INAUGURAL-DISSERTATION

zur

Erlangung der Doktorwürde

der

Naturwissenschaftlich-Mathematischen Gesamtfakultät

der

Ruprecht-Karls-Universität Heidelberg

vorgelegt von

Diplom-Mathematikerin Ann-Kristin Juschka

aus Düsseldorf

Tag der mündlichen Prüfung: ______________

Geometric Properties of Versal Deformation Rings and Universal Pseudodeformation Rings

Geometrische Eigenschaften von versellen Deformationsringen und universellen Pseudodeformationsringen

Gutachter: Prof. Dr. Gebhard Böckle

ABSTRACT

Consider the absolute Galois group G_K of an extension K of $\mathbb{Q}_p$ finite degree d, and a finite field $\mathbb{F}$ of prime characteristic p. Following Mazur [Maz89], we define the versal deformation ring $R^{\psi}_{\bar\rho}$ with fixed determinant of a Galois representation $\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F})$. Then for $n = 2$ and $p > 3$ our first main result states that $R^{\psi}_{\bar\rho}$ is an integral domain so that the associated versal deformation space $\mathfrak{X}(\overline{\rho})$ is irreducible. For this, we use the explicit relations of $R^{\psi}_{\bar\rho}$ computed in [Böc00] and a commutative algebra criterion. We deduce from [Nak13] that for $n = 2$ and any K the benign crystalline points are Zariski dense in $\mathfrak{X}(\overline{\rho})$. This is expected to be useful for the surjectivity of the p-adic local Langlands correspondence. Furthermore, for arbitrary n and p we show that the refined quadratic parts of the relations of $R^{\psi}_{\bar\rho}$ can be obtained cohomologically from a cup product and a Bockstein homomorphism if a certain lift of $\overline{\rho}$ exists. Following Chenevier [Che14], we construct the universal pseudodeformation ring $R^{\mathrm{univ}}_{\overline{D}}$ of an n-dimensional pseudorepresentation $\overline{D}\colon \mathbb{F}[G_K] \to \mathbb{F}$. Motivated by the result [Che11] on the equidimensionality of the generic fiber of the universal pseudorepresentation ring in characteristic 0, our second main result says that the special fiber $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ of $R^{\mathrm{univ}}_{\overline{D}}$ is equidimensional of dimension $dn^2 + 1$ if $p > n$ or if K does not contain a primitive p^{th} root of unity ζ_p. In the latter case, if either $n > 2$ or $n = 2$ and $d > 1$ we prove that the regular locus of $\operatorname{Spec} \overline{R}^{\mathrm{univ}}_{\overline{D}}$ consists of certain irreducible pseudodeformations and that $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ satisfies Serre's condition (R_2).

ZUSAMMENFASSUNG

Betrachte die absolute Galoisgrouppe G_K einer Erweiterung K von $\mathbb{Q}_p$ von endlichem Grad d und einen endlichen Körper $\mathbb{F}$ von Primzahlcharacteristik p. Mazur [Maz89] folgend, definieren wir den versellen Deformationsring $R^{\psi}_{\bar\rho}$ mit fester Determinante einer Galoisdarstellung $\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F})$. Dann sagt unser erstes Hauptresultat für $n = 2$ und $p > 3$ aus, dass $R^{\psi}_{\bar\rho}$ ein Integritätsring ist, sodass der dazugehörige verselle Deformationsraum $\mathfrak{X}(\overline{\rho})$ unzerlegbar ist. Dafür benutzen wir die expliziten Relationen von $R^{\psi}_{\bar\rho}$, die in [Böc00] berechnet wurden, und ein Kriterium aus der Kommutativen Algebra. Wir folgern aus [Nak13] für $n = 2$ und beliebiges K, dass die benignen krystallinen Punkte Zariski-dicht in $\mathfrak{X}(\overline{\rho})$ sind. Dies ist voraussichtlich nützlich für die Surjektivität der p-adischen lokalen Langlands-Korrespondenz. Des Weiteren zeigen wir für beliebiges n und p, dass die verfeinerten quadratischen Anteile der Relationen von $R^{\psi}_{\bar\rho}$ kohomologisch durch ein Cup-Produkt und einen Bockstein-Homomorphismus erhalten werden können – falls ein geeigneter Lift von $\overline{\rho}$ existiert. Chenevier [Che14] folgend, konstruieren wir den universellen Pseudodeformationsring $R^{\mathrm{univ}}_{\overline{D}}$ einer n-dimensionellen Pseudodarstellung $\overline{D}\colon \mathbb{F}[G_K] \to \mathbb{F}$. Motiviert durch das Resultat [Che11] über die Äquidimensionalität der generischen Faser des universellen Pseudodarstellungsrings in Characteristic 0, zeigt unser zweites Hauptresultat, dass die spezielle Faser $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ von $R^{\mathrm{univ}}_{\overline{D}}$ äquidimensional von Dimension $dn^2 + 1$ ist, falls $p > n$ oder falls K keine p^{te} primitive Einheitswurzel ζ_p enthält. In letzterem Fall beweisen wir, falls entweder $n > 2$ oder $n = 2$ und $d > 1$, dass der reguläre Lokus von $\operatorname{Spec} \overline{R}^{\mathrm{univ}}_{\overline{D}}$ aus bestimmten unzerlegbaren Pseudodeformationen besteht und dass $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ Serre's Bedingung (R_2) erfüllt.

TABLE OF CONTENTS

1. INTRODUCTION

In this thesis, we prove in Chapter 4 the irreducibility of a (uni)versal deformation ring with fixed determinant, and in Chapter 3 the equidimensionality of a universal pseudodeformation ring. In the introduction we explain the meaning of these results and give some hints on their proof. We also give some motivation of our results that stems from the conjectured p-adic local Langlands correspondence. Our results may have applications to the surjectivity of this correspondence, at present for GL_2, but with more work perhaps also for GL_n. We end the introduction with an outline of the thesis.

1.1 Irreducibility of Mazur's (uni)versal deformation rings

Throughout the thesis we fix an algebraic closure K^{alg} of a finite extension K of the p-adic numbers $\mathbb{Q}_p$ of degree $d = [K : \mathbb{Q}_p]$ with absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{alg}}/K)$, a primitive p^{th} root of unity ζ_p and a finite field $\mathbb{F}$ of prime characteristic p. Consider a continuous residual Galois representation

$$\overline{\rho}\colon G_K \longrightarrow \mathrm{GL}_n(\mathbb{F}).$$

Such Galois representations arise naturally in arithmetic geometry, for instance attached to an elliptic curve defined over $\mathbb{Q}$ or associated with a modular form. Originally motivated by "big" Galois representations associated with ordinary p-adic modular forms, B. Mazur invented the study of deformations of $\overline{\rho}$ in [Maz89]. Such a "big" Galois representation parametrizes all liftings of a residual Galois representation that are of a certain type. To provide a general framework for such a family of Galois representations with the same mod p reduction, Mazur applied formal deformation theory [Sch68] as follows.

Consider the ring of Witt vectors $W(\mathbb{F})$, the ring $\mathcal{O}$ of integers of a finite totally ramified extension of $W(\mathbb{F})[1/p]$ and the category $\widehat{\mathcal{A}r}_{\mathcal{O}}$ of complete Noetherian local $\mathcal{O}$-algebras R with maximal ideal $\mathfrak{m}_R$ and residue field $\mathbb{F}$. A *deformation of* $\overline{\rho}$ *to* R is a continuous *lifting* $\rho\colon G_K \to \mathrm{GL}_n(R)$ with $\rho \otimes_R \mathbb{F} = \overline{\rho}$ up to strict equivalency, where liftings ρ_1, ρ_2 of $\overline{\rho}$ to R are called *strictly equivalent* if there is $A \in \ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(\mathbb{F}))$ such that $\rho_2(g) = A\rho_1(g)A^{-1}$ for all $g \in G_K$.

Then the *deformation functor*

$$\mathcal{D}_{\overline{\rho}}\colon \widehat{\mathcal{A}r}_{\mathcal{O}} \longrightarrow Sets, \quad R \longmapsto \{\rho\colon G_K \to \mathrm{GL}_n(R) \ : \ \rho \text{ is a deformation of } \overline{\rho}\},$$

satisfies the criteria (H1)–(H3) from [Sch68, Thm. 2.11] so that it has a versal hull $R_{\overline{\rho}} \in \mathrm{Ob}(\widehat{\mathcal{A}r}_{\mathcal{O}})$ together with a *versal deformation* $\rho_{\overline{\rho}}^{\mathrm{ver}}\colon G \to \mathrm{GL}_n(R_{\overline{\rho}})$ that parametrizes all deformations of $\overline{\rho}$. If $\overline{\rho}$ is absolutely irreducible, then the functor $\mathcal{D}_{\overline{\rho}}$ is representable by a *universal deformation ring* $R_{\overline{\rho}}^{\mathrm{univ}} \in \mathrm{Ob}(\widehat{\mathcal{A}r}_{\mathcal{O}})$ together with a *universal deformation* $\rho_{\overline{\rho}}^{\mathrm{univ}}\colon G \to \mathrm{GL}_n(R_{\overline{\rho}}^{\mathrm{univ}})$ [Maz89, Prop. 1].

Mazur also realized that, at least in some specific cases, universal deformation rings should be given by Hecke rings. This inspired A. Wiles for his famous proof of *Fermat's last theorem* [Wil95, TW95] to put his observations on Galois representations and modular forms into a ring-theoretic language. Roughly speaking, if $\overline{\rho}$ is modular and further deformation conditions of modular Galois representations are imposed on the p-adic liftings of $\overline{\rho}$, then Wiles' *modularity theorem* states that the natural homomorphism from $R_{\overline{\rho}}^{\mathrm{univ}}$ to a certain Hecke algebra T is an isomorphism. Such $R = T$ *theorems* have later been widely used in e.g. the proof of the general modularity theorem [BCDT01, Thm. A] and the Serre conjecture [KW09a, KW09b].

For these proofs ring-theoretic properties of certain universal deformation rings are established. For instance being a complete intersection, flat over $\mathcal{O}$, an integral domain or of a specific dimension. We explain in Section 1.3 which application the following ring-theoretic results on (uni)versal deformation rings from [BJ15] by G. Böckle and the author have on the p-adic local Langlands correspondence. To state the results, consider a character $\psi\colon G_K \to \mathcal{O}^\times$ that lifts $\det \overline{\rho}$ and the following subfunctor

$$\mathcal{D}^{\psi}_{\overline{\rho}}\colon \widehat{\mathcal{A}r}_{\mathcal{O}} \to \mathit{Sets}, \quad R \mapsto \{\rho\colon G_K \to \mathrm{GL}_n(R) \,:\, \rho \text{ is a deformation of } \overline{\rho} \text{ and } \det \rho = \psi \otimes_{\mathcal{O}} R\}.$$

Then $\mathcal{D}^{\psi}_{\overline{\rho}} \subset \mathcal{D}_{\overline{\rho}}$ is relatively representable and has a versal hull $R^{\psi}_{\overline{\rho}}$ with maximal ideal $\mathfrak{m}^{\psi}_{\overline{\rho}}$ [Maz97, § 24 Prop.]. The following is our first main result.

Theorem A (Theorem 4.1.5)**.** *Suppose $n = 2$ and $p > 2$. Then the following hold:*

(i) The ring $\overline{R}^{\psi}_{\overline{\rho}} := R^{\psi}_{\overline{\rho}}/\mathfrak{m}_{\mathcal{O}} R^{\psi}_{\overline{\rho}}$ is a complete intersection;

(ii) the ring $R^{\psi}_{\overline{\rho}}$ is a complete intersection and flat over $\mathcal{O}$;

(iii) the ring $R^{\psi}_{\overline{\rho}}$ is an integral domain and in particular irreducible.

The first two assertions are proven already in [Böc00] and the proof of Theorem A uses the explicit relations of $R^{\psi}_{\overline{\rho}}$ computed in Section 4.3 following [Böc00] where $n = 2$ and $p > 2$. As explained in Section 4.1, by Theorem 4.1.4 the refined quadratic parts of the explicit relations of $R^{\psi}_{\overline{\rho}}$ in a certain associated graded ring form part of a regular sequence in an integral domain and by applying tools from commutative algebra shown in Proposition 4.2.2 we deduce Theorem A.

We remark that for $n \in \mathbb{N}_{\geq 1}$ arbitrary we also prove in Theorem 4.1.14 from [BJ15] that the refined quadratic parts of the relations of $R^{\psi}_{\overline{\rho}}$ can be obtained from the bracket cup product and the Bockstein homomorphism from Section 4.5 and Section 4.6, respectively. We refer to the introduction of [BJ15] in Section 4.1 for more details.

We point out here that the idea to use the bracket cup product to determine the quadratic parts of the relations goes back to Mazur [Maz89, § 1.6 Rem.], and it is standard in deformation theory. The computation of the refined quadratic parts of the explicit relations makes use of the fact that the (uni)versal deformation $\rho^{\psi}_{\overline{\rho}}\colon G \to \mathrm{GL}_n(R^{\psi}_{\overline{\rho}})$ factors via either a free pro-p group or a Demushkin group, whose classification in Example 2.4.5 for $p > 2$ describes the Demushkin group as the quotient of a free pro-p group with generators $x_1, \ldots, x_{d+2}$ by a relation

$$r = x_1^q (x_1, x_2)(x_3, x_4) \ldots (x_{d+1}, x_{d+2}) \quad \text{with } (x_i, x_j) = x_i^{-1} x_j^{-1} x_i x_j \text{ for all } i, j.$$

Then the similarity with classical deformation theory of representations of fundamental groups of compact Kähler manifolds [GM88a, GM88b] becomes apparent by the fact that such a fundamental group of a compact Riemann surface of genus g is the quotient of the free group with basis $x_1, \ldots, x_{2g}$ by a relation

$$r = (x_1, x_2)(x_3, x_4) \ldots (x_{2g-1}, x_{2g}).$$

In [GM88a, GM88b] it is shown that the related universal deformation rings have quadratic relations.

Remark 1.1.1. (i) The author mentored R. Eberhardt in the writing of his Bachelor thesis [Ebe14] supervised by Böckle, which investigated the assertion of Theorem A in the case that $\overline{\rho}\colon G_{\mathbb{Q}_2} \to \mathrm{GL}_2(\mathbb{F})$ is trivial. Then $\rho^{\psi}_{\overline{\rho}}$ factors via a Demushkin group, which by

Example 2.4.5 is the quotient of a free pro-2 group with generators x_1, x_2, x_3 by the relation $r = x_1^2 x_2^4 (x_2, x_3)$. One checks that the refined quadratic parts of the explicit relations of the versal deformation ring $R_{\overline{\rho}}^{\psi} = \mathcal{O}[\![x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}]\!]/I^{\psi}$, where

$$I^{\psi} = \left(X_1^2 X_2^4 (X_2, X_3) - 1\right) \quad \text{with } X_i = \begin{pmatrix} 1 + x_{i,1} & x_{i,2} \\ x_{i,3} & (1 + x_{i,2}x_{i,3})/(1 + x_{i,1}) \end{pmatrix} \text{ for all } i,$$

form a regular sequence but the associated graded ring is not an integral domain so that we cannot apply Proposition 4.2.2 as done when proving Theorem A. Based on ideas of Böckle, [Ebe14] uses SageMath [Sag14] to apply Serre's criterion for normality and the Jacobian criterion for regularity [Eis95, Exc. 11.10, Thm. 16.19 and 18.15] to a related polynomial ring $R' := \mathbb{F}[x_1, \ldots, x_8]/I$. From [Mat89, Thm. 32.2(i)] follows that the completion $\widehat{R'}$ of R' is also normal and this is used to show that $\overline{R_{\overline{\rho}}^{\psi}} \cong \widehat{R'}[\![x]\!]/(f)$, where f is an irreducible polynomial of degree 2, is an integral domain and thus irreducible.

(ii) If $d > 1$, $p = 2$ and $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\mathbb{F})$ is trivial, the Master thesis [Kre13] of M. Kremer supervised by Böckle determines the explicit relations defining $R_{\overline{\rho}}^{\psi}$ using Demushkin's Example 2.4.5. Using Singular [DGPS18], [Kre13] checks that the quadratic parts of the explicit relations form a regular sequence and that the associated graded ring is an integral domain so that Proposition 4.2.2 shows that $R_{\overline{\rho}}^{\psi}$ is an integral domain and thus irreducible.

1.2 Equidimensionality of Chenevier's universal pseudodeformation rings

As the proof of the irreducibility of the versal deformation ring $R_{\overline{\rho}}^{\psi}$ in Theorem A is based on the knowledge of the explicit relations of $R_{\overline{\rho}}^{\psi}$ from [Böc00] where $n = 2$ and $p > 2$, we seek to investigate the (uni)versal deformation ring for arbitrary $n \in \mathbb{N}_{\geq 1}$ and prime number p through G. Chenevier's universal pseudodeformation ring of the residual pseudorepresentation attached to $\overline{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$.

More precisely, by Definition 3.1.13 a *pseudorepresentation* of G_K of dimension n with values in a commutative ring A is an A-polynomial law $D\colon A[G_K] \to A$ that is multiplicative and homogeneous of degree n. Then by Example 3.1.8 the determinant $\det \rho$ of a Galois representation $\rho\colon G_K \to \mathrm{GL}_n(R)$ defines a pseudorepresentation $D\colon R[G_K] \to R$ that by Amitsur's formular [Che14, (1.5)] encodes the data of the characteristic polynomial of ρ. Using the well-known Brauer-Nesbitt theorem [CR62, (30.16) Thm.], Chenevier proves in Theorem 3.1.26 that for any pseudorepresentation $D\colon k[G_K] \to k$ over an algebraically closed field k there exists a unique semisimple Galois representation $\rho\colon G_K \to \mathrm{GL}_n(k)$ such that $D = \det \rho$.

Similarly to the deformation theory of Galois representations, Chenevier then studies in [Che14, § 3.1] the *pseudodeformation functor*

$$\mathcal{P}sD_{\overline{D}}\colon \widehat{\mathcal{A}r}_{\mathcal{O}} \to Sets, \quad R \longmapsto \{D\colon R[G_K] \longrightarrow R \text{ is a pseudodeformation of } \overline{D}\},$$

of the residual pseudorepresentation $\overline{D} := \det \overline{\rho}$, where a continuous pseudorepresentation $D\colon R[G_K] \to R$ satisfying $D \hat{\otimes}_R \mathbb{F} \cong \overline{D}$ is called a *pseudodeformation of* $\overline{D}$; see Definition 3.1.53. As stated in Proposition 3.1.57 (and Proposition 3.1.60) by Wang Erickson, Chenevier ([Che14, Prop. 3.3 and 3.7] when $\mathcal{O} = W(\mathbb{F})$) proves that $\mathcal{P}sD_{\overline{D}}$ is always representable by a *universal pseudodeformation ring* $R_{\mathcal{O},\overline{D}}^{\mathrm{univ}}$ together with a *universal pseudodeformation* $D_{\overline{D}}^{\mathrm{univ}}\colon R_{\mathcal{O},\overline{D}}^{\mathrm{univ}}[G_K] \to R_{\mathcal{O},\overline{D}}^{\mathrm{univ}}$ of $\overline{D}$. Then by universality of $R_{\mathcal{O},\overline{D}}^{\mathrm{univ}}$ there is a homomorphism

$$R_{\mathcal{O},\overline{D}}^{\mathrm{univ}} \to R_{\overline{\rho}}$$

corresponding to the pseudodeformation $\det \rho_{\overline{\rho}}^{\mathrm{ver}}$ of $\overline{D}$, which by Chenevier's Proposition 3.2.14 is an isomorphism if $\overline{\rho}$ is absolutely irreducible. Our second main result is the following.

Theorem B (Theorem 3.3.12)**.** *Suppose that $p > n$ or $\zeta_p \notin K$. Then we have for any n-dimensional pseudorepresentation $\overline{D}\colon \mathbb{F}[G_K] \to \mathbb{F}$:*

(i) the special fiber $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ of $X_{\overline{D}}^{\mathrm{univ}} := \operatorname{Spec} R_{\mathcal{O},\overline{D}}^{\mathrm{univ}}$ is equidimensional of dimension dn^2+1;

(ii) if $\zeta_p \notin K$ and χ is the Teichmüller lift of the mod p cyclotomic character of G_K, then the locus

$$(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D} := \{D \in \overline{X}_{\overline{D}}^{\mathrm{univ}} \ : \ D \text{ is irreducible and } D \neq D \otimes \chi\}$$

of nonspecial *irreducible points is open, regular and Zariski dense in the* universal mod p pseudodeformation space $\overline{X}_{\overline{D}}^{\mathrm{univ}}$*;*

(iii) if $\zeta_p \in K$, then the regular locus of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ is empty and

$$(\overline{X}_{\overline{D},\mathrm{red}}^{\mathrm{univ}})^{\mathrm{irr}} := \{D \in \overline{X}_{\overline{D},\mathrm{red}}^{\mathrm{univ}} \ : \ D \text{ is irreducible}\}$$

is open, regular and Zariski dense in the nilreduction $\overline{X}_{\overline{D},\mathrm{red}}^{\mathrm{univ}}$ of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$.

This result is motivated by Chenevier's result [Che11, Thm. 2.1] that the character variety $\mathfrak{X}_n$ of continuous pseudocharacters of G_K of dimension n and with values in $\mathbb{Q}_p^{\mathrm{alg}}$ is equidimensional of dimension dn^2+1 and that the locus $\mathfrak{X}_n^{\mathrm{irr}}$ of irreducible pseudocharacters is regular and Zariski dense in $\mathfrak{X}_n$. The definition of an n-dimensional *pseudocharacter* $\tau\colon A[G_K] \to A$ with $n! \in A^\times$, which is obtained as the trace of a representation by Example 3.1.39, is given in Definition 3.1.38. Chenevier introduced pseudorepresentations in [Che14] to overcome the restriction $n! \in A^\times$, and in order to show Theorem B in characteristic p we mimic Chenevier's inductive approach to [Che11, Thm. 2.1] with pseudorepresentations. For the induction step, we prove the following:

Theorem C (Theorem 3.3.1)**.** *Suppose that $n \geq 2$, and that for every pseudorepresentations $\overline{D}'\colon \mathbb{F}[G_K] \to \mathbb{F}$ of dimension $n' < n$ the following hold:*

(i) $\overline{X}_{\overline{D}'}^{\mathrm{univ}}$ is equidimensional of dimension $d(n')^2+1$;

(ii) If $\zeta_p \in K$, then the regular locus of $(\overline{X}_{\overline{D}',\mathrm{red}}^{\mathrm{univ}})^{\mathrm{irr}}$ is Zariski dense in $\overline{X}_{\overline{D}',\mathrm{red}}^{\mathrm{univ}}$;

(iii) If $\zeta_p \notin K$, then the locus $(\overline{X}_{\overline{D}'}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D}$ is Zariski dense.

Then for all n-dimensional $\overline{D}\colon \mathbb{F}[G_K] \to \mathbb{F}$ the irreducible locus $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ is Zariski dense in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$, unless $n = 2$ and $K = \mathbb{Q}_2$ and $\overline{D}$ is reducible.

As explained in Remark 3.3.2 if $n = 2$, $K = \mathbb{Q}_2$ and $\overline{D}$ is reducible, by computations of V. Paškūnas in [Paš17, Prop. 3.6] the irreducible locus $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ is also Zariski dense in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ in this case.

Next Chenevier shows that the reducible locus $\mathfrak{X}_n^{\mathrm{red}}$ is the singular locus of $\mathfrak{X}_n$ if $n > 2$ or $d > 1$ [Che11, Thm. 2.3]. Here lies a difference to our setting since we have:

Theorem D (Theorem 3.3.13)**.** *Suppose $\zeta_p \notin K$. Then the following hold:*

(i) The locus $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)=D} := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \smallsetminus (\overline{X}_{\overline{D}'}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D}$ of special *irreducible points lies in the singular locus of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$.*

(ii) If $n > 2$ or $d > 1$, then the reducible locus of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ lies in the singular locus of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$.

For the missing case $n = 2$ and $d = 1$, we refer once again to the computations of Paškūnas [Paš17, Prop. 3.6] summarized in Remark 3.3.2: If in addition $p > 2$, then we have $\overline{X}_{\overline{D}}^{\mathrm{univ}} = \operatorname{Spec} \mathbb{F}[\![x_1, \ldots, x_5]\!]$ so that the reducible locus lies in the regular locus.

Theorem E (Corollary 3.3.15)**.** *Suppose that $\zeta_p \notin K$ and that either $n > 2$, or that $n = 2$ and $d > 1$. Then $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}, D(1) \neq D}$ constitutes the regular locus of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ and $\overline{R}_{\overline{D}}^{\mathrm{univ}}$ satisfies Serre's condition (R_2).*

To prove Theorem B(ii) and Serre's condition (R_2) in Theorem E when $\zeta_p \notin K$, we determine an upper bound for the dimension of the locus $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}, D(1) = D}$ in Theorem 3.3.6 as follows. Corollary 2.3.6 from Clifford theory says that a semisimple representation $\rho \colon G_K \to \mathrm{GL}_n(k)$ with values in an algebraically closed field k satisfies

$$\rho \cong \rho \otimes \chi \quad \Longleftrightarrow \quad \rho \cong \operatorname{Ind}_{G_{K(1)}}^{G_K} \rho' \quad \text{for some } \rho' \colon G_{K(1)} := \ker \chi \to \mathrm{GL}_{n/\operatorname{ord}\chi}(k).$$

Based on ideas of Böckle, Theorem 3.2.23 constructs a suitable induction $\operatorname{Ind}_{G_{K(1)}}^{G_K} D' \colon B[G_K] \to B$ of a pseudorepresentation $D' \colon B[G_{K(1)}] \to B$ under Assumption 3.2.21 on B an D', and Theorem 3.3.6 then gives as desired

$$\dim(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}, D(1) = D} \leq \max_{\overline{D}' \colon \mathbb{F}[G_{K(1)}] \to \mathbb{F}} \{\dim Y_{\overline{D}'} \mid \operatorname{Ind}_{G_{K(1)}}^{G_K} \overline{D}' = \overline{D}\} \leq \frac{dn^2}{\operatorname{ord}\chi} + 2, \text{[1]} \qquad (1)$$

where $Y_{\overline{D}'}$ is the closure in $\overline{X}_{\overline{D}'}^{\mathrm{univ}}$ of the set of irreducible points.

Remark 1.2.1. (i) Contrarily to versal deformation rings, we do not know which number of variables and relations define a universal pseudodeformation ring so that we cannot show that the latter is a complete intersection and thus satisfies Serre's condition (S_2).

(ii) In [Sim94] C. Simpson succesfully applies Serre's criterion for normality to the representation space $\operatorname{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))$ of the fundamental group Γ of a smooth connected projective curve of genus $g \geq 2$. He similarly proceeds by showing inductively that the irreducible locus is Zariski dense and regular of dimension $(2g-1)n^2 + 1$. For Serre's condition (R_1) he inductively proves that the reducible locus $\operatorname{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))^{\mathrm{red}}$ has codimension at least two unless $g = 2$ and $n = 2$. He uses the fact that

$$\operatorname{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))^{\mathrm{red}} \cong \bigcup_{k=1}^{n-1} \sigma\big(\mathrm{GL}_n(\mathbb{C})/P_k \times \operatorname{Hom}(\Gamma, P_k)\big), \qquad (2)$$

where $\sigma \colon \mathrm{GL}_n(\mathbb{C})/P_k \times \operatorname{Hom}(\Gamma, P_k), (y, \rho) \mapsto \varphi(y)\rho\varphi(y)^{-1}$, with $\varphi \colon \mathrm{GL}_n(\mathbb{C})/P_k \to \mathrm{GL}_n(\mathbb{C})$ a constructible section and $P_k \subset \mathrm{GL}_n(\mathbb{C})$ the parabolic subgroup consisting of 2 blocks of respective size k and $n-k$.

(iii) [Ger10, § 3.2] studies the universal deformation ring parametrizing deformations into the Borel subgroup of upper triangular matrices in GL_n. To obtain that a universal deformation ring satisfies Serre's condition (R_1) or is a complete intersection, it might be interesting to study with Geraghty's methods the universal deformation ring parametrizing deformations into parabolic subgroups analog to (2).

(iv) Following [WE17, § 2] consider the affine scheme $\mathrm{Rep}_{G_K}^{\square,n}$ parametrizing n-dimensional representations of G_K and the *GIT quotient* $\mathrm{Rep}_{G_K}^{\square,n} /\!/ \mathrm{GL}_n := \mathcal{O}(\mathrm{Rep}_{G_K}^{\square,n})^{\mathrm{GL}_n}$ of the adjoint action of GL_n on $\mathrm{Rep}_{G_K}^{\square,n}$. Then by [WE17, Thm. 2.20] there is a canonical *adequate homeomorphism* ν from $\mathrm{Rep}_{G_K}^{\square,n} /\!/ \mathrm{GL}_n$ to the universal pseudorepresentation space $X_{G_K,n}^{\mathrm{univ}}$ from Definition 3.1.24; i.e., ν is an integral universal homeomorphism that by [Che13a, Prop. 2.3] is an isomorphism in characteristic zero. It may be possible to translate properties between the universal objects via the adequate homeomorphism ν.

1.3 *Application: The p-adic local Langlands correspondence and Zariski density of benign crystalline points*

Consider a finite extension L of $\mathbb{Q}_p$. The p-adic local Langlands correspondence is provided by Colmez' exact functor $\mathbf{V}$ from admissible unitary L-Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, residually of finite length, to continuous 2-dimensional L-representations of $G_{\mathbb{Q}_p}$ [Col10]. This correspondence has also important applications in arithmetic geometry such as the Fontaine-Mazur conjecture [Kis09, Eme06]. Furthermore, it encodes the classical local Langlands correspondence [CDP14, Thm. 1.3] and is compatible with class field theory [CDP14, Cor. 1.2].

The surjectivity of the p-adic local Langlands correspondence was proven following a strategy of Kisin in [Col10, Kis10a, CDP15]: First, benign crystalline representations of $G_{\mathbb{Q}_p}$[2] lie in the image of $\mathbf{V}$ [Col10, Prop. II.3.8]. Second, if the benign crystalline points are Zariski dense in the (uni)versal deformation space $\mathfrak{X}_V$ of an arbitrary finite-dimensional L-representation V of $G_{\mathbb{Q}_p}$ then one deduces that each point of $\mathfrak{X}_V$ is in the image of $\mathbf{V}$ and thus in particular V; see [CDP15, § 10].

As explained in the introduction of [BJ15], we show using Theorem A and [Nak13, Thm. 1.4] the following:

Theorem F (Theorem 4.1.11)**.** *Let $\mathfrak{X}(\bar\rho)$ be the rigid analytic space associated to the formal scheme $\mathrm{Spf}\, R_{\bar\rho}^{\psi}$. If $n = 2$ and $p > 2$, then the benign crystalline points are Zariski dense in $\mathfrak{X}(\bar\rho)$.*

In order to be able to apply [Nak13, Thm. 1.4], we check in Lemma 4.4.2 that each component of $\mathfrak{X}(\bar\rho)$ contains a regular crystalline point with the help of following theorem.

Theorem G (Theorem 4.1.9)**.** *Suppose $p > 2$ and $n = 2$. Consider the canonical map $D\colon \mathfrak{X}(\bar\rho) \to \mathfrak{X}(\det\bar\rho)$ given by sending a deformation of $\bar\rho$ to its determinant. Then D induces a bijection between the irreducible components of $\mathfrak{X}(\bar\rho)$ and those of $\mathfrak{X}(\det\bar\rho)$. Moreover, for both spaces, irreducible and connected components coincide. Lastly, the connected components of $\mathfrak{X}(\det\bar\rho)$ form a principal homogeneous space over the set $\mu_{p^\infty}(K)$ of p-power roots contained in K.*

We refer to Remark 4.1.13 for other cases where the assertions of Theorem F and G are shown. For the rigid analytic space $\mathfrak{X}^{\mathrm{ps}}(\overline{D})$ associated with $\mathrm{Spf}\, R_{\overline{D}}^{\mathrm{univ}}$ we show the following corresponding result.

Theorem H (Corollary 4.4.3)**.** *Suppose that $n = 2$, $p > 2$ and $\bar\rho$ is semisimple. Consider the natural functors*

$$\mathfrak{X}(\bar\rho) \xrightarrow{\pi_1} \mathfrak{X}^{\mathrm{ps}}(\overline{D}) \xrightarrow{\pi_2} \mathfrak{X}(\det\bar\rho), \tag{3}$$

where π_1 is defined by sending a deformation to the associated pseudodeformation, and π_2 by sending a pseudodeformation to its determinant from Example 3.1.9.

[2] See Definition 4.1.8.

(a) *The morphisms of connected components*

$$\pi_0(\mathfrak{X}(\bar\rho)) \xrightarrow{\pi_0(\pi_1)} \pi_0(\mathfrak{X}^{\mathrm{ps}}(\overline{D})) \xrightarrow{\pi_0(\pi_2)} \pi_0(\mathfrak{X}(\det\bar\rho)),$$

induced from (3) *are bijective.*

(b) *The benign crystalline points are Zariski dense in* $\mathfrak{X}^{\mathrm{ps}}(\bar\rho)$.

1.4 Outline

In Chapter 2 we fix notation and implement tools that are used later in this thesis. These tools steam mainly from commutative algebra and algebraic geometry, Mazur's deformation theory of Galois representations, Clifford's theory on inductions of representations and the classification of Demushkin groups. We will use these tools to study ring-theoretic and geometric properties of (uni)versal (pseudo)deformation rings in Chapter 3 and Chapter 4.

The goal of Chapter 3 is Theorem B (Theorem 3.3.12) on the equidimensionality of universal mod p pseudodeformation rings. Section 3.1 contains an exposition to Chenevier's pseudorepresentations and their universal pseudorepresentation and pseudodeformation rings following [Che14] and [WE13]. In Section 3.2 we investigate properties of certain loci of pseudodeformations. Section 3.3 contains the inductive proof of Theorem B. At first, Theorem C (Theorem 3.3.1) shows Zariski density of the irreducible locus under an induction hypothesis. Using this and the upper bound (1) from Theorem 3.3.6, the main theorem is proven in Subsection3.3.3. We finish Chapter 3 with Theorem D (Theorem 3.3.13) and Theorem E (Corollary 3.3.15).

Finally, Chapter 4 consists of the published article "Irreducibility of versal deformation rings in the (p, p)-case for 2-dimensional representations", written jointly by Böckle and the author [BJ15]. In particular, Theorem A (Theorem 4.1.5) on the irreducibility of a versal deformation ring is proved for $n = 2$ and $p > 2$ by applying Proposition 4.2.2 from commutative algebra. For further details on the structure and results of Chapter 4, we refer to the article's introduction and outline in Section 4.1.

1.5 Acknowledgements

I am very grateful for the great support of my advisor Gebhard Böckle who invested a big amount of his time to share his ideas and knowledge with me. I would like to thank Gaëtan Chenevier for pointing out that the proof of [Che11, Thm.] should also work for the special fiber of the universal pseudodeformation ring. I thank Carl Wang Erickson for the friendly and helpful email contact on the determinant of a generalized matrix algebra. Thanks also go to Alain Muller for helpful discussions on the construction in [Ger10], and to David-Alexandre Guiraud for proofreading my thesis. It is a pleasure to also thank my colleagues from the research group "Computational Arithmetic Geometry" of Heidelberg university for a stimulating work environment. I am grateful for the PhD grant of the Landesgraduiertenförderung Baden-Württemberg. Last but not least, I am very thankful for numerous discussions, proofreading and support from Hendrik Verhoek.

2. PRELIMINARIES

In this chapter we collect various notions and results in order to study ring-theoretic and geometric properties of (uni)versal (pseudo)deformation rings in Chapter 3 and Chapter 4.

Suited for this purpose, this chapter begins in Section 2.1 with tools from commutative algebra and algebraic geometry like Serre's criterion for normality from Proposition 2.1.6.

Next Section 2.2 investigates properties of Mazur's (uni)versal deformation rings.

Section 2.3 discusses results from Clifford theory on induced representations that will be a crucial ingredient for finding an upper bound for the dimension of the locus of special pseudodeformations in Subsection 3.3.2.

The chapter ends in Section 2.4 with a short summary on the classification of Demushkin groups, whose classification in Example 2.4.5 leads to the explicit description of (uni)versal deformation ring of a residual 2-dimensional Galois representation in Section 4.3.

We assume that the reader is familiar with standard topics from algebraic number theory, commutative algebra and algebraic geometry (see e.g. [Neu99, Eis95, Har77], respectively).

2.1 Commutative algebra and algebraic geometry

This section introduces important basics from commutative algebra and algebraic geometry. Many of the (uni)versal (pseudo)deformation rings that we will consider are complete Noetherian local rings. Such rings are given by the Cohen structure theorem as a quotient of a power series ring. After stating this theorem in Subsection 2.1.1, we concentrate on ring-theoretic assertions on complete Noetherian local rings such as Serre's criterion for normality in from Proposition 2.1.6. In Subsection 2.1.2 we discuss regularity and formal smoothness, and in Subsection 2.1.3 étale morphisms and étale neighbourhoods. Finally, we consider the density of 1-dimensional points.

2.1.1 Basic results on complete Noetherian local rings

We begin by introducing two categories occuring in Schlessinger's formal deformation theory set-up.

Definition 2.1.1. Let Λ be a complete Noetherian local ring and let k be its residue field.

(i) By $\widehat{\mathcal{A}r}_\Lambda$ we denote the subcategory of the category of commutative rings with:

- Objects: complete Noetherian local Λ-algebra R together with a fixed Λ-algebra isomorphism $R/\mathfrak{m}_R \xrightarrow{\sim} k$, where $\mathfrak{m}_R$ denotes the maximal ideal of R.
- Morphisms: local Λ-algebra homomorphisms $f\colon R_1 \to R_2$ such that the following diagram commutes:

$$\begin{array}{ccc} R_1 & \xrightarrow[f \bmod \mathfrak{m}_{R_1}]{} & R_2 \\ & {\scriptstyle\sim}\searrow \qquad \swarrow{\scriptstyle\sim} & \\ & k. & \end{array}$$

(ii) By $\mathcal{A}r_\Lambda$ we denote the strictly full subcategory of $\widehat{\mathcal{A}r}_\Lambda$ on the Artinian rings in $\widehat{\mathcal{A}r}_\Lambda$.

Note that every object of $\widehat{\mathcal{A}r}_\Lambda$ is a limit of objects of $\mathcal{A}r_\Lambda$. For further basic properties of these categories we refer to [Sta18, § 06GB] and [Sta18, § 06GV].

Recall that a topological ring R is called *linearly topologized* if 0 has a fundamental system of open neighbourhoods consisting of ideals of R, and an R-module M is called *linearly topologized* if 0 has a fundamental system of open neighbourhoods consisting of R-submodules of M.

Definition 2.1.2. Let R be a topological ring and M and N be linearly topologized R-modules. Then the *tensor product* of M and N is the R-module $M \otimes_R N$ equipped with the linear topology given by declaring

$$\operatorname{im}(M_\mu \otimes_R N + M \otimes_R N_\nu \longrightarrow M \otimes_R N)$$

to be a fundamental system of open submodules, where $M_\mu \subset M$ and $N_\nu \subset N$ run through fundamental systems of open submodules in M and N. The *completed tensor product*

$$M \widehat{\otimes}_R N = \lim M \otimes_R N/(M_\mu \otimes_R N + M \otimes_R N_\nu) = \lim M/M_\mu \otimes_R N/N_\nu$$

is the completion of the tensor product with respect to this topology.

Lemma 2.1.3 (Cf. [EGA IV$_1$, Lem. 0_{IV}.(19.7.1.2)])**.** *Suppose that R, R' are complete Noetherian local Λ-algebras, and $R/\mathfrak{m}_R$ is a finite extension of k. Then $R\hat{\otimes}_\Lambda R'$ is a semilocal Noetherian ring, whose maximal ideals correspond to the maximal ideals of $R/\mathfrak{m}_R \otimes_k R'/\mathfrak{m}_{R'}$.*

In particular, if R, R' are in $\widehat{\mathcal{A}r}_\Lambda$, then $R\hat{\otimes}_\Lambda R'$ lies also in $\widehat{\mathcal{A}r}_\Lambda$.

Theorem 2.1.4 (Cohen structure theorem [Sta18, § 0323])**.** *Let R be a complete Noetherian local ring whose residue field $\kappa = R/\mathfrak{m}_R$ is of characteristic p. There exists a complete discrete valuation ring W with uniformizer π and $W/\pi W \cong \kappa$ such that R is isomorphic to a quotient of $W[\![x_1, \ldots, x_n]\!]$.*

If κ is a perfect field, by [EGA IV$_1$, Rem. 0_{IV}.(19.8.7)] the complete discrete valuation ring W in the above theorem can be taken as the ring of Witt vectors $W(\kappa)$.

For a prime $\mathfrak{p}$ of a commutative ring R, the *height of* $\mathfrak{p}$ is defined as $\operatorname{ht}\mathfrak{p} = \dim R_\mathfrak{p}$.

Definition 2.1.5 (Serre's conditions)**.** Let R be a ring and $i \in \mathbb{N}_0$.

(i) *R satisfies Serre's condition* (S_i) if $\operatorname{depth}(R_\mathfrak{p}) \geq \min\{i, \operatorname{ht}\mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

(ii) *R satisfies Serre's condition* (R_i) if for every prime $\mathfrak{p}$ of height $\leq i$ the local ring $R_\mathfrak{p}$ is regular. We also say that R is *regular in codimension* $\leq i$.

In Theorem E (Corollary 3.3.15) we show under a certain hypothesis that the special fiber of a universal pseudodeformation ring satisfies Serre's condition (R_2). Unfortunately, we do not know if (S_2) also holds in order to apply the following:

Proposition 2.1.6 (Serre's criterion for normality [Sta18, Lem. 031S])**.** *A Noetherian ring R is normal if and only if R satisfies Serre's conditions (R_1) and (S_2).*

Example 2.1.7 (Cf. [Sta18, Lem. 0567])**.** Serre's criterion for normality from Proposition 2.1.6 shows that any regular ring is normal: It satisfies Serre's condition (R_1) trivially, and also Serre's condition (S_2) because it is Cohen-Macaulay.

The following proposition shows why we would like to show that the special fiber of a universal pseudodeformation ring is an integrally closed domain.

Proposition 2.1.8 (Cf. [EGA IV$_2$, Cor. (5.12.7)])**.** *Let R be a complete Noetherian local ring, and $t \in \mathfrak{m}_R$ be a regular element of R. If R/tR is an integrally closed domain, then the same holds for R.*

The following result shows why we would like to show integrality of the nilreduction of the special fiber of a universal pseudodeformation ring.

Proposition 2.1.9. *A scheme X is irreducible if and only if the reduced scheme X_{red} underlying X is an integral scheme.*

Proof. Since the natural morphism $X_{\mathrm{red}} \to X$ is a homeomorphism of topological spaces, X_{red} and X have the same irreducible components. Thus X is irreducible if and only if X_{red} is integral. □

2.1.2 Regularity and formal smoothness

In Corollary 2.2.18 we state that the natural map from the universal deformation space of a certain Galois representation ρ to the universal deformation space of $\det \rho$ is formally smooth under certain hypothesis on ρ. This subsection introduces formal smoothness and its relation to regularity.

Definition 2.1.10 (Cf. [Sta18, Def. 07EB] and [Sch68, Def. 2.2])**.** (i) Let $R_1 \to R_2$ be a homomorphism of topological rings with R_1 and R_2 linearly topologized. We say R_2 is *formally smooth over R_1* if for every commutative solid diagram

$$\begin{array}{ccc} R_2 & \longrightarrow & R/I \\ \uparrow & \searrow & \uparrow \\ R_1 & \longrightarrow & R \end{array}$$

of homomorphisms of topological rings, where R is a discrete ring and $I \subset R$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

(ii) A morphism $\operatorname{Spec} R \to \operatorname{Spec} S$ of affine schemes is called *formally smooth* if the corresponding homomorphism $S \to R$ is formally smooth.

(iii) A natural transformation $F_1 \to F_2$ of functors $\mathcal{C} \to \mathit{Sets}$, for $\mathcal{C} \in \{\widehat{\mathcal{A}r}_\Lambda, \mathcal{A}r_\Lambda\}$, is called *smooth* if for any surjection $R_2 \twoheadrightarrow R_1$ in $\mathcal{C}$, the morphism $F_1(R_2) \to F_1(R_1) \times_{F_2(R_1)} F_2(R_2)$ is surjective.

We make use of the following equivalence between a regular and formally smooth homomorphism in Corollary 3.2.15.

Proposition 2.1.11 ([Sta18, Prop. 07PM])**.** *Let $f\colon R_1 \to R_2$ be a local homomorphism of complete Noetherian local rings, let k be the residue field of R_1 and let $\overline{R}_2 = R_2 \otimes_{R_1} k$. The following are equivalent:*

(i) f is regular,

(ii) f is flat and $\overline{R}_2$ is geometrically regular over k,

(iii) f is flat and $k \to \overline{R}_2$ is formally smooth in the $\mathfrak{m}_{\overline{R}_2}$-adic topology, and

(iv) f is formally smooth in the $\mathfrak{m}_{R_2}$-adic topology.

We shall also make use of the following result:

Proposition 2.1.12 ([Sch68, Prop. 2.5(i)])**.** *If* $R_1 \to R_2 \in \mathrm{Mor}(\widehat{\mathcal{A}r}_\Lambda)$*, then*

$$h_{R_2} \to h_{R_1}, \qquad \textit{where} \quad h_R\colon \widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets},\ R' \mapsto \mathrm{Hom}_{\widehat{\mathcal{A}r}_\Lambda}(R, R') \textit{ for } R \in \mathrm{Ob}(\widehat{\mathcal{A}r}_\Lambda),$$

is smooth if and only if R_2 *is a power series ring over* R_1*.*

The above justifies the following definition.

Definition 2.1.13. We call a morphism $R \to S$ in $\widehat{\mathcal{A}r}_\Lambda$ formally smooth of relative dimension h if S is a power series ring over R in h formal variables.

Note that $h = \dim S - \dim R = \dim_{\mathbb{F}} \mathfrak{m}_S/(\mathfrak{m}_S, \mathfrak{m}_\Lambda) - \dim_{\mathbb{F}} \mathfrak{m}_R/(\mathfrak{m}_R, \mathfrak{m}_\Lambda)$.

2.1.3 Étale morphisms and étale neighbourhoods

For proving the Zariski density of the irreducible locus in Theorem C (Theorem 3.3.1), we show in Lemma 3.3.3 the existence of a suitable étale neighbourhood of a reducible point. This subsection deals with the related notions and states a result needed for the proof of Theorem C (Theorem 3.3.1).

Definition 2.1.14 ([Sta18, § 00U0 and Def. 02GI])**.** (i) Let $f\colon R \to S$ be a ring map. We call f *étale* if f is a smooth ring map of relative dimension zero.

(ii) A morphism $f : X \to S$ of schemes is *étale at* $x \in X$ if there is an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ so that the induced ring map $R \to A$ is étale. We say that f is *étale* if it is étale at each point of X.

We now define étale neighbourhoods.

Definition 2.1.15 ([Sta18, Def. 03PO])**.** Let X be a scheme.

(i) A *geometric point of* X is a morphism $\overline{x}\colon \mathrm{Spec}\, k \to X$ for k an algebraically closed field.

(ii) We call $\overline{x}$ *is lying over* x to indicate that $x \in X$ is the image of $\overline{x}$.

(iii) An *étale neighbourhood* $(U, \overline{u})$ of a geometric point $\overline{x} \in X$ is a commutative diagram

$$\begin{array}{ccc} & & U \\ & \overset{\bar{u}}{\nearrow} & \downarrow{\scriptstyle\varphi} \\ \mathrm{Spec}\, k & \xrightarrow{\overline{x}} & X, \end{array}$$

where φ is an étale morphism of schemes.

As mentioned earlier, in the proof of Theorem C (Theorem 3.3.1) we shall need the following result.

Lemma 2.1.16. *Let* $\varphi\colon U \to X$ *be an étale morphism between schemes* U *and* X*. Let* u *be a point of* U *and denote by* x *its image* $\varphi(u)$*. Consider the local homomorphism* $\varphi_u\colon \mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ *induced from* φ*. Then*

(i) The completion $\widehat{\varphi}_u\colon \widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{U,u}$ *of* φ_u *is finite étale; its degree is equal to* $[\kappa(u) : \kappa(x)]$*.*

(ii) *The ring $\widehat{\mathcal{O}}_{X,x}$ is regular if and only if $\widehat{\mathcal{O}}_{U,u}$ is regular, and in this case both have the same dimension.*

Proof. Part (i) is [Sta18, Lem. 039M] and the remark following it. For part (ii) note that by étaleness the tangent spaces at the closed point have the same dimension, and by finite étaleness the ring $\widehat{\mathcal{O}}_{U,u}$ is free of finite rank over $\widehat{\mathcal{O}}_{X,x}$ and hence they have the same dimension. From this (ii) follows easily. □

2.1.4 Density of 1-dimensional points

In Chapter 3 we study equidimensionality of the special fiber of a universal pseudodeformation space, which contains only one closed point. We will make use of Lemma 2.1.20, by which then 1-dimensional points are very dense in the special fiber.

Definition 2.1.17 ([Mat80, p. 38f.], [EGA IV$_3$, Def. 10.1.3], [Sta18, Def. 0055])**.** Let X be a topological space.

(i) A subset Z of X is called *locally closed in* X if for any point $z \in Z$ there exists an open neighbourhood $U \subset Z$ of z such that $U \cap Z$ is closed in U. Equivalently, Z is an intersection of an open and a closed set in X.

(ii) A subset X_0 of X is called *very dense in* X if every nonempty locally closed subset $Z \subset X$ satisfies $Z \cap X_0 \neq \varnothing$.

(iii) X is called *Noetherian* if the descending chain condition holds for the closed subsets of X.

(iv) A subset Z of a Noetherian space X is called *constructible* if it is a finite union of locally closed sets in X.

Proposition 2.1.18 ([EGA IV$_3$, Prop. 10.1.2])**.** *Let X be a topological space, and let X_0 be a subset of X. The following conditions are equivalent:*

(i) X_0 is very dense in X;

(ii) The map $X' \mapsto X_0 \cap X'$ defines a bijection between the open sets in X and the ones in X_0;

(iii) The map $X' \mapsto X_0 \cap X'$ defines a bijection between the closed sets in X and the ones in X_0;

In particular, a very dense subset $X_0 \subset X$ is dense in X. If moreover X is Noetherian, then for every constructible subset Z of X the set $X_0 \cap Z$ is dense in Z.

One easily deduces the following consequence.

Corollary 2.1.19 ([EGA IV$_3$, Cor. (10.1.4)])**.** *If X_0 is very dense in X and $U \subset X$ is an open subset, then $U \cap X_0$ is very dense in U.*

Lemma 2.1.20 ([Mat80, (33.F) Lem. 5])**.** *Let $X = \operatorname{Spec} A$ for a Noetherian ring A. Then the set $X_0 := \{\mathfrak{p} \in X \,:\, \dim A/\mathfrak{p} \leq 1\}$ is very dense in X.*

Corollary 2.1.21. *Suppose $X = \operatorname{Spec} A$ with A Noetherian. Let $Z \subset X$ be closed and irreducible. Then for every open subset U of X with $Z \cap U \neq \varnothing$, the set*

$$Z_{U,\leq 1} := \{x \in X \,:\, \dim x \leq 1, x \in Z \cap U\}$$

is dense in $Z \cap U$ and hence also in Z.

