theorem (3.1)) lie in $\mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$. Moreover, we will see that these spaces allows us to make two fundamental estimates, one on the asymptotics at the cusp and on the Fourier coefficients of elements therein.

The reason for our choice of name is simple. The spaces just constructed have a striking similarity in manner of construction and behavior to the Sobolev spaces of functional analysis, as we will see in the next section. However, Petersson-Sobolev spaces are not truly Sobolev spaces; one can, however, consider them as a generalization of Sobolev spaces to complex analysis, i.e. they deal with manifolds over complex variables instead of real analytic manifolds.

2.1.2 A Note on Classical Sobolev Spaces

The reason we speak of Sobolev spaces is a fundamental one that directly relates to the Sobolev spaces from functional analysis, which we will relate our definition to for a better understanding of the intuition behind the just introduced spaces.

All of the following is found in any book on functional analysis dealing with Sobolev spaces, for example [2].

In functional analysis on a manifold, let us say $\mathbb{R}$, the Sobolev spaces arise in a similar manner. Our notion of Petersson-Sobolev spaces for automorphic forms is a generalization of this concept: Consider the space of square integrable (complex valued functions) $L^2(\mathbb{R})$. One defines the scalar product

$$\langle f, g\rangle := \int dx\, f\bar{g}.$$

Next, one considers the set

$$Y_0 = \{f \in C^\infty(\mathbb{R}) |\, \langle f, f\rangle < \infty\}.$$

The space $L_2(\mathbb{R})$ then is the completion of the pre-Hilbert space Y_0 with respect to the scalar product. To define the Sobolev spaces, one proceeds inductively: We define $H^0(\mathbb{R}) := L^2(\mathbb{R})$ and defines the pre-Hilbert spaces Y_n

$$Y_n := \{f \in C^\infty(\mathbb{R}) |\, f \in L^2,\, \frac{\partial f}{\partial x} \in H^{n-1}\}$$

with the scalar product

$$\langle f, g\rangle_n = \langle f, g\rangle + \langle \frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}\rangle_{n-1}$$

and sets the Sobolev spaces H^n to be the completion of Y_n with respect to the scalar product $\langle , \rangle_n$. Our definition for Petersson-Sobolev spaces of automorphic forms is directly analogous. The statement of smoothness is equivalent to our concept of automorphic forms, and the role of the derivative in complex coordinates is taken by l_k (or, as we have seen, equivalently r_k). The classical Sobolev spaces satisfy various well known embedding theorems, the so called Sobolev inequalities, which yield inclusions, for example

$$H^1(\mathbb{R}) \subseteq C^0(\mathbb{R}),$$

i.e. all elements of H^1 are, up to definition on a set of measure zero, continuous functions. The notion of continuity for Fréchet spaces implies that for each compact set K and $x \in K$ there is some $C(K)$ such that

$$|f(x)| \leq C(K)\sqrt{\langle f, f\rangle_1}.$$

To see why being in a Sobolev spaces of higher degree imposes regularity, consider a function that has compact support, is smooth everywhere except in $x = 0$ and in some small neighborhood of zero is given by

$$f_\alpha(x) = |x|^\alpha\,. \tag{2.1}$$

Clearly, we see that $f_\alpha \in L^2$ whenever $\alpha > -1/2$. However, while $-1/2 < \alpha < 0$, f_α is not smooth. Evidently, in a small neighborhood of zero,

$$\frac{\partial f_\alpha}{\partial x}(x) = \pm\alpha\, |x|^{\alpha-1}\,,$$

which is in L^2 while $\alpha > 1/2$. Hence, $\frac{\partial f_\alpha}{\partial x} \in L^2$ imposes continuity on f_α. The lemmas in the next section will be in the spirit of the Sobolev inequalities, providing us with estimates on the growth of functions in our Sobolev spaces of automorphic forms. The more derivatives are integrable, the more regularity we can impose on the functions.

2.2 Density of Petersson-Sobolev spaces

To simplify the discussion in the following chapters, it becomes necessary to classify dense subspaces of the Petersson-Sobolev spaces. Definition (2.1) implies the obvious inclusion

$$\mathcal{H}_k^{n+1}(\Gamma\backslash\mathbb{H}, L) \subseteq \mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L).$$

As this inclusion is dense in the case of classical Sobolev spaces, a natural question to ask is whether the respective generalizes, i.e. we have

Lemma 2.3. *The inclusion $\mathcal{H}_k^{n+1}(\Gamma\backslash\mathbb{H}, L) \subseteq \mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$ is dense.*

Following the analogy to classical Sobolev spaces, one state a stronger density lemma which has the previous one as a direct corollary.

Lemma 2.4. *The compactly supported automorphic forms $\mathcal{A}_{k,L}^c(\Gamma\backslash\mathbb{H}) := \{f \in \mathcal{A}_{k,L}(\Gamma\backslash\mathbb{H}) \mid f \text{ has compact support on } \Gamma\backslash\mathbb{H}\}$ are a dense subspace of $\mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$.*

Proof. By compactness of the support, it is clear that $\mathcal{A}_{k,L}^c(\Gamma\backslash\mathbb{H}) \subseteq \mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$. For $\phi \in \mathcal{A}_{k'}^c(\Gamma\backslash\mathbb{H})$ define

$$|\phi|_{k',n}^2 := \sum_{r \leq n} \sup_{\tau \in \Gamma\backslash\mathbb{H}} \left| y^{k'/2} \left(y \left(\frac{\partial}{\partial \bar{\tau}} \right) \right)^r \phi(\tau) \right|^2 .$$

The proof now proceeds by twofold induction in n. At first, let ϕ be as above. Then there is some constant $C(k, k', n)$ such that for any $f \in \mathcal{A}_{k,L}(\Gamma\backslash\mathbb{H}) \cap \mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$

$$\|f\phi\|_{\mathcal{H}_{k+k'}^n} \leq C(k, k', n)\, |\phi|_{k',n}\, \|f\|_{\mathcal{H}_k^n} .$$

The induction start is clear for $n = 0$ as $f\phi$ is in $\mathcal{A}_{k+k',L}^c(\Gamma\backslash\mathbb{H})$. If f is in $\mathcal{A}_k \cap \mathcal{H}_k^{n+1}$ then, by Cauchy-Schwartz

$$\begin{aligned}
\|f\phi\|_{\mathcal{H}_{k+k'}^n}^2 &= \langle f\phi, f\phi\rangle_{k+k'} + \langle (l_k f)\phi, (l_k f)\phi\rangle_{k+k'-2,n-1} \\
&+ \langle f(l_{k'}\phi), f(l_{k'}\phi)\rangle_{k+k'-2,n-1} \\
&+ 2\mathrm{Re}(\langle f(l_{k'}\phi), (l_k f)\phi\rangle_{k+k'-2,n-1}) \\
&\leq \langle f\phi, f\phi\rangle_{k+k'} + 2\langle (l_k f)\phi, (l_k f)\phi\rangle_{k+k'-2,n-1} \\
&+ 2\langle f(l_{k'}\phi), f(l_{k'}\phi)\rangle_{k+k'-2,n-1} \\
&\leq C(k, k', 0)\, |\phi|_{k',0}^2\, \|f\|_{\mathcal{H}_k^0}^2 + 2C(k, k', n-1)\, |\phi|_{k',n-1}^2\, \|l_k f\|_{\mathcal{H}_{k-2}^{n-1}}^2 \\
&+ 2C(k, k', n-1)\, |l_{k'}\phi|_{k'-2,n-1}^2\, \|f\|_{\mathcal{H}_k^{n-1}}^2 \\
&\leq (C(k, k', 0) + 4C(k, k', n-1))\, |\phi|_{k',n}^2\, \|f\|_{\mathcal{H}_k^n}^2 .
\end{aligned}$$

Next, let m_0 be big enough such that for $y > 2^{m_0}$ there are only two equivalent points on the fundamental domain. Let $\varphi(y)$ be a smooth function satisfying

$$\varphi(y) := \begin{cases} 0 & \text{if } y < 2, \\ 1 & \text{if } y > 3. \end{cases}$$

Define

$$\varphi_m(y) := \varphi(y^{1/m}).$$

Then, if $m > m_0$, φ_m is in $\mathcal{A}_0(\Gamma\backslash\mathbb{H})$. Let $\{M_i\} \subseteq \mathrm{Mp}_2(\mathbb{Z})$ be such that $\{M_i^{-1} i\infty\}$ is the set of cusps. Then

$$\psi_m(\tau) := 1 - \sum_i \varphi_m(y)|_k\, M_i$$

vanishes in a neighborhood of any cusp, hence is in $\mathcal{A}_0^c(\Gamma\backslash\mathbb{H})$. Let $f \in \mathcal{A}_{k,L}(\Gamma\backslash\mathbb{H}) \cap \mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$. We claim that $f\psi_m$ converges to f in $\mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$, hence proving the lemma, as, by construction, $\mathcal{A}_{k,L}(\Gamma\backslash\mathbb{H}) \cap \mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$ is a dense subspace of $\mathcal{H}_k^n(\Gamma\backslash\mathbb{H}, L)$. The proof of this is again done by induction. If $n = 0$, we have

$$|f\psi_m(\tau)| \leq |f(\tau)|$$

and $f\psi_m$ converges pointwise to f. Hence, by the dominated convergence theorem, $f\psi_m \to f$ in $\mathcal{H}_k^0$. Now, for $f \in \mathcal{A}_k \cap \mathcal{H}_k^{n+1}$

$$\begin{aligned}
\|f(1-\psi_m)\|^2_{\mathcal{H}_k^{n+1}} &= \|f(1-\psi_m)\|^2_{\mathcal{H}_k^0} + \|l_k(f(1-\psi_m))\|^2_{\mathcal{H}_{k-2}^n} \\
&\leq \|f(1-\psi_m)\|^2_{\mathcal{H}_k^0} + 2\,\|(l_k f)(1-\psi_m)\|^2_{\mathcal{H}_{k-2}^n} \\
&+ \quad 2\,\|f(l_0(1-\psi_m))\|^2_{\mathcal{H}_{k-2}^n}.
\end{aligned}$$

By the induction hypothesis, $f\psi_m \to f$ in $\mathcal{H}_{k-2}^0$ and $(l_k f)\psi_m \to l_k f$ in $\mathcal{H}_{k-2}^n$, hence the first two terms vanish as $m \to \infty$. By the previous induction,

$$\|f(l_0(1-\psi_m))\|^2_{\mathcal{H}_{k-2}^n} \leq C(n)\,\|f\|^2_{\mathcal{H}_k^n}\,|(l_0(1-\psi_m))|^2_{-2,n}.$$

Since

$$l_0(1-\psi_m) = \sum_i (l_0\varphi_m)|_{-2}\, M_i$$

it is in $\mathcal{A}_{-2}^c(\Gamma\backslash\mathbb{H})$ and further

$$|l_0(1-\psi_m)(\tau)|^2_{-2,n} \quad \leq \quad \mathrm{Const}\cdot|l_0\varphi_m|^2_{-2,n}.$$

Note that by definition

$$\begin{aligned}
|l_0\varphi_m|^2_{-2,n} &\leq \mathrm{Const}\cdot\sum_{0<r\leq n}\sup_{\tau\in-\backslash\mathbb{H}}\left|\frac{\partial^r}{\partial\ln(y)^r}\varphi_m(y)\right|^2. \\
&= \mathrm{Const}\cdot\sum_{0<r\leq n} m^{-r}\sup_{\tau\in-\backslash\mathbb{H}}\left|\frac{\partial^r\varphi}{\partial\ln(y)^r}(y^{1/m})\right|^2 \\
&\leq \frac{\mathrm{Const}}{m}.
\end{aligned}$$

Thus

$$\|f(l_0(1-\psi_m))\|^2_{\mathcal{H}_{k-2}^n} \leq \frac{\mathrm{Const}}{m}.$$

and hence $f(1-\psi_m) \to 0$ in $\mathcal{H}_k^{n+1}(\Gamma\backslash\mathbb{H}, L)$. □

2.3 Estimates on Cuspidal Growth

In the following section, we fix some lattice L. For any topological space U, we denote the space of continuous, $\mathbb{C}[L'/L]$ valued functions on U by $C^0(U)$. We equip this space with the topology of uniform convergence on compact subsets. Thus, $C^0(U)$ is a Fréchet space. As previously discussed, we will show that the concept of Petersson-Sobolev spaces provides us with estimates on the cuspidal growth of elements therein. The first estimate we have on Petersson-Sobolev spaces is the one pertaining to the Fourier coefficients.

Lemma 2.5. *The Fourier coefficients as a map between topological vector spaces*

$$\begin{aligned} \psi^m : \mathcal{H}_k^1 &\to C^0(\mathbb{R}_{>0}) \\ f &\mapsto y^{\frac{k-3}{2}} \int_0^N f(x,y) e(-mx)\, dx \end{aligned}$$

are continuous and bounded as $y \to \infty$.

Proof. Let k be fixed. Suppose f is in the space of compactly supported automorphic forms $\mathcal{A}_k^c$. As, by Lemma (2.4), the space $\mathcal{A}_k^c$ is dense in $\mathcal{H}_k^1$ we only need to consider such elements, as it suffices to show that the Fourier coefficients define smooth maps on a dense subset. It is evident that $\psi^m(f)$ is indeed continuous, i.e. an element of $C^0(\mathbb{R}_{>0})$. All that is left to show is that the map $\psi^m(f)$ is bounded by $\|f\|_{k,1}$. Take c and$y_0 \geq c$ such that $\mathcal{F}_{y_0} := \mathcal{F} \cap \{\tau \mid \mathrm{Im}(\tau) \geq y_0\} = \{\tau \mid \mathrm{Im}(\tau) \geq y_0,\, 0 \leq \mathrm{Re}(\tau) \leq N\}$ for all such y_0. Then, for $\epsilon > 0$, since f has compact support on the fundamental domain,

$$\begin{aligned} y_0^{\frac{k-3}{2}-\epsilon} \int_0^N f(x,y_0) e(-mx)\, dx &= i \int_0^N \int_{y_0}^\infty \frac{\partial}{\partial \bar\tau} (y^{\frac{k-3}{2}-\epsilon} f(x,y) \\ &\cdot\ e(-mx))\, dy\, dx \\ &= i \int_{\mathcal{F}_{y_0}} y^{\frac{k-3}{2}-\epsilon} \frac{\partial f}{\partial \bar\tau} e(-mx)\, dy\, dx \\ &+\ 2\pi m \int_{\mathcal{F}_{y_0}} y^{\frac{k-3}{2}-\epsilon} f(x,y) e(-mx)\, dy\, dx \\ &+\ \frac{k-3-2\epsilon}{2} \\ &\cdot\ \int_{\mathcal{F}_{y_0}} y^{\frac{k-5}{2}-\epsilon} f(x,y) e(-mx)\, dy\, dx. \end{aligned}$$

Further, Cauchy-Schwarz implies

$$\int_{y_0}^\infty dy\, y^{\frac{k-3}{2}-\epsilon} \int_0^N \mid f(x,y) \mid\, dx$$

$$\leq \sqrt{\int_{\mathcal{F}_{y_0}} y^{k-2} \mid f \mid^2 \; dy\, dx \cdot \int_{\mathcal{F}_{y_0}} y^{-1-2\epsilon}\, dy\, dx}$$
$$\leq \frac{1}{\sqrt{2\epsilon}} \parallel f \parallel_{k,1} y_0^{-\epsilon}.$$

Using Cauchy-Schwarz again for the different terms allows us to estimate

$$\begin{aligned}
& \mid \int_{\mathcal{F}_{y_0}} y^{\frac{k-3}{2}-\epsilon} \frac{\partial f}{\partial \bar{\tau}} e(-mx)\, dy\, dx \mid \\
\leq \; & \frac{1}{\sqrt{2+2\epsilon}} \parallel f \parallel_{k,1} y_0^{-1-\epsilon}, \\
& \mid \int_{\mathcal{F}_{y_0}} y^{\frac{k-3}{2}-\epsilon} f(x,y) e(-mx)\, dy\, dx \mid \\
\leq \; & \frac{1}{\sqrt{2\epsilon}} \parallel f \parallel_{k,1} y_0^{-\epsilon}, \\
& \mid \int_{\mathcal{F}_{y_0}} y^{\frac{k-5}{2}-\epsilon} f(x,y) e(-mx)\, dy\, dx \mid \\
\leq \; & \frac{1}{\sqrt{2+2\epsilon}} \parallel f \parallel_{k,1} y_0^{-1-\epsilon}.
\end{aligned}$$

Hence

$$|\psi^m(f)(y_0)| \leq C(m,\epsilon) \parallel f \parallel_{k,1}, \tag{2.2}$$

for $y_0 \geq c$. For arbitrary $y > 0$, Now fix arbitrary $c > 0$. For $c \leq y < y_0$, note that finitely many copies of the fundamental domain cover the set $\{\tau \mid \mathrm{Im}(\tau) \geq c,\, 0 \leq \mathrm{Re}(x) \leq N\}$. The proof proceeds analogously, albeit with some prefactor relating to the number of these copies. This proves the continuity of the map ψ^m and moreover

$$\int_0^N f(x,y) e(-mx)\, dx = O(y^{(3-k)/2}).$$

$\square$

Remark 2.6. If $m = 0$, we can improve the estimate to

$$\int_0^N f(x,y)\, dx = O(y^{(1-k)/2})$$

as we can set $\epsilon = -1 + \epsilon'$ in above equations.

