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Infrared Nonlocal Gravity Theories

Optimizing Science Return to Euclid Satellite Mission

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Abstract

Deutsch

Nichtlokale Gravitationstheorien sind Versuche Quantenkorrekturen in die Einstein-Hilbert-Wirkung miteinzubeziehen. Dies ist ein eleganter Weg, um offene Fragen der Allgemeinen Relativitätstheorie anzugehen. In dieser Dissertation haben wir hauptsächlich infrarot-relevante, nichtlokale Modifikationen von Gravitation als mögliche Quelle für die beschleunigte Expansion des Universums zu späten Zeiten untersucht. Wir zeigen, dass, wenn diese Korrekturen in der Effektiven Wirkung mit einer infrarot-relevanten Größenordnung auftreten, sie zu einer validen Kosmologie zu späten Zeiten führen können, die sowohl auf dem Hintergrund- als auch den Störungsniveaus mit allen beobachteten Daten übereinstimmt. Überdies können manche dieser Modelle besser mit den beobachteten Daten übereinstimmen als das ΛCDM Standardmodell. Wir haben auch Probleme untersucht, die mit der theoretischen Formulierung nichtlokaler Gravitationsmodelle assoziiert werden, wie beispielsweise die Rolle der Anfangsbedingungen für die Lösungsräume. Wir haben herausgefunden, dass eine unvorsichtige Behandlung der Anfangsbedingungen zu einem Übersehen mancher physikalisch sinnvoller Lösungen führen kann. Desweiteren haben wir die Frage nach den klassischen Instabilitäten in tensoriellen nichtlokalen Gravitationsmodelle untersucht, woraus wir schließen, dass neue Mechanismen oder Symmetrien eingeführt werden müssen, um die Gültigkeit dieser Modelle zu etablieren.

English

Theories of nonlocal gravity are attempts to include quantum corrections into the Einstein-Hilbert action. Inclusion of these corrections is an elegant way to address open issues within General Relativity. In this thesis, we have primarily studied infrared-relevant nonlocal gravity modifications as a possible source for the late time accelerated expansion of the Universe. We show that if these correction appear in the effective action with an infrared-relevant scale, they can lead to a valid late time cosmology, which is in agreement with all observational data, both at background and perturbation levels. Moreover, some of these models can have even better agreement with observational data than the standard ΛCDM model. We have also investigated problems associated with the theoretical formulation of nonlocal gravity models, namely the role of initial conditions on their solution spaces. We have found that if not treated carefully this can lead to the overlook of some valid physical solutions. The question of classical instabilities in tensorial nonlocal gravity models has also been studied, where we conclude that inclusion of some new mechanisms or symmetries is needed to render these models valid.

Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. The published works of others which have been consulted in this dissertation are always clearly attributed. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such.

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List of Abbreviations

AS	Asymptotic Safety
BAO	Baryon Acoustic Oscillations
CC	Cosmological Constant
CDM	Cold Dark Matter
CMB	Cosmic Microwave Background
DE	Dark Energy
DM	Dark Matter
DW	Deser-Woodard
EH	Einstein-Hilbert
EFT	Effective Field Theory
EoM	Equations of Motion
FP	Fixed Point
FLRW	Friedmann–Lemaître–Robertson–Walker
GR	General Relativity
HD	Higher Derivative
IR	Infrared
LSS	Large Scale Structure
MG	Modified Gravity
NGFP	non-Gaussian Fixed Point
PF	Perfect Fluid
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
RG	Renormalization Group
RSD	Redshift Space Distortions
SNIa	Supernovae of Type I a
UV	Ultraviolet
VEV	Vacuum Expectation Value
ΛCDM	Λ Cold Dark Matter

CHAPTER 1

INTRODUCTION

It is already more then hundred years as Einstein in 1915 has formulated his theory of General Relativity (GR) [1] which is one of the most elegant and successful theories in the history of physics. It has been able to explain with a grate precision all gravitational phenomenon from sub-millimeter scales up to astrophysical distances, enlarging our understanding of the world surrounding us.

Despite this tremendous success, Einstein's theory of General Relativity is not able to explain the physics at cosmological distances, where one of biggest mysteries of the modern science enters into the game. This is the so-called Dark Energy (DE) component responsible for the late time accelerated expansion of the Universe [2, 3]. Together with Dark Matter (DM), they contribute almost 96% to the today's energy budget of the Universe [4].

The existence of these, physically yet unknown components has been proved by series of observational missions which have also put strong constraints on their physical properties [5, 6]. These constraints will be even more improved by upcoming surveys, such as *Euclid* satellite mission [7], which aim to shade light on physical processes at time and energy (distance) scales which have been never accessed before. This opens up new prospects for scientists to get a better understanding about fundamental mechanisms and processes governing the physical evolution of our Universe. Hence, this is the right time for the development of different theoretical models which can address the question of DE and DM and can be possibly distinguished by cosmological observations.

In this respect, the ΛCDM model is maybe the simplest one. It is built upon GR plus Cold Dark Matter (CDM) component and the cosmological constant (CC) term Λ. With a tiny value for the cosmological constant, $\Lambda \approx 3\times 10^{-122}$ in reduced Planck mass units, this model fits the current observational data remarkably well becoming a phenomenological *"étalon"* model. However, despite its phenomenological success, the ΛCDM model lacks a fundamental understanding. Indeed, the cosmological constant Λ is assumed to account for the vacuum energy of the Universe and once we make a step into Quantum Field Theory (QFT) to understand how is this value generated, we observe that the Λ is not technically natural. This means that it will receive uncompensated quantum corrections at each level of loop expansion. The main reason behind this issue is the incompatibility of classical GR with another pillar of the modern science, quantum physics, which dictates the reality at microscopic scales. At first glance this reasoning can seem a bit naive, as it is hard to imagine how those two distinct microscopic and macroscopic (cosmological) scales can influence each other. However, a consistent (renormalizable) quantum gravity theory can become *"the tool"* which will link those two physical scales. We believe that the existence of a viable quantum gravity theory would allow us self-consistently calculate the cosmological constant term and possibly find a solution to the cosmological constant problem. Unfortunately, the final theory of quantum gravity is not available yet, although, there are several well motivated candidates for this role [8–15].

Based on the above discussion, it is justified to search for theoretical models which can perform phenomenologically at the same level as the ΛCDM but on the theoretical side are free from potential dangers. But once we have decided to modify GR it opens up a plethora of possibilities.

Modifications which will be discussed in this thesis are attempts to consider GR as an effective field theory and encode possible quantum corrections into its action. For this purpose, the first thing which one can do is to analyze quantum corrections arising in the semi-classical picture. With this we refer to the studies of backreaction effects induced by quantum corrections in the matter sector, which then influence the classical gravitational background. In the effective action, these effects usually lead to a running of physical coupling constants as well as to an inclusion of higher curvature terms [16–24].

To study cosmological properties of these corrections we need to write the effective action in a coordinate space. By doing so, the running of coupling constants are generally expressed through particular integral operators that are consistent with symmetries of the theory. The appearance of integral operators in the effective action renders it to be nonlocal [22, 25–29]. These modifications of GR are presumably the most natural ones. Indeed, their inclusion does not require an introduction of new physical concepts or symmetries beyond those existing in initial theories. Nonlocal corrections will always arise in a quantum effective action, once we take into account possible quantum corrections. Moreover, if those corrections become relevant at some energy scales they can have a significant impact on the cosmic evolution of the Universe. Hence, it is natural to ask whether these corrections can target open questions which cannot be answered within GR. Motivated by this, recently there has been a lot of investigations about the role of nonlocal corrections at different time and energy (length) scales of the Universe [30–36].

In this thesis we have been mostly interested in the role of possible nonlocal corrections in the late time cosmology as DE candidates. In Chapters 4 and 5, we will see that if these correction are generated with an infrared (IR) relevant scale, they can serve as a valid DE candidate consistent with all observations both at background and perturbation levels. Furthermore, differently from most of modifications of GR, nonlocal gravity theories do not suffer from ghost instabilities at quantum level. This directly follows from the fact that theories defined with a quantum effective action, which arises from a healthy fundamental theory after integrating out some degrees of freedom, cannot accommodate any pathological degree of freedom [19, 20, 24]. In Chapter 7, we will make a step further and analyze late time cosmological predictions of a model, where quantum corrections are induced in the context of a pure nonperturbative quantum gravity theory, namely, asymptotically safe lattice quantum gravity [13, 37, 38]. In this class of quantum gravity models, the ultraviolet (UV) completion of the theory is achieved by assuming an existence of a non-Gaussian fixed point (NGFP), which controls the UV convergence of the theory [8, 9, 39]. This example is important because it shows that if quantum gravity effects lead to a strong IR relevant running of physical couplings, these running of physical quantities can result in observationally valid and theoretically well motivated DE candidates.

In addition to phenomenological studies of several nonlocal gravity models, in Chapter 6 we will investigate classical instability issues common for the subclass of nonlocal gravity models referred to as *tensorial* nonlocal gravity theories [40, 41]. Our results confirm that besides very limited cases, an introduction of additional mechanisms is needed to make theories which contain tensorial nonlocal terms theoretically viable.

At the practical level, nonlocal gravity models are usually studied by implementing a special localization procedure. This is done by introducing several auxiliary fields which translates initial nonlocal theory into a local multiscalar one. In the literature, while implementing this procedure, there was a common confusion about initial conditions of auxiliary fields. This fact has led to an incomplete investigation of solution spaces of the theories. In Chapter 4, with the use of dynamical system analysis methods, we will address this issue for a particular nonlocal gravity model and find new physical solutions of the model which previously have been overlooked. Though, our analysis are implemented for a particular nonlocal model, the approach is very general and can be easily extended for other nonlocal models too.

Throughout this thesis we will work in natural unites where the speed of light c and Planck constant $\hbar$ are set to unity, i.e. $c = \hbar = 1$. Furthermore, we will denote with a dot a derivative with respect to (w.r.t.) the cosmological time t and with a prime a derivative w.r.t. e-folding time $N = \ln a$. Metric quantities, unless explicitly specified, will be always given with the signature $(-,+,+,+)$.

1.1 Background

1.1.1 The Equivalence principle and Einstein equations

In this section we will give a basic introduction to the theory of General Relativity, by introducing the main quantities and notations used throughout this thesis. Here we mainly follow the discussion done in Ref. [42]. So, let us begin by declaring that General Relativity is a theory built on the Equivalence Principle (EP). The EP states that in a small surrounding of any spacetime point in a general gravitational field, there exist always a local "inertial" coordinate system in which the effects of gravity are absent. The EP allows us to find the equations of motion (EoM) controlling the behavior of the physical system in any small spacetime region where we can neglect gravity. To include the gravitational effects we just need to covariantize those "locally" defined equations. The general covariant equations will not depend on the choice of coordinate system and are valid in any gravitational field. In the case when the gravitational field is absent these covariant equations should reduce to those defined in a local inertial coordinate system. Within Einstein's formulation of GR the gravitational effects are encoded into the geometry of spacetime, which is characterized by a general metric $g_{\mu\nu}$. The distance between two spacetime points of a differentiable manifold, in the case of a general geometry, is given by the following line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.1}$$

where dx^μ is an infinitesimal spacetime element. For the later cosmological purposes we also need to introduce the concept of a perfect fluid (PF). The PF is a medium at every point of which there exists a local co-moving inertial Cartesian frame, where the fluid looks the same in every direction. In a corresponding local frame the components of the PF's energy momentum tensor $T_{\mu\nu}$ are

$$T_{ij} = \delta_{ij} P, \qquad T^{0i} = T_{0i} = 0, \qquad T^{00} = \rho, \tag{1.2}$$

where ρ and P are the energy density and pressure of a perfect fluid (PF), respectively. Using Eq. (1.2) we find that the energy momentum tensor $T^{\mu\nu}$ for the PF in a Cartesian frame is

$$T^{\mu\nu} = P\eta^{\mu\nu} + (P+\rho) u^\mu u^\nu. \tag{1.3}$$

To find the expression for the PF energy momentum tensor for any geometry, we write Eq. (1.3) in a covariant form, such that

$$T^{\mu\nu} = P g^{\mu\nu} + \left(P+\rho\right) u^\mu u^\nu, \tag{1.4}$$

where u^μ is a four-velocity vector of the fluid defined as

$$u^\mu \equiv \frac{dx^\mu}{d\tau}, \tag{1.5}$$

with τ standing for the proper time. The equations that establish a connection between spacetime geometry (gravity) and the matter content of the Universe are known as Einstein equations and have the following structure

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.6}$$

with the Newton constant G. The tensor $G_{\mu\nu}$ in the left hand side of Eq. (1.6) has a pure geometrical origin and is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.7}$$

The Ricci scalar R and Ricci tensor $R_{\mu\nu}$ are derived from the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ by contracting it with the metric $g_{\mu\nu}$, such that $R = g^{\mu\nu} R_{\mu\nu}$ and $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$. The Riemann tensor in its turn is given by

$$R^{\rho}_{\mu\nu\sigma} = \frac{\partial \Gamma^{\rho}_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\lambda}_{\mu\sigma} \Gamma^{\rho}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu} \Gamma^{\rho}_{\lambda\sigma}, \tag{1.8}$$

where the Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right). \tag{1.9}$$

The energy momentum tensor $T^{\mu\nu}$, written in the right hand side of Eq. (1.6), for an arbitrary matter content ψ with the Lagrange density $\mathscr{L}_{\mathrm{m}}[g_{\mu\nu}, \psi]$, is obtained by taking a functional derivative of the Lagrange density w.r.t. the metric $g_{\mu\nu}$, i.e.,

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \mathscr{L}_{\mathrm{m}} \right)}{\delta g_{\mu\nu}}, \tag{1.10}$$

with $g = \det(g_{\mu\nu})$. In particular, for the case of a perfect fluid the energy-momentum tensor $T^{\mu\nu}$ defined by Eq. (1.10) takes the form (1.4). Finally, let us introduce a full covariant action which leads to the gravitational field equations (1.6)

$$S = S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\mathrm{m}}[\psi, g_{\mu\nu}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathscr{L}_{\mathrm{m}}. \tag{1.11}$$

The first term in the action (1.11) is usually referred as the Einstein-Hilbert action, which is responsible for the gravitational part of Eq. (1.6). The second term $S_{\mathrm{m}}[\psi, g_{\mu\nu}]$ is the action corresponding to the general matter sector sourcing the right hand side of Eq. (1.6). So, as we can see the matter sector is determining the geometry of the spacetime; in other words it gravitates. The curved spacetime in turn determines how matter moves in it. For a test particle in the absence of all forces besides gravity, the trajectories will be governed by the following geodesic equation

$$\frac{du^{\mu}}{d\tau} + \Gamma^{\mu}_{\sigma\lambda} u^{\sigma} u^{\lambda} = 0. \tag{1.12}$$

Let us close this section by introducing a couple of important relations which will be used in future discussions. As we know from special relativity the energy momentum tensor $T_{\mu\nu}$ satisfies the following conservation law

$$\partial^\mu T_{\mu\nu} = 0, \tag{1.13}$$

and in general relativity this will be promoted to the covariant conservation law

$$\nabla^\mu T_{\mu\nu} = 0, \tag{1.14}$$

with ∇^μ standing for a covariant derivative.

Another useful identity, which is also named as Bianchi identity, has a pure geometrical nature and leads to the covariant conservation of the Einstein tensor $G_{\mu\nu}$, i.e.,

$$\nabla^\mu G_{\mu\nu} \equiv 0. \tag{1.15}$$

The self-consistency of the relations (1.14), (1.15) can be directly checked at the level of the Einstein equations (1.6) by acting on the both sides of it with the covariant derivative ∇^μ.

1.1.2 Homogeneous and Isotropic Universe

If we look at any direction in the Sky at distances larger than 300 billion light years, the observable Universe seems to be the same. The most important observational evidence supporting this argument is the cosmic microwave background (CMB), first observed by the COBE satellite [5]. The small temperature fluctuation spectrum in the CMB shows that the Universe was homogeneous and isotropic already 14 billion years ago. The possible existence of global homogeneity and isotropy of the Universe tells us that there is no any special location in the Universe and that it will look the same in every direction for all static observers, regardless their location in the space. The assumption of a homogenous and isotropic universe allows us to choose a coordinate system in which spacetime metric takes a simple form. This type of a metric was first studied by Friedman [43] as a solution of Einstein equations (1.6) and then derived by Robertson and Walker from first principles of isotropy and homogeneity [44]. A vast part of the modern cosmology, at least as a first approximation, is based on this choice of a metric. In a four dimensional spacetime the Friedman-Lemaître-Robertson-Walker (FLRW) metric is defined in the following way:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \left(\frac{1}{1-Kr^2} + r^2 d\Omega^2 \right), \tag{1.16}$$

with the scale factor $a(t)$ and the differential solid angle

$$d\Omega \equiv d\theta^2 + \sin(\theta)^2 d\phi^2. \tag{1.17}$$

The quantity K is the so-called curvature constant and can take values $K \subset \{-1, 0, 1\}$, corresponding to a closed, flat or an open universe, respectively. All of these three cases are representing a model with a homogeneous and isotropic structure and to know which case occurs in reality we should look at observations.

The conditions of isotropy and homogeneity constrain the structure of the PF energy momentum tensor (1.4), making it diagonal

$$T^{\mu}_{\nu} \equiv \mathrm{diag}\{-\rho, P, P, P\}. \tag{1.18}$$

In the FLRW metric, from the 0th component of the conservation law (1.14), we get

$$\nabla_{\mu} T^{0\mu} = \frac{\partial \rho}{\partial t} + \frac{3\dot{a}}{a}(\rho + P) = 0. \tag{1.19}$$

We can solve Eq. (1.19) for different matter species constituting the Universe. To do this we need to introduce a thermodynamical relation between the pressure and the energy density of individual particle species. This relation is usually written as

$$P = w\rho, \tag{1.20}$$

where w is the equation of state parameter. Plugging the equation of state (1.20) for a constant w into Eq. (1.19) we find the following general solution

$$\rho \propto a^{-3-3w}. \tag{1.21}$$

Using Eq. (1.21) we can understand how the energy density of the Universe will evolve during different stages of cosmic evolution. For example in the case of radiation-domination, where the matter content of the Universe consist mostly of a relativistic gas with $w = 1/3$, Eq. (1.21) simplifies to

$$\rho \propto a^{-4}. \tag{1.22}$$

This result is easy to understand. Indeed, as in the case of any conserved density, when we increase the volume of space, it will decrease proportional to an inverse of the volume, which in our case is proportional to a^{-3}. On the other hand, because all length scales in the universe will change proportional to the scale factor a, we should also change wavelengths of relativistic particles with the same factor a. Thus, we get in total that the energy density of a relativistic fluid will scale as a^{-4}. After the radiation-domination era, when the particles due to the cosmic expansion cool down and become non-relativistic, we enter into the matter-domination period. During this era baryonic matter in the Universe is well described by a dust-like equation of state $w = 0$. Therefore, we obtain from Eq. (1.21) the energy density

$$\rho \propto a^{-3}, \tag{1.23}$$

which is proportional, as it is expected, to an inverse of the volume of the visible universe. Another important case to be mentioned separately is the evolution of the Universe with the CC Λ. For the CC we have that $T_{\mu\nu} \propto g_{\mu\nu}$ and thus $P = -\rho$, which means that in this case the equation of state parameter is $w = -1$. Inserting this value of w into the general solution (1.21) we obtain for the CC

$$\rho \propto \text{const.} \tag{1.24}$$

All this results hold also when we have DM or early DE components in the Universe (to be introduced in the next section), provided that there is no direct interaction between the different components.

1.1.3 Dynamics of the cosmic evolution

The cosmic evolution of the Universe is governed by the Einstein equations (1.6). Using the FLRW metric (1.16) in Eq. (1.6) we find for its 00 component

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{K}{a^2} = -8\pi G\rho, \tag{1.25}$$

and for the spacial (ij) components we have

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = 8\pi G P. \tag{1.26}$$

For the cosmological implementations it is sometimes useful to write down these equations through dimensionless quantities. This task can be completed by dividing Eq. (1.25) by $3H^2$, such that we get

$$1 = \frac{8\pi G\rho}{3H^2} - \frac{K}{a^2H^2} = \Omega + \Omega_{\mathrm{K}}, \tag{1.27}$$

where we have defined

$$\Omega \equiv \frac{8\pi G\rho}{3H^2} \qquad \Omega_{\mathrm{K}} \equiv -\frac{K}{a^2H^2}. \tag{1.28}$$

The quantities defined in Eq. (1.28) are called cosmological critical densities. If the Universe consists of several not-directly interacting components such as DM, radiation, dust-like matter, we can write the total energy density ρ_{tot} as a sum of energy densities of these components ρ_i, i.e.

$$\rho_{\mathrm{tot}} \equiv \sum_i \rho_i, \tag{1.29}$$

which also implies

$$\Omega_{\mathrm{tot}} \equiv \sum_i \Omega_i. \tag{1.30}$$

From current observational data [4] we know that nowadays $|\Omega_{\mathrm{K}}| < 0.005$, which means that currently the curvature constant K is almost vanishing which in turn signals that our geometry is a flat one. Based on this, from now on we will assume that the curvature constant K is always vanishing and we have a spatially flat geometry. In this case, by inserting the scaling law of an energy density from Eq. (1.21) into the left hand side of the Friedman equation (1.25) and solving it analytically, one gets for the scale factor a

$$a \propto t^{\frac{2}{3+3w}}. \tag{1.31}$$

The solution (1.31) characterizes how the scale factor a changes during the cosmic time t for a single component universe. For the CC with $w = -1$ this solution does not hold anymore. To find the evolution of a for the CC, we solve Eq. (1.25) by setting $\rho = 3H^2/8\pi G$ to be a *constant*. The solution in this case is the following

$$a \propto \exp(Ht). \tag{1.32}$$

The FLRW metric with the scale factor (1.32) is referred to as the *de Sitter* metric. As we can see from Eqs. (1.31)-(1.32) the scale factor a is increasing with the time in all cases of our interest, e.g. radiation, matter and CC dominated periods. At this stage another important question arises, namely, what is the rate of cosmic expansion? To answer this question we introduce the deceleration parameter q defined as

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2}. \tag{1.33}$$

If the Universe expands with an acceleration $\ddot{a} > 0$ we will have that the deceleration parameter is negative. In the opposite case of a decelerating expansion this parameter is positive. It is useful to express the deceleration parameter q through the pressure and energy density of the PF. To do this we can subtract Eq. (1.25) from Eq. (1.26) and divide by H^2. After some simple algebraic manipulations we get

$$q = \frac{4\pi G(\rho + 3P)}{3H^2}. \tag{1.34}$$

From Eq. (1.34) we immediately notice that in order to have an accelerated expansion of the Universe the following condition must be satisfied

$$P + 3\rho < 0 \Rightarrow w < -\frac{1}{3}. \tag{1.35}$$

As it will become clear later, this criterion is playing an important role in the process of analyzing different Dark Energy and Modified Gravity models.

1.2 Inflation

The discussion of the FLRW cosmology in the previous section does not explain why the Universe is homogeneous and isotropic on large scales. These questions are addressed in the context of inflation, where an exponential expansion at the early stages of cosmic history drives an initially generic state towards a homogeneous and isotropic one. But maybe the most important function of inflation is to provide a mechanism, that explains how early universe quantum perturbations can seed large scale structure formation.

1.2.1 Conditions for Inflation and a Simple Realization

In this section, we will follow the discussion in Ref. [45, 46] and summarize all conditions needed to have a cosmologically viable inflationary scenario. The first ingredient is to have a period of accelerated expansion. In Sec. 1.1.3 we have already mentioned that negative values of the deceleration parameter q correspond to an accelerated expansion. The deceleration parameter q is closely related to another physical quantity called co-moving Hubble radius defined as $(aH)^{-1}$. An accelerated expansion in the language of the co-moving Hubble radius can be realized when this radius decreases. Indeed, we can write that

$$\frac{d}{dt}(aH)^{-1} = qa. \tag{1.36}$$

As we can see from Eq. (1.36), when q is negative that corresponds to a decreasing Hubble radius. In the physics of inflation in general there is another set of parameters which is used to characterize the expansion during inflation. These parameters are the so-called slow-roll parameters and are introduced in the following way:

$$\frac{d}{dt}(aH)^{-1} = -\frac{1}{a}(1-\epsilon), \tag{1.37}$$

where

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{1.38}$$

Therefore, the shrinking Hubble radius also corresponds to $\epsilon < 1$. To solve the flatness and horizon problems it is not enough to have a period with an accelerated expansion; it should also last relatively long from $N \sim 40$ to 60 e-folds to be able to set the right initial conditions for the cosmic evolution afterwards. To achieve this condition we require the relative change of ϵ per Hubble time to remain sufficiently small, i.e.

$$|\eta| \equiv |\dot{\epsilon}/H\epsilon| < 1. \tag{1.39}$$

In the rest of this section we will introduce and discuss one of the simplest inflationary models, namely, the single scalar field model. This model is given by the following action

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{1.40}$$

where R is the Ricci scalar and $V(\phi)$ is an arbitrary potential. In the action (1.40), the mass parameter $M_{\rm pl}$ stands for the reduced Planck mass defined as $M_{\rm pl} \equiv (8\pi G)^{-1/2}$. After varying the action (1.40) w.r.t the inflaton field ϕ we get

$$\ddot{\phi} + 3H\dot{\phi} = -V'. \tag{1.41}$$

Here, the prime denotes the derivative w.r.t the field ϕ. On the other side after varying the action (1.40) w.r.t the metric $g_{\mu\nu}$ we find

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]. \tag{1.42}$$

Using Eqs. (1.41)-(1.42) we derive for the time variation of H

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{\rm pl}^2}. \tag{1.43}$$

Inserting this expression into Eq. (1.38) one gets

$$\epsilon = \frac{\dot{\phi}^2}{2H^2M_{\rm pl}^2}. \tag{1.44}$$

Therefore inflation occurs ($\epsilon \ll 1$) when the kinetic energy of a scalar field $\dot{\phi}^2/2$ is much smaller then a potential energy V. This in turn implies that Eq. (1.42) can be simplified to

$$H^2 \approx \frac{V}{3M_{\rm pl}^2}. \tag{1.45}$$

By demanding that the relative change of ϵ per Hubble time is small (or equivalently that $\eta \ll 1$) we find from the Klein-Gordon equation (1.41) that

$$3H\dot{\phi} \approx -V'. \tag{1.46}$$

Plugging Eqs. (1.45)-(1.46) into Eq. (1.44) we find

$$\epsilon = \frac{\dot{\phi}^2}{2H^2M_{\rm pl}^2} \approx \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_v. \tag{1.47}$$

Finally by taking the time derivative of Eq. (1.46)

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}, \tag{1.48}$$

and plugging into Eq. (1.39) one achieves

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} \approx M_{\rm pl}^2\frac{V''}{V} \equiv \eta_v. \tag{1.49}$$

Equations (1.47)-(1.49) play a very important role in the construction procedure of different effective theories to explain inflation. In particular, as it will be shortly discussed in Sec. 2.2, UV relevant quantum corrections to the inflationary potential V can lead in some cases to a violation of the slow-roll conditions $\epsilon_v, \eta_v \ll 1$ and thus making the model physically not viable.

1.3 Dark Matter

The main part of the matter content in our Universe ($\approx 25\%$) does not consist of ordinary baryonic matter, but rather of some unknown component. There are several indirect observational evidences which confirm the existence of a non-relativistic (cold) and very weakly or non-interacting sector in our Universe, which exhibits itself only gravitationally. These observational tests span over different astrophysical length scales, from galaxy to galaxy cluster scales [42, 47, 48]. This unknown matter sector is usually referred to as Dark Matter (DM). It is called "Dark" because it does not radiate photons and therefore is completely non-luminous. Elementary particle physics theory offers several cold DM candidates. Among the widely discussed candidates are weakly interacting massive particles (WIMPs) and in particular those coming from physics beyond the Standard Model, such as string theory, SUSY, etc. . There are also several models, which in order to explain the gravitational influence of the DM sector, do not require the inclusion of any new particle species. In these models, the physical effects of DM are promoted instead to a modification of gravity (see e. g. Ref. [49–51]).

1.4 Dark Energy and Modified Gravity

The discovery of the accelerated cosmic expansion made by two independent groups [2, 3] has provoked another fascinating challenge for the modern science. The accelerated expansion of the Universe has also been conformed by many ground-based and satellite observations [4, 6] and is still to be tested by upcoming surveys, such as *Euclid* [52]. This discovery was surprising because ordinary fluids which consist of matter or radiation can only lead to a decelerating expansion, due to the fact that gravitational interaction between them is attractive. This also can be seen from Eq. (1.34) where in both cases of radiation $P = 1/3\rho$ and dust-like matter $P = 0$, the deceleration parameter q is positive, which means we have a decelerating expansion. To address this problem, one can think of two possible ways. First, we can modify the matter sector of the Universe by adding a component with an equation of state parameter smaller than $-1/3$. This type of modification will alter the right hand side of the Einstein equation (1.6) and in literature are usually called "Dark Energy" (DE) models. Another possibility will be to modify the Einstein-Hilbert action, thus changing properties of a gravitational field itself. Modifications belonging to this category are called "Modified Gravity" (MG) theories and their contribution will be encoded in the left hand side of the Einstein equations (1.6). The current observations [4] show that DE or MG components amount for approximately 68% of the Universe energy budget today and the value of their equation of state parameter is constrained to be $w = -1.019^{+0.075}_{-0.080}$.

These observational results are consistent with the simplest DE candidate one can think about, which is the standard ΛCDM model. In this model Λ stands for the cosmological constant and CDM for a cold DM component. The Einstein equation (1.6) in the presence of the CC takes the form

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{1.50}$$

Written in this way Eq. (1.50) allows us to interpret the CC as a constant source in the energy budget of the model. The energy density and pressure of the CC term are

$$\rho_\Lambda \equiv T_{00} = \frac{\Lambda}{8\pi G} \qquad P_\Lambda \equiv T^i_i/3 = -\frac{\Lambda}{8\pi G}, \tag{1.51}$$

respectively. From Eq. (1.51) it follows that the CC energy contribution does not dilute and remains always constant. This is clearly not a property of a particle fluid and thus the CC should have some other "microscopic" origin. Usually, it is associated with the vacuum energy of the Universe. Though being the most economical possibility of addressing the question of the late time expansion, the cosmological constant has its own theoretical and naturalness problems [53–55]. We will discuss these pathologies in the next section.

1.4.1 Cosmological Constant Problem

The Einstein-Hilbert action with the CC term Λ, which encounters for vacuum energy contribution in the gravity, is written as

$$S_{\rm EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\Lambda) \tag{1.52}$$

As any type of energy, the vacuum energy of matter fields gravitates. Of course, to address the question of vacuum energy at all energy scales, one has to have some knowledge about Quantum Gravity. However, if we are interested in energy scales way below the Planck scale, the interaction with gravitons will be very suppressed, so we can assume that gravity is classical whereas the matter sector can be fully quantized. In this case the effective Einstein equations (1.6) are

$$G_{\mu\nu} = 8\pi G \langle 0|T_{\mu\nu}|0\rangle. \tag{1.53}$$

On the right hand side of this equation, we have the vacuum contribution of the matter energy momentum tensor (1.10). A self-consistent way of calculating this contribution is to compute vacuum-loop diagrams for all the particle content of the Universe. As an example, using dimensional regularization, the one loop vacuum contribution from a canonical scalar field ϕ, with mass m, is [56]

$$\begin{aligned}\Gamma^1_{\rm vac} \sim \frac{i}{2}{\rm Tr}\left[\log\left(-i\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right)\right] &= -\frac{1}{2}\int d^4x\int\frac{d^4k_{\rm E}}{(2\pi)^4}\log(k_{\rm E}^2+m^2)\\ &= -\frac{m^4}{(8\pi)^4}\left[-\frac{2}{\varepsilon}+\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-\frac{3}{2}\right]\int d^4x\\ &\subset -\rho_{\rm vac}\int d^4x,\end{aligned} \tag{1.54}$$

where $\varepsilon = 4 - D$ within dimensional regularization. Furthermore, γ is the Euler-Mascheroni constant and μ is the renormalization group scale. Adding counter terms to remove the divergences and generalizing this result to over larger particle contents, one might await for the total vacuum energy to be given by [56]

$$\rho_{\text{vac}} \sim \sum_i O(1) m_i^4, \tag{1.55}$$

where the index i runs over all massive particles. Already in the case of the Standard Model we have particles almost up to TeV scale, thus the vacuum energy ρ_{vac} in this case can be estimated to be of the order $\rho_{\text{vac}} \gtrsim (\text{TeV})^4$. Now let us see how this vacuum energy will effect the dynamics of our Universe [56]. The energy momentum tensor of vacuum contributions will be

$$T_{\mu\nu}^{\text{vac}} = -\rho_{\text{vac}} g_{\mu\nu}. \tag{1.56}$$

Inserting the vacuum energy momentum tensor (1.56) into Eq. (1.53) we get for the Hubble parameter

$$H_{\text{vac}}^2 = \frac{\rho_{\text{vac}}}{3M_{\text{pl}}^2}. \tag{1.57}$$

If ρ_{vac} is positive this will lead to an accelerated expansion of the Universe. As it was already mentioned before current observations put strong constraints on the value of ρ_{vac} and they require $\rho_{\text{vac}}^{\text{obs}} \approxeq (\text{MeV})^4$ which is a much smaller value than the one obtained from vacuum one-loop results. So, now how we can reconcile this problem. The most straightforward way is to introduce a constant $\tilde{\Lambda}$ counter term in the Einstein Hilbert action, such that it will absorb the large value of the vacuum energy (1.55). Implementing this idea, we get for a renormalized cosmological constant Λ_{ren}

$$\Lambda_{\text{ren}} = \tilde{\Lambda} + \Lambda_{\text{vac}}. \tag{1.58}$$

Now, since $\Lambda_{\text{vac}} \approx (\text{TeV})^4 \sim 10^{60} (\text{MeV})^4$, to achieve the observational value $\Lambda_{\text{obs}} = \Lambda_{\text{ren}} \approx (\text{MeV})^4$ we have to make an extreme fine-tuning of the order 10^{-60} between $\tilde{\Lambda}$ and Λ_{vac}. Usually when people refer to the CC problem they have in mind this fine-tuning issue. But this fine-tuning itself cannot be a problem, because we are always allowed to add counter terms and absorb the large value of the vacuum energy. The important question here is how sensitive is this fine-tuning upon higher order radiative corrections? If the Einstein-Hilbert theory with the Standard Model particle content were perturbative renormalizable we would be able, by adding a finite number of counterterms, to stabilize our result at any loop order. Unfortunately, this is not the case and higher loop corrections in this model cannot be suppressed. For that reason, we have to tune the value of the observed cosmological constant at each level of perturbative loop expansion[1]. This

[1] In Ref. [57] one can find an estimate of the minimal amount of loops needed to be tuned.

tells us that the vacuum energy is sensitive to the details of UV physics of which we are oblivious. This is the real cosmological constant problem. There is another largely discussed statement that the CC problem can be just an artifact of a truncation in the loop expansion series. Namely, if we were able to sum all the loops we might obtain an exact expression for the vacuum energy. This is *per se* a very optimistic consideration. Indeed, by taking into account the fact that in effective field theories, which are not UV complete, beyond leading order one has higher dimensional terms with unknown coefficients, there is not much hope that we will ever be able to sum all the loops [56].

To summarize, if we want to address the CC problem within a QFT framework we first need to find a way which makes the CC safe from possible quantum corrections. At this point it is worthwhile to investigate possible alternatives to the CC term which can also generate late time acceleration of the Universe. In the current cosmological literature there are dozens of suggestions for alternatives to the CC, which are classified either as a DE model or as a MG one. Below, follwing the discussion in Refs. [48, 58] we will briefly describe some of those models. Here we have chosen the models which are most discussed in the literature and that are well-motivated from a theoretical point of view. Another important thing to be clarified here is the following: all the models which will be mentioned bellow do not solve the CC problem as such and one has to still invoke a mechanism (symmetry) to explain the absence of the vacuum energy contribution into the gravitational sector.

1.4.2 Dark Energy Models

One of the simplest models within this category which one can think of is the *quintessence* model [59–64]. This model is very similar to the simple scalar field inflationary model introduced in Sec. 1.2.1 and is given by the same action (1.40) plus the contribution from the matter sector, i.e.

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S_{\rm m}. \tag{1.59}$$

The equation of state parameter for the quintessence field ϕ is

$$w_{\rm q} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{1.60}$$

As one can see during the period when the potential energy of the quintessence field dominates over its kinetic energy $V(\phi) \gg \dot{\phi}^2/2$, this model will resemble a CC type evolution with $w \approx -1$. In this respect, the late time acceleration in quintessence models resembles to a period of very low energy scale inflation. In general, the quintessence equation of state $w_{\rm q}$ is a time-dependent quantity and provides a dynamical DE candidate. Another important feature, which is inherit to a wide class of quintessence models, is the existence of tracking solutions [65, 66]. Within the tracking regime the quintessence field energy density closely traces the background fluid density and becomes dominant over it only recently. This type of behavior can explain the so-called "coincidence" problem, which is: why are the DE and DM energy densities compatible in nowadays Universe. Finally, let us also mention that in cases when the potential $V(\phi)$ is very flat, we will effectively reintroduce the CC term which should be reabsorbed into the bare CC term. A quite simple generalization of the quintessence model, is to consider non-canonical terms for the scalar field ϕ, which will introduce derivative self interactions for ϕ. A possible action embedding these properties can be written as [58]

$$S = \int d^4\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R + K\left(\phi, X\right)\right] + S_{\rm m}, \tag{1.61}$$

where K is a general function depending on the field ϕ and its kinetic energy X defined as $X \equiv -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. These models are usually referred to as k-essence [67–69]. The equation of state parameter for k-essence is given by the following expression

$$w_{\rm K} = \frac{K}{2XK_{,X} - K}. \tag{1.62}$$

From Eq. (1.62) it is not difficult to see that, by appropriately choosing the form of the non-canonical function K, one can attain a situation with $w_{\rm K} < -1/3$, which will correspond to an accelerated expansion. An important difference between this type of non-canonical DE models as compared with canonical ones manifests itself at the level of perturbations. Indeed, as it has been shown in Ref. [70], the cosmic perturbations in k-essence models do not longer propagate with the speed of light, but rather have a K dependent sound speed $c_{\rm s}$ given

$$c_{\rm s}^2 = \frac{K_{,X}}{K_{,X} + 2XK_{,XX}}. \tag{1.63}$$

Due to this property, cosmic structure formation (to be discussed in Sec. 1.5) in k-essence models is noticeably different from that in canonical DE models, where perturbations usually propagate at the speed of light.

1.4.3 Modified Gravity Theories

Let us now turn our discussion towards theories of MG. Here we will mostly talk about MG theories which have been constructed as consistent IR modifications of GR , allowing us to explain the late time acceleration of the Universe. These IR modifications usually introduce new degrees of freedom into the physical spectrum of the theory. One important thing which one needs to check before studying the phenomenology of these models, refers to the question of their stability. Already long time ago in 1850, Ostrogradsky has formulated his famous theorem [71] according to which, theories which have EoM higher then second order contain instabilities. The modes which cause those instabilities are usually referred to as *Ostrogradsky Ghosts*. We will talk about ghost-like degrees of freedom and issues related to them in Sec. 3.3. Coming back to the discussion of MG theories and keeping in mind the Ostrogradsky theorem we can ask a natural question, namely, what is the most general single scalar-tensor field theory in a four dimensional spacetime (4D) which has at most second order EoM. The theory satisfying the above mentioned condition has been constructed by Horndeski [72] and carries his name. The Lagrangian density for Horndeski theory is [73]

$$\mathscr{L} = K(\phi,X) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R + G_{4,X}(\phi,X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] + G_5(\phi,X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}(\phi,X)\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu)^2 + 2\nabla^\mu\nabla_\alpha\phi\nabla^\alpha\nabla_\beta\phi\nabla^\beta\nabla_\mu\phi\right]. \tag{1.64}$$

Written in this form, this theory contains four arbitrary functions, K, G_3, G_4, G_5 and X is the usual kinetic term for the scalar field ϕ. In general, to fix the form of arbitrary functions in Eq. (1.64) one has to make some additional assumptions about the foundations of the theory [58], for example choose a particular symmetry of the model, see e.g. the Galileon theory [74–76]. During the last years there has been some attempts to relax the assumption of second order EoM in order to find the most general local scalar-tensor theory in 4D which has three healthy propagating degrees of freedom [58]. In these models ghostly degrees of freedom are eliminated by model-relevant constraints [77].

Another well studied class of models, within the family of scalar-tensor theories, are higher curvature $f(R)$ gravity theories. The general action corresponding to this class of models is given by

$$S = \frac{M_{\rm pl}}{2}\int d^4x\sqrt{-g}\,(R + f(R)) + S_{\rm m}(g_{\mu\nu},\psi), \tag{1.65}$$

where $f(R)$ is a general function of the Ricci scalar and $S_{\rm m}$ stands for a standard matter action. Depending on the structure of the $f(R)$ function, these corrections to the Einstein-Hilbert action become relevant either at early times when $R/H^2 \gg 1$ for UV scales or at late times when $R/H^2 \ll 1$ corresponding to IR energy scales. Based on this fact, these models have been extensively studied both as candidates for inflation physics as well as to explain the late time cosmic behavior of our Universe [48, 78, 79]. At first glance it seems that $f(R)$ theories are completely different from the above mentioned scalar-tensor theories, but as it has been shown in Refs. [80, 81], at least at the classical level, these are just scalar-tensor theories written in the Jordan frame. To see this one can perform a conformal transformation with subsequent field redefinitions after which the action (1.65) turns into

$$S = \int d^4x\left(\frac{M_{\rm pl}^2R}{2} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right) + S_{\rm m}\left[A(\phi)^2 g_{\mu\nu},\psi\right], \tag{1.66}$$

where we have defined

$$V(\phi) \equiv \frac{M_{\rm pl}}{2}\frac{\left(\phi f_{,\phi} - f\right)}{\left(1 + f_{,\phi}\right)^2}, \qquad A(\phi) = \exp\left(\frac{\phi}{\sqrt{6}}M_{\rm pl}\right), \tag{1.67}$$

where $f_{,\phi}$ stands for the first derivative of the function f w.r.t. the field ϕ. Later on, as it has been shown in Ref. [82], the local gravity tests [83] put strong constraints on the functional form of $f(R)$ models. The tension with the local gravity tests, due to the appearance of additional degrees of freedom, is also inherited to most of scalar-tensor theories. One usually avoids those problems by introducing a screening mechanism [48, 84]. Different screening mechanisms, such as chameleon [85, 86], Damour-Polyakov [87, 88] and Vainshtein [89] types, allow to screen effects of additional fields within local gravity systems, so we can recover the consistency with all local tests. Finally, let us mention that there are several $f(R)$ gravity models which can pass all local constraints and lead to a viable cosmology [90, 91]. We refer to Ref. [79] for a comprehensive review on these models.

Before starting our discussion on the main subject of this thesis, nonlocal gravity theories, let us close the current section by briefly introducing related MG models which have triggered big scientific interest in recent years. Namely, those are dRGT [92–95] and massive bigravity [96, 97] theories. These theories try to address whether it is possible or not to construct a healthy theory for massive gravitons in the effective field theory limit. Moreover, if we will be able to construct such a theory, what are the subsequent phenomenological consequences. In order to construct a mass term for a graviton we could follow the standard procedure in particle physics, i.e add a square term of the metric field $g_{\mu\nu}$ with a mass-coefficient into the Einstein-Hilbert action. But we immediately realize that this will not work in the case of gravity, as for the metric field in 4D we have that $g^{\mu\nu}g_{\mu\nu} = 4$, which is just a number. An interesting way-out of this situation is to introduce a second metric $f_{\mu\nu}$ which is referred to as a reference metric. With the use of this metric one is able to construct a unique action. For dRGT massive gravity we have [92]

$$S_{\rm dRGT} = -\frac{M_{\rm pl}^2}{2}\int d^4x\sqrt{-g}\,(R - I) + S_{\rm m}\left[g_{\mu\nu},\psi\right], \tag{1.68}$$

while for bigravity we have [96]

$$S_{\rm BG} = -\frac{M_{\rm g}^2}{2}\int d^4x\sqrt{-g}\,(R_{\rm g} - I) - \frac{M_{\rm f}^2}{2}\int d^4x\sqrt{-f}R_{\rm f} + S_{\rm m}\left[g_{\mu\nu},\psi\right], \tag{1.69}$$

with the ghost-free interaction term I between two metrics given by the following polynomial structure

$$I = 2m^2\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right). \tag{1.70}$$

In Eq. (1.70), m is the graviton mass and e_n are elementary polynomials of $\sqrt{g^{-1}f}$, defined as $\sqrt{g^{-1}f}\sqrt{g^{-1}f} \equiv g^{\mu\nu}f_{\mu\nu}$. The parameters $M_{\rm g}$ and $M_{\rm f}$ are Planck masses for g and f metrics, respectively. In both theories the matter fields, through the matter action $S_{\rm m}$, are directly coupled only to the metric $g_{\mu\nu}$ and follow its geodesics. The influence of the metric $f_{\mu\nu}$ on particle trajectories is only indirect, via the interaction term (1.70). The structural difference between dRGT and bigravity theories is triggered by the fact that in the case of dRGT the reference metric

$f_{\mu\nu}$ is not a solution of any dynamical equations and should be fixed by hand. On the other hand, in bigravity this metric will be a self-consistent solution of dynamical equations, because of its own Einstein-Hilbert term in the action (1.69).