Proof. The density of $Z_{U,\leq 1}$ in $Z \cap U$ follows from Lemma 2.1.20 combined with the last assertion of Proposition 2.1.18. Since Z is closed irreducible, any open nonempty subset of Z is dense in Z, and this shows the last assertion. □

Definition 2.1.22 ([EGA IV$_3$, § 10])**.** Let X be a topological space.

(i) The space X is called *Jacobson* if its subset of closed points is very dense in X.

(ii) A scheme is called *Jacobson* if the underlying topological space is Jacobson.

(iii) A ring R is called *Jacobson* if the scheme $\operatorname{Spec} R$ is Jacobson.

Recall that a subspace of a topological space is a subset together with the induced topology.

The following proposition applies in particular to the generic fiber of a versal deformation ring.

Proposition 2.1.23 (Cf. [EGA IV$_3$, Cor. (10.5.8)])**.** *Let R be a Noetherian ring with Jacobson radical $J(R)$. For any $f \in J(R)$, the localization R_f is Jacobson and the open subscheme $\operatorname{Spec} R \smallsetminus V\big(J(R)\big)$ of $\operatorname{Spec} R$ is a Jacobson scheme.*

Proposition 2.1.24 (Cf. [EGA IV$_3$, Cor. (10.5.9)])**.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}_R$. Then the open subscheme $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}_R\}$ of $\operatorname{Spec} R$ is a Jacobson scheme, and the closed points in $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}_R\}$ are the one-dimensional points in $\operatorname{Spec} R$.*

Proposition 2.1.25. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}_R$. Suppose all irreducible components of $\operatorname{Spec} R$ have dimension at least 2. Let $U \subset \operatorname{Spec} R \smallsetminus \{\mathfrak{m}_R\}$ be open nonempty. Then no finite subset S of dimension 1 points of U is dense in U.*

Proof. Let $S \subset U$ be a finite subset of dimension 1 points. The latter implies that for any $s \in S$ the subset $\{s, \mathfrak{m}_R\}$ is closed in $\operatorname{Spec} R$. Hence S, as a finite union of closed subsets $\{s\}$ of U is closed in U, and it follows that $U \smallsetminus S$ is open in U. Assume now that S is dense in U. Then necessarily we would have $U = S$. But U has to contain a generic point of $\operatorname{Spec} R$ and all such have dimension at least 2. We reach a contradiction. □

2.2 Galois representations and their (uni)versal deformation rings

Throughout Section 2.2 we let Λ be a complete Noetherian local ring with maximal ideal $\mathfrak{m}_\Lambda$ and residue field $k := \Lambda/\mathfrak{m}_\Lambda$. Recall the categories $\widehat{\mathcal{A}r}_\Lambda$ and $\mathcal{A}r_\Lambda$ from Definition 2.1.1. The residue field k will either be discrete or a *local field*, i.e., a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p(\!(t)\!)$ for some prime p. If k is discrete, all rings in $\mathcal{A}r_\Lambda$ will be equipped with the discrete topology and those in $\widehat{\mathcal{A}r}_\Lambda$ with the inverse limit topology.

If k is a local field, we only consider $\Lambda = k$ and $\mathcal{A}r_k$. Recall that in this case any finite-dimensional k-vector space V carries a unique topology so that it is a locally compact vector space (over k). If V is identified with k^n, it is the product topology on k^n and one shows that the topology is independent of the chosen isomorphism. Hence any A in $\mathcal{A}r_k$ carries a unique topology for which it is a locally compact topological vector space, and one easily verifies that then A is also a continuous topological k-algebra. Thus in the sequel, whenever k is a local field, we regard any $A \in \mathcal{A}r_k$ as a topological ring via this topology.

Following a convention of Kisin, see [Kis03, p. 433], for a representation ρ into $\mathrm{GL}_n(A_1)$ and a ring homomorphism $A_1 \to A_2$ we write $\rho \otimes_{A_1} A_2$ for the composition of ρ with $\mathrm{GL}_n(A_1) \to \mathrm{GL}_n(A_2)$.

2.2.1 Schlessinger's and Mazur's deformation theory

We start with giving a short overview of Schlessinger's formal deformation theory [Sch68]; see also [Sta18, Chapter 06G7]. Next we will define Mazur's (uni)versal deformation rings of Galois representations, whose existence follows from Schlessinger's representability criteria. Throughout Subsection 2.2.1 we fix a profinite group G.

In this subsection we consider functors $F\colon \mathcal{A}r_\Lambda \to \mathit{Sets}$ such that $F(k)$ contains just one element, and their extensions

$$\widehat{F}\colon \widehat{\mathcal{A}r}_\Lambda \longrightarrow \mathit{Sets}, \quad R \longmapsto \varprojlim_i F(R/\mathfrak{m}_R^i).$$

Definition 2.2.1 ([Sch68, Def. 2.2– 2.7])**.** (i) Denote by $k[\varepsilon] := k[X]/(X^2)$ the *ring of dual numbers over* k. Then $t_F := F(k[\varepsilon])$ is called the *tangent space of* F.

(ii) A pair (R, ξ) consisting of $R \in \mathrm{Ob}(\widehat{\mathcal{A}r}_\Lambda)$ and $\xi \in \widehat{F}(R)$ is called a *hull* if the map

$$h_R \longrightarrow F$$

corresponding to $\xi \in \widehat{F}(R) \cong \mathrm{Hom}(h_R, F)$ is smooth and the induced map $t_R \to t_F$ of tangent spaces is bijective.

(iii) A pair (R, ξ) consisting of $R \in \mathrm{Ob}(\widehat{\mathcal{A}r}_\Lambda)$ and $\xi \in \widehat{F}(R)$ *(pro-)represents* the functor F if the morphism $h_R \to F$ corresponding to $\xi \in \widehat{F}(R) \cong \mathrm{Hom}(h_R, F)$ is an isomorphism.

Proposition 2.2.2 ([Sch68, § 2.8–Prop. 2.9])**.** *(a) If (R, ξ) is a hull for a functor F, then (R, ξ) is unique up to isomorphisms.*

(b) If (R, ξ) pro-represents a functor F, then (R, ξ) is unique up to canonical isomorphism.

Recall that a *small extension in* $\mathcal{A}r_\Lambda$ is a surjection $f\colon R' \to R$ in $\mathcal{A}r_\Lambda$ whose kernel $\ker f$ is isomorphic to k as an R'-module, and in particular $\ker f$ is annihilated by $\mathfrak{m}_{R'}$ and $(\ker f)^2 = 0$.

Theorem 2.2.3 (Schlessinger's criteria [Sch68, Thm. 2.11])**.** *Let $F\colon \mathcal{A}r_\Lambda \to \mathit{Sets}$ be a functor such that $F(k)$ consists of one element. For the canonical map*

$$\psi\colon F(R' \times_R R'') \longrightarrow F(R') \times_{F(R)} F(R'')$$

for morphisms $f\colon R' \to R$ and $f''\colon R'' \to R$ in $\mathcal{A}r_\Lambda$ consider the following conditions on:

(H_1) The map ψ is surjective whenever the morphism f'' is a small extension.

(H_2) The map ψ is bijective if $R = k$ and $R'' = k[\epsilon]$.

(H_3) The tangent space t_F is finite-dimensional as a k-vector space.[1]

(H_4) The map ψ is an isomorphism whenever f' is a small extension.

Then the following hold:

(i) The functor F has a hull if and only if conditions (H_1) – (H_3) are satisfied;

(ii) The functor F is representable if and only if conditions (H_1) – (H_4) are satisfied.

[1] It is shown in [Sch68, Lem. 2.10] that under hypothesis H_2 the set t_F carries a natural k-vector space structure.

Mazur used Schlessinger's criteria to show the existence of (uni)versal deformation rings of representations of certain profinite groups such as the local Galois groups, which we are interested in.

Definition 2.2.4 ([Gou01, Defs. 2.1 and 2.2]). Let $\overline{\rho}\colon G \to \mathrm{GL}_n(k)$ be a continuous representation and $R \in \mathrm{Ob}(\mathcal{A}r_\Lambda)$.

(i) A *lifting of* $\bar{\rho}$ *to* R is a continuous representation $\rho\colon G \to \mathrm{GL}_n(R)$ with $\rho \otimes_R k = \overline{\rho}$.

(ii) If $\Gamma_n(R) \subset \mathrm{GL}_n(R)$ denotes the kernel of the canonical homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_n(k)$, then two liftings ρ_1, ρ_2 of $\bar{\rho}$ to R are called *strictly equivalent* if there exists $A \in \Gamma_n(R)$ such that $\rho_2(g) = A\rho_1(g)A^{-1}$ for all $g \in G$.

(iii) A *deformation of* $\overline{\rho}$ *to* R is a strict equivalence class of liftings of $\bar{\rho}$ to R.

(iv) The functor

$$\mathcal{D}_{\overline{\rho}}\colon \mathcal{A}r_\Lambda \longrightarrow \mathit{Sets}, \quad R \longmapsto \{\rho\colon G \longrightarrow \mathrm{GL}_n(R) \;:\; \rho \text{ is a deformation of } \overline{\rho}\},$$

is called the *deformation functor of* $\overline{\rho}$.

(v) If k is discrete, we extend (i)–(iv) also to $R \in \mathrm{Ob}(\widehat{\mathcal{A}r}_\Lambda)$, cf. the beginning of Section 2.2.

(vi) The representation $\mathrm{ad}_{\bar{\rho}}$ is defined as $\mathrm{Mat}_n(k)$ with the adjoint action of G via $\bar{\rho}$. Its subrepresentation on trace zero matrices is denoted by $\mathrm{ad}^0_{\bar{\rho}}$, its quotient representation modulo the center k by $\overline{\mathrm{ad}_{\bar{\rho}}}$.

Definition 2.2.5 ([Maz89, § 1.1]). A profinite group G *satisfies the finiteness condition* Φ_p if for every open subgroup $G_0 \subset G$ there are only finitely many continuous homomorphisms $G_0 \to \mathbb{F}_p$.

Example 2.2.6. The finiteness condition Φ_p is satisfied both by the absolute Galois group of a local field of characteristic 0, and by the Galois group $\mathrm{Gal}(F_S/F)$ for F a number field, and F_S a maximal algebraic extension of F that is unramified outside a finite set S of places of F.

Mazur [Maz89, Prop. 1, p. 389] proved the following theorem in case that the residual representation is absolutely irreducible and k is finite. We state the slightly generalized version by Ramakrishna following from Schur's lemma and as extended in [Gou01, § 9].

Theorem 2.2.7 ([Gou01, Thm. 3.3, Lem. 9.5, Prop. 9.6]). *Let Λ be a complete Noetherian local ring and consider any $R \in \mathrm{Ob}(\widehat{\mathcal{A}r}_\Lambda)$ as a topological ring via the $\mathfrak{m}_R$-adic topology, so that all $R \in \mathrm{Ob}(\mathcal{A}r_\Lambda)$ are discrete. Let G be a profinite group and $\overline{\rho}\colon G \to \mathrm{GL}_n(k)$ a continuous representation. Suppose in* (i) *and* (ii) *that $\mathcal{D}_{\bar{\rho}}$ has a finite-dimensional tangent space. Then the following hold:*

(i) The functor $\mathcal{D}_{\overline{\rho}}\colon \widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets}$ has a hull $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \in \mathrm{Ob}(\widehat{\mathcal{A}r}_\Lambda)$ together with a versal deformation

$$\rho^{\mathrm{ver}}_{\overline{\rho}}\colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}).$$

(ii) If $\mathrm{Cent}(\overline{\rho}) = k$, then the functor $\mathcal{D}_{\overline{\rho}}\colon \mathcal{A}r_\Lambda \to \mathit{Sets}$ is representable by $R^{\mathrm{univ}}_{\Lambda,\overline{\rho}} \in \mathrm{Ob}(\widehat{\mathcal{A}r}_\Lambda)$ together with a universal deformation

$$\rho^{\mathrm{univ}}_{\overline{\rho}}\colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{univ}}_{\Lambda,\overline{\rho}}).$$

The ring $R^{\mathrm{univ}}_{\Lambda,\overline{\rho}}$ is a quotient of $\Lambda[\![x_1,\ldots,x_h]\!]$ for $h = \dim_k t_{\mathcal{D}_{\bar{\rho}}}$, and $t_{\mathcal{D}_{\bar{\rho}}} \cong H^1(G, \mathrm{ad}_{\bar{\rho}})$.

(iii) If k is finite and if G satisfies the finiteness condition Φ_p, then $t_{\mathcal{D}_{\bar\rho}}$ is finite-dimensional.

We call $R^{\mathrm{ver}}_{\Lambda,\bar\rho}$ a versal deformation ring of $\bar\rho$ *and $R^{\mathrm{univ}}_{\Lambda,\bar\rho}$ a* universal deformation ring of $\bar\rho$*.*

The following result describes the effect of the change of the coefficient algebra.

Lemma 2.2.8 (Cf. [Wil95, p. 457])**.** *Let $\Lambda \to \Lambda'$ be a finite injective homomorphism of complete Noetherian local rings with finite residue fields $\mathbb{F}$ and $\mathbb{F}'$, respectively. Let R_Λ represent the deformation functor $\mathcal{D}_{\Lambda,\bar\rho}\colon \widehat{\mathcal{A}r}_\Lambda \to$ Sets of $\bar\rho$. Then $R_{\Lambda'} := R_\Lambda \otimes_\Lambda \Lambda'$ represents the deformation functor $\mathcal{D}_{\Lambda',\bar\rho}\colon \widehat{\mathcal{A}r}_{\Lambda'} \to$ Sets of $\bar\rho \otimes_{\mathbb{F}} \mathbb{F}'$. The assertion also holds if R_Λ and $R_{\Lambda'}$ are versal rings.*

We shall need Theorem 2.2.7 only in the case where k is finite. However we shall need a variant of it where k is a local non-archimedian field with its natural topology. For this, let us first make some remarks on continuous cohomology. Write G as a limit $G = \varprojlim_j G/H_j$ where the H_j range over open normal subgroups of G that form a neighbourhood basis of the identity. Suppose first that M is a G-module (i.e., a $\mathbb{Z}[G]$-module) which is fixed by an open subgroup H of G. Then one defines

$$H^i(G,M) := \varinjlim_{H_j \subset H} H^i(G/H_j, M).$$

Such M arise for instance if M is a finite type R-module for a ring R carrying the discrete topology and where M is equipped with a continuous R-linear action of G. A special case being $R = k$ with the discrete topology.

Suppose now k is a local field with its natural topology and that M is a finite dimensional k-vector space carrying the natural topology induced from k and a continuous k-linear G-action. Let $\mathcal{O}$ be the valuation ring of k with maximal ideal $\mathfrak{m}_\mathcal{O}$. Because G is compact a standard argument shows that M contains a G-stable $\mathcal{O}$-lattice L. In this case one defines

$$H^i_{\mathrm{cont}}(G,M) := \varprojlim_n H^i(G, L/\mathfrak{m}_\mathcal{O}^n L) \otimes_L k,$$

and one shows that this definition is independent of any choices. Note that even if M is discrete, we occasionally write $H^i_{\mathrm{cont}}(G,M)$ for $H^i(G,M)$ to have a unified notation.

Theorem 2.2.9 (Kisin)**.** *Let k be a local field and let $\bar\rho\colon G \to \mathrm{GL}_n(k)$ be a continuous representation. Assume in* (ii) – (v) *that $t_{\mathcal{D}_{\bar\rho}}$ is finite-dimensional. Then the following hold:*

(i) One has $t_{\mathcal{D}_{\bar\rho}} \cong H^1_{\mathrm{cont}}(G, \mathrm{ad}_{\bar\rho})$.

(ii) The functor $\mathcal{D}_{\bar\rho}\colon \mathcal{A}r_k \to$ Sets has a hull $R^{\mathrm{ver}}_{k,\bar\rho} \in \mathrm{Ob}(\widehat{\mathcal{A}r}_k)$ together with a versal deformation

$$\rho^{\mathrm{ver}}_{\bar\rho}\colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{ver}}_{k,\bar\rho}).$$

(iii) If $\mathrm{Cent}(\bar\rho) = k$*, then the functor $\mathcal{D}_{\bar\rho}\colon \mathcal{A}r_k \to$ Sets is pro-representable by $R^{\mathrm{univ}}_{k,\bar\rho} \in \mathrm{Ob}(\widehat{\mathcal{A}r}_k)$ together with a* universal deformation

$$\rho^{\mathrm{univ}}_{\bar\rho}\colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{univ}}_{k,\bar\rho}).$$

(iv) The rings $R^{\mathrm{ver}}_{k,\bar\rho}$ and $R^{\mathrm{univ}}_{k,\bar\rho}$, respectively, are quotients of $k[\![x_1,\ldots,x_h]\!]$ for $h = \dim_k t_{\mathcal{D}_{\bar\rho}}$.

(v) If $H^2_{\mathrm{cont}}(G,\mathrm{ad}_{\bar\rho}) = 0$, then $R^{\mathrm{ver}}_{k,\bar\rho}$ and $R^{\mathrm{univ}}_{k,\bar\rho}$ are formally smooth over k of dimension $\dim_k t_{\mathcal{D}_{\bar\rho}}$.

Proof. For k of characteristic zero, the proof is given in [Kis03, Lem. 9.3]. The proof for local fields of positive characteristic is analogous. $\square$

Remark 2.2.10. Independently of whether k is a field carrying the discrete topology or whether k is a local field, we shall use the notation $R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}$ and $R^{\mathrm{univ}}_{\Lambda,\bar{\rho}}$ for a versal and the universal deformation ring, since the distinction is determined by the topology on Λ. In the former case Λ carries the $\mathfrak{m}_\Lambda$-adic topology, in the latter case $\Lambda = k$ and k carries the topology of the local field.

To explain the usefulness of the theorem just stated, we state the following theorem that asserts that for a point $x\colon \operatorname{Spec} k \to X^{\mathrm{univ}}_{\Lambda,\bar{\rho}} := \operatorname{Spec} R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}$ of $X^{\mathrm{univ}}_{\Lambda,\bar{\rho}}$, where k is some local field, the completion of (a modification of) $\mathcal{O}_{X^{\mathrm{univ}}_{\Lambda,\bar{\rho}},x}$ at x has itself an interpretation as a (uni)versal deformation ring. We will need an analog statemnt for universal pseudodeformation spaces that we prove in Corollary 3.2.13. Let $\mathbb{F}$ be a finite field of characteristic p and let $\bar{\rho}\colon G \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation. Let Λ be the ring of integers of a finite totally ramified extension E of $W(\mathbb{F})[1/p]$. Consider a continuous homomorphism $f\colon R^{\mathrm{ver}}_{\Lambda,\bar{\rho}} \to k$ for some local field k, and suppose that the kernel of f is a prime ideal $\mathfrak{p}$ such that k is a finite extension of the fraction field of $R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}/\mathfrak{p}$. Let $\rho_k\colon G \to \mathrm{GL}_n(k)$ be the representation induced from $\rho^{\mathrm{ver}}_{\bar{\rho}}$ via f.

Suppose first that k is of characteristic 0, in which case we follow [Kis03, § 9]. Then f factors via a map $f[1/p]\colon R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}[1/p] \to k$ which is an E-algebra homomorphism, and k is a finite extension field of E. We denote by $\widehat{R}$ the completion of $R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}[1/p]$ at the kernel of $f[1/p]$. Then k is the residue field of $\widehat{R}$. From the finiteness of $E \to k$ one easily deduces that in fact $\widehat{R}$ is naturally a k-algebra. Moreover we have a continuous homomorphism $\widehat{\rho}\colon G \to \mathrm{GL}_n(\widehat{R})$ induced from $\rho^{\mathrm{ver}}_{\bar{\rho}}$. Clearly $\widehat{\rho}$ is a deformation of ρ_k. This provides one with a homomorphism

$$\varphi\colon R^{\mathrm{ver}}_{k,\rho_k} \longrightarrow \widehat{R}.$$

Suppose now that k is of characteristic p. The field k is then isomorphic to a Laurent series field $\mathbb{F}'(\!(x)\!)$ over a finite extension $\mathbb{F}'$ of the finite field $\mathbb{F}$. By passing to a suitable representative in its strict equivalence class, we may assume that ρ_k takes its image in $\mathrm{GL}_n(\mathbb{F}'[\![x]\!])$, and we denote this representation by $\rho_{\mathbb{F}'[\![x]\!]}$. It is a deformation of $\bar{\rho} \otimes_{\mathbb{F}} \mathbb{F}'$. Let $\bar{\rho}' = \bar{\rho} \otimes_{\mathbb{F}} \mathbb{F}'$ and $\Lambda' = \Lambda \otimes_{W(\mathbb{F})} W(\mathbb{F}')$ and consider now the map

$$f_k\colon R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \otimes_{\mathbb{F}'} k \overset{\text{Lem. 2.2.8}}{\cong} R^{\mathrm{ver}}_{\Lambda,\bar{\rho}} \otimes_{\mathbb{F}} k \overset{f\otimes_{\mathbb{F}}\mathrm{id}_k}{\longrightarrow} k.$$

In the present case we define $\widehat{R}$ as the completion of $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \otimes_{\mathbb{F}'} k$ at $\ker f_k$. Clearly, $\widehat{R}$ is a k-algebra with residue field k. Note that now $\rho^{\mathrm{ver}}_{\bar{\rho}} \otimes_{R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}} \widehat{R}$ defines a continuous representation

$$\hat{\rho}\colon G \longrightarrow \mathrm{GL}_n(\widehat{R})$$

which is a deformation of ρ_k. Again this yields a homomorphism

$$\varphi\colon R^{\mathrm{ver}}_{k,\rho_k} \longrightarrow \widehat{R}.$$

Theorem 2.2.11. *The map φ is formally smooth. If $R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}$ is universal, it is an isomorphism.*

Proof. If $\operatorname{Char} k = 0$, then this is [Kis03, Prop. 9.5]. In the case $\operatorname{Char} k > 0$ our proof closely follows loc.cit. Let $\mathcal{O}$ be the valuation ring of k. We consider a commutative diagram

$$\begin{array}{ccc} R^{\mathrm{ver}}_{k,\rho_k} & \longrightarrow & A \\ \downarrow & \overset{g}{\nearrow} & \downarrow \\ \widehat{R} & \longrightarrow & A/I \end{array} \qquad (1)$$

with $A \in \mathrm{Ob}(\mathcal{A}r_k)$ and $I \subset A$ is a square zero ideal, with the solid arrows given, and we seek to construct a dashed arrow g so that the two triangular subdiagrams commute. If $R^{\mathrm{ver}}_{k,\rho_k}$ is universal, we also have to show that the dashed arrow is unique. Note that A and I are finite-dimensional k-vector spaces. Also, the bottom arrow induces a pair of homomorphism $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A/I$ and $k \to A/I$, where the second one is simply the k-algebra structure map.

By possibly conjugating $\hat{\rho}$ by some matrix in $\Gamma_n(\widehat{R})$ we can assume that $\rho^{\mathrm{ver}}_{\rho_k} \otimes_{R^{\mathrm{ver}}_{k,\rho_k}} \widehat{R} = \hat{\rho}$. Following the proof in loc.cit., one shows that there exists an $\mathcal{O}$-subalgebra A° of A such that

(a) A° is free over $\mathcal{O}$ of rank equal to $\dim_k A$ and $A^\circ \otimes_{\mathcal{O}} k = A$,

(b) the image of A° under $A \to k$ is $\mathcal{O}$, and so $A^\circ \in \mathrm{Ob}(\widehat{\mathcal{A}r}_{\mathbb{F}'})$,

(c) the image of $\rho^{\mathrm{ver}}_{\rho_k} \otimes_{R^{\mathrm{ver}}_{k,\rho_k}} A$ lies in $\mathrm{GL}_n(A^\circ)$,

(d) the homomorphism $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A/I$ factors via A°/I° where $I^\circ = I \cap A^\circ$.

Write ρ_{A° for $\rho^{\mathrm{ver}}_{\rho_k} \otimes_{R^{\mathrm{ver}}_{k,\rho_k}} A$ considered with its image in $\mathrm{GL}_n(A^\circ)$. Then ρ_{A° reduces to $\rho^{\mathrm{ver}}_{\bar{\rho}'} \otimes_{R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}} A^\circ/I^\circ$ modulo I°, and thus by the versality of $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}$ there is a homomorphism $g^\circ\colon R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A^\circ$ such that $\rho^{\mathrm{ver}}_{\bar{\rho}'} \otimes_{R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}} A^\circ$ is strictly equivalent to ρ_{A°. Let $g\colon \widehat{R} \to A$ be the the homomorphism obtained from $g^\circ \otimes \mathrm{id}$ under completion. It is now not difficult to see that both triangles in (1) commute with this choice of g.

It remains to show the uniqueness of g if $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}$ is universal. The argument in [Kis03, Prop. 9.5] shows that there is in fact a directed system A°_n, $n \in \mathbb{N}_{\geq 1}$, satisfying (a) – (d) such that $\bigcup_n A^\circ_n = A$. Now if one has g_1, g_2 completing the diagram (1) to two commutative diagrams, there have to be homomorphisms $g^\circ_1, g^\circ_2\colon R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A^\circ_n$ for n sufficiently large that give rise to g_1 and g_2, respectively. The corresponding deformations $G \to \mathrm{GL}_n(A^\circ_n)$ of $\bar{\rho}'$ do agree over A and then they will agree for n sufficiently large. Hence they represent the same strict equivalence class. Because $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}$ is universal, they define the same ring maps $g^\circ_1 = g^\circ_2$ and hence $g_1 = g_2$. □

By carefully choosing ρ_k it is often possible to control $R^{\mathrm{ver}}_{k,\rho_k}$ and hence $\widehat{R}$.

The following result helps to derive consequences on $\mathrm{Spec}\, R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}$. For later applications we will focus on special fibers.

Lemma 2.2.12. *Let R be in $\widehat{\mathcal{A}r}_{\mathbb{F}}$, let $\mathfrak{p} \in \mathrm{Spec}\, R$ be a 1-dimensional point, i.e., $\dim R/\mathfrak{p} = 1$. Let $\kappa(\mathfrak{p}) = \mathrm{Quot}(R/\mathfrak{p})$, consider the homomorphism*

$$\varphi\colon R \otimes_{\mathbb{F}} \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}), \quad r \otimes \alpha \mapsto (r \mod \mathfrak{p}) \cdot \alpha,$$

set $\mathfrak{q} := \ker \varphi$ and denote by $\widehat{R}$ the completion of $R \otimes_{\mathbb{F}} \kappa(\mathfrak{p})$ at the maximal ideal $\mathfrak{q}$ and by $\widehat{R}_{\mathfrak{p}}$ the completion of $R_{\mathfrak{p}}$ at $R_{\mathfrak{p}}\mathfrak{p}$. Then the following hold:

(a) One has an isomorphism $\widehat{R}_{\mathfrak{p}}[\![T]\!] \cong \widehat{R}$.

(b) If $\widehat{R}$ is formally smooth over $\kappa(\mathfrak{p})$ of dimension d, then $R_{\mathfrak{p}}$ is regular of dimension $d-1$.

Proof. Consider $R \to R_{\mathfrak{p}} \to \widehat{R}_{\mathfrak{p}}$. Tensoring with $\kappa(\mathfrak{p})$ over $\mathbb{F}$, it yields a diagram

$$\begin{array}{ccccc} R \otimes_{\mathbb{F}} \kappa(\mathfrak{p}) & \longrightarrow & R_{\mathfrak{p}} \otimes_{\mathbb{F}} \kappa(\mathfrak{p}) & \longrightarrow & \widehat{R}_{\mathfrak{p}} \otimes_{\mathbb{F}} \kappa(\mathfrak{p}) = \left(\varprojlim R_{\mathfrak{p}}/R_{\mathfrak{p}}\mathfrak{p}^n\right) \otimes_{\mathbb{F}} \kappa(\mathfrak{p}) \\ \downarrow \iota & \swarrow \iota' & & \swarrow \iota'' & \\ \widehat{R} = \varprojlim (R \otimes_{\mathbb{F}} \kappa(\mathfrak{p}))/\mathfrak{q}^n, & & & & \end{array}$$

where ι is completion and where initially the dashed arrows ι' and ι'' do not exist. For the existence of ι', we use the universal property of localization. Thus we need to show that $R\smallsetminus\mathfrak{p}\otimes 1$ is mapped under ι to the units in $\widehat{R}$. The ring $\widehat{R}$ is local with residue map induced from φ, and therefore we need to show that $\varphi\circ\iota(R\smallsetminus\mathfrak{p}\otimes 1)$ lies in $\kappa(\mathfrak{p})^\times$, but this is clear from the definitions and the inclusion $R/\mathfrak{p}\hookrightarrow\kappa(\mathfrak{p})$. Regarding ι'', we first note that $\mathfrak{p}\otimes_{\mathbb{F}}\kappa(\mathfrak{p})$ maps to $\mathfrak{q}$ under ι and hence $\mathfrak{p}^n\otimes_{\mathbb{F}}\kappa(\mathfrak{p})$ to $\mathfrak{q}^n$. Hence the existence of ι' gives a compatible system of homomorphisms $R_{\mathfrak{p}}/R_{\mathfrak{p}}\mathfrak{p}^n\to(R\otimes_{\mathbb{F}}\kappa(\mathfrak{p}))/\mathfrak{q}^n$ und this provides the construction of ι''.

Let π denote the reduction map $\pi\colon\widehat{R}\to\kappa(\mathfrak{p})$, set $\varphi'=\pi\circ\iota'$ and $\varphi''=\pi\circ\iota''$, and define $\mathfrak{q}'=\ker\varphi'$ and $\mathfrak{q}''=\ker\varphi''$. Then the arguments just given provide a commutative diagram with canonical isomorphisms in the bottom row

$$\begin{array}{ccccc}
R\otimes_{\mathbb{F}}\kappa(\mathfrak{p}) & \longrightarrow & R_{\mathfrak{p}}\otimes_{\mathbb{F}}\kappa(\mathfrak{p}) & \longrightarrow & \widehat{R}_{\mathfrak{p}}\otimes_{\mathbb{F}}\kappa(\mathfrak{p})=\left(\varprojlim R_{\mathfrak{p}}/R_{\mathfrak{p}}\mathfrak{p}^n\right)\otimes_{\mathbb{F}}\kappa(\mathfrak{p}) \\
\downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota'} & & \downarrow{\scriptstyle\iota''} \\
\widehat{R}=\varprojlim(R\otimes_{\mathbb{F}}\kappa(\mathfrak{p}))/\mathfrak{q}^n & \xrightarrow{\simeq} & \widehat{R}':=\varprojlim(R_{\mathfrak{p}}\otimes_{\mathbb{F}}\kappa(\mathfrak{p}))/\mathfrak{q}'^n & \xrightarrow{\simeq} & \widehat{R}''=\varprojlim(\widehat{R}_{\mathfrak{p}}\otimes_{\mathbb{F}}\kappa(\mathfrak{p}))/\mathfrak{q}''^n,
\end{array}$$

where by slight abuse of notation we denote the middle and right vertical maps again ι' and ι''. Note that by the Cohen structure theorem in equal characteristic the ring $\widehat{R}_{\mathfrak{p}}$ contains $\kappa(\mathfrak{p})$ as a subfield. Focussing on the right must arrow and using that $R_{\mathfrak{p}}$ is regular if and only if $\widehat{R}_{\mathfrak{p}}$ is so, it will suffice to prove the following assertion:

Let $\mathcal{R}$ be a complete Noetherian local $\kappa(\mathfrak{p})$-algebra with residue field $\kappa(\mathfrak{p})$ and residue homomorphism $\pi\colon\mathcal{R}\to\kappa(\mathfrak{p})$, let $\psi\colon\mathcal{R}\otimes_{\mathbb{F}}\kappa(\mathfrak{p})\to\kappa(\mathfrak{p})$ be the homomorphism $r\otimes x\mapsto\pi(r)\cdot x$, let $\mathfrak{Q}=\ker\psi$ and let $\widehat{\mathcal{R}}$ be the completion of $\mathcal{R}\otimes_{\mathbb{F}}\kappa(\mathfrak{p})$ at $\mathfrak{Q}$. Then $\widehat{\mathcal{R}}\cong\mathcal{R}[\![t]\!]$.

To prove the assertion, note first that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\kappa(\mathfrak{p})$-algebras with maximal ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ such that $\kappa(\mathfrak{p})$ is in both cases the residue field, then the completion of $\mathcal{S}:=\mathcal{S}_1\otimes_\kappa(\mathfrak{p})\mathcal{S}_2$ at the maximal ideal $\mathfrak{m}:=\mathfrak{P}_1\otimes_\kappa(\mathfrak{p})\mathcal{S}_2+\mathcal{S}_1\otimes_\kappa(\mathfrak{p})\mathfrak{P}_2$ is isomorphic to

$$\varprojlim\mathcal{S}_1/\mathfrak{P}_1^n\widehat{\otimes}_\kappa(\mathfrak{p})\varprojlim\mathcal{S}_2/\mathfrak{P}_2^n.$$

If furthermore $\mathcal{S}_1$ is complete with respect to $\mathfrak{P}_1$ and if $\varprojlim\mathcal{S}_2/\mathfrak{P}_2^n\cong\kappa(\mathfrak{p})[\![T]\!]$, then the completion of $\mathcal{S}$ at $\mathfrak{m}$ is $\mathcal{S}_1[\![T]\!]$. We apply this to $\mathcal{S}_1=\mathcal{R}$, $\mathcal{S}_2=\kappa(\mathfrak{p})\otimes_{\mathbb{F}}\kappa(\mathfrak{p})$, $\mathfrak{P}_2=\ker(\kappa(\mathfrak{p})\otimes_{\mathbb{F}}\kappa(\mathfrak{p})\to\kappa(\mathfrak{p}),x\otimes y\mapsto xy)$. Then by the following lemma we have $\varprojlim\mathcal{S}_2/\mathfrak{P}_2^n\cong\kappa(\mathfrak{p})[\![T]\!]$, and we deduce $\widehat{\mathcal{R}}\cong\mathcal{R}[\![T]\!]$. □

Lemma 2.2.13. *Let $\mathbb{F}'$ be a finite extension of $\mathbb{F}$ and let L be the Laurent series field over $\mathbb{F}'$. Let $\mathfrak{q}$ be the kernel of the multiplication map $L\otimes_{\mathbb{F}}L\to L$. Then there is an isomorphism*

$$L\widehat{\otimes}L:=\varprojlim_n(L\otimes_{\mathbb{F}}L)/\mathfrak{q}^n\xrightarrow{\simeq}L[\![T]\!].$$

Proof. We think this result ought to be known. But in lack of a reference, we give a proof. We first explain why one can assume $\mathbb{F}=\mathbb{F}'$.

For this observe that $L\cong\mathbb{F}'(\!(s)\!)\cong\mathbb{F}(\!(s)\!)\otimes_{\mathbb{F}}\mathbb{F}'$. Hence, $L\otimes_{\mathbb{F}}L\to L$ can be written as the map

$$\mathbb{F}(\!(s)\!)\otimes_{\mathbb{F}}\mathbb{F}(\!(s)\!)\otimes_{\mathbb{F}}(\mathbb{F}'\otimes_{\mathbb{F}}\mathbb{F}')\to\mathbb{F}'(\!(s)\!),\quad f\otimes g\otimes\alpha\otimes\beta\mapsto fg\alpha\beta.$$

Since $\mathbb{F}'$ is a finite field, the ring $T=\mathbb{F}'\otimes_{\mathbb{F}}\mathbb{F}'$ is a finite product of fields isomorphic to $\mathbb{F}'$, i.e., T contains $[\mathbb{F}':\mathbb{F}]$ elementary idempotents, and one easily checks that all but one of them map to zero under the multiplication map $T\to\mathbb{F}'$. Hence all but one of these idempotents lie in $\mathfrak{q}$ and therefore they also lie in all powers of $\mathfrak{q}$. Thus under completion these components will vanish. Hence from now on, we shall assume $\mathbb{F}' = \mathbb{F}$.

The first observation we make is that $\mathfrak{q}/\mathfrak{q}^2$ is isomorphic to the module of differentials $\Omega_{L/\mathbb{F}}$, by one of the definitions of the latter. Now the single element s is a p-basis of L over $\mathbb{F}$, i.e., L is a vector space over $L^p\mathbb{F} = L^p$ in the basis $1, s, \ldots, s^p$. It follows from [Eis95, Thm. 16.14.b] that $\Omega_{L/\mathbb{F}}$ is a vector space of dimension 1 over L. Consequently, we have $\dim_L \mathfrak{q}^n/\mathfrak{q}^{n+1} \leq 1$ for all $n \geq 0$, and by smoothness of $L[\![T']\!]$ there is a surjective ring homomorphism $\psi\colon L[\![T']\!] \to L\widehat{\otimes}L$.

We will now construct explicit surjective homomorphisms

$$\varphi_n\colon \mathbb{F}(\!(s)\!) \otimes_{\mathbb{F}} \mathbb{F}(\!(s)\!) \to \mathbb{F}(\!(s)\!)[T]/(T^n),$$

and verify that $\mathfrak{q}$ lies in the kernel of φ_n. The idea will be that T should be the image of $s \otimes 1 - 1 \otimes s$ and that morally $\mathfrak{q}^n$ is generated by $(s \otimes 1 - 1 \otimes s)^n$. However we think that in fact the $\mathfrak{q}^n$ are infinitely generated. So we provide an explicit construction. For a formal Laurent series $f = \sum_{i \gg -\infty} a_i s^i$ and $j \in \mathbb{N}_0$ we define the hyperderivatives

$$D^j f := \sum_{i \gg -\infty} a_i \binom{i}{j} s^{i-j}.$$

The operators D^j are continuous in the s-adic topology. We observe that

$$D^j(fg) = \sum_{k=0}^{j} D^k f D^{j-k} g. \tag{2}$$

By continuity it reduces to verifying this for f and g being powers of s, and this comes down to the Vandermonde convolution for binomials $\sum_{k=0}^{j} \binom{i_1}{k}\binom{i_2}{k} = \binom{i_1+i_2}{k}$. We now define the map $\varphi_n\colon L\widehat{\otimes}L \to L[\![T]\!]/(T^n)$ by

$$f \otimes g \longmapsto \sum_{j=0}^{n-1} (-1)^j T^j \cdot f \cdot D^j g.$$

The map is well-defined, and hence additive, since the D^j are $\mathbb{F}$-linear. It is also clear that it is L-linear with L acting from the left. Using (2) and $T^l = 0$ for $l \geq n$, one verifies that the map is a ring homomorphism. To see that φ_n is surjective, one computes the images of elements of the form $f \otimes s^i$ for $i = 0, \ldots, n-1$. This results in an L-linear homomorphism $\oplus_{i=0}^{n-1} L \otimes s^i \to \oplus_{i=0}^{n-1} L \cdot T^i$ of which the obvious matrix representative is upper triangular with ± 1 on the diagonal.

It is also rather straightforward to see that $\mathfrak{q}^n$ lies in the kernel of φ_n: the ideal $\mathfrak{q}$ is generated as an $L-$vector space by the expressions $g \otimes 1 - 1 \otimes g$, $g \in L$. Therefore $\mathfrak{q}^n$ is the L-linear span of expressions $\prod_{k=1}^{n} (g_k \otimes 1 - 1 \otimes g_k)$. Their image under φ_n is

$$\prod_{k=1}^{n} \Big(g_k - \sum_{j=0}^{n-1} (-1)^j T^j D^j g_k \Big) = \prod_{k=1}^{n} \Big(-T \sum_{j=1}^{n-1} (-1)^j T^{j-1} D^j g_k \Big),$$

and the right hand side is a multiple of T^n and hence 0 in $L[\![T]\!]/(T^n)$.

Now the composition $\varphi_n \circ \psi\colon L[\![T']\!] \to L\widehat{\otimes}L \to L[\![T]\!]/(T^n)$ is a surjective L-algebra homomorphism for all $n \geq 0$ with the first and second arrows being surjective. In the limit we therefore obtain an isomorphism $L[\![T']\!] \to L\widehat{\otimes}L \to L[\![T]\!]$, as asserted. □

2.2.2 Smoothness of (uni)versal deformation rings

In this subsection, we show under certain hypotheses on the residual Galois representation ρ that the natural transformation $D_\rho \to D_{\det\rho}$ is smooth.

Theorem 2.2.14 ([Gou01, Thm. 4.2])**.** *Assume that $\overline{\rho}\colon G \to \mathrm{GL}_n(k)$ is a residual representation such that $\mathrm{Cent}(\overline{\rho}) = k$. Consider the universal deformation ring $R^{\mathrm{univ}}_{\overline{\rho}}$. Then*

$$\dim R^{\mathrm{univ}}_{\overline{\rho}}/\mathfrak{m}_\Lambda R^{\mathrm{univ}}_{\overline{\rho}} \geq h_1 - h_2, \qquad \text{where } h_i := \dim_k H^i(G_K, \mathrm{ad}_{\overline{\rho}}) \text{ for } i = 1, 2,$$

and if $h_2 = 0$, then $R^{\mathrm{univ}}_{\overline{\rho}} \cong \Lambda[\![x_1, \ldots, x_{h_1}]\!]$.

Remark 2.2.15. One can get a strengthening of the above using Krull's principal ideal theorem, [Eis95, Thm. 10.2]. It says that if $\mathcal{R}$ be a Noetherian ring and if $I = (f_1, \ldots, f_d)$, then any $\mathfrak{P} \in \operatorname{Spec}\mathcal{R}$ minimal above I satisfies $\operatorname{codim}\mathfrak{P} \leq d$, i.e., $\dim \mathcal{R}_{\mathfrak{P}} \leq d$. So if $\mathcal{R}$ is local, catenary and equidimensional (for instance a power series ring over $\mathbb{F}$ or $W(\mathbb{F})$) of dimension $e \geq d$, then it follows that every component of $\operatorname{Spec}\mathcal{R}/I$ has dimension at least $e - d$.

For the next result we require an extension of local Tate duality from Nekovář:

Theorem 2.2.16 (Tate and Nekovář)**.** *Recall that K is an extension of $\mathbb{Q}_p$ of finite degree d. Let k be a finite field or a local field of residue characteristic p with its natural topology. Let V be a finite-dimensional k-vector space with the topology induced from k, and suppose that V carries a continuous k-linear action by the absolute Galois group G_K of K. Write $V^\vee(1)$ for the twist of* $\mathrm{Hom}_k(V, k)$ *by the cyclotomic character. Then*

(a) For all $j \in \mathbb{Z}$ the k-vector space $H^j_{\mathrm{cont}}(G_K, V)$ is finite-dimensional. It vanishes for $j \notin \{0, 1, 2\}$;

(b) For $j \in \{0, 1, 2\}$ one has natural isomorphisms

$$H^{2-j}_{\mathrm{cont}}(G_K, V^\vee(1)) \xrightarrow{\simeq} H^j_{\mathrm{cont}}(G_K, V)^\vee;$$

(c) One has the Euler characteristic formula

$$\sum_{j\geq 0} (-1)^j \dim_k H^j_{\mathrm{cont}}(G_K, V) = d \cdot \dim_k V.$$

Proof. If k is finite, the above statement is just the usual Tate local duality. If k is local, let $\mathcal{O}$ be its valuation ring. Because G_K is compact one can find an $\mathcal{O}$-lattice T in V that is stable under G_K. Let $j \geq 0$. Then [Nek06, Thm. 5.2.6] asserts that each $H^j_{\mathrm{cont}}(G_K, V)$ is a finitely generated $\mathcal{O}$-module and moreover it gives a spectral sequence

$$\mathrm{Ext}^i_{\mathcal{O}}(H^{2-j}_{\mathrm{cont}}(G_K, T^\vee(1)), \mathcal{O}) \Longrightarrow H^{i+j}_{\mathrm{cont}}(G_K, T).$$

Because $\mathcal{O}$ is regular and of dimension 1, the groups $\mathrm{Ext}^1_{\mathcal{O}}(\cdot, \mathcal{O})$ are finitely generated $\mathcal{O}$-torsion modules. After tensoring the results just quoted with k over $\mathcal{O}$ part (b) and (a) are clear. Part (c) follows from [Nek06, Thm. 4.6.9 and 5.2.11] applied to T, again after tensoring with k over $\mathcal{O}$. ∎

Let k be either finite or a local field and let $\rho\colon G_K \to \mathrm{GL}_n(k)$ be a continuous absolutely irreducible representation. Consider the short exact sequence

$$0 \longrightarrow \mathrm{ad}^0_\rho \longrightarrow \mathrm{ad}_\rho \xrightarrow{\mathrm{tr}} \mathrm{ad}_{\det\rho} \cong k \longrightarrow 0. \tag{3}$$

Using that ad_ρ is self-dual it is easy to see that the sequence dual to (3) is

$$0 \longrightarrow k \xrightarrow{\mathrm{diag}} \mathrm{ad}_\rho \longrightarrow \overline{\mathrm{ad}_\rho} \longrightarrow 0 \tag{4}$$

with $\overline{\mathrm{ad}_\rho} = \mathrm{ad}_\rho / k$. Let $\delta_{p|n}$ be 0 if $p \nmid n$ and 1 if p divides n. We have the following result.

Lemma 2.2.17. *Suppose in the above situation that $H^0(G_K, \overline{\mathrm{ad}_\rho}(1)) = 0$. Then the natural transformation $\mathcal{D}_\rho \to \mathcal{D}_{\det\rho}$ is smooth of relative dimension $\dim_k H^1(G_K, \mathrm{ad}^0_\rho) - \delta_{p|n}$. The hypothesis holds in particular if $p \nmid n$.*

Proof. Let $A \to B$ be a small extension in $\mathcal{A}r_\Lambda$. Let I be its kernel so that $I^2 = 0$. For the relative smoothness, we need to show the surjectivity of

$$\mathcal{D}_\rho(A) \longrightarrow \mathcal{D}_\rho(B) \times_{\mathcal{D}_{\det\rho}(B)} \mathcal{D}_{\det\rho}(A).$$

So suppose we are given deformations $\rho_B \in \mathcal{D}_\rho(B)$ and $\tau_A \in \mathcal{D}_{\det\rho}(A)$ with $\det\rho_B = \tau_A \otimes_A B \in \mathcal{D}_{\det\rho}(B)$. We need to find a deformation $\rho_A \in \mathcal{D}_\rho(A)$ such that $\rho_A \otimes_A B = \rho_B$ and $\det\rho_A = \tau_A$.

Recall that there is a canonical obstruction class $\mathcal{O}(\rho_B) \in H^2(G_K, \mathrm{ad}_\rho) \otimes_k I$, which vanishes if and only if there exists a deformation of ρ to A that lifts ρ_B. Because of the existence of the deformation τ_A that maps to $\det\rho_B$, the obstruction class $\mathcal{O}(\det\rho_B) \in H^2(G_K, \mathrm{ad}_{\det\rho}) \otimes_k I$ vanishes. By Theorem 2.2.16 the long exact sequence of Galois cohomology arising from (3) gives the left exact sequence

$$H^2(G_K, \mathrm{ad}^0_\rho) \longrightarrow H^2(G_K, \mathrm{ad}_\rho) \xrightarrow{H^2(\mathrm{tr})} H^2(G_K, k) \longrightarrow 0$$

By Theorem 2.2.16 the sequence is dual to the right exact sequence

$$0 \longrightarrow H^0(G_K, k(1)) \xrightarrow{H^0(\mathrm{diag}(1))} H^0(G_K, \mathrm{ad}_\rho(1)) \longrightarrow H^0(G_K, \overline{\mathrm{ad}_\rho}(1)),$$

that arises from (4). By our hypothesis the map $H^0(\mathrm{diag}(1))$ is an isomorphism, and so by duality the same holds for $H^2(\mathrm{tr})$. We claim that $\mathcal{O}(\rho_B)$ maps to $\mathcal{O}(\det\rho_B) = 0$ under $H^2(\mathrm{tr}) \otimes_k \mathrm{id}_I$, which will then imply the vanishing of $\mathcal{O}(\rho_B)$.

To see the claim, choose a set-theoretic lift $\tilde\rho\colon G_K \to \mathrm{GL}_n(A)$ of ρ_B. Consider

$$c_\rho(g_1, g_2) = \tilde\rho(g_1g_2)\tilde\rho(g_2)^{-1}\tilde\rho(g_1)^{-1} - 1_n \in \mathrm{Mat}_n(I) \cong \mathrm{ad}_\rho \otimes_k I,$$

and

$$\begin{aligned} c_{\det\rho}(g_1, g_2) &= \det\tilde\rho(g_1g_2)\det\tilde\rho(g_2)^{-1}\det\tilde\rho(g_1)^{-1} - 1 \\ &= \det\left(\tilde\rho(g_1g_2)\tilde\rho(g_2)^{-1}\tilde\rho(g_1)^{-1}\right) - 1 \\ &= \det(c_\rho(g_1, g_2) + 1_n) - 1 \end{aligned}$$

The claim now follows from

$$\det(c_\rho(g_1, g_2) + 1_n) - 1 = \mathrm{tr}\left(c_\rho(g_1, g_2)\right),$$

which is obtained by setting $t = 1$ and $c = -c_\rho(g_1, g_2)$ in the equation

$$\det(t \cdot 1_n - c) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}\left(\bigwedge^k c\right) t^{n-k} = 1 - \operatorname{tr}(c)t \qquad \text{for any } c \in \mathrm{Mat}_n(I)$$

that follows from the vanishing of the k^{th}-exterior power $\bigwedge^k c \in \mathrm{Mat}_n(I^k)$ if $k \geq 2$.

We have now proved that there exists $\rho'_A \in \mathcal{D}_\rho(A)$ mapping to $\rho_B \in \mathcal{D}_\rho(B)$. However, this lift need not satisfy $\det \rho'_A = \tau_A$. At this point we note that our hypothesis in fact implies that $H^2(G_K, \mathrm{ad}^0_\rho) = 0$, so that

$$H^1(G_K, \mathrm{ad}_\rho) \longrightarrow H^1(G_K, \mathrm{ad}_{\det \rho}) = H^1(G_K, k) \tag{5}$$

is surjective. Now $\det \rho'_A$ and τ_A are deformations of τ_B and the space of all such deformations is a principal homogeneous space under $H^1(G_K, k)$, i.e., the tangent space of the deformatin problem, by [Sch68, Rem. 2.15], and likewise the deformations of ρ_B form a principal homogeneous space under $H^1(G_K, \mathrm{ad}_\rho)$. Since (5) is surjective we can thus alter ρ'_A by a class in $H^1(G_K, \mathrm{ad}_\rho)$ into some other deformation ρ_A of ρ_B that also satisfies $\det \rho_A = \tau_A$. This completes the proof of the formal smoothness.