Secondly, we want to obtain an estimate on the growth of functions f for $y \to \infty$. Fourier coefficients, with one integral in their definition, admit a bound in $\mathcal{H}_k^1$ and we will see that the limit $y \to \infty$ is nice for elements of $\mathcal{H}_k^2$. Let us state

Lemma 2.7. *There is a continuous inclusion of Fréchet spaces $\mathcal{H}_k^2 \hookrightarrow C^0(\mathbb{H})$. Moreover, any $f \in \mathcal{H}_k^2$ is $O(y^{\frac{1-k}{2}})$ at each cusp.*

Proof. Take an arbitrary $f \in \mathcal{A}_k^c$. Clearly, f is continuous, i.e. in $C^0(\mathbb{H})$. To prove the continuity of the inclusion, first, note that

$$\begin{aligned} 2y^2\frac{\partial}{\partial y} &= l_k + y^2 r_k - ky, \\ 2iy^2\frac{\partial}{\partial x} &= y^2 r_k - l_k - ky. \end{aligned}$$

It suffices to assume that $y_0 \geq c$ such that there are only two equivalent points on the fundamental domain, as in the previous Lemma. Now set

$$g(x,y) := (\frac{\partial}{\partial y} y^{\frac{k+1}{2}-\epsilon} f - \frac{1}{N}\int_0^N \frac{\partial}{\partial y} y^{\frac{k+1}{2}-\epsilon} f\, dx).$$

Then

$$\begin{aligned} \left|\int_{y_0}^{\infty} g(x,y)\, dy\right| &\leq \int_{y_0}^{\infty} |g(x,y)|\, dy \\ &\leq 2\int_{\mathcal{F}_{y_0}} \left|\frac{\partial}{\partial x} g(x,y)\right| dy\, dx \\ &= 2\int_{\mathcal{F}_{y_0}} y^{-1/2-\epsilon} y^{k/2-3} \left|y^2\frac{\partial}{\partial x}(y^2\frac{\partial}{\partial y} f)\right| dy\, dx \\ &+ |k+1-2\epsilon| \\ &\cdot \int_{\mathcal{F}_{y_0}} y^{-1/2-\epsilon} y^{k/2-2} \left|y^2\frac{\partial}{\partial x}(f)\right| dy\, dx. \end{aligned}$$

For the first inequality, we used that

$$\int_0^N g(x,y)\, dx = 0,$$

hence there exist $x_0(y)$, $x_1(y)$ such that $\mathrm{Re}(g(x_0(y),y) = \mathrm{Im}(g(x_1(y),y) = 0$ and hence we can bound g by the integral over the absolute value of its derivative.

Now, applying Cauchy Schwartz yields:

$$
\begin{aligned}
& 2\left|\int\limits_{\mathcal{F}_{y_0}} y^{-1/2-\epsilon} y^{k/2-2}\left|y^2\frac{\partial}{\partial x}(f)\right|\,dy\,dx\right|^2 \\
\leq\ & \frac{y_0^{-2\epsilon}}{\epsilon}\int\limits_{\mathcal{F}_{y_0}} y^{(k-2)-2}\left|y^2\frac{\partial}{\partial x}(f)\right|^2\,dy\,dx \\
\leq\ & \frac{y_0^{-2\epsilon}}{\epsilon}\int\limits_{\mathcal{F}_{y_0}} y^{(k-2)-2}\left|(y^2 r_k - l_k - ky)(f)\right|^2\,dy\,dx \\
\leq\ & C\frac{y_0^{-2\epsilon}}{\epsilon}(\|l_k f\|_{k-2}^2 + \|r_k f\|_{k+2}^2 + \|f\|_k^2) \\
\leq\ & C'\frac{y_0^{-2\epsilon}}{\epsilon}\,\|f\|_{k,2}^2
\end{aligned}
$$

and in a similar manner we find

$$
\begin{aligned}
& 2\left|\int\limits_{\mathcal{F}_{y_0}} y^{-1/2-\epsilon} y^{k/2-3}\left|y^2\frac{\partial}{\partial x}(y^2\frac{\partial}{\partial y}f)\right|\,dy\,dx\right|^2 \\
\leq\ & \frac{y_0^{-2\epsilon}}{\epsilon}\int\limits_{\mathcal{F}_{y_0}} y^{k-6}\left|y^2\frac{\partial}{\partial x}y^2(\frac{\partial}{\partial y}f)\right|^2\,dy\,dx \\
\leq\ & \frac{y_0^{-2\epsilon}}{\epsilon}\int\limits_{\mathcal{F}_{y_0}} y^{k-6}\left|y^2\frac{\partial}{\partial x}(l_k + y^2 r_k - ky)f\right|^2\,dy\,dx \\
\leq\ & C(k)\frac{y_0^{-2\epsilon}}{\epsilon}(\int\limits_{\mathcal{F}_{y_0}} y^{k-6}\left|y^2\frac{\partial}{\partial x}l_k f\right|^2\,dy\,dx \\
+\ & \int\limits_{\mathcal{F}_{y_0}} y^{k-2}\left|y^2\frac{\partial}{\partial x}r_k f\right|^2\,dy\,dx \\
+\ & \int\limits_{\mathcal{F}_{y_0}} y^{k-4}\left|y^2\frac{\partial}{\partial x}f\right|^2\,dy\,dx) \\
\leq\ & C(k)\frac{y_0^{-2\epsilon}}{\epsilon} \\
\cdot\ & (\int\limits_{\mathcal{F}_{y_0}} y^{k-6}\left|(y^2 r_{k-2} - l_{k-2} - (k-2)y)l_k f\right|^2\,dy\,dx \\
+\ & \int\limits_{\mathcal{F}_{y_0}} y^{k-2}\left|(y^2 r_{k+2} - l_{k+2} - (k+2)y)r_k f\right|^2\,dy\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \int\limits_{\mathcal{F}_{y_0}} y^{k-4} \left|(y^2 r_k - l_k - ky) f\right|^2 \, dy \, dx \\
&\leq C(k) \frac{y_0^{-2\epsilon}}{\epsilon} \|f\|_{k,2}^2 .
\end{aligned}
$$

By applying Cauchy-Schwarz sufficiently often, we see that the right hand side is bounded by expressions of types $\| \; r_{k+2} r_k f \; \|_{k+4}$, $\| \; r_{k-2} l_k f \; \|_k$ etc., all of which are bounded by $\|f\|_{k,2}$ yielding the estimate

$$
\int\limits_{y_0}^{\infty} |g(x,y)| \, dy \leq C \cdot \frac{y_0^{-\epsilon}}{\sqrt{\epsilon}} \|f\|_{k,2} .
$$

Now, we recall the previous lemma, (2.2), and continue by considering

$$
\begin{aligned}
& \left| y_0^{\frac{k+1}{2}-\epsilon} f(x, y_0) \right| \\
= | & \int\limits_{y_0}^{\infty} g(x,y) \, dy \\
+ & \frac{1}{N} \int\limits_{y_0}^{\infty} \int\limits_{0}^{N} \frac{\partial}{\partial y} y^{\frac{k+1}{2}-\epsilon} f \, dx \, dy | \\
\leq & \int\limits_{y_0}^{\infty} |g(x,y)| \, dy \\
+ & \left| \frac{1}{N} \int\limits_{0}^{N} y_0^{\frac{k+1}{2}-\epsilon} f(x, y_0) \, dx \right| \\
\leq & \text{Const.} \cdot \|f\|_{k,2} \cdot y_0^{1-\epsilon}.
\end{aligned}
$$

If, for $y_0 < c$, we need more than a single copy of the fundamental domain to cover the set $\{\tau \mid \mathrm{Im}(\tau) \geq c, \, 0 \leq \mathrm{Re}(x) \leq N\}$, the proof proceeds analogously noting that we again obtain a prefactor relating to the finite number of such copies. Thus the inclusion $\mathcal{A}_k^c \hookrightarrow C^0(\mathbb{H})$ is continuous in the norm on $\mathcal{H}_k^2$, hence extends to an inclusion of $\mathcal{H}_k^2$, i.e. any element of $\mathcal{H}_k^2$ is continuous, up to redefinition on a set of measure zero. The growth estimate for the cusp $i\infty$ follows directly from the proof. It holds on the other cusps as well as the Petersson-Slash operator defines an isometry on $\mathcal{H}_k^2$. □

Remark 2.8. One can interpret the lemmas in the sense of Petersson-Sobolev spaces. Elements of $\mathcal{L}_k^2$ are limits of sequences of integrable smooth functions, however, they might not be smooth themselves, similar to the "classical" example (2.1) discussed in the previous section. However, as we consider elements of $\mathcal{H}_k^1$, their Fourier coefficients satisfy certain regularity conditions, the function itself may not be smooth enough to allow estimates on its growth, i.e. its Fourier expansion might not sufficiently converge. Only once we consider elements of $\mathcal{H}_k^2$ we have sufficient regularity to allow estimates on the function itself.

2.4 The Laplace Operator

By definition, the map

$$l_k : \mathcal{H}_k^1 \quad \to \quad \mathcal{L}_{k-2}^2$$

is continuous as for any $f \in \mathcal{H}_k^1(\Gamma\backslash\mathbb{H}, L)$,

$$\langle l_k f, l_k f\rangle_{k-2} \leq \langle f, f\rangle_{k,1}.$$

Recall (see (1.2))

$$\begin{aligned} l_{k+2} r_k &= -\Delta_k - k, \\ r_{k-2} l_k &= -\Delta_k, \\ l_k^\dagger &= -r_{k-2}, \end{aligned} \tag{2.3}$$

where by † we denote the adjoint operator. In the following, we will denote the fundamental domain for a congruence subgroup Γ by $\Gamma\backslash\mathbb{H}$ to signify the dependence of the fundamental domain on Γ. This contrasts the previous sections where the fundamental domain and subgroup were fixed, hence we used $\mathcal{F}$ to denote the fundamental domain therein. As for all $f, g \in \mathcal{H}_k^1$ sufficiently smooth we have, by partial integration,

$$\begin{aligned} \langle \Delta_k f, f\rangle_k &= \langle l_k f, l_k f\rangle_{k-2} \\ &= 4 \int\limits_{\Gamma\backslash\mathbb{H}} \frac{\partial f}{\partial \bar\tau} \frac{\partial \bar f}{\partial \tau} \cdot y^{-k} \cdot dx \cdot dy \\ &= \int\limits_{\Gamma\backslash\mathbb{H}} ((\Delta_k f)\bar f \cdot y^k \frac{dx \cdot dy}{y^2} \\ &= \langle r_k f, r_k f\rangle_{k+2} - k\langle f, f\rangle_k, \end{aligned}$$

Δ_k is a positive operator and hence $\mathrm{Spec}(\Delta_k) \subseteq [\max\{0, -k\}, \infty)$. As a direct consequence, there are no integrable modular forms of weight $k < 0$. Furthermore, Δ_k as a map

$$\Delta_k : \mathcal{H}_k^2 \quad \to \quad \mathcal{L}_k^2$$

is evidently continuous. Note that the adjoint of this yields a map (which we denote by Δ_k^*)

$$\begin{aligned} \Delta_k^* : \mathcal{L}_k^2 &\to \mathcal{H}_k^2 \\ f &\mapsto \{h \mapsto \langle f, \Delta_k h\rangle_k\}. \end{aligned}$$

We thus have a notion of applying Δ_k to elements of $\mathcal{L}_k^2$. Let f be in $\mathcal{L}_k^2$. We say that $\Delta_k f$ is in $\mathcal{L}_k^2$ if there is some g in $\mathcal{L}_k^2$ such that for all $h \in \mathcal{H}_k^2$

$$\langle f, \Delta_k h \rangle_k = \langle g, h \rangle_k.$$

We then write

$$\Delta_k f := g.$$

By Lemma (2.4), $\mathcal{H}_k^2$ is dense in $\mathcal{H}_k^0$, hence, if it exists, g is unique. This provides an alternative definition of $\mathcal{H}_k^2$:

Lemma 2.9. *We have* $\mathcal{H}_k^2 = \left\{ f \in \mathcal{L}_k^2 | \, \Delta_k f \in \mathcal{L}_k^2 \right\}$.

Proof. The inclusion "$\subseteq$" follows directly from the definition. First of all, note that the right hand side is a Hilbert space with respect to the scalar product

$$\langle f, g \rangle'_{k,2} = \langle f, g \rangle_k + \langle \Delta_k f, \Delta_k g \rangle_k.$$

To prove this, we have to show that it is complete with respect to the norm. Suppose that $\{v_a\}_{a \in \mathbb{N}}$ is a Cauchy sequence in the set $\left\{ f \in \mathcal{L}_k^2 | \, \Delta_k f \in \mathcal{L}_k^2 \right\} =: \tilde{\mathcal{H}}_k^2$. Then , by definition, both v_a and $\Delta_k v_a$ are Cauchy sequences in $\mathcal{L}_k^2$. Hence there exists a v and w in $\mathcal{L}_k^2$ such that v_a converges to v and $\Delta_k v_a$ converges to w in $\mathcal{L}_k^2$. As

$$\langle v_a, \Delta_k h \rangle_k = \langle \Delta_k v_a, h \rangle_k$$

for all a and $h \in \mathcal{H}_k^2$, we must have

$$\langle v, \Delta_k h \rangle_k = \langle w, h \rangle_k$$

and thus $v \in \tilde{\mathcal{H}}_k^2$. Hence, it is complete with respect to the norm and as such, a Hilbert space. Note that this is also an equivalent scalar product on $\mathcal{H}_k^2$ as

$$\langle l_k f, l_k f \rangle_{k-2} = \langle f, \Delta_k f \rangle_k.$$

Now let $f \in \tilde{\mathcal{H}}_k^2$ be in the orthogonal complement of $\mathcal{H}_k^2$ as a subspace of $\tilde{\mathcal{H}}_k^2$. It suffices to show that $f = 0$. As $\mathcal{H}_k^2$ is dense in $\mathcal{L}_k^2$, there is a sequence $f_a \in \mathcal{H}_k^2$ such that $f_a \to f$ in $\mathcal{L}_k^2$. Since

$$\langle \Delta_k f_a, h \rangle_k = \langle f_a, \Delta_k h \rangle_k \to \langle f, \Delta_k h \rangle_k = \langle \Delta_k f, h \rangle_k$$

for all $h \in \mathcal{H}_k^2$, $\Delta_k f_a$ weakly converges to $\Delta_k f$. Now, by orthogonality of f

$$\begin{aligned} \langle f, f \rangle'_{k,2} + \langle f_a, f_a \rangle'{}_{k,2} &= \langle f - f_a, f - f_a \rangle'_{k,2} \\ &= \langle f - f_a, f - f_a \rangle_k + \langle \Delta_k (f - f_a), \Delta_k (f - f_a) \rangle_k \\ &= \langle f - f_a, f - f_a \rangle_k + \langle \Delta_k f, \Delta_k f \rangle_k + \langle \Delta_k f_a, \Delta_k f_a \rangle_k \\ &- 2\mathrm{Re}(\langle \Delta_k f_a, \Delta_k f \rangle_k). \end{aligned}$$

Rearranging the terms yields

$$\langle f, f \rangle_k + \langle f_a, f_a \rangle_k = \langle f - f_a, f - f_a \rangle_k - 2\mathrm{Re}(\langle \Delta_k f_a, \Delta_k f \rangle_k).$$

Since $\Delta_k f_a$ converges weakly to $\Delta_k f$ in $\mathcal{L}_k^2$, the right hand side is non positive as $a \to \infty$, hence we have a contradiction unless $f = 0$. $\square$

Thus, $\mathcal{H}^2_k$ is the maximal space which maps subsets of $\mathcal{L}^2_k$ into $\mathcal{L}^2_k$ by Δ_k, hence we can consider its spectrum as defined via the map $\Delta_k : \mathcal{H}^2_k \to \mathcal{L}^2_k$ and view $\mathcal{H}^2_k$ as the natural domain of Δ_k.