In quantum physics, a massive particle with a spin S in total will have $2S+1$ degrees of freedom. In this respect, one expects for a massive spin-2 graviton to have five degrees of freedom. It has been shown that indeed in the physical spectrum of dRGT theory one has five propagating degrees of freedom. Meanwhile for bigravity there are seven propagating degrees of freedom corresponding to a situation with one massive and one massless graviton. This difference between the physical content of the two theories is due to an additional kinetic term R_{f} for the metric $f_{\mu\nu}$ in the bigravity action (1.69). From the phenomenological side of those models [98–104] one arrives to the conclusion that all the consistent models which are free from gradient and ghost instabilities are almost indistinguishable from the standard ΛCDM model [105]. Together with my collaborators I have been also involved in searches of possible way-outs which could help to preserve the nice original properties of the model by providing a mechanisms to cure the bigravity model from existing instabilities. One of these possibilities discussed in our work [106] suggests instead of taking *ad hoc* FLRW-FLRW metric combinations for the metric fields $g_{\mu\nu}$ and $f_{\mu\nu}$, which is the case for most of bigravity literature, one should also investigate other possible metric combinations which could in principle predict a model free of instabilities present in initial models. In the same work, on a pure mathematical ground, we have found that not all the metric combinations can be realized in the nature and we separated those combinations which at least guarantee mathematical consistency of the theory. Whether those combinations lead to viable models free of instabilities is still unknown and is a task for future research. In another work with my collaborators [107] we have shown that allowing Lorentz invariance violation in the gravity sector one can achieve a consistent theory on both quantum and classical levels, even though the model has a ghost mode in its spectrum.

We close this section by referring interested readers to the review papers [108, 109], where one can find information about all the recent developments in the fields of Massive Gravity and Bigravity.

1.5 Cosmic Perturbation Theory

So far we have considered the Universe to be perfectly homogeneous, but if we want to see how the observed astrophysical structures form in our Universe we need to go beyond this approximation. To do this we can expand the Einstein equations (1.6) order by order in perturbations of the metric and energy-momentum tensor. Of course, this procedure will be only justified as long as we can treat perturbations to be small. For small fluctuations we can write a general metric $g_{\mu\nu}$ as a combination of a background part $\bar{g}_{\mu\nu}$ plus small perturbations $\delta g_{\mu\nu} \equiv h_{\mu\nu}$. Under this splitting the FLRW background metric can be written as

$$ds^2 = -(1+h_{00})dt^2 + 2h_{0i}dtdx^i + a^2\left(\delta_{ij} + h_{ij}\right)dx^i dx^j. \tag{1.71}$$

The spatial indices in Eq. (1.71) should be raised or lowered with the use of the Kronecker delta symbol δ_{ij}. The homogeneity and isotropy of the background metric $\bar{g}_{\mu\nu}$ allows us to decompose perturbations into scalars, divergenceless vectors and divergenceless traceless symmetric tensors, which are not coupled to each other by the field equations or conservation equations. Hence, perturbations to the metric can be written as

$$\begin{aligned} h_{00} &= 2\Psi, \\ h_{0i} &= a\left(\partial_i\tilde{\omega} + \omega_i\right), \\ h_{ij} &= a^2\left(2\Phi\delta_{ij} + \partial_i\partial_j\Theta + \partial_i\tilde{\chi}_j + \partial_j\tilde{\chi}_i + \chi_{ij}\right), \end{aligned} \tag{1.72}$$

where the perturbation functions ω_i, $\tilde{\chi}_i$ and $\chi_{ij} = \chi_{ji}$ are satisfying the following conditions

$$\frac{\partial\omega_i}{\partial x^i} = \frac{\partial\tilde{\chi}_i}{\partial x^i} = 0, \qquad \frac{\partial\chi_{ij}}{\partial x^i} = 0, \qquad \chi_{ii} = 0. \tag{1.73}$$

Now let us perturb the energy momentum tensor for the PF (1.4). Again, decomposing it into a background ($\bar{T}_{\mu\nu}$) and perturbation ($\delta T_{\mu\nu}$) part, we find for the components of the energy momentum tensor

$$\begin{aligned} T^0_0 &= -\left(\bar{\rho} + \delta\rho\right), \\ T^0_i &= \left(\bar{\rho} + \bar{P}\right)v_i, \\ T^i_j &= \left(\bar{P} + \delta P\right)\delta^i_j + \Sigma^i_j, \end{aligned} \tag{1.74}$$

where Σ^i_j stands for a traceless part of the energy momentum tensor, defined as $\Sigma^i_j \equiv T^i_j - \frac{\delta^i_j}{3}T^k_k$.

Among above mentioned perturbation quantities not all are physical modes. Some of them are connected with each other through gauge (coordinate) transformations

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x). \tag{1.75}$$

The Einstein equations are invariant under this gauge transformation. Depending on the problem one can choose a particular gauge which will allow to simplify the problem and to remove redundant modes from the theory's spectrum.

In practice for linear large scale structure formation it is sufficient to consider only scalar perturbations. This can be done in a general manner due to the fact that at the linear level scalar, vector and tensor perturbations are decoupled from each other and also in most of the standard cosmological scenarios, the vector perturbations vanish rapidly during the cosmic evolution. Tensor perturbations, on the other hand, are responsible for the gravitational waves, which have been recently directly detected [110]. For the discussion in the next chapters it is convenient to use the so-called *Conformal Newtonian Gauge*, which corresponds to $\Theta = \tilde{\omega} = 0$. The FLRW line element, in the case of scalar perturbations, is given by

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)\delta_{ij}dx^i dx^j, \tag{1.76}$$

where the scalar perturbative functions Φ and Ψ are also known as the gravitational *Bardeen* potential. With the use of Eq. (1.76), the perturbed gravitational field equations (1.6) take the form

$$\nabla^2\Phi - 3\mathcal{H}(\partial_\tau\Phi - \mathcal{H}\Psi) = -4\pi G a^2 \delta\rho, \tag{1.77}$$

$$\nabla^2(\partial_\tau\Phi - \mathcal{H}\Psi) = 4\pi G a^2 \left(\bar{\rho} + \bar{P}\right)\theta, \tag{1.78}$$

$$\partial_\tau^2 + \mathcal{H}(\partial_\tau\Phi - 2\partial_\tau\Psi) - \left(2\partial_\tau^2\mathcal{H} + \mathcal{H}^2\right)\Psi - \frac{1}{3}\nabla^2(\Psi+\Phi) = -4\pi G a^2 \delta P, \tag{1.79}$$

$$\nabla^2(\Psi+\Phi) = 12\pi G a^2 \left(\bar{\rho} + \bar{P}\right)\sigma, \tag{1.80}$$

where θ and $\mathcal{H}$ stand for the divergence of the peculiar velocity $\theta \equiv \partial_i v^i$ and for the conformal Hubble function $\mathcal{H} \equiv aH$, respectively. The anisotropic stress σ is defined through $\left(\bar{P}+\bar{\rho}\right)\nabla^2\sigma = -\left(\partial_i\partial^j - \frac{1}{3}\delta^i_j\nabla^2\right)\Sigma$.

Additionally, to the set of perturbation equations (1.77) we also need to add perturbations of the covariant conservation law of the energy-momentum tensor (1.14), this yields

$$\partial_\tau\delta = -(1+w)(\theta + 3\partial_\tau\Phi) - 3\mathcal{H}\left(c_s^2 - w\right)\delta, \tag{1.81}$$

$$\partial_\tau\theta = -\mathcal{H}(1-3w)\theta - \frac{\partial_\tau w}{1+w}\theta - \frac{c_s^2}{1+w}\nabla^2\delta + \nabla^2\sigma - \nabla^2\Psi, \tag{1.82}$$

where we have introduced the sound speed of a perfect fluid c_s defined as $c_\text{s}^2 \equiv \delta P/\delta\rho$ and the matter density contrast δ defined as $\delta \equiv \delta\rho/\bar{\rho}$. To close the set of perturbation equations (1.77)-(1.81) one needs to specify an equation of state for the PF and fix the value of the anisotropic stress perturbation σ. The anisotropic stress perturbation σ is usually sourced only by relativistic components such as, radiation and neutrinos in the early Universe. As structure formation mostly happens during the matter-domination period, so for the Universe with cold DM the above mentioned physical quantities are $c_\text{s}^2 = w = \sigma = 0$. This assumption simplifies the perturbation equations considerably. In general, it is impossible to find analytic solutions for Eqs. (1.77)-(1.81), expect we do some additional assumptions on the scales where equations are applied for. To see how this actually works, the equations are going to be written in Fourier space. To go to Fourier space we need to replace $\nabla^2 \to -k^2$ and $\partial_i v^i \to i k_i v^i$. Inserting these replacements into Eqs. (1.77)-(1.81) a particular dimensionless combination $k/\mathcal{H}$ arises on both sides of the perturbation equations. Here we separate two limits first when $k \ll \mathcal{H}$ known as super-horizon limit and $k \gg \mathcal{H}$ known as sub-horizon limit. For structure formation we are mostly interested in wave lengths which are well inside Hubble radius, i.e. $\lambda \sim k^{-1} \ll \mathscr{H}^{-1}$. Under this choice Eqs. (1.77) turn to be

$$\nabla^2\Phi = -4\pi G a^2 \bar{\rho}\delta, \tag{1.83}$$

$$\Psi = -\Phi, \tag{1.84}$$

where we have neglected the derivative terms $\partial_\tau\Psi$ and $\partial_\tau\Phi$, compare with those which have a prefactor $k/\mathscr{H}$. The energy-momentum conservation equation (1.81) simplifies to

$$\partial_\tau^2\delta + \mathscr{H}\partial_\tau\delta = k^2\Psi. \tag{1.85}$$

As a final remark to this section let us mention that, in the case of DE and MG theories the standard linear parturbation equations do not hold anymore. A convenient way to parametrize these changes upon the standard case is to introduce two new MG parameters $\eta(a,k)$ and $\mu(a,k)$, which are defined as

$$\eta = \frac{\Psi + \Phi}{\Phi}, \qquad \Psi = [1+\mu]\,\Psi_{\rm GR}, \tag{1.86}$$

where $\Psi_{\rm GR}$ is the *Bardeen* potential for GR. In this formulation, η, which is effectively also an observable parameter [111] shows how much we diverge from GR, for which it exactly vanishes $\eta_{\rm GR} = 0$. On the other hand, the parameter μ enters into right hand side of the growth rate Eq. (1.85), which can affect the overall process of structure growth in the Universe. One can find a comprehensive review on the physics of cosmological perturbations in Ref. [112].

CHAPTER 2

ORIGIN OF NONLOCAL CORRECTIONS

In this chapter, we start the discussion of the main subject of this thesis, which is nonlocal gravity theories. Throughout this chapter, we will explain how nonlocal effective actions arise at the first place. In particular, we will start from the phenomenological derivation of nonlocal theories which are suited to yield valid cosmic evolution of the Universe and to find possible solutions to some open fundamental questions in cosmology, such as the CC problem, the Big Bang singularity problem, etc. Later we will focus on questions of the origin of several nonlocal corrections to the Einstein-Hilbert action, discussing their connection with different fundamental theories and their derivation from first principles of QFT.

2.1 Nonlocality and degravitation

In this section, we introduce nonlocal theories, where nonlocality is associated with an inclusion of the mass term. These theories have the property to screen the contribution of the CC term in the Friedman equations and thus provide a possible solution to the CC problem. Before showing how the screening is realized in the gravitational context, we start from a similar example in QED, namely the Proca theory for massive photons. As we know from classical Electrodynamics, the Maxwell action for massless photons in the presence of an external source takes the form

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \right], \tag{2.1}$$

where A_μ is an electromagnetic four-vector, j_μ is a conserved ($\partial_\mu j^\mu = 0$) external current and $F_{\mu\nu}$ is the electromagnetic tensor defined as: $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The action (2.1) is invariant under $U(1)$ local gauge transformations, which amounts to a transformation with a local phase $\alpha(x)$, i.e.

$A'_\mu = e^{i\alpha(x)} A_\mu$. Now, let us consider the Proca action where one introduces an explicit mass term for photons

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m_\gamma^2 A_\mu A^\mu - j_\mu A^\mu \right], \tag{2.2}$$

In this action m_γ stands for the mass of the photon. The modified Maxwell equations for Proca theory are

$$\partial_\mu F^{\mu\nu} - m_\gamma^2 A^\nu = j^\nu. \tag{2.3}$$

By acting on both sides of Eq. (2.3) with the partial-differential operator ∂_ν and taking into account that $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ and $\partial_\nu j^\nu = 0$ we find the constrain equation

$$m_\gamma^2 \partial_\nu A^\nu = 0 \tag{2.4}$$

For $m_\gamma \neq 0$, Eq. (2.4) as a constraint equation allows us to remove one of the components of the electromagnetic field A_μ so that, at the end, we are left with three propagating degrees of freedom, as it should be for massive spin-1 fields [19]. With the use of Eq. (2.4), the Maxwell-Proca Eq. (2.3) reduces to

$$\left(\Box - m_\gamma^2 \right) A^\mu = j^\mu. \tag{2.5}$$

The Proca action (2.2) is manifestly local but it violates the $U(1)$ gauge symmetry of the original theory. The mass and interaction terms in the Proca action can be made gauge-invariant by using the Stückelberg trick [113, 114]. This method is implemented by introducing a Stückelberg field φ in the following way

$$A_\mu \to A_\mu + \frac{1}{m_\gamma} \partial_\mu \varphi, \tag{2.6}$$

so the action (2.2) becomes

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m_\gamma^2 A_\mu A^\mu - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - m_\gamma A^\mu \partial_\mu \varphi - j_\mu A^\mu \right]. \tag{2.7}$$

In its current form, the action (2.7) is invariant under the gauge transformation

$$A_\mu \to A_\mu - \partial_\mu \alpha, \qquad \varphi \to \varphi + m_\gamma \alpha. \tag{2.8}$$

The EoM corresponding to the action (2.7) are

$$\partial_\mu F^{\mu\nu} = m_\gamma^2 A^\nu + m_\gamma \partial^\nu \varphi + j^\nu, \tag{2.9}$$

$$\Box \varphi + m_\gamma \partial_\mu A^\mu = 0. \tag{2.10}$$

In the case when the field φ is vanishing, we recover back the initial EoM for Proca theory. At this point an interesting question arises: can we rewrite the action (2.7) in such a way that it will contain only the electromagnetic four-vector A_μ ultimately preserving the gauge invariance. This possibility has been explored in Ref. [115]. In that work, the goal has been achieved by formally solving Eq. (2.9) for φ, which gives $\varphi(x) = -m_\gamma \Box^{-1}(\partial_\mu A^\mu)$. Then by plugging this solution into the action (2.7), we finally get [19]

$$S = -\frac{1}{4}\int d^4x \left[F_{\mu\nu}\left(1-\frac{m_\gamma^2}{\Box}\right)F^{\mu\nu} - j_\mu A^\mu\right]. \tag{2.11}$$

For the discussion of this chapter it is enough to introduce the standard definition of an inverse operator: *An inverse operator L^{-1} is defined such that, for any differentiable function $f(x)$, the following relation takes place: $LL^{-1}f \equiv f$, i.e. $LL^{-1} = id$.* This definition allows us to write for $\left(L^{-1}f\right)(x)$ the following integral representation

$$\left(L^{-1}f\right)(x) = \int d^4\sqrt{-g(y)}\,G(x,y)f(y), \tag{2.12}$$

where the Green's function $G(x,y)$ of the integral representation satisfies the equation

$$L_x G(x,y) = \delta^4(x-y). \tag{2.13}$$

In Chapter 3, we will discuss mathematical and physical properties of inverse operators in a more detailed way. Written in its current form, the action (2.11) for massive photons is manifestly gauge invariant at the price of nonlocality which is revealed by the term $F^{\mu\nu}\Box^{-1}F_{\mu\nu}$. The EoM corresponding to this nonlocal action are

$$\left(1-\frac{m_\gamma^2}{\Box}\right)\partial_\mu F^{\mu\nu} = j^\nu \Longrightarrow \partial_\mu F^{\mu\nu} = \left(1-\frac{m_\gamma^2}{\Box}\right)^{-1} j^\nu, \tag{2.14}$$

or, in the language of A_μ,

$$\left(\Box - m_\gamma^2\right)A^\nu = \left(1-\frac{m_\gamma^2}{\Box}\right)\partial^\nu\partial_\mu A^\mu + j^\nu. \tag{2.15}$$

Let us now study the behavior of Eq. (2.14) in two different energy regimes. First, when the energy scales of the interest are much larger then the mass of the photon, i.e. $m_\gamma^2/\Box \sim m_\gamma^2/k^2 \ll 1$ and second, when the scales are much smaller, so that $m_\gamma^2/\Box \sim m_\gamma^2/k^2 \gg 1$. In the first case, the contribution of the operator $m_\gamma^2/\Box$ in the l.h.s. of Eq. (2.14) can be neglected so that we recover the standard Maxwell equations for massless photons. This is not a surprising result because if we are interested in effects happening at very high energies the mass of photons will not give any contribution to the total energy budget so that the situation will be very close to the one for massless photons. However, this is not true anymore in the second case when we are interested in energy scales much below the photon mass. Under this condition Eq. (2.14) formally reduces to

$$\partial_\mu F^{\mu\nu} = -\frac{\Box}{m_\gamma^2} j^\nu, \tag{2.16}$$

where now the contribution of any external current j^μ to the r.h.s. of the Maxwell equations will be suppressed by a factor $k^2/m_\gamma^2 \ll 1$ and hence will not affect the electromagnetic-field A_μ. This result leads us to the conclusion that the operator $\left(1-\frac{m_\gamma^2}{\Box}\right)^{-1}$ acts as a screening operator. The screening properties of this operator are very similar to those of Yukawa screening, which is implemented by the operator e^{-mr}/r. Sometimes, this type of screening in Electrodynamics is also referred to as "de-electrification". Now it is natural to ask whether it is possible or not to extend the same procedure to the case of gravity. First of all, this will allow us to construct a theory for massive gravitons which fully respects the diffeomorphism invariance of GR. Indeed, the latter one is completely broken in Massive Gravity [108] and only partially preserved in bigravity [109]. A second important achievement would be the inclusion of a possible screening mechanism, which can be then used to screen the contribution of vacuum energy thus giving a solution to the CC problem. First steps into this direction have been done in Ref. [18], where the authors, analogously to the modified Maxwell Eq. (2.14), postulated modified Einstein equations for massive gravitons as a first approximation

$$\left(1-\frac{m_\mathrm{g}^2}{\Box}\right)G_{\mu\nu} = 8\pi G T_{\mu\nu} \Longrightarrow G_{\mu\nu} = 8\pi G\left(1-\frac{m_\mathrm{g}^2}{\Box}\right)^{-1} T_{\mu\nu}, \tag{2.17}$$

where m_g is the mass of a graviton. As in the case of massive photons, we can separate out two scales: first when $m_\mathrm{g}^2/\Box \ll 1$, i.e. at scales much higher than the mass of graviton, or equivalently at distances much smaller than the graviton wavelength $\lambda_\mathrm{g} \sim m_\mathrm{g}^{-1}$. In this case, as it is expected we recover the Einstein equations (1.6) for GR. This also means that at small scales (such as solar system scales) we retain all the usual successes of GR. Secondly, in the opposite limit when $m_\mathrm{g}^2/\Box \gg 1$, Eq. (2.17) effectively becomes

$$G_{\mu\nu} \approx -8\pi G\frac{\Box}{m_\mathrm{g}^2}T_{\mu\nu}, \tag{2.18}$$

from which it is obvious that any contribution to the energy momentum tensor of the structure $\Lambda g_{\mu\nu}$ will be filtered out due to the fact that $\Box(\Lambda g_{\mu\nu}) = g_{\mu\nu}\Box\Lambda = 0$. This also means that the vacuum energy Λ does not gravitate. The problems associated with its value (or the so-called technical naturalness) will decouple from the physical scope of large scale cosmology. Despite all these remarkable properties, the model given by Eq. (2.17) has several principal drawbacks. The most important one is that this model does not follow from an action based formulation, which in turn does not give us the opportunity to understand from which fundamental theory these modification could in principle originate from. This difficulty is closely related to the fact that Eq. (2.17) is not consistent with the Bianchi identities. Indeed, applying the covariant derivative ∇_μ on both sides of Eq. (2.17) and remembering that $\nabla^\mu G_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0$, Eq. (2.17) reduces to a constrain equation

$$\nabla^\mu\left(\Box^{-1}G_{\mu\nu}\right) = 0. \tag{2.19}$$

This condition cannot be satisfied for a general background $g_{\mu\nu}$, since the covariant derivative ∇^μ and the d'Alembert operator $\Box$ do not generically commute [19], i.e. for a differentiable function f we have that $\nabla^\mu \Box f \neq \Box \nabla^\mu f$. This condition, therefore, translates also into $\nabla^\mu \Box^{-1} f \neq \Box^{-1} \nabla^\mu f$. To overcome this problem, in Ref. [116] it has been suggested to replace the nonlocal term $(\Box^{-1} G_{\mu\nu})$ in Eq. (2.17) with the term $\left(\Box^{-1} G_{\mu\nu}\right)^{\mathrm{T}}$ which transforms Eq. (2.17) into

$$G_{\mu\nu} - m_{\mathrm{g}}^2 \left(\Box^{-1} G_{\mu\nu}\right)^{\mathrm{T}} = 8\pi G T_{\mu\nu}, \tag{2.20}$$

where T denotes the extraction of the transverse part. Let us see how this change would help us to make our model consistent with the Bianchi identities. As we know, any symmetric tensor $S_{\mu\nu}$ can be decomposed as

$$S_{\mu\nu} = S_{\mu\nu}^{\mathrm{T}} + 1/2 \left(\nabla_\mu S_\nu + \nabla_\nu S_\mu\right), \tag{2.21}$$

where the transverse part $S_{\mu\nu}^{\mathrm{T}}$ satisfies the condition $\nabla^\mu S_{\mu\nu}^{\mathrm{T}} = 0$. In our case $S_{\mu\nu} = \Box^{-1} G_{\mu\nu}$, so then the covariant conservation of Eq. (2.20) is automatically fulfilled. The model (2.20) still permits a degravitating solution. This can be checked by considering a modification of Eq. (2.20) of the form [19]

$$G_{\mu\nu} - m^2 \left[\left(\Box - \mu^2\right)^{-1} G_{\mu\nu}\right]^{\mathrm{T}} = 8\pi G T_{\mu\nu}, \tag{2.22}$$

where μ is a regularization parameter to be set to zero at the end of the discussion. Now, by plugging the energy momentum tensor $T_{\mu\nu} = \Lambda g_{\mu\nu}$ for the CC into Eq. (2.22) we find that it admits a de Sitter solution $G_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu}$, where $\tilde{\Lambda} = 8\pi G \left[\mu^2/(m^2 + \mu^2)\right]$. Taking the limit where $\mu \to 0$, we observe that $\tilde{\Lambda} \to 0$, which signals degravitation of vacuum energy [19]. Unfortunately, this model does not lead to a valid cosmology and turns out to be unstable already at the background level [117, 118]. In Ref. [117], it has been also observed that these instabilities are due to tensorial nonlocal structures, such us $(\Box^{-1} G_{\mu\nu})$ or $(\Box^{-1} R_{\mu\nu})$. Later in our work [119], which will be discussed in Chapter 6, we prove that these instabilities have a more general ground and are present in a big class of tensorial nonlocal gravity models.

The authors in Ref. [117] have noticed that, if we replace in Eq. (2.20) the tensorial nonlocal term $\Box^{-1} G_{\mu\nu}$ by a scalar nonlocal term $\Box^{-1} R$, the instabilities disappear. Based on this observation the following model has been proposed

$$G_{\mu\nu} - \frac{m^2}{3} (g_{\mu\nu} \Box^{-1} R)^{\mathrm{T}} = 8\pi G T_{\mu\nu}. \tag{2.23}$$

The model (2.23) is known as the "RT" model [19], where R stands for the Ricci scalar and T for the transverse part. At the phenomenological level [117], the RT model has been shown to be able to provide a valid dynamical DE candidate. The amount of DE can be matched to the today's observed value ($\Omega_{\mathrm{DE}} \simeq 0.68$) by appropriately choosing the value of the mass parameter m in Eq. (2.23). It is found to be of the order $m \simeq 0.67 H_0$. The equation of state parameter w_{DE} for this model, in the parametrization $w_{\mathrm{DE}}(a) = w_0 + (1-a) w_{\mathrm{a}}$, has been calculated in Ref. [117]

where the authors have found that $w_0 \simeq -1.04$ and $w_{\rm a} \simeq -0.02$. These results are consistent with the latest *Planck* data [4, 6]. The cosmological perturbations of the RT model have been studied in Ref. [120] and compared to CMB, BAO, SNIa and LSS data. As a result it has been noticed that in the case when one chooses a higher prior on $h_0 \gtrsim 0.7$, which is the case for the local measurements [121], the data strongly favor the RT model over ΛCDM [122]. On the other hand, in the case of a lower prior on h_0 ($0.67 \lesssim h_0 \lesssim 0.70$), as suggested by the *Planck* data [4] the RT model and ΛCDM are statistically comparable [122]. Despite all of these phenomenological achievements, there are several open questions in the RT model still to be addressed.

The most important question is related to the possible origin of the RT nonlocal gravity model. Being derived purely on the phenomenological ground, this model lacks some connections with fundamental theories. The main obstacle to find these links is the fact that the RT model is defined only at the level of EoM and, so far, no action based formulation is known. In the early stages of the development of the RT model, some of the attempts to construct an action for this model led to the discovery of another well known nonlocal gravity model, namely the RR nonlocal gravity model. Below, we will have a brief look inside this procedure, first executed in Refs. [116, 123]. The main idea behind this procedure is to find the graviton propagator for the RT model at the level of Eq. (2.23), and to construct then a corresponding action which leads to this type of graviton propagation. To do this, we linearize Eq. (2.23) over the flat background $\eta_{\mu\nu}$ by decomposing the metric $g_{\mu\nu}$ into $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small perturbation. If we plug this metric decomposition into Eq. (2.23) and keep only terms which are at most first order in $h_{\mu\nu}$, Eq. (2.23) turns to [19]

$$\mathcal{E}^{\mu\nu,\rho\sigma} h_{\rho\sigma} - \frac{2}{3} m^2 P^{\mu\nu} P^{\rho\sigma} h_{\rho\sigma} = -16\pi G T^{\mu\nu}, \tag{2.24}$$

where we have defined $P^{\mu\nu}$ as

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\overline{\Box}}, \tag{2.25}$$

and $\mathcal{E}^{\mu\nu,\rho\sigma}$ stands for the Lichnerowicz operator [124]

$$\begin{aligned}\mathcal{E}^{\mu\nu,\rho\sigma} \equiv{}& \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - 2\eta^{\mu\nu}\eta^{\rho\sigma}\right)\overline{\Box} + \left(\eta^{\rho\sigma}\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial^\rho\partial^\sigma\right) \\ & -\frac{1}{2}\left(\eta^{\mu\rho}\partial^\sigma\partial^\nu + \eta^{\nu\rho}\partial^\sigma\partial^\mu + \eta^{\mu\sigma}\partial^\rho\partial^\nu + \eta^{\nu\sigma}\partial^\rho\partial^\mu\right).\end{aligned} \tag{2.26}$$

Here the operator $\overline{\Box}$ stands for the flat-space d'Alembertian. It is important to mention that during the process of linearization for the transverse term $(g_{\mu\nu}\Box^{-1}R)^{\rm T}$ in Eq. (2.23) we used the fact that in flat space the covariant derivative ∇_μ goes to the standard partial derivative ∂_μ and acting on both sides of Eq. (2.21) with partial derivatives ∂^μ and $\partial^\mu\partial^\nu$ we find[1] [19]

$$S^{\rm T}_{\mu\nu} = S_{\mu\nu} - \overline{\Box}^{-1}\left(\partial_\mu\partial^\rho S_{\rho\nu} + \partial_\nu\partial^\rho S_{\rho\mu}\right) + \overline{\Box}^{-2}\partial_\mu\partial_\nu\partial^\rho\partial^\sigma S_{\rho\sigma}. \tag{2.27}$$

[1] This expression for $S^{\rm T}_{\mu\nu}$ is true only for the Minkowski background $\eta_{\mu\nu}$. In a generic curved background, due to the fact that $[\nabla_\mu, \nabla_\nu] \neq 0$, it does not hold.

Here, we can already write the corresponding flat space action for Eq. (2.24), which is

$$\Gamma^{(2)}_{\rm RT} = \frac{1}{16\pi G}\int d^4x\left[\frac{1}{4}h_{\mu\nu}\mathscr{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - \frac{1}{6}m^2 h_{\mu\nu}P^{\mu\nu}P^{\rho\sigma}h_{\rho\sigma}\right] + \int d^4x h_{\mu\nu}T^{\mu\nu}. \tag{2.28}$$

Now we can try to covariantize the action (2.28). This will allow us to construct a fully covariant, nonlinear nonlocal theory whose equations at the linear level coincide with those for the RT model. To covariantize, we first observe that the Riemann tensor linearized around flat space has the structure $R_{\mu\nu} = R^{(1)}_{\mu\nu} + \mathscr{O}(h^2)$, where $R^{(1)}_{\mu\nu}$ in [125] reads

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\partial_\mu\left(\partial^\lambda h_{\nu\lambda} - \frac{1}{2}\partial_\nu h\right) + \frac{1}{2}\partial_\nu\left(\partial^\lambda h_{\mu\lambda} - \frac{1}{2}\partial_\mu h\right). \tag{2.29}$$

By contracting Eq. (2.29) with the flat metric $\eta^{\mu\nu}$ we find for the linearized Ricci scalar $R^{(1)}$

$$R^{(1)} = \partial_\mu\partial_\nu\left(h^{\mu\nu} - \eta^{\mu\nu}h\right) = \frac{1}{4}h_{\mu\nu}\mathscr{E}^{\mu\nu,\sigma\rho}h_{\sigma\rho}. \tag{2.30}$$

Let us now plug Eq. (2.30) into Eq. (2.28) so we finally obtain for $\Gamma^{(2)}_{\rm RT}$

$$\Gamma^{(2)}_{\rm RT} = \frac{1}{16\pi G}\int d^4x\left[R^{(1)} - \frac{1}{6}m^2 R^{(1)}\frac{1}{\Box^2}R^{(1)}\right] + \int d^4x h_{\mu\nu}T^{\mu\nu}. \tag{2.31}$$

Hence, the natural covariantization of the action (2.31) is

$$S = \frac{1}{16\pi G}\int\sqrt{-g}\,d^4x\left[R - \frac{1}{6}m^2 R\frac{1}{\Box^2}R\right] + \int\sqrt{-g}\,d^4x\mathscr{L}_m. \tag{2.32}$$

At this point, it is very important to mention one more time that the EoMs derived from the covariant action (2.32) do coincide with Eq. (2.23) for the RT model only at the linearized level around flat space. Nevertheless, at the full nonlinear level or linearized around other backgrounds the two models are completely different [19]. The theory defined with the action (2.32) was first introduced in Ref. [123] and is usually referred to as either RR or the Maggiore-Mancarella (MM) nonlocal gravity model. This model has successfully passed all the phenomenological tests and is a consistent model describing the late time acceleration of the Universe. We will come back to the phenomenology achievements of the RR model later in Chapter 4. For now, since we have an action based formulation for the RR nonlocal gravity model, we will investigate how in general these type of nonlocal corrections emerge into the Einstein-Hilbert action.

2.2 Nonlocality and quantum effective actions

In the general framework of QFT, nonlocalities appear when we take into account the running of physical couplings due to different loop corrections. For illustration purposes, we will start our discussion with a simple Quantum Electrodynamics (QED) example. In particular, we will consider the running of the electric charge induced by one loop electron corrections [126, 127]. The bare action for this case is given by

$$e^{i\Gamma_{\rm QED}(A_\mu)} = \int\mathscr{D}\Psi\mathscr{D}\overline{\Psi}\exp\left\{i\int d^4x\left[-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\not{D} - m_{\rm e} + i\epsilon)\Psi\right]\right\}, \tag{2.33}$$

where e is the electric charge and m_{e} is an electron mass. Taking into account the one-loop electron correction in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme, one gets for the quantum effective action up to quadratic order in A_μ the following expression [126, 128]

$$\Gamma_{\mathrm{QED}}\left(A_\mu\right) = -\frac{1}{4}\int d^4x \left[F_{\mu\nu}\frac{1}{e^2(\Box)}F^{\mu\nu} + \mathcal{O}(F^4)\right], \tag{2.34}$$

where the running of the electric charge in the coordinate space is given by

$$\frac{1}{e^2(\Box)} = \frac{1}{e^2(\mu)} - \frac{1}{8\pi^2}\int_0^1 dt(1-t^2)\log\left[\frac{m_e^2 - \frac{1}{4}(1-t^2)\Box}{\mu^2}\right]. \tag{2.35}$$

In this expression μ stands for the renormalization scale and $e(\mu)$ is the value of the renormalized electric charge at that scale. The logarithm in Eq. (2.35) is a nonlocal operator defined as [19, 28]

$$\log\left(\frac{-\Box}{\mu^2}\right) = \int_0^\infty dm^2\left[\frac{1}{m^2+\mu^2} - \frac{1}{m^2-\Box}\right]. \tag{2.36}$$

As we can see from this example, nonlocality is nothing else than a running of the effective coupling constant e in coordinate space. Let us now return back to the question of nonlocal corrections in gravity. In analogy with the above example, we can try to calculate the running of coupling constants induced in the gravitational sector. For this purpose, we will discuss an important case realized in higher curvature gravity theories. Namely, we will consider a model given by the following bare vacuum action in Euclidean signature [23, 29, 129]

$$S_{\mathrm{vac}} = S_{\mathrm{EH}} + S_{\mathrm{HD}}, \tag{2.37}$$

where the first term is the Einstein-Hilbert action (1.52) with the cosmological constant, whereas the second term is the higher order gravity term and contains higher derivative terms quadratic in the curvature, i.e. [19, 24, 129]

$$S_{\mathrm{HD}} = \int d^4x\sqrt{g}\left(c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right), \tag{2.38}$$

or in the Weyl basis $\{C^2, R^2, E\}$

$$S_{\mathrm{HD}} = \int d^4x\sqrt{g}\left(a_1C^2 + a_2R^2 + a_3E\right), \tag{2.39}$$

where

$$C^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + 1/3R^2, \tag{2.40}$$

is the square of the Weyl tensor and

$$E = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2, \tag{2.41}$$

is the integrand of the Gauss-Bonnet topological invariant. In the above expressions the coefficients a_i and c_i are dimensionless parameters. In a realistic physical situation, the matter sector is always present. A sensible EFT therefore must also include relevant matter field contributions. For the sake of simplicity we will only consider the case of a single massive scalar field non-minimally coupled to gravity, which is given by the following action

$$S_{\rm s} = \frac{1}{2}\int d^4x g^{1/2}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m_{\rm s}^2\phi^2 + \xi R\phi^2\right), \tag{2.42}$$

where $m_{\rm s}$ is the mass of the scalar field and ξ is the nonminimal coupling to gravity. The idea we want to present here is: due to quantum corrections the couplings in an effective action are not constants but rather scale dependent quantities. These couplings are exposed to a running governed by renormalization group equations. Depending on the sign of the corresponding β-functions they will either increase or decrease with energy. In the case of the current model (2.37), using heat-kernel techniques [130] combined with expansion in powers of the curvature, one gets the following one-loop effective action in the Weyl basis [21–24, 26, 129]

$$\begin{aligned} S^{\text{1-loop}} = -\frac{1}{2(4\pi)^2}\int d^4x g^{1/2} \frac{m_{\rm s}^2}{2}\left(\frac{1}{\epsilon}+\frac{3}{2}\right) + \bar{\xi} m_{\rm s}^2 R\left(\frac{1}{\epsilon}+1\right) \\ + \frac{1}{4}C_{\mu\nu\sigma\rho}\left[\frac{1}{60\epsilon}+k_{\rm W}\left(\frac{-\Box}{m_{\rm s}^2}\right)\right]C^{\mu\nu\rho\sigma} + R\left[\frac{1}{2\epsilon}\bar{\xi}^2 + k_R\left(\frac{-\Box}{m_{\rm s}^2}\right)\right]R, \end{aligned} \tag{2.43}$$

where $\epsilon^{-1} = 2(4-D)^{-1} + \log(4\pi\mu^2 m_{\rm s}^{-2}) - \gamma$ within dimensional regularization. The $k_{\rm W}(\Box)$ and $k_{\rm R}(\Box)$ are running coefficients in the finite part of the one-loop effective action and are given by [19, 24, 131]

$$\begin{aligned} k_{\rm W}\left(\frac{-\Box}{m_{\rm s}^2}\right) &= \frac{8A}{15a^4} + \frac{2}{45a^2} + \frac{1}{150}, \\ k_{\rm R}\left(\frac{-\Box}{m_{\rm s}^2}\right) &= \bar{\xi}^2 A + \left(\frac{2A}{3a^2} - \frac{A}{6} + \frac{1}{18}\right)\bar{\xi} + A\left(\frac{1}{9a^4} - \frac{1}{18a^2} + \frac{1}{144}\right) \\ &\quad + \frac{1}{108a^2} - \frac{7}{2160}, \end{aligned} \tag{2.44}$$

where

$$A = 1 - \frac{1}{a}\log\left(\frac{2+a}{2-a}\right), \qquad a^2 = \frac{4\Box}{\Box - 4m_{\rm s}^2}, \tag{2.45}$$

with $\bar{\xi} = \xi - 1/6$. To get a further inside into the running of the form factors (2.44) we can discuss two asymptotic regimes, namely the UV limit $\Box/m_{\rm s}^2 \sim k^2/m_{\rm s}^2 \gg 1$ and the IR limit $\Box/m_{\rm s}^2 \sim k^2/m_{\rm s}^2 \ll 1$. For simplicity, we will assume that the matter field is conformally coupled to gravity, i.e. $\xi = 1/6$. For $\Box/m_{\rm s}^2 \sim k^2/m_{\rm s}^2 \gg 1$ we find that [19, 24, 131, 132]

$$k_{\rm W}\left(\frac{-\Box}{m_{\rm s}^2}\right) \simeq -\frac{1}{60}\log\left(\frac{-\Box}{m_{\rm s}^2}\right) + \frac{5}{18}\left(\frac{m_{\rm s}^2}{-\Box}\right) - \frac{1}{6}\left(\frac{m_{\rm s}^2}{-\Box}\right)\log\left(\frac{-\Box}{m_{\rm s}^2}\right) + \frac{1}{4}\frac{m_{\rm s}^4}{\Box^2} + \mathcal{O}\left(\frac{m_{\rm s}^4}{\Box^2}\right), \tag{2.46}$$

$$k_{\rm R}\left(\frac{-\Box}{m_{\rm s}^2}\right) \simeq -\frac{1}{36}\log\left(\frac{-\Box}{m_{\rm s}^2}\right) - \frac{5}{27}\left(\frac{m_{\rm s}^2}{-\Box}\right) + \frac{1}{18}\left(\frac{m_{\rm s}^2}{-\Box}\right)\log\left(\frac{-\Box}{m_{\rm s}^2}\right) + \frac{1}{36}\frac{m_{\rm s}^4}{\Box^2} + \mathcal{O}\left(\frac{m_{\rm s}^4}{\Box^2}\right). \tag{2.47}$$

Then, by subtracting the divergent part in Eq. (2.43) and using the expansion from Eq. (2.46) for the limit $\Box/m_{\rm s}^2 \gg 1$, we observe that in the effective action (2.43) we get several nonlocal terms and in particular the one corresponding to the RR model, i.e. $R\left(m_{\rm s}^4\Box^{-2}\right)R$. In the opposite limit when $\Box/m_{\rm s}^2 \ll 1$ from Eq (2.44) we find that these form factors become local [19],

$$k_{\rm W}(-\Box/m_{\rm s}^2), k_{\rm R}(-\Box/m_{\rm s}^2) = \mathscr{O}(\Box/m_{\rm s}^2). \tag{2.48}$$

Indeed, this result is expected. If the particle is heavy compared to the relevant energy scale the radiative corrections in a mass-dependent subtraction scheme are local and suppressed by the mass of the corresponding particle. As we can observe, nonlocal radiative corrections due to massive particle loops can appear in the effective action for the late time cosmology only if the condition $\Box/m_{\rm s}^2 \gg 1$ is satisfied. This condition for IR scales of the late time universe can be satisfied if the particles running in the loops are either massless or light as compared to the energy scales of interest. Before closing this section let us also briefly mention that the radiative corrections, as those given in Eq. (2.46) can also have relevant contributions at UV scales, where the condition $\Box/m_{\rm s}^2 \gg 1$ must be satisfied. Opposite to the IR case, in the UV regime this condition instead will be satisfied for massive particles [19, 24]. UV relevant nonlocal corrections can contribute to the inflationary potential [36, 133, 134] and therefore one has to be careful that these corrections do not spoil its flatness, which is needed for slow-roll inflation. Nonlocal corrections at UV scales have also been used in attempts to construct UV complete theories of gravity [135–139].