From what we just proved it follows that $R^{\text{univ}}_\rho \cong R^{\text{univ}}_{\det \rho}[\![X_1, \ldots, X_h]\!]$ for some $h \in \mathbb{N}_0$ that is the relative dimension between the two rings; see Proposition 2.1.12. It follows that h is the difference of tangent space dimensions, i.e.,

$$h = \dim_k \mathcal{D}_\rho(k[\varepsilon]) - \dim_k \mathcal{D}_{\det \rho}(k[\varepsilon]) = \dim_k H^1(G_K, \mathrm{ad}_\rho) - \dim_k H^1(G_K, \mathrm{ad}_{\det \rho}),$$

where $k[\varepsilon]$ is the ring of dual numbers of k. Since (5) is surjective, the right hand side is equal to

$$\dim_k H^1(G_K, \mathrm{ad}^0_\rho) - \dim_k H^0(G_K, \mathrm{ad}_{\det \rho}) + \dim_k H^0(G_K, \mathrm{ad}_\rho) - \dim_k H^0(G_K, \mathrm{ad}^0_\rho), \tag{6}$$

and the latter expression is easily identified with the expression given in the lemma; if $p \nmid n$, then (3) is split, the right most term of (6) vanishes and the two middle terms evaluate to 1 and thus cancel. If $p|n$, then the two terms on the right of (6) evaluate to 1 and cancel, and we clearly have $\dim H^0(G_K, \mathrm{ad}_{\det \rho}) = 1$. □

Corollary 2.2.18. *Let k be a finite or local field and let $\rho\colon G_K \to \mathrm{GL}_n(k)$ be a representation with $p \nmid n$. Suppose $q := \operatorname{ord} \mu_{p^\infty}(K)$ and $H^0(G_K, \mathrm{ad}_\rho) \cong k$. Let $\mu_{q,\Lambda} \cong \operatorname{Spec} \Lambda[x]/(x^q - 1)$. Then the composition $X^{\text{univ}}_\rho \to X^{\text{univ}}_{\det \rho} \to \mu_{q,\Lambda}$ for the natural maps is formally smooth and*

$$R^{\text{univ}}_\rho \cong \Lambda[\![x_1, \ldots, x_{h+[K:\mathbb{Q}_p]+1}]\!][x]/\big((1+x)^q - 1\big)$$

for $h = \dim_k H^1(G_K, \mathrm{ad}^0_{\bar\rho})$. In particular, the nilreduction $(\overline{R}^{\text{univ}}_\rho)_{\text{red}} \cong k[\![x_1, \ldots, x_{h+[K:\mathbb{Q}_p]+1}]\!]$ of the special fiber $\overline{R}^{\text{univ}}_\rho := R^{\text{univ}}_\rho / \mathfrak{m}_\Lambda$ is regular.

Proof. By Lemma 2.2.17 the natural maps

$$X^{\text{univ}}_\rho \longrightarrow X^{\text{univ}}_{\det \rho} \quad \text{and} \quad R^{\text{univ}}_{\det \rho} \longrightarrow R^{\text{univ}}_\rho$$

are formally smooth of relative dimension $h = \dim_k H^1(G_K, \mathrm{ad}^0_{\bar\rho})$. Further, let Π be the abelianization of the pro-p completion of G_K. We know from e.g. [Gou01, Prop. 3.13] that $R^{\text{univ}}_{\det \rho} \cong \Lambda[\![\Pi]\!]$ and from local class field theory that $\Pi \cong (\mathbb{Z}_p, +) \times (1 + \mathfrak{m}_K, \cdot)$. Thus

$$R^{\text{univ}}_{\det \rho} \cong \Lambda[\![\Pi]\!] \cong \Lambda[\![x_1, \ldots, x_{[K:\mathbb{Q}_p]+1}]\!][x]/\big((1+x)^q - 1\big).$$

From the formal smoothness property proved in Lemma 2.2.17 and from Proposition 2.1.12, we deduce the isomorphism $R_\rho^{\mathrm{univ}} \cong R_{\det\rho}^{\mathrm{univ}}[\![y_1,\dots,y_h]\!]$. This shows that R_ρ^{univ} has the shape claimed. The assertion on $(\overline{R}_\rho^{\mathrm{univ}})_{\mathrm{red}}$ is now immediate. □

The following lemma enables us to also apply Lemma 2.2.17 to certain non-split extensions in Section 3.3.

Lemma 2.2.19. *Let k be a field, $\rho_i\colon G_K \to \mathrm{GL}_{n_i}(k)$ be a Galois representations for $i = 1,2$ and $\rho = \begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix}$ be an extension of ρ_1 by ρ_2. Suppose that*

(a) The class $c \in \mathrm{Ext}^1_{G_K}(\rho_2,\rho_1)$ is nontrivial,

(b) $H^0(G_K, \mathrm{ad}_{\rho_i}) \cong k$ for $i = 1,2$,

(c) $\mathrm{Hom}_{G_K}(\rho_1,\rho_2) = 0$ and $\mathrm{Hom}_{G_K}(\rho_2,\rho_1) = 0$.

Then $H^0(G_K, \mathrm{ad}_\rho) \cong k$.

Proof. Consider $A_{ij} \in \mathrm{Mat}_{n_i,n_j}(k)$ for $1 \le i,j \le 2$ and the equalities

$$\begin{aligned} 0 &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix} - \begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}\rho_1 & A_{11}c + A_{12}\rho_2 \\ A_{21}\rho_1 & A_{21}c + A_{22}\rho_2 \end{pmatrix} - \begin{pmatrix} \rho_1 A_{11} + cA_{21} & \rho_1 A_{12} + cA_{22} \\ \rho_2 A_{21} & \rho_2 A_{22} \end{pmatrix} \\ &= \begin{pmatrix} [A_{11},\rho_1] - cA_{21} & (A_{11}c - cA_{22}) + A_{12}\rho_2 - \rho_1 A_{12} \\ A_{21}\rho_1 - \rho_2 A_{21} & [A_{22},\rho_2] - A_{21}c \end{pmatrix}. \end{aligned}$$

From hypothesis (c) and the vanishing of the (2,1)-entry we deduce $A_{21} = 0$. From hypothesis (b) and the vanishing of the (1,1)- and (2,2)-entries it follows that A_{ii} are scalar for $i = 1,2$, say equal to $\lambda_i 1_{n_i}$ for some $\lambda_i \in k$, respectively. Finally, the vanishing of the (1,2)-entry gives $(\lambda_1 - \lambda_2)c = \rho_1 A_{12} - A_{12}\rho_2$. Now $g \mapsto \rho_1(g)A_{12} - A_{12}\rho_2(g)$ is a 1-coboundary with values in $\mathrm{Hom}_{G_K}(\rho_2,\rho_1)$, and so if $\lambda_1 \ne \lambda_2$, the last condition implies that c is the trivial class in $\mathrm{Ext}^1_{G_K}(\rho_2,\rho_1)$ which is excluded by ((a)). This shows $\lambda_1 = \lambda_2$, and $A_{12} \in \mathrm{Hom}_{G_K}(\rho_2,\rho_1)$, and hence $A_{12} = 0$, again by ((c)). This completes the proof. □

Lemma 2.2.20. *Let k be a field, $\rho_i\colon G_K \to \mathrm{GL}_{n_i}(k)$ be a Galois representations for $i = 1,2$, let $\chi\colon G_K \to k^\times$ be a character, and $\rho = \begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix}$ be an extension of ρ_1 by ρ_2. Suppose that*

(a) The class $c \in \mathrm{Ext}^1_{G_K}(\rho_2,\rho_1)$ is nontrivial,

(b) ρ_1 and ρ_2 are absolutely irreducible.

(c) $\mathrm{Hom}_{G_K}(\rho_1,\rho_2\otimes\chi) = 0$ and $\mathrm{Hom}_{G_K}(\rho_2,\rho_1\otimes\chi) = 0$.

Then $\mathrm{Hom}_{G_K}(\rho,\rho\otimes\chi) \cong k$, if $\rho_i \cong \rho_i \otimes \chi$ for $i = 1,2$ and c and $c\otimes\chi$ are linearly dependent in $\mathrm{Ext}^1_{G_K}(\rho_2,\rho_1)$. In all other cases $\mathrm{Hom}_{G_K}(\rho,\rho\otimes\chi) = 0$.

2.2.3 Classification of absolutely irreducible mod p Galois representations

The goal of this subsection is Lemma 2.2.23, where we show that there are only finitely many isomorphism classes of absolutely irreducible mod p Galois representatio

We recall the classification of the characters of the tame inertia group I_{t} of G_K from [Ser72], where K is the fixed finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^f}$. Let K^{sep} be a fixed separable closure of K with residue field k^{sep} and K^{nr} be the maximal unramified extension of K. If ϖ is a fixed choice of uniformizer of K^{nr} and $K_n^{\mathrm{nr}} = K^{\mathrm{nr}}(\sqrt[p^n-1]{\varpi})$ for $n \in \mathbb{N}_{\geq 1}$, then $K^{\mathrm{t}} = \varinjlim_{n\in\mathbb{N}_{\geq 1}} K_n^{\mathrm{nr}}$ is the tamely ramified extension of K. For $n \in \mathbb{N}_{\geq 1}$ the character

$$\begin{aligned} \omega_n \colon I_{\mathrm{t}} := \mathrm{Gal}(K^{\mathrm{t}}/K^{\mathrm{nr}}) \longrightarrow \mathrm{Gal}(K_n^{\mathrm{nr}}/K^{\mathrm{nr}}) &\xrightarrow{\sim} \mu_{p^n-1}(K^{\mathrm{nr}}) = \mu_{p^n-1}(k^{\mathrm{sep}}) = \mathbb{F}_{p^n}^{\times}, \\ \sigma &\longmapsto \frac{\sigma(\sqrt[p^n-1]{\varpi})}{\sqrt[p^n-1]{\varpi}}, \end{aligned}$$

gives rise to an inverse system $\{\omega_n\}_{n\in\mathbb{N}_{\geq 1}}$ so that $I_{\mathrm{t}} = \mathrm{Gal}(K^{\mathrm{t}}/K^{\mathrm{nr}}) \cong \varprojlim_{n\in\mathbb{N}_{\geq 1}} \mathbb{F}_{p^n}^{\times}$ [Ser72, Prop. 1 – 2].

Definition 2.2.21 ([Ser72])**.** Let $n \in \mathbb{N}_{\geq 1}$ and $\mathcal{P}_{\mathbb{F}_{p^n}}$ be the set of $\mathbb{F}_p$-embeddings $\mathbb{F}_{p^n} \hookrightarrow \mathbb{F}_p^{\mathrm{alg}}$.

(i) A continuous character $\omega \colon I_{\mathrm{t}} \to (\mathbb{F}_p^{\mathrm{alg}})^{\times}$ is called *of level* n if n is the smallest integer such that ω factors as $\omega \colon I_{\mathrm{t}} \to \mathbb{F}_{p^n}^{\times} \to (\mathbb{F}_p{}^{\mathrm{alg}})^{\times}$.

(ii) The composite of $\omega_n \colon I_{\mathrm{t}} \to \mathbb{F}_{p^n}^{\times}$ with an $\mathbb{F}_p$-embedding $\tau \colon \mathbb{F}_{p^n} \to \mathbb{F}_p{}^{\mathrm{alg}}$ is called a *fundamental character of level* n and denoted by $\omega_{n,\tau} \colon I_{\mathrm{t}} \to (\mathbb{F}_p^{\mathrm{alg}})^{\times}$.

The n fundamental characters $\omega \colon I_{\mathrm{t}} \to (\mathbb{F}_p^{\mathrm{alg}})^{\times}$ of level n are $\omega_n^{p^i}$ for $i \in \{0, \ldots, n-1\}$, and their name is justified by the following proposition.

Proposition 2.2.22 ([Ser72, Prop. 5])**.** *Let $\mathbb{F}$ be a finite field of characteristic p. There is an isomorphism between the set $(\mathbb{Q}/\mathbb{Z})' = \{\frac{i}{j} \in \mathbb{Q}/\mathbb{Z} \,:\, i, j \in \mathbb{Z},\ p \nmid j\}$ and the group of continuous characters $I_{\mathrm{t}} \to (\mathbb{F}^{\mathrm{alg}})^{\times}$ given by $\frac{i}{j} \mapsto \omega_n^i$, where $n \in \mathbb{N}_{\geq 1}$ is the unique minimal integer satisfying $p^n \equiv 1 \mod j$.*

Hence, there are only finitely many isomorphism classes of continuous characters $I_{\mathrm{t}} \to (\mathbb{F}^{\mathrm{alg}})^{\times}$ factoring via ω_n and this is used to show that there are also only finitely many isomorphism classes of absolutely irreducible representations $G_K \to \mathrm{GL}_n(\mathbb{F})$. To see this, consider the unramified extension K_n of K of degree n with residue field $\mathbb{F}_{q^n}$. We extend the fundamental character $\omega_{n,\tau}$ to G_{K_n} using the local Artin map $\mathrm{rec}_{K_n} \colon \hat{\mathbb{Z}} \times \mathcal{O}_{K_n}^{\times} \xrightarrow{\sim} G_{K_n}^{\mathrm{ab}}$ and the induced projection $\mathrm{pr}_2 \colon G_{K_n}^{\mathrm{ab}} \to \mathcal{O}_{K_n}^{\times}$. Then for any $\tau \in \mathcal{P}_{\mathbb{F}_{q^n}}$ the composite $I_{K_n} \cong I_K \twoheadrightarrow I_{\mathrm{t}} \xrightarrow{\omega_{fn,\tau}} (\mathbb{F}_p^{\mathrm{alg}})^{\times}$ extends as follows to G_{K_n}:

$$\omega_{fn,\tau} \colon G_{K_n} \longrightarrow G_{K_n}^{\mathrm{ab}} \xrightarrow{\mathrm{pr}_2} \mathcal{O}_{K_n}^{\times} \longrightarrow \mathbb{F}_{q^n}^{\times} \xrightarrow{\tau} (\mathbb{F}_p^{\mathrm{alg}})^{\times}.$$

We introduce the following useful notation: If $n > 1$ and if $1 \leq h \leq q^n - 2$, we say that h is *primitive* if there is no strict divisor j of n such that h is a multiple of $(q^n - 1)/(q^j - 1)$. This condition is equivalent to requiring that if we write $h = e_{n-1}e_{n-2}\ldots e_1e_0$ in base q (with digits $e_j \in \{0, \ldots, q-1\}$), then the only cyclic permutation of the digits that gives again h in base p is the identity. Also, for $\lambda \in \mathbb{F}_p^{\mathrm{alg}}$, we write $\overline{\mu}_{\lambda} \colon G_K \to \mathbb{F}'$ for the unramified character that sends the Frobenius automorphism in $\mathrm{Gal}(K^{\mathrm{nr}}/K)$ to $\lambda^{-1} \in \mathbb{F}_p^{\mathrm{alg}}$.

Lemma 2.2.23 (Cf. [Ber10, Cor. 2.1.5], [Con99, Thm. 2.2.2])**.** *If $\mathbb{F}$ is a finite field of characteristic p and $\overline{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ is an n-dimensional absolutely irreducible representation, then there exists $\tau \in \mathcal{P}_{\mathbb{F}_{q^n}}$, some primitive number $1 \le i \le q^n - 2$, and some $\lambda \in \mathbb{F}_q^{\mathrm{alg}}$ with $\lambda^n \in \mathbb{F}$, such that*

$$\overline{\rho} \otimes_{\mathbb{F}} \mathbb{F}' \cong \left(\mathrm{Ind}_{G_{K_n}}^{G_K} \omega_{fn,\tau}^i \otimes_{\mathbb{F}_{q^n}} \mathbb{F}'\right) \otimes \overline{\mu}_\lambda,$$

where $\mathbb{F}'$ is the smallest extension of $\mathbb{F}_p$ containing $\mathbb{F}$ and λ. In particular, there are only finitely many isomorphism classes of absolutely irreducible representations $G_K \to \mathrm{GL}_n(\mathbb{F})$.

Proof. We know from [Mul13, Prop. 2.1.1] that there exists such $\tau \in \mathcal{P}_{\mathbb{F}_{q^n}}$, $\lambda \in (\mathbb{F}^{\mathrm{alg}})^\times$ and $i \in \mathbb{N}_{\ge 1}$ so that

$$\overline{\rho} \otimes_{\mathbb{F}} \mathbb{F}_p^{\mathrm{alg}} \cong \left(\mathrm{Ind}_{G_{K_n}}^{G_K} \omega_{fn,\tau}^i \otimes_{\mathbb{F}_{q^n}} \mathbb{F}_p^{\mathrm{alg}}\right) \otimes \overline{\mu} \tag{7}$$

for some unramified character $\overline{\mu}\colon G_K \to (\mathbb{F}_p^{\mathrm{alg}})^\times$. Then $\overline{\mu}$ factors via $\mathrm{Gal}(K^{\mathrm{nr}}/K)$ so that $\overline{\mu}$ is uniquely determined by the image $\lambda^{-1} \in (\mathbb{F}^{\mathrm{alg}})^\times$ of the Frobenius automorphism in $\mathrm{Gal}(K^{\mathrm{nr}}/K)$ under $\overline{\mu}$. Moreover, $\mu_\lambda = \mu$ is defined over the finite field extension $\mathbb{F}(\lambda)$ of $\mathbb{F}$. Since the Frobenius homomorphism φ_q generates $\mathrm{Gal}(K_n/K)$, we have by Mackey's formula [Ser95, Prop. 22]) for any $k \in \mathbb{N}_{\ge 1}$ that

$$\mathrm{Res}_{G_{K_n}}^{G_K} \mathrm{Ind}_{G_{K_n}}^{G_K} \omega_{n,\tau}^k \cong \oplus_{\overline{g} \in \mathrm{Gal}(K_n/K)} (\omega_{fn,\tau}^k)^{\overline{g}} \cong \oplus_{j=0}^{n-1} \omega_{fn,\tau}^{q^j k}$$

is semisimple. Its characteristic polynomial is the cyclotomic polynomial $g \mapsto \prod_{j=0}^{n-1}\left(t - \omega_{fn,\tau}^{q^j k}(g)\right)$ which as minimal polynomial of the q^n-th root of unity $\omega_{fn,\tau}^k(g)$ takes values in $\mathbb{F}_q[t]$. By the theorem of Brauer-Nesbitt [CR62, (30.16) Thm.] its characteristic polynomial uniquely determines $\mathrm{Ind}_{G_{K_n}}^{G_K} \omega_{fn,\tau}^k$ and the triviality of the Brauer group of a finite field implies by [DS74, Lem. 6.13] that $\mathrm{Res}_{G_{K_n}}^{G_K} \mathrm{Ind}_{G_{K_n}}^{G_K} \omega_{n,\tau}^k$ is defined over $\mathbb{F}_q$. This shows that also $\mathrm{Ind}_{G_{K_n}}^{G_K} \omega_{n,\tau}^k$ is defined over $\mathbb{F}_q$, and the existence of the finite extension $\mathbb{F}'$ of $\mathbb{F}$ follows from taking the determinant in (7). □

2.3 *Clifford theory of induced representations and twist-invariance*

Throughout Section 2.3, G denotes a (possibly infinite) group and H a normal subgroup of finite index.

Using Clifford theory, we show in Corollary 2.3.6 that a semisimple representation $\rho\colon G \to \mathrm{GL}_n(k)$ with values in an algebraically closed field k is invariant under twisting by a character $\chi\colon G \to k^\times$ of finite order if and only if ρ is induced from a representation of $\ker\chi$. We use this to investigate the locus of special pseudodeformations in e.g. Subsection 3.3.2.

Definition 2.3.1. For a representation $\rho\colon H \to \mathrm{GL}_m(R)$ over a ring R and $g \in G$, we define the twist of ρ by g as

$$\rho^g\colon H \longrightarrow \mathrm{GL}_m(R), \qquad h \longmapsto \rho(ghg^{-1}).$$

Remark 2.3.2. Denote by $\overline{}\colon G \to G/H$, $g \mapsto \overline{g}$, the canonical projection, and suppose that $g, g' \in G$ satisfy $\overline{g} = \overline{g}'$, so that $g'g^{-1} \in H$. Then $\rho^g \cong \rho^{g'}$ because

$$\rho^{g'}(h) = \rho(g'hg'^{-1}) = \rho(g'g^{-1}ghg^{-1}(g'g)^{-1}) = \rho(g'g^{-1})\rho^g(h)\rho(g'g^{-1})^{-1} \quad \forall h \in H.$$

In particular, the number of isomorphism classes in $\{\rho^g \ : \ g \in G\}$ is finite.

Lemma 2.3.3. *Suppose that the index $[G : H]$ is invertible in a field k, and that $\rho_1, \rho_2\colon H \to \mathrm{GL}_m(k)$ are semisimple representations. Then the following hold*

(a) *The representations* $\operatorname{Ind}_H^G \rho_i$, $i = 1, 2$, *are semisimple.*

(b) $\operatorname{Ind}_H^G \rho_1 \cong \operatorname{Ind}_H^G \rho_2$ *if and only if*

$$\bigoplus_{g \in G/H} \rho_1^g \cong \bigoplus_{g \in G/H} \rho_2^g. \tag{8}$$

(c) *If* ρ_1 *is irreducible, then* (8) *is equivalent to* $\rho_2 \cong \rho_1^g$ *for some* $g \in G$.

Proof. Part (a) is immediate from [Web16, Ch. 5, Exerc. 8]. Part (c) follows from the uniqueness of composition factors and the irreducibility of ρ_1 (and hence all ρ_1^g). We now prove Part (b).

First, $\operatorname{Ind}_H^G \rho_1 \cong \operatorname{Ind}_H^G \rho_2$ implies by Mackey's formula [Ser95, Prop. 22] that

$$\bigoplus_{g \in G/H} \rho_1^g \cong \operatorname{Res}_H^G \operatorname{Ind}_H^G \rho_1 \cong \operatorname{Res}_H^G \operatorname{Ind}_H^G \rho_2 \cong \bigoplus_{g \in G/H} \rho_2^g$$

For the other direction, note first that by [CR81, Lem. 10.12] we have $\operatorname{Ind}_H^G \rho_i \cong \operatorname{Ind}_H^G \rho_i^g$ for all $g \in G$. Thus using Mackey's tensor product theorem for induced representations [CR81, (10.20) Cor.], we obtain

$$\operatorname{Ind}_H^G \Big(\bigoplus_{g \in G/H} \rho_i^g \Big) = \bigoplus_{g \in G/H} \big(\operatorname{Ind}_H^G \rho_i^g \big) = \big(\operatorname{Ind}_H^G \rho_i \big)^{\oplus [G:H]}.$$

By hypothesis, the left hand side is, up to isomorphism, independent of $i \in \{1, 2\}$. We deduce the isomorphism $(\operatorname{Ind}_H^G \rho_1)^{\oplus m} \cong (\operatorname{Ind}_H^G \rho_2)^{\oplus m}$. By Part (a) the representations $\operatorname{Ind}_H^G \rho_i$, $i = 1, 2$, are semisimple. It follows from the theorem of Brauer-Nesbitt [CR62, (30.16) Thm.] that $\operatorname{Ind}_H^G \rho_1 \cong \operatorname{Ind}_H^G \rho_2$. □

From now on, in this section we also fix some field k (that is often algebraically closed), a character $\chi \colon G \to k^\times$ of finite order $m \geq 1$ and we assume that $H = \ker \chi$. In particular, *m is invertible in k*. Using Clifford theory we show in Corollary 2.3.6 that a semisimple representation $\rho \colon G \to \mathrm{GL}_n(k)$ is invariant under twisting by χ if and only if ρ is induced from a representation of H.

The following is a standard result of Clifford Theory, e.g. [CR62, Thm. 49.2, Cor. 50.6].

Theorem 2.3.4. *Let k be algebraically closed and let $\rho \colon G \to \mathrm{GL}_n(k)$ be an irreducible representation such that $\rho \cong \rho \otimes \chi$. Then m divides n and there is an irreducible representation $\rho' \colon H \to \mathrm{GL}_{n/m}(k)$, such that*

$$\rho \cong \operatorname{Ind}_H^G \rho'$$

Moreover, ρ' satisfies

$$\operatorname{Res}_H^G \rho \cong \bigoplus_{g \in G/H} (\rho')^g,$$

and the representations $(\rho')^g$, $g \in G/H$, are irreducible and pairwise non-isomorphic.

Proof. Let A be an invertible $n \times n$-matrix over k such that

$$A\rho(g)A^{-1} = \chi(g)\rho(g) \text{ for all } g \in G. \tag{9}$$

From Equation (9) one deduces $A^m \rho(g) A^{-m} = \chi^m(g)\rho(g) = \rho(g)$ for all $g \in G$. Since ρ is irreducible, [CR81, (3.17) Schur's lemma] implies that $A^m = \lambda \cdot 1_n$ for some $\lambda \in k$. Since k is algebraically closed, we may replace A by $\sqrt[m]{\lambda}^{-1} A$ so that $A^m = 1_n$ and find $P \in \mathrm{GL}_n(k)$ such that $J := P^{-1}AP$ is a Jordan matrix. Then $1_n = A^m = (PJP^{-1})^m$ implies $1_n = J^m$. Because $m = \operatorname{ord}\chi$ is a unit in k, the matrix A must be semisimple by [CR81, Maschke's theorem (3.14)] and its eigenvalues must be m^{th}-roots of unity.

Let ζ be a primitive m^{th}-root of unity. Then after a change of basis we may write A as a block diagonal matrix with diagonal blocks $A_1, \ldots, A_{m'}$ for some $m' \in \mathbb{N}_{\geq 1}$ such that for $i = 1, \ldots, m'$ A_i is a scalar matrix $\zeta^{k_i} 1_{n_i}$ with $1 \leq k_1 < k_2 < \ldots < k_{m'} \leq m$. For all $g \in G$ and $i, j = 1, \ldots, m'$ we decompose $\rho(g)$ correspondingly into blocks $\rho_{i,j}(g)$ so that equation (9) provides

$$\zeta^{k_i - k_j} \rho_{i,j}(g) = \chi(g)\rho_{i,j}(g). \tag{10}$$

Choose $g \in H$ such that $\chi(g) = 1$. Then $\rho_{i,j}(g)$ is zero unless $k_i - k_j \equiv 1 \pmod m$. Since $\rho(g)$ is invertible for each j there must be an i such that $\rho_{i,j}(g)$ is nonzero, and hence, since the k_i are strictly increasing we must have $k_{i+1} = k_i + 1$ for $i = 1, \ldots, m' - 1$, and $k_{m'} + 1 - m = k_1$, so that $m' = m$ and $k_i = i$ for $i = 1, \ldots, m$. Moreover for $\rho(g)$ to be invertible it is necessary that the nonzero blocks $\rho_{i+1,i}(g)$, $i = 1, \ldots, m - 1$, together with $\rho_{1,m}(g)$ are square matrices, and thus of the same size. Hence m divides n and $n_i = n/m$ for all i.

For all $i, j = 1, \ldots, m$ and $h \in H$ Equation (10) becomes $\zeta^{i-j}\rho_{i,j}(h) = \rho_{i,j}(h)$ so that $\rho(h) = \bigoplus_{i=1}^{m} \rho_{i,i}(h)$ is a block diagonal matrix and $\rho_{i,i} \colon H \to \mathrm{GL}_{n/m}(k)$, $h \mapsto \rho_{i,i}(h)$, is a representation of dimension n/m. In particular, the restriction satisfies

$$\operatorname{Res}_H^G \rho = \bigoplus_{i=1}^{m} \rho_{i,i}.$$

We choose $\rho' = \rho_{1,1}$ and consider $\operatorname{Ind}_H^G \rho'$. By [CR62, (10.8) Frobenius Reciprocity Theorem] we have

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G \rho', \rho) = \operatorname{Hom}_H(\rho', \operatorname{Res}_H^G \rho) \neq 0.$$

Let $f \colon \operatorname{Ind}_H^G \rho' \to \rho$ be a nonzero G-homomorphism. Since ρ is irreducible, it must be surjective, and because $\dim \rho = n = m \cdot n/m = \dim \operatorname{Ind}_H^G \rho'$, its kernel must be zero, so that f is an isomorphism. Next note that Ind_H^G is an exact functor, see [CR81, § 10, Exerc. 20]. Hence ρ' is irreducible, because ρ is so. Note also that ρ' is irreducible if and only if $(\rho')^g$ is so. Moreover since H is a normal subgroup, we have by Mackey's formula [Ser77, Prop. 22] that

$$\operatorname{Res}_H^G \rho \cong \operatorname{Res}_H^G \operatorname{Ind}_H^G \rho' = \bigoplus_{g \in G/H} (\rho')^g.$$

Since ρ is irreducible, Mackey's irreducibility criterion [Ser77, Prop. 23] states that for all $g \in G/H$ the irreducible representations $(\rho')^g$ need to be pairwise non-isomorphic. □

Remark 2.3.5. The above proof uses that k is algebraically closed in two instances: First, to deduce from Schur's lemma that $A^m \in \operatorname{End}(\rho) \cong k$ is equal to $\lambda \cdot 1_n$ for some $\lambda \in k$. Second, we use it to replace A by $\sqrt[m]{\lambda} \cdot A$ so that $A^m = 1_n$ and the eigenvalues of A are m^{th}-roots of unity. The fact that A^m is scalar would also follow if one requires that ρ be absolutely irreducible.

The second use of algebraic closedness of k cannot be avoided. There are simple examples where $\operatorname{Ind}_H^G \rho'$ can be defined over a smaller field than ρ': For instance take G the dihedral group D_n of order $2n$ with $n > 1$ odd and H its cyclic subgroup C_n of order n, and let $\chi \colon C_n \to \mathbb{Q}(\zeta_n)^\times$ be a character of order n. Then $\mathbb{Q}(\zeta_n)$ is the minimal field over which χ is defined. However $\operatorname{Ind}_H^G \chi$ can be defined over $\mathbb{Q}(\zeta_n)^+$, the maximal totally real subfield of $\mathbb{Q}(\zeta_n)$.

Finally observe that the above proof shows that for k not algebraically closed, Theorem 2.3.4 applies after passing to a suitable finite extension k' of k, provided that ρ is absolutely irreducible. It suffices that over k' all irreducible factors of ρ and of $\operatorname{Res}_H^G \rho$ are absolutely irreducible.

Corollary 2.3.6. *Suppose that k is algebraically closed and that $\rho\colon G \to \mathrm{GL}_n(k)$ is semisimple. Then $\rho \cong \rho \otimes \chi$ holds if and only if there is a representation $\rho'\colon H \to \mathrm{GL}_m(k)$ such that $\rho \cong \mathrm{Ind}_H^G \rho'$. Moreover, any such ρ' is semisimple, and one also has $\mathrm{Res}_H^G \rho = \bigoplus_{g \in G/H} (\rho')^g$.*

Proof. If $\rho \cong \mathrm{Ind}_H^G \rho'$, then Mackey's tensor product theorem for induced representations implies

$$\rho \otimes \chi \cong (\mathrm{Ind}_H^G \rho') \otimes \chi \cong \mathrm{Ind}_H^G(\rho' \otimes \mathrm{Res}_H^G \chi) \cong \mathrm{Ind}_H^G \rho' \cong \rho \qquad \text{[CR81, Cor. (10.20)].}$$

Conversely, suppose that $\rho \cong \rho \otimes \chi$ and write $\rho = \oplus_{j \in J} \rho'_j$ with irreducible representations ρ'_j for $j \in J$. We regroup this decomposition according to orbits under iterated twisting by χ. This gives rise to a decomposition

$$\rho \cong \bigoplus_{i \in I} \Big(\bigoplus_{j=0}^{m_i - 1} \rho_i \otimes \chi^j \Big)^{\oplus r_i} \tag{11}$$

for irreducible representations $\rho_i\colon G \to \mathrm{GL}_{n_i}(k)$, $i \in I$, and divisors m_i of m so that

$$\rho_i \otimes \chi^{m_i} \cong \rho_i \qquad \text{for } i \in I,$$

and no ρ_i is isomorphic to $\rho_{i'} \otimes \chi^j$ for some $j \in \{0, \dots, m_{i'} - 1\}$ and $i' \in I$.

Under the isomorphism $G/H \cong \mathbb{Z}/(m)$ we have for $H_i = \ker \chi^{m_i} \supset H$ that $H_i/H \cong (m/m_i)\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/m_i\mathbb{Z}$, which shows that $\mathrm{Res}_{H_i}^G \chi$ is a character of order m_i. By Theorem 2.3.4 we find representations $\rho''_i\colon H_i \to \mathrm{GL}_{n_i/m_i}(k)$ such that $\mathrm{Ind}_{H_i}^G \rho''_i \cong \rho_i$. Let k_H be the trivial representation of H on k. Then

$$\begin{aligned}
\bigoplus_{j=0}^{m_i-1} \rho_i \otimes \chi^j &\cong \mathrm{Ind}_{H_i}^G \rho''_i \otimes \Big(\bigoplus_{j=0}^{m_i-1} \chi^j \Big) \cong \mathrm{Ind}_{H_i}^G \Big(\rho''_i \otimes \bigoplus_{j=0}^{m_i-1} \mathrm{Res}_{H_i}^G \chi^j \Big) \\
&\cong \mathrm{Ind}_{H_i}^G \big(\rho''_i \otimes \mathrm{Ind}_H^{H_i} k_H \big) \cong \mathrm{Ind}_{H_i}^G \mathrm{Ind}_H^{H_i} \big(\mathrm{Res}_H^{H_i} \rho''_i \otimes k_H \big) \\
&\cong \mathrm{Ind}_H^G \big(\mathrm{Res}_H^{H_i} \rho''_i \big),
\end{aligned}$$

where the second and fourth isomorphism follows from Mackey's tensor product theorem for induced representations [CR81, (10.20) Cor.]. Together with the canonical decomposition (11) this proves

$$\rho \cong \bigoplus_{i \in I} \Big(\bigoplus_{j=0}^{m_i-1} \rho_i \otimes \chi^j \Big)^{\oplus r_i} \cong \mathrm{Ind}_H^G \Big(\bigoplus_{i \in I} (\mathrm{Res}_H^{H_i} \rho''_i)^{\oplus r_i} \Big). \qquad \square$$

The two assertions at the end are immediate from Mackey's formula [Ser95, Prop. 22]; see also the proof of Lemma 2.3.3(b).

We have the following integral refinement of Theorem 2.3.4:

Theorem 2.3.7. *Let R be a complete discrete valuation ring with (not necessarily algebraically closed) residue field and fraction field L. Suppose $\rho\colon G \to \mathrm{GL}_n(R)$ is a continuous representation of a compact group G such that $\rho \otimes_R L$ is absolutely irreducible and $\rho \cong \rho \otimes \chi$ for some character $\chi\colon G \to L^\times$ of finite order m.*

Then there is a finite Galois extension L' of L with ring of integers R' and a continuous representation $\rho'\colon H \to \mathrm{GL}_{n/m}(R')$ such that $\rho \otimes_R L' \cong \mathrm{Ind}_H^G \rho' \otimes_{R'} L'$.

Proof. We follow the argument of the proof of Theorem 2.3.4. We define L' as $L(\sqrt[m]{\lambda})$. Because m^{th} roots of unity lie in $\chi(G) \subset L^\times$, this is a finite Galois extension. Then over L' one can define $\rho'\colon H \to \mathrm{GL}_{n/m}(L')$ such that $\mathrm{Ind}_H^G \rho' \cong \rho \otimes_R L'$. Observe that by its construction ρ' is continuous, as it is a direct factor of the continuous representation $\mathrm{Res}_H^G \rho$. Now use the compactness of G and the continuity of ρ' to find a change of basis of $(L')^{n/m}$ such that the image of ρ' after this base change lies in $(R')^{n/m}$. □

Remark 2.3.8. (a) It would be interesting to prove more general integral versions of Theorem 2.3.4 than Theorem 2.3.7.

(b) Another integrality type question in the spirit of Theorem 2.3.7 is the following: Suppose the pseudorepresentation of ρ is definable over a ring R. Can one describe a finite extension of R over which the pseudorepresentation of ρ' is definable?

2.4 *Cohomology of profinite groups and Demushkin groups*

This section gives a short introduction to Demushkin groups. Using their classification in Example 2.4.5 we give an explicit description of (uni)versal deformation rings of 2-dimensional Galois representations in Section 4.3. The references best suited for our purposes are [Koc00, Chapter 3], [NSW00, Chapter III] and [Lab67].

We fix a *pro-p-group* G; i.e., an inverse limit $\lim G_i$ of finite p-groups G_i; i.e., $\operatorname{ord} G_i = p_i^r$ for some integer r_i.

Definition 2.4.1. (i) A *generator system of* G is a subset $S \subseteq G$ such that S generates G as topological group and every open subgroup of G contains almost all elements of S. The generator system S is *minimal* if no proper subset of S is a generator system of G.

(ii) A *presentation of* G by a free pro-p group F with generator system $\{s_i : i \in I\}$ of F is an exact sequence

$$1 \longrightarrow R \longrightarrow F \stackrel{\varphi}{\longrightarrow} G \longrightarrow 1.$$

The presentation is *minimal* if $\{\varphi(s_i) : i \in I\}$ is a minimal generator system of G.

(iii) A *relation system of* G with respect to a presentation $\{1\} \to R \stackrel{\psi}{\to} F \to G \to \{1\}$ is a subset $\mathcal{R} \subseteq R$ so that $\psi(R)$ is the smallest closed normal subgroup of F containing $\psi(\mathcal{R})$. The relation system is called *minimal* if no proper subset of $\mathcal{R}$ is a relation system of G.

(iv) For q a power of p the *descending q-series* $\{G^{(i,q)}\}_{i\geq 1}$ *of* G is defined by

$$G^{(1,q)} := G \qquad \text{and} \qquad G^{(i+1,q)} := (G^{(i,q)})^q(G^{(i,q)}, G) \quad \text{for } i \geq 1,$$

where $(G^{(i,q)})^q(G^{(i,q)}, G)$ is the closed subgroup of G generated by the elements g^q and the commutators $(g, h) = g^{-1}h^{-1}gh$ with $g \in G^{(i,q)}, h \in G$.

By [NSW00, Prop. (3.8.2)], the descending q-power series form a fundamental system of open neighbourhoods of 1. Using Pontryagin duality, one shows the following.

Proposition 2.4.2 ([NSW00, Prop. (3.9.1) and (3.9.5)])**.** *For $i \in \mathbb{N}_0$ consider the $\mathbb{F}_p$-vector space $H^i(G) := H^i(G, \mathbb{Z}/p\mathbb{Z})$ and its dimension $h_i := \dim_{\mathbb{F}_p} H^i(G)$. Then the following hold.*

(i) (Burnside basis theorem) *Let S be a minimal generator system of G. Then the* generator rank $d(G) := \mathrm{card}(S)$ *is equal to* $\mathrm{Hom}(G/G^{(2,p)}, \mathbb{Q}/\mathbb{Z}) = h_1$ *and thus independent of the choice of S. In particular, G is finitely generated if and only if h_1 is finite.*

(ii) *Let $\mathcal{R}$ be a minimal relation system with respect to some presentation of G. Then the* relation rank $r(G) := \operatorname{card}(\mathcal{R})$ *is equal to* $\dim_{\mathbb{F}_p} \operatorname{Hom}(R/R^p(R,F), \mathbb{Q}/\mathbb{Z}) = \dim_{\mathbb{F}_p} H^1(R)^F = h_2$ *and thus independent of the choice of $\mathcal{R}$.*

Demushkin studied the following pro-p groups with a finite number of topological generators and only one relation between them.

Definition 2.4.3. The pro-p-group G is a *one relator pro-p group* if h_1 is finite and $h_2 = 1$. If in addition the cup product $H^1(G) \times H^1(G) \to H^2(G)$ is a non-degenerate bilinear form, then G is called a *Demushkin group.*

In order to classify Demushkin groups, one determines invariants of a Demushkin group.

Theorem 2.4.4 ([Lab67])**.** *Consider $U_f := 1 + p^f\mathbb{Z}_p$ and the group $U = \mathbb{Z}_p^\times \cong \mu_{p-1} \times U_1$ of p-adic units equipped with the p-adic topology. There exists a unique continuous homomorphism $\chi : G \to U$ such that the canoncical homomorphism*

$$H^1(G, I(\chi)/p^i I(\chi)) \to H^1(G, I(\chi)/pI(\chi))$$

is surjective for $i \geq 1$, where $I(\chi) := \mathbb{Z}_p$ is the topological G-module with G-action given by χ. Moreover, $\operatorname{im}\chi$ *is an invariant of G and χ defines further invariants as follows:*

(i) *the highest power $q(G)$ of p such that $\operatorname{im}\chi \subset 1 + q\mathbb{Z}_p$ (with equality if $q(G) \neq 2$);*

(ii) *$a(G) := [\operatorname{im}\chi : (\operatorname{im}\chi)^2] \in \{2, 4\}$;*

(iii) *$f(G) \in \{2, \ldots, \infty\}$ is determined by $\operatorname{im}\chi = \{\pm 1\} \times U_f$ if $q(G) \neq 2$, h_1 is odd; $\operatorname{im}\chi = (-1+2^f)$ if $q(G) = 2$, h_1 even, $a(G) = 2$;$\operatorname{im}\chi = \{\pm 1\} \times U_f$ if $q(G) = 2$, h_1 even, $a(G) = 4$; and else one sets $f(G) = \infty$.*

The invariant $q(G)$ can be alternatively described as the unique power q of p such that $G^{\mathrm{ab}} = G/(G,G) \cong \mathbb{Z}_p^{h_1 - 1} \times (\mathbb{Z}_p/q\mathbb{Z}_p)$. Together with the invariant h_1, this classifies Demushkin groups completely. We first give Demushkin's classical example that we are interested in.

Example 2.4.5 (The group of the maximal p-extension of a local field [Lab67, § 5])**.** Recall that K is a finite extension of $\mathbb{Q}_p$ with $d = [K : \mathbb{Q}_p]$. Consider the maximal p-extension $K(p)$ of K; i.e., the largest Galois extension of K whose Galois group G is a pro-p-group. If K does not contain a p^{th} root of unity, then G is a free pro-p-group of rank $d + 1$. If K does contain a p^{th} root of unity, then G is a Demushkin group with generator rank $d + 2$. If $q \neq 2$, then its relation reads

$$r = x_1^q(x_1, x_2)(x_3, x_4)\ldots(x_{d+1}, x_{d+2}) = 1.$$

If $q = 2$ and d odd, then its relation reads

$$r = x_1^2 x_2^4 (x_2, x_3)(x_4, x_5)\ldots(x_{d+1}, x_{d+2}) = 1.$$

If $q = 2$ and d even, then its relation reads

$$r = x_1^{2+2f}(x_1, x_2)(x_3, x_4)(x_5, x_6)\ldots(x_{d+1}, x_{d+2}) = 1,$$

or, depending on the invariant $\operatorname{im}\chi$,

$$r = x_1^2(x_1, x_2)x_3^{2^f}(x_3, x_4)(x_5, x_6)\ldots(x_{d+1}, x_{d+2}) = 1$$

for some $f \geq 2$.

In general, Demushkin groups are classified as follows.

Theorem 2.4.6. *Let G be a Demushkin group with a presentation $1 \to (r) \to F \to G \to 1$ and invariants h_1, q, $\operatorname{im}\chi$ and*

(i) [Dem61, Dem63] If $q \neq 2$, then h_1 is even, then there is a basis $x_1, \ldots, x_{h_1}$ of F such that

$$r = x_1^q (x_1, x_2)(x_3, x_4) \ldots (x_{h_1-1}, x_{h_1}).$$

(ii) [Ser95] If $q = 2$ and h_1 odd, then there is a basis $x_1, \ldots, x_{h_1}$ of F such that

$$r = x_1^2 x_2^{2^f} (x_2, x_3)(x_4, x_5) \ldots (x_{h_1-1}, x_{h_1})$$

for some $f = 2, 3, \ldots, \infty$ (with $f = \infty$ meaning $2^f = 0$).

(iii) [Lab67, Thm. 1] If $q = 2$ and h_1 even, then there is a basis $x_1, \ldots, x_{h_1}$ of F such that

$$r = x_1^{2+\alpha} (x_1, x_2) x_3^{2^f} (x_3, x_4)(x_5, x_6) \ldots (x_{h_1-1}, x_{h_1})$$

for some $f = 2, 3, \ldots, \infty$ and $\alpha \in 4\mathbb{Z}_2$.

3. EQUIDIMENSIONALITY OF UNIVERSAL PSEUDODEFORMATION RINGS

Recall that throughout the thesis we fix an algebraic closure K^{alg} of a finite extension K of $\mathbb{Q}_p$ of degree $d = [K : \mathbb{Q}_p]$ with absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{alg}}/K)$, and a finite field $\mathbb{F}$ of prime characteristic p. We write $\zeta_p \in K^{\mathrm{alg}}$ for a primitive p^{th} root of unity.

The aim of this chapter is Theorem B (Theorem 3.3.12) on the equidimensionality of universal mod p pseudodeformation rings.

To define these universal objects, the chapter starts in Section 3.1 with an introduction to Chenevier's pseudorepresentations and their universal pseudorepresentation and pseudodeformation rings following the original source [Che14] and Wang Erickson's PhD thesis [WE13].

In Section 3.2 we investigate properties of certain loci of pseudodeformations in universal mod p pseudodeformation spaces. In particular, by Proposition 3.2.41 certain irreducible points are regular and form open loci if $\zeta_p \notin K$, and the regular locus is empty if $\zeta_p \in K$.

Section 3.3 contains the inductive proof of Theorem B. For the induction step, the Zariski density of the irreducible locus in is proven in Theorem C (Theorem 3.3.1) under a certain induction hypothesis. When $\zeta_p \notin K$ Theorem D (Theorem 3.3.13) says that the reducible locus is contained in the singular locus. We finish by describing the regular locus of a universal mod p deformation ring and showing that it satisfies Serre's condition (R_2) if $\zeta_p \notin K$, and either $n > 2$, or $n = 2$ and $d > 1$, as stated in Theorem E (Corollary 3.3.15).

3.1 Pseudorepresentations and their universal pseudodeformation rings

This section introduces the theory of pseudorepresentations and their (pseudo)deformations that was developed by Chenevier in [Che14]. Pseudorepresentations naturally arise from the characteristic polynomial of a representation by Example 3.1.8. Conversely, if the representation takes values in a field the well-known Brauer-Nesbitt Theorem [CR62, (30.16) Thm.] states that the zeroes of the characteristic polynomial determine a semisimple finite group representation. Using this, Chenevier proves that any pseudorepresentation over an algebraically closed field corresponds to a semisimple representation and that their universal (pseudo)deformation rings coincide if they are irreducible; see Theorem 3.1.26 and Proposition 3.2.14 respectively.

Pseudorepresentations are by definition multiplicative homogeneous polynomial laws that were studied first by Roby in [Rob63, Rob80] and later also by e.g. Ziplies [Zip86, Zip87] and, as so-called *determinants*, by Vaccarino [Vac09]. For more details on pseudorepresentations we refer the reader to the original source [Che14] and the PhD thesis of Wang Erickson [WE13]. The latter source contains in particular a detailed exposition of Chenevier's theory with some generalizations and further references.

Throughout this section, A will be a commutative ring and S an A-algebra that is not necessarily commutative. We assume that A is a unital ring such that $0 \neq 1$, i.e., $A \neq 0$. If A is local, we write $\kappa(A)$ for its residue field. The category of commutative A-algebras will be denoted by $\mathcal{CA}lg_A$. If X is a scheme and $x \in X$, we write $\kappa(x)$ for $\kappa(\mathcal{O}_{X,x})$.

3.1.1 Pseudorepresentations

In this subsection, we introduce pseudoreopresentations, Azumaya algebras and Cayley-Hamilton A-algebras. Of importance is Proposition 3.1.14, which says that the characteristic polynomial cofficients of a pseudorepresentation determine the pseudorepresentation.

For an A-module M consider the functor $\underline{M}\colon \mathcal{CAlg}_A \to \mathcal{S}ets,\ B \mapsto M \otimes_A B$.

Definition 3.1.1 ([Che14, § 1.1])**.** Let M and N be A-modules and $n \in \mathbb{N}_0$.

(i) An A-*polynomial law* $P\colon M \to N$ is a natural transformation $\underline{M} \to \underline{N}$; i.e., for all $B, B' \in \mathrm{Ob}(\mathcal{CAlg}_A)$ and $f\colon B \to B' \in \mathrm{Mor}(\mathcal{CAlg}_A)$ the A-polynomial law P is a collection of maps $P_B\colon M \otimes_A B \to N \otimes_A B$ such that the following diagram is commutative:

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{P_B} & N \otimes_A B \\ {\scriptstyle \mathrm{id}_M \otimes f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_N \otimes f} \\ M \otimes_A B' & \xrightarrow{P'_B} & N \otimes_A B'. \end{array}$$

By abuse of notation, we often write P instead of P_B for all $B \in \mathrm{Ob}(\mathcal{CAlg}_A)$.

(ii) An A-polynomial law $P\colon M \to N$ is called *homogeneous of degree n* if

$$P_B(bx) = b^n P_B(x) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CAlg}_A),\ b \in B \text{ and } x \in M \otimes_A B.$$

(iii) By $\mathcal{P}ol^n_A(M, N)$ we denote the set of homogeneous polynomial laws of degree n.

Remark 3.1.2 ([Che14, after Exmp. 1.2])**.** A homogeneous polynomial law P of degree n need not be determined by P_A, as shown in [Che14, Exmp. 1.2]. It is however uniquely determined by $P_{A[T_1,\ldots,T_n]}\colon M[T_1,\ldots,T_n] \to N[T_1,\ldots,T_n]$: Suppose X generates M as an A-module. Then such a P is uniquely determined by the (finite) set of functions $P^{[\alpha]}\colon X^n \to N$, with $\alpha \in I_n = \{(\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n \ :\ \alpha_1 + \ldots + \alpha_n = n\}$, defined by the relation

$$P\Big(\sum_{i=1}^n x_i T_i\Big) = \sum_{\alpha \in I_n} P^{[\alpha]}(x_1,\ldots,x_n) T^\alpha, \qquad \text{where } x_1,\ldots,x_n \in X \text{ and } T^\alpha = \prod_{i=1}^n T_i^{\alpha_i}.$$

Example 3.1.3. (i) If $\varphi\colon M \to N$ is an A-module homomorphism, then the natural maps $\varphi \otimes_A B\colon M \otimes_A B \to N \otimes_A B$ define a homogeneous polynomial law of degree 1. Conversely, any homogeneous polynomial law $P\colon M \to N$ of degree 1 arises in this way from the A-module homomorphism $\varphi = P_A\colon M \to N$.

(ii) Homogeneous polynomial laws of degree 0 are constant maps; see [Rob63, Prop. I.5].

(iii) If $P_1\colon L \to M$ and $P_2\colon M \to N$ are polynomial laws, then so is $P_2 \circ P_1\colon L \to N$. If both are homogeneous of degrees m and n, then the composition is homogeneous of degree mn.

Definition 3.1.4 ([Che14, § 1.1])**.** Let R and S be A-algebras and $n \in \mathbb{N}_0$.

(i) An A-polynomial law $P\colon S \to R$ is called *multiplicative* if

$$P_B(1) = 1 \quad \text{and} \quad P_B(xy) = P_B(x)P_B(y) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CAlg}_A) \text{ and } x, y \in S \otimes_A B.$$

(ii) An *n-dimensional A-valued pseudorepresentation on S* is an A-polynomial law $D : S \to A$ that is multiplicative and homogeneous of degree n.