2.4.1 The Scalar Spectrum of Δ_k

The spectrum of Δ_k in $\mathcal{L}^2_k(\Gamma\backslash\mathbb{H})$, i.e. the space of scalar integrable automorphic forms, has been analyzed in great detail in [9] and [10, Satz 12.3], however, as far as we can say, the result has not been generalized to $\mathcal{L}^2_k(\Gamma\backslash\mathbb{H}, L)$, i.e. vector valued integrable automorphic forms, yet. However, the Hilbert space and Laplace operator treated therein are not quite the one we study, yet they are related by a simple transformation. We define

$$-\tilde{\Delta}_k = 4y^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} - iky\frac{\partial}{\partial x}.$$

It is densely defined on the Hilbert space

$$\mathcal{L}^2_k\widetilde{(\Gamma\backslash\mathbb{H}, L)} \coloneqq y^{k/2}\mathcal{L}^2_k(\Gamma\backslash\mathbb{H}, L)$$

with the scalar product chosen such that the relation is an isometry. We further have

$$y^{k/2}\Delta_k y^{-k/2} = \tilde{\Delta}_k - \frac{k}{2}(1-\frac{k}{2})$$

as

$$y^{k/2}[-y^2\frac{\partial^2}{\partial y^2} - ky\frac{\partial}{\partial y}, y^{-k/2}] \quad = \quad -\frac{k}{2}(1-\frac{k}{2}) + ky\frac{\partial}{\partial y}$$

where $[\cdot,\cdot]$ is the commutator bracket for two operators. Thus,

$$\mathrm{Spec}(\Delta_k) = \mathrm{Spec}(\tilde{\Delta}_k) + \frac{k}{2}(1-\frac{k}{2}).$$

The full result of [9, 10] pertains to $\mathrm{Spec}(\tilde{\Delta}_k)$, but since both Hilbert spaces are isometric, it is readily adapted as follows.

Theorem 2.10. *Let $\mathcal{L}^{2,\text{dis.}}_k(\Gamma\backslash\mathbb{H})$ be the subspace of $\mathcal{L}^2_k(\Gamma\backslash\mathbb{H})$ spanned by the eigenfunctions of Δ_k. The spectrum of Δ_k in its orthogonal complement, $\mathcal{L}^{2,\text{cont.}}_k(\Gamma\backslash\mathbb{H})$, which is closed under the operation of Δ_k, is continuous and contained in $[(k-1)^2/4, \infty)$.*

We now want to generalize this to lattices.

2.4.2 The Vector-valued Case

We prove this by reducing the vector to the scalar case. Let

$$\Gamma' = \tilde{\Gamma}(N) \coloneqq \ker(\mathrm{Mp}_2(\mathbb{Z})) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

such that $N < \gamma, \gamma > \in 2\mathbb{Z}\,\forall \gamma \in L'$ and $8|N$. If $2k$ is odd, the Weil representation on Γ' is inferred from [14, Proposition 1.6]. If $2k$ is even, we have a stronger result available; the Weil representation is known for arbitrary matrices. The full representation in the even case was first derived in [11]. Using both, for $M = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm 1\right) \in \Gamma''$ if $2k + b_+ - b_- \equiv 0 \bmod 2$, we find

$$f e_\gamma|_k\, M = \left(\frac{c}{d}\right)^{2k} \frac{1}{\sqrt{c\tau + d}^{2k}} f(M\tau) e_\gamma.$$

Here $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol as defined in [13]. Without loss of generality we choose N such that $\tilde{\Gamma}(N) \subseteq \Gamma$. If we denote by $\mathcal{A}_{k,L}(\Gamma)$ the space of $\mathbb{C}[L'/L]$- valued modular forms with respect to Γ and by $\mathcal{A}_k(\Gamma)$ the space of $\mathbb{C}$- valued modular forms, this implies an isomorphism

$$\mathcal{A}_{k,L}(\tilde{\Gamma}(N)) \cong \mathcal{A}_k(\tilde{\Gamma}(N))^{|L'/L|}. \tag{2.4}$$

As we further have the inclusion

$$\mathcal{A}_{k,L}(\Gamma) \hookrightarrow \mathcal{A}_{k,L}(\tilde{\Gamma}(N))$$

and the trace

$$\mathcal{A}_{k,L}(\tilde{\Gamma}(N)) \twoheadrightarrow \mathcal{A}_{k,L}(\Gamma)$$

we have the following

Lemma 2.11. *The maps above induce continuous maps*

$$\begin{aligned} \mathcal{H}_k^l(\tilde{\Gamma}(N)\backslash\mathbb{H}, L) \quad &\cong \quad \mathcal{H}_k^l(\tilde{\Gamma}(N)\backslash\mathbb{H})^{|L'/L|}, \\ \iota:\ \mathcal{H}_k^l(\Gamma\backslash\mathbb{H}, L) \quad &\hookrightarrow \quad \mathcal{H}_k^l(\tilde{\Gamma}(N)\backslash\mathbb{H}, L), \\ \pi:\ \mathcal{H}_k^l(\tilde{\Gamma}(N)\backslash\mathbb{H}, L) \quad &\twoheadrightarrow \quad \mathcal{H}_k^l(\Gamma\backslash\mathbb{H}, L). \end{aligned} \tag{2.5}$$

Proof. Note that it suffices to prove the statements for $l = 0$ by the usual commutation rules for l_k. To prove the first isomorphy, note that for $f = \sum_h f_h \mathfrak{e}_h \in \mathcal{H}_{0,2}^k(\tilde{\Gamma}(N)\backslash\mathbb{H}, L)$, the isomorphy is simply given by

$$f \to (f_h)_h$$

which is an isometry of Hilbert spaces as

$$\langle f, f\rangle_{k,\Gamma} = \sum_h \int_{\Gamma\backslash\mathbb{H}} f_h \bar{f}_h \cdot y^k \frac{dx \cdot dy}{y^2}.$$

The inclusion ι is evidently injective and continuous as, by choosing a system of representatives $\{\gamma_i\}$ of $\tilde{\Gamma}(N)\backslash\Gamma$,

$$\langle f, f\rangle_{k,\tilde{\Gamma}(N)} \quad = \quad \int_{\tilde{\Gamma}(N)\backslash\mathbb{H}} f\bar{f} \cdot y^k \frac{dx \cdot dy}{y^2}$$

$$
\begin{aligned}
&= \sum_i \int\limits_{\gamma_i(\Gamma\backslash\mathbb{H})} f\bar{f}\cdot y^k \frac{dx\cdot dy}{y^2} \\
&= \sum_i \int\limits_{\Gamma\backslash\mathbb{H}} f\bar{f}\cdot y^k \frac{dx\cdot dy}{y^2} \\
&= \left|[\Gamma:\tilde{\Gamma}(N)]\right| \int\limits_{\Gamma\backslash\mathbb{H}} f\bar{f}\cdot y^k \frac{dx\cdot dy}{y^2} \\
&= \left|[\Gamma:\tilde{\Gamma}(N)]\right| \langle f,f\rangle_{k,\Gamma}.
\end{aligned}
$$

The continuity of the trace follows from (where we use Cauchy-Schwarz)

$$
\begin{aligned}
\langle \pi f,\pi f\rangle_{k,\Gamma} &= \int\limits_{\Gamma\backslash\mathbb{H}} \pi f\overline{\pi f}\cdot y^k \frac{dx\cdot dy}{y^2} \\
&\le \int\limits_{\tilde{\Gamma}(N)\backslash\mathbb{H}} \pi f\overline{\pi f}\cdot y^k \frac{dx\cdot dy}{y^2} \\
&= \frac{1}{\left|[\Gamma:\tilde{\Gamma}(N)]\right|^2} \\
&\cdot \sum_{i,j} \int\limits_{\tilde{\Gamma}(N)\backslash\mathbb{H}} f|_k\,\gamma_i\cdot\overline{f|_k\,\gamma_j}\cdot y^k \frac{dx\cdot dy}{y^2} \\
&\le \frac{1}{\left|[\Gamma:\tilde{\Gamma}(N)]\right|^2} \sum_{i,j} \sqrt{\langle f|_k\,\gamma_i, f|_k\,\gamma_i\rangle_{k,\tilde{\Gamma}(N)}} \\
&\cdot \sqrt{\langle f|_k\,\gamma_j, f|_k\,\gamma_j\rangle_{k,\tilde{\Gamma}(N)}} \\
&= \frac{1}{\left|[\Gamma:\tilde{\Gamma}(N)]\right|^2} \sum_{i,j} \langle f,f\rangle_{k,\tilde{\Gamma}(N)} \\
&= \langle f,f\rangle_{k,\tilde{\Gamma}(N)}.
\end{aligned}
$$

□

Having proven this, we can show

Theorem 2.12. *(Generalization of Theorem (2.10) to lattices.) The continuous spectrum of Δ_k in $\mathcal{L}_k^2(\Gamma\backslash\mathbb{H},L)$ is contained in $[(k-1)^2/4,\infty)$. Furthermore, the discrete spectrum is spanned by eigenvalues of Δ_k, which form a discrete subset of $[0,\infty)$.*

Proof. Now, as $\Delta_k\pi = \pi\Delta_k$, we must have

$$\pi(\mathcal{L}_k^{2,\text{dis.}}(\tilde{\Gamma}(N)\backslash\mathbb{H},L)) = \mathcal{L}_k^{2,\text{dis.}}(\Gamma\backslash\mathbb{H},L)$$

and

$$\pi(\mathcal{L}_k^{2,\text{cont.}}(\tilde{\Gamma}(N)\backslash\mathbb{H}, L)) = \mathcal{L}_k^{2,\text{cont.}}(\Gamma\backslash\mathbb{H}, L),$$

To see this, suppose g is an eigenfunction in $\mathcal{L}_k^{2,\text{dis.}}(\Gamma\backslash\mathbb{H}, L)$, $h \in \mathcal{L}_k^{2,\text{cont.}}(\tilde{\Gamma}(N)\backslash\mathbb{H}, L)$ and $\{\gamma_i\}$ is our usual system of representatives. Then

$$\begin{aligned}
\langle g, \pi h\rangle_{\mathcal{L}_k^2(\Gamma\backslash\mathbb{H},L)} &= \frac{1}{\left|[\Gamma : \tilde{\Gamma}(N)]\right|} \sum_i \int\limits_{(\Gamma\backslash\mathbb{H})} g \cdot \overline{f|_k \gamma_i} \cdot y^k \frac{dx \cdot dy}{y^2} \\
&= \frac{1}{\left|[\Gamma : \tilde{\Gamma}(N)]\right|} \sum_i \int\limits_{\gamma_i(\Gamma\backslash\mathbb{H})} g \cdot \overline{f} \cdot y^k \frac{dx \cdot dy}{y^2} \\
&= \frac{1}{\left|[\Gamma : \tilde{\Gamma}(N)]\right|} \langle g, h\rangle_{\mathcal{L}_k^2(\tilde{\Gamma}(N)\backslash\mathbb{H},L)} \\
&= 0.
\end{aligned}$$

Now further suppose that λ is not in the spectrum of Δ_k as an operator on $\mathcal{L}_k^{2,\text{cont.}}(\tilde{\Gamma}(N)\backslash\mathbb{H}, L)$. Then

$$\begin{aligned}
& ((\Delta_k - \lambda)\,|_{\mathcal{L}_k^{2,\text{cont.}}(\Gamma\backslash\mathbb{H},L)})^{-1} \\
= \;& \pi(\Delta_k - \lambda)^{-1}\iota,
\end{aligned}$$

i.e. λ is not in the spectrum of Δ_k as an operator on $\mathcal{L}_k^{2,\text{cont.}}(\Gamma\backslash\mathbb{H}, L)$, i.e.

$$\begin{aligned}
\operatorname{Spec}(\Delta_k|_{\mathcal{L}_k^{2,\text{cont.}}(\Gamma\backslash\mathbb{H},L)}) &\subseteq \operatorname{Spec}(\Delta_k|_{\mathcal{L}_k^{2,\text{cont.}}(\tilde{\Gamma}(N)\backslash\mathbb{H},L)}) \\
&\subseteq [(k-1)^2/4, \infty),
\end{aligned}$$

proving the theorem. □

This allows us to invert expressions involving the Laplace operator in the next section.

2.4.3 The Meromorphic Operator $(\Delta_k - s(1-s-k))^{-1}$

At first, we briefly recapitulate the concept of a meromorphic operator in a Hilbert space.

Definition 2.13. Let $U \subseteq \mathbb{C}$ be an open set and H a Hilbert space. Let $S \subseteq U$ be discrete and $T : U\backslash S \times H \to H$ be a continuous map. T is *meromorphic* if for any holomorphic function $f : U \to H$ and any $g \in H$ the function $\langle T(s, f(s)), g\rangle_H : U\backslash S \to \mathbb{C}$ is meromorphic.

Now, set $U = \left\{s \in \mathbb{C} |\operatorname{Re}(s) > \frac{(k-1)^2}{4}\right\}$ and $S = \{s \in U |\, s(1-s-k) \in \operatorname{Spec}(\Delta_k)\}$. By the previous section, S is a finite subset of U and entirely on the real axis as

$$s(1-s-k) \in \mathbb{R} \quad \Leftrightarrow \quad (\operatorname{Re}(s) = \frac{(1-k)^2}{4}$$

$$\text{or} \quad \mathrm{Im}(s) = 0)$$

Thus, $(\Delta_k - s(1-s-k))^{-1}$ is well defined and continuous as a map from $U \setminus S \times \mathcal{L}_k^2$ into $\mathcal{L}_k^2$. Now, take $f : U \to \mathcal{L}_k^2$ be holomorphic. Let $\mathcal{L}_k^{2,\mathrm{cont.}}$ again be the orthogonal complement of the eigenfunctions in $\mathcal{L}_k^2$. We further set $\mathcal{L}_k^{2,\mathrm{dis.}}$ to be the space spanned by the eigenfunctions. As the continuous spectrum is contained in $[(k-1)^2/4, \infty)$, for any $g \in \mathcal{L}_k^{2,\mathrm{cont.}}$, $(\Delta_k - \bar{s}(1-\bar{s}-k))^{-1} g \in \mathcal{L}_k^2$ is anti-holomorphic while $s \in U$. Thus

$$\begin{aligned} & \langle (\Delta_k - s(1-s-k))^{-1} f(s), g \rangle_k \\ = \quad & \langle f(s), (\Delta_k - \bar{s}(1-\bar{s}-k))^{-1} g \rangle_k \end{aligned}$$

for all $s \in U \backslash S$ and the meromorphicity is obvious. Now, suppose that $g \in \mathcal{L}^k_{2,\mathrm{dis.}}$. We may further suppose that it is an eigenfunction, i.e. $\Delta_k g = \lambda g$. Then

$$(\Delta_k - \bar{s}(1-\bar{s}-k))^{-1} g = \frac{1}{\lambda - \bar{s}(1-\bar{s}-k)} g$$

and hence

$$\begin{aligned} & \langle (\Delta_k - s(1-s-k))^{-1} f(s), g \rangle_k \\ = \quad & \langle f(s), (\Delta_k - \bar{s}(1-\bar{s}-k))^{-1} g \rangle_k \\ = \quad & \frac{1}{\lambda - s(1-s-k)} \langle f(s), g \rangle_k, \end{aligned}$$

which is meromorphic in U as well. Hence, $(\Delta_k - s(1-s-k))^{-1}$ is a well defined meromorphic operator mapping $\mathcal{L}_k^2$ onto itself. However, we could, by the same principles, also regard it as a meromorphic operator mapping $\mathcal{L}_k^2$ onto $\mathcal{H}_k^2$, simply by noting that

$$\begin{aligned} & \Delta_k (\Delta_k - s(1-s-k))^{-1} \\ = \quad & \mathrm{id} + s(1-s-k)(\Delta_k - s(1-s-k))^{-1}, \end{aligned}$$

the proof of which is analogous to the above.