2.3 Nonlocalities from anomalous symmetry breaking

As explained in the previous section, the IR relevant nonlocal corrections in the quantum effective action are induced by massless or very light particles. In this context, another important example with IR-relevant nonlocal structure is the conformal-anomaly induced effective action. Let us start our discussion with an example in two dimensional (2D) gravity. For the matter content of the model we will consider $N_{\rm s}$ conformally-coupled massless scalar and $N_{\rm f}$ fermionic fields. In the classical limit, because of the conformal symmetry the trace of the energy-momentum tensor for conformal matter fields vanishes, i.e. $T^a_a = 0$. However, the situation changes at quantum level. Indeed, the counter-terms which we add to the bare action during the standard regularization procedure do not preserve conformal symmetry such as the trace of the quantum corrected energy momentum tensor does not vanish anymore. This phenomenon is known as *conformal-anomaly* or *trace-anomaly* [140, 141]. In two-dimensional spacetime the conformal-anomaly induced non-vanishing vacuum expectation value (VEV) of the trace T^a_a is given by [19, 142, 143]

$$\langle 0|T^a_a|0\rangle = \frac{N}{24\pi}R, \tag{2.49}$$

where $N = N_{\rm s} + N_{\rm f}$ is the total number massless particles. As shown in Ref. [130] even though the trace-anomaly (2.49) has been obtained at the one-loop level, it is in fact exact, so that the higher loops do not give any contribution to this result. Next, we try to find an effective action which will reproduce the trace-anomaly (2.49). To do this we can make the following conformal transformation $g_{ab} = e^{2\sigma}\overline{g}_{ab}$, where $\overline{g}_{ab}$ is the reference metric. Under this transformation the Ricci scalar R transforms as

$$R = e^{-2\sigma}\left(\overline{R} - 2\overline{\Box}\sigma\right). \tag{2.50}$$

The conformal-anomaly induced effective action $\Gamma_{\rm an}$ is defined as [19, 24]

$$\frac{\delta\Gamma_{\rm an}}{\delta\sigma} = 2g_{ab}\frac{\delta\Gamma_{\rm an}}{\delta g_{ab}} = \sqrt{-g}\,\langle 0|T^a_a|0\rangle \tag{2.51}$$

In a two-dimensional spacetime we can always choose the reference metric to be locally flat, i.e. $\overline{g}_{ab} = \eta_{ab}$. In this case Eq. (2.50) becomes

$$R = -2e^{-2\sigma}\overline{\Box}\sigma, \tag{2.52}$$

where $\overline{\Box}$ is the flat-space d'Alembertian. Inserting (2.52) into Eq. (2.51) and integrating the result over σ we finally obtain [19, 24]

$$\Gamma_{\rm an}[\sigma] = -\frac{N}{24\pi}\int d^2x\sigma\overline{\Box}\sigma, \tag{2.53}$$

where we have used the fact that for vanishing curvature invariants ($\sigma = 0$) the effective action $\Gamma_{\rm an}$ also vanishes $\Gamma_{\rm an}[0] = 0$. We can covariantize the action (2.53) for a general metric g_{ab} at the price of locality. This provides [19, 142, 144]

$$\Gamma_{\rm an}[g_{ab}] = -\frac{N}{24\pi}\int d^2x e^{2\sigma}\sigma\Box\sigma = -\frac{N}{96\pi}\int d^2x\sqrt{-g}R\frac{1}{\Box}R. \tag{2.54}$$

In this expression we have used the relations $\Box = e^{-2\sigma}\overline{\Box}$ and $R = -2\Box\sigma$. The nonlocal action (2.54) is known as the Polyakov quantum effective action, and exact at any order of a perturbative theory. The next step will be, to construct the anomaly-induced effective action in four-dimensional spacetime by using the same strategy as above. In 4D the trace-anomaly is [24, 130, 145, 146]

$$\langle 0|T^\mu_\mu|0\rangle = b_1C^2 + b_2\left(E - \frac{2}{3}\Box R\right) + b_3\Box R, \tag{2.55}$$

where C^2 and E are the Weyl squared (2.40) and Gauss-Bonnet (2.41) terms, respectively. The parameters b_1, b_2 and b_3 are theory-level fixed constants depending on the number of conformally-coupled massless particle content [19, 24]. Just as in the case of 2D we write that

$$\frac{\delta\Gamma_{\rm an}}{\delta\sigma} = \sqrt{-g}\left[b_1C^2 + b_2\left(E - \frac{2}{3}\Box R\right) + b_3\Box R\right]. \tag{2.56}$$

Let us now apply the conformal transformation $g_{\mu\nu} = e^{2\sigma}\overline{g}_{\mu\nu}$. The crucial difference between 2D and 4D cases relies on the fact that in 4D, due to diffeomorphism invariance, one cannot choose the reference metric be flat anymore, i.e. to be of the form $\overline{g}_{\mu\nu} = \eta_{\mu\nu}$ [147]. Under the conformal transformation $g_{\mu\nu} = \exp(2\sigma)\overline{g}_{\mu\nu}$ the geometrical quantities of the model transform as [19, 24]

$$R = e^{-2\sigma}\left(\overline{R} - 6\overline{\Box}\sigma - 6\overline{\nabla}_\mu\sigma\overline{\nabla}^\mu\sigma\right), \tag{2.57}$$

and

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2, \tag{2.58}$$

$$\sqrt{-g}\left(E - \frac{2}{3}\Box R\right) = \sqrt{-\bar{g}}\left(\overline{E} - \frac{2}{3}\overline{\Box}\overline{R} + 4\bar{\Delta}_4\sigma\right), \tag{2.59}$$

where the fourth-order differential operator Δ_4 is called Paneitz operator [148] and is defined as

$$\Delta_4 \equiv \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}g^{\mu\nu}\nabla_\mu R\nabla_\nu. \tag{2.60}$$

So, now we can already integrate Eq. (2.56), which gives us [19, 144]

$$\Gamma_{\rm an}(g_{\mu\nu}) = \Gamma_{\rm an}(\overline{g}_{\mu\nu}) - \frac{b_3}{12}\int d^4x\sqrt{-g}R^2 + \int d^4x\sqrt{-\bar{g}}\left[b_1\sigma\overline{C}^2 + b_2\sigma\left(\overline{E} - \frac{2}{3}\overline{\Box}\overline{R}\right) + 2b_2\sigma\bar{\Delta}_4\sigma\right]. \tag{2.61}$$

The first term in Eq. (2.61) is the conformally-invariant part of the effective action ($\Gamma_{\rm an}(\overline{g}_{\mu\nu}) \equiv \Gamma_c(\overline{g}_{\mu\nu})$), which satisfies

$$\Gamma_c\left(e^{2\sigma}\overline{g}_{\mu\nu}\right) = \Gamma_c\left(\overline{g}_{\mu\nu}\right). \tag{2.62}$$

As we have already done for the Polyakov action (2.53), we can try to covariantize the action (2.61) to get the effective action for an arbitrary metric $g_{\mu\nu}$. One of the possible covariantizations of the action (2.61) is given by the Riegert action, first derived in Ref. [149]

$$\Gamma_{\rm an}(g_{\mu\nu}) = \Gamma_{\rm c}(g_{\mu\nu}) - \frac{b_3}{12}\int d^4x\sqrt{-g}R^2 + \frac{1}{8}\int d^4x\sqrt{-g}\left(E - \frac{2}{3}\Box R\right)\Delta_4^{-1}\left[b_2\left(E - \frac{2}{3}\Box R\right) + 2b_1C^2\right]. \tag{2.63}$$

The Riegert action (2.63) is generally-covariant and has a nonlocal structure. Of course, this action is only one part of the full effective action and is induced by the trace-anomaly. The full effective action is obtained by adding this part to the Einstein-Hilbert action. On the phenomenological side, one of the intriguing properties of anomaly induced corrections is related to their possible relevance at the IR scales. Indeed, as it has been stressed in Refs. [150, 151], conformal-anomaly induced effects imply that quantum fluctuations grow logarithmically and remain relevant at large scales and hence can modify the IR behavior of the classical theory. Hereby, when talking about quantum fluctuations, we mean those related to the conformal factor σ. A final observation which we would like to make at the end of this section refers to the question of possible connections between the RR nonlocal gravity model and the anomaly-induced nonlocal effective action. Now, by using Eq. (2.57) for the transformed Ricci scalar and choosing a locally flat reference metric $\overline{g}_{\mu\nu} = \eta_{\mu\nu}$ for the sake of simplicity, we get

$$R = -6e^{-2\sigma}\left(\Box\sigma + \partial_\mu\sigma\partial^\mu\sigma\right), \tag{2.64}$$

Accordingly, up to linear order in σ, we have

$$R = -6\sigma + \mathcal{O}\left(\sigma^2\right) \tag{2.65}$$

Finally, plugging Eq. (2.65) into the nonlocal part of the action (2.32) and upon integration by parts one obtains [19]

$$R\frac{1}{\Box^2}R = 36\sigma^2 + \mathcal{O}\left(\sigma^3\right). \tag{2.66}$$

This expression tells us that the nonlocal term in the RR model corresponds to a mass term for the conformal mode. This observation is particularly interesting in the context of the trace-anomaly. Indeed, in pure GR the conformal mode is a massless and non-propagating degree of freedom. The situation is completely different at the quantum level where due to the trace-anomaly, the conformal mode has a kinetic term (see e.g. Eqs. (2.54) and (2.61)) and is fully dynamical. All this allows us in the context of R^2 gravity to discuss the possibility of a dynamical mass generation of a massless conformal mode σ [27].

2.4 Nonlocalities from nonperturbative Quantum Gravity

In this section we will discuss possible nonlocal modifications of the Einstein-Hilbert action motivated by the nonperturbative Quantum Gravity studies. The nonlocal gravity models we study arise in the context of asymptotically safe Quantum Gravity theories. This candidate of Quantum Gravity theory is usually referred to as *Asymptotic Safety* (AS). Here, we just give a brief introduction to AS, which will be sufficient for later cosmological discussions. The interested reader can find comprehensive studies on fundamental aspects of AS in Refs. [8, 9].

As we know the Einstein-Hilbert theory of gravity is not *perturbatively* renormalizable [152–154]. On the other side, the phenomenological success of GR motivates us to look at the Einstein-Hilbert theory as an effective theory at low energy scales and to try to find a possible UV completion at high energies [155]. The UV completion in the AS scenario might be achieved by an existence of a UV non-Gaussian fixed point (NGFP) for the gravitational renormalization group (RG) flow [156]. This fixed point controls the running of physical quantities at high energies and preserves them from unphysical divergences. Another important concept in the AS scenario is the UV critical surface corresponding to the NGFP. It is a surface which consists of all points in the theory space that are attracted into the NGFP by the inverse RG flow, which means from IR to UV scales [8]. The dimensionality of the critical surface corresponds to the number of attractive directions in the space of couplings. For the AS paradigm to be valid one needs to prove the existence of the UV NGFP. Additionally, it has to be shown that there is a regime where classical gravity is reproduced. This question has been investigated widely, putting the AS scenario on firm grounds [39, 132, 157, 158]. The main mathematical tool on which AS computations rely is the functional renormalization group (FRG) equations [159] for the gravitational effective action

Γ_k at some energy scale k, also known as Wetterich equations, given by

$$\partial_k \Gamma_k [g, \bar{g}] = \frac{1}{2} \mathrm{Tr} \left[\frac{\partial_k \mathscr{R}_k}{\Gamma_k^{(2)} + \mathscr{R}_k} \right], \tag{2.67}$$

The term $\Gamma_k^{(2)}$ is the Hessian of the effective action and is defined as the second functional derivative of it w.r.t. the fluctuation field on a fixed background. The quantity $\mathscr{R}_k$ is an IR regulator, which provides a k-dependent mass term for the fluctuations with momenta $p^2 \ll k^2$, while vanishing for $p^2 \gg k^2$. The appearance of the regulator $\mathscr{R}_k$ both in the nominator and denominator of Eq. (2.67) renders the trace finite both at the IR and UV scales and ensures that the flow is driven by fluctuations with momenta $p^2 \approx k^2$ [159–161]. By construction, Eq. (2.67) uses the background formalism where one splits the general metric $g_{\mu\nu}$ into the combination of a background metric $\bar{g}_{\mu\nu}$ and a perturbation $h_{\mu\nu}$. From this one could conclude that the RG flow may actually strongly depend on the background metric choice. As is shown in Ref. [162], this is not the case and physical results are background independent. The simplest effective action to start with in the gravitational field space is the Einstein-Hilbert effective action

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} (-R + 2\Lambda_k) + \Gamma_{\mathrm{gf}} + \Gamma_{\mathrm{gh}}. \tag{2.68}$$

This effective action is written in Euclidean signature, where the physical constants are now scale dependent quantities. The Γ_{gf} and Γ_{gh} are the usual gauge-fixing and ghost terms, respectively. The scale dependence of the coupling-constants G_k and Λ_k is governed by the following flow equations

$$k\partial_k g_k = \beta_g(g_k, \lambda_k), \qquad k\partial_k \lambda_k = \beta_\lambda(g_k, \lambda_k), \tag{2.69}$$

where g_k and λ_k are the dimensionless counterparts of G_k and Λ_k, respectively, which are defined as

$$g_k \equiv k^2 G_k, \qquad \lambda_k \equiv k^{-2} \Lambda_k. \tag{2.70}$$

The beta functions fully characterize the scale dependence of the coupling constants. They also contain information about fixed points u^* of the RG flow, which are the points where the beta function vanishes, i.e. $\beta(u^*) = 0$. To find the RG flow of the coupling constants in the vicinity of a fixed point, we can linearize beta functions in the right hand side of Eqs. (2.69) around that point, so that

$$k\partial_k u_\alpha(k) = \sum_\gamma B_{\alpha\gamma} \left(u_\gamma(k) - u_\gamma^* \right), \tag{2.71}$$

where $u_\alpha(k)$ stands for the set of running coupling constants and u_α^* are the corresponding fixed points. The $B_{\alpha\gamma}$ is the Jacobi matrix for beta functions and is defined as $B_{\alpha\gamma} \equiv \partial_\gamma \beta_\alpha(u^*)$. The general solution of Eq. (2.71) can be written as

$$u_\alpha(k) = u_\alpha^* + \sum_{\mathrm{I}} C_{\mathrm{I}} V_\alpha^{\mathrm{I}} \left(\frac{k_0^2}{k^2} \right)^{\theta_I/2}, \tag{2.72}$$

where the V^{I} are the eigenvectors of $B_{\alpha\gamma}$ with eigenvalues $-\theta_{\mathrm{I}}$. The C_{I} in Eq. (2.72) are constants of integration and k_0 is a fixed reference scale. The quantities θ_I are also called as *critical exponents*. If $u_\alpha(k)$ describes a real RG trajectory on the UV critical surface, one has to show that when $k \to \infty$ they approach UV fixed point u^*_α.

Now, let us try to embed the running (2.72) of the coupling constants into the effective action (2.68). To do this, we first need to specify how we are going to write the RG improved effective action in a coordinate space. If we want to preserve the general covariance of the theory at the level of action, we should identify the cutoff scale k of the theory with a covariant quantity which has the same scaling as k^2. Since, there is no an unique way of doing so, we should consider different possible identifications. In the literature the most discussed ones are $k^2 \sim R$ [155, 163], $k^2 \sim \chi$ [164] (where χ is a *"cosmon"* field) and $k^2 \sim \Box$ [37, 38]. In our studies, we have mostly focused on the third choice, which is to identify the momentum dependence of the RG running of coupling constants with the d'Alembert operator, ultimately promoting them to a nonlocal form. Pioneering works in this direction are Refs. [37, 38]. In these works, the authors have shown that the RG improvement of Newton's constant can lead to strong IR-relevant effects, providing a valid DE model. The nonlocal effective action, which encodes the RG running of Newton's constant in it, is written as

$$\frac{1}{16\pi G_0}\int d^4x\sqrt{-g}\left(1-c_\zeta\left(\frac{1}{\zeta^2\Box}\right)^{1/2\nu}+\mathcal{O}\left(\left(\zeta^2\Box\right)^{-1/\nu}\right)\right)R, \tag{2.73}$$

where G_0 is the local value of Newton's constant and ν stands for the critical exponents and is related to θ_g in Eq. (2.72) as $\nu^{-1} \equiv \theta_g$. The coefficient c_ζ is an order one dimensionless quantity. The only component left to be specified in the action (2.73) is the non-perturbative reference scale $k_0 = \zeta^{-1}$. This scale has an important role as it quantifies the deviation of the quantum theory from the classical one at the large scales. As discussed in Ref. [37], the magnitude of ζ is proportional to

$$\zeta^{-1} \propto \Lambda_{\mathrm{UV}}\exp\left(-\int^g \frac{dg'}{\beta(g')}\right) \sim \lim_{g\to g^*}|g-g^*|^\nu, \tag{2.74}$$

where Λ_{UV} is the UV cutoff of the theory and g^* is a UV NGFP. To find the actual value of the nonperturbative scale ζ, we need to have some external input, as the underlying theory can not fix it. But some properties of the overall behavior of ζ can be still reflected from Eq. (2.74). Indeed, for positive values of the critical exponent $\nu > 0$, the magnitude of ζ^{-1} increases if we move away from the UV fixed point along a RG trajectory. So, the total coefficient at the front of $\Box^{-1/2\nu}$ can be sufficiently big at scales away from the UV fixed point. In Ref. [37] it has been argued that it is natural to identify ζ at the cosmological scales either with the spacial average of the Ricci scalar $\langle R\rangle$ or with the macroscopic expansion rate of the Universe, given by the Hubble function, i.e. $\zeta \sim H^{-1}$. In our work, which will be presented in Chapter 7, we have chosen the latter scale identification. In previous works [37, 38, 165], the phenomenological studies of the model (2.73) have been carried out at the level of effective Einstein equations, where the bare Newtonian constant was directly replaced with the RG improved one, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G(1 + A(\Box)) T_{\mu\nu}, \tag{2.75}$$

with

$$A(\Box) \equiv c_\zeta \left(\frac{1}{\zeta^2 \Box} \right)^{1/2\nu}. \tag{2.76}$$

In Refs. [37, 165] it has been shown that Eq. (2.73) leads to a late time acceleration for positive values of the critical exponent ν. The studies based on Eq. (2.73) have two important drawbacks. First, the effective Eq. (2.75) does not correspond to the one which we would get by varying the action (2.73) with respect to the metric $g_{\mu\nu}$. Indeed, in this case one also has to vary the d'Alembert operator, which gives rise to additional terms in Eq. (2.75). Secondly, written in its current form Eq. (2.75) is not consistent with the Bianchi identities (1.15). This is a direct consequence of the fact that in a general curved background the d'Alembert operator $\Box$ and the covariant derivative ∇_μ do not commute. As a result by acting with the covariant derivative on the right hand side of Eq. (2.75) we will in general have that $\nabla^\mu \left[(1 + A(\Box))T_{\mu\nu}\right] \neq 0$, which is indeed not consistent with the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$. In our work discussed in Chapter 7, we have studied the model in a self-consistent way, which is to get the EoMs directly from the level of the action (2.73). Due to the covariant formulation of this action, the EoMs derived from it will automatically be consistent with the Bianchi identities.

2.5 Other scalar nonlocal gravity models

In the previous sections, we have discussed several mechanisms for the generation of nonlocalities from QFT. There are nonlocal gravity models, for example the RR or Hamber models, which may originate from those mechanisms. On the other side, in the literature on nonlocal gravity there are also other phenomenologically well established models, whereby the question of their origin still remains widely open. Within these models, the Deser-Woodard (DW) model [166] is worth to be mentioned separately. This model was the first nonlocal model suggested to be able to explain the late time acceleration of the Universe and was crucial for understanding different aspects of nonlocal gravity models. This model generalizes the Einstein-Hilbert action in the following way:

$$S_{\mathrm{DW}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \left[1 + f\left(\Box^{-1} R\right)\right], \tag{2.77}$$

where f is a free function also known as the nonlocal distortion function [167]. This model, being the first in its class, provides a natural delay from the onset of the cosmic acceleration, addressing therefore the coincidence problem in cosmology. The delay is due to the fact that during the radiation-domination period the Ricci scalar R vanishes (or more precisely it is much smaller than the Hubble rate, i.e. $R/H^2 \ll 1$), so $\Box^{-1} R$ cannot grow until the beginning of the matter-domination period, and even then its growth is only logarithmic [168]. Of course, these arguments can be equally applied to other nonlocal models which also have a $\Box^{-1}R$ structure in their construction. This model has also another important property, which is the absence of a new scale in the theory. The only physical scale present in the model is the Planck mass. This is important because the corrections to Einstein-Hilbert action will not be suppressed by an additional $M_{\rm pl}^2$ term as is the case for $f(R)$ theories introduced in Sec. 1.4.3. Moreover, in the case of the action (2.77), the nonlocal term we have added is not prohibited by any symmetry and has the same physical relevance as the Einstein-Hilbert term. The precise shape of the function f is of course beyond our control and ideally has to be derived from some fundamental theory. Although there have been some suggestions that this type of nonlocal corrections can appear in different string compactification scenarios [169], this question needs further extensive investigation. So, at this stage one should assume that Eq. (2.77) just provides a simple parametrization of the physics at particular energy scales and can be constrained by observations. As it has been shown in Ref. [167] the function $f(\Box^{-1}R)$ can be always tuned to reproduce any behavior of the scale factor $a(t)$. In the same work, the authors have worked out the form of the function f needed to reproduce the exact ΛCDM evolution. The phenomenology of this model both on the background and perturbative levels is discussed in Chapter 5, where we will also investigate the observational validity of the model.

2.6 Tensorial nonlocal gravity models

The nonlocal gravity models discussed so far belong to the family of so-called scalar nonlocal gravity theories. This name represents the point that in all the models above we have that nonlocal operators are located between two scalar functions, namely the Ricci scalar R. However, as we can observe from Eqs. (2.43) and (2.63) a general EFT action in addition to scalar nonlocal terms will also contain some nonlocal terms of the tensorial nature, i.e. a nonlocal operator located between two tensors. In particular, these terms are of the structure where a nonlocal operator is directly applied on tensors, such as the Ricci tensor $R_{\mu\nu}$ or the Riemann tensor $R_{\mu\nu\sigma\rho}$. Importantly, these tensorial nonlocal corrections in an effective action are of the same order as their scalar companions and should be discussed simultaneously in a consistent EFT framework. In this respect, a natural generalization of the RR scalar nonlocal gravity model including tensorial nonlocalities would be [19]

$$S = \frac{M_{\rm pl}^2}{2}\int d^4x\sqrt{-g}\left[R + \alpha_1 R\frac{1}{\Box^2}R + \alpha_2 R^{\mu\nu}\frac{1}{\Box^2}R_{\mu\nu} + \alpha_3 R^{\mu\nu\sigma\rho}\frac{1}{\Box^2}R_{\mu\nu\sigma\rho}\right]. \tag{2.78}$$

It is also useful to write the action (2.78) in the Weyl basis which will give us

$$S = \frac{M_{\rm pl}^2}{2}\int d^4x\sqrt{-g}\left[R + \beta_1 R\frac{1}{\Box^2}R + \beta_2 R^{\mu\nu}\frac{1}{\Box^2}R_{\mu\nu} + \beta_3 C^{\mu\nu\sigma\rho}\frac{1}{\Box^2}C_{\mu\nu\sigma\rho}\right]. \tag{2.79}$$

Here $\{\alpha_i\}$ and $\{\beta_i\}$ are parameters with mass squared dimension. As in the case of scalar nonlocal theories, we will study the phenomenology of tensorial models at the background and perturbation levels. Let us start our discussion for the background level. If we choose our cosmological background to be of FLRW type, the Weyl tensor on that background is vanishing ($C_{\mu\nu\rho\sigma} \equiv 0$). As a result, the Weyl-square term in the action (2.79) will not make any contribution. As we can see, the only relevant tensorial term at the background level is an arbitrary nonlocal $\mathscr{F}(\Box)$ operator placed between two Ricci tensors, i.e. $R_{\mu\nu}\mathscr{F}(\Box)R^{\mu\nu}$.

Before investigating more complicated choices of the operator $\mathscr{F}$, let us start with the simplest case when $\mathscr{F}(\Box) \equiv \Box^{-1}$. For this choice the corresponding action was first fully studied in Ref. [40] and is given by

$$S = \frac{M_{\rm pl}^2}{2}\int d^4x\sqrt{-g}\left(R + \alpha R^{\mu\nu}\frac{1}{\Box}R_{\mu\nu}\right), \tag{2.80}$$

where we have dropped the scalar nonlocal part $R\Box^{-1}R$ for simplicity and because we are mostly interested in the individual contribution of the tensorial terms. To study the behavior of the model at the background level we need to derive the EoM. By varying the action (2.80) with respect to the metric $g_{\mu\nu}$ and using the relation $\delta(\Box^{-1}) = -\Box^{-1}\delta(\Box)\Box^{-1}$ [19], we find for the modified Einstein equations [40]

$$G_{\mu\nu} = 8\pi G\left(T^{\rm NL}_{\mu\nu} + T_{\mu\nu}\right), \tag{2.81}$$

where the term

$$\begin{aligned} T^{\rm NL}_{\mu\nu} = \frac{\alpha}{8\pi G}\Big(&-g_{\mu\nu}R^{\sigma\rho}S_{\sigma\rho} + R_{\mu\nu} + g_{\mu\nu}\nabla_\sigma\nabla_\rho S^{\sigma\rho} + \nabla_\mu H^{\sigma\rho}\nabla_\nu S_{\sigma\rho} \\ &-\frac{1}{2}g_{\mu\nu}\nabla_\lambda S^{\sigma\rho}\nabla^\lambda S_{\sigma\rho} + \Big[S^{\sigma\rho}\nabla_\sigma\nabla_\mu S_{\rho\nu} + \nabla_\sigma S^{\sigma\rho}\nabla_\mu S_{\nu\rho} - \nabla_\sigma S^\rho_\mu \nabla_\nu S^\sigma_\rho \\ &-S^\rho_\mu\nabla_\rho\nabla_\nu S^\rho_\sigma + R^\sigma_\mu S_{\sigma\nu} - \nabla_\sigma\nabla_\mu S^\sigma_\nu + (\mu\leftrightarrow\nu)\Big]\Big), \end{aligned} \tag{2.82}$$

stands for the contribution of the tensorial nonlocalities. Here we have introduced the auxiliary tensorial field $S_{\mu\nu}$ which is defined as a solution of the local differential equation

$$\Box S_{\mu\nu} = R_{\mu\nu}. \tag{2.83}$$

The natural expectation from any type of modification of the Einstein-Hilbert theory is, to preserve all the successes of GR for the cases where it is considered to be a valid theory. This logic should also hold for nonlocal tensorial models. A first requirement for the model (2.80) is not to differ much from GR during the radiation- and matter-domination periods where the GR picture is very accurate. For this to be the case, the energy momentum tensor for the nonlocal contribution $T^{\rm NL}_{\mu\nu}$ in Eq. (2.81) must be smaller from the one of the matter content, i.e. $T^{\rm NL}_{\mu\nu} \ll T_{\mu\nu}$. Let us now assume that this condition is indeed satisfied, so we can consider nonlocal terms as a perturbation on the GR background. In this case, the FLRW line element in conformal time τ is

$$ds^2 = a(\tau)^2\left(d\tau^2 - d\vec{x}^2\right), \tag{2.84}$$

where the scale factor a takes the form $a(\tau) = a_0\tau$ for the radiation-domination period and $a(\tau) = a_0\tau^2$ for the matter-domination period. To solve Eq. (2.83) for the field $S_{\mu\nu}$, we first notice that the symmetry of the FLRW metric allows us to reduce the tensor $S_{\mu\nu}$ to a simple diagonal form

$$S^{\mu}_{\nu} = \text{diag}\{S_1, -S_2, -S_2, -S_2\}, \tag{2.85}$$

where S_1 and S_2 are homogenous scalar functions. Taking this into account, the set of equations (2.83) becomes

$$\ddot{S}_+ + 2\mathcal{H}\dot{S}_+ - 8\mathcal{H}^2 S_+ = 4\mathcal{H}^2 - 2\frac{\ddot{a}}{a}, \tag{2.86}$$

$$\ddot{S}_- + 2\mathcal{H}\dot{S}_- = -6\frac{\ddot{a}}{a}, \tag{2.87}$$

with the conformal Hubble function $\mathcal{H} \equiv aH$ and the scalar functions $S_\pm$ defined as

$$S_+ \equiv S_1 + S_2, \qquad S_- \equiv S_1 - 3S_2. \tag{2.88}$$

The set of equations (2.86)-(2.87) for the pure radiation ($a(\tau) = a_0\tau$) and matter-domination ($a(\tau) = a_0\tau^2$) periods can be solved analytically. By inserting the expression for the scale factor during the radiation-domination period into Eqs. (2.86)-(2.87) we find for $S_\pm$ the following power-law solutions

$$S_+ = \alpha_{1+}\tau^{p_+} + \alpha_{2+}\tau^{p_-} - \frac{1}{2}, \qquad S_- = \frac{\alpha_{1-}}{\tau} + \alpha_{2-}, \tag{2.89}$$

with $p_\pm = -\frac{1}{2} \pm \frac{\sqrt{33}}{2}$, and the corresponding solutions for the pure matter-domination period are

$$S_+ = \beta_{1+}\tau^{\tilde{p}_+} + \beta_{2+}\tau^{\tilde{p}_-} - \frac{3}{8}, \qquad S_- = \frac{\beta_{1-}}{\tau^3} - 4\ln\tau + \beta_{2-}, \tag{2.90}$$

with $\tilde{p}_\pm = -\frac{3}{2} \pm \frac{\sqrt{137}}{2}$. In the solutions (2.89)-(2.90), the parameters $\{\alpha_{i\pm}\}$ and $\{\beta_{i\pm}\}$ are arbitrary constants that need to be fixed from initial conditions. Now, when we have derived the evolution of the tensorial field $S_{\mu\nu}$ during pure radiation and matter-domination eras, we can check whether the initial assumption $T^{\text{NL}}_{\mu\nu} \ll T_{\mu\nu}$, holds. After all, we notice that in the solutions (2.89)-(2.90) the dominant contribution is given by the terms with a positive power-law. During the radiation-domination period the growth is governed by the term $S_+ \propto \tau^{p_+} = \tau^{2.37} \propto a^{2.37}$ and during the matter-domination period by $S_+ \propto \tau^{\tilde{p}_+} = \tau^{4.35} \propto a^{2.18}$. Finally, inserting these leading modes into Eq. (2.82) we get $a^{-2}T^{\text{NL}}_{00} = \rho_{\text{NL}} \propto a^{0.74}$ and $a^{-2}T^{\text{NL}}_{00} = \rho_{\text{NL}} \propto a^{1.35}$ for the radiation and matter-domination periods, respectively.

This result is of crucial importance which demonstrates that the above-discussed tensorial nonlocalities do not lead to a valid cosmic evolution. In fact, during radiation- and matter-domination periods the assumption $T^{\text{NL}}_{\mu\nu} \ll T_{\mu\nu}$ will not be satisfied. Indeed, as we see during the radiation-domination period the nonlocal energy density $\rho_{\rm NL}$ increases with the scale factor a as $\rho_{\rm NL} \propto a^{0.74}$, whereas the standard radiation density decreases with the scale factor as $\rho_{\rm R} \propto a^{-4}$. This shows that, even if initially $\rho_{\rm NL} \ll \rho_{\rm R}$, after a short time $\rho_{\rm NL}$ becomes dominant thus violating the condition $T^{\rm NL}_{\mu\nu} \ll T_{\mu\nu}$. These arguments also hold for the matter-domination period where the matter density $\rho_{\rm M}$ decreases as $\rho_{\rm M} \propto a^{-3}$. At this point the following remark is in place. As one can observe, in Eq. (2.89) the growing mode is multiplied with the constant α_{1+}, so we can ask whether it would be possible to choose initial conditions such that this constant will vanish. If α_{1+} is set to zero, the contribution of the leading τ^{p_+} term vanishes. Moreover, as is shown in Ref. [40] the next to leading terms in Eq. (2.89) do not lead to a growing $\rho_{\rm NL}$ and therefore the condition $T^{\rm NL}_{\mu\nu} \ll T_{\mu\nu}$ will be satisfied. However, the price to pay is an extreme fine tuning. Namely, we can choose initial conditions so that α_{1+} vanishes, but because solution (2.89) is not of an attractor type, any small perturbation around the chosen initial conditions will again reactivate the dangerous τ^{p_+} term. Moreover, for the case of the matter-domination period, in Eq. (2.90) we do not have the freedom to choose initial conditions. They will be dictated by the evolution of the system during the radiation-domination period, and in general the parameter β_{1+} will not vanish. Hence, the main conclusion of this section is, due to the existence of fast growing modes, the tensorial nonlocal model given by the action (2.80) will not lead to a viable cosmology. Already in Ref. [40] it has been argued that the instabilities in the model (2.80) might be cured by promoting the nonlocal operator $\Box^{-1}$ to some other form, which could possibly include two physically very relevant cases, such as the cases of a massive propagator $\left(\Box + m^2\right)^{-1}$ and the conformal anomaly operator Δ_4 (2.60). These possibilities have been firmly addressed in our work [119]. We devote Chapter 6 to the discussion of this work.

CHAPTER 3

Nonlocal gravity as an EFT

In the previous section we have seen how nonlocal corrections arise in the effective field theory action once we take into account contributions coming from different radiative corrections. To proceed, we first need to understand how to deal with these nonlocal terms. In order to obtain the spectrum of solutions for the models given by nonlocal actions, we should find the corresponding EoMs. The main task of this chapter is to elucidate how the general variational principle is implemented for nonlocal models and what are the important differences compare to local theories.

3.1 Nonlocal actions

The first observation we make is: a variation of nonlocal actions leads to EoM which have both causal and acausal parts. Let us see why this is the case. For simplicity let us consider the following correction [168] to the scalar field action (1.40)

$$\Delta\Gamma = -\frac{1}{2}\int d^4x\phi(x)\left(\Box^{-1}\phi\right)(x) = -\frac{1}{2}\int d^4x\int d^4x'\phi(x)G\left(x,x'\right)\phi(x'). \tag{3.1}$$

Here $G\left(x,x'\right)$ is the Green's function of the nonlocal operator $\Box^{-1}$ defined as

$$\left(\Box^{-1}\phi\right)(x) = \int d^4x' G(x,x')\phi(x'). \tag{3.2}$$

The Green's function $G\left(x,x'\right)$ can be written as a combination of retarded and advanced Green's function, i.e.

$$G\left(x,x'\right) \equiv G_{\rm R}\left(x,x'\right) + G_{\rm A}\left(x',x\right). \tag{3.3}$$

From the symmetry condition of a general Green's function, $G(x,x') = G(x',x)$, we see that under coordinate permutations a retarded Green's function $G_{\mathrm{R}}(x,x')$ transforms to an advanced Green's function $G_{\mathrm{A}}(x',x)$ and vice versa. Now by varying the action (3.1) w.r.t. the field ϕ we obtain

$$\frac{\delta\Delta\Gamma(\phi)}{\delta\phi(y)} = -\frac{1}{2}\int d^4x\int d^4x'\left(\delta^4(x-y)G\left(x,x'\right)\phi(x') + \phi(x)G\left(x,x'\right)\delta^4(x'-y)\right)$$

$$= -\frac{1}{2}\int d^4x(G(y,x)+G(x,y)). \tag{3.4}$$

Thus, from Eq. (3.4) we can see that EoM arising from the action (3.1) will contain a part with the advanced Green's function. Even if initially the action (3.1) would be defined only through a retarded Green's function, due to the fact that the variation affects both the field $\phi(x)$ in the future and the field $\phi(x')$ in the past, we will get a term with $G_{\mathrm{R}}(x,x')$ and a term with $G_{\mathrm{R}}(x',x)$, which is just a corresponding advanced Green's function $G_{\mathrm{A}}(x,x') = G_{\mathrm{R}}(x',x)$ [168].

Now let us discuss why the appearance of an advanced Green's function in EoM is not desired. In the definition of a retarded Green's function $G_{\mathrm{R}}(x,x')$ the four-coordinate x'^{μ} is on or within the past like cone of the four-coordinate x^{μ}, which means that x'^{μ} is the point where initial conditions on $G_{\mathrm{R}}(x,x')$ are provided. On the other hand, in the case of an advanced Green's function the four-coordinate x'^{μ} is located within the future light-cone of the point x^{μ}. In this case, to identify an advanced Green's function $G_{\mathrm{A}}(x,x')$ at the current time t we need to provide some information about the state of our physical system at some future point t_{f}. This situation is in disagreement with the causality conjuncture in physics, which states that physical conditions in a physical system at a given time-point t should depend only on the processes happening in the past light-cone of that point. From the arguments mentioned above one can conclude that in a fundamental action nonlocality implies the loss of causality already at the classical level, unless there are some symmetries which protect the model from acausal solutions.

However, the situation changes when we move towards quantum effective actions, which as we have seen in the previous section are in general nonlocal. Here a natural question arises: can quantum effective actions describe a system with causal EoMs. The answer of this question is positive, and is obtained by implementing the so-called Schwinger-Keldysh path integral formalism [170–173]. We know that the variation of a quantum effective action does not yield classical EoMs of the field, but rather EoMs for the vacuum expectation values of corresponding field operators [174]. To obtain these EoMs we use either *in-in* or *in-out* matrix elements. In order to clearly distinguish these two cases, let us first introduce the scattering amplitude $\langle\Psi_{\mathrm{out}}|\hat{\phi}|\Psi_{\mathrm{in}}\rangle$, which is given by the following expression [175]

$$\langle\Psi_{\mathrm{out}}|\hat{\phi}|\Psi_{\mathrm{in}}\rangle \sim \int \mathscr{D}\phi(t)\Psi^*_{\mathrm{out}}\left[\phi(t_{\mathrm{f}})\right]\phi(t)\Psi_{\mathrm{in}}\left[\phi(t_{\mathrm{i}})\right]e^{i\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}dtL[\phi(t)]}, \tag{3.5}$$

where $\mathscr{D}\phi$ stands for an integral over the space of all possible trajectories for the field ϕ. The physical states $|\Psi_{\mathrm{in}}\rangle$ and $\langle\Psi_{\mathrm{out}}|$ are defined at the initial and final times t_{i} and t_{f}, respectively. The expression (3.5) for the scattering amplitude is acausal because it counts processes happening before and after the time t. Moreover, if one computes vacuum to vacuum amplitudes on non-trivial backgrounds as we have in the case of cosmology, the final vacuum state is no longer equal to an initial one and the matrix element $\langle 0_{\rm out}|\hat{\phi}|0_{\rm in}\rangle$ is not real-valued [174][1]. In fact, there is nothing wrong with this result. The *in-out* matrix elements are not observable quantities, but just auxiliary objects used in intermediate steps to calculate scattering amplitudes and Feynman propagators. Indeed, as explained in Ref. [176] in the case of physical scattering amplitudes, the real observable quantity is not $\langle\Psi_{\rm out}|\hat{\phi}|\Psi_{\rm in}\rangle$ but rather the corresponding probability

$$|\langle\Psi_{\rm out}|\hat{\phi}|\Psi_{\rm in}\rangle|^2=\langle\Psi_{\rm in}|\left(\hat{\phi}^{\dagger}|\Psi_{\rm out}\rangle\langle\Psi_{\rm out}|\hat{\phi}\right)|\Psi_{\rm in}\rangle, \tag{3.6}$$

which can be understood as an *in-in* matrix element of the new operator $\left(\hat{O}=\hat{\phi}^{\dagger}|\Psi_{\rm out}\rangle\langle\Psi_{\rm out}|\hat{\phi}\right)$. As will become clear below this operator is well defined and causal. In order to get causal EoM for physical observables we inspect the quantum effective action for the expectation value $\langle\Psi_{\rm in}|\hat{\phi}|\Psi_{\rm in}\rangle$, where both states $|\Psi\rangle$ are given at the same time point $t_{\rm i}$. To construct this matrix element we can use Eq. (3.5) where instead of connecting $|\Psi_{\rm in}\rangle$ from $t_{\rm i}$ to $\hat{\phi}$ at t and then the latter to $\langle\Psi_{\rm out}|$ at $t_{\rm f}$, we now connect $|\Psi_{\rm in}\rangle$ from $t_{\rm i}$ to $\hat{\phi}$ at t and then the latter again back to $|\Psi_{\rm in}\rangle$ at $t_{\rm i}$. Under these conditions Eq. (3.5) becomes

$$\begin{aligned}\langle\Psi_{\rm in}|\hat{\phi}|\Psi_{\rm in}\rangle\sim&\int\mathscr{D}\phi_+(t)\mathscr{D}\phi_-(t)\Psi^*_{\rm in}\left[\phi_-(t_{\rm i})\right]\Psi_{\rm in}\left[\phi_+(t_{\rm i})\right]\\&\times\delta\left(\phi_+(t)-\phi_-(t)\right)\exp\left[i\int_{t_{\rm i}}^{t}dt'\left(L\left[\phi_+(t')\right]-L\left[\phi_-(t')\right]\right)\right],\end{aligned} \tag{3.7}$$

where $\phi_+(t)$ and $\phi_-(t)$ correspond to modes going forward and backward in time, respectively, within the same time interval $[t_{\rm i},t]$. From Eq. (3.7) it is already obvious that the dynamics of the matrix element $\langle\Psi_{\rm in}|\hat{\phi}|\Psi_{\rm in}\rangle$ will be determined by the physics in the time-interval $[t_{\rm i},t]$ implying a causal evolution. The generating functional $W_{\text{in-in}}[J_+,J_-]$ for this case has the following form [175]

$$\begin{aligned}W_{\text{in-in}}[J_+,J_-]=-i\log&\int\mathscr{D}\phi_+(t)\mathscr{D}\phi_-(t)\Psi^*_{\rm in}\left[\phi_-(t_{\rm i})\right]\Psi_{\rm in}\left[\phi_+(t_{\rm i})\right]\\&\times\delta\left(\phi_+(t)-\phi_-(t)\right)\exp\left[i\int_{t_{\rm i}}^{t}dt'\left(L\left[\phi_+(t')\right]-L\left[\phi_-(t')\right]-\phi_+J_++\phi_-J_-\right)\right],\end{aligned} \tag{3.8}$$

and the corresponding effective action upon Legendre transform of the generating functional is given by

$$\Gamma_{\text{in-in}}=W_{\text{in-in}}[J_+,J_-]-\int dt'\left(\phi_+J_+-\phi_-J_-\right). \tag{3.9}$$

This effective action generates the following EoMs for the field expectation values,

$$\frac{\delta\Gamma_{\text{in-in}}}{\delta\varphi_+(t')}=-J_+(t'),\qquad\frac{\delta\Gamma_{\text{in-in}}}{\delta\varphi_-(t')}=-J_-(t'). \tag{3.10}$$

[1] In the case of Minkowski spacetime one has that $|0_{\rm in}\rangle=|0_{\rm out}\rangle=|0\rangle$.

Here, $\varphi_+(t')$ and $\varphi_-(t')$ are the vacuum expectation values of the fields $\phi_\pm$, defined as $\varphi_+(t') \equiv \langle\Psi_{\text{in}}|\hat{\phi}_+|\Psi_{\text{in}}\rangle$ and $\varphi_-(t') \equiv \langle\Psi_{\text{in}}|\hat{\phi}_-|\Psi_{\text{in}}\rangle$, respectively. Setting external sources $J_\pm$ to zero, the variation of $\Gamma_{\text{in-in}}$ also vanishes, leading to the correct effective EoMs.

In summary, in this section we have seen that by using Schwinger-Kyldish path integral formalism we can find EoM corresponding to the *in-in* effective action $\Gamma_{\text{in-in}}$, which are perfectly causal. Hence we conclude that if a nonlocal model is described with an effective quantum action of some underlying fundamental local theory, by using the Schwinger-Keldysh formalism we get causal and well defined dynamics for the theory.

3.2 Localization procedure

Another subtle issue to be investigated concerns the number of degrees of freedom in nonlocal gravity. To address this question we will first discuss localization procedure in nonlocal gravity. Nonlocal gravity theories are generally described by integro-differential equations, which contain both derivatives and integrals of some function [16–19, 128, 176–178]. In general, these equations can be written as

$$\mathscr{L}[f(x)] - \lambda \int_a^b K(x, y, \mathscr{P}(f(y)))\,dy = f(x), \tag{3.11}$$

here $\mathscr{L}$ is called external differential operator, K is the so-called kernel of integro-differential equation and $\mathscr{P}$ is another differential operator which is also referred to as an internal differential operator. In physical context we will mostly deal with linear integro-differential equations, where the internal differential operator $\mathscr{P}$ enters linearly, i.e.

$$\mathscr{L}[f(x)] - \lambda \int_a^b K(x, y)\mathscr{P}(f(y))dy = f(x). \tag{3.12}$$

Equation (3.12) is usually called Fredholm equation of the second kind [179]. When the source function $f(x)$ in Eq. (3.12) vanishes, the equation is known as Fredholm equation of first kind. It is also possible to have situations when one of the limits of integration in Eq. (3.12) is not a constant but a variable. In this case the integro-differential equations are referred to as Voltera type integro-differential equations [179]. Depending on the choice of the integral kernel K it is sometimes possible to introduce a finite number of auxiliary fields and express an integro-differential equation through a set of differential equations of these auxiliary fields [180–184]. In this case the corresponding nonlocal theory can be written as a local theory of multiple auxiliary fields. We call this type of nonlocal theories as *localizable*. Later we will see how this works when discussing particular nonlocal models and their localization procedure. For a moment let us assume that we have a nonlocal theory of the scalar field φ, which leads to the following equation of motion [175]

$$\Box\varphi + m^4\Box^{-1}\varphi = J, \tag{3.13}$$

where m is the mass parameter of the model and J is some source term. Here, and in the reminder of this section, in agreement with the previous section, we will assume that $\Box^{-1} = \Box_R^{-1}$, i.e. we consider only retarded Green's functions. Now let us introduce a new function ϕ which is defined as

$$\phi \equiv m^2 \Box^{-1} \varphi. \tag{3.14}$$

Using the definition of the function ϕ we can write Eq. (3.13) in a local form, i.e.

$$\Box \varphi + m^2 \phi = J. \tag{3.15}$$

To close the set of local differential equations describing our model, one must supplement Eq. (3.15) with the following equation for ϕ

$$\Box \phi = m^2 \varphi. \tag{3.16}$$

Now let us check whether the final set of Eqs. (3.15)-(3.16) is equivalent to the initial nonlocal equation (3.13) or not. To do this we need to solve Eq. (3.16) for ϕ and plug the solution back into Eq. (3.15). By solving Eq. (3.16) analytically we find the following general solution

$$\phi = \phi_{\text{hom}} + m^2 \Box^{-1} \varphi, \tag{3.17}$$

where ϕ_{hom} is the solution of the homogeneous equation $\Box \phi = 0$. From Eq. (3.17) we see that the solution for ϕ differs from its initial definition by an addition of the homogeneous solution ϕ_{hom}. In this case the local and nonlocal descriptions coincide if and only if $\phi_{\text{hom}} = 0$ at the initial time t_{i} when we start the convolution in Eq. (3.13). This leads to an important conclusion that the dynamical field ϕ satisfying differential Eq. (3.16) is not a free field. This means that the initial conditions on the field ϕ are not free to be chosen, as is the case for usual local theories, but are uniquely fixed by boundary conditions of the original nonlocal theory. The choice of the homogenous solution is part of the definition of $\Box^{-1}$ operator and therefore of the original nonlocal effective theory. Let us clarify this point in a more detailed way. By using the definition of an inverse operator L^{-1} satisfying $LL^{-1} = \text{id}$, one can easily show that it does not uniquely specify the inverse operator L^{-1}. Indeed, we can have several inverse operators which are related to each other by adding a homogenous solution and as such if we pick a L^{-1} once and for all, other inverse operators can be found by adding a homogenous solution to L^{-1} [175]. Indeed one can write for any inverse operator

$$L_{\text{gen}}^{-1} \varphi = L^{-1} \varphi + \phi_{\text{hom}}, \tag{3.18}$$

and now if we let the operator L act on both sides of this equation, keeping in mind that $LL^{-1} = \text{id}$ and $L\phi_{\text{hom}} = 0$, we find an identity. On the other hand, the inverse operator L^{-1} in a Green's function representation is

$$\left(L^{-1} \varphi\right)(x) = \int d^D y G(x, y). \tag{3.19}$$

By combining Eqs. (3.18) and (3.19) we can conclude that different choices of L^{-1} correspond to different spacetime boundary conditions on the Green's function. Coming back to our initial discussion we can state that different choices of the homogenous solution $\phi_{\rm hom}$ correspond to different choices of the $\Box^{-1}$ operator, thus different original theories. Importantly, this also implies that the choice of initial conditions on ϕ is a theory-level data.