(iii) By $\mathcal{M}^n_A(S, R)$ we denote the set of homogeneous multiplicative A polynomial laws $S \to R$ of degree n, and we write $\mathcal{P}s\mathcal{R}^n_S(A)$ for the set of n-dimensional A-valued pseudorepresentations in $\mathcal{M}^n_A(S, A)$.

Similarly as the polynomial laws given by Example 3.1.3, we have the following examples of multiplicative polynomial laws.

Example 3.1.5. (i) If $\varphi\colon R \to S$ is an A-algebra homomorphism, then the natural homomorphisms $\varphi \otimes_A B\colon R \otimes_A B \to S \otimes_A B$ define a multiplicative polynomial law of dimension 1. Conversely, any multiplicative polynomial law of dimension 1 arises in this way from the A-algebra homomorphism $\varphi_A\colon R \to S$.

(ii) The only multiplicative polynomial law of dimension zero is the constant map with value 1.

(iii) Suppose that S, S', S'' are A-algebras, and that $P\colon S \to S'$ and $P'\colon S' \to S''$ are multiplicative polynomial laws of dimensions n and n', respectively. Then $P' \circ P\colon S \to S''$ is a multiplicative polynomial law of dimension nn'.

Before giving the example of a pseudorepresentation attached to a representation, we introduce the following generalization of a central simple algebra and its reduced norm.

Definition 3.1.6 ([Mil80, § IV.1 – IV.2])**.** Suppose that A is a local commutative ring.

(i) An *Azumaya algebra over* A is a ring R free of finite rank as an A-module such that the map

$$R \otimes_A R^\circ \longrightarrow \mathrm{End}_A(R), \quad r \otimes r' \longmapsto (x \mapsto rxr'),$$

is an isomorphism, where R° denotes the algebra with the multiplication of R reversed.

(ii) An extension $A \subset B$ is called a *neutralizing* A*-algebra* for an Azumaya algebra C over A if there exists a faithful projective B-module P and an isomorphism $\sigma\colon C \otimes_A B \xrightarrow{\sim} \mathrm{End}_B(P)$ of B-algebras; cf. [KO74, § III.6]. Since A is local, such B always exist; cf. Remark 3.1.7.

(iii) The *reduced norm* of an Azumaya algebra C over A is $\det\colon C \to A,\ c \mapsto \det\big(\sigma(c \otimes 1_B)\big)$, where B is a neutralizing A-algebra for C with corresponding isomorphism σ; the reduced norm is independent of the choice of B and σ by [KO74, § IV.2].

(iv) Let X be a scheme and C an $\mathcal{O}_X$-algebra. The $\mathcal{O}_X$-algebra C is called an *Azumaya algebra over* X if C is a coherent $\mathcal{O}_X$-module, and for all closed points $x \in X$ C_x is an Azumaya algebra over $\mathcal{O}_{X,x}$.

Remark 3.1.7. Let C be an $\mathcal{O}_X$-algebra. By [Mil80, beginning of § IV.2] we have the following equivalences:

(a) C is an Azumaya algebra over X;

(b) C is locally free of finite rank as an O_X-module and $C_x \otimes \kappa(x)$ is a central simple algebra over $\kappa(x)$ for all points $x \in X$;

(c) there is a Zariski cover $\{U_i\}$ of X and for each i a finite étale surjective cover $U'_i \to U_i$ such that one has an isomorphism $C \otimes_{\mathcal{O}_X} \mathcal{O}_{U'_i} \xrightarrow{\sim} \mathrm{Mat}_{n_i}(\mathcal{O}_{U'_i})$ for suitable $n_i \in \mathbb{N}_{\geq 1}$.

By (b) there is a locally constant function $\underline{n}\colon X \to \mathbb{N}_{\geq 1}$ such that $\mathrm{rank}_{\mathcal{O}_X} C = \underline{n}^2$. Moreover, for the cover in (c) the function $\underline{n}$ is constant on U_i and takes the value n_i.

Example 3.1.8. Let C be an Azumaya algebra over A of rank n^2 and denote by $\det_C$ the reduced norm for C over A.

(i) By [Che14, § 1.5], the family of maps

$$(\det_{C\otimes_A B}\colon C\otimes_A B \to B)_{B\in \mathrm{Ob}(\mathcal{CAlg}_A)}$$

defines a pseudorepresentation det of dimension n. Hence for any A-algebra homomorphism $r\colon S \longrightarrow C$, the map $\det\circ r$ defines an A-valued pseudorepresentation of dimension n; cf. Example 3.1.5(iii).

(ii) Let $D\colon C \to A$ be a pseudorepresentation of dimension n'. Then by [Che14, Lem. 2.15][1], we have $n|n'$ and $D = \det_C^{n'/n}$, or, in the notation introduced in Definition 3.1.29, $D = \det_C^{\oplus n'/n}$.

Example 3.1.9 (Determinant of a pseudorepresentation)**.** Let G be a group and $D : A[G] \to A$ be an n-dimensional A-valued pseudorepresentation on $A[G]$. Then the restriction $D_A|_G\colon G \to A^\times$ is a group homomorphism. Define

$$\det D_B\colon B[G] \to B, \quad \sum_i b_i g \mapsto \sum_i b_i D_A(g),$$

for any $B \in \mathrm{Ob}(\mathcal{CAlg}_A)$. Then this defines a 1-dimensional pseudorepresentation $\det D\colon A[G] \to A$ that we call the *determinant of D*.

Lemma 3.1.10 ([Che14, § 1.10])**.** *Let $D : S \to A$ be an n-dimensional pseudorepresentation. Consider for each $B \in \mathrm{Ob}(\mathcal{CAlg}_A)$ the map*

$$\chi_{D,B}(\cdot,t)\colon S\otimes_A B \longrightarrow B[t], \quad s \longmapsto \chi_{D,B}(s,t) := \sum_{i=0}^{n}(-1)^i\Lambda_{D,i,B}(s)t^{n-i} := D_{B[t]}(t-s).$$

(i) For $i = 0,\dots,n$ the maps $\Lambda_{D,i,B}\colon S\otimes_A B \to B$ define a homogeneous A-polynomial law $\Lambda_{D,i}\colon S \to A$ of degree i.

(ii) $\Lambda_{D,0,B}(s) = 1_B$ and $\Lambda_{D,n} = D$.

(iii) The maps $\chi_{D,B}(\cdot,t)\colon S\otimes_A B \to B[t]$ form an A-polynomial law $\chi_D = \chi_D(\cdot,t)\colon S \to A[t]$.

(iv) The maps $\chi_{D,B}(s,s)\colon S\otimes_A B \longrightarrow B$ define an A-polynomial law

$$\chi_D\colon S \to S, \quad s \longmapsto \sum_{i=0}^{n}(-1)^i\Lambda_{D,i}(s)s^{n-i},$$

that is homogeneous of degree n.

Definition 3.1.11. [Che14, § 1.10] Let S be an A-algebra, and $D\colon S \to A$ be an n-dimensional pseudorepresentation.

(i) The polynomial law χ_D is called the *characteristic polynomial associated with D*, and the polynomial law $\Lambda_{D,i}$ is called the i^{th} *characteristic polynomial coefficient* for $i = 0,\dots,n$.

(ii) The A-linear map $\tau_D := \Lambda_{D,1}$ is called the *trace associated with D*.

Lemma 3.1.12 ([Che14, Lem. 1.12(i)])**.** *For all $s, s' \in S$, $D(1+ss') = D(1+s's)$.*

[1] Alternatively, one can solve Exercise 2.5 in [Che14], where it should read n *divides* d in line 4.

Definition 3.1.13 ([Che14, p. 3])**.** Let G be a group. An *A-valued pseudorepresentation of G of dimension n* is an A-valued n-dimensional pseudorepresentation $D\colon A[G] \to A$.

Proposition 3.1.14 ([Che14, Lem. 1.12, Cor. 1.14])**.** *Consider a pseudorepresentation $D\colon G \to A$ of dimension n on a group G.*

(i) D satisfies Amitsur's formula [Che14, (1.4)]. In particular, the characteristic polynomial coefficients determine D.

(ii) If $C \subset A$ is the subring generated by the coefficients $\Lambda_{D,i}(g)$ of $\chi_D(g,t)$ for all $g \in G$ and $i \in \mathbb{N}_{\geq 1}$, then D factors through a unique n-dimensional pseudorepresentation $G \to C$.

Next we define some important properties of pseudorepresentations, which hold for the pseudorepresentation attached to a representation.

Definition 3.1.15 ([Che14, § 1.17])**.** Let M and N be A-modules.

(i) For a polynomial law $P\colon M \to N$ let $\ker(P) \subset M$ be the A-submodule

$$\{x \in M \ : \ P(x \otimes b + m) = P(m) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CA}lg_A),\ b \in B \text{ and } m \in M \otimes_A B\}$$

(ii) A polynomial law $P\colon M \to N$ is called *faithful* if $\ker(P) = 0$.

(iii) For a pseudorepresentation $D\colon S \longrightarrow A$ let $\mathrm{CH}(D)$ be the two-sided ideal of S that is generated by the coefficients of

$$\chi_{D,S[t_1,\ldots,t_m]}(t_1 s_1 + \ldots + t_m s_m) \in S[t_1,\ldots,t_m],$$

for all $s_1,\ldots,s_m \in S$ and $m \in \mathbb{N}_{\geq 1}$.

(iv) An n-dimensional pseudorepresentation $D\colon S \to A$ is called *Cayley-Hamilton* and (S,D) a *Cayley-Hamilton A-algebra of degree n* if $\mathrm{CH}(D) = 0$.

Proposition 3.1.16 ([Che14, Lem. 1.19, Exmp. 1.20, Lem. 1.21])**.** *Let $D\colon S \to A$ be an n-dimensional pseudorepresentation. Then the following hold.*

(a) $\ker D$ *is a two-sided ideal of S. It is proper if $n > 0$. It is the biggest two-sided ideal $K \subset S$ such that D admits a factorization $D = \widetilde{D} \circ \pi$ with π the canonical surjection $S \to S/K$ and $\widetilde{D} \in \mathcal{P}s\mathcal{R}^n_{S/K}(A)$.*

(b) $\ker(D) \supset \mathrm{CH}(D)$.

(c) If D is Cayley-Hamilton and $S' \subset S$ is any A-subalgebra, then the restriction of D to S' is (obviously) Cayley-Hamilton.

(d) If S is an Azumaya algebra of rank n^2 over A and D is the reduced norm, then D is Cayley-Hamilton and faithful.

The Cayley-Hamilton property behaves rather well under several operations, which is in general not the case of the faithful property. For instance, Proposition 3.1.16(c) does not hold for faithful instead of Cayley-Hamilton; see [Che14, Exmp. 1.20(b)].

Corollary 3.1.17. *Any $D \in \mathcal{P}s\mathcal{R}^n_S(A)$ factors as $\widetilde{D} \circ \pi$, where π is the canonical surjection $S \to S/\mathrm{CH}(D)$ for some unique $\widetilde{D} \in \mathcal{P}s\mathcal{R}^n_{S/\mathrm{CH}(D)}(A)$, which is Cayley-Hamilton.*

3.1.2 The divided power algebra and universal pseudorepresentation rings

We start with the definition of the divided power A-algebra whose abelianization will represent the pseudorepresentation functor by Proposition 3.1.23. Next we reall [Che14, Thm. A] by which for any pseudorepresentation over an algebraically closed field there is a corresponding semisimple representation.

Definition 3.1.18. Let M be an A-module. Consider the polynomial A-algebra $G_A(M) = A[x_{m,n} : m \in M,\ n \in \mathbb{N}_0]$ and the ideal $I_A(M) \subset G_A(M)$ generated by the following relations:

(a) $x_{m,0} - 1$ for all $m \in M$,

(b) $x_{(am),n} - a^n x_{m,n}$ for all $a \in A,\ m \in M$ and $n \in \mathbb{N}_0$,

(c) $x_{m,n_1} x_{m,n_2} - \frac{(n_1+n_2)!}{n_1! n_2!} x_{m,n_1+n_2}$ for all $m \in M$ and $n_1, n_2 \in \mathbb{N}_0$,

(d) $x_{m_1 m_2,n} - \sum_{i=0}^{n} x_{m_1,i} x_{m_2,n-i}$ for all $m_1, m_2 \in M$ and $n \in \mathbb{N}_0$.

Then the *divided power A-algebra* $\Gamma_A(M)$ *of* M is the quotient $G_A(M)/I_A(M)$. For $m \in M$ and $n \in \mathbb{N}_0$ denote by $m^{[n]}$ the image of the indeterminant $x_{m,n}$ in $\Gamma_A(M)$.

There exists a unique grading on $G_A(M) = \oplus_{n\in\mathbb{N}_0} G_A^n(M)$ that respects its A-algebra structure and that assigns the degree n to the indeterminant $x_{m,n}$ for $m \in M$ and $n \in \mathbb{N}_0$. With respect to this grading, the ideal $I_A(M)$ is homogeneous so that

$$\Gamma_A(M) = \oplus_{n\in\mathbb{N}_0} \Gamma_A^n(M)$$

inherits the grading from $G_A(M)$ and $m^{[n]}$ is of degree n for $m \in M$ and $n \in \mathbb{N}_0$.

We introduce a specific polynomial law to $\Gamma_A^n(M)$ that Roby showed to be universal.

Proposition 3.1.19 ([Rob63, Thm. III.3 and Prop. IV.1])**.** *Let M be an A-module, B a commutative A-algebra and $n \in \mathbb{N}_0$.*

(i) There is a well-defined B-algebra homomorphism $\omega_B \colon \Gamma_A(M \otimes_A B) \to \Gamma_A(M) \otimes_A B$, given on generators by $(m \otimes 1)^{[n]} \mapsto m^{[n]} \otimes 1$, and ω_B is an isomorphism.

(ii) Consider for all $B \in \mathrm{Ob}(\mathcal{CAlg}_A)$ the maps $\gamma^n_{M,B} \colon M \otimes B \to \Gamma^n_A(M \otimes_A B)$, $z \mapsto z^{[n]}$, and

$$(L^n_M)_B := \omega_B \circ \gamma^n_{M,B} \colon M \otimes_A B \longrightarrow \Gamma_A(M \otimes_A B) \longrightarrow \Gamma_A(M) \otimes_A B.$$

Then this defines a polynomial law $L^n_M \colon M \to \Gamma^n_A(M)$ that is homogeneous of degree n.

Next, we study universal objects of the following covariant functors.

Definition 3.1.20. Let $M \in \mathrm{Ob}(\mathcal{Mod}_A)$ and $S \in \mathrm{Ob}(\mathcal{Alg}_A)$. Define the following functors:

(a) $\mathcal{Pol}^n_A(M, \cdot\,) \colon \mathcal{Mod}_A \to \mathcal{Sets},\ N \mapsto \mathcal{Pol}^n_A(M, N)$;

(b) $\mathcal{M}^n_A(S, \cdot\,) \colon \mathcal{Alg}_A \to \mathcal{Sets},\ R \mapsto \mathcal{M}^n_A(S, R)$;

(c) $\mathcal{PsR}^n_S(\,\cdot\,) \colon \mathcal{CAlg}_A \to \mathcal{Sets},\ B \mapsto \mathcal{PsR}^n_{S\otimes_A B}(B)$.

Theorem 3.1.21 ([Rob63, Thm. IV.1])**.** *The functor $\mathcal{Pol}^n_A(M, \cdot\,)$ is represented by the pair $(\Gamma^n_A(M), L^n_M)$. In particular, for all $N \in \mathrm{Ob}(\mathcal{Mod}_A)$ there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{Mod}_A}(\Gamma^n_A(M), N) \xrightarrow{\sim} \mathcal{Pol}^n_A(M, N), \quad f \longmapsto f \circ L^n_M.$$

Let $n \in \mathbb{N}_0$ and $S \in \mathrm{Ob}(\mathcal{A}lg_A)$. In [Rob80], Roby defines an A-algebra structure on $\Gamma_A^n(S)$ as follows: There exists a linear map $\widetilde{\alpha}_S \colon \Gamma_A^n(S) \otimes_A \Gamma_A^n(S) \to \Gamma_A^n(S \otimes_A S)$, given on generators by $s^{[n]} \otimes (s')^{[n]} \mapsto (s \otimes s')^{[n]}$. Let $\theta \colon S \otimes_A S \to S$ be the A-linear map defined by the bilinear multiplication on S, and $\Gamma^n(\theta)$ the restriction of the induced map $\Gamma(\theta) \colon \Gamma(S \otimes_A S) \to \Gamma(S)$ to $\Gamma^n(S \otimes_A S)$. Then the composition

$$\theta_n \colon \Gamma_A^n(S) \otimes_A \Gamma_A^n(S) \xrightarrow{\widetilde{\alpha}_S} \Gamma_A^n(S \otimes_A S) \xrightarrow{\Gamma^n(\theta)} \Gamma_A^n(S)$$

defines an A-algebra structure on $\Gamma_A^n(S)$. Further, θ_1 recovers the A-algebra structure on S, and $\Gamma_A^n(S)$ is unital, associative or commutative if S has the corresponding property.

The polynomial law $L_S^n \colon S \to \Gamma_A^n(S)$ is multiplicative with respect to θ_n.

Theorem 3.1.22 ([Rob80])**.** *Let $n \in \mathbb{N}_0$ and $S \in \mathrm{Ob}(\mathcal{A}lg_A)$. The functor $\mathcal{M}_A^n(S, \cdot\,)$ is represented by $(\Gamma_A^n(S), L_S^n)$.*

Recall that the *abelianization* of a ring R' is the quotient of R' by the two-sided ideal generated by $r_1 r_2 - r_2 r_1$ for all $r_1, r_2 \in R'$.

Proposition 3.1.23 ([Che14, Prop. 1.6])**.** *The functor $\mathcal{P}s\mathcal{R}_S^n(\,\cdot\,)$ is represented by the abelianization $R_{S,n}^{\mathrm{univ}} := \Gamma_A^n(S)^{\mathrm{ab}}$ of $\Gamma_A^n(S)$ together with the natural pseudorepresentation*

$$D_S^{\mathrm{univ}} \colon S \otimes_A R_{S,n}^{\mathrm{univ}} \longrightarrow R_{S,n}^{\mathrm{univ}} = \Gamma_A^n(S)^{\mathrm{ab}}$$

that is constructed using the universal property of the tensor product from the composition of $L_S^n \colon S \to \Gamma_A^n(S)$ with the A-algebra homomorphism $\Gamma_A^n(S) \to \Gamma_A^n(S)^{\mathrm{ab}}$.

Definition 3.1.24. We call $R_{S,n}^{\mathrm{univ}} = \Gamma_A^n(S)^{\mathrm{ab}}$ the *n-dimensional universal pseudorepresentation ring of S* and $D_{S,n}^{\mathrm{univ}} \colon S \to R_{S,n}^{\mathrm{univ}}$ the *n-dimensional universal pseudorepresentation.*

If G is a group and $S = \mathbb{Z}[G]$, then we abbreviate $R_{G,n}^{\mathrm{univ}} := R_{\mathbb{Z}[G],n}^{\mathrm{univ}}$ and $D_{G,n}^{\mathrm{univ}} := D_{\mathbb{Z}[G],n}^{\mathrm{univ}}$.

If X is an A-scheme, we can extend the notion of a pseudorepresentation and define an $\mathcal{O}(X)$-*valued pseudorepresentation $S \to \mathcal{O}(X)$ of dimension $n \in \mathbb{N}_0$*. Then the *n-dimensional universal pseudorepresentation space*

$$X_{S,n}^{\mathrm{univ}} := \operatorname{Spec} R_{S,n}^{\mathrm{univ}} = \operatorname{Spec} \Gamma_A^n(S)^{\mathrm{ab}}$$

of S represents the obvious pseudodeformation functor on the category of A-schemes. If G is a group and $S = \mathbb{Z}[G]$, then we write $X_{G,n}^{\mathrm{univ}} := X_{\mathbb{Z}[G],n}^{\mathrm{univ}}$.

Example 3.1.25 (Determinant of a pseudorepresentation)**.** Let G be a group. By Proposition 3.1.23 we have universal pseudorepresentations

$$D_{A[G],n}^{\mathrm{univ}} \colon A[G] \otimes_A R_{A[G],n}^{\mathrm{univ}} \longrightarrow R_{A[G],n}^{\mathrm{univ}} \quad \text{and} \quad D_{A[G],1}^{\mathrm{univ}} \colon A[G] \otimes_A R_{A[G],1}^{\mathrm{univ}} \longrightarrow R_{A[G],1}^{\mathrm{univ}}.$$

Now $\det D_{A[G],n}^{\mathrm{univ}} \colon A[G] \otimes_A R_{A[G],n}^{\mathrm{univ}} \longrightarrow R_{A[G],n}^{\mathrm{univ}}$ defined as in Example 3.1.9 is a 1-dimensional pseudorepresentation. By universality of $R_{A[G],1}^{\mathrm{univ}}$ we obtain a ring homomorphism

$$\det \colon R_{A[G],1}^{\mathrm{univ}} \to R_{A[G],n}^{\mathrm{univ}}.$$

and an induced morphism of schemes $\det \colon X_{A[G],n}^{\mathrm{univ}} \to X_{A[G],1}^{\mathrm{univ}}$ (which we both denote by det).

Theorem 3.1.26 ([Che14, Thm. A]). *Suppose that k is an algebraically closed field and S is a k-algebra. If $D\colon S \to k$ is an n-dimensional pseudorepresentation, then there is a semisimple representation $\rho_D\colon S \to \mathrm{Mat}_n(k)$ with associated pseudorepresentation D.*

Furthermore, ρ_D is unique up to isomorphism and $\ker \rho_D = \ker D$.

We use Theorem 3.1.26 to define the following notions for pseudorepresentations.

Definition 3.1.27. Let k be a field, S a k-algebra and $D\colon S \to k$ a pseudorepresentation of dimension n. Fix an algebraic closure k^{alg} of k, and consider the unique semisimple representation $\rho_{D\otimes_k k^{\mathrm{alg}}}\colon S \otimes_k k^{\mathrm{alg}} \to \mathrm{Mat}_n(k^{\mathrm{alg}})$ satisfying $D \otimes_k k^{\mathrm{alg}} = \det \circ \rho_{D\otimes_k k^{\mathrm{alg}}}$ from Theorem 3.1.26.

(i) D is *irreducible* if $\rho_{D\otimes_k k^{\mathrm{alg}}}$ is irreducible.

(ii) D is *reducible* if $\rho_{D\otimes_k k^{\mathrm{alg}}}$ is reducible.

(iii) D is *multiplicity free* if $\rho_{D\otimes_k k^{\mathrm{alg}}}$ is a direct sum of pairwise non-isomorphic irreducible k^{alg}-linear representations of $S \otimes_k k^{\mathrm{alg}}$.

(iv) D is *split* if D is the determinant of a representation $S \to \mathrm{Mat}_n(k)$.

For later use we shall also need the following refinement of Theorem 3.1.26. It is a fundamental result of Chenevier for pseudorepresentations over a field. We need to recall the exponent, defined for certain field extensions: Let $k' \supset k$ be a field extension and denote by $k^{\mathrm{sep}} \subset k'$ the maximal separable extension of k in k'. Assume that k^{sep} is finite over k and that there exists a power q of $p := \mathrm{Char}\, k > 0$ such that $(k')^q \subset k^{\mathrm{sep}}$. The *exponent* (f, q) *of* $k' \supset k$ is defined be setting $f := [k^{\mathrm{sep}} : k]$ and taking for q the minimal p-power such that $(k')^q \subset k^{\mathrm{sep}}$.

Theorem 3.1.28 ([Che14, Thm. 2.16]). *Let k be a field, let $D\colon S \to k$ be a pseudorepresentation of dimension n. Then as a k-algebra*

$$S/\ker D \xrightarrow{\simeq} \prod_{i=1}^{s} S_i,$$

where S_i is a simple k-algebra which is of finite dimension n_i^2 over its center k_i, and where k_i/k has a finite exponent (f_i, q_i).

Moreover, under such an isomorphism, D coincides with the product determinant

$$D = \prod_{i=1}^{s} \det\nolimits_{S_i}^{m_i},$$

$n = \sum_i m_i n_i q_i f_i$, where m_i are some uniquely determined integers.

In particular, $S/\ker D$ is semisimple. It is finite dimensional over k if and only if each k_i is. This always occurs in each of the following three cases : k is perfect, or k has characteristic $p > 0$ and $[k : k^p] < \infty$, or $n < p$.

We define a direct sum of two pseudorepresentations, which in particular is useful when studying reducible pseudorepresentations later.

Definition 3.1.29 ([WE13, § 1.1.11]). Let S_1, S_2 and S be A-algebras and B a commutative A-algebra. For $i = 1, 2$ consider a multiplicative A-polynomial law $P_i\colon S_i \longrightarrow B$ that is homogeneous of degree $n_i \in \mathbb{N}_0$.

(i) The multiplicative homogeneous A-polynomial law

$$P_1 \oplus P_2 \colon S_1 \times S_2 \to B, \quad (x_1, x_2) \mapsto P_1(x_1)P_2(x_2),$$

of degree $n_1 + n_2$ is called the *direct sum of* P_1 *and* P_2.

(ii) If $D_i := P_i \colon S \to A$ is a pseudorepresentation for $i = 1, 2$, then the pseudorepresentation

$$D_1 \oplus D_2 \colon S \times S \longrightarrow A, \quad (x_1, x_2) \mapsto D_1(x_1)D_2(x_2),$$

of dimension $n_1 + n_2$ is called the *direct sum of* D_1 *and* D_2.

We remark that this direct sum operation is called a product in [Che14].

Theorem 3.1.30 ([Che14, Lem. 2.2]; cf. [Rob63, Thm. III.4]). *(i) The canonical map*

$$\Gamma_A^n(S_1 \times S_2)^{\mathrm{ab}} \longrightarrow \bigoplus_{i=0}^{n} \Gamma_A^i(S_1)^{\mathrm{ab}} \otimes_A \Gamma_A^{n-i}(S_2)^{\mathrm{ab}}, \quad (s_1, s_2)^{[n]} \mapsto \sum_{i=0}^{n} s_1^{[i]} \otimes s_2^{[n-i]}, \qquad (1)$$

is an A-algebra isomorphism.

(ii) Let $P \colon S_1 \times S_2 \longrightarrow B$ be a multiplicative A-polynomial law that is homogeneous of degree n. Suppose that $\operatorname{Spec} B$ *is connected and $B \neq 0$. Then there exists for $i = 1, 2$ a unique multiplicative homogeneous A-polynomial law $P_i \colon S_i \to B$ of degree n_i such that $n_1 + n_2 = n$ and $P = P_1 \oplus P_2$. In other words, the A-algebra homomorphism*

$$\Gamma_A(S_1 \times S_2)^{\mathrm{ab}} \to B$$

corresponding to P factors through $\Gamma_A^{n_1}(S_1)^{\mathrm{ab}} \otimes_A \Gamma_A^{n_2}(S_2)^{\mathrm{ab}}$ in (1).

In the case $S_1 = S_2$, Theorem 3.1.30 implies the following:

Corollary 3.1.31 ([WE13, Lem. 1.1.11.7]). *(i) The map*

$$\iota_{n_1,n_2} \colon X_{S,n_1}^{\mathrm{univ}} \times_A X_{S,n_2}^{\mathrm{univ}} \longrightarrow X_{S,n_1+n_2}^{\mathrm{univ}}$$

defined by $(D_1, D_2) \mapsto D_1 \oplus D_2$ is a morphism of affine A-schemes that corresponds to the homomorphism

$$\iota_{n_1,n_2}^* \colon \Gamma_A^{n_1+n_2}(S)^{\mathrm{ab}} \overset{\Gamma_A^{n_1+n_2}(\mathrm{diag})}{\longrightarrow} \Gamma_A^{n_1+n_2}(S \times S)^{\mathrm{ab}} \twoheadrightarrow \Gamma_A^{n_1}(S)^{\mathrm{ab}} \otimes_A \Gamma_A^{n_2}(S)^{\mathrm{ab}},$$

where $\Gamma_A^{n_1+n_2}(\mathrm{diag})$ is induced by the diagonal map $\mathrm{diag} \colon S \to S \times S$.

(ii) For $i = 1, 2$ let $\rho_i \colon S \to \mathrm{GL}_{n_i}(A)$ be a representation and D_{ρ_i} be the associated pseudorepresentation. If $D_{\rho_1 \oplus \rho_2}$ is the pseudorepresentation defined by $\det(\rho_1 \oplus \rho_2)$, then

$$D_{\rho_1 \oplus \rho_2} = D_{\rho_1} \oplus D_{\rho_2}.$$

We need the following pseudorepresentations when studying the ideal of total reducibility in A in Proposition 3.1.48.

Lemma 3.1.32 ([Che14, Lem. 2.4]). *Let S be an A-algebra, $e \in S$ be an idempotent, and $D \colon S \to A$ be a pseudorepresentation of dimension n. Suppose that* $\operatorname{Spec}(A)$ *is connected.*

(i) The polynomial law $D_e\colon eSe \to A$, $s \mapsto D(s+1-e)$, *is a pseudorepresentation of dimension* $r(e) \leq n$. *One has* $r(1-e)+r(e)=n$.

(ii) The restriction of D *to the* A*-subalgebra* $eSe \oplus (1-e)S(1-e)$ *is the sum* $D_e \oplus D_{1-e}$. *It is a pseudorepresentation of dimension* n.

(iii) If D *is faithful or Cayley-Hamilton, then* D_e *is faithful or Cayley-Hamilton, respectively.*

(iv) Suppose that D *is Cayley-Hamilton. Then* $e=1$ *if and only if* $D(e)=1$, *and* $e=0$ *if and only if* $r(e)=0$. *If* $e_1,\dots,e_s$ *is a family of nonzero orthogonal idempotents of* S, *then* $s \leq n$ *and* $\sum_{i=1}^{s} r(e_i) \leq n$. *Further,* $\sum_{i=1}^{s} r(e_i) = n$ *if and only if* $e_1+e_2+\dots+e_s=1$.

3.1.3 Generalized matrix algebras (GMAs) and pseudocharacters

Generalized matrix algebras are a generalization of matrix algebras and are also equipped with a trace map. Such generalized matrix algebras were introduced as trace algebras by Procesi in [Pro87]. Next we define pseudocharacters that were studied by various authors and arise from the trace of a representation: At first attached to 2-dimensional Galois representations by Wiles [Wil88], and then in a more general notion by e.g. Taylor [Tay91] and Rouquier [Rou96]. We mostly follow the exposition in [BC09] for pseudocharacters. In [BC09, § 1.3] pseudocharacters are further studied as the trace of a generalized matrix algebra. In Proposition 3.1.40 we also mention the relation to the previously defined pseudorepresentations following [Che14, WE13]. Finally, we associate a pseudorepresentation with a generalized matrix algebra via the Leibniz formula in Definition 3.1.47 and study the ideal of total reducibility of A in Proposition 3.1.48.

Definition 3.1.33 (Cf. [BC09, Def. 1.3.1], [WE13, Rem. 2.3.0.4 and 2.3.3.6])**.** Suppose that $n_1,\dots,n_r$ are positive integers and $n := \sum_{i=1}^{r} n_i$. We call S a *generalized matrix algebra (GMA) of type* $(n_1,\dots,n_r)$ if there exist

(a) a family of orthogonal idempotents $e_1,\dots,e_r \in S$ with $\sum_{i=1}^{r} e_i = 1_S$, and

(b) a family of A-algebra isomorphisms $\psi_i : e_iSe_i \xrightarrow{\sim} \mathrm{Mat}_{n_i}(A)$ for $i=1,\dots,r$

such that the associated trace map $\tau : S \to A, x \mapsto \sum_{i=1} \mathrm{tr}(\psi_i(e_ixe_i))$ satisfies $\tau(xy)=\tau(yx)$ for all $x,y \in S$. We call $\mathcal{E} := \{e_i,\psi_i\}_{i=1,\dots,r}$ the *data of idempotents of* S.

Example 3.1.34 (The standard GMA of type $(n_1,\dots,n_r)$ [BC09, Exmp. 1.3.4])**.** We suppose that B is a commutative A-algebra and $(A_{i,j})_{1\leq i,j\leq r}$ is a family of A-submodules of B such that

$$A_{i,i} = A \quad \text{and} \quad A_{i,j}A_{j,k} \subset A_{i,k} \qquad \text{for all } 1 \leq i,j,k \leq r.$$

Let S be the A-submodule

$$\begin{pmatrix} \mathrm{Mat}_{n_1}(A_{1,1}) & \cdots & \mathrm{Mat}_{n_1,n_r}(A_{1,r}) \\ \vdots & \ddots & \vdots \\ \mathrm{Mat}_{n_r,n_1}(A_{r,1}) & \cdots & \mathrm{Mat}_{n_r}(A_{r,r}) \end{pmatrix}$$

of $\mathrm{Mat}_n(B)$. Then S is an A-subalgebra of $\mathrm{Mat}_n(B)$. Further, let $e_i \in \mathrm{Mat}_n(B)$ with diagonal entries 1 on the i^{th} diagonal block and everywhere else 0. Then $e_i \in S$ and $\sum_{i=1}^{r} e_i = 1_S$. Together with the canonical isomorphisms $\psi_i : e_iSe_i \xrightarrow{\sim} \mathrm{Mat}_{n_i}(A)$, S is a GMA of type $(n_1,\dots,n_r)$ that we call the *standard GMA of type* $(n_1,\dots,n_r)$ *associated with* $(A_{i,j})_{1\leq i,j\leq r}$.

We need the following notation to describe the structure of a GMA and to define its determinant.

Definition 3.1.35. Let S be a GMA of type $(n_1, \dots, n_r)$. For $1 \le i \le r$ and $1 \le k, l \le n_i$ we denote by $E_i^{k,l}$ the unique element in $e_i S e_i$ that maps under ψ_i to the matrix in $\mathrm{Mat}_{n_i}(A)$ that has 1 in the (k,l)-entry and everywhere else 0.

Lemma 3.1.36 (Structure of a GMA [BC09, p. 21ff.]). *(i) Let $(S, \mathcal{E})$ be a generalized matrix algebra of type $(n_1, \dots, n_r)$. Consider the canonical family $(\mathcal{A}_{i,j})_{1 \le i,j \le r}$ of A-modules with $\mathcal{A}_{i,j} := E_i^{1,1} S E_j^{1,1}$. Then for $1 \le i, j, k \le r$ the associated trace τ defines canonical isomorphisms $\mathcal{A}_{i,i} \cong A$ and since $\mathcal{A}_{i,j}\mathcal{A}_{j,k} \subset \mathcal{A}_{i,k}$ the product in S induces A-linear maps $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}$ that satisfy the following conditions:*

(UNIT) For $1 \le i, j \le r$ we have $\mathcal{A}_{i,i} = A$ and both $\varphi_{i,i,j}$ and $\varphi_{i,j,j}$ agree with the A-module structure on $A_{i,j}$.

(ASSO) For $1 \le i, j, k, l \le r$ and $x \otimes y \otimes z \in \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \otimes_A \mathcal{A}_{k,l}$ we have

$$\varphi_{i,k,l}\big(\varphi_{i,j,k}(x \otimes y) \otimes z\big) = \varphi_{i,j,l}\big(x \otimes \varphi_{j,k,l}(y \otimes z)\big) \quad \text{in } \mathcal{A}_{i,l}.$$

(COMM) For $1 \le i, j \le r$, $x \in \mathcal{A}_{i,j}$ and $y \in \mathcal{A}_{j,i}$ we have $\varphi_{i,j,i}(x \otimes y) = \varphi_{j,i,j}(y \otimes x)$.

Thus the A-module $\bigoplus_{i,j=1}^r \mathrm{Mat}_{n_i,n_j}(\mathcal{A}_{i,j})$ is an A-algebra via

$$x \cdot y = \sum_{i,j=1}^{r} \sum_{k=1}^{r} x_{i,k} \cdot y_{k,j} \quad \text{with } (x_{i,k} \cdot y_{k,j})_{l,n} := \sum_{m=1}^{n_k} \varphi_{l,m,n}\big((x_{i,k})_{l,m} \otimes (y_{k,j})_{m,n}\big)$$

for $1 \le l \le n_i$, $1 \le n \le n_j$ and $x = \sum_{i,j=1}^r x_{i,j}$, $y = \sum_{i,j=1}^r y_{i,j} \in \bigoplus_{i,j=1}^r \mathrm{Mat}_{n_i,n_j}(\mathcal{A}_{i,j})$, and there is a canonical isomorphism of A-algebras

$$S \cong \begin{pmatrix} \mathrm{Mat}_{n_1}(\mathcal{A}_{1,1}) & \cdots & \mathrm{Mat}_{n_1,n_r}(\mathcal{A}_{1,r}) \\ \vdots & \ddots & \vdots \\ \mathrm{Mat}_{n_r,n_1}(\mathcal{A}_{r,1}) & \cdots & \mathrm{Mat}_{n_r}(\mathcal{A}_{r,r}) \end{pmatrix} := \bigoplus_{i,j=1}^{r} \mathrm{Mat}_{n_i,n_j}(\mathcal{A}_{i,j}). \tag{2}$$

(ii) Conversely, suppose we are given a family $(\mathcal{A}_{i,j})_{1 \le i,j \le r}$ of A-modules together with A-linear maps $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}$ for $1 \le i, j, k \le r$ satisfying the above conditions (UNIT), (ASSO) and (COMM). Then there is a unique structure of a GMA of type $(n_1, \dots, n_r)$ on the A-module $S := \oplus_{i,j=1}^r \mathrm{Mat}_{n_i,n_j}(\mathcal{A}_{i,j})$.

Lemma 3.1.37. *Let $(S, \mathcal{E})$ be a GMA and let $D \colon S \to A$ a pseudorepresentation. Then for any $x \in \mathrm{Mat}_{n_i \times n_j}(\mathcal{A}_{i,j})$ for some $1 \le i, j \le r$ with $i \ne j$, we have $D(1 + e_i x e_j) = 1$.*

Proof. By Lemma 3.1.12 we have $D(1 + e_i x e_j) = D(1 + e_j e_i x) = D(1) = 1$. □

Definition 3.1.38. A *pseudocharacter on S* is an A-linear map $\tau \colon S \to A$ satisfying

(i) τ is central; i.e., $\tau(s_1 s_2) = \tau(s_2 s_1)$ for all $s_1, s_2 \in S$

(ii) there exists an integer $n \in \mathbb{N}_{\ge 1}$ such that $n! \in A^\times$ and the map

$$\begin{aligned} S_{n+1}(\tau) \colon S^{n+1} &\longrightarrow A, \\ x &\longmapsto \sum_{\sigma \in \mathfrak{S}_{n+1}} \epsilon(\sigma)\, \tau^\sigma(x), \end{aligned} \tag{3}$$

vanishes, where for all $\sigma \in \mathfrak{S}_{n+1}$ we set

$$\tau^\sigma : S^{n+1} \longrightarrow A, \quad x = (x_1, \ldots, x_{n+1}) \longmapsto \prod_{i=1}^r \tau^{\sigma_i}(x),$$

with $\sigma = \prod_{i=1}^r \sigma_i$ is the cycle decomposition and $\tau^{\sigma_i}(x) := \tau(x_{i_1} \cdots x_{i_k})$ for $\sigma_i = (i_1 \cdots i_k)$.

The smallest integer n such that $S_{n+1}(\tau) = 0$ is called the *dimension of* τ.

Example 3.1.39 ([BC09, § 1.2.3, Main Example 1.2.2, Cor. 1.3.16])**.** ,

(i) The trace of a representation $r : S \to \mathrm{Mat}_n(A)$ is an A-valued n-dimensional pseudocharacter.

(ii) If $(S, \mathcal{E})$ is a GMA of type $(n_1, ..., n_r)$, $n := \sum_{i=1}^r n_i$ and $n! \in A^\times$ holds, then the trace τ of S is a pseudocharacter of dimension n.

We remark that these pseudocharacters satisfy a Cayley-Hamilton identity, of which the map (3) is a polarization. As for pseudorepresentations, the Cayley-Hamilton identity implies that the kernel of the pseudocharacters vanish so that they are called faithful.

It is the restriction on the characteristic by the condition $n! \in A^\times$, which motivated Chenevier to introduce the more general notion of a pseudorepresentation. As mentioned earlier, a pseudorepresentation encodes not only the data given by the trace of a representation but instead the data given by all characteristic polynomial coefficients.

Proposition 3.1.40 ([Che14, Prop. 1.27– 1.29])**.** *The map*

$$\{n\text{-dimensional pseudorepresentations } S \to A\} \longrightarrow \{n\text{-dimensional pseudocharacters } S \to A\},$$
$$D \longmapsto \tau_D,$$

is an injection. If either $n = 2$ and $2 \in A^\times$, or if $n > 2$ and $(2n)! \in A^\times$, then it is a bijection.

For the remainder of this subsection, we fix a generalized matrix algebra $(S, \mathcal{E})$ of type $(n_1, \ldots, n_r)$. In order to define a determinant of the GMA $(S, \mathcal{E})$ using the Leibniz formula and to associate a pseudorepresentation with $(S, \mathcal{E})$, we need to embed the A-modules $\mathcal{A}_{i,j}$ in a commutative A-algebra B. More precisely, we define a universal object among such A-algebras.

Definition 3.1.41 ([BC09, § 1.3.3])**.** Let B be a commutative A-algebra.

(i) A representation $\rho : S \to \mathrm{Mat}_n(B)$ is called *adapted to* $\mathcal{E}$ if its restriction to the A-subalgebra $\bigoplus_{i=1}^r e_i S e_i$ is the representation $\oplus_{i=1}^r \psi_i$ composed with the natural diagonal map $\mathrm{Mat}_{n_1}(A) \oplus \ldots \oplus \mathrm{Mat}_{n_r}(A) \to \mathrm{Mat}_n(B)$.

(ii) We call G the functor that sends a commutative A-algebra B to the set of representations $\rho : S \to \mathrm{Mat}_n(B)$ adapted to $\mathcal{E}$.

(iii) We call F the functor that sends a commutative A-algebra B to the set $\big\{(f_{i,j})_{1 \le i,j \le r} : f_{i,j} : \mathcal{A}_{i,j} \to B \text{ is an } A\text{-linear map}\big\}$ such that

  (i) $f_{i,i}$ coincides with the A-algebra structure on B,

  (ii) $f_{i,k}\big(\varphi_{i,j,k}(x \otimes z)\big) = f_{i,j}(x) \cdot f_{j,k}(y)$ for all $x \in \mathcal{A}_{i,j}$, $y \in \mathcal{A}_{j,k}$ and $i, j, k = 1, \ldots, r$.

Both F and G are covariant functors $\mathcal{CA}lg_A \to Sets$.

Proposition 3.1.42 ([BC09, Prop. 1.3.9]). *(i) The functor F is representable by a commutative A-algebra B^{univ} together with universal maps*

$$\left(f_{i,j}^{\mathrm{univ}} : \mathcal{A}_{i,j} \to B^{\mathrm{univ}}\right)_{1\le i,j\le r} \in F(B^{\mathrm{univ}}).$$

(ii) There is a natural isomorphism of functors $G \to F$, and so G is represented by B^{univ}.

Proof. Let $\mathcal{B} := \mathrm{Sym}(\bigoplus_{i\neq j} \mathcal{A}_{i,j})$ be the symmetric algebra over A of the A-module $\bigoplus_{i\neq j} \mathcal{A}_{i,j}$. Finally, let B^{univ} be its quotient by the ideal generated by all differences of the form $x \otimes y - \varphi_{i,j,k}(x \otimes y)$ for $x \in \mathcal{A}_{i,j}$, $y \in \mathcal{A}_{j,k}$ and all $i, j, k \in \{1, \dots, r\}$. It is obvious that B^{univ}, equipped with the canonical element $(f_{i,j} \colon \mathcal{A}_{i,j} \to B^{\mathrm{univ}})_{i,j} \in F(B^{\mathrm{univ}})$ is the universal object for F. This proves (i). For (ii) see [BC09, Prop. 1.3.9]. □

Definition 3.1.43 ([BC09, § 1.3.6]). Let $\Omega = \{(i, j) \in \{1, \dots, r\}^2 : i \neq j\}$, and write $\mathbf{i}, \mathbf{j} \colon \Omega \to \{1, \dots, r\}$ for the projections on the first and second component, respectively.

(i) We identify any tuple $\tau = (\tau_{i,j})_{(i,j)\in\Omega} \in \mathbb{N}_0^\Omega$ with the directed graph on the vertex set $\{1, \dots, r\}$ where the edge from i to j has multiplicity $\tau_{i,j}$ and where there are no edges from i to itself. The degree of this graph, i.e., of $(\tau_{i,j})$, is the tuple $\deg \tau = \big((\deg \tau)_i\big)_{1\le i\le r} \in \mathbb{Z}^r$, where $(\deg \tau)_i$ is the number of edges arriving at i minus the number of the edges leaving i.

(ii) For any $(i, j) \in \Omega$, let $\tau(i, j)$ be the graph with a single edge from i to j. For $i \in \{1, \dots, r\}$, let $\tau(i, i)$ be the edgeless graph $(0, \dots, 0) \in \mathbb{N}_0^\Omega$.

(iii) Suppose that $x_1, \dots, x_s \in \Omega$ satisfy $\mathbf{j}(x_k) = \mathbf{i}(x_{k+1})$ for $k = 1, \dots, s-1$. Then $\gamma = (x_1, \dots, x_s)$ is called a *path from* $\mathbf{i}(x_1)$ *to* $\mathbf{j}(x_s)$. If further $\mathbf{j}(x_s) = \mathbf{i}(x_1)$, then γ is called a *cycle*.

(iv) If $c_1, \dots, c_m$ is a (possibly empty) sequence of cycles, and γ is a path from i to j, then $\Gamma = (c_1, \dots, c_m, \gamma)$ is called an *extended path from i to j*. If γ is a cycle, then Γ is called an *extended cycle*.

(v) To an extended path Γ as in (iv) one attaches a graph $\tau(\Gamma) \in \mathbb{N}_0^\Omega$ by setting $\tau(\Gamma)_{i,j}$ to be the number of times that the sequence (i, j) occurs in γ or any of the c_k.

Lemma 3.1.44. *Let Γ be an extended path. Then $\deg \tau(\Gamma) = (0, \dots, 0) \in \mathbb{Z}^r$ if and only if γ is an extended cycle.*

Proof. This follows from [BC09, Lem. 1.3.14 (i)]. □

Proposition 3.1.45 (Cf. proof of [BC09, Prop. 1.3.13]). *(i) For $n \in \mathbb{Z}^r$ set*

$$\mathcal{B}_n := \bigoplus_{\substack{\tau\in\mathbb{N}_0^\Omega \\ \deg\tau = n}} \bigotimes_{(i,j)\in\Omega} \mathrm{Sym}^{\tau_{i,j}} \mathcal{A}_{i,j},$$

Then the $\mathbb{Z}^r$-grading on $\mathcal{B} = \bigoplus_{n\in\mathbb{Z}^r} \mathcal{B}_n$ induces a $\mathbb{Z}^r$-grading on $B^{\mathrm{univ}} = \bigoplus_{n\in\mathbb{Z}^r} B_n^{\mathrm{univ}}$ such that for any $i, j \in \{1, \dots, r\}$ the image of $f_{i,j}^{\mathrm{univ}}$ lies in $B^{\mathrm{univ}}_{\deg \tau(i,j)}$.

(ii) For any $i, j \in \{1, \dots, r\}$ and $n = \det \tau(i, j)$, there exists an A-linear map $\psi_n : B_n^{\mathrm{univ}} \to \mathcal{A}_{i,j}$ such that $\psi_n \circ f_{i,j}^{\mathrm{univ}}$ is the identity map on $\mathcal{A}_{i,j}$.

Note that Lemma 3.1.44 implies that $B^{\mathrm{univ}}_{(0,\dots,0)} = A$.

Recall that $n = \sum_{i=1}^r n_i$. Let $1 \le i \le r$ and $1 \le j \le n_i$ and write $E_{i,j}$ for the elements E_i^{jj} from Definition 3.1.35. Let $J := \{(i,j) : i \in \{1,\dots,r\}, j \in \{1,\dots,n_i\}\}$. Then we have a bijection $\{1,\dots,n\} \to J$ defined by associating to $m \in \{1,\dots,n\}$ the unique pair (i,j) satisfying $m = \sum_{k=1}^{i-1} n_k + j$. Through this identification the symmetric group $\mathfrak{S}_n$ acts on elements (i,j) in J. For $\sigma \in \mathfrak{S}_n$ we denote the tuple $\sigma(i,j)$ by $\big(\mathbf{i}(\sigma(i,j)), \mathbf{j}(\sigma(i,j))\big)$.

Proposition 3.1.46 (Cf. [WE17, Prop. 2.23]). *Let $f^{\mathrm{univ}} : S \to \mathrm{Mat}_n(B^{\mathrm{univ}})$ be the map defined by $f^{\mathrm{univ}}_{i,j} : \mathcal{A}_{i,j} \to B^{\mathrm{univ}}$ for $1 \le i,j \le r$. Then the composition of f^{univ} with the usual determinant* $\det : \mathrm{Mat}_n(B^{\mathrm{univ}}) \to B^{\mathrm{univ}}$ *is explicitely given by*

$$\det \circ f^{\mathrm{univ}} \colon S \longrightarrow B^{\mathrm{univ}}, \quad x \longmapsto \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \otimes_{i=1}^r \otimes_{j=1}^{n_i} f^{\mathrm{univ}}_{i,\mathbf{i}(\sigma(i,j))}(E_{i,j} x E_{\sigma(i,j)}), \tag{4}$$

and takes values in A.

Proof. The explicit formula follows from the Leibniz formula of the determinant det. Write $\sigma \in \mathfrak{S}_n$ as a product $c_1 \circ \ldots \circ c_s$ of unique disjoint cycles with $c_k = \big((i_{k,1}, j_{k,1}) \cdots (i_{k,t_k}, j_{k,t_k})\big)$ for $k = 1,\dots,s$. For ease of notation we set $(i_{k,t_k+1}, j_{k,t_k+1}) := (i_{k,1}, j_{k,1})$. By commutativity of B^{univ}, we can sort the factors in (4) according to the cycle decomposition so that for $x \in S$ we have

$$\otimes_{i=1}^r \otimes_{j=1}^{n_i} f^{\mathrm{univ}}_{i,\mathbf{i}(\sigma(i,j))}(E_{i,j} x E_{\sigma(i,j)}) = \otimes_{k=1}^s \otimes_{m=1}^{t_k} f^{\mathrm{univ}}_{i_{k,m}, i_{k,m+1}}(E_{i_{k,m}, j_{k,m}} x E_{i_{k,m+1}, j_{k,m+1}}).$$

Using the condition (ASSO) and the relations defining B^{univ}, we find that the latter term lies in A. □

We use the previous proposition to associate a pseudorepresentation with the GMA $(S, \mathcal{E})$.