3 Poincaré Series

3.1 Continuation of Poincaré Series

We formulate our first theorem pertaining to Poincaré series, but first we need to introduce a little more notation. If Γ is some congruence subgroup, we define

$$\Gamma_\infty = \{M \in \Gamma |\, M i\infty = i\infty\}\,.$$

This allows us to formulate our first

Theorem 3.1. *For any smooth function $f : \mathbb{R} \to \mathbb{C}$ real-analytic around zero and any congruence subgroup*

$$\Gamma \leq \mathrm{Mp}_2(\mathbb{Z})$$

of level N and any lattice L, the related Poincaré series for $m \in \frac{1}{N}\mathbb{Z}\backslash\{0\}$, given by

$$P_k(f, m, \tau, s, \gamma) := \sum_{M \in \Gamma_\infty \backslash \Gamma} (y^s f(y) e_\gamma(mx)) \mid_k M,$$

converges absolutely for $2\sigma + k > 2$. It has a meromorphic continuation to $2\sigma + k > 1$ with at most finitely many simple poles on the real axis for both $\sigma(1 - \sigma - k) \geq \max\{0, -k\}$ and $2\sigma + k \leq 2$.

3.1.1 Equivalent Statements to Theorem (3.1).

Since f as in the statement of the theorem above is analytic around zero, we can expand it

$$f(y) = C + y r_f(y)$$

such that

$$|r(y)| \leq D$$

for some positive constants C, D while y is sufficiently small. Hence

$$\begin{aligned} P_k(f, m, \tau, s, \gamma) &= C \sum_{M \in \Gamma_\infty \backslash \Gamma} (y^s e_\gamma(mx)) \mid_k M \\ &+ \sum_{M \in \Gamma_\infty \backslash \Gamma} (y^{s+1} r_f(y) e_\gamma(mx)) \mid_k M. \end{aligned}$$

We do note the absolute convergence of the first term for $2\sigma + k > 2$. Moreover, the second term converges absolutely while $2\sigma + k > 0$ as r_f is bounded as $y \to 0$. As the analytic properties of the first term do not depend on f and the holomorphicity of the second term is guaranteed by absolute convergence, P_k has a meromorphic continuation if and only if it the theorem holds for one arbitrary but fixed $f(y)$. An ideal candidate is $f(y) = e(i\,|m|\,y)$, where we refer the reader to equation (1.1) for notation. Thus we have shown that Theorem (3.1) is equivalent to

Theorem 3.2. *For any congruence subgroup $\Gamma \leq \mathrm{Mp}_2(\mathbb{Z})$ and any lattice L, the related Poincaré series form $\in \frac{1}{N}\mathbb{Z}\backslash\{0\}$, given by*

$$P_k(m,\tau,s,\gamma) := \sum_{M \in \Gamma_\infty\backslash\Gamma} (y^s e_\gamma(mx + i\,|m|\,y))\ |_k\ M$$

has a meromorphic continuation to $2\sigma+k>1$ with at most finitely many simple poles on the real axis for $\sigma(1-\sigma-k) \geq \max\{0,-k\}$ and $2\sigma+k \leq 2$.

Furthermore, note that while $m \neq 0$ we estimate

$$|P_k(m,\tau,s,\gamma)|_k\,M| \leq C(M)\,y^{1-\sigma-k}$$

as $y \to \infty$ for all $M \in \mathrm{Mp}_2(\mathbb{Z})$ and $2\sigma+k>2$. This estimate is crucial! If $m=0$, the case of Eisenstein series, the best estimate is

$$|P_k(0,\tau,s,\gamma)|_k\,M| \leq C(M)\max\{y^{1-\sigma-k}, y^\sigma\}$$

which will make it far more difficult to employ spectral theory as the Eisenstein series are not square integrable.

Remark 3.3. To prove the estimate for $m \neq 0$ and M such that $Mi\infty \neq i\infty$ note that we can write

$$y^{k/2}\,|P_k(m,\tau,s,\gamma)|_k\,M| \leq \{\sum_{M \in \mathrm{Mp}_2(\mathbb{Z})} \mathrm{Im}(My)^{\sigma+k/2} - y^{\sigma+k/2}\}. \tag{3.1}$$

If however $Mi\infty = i\infty$ we can bound the sum by

$$\begin{aligned} y^{k/2}\,|P_k(m,\tau,s,\gamma)|_k\,M| \quad &\leq \quad y^{\sigma+k/2}e^{-|m|y} \\ &+ \sum_{M \in \mathrm{Mp}_2(\mathbb{Z})} \mathrm{Im}(My)^{\sigma+k/2} - y^{\sigma+k/2}. \end{aligned} \tag{3.2}$$

3.1.2 Continuing Poincaré Series

Recall

$$P_k(m,\tau,s,\gamma) = \sum_M (y^s e_\gamma(mx + i\,|m|\,y))|_k\,M,$$

where $m \neq 0$. By the equations (3.1) and (3.2), we have the estimate

$$|P_k(m,\tau,s,\gamma)|_k\,M| \leq C(M)y^{1-k-\sigma}$$

for all $M \in \mathrm{Mp}_2(\mathbb{Z})$. Next, let us momentarily denote by

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

the heavyside step function. A straightforward calculation shows

$$l_k P_k(m,\tau,s,\gamma) \quad = \quad sP_{k-2}(m,\tau,s+1,\gamma)$$

$$- \quad 4\pi \mid m \mid (1-\theta(m))P_{k-2}(m,\tau,s+2,\gamma), \qquad (3.3)$$

$$\begin{aligned} r_k P_k(m,\tau,s,\gamma) &= (s+k)P_{k+2}(m,\tau,s-1,\gamma) \\ &- 4\pi \mid m \mid \theta(m)P_{k+2}(m,\tau,s,\gamma). \end{aligned}$$

Thus, P_k is in the Petersson-Sobolev space $\mathcal{H}_k^2$ for $2\sigma + k > 2$ since

$$y^{k-2} \left| P_k(m,\tau,s,\gamma)|_k M \right|^2 = O(y^{-2\sigma-k})$$

in all cusps. Lastly (by using equation (1.2)), we have

$$\begin{aligned} -\Delta_k P_k(m,\tau,s,\gamma) &= s(s+k-1)P_k(m,\tau,s,\gamma) \\ &- 4\pi \mid m \mid (s+k(1-\theta(m))) \\ &\cdot P_k(m,\tau,s+1,\gamma). \end{aligned} \qquad (3.4)$$

Let us define

$$Q_k(m,\tau,s,\gamma) = 4\pi \mid m \mid (s+k(1-\theta(m))) \cdot P_k(m,\tau,s+1,\gamma),$$

which by the previous arguments again must be in $\mathcal{H}_k^2$ for $2\sigma + k > 1$. This allows us to state the

Proof. (Of Theorem (3.1)) As $Q_k(m,\tau,s,\gamma)$ is square integrable, it satisfies

$$P_k(m,\tau,s,\gamma) = (\Delta_k - s(1-k-s))^{-1} Q_k(m,\tau,s,\gamma).$$

The right hand side is holomorphic while $s(1-s-k)$ is not in $\mathrm{Spec}(\Delta_k) \subseteq [\max\{0,-k\},\infty)$ and $2\sigma+k>0$. This can happen at most if $s=\sigma$ or $2\sigma+k=1$. Hence, the right hand side and thus the left hand side has a continuation to $2\sigma + k > 1$. Furthermore, it is obvious that, by definition, the Poincaré series are holomorphic in s while $2\sigma + k > 2$. To see that the continuation indeed has poles, by discreteness of the spectrum, we can expand P_k

$$\begin{aligned} P_k(m,\tau,s,\gamma) &= P_k'(m,\tau,s,\gamma) \\ &+ \sum_l \alpha_{k,l}(m,s,\gamma) f_{k,l}(\tau) \end{aligned}$$

such that the $f_{k,l}$ are an orthonormal basis for the eigenvalues $< (k-1)^2/4$ and P_k' lies in the orthogonal complement. As $(\Delta_k - s(1-s-k))^{-1}$ is holomorphic on the orthogonal complement, since $2\sigma + k > 1$, and by the uniqueness of the decomposition, $P_k'(m,\tau,s)$ must be holomorphic for $2\sigma + k > 1$. Now,

$$(\Delta_k - s(1-s-k)) f_{k,l} = (\lambda_l - s(1-s-k)) f_{k,l}.$$

The decomposition then implies

$$(\lambda_l - s(1-s-k))\alpha_{k,l}(m,s,\gamma) = 4\pi \mid m \mid (s+k(1-\theta(m)))$$

$$\cdot \quad \alpha_{k,l}(m, s+1, \gamma).$$

Since

$$\frac{d}{ds}(\lambda_l - s(1-s-k)) = 2s + k - 1$$

we see that the $\alpha_l^k(m,s,\gamma)$ can have at most simple poles at $\lambda_l - s(1-s-k) = 0$, and hence the same is valid for P_k proving our theorem. $\square$

3.2 The Case of Small k

Since one of our goals is constructing holomorphic Poincaré series, we will experience some difficulties defining them whenever $0 \leq k \leq 2$. To deal with this, we will show

Theorem 3.4. *If $3/2 \leq k \leq 2$, there is no pole at $s = 0$. If $0 \leq k \leq 1/2$, there is a simple pole at $s = 1-k$ only if $m > 0$.*

Proof. The proof is rather straightforward. At first, we recall isometry $*$ and the ξ-operator

$$\begin{array}{rcl} *_k : \mathcal{L}_k^2 & \to & \mathcal{L}_{-k}^2 \\ \xi_k(f) & := & y^k \bar{f}, \\ \xi_k : \mathcal{H}_2^1 & \to & \mathcal{L}_{2-k}^2 \\ f & \to & *_{k-2}(l_k f). \end{array}$$

This induces a map

$$\xi_k : \mathcal{H}_k^1 \to \mathcal{L}_{2-k}^2.$$

Since

$$*_k(P_k(m,\tau,s,\gamma)) = P_{-k}(-m,\tau,\bar{s}+k,\gamma),$$

we have

$$\begin{array}{rcl} (\xi_k P_k)(m,\tau,s,\gamma) & = & \bar{s} P_{2-k}(-m,\tau,\bar{s}+k-1,\gamma) \\ & - & 4\pi \mid m \mid (1-\theta(m)) \\ & \cdot & P_{2-k}(-m,\tau,\bar{s}+k,\gamma). \end{array} \tag{3.5}$$

If $2 \geq k \geq 3/2$, as $s \to 0$, the right hand side has no pole (since the poles of P_{2-k} are all simple). Since [10] implies invertibility of Δ_k on the subspace orthogonal to $\ker \xi_k$, we have that

$$\xi_k^{-1} = \xi_{2-k} \Delta_{2-k}^{-1}.$$

Thus, ξ_k is invertible on the same subspace and P_k can have poles only parallel to its holomorphic coefficients. We decompose

$$\begin{array}{rcl} P_k(m,\tau,s,\gamma) & = & P_k''(m,\tau,s,\gamma) \\ & + & \displaystyle\sum_l \gamma_{k,l}(m,s,\gamma) f_{k,l}(\tau) \end{array}$$

such that the $f_{k,l}$are an orthonormal basis of the zero eigenvalues, i.e. integrable holomorphic modular forms. We know that $P''_k(m,\tau,s,\gamma)$ can not have a pole in $s=0$, thus all poles in that point must be poles of the $\gamma_l^k(m,s,\gamma)$. However, the relation

$$\begin{aligned}
\gamma_l^k(m,s,\gamma) &= \langle P_k(m,\cdot,s,\gamma), f_l^k\rangle \\
&= \int\limits_{\mathcal{F}} \sum_M (y^s e_\gamma(mx+i\,|m|\,y))|_k\, M \cdot \overline{f_l^k}\, dx\, dy \\
&= \int\limits_0^\infty \int\limits_0^N e_\gamma(mx+i\,|m|\,y)\overline{f_l^k}\,\frac{dx\,dy}{y^{2-k-s}} \\
&= \overline{f_{l,m,\gamma}^k} \int\limits_0^\infty e(2i\,|m|\,y)\,\frac{dy}{y^{2-k}} = \\
&= \overline{f_{l,m,\gamma}^k}\Gamma(s+k-1)(4\pi\,|m|)^{1-s-k} \qquad (3.6)
\end{aligned}$$

for the Fourier coefficients $f_{l,m,\gamma}^k$ of f_l^k holds. We also note that the integral vanishes if $m<0$, since there are no non trivial holomorphic modular forms. The coefficients are thus obviously holomorphic for $\sigma > 1-k$ and hence $P_{3/2}$ has no pole in $s=0$. We also see that there is a pole if $s=1-k$ if $0 \le k \le 1/2$ and $m>0$ if there are non vanishing integrable holomorphic modular forms. □

Corollary 3.5. *If $3/2 \le k \le 2$ the Poincaré series $\{P_k(m,\tau,0,\gamma)\,|\,m>0\}$ generate $\ker(\Delta_{3/2}) = \ker(l_{3/2})$, i.e. they are a generating system for the holomorphic integrable modular forms. If $k=1/2$, the same holds true for the holomorphic Poincaré series $\{\operatorname{Res}_{s=1/2} P_{1/2}(m,\tau,s,\gamma)|\,m>0\}$. If $k=0$, the set $\{1\}\cup\{\operatorname{Res}_{s=1/2} P_{1/2}(m,\tau,s,\gamma)|\,m>0\}$ provides a basis.*

Proof. If $m>0$ and $3/2 \le k \le 2$, by (3.5), the $P_k(m,\tau,0,\gamma)$ are certainly holomorphic as $P_{2-k}(-m,\tau,k-1,\gamma)$ has no pole. For $0 \le k \le 1/2$, the holomorphicity of $\operatorname{Res}_{s=1-k} P_k(m,\tau,s,\gamma)$ is obtained by applying the residue to (3.4) and noting the simplicity of the pole. (3.6) implies that the space generated by them is dense in $\ker(\Delta_k)$ as a holomorphic modular form vanishes iff all non constant Fourier coefficients vanish, unless $k=0$, in which case the modular form must be a constant. □

3.3 Further Continuation in $2\sigma + k < 1$

The continuation theorem in the penultimate section is not yet the best we can do. Consider the equation (3.4) once more:

$$\begin{aligned}
P_{k,L}(m,\tau,s,\gamma) &= 4\pi\,|m|\,(s+k(1-\theta(m))) \\
&\cdot\ (\Delta_k - s(1-s-k))^{-1}
\end{aligned}$$

$$\cdot \quad P_{k,L}(m,\tau,s+1,\gamma). \tag{3.7}$$

We have just shown that $P_{k,L}$ has a natural extension to $1 < 2\sigma + k$, hence $P_{k,L}(m,\tau,s+1,\gamma)$ is well defined for $-1 < 2\sigma + k$. On the other hand, $(\Delta_k - s(1-s-k))^{-1}$ is well defined for $2\sigma+k \neq 1$, hence the left side is well defined for $-1 < 2\sigma + k < 1$. By induction, this yields a continuation to $2\sigma + k \notin 1 - 2\mathbb{N}_0$. This allows us to state

Lemma 3.6. *The Poincaré series $P_{k,L}(m,\tau,s,\gamma)$ has a natural extension to $\{z\in\mathbb{C}|\, 2\sigma + k \notin 1 - 2\mathbb{N}_0\}$, given by (3.4) satisfying (3.3).*