3.3 Degrees of freedom and ghosts

We begin our discussion of this section by considering first nonlocal theories which do not have local representation. Those, as it was also mentioned before, are theories which can not be represented by a finite set of second-order differential equations of auxiliary fields. These models usually can be shown to be equivalent to ones with infinitely many derivatives [31, 183]. The question whether the theories with infinity many derivatives contain ghosts is directly related with the structure of the theory's propagator. In particular, if the propagator of the theory contains only a single pole, the theory will be ghost-free [183]. The reason for this is that theories with only one pole in the propagator describe a single degree of freedom and thus do not contain any spurious degrees of freedom. Now let us see how this statement goes along with the famous Ostrogradski theorem [71, 185], stating that: *"Every non-degenerated Lagrangian that contains derivatives of order two or higher describes a theory that propagates at least one ghost degree of freedom"* [186]. As an example let us consider a higher-derivative theory with a Lagrangian that depends non-degenerately on some field and its first N derivatives. In this case the corresponding Hamiltonian will depend on $2N$ canonical variables corresponding to $2N$ initial data which are needed to specify the solutions of the EoM. It can be easily shown that the Hamiltonian in this case will depend linearly on $N-1$ conjugate momenta. From this it follows that the Hamiltonian is unbounded from below and in the limit of large N half of modes in the physical phase-space will correspond to unstable degrees of freedom [183]. One could naively think that in the case of all infinite-derivative theories the situation is even worse and by taking the limit $N \to \infty$ in Ostrogradski theorem we would get that this models contain infinitely many ghost-like degrees of freedom. But this is not the case [183]. Indeed, as far as the propagator of infinite-derivative theories contains only one pole, one will only need two initial conditions to specify the solutions of the EoM. This situation is the same as in the case of the theories with two canonical variables, which are perfectly in agreement with Ostrogradski theorem. The breakdown of the Ostrogradski theorem for the case of infinite derivative theories has been first noted in Ref. [187].

We will now proceed and discuss the question of degrees of freedom in the context of *"localizable"* nonlocal theories. As we have discussed in the previous sections these theories are local but the auxiliary fields therein are constrained. As we will see later usually at least one of these auxiliary fields has a ghost-like signature in the diagonalized action, i.e. has a negative coefficient at the front of its kinetic energy. At this point it is important to understand if these constrained fields destabilize the theory or not. Here we need to separate two distinct cases, namely, the stability of the model at classical and quantum levels. Following Refs. [20, 24, 123] let us start our discussion from stability questions at quantum level. In our case, this question will be translated into the problem of quantization of a theory with constrained auxiliary fields. We will illustrate the details of this procedure on a particular nonlocal gravity model, namely the RR nonlocal gravity model introduced in Sec. 2.1. This model is given by the effective action (2.32). For the moment we will assume that this action correspond to some fundamental theory. To find the propagator structure of the model we first decompose the metric $g_{\mu\nu}$ as a combination of the background metric $\eta_{\mu\nu}$ and the small perturbation $h_{\mu\nu}$ such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Now, plugging this decomposition back into the action (2.32) and expanding up to quadratic order in powers of $h_{\mu\nu}$ we find [20, 123]

$$\Gamma^{(2)}_{\rm RR} = \frac{1}{16\pi G}\int d^4x \left[\frac{1}{4}h_{\mu\nu}\mathcal{E}^{\mu\nu,\rho\sigma}h_{\rho\sigma} - \frac{1}{6}m^2 h_{\mu\nu}P^{\mu\nu}P^{\rho\sigma}h_{\rho\sigma}\right] + \int d^4x h_{\mu\nu}T^{\mu\nu}, \tag{3.20}$$

where $\mathcal{E}^{\mu\nu,\rho\sigma}$ and $P^{\mu\nu}$ are defined in Eqs. (2.26) and (2.25), respectively. The linearized action (3.20) around a flat background is the same as the action (2.28) for the RT model. This is not surprising because, as we already mentioned in Sec. 2.1, the RR and RT models linearized around the Minkowski background coincide. Therefore, the outcomes for this section regarding the RR model will also hold for the RT model [19]. Now, to find the propagator corresponding to the action (3.20), we still need to add the gauge fixing term for linearized gravity given by [152]

$$\mathcal{L}_{\rm gf} = -\left(\partial^\nu \bar{h}_{\mu\nu}\right)\left(\partial_\rho \bar{h}^{\rho\mu}\right), \tag{3.21}$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)h\eta_{\mu\nu}$. Finally, inverting the quadratic action $\Gamma^{(2)}_{\rm RR} + \Gamma_{\rm gf}$ we obtain the theory's propagator $\tilde{D}^{\mu\nu\rho\sigma}(k) = -i\Delta^{\mu\nu\rho\sigma}(k)$ [19, 20], where $\Delta^{\mu\nu\rho\sigma}(k)$ is

$$\begin{aligned}\Delta^{\mu\nu\rho\sigma}(k) = &\frac{1}{2k^2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)\\ &+\frac{1}{6}\left(\frac{1}{k^2} - \frac{1}{k^2 - m^2}\right)\eta^{\mu\nu}\eta^{\rho\sigma}.\end{aligned} \tag{3.22}$$

The second term in this propagator shows that we have an exchange between a healthy massless scalar mode plus a ghostlike massive scalar mode. In a standard QFT picture, if our theory would had such a ghost mode in its spectrum we would immediately state that the model is ill-defined and physically nonviable at quantum level. Is the situation the same in the case of local theories with constrained auxiliary fields? To be able to answer to this question, we first need to localize the RR model (2.32). We can do this by introducing two auxiliary fields U and S defined as [123]

$$U \equiv -\Box^{-1}R, \qquad S \equiv -\Box^{-1}U. \tag{3.23}$$

Now, with the help of these functions, we can write the action (2.32) in the following local form

$$S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\left[R\left(1 - \frac{m^2}{6}S\right) - \xi_1(\Box U + R) - \xi_2(\Box S + U)\right], \tag{3.24}$$

where ξ_1 and ξ_2 are Lagrange multipliers. Varying this localized action w.r.t. to the metric $g_{\mu\nu}$ we find the modified Einstein equations [123]

$$G_{\mu\nu} = \frac{m^2}{6} K_{\mu\nu} + 8\pi G T_{\mu\nu}, \tag{3.25}$$

where

$$K_{\mu\nu} = 2SG_{\mu\nu} - 2\nabla_\mu \partial_\nu S + g_{\mu\nu}\left[-2U + \partial_\rho S \partial^\rho U - (1/2)U^2\right] - \left(\partial_\mu S \partial_\nu U + \partial_\nu S \partial_\mu U\right). \tag{3.26}$$

In order to find which field in this localized language corresponds to the massive ghost mode mentioned above we need to perturb Eq. (3.26) to linear order. For the scalar perturbations in the Newtonian gauge one finds [19, 123]

$$\nabla^2 \left[\Phi - \frac{m^2}{6} S\right] = -4\pi G \rho, \tag{3.27}$$

$$\Phi - \Psi - \frac{m^2}{3} S = -8\pi G \sigma. \tag{3.28}$$

To this set of equations we need to add the ones which we get after linearly perturbing the localization equations (3.23). These equations first derived in Ref. [123] are

$$\left(\overline{\Box} + m^2\right) U = -8\pi G(\rho - 3P) \qquad \overline{\Box} S = -U. \tag{3.29}$$

From Eqs. (3.27)-(3.29) we immediately recognize that the *Bardeen* potentials Φ and Ψ remain non-radiative degrees of freedom. Secondly, we note that the massive ghost mode in Eq. (3.22) corresponds to the auxiliary field U, whereas the massless healthy pole belongs to the field S. Now that we have identified the ghost-like mode we can investigate how it will behave at the quantum level. The canonical quantization of the theory is done by promoting classical fields of the model to quantum operators imposing particular conditions on their commutators [174]. In order to discuss particle creation processes in a quantum theory, one needs to introduce particle creation and annihilation operators. This is done by solving the classical equation for the quantizable field and promoting its free coefficients to creation and annihilation operators. Returning back to the RR model for the field U as a solution of the inhomogeneous equation (3.29) we get

$$U = U_{\text{hom}} + \widetilde{U}, \tag{3.30}$$

where U_{hom} is the general solution of the homogenous equation $\overline{\Box} U = 0$ and $\widetilde{U}$ is a particular solution of the inhomogeneous equation. The flat-space homogenous solution U_{hom} can be represented as a combination of plane waves with the free coefficients which then, in the context of QFT, will be promoted to annihilation and creation operators. In general, this procedure is valid only for dynamical fields. However, in our case the situation is different. As it was discussed in the previous Sec. 3.2, the initial conditions for U are fixed once we have specified our definition of the $\Box^{-1}$ operator, hence the homogenous solution U_{hom} will not have any more free parameters. Therefore, at quantum level we cannot promote them to annihilation and creation operators [19, 140, 174, 188].

From the above discussion we conclude that at quantum level one cannot associate any quantum degree of freedom to the auxiliary field U and, as such, it is just an artificial field (spurious degree of freedom) which has been introduced to localize our theory.

The Polyakov action (2.54) serves as another useful example which again illustrates that the spectrum of the theory cannot be read naively from the quantum effective action [19, 20]. This action is an effective quantum action of a perfectly healthy fundamental theory. We can localize this model in analogy with the RR model by introducing an auxiliary field U defined again as $U \equiv -\Box^{-1}R$. Inserting this field with the corresponding Lagrange multiplier ξ into the action (2.54) we get

$$S = \int d^2x\sqrt{-g}\,[-cRU + \xi(\Box U + R)], \tag{3.31}$$

with $c = -N/(96\pi)$, where N is the number of conformally coupled massless particles. Varying this local action with respect to the fields ξ and U one finds

$$\Box U = -R \qquad\qquad \Box\xi = cR, \tag{3.32}$$

respectively. Using Eq. (3.32) we can write $\xi = c\Box^{-1}R = -cU$, so the local action (3.31) finally simplifies to

$$S = c\int d^2x\,[\partial_a U\partial^a U - 2UR]\,. \tag{3.33}$$

Classical equivalence of the actions (3.33) and (2.54) has been shown in Ref. [150], where the authors have checked that the trace of the energy-momentum tensor computed from the action (3.33) correctly reproduces the quantum trace anomaly (2.49). Furthermore, by applying a conformal transformation $g_{ab} = e^{2\sigma}\eta_{ab}$ to the action (3.33) and introducing the field $\chi = U/2 - \sigma$ to diagonalize it, we eventually arrive at [19, 24]

$$S = 4c\int d^2x\left(\eta^{ab}\partial_a\chi\partial_b\chi - \eta^{ab}\partial_a\sigma\partial_b\sigma\right). \tag{3.34}$$

This action with its current structure suggests that one of the fields is a ghost. For a negative c, the field σ is a ghost, while χ is a healthy mode[2]. This naive outcome is not correct since, as we know the Ployakov effective action is originally derived from a healthy theory which consists of gravity plus a conformally coupled healthy massless matter sector, and does not contain any pathological degree of freedom in its spectrum [19, 20, 24]. Here, we emphasize again that, as in the case of the previous example for the field U, the field σ does not have any associated quantum degree of freedom and is eliminated by the physical state condition [189]. At the end of this section let us make some remarks about the question of the classical stability of nonlocal

[2] In our case $c = -N/(96\pi)$, i.e. it is a negative number.

gravity models. At quantum level a possible existence of *unconstrained* ghost modes is disastrous, leading to an immediate decay of the physical vacuum. On the other hand, even if we consider nonlocal models purely classically, the existence of constraints on ghost modes does not guarantee classical stability of those models. At classical level both constrained and unconstrained ghosts can be equally dangerous [19, 176, 190]. In order to be sure that the classical evolution of our models, both at background and perturbation levels, is not damaged by ghost modes, we need to analyze these models case by case to find out about possible issues. In this respect, it is important to mention that since the scale of nonlocalities for IR relevant effects is of the order of the Hubble parameter today, H_0, any such instabilities would only develop on cosmological timescales, so they must be studied on the FLRW background [19]. This has been implemented for the studies conducted in the proceeding chapters.

Chapter 4

Publication 1

Dynamical analysis of the RR nonlocal gravity model

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Dynamical analysis of $R\frac{1}{\Box^2}R$ cosmology: Impact of initial conditions and constraints from supernovae

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We discuss the cosmological implications of the $R\Box^{-2}R$ nonlocal modification to standard gravity. We relax the assumption of special initial conditions in the local formulation of the theory, perform a full phase-space analysis of the system, and show that the late-time cosmology of the model exhibits two distinct evolution paths, on which a large range of values for the present equation of state can be reached. We then compare the general solutions to supernovae data and place constraints on the parameters of the model. In particular, we find that the mass parameter of the theory should be smaller than 1.2 in Hubble units.

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I. INTRODUCTION

The current standard model of cosmology, called ΛCDM (CDM for cold dark matter), cannot be reconciled with general relativity (GR) and the Standard Model of particle physics without extreme fine-tuning. In particular, the ratio $\sqrt{\Lambda}/M_{\rm Pl}^2$ derived from observations [with Λ the notorious cosmological constant (CC) and $M_{\rm Pl}$ the reduced Planck mass] is almost infinitesimal compared to the value obtained by the most straightforward extrapolations of GR and quantum field theory, to the infrared scale $\sqrt{\Lambda}/M_{\rm Pl}$ and high-energy scales approaching $M_{\rm Pl}$, respectively. This calls both for the observational pursuit of signatures that could provide hints on the possible physics beyond the ΛCDM model, and for theoretical extensions that could explain the cosmological data in a more natural way [1,2].

Various attempts at such extensions have been undertaken in the context of nonlocal gravity [3,4]. In a top-bottom approach, the possibility that gravitational interactions become nonlocal near the Planck scale is suggested, among others, by string theory [5,6]. From a bottom-up perspective, nonlocal theories are appealing because of their potential to provide an ultraviolet completion of the metric gravity theory [7–9], but there are also motivations to contemplate nonlocal terms in the infrared as well. Such infrared nonlocal terms arise generically in effective field theories after integrating out light degrees of freedom [4,10,11], but may also feature in more fundamental actions in Euclidean quantum gravity [12,13]. Nonlocal effective formulations have been found for gravity models with a massive graviton [14,15], multiple metrics [16], and post-Riemannian, affine geometry [17]. In passing, we note that indeed the recent development of a conformal affine gauge theory of gravity [18] introduces a novel holography that, along the lines of Ref. [19], may naturally provide a nonlocal link between the value of the cosmological constant and the amount of information contained in the emergent spacetime.

Nonlocal gravity models are typically written as an Einstein-Hilbert term supplemented with integral or infinite-derivative curvature terms. The first proposal for a nonlocal dark-energy model was put forward by Deser and Woodard (DW) and has the form [20]

$$\mathcal{L}^{\rm DW} = \frac{M_{\rm Pl}^2}{2} R\left[1 - f\left(\frac{R}{\Box}\right)\right], \tag{1}$$

where R is the Ricci curvature scalar and $1/\Box$ is the inverse d'Alembertian, an integral operator such that $\Box(1/\Box) = 1$, with $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ and ∇_μ the Christoffel covariant derivative. With the dimensionless combination $R/\Box$, one could in principle construct models without introducing new scales. The integral dependence of the corrections could generate the observed acceleration at the present cosmological epoch dynamically and without special fine-tunings. However, detailed investigations have shown that, although the function f can be chosen in such a way that the background expansion is consistent with the data [21–23] and the model has a viable Newtonian limit [24,25], the impact of the nonlocal corrections on the evolution of perturbations is strong and utterly rules the model out when this is confronted with large-scale structure data [26]. On top of that, nonlocal modifications of gravity result generically in instabilities at the level of perturbations, at least if they involve tensorial terms such as $(W_{\mu\nu\rho\sigma}/\Box^2)W^{\mu\nu\rho\sigma}$ [27] with $W_{\mu\nu\rho\sigma}$ the Weyl tensor appearing in models inspired by the conformal anomaly [28,29].

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One of the remarkable features of the model,

$$\mathcal{L}^{\rm MM} = \frac{M_{\rm Pl}^2}{2} R \left[1 - \frac{m^2}{6}\left(\frac{1}{\Box}\right)^2 R\right] = \frac{M_{\rm Pl}^2}{2}\left[R - \frac{m^2}{6}\left(\frac{R}{\Box}\right)^2\right], \quad (2)$$

proposed by Maggiore and Mancarella (MM) [30] is that it can produce nonlocal dark energy able to fit the background data while retaining a matter power spectrum compatible with observations (see Refs. [31–34] and [35–38] for studies of the background expansion and of structure formation, respectively). It is also notable that the $(R/\Box)^2$-correction to GR has indeed been obtained in an effective field theory for gravity at the second order curvature expansion[1] [10] and that the MM model appears to have only one new parameter m at the level of the gravitational Lagrangian, i.e. none more than ΛCDM.[2] It has also been argued that ghost fields do not destabilize the model [30] (see also Ref. [40]). Spherically symmetric solutions have also been considered [25,41].

In this paper, we study the cosmological dynamics of the MM model, with special attention to the problem of initial conditions. Nonlocal theories with infinite order derivative operators require the specification of an infinite number of initial conditions for the formulation of the Cauchy problem. Analogously, nonlocal integral operators, such as the one featured in the MM model, are strictly defined only by specifying the boundary conditions for each of the infinite number of modes in the continuum limit of the Fourier space. Various techniques have been considered to deal with such theories, see Refs. [42–49]. The MM model (2) can be reformulated in terms of two scalar fields [24], which should not be considered however as local dynamical fields evolving freely in time, but as auxiliary fields whose configuration at each spatial hypersurface is dictated by the other fields and the boundary conditions of the $1/\Box$-operator. In the phase space of the homogeneous cosmological dynamics, the trajectories of the two (fake) scalar degrees of freedom are uniquely fixed given four numbers at any given cosmological epoch. The cosmology of the MM model seems to offer a natural or "minimal" assumption for the choice of these numbers: at a sufficiently early epoch in the standard cosmology, the Universe is filled with radiation only, for which $R \approx 0$. It therefore seems an obvious choice to set $R/\Box = R/\Box^2 = 0$ at such an epoch.[3] However, already at the linear order in the inhomogeneous fluctuations, both the inverse- and the double-inverse-d'Alembertian operators bring forth scale-dependent functions in the momentum space. Unless finely adjusted and compensating scale dependence is encoded into the boundary conditions of the $1/\Box$-operators, the initial conditions for cosmological perturbations would feature additional scale dependence (compared to ΛCDM). The minimal boundary conditions, that is $\delta(R/\Box) = 0$ when $\delta R = 0$ (we denote perturbations with δ), would require scale dependence in the initial conditions for the auxiliary fields. An important point is that due to their assumed nonlocal origin, they impose constraints rather than adding dynamics. Thus one expects the nonminimal scale dependence of the initial conditions to be directly projected (or, if set in terms of the auxiliary fields, to effectively propagate) to the smaller redshifts of the crucial observables, where especially the matter power spectrum is very sensitive to the possible scale dependence in the dark sector, as that is reflected through the gravitational interaction in the baryon distribution. Since the confrontation with large-scale structure is crucial for distinguishing the MM (2) and the earlier proposal (1), the issue of (scale-dependent) linear boundary conditions calls for clarification.

In this paper we undertake a comprehensive study of the expansion dynamics in the MM model. In Sec. II we rewrite the model (2) in terms of two (effective) auxiliary scalar fields, and set up the phase space spanned by convenient dimensionless variables whose dynamical system can be closed into an autonomous form. In Sec. III we perform a full dynamical system analysis in order to identify the critical points in the cosmological phase space and determine their stability. Each set of initial conditions fixes a trajectory in the phase space, corresponding to a particular family of MM models with the same mass parameter m and the same four cosmological background boundary conditions. By exploring the global structure of the phase space we can thus map the cosmology of different models and investigate the sensitivity of the predictions to changing the parameters of the model (i.e. to the initial conditions that have been previously assumed minimal). In Sec. IV we confront the model with supernovae data constraining the background expansion, in such a way that we do not fix all the initial conditions but marginalize over them. Our findings are then summarized in Sec. V.

II. THE COSMOLOGY OF $R\frac{1}{\Box^2}R$ GRAVITY MODEL

The full action, including both gravity and matter sectors, for the MM nonlocal theory introduced in Eq. (2) has the form

$$S^{\rm MM} = \frac{M_{\rm Pl}^2}{2}\int {\rm d}^4x\sqrt{-g}\left(R - \frac{m^2}{6}R\frac{1}{\Box^2}R\right) + \int {\rm d}^4x\sqrt{-g}\mathcal{L}_m, \quad (3)$$

---

[1] As shown in Ref. [39], the coefficient of the $R\Box^{-2}R$ obtained by this procedure should satisfy $M^2/H^2 \ll 1$ with $M^4 \sim (M_{\rm Pl}m)^2$. Unfortunately, this condition is not compatible with the value of m required to obtain a realistic cosmology ($m \sim H_0$).

[2] Expectedly, viable dark energy models require $m \sim \Lambda/M_{\rm Pl} \sim H_0$, where H_0 is the present Hubble rate.

[3] Note however that the cosmology obviously depends on the thermal history. In Appendix B, we check the impact of setting $R/\Box = R/\Box^2 = 0$ either at the matter-radiation equality or at an earlier period.

with the mass scale m the only free parameter of the theory, to be determined observationally, and $\mathcal{L}_m$ the matter Lagrangian minimally coupled to gravity.

In order to derive the modified Einstein equations, we vary the action (3) with respect to the metric $g_{\mu\nu}$:

$$\delta S^{\rm MM} = \frac{M_{\rm Pl}^2}{2}\int d^4x\delta(\sqrt{-g})\left(R - \frac{m^2}{6}R\frac{1}{\Box^2}R\right) + \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left(\delta R - \frac{m^2}{3}\delta R\frac{1}{\Box^2}R + \frac{m^2}{3}R\frac{1}{\Box}\delta\Box\left(\frac{1}{\Box^2}R\right)\right) + \delta\int d^4x\sqrt{-g}\mathcal{L}_m, \quad (4)$$

where we have used $\delta(\Box^{-2}) = -2\Box^{-1}(\delta\Box)\Box^{-2}$. Denoting the conserved stress-energy tensor of matter by T^μ_ν, the gravitational field equations turn out to be [30]

$$G^\mu_\nu - \frac{1}{6}m^2K^\mu_\nu = 8\pi G T^\mu_\nu, \quad (5)$$

where we have defined

$$K^\mu_\nu \equiv 2SG^\mu_\nu - 2\nabla^\mu\partial_\nu S + 2\delta^\mu_\nu\Box S + \delta^\mu_\nu\partial_\rho S\partial^\rho U - \frac{1}{2}\delta^\mu_\nu U^2 - (\partial^\mu S\partial_\nu U + \partial_\nu S\partial^\mu U), \quad (6)$$

and introduced the two auxiliary fields U and S through the equations

$$\Box U \equiv -R, \quad (7)$$

$$\Box S \equiv -U. \quad (8)$$

Writing the field equations in terms of U and S allows us to work with a local formulation of the theory [30]. In order to solve Eq. (5) we need to first solve Eqs. (7) and (8). The general solutions for U and S are given by

$$U \equiv U_{\rm hom} - \Box^{-1}_{\rm ret}R, \quad (9)$$

$$S \equiv S_{\rm hom} - \Box^{-1}_{\rm ret}U, \quad (10)$$

with $U_{\rm hom}$ and $S_{\rm hom}$ the solutions to the homogeneous equations

$$\Box U_{\rm hom} = 0, \qquad \Box S_{\rm hom} = 0, \quad (11)$$

and $\Box^{-1}_{\rm ret}$ the inverse of the retarded d'Alembertian operator. The equivalent local form of the theory then depends on the choice of $U_{\rm hom}$ and $S_{\rm hom}$. The *ad hoc* choice of a retarded Green function in the definition of inverse d'Alembertian operator $\Box^{-1}$ will ensure causality (for details see e.g. Ref. [49]). Note, however, that it has been argued that causality can emerge automatically if one considers only in-in (observable) vacuum expectation values [40,50,51].

Let us now turn to our studies of the cosmology of the model. We will assume a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2, \quad (12)$$

wtih t the cosmic time and a the scale factor.

Solving the field equations for this metric yields the evolution equations (equivalent to the Friedmann equation) [30]

$$h^2 = \frac{\Omega^0_{\rm M}e^{-3N} + \Omega^0_{\rm R}e^{-4N} + (\gamma/4)U^2}{1 + \gamma(-3V - 3V' + (1/2)V'U')}, \quad (13)$$

$$U'' = 6(2+\xi) - (3+\xi)U', \quad (14)$$

$$V'' = h^{-2}U - (3+\xi)V', \quad (15)$$

in terms of the auxiliary fields U and $V \equiv H_0^2 S$, and their derivatives, with H_0 the present Hubble rate. Additionally, we have assumed the Universe to be filled with matter and radiation, with present density parameters $\Omega^0_{\rm M}$ and $\Omega^0_{\rm R}$, respectively, and have defined the quantities

$$\gamma \equiv \frac{m^2}{9H_0^2}, \qquad h \equiv \frac{H}{H_0}, \qquad \xi \equiv \frac{h'}{h}, \quad (16)$$

where a prime denotes a derivative with respect to the number of e-foldings $N \equiv \ln a$.

The evolution of the total energy density can be parametrized in terms of an effective equation of state [1]

$$w_{\rm eff} = -1 - \frac{2}{3}\frac{h'}{h} = -1 - \frac{2}{3}\xi. \quad (17)$$

The evolutions of the matter, radiation and dark energy components contributing to $w_{\rm eff}$ follow from the conservation of the energy-momentum tensor,

$$\Omega'_{\rm M} + (3+2\xi)\Omega_{\rm M} = 0,$$
$$\Omega'_{\rm R} + (4+2\xi)\Omega_{\rm R} = 0,$$
$$\Omega'_{\rm DE} + (3 + 3w_{\rm DE} + 2\xi)\Omega_{\rm DE} = 0, \quad (18)$$

with

$$\xi = \frac{-4\Omega_{\rm R} - 3\Omega_{\rm M} + 3\gamma(h^{-2}U + U'V' - 4V')}{2(1-3\gamma V)}. \quad (19)$$

Combining the conservation equations (18) and taking into account the cosmic sum rule,

$$\Omega_{\rm DE} = 1 - h^{-2}(\Omega^0_{\rm M}e^{-3N} + \Omega^0_{\rm R}e^{-4N}) = \gamma\left(\frac{1}{4}h^{-2}U^2 + 3V + 3V' - \frac{1}{2}V'U'\right), \quad (20)$$

we obtain the dark energy equation of state,

$$w_{\rm DE} = \frac{\gamma(4(U+3) - (U+2)U')V' + U(\gamma V(\Omega_{\rm R}+3) - \Omega_{\rm DE}) + 4(3\gamma V - \Omega_{\rm DE})}{U(1-3\gamma V)\Omega_{\rm DE}}. \tag{21}$$

III. PHASE SPACE AND DYNAMICAL ANALYSIS

In order to perform the dynamical analysis of the model, it is convenient to rewrite the second order differential equations (14) and (15) in a first order form. To do this, we introduce two new fields Y_1 and Y_2 defined as $Y_1 \equiv U'$ and $Y_2 \equiv V'$. We can now rewrite the system as a set of six autonomous first order differential equations,

$$U' = Y_1, \tag{22}$$

$$V' = Y_2, \tag{23}$$

$$Y_1' = -\frac{3((U+2)Y_1 - 4(U+3))(2(3\gamma V-1) - \gamma(Y_1-6)Y_2) + 3(U+4)(Y_1-6)\Omega_{\rm M} + 4(U+3)(Y_1-6)\Omega_{\rm R}}{2U(3\gamma V-1)}, \tag{24}$$

$$Y_2' = -\frac{Y_2(6(3\gamma V-1) - 3\gamma(Y_1-4)Y_2 + 3\Omega_{\rm M} + 4\Omega_{\rm R})}{2(3\gamma V-1)} - \frac{(2(3\gamma V-1) + 3\gamma Y_2)(2(3\gamma V-1) - \gamma(Y_1-6)Y_2 + 2(\Omega_{\rm M}+\Omega_{\rm R}))}{\gamma U(3\gamma V-1)}, \tag{25}$$

$$\Omega_{\rm M}' = -\frac{\Omega_{\rm M}(U(3(3\gamma V-1) - 3\gamma(Y_1-4)Y_2 + 3\Omega_{\rm M} + 4\Omega_{\rm R}) + 12(3\gamma V-1) + 12(\Omega_{\rm M}+\Omega_{\rm R}) - 6\gamma(Y_1-6)Y_2)}{U(3\gamma V-1)}, \tag{26}$$

$$\Omega_{\rm R}' = -\frac{\Omega_{\rm R}(U(4(3\gamma V-1) - 3\gamma(Y_1-4)Y_2 + 3\Omega_{\rm M} + 4\Omega_{\rm R}) + 12(3\gamma V-1) + 12(\Omega_{\rm M}+\Omega_{\rm R}) - 6\gamma(Y_1-6)Y_2)}{U(3\gamma V-1)}. \tag{27}$$

A quick look at Eqs. (22)–(27) reveals that they are not invariant under $U \to U + U_{\rm hom}$ and $\rho \to \rho + \Lambda$, where ρ is the energy density of the system. Contrary to the nonlocal models considered in Ref. [40], nonzero and constant values of $U_{\rm hom}$ *are not* equivalent to a cosmological constant. The main purpose of this work is a complete characterization of the system (22)–(27) for arbitrary values of $U_{\rm hom}$ and $V_{\rm hom}$. As argued in Ref. [30], each choice of $U_{\rm hom}$ and $V_{\rm hom}$ in Eq. (11) (note that S and V are the same up to a constant factor) corresponds to the choice of one and only one boundary condition in the nonlocal formulation of the theory. Different initial conditions, and therefore different solutions, should be associated with different nonlocal models. The qualitative analysis of Eqs. (13)–(15) will allow us to understand which of these models are phenomenologically viable.

A. Critical points and evolution paths: Numerical analysis

The fixed points of the dynamical system (22)–(27) are those at which all the first derivatives on the left-hand side of the equations vanish. In some cases though, one can have fixed surfaces instead of fixed points, that is, only a subset of variables is constant. In order to go from the fixed surfaces to fixed points (in a lower dimensional phase space) one has to perform an appropriate variable transformation (cf. Appendix A for details regarding the

TABLE I. Critical points of the dynamical system (22)–(27). The quantities $\tilde{\Omega}_{\rm M}$, $\tilde{\Omega}_{\rm R}$, and $\tilde{U}$ stand, respectively, for some constant values of $\Omega_{\rm M}$, $\Omega_{\rm R}$ and U.

Point	U	V	U'	V'	$\Omega_{\rm M}$	$\Omega_{\rm R}$	$w_{\rm eff}$	Type
I	$\tilde{U}$	$(1-\tilde{\Omega}_{\rm R})/(3\gamma)$	0	0	0	$\tilde{\Omega}_{\rm R}$	$1/3$	Saddle
II	$2N+\tilde{U}$	$(1-\tilde{\Omega}_{\rm M})/(3\gamma)$	2	0	$\tilde{\Omega}_{\rm M}$	0	0	Saddle
III	$+\infty$	$1/(3\gamma)$	4	0	0	0	-1	Attractor
IV	$4N+\tilde{U}$	$\pm\infty$	4	$\pm\infty$	0	0	-1	Saddle
V	-3	$\pm\infty$	0	$4V$	$\mp\infty$	0	$1/3$	Attractor

treatment of fixed lines). By following this procedure, we obtain the five nontrivial fixed points/surfaces I–V listed in Table I. The values of the dynamical variables of the system (i.e. the quantities U, V, U', V', $\Omega_{\rm M}$, and $\Omega_{\rm R}$) are given for each point, as well as the value of the effective equation of state parameter $w_{\rm eff}$. As reflected in the table, we find two attractors and three saddle points.[4]

The behavior of the solutions around each of the critical points can be determined by using the standard phase-space analysis methods. Although the set of equations (22)–(27) is nonlinear, the system behaves linearly in the vicinity of each critical point, provided that the point is isolated and the Jacobian at the point is invertible.[5] The linearization of Eqs. (22)–(27) in the vicinity of each fixed point gives rise to a set of linear equations, which can be generically written in a matrix form $\mathbf{X}' = \mathbf{A}\cdot\mathbf{X}$, with $\mathbf{A}$ a 6×6 matrix and $\mathbf{X} = \{U, V, U', V', \Omega_{\rm M}, \Omega_{\rm R}\}$. The behavior of the system around each critical point is determined by the eigenvalues of the corresponding $\mathbf{A}$ matrix. The results are summarized in the last column of Table I (cf. Appendix A for details).

The precise interpolation of the solutions between the critical points I–V depends on the initial conditions, and in particular, on the relation between the initial value U_0 and a γ-dependent critical value $\bar{U}$ that we obtain numerically for the case $\Omega_{\rm M}^0 = 0.3$,[6]

$$\bar{U}(\gamma) \simeq -14.82 + 0.67\log\gamma, \tag{28}$$

and that is valid in the range illustrated in Fig. 1. We can distinguish two kinds of trajectory. If U_0 (the initial value of U) is bigger than $\bar{U}$, the system follows the sequence I → II → III. In the opposite case, it follows the I → II → IV → V sequence. We will refer to these two possibilities as path A and path B, respectively (see Fig. 1). The previous work on this model, i.e. Ref. [32], has focused on the particular case of path A, as we discuss in detail below.

1. Path A

The numerical behavior of the dynamical system along path A is shown in Figs. 2 and 3. Note that in Fig. 2 we have fixed $V_0 = 0$. This choice can be made without loss of generality due to the attractor behavior of point III.

As can be clearly seen in Fig. 2, the saddle points I and II correspond to intermediate radiation- and matter-dominated eras. The transition to the attractor point III proceeds through a transient phantom regime with

[4]One should note that for point III, Eqs. (22)–(27) may seem to be singular in the limit $V \to 1/3\gamma$. This is, however, not the case, as in the limit $V' = \Omega_{\rm M} = \Omega_{\rm R} = 0$ and $U' = 4$, the divergent factor is canceled out.

[5]This argumentation holds only if the fixed point of the linearized system is not a center-type point.

[6]We will recover analytically the γ-dependent part of this equation in Sec. III B.

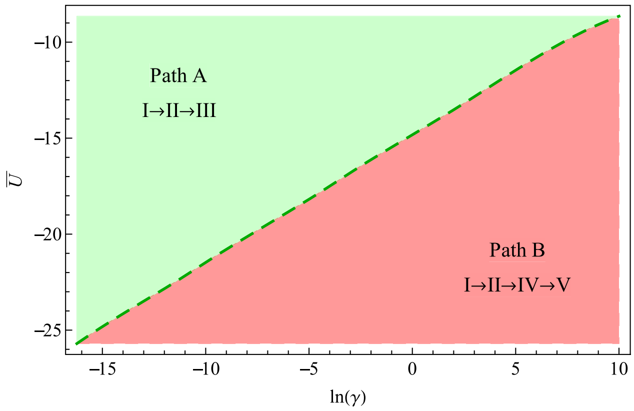

FIG. 1. The two evolution paths A and B for the background cosmology of $R\Box^{-2}R$ gravity, in terms of the initial value U_0 of the auxiliary field U and the value of $\gamma \equiv \frac{m^2}{9H_0^2}$. The diagonal line depicts the critical value $\bar{U}$ as a function of γ. The green (red) region corresponds to the realizations of path A (B).

$w_{\rm eff} < -1$. This kind of behavior was first recognized in Ref. [32] where the authors considered the solution of the dynamical system (13)–(15) for a specific choice of the initial conditions ($U_0 = 0$, $V_0 = 0$) and derived a lower bound for the effective equation of state ($-1.14 \leq w_{\rm eff} < -1$). As shown in Fig. 2, this bound is not robust under variations of the initial conditions. General choices of U_0 can lead to a stronger phantom regime (or even to its complete disappearance, cf. Sec. III A 2). Note also that the particular choice of initial conditions in the MM model rests on the assumption of a vanishing Ricci scalar prior to matter-radiation equality, or in others words, on the existence of a perfect radiation-dominated era. However, the accuracy and redshifts for which this assumption holds depend on the thermal history of the Universe. As shown in Appendix B, if the initial MM conditions were set for instance at the end of inflation/reheating, one should expect nonvanishing values of U_0 at the number of e-foldings at which the MM initial conditions are usually implemented ($N \simeq -14$) [35].

In spite of the asymptotic approach of the effective equation of state to $w_{\rm eff} = -1$, the attractor point III *should not* be identified, *sensu stricto*, with a de Sitter point. For a solution to be de Sitter, the Hubble parameter around this solution should remain constant (or, more generally, the Ricci scalar R should be constant). This is certainly not the case here. Indeed, the Hubble rate $H(N)$ becomes infinitely large when $N \to \infty$. This unusual behavior can be easily understood by considering the consistency of Eqs. (14), (15), and (17) at the fixed point III.[7]

[7]In order for Eq. (15) to be consistent at III, we must require $H \to \infty$ faster than U.

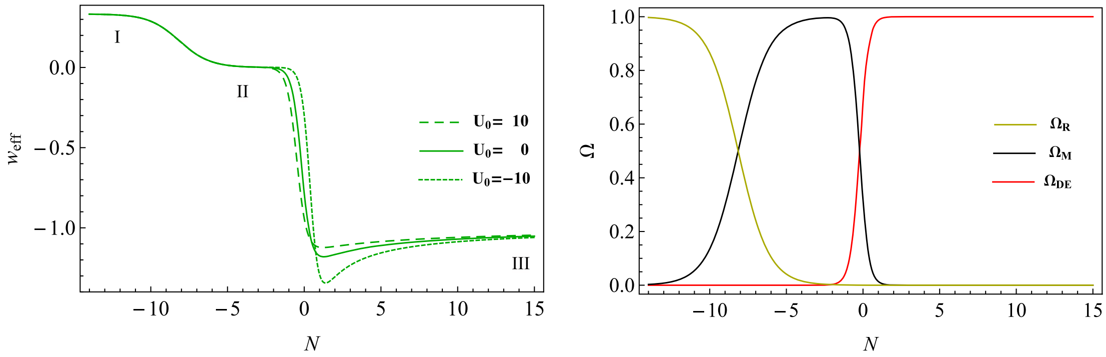

FIG. 2. (Left) Evolution of the effective equation of state $w_{\rm eff}$ as a function of $N \equiv \ln a$ for path A. (Right) Evolution of the density parameters $\Omega_{\rm M}, \Omega_{\rm R}$, and $\Omega_{\rm DE}$ for the same path with $U_0 = 0$. In both plots we have fixed $V_0 = 0$.

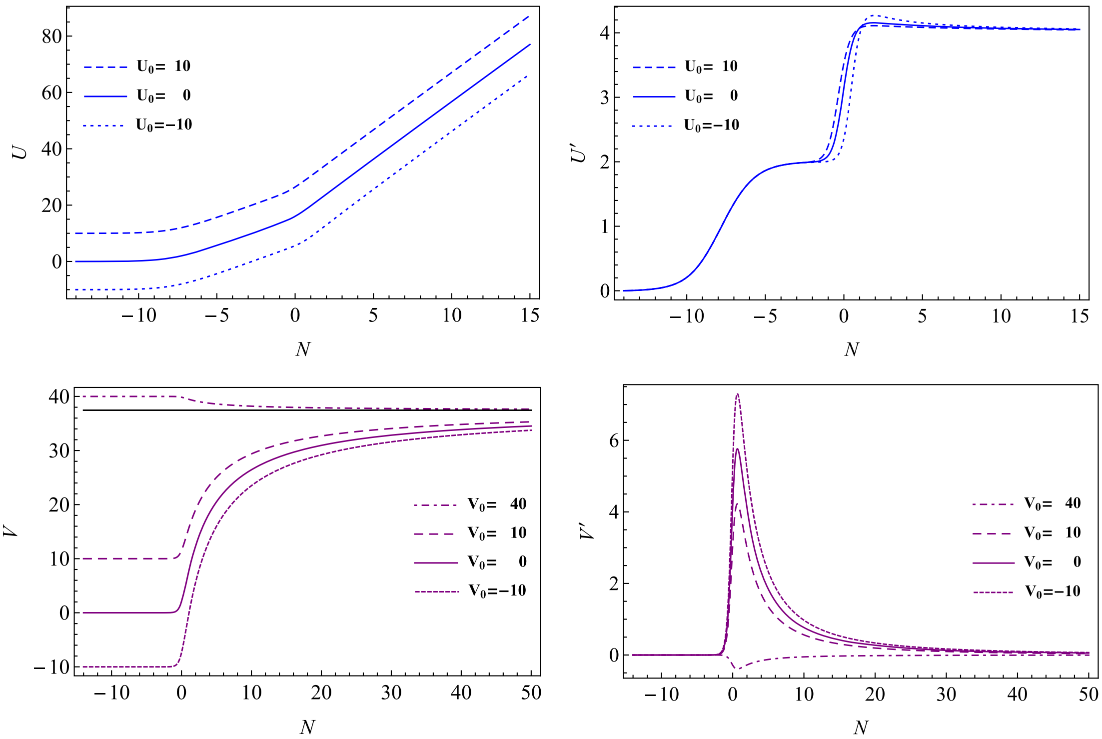

FIG. 3. Evolution of the auxiliary fields U and V, and their derivatives with respect to $N \equiv \ln a$, U' and V', for path A. In the plots of U and U' we have fixed $V_0 = 0$, and in the plots of V and V' we have fixed $U_0 = 0$.

2. Path B

The numerical evolution of the dynamical system along path B is shown in Figs. 4 and 5. Note that in Fig. 4 we have fixed $V_0 = 0$. This can be done without loss of generality, provided that $V_0 < 1/(3\gamma) - V_0{}'$ (see the discussion below).

The initial behavior of the system coincides with that in path A. In particular, the Universe undergoes radiation- and matter-dominated eras while passing through the saddle points I and II. The differences appear only when the system approaches the fixed point IV. As shown on the left-hand side of Fig. 4, this point gives rise to a *true* de Sitter epoch with $w_{\rm eff} \simeq -1$ and $H(N) \simeq$ constant. Note, however, that this point is not an attractor but rather a saddle point. This means that the solution stays close to the point for some period of time but eventually moves to the final attractor, the fixed point V. In particular, the late-time evolution depends on the value of V_0 and V_0'. As discussed in Appendix A, if $V_0 > 1/(3\gamma) - V_0'$ then the system approaches the fixed point V with $\Omega_{\rm M} \to -\infty$. Since $\Omega_{\rm M}$ takes negative values when $V_0 > 1/(3\gamma) - V_0'$, this set of initial conditions should be discarded on general physical grounds. On the contrary, if $V_0 < 1/(3\gamma) - V_0'$ we can obtain a physically viable scenario. As shown on the right-hand side of Fig. 4, the matter density parameter in this case

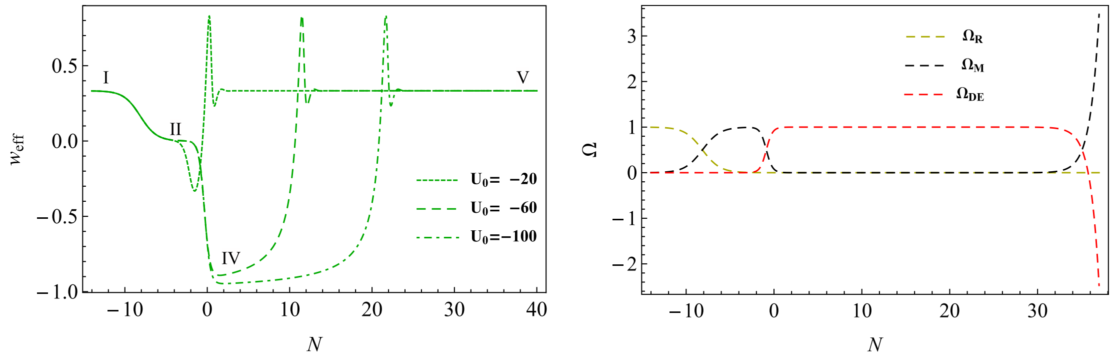

FIG. 4. (Left) Evolution of the effective equation of state $w_{\rm eff}$ as a function of $N \equiv \ln a$ for path B. (Right) Evolution of the density parameters Ω_M, Ω_R, and Ω_{DE} for the same path with $U_0 = -60$. In both plots we have fixed $V_0 = 0$.