Definition 3.1.47. (i) The *determinant map* $\det_{(S,\mathcal{E})}$ *attached to the fixed GMA* $(S,\mathcal{E})$ is

$$\det \circ f^{\mathrm{univ}} : S \longrightarrow A, \quad x \longmapsto \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \otimes_{i=1}^r \otimes_{j=1}^{n_i} f^{\mathrm{univ}}_{i,\mathbf{i}(\sigma(i,j))}(E_{i,j} x E_{\sigma(i,j)}).$$

(ii) The *ideal of total reducibility* in A is $I = \sum_{i \ne j} \mathcal{A}_{i,j}\mathcal{A}_{j,i} \subset A$, and the *locus of total reducibility* is $\mathrm{Spec}(A/I)$.

For the following result recall the notation D_e from Lemma 3.1.32.

Proposition 3.1.48 (Cf. [BC09, Prop. 1.5.1]). *Let $I = \sum_{i \ne j} \mathcal{A}_{i,j}\mathcal{A}_{j,i}$ be the ideal of total reducibility in A.*

(i) (1) If $I = 0$, then the map $\pi \colon S \to \sum_i e_i S e_i \subset S, x \mapsto \sum_i e_i x e_i$ is a ring homomorphism.

(2) Denoting by D_i the map $e_i S e_i \xrightarrow{\psi_i} \mathrm{Mat}_{n_i}(A) \xrightarrow{\det} A$ for $i = 1,\dots,r$, one has

$$\det_{(S,\mathcal{E})} = \oplus_{i=1}^r D_i \circ \pi \mod I.$$

(ii) Suppose that there exist m_i-dimensional pseudorepresentations $D'_i \colon S \to A$ with $m_i > 0$ for $i \in \{1,\dots,r\}$ such that one has $\det_{(S,\mathcal{E})} = \oplus_{i=1}^r D'_i$. Then $I = 0$ and for a unique permutation $\sigma \in \mathfrak{S}_r$ we have $D'_{\sigma(i)} = D_i \circ \pi$ with D_i and π from (i).

Proof. Part (1) of (i) is a straightforward matrix calculation using $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = 0$ for all $i \neq j$ from $\{1, \dots, r\}$. To see part (2) of (i) note that by our definitions we have the explicit formula

$$D_i \mod I \colon e_iSe_i \longrightarrow A/I, \quad x \longmapsto \sum_{\sigma_i \in \mathfrak{S}_{n_i}} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} E_{i,j}xE_{i,\sigma_i(j)} \mod I,$$

and using distributivity for $x \in S$

$$\begin{aligned}\prod_{i=1}^{r}(D_i \circ \pi)(x) \mod I &= \prod_{i=1}^{r} \sum_{\sigma_i \in \mathfrak{S}_{n_i}} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} E_{i,j}xE_{i,\sigma_i(j)} \mod I \\ &= \sum_{\sigma_1 \in \mathfrak{S}_{n_1}} \dots \sum_{\sigma_r \in \mathfrak{S}_{n_r}} \prod_{i=1}^{r} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} E_{i,j}xE_{i,\sigma_i(j)} \mod I.\end{aligned}$$

Now in the sum $\operatorname{sgn}(\sigma) \prod_{i=1}^{r} \prod_{j=1}^{n_i} f^{\text{univ}}_{i,\mathbf{i}(\sigma(i,j))}(E_{i,j}xE_{\sigma(i,j)})$, modulo I only those summands are nonzero for which $\sigma \in \mathfrak{S}_n$ satisfies $\mathbf{i}(\sigma(i,j)) = i$. Therefore, in a nonzero summand we can write $\sigma = (\sigma_1, \dots, \sigma_r) \in \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_r}$ and

$$\begin{aligned}\det_{(S,\mathcal{E})}(x) &= \sum_{(\sigma_1,\dots,\sigma_r) \in \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_r}} \prod_{i=1}^{r} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} f_{i,i}(E_{i,j}xE_{i,\sigma_i(j)}) \mod I \\ &= \sum_{\sigma_1 \in \mathfrak{S}_{n_1}} \dots \sum_{\sigma_r \in \mathfrak{S}_{n_r}} \prod_{i=1}^{r} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n_i} f_{i,i}(E_{i,j}xE_{i,\sigma_i(j)}) \mod I.\end{aligned}$$

This completes the proof of (i).

We now prove (ii). In a first step, we show the **claim** that there is a unique permutation $\sigma \in \mathfrak{S}_r$ such that $D_i = (D'_{\sigma(i)})_{e_i}$ and $(D'_{i'})_{e_i} = 1$ for $i' \neq \sigma(i)$. For this, we restrict $\oplus_{i'=1}^{r} D'_{i'}$ to e_iSe_i, so that

$$D_i = (\det_{(S,\mathcal{E})})_{e_i} = \oplus_{i'}(D'_{i'})_{e_i}.$$

By Lemma 3.1.32 the $(D'_{i'})_{e_i}$ are pseudorepresentations, and one has $m_{i'} \geq m_{i',i} := \dim(D'_{i'})_{e_i}$. Now under the direct sum in the sense of Corollary 3.1.31 dimensions are added, and it follows that

$$n_i = \sum\nolimits_{i'=1}^{r} m_{i',i}.$$

Since $e_iSe_i = \mathrm{Mat}_{n_i}(A)$ it follows from Example 3.1.8(ii) that each $m_{i',i}$ is divisible by n_i. Hence there is a map $\sigma \colon \{1, \dots, r\} \to \{1, \dots, r\}$ such that $m_{\sigma(i),i} = n_i$ and $m_{i',i} = 0$ for $i' \neq \sigma(i)$, and moreover $D_i = (D'_{\sigma(i)})_{e_i}$. The uniqueness of σ is clear from the construction. It remains to show that σ is bijective. It will suffice to show that σ is surjective.

For this, let $S'_{i'} := \oplus_{i \in \sigma^{-1}(i')} e_iSe_i$, so that $S = \oplus_{i'} S'_{i'}$. The restriction of $D'_{i''}$ to $S'_{i'}$ is zero if $i'' \neq i'$, and the restriction of $D'_{i'}$ to $S'_{i'}$ is a pseudorepresentation with

$$m_{i'} \overset{3.1.32}{\geq} \dim D'_{i'}|_{S'_{i'}} = \dim \oplus_{i''=1}^{r} D'_{i''}|_{S'_{i'}} = \dim \det_{(S,\mathcal{E})}|_{S'_{i'}} = \sum_{i \in \sigma^{-1}(i')} n_i.$$

Summing over all i' in the image of σ implies $\sum_{i' \in \sigma(\{1,\dots,r\})} m_{i'} \geq n$. However, all $m_{i'}$ are strictly positive and $\sum_{i'=1}^{r} m_{i'} = n$, and this implies that σ is surjective, and hence the claim is proved.

For simplicity of notation we assume from here on, without loss of generality, that $\sigma = \mathrm{id}$. We now show that $I = 0$. For this, it suffices to show that $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = 0$ for all $i \neq j$. By restricting to the subalgebra $S' = e_iSe_i + e_jSe_j + e_iSe_j + e_jSe_i$ with $\mathcal{E}' = (e_i, \psi_i, e_j, \psi_j,)$, i.e., by considering $D_{e_i+e_j}$, and using $\det_{(S,\mathcal{E})}|_{S'} = \det_{(S',\mathcal{E}')}$, we may assume $r = 2$ for the proof of $I = 0$.

Let b be in $\mathcal{A}_{1,2}$ and c in $\mathcal{A}_{2,1}$, and write x for $e_1E_1^{1,1}bE_2^{1,1}e_2$ and y for $e_2E_2^{1,1}cE_1^{1,1}e_1$ with $E_i^{k,l}$ from Definition 3.1.35. Using the description of GMA's from Lemma 3.1.36 one easily verifies that

$$1 + xy = 1 + E_1^{1,1}bc \in e_1Se_1 + (1 - e_1),\ 1 + yx = 1 + E_2^{1,1}bc \in (1 - e_2) + e_2Se_2.$$

Note moreover that by Lemma 3.1.12 we have $D(1 + xy) = D(1 + yx)$ for every pseudorepresentation $D\colon S \to A$. If we apply this to D_i' and our earlier observations on $(D_i')_{e_{i'}}$, we find that

$$D_i'(1 + xy) = D_i'(1 + yx) = 1$$

for $i = 1, 2$ and hence from hypothesis (2) that $\det_{(S,\mathcal{E})}(1 + E_1^{1,1}bc) = 1$. From the formula for $\det_{(S,\mathcal{E})}$ on $e_1Se_1 + e_2Se_2 \cong \mathrm{Mat}_{n_1}(A) \times \mathrm{Mat}_{n_2}(A)$, we deduce that

$$\det_{(S,\mathcal{E})}(1 + E_1^{1,1}bc) = 1 + bc,$$

and hence that $bc = 0$, as was to be shown.

For the second assertion, observe that by Lemma 3.1.37 we have $D_i'(1 + e_ixe_j) = 1$ for any $i \neq j$ and $x \in \mathrm{Mat}_{n_i,n_j}(\mathcal{A}_{i,j})$. It follows that $D_i'(1 + u) = 1$ for any u in the kernel of π. And now the second assertion follows from knowing the restriction of D_i' to $\sum_i e_iSe_i$ given in the first claim of the proof of (ii). □

Lemma 3.1.49. *Let S be a generalized matrix algebra of type $(n_1, \ldots, n_r)$, and $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$. Then $S \otimes_A B$ is a generalized matrix algebra of type $(n_1, \ldots, n_r)$.*

Proof. Choose a family of orthogonal idempotents $e_1, \ldots, e_r \in S$ with $\sum_{i=1}^r e_i = 1_S$, and a family of A-algebra isomorphisms

$$\psi_i : e_iSe_i \xrightarrow{\sim} \mathrm{Mat}_{n_i}(A), \qquad i = 1, \ldots, r,$$

such that the associated trace map $\tau : S \to A,\ s \mapsto \sum_{i=1}^r \mathrm{tr}\left(\psi_i(e_ise_i)\right)$ satisfies $\tau(s_1s_2) = \tau(s_2s_1)$ for all $s_1, s_2 \in S$. Then $e_{B,i} := e_i \otimes 1_B \in S \otimes_A B$, $i = 1, \ldots, r$, form a family of orthogonal idempotents in $S \otimes_A B$ such that $\sum_{i=1}^r e_{B,i} = \sum_{i=1}^r e_i \otimes 1_B = 1_{S\otimes_A B}$. Then there is a family of A-algebra isomorphisms

$$\psi_{B,i}\colon e_{B,i}(S \otimes_A B)e_{B,i} \cong e_iSe_i \otimes_A B \xrightarrow{\psi_i\otimes \mathrm{id}_B} \mathrm{Mat}_{n_i}(A) \otimes_A B \cong \mathrm{Mat}_{n_i}(B), \qquad i = 1, \ldots, r,$$

such that the associated trace map

$$\tau_B : S \otimes_A B \to A,\ s \otimes b \mapsto \sum_{i=1}^r \mathrm{tr}\left(\psi_{B,i}\big(e_{B,i}(s \otimes b)e_{B,i}\big)\right) = \sum_{i=1}^r \mathrm{tr}\left(\psi_i(e_ise_i)\right) \otimes b = \tau(s) \otimes b$$

satisfies $\tau_B(s_1\otimes b_1 \cdot s_2\otimes b_2) = \tau_B(s_1s_2\otimes b_1b_2) = \tau(s_1s_2)\otimes b_1b_2 = \tau(s_2s_1)\otimes b_2b_1 = \tau_B(s_2\otimes b_2 \cdot s_1\otimes b_1)$ for all $s_1 \otimes b_1, s_2 \otimes b_2 \in S \otimes_A B$. □

Chenevier uses the pseudorepresentations D_e from Lemma 3.1.32 to show the following.

Theorem 3.1.50 ([Che14, Thm. 2.22])**.** *Assume that A is a henselian local ring with maximal ideal m_A and residue field $k := A/m_A$, that S is an A-algebra and that $D\colon S \to A$ is an n-dimensional Cayley-Hamilton pseudorepresentation.*

(i) If the residual pseudorepresentation $\overline{D} = D \otimes_A k \colon S/m_A S \longrightarrow k$ is split and irreducible, then there is an A-algebra isomorphism $\rho\colon S \xrightarrow{\sim} \mathrm{Mat}_n(A)$ such that $D = \det \circ \rho$.

(ii) If $\overline{D}$ is split and multiplicity free, then S is a generalized matrix algebra with determinant D.

3.1.4 Universal pseudodeformation rings

This subsection constructs in Proposition 3.1.57 the object of our interest, the universal pseudodeformation ring of a residual pseudorepresentation $\overline{D}$, as the completion of the universal pseudorepresentation ring with respect to its $\overline{D}$-open ideals defined in Definition 3.1.55. Then this universal pseudodeformation rings parametrizes continuous liftings of $\overline{D}$.

Definition 3.1.51 (Cf. [Che14, § 2.30], [WE13, Def. 3.1.0.10])**.** Let A be a commutative topological ring and S be a topological continuous A-algebra. Then an n-dimensional pseudorepresentation $D\colon S \to A$ is called *continuous* if and only if either of the two following equivalent conditions is satisfied:

(a) the characteristic polynomial functions $\Lambda_{D,i}$, $i = 1, \dots, n$, are continuous;

(b) for every commutative continuous A-algebra B, the function $D_B\colon S \otimes_A B \to B$ is continuous;

(c) the functions $P^{[\alpha]}\colon R^n \to A$ from Remark 3.1.2 are continuous for all $\alpha \in I_n$.

We now consider the case of continuous pseudorepresentations of the group algebra of a profinite group.

Example 3.1.52. Let A be a topological ring that contains an open subring A_0 that is linearly topologized, and let G be a profinite group. Then $A[G]$ is a topological ring with a basis of open neighbourhoods of 0 given by the sets

$$I[H] := \Big\{ \sum_{h \in H} a_h h \ :\ a_h \in I,\ \text{almost all } a_h = 0 \Big\},$$

where $I \subset A_0$ is an open ideal and $H \subset G$ is an open normal subgroup of G. Then the $P^{[\alpha]}\colon A[G]^n \to A$ are continuous if and only if their restriction to G^n is continuous. Hence, cf. [Che14, § 2.30], an n-dimensional pseudorepresentation $D\colon A[G] \to A$ is continuous if and only if $\Lambda_{D,i}\colon G \to A$ is continuous for all $i \leq n$.

A particular case of the above is that when A is profinite so that one can take $A_0 = A$. Then the rings $A[G]/I[H] \cong A/I[G/H]$ are finite, and their inverse limit (simultaneously over I and N) is the profinite completion $A[\![G]\!]$ of $A[G]$. Then an n-dimensional pseudorepresentation $D\colon A[G] \to A$ is continuous if and only if there exists an induced n-dimensional pseudorepresentation $D\colon A[\![G]\!] \to A$ that is furthermore continuous.

Another case relevant to us is that when A lies in $\mathcal{A}r_k$ and k is a local field.

We let Λ be a Noetherian local commutative $W(\mathbb{F})$-algebra with finite residue field $\mathbb{F}$. Consider the category $\hat{\mathcal{C}}_\Lambda$ of profinite local Λ-algebras with residue field $\mathbb{F}$. The category $\mathcal{A}r_\Lambda$ from Subsection 2.1.1 is a full subcategory of $\hat{\mathcal{C}}_\Lambda$, and objects in $\hat{\mathcal{C}}_\Lambda$ are projective limits of objects in $\mathcal{A}r_\Lambda$. We remark that profinite Λ-modules are linearly topologized [Coh73, Prop. 2.7] and the completed tensor product $A \hat{\otimes}_\Lambda B$ of $A, B \in \mathrm{Ob}(\hat{\mathcal{C}}_\Lambda)$ from Definition 2.1.2 also lies in $\hat{\mathcal{C}}_\Lambda$.

Definition 3.1.53 ([WE13, § 3.1.4.3])**.** (i) A continuous pseudorepresentation $D\colon S \hat{\otimes}_\Lambda R \to R$ satisfying $D \hat{\otimes}_R \mathbb{F} \cong \overline{D}$ is called a *pseudodeformation of* $\overline{D}$.

(ii) The functor

$$\mathcal{P}sD_{\overline{D}}\colon \hat{\mathcal{C}}_\Lambda \to Sets, \quad R \longmapsto \{D\colon S\hat{\otimes}_\Lambda R \longrightarrow R \text{ is a pseudodeformation of } \overline{D}\},$$

is called the *pseudodeformation functor* of the *residual pseudorepresentation* $\overline{D}$.

Wang Erickson also uses the term pseudorepresentation for D which are not continuous. The meaning depends on the part of [WE13] where the term is used.

For later use, we also note the following result:

Proposition 3.1.54 ([WE13, Lem. 3.1.2.2, Rem. 3.1.4.1])**.** *Suppose that $\overline{D}\colon G \to \mathbb{F}$ is n-dimensional and continuous. Then the associated representation $\rho_{\overline{D}}\colon G_K \to \mathrm{GL}_n(k^{\mathrm{alg}})$ is defined over a finite extension of $\mathbb{F}$ and continuous.*

Proof. By [Che14, Observation after Lem. 1.19], for $S = \Lambda[G]$ one has

$$\ker \overline{D} = \{r \in R \,:\, \forall B \in \mathrm{Ob}(\mathcal{CA}lg_\Lambda), \forall r' \in S \otimes_A B, \forall i \geq 1 : \Lambda_{\overline{D},i}(rr') = 0.\}$$

This shows that $\ker \overline{D}$ is closed in S, and this implies that $\{g \in G \,:\, g \in \ker \overline{D}\}$ is closed in G. By Theorem 3.1.26 we have $\ker \overline{D} = \ker \rho_{\overline{D}}$ (viewed as ideals in S), and hence the kernel of the representation $\rho_{\overline{D}}$ is closed. Now from Theorem 3.1.28, one deduces that $\rho_{\overline{D}}$ is in fact defined over a finite extension of $\mathbb{F}$ and so $\ker \rho_{\overline{D}}$ has finite index in G. It follows that $\ker \rho_{\overline{D}}$ is open in G and this completes the proof. □

For the construction of the universal pseudodeformation rings we make a definition.

Definition 3.1.55. Let Λ be either in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ or a local field that is a $W(\mathbb{F})$-algebra; in the former case set $k = \mathbb{F}$ in the latter $k = \Lambda$. Let S be a topological Λ-algebra, let $\pi\colon A \to k$ be a surjection in $\mathcal{CA}lg_\Lambda$.

Let $D\colon S \otimes_\Lambda A \to A$ be a pseudorepresentation, which is not necessarily continuous, such that $\overline{D} := D \otimes_A k\colon S\hat{\otimes}_\Lambda k \to k$ is continuous.

An ideal I of A is called $\overline{D}$*-open* if the following conditions hold:

(a) $I \subset \ker \pi$ and A/I is Artinian local;

(b) the representation $D_I := D \otimes_A A/I$ is continuous – recall that A/I is discrete if $k = \mathbb{F}$ and that it carries the canonical k-vector space topology if k is a local field;

Lemma 3.1.56. *Let the notation be as in Definition 3.1.55. Then the $\overline{D}$-open ideals of A form a basis of a topology on A.*

Proof. (Cf. [WE13, Thm. 3.1.4.6]) One has to show that if I, I' are $\overline{D}$-open ideals, then so is $I \cap I'$. For this, one considers the injection

$$\iota\colon A/(I \cap I') \longrightarrow A/I \times A/I'.$$

It is straightforward (for the two cases of Λ we consider) to verify that ι is an isomorphism onto its image. Now a pseudorepresentation is continuous if and only if this holds for its characteristic polynomial functions; cf. Definition 3.1.51. It now follows easily that $I \cap I'$ is $\overline{D}$-open if both I and I' are $\overline{D}$-open. □

Let Λ be in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ and let S be a topological continuous Λ-algebra. The following is due to Chenevier in [Che14, Prop. 3.3] for $\Lambda = W(\mathbb{F})$. We quote the general result.

Proposition 3.1.57 (Cf. [WE13, Thm. 3.1.4.6])**.** *Let $\overline{D}\colon S\hat{\otimes}_\Lambda\mathbb{F}\to\mathbb{F}$ be a continuous residual pseudorepresentation. The pseudodeformation functor $\mathcal{P}sD_{\overline{D}}$ is representable by a profinite local Λ-algebra $R^{\mathrm{univ}}_{\Lambda,\overline{D}}\in\mathrm{Ob}(\hat{\mathcal{C}}_\Lambda)$ together with a universal pseudodeformation*

$$D^{\mathrm{univ}}_{\overline{D}}\colon S\hat{\otimes}_\Lambda R^{\mathrm{univ}}_{\Lambda,\overline{D}}\longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}}.$$

Proof. We give an indication of the proof since we want to later use similar arguments. Consider the universal ring $R^{\mathrm{univ}}_{S,n}=\Gamma^n_A(S)^{\mathrm{ab}}$ from Proposition 3.1.23 with its universal pseudorepresentation

$$D^{\mathrm{univ}}_S\colon S\otimes_\Lambda R^{\mathrm{univ}}_{S,n}\longrightarrow R^{\mathrm{univ}}_{S,n}$$

By definition $R^{\mathrm{univ}}_{S,n}$ is a Λ-algebra. The map $\overline{D}$ induces a homomorphism $\pi\colon R^{\mathrm{univ}}_{S,n}\to\mathbb{F}$ of Λ-algebras. By Lemma 3.1.56, the $\overline{D}$-open ideals of $A:=R^{\mathrm{univ}}_{S,n}$ form the basis of a topology on A, and one defines $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ as the completion of $R^{\mathrm{univ}}_{S,n}$ with respect to its $\overline{D}$-open ideals. It is then straightforward to prove the wanted universal property for $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ together with the pseudorepresentation $D^{\mathrm{univ}}_S\otimes_{R^{\mathrm{univ}}_{S,n}}R^{\mathrm{univ}}_{\Lambda,\overline{D}}$, by verifying it for the restriction of $\mathcal{P}sD_{\overline{D}}$ to $\mathcal{A}r_\Lambda$. □

Definition 3.1.58. We call $R^{\mathrm{univ}}_{\overline{D}}:=R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ the *universal (Λ-)pseudodeformation ring of $\overline{D}$*, $X^{\mathrm{univ}}_{\overline{D}}:=X^{\mathrm{univ}}_{\Lambda,\overline{D}}:=\operatorname{Spec}R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ the *universal (Λ-)pseudodeformation space of $\overline{D}$* and

$$D^{\mathrm{univ}}_{\overline{D}}\colon S\hat{\otimes}_\Lambda R^{\mathrm{univ}}_{\Lambda,\overline{D}}\longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}}$$

the *universal (Λ-)pseudodeformation of $\overline{D}$.*

If $S=\mathbb{Z}[G]$ for a group G, we often set $R^{\mathrm{univ}}_{\overline{D}}:=R^{\mathrm{univ}}_{G,\overline{D}}:=R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ and $D^{\mathrm{univ}}_{\overline{D}}:=D^{\mathrm{univ}}_{G,\overline{D}}:=D^{\mathrm{univ}}_{\Lambda,\overline{D}}$.

The argument indicated in the proof of Proposition 3.1.57 also shows in the case that Λ is a local field and a $W(\mathbb{F})$-algebra the following:

Proposition 3.1.59. *Let k be a local field, let S be a topological continuous k-algebra and let $\overline{D}\colon S\to k$ be a continuous pseudorepresentation. Then the pseudodeformation functor*

$$\mathcal{P}sD_{\overline{D}}\colon\mathcal{A}r_k\to\mathit{Sets}$$

is pro-representable by a profinite local k-algebra $R^{\mathrm{univ}}_{k,\overline{D}}\in\mathrm{Ob}(\hat{\mathcal{C}}_k)$ together with a universal pseudodeformation

$$D^{\mathrm{univ}}_{\overline{D}}\colon S\hat{\otimes}_\Lambda R^{\mathrm{univ}}_{k,\overline{D}}\longrightarrow R^{\mathrm{univ}}_{k,\overline{D}}.$$

The following assertion summarizes conditions when the tangent space of the pseudodeformation functor $\mathcal{P}sD_{\overline{D}}$ is finite-dimensional, thereby implying Noetherianness of the universal pseudodeformation ring.

Proposition 3.1.60 (Cf. [Che14, Prop. 3.7], [WE13, Thm. 3.1.5.3])**.** *Suppose that Λ is a complete Noetherian local $W(\mathbb{F})$-algebra and S a profinite continuous Λ-algebra. If $\overline{D}\colon S\hat{\otimes}_\Lambda\mathbb{F}\to\mathbb{F}$ is a continuous residual pseudorepresentation of dimension n, then the complete local profinite Λ-algebra $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ is Noetherian if and only if one of the following hold:*

(i) S is a topologically finitely generated Λ-algebra;

(ii) $\dim_{\mathbb{F}}\operatorname{Ext}^1_S(S/\ker(\overline{D}),S/\ker(\overline{D}))<\infty$*;*

(iii) $S = \Lambda[\![G]\!]$ *for G a profinite group, and* $\dim_{\mathbb{F}^{\mathrm{alg}}} H^1_{\mathrm{cont}}(G, \mathrm{ad}_{\rho_{\overline{D}}}) < \infty$*, where $\rho_{\overline{D}}$ is the representation associated with $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}^{\mathrm{alg}}$ from Theorem 3.1.26;*

(iv) $S = \Lambda[\![G]\!]$ *for G a profinite group that satisfies the Φ_p-condition from Definition 2.2.5.*

Proposition 3.1.61 (Cf. [CDT99, § A.1])**.** *Let $\Lambda \to \Lambda'$ be a local homomorphisms of local Noetherian rings with respective finite residue fields k and k'. Then*

$$R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda'$$

is the universal Λ'-pseudodeformation ring of $\overline{D}' := \overline{D} \otimes_k k' \colon S\hat{\otimes}_\Lambda \Lambda' \otimes_{\Lambda'} k' \to S \otimes_\Lambda k' \to k'$.

Proof. Our proof mimics that given in [Wil95, p. 457].[2] By tensoring $D^{\mathrm{univ}}_{\overline{D}}$ with Λ' over Λ we obtain a pseudorepresentation

$$D^{\mathrm{univ}}_{\overline{D}} \hat{\otimes}_\Lambda \Lambda' \colon S' \hat{\otimes}_{\Lambda'} (R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda') \longrightarrow (R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda'),$$

which is a deformation of $\overline{D} \otimes_\Lambda \Lambda'$ and hence of $\overline{D} \otimes_\Lambda k'$. The universality of $R^{\mathrm{univ}}_{\Lambda',\overline{D}'}$ then yields a unique homomorphism

$$R^{\mathrm{univ}}_{\Lambda',\overline{D}'} \to R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda'$$

in $\hat{\mathcal{C}}_{\Lambda'}$ that maps $D^{\mathrm{univ}}_{\overline{D}\otimes_k k'}$ to $D^{\mathrm{univ}}_{\overline{D}} \otimes_\Lambda \Lambda'$. Next consider the subring R' of $R^{\mathrm{univ}}_{\Lambda',\overline{D}'}$ of elements that map to $k \subset k'$ under the reduction modulo the maximal ideal of $R^{\mathrm{univ}}_{\Lambda',\overline{D}'}$. Then R' lies in $\hat{\mathcal{C}}_\Lambda$ and the pseudodeformation $D^{\mathrm{univ}}_{\overline{D}\otimes_k k'}$ is defined over R' using Proposition 3.1.14. By the universal property of $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ we obtain a unique map

$$R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to R'$$

in $\hat{\mathcal{C}}_\Lambda$ mapping $D^{\mathrm{univ}}_{\overline{D}} \otimes_\Lambda \Lambda'$ to $D^{\mathrm{univ}}_{\overline{D}\otimes_k k'}$. Embedding R' into $R^{\mathrm{univ}}_{\Lambda',\overline{D}'}$ and using the Λ'-algebra structure of the latter, we obtain homomorphisms

$$R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda' \to R^{\mathrm{univ}}_{\Lambda',\overline{D}'} \to R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda'$$

in $\hat{\mathcal{C}}_{\Lambda'}$. From the construction it is not hard to check that the composition of the two maps is the identity. We wish to show that the left map is surjective, from which then our assertion is straight forward.

To see surjectivity, it suffices to show surjectivity on tangent spaces modulo $\mathfrak{m}_\Lambda$. We write R'_0 for the left and right ring and R'_1 for the middle ring and $\mathfrak{m}'_i$, $i = 0, 1$ for the respective maximal ideals. Then we have an induced map

$$k' = R'_0/\mathfrak{m}_0 \to R'_1/\mathfrak{m}_0 R'_1,$$

and we need to show that it is surjective. Now observe that the pseudorepresentation on $R'_1/\mathfrak{m}_0 R'_1$ induced from this map is the trivial deformation of $D^{\mathrm{univ}}_{\overline{D}\otimes_k k'}$. From the universality of $R'_1 = R^{\mathrm{univ}}_{\Lambda',\overline{D}'}$ it follows that $R'_1/\mathfrak{m}_0 R'_1$ has to be k', as was to be shown. □

Hence, it makes sense to define the following.

Definition 3.1.62. We call $\overline{R}^{\mathrm{univ}}_{\overline{D}} := R^{\mathrm{univ}}_{\mathbb{F},\overline{D}} \cong R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \mathbb{F}$ the *universal mod p pseudodeformation ring of $\overline{D}$* and the special fiber $\overline{X}^{\mathrm{univ}}_{\overline{D}} := X^{\mathrm{univ}}_{\mathbb{F},\overline{D}} \cong X^{\mathrm{univ}}_{\Lambda,\overline{D}} \times_\Lambda \mathbb{F}$ the *universal mod p pseudodeformation space of $\overline{D}$.*

If $S = \mathbb{Z}[G]$ for a group G, we also write $\overline{R}^{\mathrm{univ}}_{G,\overline{D}} := \overline{R}^{\mathrm{univ}}_{\overline{D}}$ and $\overline{X}^{\mathrm{univ}}_{G,\overline{D}} := \overline{X}^{\mathrm{univ}}_{\overline{D}}$.

[2] One can also apply arguments as in [CDT99, Appendix A] to Proposition 3.1.68 to obtain a proof.

3.1.5 Group pseudorepresentations

We show in Corollary 3.1.71 that there are only finitely many n-dimensional continuous pseudorepresentations $\overline{D}\colon G_K \to \mathbb{F}$. We also introduce slightly more general pseudodeformation functors.

Consider a profinite group G, the finite field $\mathbb{F}$ of characteristic p and further the fixed commutative ring A and each $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$, where we equip all with the discrete topology.

Lemma 3.1.63 ([Che14, Lem. 2.33])**.** *An n-dimensional pseudorepresentation $D\colon G \longrightarrow B$, considered as a polynomial law $P \in \mathcal{M}_A^n(A[G], B)$, is continuous if and only if* $\ker(P) \subset A[G]$ *is open. In this case, the natural representation*

$$G \longrightarrow (B[G]/\ker(D))^\times$$

factors through a finite quotient G/H of G for some open subgroup H.

For later purposes, we enlarge the base category of the pseudodeformation functor. Chenevier [Che14, § 3.9] refers to [EGA I, Ch. 0 § 7, Ch. 1 § 10] for an introduction to topological rings and formal schemes.

Definition 3.1.64 ([Che14, § 3.9])**.** Consider the ring $W(\mathbb{F})$ of Witt vectors over $\mathbb{F}$ as a topological ring.

(i) A topological ring A is *admissible* if there is a topological isomorphism

$$A \xrightarrow{\sim} \varprojlim A_\lambda,$$

where the limit is taken over a directed ordered set S with minimal element 0, each A_λ is a discrete ring, and each $A_\lambda \to A_0$ is surjective with nilpotent kernel.

(ii) Let $\mathcal{A}dm$ be the category consisting of:

- Objects: admissible topological rings A together with a continuous homomorphism $W(\mathbb{F}) \to A$;
- Morphisms: continuous ring homomorphisms.

(iii) A ring $A \in \mathrm{Ob}(\mathcal{A}dm)$ is *topologically of finite type over $W(\mathbb{F})$* if there are $i, j \in \mathbb{N}_{\geq 1}$ such that A is a quotient of $W(\mathbb{F})[[t_1, \ldots, t_i]]\langle x_1, \ldots, x_j\rangle$ together with its I-adic topology given by $I = (t_1, \ldots, t_i, p)$. [3]

Lemma 3.1.65 ([Che14, Lem. 3.10], [WE13, Lem. 3.1.6.7])**.** *Consider $A \in \mathrm{Ob}(\mathcal{A}dm)$, a continuous pseudorepresentation $D\colon A[G] \to A$, and the closure $C \subset A$ of the $W(\mathbb{F})$-algebra generated by the characteristic polynomial coefficients $\Lambda_{D,i}(g)$ for $g \in G$ and $i \in \mathbb{N}_{\geq 1}$.*

(i) The ring C is an admissible profinite subring of A. In particular, $C = \varprojlim_i C_i$ is a finite product of local $W(\mathbb{F})$-algebras with finite residue fields.

(ii) If further $\iota\colon A \longrightarrow A'$ is a continuous $W(\mathbb{F})$-algebra homomorphism, $D'\colon A'[G] \to A'$ is the induced determinant and $C' \subset A'$ is the closure $C' \subset A'$ of the $W(\mathbb{F})$-algebra generated by the characteristic polynomial coefficients $\Lambda_{D',i}(g)$ for $g \in G$ and $i \in \mathbb{N}_{\geq 1}$, then ι induces a continuous surjection $C \to C'$.

[3] Recall that for an admissible ring A the ring $A\langle t\rangle$ is the A-subalgebra of $A[[t]]$ of formal power series $\sum a_n t^n$ such that $a_n \to 0$ for $n \to \infty$.

We use the above lemma to make the following definition.

Definition 3.1.66. (i) Let $|G(n)| \subset \operatorname{Spec}(\Gamma^n_{W(\mathbb{F})}(W(\mathbb{F})[G])^{\mathrm{ab}})$ be the subset of closed points $z \in \operatorname{Spec}(\Gamma^n_{W(\mathbb{F})}(W(\mathbb{F})[G])^{\mathrm{ab}})$ with finite residue field $\kappa(z)$, and $D_z\colon G \to \kappa(z)$ be the corresponding pseudorepesentation of dimension n. We also write $\kappa(D_z)$ for $\kappa(z)$.[4]

(ii) Let A, $D\colon A[G] \to A$ and $C \subset A$ be as in Lemma 3.1.65. If C is local, then D is called *residually constant*.

(iii) In case (ii), the field $\kappa(C)$ is finite by Lemma 3.1.65(i), and by definition of C there exists $z \in |G(n)|$ such that $\kappa(C) = \kappa(z)$ and $D \otimes_C \kappa(C) \cong D_z$. If $C_0 = \kappa(C)$ (which can always be assumed by altering $A = \varprojlim_\lambda A_\lambda$), then $D \otimes_A A_0 = D_z \otimes_{C_0} A_0$. One says that D is *residually equal* to D_z.

Definition 3.1.67 ([Che14, § 3.9])**.** (i) Consider for $n \in \mathbb{N}_0$ the covariant *pseudorepresentation functor on the category $\mathcal{A}dm$*

$$\begin{aligned}\mathcal{P}s\mathcal{R}^{\mathcal{A}dm,n}\colon \mathcal{A}dm &\longrightarrow \mathcal{S}ets,\\ A &\longmapsto \{\text{continuous } n\text{-dimensional pseudorepresentations } A[G] \to A\},\end{aligned}$$

and for each $z \in |G(n)|$ its subfunctor $\mathcal{P}s\mathcal{R}^{\mathcal{A}dm}_z\colon \mathcal{A}dm \longrightarrow \mathcal{S}ets$,

$$A \longmapsto \{D\colon A[G] \to A \in \mathcal{P}s\mathcal{R}^{\mathcal{A}dm,n}(A) \ : \ D \text{ is residually equal to } D_z\}.$$

(ii) Consider for $n \in \mathbb{N}_0$ the contravariant *pseudorepresentation functor $\mathcal{P}s\mathcal{R}^{\mathcal{FS}/W(\mathbb{F}),n}$ on the category $\mathcal{FS}/W(\mathbb{F})$ of formal schemes over* $\operatorname{Spf} W(\mathbb{F})$ that is given by

$$\mathcal{X} \longmapsto \{\text{continuous } n\text{-dimensional pseudorepresentations } \mathcal{O}(\mathcal{X})[G] \to \mathcal{O}(\mathcal{X})\},$$

and for each $z \in |G(n)|$ its subfunctor $\mathcal{P}s\mathcal{R}^{\mathcal{FS}/W(\mathbb{F})}_z\colon \mathcal{FS}/W(\mathbb{F}) \longrightarrow \mathcal{S}ets$,

$$\begin{aligned}\mathcal{X} \longmapsto \{D\colon \mathcal{O}(\mathcal{X})[G] \to \mathcal{O}(\mathcal{X}) \in \mathcal{P}s\mathcal{R}^{\mathcal{FS}/W(\mathbb{F}),n}(A) \ : \ &\text{for all open affine } \mathcal{U} \subset \mathcal{X}\\ D \otimes_{\mathcal{O}(\mathcal{X})} \mathcal{O}(\mathcal{U}) \in \mathcal{P}s\mathcal{R}^{\mathcal{FS}/W(\mathbb{F}),n}(\mathcal{U}) = \mathcal{P}s\mathcal{R}^{\mathcal{A}dm,n}(\mathcal{O}(\mathcal{U}))\ &\text{lies in } \mathcal{P}s\mathcal{R}^{\mathcal{A}dm}_z(\mathcal{O}(\mathcal{U})\}.\end{aligned}$$

Chenevier notes that the restriction of $\mathcal{P}sD^{\mathcal{FS}/W(\mathbb{F})}_z$ to the full subcategory of affine formal schemes coincides with the opposite functor of $\mathcal{P}sD^{\mathcal{A}dm}_z$.

Proposition 3.1.68 ([Che14, Prop. 3.13],[WE13, Cor. 3.1.6.11])**.** *For $z \in |G(n)|$ the following hold:*

(i) $\mathcal{P}s\mathcal{R}^{\mathcal{A}dm}_z$ is representable by a local ring R^{univ}_z in $\mathcal{A}dm$.

(ii) The ring R^{univ}_z is canonically isomorphic to $R^{\mathrm{univ}}_{W(\mathbb{F}),D_z}$ from Proposition 3.1.57.

(iii) If D_z satisfies one of the conditions in Proposition 3.1.60, then R^{univ}_z lies in $\widehat{\mathcal{A}r}_{W(\kappa(z))}$.

Corollary 3.1.69 ([Che14, Cor. 3.14])**.** *Let G be a profinite group that satisfies Mazur's finiteness condition Φ_p from Definition 2.2.5. Then $\mathcal{P}s\mathcal{R}^{\mathcal{FS}/W(\mathbb{F})}_z$ is representable by $\operatorname{Spf}(R^{\mathrm{univ}}_z)$ and $\mathcal{P}s\mathcal{R}^{\mathcal{FS}/W(\mathbb{F}),n}$ is representable by the formal scheme*

$$\mathcal{X}^{\mathrm{univ}}_n := \coprod_{z \in |G(d)|} \operatorname{Spf}(R^{\mathrm{univ}}_z).$$

[4] Note that since unlike Chenevier our base ring is $W(\mathbb{F})$ and not $\mathbb{Z}_p$, the residue fields $\kappa(z)$ are all (finite) extensions of $\mathbb{F}$.

Lemma 3.1.70. *Let $\overline{D}\colon G \to \mathbb{F}$ be an n-dimensional pseudorepresentation. Then there are natural numbers r, m_i and n_i for $i = 1, \ldots, r$, field extensions $\mathbb{F}_i$ of $\mathbb{F}$ and irreducible pseudorepresentations $\overline{D}_i\colon G \to \mathbb{F}_i$ of dimension n_i over $\mathbb{F}_i$ for $i = 1, \ldots, r$, such that $n = \sum_i [\mathbb{F}_i : \mathbb{F}] m_i n_i$ and for $\mathbb{F}'$ the composition of the $\mathbb{F}_i$ in an algebraic closure $\mathbb{F}^{\mathrm{alg}}$ of $\mathbb{F}$ one has*

$$\overline{D} \otimes_{\mathbb{F}} \mathbb{F}' = (\overline{D}_1^{m_1} \otimes_{\mathbb{F}_1} \mathbb{F}') \oplus \ldots \oplus (\overline{D}_r^{m_r} \otimes_{\mathbb{F}_r} \mathbb{F}'). \tag{5}$$

In particular, $[\mathbb{F}' : \mathbb{F}]$ divides $n!$ and each $\overline{D}_i$ can be defined over $\mathbb{F}'$.

Proof. By Theorem 3.1.28 there is an integer $r \in \mathbb{N}_{\geq 1}$, simple $\mathbb{F}$-algebras S_i of finite dimension n_i^2 over its center $\mathbb{F}_i$ for $i = 1, \ldots, r$ and an $\mathbb{F}$-algebra isomorphism

$$\mathbb{F}[G]/\ker \overline{D} \xrightarrow{\sim} \prod_{i=1}^{r} S_i$$

such that the following holds because $\mathbb{F}$ is finite and hence perfect: the algebra S_i is finite-dimensional over $\mathbb{F}$, and hence finite, and hence isomorphic to $\mathrm{Mat}_{n_i}(\mathbb{F}_i)$; the fields $\mathbb{F}_i$ are finite and hence finite separable over $\mathbb{F}$; the q_i of loc.cit. are therefore all equal to 1; one has $\overline{D} = \bigoplus_{i=1}^{r} \det_{S_i}^{m_i}$ for unique $m_i \in \mathbb{N}$; finally with $f_i = [\mathbb{F}_i : \mathbb{F}]$ one has $n = \sum_i f_i n_i m_i$.

Let $\mathbb{F}'$ be the compositum of the $\mathbb{F}_i$ in an algebraic closure $\mathbb{F}^{\mathrm{alg}}$ of $\mathbb{F}$. By the above formula, $f_i \leq n$ for all i and hence $[\mathbb{F}' : \mathbb{F}]$ divides $n!$. Let $\overline{D}_i := \det_{S_i}$. Then $\overline{D}_i$ is an $n_i f_i$ dimensional pseudorepresentation over $\mathbb{F}$ that is split over $\mathbb{F}_i$ and hence over $\mathbb{F}'$. We obtain

$$\overline{D} \otimes_{\mathbb{F}} \mathbb{F}'\colon \mathbb{F}'[G] \longrightarrow \left(\mathbb{F}[G]/\ker \overline{D}\right) \otimes_{\mathbb{F}} \mathbb{F}' \xrightarrow{\sim} \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\mathbb{F}') \xrightarrow{\bigoplus_i \overline{D}_i^{m_i}} \mathbb{F}',$$

and this yields (5). The other claims are clear by construction. □

Corollary 3.1.71. *There exist only finitely many continuous pseudorepresentations $\overline{D}\colon G_K \to \mathbb{F}$ of dimension n.*

If moreover $\mathbb{F}' \supset \mathbb{F}$ denotes the unique field extension of degree $n!$, then $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}'$ for any $\overline{D}$ as above is a direct sum of irreducible pseudorepresentations $\overline{D}_i\colon G_K \to \mathbb{F}'$.

Proof. The second part is immediate from Lemma 3.1.70. Hence it suffices to prove the first part for irreducible $\overline{D}$. Let $\mathbb{F}^{\mathrm{alg}}$ be an algebraic closure of $\mathbb{F}$ and denote by $\rho_{\overline{D}}\colon G_K \to \mathrm{GL}_n(\mathbb{F}^{\mathrm{alg}})$ the absolutely irreducible representation attached to $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}^{\mathrm{alg}}$ by Theorem 3.1.26. Since the traces of $\rho_{\overline{D}}$ lie in $\mathbb{F}$, the field $\mathbb{F}$ is its field of definition, and so we may assume that $\rho_{\overline{D}}$ takes values in $\mathrm{GL}_n(\mathbb{F})$. By Lemma 2.2.23 there are only finitely many absolutely irreducible representations $G_K \to \mathrm{GL}_n(\mathbb{F})$, and this completes the proof of the first part. □

Recall from Corollary 3.1.31 for $n_1, n_2 \in \mathbb{N}_0$ the morphism

$$\iota_{n_1,n_2}\colon X^{\mathrm{univ}}_{\mathbb{F}[G],n_1} \times_{\mathrm{Spec}\,\mathbb{F}} X^{\mathrm{univ}}_{\mathbb{F}[G],n_2} \longrightarrow X^{\mathrm{univ}}_{\mathbb{F}[G],n_1+n_2}$$

of affine $\mathbb{F}$-schemes defined by $(D_1, D_2) \mapsto D_1 \oplus D_2$ that corresponds to the $\mathbb{F}$-algebra homomorphism

$$\Gamma^{n_1+n_2}_{\mathbb{F}}(\mathbb{F}[G])^{\mathrm{ab}} \xrightarrow{\Gamma^{n_1+n_2}_{\mathbb{F}}(\mathrm{diag})} \Gamma^{n_1+n_2}_{\mathbb{F}}(\mathbb{F}[G] \times \mathbb{F}[G])^{\mathrm{ab}} \twoheadrightarrow \Gamma^{n_1}_{\mathbb{F}}(\mathbb{F}[G])^{\mathrm{ab}} \otimes_{\mathbb{F}} \Gamma^{n_2}_{\mathbb{F}}(\mathbb{F}[G])^{\mathrm{ab}}.$$

When we prove in Section 3.3 the equidimensionality of the universal mod p pseudodeformation ring of a residual pseudorepresentation of dimension $n = n_1 + n_2$ inductively, we make use of the homomorphism induced by ι_{n_1,n_2} on the complete and local universal pseudodeformation rings: For this, we consider for all $n \in \mathbb{N}_0$ the special fiber $\overline{\mathcal{X}}_n^{\mathrm{univ}} := \mathcal{X}_n^{\mathrm{univ}} \times_{\mathrm{Spf}\, W(\mathbb{F})} \mathrm{Spf}\, \mathbb{F}$ of the formal scheme $\mathcal{X}_n^{\mathrm{univ}}$ from Corollary 3.1.69, and the induced morphism

$$\iota_{n_1,n_2} \colon \overline{\mathcal{X}}_{n_1}^{\mathrm{univ}} \times_{\mathrm{Spf}\, \mathbb{F}} \overline{\mathcal{X}}_{n_2}^{\mathrm{univ}} \longrightarrow \overline{\mathcal{X}}_{n_1+n_2}^{\mathrm{univ}}$$

of formal $\mathbb{F}$-schemes defined by $(D_1, D_2) \mapsto D_1 \oplus D_2$. Recall from [EGA I, § 10.7] that the fiber product of two affine formal schemes $\mathrm{Spf}\, A$ and $\mathrm{Spf}\, B$ over $\mathbb{F}$ is $\mathrm{Spf}(A \hat{\otimes}_{\mathbb{F}} B)$.

Now Corollary 3.1.69 and Corollary 3.1.71 yield the following.

Corollary 3.1.72 (Cf. [Che14, Cor. 3.14])**.** *Let $\overline{D} \colon G \to \mathbb{F}$ be a pseudorepresentation of dimension $n = n_1 + n_2$ valued in a finite field $\mathbb{F}$, $\overline{\mathcal{X}}_{\overline{D}}^{\mathrm{univ}} := \mathrm{Spf}\, \overline{R}_{\overline{D}}^{\mathrm{univ}}$ and*

$$\overline{\mathcal{X}}_{\overline{D},n_1,n_2}^{\mathrm{univ}} := \overline{\mathcal{X}}_{\overline{D}}^{\mathrm{univ}} \times_{\overline{\mathcal{X}}_n^{\mathrm{univ}}} (\overline{\mathcal{X}}_{n_1}^{\mathrm{univ}} \times_{\mathrm{Spf}\, \mathbb{F}} \overline{\mathcal{X}}_{n_2}^{\mathrm{univ}}) = \iota_{n_1,n_2}^{-1}(\overline{\mathcal{X}}_{\overline{D}}^{\mathrm{univ}})$$

Then there is a finite extension $\mathbb{F}'$ of $\mathbb{F}$ such that

$$\overline{\mathcal{X}}_{\overline{D},n_1,n_2}^{\mathrm{univ}} \times_{\mathrm{Spf}\, \mathbb{F}} \mathrm{Spf}\, \mathbb{F}' \cong \coprod_{\overline{D}_1,\overline{D}_2 \,:\, \overline{D}_i \in \overline{\mathcal{X}}_{n_i}^{\mathrm{univ}}(\mathrm{Spf}\, \mathbb{F}') \text{ for } i=1,2 \text{ and } \overline{D}_1 \oplus \overline{D}_2 = \overline{D}} \mathrm{Spf}(\overline{R}_{\overline{D}_1}^{\mathrm{univ}} \hat{\otimes}_{\mathbb{F}'} \overline{R}_{\overline{D}_2}^{\mathrm{univ}}).$$

The disjoint union is over a finite index set. The morphism $\overline{\mathcal{X}}_{\overline{D},n_1,n_2}^{\mathrm{univ}} \to \overline{\mathcal{X}}_{\overline{D}}^{\mathrm{univ}}$ induced from i_{n_1,n_2} is a closed immersion if $\overline{D}$ is split and multiplicity free.

Proof. By what was said above, it remains to prove that the morphism $\overline{\mathcal{X}}_{\overline{D},n_1,n_2}^{\mathrm{univ}} \to \overline{\mathcal{X}}_{\overline{D}}^{\mathrm{univ}}$ is closed. For this, we may pass from $\mathbb{F}$ to $\mathbb{F}'$, and hence without loss of generality we assume $\mathbb{F}' = \mathbb{F}$. Now since the union is finite, it suffices to show that for each pair $\overline{D}_1, \overline{D}_2$ with $\overline{D}_1 \oplus \overline{D}_2 = \overline{D}$ the induced map of rings

$$\overline{R}_{\overline{D}}^{\mathrm{univ}} \longrightarrow \overline{R}_{\overline{D}_1}^{\mathrm{univ}} \hat{\otimes}_{\mathbb{F}'} \overline{R}_{\overline{D}_2}^{\mathrm{univ}}$$

is surjective. Since both are complete Noetherian local and have isomorphic residue field, it suffices to show the surjectivity for the induced map of the duals of their tangent spaces; i.e., the injectivity of

$$\mathcal{P}s D_{\overline{D}_1}(\mathbb{F}[\varepsilon]) \times \mathcal{P}s D_{\overline{D}_2}(\mathbb{F}[\varepsilon]) \longrightarrow \mathcal{P}s D_{\overline{D}}(\mathbb{F}[\varepsilon]), \quad (D_1, D_2) \longmapsto D_1 \oplus D_2. \tag{6}$$

Consider n_i-dimensional pseudodeformations $D_i, D_i' \in \mathcal{P}s D_{\overline{D}_i}(\mathbb{F}[\varepsilon])$ for $i = 1, 2$ such that $D_1 \oplus D_2 = D_1' \oplus D_2'$. By hypothesis, $\overline{D}_1 \oplus \overline{D}_2$ is split and multiplicity free so that we have isomorphisms

$$\mathbb{F}[\varepsilon][G]/\ker(\overline{D}_i) \cong \prod_{j=1}^{s_i} \mathrm{Mat}_{n_{i,j}}(\mathbb{F}) \qquad \text{with } \sum_{j=1}^{s_i} n_{i,j} = n_i \text{ for } i = 1, 2.$$

As discussed in the proof of [Che14, Thm. 2.22], we can lift the canonical family of central orthogonal idempotents of $\mathbb{F}[\varepsilon][G]/\ker(\overline{D}_i)$ to a family of orthogonal idempotents $e_{i,1} + \ldots + e_{i,s_i} = 1$ in $\mathbb{F}[\varepsilon][G]$, and we further have a family of A-algebra isomorphisms $\psi_{i,j} : e_{i,j}\mathbb{F}[\varepsilon][G]e_{i,j} \xrightarrow{\sim} \mathrm{Mat}_{n_{i,j}}(\mathbb{F}[\varepsilon])$ for $j = 1, \ldots, s_i$ and $i = 1, 2$. Putting this together, we obtain by Theorem 3.1.50 applied to D_i and D_i' that $(\mathbb{F}[\varepsilon][G], \mathcal{E}_i)$ is a generalized matrix algebra with data of idempotents $\mathcal{E}_i := \{e_{i,j}, \psi_{i,j}\}_{j=1,\ldots,r}$ and determinant $D_i = \det_{(\mathbb{F}[\varepsilon][G],\mathcal{E}_i)} = D_i'$ for $i = 1, 2$, which implies the assertion on the map (6). □

3.2 *Geometric loci of universal pseudodeformation spaces*

Recall that throughout the thesis we fix an algebraic closure K^{alg} of a finite extension K of $\mathbb{Q}_p$ of degree $d = [K : \mathbb{Q}_p]$ with absolute Galois group G_K, a primitive p^{th} root of unity ζ_p and a finite field $\mathbb{F}$ of characteristic p.