Proof. We use (3.4) to define $P_{k,L}(m,\tau,s,\gamma)$. In the next step, we need to show that (3.3) is compatible with the extension. The reason for this is that the set $2\sigma + k \notin 1 - 2\mathbb{Z}$ is not path connected and hence the uniqueness principles of the holomorphic extension do not apply, hence the equation might be violated. However, note that

$$l_k\Delta_k = \Delta_{k-2}l_k - (k-2)l_k.$$

And hence

$$(\Delta_{k-2} - (s+1)(2-s-k))l_k = l_k(\Delta_k - s(1-s-k)).$$

Hence, for $2\sigma + k \notin 1 - 2\mathbb{Z}$,

$$\begin{aligned}
l_kP_{k,L}(m,\tau,s,\gamma) &= 4\pi\,|m|\,(s+k(1-\theta(m))) \\
&\cdot\ (\Delta_{k-2} - (s+1)(2-s-k))^{-1} \\
&\cdot\ l_kP_{k,L}(m,\tau,s+1,\gamma) \\
&= 4\pi\,|m|\,(s+k(1-\theta(m))) \\
&\ (\Delta_{k-2} - (s+1)(2-s-k))^{-1} \\
&\cdot\ ((s+1)P_{k-2}(m,\tau,s+2,\gamma) \\
&-\ 4\pi\mid m\mid(1-\theta(m))P_{k-2}(m,\tau,s+3,\gamma)) \\
&\stackrel{!}{=} sP_{k-2}(m,\tau,s+1,\gamma) \\
&-\ 4\pi\,|m|\,(1-\theta(m))P_{k-2}(m,\tau,s+2).
\end{aligned} \tag{3.8}$$

To check this, apply $(\Delta_{k-2} - (s+1)(2-s-k))^{-1}$ to both sides and note that

$$\begin{aligned}
&(\Delta_{k-2} - (s+1)(2-s-k)) \\
\cdot\ &P_{k-2}(m,\tau,s+1,\gamma) \\
=\ &4\pi\,|m|\,(s+1+(k-2)(1-\theta(m))) \\
\cdot\ &P_{k-2}(m,\tau,s+2,\gamma)
\end{aligned}$$

on $-1 < 2\sigma + k < 1$ by definition. Next,

$$\begin{aligned}
&(\Delta_{k-2} - (s+1)(2-s-k))P_{k-2}(m,\tau,s+2,\gamma) \\
=\ &(\Delta_{k-2} - (s+2)(1-s-k))P_{k-2}(m,\tau,s+2,\gamma)
\end{aligned}$$

$$
\begin{aligned}
&- \quad (2s+k)P_{k-2}(m,\tau,s+2,\gamma)\\
&= \quad 4\pi\,|m|\,((s+2)+(k-2)(1-\theta(m))\\
&\cdot \quad P_{k,L}(m,\tau,s+3,\gamma)\\
&- \quad (2s+k)P_{k-2}(m,\tau,s+2,\gamma).
\end{aligned}
$$

Lastly, we note that

$$
\begin{aligned}
& \quad 4\pi\,|m|\,(1-\theta(m))\\
\cdot & \quad 4\pi\,|m|\,((s+2)+(k-2)(1-\theta(m))\\
= & \quad (4\pi\,|m|)^2(1-\theta(m))((s+k))\\
= & \quad (4\pi\,|m|)^2((s+k(1-\theta(m))(1-\theta(m))
\end{aligned}
$$

and

$$
\begin{aligned}
& \quad 4\pi\,|m|\,(s+k(1-\theta(m)))(s+1)\\
= & \quad s(s+1+(k-2)(1-\theta(m)))\\
+ & \quad (2s+k)(1-\theta(m)),
\end{aligned}
$$

hence all the terms in (3.8) match up, finishing the proof. □

Remark 3.7. The continuation is not an analytic one in the classical sense since the continued function is not defined in a domain, as there is no definition on the lines where $2\sigma + k \in 1 - 2\mathbb{Z}$. However, the definition "leaks" through the lines and has a continuation beyond.

4 Analytic Continuation of Eisenstein Series

In this section, we will discuss the analytic continuation of Eisenstein series. The issue has been treated in the literature before, in [8] or [12], who also employ spectral theoretic proofs. In this section, we will offer a new method to prove the continuation while outlining a close connection between Eisenstein series and Poincaré series. We will see that the Fourier coefficients of Eisenstein series correspond to the constant coefficients of Poincaré series. To show this, let us note that Eisenstein series are essentially Poincaré series for $m = 0$. Let us briefly define the Poincaré series

$$\mathcal{P}_k(m,\tau,s,\gamma) = \sum_{M\in\tilde{\Gamma}(N)_\infty\backslash\tilde{\Gamma}(N)} (y^s e_\gamma(m\tau))|_k\, M$$

where we allow $m \in \frac{1}{N}\mathbb{Z}$. By Theorem (3.1), the series have a meromorphic continuation to $2s+k>1$ if $m \neq 0$. Since Eisenstein series are the case $m = 0$, we want to show that the continuation theorems hold for them as well. To prove this, we need to examine their Fourier coefficients.

4.1 Fourier Coefficients of Poincaré Series

Let $\alpha \in \mathrm{Mp}_2(\mathbb{Z})$ and $c_k^{n,\alpha}(m,y,s,\gamma)$ denote the $n-$th Fourier coefficient of $\mathcal{P}_k(m,\tau,s,\gamma)|_k\,\alpha$ where $n \in \frac{1}{N}\mathbb{Z}$. Denoting by $\begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix}$ the matrix components of $M \in \mathrm{Mp}_2(\mathbb{Z}))$ and defining

$$\delta(M) = \begin{cases} 1 & \text{if } Mi\infty = i\infty, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

we compute

$$\begin{aligned}
& Nc_k^{n,\alpha}(m,y,s,\gamma) \\
= & \sum_{M\in\tilde{\Gamma}(N)_\infty\backslash\tilde{\Gamma}(N)\alpha} \int_0^N (y^s e_\gamma(m\tau))|_k\, M\, e(-nx)\,dx \\
= & \sum_{\substack{M\in\tilde{\Gamma}(N)_\infty\backslash\tilde{\Gamma}(N)\alpha/\tilde{\Gamma}(N)_\infty \\ M\notin\tilde{\Gamma}(N)_\infty}} \\
\cdot & \int_{-\infty}^{+\infty} (y^s e_\gamma(m\tau))|_k\, M\, e(-nx)\,dx \\
+ & \delta(\alpha)\delta_{m,n} y^s e_\gamma(imy) \\
= & \sum_{\substack{M\in\Gamma(N)_\infty\backslash\Gamma(N)\alpha/\Gamma(N)_\infty \\ M\notin\Gamma(N)_\infty}} e(\frac{nd_M + ma_M}{c_M})(\rho_L(M)^{-1}\mathfrak{e}_\gamma)
\end{aligned}$$

$$
\begin{aligned}
\cdot & \int_{-\infty}^{+\infty} \frac{1}{\sqrt{c_M\tau}^{2k}} \frac{1}{|c_M\tau|^{2s}} e(-nx) e(\frac{-m}{c_M^2\tau})\, dx \\
+ & \delta(\alpha)\delta_{m,n} y^s e_\gamma(imy) \\
= & y^{1-s-k} \sum_{M\in\Gamma(N)_\infty\backslash\Gamma(N)\alpha/\Gamma(N)_\infty} \\
\cdot & e(\frac{nd_M+ma_M}{c_M})(\rho_L(M)^{-1}\mathfrak{e}_\gamma)\frac{i^{-k(1-\mathrm{sgn}(c_M))}}{|c_M|^{2s+k}} \\
\cdot & \int_{-\infty}^{+\infty} \frac{1}{\sqrt{x+i}^{2k}} \frac{1}{|x+i|^{2s}} e(-nyx) e(\frac{-m}{c_M^2 y(x+i)})\, dx \\
+ & \delta(\alpha)\delta_{m,n} y^s e_\gamma(imy).
\end{aligned}
$$

Note that

$$
e(\frac{-m}{c_M^2 y(x+i)}) =: 1 + \frac{1}{c_M^2 y(x+i)} f_m(\frac{1}{c_M^2 y(x+i)}),
$$

where we define f_m such that the equation holds. It is analytic and globally bounded in x. We abbreviate

$$
\begin{aligned}
& K_k(\alpha,s,m,n,\gamma) \\
:= & \sum_{M\in\Gamma(N)_\infty\backslash\Gamma(N)\alpha/\Gamma(N)_\infty} e(\frac{nd_M+ma_M}{c_M}) \\
\cdot & (\rho_L(M)^{-1}\mathfrak{e}_\gamma)\frac{i^{-k(1-\mathrm{sgn}(c_M))}}{|c_M|^{2s+k}}.
\end{aligned}
$$

The K here stands for Kloosterman as the sum is indeed a Kloosterman sum in the classical sense. This allows us to write

$$
\begin{aligned}
& N c_k^{n,\alpha}(m,y,s,\gamma) \\
= & \delta(\alpha)\delta_{m,n} y^s e_\gamma(imy) \\
+ & y^{1-s-k} K_k(\alpha,s,m,n,\gamma) \\
\cdot & \int_{-\infty}^{+\infty} \frac{1}{\sqrt{x+i}^{2k}} \frac{1}{|x+i|^{2s}} e(-nyx)\, dx \\
+ & y^{-s-k} K_k(\alpha,s+1,m,n,\gamma) \\
\cdot & \int_{-\infty}^{+\infty} \frac{1}{\sqrt{x+i}^{2k+2}} \frac{1}{|x+i|^{2s}} \\
\cdot & e(-nyx) f_m(\frac{1}{c_M^2 y(x+i)})\, dx.
\end{aligned}
$$

The definition also implies

$$
K_k(\alpha,s,m,n,\gamma) = K_{k+2}(\alpha,s-1,m,n,\gamma), \tag{4.2}
$$

as well as

$$\begin{aligned}
& \langle \mathfrak{e}_{\gamma'}, K_k(\alpha, s, m, n, \gamma)\rangle \\
= & \sum_{M \in \Gamma(N)_\infty \backslash \Gamma(N)\alpha/\Gamma(N)_\infty} e(\frac{nd_M + ma_M}{-c_M}) \\
\cdot & \frac{i^{k(1-\mathrm{sgn}(c_M))}}{|c_M|^{2s+k}} \langle \mathfrak{e}_{\gamma'}, \rho_L(M)^{-1}\mathfrak{e}_\gamma \rangle \\
= & i^{2k} \sum_{M \in \Gamma(N)_\infty \backslash \Gamma(N)\alpha/\Gamma(N)_\infty} e(\frac{nd_M + ma_M}{-c_M}) \\
\cdot & \frac{i^{-k(1-\mathrm{sgn}(-c_M))}}{|c_M|^{2s+k}} \langle \rho_L(M^{-1})^{-1}\mathfrak{e}_{\gamma'}, \mathfrak{e}_\gamma \rangle \\
= & i^{2k} \langle K_k(\alpha^{-1}, s, n, m, \gamma'), \mathfrak{e}_{\gamma'} \rangle.
\end{aligned} \tag{4.3}$$

Here, we used that the mapping $M \mapsto M^{-1}$ is a bijection of the residue classes in $\Gamma(N)_\infty \backslash \Gamma(N)\alpha/\Gamma(N)_\infty$ to those of $\Gamma(N)_\infty \backslash \Gamma(N)\alpha^{-1}/\Gamma(N)_\infty$. To further simplify the expressions for the Fourier coefficients, let us define

$$\begin{aligned}
\epsilon_k(s, y, n, f) \quad := & \int_{-\infty}^{+\infty} \frac{1}{\sqrt{x+i}^{2k}} \frac{1}{|x+i|^{2s}} \\
\cdot & \quad e(-nyx) f(\frac{1}{c_M^2 y(x+i)})\, dx
\end{aligned}$$

for an arbitrary f which is globally smooth and analytic at 0. Then, we can further simplify to

$$\begin{aligned}
Nc_k^{n,\alpha}(m, y, s, \gamma) \quad = & \quad \delta(\alpha)\delta_{m,n} y^s e_\gamma(imy) \\
+ & \quad y^{1-s-k} K_k(\alpha, s, m, n, \gamma)\epsilon_k(s, y, n, 1) \\
+ & \quad y^{-s-k} K_k(\alpha, s+1, m, n, \gamma) \\
\cdot & \quad \epsilon_{k+1}(s, y, n, f_m).
\end{aligned} \tag{4.4}$$

Ideally, we want a result about the absolute convergence of the Fourier series. To prove this, we need some results that bound each term independently of n, which will be partly done in the next subsection. However, the bounds for the function ϵ are easily derived.

Lemma 4.1. *There is some constant $C(\epsilon, \delta, l, f) > 0$ such that for $2\sigma + k \geq \epsilon$, $y \geq \delta$ and $n \neq 0$ the estimate*

$$\epsilon_k(s, y, n, f) \leq C(k, \epsilon, \delta, l, f)(|n|\, y)^{-l} \tag{4.5}$$

holds.

Proof. We show this by induction on l. The hypothesis holds for $l = 0$ and $2\sigma + k > 2$ by boundedness of the integral. The induction step follows by partial integration:

$$\begin{aligned} & \epsilon_k(s, y, n, f) \\ = & \frac{1}{(2\pi i n y)} \int_{-\infty}^{+\infty} e(-nyx) \\ \cdot & \frac{\partial}{\partial x} \left\{ \frac{1}{\sqrt{x+i}^{2k}} \frac{1}{|x+i|^{2s}} f(\frac{1}{c_M^2 y(x+i)}) \right\} dx \\ = & \frac{1}{(2\pi i n y)} \{-k\epsilon_{k+1}(s, y, n, f) \\ - & 2s(\epsilon_{k-1}(s+1, y, n, f) - i\epsilon_k(s+1, y, n, f)) \\ - & \frac{1}{c_M^2 y} \epsilon_{k+2}(s, n, y, f')\}. \end{aligned}$$

Since all the arguments on the right hand side converge uniformly for $2\sigma + k > 1$ and satisfy a sufficient estimate therein as f' is also globally smooth and analytic in zero, we are done. □

4.2 Continuation theorems for Eisenstein series

This analysis now allows us to construct analytic Eisenstein series. To do that, at first we apply (4.4) to $m = 0$, i.e. the $n-$th coefficient of the Eisenstein series

$$E_k(\tau, s, \gamma)|_k\, \alpha = \sum_{M \in \Gamma(N)_\infty \backslash \Gamma(N)} (y^s \mathfrak{e}_\gamma)|_k\, M\alpha,$$

which is given by

$$\begin{aligned} N c_k^{n,\alpha}(0, y, s, \gamma) &= \delta(\alpha)\delta_{0,n} y^s \mathfrak{e}_\gamma \\ &+ y^{1-s-k} K_k(\alpha, s, 0, n, \gamma)\epsilon_k(s, y, n, 1) \\ &+ y^{-s-k} K_k(\alpha, s+1, 0, n, \gamma) \\ &\cdot \epsilon_{k+1}(s, y, n, 1). \end{aligned}$$

By equation (4.3), the function $K_k(\alpha, s, 0, n, \gamma)$ is bounded by $K_k(\alpha, s, n, 0, \gamma)$, which is a term appearing in the constant coefficient of the Fourier series $\mathcal{P}_k(m, \tau, s, \gamma)|_k\, \alpha$. Hence, we can relate the Fourier coefficients of the Eisenstein series to the constant coefficients of Poincaré series. By definition

$$K_k(\alpha, s, 0, n, \gamma) \leq C(\epsilon)$$

while $2\sigma + k \geq 2 + \epsilon$ as the sum absolutely converges. Hence

$$|K_k(\alpha, s+1, 0, n, \gamma) \cdot \epsilon_{k+1}(s, y, n, 1)| \leq C(k, \epsilon, \delta, l)(|ny|)^{-l}$$

by (4.5) while $2\sigma + k \geq 1 + \epsilon$, $y \geq \delta$. If we can now bound $K_k(\alpha, s, 0, n, \gamma)$ uniformly for $2\sigma + k \geq 1 + \epsilon$, (4.5) also implies convergence of the sum over the non constant Fourier coefficients of the Eisenstein series. We can now bound the Kloosterman sum $K_k(\alpha, s, 0, n, \gamma)$ in $2\sigma + k > 1$ by our theory of Petersson-Sobolev spaces. The bound is given by the following

Lemma 4.2. *The Kloosterman sums are bounded in compact subsets $s \in D$ of $2\sigma + k > 1$ via*

$$|K_k(\alpha, s, n, 0, \gamma)| \quad \leq \quad C \frac{\left\|(\Delta_{k'} - s'(1 - s' - k'))^{-1}\right\|}{|\epsilon_{k'}(s', 1, 0, 1)|} \tag{4.6}$$

where k' denotes the unique element in $\{-1, -1/2, 0, 1/2\}$ such that $k \equiv k'$ mod 2 and $s' = s + (k - k')/2$. Moreover, the constant C only depends on k and the subset D.