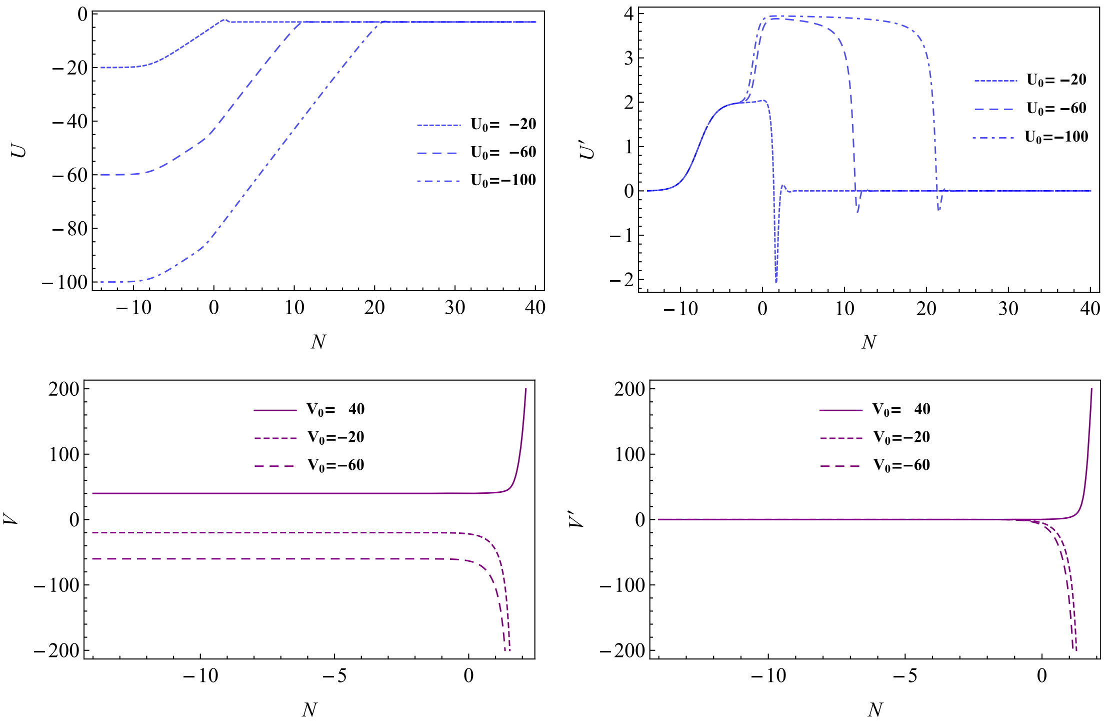

FIG. 5. Evolution of the auxiliary fields U and V, and their derivatives with respect to $N \equiv \ln a$, U' and V', for path B. In the plots of U and U' we have fixed $V_0 = 0$, and in the plots of V and V' we have fixed $U_0 = -20$.

is driven to $+\infty$ while the dark energy one goes to $-\infty$. This limit is acceptable since Ω_{DE} does not represent a proper matter content but rather an effective description of the gravitational degrees of freedom. Note that the effective equation of state at point V approaches the radiation-domination value $w_{\rm eff} = 1/3$, even though there is no radiation left.[8]

The cosmological evolution along path B requires only that $U_0 < \bar{U}$. The value of U_0 is in principle unbounded from below. Could it be possible to obtain a phantom regime similar to that occurring for path A by choosing $U_0 \ll \bar{U}$? The answer to this question turns out to be negative. As shown in Fig. 5, when we increase the absolute value of U_0, the variable U' approaches a maximal value $U'_{\rm max} = 4$, stays there for some time interval $\Delta N^{U'}_{\rm max}(U_0)$, and eventually falls into its future attractor regime $U' = 0$. The maximum value of U' ($U'_{\rm max} = 4$) translates, through Eq. (14), into a value $\xi_{\rm max} = 0$, and as a result, $w_{\rm eff} = -1 - \frac{2}{3}\xi$ cannot be smaller than -1. In other words, path B is never phantom.

B. Evolution paths: Analytical results

The novel ingredient of the local formulation of the $R\Box^{-2}R$ model with respect to general relativity is the

[8]In fact, for all the points III, IV, and V, $\Omega_R \to 0$.

presence of two "integral fields" U and V arising from the nonlocal structure of the theory, cf. Eqs. (9) and (10). In this subsection, we take an in-depth look at the evolution of the $2+2$ homogeneous and $1+1$ inhomogeneous modes and analytically confirm the results obtained in Secs. III A 1 and III A 2.

The basic building blocks of cosmological model construction are solutions with constant effective equation of state $w_{\rm eff}$. Assuming $w_{\rm eff} = w_c$ with w_c a constant, and using Eqs. (9) and (10), we obtain the following equations for the U and V fields:

$$U(N) = u_0 + u_1 e^{-\frac{3}{2}(1-w_c)N} + \frac{2(1-3w_c)}{1-w_c}N, \qquad (29)$$

$$V(N) = \frac{2e^{3(1+w_c)N}}{9(1+w_c)(3+w_c)} \times \left(u_0 - \frac{2(1-3w_c)(5+3w_c)}{3(1-w_c^2)(3+w_c)} - \frac{2(1-3w_c)}{1-w_c}N\right) + \frac{2e^{\frac{3}{2}(1+3w_c)N}}{9(1+w_c)(1+3w_c)}u_1 + v_0 + v_1 e^{-\frac{3}{2}(1-w_c)N}, \qquad (30)$$

with u_0, u_1, v_0, and v_1[9] integration constants set at $N = 0$. These equations reveal that the inhomogeneous modes disappear if and only if $w_c = 1/3$. As a default, $w_c = 1/3$ is the only constant equation of state giving rise to an attractor solution.[10] Note also that for $-3 < w_c < 1$ the fastest-growing exponent in Eq. (30) appears in the first term, which is controlled by u_0 only. Taking this into account, we will mostly focus on variations of u_0 in what follows.

Let us first consider a solution within radiation domination, like that taking place around the fixed point I. The growing modes in this case are given by $U \to u_0$ and $V \to \frac{1}{20}e^{4N}u_0$. If we start with an initial condition $\Omega_R = 1$, the numerator of Eq. (13) tells us that the nonlocal corrections take over at $N_{\rm NL} = -\frac{1}{4}\log(\frac{\gamma u_0}{4})$ e-foldings. Thus, the radiation-dominated Universe in the MM model is stable if and only if we have exactly the minimal boundary condition prescription.

In a realistic cosmology we should also consider a matter-dominated epoch following the radiation-domination era. This matter-dominated era happens around the critical point II. For $w_c = 0$, the inhomogeneous modes in Eqs. (29) and (30) survive and the solution is necessarily unstable. The number of e-foldings $N_{\rm NL}$ at which the nonlocal corrections take over is again dictated by the numerator of Eq. (13). At $N_{\rm NL}$ e-foldings, the sign of the fastest-growing mode is positive if[11]

$$u_0 > -\left(\frac{10}{9} + \frac{4}{3}\log\frac{9}{5}\right) + \frac{2}{3}\log\gamma. \qquad (31)$$

As we will confirm below, one should expect this sign to determine the evolution of the system beyond point II. Note that Eq. (31) can be translated into a bound on the value of $U(N_*)$ at any given number of e-foldings N_* by noticing that

$$u_0 = U(N_*) - 2\frac{(1-3w_c)N_*}{1-w_c} - \frac{4(1-3w_c)}{3(1-w_c)^2}. \qquad (32)$$

In particular, for matter-radiation equality ($N_* = -8.1$), we get $U(N_*) > -15.65 + \frac{2}{3}\log\gamma$. Note that this is close to the numerical value of $\bar{U}$ found in Eq. (28).

The above two cases constitute the only possibilities for realizing a constant equation of state $w_{\rm eff} = w_c$ in a universe with nonvanishing and minimally coupled radiation and dust components. In what follows, we will consider vacuum solutions with $\Omega_R = \Omega_M = 0$. Combining Eqs. (13) and (15) we get

$$UV'' - 2U'V' = -\frac{3}{2}(U - w_c U + 8)V' - 12V + \frac{4}{\gamma}. \qquad (33)$$

As in the nonvacuum case, the attractor solutions can be associated only to an effective equation of state $w_c = 1/3$. The fixed point V falls into this category. Indeed, when we set $u_1 = v_1 = 0$, Eq. (33) reduces to

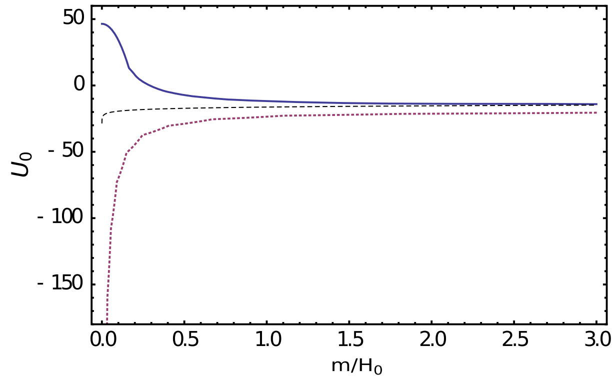

FIG. 6. Value of U_0 needed to reach path A (blue curve) or path B (red dotted curve) when $\Omega_M^0 = 0.3$. For large m, the two curves converge to $\bar{U}$, represented by the intermediate dashed curve, which follows Eq. (28).

[9]The values u_0 and v_0, that are set at $N = 0$ according to the solutions (29) and (30), should not in general be confused with U_0 and V_0, the initial values for U and V set at an early radiation-dominated epoch.

[10]Equations (29) and (30) are exact and model-independent solutions as long as we can assume that $w_{\rm eff}$ is a constant.

[11]This formula is approximate because when $\Omega_M = 1/2$, $w_c = 0$ is not exact.

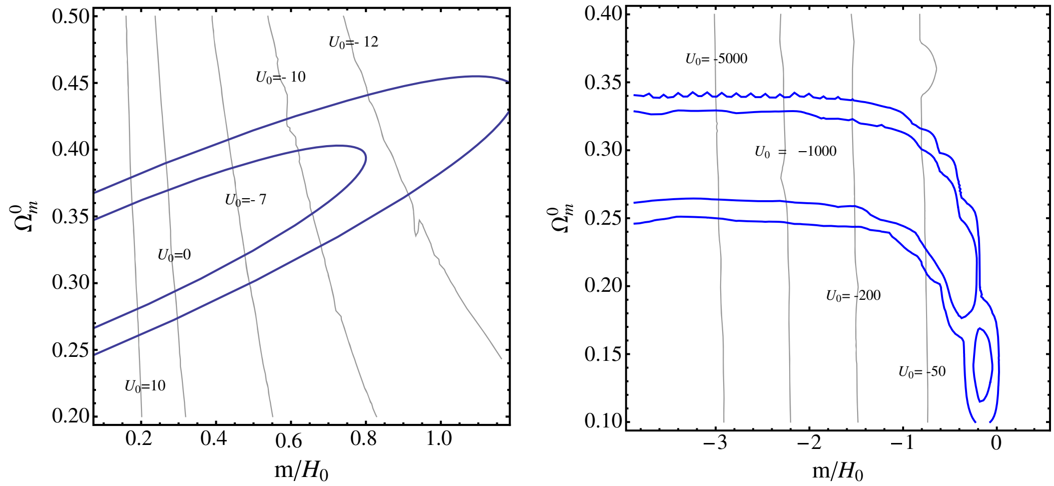

FIG. 7. Supernovae likelihood contours at 2-σ level for path A (left panel) and path B (right panel). The associated values of U_0 are also displayed.

$$e^{4N}(u_0+3) - \frac{\gamma}{4u_0}(1-3\gamma v_0) = 0, \tag{34}$$

which allows for the solutions

$$U(N) = -3, \quad V(N) = \frac{1}{3\gamma} - \frac{3}{20}e^{4N}. \tag{35}$$

Equations (35) are exact solutions of the system of Eqs. (29), (30), and (33). Note, however, that there can still be approximate solutions. Consider temporary regions with $w_c \approx -1$, as those appearing around points III and IV. In these regions, Eqs. (29) and (33) give $U'(N) \to 4$ and $V(N) = 1/(3\gamma)$, but one should be aware that Eq. (30) is only valid for a constant $w_{\rm eff}$. If $U(N_{\rm NL}) < -3$ when this solution is reached at $N_{\rm NL}$, the trajectory will hit the aforementioned attractor with $w_c = 1/3$ and stay there; this is then part of what we called path B, cf. Sec. III A 2. If $U(N_{\rm NL}) > -3$, the evolution will continue in the phase with $w_c \approx -1$ and $U' = 4$; this phase belongs to path A, cf. Sec. III A 1.

To summarize, the post-matter-dominated Universe reaches an accelerating stage with $w_c \approx -1$ which goes on until $U(N) = u_0 + 4N = -3$. If $u_0 > -3$, this period extends forever. Otherwise, one can prolong the transient acceleration for ΔN e-foldings by lowering the initial value of $U_0 \to U_0 - 4\Delta N$.

The results of this subsection are in agreement with what we studied in greater detail in the previous subsections, namely the stability analysis in the phase-space formulation and the numerical integration of the field equations.

IV. CONSTRAINTS FROM SUPERNOVAE DATA

Since both paths A and B realize cosmologies that are in principle viable (i.e., they contain a sequence of proper radiation-, matter-, and dark-energy-dominated eras), we need to compare both to observations. Here, we assume as free parameters, m in units of H_0 and the present matter density parameter $\Omega_{\rm M}^0$, and fix $\Omega_{\rm R}^0 = 4.15 \times 10^{-5}h^{-2}$ and $V_0 = 0$. The initial condition deep in the radiation era, that we choose arbitrarily as $U_0 \equiv U(N=-14)$, is fixed by the requirement that we reach $\Omega_{\rm M}^0$ today. In practice, for every point $\{m, \Omega_{\rm M}^0\}$ in the parameter space, we vary iteratively U_0 until we find $\Omega_{\rm M}^0$ at $N=0$. Since there are two possible paths, we find two values of U_0 for every choice of parameters. The particular choice of U_0 as a function of m when $\Omega_{\rm M}^0 = 0.3$ is presented in Fig. 6. For large m, paths A and B lead to a common behavior, and their initial condition U_0 also converges.

Once the two trajectories are found, we evaluate the Hubble rate $H(z)$ for each path and compare the associated luminosity distance $d_L(H(z))$ to the Joint Light-curve Analysis (JLA) supernovae data set [52] in order to obtain two independent likelihoods over m and $\Omega_{\rm M}^0$, one for each path. When the pair $\{m, \Omega_{\rm M}^0\}$ is specified, the effective equation of state $w_{\rm eff}$ is completely determined. The results are shown in Figs. 7 and 8. Focusing on $\Omega_M^0 \approx 0.3$, one sees that all the values of m up to 0.5 are roughly compatible with supernovae. Note, however, that the expectation value $\Omega_{\rm M}^0 \approx 0.3$ comes from standard cosmology and it should not be directly applied to modified gravity cases. In fact, the supernovae data set is roughly compatible with all values of $\Omega_{\rm M}^0 < 0.45$, so a more robust upper limit for m is around 1.2. For very small m, the trajectories of both path A and path B become observationally indistinguishable from ΛCDM.[12] Note, however, that this may change in the

[12]Indeed, when m is small, the dynamical part associated with nonlocal contributions in Eq. (13) is suppressed. The leading contribution at early times is of order $mU_{\rm hom}$, which is a constant in our case. Note that this is in agreement with the curve corresponding to path B in Fig. 6.

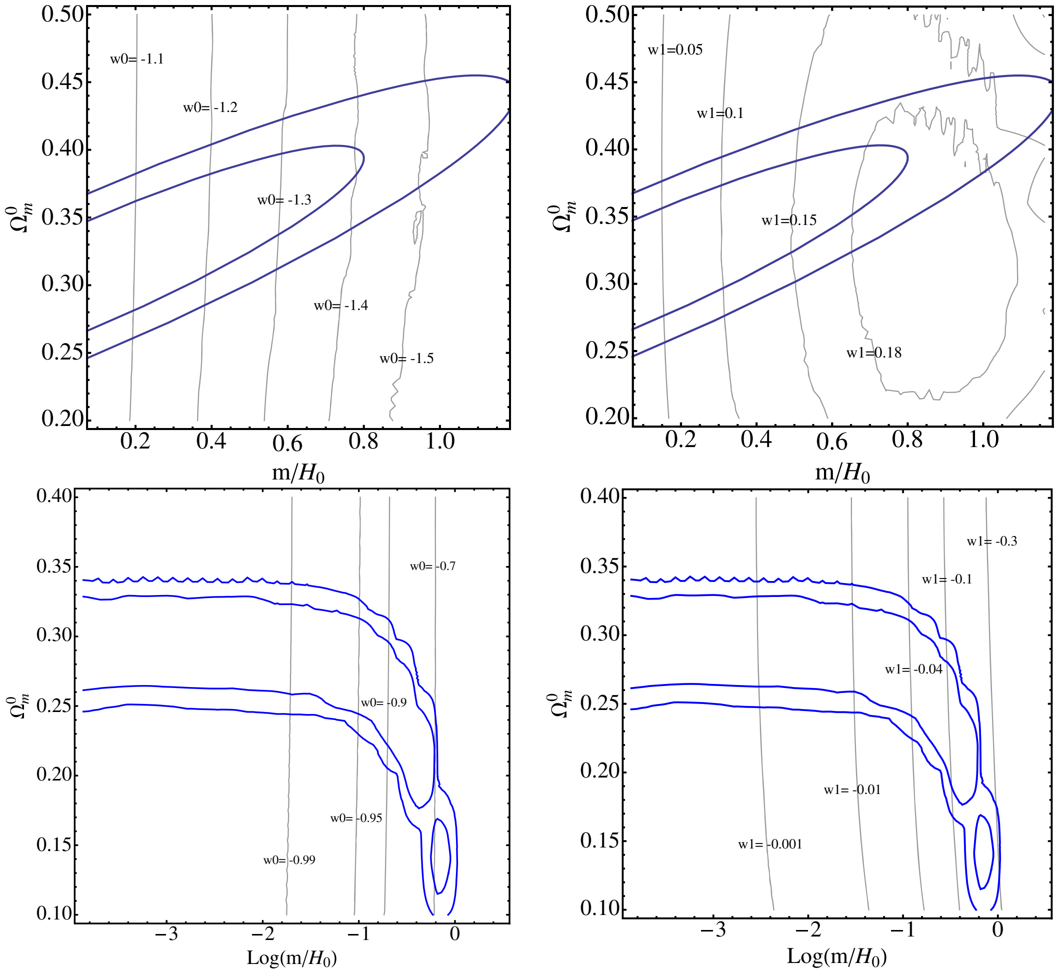

FIG. 8. Supernovae likelihood contours at 2-σ level for path A (upper panels) and path B (lower panels). The associated values of w_0 and w_1 in the standard parametrization $w_{\rm eff} = w_0 + (1 - a)w_1$ are also displayed.

future when the dynamical part associated with nonlocal contributions in Eq. (13) becomes dominant again.

V. SUMMARY AND CONCLUSIONS

Nonlocality can emerge from local theories. If one focuses on classical physics at long wavelengths, there can appear nonlocal constraints due to the effect of short wavelengths that have been integrated out. In quantum field theories, nonlocality is introduced in the computation of the effective action via the integration of the radiative corrections due to massless or light particles.

It is thus natural to consider that nonlocal, infrared modifications of gravity at cosmological scales, such as in the model described by (2), could provide a useful effective approach to study the problems of cosmological constant and of dark energy. An important subtlety, not arising in local modifications of gravity, has to then be taken into account in such studies: in addition to the mass parameter in the Lagrangian (2), the nonlocal model is understood to be specified by the boundary conditions implied by the presence of the inverse-d'Alembertian.

In this work, we have considered the effect of general initial conditions on the dynamical system (22)–(27) for the background evolution of the $R\Box^{-2}R$ cosmological model, as well as the constraints from supernovae data on the parameters m and $\Omega_{\rm M}^0$. The system exhibits two distinct classes of late-time behavior, which lead to two different types of cosmological evolution, dubbed path A and path B.

Path A (path B) is realized above (below) a certain threshold $\bar{U}$ for the initial condition of the auxiliary field U, U_0. The case $U_0 = 0$, belonging to path A, is the one already discussed in Ref. [32]. Note, however, that the initial conditions in this theory are sensitive to the thermal history of the Universe. In particular, the value of U at a given number of e-foldings cannot be unambiguously set to zero by setting $U_{\rm hom} = 0$. The main result of this work is to extend the cosmological analysis to the full range of initial conditions.

We found that although both paths possess well-behaved radiation and matter eras, the subsequent evolution is in general radically different. Along path A, the system goes through a phantom regime and finally reaches an attractor on which the effective equation of state $w_{\rm eff}$ remains frozen at the CC value -1. This final state, however, is not a de Sitter stage since H (and therefore the Ricci scalar R) is not constant, but rather grows indefinitely. Along path B, instead, the evolution remains always nonphantom; the system reaches generically a $w_{\rm eff} = -1$ stage which approaches a true de Sitter stage. This is however a temporary stage in cosmic evolution, as the solution is not an attractor but, rather, a saddle point. After a transient period the system reaches a final configuration represented by a decelerated, radiationlike, $w_{\rm eff} = 1/3$ state (and therefore with $R = 0$), in which, however, no radiation is present. The present value of the nonlocal-term equation of state, $w_{\rm DE}$, can take essentially any values around -1. The impact of initial conditions on the final evolution is therefore important.

Both paths are in principle cosmologically viable. When compared to supernovae observations, we find the regions in the $\{m, \Omega_{\rm M}\}$ parameter space that satisfy observational constraints. It is interesting to note that small, even vanishing, values of m are perfectly acceptable. This means that cosmologically viable nonlocal terms can be generated from standard loop corrections, which require $m/H_0 \ll 1$ (in Planck units). However, in this case the evolution becomes indistinguishable from ΛCDM.

The methods employed in this paper can be applied to more general nonlocal models. We saw that the possible cosmological dynamics of a given model can be conveniently derived from the behavior of the additional scalar modes carried by the nonlocal integral operators. Assuming a background evolution with a power-law expansion of the scale factor, we solved for the mode functions in the model (2), and from the general solutions (29) and (30) deduced the handful of fixed points and their basic properties. A similar analysis should be even more transparent for actions of the type (1), since there the nontrivial modes are given solely by Eq. (29). In models featuring the conformal Weyl curvature [18,28,29], a substantial simplification is that only the homogeneous modes are nonvanishing.

In conclusion, the cosmological background dynamics of the nonlocal model studied here depend qualitatively upon the initial conditions. We discussed the initial conditions in terms of U_0 set at an early radiation-dominated epoch and showed that different values for this parameter can result in the Universe ending up eventually in drastically different stages: in the two most typical cases studied here, either a phantomlike approach towards asymptotic singularity, or an eternal conformal expansion. The initial value $U_0 = 0$ is a natural choice, but its implementation in fact depends on the thermal history of the Universe. As shown in Appendix B, however, the ambiguity is not large enough to change the evolution from the former track to the latter.

We can thus regard the interesting dark energy behavior as a robust background prediction of the model (2). It remains to be seen whether one can learn further about the structure formation in nonlocal cosmology by revisiting the boundary conditions of the integral operators at the level of the inhomogeneous perturbation modes.

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APPENDIX A: SOME CLARIFICATION ON FIXED POINTS AND PATHS

1. From fixed surfaces to fixed points

As explained in Sec. III A, when the first derivative of a variable on the left-hand side of Eqs. (22)–(27) takes a constant value, the dynamical system contains a fixed surface rather than a fixed point. To illustrate how to deal with this situation, we present below the complete phase-space analysis for the critical point II in Table I. A similar analysis can be done for points III and IV.

As follows directly from Eq. (22), when $Y_1 = 2$ the variable U satisfies the equation of a line, $U' = 2$. In order to go from this fixed line to a fixed point we can consider a field redefinition,

$$U = \tilde{U} + 2N, \tag{A1}$$

with N the number of e-foldings. Inserting this relation into Eqs. (22)–(27), one immediately realizes that $Y_1 = 2$ corresponds to a fixed point $\tilde{U}' = 0$. Written in the new variables, the analysis proceeds along the lines discussed in Sec. III A. The behavior of the system around the fixed point is determined by the eigenvalues of the characterizing matrix, which are given by

$$\lambda_i = \left\{0, 0, -1, -\frac{3}{2}, -\frac{3}{2}, \frac{3\tilde{U}_0 + 6N + 4}{\tilde{U}_0 + 2N}\right\}, \tag{A2}$$

with $\tilde{U}_0$ an arbitrary constant. From the theory of dynamical systems we know that the so-called Lyapunov coefficients s_i ($i = 1, 2, \ldots, 6$) are equal to the real part of the eigenvalues λ_i, provided that these eigenvalues are constant. Note however that, due to the field redefinition (A1), the last eigenvalue in Eq. (A2) depends on the number of e-foldings N. The Lyapunov coefficient in this case is defined by the upper limit

$$s_6 = \lim_{N\to\infty} \frac{1}{N-N_0}\int_{N_0}^{N} \mathrm{Re}\{\lambda_6(N')\}\mathrm{d}N', \tag{A3}$$

with N_0 some initial value for N. Taking into account (A2), we get

$$s_6 = \lim_{N\to\infty} \frac{1}{N-N_0}\int_{N_0}^{N} \frac{3\tilde{U}_0 + 6N' + 4}{\tilde{U}_0 + 2N'}\mathrm{d}N' = 3. \tag{A4}$$

The resulting spectrum of Lyapunov coefficients

$$s_i = \left\{0, 0, -1, -\frac{3}{2}, -\frac{3}{2}, 3\right\} \tag{A5}$$

shows that the fixed point under consideration is a saddle point.

2. On the two realizations of the fixed point V

Note that point V in Table I has two different realizations. The first one is obtained for $V = +\infty$ and $\Omega_M = -\infty$, while the second case corresponds to $V = -\infty$ and $\Omega_M = +\infty$. In this Appendix, we discuss the set of initial conditions giving rise to each one of these configurations.

Around the fixed point V, we have $U' = U'' = 0$. These two conditions restrict the ξ parameter in Eq. (14) to a fixed value $\xi = -2$. Inserting this constant value into Eq. (15) and taking into account the large N limit of Eq. (13), we get

$$V'' - 3V' - 4V + \frac{4}{3\gamma} = 0. \tag{A6}$$

For $N \gg 1$, the solution of this differential equation reads

$$V \approx \frac{1}{3\gamma} + \left(V_0 - \frac{1}{3\gamma} + {V_0}'\right)e^{4(N-N_0)}, \tag{A7}$$

with V_0 and V_0' the values of V and V' at some initial time N_0. In view of this solution, we can distinguish two possibilities. If $V_0 > 1/(3\gamma) - {V_0}'$, the system approaches the fixed point V with $V \to +\infty$.[13] In the opposite case, the fixed point V is realized with $V \to -\infty$.

APPENDIX B: INITIAL CONDITIONS AND THERMAL HISTORY OF THE UNIVERSE

In this Appendix we estimate the robustness of the MM initial conditions when the detailed particle content of the Universe prior to matter-radiation equality is taken into account.

For some purposes, the transition from radiation to matter domination can be approximated by an instant transition at $N_{\mathrm{eq}} \simeq -8.1$ in which the trace of the energy-momentum tensor changes abruptly from zero to $e^{3N_{\mathrm{eq}}}$ times its present value. This approximation implicitly assumes that all the particles in the early Universe have roughly the same mass, and transit simultaneously from a relativistic to a nonrelativistic state. A detailed analysis of the thermal history of the Universe allows us to go beyond this approximation and to account for the fact that particles with different masses become nonrelativistic at different temperatures, or equivalently, at different cosmic times.

The change in the trace of the energy-momentum tensor can be parametrized as

$$\mathrm{tr}(T_{\mu\nu}) = \frac{\rho - 3P}{M_{\mathrm{Pl}}^2} \equiv \frac{\rho_{\mathrm{R}}}{M_{\mathrm{Pl}}^2}\Sigma(T), \tag{B1}$$

with T and ρ_{R} the temperature and energy density of the radiation bath.[14] The so-called "kick" function $\Sigma(\mathrm{T})$ is computed by summing the individual contributions to the bath of the particles with mass m_i, temperature T_i and g_i degrees of freedom [53],[15]

$$\Sigma_i(T) = \frac{\rho_i - 3p_i}{\rho_i} = \frac{15}{\pi^4}\frac{g_i}{g_*(T)}\left(\frac{m_i}{T}\right)^2 \int_{m_i/T}^{\infty} \frac{\sqrt{u^2 - (m_i/T)^2}}{e^u \pm 1}\mathrm{d}u, \tag{B2}$$

where $g_*(T) \equiv \rho_{\mathrm{R}}[(\pi^2/30)T^4]^{-1}$ is the total number of relativistic degrees of freedom in the bath.

Equation (B1) translates, via Einstein equations, into a change of the Ricci scalar, $R = \mathrm{tr}(T_{\mu\nu})$. Integrating Eq. (9) with $a(t) \simeq t^{1/2}$ and taking into account the relation between time and temperature at radiation domination,

$$t \simeq \sqrt{\frac{45}{2\pi^2}} g_*^{-1/2}\frac{M_{\mathrm{Pl}}}{T^2}, \tag{B3}$$

we get

$$\Delta U \equiv U - U_{\mathrm{hom}} = \frac{45}{2\pi^2}M_{\mathrm{Pl}}^2 \int_{T_{\mathrm{i}}}^{T_{\mathrm{f}}} \mathrm{d}T'\mathcal{D}(T')g_*^{1/4}(T')T' \times \int_{T_{\mathrm{i}}}^{T'} \mathrm{d}T''\mathcal{D}(T'')\frac{R(T'')}{(T'')^5 g_*^{5/4}(T')}, \tag{B4}$$

with

[13] Note that realizations with $V_0 > 1/(3\gamma) - {V_0}'$ are not physically acceptable. As can be easily deduced from Eq. (13), these configurations give rise to negative values of h^2 around the fixed point V.

[14] Note that if all the particles prior to recombination were completely massless, $\Sigma(T)$ would be zero.

[15] The + and − signs in the denominator of the integrand apply, respectively, to fermions and bosons.

$$\mathcal{D}(T) \equiv \left(-\frac{2}{T} - \frac{1}{2g_*(T)}\frac{\partial g_*(T)}{\partial T}\right). \tag{B5}$$

Here, T_i and T_f are the higher and lower temperatures for which radiation domination is a reasonable first order approximation for the background evolution of the Universe. Note that, due to the integration between T_i and T_f, even a tiny value of the scalar curvature at early times ($T_\mathrm{f} \ll T_\mathrm{i}$) can give rise to a sizable modification of U at matter-radiation equality.[16]

The radiation-domination requirement ($R = 0$) giving rise to the MM initial conditions should be understood only as an approximation of the actual dynamics. The value of U at radiation domination cannot be unambiguously set to zero by simply setting $U_\mathrm{hom} = 0$. Indeed, if the initial conditions are set at the end of inflation/reheating, one should expect nonvanishing values of U_0 at the number of e-foldings at which the MM initial conditions are usually implemented ($N \simeq -14$). Note also that, even if the MM initial conditions are taken for granted, the detailed thermal history of the Universe will inevitably affect the subsequent evolution of U. The uncertainty associated to this effect depends on the particle content of the early Universe. Assuming radiation domination between 1000 GeV and 0.75 eV and considering only the contribution of Standard Model particles, we can numerically integrate Eq. (B4) to obtain a correction

$$\Delta U \approx 1.6, \tag{B6}$$

to be added on top of the nonvanishing MM value at matter-radiation equality [35]. The assumptions leading to the uncertainty (B6) are indeed quite conservative. Larger values of ΔU should be expected if we accept the existence of new physics beyond the Standard Model.

[16]For $T_\mathrm{i} \gg T_\mathrm{f}$ and constant values of g_* and Σ we have

$$\Delta U \sim \Sigma \int_{T_\mathrm{i}}^{T_\mathrm{f}} \mathrm{d}T' \int_{T_\mathrm{i}}^{T'} \mathrm{d}T'' \frac{1}{(T'')^2} \sim \Sigma\left[\ln\frac{T_\mathrm{i}}{T_\mathrm{f}} - \frac{T_\mathrm{i} - T_\mathrm{f}}{T_\mathrm{i}}\right] \sim \Sigma \ln\frac{T_\mathrm{i}}{T_\mathrm{f}}.$$

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Chapter 5

Publication 2

Structure formation in the Deser-Woodard nonlocal gravity model

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Structure formation in the Deser-Woodard nonlocal gravity model: a reappraisal

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Abstract. In this work, we extend previous analyses of the structure formation in the $f(\Box^{-1}R)$ model of nonlocal gravity proposed by Deser and Woodard (DW), which reproduces the background expansion of ΛCDM with no need of a cosmological constant nor of any dimensional constant beside Newton's one. A previous analysis based on redshift-space distortions (RSD) data concluded that the model was ruled out. In this work we revisit the issue and find that, when recast in a localized model, the DW model is not ruled out and actually gives a better fit to RSD data than ΛCDM. In fact, the DW model presents a suppressed growth of matter perturbations with respect to ΛCDM and a slightly lower value of σ_8, as favored by observations. We also produce analytical approximations of the two modified gravity functions, i.e. the anisotropic stress η and the relative change of Newton's constant Y, and of $f\sigma_8(z)$ as a function of redshift. Finally, we also show how much the fit depends on initial conditions when these are generalized with respect to a standard matter-dominated era.

Keywords: cosmological parameters from LSS, modified gravity

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1 Introduction

The late-time accelerated expansion of the universe [1–8] is attributed in the standard cosmological model, or ΛCDM, to the influence of dark energy in the form of a cosmological constant Λ, interpreted as the energy density of the vacuum. However, this otherwise formally and observationally consistent model carries two unsolved puzzles: the so-called coincidence and the fine-tuning problems. The former issue refers to ΛCDM not explaining the fact that the accelerated phase in the expansion began only recently in the cosmological time, while the latter expresses the enormous disagreement between the energy scale introduced by Λ and the predictions of the Standard Model of particle physics for the vacuum energy density. Consequently, a wealth of alternative, more complicated cosmological models are continuously developed and proposed with the purpose of providing a more accurate and robust description of our universe, the majority of which may be classified as dark energy (if they introduce new matter content) or modified gravity (if they depart from Einstein's general relativity) models (although of course from a purely gravitational point of view there is no fundamental distinction between these two classes). Typically, these new models are required to emulate the background expansion history of the universe given by ΛCDM, which provides a good fit to data. The imposition of this condition is called the reconstruction problem. Once this step is fulfilled, one can observationally distinguish among models by looking at their predictions beyond the background, such as solar system tests and the structure formation in the universe.

Within the class of modified gravity models, nonlocal gravity theories have recently gained remarkable interest. In this direction, pioneering works are [9, 10], where the authors attempt to construct a viable alternative to the standard ΛCDM cosmology through nonlocal modifications of the form $f(\Box^{-1}R)$. This model, at the background level, has the advantage, over ΛCDM, that it exactly reproduces the same evolution without introducing a new energy scale. The price to pay is the loss of structural simplicity. Indeed, in order to exactly duplicate the ΛCDM behavior, the function $f(\Box^{-1}R)$ must be of a somewhat contrived form [10]. On

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a phenomenological basis, the DW nonlocal gravity model has been shown to be ghost-free[1] and close to GR in gravitationally-bound systems [13]. The behavior of the model at the perturbation level was studied in [11] and in [14, 15]. The authors of the last two papers found that, according to the redshift-space distortions (RSD) observations available at the time, the DW model was disfavored over ΛCDM by 7.8σ.

In this work we revisit this problem and show that the localized version of DW model shows a different picture according to which the DW model is not anymore disfavored over ΛCDM and actually gives a significantly better fit to the RSD data. At the same time, the model predicts a slightly lower value of σ_8 than ΛCDM, in agreement with recent lensing results [16, 17]. It is important to remark that once the background is fixed to reproduce ΛCDM, no more free parameters are left to adjust to the RSD data. Our results disagree with those in [14, 15]. Despite intensive testing, we have been unable to identify the reasons for this discrepancy; we discuss some conjectures below.

We also make one step further and relax the model-dependent assumptions implicit in previous works concerning the initial conditions for the perturbation equation in the matter era. More precisely, we allow the two initial conditions for the linear growth equation to vary (as opposed to fixing them to their standard CDM values). As we will show, however, this improves the fit only marginally.

Throughout the paper, we work in flat space and natural units, i.e. units such that $c = \hbar = 1$.

2 The model

In ref. [9] the authors proposed a model in which the Einstein-Hilbert action is nonlocally modified as

$$S_{\rm DW} = \frac{1}{16\pi G}\int d^4x\, \sqrt{-g}R\Big[1 + f(\Box^{-1}R)\Big], \tag{2.1}$$

where the nonlocal distortion function f is a free function of the inverse d'Alembertian acting on the Ricci scalar, $\Box^{-1}R$. Since this combination is dimensionless, the Lagrangian does not introduce any new energy scale. Variation of (2.1) with respect to the metric $g_{\mu\nu}$ yields the modified Einstein equations,

$$G_{\mu\nu} + \Delta G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.2}$$

which in a Friedman-Lemaître-Robertson-Walker (FLRW) background

$$ds^2 = -dt^2 + a^2 d\vec{x}^2, \tag{2.3}$$

can be written as

$$\begin{aligned} 3H^2 + \Delta G_{00} &= 8\pi G\rho, \\ -2\dot{H} - 3H^2 + \frac{1}{3a^2}\delta^{ij}\Delta G_{ij} &= 8\pi G p. \end{aligned} \tag{2.4}$$

[1]The localized version of the DW model has been shown [11, 12] to be ghost-free only when the function $f(\Box^{-1}R)$ satisfies particular ghost-freeness conditions.

Here, the tensor $\Delta G_{\mu\nu}$ corresponds to the nonlocal contribution and is given, for the FLRW metric, by the following expressions [9],

$$\Delta G_{00} = \Big[3H^2 + 3H\partial_t\Big]\Big\{f\Big(\Box^{-1}R\Big)+\frac{1}{\Box}\Big[Rf,\Big(\Box^{-1}R\Big)\Big]\Big\} + \frac{1}{2}\partial_t\Big(\Box^{-1}R\Big)\partial_t\Big(\frac{1}{\Box}\Big[Rf,\Big(\Box^{-1}R\Big)\Big]\Big), \tag{2.5}$$

$$\Delta G_{ij} = a^2\delta_{ij}\Big[\frac{1}{2}\partial_t\Big(\Box^{-1}R\Big)\partial_t\Big(\frac{1}{\Box}\Big[Rf,\Big(\Box^{-1}R\Big)\Big]\Big) - \Big(2\dot{H}+3H^2+2H\partial_t+\partial_t^2\Big)\Big(f+\frac{1}{\Box}\Big[Rf,\Big(\Box^{-1}R\Big)\Big]\Big)\Big], \tag{2.6}$$

where ρ and p are respectively the energy density and pressure of a perfect fluid. From now on, a comma next to f represents a derivative of the function w.r.t. its argument. Equations (2.5)–(2.6) can be localized by introducing the auxiliary variables X and U defined as

$$\Box X \equiv R, \tag{2.7}$$
$$\Box U \equiv f,R \tag{2.8}$$

With the use of the auxiliary functions X and U, eqs. (2.5)–(2.6) can be rewritten as

$$\Delta G_{00} = (3H^2 + 3H\partial_t)(f+U) + \frac{1}{2}\dot{X}\dot{U}, \tag{2.9}$$
$$\Delta G_{ij} = a^2\delta_{ij}\Big[\frac{1}{2}\dot{X}\dot{U} - \Big(2\dot{H}+3H^2+2H\partial_t+\partial_t^2\Big)(f+U)\Big], \tag{2.10}$$

where an overdot stands for a derivative w.r.t. cosmological time.

The DW model has been shown to be capable of reproducing the background evolution given by ΛCDM with $\Omega_M \approx 0.28$ [10] by fixing the nonlocal function to

$$f(X) = 0.245\Big[\tanh(0.350Y + 0.032Y^2 + 0.003Y^3) - 1\Big], \tag{2.11}$$

with $Y \equiv X + 16.5$. This choice fully determines the model and no more free parameters are left.

The dependence of the nonlocal modification on X is suggested by quantum radiative corrections [18] and is triggered mainly at the end of the radiation domination era, where the Ricci scalar in units of H^2 can be taken to be very small, $R/H^2 \approx 0$, with a slow evolution afterwards. The interesting question arises then, whether the DW model that gives the same background evolution as ΛCDM, produces also the same behavior at perturbation level. The answer is no, and in the following sections we will see why is it so.

It is useful to write down modified Einstein equations (2.2) as well as auxiliary field equations (2.7)–(2.8) through e-folding time $N = \ln a$

$$1 + f + U + f' + U' + \frac{1}{6}X'U' = \Omega_M + \Omega_R \tag{2.12}$$

$$-2\xi-3+\frac{1}{2}X'U'-(2\xi+3)(f+U)-2(f'+U')-f''-U''-\xi(f'+U') = 3w_M\Omega_M+3w_R\Omega_R \tag{2.13}$$

and

$$X'' + (3+\xi)X' = -RH^{-2} = -6\xi - 12 \tag{2.14}$$
$$U'' + (3+\xi)U' = -Rf,H^{-2} = -f,(6\xi + 12) \tag{2.15}$$

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where the prime stands for a derivative w.r.t. N, $\xi \equiv H'/H$ and Ω_M and Ω_R are the matter and radiation fractional densities, respectively. We will use these equations later.

3 Perturbation equations

In this section, we introduce the linear scalar perturbation equations for the DW model. Our method of getting perturbation equations is similar to one implemented in ref. [11] and the results are consistent up to some conventions. Here, we work in the Newtonian gauge, in which scalar perturbations of the metric are given by

$$ds^2 = -\left(1+2\Psi\right)dt^2 + a^2\left(t\right)\left(1+2\Phi\right)\delta_{ij}dx^i dx^j \tag{3.1}$$

We expand the auxiliary fields as $X + \delta X$ and $U + \delta U$. In general for the anisotropic fluid in the first order of perturbation we have

$$T_0^0 = -\left(\rho + \delta\rho\right) \tag{3.2}$$
$$T_i^0 = \left(\rho + p\right)v_i \tag{3.3}$$
$$T_j^i = \left(p + \delta p\right)\delta_j^i + \Sigma_j^i \tag{3.4}$$

Here we write the pressure perturbation δp as $\delta p = c_s^2 \delta\rho$, where c_s^2 is the sound speed of the perfect fluid. The density contrast δ is defined as $\delta\rho/\rho$ and v_i is the peculiar velocity field. In the case where the matter content consists of radiation and non-relativistic matter, we have a vanishing anisotropic stress tensor $\Sigma_j^i \simeq 0$. Below we will write down the linearly perturbed field equations

$$\delta\left(G_{00} + \Delta G_{00}\right) = 8\pi G \delta T_{00}, \tag{3.5}$$
$$\delta\left(G_{ij} + \Delta G_{ij}\right) = 8\pi G \delta T_{ij}\,, \tag{3.6}$$

in Fourier space. The first order perturbation of the (00) component of Friedman equations is given by the following expression:

$$\delta G_{00} = 6H^2\Phi' + 2\frac{k^2}{a^2}\Phi \tag{3.7}$$
$$\delta\Delta G_{00} = \frac{k^2}{a^2}f,\delta X + 2\frac{k^2}{a^2}\Phi f + \frac{3}{2}H^2\left(X'\delta U' + U'\delta X'\right) + \frac{k^2}{a^2}\delta U + \frac{2k^2}{a^2}\Phi U \tag{3.8}$$
$$\delta T_{00} = \rho\delta \tag{3.9}$$

In the following equations we often put ourselves in the sub-horizon limit ($k/aH \gg 1$). To do this, we assume that $\Phi, \Psi, \delta U, \delta X, k^{-2}\delta$, and their $\ln a$ derivatives, are all of the same order (as indeed can be verified a posteriori) and systematically take the limit of large k/aH. For the (ij) component, after contracting it with the projecting operator $\left(\frac{k^i}{k}\frac{k^j}{k} - \frac{1}{3}\delta^{ij}\right)$, we get

$$\frac{2}{3}k^2\left(\Psi + \Phi\right) + \left(\frac{k^i}{k}\frac{k^j}{k} - \frac{1}{3}\delta^{ij}\right)\delta\Delta G_{ij} = -8\pi G\left(\rho + p\right)\sigma \tag{3.10}$$

where σ represents the anisotropic stress, and where $\delta\Delta G_{ij}$ in the sub-horizon limit is

$$\begin{aligned}\delta\Delta G_{ij} = {} & D_{ij}\left(f, \delta X + \delta U\right) + \delta D_{ij}\left(f + U\right) \\ & + \frac{1}{2}H^2 a^2\left(X'\delta U' + U'\delta X'\right)\delta_{ij} + H^2 a^2\left(\Phi - \Psi\right)X'U'\delta_{ij}\end{aligned} \tag{3.11}$$

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where D_{ij} and δD_{ij} , also in the sub-horizon limit, are respectively

$$D_{ij} = -\delta_{ij} k^2 + k_i k_j \tag{3.12}$$

$$\delta D_{ij} = \left(-\delta_{ij} k^2 + k_i k_j\right) (\Psi + \Phi) . \tag{3.13}$$

Now to complete the set of equations we need also to perturb eqs. (2.7)–(2.8). We get

$$\delta X'' + (3+\xi)\,\delta X' + \hat{k}^2(\delta X + 2\Psi + 4\Phi) - 2\Psi(X'' + 3X' + \xi X' + 6 + 6\xi) \tag{3.14}$$
$$-\Psi'(X'+6) + 3\Phi' X' + 6\Phi'' + 6\,(4+\xi)\,\Phi = 0$$

and

$$\delta U'' + (3+\xi)\,\delta U' + \hat{k}^2 \delta U - 2\Psi U'' - \left(2\,(3+\xi)\,\Psi + \Psi' - 3\Phi'\right) U' \tag{3.15}$$
$$= -6f,,\delta X\,(\xi+2) + 6f,\left(\Psi' + 2\,(2+\xi)\,\Psi\right)$$
$$-6f,\left(\Phi'' + (4+\xi)\,\Phi\right) - 2f,\hat{k}^2\,(\Psi + 2\Phi) .$$

where $\hat{k} = k/aH$. Moving to the sub-horizon limit we find, for the (00) component,

$$\Phi + \frac{f,\delta X}{2} + \Phi f + \frac{\delta U}{2} + \Phi U = 4\pi G \frac{a^2 \rho \delta}{k^2}$$
$$= \frac{3H_0^2}{2k^2}\left(\Omega_R^0 a^{-2}\delta_R + \Omega_M^0 a^{-1}\delta_M\right) . \tag{3.16}$$

For the (ij) component, after acting with the projection operator, we have

$$\frac{2}{3a^2} k^2\,(\Psi+\Phi) + \frac{2}{3a^2} k^2 [f,\delta X + \delta U + (\Psi+\Phi)\,(f+U)] = -8\pi G\,(\rho+p)\,\sigma, \tag{3.17}$$

At late times, when the relativistic contribution is small, we can neglect the contribution coming from the anisotropic stress, $\sigma \approx 0$. So we get

$$\Psi + \Phi + f,\delta X + \delta U + (\Psi+\Phi)\,(f+U) = 0 \tag{3.18}$$

Eqs. (3.14)–(3.15) reduce to

$$\delta X = -2\,(\Psi + 2\Phi) , \tag{3.19}$$

$$\delta U = -2f,(\Psi + 2\Phi) . \tag{3.20}$$

From the covariant conservation law of the energy-momentum tensor, $\nabla^\mu T_{\mu\nu} = 0$, we get finally the following equations for the matter density perturbation δ_M in the sub-horizon limit:

$$\delta_M'' + (2+\xi)\,\delta_M' = -\hat{k}^2 \Psi. \tag{3.21}$$

In order to solve this equation we need to find an expression for Ψ. This can be done by combining eqs. (3.16)–(3.18) and eqs. (3.19)–(3.20). After simple algebraic manipulations we find for the modified gravity function η and for the potentials Ψ and Φ the following expressions:

$$\eta = \frac{\Phi+\Psi}{\Phi} = \frac{4f,}{1+U+f-4f,}, \tag{3.22}$$

$$\Psi = -\frac{3H_0^2\,(1+U-8f,+f)\,\Omega_M^0 \delta_M}{2ak^2\,(1+U-6f,+f)\,(1+f+U)}, \tag{3.23}$$

$$\Phi = \frac{3H_0^2\,(1+U-4f,+f)\,\Omega_M^0 \delta_M}{2ak^2\,(1+U-6f,+f)\,(1+f+U)}. \tag{3.24}$$

Finally, by plugging the expression for Ψ from eq. (3.23) into eq. (3.21), we obtain the k-independent growth equation

$$\delta_M'' + (2+\xi)\,\delta_M' = \frac{3H_0^2\,(1+U-8f,+f)\,\Omega_M^0\delta_M}{2a^3H^2\,(1+U-6f,+f)\,(1+f+U)}. \tag{3.25}$$

In order to solve numerically eqs. (2.7)–(2.8), we set the following initial conditions deep inside radiation-dominated period ($N_{\rm in} = ln a^*_{\rm in} = -16$):

$$X(a^*_{\rm in}) = U(a^*_{\rm in}) = X'(a^*_{\rm in}) = U'(a^*_{\rm in}) = 0. \tag{3.26}$$

At this point it is important to mention that for the function $f(X)$ at hand, the denominators in eqs. (3.22)–(3.25) never vanish. In ref. [19] it was argued that these initial conditions force the homogenous solutions of the localized model to vanish, rendering it equivalent to the nonlocal versions of the DW model. For the growth equation (3.21), the initial conditions deep into the matter era are taken to be as in pure CDM

$$\delta_M(a_{\rm in}) = a_{\rm in}, \qquad \frac{\delta_M'(a_{\rm in})}{\delta_M(a_{\rm in})} = 1, \tag{3.27}$$

where the initial scale factor $a_{\rm in}$ is taken at redshift $z_{\rm in} = 9$. In the next section we generalize the initial conditions.