In this section, we start by showing openness of the irreducible locus of the special fiber of a universal pseudodeformation space following [Che14, Exmp. 2.20]. Chenevier's Corollary 3.2.13 says that the universal Cayley-Hamilton sheaf is an Azumaya algebra over the irreducible locus.

In Subsection 3.2.2 we define an induction for certain pseudorepresentations following ideas of Böckle: At first, the characteristic polynomial of an induced representation with values in an Azumaya algebra is described in Lemma 3.2.20. As the characteristic polynomial coefficients determine a pseudorepresentation by Proposition 3.1.14, Lemma 3.2.20 allows us to define an induced pseudorepresentation in Theorem 3.2.23 under Assumption 3.2.21.

In Subsection 3.2.3 we define the twist of a pseudorepresentation with a character and show that if $\zeta_p \notin K$ nonspecial irreducible points are regular and form open loci. If $\zeta_p \in K$, then the regular locus is empty and we instead consider regular points in the nilreduction if $n \nmid p$.

3.2.1 *The locus of irreducibility and the universal Cayley-Hamilton algebra*

In this subsection, we summarize properties of the locus of irreducible pseudodeformations in a universal deformation space. In particular, Chenevier shows that over this locus the universal Cayley-Hamilton algebra is an Azumaya algebra. We later investigate pseudodeformations with values in local fields corresponding to 1-dimensional points x in universal pseudodeformation spaces. We show in Corollary 3.2.13 that the (slightly modified) local rings at such points x are universal pseudodeformation spaces. Let now G denote a group.

Definition 3.2.1. Let $D^{\mathrm{univ}}\colon R^{\mathrm{univ}}_{G,n}[G] \to R^{\mathrm{univ}}_{G,n}$ be the universal n-dimensional pseudorepresentation and $X^{\mathrm{univ}}_{G,\overline{D}} := \operatorname{Spec} R^{\mathrm{univ}}_{G,\overline{D}}$ the universal n-dimensional pseudorepresentation space. Let $x \in X^{\mathrm{univ}}_{G,n}$ be a point and $f_x\colon R^{\mathrm{univ}}_{G,n} \to \kappa(x)$ be the morphism corresponding to x, where $\kappa(x)$ is the residue field of x with algebraic closure $\kappa(x)^{\mathrm{alg}}$.

(i) $D_x := f_x \circ D^{\mathrm{univ}}\colon G \overset{D^{\mathrm{univ}}}{\to} R^{\mathrm{univ}}_{G,n} \overset{f_x}{\to} \kappa(x)$ is called the *pseudorepresentation of* D^{univ} *at* x;

(ii) the representation $\rho_x := \rho_{D_x}\colon G \to \mathrm{GL}_n(\kappa(x)^{\mathrm{alg}})$ corresponding to $D_x \otimes_{\kappa(x)} \kappa(x)^{\mathrm{alg}}$ from Theorem 3.1.26 is called the *(semisimple) representation attached to* D_x;

(iii) We say that x and D_x have a property if this property holds for ρ_x.

(iv) The *irreducible locus* $(X^{\mathrm{univ}}_{G,n})^{\mathrm{irr}}$ *and the reducible locus* $(X^{\mathrm{univ}}_{G,n})^{\mathrm{red}}$ *in* $X^{\mathrm{univ}}_{G,n}$ consists of the points with the respective property in $X^{\mathrm{univ}}_{G,n}$; cf. Definition 3.1.27. The same notation is used for other spaces parameterizing pseudorepresentations, such as the universal pseudodeformation space $X^{\mathrm{univ}}_{\overline{D}}$ of a residual psudorepresentation $\overline{D}$.

Lemma 3.2.2 (Cf. [Che14, Example 2.20.])**.** *Consider an n-dimensional residual pseudorepresentation $\overline{D}\colon G \to \mathbb{F}$ with its universal pseudodeformation $D^{\mathrm{univ}}_{\overline{D}}$.*

Then the subsets $(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \subset X^{\mathrm{univ}}_{\overline{D}}$ and $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ are Zariski open.

Proof. The first assertion is proved in [Che14, Example 2.20.], the second follows from the first since

$$(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} = \overline{X}^{\mathrm{univ}}_{\overline{D}} \cap (X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \subset X^{\mathrm{univ}}_{\overline{D}}.$$

Since the proof in [Che14, Example 2.20.] is somewhat sketchy, we give a complete direct proof of the second result, following loc.cit., which is the assertion of main interest to us: For any sequence of elements $(g_1, \dots, g_{n^2}) \in G$, one considers its discriminant

$$\Delta(g_1, \dots, g_{n^2}) := \det \begin{pmatrix} \tau_{\overline{D}}^{\mathrm{univ}}(g_1 g_1) & \cdots & \tau_{\overline{D}}^{\mathrm{univ}}(g_1 g_{n^2}) \\ \vdots & \ddots & \vdots \\ \tau_{\overline{D}}^{\mathrm{univ}}(g_{n^2} g_1) & \cdots & \tau_{\overline{D}}^{\mathrm{univ}}(g_{n^2} g_{n^2}) \end{pmatrix} \in \overline{R}_{\overline{D}}^{\mathrm{univ}},$$

where $\tau_{\overline{D}}^{\mathrm{univ}}$ denotes the trace of the universal pseudodeformation $D_{\overline{D}}^{\mathrm{univ}}$ (see e.g. [Nak00, Rem. 3.2] why this is the (square of the) discriminant up to sign).

Let $x \in \overline{X}_{\overline{D}}^{\mathrm{univ}}$ be a point corresponding to a prime $\mathfrak{p}_x \subset \overline{R}_{\overline{D}}^{\mathrm{univ}}$ and $\kappa(x)$ be its residue field $\mathcal{O}_{\overline{X}_{\overline{D}}^{\mathrm{univ}},x}/\mathfrak{p}_x\mathcal{O}_{\overline{X}_{\overline{D}}^{\mathrm{univ}},x}$. Consider the pseudorepresentation $D_x := D_{\overline{D}}^{\mathrm{univ}} \otimes_{\overline{R}_{\overline{D}}^{\mathrm{univ}}} \kappa(x)$. Let $\rho_x \colon G \to \mathrm{GL}_n(\kappa(x)^{\mathrm{alg}})$ be the representation attached to $D_{\overline{D}}^{\mathrm{univ}}$ at x. Let $r_x \colon \kappa(x)^{\mathrm{alg}}[G] \to \mathrm{Mat}_n(\kappa(x)^{\mathrm{alg}})$ be the induced representation of the group algebra $\kappa(x)^{\mathrm{alg}}[G]$. If ρ_x is absolutely irreducible, then r_x is surjective [CR62, Burnside's theorem (27.4)]. Conversely, if ρ_x is reducible, then $\mathrm{im}\,(\rho_x)$ lies in a parabolic subgroup of $\mathrm{GL}_n(\kappa(x)^{\mathrm{alg}})$ and so $\mathrm{im}\,(r_x)$ is properly contained in $\mathrm{Mat}_n(\kappa(x)^{\mathrm{alg}})$. If $(g_1, \dots, g_{n^2}) \in G$ is an arbitrary sequence of elements, then $r_x(g_1), \dots, r_x(g_{n^2})$ generate $\mathrm{im}\,(r_x) = \mathrm{Mat}_n(\kappa(x)^{\mathrm{alg}})$ if and only if $\Delta(g_1, \dots, g_{n^2}) \neq 0$ since the trace $\mathrm{tr} : \mathrm{Mat}_n(\kappa(x)^{\mathrm{alg}}) \to \kappa(x)^{\mathrm{alg}}$ is nondegenerate.

Define $I \subset \overline{R}_{\overline{D}}^{\mathrm{univ}}$ as the ideal generated by $\Delta(g_1, \dots, g_{n^2})$ for all $(g_1, \dots, g_{n^2}) \in G$. Then $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} = \overline{X}_{\overline{D}}^{\mathrm{univ}} \smallsetminus V(I)$ is a Zariski open. □

Definition 3.2.3. Let $D^{\mathrm{univ}} \colon R_{G,n}^{\mathrm{univ}}[G] \to R_{G,n}^{\mathrm{univ}}$ be the universal n-dimensional pseudorepresentation from Definition 3.1.24. (Note that here G is considered as a discrete group.)

(i) The n-dimensional Cayley-Hamilton $R_{G,n}^{\mathrm{univ}}$-algebra $S_{G,n}^{\text{CH-univ}} := R_{G,n}^{\mathrm{univ}}[G]/\,\mathrm{CH}(D_n^{\mathrm{univ}})$ is called the *universal Cayley-Hamilton algebra* and the natural representation

$$\rho^{\text{CH-univ}} \colon G \longrightarrow (S_{G,n}^{\text{CH-univ}})^{\times}$$

the *universal Cayley-Hamilton representation.*

(ii) The quasi-coherent sheaf $\mathcal{S}_{G,n}^{\text{CH-univ}}$ of Cayley-Hamilton algebras on $X_{G,n}^{\mathrm{univ}}$ defined by the universal Cayley-Hamilton algebra $S_{G,n}^{\text{CH-univ}}$ is called the *universal Cayley-Hamilton sheaf.*

Remark 3.2.4 ([Che14, § 1.22]). (i) Chenevier also introduces the notion of a Cayley-Hamilton representation and shows in [Che14, Prop. 1.23] that the initial object of the category of Cayley-Hamilton representations is given by the triple $\big(R_{G,n}^{\mathrm{univ}}, (S_{G,n}^{\text{CH-univ}}, D^{\mathrm{univ}}), \rho^{\text{CH-univ}}\big)$.

(ii) If $D \colon A[G] \to A$ is a pseudorepresentation defined by a morphism $\mathrm{Spec}(A) \to X_{G,n}^{\mathrm{univ}}$, then there need not be a representation $\rho : A[G] \to \mathrm{Mat}_n(A)$ with attached pseudorepresentation D since e.g. in the settings of pseudocharacters this implies that A is factorial [BC09, Thm. 1.6.3]). The universal Cayley-Hamilton representation is a natural candidate for a substitute.

(iii) The formation of the universal Cayley-Hamilton algebra commutes with arbitrary base change; i.e., for any morphism $f \colon \mathrm{Spec}(A) \to X_{G,n}^{\mathrm{univ}}$ with corresponding pseudorepresentation $D_f \colon A[G] \to A$ the natural surjective map $A[G] \to \mathcal{S}_{G,n}^{\text{CH-univ}} \otimes_{R_{G,n}^{\mathrm{univ}},f} A$ provides an isomorphism

$$A[G]/\,\mathrm{CH}(D_f) \xrightarrow{\sim} \mathcal{S}_{G,n}^{\text{CH-univ}} \otimes_{R_{G,n}^{\mathrm{univ}},f} A.$$

One has the following important result on $\mathcal{S}^{\text{CH-univ}}_{G,n}$ over the irreducible locus of $X^{\text{univ}}_{G,n}$ which is derived from Theorem 3.1.50.

Lemma 3.2.5 (Cf. [Che14, Cor. 2.23]). *The $\mathcal{O}_{(X^{\text{univ}}_{G,n})^{\text{irr}}}$-algebra $\mathcal{S}^{\text{CH-univ}}_{G,n} \otimes_{\mathcal{O}_{X^{\text{univ}}_{G,n}}} \mathcal{O}_{(X^{\text{univ}}_{G,n})^{\text{irr}}}$ is an Azumaya algebra of rank n^2 equipped with its reduced norm.*

Remark 3.2.6. We are not clear whether [Che14, Cor. 2.23(ii)] is correct as stated there. We think that this is only the case if $\kappa(x)$ is finite or finite over $\mathbb{Q}_p$. We give a formulation in our context below in Corollary 3.2.13. We think that [Che14, Cor. 2.23(ii)] has to be modified in a similar way.

Remark 3.2.7. The proof of Corollary 3.2.13(i) in fact shows the following. Let S be an A-algebra and let $D\colon S \to A$ be an n-dimensional Cayley-Hamilton representation. If D_x is irreducible for all $x \in \operatorname{Spec} A$, then S is an Azumaya algebra over A of rank n^2 and D is equal to the reduced norm $\det_S$ of S recalled in Example 3.1.8.

In our applications, we will need $\mathcal{S}^{\text{CH-univ}}_{G,n}$ in a profinite context. We follow [WE17].

Definition 3.2.8. Let G be a profinite group. Let $D^{\text{univ}}_{\overline{D}}\colon G \to R^{\text{univ}}_{G,\overline{D}}$ be the universal pseudodeformation of a residual pseudorepresentation $\overline{D}\colon G \to \mathbb{F}$. Recall $R^{\text{univ}}_{G,\overline{D}}[\![G]\!]$ from Example 3.1.52.

(i) The n-dimensional Cayley-Hamilton $R^{\text{univ}}_{G,\overline{D}}$-algebra

$$S^{\text{CH-univ}}_{G,\overline{D}} := R^{\text{univ}}_{G,\overline{D}}[\![G]\!]/\operatorname{CH}(D^{\text{univ}}_{\overline{D}})$$

is called the *universal Cayley-Hamilton algebra of $\overline{D}$*, and we write $D^{\text{CH-univ}}_{G,\overline{D}}$ for the pseudorepresentation of $S^{\text{CH-univ}}_{G,\overline{D}}$ induced from $D^{\text{univ}}_{\overline{D}}$.

(ii) The quasi-coherent sheaf $\mathcal{S}^{\text{CH-univ}}_{G,\overline{D}}$ of Cayley-Hamilton algebras on $X^{\text{univ}}_{G,\overline{D}}$ defined by $S^{\text{CH-univ}}_{G,\overline{D}}$ is called the *universal Cayley-Hamilton sheaf of $\overline{D}$*.

Note that because of Remark 3.2.4(iii) the algebra $S^{\text{CH-univ}}_{G,\overline{D}}$ is isomorphic to the profinite completion of $\mathcal{S}^{\text{CH-univ}}_{G,n} \otimes_{R^{\text{univ}}_{G,n}} R^{\text{univ}}_{G,\overline{D}}$. The following result summarizes the basic properties of $S^{\text{CH-univ}}_{G,\overline{D}}$:

Proposition 3.2.9 ([WE17, Prop. 3.6]). *Let $\overline{D}\colon G \to \mathbb{F}$ be a residual pseudorepresentation of a profinite group G and suppose that $\dim_{\kappa(\overline{D})} H^1_{\text{cont}}(G, \operatorname{ad}_{\rho_{\overline{D}}}) < \infty$, where the associated semisimple representation $\rho_{\overline{D}}$ is defined over a finite extension $\kappa(\overline{D})$ of $\mathbb{F}$ by Theorem 3.1.28. Then the following hold:*

(a) The natural quotient map $\pi\colon R^{\text{univ}}_{G,\overline{D}}[\![G]\!] \longrightarrow S^{\text{CH-univ}}_{G,\overline{D}}$ is continuous.

(b) $S^{\text{CH-univ}}_{G,\overline{D}}$ is module-finite as an $R^{\text{univ}}_{G,\overline{D}}$-algebra, and therefore Noetherian.

(c) On $S^{\text{CH-univ}}_{G,\overline{D}}$ the profinite topology, the $\mathfrak{m}_{\overline{D}}$-adic topology, and the quotient topology from the surjection π are equivalent.

(d) When $\overline{D}$ is multiplicity-free, $S^{\text{CH-univ}}_{G,\overline{D}}$ is a generalized matrix algebra with canonical pseudorepresentation equivalent to its determinant (cf. Definition 3.1.47).

(e) When $\overline{D}$ is irreducible, $S^{\text{CH-univ}}_{G,\overline{D}} = \operatorname{Mat}_n(R^{\text{univ}}_{G,\overline{D}})$.

The following result is a generalization of Proposition 3.1.54.

Corollary 3.2.10. *Let $\overline{D}\colon G \to \mathbb{F}$ be a residual pseudorepresentation of a profinite group G and suppose that $\dim_{\kappa(\overline{D})} H^1_{\mathrm{cont}}(G, \mathrm{ad}_{\rho_{\overline{D}}}) < \infty$. Let $x \in X^{\mathrm{univ}}_{G,\overline{D}}$ be a point of dimension 1 so that $\kappa(x)$ is a local field. Then ρ_{D_x} is continuous.*

Proof. Let $\mathfrak{p}_x \subset R^{\mathrm{univ}}_{G,\overline{D}}$ be the prime ideal corresponding to x and denote by R_x and S_x the respective reductions mod $\mathfrak{p}_x$ of $R^{\mathrm{univ}}_{G,\overline{D}}$ and $S^{\mathrm{CH\text{-}univ}}_{G,\overline{D}}$, and set $S(x) := S_x \otimes_{R_x} \kappa(x)$. By Proposition 3.1.16 the pseudorepresentation D_x factors via $G \to S(x) \to \kappa(x)$, via a pseudorepresentation $\widetilde{D}_x\colon S(x) \to \kappa(x)$. Because of our assumption, the ring $S(x)$ is finite-dimensional as a $\kappa(x)$-vector space by Proposition 3.2.9. It is a topological vector space for the natural topology of $\kappa(x)$, since its topology is induced from the $\mathfrak{m}_{\overline{D}}$-adic topology on $R^{\mathrm{univ}}_{G,\overline{D}}$; the homomorphism $\psi_x\colon G \to S(x)^\times$ and the pseudorepresentations $\widetilde{D}_x$ are continuous. As recalled at the beginning of Section 2.2, this topology is unique and every sub $\kappa(x)$-vector space of $S(x)$ is closed; this still holds after base change to $\kappa(x)^{\mathrm{alg}}$. Set $S(x)^{\mathrm{ss}} := S(x) \otimes_{\kappa(x)} \kappa(x)^{\mathrm{alg}} / \ker(\widetilde{D}_x \otimes_{\kappa(x)} \kappa(x)^{\mathrm{alg}})$. It follows that the induced homomorphism

$$\rho\colon G \to (S(x)^{\mathrm{ss}})^\times$$

and the pseudorepresentation

$$S(x)^{\mathrm{ss}} \to \kappa(x)^{\mathrm{alg}},$$

which we also denote by $\widetilde{D}_x$, are continuous. It follows from Theorem 3.1.28 that $S(x)^{\mathrm{ss}} \cong \prod_{i=1}^{s} \mathrm{Mat}_{n_i}(\kappa(x)^{\mathrm{alg}})$ and that $\widetilde{D}_x$ is of the form $\prod_i \det^{m_i}_{\mathrm{Mat}_{n_i}(\kappa(x)^{\mathrm{alg}})}$ for suitable integers $n_i, m_i > 0$. It follows that $\widetilde{D}_x \circ \rho$ is attached to a continuous semisimple finite-dimensional representation of G defined over $\kappa(x)$. By uniqueness of semisimple representations over fields having both the same pseudorepresentation, it follows that ρ_{D_x} and $\widetilde{D}_x \circ \rho$ are isomorphic over $\kappa(x)^{\mathrm{alg}}$, and this proves the continuity of ρ_{D_x}. ☐

For later use, we deduce the following result:

Lemma 3.2.11. *Let G be a profinite group, let L be a local field with valuation ring $\mathcal{O}_L$, and let $D\colon G \to L$ be a continuous n-dimensional pseudorepresentation. Then the following hold, where in (ii) and (iii) we assume that G satisfies Condition Φ_p from Definition 2.2.5.:*

(i) D takes values in $\mathcal{O}_L$, and $D\colon G \to \mathcal{O}_L$ is residually equal to $\overline{D} := D \otimes_{\mathcal{O}_L} \kappa(\mathcal{O}_L)$.

(ii) The representation $\rho_D\colon G \to \mathrm{GL}_n(L^{\mathrm{alg}})$ from Theorem 3.1.26 is continuous.

(iii) If D is reducible, then there exists a finite extension L'/L and irreducible pseudorepresentations $D_1, \ldots, D_r\colon G_K \to \mathcal{O}_{L'}$ such that

$$D \otimes_{\mathcal{O}_L} \mathcal{O}_{L'} = D_1 \oplus \ldots \oplus D_r \tag{7}$$

Hence $D \otimes_{\mathcal{O}_L} \kappa(\mathcal{O}_{L'}) = \overline{D}_1 \oplus \ldots \oplus \overline{D}_r$ and $\overline{D}_i := D_i \otimes_{\mathcal{O}_{L'}} \kappa(\mathcal{O}_{L'})$ for $1 \le i \le r$.

Proof. Let ρ_D be the representation from Theorem 3.1.26 attached to $D \otimes_L L^{\mathrm{alg}}$. For (i) observe first that the characteristic polynomial coefficients $\Lambda_{D,i}$ of $\chi_D(g, \cdot)$ are continuous for $1 \le i \le n$, and hence the sets $\Lambda_{D,i}(G)$ are compact in L. Assume that for some $g \in G$, $\Lambda_{D,i}(g)$ does not lie in $\mathcal{O}_L$. Then at least one eigenvalue of $\rho_D(g)$ in L^{alg} has valuation different from 0, and, since we can pass to g^{-1}, we may assume that this valuation is negative. Let $\lambda_1, \ldots, \lambda_n \in L^{\mathrm{alg}}$ denote the eigenvalues of $\rho_D(g)$ and index them so that $\lambda_1, \dots, \lambda_j$ are precisely those with negative valuation. Then for $n > 0$, the valuation of $\Lambda_{D,j}(g^n)$ is the valuation of $(\lambda_1 \cdot \ldots \cdot \lambda_j)^n$. The latter valuations are unbounded. This contradicts the compactness of $\Lambda_{D,j}(G)$, proving the first part of (i).

To see the second part of (i), let $C \subset \mathcal{O}_L$ be the smallest closed $W(\kappa(\mathcal{O}_L))$-subalgebra generated by the characteristic polynomial coefficients of D. Its residue field must contain $\kappa(\mathcal{O}_L)$. Hence if we write $C = \lim_n C_n$ with C_n the finite image of C in $\mathcal{O}_L/(z^{n+1})$, we have $C_0 = \kappa(\mathcal{O}_L)$ and so D is residually constant and residually equal to $D \otimes_{\mathcal{O}_L} \kappa(\mathcal{O}_L) = D \otimes_C C_0$.

We now prove (ii). By (i) there is a continuous homomorphism $R^{\mathrm{univ}}_{G,\overline{D}} \to \mathcal{O}_L$, that induces D. Hence $\rho_D \cong \rho_x$ for the corresponding point x of $X^{\mathrm{univ}}_{G,\overline{D}}$. Since L is a local field containing $\kappa(x)$, the dimension of x is at most 1. If $\dim x = 1$, we deduce (ii) from Corollary 3.2.10, because $\dim_{\kappa(\overline{D})} H^1_{\mathrm{cont}}(G, \mathrm{ad}_{\rho_{\overline{D}}}) < \infty$ is implied by the Condition Φ_p. If $\dim x = 0$, then (ii) follows from Proposition 3.1.54.

For (iii) write $\rho_D = \bigoplus_{i=1}^r \rho_i$ for irreducible representations ρ_i of G_K. Because ρ_D is continuous, so are the ρ_i. Let $D_1, \dots, D_r$ be the continuous pseudorepresentations associated with $\rho_1, \dots, \rho_r$, and now (iii) is straightforward from (i). □

Remark 3.2.12. It would be nice to have a more direct and possibly simpler proof of the continuity assertion in part (ii) of the previous lemma.

We now give an analog of Lemma 3.2.5 in a topological context, and a local consequence. The local consequence asserts that for an equi-characteristic dimension 1 point x the completion of (a modification of) $\mathcal{O}_{X^{\mathrm{univ}}_{G,\overline{D}},x}$ at x has itself an interpretation as a universal pseudodeformation ring.

Corollary 3.2.13 (Cf. [Che14, Cor. 2.23]). *Let $\overline{D}\colon G \to \mathbb{F}$ be an n-dimensional pseudorepresentation of a profinite group G.*

(i) Over the locus $(X^{\mathrm{univ}}_{G,\overline{D}})^{\mathrm{irr}}$, the $\mathcal{O}_{X^{\mathrm{univ}}_{G,\overline{D}}}$-algebra $\mathcal{S}^{\text{CH-univ}}_{G,n} \otimes \mathcal{O}_{X^{\mathrm{univ}}_{G,\overline{D}}}$ is an Azumaya $\mathcal{O}_{X^{\mathrm{univ}}_{G,\overline{D}}}$-algebra of rank n^2 equipped with its reduced norm.

(ii) Let $x \in X^{\mathrm{univ}}_{G,n}$ be such that $\kappa(x)$ is a local field. Denote by $\pi'_x\colon R^{\mathrm{univ}}_{\mathbb{Z}[G],n} \to \kappa(x)$ the corresponding residue map, and by

$$\pi' := \mathrm{id}_{\kappa(x)} \otimes \pi'_x \colon R' := \kappa(x) \otimes_{\mathbb{Z}} R^{\mathrm{univ}}_{\mathbb{Z}[G],n} \longrightarrow \kappa(x)$$

the induced surjection. Let D'_x be the pseudorepresentation $G \to \kappa(x), g \mapsto 1 \otimes_{W(\mathbb{F})} D_x(g)$. By Proposition 3.1.59, the completion of R' at the D'_x-open ideals represents the universal pseudodeformation ring $R^{\mathrm{univ}}_{D'_x}$ for pseudodeformations of D'_x. There also is a residue map $\pi_x\colon \mathcal{O}_{X^{\mathrm{univ}}_{G,\overline{D}},x} \to \kappa(x)$ and a second canonical surjection

$$\pi := \mathrm{id}_{\kappa(x)} \otimes \pi_x \colon R := \kappa(x) \otimes_{W(\mathbb{F})} \mathcal{O}_{X^{\mathrm{univ}}_{G,\overline{D}},x} \longrightarrow \kappa(x).$$

Write $\widehat{R}^{\mathfrak{p}}$ for the completion of R at $\mathfrak{p} := \ker \pi$. Then for any $\mathfrak{p}$-primary ideal $I \subset \widehat{R}^{\mathfrak{p}}$ the induced pseudorepresentation $G \to \widehat{R}^{\mathfrak{p}}/I$ is continuous, so that one has a natural homomorphism $R^{\mathrm{univ}}_{D'_x} \to \widehat{R}^{\mathfrak{p}}$. The latter map is an isomorphism.

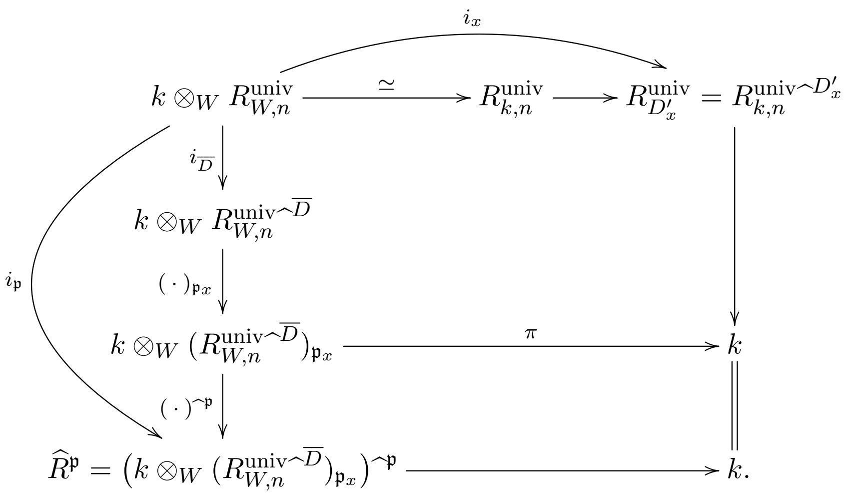
(iii) Suppose that x in (ii) is split and irreducible, and that $\dim_{\kappa(\overline{D})} H^1_{\mathrm{cont}}(G, \mathrm{ad}_{\rho_{\overline{D}}}) < \infty$. Denote by $\rho_x\colon G \to \mathrm{GL}_n(\kappa(x))$ the representation attached to D_x, and by $R^{\mathrm{univ}}_{\rho_x}$ the universal deformation ring for deformations to $\mathcal{A}r_{\kappa(x)}$ of ρ_x. Then the natural map $R^{\mathrm{univ}}_{D'_x} \to R^{\mathrm{univ}}_{\rho_x}$ induced from $\rho \mapsto \det_\rho$ is an isomorphism.

Proof. The proof of (i) is exactly as that of [Che14, Cor. 2.23(i)]: Let x be in $(X^{\mathrm{univ}}_{G,\overline{D}})^{\mathrm{irr}}$, and let A be the strict Henselianization of the local ring $\mathcal{O}_x$ at x. Recall that the formation of the Cayley-Hamilton quotient commutes with arbitrary base change. In particular,

$$S^{\text{CH-univ}}_{G,\overline{D}} \otimes_{R^{\mathrm{univ}}_{G,\overline{D}}} A \cong A[\![G]\!]/\,\mathrm{CH}(D^{\text{CH-univ}}_{G,\overline{D}} \otimes A)$$

Theorem 3.1.50(3.1.50) shows that the A-algebra on the right side is isomorphic to $\mathrm{Mat}_n(A)$ for some $n \in \mathbb{N}_{\geq 1}$, thus $S^{\text{CH-univ}}_{G,\overline{D}} \otimes_{R^{\mathrm{univ}}_{G,\overline{D}}} \mathcal{O}_x$ is an Azumaya algebra of rank n^2, as $\mathcal{O}_x \to A$ is faithfully flat. Part (i) follows then from the following abstract result: Let C be a commutative ring, $n \geq 1$ an integer, and R a C-algebra. Assume that for all $x \in \operatorname{Spec} C$, the localization R_x is an Azumaya algebra of rank n^2 over C_x. Then R is an Azumaya C-algebra (locally free) of rank n^2.

For (ii) we shall use the diagram below for which we need to introduce some notation. We write W for $W(\mathbb{F})$ and k for $\kappa(x)$. For any commutative ring A, we set $R^{\mathrm{univ}}_{A,n} := R^{\mathrm{univ}}_{A[G],n}$, which is naturally isomorphic to $A \otimes_{\mathbb{Z}} R^{\mathrm{univ}}_{G,n}$. The symbol ${}^{\frown\overline{D}}$ denotes the completion of a ring at its $\overline{D}$-open ideals, cf. Definition 3.1.55, and similarly ${}^{\frown D'_x}$ for the completion at the D'_x-open ideals – when this makes sense. Then $R^{\mathrm{univ}}_{G,\overline{D}} = R^{\mathrm{univ}\frown\overline{D}}_{W,n}$, and the universal pseudodeformation ring for continuous pseudodeformations of D'_x is $R^{\mathrm{univ}}_{D'_x} = R^{\mathrm{univ}\frown D'_x}_{k,n}$. Let $\mathfrak{p}_x$ denote the kernel of the homomorphism $R^{\mathrm{univ}}_{G,\overline{D}} \to k$ corresponding to D'_x, and $\mathfrak{p}$ the kernel of $\pi\colon R \to k$ or of π restricted to $k \otimes_W R^{\mathrm{univ}}_{G,\overline{D}}$, and write ${}^{\frown\mathfrak{p}}$ for the completion at $\mathfrak{p}$ and similarly ${}^{\frown\mathfrak{p}_x}$ for that at $\mathfrak{p}_x$. Then we have the following diagram:



Let $\varphi_k\colon k \otimes_W R^{\mathrm{univ}}_{W,n} \to k$ be the diagonal homomorphism from the top left to the bottom right. To show the assertion of (ii), let A be in $\mathcal{A}r_k$ with residue homomorphism $\psi\colon A \to k$, and let $\varphi_A\colon k \otimes_W R^{\mathrm{univ}}_{W,n} \to A$ be a surjective homomorphism with $\psi \circ \varphi_A = \varphi_k$. Let $D_A\colon G \to A$ be the induced pseudorepresentation. We need to show that φ_A factors via i_x if and only if it factors via $i_{\mathfrak{p}}$.

Note first that from the definition of ${}^{\frown\mathfrak{p}}$ it is clear that φ_A factors via $i_{\mathfrak{p}}$ if and only if it factors via $(\,\cdot\,)_{\mathfrak{p}_x} \circ i_{\overline{D}}$. Since φ_k maps the elements of $R^{\mathrm{univ}}_{W,n} \smallsetminus \mathfrak{p}_x$ to units in k, so does φ_A since A is local with residue field k. Hence we need to show that φ_A factors via $i_{\mathfrak{p}}$ if and only if it factors via $i_{\overline{D}}$. Let I be the kernel of the compositum of φ_A with $R^{\mathrm{univ}}_{W,n} \to k \otimes_W R^{\mathrm{univ}}_{W,n}, r \mapsto 1 \otimes r$, and let $D_I\colon G \to R^{\mathrm{univ}}_{W,n}/I$ be the induced pseudorepresentation. Because of $\psi \circ \varphi_A = \varphi_k$ we have

$I \subset \mathfrak{p}_x$, and because A is Artinian it follows that $\mathfrak{p}_x$ is the radical of I. Let $\mathfrak{m}_{\overline{D}}$ be the kernel of $R^{\mathrm{univ}}_{W,n} \to \mathbb{F}_{\overline{D}}$ given by $\overline{D}$. That φ_A factors via $i_{\overline{D}}$ means that the ideals $I_n = I + \mathfrak{m}^n_{\overline{D}}$ are $\overline{D}$-open for $n \in \mathbb{N}_{\geq 1}$ and that $I = \bigcap_n I_n$. For each I_n (and for I) let $D_{I_n} : G \to R^{\mathrm{univ}}_{W,n}/I_n$ (and D_I, resp.,) be the induced pseudorepresentation. That all D_{I_n} are continuous is therefore equivalent to D_I being continuous. On the other hand, that φ_A factors via i_x means that $\ker \varphi_A$ is D'_x-open.

Define $\widehat{R}^{\mathrm{univ}}_{W,n}/I$ as the completion of $R^{\mathrm{univ}}_{W,n}/I$ with respect to the I_n (i.e., with respect to $\mathfrak{m}^n_{\overline{D}}$). Hence we need to show that the following two conditions are equivalent:

(a) $D_A : G \to A$ is continuous, i.e., all of its characteristic polynomial coefficients are;

(b) $D_I : G \to \widehat{R}^{\mathrm{univ}}_{W,n}/I$ is continuous with respect to the profinite topology on $\widehat{R}^{\mathrm{univ}}_{W,n}/I$.

By definition of D'_x, as a pseudodeformation of $\overline{D}$, it is continuous as a pseudorepresentation $G \to k$ with respect to the natural topology on k, and continuous as a pseudorepresentation $G \to \widehat{R}^{\mathrm{univ}}_{W,n}/\mathfrak{p}_x$ with the profinite topology on the latter. We also note that by construction, we start from the map φ_A, the pseudorepresentation D_A factors as D_I composed with the homomorphism $\widehat{R}^{\mathrm{univ}}_{W,n}/I \to A$. So we need to show that $\widehat{R}^{\mathrm{univ}}_{W,n}/I \to A$, which by construction is injective, identifies $\widehat{R}^{\mathrm{univ}}_{W,n}/I$ with a compact open subring of A (in the topology of A).

We shall induct over the length of A to show that $\widehat{R}^{\mathrm{univ}}_{W,n}/I \subset A$ is a compact open subring. As observed in the previous paragraph, by hypothesis we know that $R/\mathfrak{p}_x \subset k$ is a compact open subring that is contained in the valuation ring of k. This completes the case where A has length 1. In the induction step, let $J \subset A$ be an ideal with quotient $A' = A/J$ such that $\dim_k J = 1$. Let I' be the corresponding ideal of $R^{\mathrm{univ}}_{W,n}$, and consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I/I' & \longrightarrow & \widehat{R}^{\mathrm{univ}}_{W,n}/I & \longrightarrow & \widehat{R}^{\mathrm{univ}}_{W,n}/I' & \longrightarrow & 0.
\end{array}$$

By the surjectivity of φ_A, it is clear that A is the k-span of it subring $\widehat{R}^{\mathrm{univ}}_{W,n}/I$. By induction hypothesis, the right hand inclusion identifies $\widehat{R}^{\mathrm{univ}}_{W,n}/I'$ with a compact open subring of A' that spans A' over k. Denoting by $\mathcal{O}$ the ring of integers of k, this is equivalent to $\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{W,n}/I'$ being an $\mathcal{O}$-lattice in A' and to $(\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{W,n}/I')/(\widehat{R}^{\mathrm{univ}}_{W,n}/I')$ being finite. We need to show the analog for A and I.

We know that $J \cong k$ as a k-module and that I/I' is a finitely generated $\widehat{R}^{\mathrm{univ}}_{W,n}/\mathfrak{p}_x$-submodule. Since $R/\mathfrak{p}_x \subset k$ is compact open, we find that $\mathcal{O} \cdot I/I'$ is a lattice in J and that $\mathcal{O} \cdot (I/I')/(I/I')$ is finite. Let $b_0 \in I/I'$ be an $\mathcal{O}$-basis of $\mathcal{O} \cdot I/I'$. Choose an $\mathcal{O}$-basis of $\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{W,n}/I'$ in $\widehat{R}^{\mathrm{univ}}_{W,n}/I'$ (this is possible by Nakayama's Lemma by first working in the reduction modulo $\mathfrak{m}_{\mathcal{O}}$) and lift these basis elements to elements $b_1, \ldots, b_t$ in $\widehat{R}^{\mathrm{univ}}_{W,n}/I$. Then one verifies that the $\mathcal{O}$-span of $\{b_0, \ldots, b_t\}$ contains $\widehat{R}^{\mathrm{univ}}_{W,n}/I$, and that $(\mathcal{O} \cdot \widehat{R}^{\mathrm{univ}}_{W,n}/I)/(\widehat{R}^{\mathrm{univ}}_{W,n}/I)$ is finite. This completes the induction step and the proof of (ii).

To prove (iii), we need to show that natural transformation of functors $\mathcal{A}r_k \to \mathit{Sets}$ defined by

$$A \mapsto \begin{array}{c}
\{\text{continuous deformations } \rho_A \text{ of } \rho_x \text{ to } A\} \\
\Big\downarrow {\scriptstyle \rho_A \mapsto \det_{\rho_A}} \\
\{\text{continuous pseudodeformations } D_A \text{ of } D'_x \text{ to } A\}
\end{array}$$

is an isomorphism. Well-definedness is clear. Injectivity follows from Theorem 3.1.50(3.1.50) (due to Chenevier) since ρ_x is absolutely irreducible. To prove surjectivity, consider a pseudodeformation $D_A\colon G \to A$ of D'_x and note that by the just quoted theorem there exists a deformation ρ_A of ρ_x to A with $D_A = \det \rho_A$. However, it remains to show that ρ_A is continuous. For this, one proceeds as in the proof of Corollary 3.2.10 using Proposition 3.2.9 by Wang Erickson. The situation is simplified by the fact that ρ_x is absolutely irreducible. We omit details. ☐

Proposition 3.2.14 (Cf. [Che14, Exmp. 3.4])**.** *Suppose that k is either a finite field or a local field of equi-characteristic p. Let $\overline{\rho}\colon G \to \mathrm{GL}_n(k)$ be an absolutely irreducible residual representation with associated pseudorepresentation $\overline{D}$. Then the deformation functor $\mathcal{D}_{\overline{\rho}}$ of $\overline{\rho}$ is canonically isomorphic to $\mathcal{P}sD_{\overline{D}}$.*

Proof. The assertion for k finite is [Che14, Exmp. 3.4]. For k a local field, this follows from Corollary 3.2.13(iii). ☐

Corollary 3.2.15. *Let $\overline{D}\colon G_K \to \mathbb{F}$ be an n-dimensional pseudorepresentation and $x \in \overline{X}^{\mathrm{univ}}_{\overline{D}}$ be of dimension 1 so that $\kappa(x)$ is a local field. Let $D_x\colon G_K \to \kappa(x)$ be the pseudorepresentation from Definition 3.2.1 and assume that D_x is irreducible. Let C_x be the Azumaya-$\kappa(x)$-algebra $\kappa(x)[G]/\operatorname{CH}(D_x)$ of rank n^2 over $\kappa(x)$ and $\psi_x\colon G \to C_x^\times$ the natural homomorphism, so that $D_x = \det \circ \rho_x$.*

Then we have:

(a) There is a finite extension $L/\kappa(x)$ and a representation $\rho_x\colon G_K \to \mathrm{GL}_n(L)$ such that $\rho_x = \psi_x \otimes_{\kappa(x)} L$.

(b) The representations ρ_x and ψ_x are continuous.

(c) Let $\widehat{R}^{\mathfrak{p}}$ be as in Corollary 3.2.13(ii). If $H^2(G_K, \mathrm{ad}_{\rho_x}) = 0$ for ρ_x from ((b)), then $\widehat{R}^{\mathfrak{p}} \otimes_{\kappa(x)} L$ is formally smooth over L of dimension $\dim H^1(G_K, \mathrm{ad}_{\rho_x})$.

Proof. Part ((a)) is clear by taking for L any splitting field of C_x that is finite over $\kappa(x)$. Part ((b)) follows from Lemma 3.2.11. Regarding ((c)) note first that by Proposition 3.1.61 and Corollary 3.2.13 the ring $\widehat{R}^{\mathfrak{p}} \otimes_{\kappa(x)} L$ is the universal deformation ring of ρ_x. By the analog of Theorem 2.2.14 for representations to local fields, the condition $H^2(G_K, \mathrm{ad}_{\rho_x}) = 0$ implies that $\widehat{R}^{\mathfrak{p}} \otimes_{\kappa(x)} L$ is regular and that it is a power series ring over L of Krull dimension $\dim H^1(G_K, \mathrm{ad}_{\rho_x})$. This is equivalent to $\widehat{R}^{\mathfrak{p}} \otimes_{\kappa(x)} L$ being formally smooth over L by Proposition 2.1.11. ☐

3.2.2 Induction for pseudorepresentations

In this subsection we fix a profinite group G and a normal open subgroup H of index m. Under suitable irreducibility hypothesis on a given pseudorepresentation of H over a profinite ring we shall define its induction to G. A main tool is the universal Cayley-Hamilton sheaf introduced in the previous subsection, which is an Azumaya algebra over the irreducible locus.

Lemma 3.2.16. *Let A be a commutative ring and C an Azumaya A-algebra. Consider a representation $\rho\colon H \longrightarrow C^\times$. There exists a representation $\rho^*\colon G \to \mathrm{Mat}_m(C)^\times$ such that for any étale extension $A \to A'$ that splits C, there is an isomorphism $\rho^* \otimes_A A' \cong \mathrm{Ind}_H^G(\rho \otimes_A A')$ of G-representations over A.*

Its induced algebra representation $A[G] \to \mathrm{Mat}_m(C)$ takes values in an Azumaya algebra, and by Example 3.1.8 therefore D_{ρ^*} is a pseudorepresentation with values in A.

Proof. To prove the lemma, we adapt the description of the induced matrix representation from [CR81, pp. 227-230] to the setting of Azumaya-algebras. Let $g_1, \dots, g_m$ be a set of representatives of left cosets of G/H such that $G = \bigsqcup_{i=1}^m g_i H$. For $g \in G$ we define for each $j \in \{1, \dots, m\}$ an $i = i_j$ in $\{1, \dots, m\}$ by the condition

$$gg_j \in g_i H.$$

The assignment $j \to i_j$ is a permutation of $\{1, \dots, m\}$. We extend ρ from H to G by defining

$$\widetilde{\rho}\colon G \longrightarrow C, \quad g \longmapsto \begin{cases} \rho(g) & \text{if } g \in H, \\ 0 & \text{if } g \in G \smallsetminus H. \end{cases}$$

Consider the map

$$\rho^*\colon G \longrightarrow \mathrm{Mat}_m(C), \quad g \longmapsto \begin{pmatrix} \widetilde{\rho}(g_1^{-1}gg_1) & \cdots & \widetilde{\rho}(g_1^{-1}gg_m) \\ \vdots & \ddots & \vdots \\ \widetilde{\rho}(g_m^{-1}gg_1) & \cdots & \widetilde{\rho}(g_m^{-1}gg_m) \end{pmatrix}.$$

Then for all $g \in G$ the image $\rho^*(g)$ is a monomial matrix over the skewfield C since for $1 \leq i, j \leq m$ the only nonzero entry in the i^{th} row and j^{th} column of $\rho^*(g)$ is $\rho(g_{i_j}^{-1}gg_j) \in C^\times$. In particular, this shows that $\rho^*(g)$ lies in $\mathrm{GL}_m(C)$.

We claim that ρ^* has the properties asserted in the lemma. Let $A \to A'$ be finite étale so that $C \otimes_A A' = \mathrm{Mat}_r(A')$ for a suitable $r \in \mathbb{N}_{\geq 1}$. Then $\rho^* \otimes_A A'$ is the matrix representation of the induced representation of

$$\rho \otimes_A A'\colon H \longrightarrow \mathrm{GL}_r(A')$$

simply by our construction following [CR81]. This implies the multiplicativity of the map ρ^*, i.e., that it is a homomorphism. Moreover, it shows that $\rho^* \otimes_A A'$ is the usual induced representation of $\rho \otimes_A A'$. □

Remark 3.2.17. It can be shown that $\rho \mapsto \rho^*$ in Lemma 3.2.16 is uniquely characterized as the right adjoint of the restriction homomorphism from G-representations to H-representations on Azumaya algebras.

Definition 3.2.18. We call ρ^* in Lemma 3.2.16 the *representation induced from ρ under $H \subset G$* and denote it by $\mathrm{Ind}_H^G \rho$.

Below we want to have a rather explicit description of the characteristic polynomial of $\mathrm{Ind}_H^G \rho$. This is prepared in the following lemmas. We could not locate these presumably well-known results in the literature, so we indicate some proofs.

Let A be a commutative ring, let C be an Azumaya A-algebra. Recall that for elements $c \in C$ one has a notion of characteristic polynomial χ_c. It is a monic polynomial in $A[t]$ of degree n where n^2 is the rank of C over A; it is defined by first passing from A to an étale splitting extension A' and then taking the usual characteristic polynomial over A'. We also write $\chi_c(t) = \sum_{i=0}^n (-1)^i \Lambda_{c,i}(c) t^{n-i}$, similarly to Definition 3.1.11. Recall also that if C is an Azumaya A-algebra, then so is $\mathrm{Mat}_m(C)$. We write χ_c^m for the characteristic polynomial (of degree nm) of $c \in \mathrm{Mat}_m(C)$.

Lemma 3.2.19. *Let $c = (c_{i,j})$ be in $\mathrm{Mat}_m(C)$. Suppose that there is a permutation $\sigma \in \mathfrak{S}_m$ such that $c_{i,j} = 0$ for $i \neq \sigma(j)$ and such that $c_{\sigma(j),j}$ lies in $C^\times$ for all j. Then χ_c^m has the following description:*

Write σ in its cycle decomposition $\sigma = \sigma_1 \cdot \ldots \cdot \sigma_v$, where the σ_l are disjoint cycles of length m_l such that $\sum_{l=1}^{v} m_l = m$ and let j_l be in the support of σ_l such that $\sigma_l = (j_l, \sigma(j_l), \ldots, \sigma^{m_l-1}(j_l))$. Then

$$\chi_c^m(t) = \prod_{l=1}^{v} \chi_{c(l)}(t^{m_l}) \qquad \text{with } c(l) := c_{j_l,\sigma^{m_l-1}(j_l)} c_{\sigma^{m_l-1}(j_l),\sigma^{m_l-2}(j_l)} \cdot \ldots \cdot c_{\sigma(j_l),j_l}.$$

Proof. Let $s_i = m_1 + \ldots + m_{l-1}$ for $l = 1, \ldots, v$, with $m_0 = 0$, and let $\tau \in \mathfrak{S}_m$ be the permutation whose inverse is given by

$$\begin{pmatrix} s_1+1 & s_1+2 & \cdots & s_1+m_1 \\ j_1 & \sigma(j_1) & \ldots & \sigma^{m_1-1}(j_i) \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} s_v+1 & s_v+2 & \cdots & s_v+m_v \\ j_v & \sigma(j_v) & \ldots & \sigma^{m_v-1}(j_v) \end{pmatrix},$$

and let $p = p_\tau$ in $\mathrm{Mat}_m(C)$ be the permutation matrix attached to τ, i.e., with $p_{i,j} = 0$ for $i \neq \tau(j)$ and $p_{\tau(j),j} = 1_C$ for all j. Then one verifies that $p_\tau c p_\tau^{-1}$ is a block diagonal matrix in $\mathrm{Mat}_m(C)$ with v blocks on the diagonal, the l^{th} block lies in $\mathrm{Mat}_{m_l}(C)$ and is of the form

$$\begin{pmatrix} 0 & 0 & & \cdots & c_{j_l,\sigma^{m_l-1}(j_l)} \\ c_{\sigma(j_l),j_l} & 0 & \ddots & & 0 \\ 0 & c_{\sigma^2(j_l),\sigma(j_l)} & \ddots & & \ddots \\ \vdots & \ddots & & & \\ 0 & \cdots & 0 & c_{\sigma^{m_l-1}(j_l),\sigma^{m_l-2}(j_l)} & 0 \end{pmatrix}.$$

We leave it as a simple exercise in matrix manipulations to complete the result in this case. □

Lemma 3.2.20. *Let the hypotheses be as in Lemma 3.2.16. Fix $g' \in G$ and denote by m' its order in the group G/H. Then for $h \in H$ one has*

$$\chi_{\mathrm{Ind}_H^G \rho(g'h)}(t) = \prod_{g \in G/H\langle g' \rangle} \chi_{\rho^{(g^{-1})}((g'h)^{m'})}(t^{m'}).$$

If $G/H\langle g' \rangle$ is a group (and not only a coset), then the inversion in $\rho^{(g^{-1})}$ can be omitted.

Recall that the twist $\rho^{(g^{-1})}$ as defined in Definition 2.3.1 also applies to the present situation. Note also that $(g'h)^{m'}$ lies in H so that the above formula is well-defined, since $H \subset G$ is a normal subgroup and $m' = \mathrm{ord}_{G/H}(g')$.