We briefly remark that our choice of k' will make the following proof easier as it is short to argue that the factor $\epsilon_{k'}(s', 1, 0, 1)$ is meromorphic and not zero if $k' \in \{-1, -1/2, 0, 1/2\}$.

Proof. We know that by (4.2), $|K_k(\alpha, s, n, 0, \gamma)| = |K_{k\pm 2}(\alpha, s \mp 1, n, 0, \gamma)|$. Hence we can replace k by k', s by s' and for simplicity assume that $k = k'$ as well as $s = s'$. Next, (4.3) it is sufficient to bound the constant coefficient of the Poincaré series, $K_k(\alpha, s, n, 0, \gamma)$, for $2\sigma + k \geq 1 + \epsilon$, as it implies that

$$|K_k(\alpha, s, 0, n, \gamma)| \leq \text{Const.} \cdot \left|K_k(\alpha^{-1}, s, n, 0, \gamma)\right|.$$

Let $n \neq 0$. Then, by (4.4),

$$\begin{aligned} & K_k(\alpha, s, n, 0, \gamma) \\ = \quad & \frac{1}{\epsilon_k(s, y, 0, 1)} \Big(\int_0^1 \mathcal{P}(n, x + iy, s, \gamma)|_k \, \alpha \, dx \\ - \quad & \frac{1}{y} K_k(\alpha, s + 1, n, 0, \gamma) \epsilon_{k+1}(s, y, 0, f_m) \Big). \end{aligned} \tag{4.7}$$

To show that this is indeed well defined, we need to show that the factor $\frac{1}{\epsilon_k(s,y,0,1)}$ is defined. We can divide by $\epsilon_k(s, y, 0, 1)$ if it is not identically zero as it is meromorphic. To see this observe that $\epsilon_k(s, y, 0, 1)$ is constant in y and has no zeroes for $-1 \leq k \leq 1/2$ on the real axis, as by definition

$$\epsilon_k(s, 1, 0, 1) := \int_{-\infty}^{+\infty} \frac{1}{\sqrt{x + i}^{2k}} \frac{1}{|x + i|^{2s}} \, dx.$$

Hence, $\sqrt{x + i}^{2k}$ is always in some half plane and hence the integral converges absolutely while $2\sigma + k > 1$ and does not vanish for real valued s, hence it is meromorphic and non vanishing, and so $\frac{1}{\epsilon_k(s,y,0,1)}$ is well defined. Now, we focus our attention on the above sum (4.7). The second term in it is obviously uniformly bounded while $2\sigma + k \geq 1 + \epsilon$ as the Kloosterman sum absolutely converges, thus we can simply ignore it. To show that

$$\int_0^1 \mathcal{P}_k(n, x+iy, s, \gamma)|_k\, \alpha\, dx$$

is sufficiently bounded requires arguments from the previous section. First, let $n > 0$. Then

$$\mathcal{P}_k(n, x+iy, s, \gamma)|_k\, \alpha \in \mathcal{H}_k^1$$

and

$$\begin{aligned} & \mathcal{P}_k(n, x+iy, s, \gamma)|_k\, \alpha \\ = \;& (\Delta_k - s(1-s-k))^{-1} \\ \cdot \;& (4\pi s n \mathcal{P}_k(n, x+iy, s+1, \gamma)|_k\, \alpha). \end{aligned}$$

Which, by (2.2), yields

$$\begin{aligned} & \left| \int_0^1 \mathcal{P}_k(n, x+iy, s, \gamma)|_k\, \alpha\, dx \right| \\ \leq \;& \left\| (\Delta_k - s(1-s-k))^{-1} \right\| \\ \cdot \;& \text{Const.} \cdot \left\| \mathcal{P}_k(n, \cdot, s+1, \gamma) \right\|_{1,2}. \end{aligned} \tag{4.8}$$

The function norm $\left\| \mathcal{P}_k(n, \cdot, s+1, \gamma) \right\|_{1,2}$ is uniformly bounded (independently of n) while $2s + k \geq \epsilon$ by absolute convergence of the Poincaré series. Note that the constant in this equation only depends on D and k. To summarize, if $n > 0$

$$\left| \int_0^1 \mathcal{P}_k(n, x+iy, s, \gamma)|_k\, \alpha\, dx \right| \leq \text{Const.} \cdot \left\| (\Delta_k - s(1-s-k))^{-1} \right\|$$

and thus

$$|K_k(\alpha, s, n, 0, \gamma)| \leq \text{Const.} \cdot \left| \frac{\left\| (\Delta_k - s(1-s-k))^{-1} \right\|}{\epsilon_k(s, y, 0, 1)} \right|. \tag{4.9}$$

Now, for $n < 0$, we can write, by recalling the notion of integrable Poincaré series P_k (see (3.4)),

$$\begin{aligned} & \mathcal{P}_k(n, \tau, s, \gamma)|_k\, \alpha \\ = \;& (\Delta_k - s(1-s-k))^{-1} \\ \cdot \;& (4\pi n(s+k)) P_k(n, \tau, s+1, \gamma)|_k\, \alpha \\ - \;& \sum_{M \in \Gamma(N)_\infty \backslash \Gamma(N)} (y^s (e_\gamma(nx)(e(-iny) - 1))|_k\, M|_k\, \alpha \end{aligned}$$

$$+ \sum_{M\in\Gamma(N)_\infty\backslash\Gamma(N)} (y^s(e_\gamma(nx)(e(+iny)-1))|_k M|_k \alpha.$$

The last two terms admit an absolute estimate for $2\sigma + k \geq \epsilon$, the first one an integral estimate similar to the previous one and hence (4.9) holds for all $n \neq 0$. $\square$

Using this lemma, we can bound the Fourier coefficients using (4.5) and (4.6) (recall that $k' \equiv k \mod 2$ such that $k' \in \{-1, -1/2, 0, 1/2\}$ and $s' = s + (k-k')/2$)

$$\begin{aligned} |c_k^{n,\alpha}(n,y,s,\gamma)| &\leq (|n|)^{-l} \frac{C(\epsilon,k,l)}{|\epsilon_{k'}(s',1,0,1)|} y^{1-\sigma-k} \\ &\cdot \left\|(\Delta_{k'} - s'(1-s'-k))^{-1}\right\|, \end{aligned}$$

which means that the sum comprising

$$E_k(\tau,s,\gamma)|_k\,\alpha - c_k^{0,\alpha}(0,y,s,\gamma) \quad =: \quad \tilde{E}_k(\tau,s,\gamma)|_k\,\alpha$$

converges absolutely except for a discrete subset in $2s + k > 1$, which corresponds to the poles of $\frac{\left\|(\Delta_{k'}-s'(1-s'-k))^{-1}\right\|}{|\epsilon_{k'}(s',1,0,1)|}$. As such, above equation defines a meromorphic function that is bounded by

$$\begin{aligned} &\left|E_k(\tau,s,\gamma)|_k\,\alpha - c_k^{0,\alpha}(0,y,s,\gamma)\right| \\ \leq\; &\frac{C'(\epsilon,k)}{|\epsilon_{k'}(s',1,0,1)|} y^{1-\sigma-k} \\ \cdot\; &\left\|(\Delta_{k'} - s'(1-s'-k))^{-1}\right\|. \end{aligned} \tag{4.10}$$

To fully prove the continuation of E_k, we still need to examine the constant coefficient for one such α. The ideal candidate is the natural choice $\alpha = \mathrm{id}$. It is well known that the set $\tilde{\Gamma}(N)_\infty\backslash\tilde{\Gamma}(N)$ has representatives uniquely classified by their second line, i.e. the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (c,d)$$

By this, it trivially follows that we obtain the representatives

$$\begin{aligned} &\tilde{\Gamma}(N)_\infty\backslash\tilde{\Gamma}(N)/\tilde{\Gamma}(N)_\infty \\ \cong\; &\{(0,1)\} \\ \dot{\cup}\; &\{(c,d)|\gcd(c,d)=1, 0 \leq d < N\,|c|\,, d \equiv 1(N), c \equiv 0(N)\}. \end{aligned}$$

As such, if N is sufficiently big (i.e. using the argumentation presented in (1.8)) we obtain by again considering (2.4)

$$K_k(\mathrm{id},s,0,0,\gamma) \quad = \quad 1 + C \sum_{\substack{0\leq d<N|c|, d\equiv 1(N), c\equiv 0(N) \\ c>0}} \left(\frac{d}{c}\right)^{2k} \frac{1}{c^{2s+k}}.$$

If $2k$ is even, this simplifies to

$$K_k(\mathrm{id}, s, 0, 0, \gamma) = 1 + \frac{C'}{N^{2s+k}} \sum_{c>0} \varphi(N^2 c) \frac{1}{(Nc)^{2s+k}},$$

whose meromorphy is easily derived from the Euler product of the sum (such that $N = \prod p_i^{a_i}$ for pairwise distinct primes p_i with $a_i > 0$)

$$\begin{aligned} \frac{1}{N^s} \sum_{c>0} \varphi(N^2 c) \frac{1}{c^s} &= \tfrac{1}{N^s} \sum_{c>0} \varphi(c) \frac{1}{c^s} \\ &\cdot \prod_i \frac{(p_i + (p_i - 1) p_i^{2a_i} \frac{p_i^{s-1}}{p_i^{s-1} - 1})}{(p_i + (p_i - 1) \frac{p_i^{s-1}}{p_i^{s-1} - 1})}. \end{aligned}$$

Now, if $2k$ is odd, we need some more work. Notice that

$$\sum_{\substack{d \equiv 1 (N) \\ 0 \le d < Nc}} \left(\frac{d}{c} \right) = \frac{1}{\varphi(N)} \sum_{\chi_N} \sum_{0 \le d < Nc} \chi_N(d) \left(\frac{d}{c} \right),$$

where we sum over all characters χ_N modulo N. Hence, the sum vanishes unless there is some character mod N such that $\chi_N(\cdot) \left(\frac{\cdot}{c} \right)$ is the trivial character mod Nc. This happens if and only if $c = p_1^{b_1} \cdot \ldots \cdot p_n^{b_n} r^2$ such that $\gcd(r, N) = 1$ and p_i defined as above with $b_i \ge 0$, in which case we have

$$\sum_{\substack{d \equiv 1 (N) \\ 0 \le d < Nc}} \left(\frac{d}{c} \right) = \frac{1}{\varphi(N)} \varphi(N p_1^{b_1} \cdot \ldots \cdot p_n^{b_n} r^2).$$

Hence

$$\begin{aligned} & \sum_{\substack{0 \le d < N|c|, d \equiv 1 (N) c \equiv 0 (N) \\ c > 0}} \left(\frac{d}{c} \right) \frac{1}{c^s} \\ = \quad & \frac{1}{N^s \varphi(N)} \sum_{\substack{b_i \ge 0, c > 0 \\ \gcd(N,c) = 1}} \varphi(N^2 p_1^{b_1} \cdot \ldots \cdot p_n^{b_n} c^2) \\ \cdot \quad & \frac{1}{(p_1^{b_1} \cdot \ldots \cdot p_n^{b_n} c^2)^s} \\ = \quad & \frac{1}{N^s \varphi(N)} \sum_{c>0} \varphi(c) \frac{1}{c^{2s-1}} \\ \cdot \quad & \prod_i \frac{(1 - p_i) p_i^{2a_i} \frac{p_i^{s-1}}{1 - p_i^{s-1}}}{(p_i + (1 - p_i) \frac{p_i^{2(s-1)}}{1 - p_i^{2(s-1)}})}, \end{aligned}$$

which is globally meromorphic as

$$\sum_{c>0}\varphi(c)\frac{1}{c^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

thus finishing the proof of analytic continuation for the constant coefficient, and hence the Eisenstein series can be continued as well.

5 The ξ-Operator and Harmonic Maass Forms

5.1 Integrable Holomorphic Modular Forms and Poincaré Series

In this section, we would like to explicitly construct integrable holomorphic modular forms, i.e. cusp forms if $k > 1$ and holomorphic modular forms for $k < 1$. While the construction could be done, in general, as in the next subsection, in this subsection we would like to outline an approach that focuses on the analytic continuation of Poincaré series.

$$M_{k,L}^{\mathrm{int}} \coloneqq \ker(\xi_k) \cap \mathcal{L}_k^2. \tag{5.1}$$

Recall the notion of weakly holomorphic modular forms $M_{k,L}^!$ in Definition (1.9). By Lemma (2.7)

$$M_{k,L}^{\mathrm{int}} \subseteq M_{k,L}^!.$$

Further recall the definition of weakly harmonic Maass forms $H_{k,L}$ as in Definition (1.8). We want to prove

Theorem 5.1. *If $k \neq 1$, $M_{2-k,L^-}^{\mathrm{int}} \subseteq \mathrm{Im}(\xi_k|_{H_{k,L}})$*

Note that we have nothing to prove if $k > 2$ as there are no integrable holomorphic modular forms of weight $2-k$. Else, for $m > 0$, define the series

$$\begin{aligned}
\mathcal{R}_{k,L}(m,\tau,s,\gamma) &= \sum_{M \in \Gamma'_\infty \backslash \Gamma'} \left(y^s f_{m,s}(y) e_\gamma(-m\bar{\tau})\right)|_{k,L}\ M,\\
\mathcal{P}_{k,L}(m,\tau,s\gamma) &= \sum_{M \in \Gamma'_\infty \backslash \Gamma'} \left(y^s e_\gamma(m\tau)\right)|_{k,L}\ M.
\end{aligned}$$

Proof. Let $f_{m,s}$ solve the differential equation

$$s f_{m,s} + y f'_{m,s} - 4\pi m y f_{m,s} = 1.$$

Then $f_{m,s}$ is analytic in y with its coefficients given by

$$f_{m,s}(y) = \sum_n a_n y^n$$

where

$$a_n = \frac{(4\pi m)^n}{s} \frac{\Gamma(s+1)}{\Gamma(s+n+1)}.$$

The equation is obtained by noting that for $n > 0$

$$0 = s a_n + n a_n - 4\pi m a_{n-1}.$$

Note that

$$f_{m,s}(y) = (4\pi m y)^{-s} e^{4\pi m y} \gamma(s, 4\pi m y),$$

where $\gamma(\cdot,\cdot)$ denotes the lower partial gamma function. Now

$$\begin{aligned}
& \xi_k(y^s f_{m,s}(y) e_\gamma(-m\bar\tau)) \\
= \;& y^{\bar s+k-1}\overline{(s f_{m,s} + y f'_{m,s} - 4\pi m y f_{m,s})} e_\gamma(m\tau) \\
= \;& y^{\bar s+k-1} e_\gamma(m\tau).
\end{aligned}$$

Hence

$$\begin{aligned}
(\xi_k \mathcal{R}_{k,L})(m,\tau,s,\gamma) &= \mathcal{P}_{2-k,L^-}(m,\tau,\bar s+k-1,\gamma), \\
(\xi_k \mathcal{P}_{k,L})(m,\tau,s,\gamma) &= \bar s \mathcal{P}_{2-k,L^-}(-m,\bar\tau,\bar s+k-1,\gamma).
\end{aligned} \tag{5.2}$$

□

Notice that since $f_{m,s}$ is analytic and exponentially bounded, both $\mathcal{R}_{k,L}$ and $\mathcal{P}_{k,L}$ are well defined. For $3/2 \le k \le 2$, we define the functions (where p.p. denotes the Cauchy principal value)

$$\begin{aligned}
\mathfrak{r}_k^L(m,\tau,\gamma) &= \operatorname*{Res}_{s=0} \mathcal{R}_{k,L}(m,\tau,s,\gamma), \\
\mathfrak{p}_{2-k}^{L^-}(m,\tau,\gamma) &= \operatorname*{Res}_{s=1-k} \mathcal{P}_{k,L^-}(m,\tau,s,\gamma),
\end{aligned}$$

whereas for $k \le 1/2$ we define

$$\begin{aligned}
\mathfrak{r}_k^L(m,\tau,\gamma) &= \operatorname*{p.p.}_{s=1-k} \mathcal{R}_{k,L}(m,\tau,s,\gamma), \\
\mathfrak{p}_{2-k}^{L^-}(m,\tau,\gamma) &= \operatorname*{p.p.}_{s=0} \mathcal{P}_{2-k,L^-}(m,\tau,s,\gamma).
\end{aligned}$$

These are well defined for all $m \neq 0$ as $2s+k>1$. Then, by construction (as taking the residue in s commutes with ξ_k, as the integral over a circle and the derivative commute)

$$\xi_k \mathfrak{r}_k^L(m,\tau,\gamma) = \mathfrak{p}_{2-k}^{L^-}(m,\tau,\gamma).$$

Note that we could extend these constructions to all $k \neq 1$. If $k<0$, the non-holomorphic Poincaré series $\mathfrak{r}_k^L$ would correspond to those in [3]. By Theorem (3.1), however, the series in [3] must have an analytic continuation to $s=0$ for all $k<1$, yet we can say little about the nature of the continuation and possible poles as their construction differs from ours.