The quantity of interest in this paper is the RSD observable, also called growth rate:

$$f\sigma_8(z) \equiv \sigma_8(z)(\ln\delta_M)', \tag{3.28}$$

with the amplitude of fluctuations $\sigma_8(z)$ defined as

$$\sigma_8(z) = \sigma_8^0 \tfrac{\delta_M(z,k)}{\delta_M(0,k)}. \tag{3.29}$$

As we mentioned, the background has been fixed to ΛCDM, so the same constraints on the background apply, namely $\Omega_M^0 \approx 0.3$, so for simplicity we fix $\Omega_M^0 = 0.3$ in the following. At perturbation level, the model is also identical to ΛCDM far into the matter era, so we take $\sigma_8(z_{\rm in} = 9)$ to be identical to ΛCDM. Then there are no more free parameters and we just need to compare the result of eq. (3.25) with the RSD data collected in table 1. The numerical results for the anisotropic stress η and for $f\sigma_8(z)$ are presented in figure 1.[2] The DW model fits the data better than ΛCDM (χ^2 per d.o.f. equal to 0.66 instead of 1.05, see table 2). The resulting lower normalization, $\sigma_8^0 = 0.78$, is in agreement to better than 1σ with the recent estimates based on lensing ($\sigma_8^0 = 0.745^{+0.038}_{-0.038}$ for $\Omega_M^0 = 0.3$, see [17], table F_2, column S_8), contrary to ΛCDM. See also [16]. It should be stressed again that the value $\sigma_8^0 = 0.78$ for the DW model was obtained by assigning the same initial $\sigma_8(z_{\rm in})$ as in ΛCDM and evolving it with the linear growth factor δ_M of the DW model. A more general treatment in this respect is provided in section 6.

Our results differ significatively from those in [15], in which $f\sigma_8(z)$ lies above the ΛCDM curve, although we agree with their results at the background level, in particular with figure 1

[2] We would like to point out the fact that $f\sigma_8(z)$ data are not completely model-independent, as they assume a scenario of specific bias, redshift space distortions and nonlinear clustering which is not necessarily shared by modified gravity theories (see e.g. [20, 21]). Therefore, the comparison made here and in [15] between ΛCDM and the DW model based on ΛCDM data should be taken carefully. However, this does not affect our extension of the analysis done in [15], as in that work the authors also assume that such comparison can be made.

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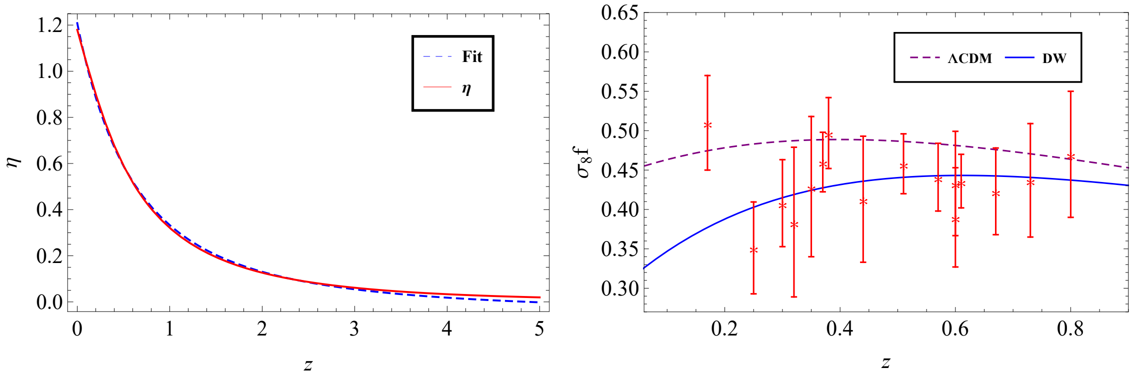

Figure 1. *Left panel:* the evolution of anisotropic stress η as a function of redshift and best fit with the polynomial $A_3a^3 + A_2a^2 + A_0$, where $\{A_3, A_2, A_0\} = \{-0.54, 1.79, -0.05\}$. *Right panel:* the growth rate $f\sigma_8(z)$ for ΛCDM and DW as functions of redshift.

of [15]. We have not been able to point out the reasons for this discrepancy. One could conjecture it might be due to the fact we are solving a localized version of the model in which the solution depends on the initial conditions on $\delta X, \delta U$, which in the sub-horizon limit are not free quantities but depend on Φ, Ψ, and therefore on δ_M. However, the quasi-static solution is the attractor solution for large k, so it is unlikely that the discrepancy depends on the localization procedure or the quasi-static approximation. Our expression (3.19) is indeed different from the corresponding quasi-static non-local expression eq. (29) in [14], but the two expressions coincide asymptotically in time in the limit in which Ψ, Φ are weakly time-dependent with respect to the rapidly varying sinusoidal function in the integrand for large k, conditions that are met in the quasi-static limit. It is still possible that the disagreement is due to the asymptotic equivalence between the local and non-local quasi-static limits not having been reached by the present time.

We have also checked the ghost-freeness conditions for the localized theory derived in ref. [11]

$$6f, > 1 + f + U > 0, \tag{3.30}$$

and we found that at all times the condition $6f, > 1 + f + U$ is violated, while the condition $1 + f + U > 0$ is always satisfied. In ref. [22] the authors have discussed a particular case when the ghost-free condition (3.30) is satisfied and leads to an interesting cosmology. In the case of tensor-scalar theories it has been shown that the appearance of a ghost mode in the theory's spectrum will lead to a situation where the effective gravitational constant G_{eff} (in units of G_{Newton}) defined as

$$G_{\text{eff}} = 1 + Y = -\frac{2k^2\Psi}{3(aH)^2\Omega_M^0\delta_M} = \frac{(1 + U - 8f, +f)}{(1 + U - 6f, +f)(1 + U + f)} \tag{3.31}$$

will become negative [23]. From figure 2, left panel, we indeed see that Y is always negative when the non-local contributions are non negligible. This explains why the perturbations grow more slowly than in ΛCDM. We see that G_{eff} goes negative near the present epoch. This however is only true in our linear approximation. Near non-linear structures, one must assume the existence of a screening mechanism in order to pass local gravity constraints, so

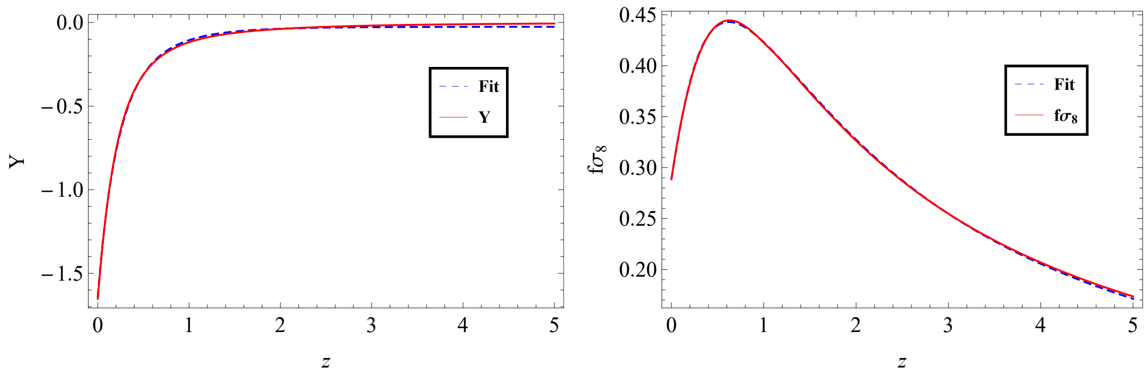

Figure 2. *Left panel:* the evolution of the effective gravitational constant correction $Y = G_{\rm eff} - 1$ as a function of red-shift and best fit with the polynomial $A_4a^4 + A_3a^3 + A_0$, where $\{A_4, A_3, A_0\} = \{-1.95, 0.33, -0.03\}$. *Right panel:* the evolution of $f\sigma_8(z)$ as a function of redshift and best fit with the polynomial $A_4a^4 + A_3a^3 + A_2a^2 + A_1a$, where $\{A_4, A_3, A_2, A_1\} = \{1.41, -3.03, 0.97, 0.94\}$.

that within the screening radius, standard gravity is recovered. This issue has been already discussed in ref. [19].[3]

The behavior of η can be approximated with an analytic fit of the form: $A_3a^3+A_2a^2+A_0$, where A_3, A_2 and A_0 are free parameters. We find the following best fit results with a percentage error up to 2% for the quantity $1 + \eta$ in the range $z \in (0, 5)$: $\{A_3, A_2, A_0\} = \{-0.54, 1.79, 0.95\}$ (see figure 1, left panel).

Similarly, the behaviors of Y and $f\sigma_8(z)$ can be approximated with analytical fits of the polynomial form $A_4a^4 + A_3a^3 + A_0$. The best fit result in the case of Y, with a percentage error up to 3% for the quantity $1 + Y$ within $z \in (0, 5)$, corresponds to $\{A_4, A_3, A_0\} = \{-1.95, 0.33, 0.97\}$ (see figure 2, left panel). For the case of $f\sigma_8(z)$ we find the best fit result $A_4a^4 + A_3a^3 + A_2a^2 + A_1a$. with a percentage error up to 1% in the same redshift range, for $\{A_4, A_3, A_2, A_1\} = \{1.41, -3.03, 0.97, 0.94\}$ (see figure 2, right panel).

4 Sound speed

In order to ascertain that the quasi-static approximation is valid, one has to ensure the absence of gradient instabilities, i.e. whether the sign of the sound-speed of the perturbative-quantities δX and δU is positive. To do this, we rewrite eqs. (3.14)–(3.15) in the following way:

$$\delta X'' + 3\delta X' + \hat{k}^2(\delta X + 2\Psi + 4\Phi) = 0 \tag{4.1}$$
$$\delta U'' + 3\delta U' + \hat{k}^2(\delta U + 2f_{,}\Psi + 4f_{,}\Phi) = 0 \tag{4.2}$$

keeping now the derivatives of $\delta X, \delta U$, but still neglecting Ψ, Φ and their derivatives with respect to $\hat{k}^2\Psi, \hat{k}^2\Phi$. Now from eqs. (3.16)–(3.18) in the case of vacuum ($\delta = \sigma = 0$) we find that

$$\Phi = \Psi = -\frac{f_{,}\delta X + \delta U}{2(1 + f + U)}. \tag{4.3}$$

[3]In this work the authors use the fact that $X \equiv \Box^{-1}R$ changes its sign from positive to negative one, when one shifts from gravitationally bound systems to large scale cosmology. On the other hand, the reconstruction procedure for the function $f(X)$ is only done for cosmology and it does not fix the value of $f(X)$ when $X > 0$. In this setup one can achieve a perfect screening by choosing $f(X) \equiv 0$ for all $X > 0$. This is not the only choice that effectively screens non-local gravity for bound systems, it is just the simplest possibility.

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Survey	z	$\sigma_8 f$	References
6dFGRS	0.067	0.423 ± 0.055	Beutler et al. (2012) [24]
LRG-200	0.25	0.3512 ± 0.0583	Samushia et al. (2012) [25]
	0.37	0.4602 ± 0.0378	
BOSS	1) 0.30	0.408 ± 0.0552, $\rho_{12} = -0.19$	Tojeiro et al. (2012) [26]
	2) 0.60	0.433 ± 0.0662	
	3) 0.38	0.497 ± 0.045, $\rho_{34} = 0.48$	Alam et al. (2016) [27]
	4) 0.51	0.458 ± 0.038, $\rho_{45} = 0.51$	
	5) 0.61	0.436 ± 0.034, $\rho_{35} = 0.17$	
WiggleZ	1) 0.44	0.413 ± 0.080, $\rho_{12} = 0.51$	Blake (2011) [28]
	2) 0.60	0.390 ± 0.063, $\rho_{23} = 0.56$	
	3) 0.73	0.437 ± 0.072	
Vipers	0.8	0.47 ± 0.08	De la Torre et al. (2013) [29]
2dFGRS	0.17	0.51 ± 0.06	Percival et al. (2004) [30, 31]
LRG	0.35	0.429 ± 0.089	Chuang and Wang (2013) [32]
LOWZ	0.32	0.384 ± 0.095	Chuang et al. (2013) [33]
CMASS	0.57	0.441 ± 0.043	Samushia et al. (2013) [34]

Table 1. Up-to-date RSD measurements from various sources, with the relevant correlation coefficients ρ_{ij}. These are the points shown in figure 1 and figure 4.

So, by inserting eq. (4.3) into eqs. (4.1)–(4.2) we get finally

$$\delta X'' + 3\delta X' + \hat{k}^2 \delta X = \frac{3\hat{k}^2 (f, \delta X + \delta U)}{(1 + f + U)} \tag{4.4}$$

and

$$\delta U'' + 3\delta U' + \hat{k}^2 \delta U = \frac{3f, \hat{k}^2 (f, \delta X + \delta U)}{(1 + f + U)} \tag{4.5}$$

We can combine these two equations and write them in a matrix form

$$X_i'' + 3X_i' + \hat{k}^2 S_{ij} X_j = 0, \tag{4.6}$$

where S_{ij} is a two by two matrix, defined as

$$S = \frac{1}{(1 + f + U)} \begin{pmatrix} 1{+}f{+}U{-}3f, & -3 \\ -3f,^2 & 1{+}f{+}U{-}3f, \end{pmatrix} \tag{4.7}$$

and X_i are the components of the vector $X = (\delta X, \delta U)$. If S is positive-definite, the perturbative quantities δX and δU have a positive sound-speed independent of the propagation direction. An arbitrary matrix A is called positive-definite when it has only positive eigenvalues. The eigenvalues of the matrix S are

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{(1 + f + U - 6f,)}{(1 + f + U)}. \tag{4.8}$$

As mentioned in the previous section, for the DW model one has $(1 + f + U) > 0$. On the other hand from the violation of the ghost-free condition (3.30) we have that $(6f, -f - U - 1) < 0$.

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Under these two conditions both eigenvalues of the matrix S are always positive and so the matrix S is positive-definite. Here, we conclude that the quasi-static approximation is a valid one, which means that the solution based on this approximation is an attractor one and any solution of eqs. (3.14)–(3.15) should approach it at some point. In ref. [11], the same procedure is carried out in the Einstein frame, with the same result.

5 Model-independent constraints

Eq. (3.21) is a second order differential equation for the density contrast δ_M and, in order to solve it, we need to specify two initial conditions. The typical choice corresponds to a standard cosmology dominated by pressureless matter at high redshifts, in which $\delta_M \sim a$. Namely, as already mentioned, one assumes

$$\delta_M(a_{\rm in}) = a_{\rm in}, \qquad \frac{\delta'_M(a_{\rm in})}{\delta_M(a_{\rm in})} = 1, \tag{5.1}$$

where $a_{\rm in}$ is some arbitrary initial value of the scale factor a outside the range of redshift for which we have observations, say at redshift $z = 9$.

These initial conditions, however, depend on several assumptions about the past: they require, in fact, that matter dominates ($\Omega_M = 1$), that matter is pressureless, that any decaying mode has been suppressed, and that gravity is Einsteinian. Broadly speaking, there is very little direct proof for any of these assumptions. Let us consider for instance two analytical toy models. In the first, one can imagine that there is a fraction Ω_h of a homogeneously distributed component along with matter in the past, just like in models of Early Dark Energy (except we are not requiring this component to lead to acceleration at the present). Then the growth of fluctuation obeys the equation

$$\delta''_M + (2 + \xi)\, \delta'_M - \frac{3}{2}\Omega_M \delta_M = 0, \tag{5.2}$$

with $\Omega_M = 1 - \Omega_h$ instead of $\Omega_M = 1$. In this case, the growth exponent is no longer $\delta_M \sim a^1$ but rather $\sim a^p$ where

$$p = \frac{1}{4}\left(-1 \pm \sqrt{1 + 24\Omega_M}\right) \tag{5.3}$$

Therefore, if for instance $\Omega_h = 0.05$, a value consistent with the analysis in [35] the total growth from $z_{\rm CMB} \approx 1100$ to $z_0 = 0$ is smaller than the corresponding pure CDM one (we are neglecting here the final accelerated epoch) by a factor

$$\frac{(z_{\rm CMB})^p}{z_{\rm CMB}} \approx 0.8 \tag{5.4}$$

So the existence of a small non-vanishing homogeneous component would produce a value of σ_8 which would be 0.8 times smaller than the Planck ΛCDM value.

The second analytical toy case comes from the simplest Brans-Dicke model, parametrized by the Brans-Dicke coupling parameter ω. In such a case in fact one has during the matter era

$$\frac{\delta'_M(a_{\rm in})}{\delta_M(a_{\rm in})} = \frac{2 + \omega}{1 + \omega} \tag{5.5}$$

rather than unity. If the Brans-Dicke gravity is universal and unscreened, then $\omega \gg 1$ because of local gravity constraints, and one recovers the standard initial condition. But if the scalar force is not universal and baryons are uncoupled or, alternatively, if the force is screened by a chameleon-like mechanism, then δ'_M/δ_M can deviate substantially from standard.

These two toy models show that if one wants to test modified gravity, and not also at the same time the entire CDM paradigm, then one needs to isolate the effects of modified gravity from those that depend on different assumptions. The simplest way to do so is to introduce then two new parameters that correspond to the two initial values of the growth equation (5.2) and marginalize the likelihood over them. Instead of $\delta_M(a_{\rm in})$ we adopt the present normalization σ_8^0 as first parameter, and

$$\alpha \equiv \frac{\delta'_M(a_{\rm in})}{\delta_M(a_{\rm in})} \tag{5.6}$$

as second free parameter. It is worth mentioning that our approach is of course not completely model independent, in the sense that we still assume that the matter content in the observational range is given by a pressureless perfect fluid and is conserved.

The current value of the amplitude of fluctuations σ_8^0 is estimated through e.g. weak lensing [36], the cosmic microwave background power spectrum [37] or cluster abundances [38]. In all these cases the estimate depends, in general, on the choice of a gravity theory. As a consequence the current value of σ_8^0 is highly model-dependent.

6 Likelihood analysis

Assuming now that σ_8^0 and α can take any value for modified gravity models, it is of interest to see how the growth rate $f\sigma_8(z)$ behaves, in terms of agreement with data, when we allow those two parameters to vary. The data we used in this analysis are displayed in table 1. Here again we fix $\Omega_M^0 = 0.3$ for simplicity.

In general, the likelihood function of a given model (represented by the parameter vector Θ) with respect to some data is given by $L(\Theta) = A\exp[-\chi^2/2]$ with

$$\chi^2 = (D-T)^T C^{-1}(D-T). \tag{6.1}$$

Here, D and T are respectively the data and theory vectors, C is the covariance matrix and A is a normalization constant. The vector D contains the measurements of the observable quantity (in our case, the growth rate $f\sigma_8(z)$) for each point (i.e. each redshift value) and T represents the corresponding predictions for that observable. Note that, as is manifest in table 1, the RDS measurements used are not independent, so we consider the full covariance matrix.

We choose to integrate the equation for density perturbations (3.21) from the redshift $z_{\rm in} = 9$. The results of the fits are shown in table 2, in which we display the χ^2 deviation given by ΛCDM, by the DW model and by the DW varying first only σ_8^0 (Case I) and then varying both σ_8^0, α (Case II).

The Case I likelihood for σ_8^0 is displayed in figure 3. For Case II, the best-fit yields a better performance compared with ΛCDM, but the χ^2 improves only marginally upon the no-free parameters DW.

In figure 4 we show the growth rate given by the three analysis and by ΛCDM.

Model	σ_8^0	α	χ^2/dof
ΛCDM	0.83	1	1.05
DW	0.78	1	0.66
DW: case I	0.80 ± 0.024	1	0.68
DW: case II	$1.04^{+0.33}_{-0.27}$	$-1.26^{+3.26}_{-0.06}$	0.65

Table 2. The χ^2/dof for RSD measurements for ΛCDM, DW and for the best-fit cases of the DW model for σ_8^0 (Case I) and $\{\sigma_8^0, \alpha\}$ (Case II), respectively.

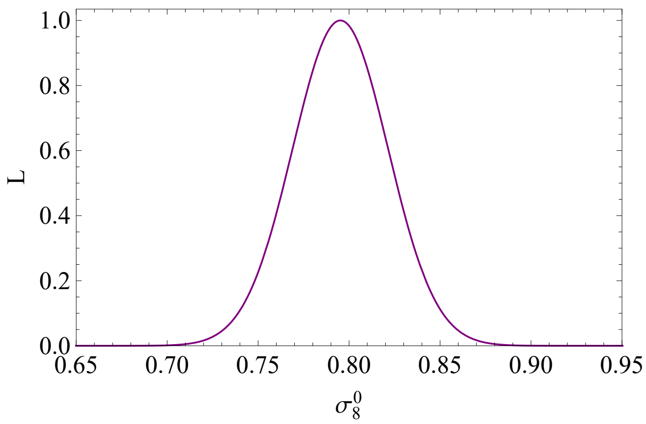

Figure 3. The likelihood for σ_8^0, when α is fixed to 1 (Case I). The maximum lies at $\sigma_8^0 = 0.80$.

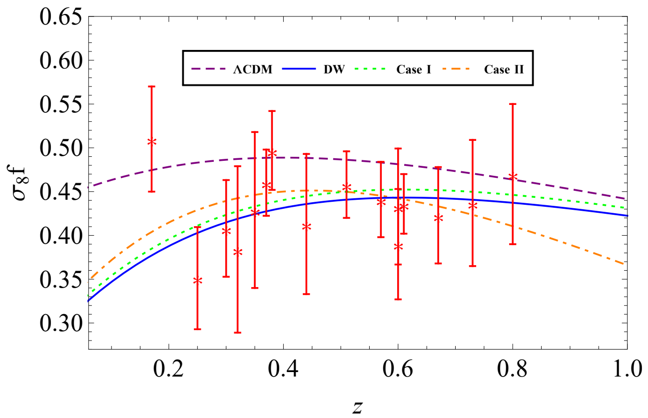

Figure 4. The growth rate $f\sigma_8(z)$ for ΛCDM and the three best-fit cases for the DW model. The data points are collected in table 1.

7 Conclusion

In this work, we have extended the analysis performed in [15] regarding the predictions of RSD given by the DW model of nonlocal gravity. We have found that the localized version of the theory is not ruled out by the RSD data, as was the case for the previous analysis, and actually leads to a better agreement than the standard ΛCDM cosmology. Ultimately, this behavior is due to the violation of the ghost-free condition. In ref. [39] it has been argued

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that the existence of ghosts in non-local theories does not necessarily rule out the model since the ghost mode is actually "frozen" due to fixed boundary conditions imposed on the model. Interestingly, the predicted value $\sigma_8^0 = 0.78$ for $\Omega_M^0 = 0.3$ is in agreement to better than 1σ with the recent estimates ($\sigma_8^0 = 0.745^{+0.038}_{-0.038}$) based on lensing, see e.g. [16, 17]. We have also investigated how much the fit improves when we generalize the initial conditions of the growth rate function, and we find that the improvement is just marginal.

We fixed $\Omega_M^0 = 0.3$ because, as we repeatedly mentioned, the background of the DW model is by construction identical to the ΛCDM one. The value of Ω_M^0 is then highly constrained by cosmological probes likes CMB, SN etc. It is true, however, that at the perturbation level the DW resembles ΛCDM only up to a redshift of order unity and therefore the constraints coming from probes that depend on perturbations, primarily CMB, will be different from those on standard cosmology. Because of this, an analysis that generalises to any Ω_M^0 would be welcome, also to see how the degeneracy direction Ω_M, σ_8 compares to the one observationally found, e.g. in KIDs. Since our main goal here was however to note that the DW model is in agreement with current data, rather that to fully explore its consequences, we leave this task to future work.

Our perturbation results do not agree with the analysis in [15], who integrated numerically the equations in their nonlocal form. This could be due to an intrinsic difference between the nonlocal and the localized versions of the DW model, for instance in the way the quasi-static limit is performed. Unfortunately, we have been unable to point out the reason for this discrepancy, notwithstanding extended testing. In any case, we believe the localized version of the DW theory gives interesting predictions on linear perturbation level and deserves further consideration.

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CHAPTER 6

PUBLICATION 3

INSTABILITIES IN TENSORIAL NONLOCAL GRAVITY

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Instabilities in tensorial nonlocal gravity

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We discuss the cosmological implications of nonlocal modifications of general relativity containing tensorial structures. Assuming the presence of standard radiation- and matter-dominated eras, we show that, except in very particular cases, the nonlocal terms contribute a rapidly growing energy density. These models therefore generically do not have a stable cosmological evolution.

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I. INTRODUCTION

Most extensions of general relativity are manifestly local. *A priori*, however, we need not impose this restriction. Like general relativity itself, most proposed theories of modified gravity are nonrenormalizable, which is often a sign of new physics at high energies. From a local high-energy theory, nonlocalities often appear in the effective theory describing low-energy physics. For example, nonlocalities appear generically when massless or light degrees of freedom are integrated out of a local fundamental theory [1–4].

Nonlocal modifications of general relativity constructed out of inverse differential operators give rise to infrared effects that become relevant at large temporal and spatial scales. The consequences of these nonlocalities are far-reaching and could provide a dynamical explanation for dark energy. Numerous examples of this line of thinking can be found in the literature [5–11]. Most of the existing nonlocal gravity models are purely phenomenological and are constructed out of nonlocal operators involving the Ricci scalar only for reasons of simplicity [12–15]. It is still an open question whether we should expect these particular nonlocal structures, as opposed to something more complicated, to arise in the low-energy limit of fundamental theories. Tensorial extensions involving elements such as the Ricci or the Riemann tensors should not be *a priori* excluded.

Adding nonlocal interactions can also improve some of general relativity's more undesirable properties, and these seem to specifically require tensorial nonlocalities. For example, in order to alleviate the ultraviolet divergences of general relativity, one has to modify the graviton propagator, which requires a tensorial term in the action [16]. Considerable recent progress has been made in ghost-free ultraviolet nonlocal gravity [17,18]. Furthermore, nonlocal modifications of general relativity could degravitate a large cosmological constant, providing an appealing solution to the problem of why a large vacuum energy does not gravitate [8]. For the purposes of degravitation, it is likely insufficient to rely on scalar degrees of freedom introduced via nonlocal scalar curvature terms. Tensorial nonlocalities, by contrast, could help implement a consistent degravitation mechanism, as is the case in the framework of massive gravity where nonlocalities modify the tensor propagator [10,19].

The cosmological consequences of tensorial nonlocalities involving inverse powers of the d'Alembertian operator were considered in Refs. [20–22]. Tensor nonlocalities in these models were shown to contain rapidly growing modes, leading to instabilities in the background expansion.[1] Note, however, that the inverse d'Alembertian operators considered in these references are certainly not the most general possibility that can be implemented at each order in curvature. It is possible that other well-motivated differential operators might lead to a somewhat different evolution that is consistent with observations.

In this work we extend the analysis of Refs. [20,21] to general nonlocal tensorial actions at quadratic order in the curvature invariants and investigate whether these

[1]Note that these instabilities are not directly related to unitarity violation. This can be seen by considering a nonlocal term of the form $G_{\alpha\beta}(m^2/\Box^2)R^{\alpha\beta}$, which results in a massive graviton propagator of a unitary form (cf. Ref. [9], where this model corresponds to $\alpha = 0$ and is shown to be unitary.). However, as the action contains, in addition to scalar terms, the tensorial term $R^{\alpha\beta}(m^2/\Box^2)R_{\alpha\beta}$, it will be cosmologically unstable, as can be seen from Ref. [20] and from the following.

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modifications are phenomenologically viable. This paper is organized as follows. In Sec. II, we introduce our tensorial nonlocal model. The cosmological consequences of this model during radiation (RD) and matter domination (MD) are discussed in Sec. III. Finally, the conclusions are presented in Sec. IV.

II. THE $R_{\alpha\beta}\triangle^{-1}R^{\alpha\beta}$ MODEL

Consider the most general action quadratic in the curvature invariants [16] for some differential operator $\triangle$,

$$S = \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}(-R + Rf(\triangle)R + R^{\alpha\beta}g(\triangle)R_{\alpha\beta} + R^{\mu\nu\alpha\beta}h(\triangle)R_{\mu\nu\alpha\beta}) + \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m, \tag{1}$$

where $M_{\rm Pl} \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass and $\mathcal{L}_m$ is the matter Lagrangian minimally coupled to gravity. Different nonlocal theories are characterized by different choices of the operator $\triangle$ and of the functions f, g and h. In the case $\triangle = \Box$, the above action is the most general parity-invariant quadratic curvature action; see Ref. [23] for derivation of the field equations. We generalize this by allowing for more general differential operators, in particular, those with curvature dependence. Note that we would recover the results of Refs. [16,23] for the quadratic truncation of the theory.

It is well motivated to consider more general forms for the operator $\triangle$; in fact the main rationale for the usual choice $\triangle = \Box$ is just simplicity. For the nonlocally modified theory to be consistent on suitable backgrounds, one may need to implement a regularization [24,25]. For example, Ref. [25] considered a curvature-dependent regularization of the form $(\Box + \hat{P})^{-1}$ with[2]

$$\hat{P} \equiv P_{\alpha\beta}{}^{\mu\nu} = aR_{(\alpha}{}^{(\mu}{}_{\beta)}{}^{\nu)} + b(g_{\alpha\beta}R^{\mu\nu} + g^{\mu\nu}R_{\alpha\beta}) + cR^{(\mu}_{(\alpha}\delta^{\nu)}_{\beta)} + dRg_{\alpha\beta}g^{\mu\nu} + eR\delta^{\mu\nu}_{\alpha\beta}, \tag{2}$$

and a, b, c, d, and e arbitrary constants. For example, in the de Donder gauge the graviton kinetic operator[3] would correspond to $a = -2$, $b = 0$, $c = 2$, $d = 1/3$, and $e = -4/3$.

In the following, we also allow the differential part of the operator to assume a more generic form, involving combinations of the curvature invariants and covariant derivatives ∇ that arise in explicit loop computations. We consider simple forms for the functions g and h,

$$g(\triangle) \equiv \frac{\bar{M}_1{}^2}{6\triangle}, \qquad h(\triangle) \equiv \frac{\bar{M}_2{}^2}{6\triangle}, \tag{3}$$

with $\bar{M}_1$ and $\bar{M}_2$ mass scales to be determined by observations. These two properties allow us to simplify the action (1) for a Friedman-Lemaître-Robertson-Walker (FLRW) background

$$\mathrm{d}s^2 = H^{-2}\mathrm{d}N^2 - a^2\mathrm{d}\mathbf{x}^2, \tag{4}$$

where $N \equiv \ln a$ is the number of e-folds, a is the scale factor, and $H \equiv \dot{a}/a$ stands for the Hubble rate with the dot denoting derivative with respect to cosmic time. Indeed, by noticing that for an FLRW metric in four dimensions the Weyl tensor

$$C_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} - (g_{\mu[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R_{\beta]\mu}) + \frac{1}{3}g_{\mu[\alpha}g_{\beta]\nu}R \tag{5}$$

vanishes, and using the fact that $\triangle$ is by construction metric compatible, we can write

$$C_{\mu\nu\alpha\beta}\triangle^{-1}C^{\mu\nu\alpha\beta} = 0 \longrightarrow R_{\mu\nu\alpha\beta}\triangle^{-1}R^{\mu\nu\alpha\beta} = -\frac{1}{3}R\triangle^{-1}R + 2R_{\alpha\beta}\triangle^{-1}R^{\alpha\beta}. \tag{6}$$

Substituting this relation into Eq. (1) we obtain the simplified action

$$S = \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}(-R + RF(\triangle)R + R^{\alpha\beta}g(\triangle)R_{\alpha\beta}) + \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m, \tag{7}$$

where we have defined $F(\triangle) \equiv f(\triangle) - \frac{\bar{M}^2}{18}\triangle^{-1}$ with $\bar{M}^2$ being a linear combination of $\bar{M}_1^2$ and $\bar{M}_2^2$. For cosmological backgrounds, the Riemann tensor does not explicitly contribute to the background evolution[4]; all the dynamical information can be encoded in nonlocal terms constructed out of Ricci scalars and Ricci tensors only.

The $RF(\triangle)R$ part of Eq. (7) has been extensively studied the literature for several choices of $F(\triangle)$ and $\triangle$ [12,15,26–28]. In this work we concentrate on the phenomenological consequences of the tensorial structure $R^{\alpha\beta}g(\triangle)R_{\alpha\beta}$. In particular, we consider the action

[2] Here and in the following, $(\mu\nu)$ denotes symmetrization over the indices and $[\mu\nu]$ denotes the antisymmetrization.

[3] In an isotropic and homogeneous background, the action of this operator on a tensor reduces to the action of the scalar operator on each component of the tensor [22], suggesting that the cosmological tensorial instability might be removed by dressing the inverse d'Alembertian into its appropriate tensor representation. Strictly speaking, this would take us beyond the starting point action (1) [or, otherwise, we would consider the four indices in the representation of the $(1/\Delta)^{\mu\nu}{}_{\alpha\beta}$ implicitly shuffling those of the $R_{\mu\nu}$]. Explicit construction of such models can be considered as a topic of future study; in this article we focus on the action of a general scalar (derivative) operator $1/\Delta$ on the (Ricci) tensor $R_{\alpha\beta}$.

[4] Note however that it contributes at the level of perturbations.

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left(-R + \frac{\bar{M}^2}{6}R_{\alpha\beta}\triangle^{-1}R^{\alpha\beta}\right) + \int d^4x\sqrt{-g}\mathcal{L}_m, \tag{8}$$

with

$$\triangle \equiv m^4 + \alpha_1\Box + \alpha_2\Box^2 + \beta_1 R_{\alpha\beta}\nabla^\alpha\nabla^\beta + \beta_2 R\Box + \gamma(\nabla^\alpha R_{\alpha\beta})\nabla^\beta, \tag{9}$$

and α_1, α_2, β_1, β_2, γ, m constant parameters. Up to the m^4 term, the differential operator (9) is the most general fourth-order operator containing at least one covariant derivative acting on the function following it. This choice of operator has a special physical motivation in the celebrated conformal anomaly [29,30], in which quantum effects break the conformal symmetry of massless fields coupled to gravity. In this case the trace of the energy-momentum tensor receives a nonvanishing contribution from the counterterms introduced by renormalization. The form of this contribution is highly nontrivial and depends on the particle content. In four dimensions, the effective action induced by the conformal anomaly is given by [30]

$$S_{\rm A} = -\frac{1}{8}\int d^4x\sqrt{-g}\left(E - \frac{2}{3}\Box R\right)\triangle_4^{-1} \times \left[b'\left(E - \frac{2}{3}\Box R\right) - 2bC^2_{\mu\nu\alpha\beta}\right], \tag{10}$$

where $E \equiv R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the Gauss-Bonnet term, $C^2_{\mu\nu\alpha\beta} = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + R^2/3$ is the square of the Weyl tensor, b and b' are numbers that depend on the particle content of the theory, and $\triangle_4$ is defined as

$$\triangle_4 = \Box^2 + 2R_{\alpha\beta}\nabla^\alpha\nabla^\beta - \frac{2}{3}R\Box + \frac{2}{3}(\nabla^\alpha R_{\alpha\beta})\nabla^\beta. \tag{11}$$

This operator is just a particular case of the operator (9) with $m = 0$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 2$, $\beta_2 = -2/3$ and $\gamma = 2/3$.[5]

The equations of motion associated to the nonlocal action (8) can be obtained by following a standard procedure for the study of nonlocal theories. We localize the action by introducing two auxiliary fields $S_{\alpha\beta}$ and $K_{\alpha\beta}$, defined as solutions of the differential equations

$$\triangle S_{\alpha\beta} = R_{\alpha\beta}, \qquad \Box S_{\alpha\beta} = K_{\alpha\beta}. \tag{12}$$

After variation of our nonlocal action (8) with respect to the metric $g_{\mu\nu}$ and taking into account the identity $\delta(\triangle^{-1}) = -\triangle^{-1}\delta(\triangle)\triangle^{-1}$ (see Refs. [25,31] for details) we get the modified Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{1}{M_{\rm Pl}^2}(T_{\alpha\beta} + T^{\rm NL}_{\alpha\beta}), \tag{13}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor associated to the matter Lagrangian $\mathcal{L}_m$, which is by construction covariantly conserved, $\nabla_\alpha T^\alpha_\beta = 0$. The interaction term $T^{\rm NL}_{\alpha\beta}$ arises from the variation of the nonlocal term $R_{\alpha\beta}\triangle^{-1}R^{\alpha\beta}$ and can be naturally split into six pieces,

$$T^{\rm NL}_{\alpha\beta} = T^{\rm NL(0)}_{\alpha\beta} + T^{\rm NL(1)}_{\alpha\beta} + T^{\rm NL(2)}_{\alpha\beta} + T^{\rm NL(3)}_{\alpha\beta} + T^{\rm NL(4)}_{\alpha\beta} + T^{\rm NL(5)}_{\alpha\beta}, \tag{14}$$

where we have defined

$$\frac{1}{2M^4}T^{\rm NL(0)}_{\alpha\beta} \equiv \frac{1}{2}R_{\mu\nu}S^{\mu\nu}g_{\alpha\beta} - 2R^\mu_\alpha S_{\mu\beta} - \Box S_{\alpha\beta} - g_{\alpha\beta}\nabla_\mu\nabla_\nu S^{\mu\nu} + 2\nabla_\mu\nabla_\alpha S^\mu_\beta, \tag{15}$$

$$\frac{1}{2\alpha_1 M^4}T^{\rm NL(1)}_{\alpha\beta} \equiv \frac{1}{2}g_{\alpha\beta}\nabla_\sigma S^{\mu\nu}\nabla^\sigma S_{\mu\nu} - \nabla_\alpha S^{\mu\nu}\nabla_\beta S_{\mu\nu} - 2S^{\mu\nu}\nabla_\nu\nabla_\alpha S_{\mu\beta} + 2S^\mu_\alpha\nabla_\nu\nabla_\beta S^\nu_\mu \tag{16}$$

$$- 2\nabla_\mu S^{\mu\nu}\nabla_\alpha S_{\beta\nu} + 2\nabla_\nu S^\mu_\alpha\nabla_\beta S^\nu_\mu + \frac{1}{2}g_{\alpha\beta}S^{\mu\nu}\nabla_\sigma\nabla^\sigma S_{\mu\nu}, \tag{17}$$

$$\begin{aligned}\frac{1}{2\alpha_2 M^4}T^{\rm NL(2)}_{\alpha\beta} \equiv\; & 2K_{\beta\nu}\nabla_\mu\nabla_\alpha S^{\mu\nu} + 2\nabla_\alpha S^{\mu\nu}\nabla_\mu K_{\beta\nu} - 2\nabla_\mu S^{\mu\nu}\nabla_\alpha K_{\beta\nu} - 2S^{\mu\nu}\nabla_\mu\nabla_\alpha K_{\beta\nu} \\ & - 2K^{\mu\nu}\nabla_\mu\nabla_\alpha S_{\beta\nu} - 2\nabla_\alpha S_{\beta\nu}\nabla_\mu K^{\mu\nu} + 2\nabla_\mu S_{\beta\nu}\nabla_\alpha K^{\mu\nu} + 2S_{\beta\nu}\nabla_\mu\nabla_\alpha K^{\mu\nu} \\ & - 2\nabla_\alpha S^{\mu\nu}\nabla_\beta K_{\mu\nu} + g_{\alpha\beta}\nabla_\sigma S^{\mu\nu}\nabla^\sigma K_{\mu\nu} + \frac{1}{2}g_{\alpha\beta}S^{\mu\nu}\Box K_{\mu\nu} + \frac{1}{2}g_{\alpha\beta}K_{\mu\nu}\Box S^{\mu\nu},\end{aligned} \tag{18}$$

[5]Note that even though the form of the operator (9) is motivated by the form of the conformal anomaly operator $\triangle_4$, the action (1) considered in this paper is not of the form of the action (10).

$$\begin{aligned}\frac{1}{2\beta_1 M^4}T^{\mathrm{NL}(3)}_{\alpha\beta} \equiv& -2R_{\alpha\sigma}\nabla_\mu S^{\mu\nu}\nabla^\sigma S_{\beta\nu} + 2R_{\beta\sigma}\nabla_\mu S_{\alpha\nu}\nabla^\sigma S^{\mu\nu} + 2R_{\beta\sigma}S_{\alpha\nu}\nabla_\mu\nabla^\sigma S^{\mu\nu} + 2R_{\alpha\sigma}S^{\mu\nu}\nabla_\beta\nabla^\sigma S_{\mu\nu}\\ &+\frac{1}{2}R_{\alpha\beta}\nabla_\sigma S^{\mu\nu}\nabla^\sigma S_{\mu\nu} + \frac{1}{2}R_{\alpha\beta}S^{\mu\nu}\Box S_{\mu\nu} - \frac{1}{2}\nabla^\sigma\nabla_\alpha(S^{\mu\nu}\nabla_\beta\nabla_\sigma S_{\mu\nu}) - 2R_{\alpha\sigma}S^{\mu\nu}\nabla_\mu\nabla^\sigma S_{\beta\nu}\\ &-\frac{1}{2}g_{\alpha\beta}\nabla^\sigma\nabla^\tau(S^{\mu\nu}\nabla_\sigma\nabla_\tau S_{\mu\nu}) + S_{\mu\alpha}S^\mu_\beta(\nabla^\sigma\nabla^\tau R_{\sigma\tau}) - \frac{1}{2}\nabla^\sigma\nabla_\beta(S^{\mu\nu}\nabla_\alpha\nabla_\sigma S_{\mu\nu})\\ &+\frac{1}{2}\Box(S^{\mu\nu}\nabla_\alpha\nabla_\beta S_{\mu\nu}) - R_{\alpha\sigma}\nabla_\beta S^{\mu\nu}\nabla^\sigma S_{\mu\nu} - 2(\nabla_\mu R^{\mu\sigma})(S^\nu_\alpha\nabla_\sigma S_{\beta\nu})\\ &-2(\nabla_\mu R_{\alpha\sigma})(S^{\mu\nu}\nabla_\sigma S_{\beta\nu}) + 2(\nabla_\mu R_{\beta\sigma})(S_{\alpha\nu}\nabla^\sigma S^{\mu\nu}) + 2(\nabla^\mu R_{\mu\sigma})\nabla^\sigma(S_{\alpha\nu}S^\nu_\beta)\\ &-(\nabla^\sigma R_{\sigma\alpha})(S_{\mu\nu}\nabla_\beta S^{\mu\nu}) + \frac{1}{2}(\nabla^\sigma R_{\alpha\beta})(S_{\mu\nu}\nabla_\sigma S^{\mu\nu}),\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{1}{2\beta_2 M^4}T^{\mathrm{NL}(4)}_{\alpha\beta} \equiv& S^\nu_\alpha\Box R S_{\beta\nu} - RS^\nu_\beta\Box S_{\alpha\nu} + S_{\beta\nu}\nabla_\mu\nabla_\alpha RS^{\mu\nu} + \nabla_\alpha RS^{\mu\nu}\nabla_\mu S_{\beta\nu} - \nabla_\mu RS^{\mu\nu}\nabla_\alpha S_{\beta\nu}\\ &-S^{\mu\nu}R\nabla_\mu\nabla_\alpha S_{\beta\nu} - S^{\mu\nu}\nabla_\mu\nabla_\alpha RS_{\beta\nu} - \nabla_\alpha RS_{\beta\nu}\nabla_\mu S^{\mu\nu}\\ &+\nabla_\mu RS_{\beta\nu}\nabla_\alpha S^{\mu\nu} + RS_{\beta\nu}\nabla_\mu\nabla_\alpha S^{\mu\nu} - \nabla_\beta RS^{\mu\nu}\nabla_\alpha S_{\mu\nu} + R_{\alpha\beta}(S^{\mu\nu}\Box S_{\mu\nu})\\ &+\frac{1}{2}g_{\alpha\beta}\nabla_\sigma RS^{\mu\nu}\nabla^\sigma S_{\mu\nu} + \frac{1}{2}g_{\alpha\beta}RS^{\mu\nu}\Box S_{\mu\nu} + g_{\alpha\beta}\Box(S^{\mu\nu}\Box S_{\mu\nu}) - \nabla_\alpha\nabla_\beta(S^{\mu\nu}\Box S_{\mu\nu}),\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{1}{2\gamma M^4}T^{\mathrm{NL}(5)}_{\alpha\beta} \equiv& \frac{1}{2}g_{\alpha\beta}\nabla_\tau(S^{\mu\nu}R^{\tau\sigma}\nabla_\sigma S_{\mu\nu}) - \frac{1}{2}\nabla_\tau(S^{\mu\nu}R_{\alpha\beta}\nabla^\tau S_{\mu\nu})\\ &+S^{\mu\nu}(\nabla^\tau R_{\tau\alpha}\nabla_\beta S_{\mu\nu}) + S^{\mu\nu}(\nabla_\beta R_{\alpha\tau}\nabla^\tau S_{\mu\nu})\\ &+\frac{1}{2}\nabla_\sigma\nabla_\alpha\nabla_\beta(S^{\mu\nu}\nabla^\sigma S_{\mu\nu}) - \frac{1}{2}g_{\alpha\beta}\nabla_\sigma\nabla_\tau\nabla^\sigma(S^{\mu\nu}\nabla^\tau S_{\mu\nu}) - \nabla_\sigma(S_{\alpha\nu}S^\nu_\beta(\nabla_\mu R^{\mu\sigma})),\end{aligned} \tag{21}$$

with $M^4 \equiv \frac{1}{12}\bar{M}^2 M^2_{\mathrm{Pl}}$.