Proof. Let the notation be as in the proof of Lemma 3.2.16, and set $v = m/m'$. Define $\sigma_l \in \mathfrak{S}_m$ as the (unique) permutation such that $g_l g_j \in g_{\sigma_l(j)} H$ for all $l \in \{1, \ldots, m\}$. Let $c \in \mathrm{Mat}_m(C)$ be the matrix with $c_{i,j} = 0$ for $i \neq \sigma_l(j)$ and $c_{\sigma_l(j),j} = \rho(g_{\sigma_l(j)}^{-1} g_l h g_j)$, so that $c = \rho^*(g_l h)$. Choose $j_1, \ldots, j_v$ such that the elements g_{j_i} are representatives of the cosets of $G/H\langle g_j \rangle$, or, equivalently, such that the orbits of the j_i under σ_l are in bijection with the orbits in $\{1, \ldots, m\}$ under σ_l. Now c is monomial, and by Lemma 3.2.19 its characteristic polynomial is given by

$$\chi_{\rho^*(g_l h)}(t) = \prod_{s=1}^{v} \chi_{\rho(g_{j_s}^{-1} g_l h g_{\sigma_l^{m'-1}(j_s)}) \rho(g_{\sigma_l^{m'-1}(j_s)}^{-1} g_l h g_{\sigma_l^{m'-2}(j_s)}) \cdot \ldots \cdot \rho(g_{\sigma_l(j_s)}^{-1} g_l h g_{j_s})}(t^{m'}),$$

where we use our explicit shape of c, so that in particular $j \mapsto m_j$ is constant with value m'. Next one one uses the multiplicativity of ρ as a representation to combine its arguments as a product in which cancellations occur. Using also the twist of ρ by some $g \in G$ defined in Definition 2.3.1, we obtain

$$\chi_{\rho^*(g_l h)}(t) = \prod_{s=1}^{v} \chi_{\rho^{(g_{j_s}^{-1})}((g_l h)^{m'})}(t^{m'}),$$

Now up to isomorphy we can replace g_{j_s} in $\rho^{(g_{j_s}^{-1})}$ by any other representative of the class $g_{j_s} H\langle g_l\rangle$. To conclude the proof of the formula in the lemma note that we may from that start assume that the g_i are chosen in such a way that g' is among them. □

In the remainder of this subsection, let G be a profinite group, $H \subset G$ be a normal subgroup of finite index m, and

$$D_H \colon H \longrightarrow B$$

be a pseudorepresentation of dimension n with values in a commutative ring B. Denote by $\mathrm{Min}(B)$ the set of minimal primes of B. For a local ring A denote by A^{sh} its strict henselization.

In order to define an induction of D_H, we additionally assume that the following hold.

Assumption 3.2.21 (Basic assumptions on B and D_H). The ring B is a complete Noetherian semilocal equidimensional ring of characteristic p satisfying the following:

(i) The homomorphism $B \to \prod_{\mathfrak{p}\in \mathrm{Min}(B)} B_{\mathfrak{p}}$ is injective.

(ii) For each $\mathfrak{p} \in \mathrm{Min}(B)$ there is an n-dimensional representation $\rho_{\mathfrak{p}}$ of H over $B_{\mathfrak{p}}^{\mathrm{sh}}$ such that $D_H \otimes_B B_{\mathfrak{p}}^{\mathrm{sh}}$ is the determinant attached to $\rho_{\mathfrak{p}}$.

Remark 3.2.22. (a) Assumption 3.2.21(i) is equivalent to B satisfying Serre's condition (S_1), which in turn is equivalent to all associated primes of B being minimal; cf. [Mat89, Thm. 6.1, Rks. above Thm. 23.8]. This uses that $Z = \bigcup\{\mathfrak{p} : \mathfrak{p} \text{ is an associated prime}\}$ is the set of zero divisors of B and that, as is elementary to see, $S = B \smallsetminus Z$ is the largest multiplicatively closed subset of B such that $B \to B_S$ is injective.

(b) A simple, in our applications sufficient, condition for B to satisfy (S_1) is that B is reduced.

(c) If $D_{H,\mathfrak{p}} := D_H \otimes_B B_{\mathfrak{p}}$ is irreducible for all $\mathfrak{p} \in \mathrm{Min}(B)$, then Assumption 3.2.21(ii) holds. This follows from [Che14, Thm. 2.22] recalled in Theorem 3.1.50(3.1.50), since in this case $D_H \otimes_B B_{\mathfrak{p}}^{\mathrm{sh}}$ factors via the Cayley-Hamilton quotient of $B_{\mathfrak{p}}^{\mathrm{sh}}[H]$ which by the results quoted is a rank n matrix algebra over $B_{\mathfrak{p}}^{\mathrm{sh}}$.

(d) Due to our treatment below and the explicit formulas we have for characteristic polynomial coefficients, we expect that eventually Assumption 3.2.21 might be superfluous in what follows.

In the following we shall write $D_{\overline{x}}$ for $D \otimes_B \kappa(\overline{x})$ for any pseudorepresentation D defined over B and any geometric point $\overline{x} \colon \operatorname{Spec} \kappa(\overline{x}) \to \operatorname{Spec} B$.

Theorem 3.2.23. *Suppose Assumption 3.2.21 holds. Then there exists a unique pseudorepresentation $D_G \colon G \to B$ whose characteristic polynomial on a coset $g'H$ is given by*

$$\chi_{D_G,B}(g'h,t) = \prod_{g\in G/H\langle g'\rangle} \chi_{D_H^{(g^{-1})},B}((g'h)^{m'}, t^{m'}). \tag{8}$$

It has the following properties.

(a) *Let* $\overline{x}\colon \operatorname{Spec}\kappa(\overline{x}) \to \operatorname{Spec} B$ *be any geometric point and denote by* $\rho_{\overline{x}}$ *the representation of* H *corresponding to* $D_{H,\overline{x}}$*. Then* $D_{G,\overline{x}}$ *is the pseudorepresentation attached to* $\operatorname{Ind}_H^G \rho_{\overline{x}}$*.*

(b) *If* D_H *is continuous, then so is* D_G*.*

(c) *One has*

$$\operatorname{Res}_H^G D_G \cong \bigoplus_{g\in G/H} D_H^g.$$

(d) *Suppose that over an affine open subset* $U = \operatorname{Spec} B' \subset \operatorname{Spec} B$ *the pseudorepresentation is irreducible, so that* $C := B[G]/\operatorname{CH}(D_H) \otimes_B B'$ *is an Azumaya* B'*-algebra and* D_H *is the determinant attached to the reduced norm of* C *composed with the natural homomorphism* $\psi\colon G \to C^\times$*. Then* $D_G \otimes_B B' = \det \circ \operatorname{Ind}_H^G \psi$*.*

(e) *Let* $i \in \{0, \ldots, nm\}$ *and let* $g' \in G$ *have order* m' *in* G/H*. Then* $\Lambda_{D_G,i}(g') = 0$ *if* $m' \nmid i$*.*

(f) *Let* $B \to B'$ *be any surjective homomorphism onto a domain* B' *(so that* B' *satisfies Assumption 3.2.21 automatically). Denote by* D'_H *the reduction* $D_H \otimes_B B'$ *and by* $D'_G\colon G \to B'$ *the unique pseudorepresentation whose characteristic polynomial is given by (8) modified so that* D_H *and* D_G *are replaced by* D'_H *and* D'_G*, respectively. Then* $D_G \otimes_B B' = D'_G$*.*

Proof. By the Cohen structure theorem we have $B_{\mathfrak{p}} \cong \kappa(\mathfrak{p})[x_1, \ldots, x_h]/I$ for some $h \in \mathbb{N}_{\geq 1}$ and some ideal I such that a power of $(x_1, \ldots, x_h)$ is a subset of I. Then $B_{\mathfrak{p}}^{\mathrm{sh}} = \kappa(\mathfrak{p})^{\mathrm{sep}}[X_1, \ldots, x_n]/I$, where the canonical inclusion $B_{\mathfrak{p}} \to B_{\mathfrak{p}}^{\mathrm{sh}}$ is the strict henselization of $B_{\mathfrak{p}}$. It follows from Assumption 3.2.21(i) that the ring homomorphism $\iota\colon B \to B_\eta^{\mathrm{sh}} := \prod_{\mathfrak{p}\in\operatorname{Min}(B)} B_{\mathfrak{p}}^{\mathrm{sh}}$ is injective. Hence we shall regard B as a subring of B_η^{sh} via ι, and by Assumption 3.2.21(ii) there exists a representation $\rho_\eta\colon H \to \mathrm{GL}_n(B_\eta^{\mathrm{sh}})$ such that $\det \circ \rho_\eta = D_H \otimes_B B_\eta^{\mathrm{sh}}$. Define D_G as $\det \circ \operatorname{Ind}_H^G \rho_\eta\colon G \to \mathrm{GL}_{nm}(B_\eta^{\mathrm{sh}})$. Then (8) holds for D_G by Lemma 3.2.20. Obviously the right hand side of (8) has coefficients in B. Thus by Proposition 3.1.14 the pseudorepresentation D_G is already defined over B, and by the same result D_G is uniquely determined by the coefficients of χ_{D_G}. It remains to prove the properties listed in (a)–(e).

To see (a) note first that formula (8) is preserved under base change to $\kappa(\overline{x})$, i.e., the formula still holds if we replace simultaneously D_G by $D_{G,\overline{x}}$ and D_H by $D_{H,\overline{x}}$. By the construction of $\rho_{\overline{x}}$ one has $\chi_{\rho_{\overline{x}}} = \chi_{D_{H,\overline{x}}}$, and by Lemma 3.2.20, the right hand side of (8) over $\kappa(\overline{x})$ is equal to $\chi_{\operatorname{Ind}_H^G \rho_{\overline{x}}}$. This proves (a).

Part (b) follows immediately from (8): it suffices to verify the continuity of the characteristic polynomial coefficients, and this may be done on the open cover gH, $g \in G$. On each open of this cover, (8) describes these coefficients. Since D_H and hence the $D_H^{(g')}$ are continuous and since $gh \mapsto (gh)^{m'}$ is continuous, the result follows.

Next, the formula in (c) clearly holds over B_η^{sh} since there $\operatorname{Res}_H^G \operatorname{Ind}_H^G \rho_\eta = \oplus_{g\in G/H} \rho_\eta^g$. Because ι is injective, taking characteristic polynomials, formula (c) holds.

The first half of (d) is proved by an argument analogous to the proof of Corollary 3.2.13(i). To see the further assertion, one proceeds similar to the proof of (c), using however the injection $\iota \otimes_B B'\colon B' \to B_\eta^{\mathrm{sh}} \otimes_B B'$. Part (e) is immediate from (8), since the right hand side is a polynomial in $t^{m'}$. Part (f) is clear, since we may tensor (8) over B with B' and since a pseudorepresentation is uniquely determined by its characteristic polynomial by Amitsur's formula; see Proposition 3.1.14(i). □

Definition 3.2.24. We call the pseudorepresentation D_G from Theorem 3.2.23 the *induced pseudorepresentation of* D_H *under* $H \subset G$ and write $\operatorname{Ind}_H^G D_H$ for it.

Lemma 3.2.25. *Suppose that Assumption 3.2.21 holds, that $U \subset \operatorname{Spec} B$ and ψ are as in Theorem 3.2.23(d), and that x is a point of U of dimension 1 with corresponding prime $\mathfrak{p} \in \operatorname{Spec} B$. Then the following holds:*

(i) $D_H \otimes_B \kappa(x)$ is equal to $\det \circ \psi_x$ for $\psi_x = \psi \otimes_B \kappa(x)$;

(ii) we have $\operatorname{Ind}_H^G(D_H \otimes_B B/\mathfrak{p}) = \left(\operatorname{Ind}_H^G D_H\right) \otimes_B B/\mathfrak{p}$;

(iii) if D_H is continuous, then so is $\operatorname{Ind}_H^G(D_H \otimes_B B/\mathfrak{p})$.

Proof. Assertion (i) follows immediately from $x \in U$. Part (ii) is a special case of Theorem 3.2.23(f). Finally, assertion (iii) follows from Theorem 3.2.23(b) applied to $B/\mathfrak{p}$ since the continuity of D_H implies the continuity of $D_H \otimes_B B/\mathfrak{p}$. □

The hypotheses in Assumption 3.2.21 clearly hold for $B = \mathbb{F}^{\mathrm{alg}}$, for instance by Remark 3.2.22. In this case for later use we record the following result.

Lemma 3.2.26. *Let $\overline{D}\colon G_K \to \mathbb{F}^{\mathrm{alg}}$ be a continuous pseudorepresentation and $S_{\overline{D}}$ be the set of pseudorepresentations $\overline{D}'\colon G_{K(1)} \to \mathbb{F}^{\mathrm{alg}}$ satisfying $\operatorname{Ind}_{G_{K(1)}}^{G_K} \overline{D}' = \overline{D}$.*

(i) $S_{\overline{D}}$ is finite.

(ii) If $\overline{D} = \overline{D}(1)$, then $S_{\overline{D}}$ is nonempty.

In particular, there is a finite extension $\mathbb{F}'$ of $\mathbb{F}$ such that $\overline{D}$ and any $\overline{D}' \in S_{\overline{D}}$ are defined over $\mathbb{F}'$.

Proof. The first assertion follows from Corollary 3.1.71. If $\overline{D}$ is irreducible, then this follows from Theorem 2.3.4. In general one uses the ideas from the proof of Corollary 2.3.6. This is quite straightforward. □

3.2.3 Loci of regular and singular pseudodeformations in special fibers

We first define the twist of a pseudorepresentation with a character. Next we consider the closed locus of pseudodeformations that are invariant under certain twists. Finally, we show that certain irreducible points are regular if $\zeta_p \notin K$ and form open loci. If $\zeta_p \in K$, then the regular locus is empty and if in addition $n \nmid p$ we instead consider regular points in the nilreduction.

Throughout this subsection, we fix an n-dimensional residual pseudorepresentation $\overline{D}\colon G_K \to \mathbb{F}$.

Definition 3.2.27. Let S, S' be A-algebras, let $D\colon S \to S'$ be a multiplicative A-polynomial law and let $r\colon S \to A$ be an A-algebra homomorphism.

(a) The *twist $D \otimes r$ of D by r* is defined by

$$(D \otimes r)_B := D_B \otimes r_B \colon S \otimes_A B \longrightarrow S' \otimes_A B, \quad s \longmapsto r(s) \cdot D_B(s), \quad \forall B \in \operatorname{Ob}(\mathcal{CA}lg_A);$$

it is indeed a multiplicative polynomial law $S \to S'$; we omit the elementary details.

(b) Suppose $S = A[G]$ and $\chi\colon G \to A^\times$ is a group homomorphism. Denote by $r_\chi\colon S \to S', \sum a_g g \mapsto \sum a_g \chi(g)$ the induced A-algebra homomorphism. Then the *twist $D \otimes \chi$ of D by χ* is the multiplicative polynomial law $D \otimes r_\chi$.

Remark 3.2.28. Let the notation be as in the above definition.

(a) If D is homogeneous of degree n, then so is $D \otimes r$. Moreover, if D is an n-dimensional pseudorepresentation, then this is also true for $D \otimes r$.

(b) If D and r are continuous, then so is $D \otimes r$.

(c) For the characteristic polynomial coefficients one has the identities

$$\Lambda_{D\otimes r,i}(s) = \Lambda_{D,i}(s) \cdot (r(s))^i \quad \text{for all } i \text{ and all } s \in S.$$

Lemma 3.2.29. *Let $D, D' \colon S \to A$ be pseudorepresentations and let $r \colon S \to A$ be an A-algebra homomorphism. Then $D' = D \otimes r$ if and only if $\Lambda_{D',i}(s) = \Lambda_{D,i}(s) \cdot (r(s))^i$ for all i and all $s \in S$.*

Proof. By the previous remark it suffices to prove the *if*-direction. However, this follows from Proposition 3.1.14(i), which says that a pseudorepresentation is determined by its characteristic polynomial coefficients. ☐

Corollary 3.2.30. *Let $D \colon G \to A$ be an n-dimensional pseudorepresentation of a group G and $\chi \colon G \to A^\times$ a character of finite order prime to p. Then $D = D \otimes \chi$ if and only if*

$$\Lambda_{D,i}(g) = 0 \quad \textit{if } \operatorname{ord}\chi(g) \nmid i \textit{ for all } i = 0, \dots, n \textit{ and } g \in G.$$

Proof. By Lemma 3.2.29 we have $D = D \otimes \chi$ if and only if

$$\Lambda_{D,i}(g) = \Lambda_{D,i}(g) \cdot \chi^i(g) \quad \text{for all } i \text{ and all } g \in G.$$

Since $1 - \chi^i(g)$ is a unit in $\mathbb{F}^{\mathrm{alg}}$ whenever $\operatorname{ord}\chi(g) \nmid i$, and is zero otherwise, the latter is clearly equivalent to the condition given in the corollary. ☐

Corollary 3.2.31. *Let $D := \operatorname{Ind}_H^G D_H \colon G \to B$ be the pseudorepresentation that was constructed in Theorem 3.2.23 under Assumption 3.2.21 on B and $D_H \colon H \to B$. Then $D = D \otimes \chi$ for any 1-dimensional character $\chi \colon G/H \to A^\times$.*

Proof. This follows from Lemma 3.2.20 and Corollary 3.2.30. ☐

Corollary 3.2.32. *Let $R \in \mathrm{Ob}(\widehat{\mathcal{A}r}_{W(\mathbb{F})})$, let $D \colon G \to R$ be an n-dimensional pseudorepresentation and let $\chi \colon G \to R^\times$ be a character of finite order prime to p. Let I be the ideal of R generated by the set*

$$\{\Lambda_{D,i}(g) \ : \ g \in G, i \in \{1, \dots, n\} \textit{ such that } \operatorname{ord}\chi(g) \not| i\}.$$

Then the locus of $\operatorname{Spec} R$ on which $D = D \otimes \chi$ is the closed subscheme $\operatorname{Spec} R/I$.

Proof. This follows as one has for any ideal J of R:

$$(D \otimes_R R/J) \otimes \chi = D \otimes_R R/J \iff I \subset J.$$ ☐

Definition 3.2.33. Let R be in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ and let $D \colon G_K \to R$ be any pseudorepresentation.

(a) For $i \in \mathbb{Z}$ we write $D(i)$ for the twist of D by the i-fold tensor power of the Teichmüller lift of the mod p cyclotomic character of G_K (which has order dividing $p-1$).

(b) If $\zeta_p \notin K$, we call D *special* if $D = D(1)$ and *nonspecial* otherwise.

Definition 3.2.34. Suppose $\zeta_p \notin K$.

(i) $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)=D} \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ denotes the locus of special points and $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)\neq D}$ its complement $(\overline{X}_{\overline{D}}^{\mathrm{univ}}) \smallsetminus (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)=D}$.

(ii) The intersections $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D} := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \cap (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)\neq D}$ and $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)=D} := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \cap (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)=D}$, where $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ as in Definition 3.2.1, denote the loci of non-special irreducible and special irreducible points of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$, respectively.

Note that if $\overline{D} \neq \overline{D}(1)$, then $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D} = (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$.

Lemma 3.2.35. *If $\zeta_p \notin K$, then we have the following:*

(i) $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)\neq D} \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ *is Zariski open.*

(ii) $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D} \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ *is Zariski open.*

Proof. By Corollary 3.2.32 the locus $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)=D} \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ is Zariski closed, and this implies (i). Part (ii) follows from (i) together with Lemma 3.2.2. □

Proposition 3.2.36. *Suppose* $\dim \overline{X}_{\overline{D}}^{\mathrm{univ}} \geq 1$. *Then in each of the nonempty components of the following spaces, points of dimension 0 and 1 are very dense. Moreover the first 3 are Zariski open, the middle one is locally closed and the last 3 are Zariski closed in* $\overline{X}_{\overline{D}}^{\mathrm{univ}}$*:*

$$(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{reg}}, (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}, (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D}, (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)=D}, (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{D(1)=D}, (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}, (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}}.$$

Proof. The assertions on openness, closedness and local closedness follow from Lemma 3.2.35 and Lemma 3.2.2, and for $(\cdot)^{\mathrm{reg}}$ and $(\cdot)^{\mathrm{sing}}$ from [EGA IV$_2$, Thm. 6.12.7]. The density assertion follows from Lemma 2.1.20. □

Remark 3.2.37. Note that $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ contains precisely one closed point, namely $\overline{D}$. So typically dimension 1 points are very dense. Moreover, $\dim \overline{X}_{\overline{D}}^{\mathrm{univ}} \smallsetminus \{\overline{D}\} = \dim \overline{X}_{\overline{D}}^{\mathrm{univ}} - 1$.

The following shows that $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D}$ is contained in the regular locus.

Lemma 3.2.38. *Suppose that* $U := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr},D(1)\neq D} \smallsetminus \{\mathfrak{m}_{\overline{R}_{\overline{D}}^{\mathrm{univ}}}\}$ *is nonempty. Let x be a dimension 1 point of U. Denote by $\widehat{R}^{\mathfrak{p}}$ the universal pseudodeformation ring of the associated pseudorepresentation $D'_x\colon G \to \kappa(x), g \mapsto 1 \otimes_{W(\mathbb{F})} D_x(g)$, from Corollary 3.2.13. Then $\widehat{R}^{\mathfrak{p}}$ is regular and* $\dim \widehat{R}^{\mathfrak{p}} = dn^2 + 1$. *Moreover, U is regular and equidimensional of dimension dn^2.*

Proof. Consider the Galois representation $\rho_x\colon G_K \to \mathrm{GL}_n(L)$ with $\det \rho_x = D'_x \otimes_{\kappa(x)} L$ from Theorem 3.1.50 that is defined over a finite extension L of $\kappa(x)$. By assumption, ρ_x satisfies $\rho_x \ncong \rho_x(1)$ so that the regularity of $\widehat{R}^{\mathfrak{p}}$ follows from

$$H^2(G_K, \mathrm{ad}_{\rho_x})^\vee = H^0(G_K, \mathrm{ad}_{\rho_x}(1)) = \mathrm{Hom}_{G_K}(\rho_x, \rho_x(1)) = 0,$$

using Theorem 2.2.16. By the Euler characteristic formula of the same theorem we also find

$$\dim \widehat{R}^{\mathfrak{p}} = \dim_L H^1(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + \dim_L H^0(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + 1.$$

It follows from Lemma 2.2.12 that x is a regular point of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ of dimension $dn^2 + 1 - 1 = dn^2$. Since x lies on U, it is also a regular point of U. To see that U is regular, let $Y \subset U$ be the closed subscheme of singular points. We know that points of dimension at most 1 will be dense in the constructible set Y. Since the unique closed point of $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ is not in U, points of dimension 1 are dense in $Y \subset U$. However as we just saw, such points are regular and cannot lie in Y. Therefore Y must be empty. And again by the density of dimension 1 points in U, it follows that U is regular and equidimensional of dimension dn^2. □

It will also be useful to have a weaker result on $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$.

Lemma 3.2.39. *Let x be a dimension 1 point of $U := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \smallsetminus \{\mathfrak{m}_{\overline{R}_{\overline{D}}^{\mathrm{univ}}}\}$. Denote by $\widehat{R}^{\mathfrak{p}}$ the universal pseudodeformation ring of the associated pseudorepresentation $D'_x \colon G \to \kappa(x), g \mapsto 1 \otimes_{W(\mathbb{F})} D_x(g)$, from Corollary 3.2.13. Then $\widehat{R}^{\mathfrak{p}}$ is a complete intersection ring with $\dim \widehat{R}^{\mathfrak{p}} \in \{dn^2+1, dn^2+2\}$. Moreover, U is of dimension at most dn^2+1.*

Proof. By exactly the same computations as in the previous result, we obtain a presentation $0 \to I \to \kappa(x)[\![X_1, \ldots, X_{dn^2+2}]\!] \to \widehat{R}_{\mathfrak{p}} \to 0$, where the ideal I is generated by at most one element over $\kappa(x)[\![X_1, \ldots, X_{dn^2+2}]\!]$. This proves the claims on $\widehat{R}_{\mathfrak{p}}$. The remaining assertion follows from the density of dimension 1 points in U and Lemma 2.2.12. □

Let us give one further variant of the previous two results.

Lemma 3.2.40. *Let x be a dimension 1 point of $U := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \smallsetminus \{\mathfrak{m}_{\overline{R}_{\overline{D}}^{\mathrm{univ}}}\}$. Denote by $\widehat{R}^{\mathfrak{p}}$ the universal pseudodeformation ring of the associated pseudorepresentation $D'_x \colon G \to \kappa(x), g \mapsto 1 \otimes_{W(\mathbb{F})} D_x(g)$, from Corollary 3.2.13. Let ρ_x be an absolutely irreducible representation over a finite extension L of $\kappa(x)$ such that $\det \circ \rho_x = D_x \otimes_{\kappa(x)} L$. Suppose that $\zeta_p \in K$ and $H^0(G_K, \overline{\mathrm{ad}_{\rho_x}}) = 0$. Then $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ is complete regular local of dimension dn^2+1. In particular, if $p \not| n$, then U_{red} is regular and equidimensional of dimension dn^2.*

Proof. It follows from Corollary 2.2.18 that $(\overline{R}_{\rho_x}^{\mathrm{univ}})_{\mathrm{red}}$ is complete regular local of dimension dn^2+1. From Proposition 3.1.61 and Proposition 3.2.14 we deduce $\overline{R}_{D_x}^{\mathrm{univ}} \otimes_{\kappa(x)} L \cong \overline{R}_{\rho_x}^{\mathrm{univ}}$, and the assertion on $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ follows. Since $p \not| n$ implies that $H^0(G_K, \overline{\mathrm{ad}_{\rho_x}})$ vanishes, the remaining assertion follows from the density of dimension 1 points in U and Lemma 2.2.12. □

The following proposition summarizes some of the results we have obtained so far:

Proposition 3.2.41. *(i) Let $\det \overline{D}$ be the determinant of $\overline{D}$ as defined in Example 3.1.9. Then the canonical morphism*

$$\det \colon X_{\overline{D}}^{\mathrm{univ}} \to X_{\det \overline{D}}^{\mathrm{univ}}$$

from Example 3.1.25 is smooth at a point $x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ with $\dim x = 1$ (and with ρ_x an absolutely irreducible representation defined over a finite extension of $\kappa(x)$ whose determinant is equal to D_x) if one of the following conditions holds:

(1) $\rho_x(1) \ncong \rho_x$;

(2) $\zeta_p \in K$ and $H^0(G_K, \overline{\mathrm{ad}_{\rho_x}}) = 0$; note that the second condition is implied by $p \not| n$;

(ii) For $x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ one has $\rho_x(1) \cong \rho_x$ if and only if ρ_x is induced from $G_{K(\zeta_p)} \subset G_K$, by Theorem 2.3.4. Moreover, if $\zeta_p \notin K$ the locus

$$(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}, D(1) \neq D} := \{x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \ : \ \rho_x(1) \ncong \rho_x\}$$

of nonspecial irreducible pseudorepresentations in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ is Zariski open in $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$.

(iii) One has

(1) $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr},D(1)\neq D} \subset (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{reg}}$ *if* $\zeta_p \notin K$, *and* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{reg}} = \varnothing$ *if* $\zeta_p \in K$.

(2) $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}} \subset (\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{reg}}$ *if* $n \nmid p$, *where* $\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$ *denotes the nilreduction of* $\overline{X}^{\mathrm{univ}}_{\overline{D}}$.

Proof. Part (i) follows from Lemma 2.2.17, Proposition 3.2.14 and Proposition 3.1.61. Part (ii) and Part (iii)(1) follow from Lemma 3.2.38, Corollary 2.3.6 and again Proposition 3.2.14. Part (iii)(2) follows from Lemma 3.2.40 and the last part also uses Corollary 2.2.18. □

3.3 Equidimensionality of special fibers and Zariski density of the regular locus

Recall that we fix an algebraic closure K^{alg} of an extension K of $\mathbb{Q}_p$ of finite degree $d = [K : \mathbb{Q}_p]$ with absolute Galois group G_K, a primitive p^{th} root of unity ζ_p, and a finite field $\mathbb{F}$ of characteristic p.

This section inductively proves Theorem B (Theorem 3.3.12) on the equidimensionality of the special fiber of a universal pseudodeformations space. Our proof proceeds with the same steps as Chenevier's proof of the equidimensionality of the generic fiber of the universal pseudorepresentation space.

In Subsection 3.3.1, Theorem C (Theorem 3.3.1) on the Zariski density of the irreducible locus is established with the help of two technical lemmas.

In Subsection 3.3.2, Theorem 3.3.6 gives an upper bound for the dimension of the locus of special (irreducible) pseudodeformations.

This enables us to prove the equidimensionality in Subsection 3.3.3: The base case is Proposition 3.3.11, and the induction step is shown with the help of the proven Theorem C (Theorem 3.3.1).

If $\zeta_p \notin K$, Theorem D (Theorem 3.3.13) says that the reducible locus and the locus of special irreducible pseudodeformations are contained in the singular locus. We finish Subsection 3.3.4 by describing the regular locus of a universal deformation ring and showing that it satisfies Serre's condition (R_2) if $\zeta_p \notin K$, and either $n > 2$, or $n = 2$ and $d > 1$, as stated in Theorem E (Corollary 3.3.15).

3.3.1 Zariski density of the irreducible locus

The aim of this subsection is an analog of the Zariski density of the irreducible locus in the generic fiber [Che11, Thme. 2.1] that we formulate as a result suitable for an induction.

Theorem 3.3.1 (Theorem C)**.** *Given* $n \geq 2$. *Suppose that for all* $\overline{D}' : G_K \to \mathbb{F}$ *of dimension* $n' < n$ *the following hold:*

(i) $\overline{X}^{\mathrm{univ}}_{\overline{D}'}$ *is equidimensional of dimension* $d(n')^2 + 1$,

(ii) if $\zeta_p \notin K$, *then* $(\overline{X}^{\mathrm{univ}}_{\overline{D}'})^{\mathrm{irr},D(1)\neq D} \subset \overline{X}^{\mathrm{univ}}_{\overline{D}'}$ *is Zariski dense,*

(iii) if $\zeta_p \in K$, *then the regular locus* $((\overline{X}^{\mathrm{univ}}_{\overline{D}'})^{\mathrm{irr}}_{\mathrm{red}})^{\mathrm{reg}}$ *of* $(\overline{X}^{\mathrm{univ}}_{\overline{D}'})^{\mathrm{irr}}_{\mathrm{red}}$ *is Zariski dense in* $\overline{X}^{\mathrm{univ}}_{\overline{D}'\mathrm{red}}$.

Then for all n-dimensional $\overline{D} : G_K \to \mathbb{F}$ *the subspace* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ *is Zariski dense, unless* $n = 2$ *and* $K = \mathbb{Q}_2$ *and* $\overline{D}$ *is reducible.*

In the missing case $n = 2$, $K = \mathbb{Q}_2$ and $\overline{D}$ reducible, we have the following:

Remark 3.3.2. Let $\overline{D}\colon G_{\mathbb{Q}_p} \to \mathbb{F}$ be a split 2-dimensional pseudorepresentation, so that $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$ for 1-dimensional pseudorepresentations $\overline{D}_i$ – we call them sometimes ρ_i to stress that they are also representations. Suppose that ρ_1 is not isomorphic to $\rho_2(l)$ for $l \in \{0, \pm 1\}$. Then $H^1(G_{\mathbb{Q}_p}, \rho_i \rho_j^{-1}) \cong \mathbb{F}$ for $i \neq j$ in $\{0,1\}$, and so up to isomorphism there are unique 2-dimensional representations $\rho_{i,j}$ that are a non-trivial extension of ρ_i by ρ_j. In this situation it is proved in [Paš17, Prop. 3.6] that one has isomorphisms

$$R^{\mathrm{univ}}_{\mathbb{Q}_p,\overline{D}} \cong R^{\mathrm{univ}}_{\mathbb{Q}_p,\rho_{i,j}}$$

for both choices of i,j. Moreover, as Paškūnas explains, it follows from [Che11, Sec. 4] that for $p = 2$ one has

$$R^{\mathrm{univ}}_{\mathbb{Q}_2,\overline{D}} \cong W(\mathbb{F})[X_0]/(X_0^2 - 2X_0)[\![X_1,\dots,X_5]\!],$$

and for $p > 2$ there is an isomorphism $R^{\mathrm{univ}}_{\mathbb{Q}_p,\rho_{i,j}} \cong W(\mathbb{F})[\![X_1,\dots,X_5]\!]$, as follows from an easy computation using Galois cohomology. This has two consequences, we would like to mention:

(a) If $p = 2$, then the ring $(\overline{R}^{\mathrm{univ}}_{\mathbb{Q}_2,\overline{D}})_{\mathrm{red}} \cong \mathbb{F}[\![X_1,\dots,X_5]\!]$ is regular. Since $(\overline{R}^{\mathrm{univ}}_{\mathbb{Q}_2,\overline{D}})^{\mathrm{red}}$ has dimension 4,[5] it follows that $(\overline{X}^{\mathrm{univ}}_{\mathbb{Q}_2\overline{D}})^{\mathrm{irr}}$ is Zariski dense in $\overline{X}^{\mathrm{univ}}_{\mathbb{Q}_2,\overline{D}}$. We cannot deduce this from Theorem C (Theorem 3.3.1). We expect that similar explicit computations in the only remaining case (for $p = 2$) when $\overline{D}_1 = \overline{D}_2$ also yield the same density result. We have no proof though.

(b) If $p > 2$ then $\overline{R}^{\mathrm{univ}}_{\mathbb{Q}_p,\overline{D}} \cong \mathbb{F}[\![X_1,\dots,X_5]\!]$ is regular. In particular, the presence of a reducible locus in a pseudodeformation space need not cause singularities if $K = \mathbb{Q}_p$; cf. Theorem D (Theorem 3.3.13).

For the proof of Theorem C (Theorem 3.3.1) we follow closely the argument of Chenevier; with some adjustments.

We first recall some constructions from Corollary 3.1.72: For $n \in \mathbb{N}_0$ we set $\overline{X}^{\mathrm{univ}}_n = X^{\mathrm{univ}}_{\mathbb{F}[G_K],n}$. For $n_1, n_2 \in \mathbb{N}_0$ with $n = n_1 + n_2$, the addition $(D_1, D_2) \mapsto D_1 \oplus D_2$ of pseudorepresentations yields a morphism $\iota_{n_1,n_2}\colon \overline{X}^{\mathrm{univ}}_{n_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{n_2} \longrightarrow \overline{X}^{\mathrm{univ}}_n$. Passing to the formal completion at $\mathbb{F}_p^{\mathrm{alg}}$-points, defines a morphism $\overline{\mathcal{X}}^{\mathrm{univ}}_{n_1} \times_{\mathrm{Spf}\,\mathbb{F}} \overline{\mathcal{X}}^{\mathrm{univ}}_{n_2} \longrightarrow \overline{\mathcal{X}}^{\mathrm{univ}}_n$ of formal $\mathbb{F}$-schemes, which we again denote by ι_{n_1,n_2}. Fix a residual pseudorepresentation

$$\overline{D}\colon G_K \longrightarrow \mathbb{F}$$

of dimension n. Then the pullback of the latter morphism under $\overline{\mathcal{X}}^{\mathrm{univ}}_{\overline{D}} := \mathrm{Spf}\,\overline{R}^{\mathrm{univ}}_{\overline{D}} \hookrightarrow \overline{\mathcal{X}}^{\mathrm{univ}}_n$ gives a morphism

$$\iota_{\overline{D},n_1,n_2}\colon \overline{\mathcal{X}}^{\mathrm{univ}}_{\overline{D},n_1,n_2} \longrightarrow \overline{\mathcal{X}}^{\mathrm{univ}}_{\overline{D}},$$

which is a closed immersion by Corollary 3.1.72 if $\overline{D}$ is split and multiplicity free. By possibly enlarging $\mathbb{F}$, using Corollary 3.1.72 we will assume that we have an isomorphism

$$\overline{\mathcal{X}}^{\mathrm{univ}}_{\overline{D},n_1,n_2} = \iota^{-1}_{n_1,n_2}(\overline{\mathcal{X}}^{\mathrm{univ}}_{\overline{D}}) \cong \bigsqcup_{\overline{D}_i \in \overline{\mathcal{X}}^{\mathrm{univ}}_{n_i}(\mathrm{Spf}\,\mathbb{F}) \text{ for } i=1,2 \text{ and } \overline{D}_1 \oplus \overline{D}_2 = \overline{D}} \mathrm{Spf}(\overline{R}^{\mathrm{univ}}_{\overline{D}_1} \hat{\otimes}_{\mathbb{F}'} \overline{R}^{\mathrm{univ}}_{\overline{D}_2})$$

[5] Arguing as in Lemma 3.3.14 one can show that the dimension is at most 4; by considering semisimple diagonal representations and invoking Proposition 3.3.11 it is at least 4.

where the union on the right is finite. For the following lemma we consider the corresponding affine scheme

$$\overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2} := \iota^{-1}_{n_1,n_2}(\overline{X}^{\mathrm{univ}}_{\overline{D}}) \cong \coprod_{\overline{D}_i \in \overline{X}^{\mathrm{univ}}_{n_i}(\mathbb{F}) \text{ for } i=1,2 \text{ and } \overline{D}_1 \oplus \overline{D}_2 = \overline{D}} \operatorname{Spec}(\overline{R}^{\mathrm{univ}}_{\overline{D}_1} \hat{\otimes}_{\mathbb{F}'} \overline{R}^{\mathrm{univ}}_{\overline{D}_2}) \tag{9}$$

together with the induced morphism

$$\iota_{\overline{D},n_1,n_2} \colon \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2} \longrightarrow \overline{X}^{\mathrm{univ}}_{\overline{D}} \tag{10}$$

that is a closed immersion if $\overline{D}$ is split and multiplicity free.

Lemma 3.3.3 ([Che11, Lem. 1.1.]). *Let $(x, x_1, x_2) \in \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2}$ be such that the pseudorepresentations D_1 and D_2 corresponding to x_1 and x_2, respectively, are irreducible. Consider a geometric point $\overline{x}$ lying over $x \in \overline{X}^{\mathrm{univ}}_{\overline{D}}$. Then there is an étale neighbourhood (U, u) of $\overline{x}$ in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ with an étale morphism $\varphi_U \colon U \to \overline{X}^{\mathrm{univ}}_{\overline{D}}$ such that the base change of $\iota_{\overline{D},n_1,n_2}$ along φ_U, i.e., the morphism*

$$U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2} \overset{\iota_{\overline{D},n_1,n_2}}{\longrightarrow} U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} \overline{X}^{\mathrm{univ}}_{\overline{D}} = U,$$

is a closed immersion with image $U^{\mathrm{red}} = U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$.

Proof. By Proposition 3.1.16, the universal pseudodeformation $D^{\mathrm{univ}}_{\overline{D}} \colon \overline{R}^{\mathrm{univ}}_{\overline{D}}[G_K] \to \overline{R}^{\mathrm{univ}}_{\overline{D}}$ factors via the universal Cayley-Hamilton pseudodeformation

$$D^{\text{CH-univ}}_{\overline{D}} \colon \overline{S}^{\text{CH-univ}}_{\overline{D}} := \overline{R}^{\mathrm{univ}}_{\overline{D}}[G_K]/\operatorname{CH}(D^{\mathrm{univ}}_{\overline{D}}) \cong \mathcal{S}^{\text{CH-univ}}_n \otimes_{\overline{R}^{\mathrm{univ}}_n} \overline{R}^{\mathrm{univ}}_{\overline{D}} \longrightarrow \overline{R}^{\mathrm{univ}}_{\overline{D}}.$$

Consider the strictly local ring at x,

$$\mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x} \cong \mathcal{O}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},\overline{x}} = \operatorname{colim}_{(V,\overline{v})} \mathcal{O}(V),$$

where $(V, \overline{v})$ runs over all connected étale neighbourhoods of $\overline{x}$ in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ [Sta18, Lem. 04HX]. Since the formation of the Cayley-Hamilton quotient $\overline{S}^{\text{CH-univ}}_{\overline{D}}$ commutes with arbitrary base change [Che14, § 1.22], for any étale neighbourhood $(V, \overline{v})$ of $\overline{x}$ there is an isomorphism

$$\mathcal{O}(V)[G_K]/\operatorname{CH}(D^{\mathrm{univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}(V)) \overset{\sim}{\longrightarrow} \overline{S}^{\text{CH-univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}(V) =: \overline{S}_V.$$

Similarly,

$$\mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x}[G_K]/\operatorname{CH}(D^{\mathrm{univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x}) \overset{\sim}{\longrightarrow} \overline{S}^{\text{CH-univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x} =: \overline{S}_x.$$

From Theorem 3.1.50 it follows that $\overline{S}_x$ is a generalized matrix algebra of type (n_1, n_2) with determinant $D_x := D^{\mathrm{univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x}$. In particular, there exists idempotents $e_1, e_2 \in \overline{S}_x$ with $e_1 + e_2 = 1$ and for $i = 1, 2$ an isomorphism $\psi_{x,i} \colon e_i \overline{S}_x e_i \to \operatorname{Mat}_{n_i}(\mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x})$, whose inverse defines an injective homomorphism

$$\psi'_{x,i} \colon \operatorname{Mat}_{n_i}(\mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x}) \to \overline{S}_x$$

of $\mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x}$-algebras. Since $\mathrm{Mat}_{n_i}(\mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x}) = \mathrm{colim}_{(V,\overline{v})}\,\mathrm{Mat}_{n_i}(\mathcal{O}(V))$ and $\overline{S}_x = \mathrm{colim}_{(V,\overline{v})}\,\overline{S}_V$, we deduce from [EGA IV$_3$, Thm. 8.5.2] that there exists an étale neighbourhood U and for $i = 1, 2$ a homomorphism

$$\psi'_{U,i}\colon \mathrm{Mat}_{n_i}(\mathcal{O}(U)) \to \overline{S}_U \quad \text{satisfying} \quad \psi'_{U,i} \otimes_{\mathcal{O}(U)} \mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x} = \psi'_{x,i}$$

such that also $\psi'_{U,i}$ is injective, since $\ker \psi'_{x,i}$ is finitely generated [EGA IV$_3$, Cor. 8.5.8(ii)].

By abuse of notation, for $i = 1, 2$ we let $e_i \in \overline{S}_U$ be the idempotent that is the image under $\psi'_{U,i}$ of the identity matrix in $\mathrm{Mat}_{n_i}(\mathcal{O}(U))$. Since $\overline{S}^{\text{CH-univ}}_{\overline{D}}$ is finitely presented as an $\overline{R}^{\mathrm{univ}}_{\overline{D}}$-module [WE13, Prop. 3.2.2.1] and $e_i\overline{S}_U e_i \otimes_{\mathcal{O}(U)} \mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x} = e_i\overline{S}_x e_i$, by [EGA IV$_3$, Thm. 8.5.2 and Cor. 8.5.2.4] we know that for $i = 1, 2$ the isomorphism $\psi_{x,i}\colon e_i\overline{S}_x e_i \to \mathrm{Mat}_{n_i}(\mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x})$ spreads out to an isomorphism $\psi_{U,i}\colon e_i\overline{S}_U e_i \to \mathrm{Mat}_{n_i}(\mathcal{O}(U))$. Consider the n-dimensional pseudorepresentation $D_U := D^{\mathrm{univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}(U)$ that satisfies $D_U \otimes_{\mathcal{O}(U)} \mathcal{O}^{\mathrm{sh}}_{\overline{X}^{\mathrm{univ}}_{\overline{D}},x} = D_x$. Since $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ is Noetherian, the kernel K_U of the homomorphism $\mathrm{tr} - \tau_{D_U} \circ \psi'_U$

$$\mathrm{tr} - \tau_{D_U} \circ (\psi'_{U,1}, \psi'_{U,2})\colon \mathrm{Mat}_{n_1}\big(\mathcal{O}(U)\big) \times \mathrm{Mat}_{n_2}\big(\mathcal{O}(U)\big) \longrightarrow \mathcal{O}(U)$$

is finitely generated and thus K_U must vanish after possibly shrinking the étale neighbourhood U [EGA IV$_3$, Cor. 8.5.8(ii)].

Therefore, we may assume that $\overline{S}_U$ is a generalized matrix algebra of type (n_1, n_2). In particular, there exist $\mathcal{O}(U)$-modules $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ such that

$$\overline{S}_U \cong \begin{pmatrix} \mathrm{Mat}_{n_1}(\mathcal{O}(U)) & \mathrm{Mat}_{n_1,n_2}(\mathcal{A}_{12}) \\ \mathrm{Mat}_{n_2,n_1}(\mathcal{A}_{21}) & \mathrm{Mat}_{n_2}(\mathcal{O}(U)) \end{pmatrix}.$$

Let $I = \mathcal{A}_{12}\mathcal{A}_{21} + \mathcal{A}_{21}\mathcal{A}_{12} = \mathcal{A}_{12}\mathcal{A}_{21}$ be the ideal of total reducibility. From Proposition 3.1.48(i) we deduce that there exist unique pseudorepresentations $D_i\colon e_i\overline{S}_U e_i \to \mathcal{O}(U)/I$ for $i = 1, 2$ such that

$$(D_U \mod I) = D_1 \oplus D_2.$$

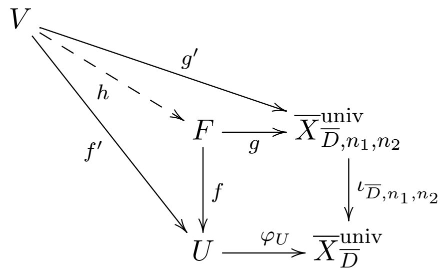
Consider the locus of total reducibility $F := \mathrm{Spec}(\mathcal{O}(U)/I)$, the natural closed immersion $f\colon F \to U$, and the morphism $\varphi_U\colon U \to \overline{X}^{\mathrm{univ}}_{\overline{D}}$ corresponding to the pseudorepresentation $D_U \mod I$. Let $g\colon F \to \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2}$ be the morphism corresponding to the $\mathcal{O}(F)$-valued pseudorepresentations $(D_U \mod I, D_1, D_2)$. Then the morphism $\varphi_U \circ f$ corresponds to the $\mathcal{O}(U)/I$-valued pseudorepresentation $D_U \mod I$ and there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\;g\;} & \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{\overline{D},n_1,n_2}} \\ U & \xrightarrow{\;\varphi_U\;} & \overline{X}^{\mathrm{univ}}_{\overline{D}} \end{array}$$

since $\varphi_U \circ f$ and $\iota_{\overline{D},n1,n_2} \circ g$ both correspond to $D_U \mod I = D_1 \oplus D_2$. To deduce the assertion of the lemma, we want to show that $F = \mathrm{Spec}(\mathcal{O}(U)/I)$ is isomorphic to $U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2}$ so that

$$U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2} \xrightarrow{\iota_{\overline{D},n_1,n_2}} U$$

is a closed immersion of affine schemes. That is, by [Sta18, Def. 01JP] given any connected affine scheme V together with morphisms $f'\colon V \to U$ and $g'\colon V \to \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2}$ such that the solid diagram



commutes, we need to check that there exists a unique dashed arrow h making the diagram commute.

The morphism $\varphi_U \circ f' = \iota_{\overline{D},n_1,n_2} \circ g'$ defines an $\mathcal{O}(V)$-valued pseudorepresentation $\overline{D}_V$, and the morphism g' a pair (D'_1, D'_2) of $\mathcal{O}(V)$-valued pseudorepresentations D'_j of dimension n_j for $j = 1, 2$. The connectedness of V together with (9) implies that there are unique $\overline{D}'_i \in \overline{X}^{\mathrm{univ}}_{n_i}(\mathbb{F})$ for $i = 1, 2$ and $D'_i \in X^{\mathrm{univ}}_{\overline{D}_i}(\mathcal{O}(V))$, such that $D_V = D'_1 \oplus D'_2$. By Lemma 3.1.49 the base change $\overline{S}_V = \overline{S}_U \otimes_{\mathcal{O}(U),(f')^*} \mathcal{O}(V)$ is a generalized matrix algebra of type (n_1, n_2). Therefore, we can apply Proposition 3.1.48(ii) and conclude that the ideal

$$I' := I \otimes_{\mathcal{O}(U),(f')^*} \mathcal{O}(V) = \mathcal{A}_{12}\mathcal{A}_{21} \otimes_{\mathcal{O}(U),(f')^*} \mathcal{O}(V)$$

of total reducibility of $\overline{S}_V$ vanishes. In particular, there exists a morphism $h\colon V \to F$ such that $(f')^*$ factors as $\mathcal{O}(U) \xrightarrow{f^*} \mathcal{O}(F) \xrightarrow{h^*} \mathcal{O}(V)$, as desired.

It remains to understand the image of the closed immersion g. Since the image of

$$U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2} \xrightarrow{\iota_{\overline{D},n_1,n_2}} U$$

is clearly contained in U^{red}, it remains to show that any point $y \in U^{\mathrm{red}}$ lies in this image. Suppose that D_y is the reducible pseudorepresentation corresponding to the homomorphism $\overline{R}^{\mathrm{univ}}_{\overline{D}} \to \mathcal{O}(U) \to \kappa(y)$. By Lemma 3.1.49 the base change $S_y := \overline{S}_U \otimes_{\mathcal{O}(U)} k(y)^{\mathrm{alg}}$ of $\overline{S}_U$ is also a generalized matrix algebra of type (n_1, n_2). Since D_y is reducible, there exists pseudorepresentations $D_1, D_2\colon G_K \to k(y)^{\mathrm{alg}}$ such that $D_y = D_1 \oplus D_2$. By again applying Proposition 3.1.48 we find that the ideal of total reducibility of the generalized matrix algebra S_y vanishes, that the two pseudorepresentations D_1 and D_2 are the unique pseudorepresentations satisfying $D_y = D_1 \oplus D_2$ and that, after possibly reindexing them, we have $\dim D_i = n_i$. This shows the assertion. □

In order to prove Zariski density of the irreducible locus, we need another technical result.

Lemma 3.3.4 (Cf. [Che11, Lem. 2.2])**.** *Consider $(x_1, x_2) \in (\overline{X}^{\mathrm{univ}}_{n_1})^{\mathrm{irr}} \times_{\mathbb{F}} (\overline{X}^{\mathrm{univ}}_{n_2})^{\mathrm{irr}}$ such that $x := \iota_{n_1,n_2}(x_1, x_2) \in \overline{X}^{\mathrm{univ}}_n$ is defined over a local field $\kappa(x)$. Let D_1, D_2 and D denote the pseudorepresentations defined by x_1, x_2 and x. Consider the corresponding representations ρ_{x_1} and ρ_{x_2} that are defined over a finite extension L of $\kappa(x)$. Suppose that $\rho_{x_1} \not\cong \rho_{x_2}(m)$ for all $m \in \mathbb{Z}$.*

(i) There exists a nontrivial extension $\rho\colon G_K \to \mathrm{GL}_n(L)$ of ρ_{x_2} by ρ_{x_1}, and $H^0(G_K, \mathrm{ad}_\rho) \cong L$. The pseudorepresentation D_ρ associated with ρ coincides with $D \otimes_{\kappa(x)} L$. The functor $\mathcal{D}_\rho$ of continuous deformations of ρ on $\mathcal{A}r_L$ is pro-representable; we shall write R_ρ for the representing universal ring and $\rho^{\mathrm{univ}}_\rho\colon G_K \to \mathrm{GL}_n(R_\rho)$ for a universal deformation.