Now, we have three lemmas finishing the proof of the theorem.

Lemma 5.2. *The estimate $\mathfrak{r}_k^L(m,\tau,\gamma) = O(e^{Cy})$ holds at each cusp.*

Proof. By definition

$$\mathcal{R}_{k,L}(m,\tau,s,\gamma) = \sum_{M\in {\Gamma'}_\infty\backslash\Gamma'} (y^s(f_{m,s}(y) - \frac{1}{s}) e_\gamma(m\bar\tau)) \mid_{k,L} M$$

$$+ \quad \frac{1}{s}\overline{\mathcal{P}_{k,L}(-m,\tau,\bar{s},\gamma)}.$$

Let α be an element of $\mathrm{Mp}_2(\mathbb{Z})$. Note that as the first term converges absolutely for $2\sigma + k > 0$, it grows as $\delta(\alpha)O(y^s(f_s(y) - \frac{1}{s}) + O(y^{1-s-k})$ at the cusp $\alpha i\infty$ per the usual estimate. Furthermore, again by definition,

$$\begin{aligned}
\mathcal{P}_{k,L}(m,\tau,s,\gamma) &= P_{k,L}(m,\tau,s,\gamma) \\
&- (1-\theta(m)) \cdot \sum_{M\in\Gamma'_\infty\backslash\Gamma'} (y^s e_\gamma(mx)(e(-imy)-1)) \mid_{k,L} M \\
&+ (1-\theta(m)) \cdot \sum_{M\in\Gamma'_\infty\backslash\Gamma'} (y^s e_\gamma(mx)(e(imy)-1)) \mid_{k,L} M.
\end{aligned}$$

By Lemma (2.5), the first term is at most polynomial at every cusp while the second and third term are absolutely converging sums for $2\sigma+k>0$ and grows like $\delta(\alpha)O(y^s e(m\tau) + e(m\bar{\tau})) + O(y^{1-s-k})$ at each cusp $\alpha i\infty$ and hence all $\mathfrak{r}_k^L(m,\tau,\gamma)$ are harmonic Maass forms if we prove the following: ☐

Lemma 5.3. *$\xi_k\mathfrak{p}_k^L(m,\tau,\gamma) = 0$ if $m > 0$.*

Proof. Consider (5.2). If $3/2 \le k \le 2$, it implies

$$\begin{aligned}
\xi_k\mathfrak{p}_k^L(m,\tau,\gamma) &= \operatorname*{p.p.}_{s=0} \bar{s}\mathcal{P}_{1/2,L^-}(-m,\bar{\tau},\bar{s}+k-1,\gamma) \\
&= \operatorname*{p.p.}_{s=0} \bar{s}P_{2-k,L^-}(-m,\tau,\bar{s}+k-1,\gamma) \\
&= 0
\end{aligned}$$

as $P_{2-k,L^-}(-m,\tau,\bar{s}+1/2,\gamma)$ has no pole for $m>0$ by (3.6) as $2-k \le 1/2$. If $k \le 1/2$,

$$\begin{aligned}
\xi_k\mathfrak{p}_k^L(m,\tau,\gamma) &= \operatorname*{Res}_{s=1-k} \bar{s}\mathcal{P}_{2-k,L^-}(-m,\bar{\tau},\bar{s}+k-1,\gamma) \\
&= \operatorname*{Res}_{s=1-k} \bar{s}P_{2-k,L^-}(-m,\tau,\bar{s}+k-1,\gamma) \\
&= 0
\end{aligned}$$

which has no pole and hence vanishes by (3.6). ☐

To finish the proof, we show two more lemmas. The first one is

Lemma 5.4. *The $\{\mathfrak{p}_k^L(m,\tau,\gamma)|m>0\}$ form a generating set of $M_{k,L}^{\mathrm{int}}$ if $k \ne 0,1$. If $k=0$, $\{f_\gamma\} \cup \{\mathfrak{p}_k^L(m,\tau,\gamma)|m>0\}$ is a generating set of $M_{k,L}^{\mathrm{int}}$, where f_γ are suitable constant functions.*

Proof. This is a direct corollary of (3.6) as any integrable holomorphic modular form that is orthogonal to all $\mathfrak{p}_k^L(m,\tau,\gamma)$ must be constant, hence vanishes unless $k=0$. ☐

All that remains now is to show that all constant elements of $M_{0,L}^{\mathrm{int}}$ are in the image of ξ_2. We state this as

Lemma 5.5. *Let $f \in M^{\text{int}}_{0,L}(\Gamma)$ be constant. Then f is in the image of ξ_2.*

Proof. Consider the non holomorphic Eisenstein series

$$G_2(\tau) \quad := \quad 2\zeta(2) + 4\frac{\zeta(2)}{\zeta(-1)}\sum_{n\geq 1}\sigma_1(n)e(n\tau) + \frac{1}{8\pi y}$$

and via, (2.4), its lifts to Eisenstein series $G_{2,\gamma}(\tau)$ by means of the trace formula as follows: We choose N big enough (recall (1.7)) such that (2.5) holds and hence $G_2(\tau)\mathfrak{e}_\gamma$ is an element of the automorphic forms $\mathcal{A}_{2,L^-}(\tilde{\Gamma}(N))$. Now, consider the trace map

$$\pi_{2,L^-} : \mathcal{A}_{2,L^-}(\tilde{\Gamma}(N)) \to \mathcal{A}_{0,L}(\Gamma).$$

We define

$$G_{2,\gamma} := G_2(\tau)\mathfrak{e}_\gamma.$$

which is $H_{2,L^-}(\tilde{\Gamma}(N))$. Then

$$\begin{aligned} \xi_2 G_{2,\gamma} &= \xi_2 G_2(\tau)\mathfrak{e}_\gamma \\ &= \xi_2(\frac{1}{8\pi y})\mathfrak{e}_\gamma \\ &= -\frac{1}{8\pi}\mathfrak{e}_\gamma. \end{aligned}$$

Now, since f is constant, we can write

$$f = \sum_\gamma \frac{-\alpha_\gamma}{8\pi}\mathfrak{e}_\gamma.$$

Hence

$$\xi_2(\sum_\gamma \alpha_\gamma G_{2,\gamma}) = f.$$

Composing this with the projection π_{2,L^-} yields

$$\begin{aligned} \xi_2\pi_{2,L^-}(\sum_\gamma \alpha_\gamma G_{2,\gamma}) &= \pi_{0,L}(\xi_2(\sum_\gamma \alpha_\gamma G_{2,\gamma})) \\ &= \pi_{0,L}(f) \\ &= f, \end{aligned}$$

finishing the proof as $\pi_{2,L^-}(\sum_\gamma \alpha_\gamma G_{2,\gamma})$ is in $H_{2,L^-}(\Gamma)$. □

5.2 More on Weakly Harmonic Maass Forms

Lastly, we want to investigate the construction of weakly harmonic Maass forms, which is a somewhat more challenging task. This section will at first deal with the construction of weakly holomorphic modular forms, then we will show that all those lie in the image of ξ_k, which will then allow us to prove the surjectivity of the latter operator.

5.2.1 Weakly Holomorphic Modular Forms

To simplify notation, let us introduce the notion of quasi analytic functions. We say a function f is quasi-analytic in zero if

$$f(y) = y^{\alpha} g(y)$$

for some $\alpha \in \mathbb{R}$ such that g is real analytic in zero. Notice that for any f *quasi-analytic* at zero such that $f = O(y^l)$ at zero, $2l + k > 2$ and $m > 0$ the series

$$R_{k,L}(f, m, \tau, \gamma) = \sum_{M \in \Gamma_\infty \backslash \Gamma} (f(y) e_\gamma(m\tau)) \mid_{k,L} M$$

converges absolutely. Moreover, we have seen that (with the δ-notation and estimates as in (4.1)

$$R_{k,L}(f, m, \tau, \gamma)|_k M = \delta(M) f(y) e_\gamma(m\tau) + O(y^{1-l-k})$$

at each cusp $Mi\infty$. Furthermore

$$\begin{aligned}
\xi_k R_{k,L}(f, m, \tau, \gamma) &= \xi_k \sum_{M \in \Gamma_\infty \backslash \Gamma} (f(y) e_\gamma(m\tau)) \mid_{k,L} M \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} \xi_k (f(y) e_\gamma(m\tau)) \mid_{2-k,L^-} M \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} (y^k \cdot f'(y) \cdot e_\gamma(-m\bar{\tau})) \mid_{2-k,L^-} M \\
&= R_{2-k.L^-}(y^k f'(y), -m, \bar{\tau}, \gamma),
\end{aligned}$$

which is in $\mathcal{L}^2_{2-k}$ if $y^k f'(y) e^{-2\pi m y}$ vanishes sufficiently fast as $y \to \infty$. Note that for any $g \in M^{\text{int}}_{2-k,L^-}$, we have

$$\begin{aligned}
\langle R_{2-k,L^-}(y^k f'(y), -m, \bar{\tau}, \gamma), g \rangle &= \int_{\mathcal{F}} \sum_{M \in \Gamma_\infty \backslash \Gamma} (y^k (f'(y) e_\gamma(-m\bar{\tau}))) \mid_{k,L} M \\
&\cdot \quad \bar{g}(\tau) y^{2-k} \frac{dx\, dy}{y^2} \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} \int_{M^{-1}\mathcal{F}} f'(y) e_\gamma(-m\bar{\tau}) \\
&\cdot \quad \bar{g}(\tau)\, dx\, dy \\
&= \int_0^\infty f'(y) (\int_0^N e_\gamma(-m\bar{\tau}) \cdot \bar{g}(, \tau) dx) dy \\
&= \overline{g_{-m,\gamma}} \int_0^\infty f'(y) dy \qquad (5.3)
\end{aligned}$$

where, as previously, $g_{m,\gamma}$ denotes the m-th Fourier coefficient w.r.t. the base vector $\mathfrak{e}_\gamma$. We now choose some constant $c > 0$ and some smooth φ_k

$$\varphi_k(y) = \begin{cases} y^{2+|k|} & \text{if } 0 < y < c/2, \\ 1 & \text{if } y > c. \end{cases}$$

The idea behind is that the Poincaré series defined by φ_k will absolutely converge. By definition, φ_k is quasi-analytic. Since $\varphi_k'(y) = 0$ for $y > c$, $R_{2-k,L^-}(y^k \varphi_k'(y), -m, \bar{\tau}, \gamma)$ is square integrable. Further, by (5.3), we have

$$\begin{aligned} \langle R_{2-k,L^-}(y^k f'(y), -m, \bar{\tau}, \gamma), g \rangle &= \overline{g_{-m,\gamma}} \int_0^\infty f'(y) dy \\ &= \overline{g_{-m,\gamma}}. \end{aligned} \tag{5.4}$$

This suggests the following lemma.

Lemma 5.6. *The series $\xi_k R_{k,L}(\varphi_k, m, \tau, \gamma)$ is in $\mathcal{H}^1_{2-k}$.*

Proof. We know that

$$\begin{aligned} \xi_k R_{k,L}(\varphi_k, m, \tau, \gamma) &= R_{2-k,L^-}(y^k \varphi_k(y)', -m, \bar{\tau}, \gamma) \\ &= \delta(M) y^k \varphi_k'(y) e_\gamma(-m\bar{\tau}) + O(y^{-2-|k|}) \\ &= O(y^{-2-|k|}) \end{aligned}$$

as $y \to \infty$, hence $\xi_k R_{k,L}(\varphi_k, m, \tau, \gamma)$ is in $\mathcal{L}^2_{2-k}$. Now,

$$\begin{aligned} \Delta_k R_{k,L}(\varphi_k, m, \tau, \gamma) &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \Delta_k(\varphi_k(y) e_\gamma(m\tau)) \mid_{k,L} M \\ &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \psi_k(y) e_\gamma(m\tau) \mid_{k,L} M \end{aligned}$$

with

$$\psi_k(y) := -y^2 \varphi_k(y)'' - k y \varphi_k(y)' + 4\pi m y^2 \varphi_k(y)'.$$

Clearly, $\psi_k(y) = 0$ for $y > c$ and $\psi_k(y) = O(y^{2+|k|})$ as $y \to 0$. Hence, for the same reasons as before, $\Delta_k R_{k,L}$ is in $\mathcal{L}^2_k$ and thus $\xi_k R_{k,L}(\varphi_k, m, \tau, \gamma)$ is in $\mathcal{H}^1_{2-k}$. □

Now, consider the space of square integrable holomorphic modular forms, $M^{\text{int}}_{2-k,L^-} \subseteq \mathcal{H}^1_{2-k}$. Denote by

$$\Pi_{2-k} : \mathcal{H}^{2-k}_{1,2} \to (M^{\text{int}}_{2-k,L^-})^\perp$$

the projection operator. By Theorem (2.10), we can apply the inverse Laplace operator Δ^{-1}_{2-k} to $\Pi_{2-k} \xi_k R_{k,L}(\varphi_\alpha, m, \tau, \gamma)$. (Note that here we use $k \neq 1$. If $k = 1$, the continuous spectrum includes 0 and we can not invert Δ_k) and define

$$\tilde{R}_{k,L}(m, \tau, \gamma) = R_{k,L}(\varphi_k, m, \tau, \gamma)$$

$$- \quad \xi_{2-k}\Delta_{2-k}^{-1}\Pi_{2-k}\xi_k R_{k,L}(\varphi_k, m, \tau, \gamma), \tag{5.5}$$

which, since $\xi_k\xi_{2-k}\Delta_{2-k}^{-1} = \mathrm{id}$, satisfies

$$\xi_k \tilde{R}_{k,L}(m,\tau,\gamma) = (\mathrm{id} - \Pi_{2-k})\xi_k R_{k,L}(\varphi_k, m, \tau, \gamma),$$

thus is an element of $M^{\mathrm{int}}_{2-k,L^-}$. We see that the $\tilde{R}_{k,L}$ are not weakly holomorphic.