III. $R_{\alpha\beta}\triangle^{-1}R^{\alpha\beta}$ COSMOLOGY

Finding exact solutions for the complicated set of equations derived in the previous section is certainly not an easy task. In what follows, we adopt the approach of Ref. [20] and assume that the energy density contributed by nonlocal effects is subdominant, so that we have the standard radiation- and matter-dominated eras ($T^{\mathrm{NL}}_{\alpha\beta} \ll T_{\alpha\beta}$). We investigate the stability of various regions of parameter space, defined as the presence or absence of growing modes in the energy density contributed by the nonlocal interactions.

We assume $\alpha_1 = m = 0$, which allows us to find certain analytic solutions. We have carried out a preliminary numerical study for nonvanishing values of α_1 and m and found that the inclusion of these parameters does not significantly modify the results presented below. A full numerical study of the parameter space is beyond the scope of this work.

A. Radiation-dominated era

During radiation domination, the Ricci scalar is 0 and the terms proportional to $R\Box$ and $(\nabla^\sigma R_{\sigma\tau})\nabla^\tau$ in Eq. (9) vanish (the latter due to the Bianchi identity). On top of that, the symmetry of the FLRW metric (4) allows us to reduce the tensor $S_{\mu\nu}$ in Eq. (12) to a simple diagonal form, $S^\nu_\mu = \mathrm{diag}(S_1, -S_2, -S_2, -S_2)$, that depends on two (homogeneous) scalar functions S_1 and S_2. Taking into account these simplifications, the set of equations (12) can be rewritten as

$$\begin{aligned}&\alpha_2 S^{(4)}_+ - 6\alpha_2 S^{(3)}_+ + 3\beta_1 S''_+ - 11\alpha_2 S''_+ + (60\alpha_2 - 9\beta_1)S'_+\\ &\quad + 8(\beta_1 - 4\alpha_2)S_+ = \frac{4a^4}{\Omega^0_{\mathrm{R}}},\end{aligned} \tag{22}$$

$$\alpha_2 S^{(4)}_- - 6\alpha_2 S^{(3)}_- + (3\beta_1 + 5\alpha_2)S''_- + (12\alpha_2 - 9\beta_1)S'_- = 0, \tag{23}$$

where $\prime \equiv \mathrm{d}/\mathrm{d}N$ denotes derivatives with respect to the number of e-folds N, Ω^0_{R} is the current value of the critical radiation density, and we have defined two dimensionless variables

$$S_+ \equiv (S_1 + S_2)H_0^2, \quad S_- \equiv (S_1 - 3S_2)H_0^2, \tag{24}$$

in terms of the Hubble parameter today, $H_0^2 = H^2a^4/\Omega^0_{\mathrm{R}}$. Note that for $\alpha_2 = 0$, the fourth-order differential equations (22) and (23) reduce to second-order differential equations admitting the simple solution

$$S_+ = a^{\frac{3}{2}}\left[c_1 \sin\left(\frac{1}{2}\sqrt{\frac{5}{3}}\ln a\right) + c_2 \cos\left(\frac{1}{2}\sqrt{\frac{5}{3}}\ln a\right)\right] + \frac{a^4}{5\beta_1\Omega_R^0}, \quad (25)$$

$$S_- = \frac{1}{3}\tilde{c}_1 a^3 + \tilde{c}_2, \quad (26)$$

where c_1, c_2, $\tilde{c}_1$, and $\tilde{c}_2$ are integration constants to be fixed by initial conditions. In the general case $\alpha_2 \neq 0$, the solution of Eqs. (22) and (23) is

$$S_+ = a^{3/2}\left(c_1 a^{-q_-} + c_2 a^{q_-} + c_3 a^{-q_+} + c_4 a^{q_+} - \frac{a^{5/2}}{\Omega_R^0(24\alpha_2 - 5\beta_1)}\right), \quad (27)$$

$$S_- = \frac{2a^{3/2-y/2}}{3-y}\tilde{c}_1 + \frac{2a^{3/2+y/2}}{3+y}\tilde{c}_2 + \frac{1}{3}\tilde{c}_3 a^3 + \tilde{c}_4, \quad (28)$$

where

$$q_\mp = \frac{\sqrt{49\alpha_2 - 6\beta_1 \mp 2\sqrt{(44\alpha_2 - 9\beta_1)(12\alpha_2 - \beta_1)}}}{2\sqrt{\alpha_2}},$$
$$y = \frac{\sqrt{25\alpha_2 - 12\beta_1}}{\sqrt{\alpha_2}}, \quad (29)$$

and c_i and $\tilde{c}_i$ ($i = 1, \ldots, 4$) are integration constants. Note that in both cases the leading contributions in S_+, S_- at large values of the scale factor a take the power-law forms

$$S_+ \approx \tilde{A}a^A, \qquad S_- \approx \tilde{B}a^B, \quad (30)$$

with A and B being *positive* constants related only to the model parameters $\{\alpha_2, \beta_1\}$, and $\tilde{A}$ and $\tilde{B}$ coefficients keeping track of the integration constants c_i and $\tilde{c}_i$ ($i = 1, \ldots, 4$), i.e., keeping track of the initial conditions. Inserting these asymptotic expressions into Eq. (14) and comparing the result with the standard form $T^\mu_\nu = \mathrm{diag}(\rho_{NL}, -p_{NL}, -p_{NL}, -p_{NL})$ for a perfect fluid, we can derive approximate expressions at the lowest order in Ω_R^0 for the nonlocal energy density ρ_{NL} and the nonlocal equation of state $w_{NL} \equiv p_{NL}/\rho_{NL}$ during radiation domination,

$$\rho_{NL} \approx -3M^4\Omega_R^0(\tilde{A}(A+4)a^{A-4} + \tilde{B}(B+1)a^{B-4}), \quad (31)$$

$$w_{NL} \approx -\frac{1}{3}\frac{(A-1)\tilde{A}(A+4)a^{A-4} + \tilde{B}(B^2-1)a^{B-4}}{\tilde{A}(A+4)a^{A-4} + \tilde{B}(B+1)a^{B-4}}. \quad (32)$$

The behavior of w_{NL} at large values of a depends on the relation between A and B, i.e., on the precise choice of the model parameters $\{\alpha_2, \beta_1\}$. For $B < A$, the equation of state asymptotically approaches $w_{NL} = -\frac{1}{3}(A-1)$, while for $B > A$ it instead evolves towards $w_{NL} = -\frac{1}{3}(B-1)$. Note that, contrary to the nonlocal energy density ρ_{NL}, the asymptotic values of w_{NL} do not depend on the initial conditions.

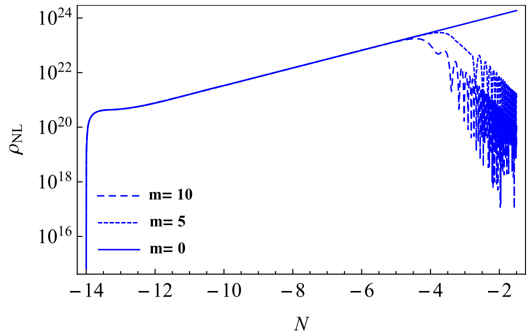

FIG. 1. Functional behavior of the nonlocal energy density ρ_{NL} versus the number of e-folds N during radiation domination for an operator $\triangle = \alpha_1\Box + m^4$ and different values of m. All quantities are expressed in units with $H_0 = 1$. Note that the dimensionful parameter α_1 is not an independent parameter: together with $\bar{M}$, it fixes the amplitude of nonlocal effects and does not modify the dynamics. In this plot, we set $\bar{M} = H_0$ and $\alpha_1 = H_0^2$. The late-time evolution of the nonlocal energy density develops a damped oscillatory pattern in the vicinity of $N = 0$ when our radiation-domination ansatz for the scale factor a is no longer applicable. The *average* of this quantity over an oscillation period scales as a^{-8}, i.e., faster than the radiation fluid ($\rho_R \sim a^{-4}$). A similar damping during radiation domination would require values of m comparable to the Hubble rate during that era.

For $\alpha_2 = 0$ we have $A = 4$ and $B = 3$ [cf. Eqs. (25) and (26)]. These asymptotic values translate into a constant nonlocal energy density ρ_{NL} and a cosmological-constantlike equation of state $w_{NL} = -1$. Therefore, nonlocal contributions with $\alpha_2 = 0$ can *in principle* lead to a viable cosmology, as long as the radiation energy density is dominant over ρ_{NL} for the entire radiation-dominated era.[6]

The situation changes completely in the $\alpha_2 \neq 0$ case. Demanding the absence of a growing mode in Eq. (31) imposes A, $B \leq 4$. By considering Eqs. (27) and (29) with the restriction $B \leq 4$, we get the constraints

$$\alpha_2 > 0, \qquad \beta_1 \in \left[0, \frac{25}{12}\alpha_2\right]. \quad (33)$$

Unfortunately, these two conditions are never satisfied for $A \leq 4$. Indeed, a simple inspection of Eq. (27) shows that in order to keep $A \leq 4$ we must have $q_+ + 3/2 \leq 4$ and $q_- + 3/2 \leq 4$, or equivalently $\beta_1 \leq 24/5\alpha_2$ and $\beta_1 \geq 24/5\alpha_2$, in clear contradiction with each other and with (33). The growing modes become rapidly dominant unless the prefactor of the nonlocal contribution in the action is largely suppressed.[7]

[6]Note that this conclusion holds only for $\alpha_1 = 0$. As shown in Ref. [20], the $\alpha_1 \neq 0$ scenario contains growing modes and leads to an unstable cosmology.

[7]Note that instabilities associated with tensorial structures appear also in ultraviolet extensions of general relativity. In the case of Starobinsky inflation, the problem of instabilities coming from the tensorial components is addressed by introducing a hierarchy between energy scales of the R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms [32].

This conclusion does not seem to be modified for an operator $\triangle = \alpha_1 \Box + m^4$ with values of m and α_1 of order H_0 and H_0^2, respectively. As shown in Fig. 1, the evolution of the nonlocal energy density in this case develops a damped oscillatory pattern in the vicinity of $N = 0$, when our radiation-domination ansatz for the scale factor a is no longer applicable. A similar damping during radiation domination would require values of m comparable to the Hubble rate at that era.

B. Matter-dominated era

Can the instabilities generated during radiation domination be suppressed during the subsequent evolution of the Universe? To answer this question we study the behavior of a subdominant nonlocal tensorial contribution during matter domination ($\rho_\mathrm{M} \gg \rho_\mathrm{NL}$). Taking into account the definitions in (24) (with $H_0^2 = H^2 a^3/\Omega_\mathrm{M}^0$), we can write the differential equations in (12) as

$$4\alpha_2 S_+^{(4)} - 12\alpha_2 S_+^{(3)} + 6\beta_1 S_+'' - 73\alpha_2 S_+'' - 12\beta_2 S_+'' - 3S_+'(9\beta_1 - 6\gamma - 41\alpha_2 + 6\beta_2) + 16 S_+(3\beta_1 + 7\alpha_2 + 6\beta_2) = \frac{12a^3}{\Omega_\mathrm{M}^0}, \tag{34}$$

$$\frac{4}{3}\alpha_2 S_-^{(4)} - 4\alpha_2 S_-^{(3)} - (3\alpha_2 - 2\beta_1 + 4\beta_2)S_-'' - (9\beta_1 - 6\gamma - 9\alpha_2 + 6\beta_2)S_-' = -\frac{4a^3}{\Omega_\mathrm{M}^0}, \tag{35}$$

with Ω_M^0 being the critical matter density today. As in the case of radiation domination, if we choose $\alpha_2 = 0$, then Eqs. (34) and (35) are reduced from fourth-order to second-order differential equations. These equations can be solved analytically,

$$S_+ = \frac{4a^3}{\Omega_\mathrm{M}^0(7\beta_1 - 22\beta_2 + 18\gamma)} + c_1 a^{p_-} + c_2 a^{p_+}, \tag{36}$$

$$S_- = \frac{4a^3}{9\Omega_\mathrm{M}^0(\beta_1 + 6\beta_2 - 2\gamma)} + \tilde{c}_1 \frac{2(\beta_1 - 2\beta_2)}{3(3\beta_1 + 2\beta_2 - 2\gamma)} a^{\tilde{y}} + \tilde{c}_2, \tag{37}$$

with

$$p_\pm = \frac{9\beta_1 + 6\beta_2 - 6\gamma \pm \sqrt{-47\beta_1^2 + 108\beta_1(\beta_2 - \gamma) + 548\beta_2^2 - 72\beta_2\gamma + 36\gamma^2}}{4(\beta_1 - 2\beta_2)}, \qquad \tilde{y} = \frac{3(3\beta_1 + 2\beta_2 - 2\gamma)}{2(\beta_1 - 2\beta_2)}. \tag{38}$$

The detailed solution of Eqs. (34) and (35) for the $\alpha_2 \neq 0$ case is cumbersome and largely irrelevant for the following discussion. On general grounds, the leading contributions to S_+ and S_- at large values of the scale factor a can be parametrized as[8]

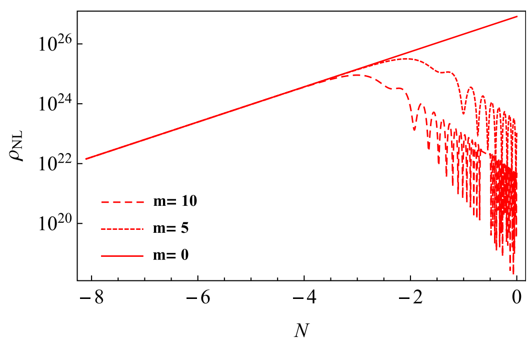

FIG. 2. Functional behavior of the nonlocal energy density ρ_NL versus the number of e-folds N during matter domination for an operator $\triangle = \alpha_1 \Box + m^4$ and different values of m. All quantities are expressed in units with $H_0 = 1$. Note that the dimensionful parameter α_1 is not an independent parameter. Together with $\bar{M}$ in the action, it fixes the amplitude of nonlocal effects and does not modify the dynamics. In this plot we set $\bar{M} = H_0$ and $\alpha_1 = H_0^2$. The late-time evolution of the nonlocal energy density develops a damped oscillatory pattern. The *average* of this quantity over an oscillation period scales as a^{-6}, i.e., faster than the matter fluid ($\rho_M \sim a^{-3}$). Note that when m is of the order of the Hubble rate at matter-radiation equality, this could alleviate the previous growth during radiation domination.

[8]Our results cover the tensorial action induced by the conformal anomaly and the extension of the Maggiore-Mancarella model considered in Ref. [21]. For the parameters associated to the conformal anomaly ($\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 2$, $\beta_2 = -2/3$, $\gamma = 2/3$), one obtains

$$S_+ = a^{\frac{3}{4}}\left(c_1 a^{-\frac{1}{4}\sqrt{133-4\sqrt{385}}} + c_2 a^{\frac{1}{4}\sqrt{133-4\sqrt{385}}} + c_3 a^{-\frac{1}{4}\sqrt{133+4\sqrt{385}}} + c_4 a^{\frac{1}{4}\sqrt{133+4\sqrt{385}}} - \frac{2a^{9/4}}{9\Omega_\mathrm{M}^0}\right),$$

$$S_- = 2\tilde{c}_1 a^{\frac{1}{2}} + \frac{2}{3}\tilde{c}_2 a^{\frac{3}{2}} + \tilde{c}_3 a + \tilde{c}_4 - \frac{2a^3}{15\Omega_\mathrm{M}^0},$$

while for the case $\triangle \propto \Box^2$ ($\alpha_1 = \beta_1 = \beta_2 = \gamma = 0$, $\alpha_2 \neq 0$) considered in Ref. [21] we find

$$S_+ = a^{-\frac{1}{4}(3+\sqrt{137})}\left(c_2 a^{\frac{\sqrt{137}}{2}} + c_3 a^3 + c_4 a^{\frac{1}{2}(6+\sqrt{137})} + c_1 - \frac{3a^{\frac{1}{4}(15+\sqrt{137})}}{44\Omega_\mathrm{M}^0}\right),$$

$$S_- = -\frac{2}{3}\tilde{c}_1 a^{-\frac{3}{2}} + \frac{2}{3}\tilde{c}_2 a^{\frac{3}{2}} + \frac{\tilde{c}_3}{3}a^3 + \tilde{c}_4 - \frac{36\ln a - 44}{243\Omega_\mathrm{M}^0}a^3.$$

TABLE I. Characteristic values of the nonlocal equation of state $w_{\rm NL}$ during RD and MD when only one of the parameters in the nonlocal operator (9) is different from 0. Note that the operators associated to β_2 and γ vanish exactly during radiation domination. The values associated to the conformal anomaly operator (11) are also presented.

Model	$w_{\rm NL}$(RD)	$w_{\rm NL}$(MD)
α_1	−1.25	−1.45
α_2	−1.79	−2.45
β_1	−1	−2
β_2	0	−1
γ	0	−1
m	5/3	1
$\triangle_4$	−1.55	−1.92

$$S_+ \approx \tilde{C} a^C, \qquad S_- \approx \tilde{D} a^D, \tag{39}$$

with the *positive* constants C and D encoding information about the model parameters, and the prefactors $\tilde{C}$ and $\tilde{D}$ tracing the initial conditions. Note that the a^3 dependence of the source term in Eqs. (34) and (35) forces C and D to be asymptotically larger or equal to 3. Using Eq. (14), we can derive the nonlocal energy density

$$\rho_{\rm NL} \approx \frac{M^4 (\Omega_{\rm M}^0)^2}{32} (\tilde{E} a^{2C-6} + \tilde{F} a^{2D-6}), \tag{40}$$

with $\tilde{E}$ and $\tilde{F}$ being some constants built from the free parameters of the theory and the initial conditions. Since the exponents C and D satisfy always the condition C, $D \geq 3$, we have either a constant or growing nonlocal energy density $\rho_{\rm NL}$. Therefore, the instabilities arising during radiation domination cannot be suppressed in the matter-dominated era. Note that this result also holds for the operator $\triangle = \alpha_1 \Box + m^4$ with nonvanishing values of m and α_1, with numerical results presented in Fig. 2.

For the sake of completeness, we present in Table I the asymptotic values of the nonlocal equation of state $w_{\rm NL}$ when only one of the parameters in the operator (9) is different from 0. The values associated to the conformal anomaly operator (11) are also displayed. This helps us to see in a qualitative way the contribution coming from the different operators in (9) when the condition $T^{\rm NL}_{\alpha\beta} \ll T_{\alpha\beta}$ is satisfied.[9]

[9]Note that this can always be achieved by fine-tuning the mass scale $\bar{M}$ in Eq. (8).

IV. CONCLUSIONS

In this paper, we have explored the stability of a general class of tensorial nonlocal extensions of general relativity. Our result is a direct answer to Ref. [20], where the authors conjectured that the instabilities arising in the tensorial $R_{\alpha\beta}\Box^{-1}R^{\alpha\beta}$ model might be cured by a generalization of the d'Alembertian operator to $\alpha_1\Box + m^4$ or to the conformal anomaly operator $\triangle_4$. We have found that the growing mode and the associated instabilities of tensorial nonlocal models cannot be generically avoided by introducing the most general nonlocal operator at second order in covariant derivatives.

This conclusion holds also for a restricted version of the operator, namely $\alpha_1\Box + m^4$, if the scale m is chosen to be of the order of the Hubble rate today. One could alternatively consider scenarios in which m is comparable to the Hubble rate at matter-radiation equality. In those cases, an oscillatory pattern arises that could be compatible with our requirement that the nonlocal contribution to the cosmic expansion be subdominant to the matter contribution. This might give rise to phenomenologically interesting features in the form of an oscillating early dark energy.

In the presence of growing modes, terms at higher and higher order in curvature are expected to become relevant, compromising the validity of the effective action (8). Although one cannot exclude the possibility of some cancellation mechanism among the various terms, a nonperturbative study within the effective nonlocal theory is quite difficult. We believe that the instabilities associated to tensorial nonlocalities should instead be addressed in the framework of local field theories by considering mechanisms able to generate well-behaved nonlocal actions in the infrared.

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Chapter 7

Publication 4

Quantum Gravity inspired nonlocal gravity model

Quantum Gravity inspired nonlocal gravity model

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We consider a nonlocal gravity model motivated by nonperturbative Quantum Gravity studies. This model, if correct, suggests the existence of strong IR relevant effects which can lead to an interesting late time cosmology. We implement the IR modification directly in the effective action. We show that, upon some assumptions on initial conditions, this model describes an observationally viable background cosmology being also consistent with local gravity tests.

Keywords: modified gravity, nonlocal gravity, dark energy, background cosmology

I. INTRODUCTION

After the discovery [1–8] of the late time acceleration of the universe, many suggestions how to explain it have been advanced. As is well-known, the most elegant and simple solution is to include the so-called cosmological constant Λ into the Einstein-Hilbert action. Although this model works extremely well at the classical level, it faces dramatic challenges once we make a step towards quantum physics. Indeed, even in the quasi-classical approximation where gravity is still classical and we quantize just the matter sector, the cosmological constant receives quantum radiative corrections of the order of m^4 for each massive species, which give a big contribution to the tiny value of Λ needed for an appropriate cosmic evolution. In other words, Λ is not a technically natural parameter.

Of course, if the theory were renormalizable, one could adjust the value of Λ to fit the observations and then not worry anymore about the questions of technical naturalness. Unfortunately, the Einstein-Hilbert theory with a cosmological constant is not perturbatively renormalizable, and at each level of loop corrections it will receive uncompensated quantum contributions which will destroy the predictability of the model. In this respect, quantum gravity could solve this problem by providing a technically natural way of cosmological constant generation. Again, in the context of quantum gravity there could be strong IR-relevant effects which can lead to an interesting modification of the standard cosmological constant scenario and provide a dynamical dark energy candidate [9]. In this direction, in Ref. [10, 11], the authors have argued that non-perturbative lattice quantum gravity calculations can lead to a situation where the gravitational interactions slowly increase with distance. This behavior is encoded in the running of the gravitational constant G. The running of G is calculated in momentum space and in order to write a corresponding running in coordinate space one has to specify what is the relevant cutoff.

The choice of the relevant cutoff is not unique and in principle if we want to have a general covariance at the level of the effective action, we can choose as a cutoff an arbitrary covariant function which scales as k^2. In literature, common choices have been either $k^2 \sim R$ (see e.g. Refs. [12, 13]) or $k^2 \sim \Box$. In this paper we explore the choice $k^2 \sim \Box$, a choice that renders the effective action nonlocal. Another important point is the inclusion of a non-perturbative scale ζ, the scaling of which can be approximated with an inverse Hubble function H^{-1}. The appearance of this scale makes the model to be relevant for IR-dynamics of our universe and can lead to an interesting phenomenology for the late-time universe. Previously, in Ref. [10, 11], the authors have studied the late time cosmology of this model by replacing Newton's constant with a running one at the level of Friedman equations. In this work, we study this model in a more self-consistent way, which is to get the equations of motions (EoMs) by varying an effective action. In this case, one has to vary the d' Alembert operator, which gives rise to additional terms in Friedman's equations. For a particular choice of a critical exponent ν this model predicts results for the late time cosmology very similar to those of Maggiore and Mancarella's (RR) nonlocal gravity model [14].

Throughout the paper, we work in flat space and natural unites, i.e. units such that $c = \hbar = 1$. Furthermore, we will denote with a "dot" derivative with respect to the cosmic time and with a "prime" derivative with respect to the number of e-foldings.

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II. THE MODEL

The structure of the model derives from the studies of non-perturbative lattice quantum gravity. Ref. [10, 11, 15] have argued that the RG improvement of the gravitational constant G leads to the following effective action in coordinate space[1]

$$\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(1-c_\zeta(\frac{1}{\zeta^2\Box})^{1/2\nu}+O((\zeta^2\Box)^{-1/\nu})\right)R, \tag{1}$$

where the relevant cutoff is provided in the form $k^2\sim\Box$. In the action (1) the constant ν stands for a critical exponent, which is defined as

$$\nu=-(\beta'(G_{\rm c}))^{-1}, \tag{2}$$

with the β function calculated in the vicinity of the UV non-Gaussian fixed point (NGF) $G_{\rm c}$. The critical exponent ν in general is a positive rational number and highly depends on the scheme of calculation. In realistic scenarios the value of ν^{-1} belongs to the interval $\nu^{-1}\in[1,4]$ [17–19]. Another parameter to be specified in the action (1) is the genuinely non-perturbative scale ζ. Indeed, ζ is defined as

$$\zeta^{-1}\approx\Lambda_{\rm cut}\exp(-\int^G\beta(G')^{-1}dG')\sim_{G\longrightarrow G_{\rm c}}\Lambda_{\rm cut}|G-G_{\rm c}|^\nu, \tag{3}$$

where $\Lambda_{\rm cut}$ is the UV cutoff of the theory[2]. To determine the real physical value of ζ we need to have some physical input since the underlying theory cannot fix it. An important property of ζ defined in Eq. (3) is that when we move away from the UV fixed point along a RG trajectory, and in the case of positive critical exponents ν, we have that ζ^{-1} grows. This behavior tells us that the corresponding QG corrections to the Einstein-Hilbert action, which are proportional to ζ^{-1}, might enter into the strongly coupled phase and become dominant at late times when we run towards IR scales along RG trajectories. Now, in order to associate the scale of ζ with a scale of some physical quantity we will follow the discussion in [11] where it has been argued that at the late-time cosmological setup it is natural to associate this non-perturbative scale with either the inverse of the average curvature $\langle R\rangle$ or with the inverse of the Hubble function, which determines the macroscopic size of the universe. As in previous works [10, 11], here we will select the second option, so that $\zeta\approx H_0^{-1}$, where H_0 is the current value of the Hubble parameter. The only parameter left in the action (1) is then c_ζ. The value of c_ζ can be estimated in a lattice quantum gravity theory and, in contrast to ν, which has a universal value, it depends on the choice of the regularization scheme and in general is estimated to be a order one parameter. So the only free parameter of the model at hand is the non-perturbative scale ζ, which has to be fixed in such a way to provide a valid cosmology.

Depending on the value of ν we can have either rational or integer powers of $(1/\Box)$ operator in the action. In general it is not trivial how to deal with a rational power of a differential operator (see e.g. Ref. [20, 21]). In this work, however, we will study only the cases when the power of the $\Box$ operator is an integer number, leaving the treatment of the general rational power of the d'Alembert operator to future works. In the range $\nu^{-1}\in[1,4]$, an integer value for $1/(2\nu)$ can be realized either when $\nu^{-1}=2$ or $\nu^{-1}=4$. A model similar to the case with $\nu^{-1}=2$ has been recently studied in Ref. [22]. So, in this work we will mainly concentrate on the case when $\nu^{-1}=4$. For this choice of ν the effective action (1) reduces to

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(1-\frac{M^4}{6}\frac{1}{\Box^2}\right)R+\int d^4x\sqrt{-g}\mathcal{L}_m, \tag{4}$$

where we have also added the general matter Lagrangian. As already mentioned, the cosmological implementation of this type of models has already been studied in [10, 11]. In these works, for simplicity, the running of the gravitational constant was directly embedded into the right hand side of the Friedman equations. Below, we will show that doing so one loses terms in equations of motion which can have a significant impact on the cosmological evolution. Another important issue which arises when we directly implement the running of the gravitational constant into the Friedman equations, is related to a violation of the covariant conservation of the energy-momentum tensor. Indeed, if the energy-momentum tensor of the matter sector is derived from a covariant action, it will be automatically covariantly conserved.

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[1] The correct effective action should also contain the running cosmological constant term Λ_k. As is argued in Refs. [10, 16] in pure lattice gravity the bare cosmological constant Λ is scaled out and does not run. So, in this case Λ is a constant which has to be properly fixed. Similarly to studies in Ref. [10], in this work will assume that the contribution from Λ is sub-dominant with respect to the one induced by the running of G, so that we can effectively approximate $\Lambda\approx 0$.

[2] For the case of lattice quantum gravity the UV cutoff corresponds to the inverse lattice spacing, i.e. $\Lambda_{\rm cut}\sim l_{\rm p}^{-1}$.

On the other hand, from the Bianchi identities, we know that the Einstein tensor is also covariantly conserved. So, if in the Friedman equations we change the gravitational constant with a running one obtained through an inverse-box structure, we will not have anymore a covariant conservation on the matter side of the equations, as the covariant derivative ∇_μ and the inverse d'Alembert operator $\Box^{-1}$ do not commute in general, $\left[\nabla_\mu, \Box^{-1}\right] \neq 0$.

III. THE $\Box^{-2}R$ MODEL

To study the cosmological evolution of this model we need to derive the Friedman equations. In Appendix (A) we have derived the EoMs for a more general model. Corresponding EoMs for the $\Box^{-2}R$ model can be deduced by inserting $p = 0$ and $n = 1$ into Eqs. (A7-A8), which will give us

$$\begin{aligned} G_{\alpha\beta} = \frac{M^4}{6}\Big\{ & LR_{\alpha\beta} - \nabla_\alpha\nabla_\beta L - g_{\alpha\beta}Q - \frac{1}{2}g_{\alpha\beta}[S + RL] + \\ & + \frac{1}{2}g_{\alpha\beta}g^{\sigma\lambda}[\nabla_\sigma Q\nabla_\lambda S + \nabla_\sigma U\nabla_\lambda L] + \\ & - \nabla_\alpha U\nabla_\beta L - \nabla_\beta Q\nabla_\alpha S - \frac{1}{2}g_{\alpha\beta}UQ\Big\} + 8\pi G T_{\alpha\beta}, \end{aligned} \tag{5}$$

with the four auxiliary fields satisfying the following set of equations

$$\begin{aligned} \Box U &= -R, & \Box Q &= -1, \\ \Box S &= -U, & \Box L &= -Q. \end{aligned} \tag{6}$$

In Eq. (5), $G_{\alpha\beta}$ stands for the Einstein tensor and $T_{\alpha\beta}$ is a perfect fluid energy-momentum tensor defined as $T^\beta_\alpha = \text{diag}(-\rho, p, p, p)$, where ρ and p are correspondingly the energy density and pressure of the fluid.

In the case of the $\Box^{-2}R$ model the auxiliary fields Q and L have a simple meaning, namely, they are the Lagrange multipliers of the constraint equations. Indeed, let us write the gravitational part of our nonlocal action (4) in a local way by introducing the constraint equations right at the level of the action. Then we will have for the gravitational part

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - \frac{M^4}{6}S + \alpha_1(\Box U + R) + \alpha_2(\Box S + U)\right], \tag{7}$$

where α_1 and α_2 are the Lagrange multipliers. Now by taking the variation of the action (7) with respect to the Lagrange multipliers and the auxiliary fields U and S, we get the following set of equations

$$\begin{aligned} \Box U &= -R, & \Box\alpha_1 &= -\alpha_2, \\ \Box S &= -U, & \Box\alpha_2 &= \frac{M^4}{6}. \end{aligned} \tag{8}$$

From Eq. (8) we see that α_1 and α_2, up to the multiplicative constant $M^4/6$, correspond to L and Q, respectively. Defining the dimensionless fields

$$V = H_0^2 S \qquad W = H_0^2 Q \qquad Z = H_0^4 L, \tag{9}$$

and assuming a flat FLRW metric, the (00) component of Eq. (5) becomes

$$h^2 = \frac{\gamma}{4}\left\{V + WU + h^2[6Z + 6Z' - U'Z' - V'W']\right\} + \Omega_R^{(0)}e^{-4N} + \Omega_M^{(0)}e^{-3N}, \tag{10}$$

where $\gamma = (1/9)(M/H_0)^4$ and $h = H/H_0$.

In Eq. (10), $\Omega_R^{(0)}$ and $\Omega_M^{(0)}$ are the current values of radiation and matter critical densities in the universe, respectively. By using the identity

$$\Omega = \Omega_R^{(0)}e^{-4N} + \Omega_M^{(0)}e^{-3N} = h^2(\Omega_R + \Omega_M), \tag{11}$$

from Eq. (10) we finally get for h^2

$$h^2 = \frac{(\gamma/4)(V+WU)}{1-\Omega_{\rm M}-\Omega_{\rm R}-(\gamma/2)[3Z+3Z'-(1/2)(U'Z'+V'W')]}. \tag{12}$$

The set of differential equations (6) for the auxiliary fields, assuming homogeneity, are now written as

$$U'' = -(3+\xi)U' + 6(2+\xi), \tag{13}$$

$$V'' = -(3+\xi)V' + \frac{U}{h^2}, \tag{14}$$

$$W'' = -(3+\xi)W' + \frac{1}{h^2}, \tag{15}$$

$$Z'' = -(3+\xi)Z' + \frac{W}{h^2}. \tag{16}$$

where ξ, defined as $\xi = h'/h$, has the following structure:

$$\xi = \frac{1}{2(1-(3/2)\gamma Z)}\left[\frac{\Omega'}{h^2} + \frac{3}{2}\gamma\left(\frac{W}{h^2} - 4Z' + U'Z' + V'W'\right)\right]. \tag{17}$$

IV. NUMERICAL SOLUTIONS

Before moving to the full numerical analysis of the system, here we can briefly comment upon the stability of the system during matter and radiation domination periods. In this case, ξ can be well approximated with a constant in each era: $\xi_0 = \{-2, -3/2\}$ in matter and radiation-dominated periods, respectively. First, we can check the consistency of the homogenous solutions of Eqs. (13-16). The homogenous solutions are the followings

$$U = u_0 + u_1 e^{-(3+\xi_0)N}, \tag{18}$$

$$\bar{V} = \bar{v}_1 e^{-(3-\xi_0)N} + \bar{v}_2 e^{2\xi_0 N}, \tag{19}$$

$$\bar{W} = \bar{w}_1 e^{-(3-\xi_0)N} + \bar{w}_2 e^{2\xi_0 N}, \tag{20}$$

$$\bar{Z} = \bar{z}_1 e^{-(3-\xi_0)N} + \bar{z}_2 e^{2\xi_0 N}, \tag{21}$$

where $\bar{V} = h^2 V$, $\bar{W} = h^2 W$ and $\bar{Z} = h^2 Z$. From the equations above we see that when ξ_0 is changing in the interval $\xi_0 \in [-2, 0]$ we have that the solutions either remain constant or decrease exponentially. This insures that the solutions are stable during matter and radiation-dominated periods.

To solve Eqs. (13-16) numerically, with the constrains (12) and (17), we first fix the present values

$$\Omega_{\rm M}^{(0)} = 0.3 \qquad \Omega_{\rm R}^{(0)} = 4.15\times 10^{-5} h^{-2} \tag{22}$$

to the standard values. Although the constraint on $\Omega_{\rm M}$ has been obtained assuming standard ΛCDM and therefore in principle should be estimated anew with the present model, we will see the background evolution turns out not to be very different from the standard one, so our choice may be considered a reasonable approximation. For later use, we also need to define the effective equation of state $w_{\rm eff}$ and the critical dark energy density $\Omega_{\rm DE}$, respectively, as

$$w_{\rm eff} = -1 - \frac{2}{3}\xi, \tag{23}$$

$$\Omega_{\rm DE} = \frac{\gamma}{4}\left[\frac{1}{h^2}(V+WU) + 6Z + 6Z' - U'Z' - V'W'\right]. \tag{24}$$

Using the definition of $\Omega_{\rm DE}$ we can rewrite Eq. (10) as

$$\Omega_{\rm DE} = 1 - \Omega_{\rm M} - \Omega_{\rm R}. \tag{25}$$

Furthermore, from the continuity equation of the dark energy critical density $\Omega_{\rm DE}$

$$\Omega'_{\rm DE} + (3 + 3w_{\rm DE} + 2\xi)\Omega_{\rm DE} = 0, \tag{26}$$

γ	0.702	0.222	0.030	0.015
$w_{\rm DE}^0$	−1.752	−1.268	−1.099	−1.086
$w_{\rm DE}^{\rm a}$	0.843	−0.170	−0.077	−0.061

Table I. Today's values of $w_{\rm DE}$ and its first derivative w.r.t. the scale factor a, for different values of γ corresponding to different initial conditions on the field W.

we find for the dark energy equation of state parameter $w_{\rm DE}$

$$w_{\rm DE} = -1 - \frac{2}{3}\xi - \frac{1}{3}\frac{\Omega'_{\rm DE}}{\Omega_{\rm DE}} = -\frac{3+\Omega_{\rm R}+2\xi}{3\Omega_{\rm DE}}. \tag{27}$$

Finally, making use of Eq. (17), we can write $w_{\rm DE}$ explicitly in terms of the auxiliary fields

$$\begin{aligned} w_{\rm DE} = &-\frac{3+\Omega_{\rm R}}{3\Omega_{\rm DE}} - \frac{2}{2-3\gamma Z}\left(1-\frac{3+\Omega_{\rm R}}{3\Omega_{\rm DE}}+2\frac{W}{V+UW}\right)+ \\ &-\left(\frac{3\gamma}{2-3\gamma Z}\right)\frac{1}{3\Omega_{\rm DE}}\left[U'Z'+V'W'-4Z'+\frac{W}{V+UW}\left(U'Z'+V'W'-6Z-6Z'\right)\right]. \end{aligned} \tag{28}$$

The value of the only dimensionless free parameter of our model, γ, should be fixed in a such way as to satisfy the condition $h(0) = 1$. This produces, for instance, the values $\gamma = \{0.702, 0.222, 0.030, 0.015\}$ for the following initial conditions on W, $W_0 = \{0, 0.5, 5, 10\}$, respectively. Now, to integrate Eqs. (13-16) we need to specify initial conditions on our auxiliary fields at the onset of integration deep inside the radiation-dominated period. We choose $N_{\rm in} = -14$ as the initial time. As is also argued in Ref. [23], the choice of initial conditions for the auxiliary fields in a deep radiation-dominated period are *per se* arbitrary. Their value highly depends on the physical content of the Universe at the epoch we start evolving the differential equations (13-16). For simplicity, we will assume that all the auxiliary fields in our model, apart from W, have vanishing initial conditions. The reason behind this particular choice of initial conditions will become clear below.

Performing the integration of Eqs. (13-16), we find the numerical results as a function of $N = \log a$ plotted in Figures (1-4) for four different choices of initial conditions for W. As we can see from Figures (1-4), the evolution of all physical quantities as well as auxiliary fields does not show singularities. Moreover, from the left panel of Figure 3 we observe that the present model predicts a well defined radiation-domination period ($w_{\rm eff} = 1/3$), followed by a matter dominated period ($w_{\rm eff} = 0$), which finally ends in a dark energy-dominated period. The transition between the matter epoch and dark energy epoch, ($N \approx -0.4$), is very well constrained by the current observational data [24], so one can already put some restrictions on the model. In this respect, as one can notice from Figure 3, in our case when the initial value of W (W_0) is set to be vanishing, the evolution of $w_{\rm eff}$ and $w_{\rm DE}$ exhibits a strongly phantom behavior. The dark energy equation of state parameter $w_{\rm DE}$ increases sharply from $w_{\rm DE} \approx -2.1$ to $w_{\rm DE} \approx -1.2$. From the observational side, constraints on $w_{\rm DE}$ are often obtained parametrizing it as a linear function of the scale factor a, i.e. $w_{\rm DE} = w_{\rm DE}^0 + (1-a)\, w_{\rm DE}^{\rm a}$. Comparing the values of $w_{\rm DE}^0$ and $w_{\rm DE}^{\rm a}$ for our model from Table I with the corresponding observational constraints (see e.g. Ref. [25], Table 7) we immediately see that our solution for vanishing W_0 is in strong tension with the constraints and is probably already ruled out[3].

As mentioned, we chose vanishing initial conditions for all auxiliary fields expect for W, which means that homogeneous solutions for those auxiliary fields are set to be zero deep in the radiation domination period. By relaxing these assumptions we can see how the overall quantitative evolution is affected. This kind of analysis of initial conditions has been performed in Ref. [23] in the case of the RR model. Here we apply the same analysis but only state the main outcome. We find that the behavior of the present model during the dark energy-dominated period highly depends on the choice of initial conditions for the auxiliary field W. Indeed, again from Figure 3, we observe that even a small non-vanishing value of W_0 can efficiently soften the strongly phantom behavior of the dark energy equation of state parameter $w_{\rm DE}$, making it compatible with current observational constraints. Indeed, in contrast to the case of the field U satisfying the equation $\Box U = -R$, the field W satisfies the equation $H_0^2 \Box W = -1$. Therefore the arguments in the literature [14] for choosing vanishing initial conditions for U, related to the fact that R is also vanishing during the radiation-dominated period, do not hold anymore. In principle W can have any initial value depending on its

[3] Here is important to mention that these observational constraints are obtained by combining the supernovae (SNe) and Cosmic Microwave Background (CMB) data. The constraints coming from CMB data can not be directly applied to our nonlocal model, both because they are based on ΛCDM and because the linear parametrization of $w_{\rm DE}$ is not a good approximation to the evolution of the dark energy in the past. Lifting the CMB constraints the error bars on $w_{\rm DE}^0$ and $w_{\rm DE}^{\rm a}$, relax considerably, but even in the most realistic case the nonlocal model with vanishing W_0 will be still highly disfavored.

early history. The dependence of the evolution of the field W on initial conditions is presented in the left panel of Figure 2.