Remark 5.7. Since $\xi_{2-k}\Delta_{2-k}^{-1}\Pi_{2-k}\xi_k R_{k,L}(\varphi_k, m, \tau, \gamma)$ is in $\mathcal{H}^{2-k}_{2,2}$, by Lemma (2.7), at each cusp $Mi\infty$ as $y \to \infty$:

$$\tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M = \delta(M)e_\gamma(m\tau) + O(y^{(1-k)/2}). \tag{5.6}$$

One might now wonder what use the $\tilde{R}_{k,L}$ are if they are not weakly holomorphic The answer to this is given by the following

Theorem 5.8. *For $k \neq 1$, any element of $M^!_{k,L}$ is a finite linear combination of elements in*

$$\begin{aligned} T_{k,L} \quad &:= \quad \{\xi_{2-k}\tilde{R}_{2-k,L}(m,\tau,\gamma)|_k\, M,\ \tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M \\ &\quad\; | \quad m \in \frac{1}{N}\mathbb{Z},\ m \leq 0,\ M \in \mathrm{Mp}_2(\mathbb{Z})\}. \end{aligned}$$

Proof. At first let us remark that by (5.4), all non constant elements in $M^{\mathrm{int}}_{k,L}$ are a finite linear combination of elements in the set $\{\xi_{2-k}\tilde{R}_{2-k,L}(m,\tau,\gamma)|\, m \in \frac{1}{N}\mathbb{Z},\ m \leq 0\}$. To obtain estimates on the dimensions involved and prove our theorem, we will need to employ Riemann-Roch, for which we need to assume that Γ operates freely on $\mathbb{H}$. This, however can be achieved by passing to a suitable $\tilde{\Gamma}(N)$, the results of this theorem then hold by surjectivity of the trace map. We split the proof into two cases, the first one being $k > 1$. For $n \in \frac{1}{N}\mathbb{Z}$, let $T_{k,n,L}$ denote the set $\{\tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M\, |M \in \mathrm{Mp}_2(\mathbb{Z}), -n \leq m \leq 0\}$ and $V(T_{k,n,L})$ the finite vector space generated by it. Let d denote the number of cusps. By (5.6), we have

$$\dim(V(T_{k,n,L})) = d(nN+1)\,|L'/L|\,,$$

since, at the cusp $M^{-1}i\infty$, the definition implies

$$\tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M \approx e_\gamma(m\tau) + O(y^{(1-k)/2}),$$

and, by (5.6), at all other cusps,

$$\tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M \approx O(y^{(1-k)/2}),$$

i.e. $\tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M$ vanishes therein. Since $(1-k) < 0$, and all $e_\gamma(m\tau)$ are linearly independent for $m \leq 0$, the $\tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M$ are linearly independent for different cusps $M^{-1}i\infty$. Clearly, ξ_k defines a map

$$\xi_k : V(T_{k,n,L}) \to M^{\mathrm{int}}_{2-k,L^-}.$$

The analysis that follows is certainly inspired from [4]. Consider the compact closure of the fundamental domain $\mathcal{F}$ as a complex manifold X. Denote by $D = \sum_i s_i$ the divisor that is 1 at each cusp s_i, $\mathcal{O}_D$ the sheaf associated to the divisor and $\mathcal{O}_{k,L}$ the sheaf of holomorphic modular forms of weight k. Then, since $2-k<1$ and any holomorphic modular form of that weight is integrable and vice versa

$$M^{\mathrm{int}}_{2-k,L^-} = H^0(X, \mathcal{O}_{2-k,L^-}).$$

Denote by $K(T_{k,n,L})$ the kernel of ξ_k in $V(T_{k,n,L})$. By (5.6),

$$\begin{aligned} & K(T_{k,n,L})\backslash\{0\} \\ \subseteq\ & H^0(X, \mathcal{O}_{(Nn+1)D} \otimes \mathcal{O}_D \otimes \mathcal{O}_{k,L})\backslash H^0(X, O_D \otimes \mathcal{O}_{k,L}), \end{aligned} \tag{5.7}$$

i.e. any non zero element in $K(T_{k,n,L})$ does non vanish in at least one cusp. Now, let Ω denote the principal bundle of X. We then have

$$\Omega \cong \mathcal{O}_2 \otimes (-D),$$

i.e. the principal bundle on X are the cusp forms of weight 2. Next, let us state Riemann-Roch for holomorphic vector bundles. If X is an algebraic curve of genus g and A, B are complex vector bundles of dimension a resp. b on X, A^{-1}, B^{-1} the corresponding dual bundles and h^0 the dimension of the cohomology groups and c_1 the Chern class. Then (see [6, Thm. 21.1.2])

$$\begin{aligned} & h^0(X, A \otimes B^{-1}) - h^0(h, A^{-1} \otimes B \otimes \Omega) \\ =\ & bc_1(A) - ac_1(B) + ab(1-g). \end{aligned}$$

Hence, for a third bundle C of dimension b,

$$\begin{aligned} h^0(X, A \otimes B^{-1}) - h^0(X, A \otimes C^{-1}) &= h^0(h, A^{-1} \otimes B \otimes \Omega) \\ &- h^0(h, A^{-1} \otimes C \otimes \Omega) \\ &+ a(c_1(C) - c_1(B)). \end{aligned}$$

Now, we set $A = \mathcal{O}_{k,L}$, $B^{-1} = \mathcal{O}_{(Nn+1)D} \otimes \mathcal{O}_{-D} = \mathcal{O}_{NnD}$, $C^{-1} = \mathcal{O}_D$. As for any divisor Y, $c_1(\mathcal{O}_Y) = \deg(Y)$, hence $\deg(D) = d$, the number of cusps, and by (5.7) and Riemann-Roch,

$$\begin{aligned} \dim(K(T_{k,n,L})) &\leq h^0(X, \mathcal{O}_{(Nn+1)D} \otimes \mathcal{O}_{-D} \otimes \mathcal{O}_{k,L}) \\ &- h^0(X, \mathcal{O}_{-D} \otimes \mathcal{O}_{k,L}) \\ &= |L'/L|\,(Nn+1)d \\ &+ h^0(X, (\mathcal{O}_{(Nn+1)D} \otimes (-D) \otimes \mathcal{O}_{k,L})^{-1} \otimes \Omega) \\ &- h^0(X, ((-D) \otimes \mathcal{O}_{k,L})^{-1} \otimes \Omega) \\ &= |L'/L|\,(Nn+1)d \\ &+ h^0(X, \mathcal{O}_{-(Nn+1)D} \otimes \mathcal{O}_{2-k,L^-}) \\ &- h^0(X, \mathcal{O}_{2-k,L^-}) \end{aligned}$$

since $k > 1$, for n big enough,

$$h^0(X, \mathcal{O}_{-(Nn+1)D} \otimes \mathcal{O}_{2-k,L^-}) = 0$$

as the number of zeroes of globally holomorphic modular forms is bounded. Further, as any holomorphic modular form of weight $2-k$ is integrable and vice versa

$$\begin{aligned} h^0(X, \mathcal{O}_{2-k,L^-}) &= \dim(M^{\text{int}}_{2-k,L^-}) \\ &= \dim(\xi_k(V(T_{k,n,L})). \end{aligned}$$

And hence

$$\begin{aligned} \dim(K(T_{k,n,L})) + \dim(\xi_k(V(T_{k,n,L})) &\leq |L'/L|\,(Nn+1)d \\ &= \dim(V(T_{k,n,L})). \end{aligned}$$

By the usual kernel and image formula for finite dimensional vector spaces, the inequality must already be an equality. Thus

$$\dim(K(T_{k,n,L})) + \dim(M^{\text{int}}_{k,L}) = h^0(X, \mathcal{O}_{NnD} \otimes \mathcal{O}_{k,L}).$$

$\square$

Since $K(T_{k,n,L}) \cap M^{\text{int}}_{k,L} = 0$, linear algebra implies that

$$H^0(X, \mathcal{O}_{NnD} \otimes \mathcal{O}_{k,L}) = K(T_{k,n,L}) \oplus M^{\text{int}}_{k,L},$$

i.e. for $k > 1$ any element of $M^!_{k,L}$ that has has at most a pole of order n in each cusp is a linear combination of elements in $T_{k,n,L}$. Now, consider the case $k < 1$. We need a slight variation of above as all holomorphic forms are integrable, i.e.

$$M^{\text{int}}_{k,L} = H^0(X, \mathcal{O}_{k,L}).$$

We proceed analogously and define the set $T_{k,n,L} \coloneqq \{\tilde{R}_{k,L}(m,\tau,\gamma)|\ -n \leq m < 0\}$ and the vector spaces $V(T_{k,n,L})$ and $K(T_{k,n,L})$ in an analogous fashion. Analogously to the previous section one argues that

$$\dim(L(T_{k,n,L})) = d\,|L'/L|\,Nn,$$

since again all elements of $T_{k,n,L}$ are linearly independent. Further, as before,

$$K(T_{k,n,L}) \backslash \{0\} \subseteq H^0(X, \mathcal{O}_{NnD} \otimes \mathcal{O}_{k,L}) \backslash H^0(X, \mathcal{O}_{k,L}),$$

i.e. all non zero elements of $K(T_{k,n,L})$ are not integrable and have a pole in $i\infty$ of at most order n, Riemann-Roch again implies

$$\begin{aligned} \dim(K(T_{k,n,L})) &\leq h^0(X, \mathcal{O}_{NnD} \otimes \mathcal{O}_{k,L}) - h^0(X, \mathcal{O}_{k,L}) \\ &= |L'/L|\,Nnd + h^0(X, (\mathcal{O}_{-NnD} \otimes \mathcal{O}_{k,L})^{-1} \otimes \Omega) \end{aligned}$$

$$
\begin{aligned}
&- \quad h^0(X, (\mathcal{O}_{k,L})^{-1} \otimes \Omega) \\
&= \quad |L'/L|\, Nnd + h^0(X, \mathcal{O}_{-NnD} \otimes \mathcal{O}_{-D} \otimes \mathcal{O}_{2-k,L^-}) \\
&- \quad h^0(X, \mathcal{O}_{-D} \otimes \mathcal{O}_{2-k,L^-}).
\end{aligned}
$$

Now, since $k < 1$, the integrable holomorphic modular forms of weight $2-k$ are precisely the cusp forms, we have

$$
\begin{aligned}
h^0(X, \mathcal{O}_{-D} \otimes \mathcal{O}_{2-k,L^-}) &= \dim(M^{\text{int}}_{2-k,L^-}) \\
&= \dim(\xi_k(V(T_{k,n,L})).
\end{aligned}
$$

And thus for n big enough

$$
\begin{aligned}
\dim(K(T_{k,n,L})) + \dim(\xi_k(V(T_{k,n,L})) &\leq |L'/L|\, Nnd \\
&= \dim(V(T_{k,n,L})).
\end{aligned}
$$

And once again equality must hold and we have

$$
H^0(X, \mathcal{O}_{NnD} \otimes \mathcal{O}_{k,L}) = K(T_{k,n,L}) \oplus M^{\text{int}}_{k,L}.
$$

proving the statement for $k < 1$ as well.

5.2.2 Harmonic Maass Forms

To construct a preimage under ξ_k of the $\tilde{R}_{k,L}$, for f quasi-analytic, define the operator

$$
B_{k,m}(f)(y) \quad := \quad e^{-4\pi m y} \int_0^y e^{4\pi m x} f(x) x^{-k} dx.
$$

By assuming y to be sufficiently small, we see that $B_{k,m}(f)$ is quasi-analytic, as for $f = O(y^\alpha)$ as $y \to 0$

$$
B_{k,m}(f) \quad = \quad O(y^{\alpha+1-k}).
$$

We also note that the integral operator $B_{k,m}(f)$ is only defined if $\alpha + 1 - k > 0$, whereas otherwise we have divergences of the integral. We will ensure this to hold later on, assume it holds in the following. Further

$$
\xi_k(B_{k,m}(f)(y) e(m\bar{\tau})) \quad = \quad f(y) e(-m\tau),
$$

and thus, if $2\alpha + (2-k) > 2$,

$$
\begin{aligned}
&\quad \xi_k R_{k,L}(B_{k,m}(f), -m, \bar{\tau}, \gamma) \\
&= \quad \sum_{M \in \Gamma_\infty \backslash \Gamma} \xi_k(B_{k,m}(f)(y) e_\gamma(-m\bar{\tau})) \,|_{2-k,L^-}\, M \\
&= \quad R_{2-k,L^-}(f, m, \tau, \gamma)
\end{aligned}
$$

for all such quasi analytic f as then the series converges absolutely and $B_k(f)$ is well defined. Now, we set

$$\begin{aligned} S_{k,L}(m,\tau,\gamma) &:= R_{k,L}(B_{k,m}(\varphi_{2-k}),-m,\bar{\tau},\gamma) \\ &- \Delta_k^{-1}\Pi_k\xi_{2-k}R_{2-k,L}(\varphi_{2-k},m,\tau,\gamma), \end{aligned}$$

which satisfies (we recall (5.5)):

$$\begin{aligned} \xi_k S_{k,L}(m,\tau,\gamma) &= \xi_k R_{k,L}(B_{k,m}(\varphi_{2-k}),m,\tau,\gamma) \\ &- \xi_k\Delta_k^{-1}\xi_{2-k}R_{2-k,L^-}(\varphi_k,m,\tau,\gamma) \\ &= R_{2-k,L^-}(\varphi_{2-k},m,\tau,\gamma) \\ &- \xi_k\Delta_k^{-1}\Pi_k\xi_{2-k}R_{2-k,L^-}(\varphi_{2-k},m,\tau,\gamma) \\ &= \tilde{R}_{2-k,L^-}(m,\tau,\gamma). \end{aligned}$$

To construct Maass forms from the $S_{k,L}$, we must first prove:

Lemma 5.9. *$S_{k,L}$ is of at most exponential growth at any cusp.*

Proof. By definition, $B_{k,m}(\varphi_{2-k}) = O(y^{3+|k|-k})$ as $y \to 0$. By definition, $B_{k,m}(\varphi_{2-k}) = O(y^{1-k}e^{4\pi|m|y})$ as $y \to \infty$. Hence

$$S_{k,L}(m,\tau,\gamma)|_k\, M = \delta(M)O(e^{Cy}) + O(y^{-2-|k|})$$

at each cusp $Mi\infty$. □

A direct consequence of this lemma is (where ξ_k is an operator as in Lemma (1.10))

Corollary 5.10. *For $k \neq 1$, ξ_k is surjective.*

Proof. Consider the set $T_{k,L}$ as in Theorem (5.8). Define the set

$$\begin{aligned} \tilde{T}_{k,L} &:= \{\tilde{R}_{k,L}(m,\tau,\gamma)|_k\, M,\, S_{k,L}(m,\tau,\gamma)|_k\, M \\ &\mid \quad m \in \frac{1}{N}\mathbb{Z},\, m \leq 0,\, M \in \mathrm{Mp}_2(\mathbb{Z})\}. \end{aligned}$$

Consider the vector spaces spanned by finite linear combinations of those sets, denoted $V_{k,L} = V(\tilde{T}_{k,L})$ and $W_{2-k,L^-} = V(T_{2-k,L^-})$. Then, ξ_k defines a sequence

$$V_{k,L} \overset{\xi_k}{\twoheadrightarrow} W_{2-k,L^-} \overset{\xi_{2-k}}{\twoheadrightarrow} M^{\mathrm{int}}_{k,L} \tag{5.8}$$

as $\xi_k(\tilde{T}_{k,L^-}) = T_{2-k,L^-}$. Since, by Theorem (5.8), $M^!_{2-k,L^-} = \ker(\xi_{2-k} : W_{2-k,L^-} \to M^{\mathrm{int}}_{k,L})$ and above lemma, we see that the vector space $\xi_k^{-1}(M^!_{2-k,L^-}) \subseteq V_{k,L}$ consists only of harmonic Maass forms. □

Furthermore, another corollary is

Corollary 5.11. *Every harmonic Maass form of weight k is a finite linear combination of elements of the set*

$$\hat{T}_{k,L} \quad := \quad T_{k,L} \cup \tilde{T}_{k,L}.$$

Proof. Consider a harmonic Maass form f of weight k. Now, by the first corollary and using the sequence (5.8) , we know there is a finite linear combination of elements in $\tilde{T}_{k,L}$, which we denote by f_0 such that $\xi_k f = \xi_k f_0$. Hence $f - f_0$ must be in $M^!_{k,L}$, i.e. $f - f_0 = f_1$, which is a finite linear combination of elements in $T_{k,L}$ by Theorem (5.8). Hence $f = f_0 + f_1$ and the corollary is proven. □

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