The high sensitivity of $w_{\rm DE}$ to the non-vanishing choice of initial conditions for W can be also inferred from Eq. (28). In the expression for $w_{\rm DE}$ we see that there are several terms which are directly proportional to W. Therefore they will affect the value of $w_{\rm DE}$ only in the cases when W is non-vanishing. On the other hand, from Figure 2, we see that W remains very close to its initial value W_0 until the matter-dark energy transition point ($N \approx -0.4$). Therefore, when $W_0 = 0$, W is also vanishing, so the terms in Eq. (28) proportional to W will never be activated and thus will not contribute to the value of $w_{\rm DE}$.

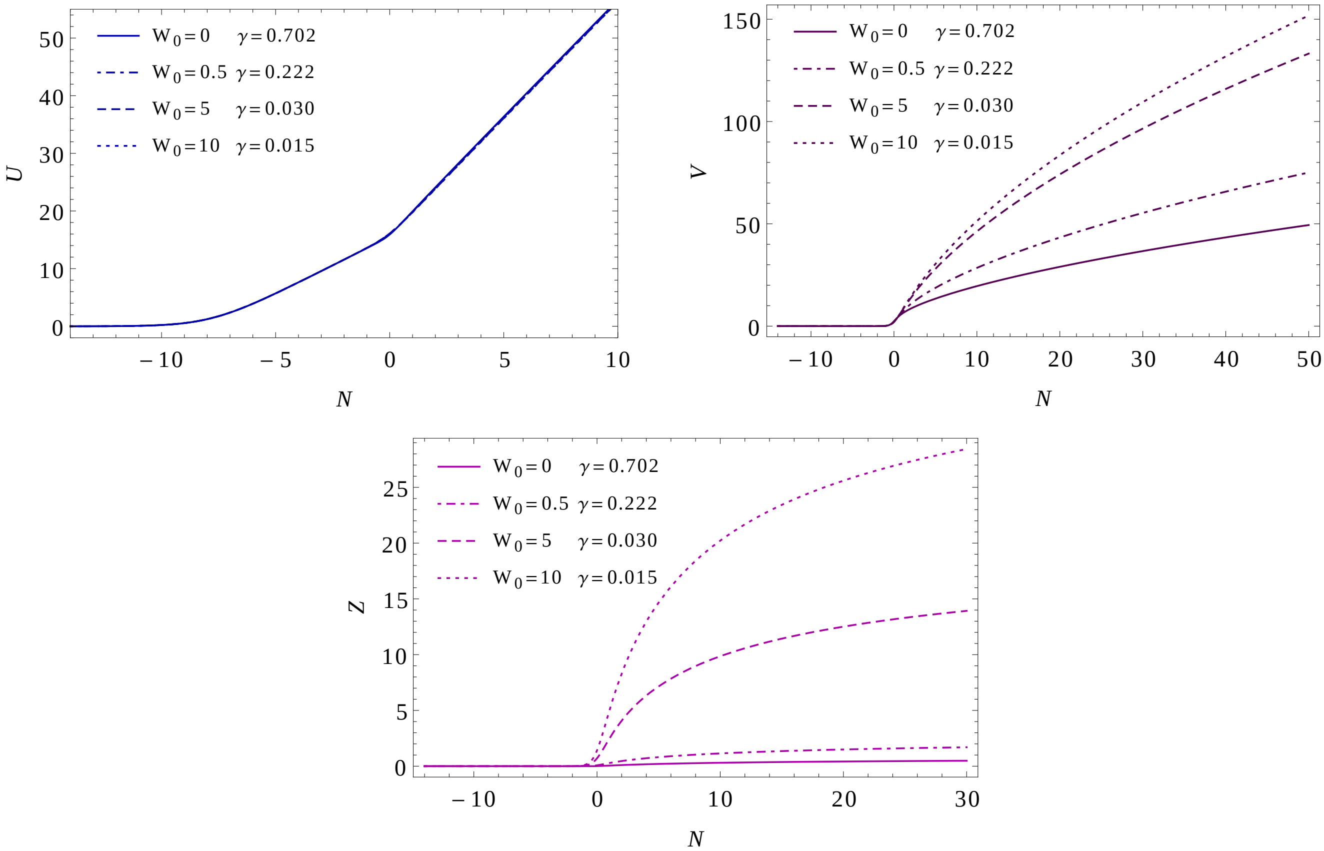

Figure 1. Evolution of auxiliary fields U, V and Z as a function of $N = \ln a$, for different values of W_0 and corresponding γ. Initial conditions: $U_0 = 0$, $V_0 = 0$, $Z_0 = 0$, $U_0' = 0$, $V_0' = 0$, $W_0' = 0$, $Z_0' = 0$.

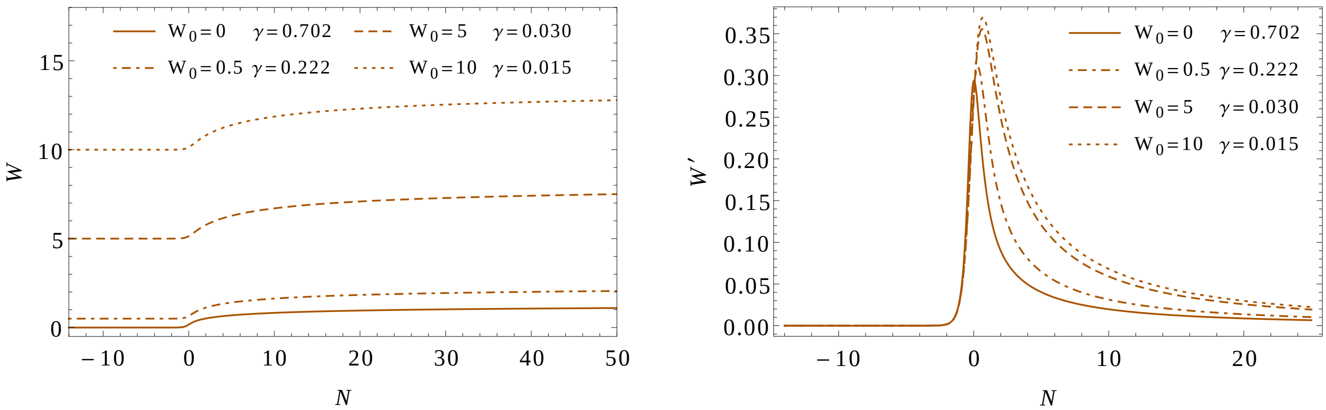

Figure 2. Evolution of the auxiliary field W, and its first derivative W' as a function of $N = \ln a$, for different values of W_0 and corresponding γ. Initial conditions: $U_0 = 0$, $V_0 = 0$, $Z_0 = 0$, $U_0' = 0$, $V_0' = 0$, $W_0' = 0$, $Z_0' = 0$.

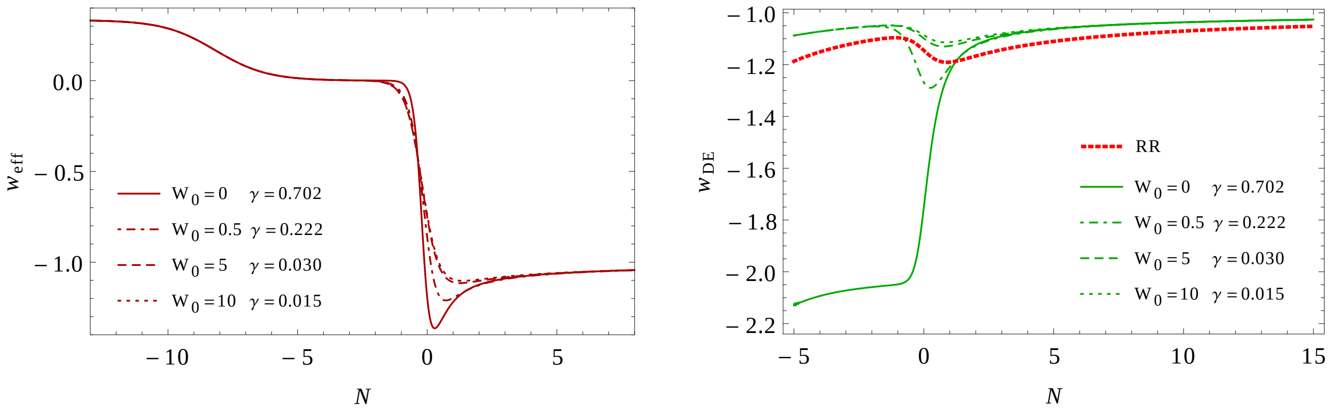

Figure 3. Left-hand panel: Evolution of $w_{\rm eff}$ as a function of $N = \ln a$, for different values of W_0 and corresponding γ. Initial conditions: $U_0 = 0$, $V_0 = 0$, $Z_0 = 0$, $U_0' = 0$, $V_0' = 0$, $W_0' = 0$, $Z_0' = 0$. Right-hand panel: Evolution of $w_{\rm DE}$ as a function of $N = \ln a$, for different values of W_0 and the corresponding γ. Initial conditions: $U_0 = 0$, $V_0 = 0$, $Z_0 = 0$, $U_0' = 0$, $V_0' = 0$, $W_0' = 0$, $Z_0' = 0$.

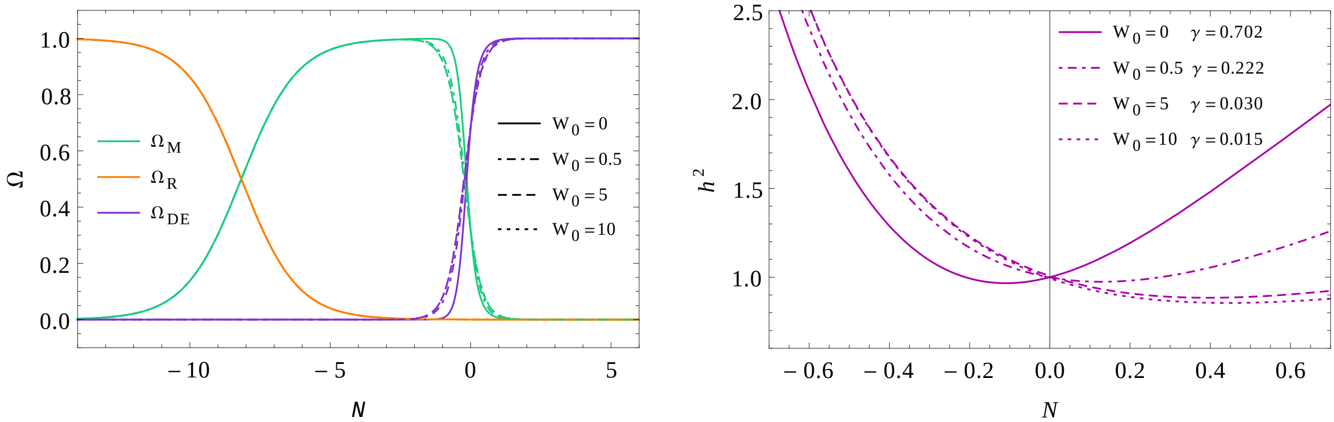

Figure 4. Left panel: Evolution of $\Omega_{\rm M}$, $\Omega_{\rm R}$, $\Omega_{\rm DE}$ as a function of $N = \ln a$, for different values of W_0 and corresponding γ. Initial conditions: $U_0 = 0$, $V_0 = 0$, $Z_0 = 0$, $U_0' = 0$, $V_0' = 0$, $W_0' = 0$, $Z_0' = 0$. Right panel: Evolution of Hubble function h .

V. RELATION BETWEEN $R\Box^{-2}R$ AND $\Box^{-2}R$ MODELS

As one can see from Figures (1-4), the numerical evolution of the auxiliary fields as well as the behavior of $w_{\rm eff}$ and $\Omega_{\rm M,R,DE}$ are very similar to those of the RR nonlocal model presented in Refs. [23, 26]. In order to understand why two, at first glance, completely different models exhibit almost the same cosmological evolution let us first compare them at the level of the actions. Here we just concentrate on the gravitational sectors:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - \frac{M^4}{6}\frac{1}{\Box^2}R\right], \tag{29}$$

$$S^{\rm RR} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - \frac{m^2}{6}R\frac{1}{\Box^2}R\right]. \tag{30}$$

By comparing the actions (29) and (30) we notice that when for these two models to predict the same behavior at late times one needs to have for that period

$$m^2 R = \beta M^4, \tag{31}$$

where β is some constant parameter of the proportionality. The relation (31) in the language of the dimensionless parameter γ can be written as

$$\beta^{-1}\gamma_{\rm RR} = B\gamma, \tag{32}$$

with $B \equiv H_0^2/R$. The value of the $\gamma_{\rm RR}$ and γ should be fixed in a way to reproduce the correct dark energy density nowadays. For the RR nonlocal gravity model the parameter $\gamma_{\rm RR}$ is estimated to be $\gamma_{\rm RR} \simeq 0.0089$ [14]. Moreover,

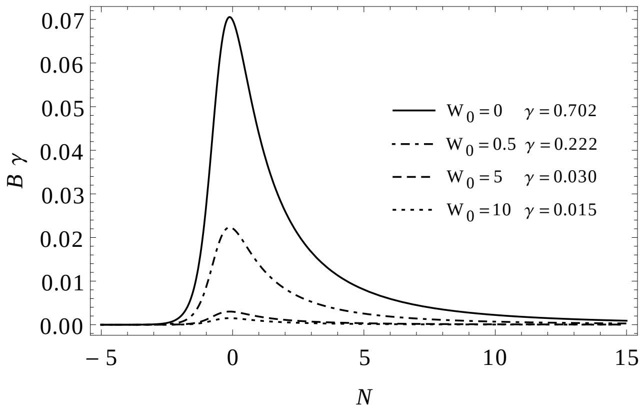

Figure 5. The evolution of $B\gamma$ as a function of e-foldings $N = \log a$ for different values of γ.

from Figure 5 we see that once we fix the value of γ to reproduce the exact matter content in the universe, due to the running of B the quantity $B\gamma$ grows initially and then after some time ($N \approx 5$) saturates and stays approximately constant. This means that the two theories then should become equivalent, i.e. the condition (32) holds. This can also be recognized from the right panel of Figure 3, where we display the evolution of the dark energy equation of state. One can see that at early times the two models behave very differently and then around $N \approx 1.5$ approach each other.

VI. LOCAL GRAVITY CONSTRAINTS

Another important point to be discussed is whether the constraints on the gravitational constant G in the solar system are satisfied. Indeed, as it was already discussed in Ref. [11], a vacuum-polarization-driven running of G can lead to serious difficulties with experimental constraints on the time variability of G. Solar system measurements put strong constraints on the time variation of G [27] $|\dot{G}/G| < 10^{-12}\mathrm{yr}^{-1}$.

It is important to mention that the above mentioned constraints on the time variation of the effective gravitational constant have been derived for the Earth-Moon system. In this respect it is important to know whether we can use the time variation of G calculated at the cosmological scales inside the Earth-Moon system. In Refs. [28, 29] it has been argued that this question should be taken with a special care. Indeed, inside the local scales such as the solar system, Earth-Moon system and etc, we do not have expansion with the Hubble rate as is the case for very large scales. This boils down to the question whether inside the solar system the scale factor a in Eq. (A14) has time dependence or not. So, if the scale factor a is time dependent, the d'Alembertian operator will also depend on time so G will vary inside the local scales. In the opposite case, the effective gravitational constant will be time independent so the constraints on it will be trivially satisfied. Let us also emphasize that even if on the background level, for local scales, the time dependence of G can be neglected it will not guarantee that the result will be the same also on perturbative level. Indeed, possible time and coordinate dependent perturbations in the local scales can reintroduce a time dependence of G which then should be consistent with all constraints. In any case, this is an open question and deserves a dedicated study which is beyond the scope of the current work.

After simple algebraic steps we get that for our model today's rate of G is

$$|\dot{G}/G|_0 \approx |\frac{12\gamma}{2-3\gamma}|H_0. \tag{33}$$

In our case valid cosmological models are obtained for non-vanishing values of W_0. Plugging the corresponding values of γ into Eq. (33) we get that indeed for all these cases $|\dot{G}/G|_{H_0} \lesssim H_0 \lesssim 10^{-12}\text{yr}^{-1}$. So, the models which have a valid cosmological evolution satisfy the local constraints too.

VII. CONCLUSIONS

Nonlocal cosmological models have been the topic of intense work in the last few years [26, 30–34]. They can be seen as an attempt at capturing quantum corrections to the Einstein-Hilbert Lagrangian and to provide, at the same time, an accelerated cosmology even in the limit of vanishing cosmological constant. In this paper we have studied in detail the background cosmological evolution of a novel nonlocal model inspired by a quantum gravity induced non-perturbative effective action in which the FRG running of the gravitational constant, in a coordinate space, is manifested by nonlocal operators. The model depends on a single dimensional constant M.

We find that, when the dimensional coupling constant is chosen appropriately, this model reproduces a viable background with a final stable acceleration compatible with current constraints. Comparing our model with the one proposed in [14], we find that the two models exhibit a different behavior in the past, but converge near the present epoch. We also observe that the background evolution of the current model is sensitive to the choice of initial conditions for the auxiliary field W (W_0). In the case of vanishing W_0 the dark energy equation of state parameter $w_{\rm DE}$ exhibits very strong phantom behavior and is in strong tension with current observational data. Furthermore, the model with vanishing W_0 does not pass the local gravity constraints on the time variation of G. The situation is completely different for non-vanishing choices of W_0, such that, even a small non-vanishing value of W_0 changes the overall behavior of the model sufficiently, making it compatible with observational constraints. As for any cosmological model, one should also investigate the growth of matter perturbations to ensure compatibility with observations. This is left to future work. Another work in progress is devoted to the investigation of cases with general (rational) values of the critical exponent ν. These studies will allow us to deal with more realistic RG improved effective actions, where the value of critical exponent ν, in general, can be arbitrary.

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Appendix A: A general case

1. The Model

We consider the general model defined by the following action:

$$S^{\rm NL} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - \mu f(R)\frac{1}{\Box^2}g(R)\right] + \int d^4x\sqrt{-g}\mathcal{L}_{\rm m}, \tag{A1}$$

where f and g are analytic functions of the Ricci scalar, μ stands for the scale of nonlocality and $\mathcal{L}_{\rm m}$ is the matter Lagrangian minimally coupled to gravity.

Taking the variation with respect to the metric tensor $g_{\mu\nu}$,

$$\begin{aligned}\delta S^{\rm NL} &= \frac{1}{16\pi G}\int d^4x\delta(\sqrt{-g})\left[R - \mu f\frac{1}{\Box^2}g\right] \\ &+ \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\delta R - \mu f_{,R}\,\delta R\frac{1}{\Box^2}g - \mu f\delta(\Box^{-2})g - \mu f\frac{1}{\Box^2}g_{,R}\,\delta R\right] + \delta\int d^4x\sqrt{-g}\mathcal{L}_{\rm m}\ ,\end{aligned} \tag{A2}$$

making use of the identity

$$\delta(\Box^{-2}) = -\Box^{-1}\delta(\Box)\Box^{-2} - \Box^{-2}\delta(\Box)\Box^{-1}, \tag{A3}$$

and introducing the four auxiliary fields

$$U = -\Box^{-1} g \,, \qquad Q = -\Box^{-1} f \,,$$
$$S = -\Box^{-1} U \,, \qquad L = -\Box^{-1} Q \,, \tag{A4}$$

we find the following equations of motion,

$$G_{\alpha\beta} = \mu K^{\rm NL}_{\alpha\beta} + 8\pi G T_{\alpha\beta}, \tag{A5}$$

where the contribution coming from the nonlocal term is the following:

$$\begin{aligned} K^{\rm NL}_{\alpha\beta} \equiv & (f,_R S + g,_R L) R_{\alpha\beta} - \nabla_\alpha \partial_\beta (f,_R S + g,_R L) + g_{\alpha\beta} \Box (f,_R S + g,_R L) + \\ & - \frac{1}{2} g_{\alpha\beta} [fS + gL] + \frac{1}{2} g_{\alpha\beta} g^{\sigma\lambda} (\partial_\sigma Q \partial_\lambda S + \partial_\sigma U \partial_\lambda L) + \\ & - \partial_\alpha U \partial_\beta L - \partial_\beta Q \partial_\alpha S - \frac{1}{2} g_{\alpha\beta} U Q. \end{aligned} \tag{A6}$$

For simplicity we investigate the action (A1) in the case when $f(R)$ and $g(R)$ have a general power-law structure. Namely, and $g(R)$ are chosen to be $f(R) = R^p$ and $g(R) = R^n$, respectively, with n and p being integer non-negative numbers. The equations of motion for this model are easily obtained from Eqs. (A5) and (A6) by a direct substitution

$$\begin{aligned} G_{\alpha\beta} = \mu \Big\{ & (R_{\alpha\beta} - \nabla_\alpha \nabla_\beta + g_{\alpha\beta} \Box)(p R^{p-1} S + n R^{n-1} L) + \\ & - \frac{1}{2} g_{\alpha\beta} (R^p S + R^n L) + \frac{1}{2} g_{\alpha\beta} g^{\sigma\lambda} (\partial_\sigma Q \partial_\lambda S + \partial_\sigma U \partial_\lambda L) + \\ & - \partial_\alpha U \partial_\beta L - \partial_\beta Q \partial_\alpha S - \frac{1}{2} g_{\alpha\beta} U Q \Big\} + 8\pi G T_{\alpha\beta}, \end{aligned} \tag{A7}$$

while the auxiliary fields satisfy the following set of coupled differential equations

$$\Box U = -R^n \,, \qquad \Box Q = -R^p \,,$$
$$\Box S = -U \,, \qquad \Box L = -Q \,. \tag{A8}$$

If we introduce in the Lagrangian a parameter M with the dimension of a mass, we can redefine the μ parameter in (A7) as follows:

$$\mu = \frac{1}{6} M^{-2(n+p)+6} \,. \tag{A9}$$

If $n = 1$ and $p = 1$ from Eqs. (A7) and (A8) we find the equations of motion of RR nonlocal gravity model [14].

$$G_{\alpha\beta} = \frac{M^2}{6} K^{\rm RR}_{\alpha\beta} + 8\pi G T_{\alpha\beta}, \tag{A10}$$
$$\Box U = -R, \tag{A11}$$
$$\Box S = -U, \tag{A12}$$

where $K^{\rm RR}_{\alpha\beta}$ tensor, which stands for the correction to Einstein equations coming from nonlocal corrections, is defined as

$$\begin{aligned} K^{\rm RR}_{\alpha\beta} \equiv \;& 2(G_{\alpha\beta} - \nabla_\alpha \nabla_\beta + g_{\alpha\beta} \Box) S + g_{\alpha\beta} g^{\sigma\lambda} \nabla_\sigma U \nabla_\lambda S + \\ & - (\nabla_\alpha U \nabla_\beta S + \nabla_\beta U \nabla_\alpha S) - \frac{1}{2} g_{\alpha\beta} U^2. \end{aligned} \tag{A13}$$

These equations fully coincide with those for the RR nonlocal gravity model [14].

2. Cosmological Equations

In this section we will study the background cosmology of our model. To do this we choose our metric to be of a flat FLRW type

$$ds^2 = -\frac{1}{H^2} dN^2 + a^2 d\vec{x}^2, \tag{A14}$$

with a being the scale factor, H the Hubble rate and $N = \log a$ the number of e-foldings.

We also introduce the following quantities

$$h = \frac{H}{H_0}, \qquad \xi \equiv \frac{H'}{H} = \frac{h'}{h}, \tag{A15}$$

where H_0 is the Hubble rate today. For the cosmological analysis it is sometimes useful to go from dimensionful quantities to dimensionless ones. To do this we multiply our dimensionful auxiliary fields by powers of H_0 and as such we define new dimensionless auxiliary fields as follows

$$X \equiv H_0^{2-2n} U, \qquad W \equiv H_0^{2-2p} Q,$$
$$V \equiv H_0^{4-2n} S, \qquad Z \equiv H_0^{4-2p} L.$$

From the (00) component of Eq. (A7) one can express h^2 through new dimensionless functions defined above:

$$h^2 = \frac{2\mu}{3H_0^{6-2(n+p)}} h^2 Y^{\rm NL} + \frac{8\pi G}{3H_0^2}\rho, \tag{A16}$$

where all terms coming from the nonlocal part are collected in the following quantity:

$$\begin{aligned} Y^{\rm NL} =& \frac{1}{4h^2}\left\{WX - h^2\left(V'W' + Z'X'\right) + h^{2p}\left[C_p\left(VB_p + pV'\right) + D_p V\xi'\right] + \right. \\ & \left. + h^{2n}\left[C_n(ZB_n + nZ') + D_n Z\xi'\right]\right\}, \end{aligned} \tag{A17}$$

where, to simplify the notation, we have defined the following coefficients

$$B_k \equiv (2k-1)(k-1)\xi - k + 2\,, \tag{A18}$$
$$C_k \equiv 6^k(\xi+2)^{k-1}\,, \tag{A19}$$
$$D_k \equiv k(k-1)6^k(\xi+2)^{k-2}\,. \tag{A20}$$

These terms only depends on ξ and on k, where the latter takes the values of either n or p. We will also express the dimensionful scale parameter of the model μ through a new dimensionless quantity defined as

$$\gamma \equiv \frac{2}{3}\mu H_0^{2(n+p)-6}\,. \tag{A21}$$

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CHAPTER 8

SUMMERY, DISCUSSION AND CONCLUSION

In this thesis we have studied the role of different IR-relevant nonlocal modifications of GR in the late evolution of our Universe. We have investigated cosmological properties of several well motivated models, both at background and perturbative levels. Some open issues in the formulation of nonlocal theories have been also addressed.

8.1 Dynamical analysis of $R\Box^{-2}R$ model: Impact of initial conditions and constraints from supernovae

Within the family of IR-relevant nonlocal gravity models, the RR model has received a lot of attention in the last years. The corresponding action of the theory is

$$S = \frac{1}{16\pi G}\int\sqrt{-g}\,d^4x\left[R - \frac{1}{6}m^2 R\frac{1}{\Box^2}R\right] + \int\sqrt{-g}\,d^4x\mathscr{L}_{\rm m}, \tag{8.1}$$

where m is the only free parameter of the model and has a dimension of mass. This model was first introduced in Ref. [123], where the authors found that the model has a background evolution consistent with current observational data [4, 6, 191]. Later in Refs. [122, 192] the RR model was analyzed at the level of perturbations. In these works, using a modified CLASS code [193], it was shown that the RR model is in good agreement with CMB, BAO, JLA supernovae and LSS data, but still distinguishable from ΛCDM.

On the fundamental side, the question of a possible origin of the RR nonlocal modification has also been widely studied. In particular, it has been argued that this type of modifications arise in higher curvature theories, after taking into account loop corrections (see discussion in Sec. 2.2). Unfortunately, as it was shown in Ref. [194] the mass scale m of the theory, estimated from perturbative loop expansion, is not sufficient to account for the DE density today ($\Omega_{\rm DE} \approx 68\%$).

This result has been derived by assuming that masses of particles running over loops are constant [23, 24, 129]. In cases when the running of mass terms for the particles is also taken into account, it might generate a value of the scale parameter m such that nonlocal modifications become relevant in the IR [27, 131]. Another possible origin of the RR type corrections refers to the dynamical mass generation for the conformal mode σ, which is defined from the conformal transformation $g_{\mu\nu} = e^{2\sigma}\overline{g}_{\mu\nu}$. As discussed in Sec. 2.2, in this language the nonlocal modification in (8.1) corresponds to the mass term of the conformal mode σ, i.e. $m^2 R\Box^{-2}R \sim m^2\sigma^2$.

From the technical point of view, physical solutions of the RR model were found [123] by localizing the action (8.1). To accomplish this, one introduces two auxiliary fields U and S as solutions of the equations $\Box U = -R$ and $\Box S = -U$, respectively, such that the localized action is

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\,d^4x\left[R - \frac{1}{6}m^2 RS + \xi_1(\Box U + R) + \xi_2(\Box S + U)\right] + \int \sqrt{-g}\,d^4x\mathscr{L}_{\rm m}, \tag{8.2}$$

where ξ_1 and ξ_2 are the corresponding Lagrange multiplier. In our work presented in Chapter 4, investigating full properties of the RR model we have shown that the previously found solution space of the model is not complete and other solutions exist leading to completely different cosmological evolution of the model at late times. The main reason behind the fact that big part of theory's solution space was previously discarded originates from misleading assumptions about initial conditions on auxiliary fields, which where set to be always vanishing. In our work, we studied different mechanisms which can generate non-trivial initial conditions for the auxiliary fields. With the methods of dynamical phase-space analysis we constructed the full solution space of the RR model. We observed that depending on the initial condition of the field U, U_0, there are two distinct scenarios for the cosmic evolution of the RR model, separated by the threshold line $\overline{U}$ that only depends on the mass parameter m. The two scenarios are: In cases when U_0 is bigger than $\overline{U}$, the system follows the trajectory referred to as path A. The numerical evolution of the effective equation of state $w_{\rm eff}$ along this path shows that the model goes from the radiation-domination period ($w_{\rm eff} = 1/3$) towards the matter-domination period ($w_{\rm eff} = 0$) which is then followed by a transient phantom regime ($w_{\rm eff} < -1$) that asymptotically approaches a quasi *de Sitter* state ($w_{\rm eff} = -1$) at late times[1]. This solution was previously discussed in the literature [122, 123]. If the value of U_0 is smaller than the threshold value $\overline{U}$, the system will evolve along a different trajectory referred to as path B. In this scenario the system will move from the radiation-domination period towards the matter-domination period as for the case of path A. Then the system moves from matter-domination to a true *de Sitter* era ($w_{\rm eff} \simeq -1$ and $H = \text{const}$), which is then followed by a radiation like state with $w_{\rm eff} = 1/3$. This type of cosmic evolution for the RR model is the novel result of our work and has not been discussed before. We compared our results with the current SNIa data and found that the evolution of the RR model along path B is cosmologically viable and placed constraints on the parameters of the model.

[1]This state is referred to as a quasi *de Sitter* state, because though $w_{\rm eff} = -1$, the Hubble parameter is not constant, which should be the case for the *de Sitter* solution.

A next step in this direction is to test path B at perturbation level by checking its consistency with CMB, BAO and LSS data. It is important to mention that even though our studies in Chapter 4 were done for a particular nonlocal model, the approach developed in this work is very general and can be easily extended also for other nonlocal models. This is indeed crucial, as it can well be that also for other nonlocal models misleading assumptions about initial conditions have led to an incomplete investigation of their solution spaces.

8.2 Structure formation in the DW nonlocal gravity model

The Deser-Woodard (DW) nonlocal gravity model is a pioneer within nonlocal gravity models trying to address IR-relevant physics of the Universe. This model was first introduced in Ref. [166] and is defined by the action

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\,d^4x \left[R + R f\left(\frac{R}{\Box}\right)\right] + \int \sqrt{-g}\,d^4x \mathscr{L}_m, \tag{8.3}$$

where f is a general analytic function of its argument. Once we relax the assumption of locality at large scales, there is no *a priori* reason that forbids adding this type of terms to the Einstein-Hilbert action. Indeed, due to the fact that the Ricci scalar R and the d'Alembert operator $\Box$ have the same physical dimension, the function $f(R/\Box)$ is dimensionless. Hence, the function f does not introduce any new physical scale into the theory and multiplied with R it has the same importance as the Einstein-Hilbert term and cannot be neglected. Since the function f respects diffeomorphism symmetry, there are no symmetry based arguments which prohibit its addition to the Einstein-Hilbert action as well. The background cosmology of the DW model has been extensively studied in Refs. [166–168, 190]. In particular, in Ref. [167] the authors have reconstructed the functional form of f as

$$f(X) = 0.245\Big[\tanh(0.350Y + 0.032Y^2 + 0.003Y^3) - 1\Big], \tag{8.4}$$

with $Y \equiv X + 16.5$, so that, it reproduces exactly ΛCDM cosmology. This choice was justified by the fact that any viable background cosmology should be sufficiently close to the one of ΛCDM, which fits all the background observational data remarkably well. On the other hand, even if two models are the same at the background level it does not guarantee that they would be the same also at the perturbation level. Indeed, as it has been shown in Refs. [195, 196] the DW model at the perturbation level behaves completely different from the ΛCDM model. Moreover, in the same papers the authors have found that the DW model is in strong tension with Redshift Space Distortions (RSD) data and is ruled out at up to 7.8σ confidence level.

In our work presented in Chapter 5, we revisited this problem. By localizing the model we have surprisingly found that the previous results which rule out the model are not true for the localized version. Furthermore, the localized version of the DW model predicts a present-time value of the amplitude of fluctuations $\sigma_8^0 = 0.78$, which is in a better (1σ) agreement with recent data ($\sigma_8^0 = 0.745^{+0.038}_{-0.038}$) [6, 197] than the ΛCDM model with $\sigma_8^0 = 0.83$. We have analyzed how much our fit with RSD data improves when we generalize initial conditions of the growth rate function. In our work we have found that the corresponding improvements are just marginal.

The next step on the phenomenological side is to implement the model into a Boltzmann code which will allow to integrate the perturbation equations of the DW model for all energy scales. Doing so we can probe our model against full CMB constraints.

On the theoretical side, we can try to simplify the functional structure of f while preserving the success of this model at the perturbation level. This can be done by relaxing the requirement that the DW model exactly resembles the background cosmology of the ΛCDM model. Indeed, there is no reason behind why at the background level the DW and ΛCDM models should be exactly the same. This was initially just a useful assumption for the reconstruction procedure. In reality, one could come up with the function f, which does not reproduce exact ΛCDM behavior, but still is consistent with observational data.

8.3 Tensorial nonlocal gravity

In general (see e.g. discussion in Sec. 2.2) nonlocal modifications of GR generated from quantum corrections have also tensorial terms in their structure. These terms are mostly composed of nonlocal operators placed between two Ricci tensors or Riemann tensors. In the context of IR-relevant nonlocal gravity theories only the phenomenology of scalar radiative corrections (those constructed from the Ricci scalar) has been studied [19, 168]. This is mainly due to the fact that tensorial nonlocal models exhibit fast growing modes which render the model nonviable already at the background level. In our work presented in Chapter 6 we tried to see whether these instabilities are due to a particular choice of the nonlocal operator or whether they have a more generic nature. In this respect, we have analyzed the following tensorial nonlocal modification of the Einstein-Hilbert action

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\, d^4x \left[R - \frac{1}{6}m^2 R_{\mu\nu}\frac{1}{\Delta}R^{\mu\nu}\right] + \int \sqrt{-g}\, d^4x \mathscr{L}_{\rm m}, \tag{8.5}$$

where Δ is the most general fourth-order operator defined as

$$\Delta \equiv m^4 + \alpha_1\Box + \alpha_2\Box^2 + \beta_1 R_{\mu\nu}\nabla^\mu\nabla^\nu + \beta_2 R\Box + \gamma\left(\nabla^\mu R_{\mu\nu}\right)\nabla^\nu, \tag{8.6}$$

with α_1, α_2, β_1, β_2 and γ as free parameters. This choice of the operator is motivated by the fact that it covers two very important cases. First, for particular values of free parameters it reduces to the conformal-anomaly induced Paneitz operator (2.60)

$$\Delta_4 \equiv \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}g^{\mu\nu}\nabla_\mu R\nabla_\nu. \tag{8.7}$$

Secondly, it covers the case of the nonlocal operator corresponding to a massive propagator $\left(\Box + m^2\right)^{-1}$. To have a valid background cosmology we need sufficiently lasting radiation and matter domination eras, which then will be followed by the DE domination period. Therefore, any modification of GR which tries to address the DE problem should not violate the standard evolution of the Universe during radiation and matter domination periods. For tensorial non-local theories this means that their energy density $\rho_{\rm NL}$ should be negligible compared to the radiation and matter energy densities during the periods of radiation and matter-domination, respectively. In our work we have found that growing modes are also present for our choice of the nonlocal operator (8.6). These modes, in general, grow so quickly that shortly after the onset of radiation-domination period the nonlocal energy density $\rho_{\rm NL}$ dominates over the radiation energy density, which terminates the regular cosmic evolution. The same occurs also during the matter-domination period. In our work we have concluded that this is a warning signal, revealing that for the case of tensorial nonlocal models one needs to introduce particular mechanisms (symmetries) which will naturally suppress or eliminates unstable modes, hence allowing to construct a consistent cosmological model.

8.4 Quantum Gravity inspired nonlocal gravity model

As we know GR is perturbatively non-renormalizable [152–154]. In this respect to construct a predictable quantum theory of Gravity, one has to implement either new techniques of renormalization or to go beyond GR. In the first case, a possible solution is to implement non-perturbative renormalization methods [8, 13, 156], while for the second case we need to add particular, usually higher curvature terms [137, 138, 198, 199], which could cancel divergences making the theory renormalizable at every loop order. Most importantly, whatever modification we make, we need to recover GR for those scales where it is estimated to give a good description of nature. In this context we can think of GR as an effective theory defined at some particular energy scale k. In this case the corresponding effective action is

$$\Gamma_k = \frac{1}{16\pi G_k}\int d^4x\sqrt{g}\,(-R+2\Lambda_k)+\int\sqrt{-g}\,d^4x\mathscr{L}_{\rm m}. \tag{8.8}$$

In this action the gravitational constant G and the cosmological constant Λ are now scale dependent quantities. Their evolution is governed by the corresponding flow equations. As argued in Ref. [156], a possible existence of a UV non-Gaussian fixed point (NGFP) u^*, i.e. such that for the beta-functions we have $\beta(u^*)=0$ for $u^*\neq 0$, can render the theory nonperturbatively renormalizable (see the discussion in Sec. 2.4). In Refs. [37, 38], based on similar ideas, it has been argued that nonperturbative lattice quantum gravity studies suggest that the strength of gravitational interactions might increase with distance. This behavior is encoded in the running of the gravitational constant G_k, which is given by

$$G_k = G_0\left(1+c_\zeta\left(\frac{1}{\zeta^2k^2}\right)^{1/2\nu}+\mathscr{O}\left(\left(1/\zeta^2k^2\right)\right)^{1/\nu}\right), \tag{8.9}$$

where ν is the critical exponent and ζ is the IR-relevant nonperturbative scale parameter of the theory. In the limit of small k, corresponding to IR scales, and for the positive ν, the effective gravitational constant G_k grows thus strengthening gravitational interactions.

The value of the critical exponent highly depends on the details of the implemented calculation scheme. In general, predictions for the value of ν suggest that it would be in the interval $\nu \in [1,4]$ [200–202]. The value of a generic nonperturbative scale parameter ζ, as is mentioned in Ref. [203], cannot be fixed within the underlying theory and one needs additional input to determine its value. In the same work it was argued that the ζ can either be associated with the inverse of the vacuum expectation value of the curvature scalar $\zeta^{-1} \sim \langle R \rangle$, or more naturally with the inverse of the Hubble parameter, i.e. $\zeta^{-1} \sim H$ (see also discussion in Sec. 2.4).

In our work presented in Chapter 7, we have studied the phenomenology of the model by inserting the running of G from (8.9) into the effective action (8.8). To write the effective action in a coordinate space we have identified the relevant cutoff k^2 with the d'Alembert operator $\Box$, which gives[2]

$$S = \frac{1}{16\pi G_0} \int d^4x \sqrt{-g} \left(1 - c_\zeta \left(\frac{1}{\zeta^2 \Box} \right)^{1/2\nu} + \mathcal{O}\left(\left(\zeta^2 \Box \right)^{-1/\nu} \right) \right) R + \int \sqrt{-g}\, d^4x \mathcal{L}_{\rm m}. \tag{8.10}$$

We derived the effective EoMs by varying this action w.r.t. the metric $g_{\mu\nu}$. Since the d'Alembert operator depends on the metric too, it will also undergo variation. This is a crucial point of our work. In previous studies [37, 38] the cosmology of the current model has been studied by directly embedding $G(\Box)$ into the right hand side of the Einstein equations. However, as discussed in our work, this modification, made directly at the level of Einstein equations, does not respect the Bianchi identities and in principle is not valid. Contrary to this, in our case, as we derive the modified Einstein equations directly from the covariant effective action, the Bianchi identities will be satisfied automatically.

Another important point to be mentioned, refers to the choice of the value for the critical exponent. Indeed, depending on its value the power of the operator $\Box^{-1}$ in (8.10) can either be rational or integer. In general it is not trivial to deal with a rational power of an inverse operator, though there are some mathematical tools [165, 183] which provide useful guidelines on how to proceed in this case. In our work in Chapter 7, as a first step we chose the value of the critical exponent such that the d' Alembert operator occurs with an integer power in the effective action. Since the exponent of the d' Alembert operator in the effective action (8.10) is $-1/2\nu$, in the interval $\nu \in [1,4]$ there are only two values of the critical exponent which lead to an integer power of the d'Alembert operator: $\nu^{-1} = 2$ or $\nu^{-1} = 4$. Recently, in Ref. [205] from completely different perspectives the authors have developed a nonlocal model which is similar to the case with $\nu^{-1} = 2$. Thus, in our work we have investigated the case with $\nu^{-1} = 4$. For this

[2]The correct effective action also contains the running of the CC term Λ_k. But as is argued in Refs. [37, 204] in pure lattice gravity the bare CC Λ does not run and has to be fixed properly. In our work in agreement with Ref. [37] we have assumed that the contribution from the CC is sub-dominant with respect to the one of the running G, so that the former can be approximated with $\Lambda \approx 0$.

choice, the d' Alembert operator enters the effective action with quadratic power. If an action contains a nonlocal operator which has a power-law structure with an integer power, it is possible to localize the theory by introducing several auxiliary fields. For our case to localize the model we have introduced four auxiliary fields. Investigating the model we found that for ζ being of the order of the inverse Hubble parameter ($\zeta \approx H_0^{-1}$), the model produces a valid background evolution with a final accelerated state compatible with current observations. Moreover, we have observed that the background evolution of the model is strongly sensitive to the value of initial conditions on one of the auxiliary fields. For vanishing initial conditions the model exhibits strong phantom behavior and is highly disfavored by current observational data. On the other hand, even a small non-vanishing value of the initial condition, changes the situation completely by efficiently softening the phantom behavior of the model and making it compatible both with local as well as large scale cosmological data.

By comparing the background evolution of this model with the one of the RR model [123], we find that although the two models have a different behavior in the past they converge near the present epoch making them indistinguishable in the future. Even if the two models predict similar behavior at late times, they have different origins and motivations from the theoretical point of view. In particular, the current model is defined at the theory level with the scale ζ, which is of the order of the inverse Hubble parameter H_0^{-1}, whereas for the RR model the scale parameter m is completely free and is fixed later to be of the order of H_0, in order to explain observational data. This circumstance, in some sense gives a preference to the current model over the RR one. As for any cosmological model one has to investigate the growth of structures to ensure that it is consistent with corresponding observations. This task is under progress. Another important direction of studies within quantum gravity inspired models are investigations of cases where the critical exponent ν is a general rational number. In this cases the effective action will consist of inverse d'Alembert operators with non-integer powers. These studies will allow us to have a better understanding of phenomenological properties of IR-relevant quantum gravity modifications, where the value of the critical exponent, in general, can be arbitrary.

8.5 Conclusion

Nonlocal gravity theories are theoretically well-motivated modifications of GR. They attempt to use different quantum corrections to GR in order to address still unsolved physical questions in cosmology. In this respect, there are already several viable nonlocal gravity models which successfully target some of those open physical issues like Dark Energy, Dark Matter, Big Bang Singularity, Black Hole Singularity, inflation.

Our results represented in this thesis first of all try to make a clarification in the theoretical formulation of these models, emphasizing the importance of initial conditions in their cosmological evolution. Secondly, we show that one of the nonlocal gravity models, namely the DW model, gives a better agreement with observational data than the ΛCDM model. This is a quite intriguing result as only a few modified gravity models are able to perform observationally better than ΛCDM model, while being fully consistent at the theoretical level. In this thesis we also studied the issue of classical instabilities of tensorial nonlocal gravity models. We found that these pathologies seem to be generic and there is a need of a comprehensive investigation of special mechanisms which may render tensorial nonlocal gravity models valid. This is very important because from a physical point of view tensorial nonlocal models are of the same relevance as their scalar analogs and in realistic physical situations they usually appear simultaneously. Finally, we also analyzed a possible nonlocal modification motivated by non-perturbative QG studies and we found that these models in general are able to lead to a valid late time cosmology of the Universe. This result gives us the hope that a possible QG theory might be able to provide a consistent and complete description of the Universe's cosmic evolution.

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