(ii) *If* $(x_1, x_2) \in (\overline{X}^{\mathrm{univ}}_{n_1})^{\mathrm{irr}, D(1) \neq D} \times_{\mathbb{F}} (\overline{X}^{\mathrm{univ}}_{n_2})^{\mathrm{irr}, D(1) \neq D}$*, then* R_ρ *is smooth over* L *of dimension*

$$\dim_L \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} = dn^2 + 1 \quad \textit{with } X^{\mathrm{univ}}_\rho = \operatorname{Spec} R_\rho.$$

(iii) *If* $\zeta_p \in K$*, then* $h := \dim_L \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} = dn^2 + 2$*, and* R_ρ *has a presentation*

$$0 \longrightarrow f\mathcal{R} \longrightarrow \mathcal{R} := L[\![x_1, \dots, x_h]\!] \longrightarrow R_\rho \longrightarrow 0$$

for some $f \in \mathcal{R}$ *(that at this point might be* 0*).*

(iv) *Denote by* $\widehat{R}^{\mathfrak{p}}$ *the universal pseudodeformation ring for* D_ρ*, by* $\varphi \colon X^{\mathrm{univ}}_\rho \to \widehat{X} := \operatorname{Spec} \widehat{R}^{\mathfrak{p}}$ *the map of* L*-schemes induced by sending* $\rho^{\mathrm{univ}}_\rho$ *to its associated pseudorepresentation* $D_{\rho^{\mathrm{univ}}_\rho} = \det \circ \rho^{\mathrm{univ}}_\rho$*, and by* $\mathrm{d}\,\varphi \colon \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} \to \mathbf{t}_{\widehat{X}, x}$ *the induced* L*-linear map on tangent spaces. Suppose that* $\rho' \in \ker \mathrm{d}\varphi$*, i.e., that*

$$\rho' \in \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} \cong \mathcal{D}_\rho(L[\varepsilon]) \textit{ satisfies } \det \circ \rho' = \det \circ \rho.$$

Then with respect to a suitable basis ρ' *has constant diagonal blocks and is upper triangular.*

(v) *For* (x_1, x_2) *as in (ii), we have*

$$\dim_L \ker \mathrm{d}\varphi = dn_1 n_2 - 1 \quad \textit{and} \quad \dim_L \operatorname{im} \mathrm{d}\varphi = dn^2 - dn_1 n_2 + 2.$$

(vi) *Suppose* $\zeta_p \in K$*. Denote by* $\varphi_{\mathrm{red}} \colon (X^{\mathrm{univ}}_\rho)_{\mathrm{red}} \to (\widehat{X})_{\mathrm{red}}$ *the morphism on reduced* L*-schemes associated to* φ *and by* $\mathrm{d}\,\varphi_{\mathrm{red}} \colon \mathbf{t}_{(X^{\mathrm{univ}}_\rho)_{\mathrm{red}}, \rho} \to \mathbf{t}_{(\widehat{X})_{\mathrm{red}}, x}$ *the induced map on tangent spaces. Then*

$$\dim_L \ker \mathrm{d}\varphi_{\mathrm{red}} = dn_1 n_2 - 1 - \delta \quad \textit{and} \quad \dim_L \operatorname{im} \mathrm{d}\varphi_{\mathrm{red}} = dn^2 - dn_1 n_2 + 2 - \delta'$$

for suitable $\delta, \delta' \in \{0, 1\}$ *such that* $\delta + \delta' \leq 1$*.*

Proof. By Theorem 2.2.16, the assumptions imply that

$$\dim_L H^2(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) = \dim_L H^0(G_K, \rho_{x_1} \otimes \rho_{x_2}(1)^\vee) = \dim_L \operatorname{Hom}_{G_K}(\rho_{x_1}, \rho_{x_2}(1)) = 0.$$

The Euler characteristic formula in Theorem 2.2.16 now gives

$$\begin{aligned}
\dim_L \operatorname{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) &= \dim_L H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) \\
&= dn_1 n_2 + \dim_L H^0(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) + \dim_L H^2(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) \\
&= dn_1 n_2.
\end{aligned}$$

Thus there exists an nonzero element $c \in \operatorname{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1})$. Setting $\rho = \begin{pmatrix} \rho_{x_1} & c \\ 0 & \rho_{x_2} \end{pmatrix}$ and applying Lemma 2.2.19 and Theorem 2.2.9 shows (i).

To show (ii), by Theorem 2.2.9 we need to show $H^2(G_K, \mathrm{ad}_\rho) = 0$ and $\dim_L H^1(G_K, \mathrm{ad}_\rho) = dn^2 + 1$. By the duality in Theorem 2.2.16, the first part is equivalent to $\operatorname{Hom}_{G_K}(\rho, \rho(1)) = 0$, and this follows from Lemma 2.2.20 with $\chi = \mathbb{F}(1)$ and the hypotheses on (x_1, x_2) in (ii). The other part follows from the Euler characteristic formula in Theorem 2.2.16:

$$\dim_L H^1(G_K, \mathrm{ad}_\rho) = dn^2 + \dim_L H^0(G_K, \mathrm{ad}_\rho) + \dim_L H^2(G_K, \mathrm{ad}_\rho) = dn^2 + 1 + 0.$$

The proof of (iii) is identical to (ii) with the same references and analogous computations. The point is that here we deduce $\mathrm{Hom}_{G_K}(\rho, \rho(1)) \cong L$ from Lemma 2.2.20. We omit further details.

For assertion (iv), let us first explain how φ is constructed. The representation $\rho_\rho^{\mathrm{univ}}$ is the inverse limit over n of the representations $\rho_{\rho,n}^{\mathrm{univ}} := \rho_\rho^{\mathrm{univ}} \pmod{\mathfrak{m}_{R_\rho}^n}\colon G_K \to \mathrm{GL}_n(R_\rho/\mathfrak{m}_{R_\rho}^n)$ to local Artin algebras. This yields an inverse system $\det \circ \rho_{\rho,n}^{\mathrm{univ}}$ of pseudodeformations of D_ρ. By the universal property of $\widehat{R}^{\mathfrak{p}}$ we obtain an inverse system of homomorphisms $\widehat{R}^{\mathfrak{p}} \to R_\rho/\mathfrak{m}_{R_\rho}^n$ in $\widehat{\mathcal{A}r}_L$, and in the limit a homomorphism $\widehat{R}^{\mathfrak{p}} \to R_\rho$. The morphism φ is its induced morphism on spectra $X_\rho^{\mathrm{univ}} \to \widehat{X}$.

For the proof of (iv), we use the canonical identifications (see [Maz97, Prop., p. 271])

$$\mathcal{D}_\rho(L[\varepsilon]) \cong \mathbf{t}_{X_\rho^{\mathrm{univ}}} \quad \text{and} \quad \mathcal{P}sD_{D_\rho}(L[\varepsilon]) \cong \mathbf{t}_{\widehat{X}} \tag{11}$$

to identify $\ker \mathrm{d}\varphi$ with the L-subspace of $\mathcal{D}_\rho(L[\varepsilon])$, which consists of the deformations of ρ to $L[\varepsilon]$ that map under $\mathrm{d}\,\varphi$ to the trivial pseudodeformation to $L[\varepsilon]$ of the residual pseudorepresentation D_ρ associated with ρ. By definition, the residual pseudorepresentation D_ρ is multiplicity free and split. Hence we can apply Theorem 3.1.50 to the trivial pseudodeformation $D_{\rho'}\colon G_K \to L[\varepsilon]$ associated with ρ'. It provides $L[\varepsilon]$-submodules $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ so that

$$\mathrm{im}\,(\rho') \cong L[\varepsilon][G_K]/\ker(D_{\rho'}) \cong \begin{pmatrix} \mathrm{Mat}_{n_1}(L[\varepsilon]) & \mathrm{Mat}_{n_1 n_2}(\mathcal{A}_{12}) \\ \mathrm{Mat}_{n_2 n_1}(\mathcal{A}_{21}) & \mathrm{Mat}_{n_2}(L[\varepsilon]) \end{pmatrix}$$

is a generalized matrix algebra of type (n_1, n_2) with determinant $D_{\rho'}$. Since ρ is a nonsplit extension, $\mathcal{A}_{12} \mod \varepsilon = L$ or $\mathcal{A}_{21} \mod \varepsilon = L$. Since furthermore $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ are ideals in $L[\varepsilon]$ and thus equal to 0, εL or $L[\varepsilon]$, we deduce $\mathcal{A}_{12} = L[\varepsilon]$ or $\mathcal{A}_{21} = L[\varepsilon]$. We assume that $\mathcal{A}_{12} = L[\varepsilon]$. By assumption $D_{\rho'}$ is a trivial deformation of $D_{\rho'} = D_{\rho_{x_1}} \oplus D_{\rho_{x_2}}$ so that by Proposition 3.1.48(ii) the ideal of total reducibility $\mathcal{A}_{12}\mathcal{A}_{21}$ vanishes. Therefore, $\mathcal{A}_{21} = 0$ and $\mathrm{im}\,\rho'$ is upper triangular. By hypothesis, ρ_{x_1} and ρ_{x_2} and their respective associated pseudorepresentations $D_{\rho_{x_1}}$ and $D_{\rho_{x_2}}$ are irreducible. By Theorem 3.1.26 constancy of $D_{\rho_{x_1}}$ and $D_{\rho_{x_2}}$ implies constancy of ρ_{x_1} and ρ_{x_2}. Since D_ρ is a trivial deformation of $D_{\rho'} = D_{\rho_{x_1}} \oplus D_{\rho_{x_2}}$ we deduce that the non-split extension ρ is constant on its diagonal blocks ρ_{x_1} and ρ_{x_2}.

To show in (v) that $\dim_L \ker \mathrm{d}\varphi = d n_1 n_2 - 1$, we consider lifts ρ_1, ρ_2 of ρ to $L[\varepsilon]$ whose associated deformation classes satisfy $[\rho_1] = [\rho_2] \in \ker \mathrm{d}\varphi \subset \mathbf{t}_{X_\rho^{\mathrm{univ}}} \cong \mathcal{D}_\rho(L[\varepsilon])$. By assertion (iv) we have $\rho_i = \rho + \varepsilon \begin{pmatrix} 0 & c_i \\ 0 & 0 \end{pmatrix}$ for some cocycle $c_i \in Z^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$. In order to obtain $\dim_L \ker \mathrm{d}\varphi$, we determine when ρ_1 is equivalent to ρ_2. In this case there exists a matrix $U \in \mathrm{Mat}_n(L)$ such that

$$\begin{aligned} \rho + \varepsilon \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} &= \rho_2 \\ &= (1 + \varepsilon U)\rho_1(1 - \varepsilon U) \\ &= (1 + \varepsilon U)(\rho + \varepsilon \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix})(1 - \varepsilon U) \\ &= \rho + \varepsilon(U\rho - \rho U + \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix}). \end{aligned}$$

If we write $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ with matrices $U_{ij} \in \mathrm{Mat}_{n_i \times n_j}(L)$ for $1 \leq i, j \leq 2$, then the above equality is equivalent to

$$\begin{pmatrix} 0 & c_2 - c_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U_{11}\rho_{x_1} & U_{11}c + U_{12}\rho_{x_2} \\ U_{21}\rho_{x_1} & U_{21}c + U_{22}\rho_{x_2} \end{pmatrix} - \begin{pmatrix} \rho_{x_1}U_{11} + cU_{21} & \rho_{x_1}U_{12} + cU_{22} \\ \rho_{x_2}U_{21} & \rho_{x_2}U_{22} \end{pmatrix}$$

Because $\dim_L H^0(G_K, \rho_{x_i} \otimes \rho_{x_j}^\vee) = 0$ and $\dim_L H^0(G_K, \rho_{x_i} \otimes \rho_{x_i}^\vee) = 1$ for $1 \le i, j \le 2$ and $i \neq j$, we deduce that $U_{21} = 0$ and that U_{11} and U_{22} are scalar matrices. Finally, the map

$$-\rho_{x_1}U_{12} + U_{12}\rho_{x_2} \in B^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$$

is a coboundary. Therefore, $c_2 = (U_{11} + U_{22})c + c_1 \in H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$ and

$$\dim_L \ker \mathrm{d}\varphi = \dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) - 1 = \dim_L H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) - 1 = dn_1n_2 - 1. \qquad (12)$$

Now (v) is immediate from (ii) and $\dim V = \dim \ker \psi + \dim \operatorname{im} \psi$ for a vector space V and a linear map ψ with domain V.

For (vi) consider the following diagram with left exact rows and where the middle and right vertical arrows are injective (by definition of $\mathbf{t}$):

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker\varphi & \longrightarrow & \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} & \xrightarrow{\mathrm{d}\varphi} & \mathbf{t}_{\widehat{X},D_\rho} \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \ker\varphi_{\mathrm{red}} & \longrightarrow & \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} & \xrightarrow{\mathrm{d}\varphi_{\mathrm{red}}} & \mathbf{t}_{(\widehat{X})_{\mathrm{red}},D_\rho}.
\end{array}$$

By a simple diagram chase one deduces $\ker\varphi_{\mathrm{red}} = \ker\varphi \cap \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} \subset \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho}$. Next consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\mathrm{d}\varphi & \longrightarrow & \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} & \longrightarrow & \operatorname{im}\mathrm{d}\varphi & \longrightarrow & 0 \\
 & & \alpha\uparrow & & \beta\uparrow & & \gamma\uparrow & & \\
0 & \longrightarrow & \ker\mathrm{d}\varphi_{\mathrm{red}} & \longrightarrow & \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} & \longrightarrow & \operatorname{im}\mathrm{d}\varphi_{\mathrm{red}} & \longrightarrow & 0
\end{array}$$

with exact rows and where the left and middle vertical arrows are injective. Because of $\ker\varphi_{\mathrm{red}} = \ker\varphi \cap \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho}$ the map γ is injective, and we deduce from the 9-Lemma that $\dim\operatorname{coker}\alpha + \dim\operatorname{coker}\gamma = \dim\operatorname{coker}\beta$. Now from (iii) we have $\dim R_\rho = \dim(R_\rho)_{\mathrm{red}} \in \{1+dn^2, 2+dn^2\}$ and $\dim \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} = dn^2+2$. One deduces that $\dim\operatorname{coker}\beta \in \{0,1\}$, and the assertions on $\dim\operatorname{coker}\alpha$ and $\dim\operatorname{coker}\gamma$ needed to complete (vi) are immediate. □

Proof of Theorem C (Theorem 3.3.1). We suppose to the contrary that there exists a nonempty open affine $V \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ such that $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \cap V = \varnothing$. Since $V \neq \operatorname{Spec}\mathbb{F}$ and the one-dimensional points are very dense in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$, by Lemma 2.1.20 there exists a 1-dimensional point $x \in V$ that defines a reducible pseudodeformation

$$D_x \colon G_K \longrightarrow \kappa(x)$$

of $\overline{D}$ such that $\kappa(x)$ is a local field. By Lemma 3.2.11(iii) there exist a finite extension $L'/\kappa(x)$ with finite residue field $\mathbb{F}' \supset \mathbb{F}$, residual pseudorepresentation $\overline{D}_i \colon G_K \to \mathbb{F}'$ of dimension n_i for some $n_i \in \mathbb{N}_0$ with $n_1 + n_2 = n$, and pseudorepresentations $D_1, D_2 \colon G_K \to \mathcal{O}_{L'}$ corresponding to points $(x_1, x_2) \in \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ such that $D_x \otimes_{\kappa(x)} L' = (D_1 \oplus D_2) \otimes_{\mathcal{O}_{L'}} L'$.

Now the inverse image of V under $\iota_{\overline{D},n_1,n_2}$ from (10) is an open neighbourhood of (x_1, x_2). It now follows from hypotheses (ii) and (iii) that we may assume the following

(a) if $\zeta_p \notin K$, then $x_i \in U_i := (\overline{X}^{\mathrm{univ}}_{\overline{D}_i})^{\mathrm{irr}, D(1)\neq D}$ for $i = 1, 2$;

(b) if $\zeta_p \in K$, then $x_i \in U_i := ((\overline{X}^{\mathrm{univ}}_{\overline{D}_i})^{\mathrm{irr}}_{\mathrm{red}})^{\mathrm{reg}}$ for $i = 1, 2$.

If $n_1 = n_2$, we may further assume $D_1 \neq D_2(m)$ for all $m \in \{0, 1, \ldots, p-2\}$, because the $\overline{X}^{\mathrm{univ}}_{\overline{D}_i}$ have dimension at least 2 and they contain exactly one closed point, and so the open U_i have to contain infinitely many dimension 1 points by Proposition 2.1.25. We also observe that by Lemma 3.2.38 the schemes U_i are regular in case (a).

Let $\overline{x}$ be a geometric point above x. By Lemma 3.3.3 there exists an étale neighbourhood (U, u) of $\overline{x}$ in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ together with an étale morphism $\varphi_U \colon U \to \overline{X}^{\mathrm{univ}}_{\overline{D}}$ such that the induced morphism

$$W := U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}, \iota_{\overline{D},n_1,n_2}} \overline{X}^{\mathrm{univ}}_{\overline{D},n_1,n_2} \xrightarrow{\iota_{\overline{D},n_1,n_2}} U = U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} \overline{X}^{\mathrm{univ}}_{\overline{D}}$$

is a closed immersion with image $U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}} (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$. We may replace U by $\varphi_U^{-1}(V)$, which is nonempty since $x \in V$, and étale over V, and we may shrink W accordingly. By further replacing U by an open subset (and accordingly W), we can assume that U is connected and affine. Since $W \to U$ is a closed immersion, the scheme W is affine. But we also have that $W \to U$ is surjective as a map of topological spaces, since all points of V are reducible. Hence the nilreduction of $W \to U$ is an isomorphism of schemes $W_{\mathrm{red}} \to U_{\mathrm{red}}$, and as a map of topological spaces $W \to U$ is a homeomorphism. Since the base change of étale morphisms is étale, so is the map $W \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ that is the base change of φ_U under $\iota_{\overline{D},n_1,n_2}$. We shrink W (and hence U) to an affine open so that the image of W in $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ lies in the image of $U_1 \times U_2$ in that scheme, and we write $W_0 \to U_1 \times U_2$ for the base change of φ_U along $U_1 \times U_2 \to \overline{X}^{\mathrm{univ}}_{\overline{D}}$. We display the situation in the following diagram:

$$\begin{array}{ccccccc}
U_1 \times U_2 & \hookrightarrow & \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2} & \xrightarrow{\iota_{\overline{D},n_1,n_2}} & \overline{X}^{\mathrm{univ}}_{\overline{D}} & \hookleftarrow & V \\
\uparrow & & \uparrow & & & \nearrow_{\varphi_U} & \\
W_0 & \longrightarrow & W & \longrightarrow & U & &
\end{array}$$

Since $U_1 \times U_2 \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ is a homeomorphism onto its image and an immersion, it follows that $W_0 \to W$ is a closed immersion and a homeomorphism. Since $U_1 \times U_2$ is regular, so is its étale cover W_0. We deduce that $W_0 \to W$ is the nilreduction morphism, and in particular $W_0 \to W_{\mathrm{red}} \to U_{\mathrm{red}}$ are isomorphisms of (regular) schemes. Let $w_0 \in W_0$ be the point corresponding to $u \in U$ under the homeomorphism $W_0 \to U$.

By Lemma 3.3.4 (i), there exists a nontrivial extension $\rho \colon G_K \to GL_n(L)$ such that

$$H^2(G_K, \mathrm{ad}_\rho)^\vee \cong H^0(G_K, \mathrm{ad}_\rho) \cong L. \tag{13}$$

Denote by $\widehat{R}^{\mathfrak{p}}$ the universal pseudodeformation ring for $D_\rho = \det \circ \rho$. Consider its universal pseudodeformation space $\widehat{X} := \operatorname{Spec} \widehat{R}^{\mathfrak{p}}$ and the universal deformation space $X^{\mathrm{univ}}_\rho := \operatorname{Spec} R_\rho$ of ρ. Then by Lemma 3.3.4 (iv) we have a canonical homomorphism

$$\varphi \colon X^{\mathrm{univ}}_\rho \longrightarrow \widehat{X}$$

By Lemma 3.3.4 (ii) and (iii) we have a good control of the dimensions R_ρ and by Lemma 3.3.4 (v) and (vi) the induced map of tangent spaces

$$\mathrm{d}\,\varphi \colon \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} \to \mathbf{t}_{\widehat{X}, x}.$$

Note also the important isomorphism $\widehat{R}^{\mathfrak{p}} = \widehat{\mathcal{O}}_{V,x}[\![T]\!]$ obtained by combining Corollary 3.2.13 and Lemma 2.2.12

We now display this data together with some completed local rings from the previous diagram in preparation for the completion of the present proof:

$$\begin{array}{ccccc} \widehat{\mathcal{O}}_{U_1\times U_2,(x_1,x_2)} & \longleftarrow & \widehat{\mathcal{O}}_{V,x} & \longrightarrow & \widehat{\mathcal{O}}_{V,x,\mathrm{red}} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_{\mathrm{red}}} \\ \widehat{\mathcal{O}}_{W_0,w_0} & \xleftarrow{\ \gamma\ } & \widehat{\mathcal{O}}_{U,u} & \longrightarrow & \widehat{\mathcal{O}}_{U,u,\mathrm{red}}, \end{array} \qquad \gamma_{\mathrm{red}}\colon \widehat{\mathcal{O}}_{U,u,\mathrm{red}} \xrightarrow{\ \simeq\ } \widehat{\mathcal{O}}_{W_0,w_0}$$

$$\begin{array}{ccc} \widehat{R}^{\mathfrak{p}} & \xrightarrow{\ \varphi^{\#}\ } & R_\rho \\ \downarrow & & \downarrow \\ \widehat{R}^{\mathfrak{p}}_{\mathrm{red}} & \xrightarrow{\ \varphi^{\#}_{\mathrm{red}}\ } & R_{\rho,\mathrm{red}}. \end{array}$$

Note that $\widehat{\mathcal{O}}_{V,x} \to \widehat{\mathcal{O}}_{U_1\times U_2,(x_1,x_2)}$ exists since the inverse image of the open V of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ under $\iota_{\overline{D},n_1,n_2}$ and the nilreduction morphism is an open neighbourhood of (x_1,x_2) in $U_1 \times U_2$. By Lemma 2.1.16, the maps α and β are finite étale. By what was observed above for $W_0 \to U$, the kernel of γ is nilpotent. Moreover the ring $\widehat{\mathcal{O}}_{U_1\times U_2,(x_1,x_2)}$ is regular by hypothesis and hence so is $\widehat{\mathcal{O}}_{W_0,w_0}$ again by Lemma 2.1.16, and both have the same dimension. From β being étale it follows that the induced map β_{red} is finite étale. Also the map $\gamma_{\mathrm{red}}\colon \widehat{\mathcal{O}}_{U,u,\mathrm{red}} \to \widehat{\mathcal{O}}_{W_0,w_0}$ is an isomorphism. Hence $\widehat{\mathcal{O}}_{U,u,\mathrm{red}}$ is regular and $\widehat{\mathcal{O}}_{V,x,\mathrm{red}}$ is regular and of the same dimension as $\widehat{\mathcal{O}}_{U_1\times U_2,(x_1,x_2)}$, again by Lemma 2.1.16. It follows that $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ is regular and that its Krull dimension is one more than that of $\widehat{\mathcal{O}}_{U_1\times U_2,(x_1,x_2)}$.

Suppose first that we are in case (a). Then from Lemma 3.3.4 (ii) we deduce that R_ρ is regular and hence isomorphic to $R_{\rho,\mathrm{red}}$. Moreover from Lemma 3.3.4 (v) we have $\dim_L \operatorname{im} \mathrm{d}\varphi = dn^2 - dn_1n_2 + 2$. Now $\varphi^{\#}$ factors via $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ which is regular and we deduce

$$dn^2 - dn_1n_2 + 2 \le \dim \widehat{R}^{\mathfrak{p}}_{\mathrm{red}} = \dim \widehat{\mathcal{O}}_{U_1\times U_2,(x_1,x_2)} + 1 = \dim \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2} = d(n_1^2 + n_2^2) + 2,$$

by the hypotheses (i) and (ii) – note that (x_1,x_2) is a regular point of dimension 1 on the equidimensional scheme $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$. This implies that $dn_1n_2 \le 0$, which is absurd. We reach a contradiction.

Suppose now that we are in case (b). Then from loc.cit. we have $\dim_L \operatorname{im} \mathrm{d}\varphi_{\mathrm{red}} = dn^2 - dn_1n_2 + 2 - \delta'$ for some $\delta' \in \{0,1\}$. Again using that $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ is regular, we deduce

$$dn^2 - dn_1n_2 + 2 - \delta' \le \dim \widehat{R}^{\mathfrak{p}}_{\mathrm{red}} = \dim \widehat{\mathcal{O}}_{U_1\times U_2,(x_1,x_2)} + 1 = d(n_1^2 + n_2^2) + 2,$$

by the hypotheses (i) and (iii). It now follows that $dn_1n_2 \le \delta'$. The only case possible is thus $\delta' = d = n_1 = n_2 = 1$, and from $\zeta_p \in K = \mathbb{Q}_p$ it also follows that $p = 2$. Hence the proof of the theorem is complete. □

3.3.2 An upper bound for the dimension of special points

In order to prove Theorem B (Theorem 3.3.12) and later Theorem E (Corollary 3.3.15), we now determine an upper bound for the dimension of the locus of special (irreducible) pseudodeformations. This makes use of the fact that for semisimple representations over algebraically closed fields being special is equivalent to being induced from H to G by Corollary 2.3.6.

Recall that K is a p-adic field with $d = [K : \mathbb{Q}_p]$. By $\chi\colon G_K \to \mathbb{F}^\times$ we denote the mod p cyclotomic character. We write $K(1) := K(\zeta_p)$ so that $G_{K(1)} = \ker \chi$ and we set $m = [G_K : G_{K(1)}]$.

We often abbreviate $G = G_K$ and $H = G_{K(1)}$. We fix an n-dimensional continuous pseudorepresentation $\overline{D}\colon G \to \mathbb{F}$, and we assume that $\mathbb{F}$ is sufficiently large so that $\overline{D}$ and its restriction $\mathrm{Res}^G_H \overline{D}$ are both split. Then both of them are the determinant of suitable representations defined over $\mathbb{F}$, cf. Theorem 3.1.26 and Definition 3.1.27. If $D = D \otimes \chi$, then it follows from the existence of these representations and from Remark 2.3.5 that for a suitable pseudorepresentation $\overline{D}'\colon H \to \mathbb{F}$ one has

$$\overline{D} = \mathrm{Ind}^G_H \overline{D}'. \tag{14}$$

In general, $\overline{D}'$ is not unique. In this subsection $\overline{D}'$ will always denote such a pseudorepresentation $H \to \mathbb{F}$, provided that it exists.

Definition 3.3.5. By $\overline{R}^{D=D(1)}_{K,\overline{D}}$ we denote the quotient of $\overline{R}^{\mathrm{univ}}_{K,\overline{D}}$ by the ideal defining the locus of points D such that $D = D(1)$ holds; cf. Corollary 3.2.32.

In this subsection we prove the following result:

Theorem 3.3.6. *One has*

$$\dim \overline{R}^{D=D(1)}_{K,\overline{D}} \leq \max_{\overline{D}'} \Big\{ \dim \overline{R}^{\mathrm{univ}}_{K(1),\overline{D}'} \mid \mathrm{Ind}^{G_K}_{G_{K(1)}} \overline{D}' = \overline{D} \Big\}$$

Denote further by Y the closure in $\mathrm{Spec}\,\overline{R}^{D=D(1)}_{K,\overline{D}}$ of the set of irreducible points and, analogously, for each $\overline{D}'$ such that (14) holds, by $Y_{\overline{D}'}$ the closure in $\overline{X}^{\mathrm{univ}}_{K(1),\overline{D}'}$ of the set of irreducible points – as topological spaces. Then one has

$$\dim Y \leq \max_{\overline{D}'} \{\dim Y_{\overline{D}'} \mid \mathrm{Ind}^{G_K}_{G_{K(1)}} \overline{D}' = \overline{D}\},$$

and moreover, the quantity on the right is bounded above by $\frac{dn^2}{m} + 2$.[6]

Lemma 3.3.7 ([Hoc07, Prop. (d),(g)])**.** *If R is a complete Noetherian local domain with perfect residue field $\mathbb{F}$ and fraction field $\mathbb{K}$, then $[\mathbb{K} : \mathbb{K}^p] < \infty$.*

Lemma 3.3.8. *Let R be a complete Noetherian local domain with finite residue field $\mathbb{F}$ and fraction field $\mathbb{K}$. Suppose that $D\colon G_K \to R$ is a pseudodeformation of $\overline{D}$ to R such that $D = D(1)$. Then the following hold:*

(i) There exists a finite extension $\mathbb{K}'$ of $\mathbb{K}$ so that $D \otimes_{\mathbb{K}} \mathbb{K}'$ is the pseudorepresentation attached to a representation $\rho\colon G_K \to GL_n(\mathbb{K}')$.

(ii) The field $\mathbb{K}'$ of (i) may be chosen so that furthermore $\mathrm{Res}^{G_K}_{G_{K(1)}} \rho = \bigoplus_{i \in \mathbb{Z}/(m)} \rho_i$ for pairwise non-isomorphic semisimple representations $\rho_i\colon G_{K(1)} \to \mathrm{GL}_{n/m}(\mathbb{K}')$ satisfying $\rho_{i+1} = \rho_i^g$ and $\rho = \mathrm{Ind}^{G_K}_{G_{K(1)}} \rho_i$ for all $i \in \mathbb{Z}/(m)$; and so that all simple summands of the ρ_i are absolutely irreducible. Up to isomorphism, the number of possible ρ_i is finite.

(iii) If $D \otimes_R \mathbb{K}$ is irreducible, then ρ in (i) and the ρ_i in (ii) are absolutely irreducible.

(iv) Denote by R' the integral closure of R in $\mathbb{K}'$. Then R' is finite over R, a complete Noetherian local domain with finite residue field $\mathbb{F}' \supseteq \mathbb{F}$, and the pseudorepresentation D_i attached to ρ_i takes values in R'.

[6] We expect the last bound to be equal to $\frac{dn^2}{m} + 1$. By Lemma 3.2.40, this bound holds if $p \nmid n$.

(v) For $i \in \mathbb{Z}/(m)$ the pseudorepresentation $\overline{D}_i := D_i \otimes_{R'} \mathbb{F}'$ satisfies $\overline{D}_{i+1} = \overline{D}_i^g$ and if $\overline{\rho}_i$ is any semisimple representation over $\mathbb{F}'$ with pseudorepresentation $\overline{D}_i$, then $\mathrm{Ind}_{G_{K(1)}}^{G_K} \overline{\rho}_i = \overline{\rho} \otimes_{\mathbb{F}} \mathbb{F}'$.

Proof. The argument for (i) is contained in [Che14, Thm. 2.12ff.]. By Theorem 3.1.28 due to Chenevier, there is an integer $r \in \mathbb{N}_{\geq 1}$, a simple $\mathbb{K}$-algebra S_i of finite dimension n_i^2 over its center k_i for $i = 1, \ldots, r$ and an isomorphism

$$\mathbb{K}[G_K]/\ker D \xrightarrow{\simeq} \prod_{i=1}^{r} S_i,$$

where each k_i is a finite field extension of $\mathbb{K}$ of degree at most n since $[\mathbb{K} : \mathbb{K}^p] < \infty$ by Lemma 3.3.7; moreover $D = \bigoplus_{i=1}^r \det_{S_i}^{m_i}$ for $m_i > 0$ such that $n = \sum_i m_i n_i f_i q_i$ as in Theorem 3.1.28. By [Gro68, Cor. 3.8] there is a finite separable extension of k_i of degree at most n_i that splits S_i. We let $\mathbb{K}'$ be a finite extension of $\mathbb{K}$ the contains splitting fields for all S_i. Then (i) follows.

Part (ii) follows from Corollary 2.3.6, after possibly enlarging $\mathbb{K}'$ according to the last paragraph of Remark 2.3.5. The finiteness assertion follows from the finiteness (up to isomorphism) of simple factors of $\mathrm{Res}_{K(1)}^{K} \rho \otimes_{\mathbb{K}'} \mathbb{K}^{\mathrm{alg}}$. Part (iii) follows from (ii), the definition of irreducible for pseudorepresentations and from Theorem 2.3.4.

For (iv), by Proposition 3.1.14 it suffices so show that the characteristic polynomial coefficients of D_i lie in R' for all $i \in \mathbb{Z}/(m)$. For all $g \in G_{K(1)}$ we have $\chi_D(g,t) = \prod_{i \in \mathbb{Z}/(m)} \chi_{D_i}(g,t)$, where $\chi_D(g,t)$ is a monic polynomial in $R[t]$ and for all $i \in \mathbb{Z}/(m)$ $\chi_{D_i}(g,t)$ is a monic polynomial in $\mathbb{K}'[t]$. Then all roots of $\chi_{D_i}(g,t)$ for any $i \in \mathbb{Z}/(m)$ are integral over R and thus the characteristic polynomial coefficients are also integral over R. By normality of R' in $\mathbb{K}'$ the characteristic polynomial coefficients lie in R'; cf. [Mat89, Thm. 9.2]. Finally, (v) follows from (ii) and (iv). □

Proof of Theorem 3.3.6. Let $D_{K,\overline{D}}^{\mathrm{univ}} \colon G_K \to \overline{R}_{K,\overline{D}}^{\mathrm{univ}}$ be the universal mod p pseudodeformation of the pseudorepresentation $\overline{D} \colon G_K \to \mathbb{F}$, and define

$$D_{\overline{D}}^{\mathrm{tw}} := D_{K,\overline{D}}^{\mathrm{univ},D=D(1)} \colon G_K \longrightarrow \overline{R}_{K,\overline{D}}^{\mathrm{univ},D=D(1)} =: \overline{R}_{\overline{D}}^{\mathrm{tw}}$$

as the composition of $D_{K,\overline{D}}^{\mathrm{univ}}$ with $\overline{R}_{K,\overline{D}}^{\mathrm{univ}} \to \overline{R}_{\overline{D}}^{\mathrm{tw}}$, where $\overline{R}_{\overline{D}}^{\mathrm{tw}}$ is the quotient of $\overline{R}_{K,\overline{D}}^{\mathrm{univ}}$ parametrizing mod p pseudodeformations with $D = D(1)$. We denote by $D_{K(1),\overline{D}'}^{\mathrm{univ}} \colon G_{K(1)} \to \overline{R}_{K(1),\overline{D}'}^{\mathrm{univ}}$ the universal mod p pseudodeformation for some $\overline{D}' \colon G_{K(1)} \to \mathbb{F}$ such that $\mathrm{Ind}_{G_{K(1)}}^{G_K} \overline{D}' = \overline{D}$. We add the subscript red to a ring to denote its reduced quotient. Then by Theorem 3.2.23 we have an induced pseudorepresentation

$$D_{\overline{D}'}^{\mathrm{ind}} := \mathrm{Ind}_H^G D_{K(1),\overline{D}'}^{\mathrm{univ}} \colon G_K \to \overline{R}_{K(1),\overline{D}',\mathrm{red}}^{\mathrm{univ}} =: \overline{R}_{\overline{D}',\mathrm{red}}.$$

By Corollary 3.2.31 we have $D_{\overline{D}'}^{\mathrm{ind}} = D_{\overline{D}'}^{\mathrm{ind}}(1)$, and so by the universal property of $D_{\overline{D}}^{\mathrm{tw}}$ there exists a unique homomorphism $\overline{R}_{\overline{D}}^{\mathrm{tw}} \to \overline{R}_{\overline{D}',\mathrm{red}}$ which when precomposed with $D_{\overline{D}}^{\mathrm{tw}}$ gives $D_{\overline{D}'}^{\mathrm{ind}}$.

Let now $\mathfrak{p}$ be a minimal prime of $\overline{R}_{\overline{D}}^{\mathrm{tw}}$, denote by $B_{\mathfrak{p}}$ the quotient $\overline{R}_{\overline{D}}^{\mathrm{tw}}/\mathfrak{p}$ and by $\kappa(\mathfrak{p})$ the quotient field of the latter. We consider the induced pseudorepresentation $D_{\mathfrak{p}} \colon G \to B_{\mathfrak{p}} \to \kappa(\mathfrak{p})$. We choose a finite extension $\kappa(\mathfrak{p})'$ of $\kappa(\mathfrak{p})$ according to Lemma 3.3.8 and denote by $B_{\mathfrak{p}}'$ the normalization of $B_{\mathfrak{p}}$ in $\kappa(\mathfrak{p})'$ such that

$$D_{\mathfrak{p}} \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})' \cong \bigoplus_{g \in G/H} (D_{\mathfrak{p}}')^g \otimes_{B_{\mathfrak{p}}'} \kappa(\mathfrak{p})' \tag{15}$$

for a suitable pseudorepresentation $D'_{\mathfrak{p}}\colon H \to B'_{\mathfrak{p}}$. Let $\mathbb{F}'$ be the residue field of $B'_{\mathfrak{p}}$; it is a finite extension of $\mathbb{F}$. Note that (15) also holds without tensoring with $\kappa(\mathfrak{p})'$ if we regard $D_{\mathfrak{p}}$ as a pseudorepresentation into $B'_{\mathfrak{p}}$. The latter formula we may reduce from $B'_{\mathfrak{p}}$ to $\mathbb{F}'$. It follows, cf. Theorem 3.2.23, that the reduction $D'_{\mathfrak{p}} \otimes_{B'_{\mathfrak{p}}} \mathbb{F}'$ is isomorphic to $\overline{D}' \otimes_{\mathbb{F}} \mathbb{F}'$ for some pseudorepresentation $\overline{D}'\colon G \to \mathbb{F}$ such that $\overline{D} = \operatorname{Ind}_H^G \overline{D}'$.

Let $B''_{\mathfrak{p}}$ be the subring of $B'_{\mathfrak{p}}$ of elements that reduce to $\mathbb{F}$. Then $B''_{\mathfrak{p}}$ lies in $\widehat{\mathcal{A}r}_{\mathbb{F}}$ and $D'_{\mathfrak{p}}$ takes its values in $B''_{\mathfrak{p}}$. By the universality of $D^{\mathrm{univ}}_{K(1),\overline{D}'}$ there is a unique homomorphism $\overline{R}_{\overline{D}',\mathrm{red}} \to B''_{\mathfrak{p}}$, by which $D'_{\mathfrak{p}}$ is induced from $D^{\mathrm{univ}}_{K(1),\overline{D}'}$. Denote by $B'''_{\mathfrak{p}}$ the image of $\overline{R}_{\overline{D}',\mathrm{red}}$ in $B''_{\mathfrak{p}}$. The ring $B'''_{\mathfrak{p}}$ is a domain and so applying again induction for pseudorepresentations, Theorem 3.2.23 gives a pseudorepresentation $\operatorname{Ind}_H^G D'_{\mathfrak{p}}\colon G \to B'''_{\mathfrak{p}}$ that deforms $\overline{D}$ by Theorem 3.2.23(f). By the universality of $D^{\mathrm{tw}}_{\overline{D}}$ and our construction of $B'_{\mathfrak{p}}$ we obtain necessarily injective ringhomomorphisms

$$B_{\mathfrak{p}} \hookrightarrow B'''_{\mathfrak{p}} \hookrightarrow B''_{\mathfrak{p}} \hookrightarrow B'_{\mathfrak{p}},$$

such that $\operatorname{Ind}_H^G D'_{\mathfrak{p}}$ is the composition of $D_{\mathfrak{p}}$ with $B_{\mathfrak{p}} \to B'''_{\mathfrak{p}}$. Since $B_{\mathfrak{p}} \to B'_{\mathfrak{p}}$ is finite by Lemma 3.3.8, the same holds for $B_{\mathfrak{p}} \to B'''_{\mathfrak{p}}$, so that $\dim B_{\mathfrak{p}} \leq \dim B'''_{\mathfrak{p}}$. The ring $B'''_{\mathfrak{p}}$ is a quotient of $R^{\mathrm{univ}}_{K(1),\overline{D}'}$ and $B_{\mathfrak{p}} = \overline{R}^{\mathrm{tw}}_{\overline{D}}/\mathfrak{p}$. Since $\mathfrak{p} \in \operatorname{Min}(\overline{R}^{\mathrm{tw}}_{\overline{D}})$ is arbitrary, this proves the first inequality.

The second inequality follows by the same argument as the first inequality: We note that the irreducible locus is open by Lemma 3.2.2, and hence it is dense open in Y and in the $Y_{\overline{D}'}$ by their very definition. This implies that the pseudorepresentations $D_{\mathfrak{p}}$ that we consider will be irreducible and by Lemma 3.3.8 that also the pseudorepresentations $D'_{\mathfrak{p}}$ will be irreducible. Hence to obtain the upper bound for $\dim Y$ it suffices to minimize over a possibly smaller set of $\overline{D}'$ and for each $\overline{D}'$ it suffices to only consider the space $Y_{\overline{D}'}$. Finally, the bound at the very end follows by applying Lemma 3.2.39 to each of the $Y_{\overline{D}'}$. ☐

By Theorem C (Theorem 3.3.1) the locus of irreducible points is Zariski dense under a certain induction hypothesis. If $\zeta_p \notin K$ we now prove that also the locus of nonspecial irreducible points defined in Definition 3.2.34 is dense:

Corollary 3.3.9. *Suppose that $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$ is dense in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$. Then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr},D(1)\neq D}$ is dense open in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$.*

Proof. Recall from Lemma 3.2.2, that the subset $U := (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$ of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ is open. By our hypothesis U is also dense. We assume that $\overline{D}(1) = \overline{D}$ since otherwise there is nothing to show – as $\overline{D} \neq \overline{D}(1)$ implies the same for all pseudodeformations. In particular this implies $2 \leq m$. We also recall that the locus $X := (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{D(1)=D}$ is closed in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ by Proposition 3.2.36.

We need to show that $X \cap U$ contains no open subset V of U. We argue by contradiction and assume that V exists. As a subset of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ the set V is locally closed and hence points of dimension 1 will be dense in it, since only the single closed point of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ has dimension zero. Let $x \in V$ be a point of dimension 1. Denote by $\widehat{R}_x$ the completion of the local ring $\mathcal{O}_{V,x}$. Because $V \subset X$ and $\dim x = 1$, the last bound in Theorem 3.3.6 implies

$$\dim \widehat{R}_x \leq \dim X - 1 \leq \frac{dn^2}{m} + 1.$$

On the other hand, V is open in U, and it follows from Lemma 3.2.39 and Lemma 2.2.12 that $\dim \widehat{R}_x \geq dn^2$. We deduce $dn^2 \leq \frac{dn^2}{m} + 1$, which is absurd since $2 \leq m$ and $m|n$. ☐

3.3.3 Equidimensionality of universal mod p pseudodeformation rings

Under certain hypotheses on the n and on $[K(\zeta_p) : K]$, we now inductively prove the equidimensionality of the universal mod p pseudodeformation ring $\overline{X}^{\mathrm{univ}}_{\overline{D}} := \operatorname{Spec} \overline{R}^{\mathrm{univ}}_{\overline{D}}$ of a fixed n-dimensional residual pseudorepresentation $\overline{D}\colon G_K \to \mathbb{F}$. Also we identify a dense open regular subspace of $\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$.

Assumption 3.3.10. By replacing $\mathbb{F}$ with a finite extension that depends only on the dimension of $\overline{D}$, we shall assume that $\overline{D}$ as well as $\overline{D}|_{G_{K(1)}}$ are split over $\mathbb{F}$; cf. Corollary 3.1.71.

We will proceed by induction on the dimension of the residual pseudorepresentation.

Proposition 3.3.11 (Base case)**.** *Suppose $\overline{D}$ has dimension 1, so that $\overline{D}$ is a 1-dimensional character and pseudodeformations are deformations and vice versa. Let $e \in \mathbb{N}_0$ be maximal such that $\zeta_{p^n} \in K$. Then the following hold:*

(i) $R^{\mathrm{univ}}_{\overline{D}} \cong W(\mathbb{F})[\zeta_{p^e}][\![X_1, \ldots, X_{d+1}]\!]$.

(ii) $\overline{R}^{\mathrm{univ}}_{\overline{D}} \cong \mathbb{F}[\![X_1, \ldots, X_{d+1}]\!]$ *if* $\zeta_p \notin K$.

(iii) $\overline{R}^{\mathrm{univ}}_{\overline{D},\mathrm{red}} \cong \mathbb{F}[\![X_1, \ldots, X_{d+1}]\!]$ *for any finite extension K of $\mathbb{Q}_p$.*

Proof. We regard $\overline{D}$ as a 1-dimensional Galois representation. Then the shape of the universal deformation ring of $\overline{D}$ is well-known; see Corollary 2.2.18 and its proof. This proves (i). Parts (ii) and (iii) are straightforward consequences. □

We now prove the main result of this work:

Theorem 3.3.12 (Theorem B)**.** *Suppose that $p > n$ or that $[K(\zeta_p) : K] > 1$. Then for any n-dimensional pseudorepresentation $\overline{D}\colon G_K \to \mathbb{F}$ the following holds:*

(i) $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ *is equidimensional of dimension $dn^2 + 1$,*

(ii) if $\zeta_p \notin K$, then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr},D(1)\neq D}$ is open, regular and Zariski dense in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$,

(iii) if $\zeta_p \in K$, then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{reg}} = \varnothing$ and $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}}$ is open, regular and Zariski dense in $\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$.

Let us note that for $p = 2$ the above theorem only carries the case $n = 1$.

Proof. Recall that we $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{reg}} = \varnothing$ if $\zeta_p \in K$ by Proposition 3.2.41. For the other assertions we proceed by induction. The base case $n = 1$ is covered by Proposition 3.3.11. Suppose now that $n < p$ or $1 < [K(\zeta_p) : K]$ and that (i)–(iii) hold for all pseudorepresentations $\overline{D}'\colon G_{K'} \to \mathbb{F}'$ of dimension $n' \in \{1, \ldots, n-1\}$ and such that $n' < p$ or $1 < [K'(\zeta_p) : K']$. It then follows from Theorem C (Theorem 3.3.1) that $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ is Zariski dense (in the case $p = 2$ we must have $n = 1$, and so there is nothing to prove).

Suppose first that $\zeta_p \notin K$. Then it follows from Corollary 3.3.9 that $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr},D(1)\neq D}$ is open and Zariski dense in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$. In this situation assertions (i) and (ii) follow from Lemma 3.2.38; note that the open U from Lemma 3.2.38 has dimension 1 less than $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ since it does not contain the unique closed point $\mathfrak{m}_{\overline{R}^{\mathrm{univ}}_{\overline{D}}}$ of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$.

Suppose now that $\zeta_p \in K$. Then $n < p$, and it follows from Lemma 3.2.40 that $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}}$ lies in $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{reg}}$ and that $U := (\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}} \smallsetminus \{\mathfrak{m}_{\overline{R}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}}\}$ is equidimensional of dimension dn^2. It follows that $\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$ is equidimensional and of dimension $dn^2 + 1$. □

3.3.4 *Zariski density of the regular locus and Serre's condition* (R_2)

Throughout this subsection we consider the universal mod p pseudodeformation space $\overline{X}^{\mathrm{univ}}_{\overline{D}} := \operatorname{Spec} \overline{R}^{\mathrm{univ}}_{\overline{D}}$ of a fixed residual pseudodeformation $\overline{D}\colon G_K \to \mathbb{F}$ of dimension n.

If $\zeta_p \in K$, then by Proposition 3.2.41 the regular locus of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ is empty. If $\zeta_p \notin K$, then we will now describe its regular locus and show that $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ satisfies Serre's condition (R_2).

Theorem 3.3.13 (Theorem D; cf. [Che11, Thm. 2.3])**.** *If $\zeta_p \notin K$, then the following hold:*

(i) The closure of $X_1 := (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr},D(1)=D}$ lies in $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{sing}}$.

(ii) If $n > 2$ or $[K : \mathbb{Q}_p] > 1$, then $X_2 := (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} \subset (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{sing}}$.[7]

Proof. We know from Proposition 3.1.60 that $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ is a complete Noetherian local ring so that by a theorem of Nagata [EGA IV$_2$, Thm. (6.12.7)] its singular locus $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{sing}}$ is closed in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$. By Proposition 3.2.36 and Corollary 2.1.21, the points of X_i of dimension at most one are dense in X_i, and since $\dim X_i > 0$, in fact the points of dimension equal to one will be dense. Note also that case (ii) is concerned with the image of the spaces $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ under the maps $\iota_{\overline{D},n_1,n_2}$ for all pairs (n_1, n_2) with $n_i \geq 1$ and $n_1 + n_2 = n$ and $(\overline{D}_1, \overline{D}_2)$ with $\overline{D}_1 \oplus \overline{D}_2 = \overline{D}$ as in (10) and (9). As in the proof of Theorem C (Theorem 3.3.1), to prove (ii) it will suffice to consider points $x = \iota_{\overline{D},n_1,n_2}(x_1, x_2)$ of dimension 1 such that the x_i represent n_i-dimensional irreducible pseudorepresentations D_i that lie in $(\overline{X}^{\mathrm{univ}}_{\overline{D}_i})^{\mathrm{irr},D(1)\neq D}$ and such that D_1 is not isomorphic to $D_2(l)$ for any $l \in \{0, 1, \ldots, p-2\}$.

For points x of the shape identified above and a corresponding absolutely irreducible representation ρ_x for D_x in case (i) and a pair of absolutely irreducible representations ρ_{x_i} for D_{x_i}, $i = 1, 2$, of dimension n_i in case (ii), defined over a finite extension L of $\kappa(x)$ so that $\det \circ \rho_x = D_x$, we now show that the local rings $\widehat{\mathcal{O}}_{X_i,x}$ are not regular. To do so we shall show that their tangent space dimension exceeds $\dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - \dim x = dn^2$. For this we shall prove that the tangent space dimension of the universal deformation ring $\overline{R}^{\mathrm{univ}}_{\rho_x}$ in case (i) and the universal pseudodeformation ring $\overline{R}^{\mathrm{univ}}_{D_x}$ in case (ii) are larger than $\dim \overline{X}^{\mathrm{univ}}_{\overline{D}} = dn^2 + 1$, which is equal to $\dim \overline{R}^{\mathrm{univ}}_{\rho_x}$ and $\dim \overline{R}^{\mathrm{univ}}_{D_x}$, respectively. To see that it is sufficient to show the singularity of the latter universal rings, we make use of Lemma 2.2.12 and Corollary 3.2.13, as well as of Theorem B (Theorem 3.3.12) and Proposition 3.2.14.

Let us first consider case (i). The required tangent space computation is standard and proceeds as in the proof of Lemma 3.3.4(ii): Using that

$$\dim_L H^2(G_K, \mathrm{ad}_{\rho_x}) = \dim_L H^0(G_K, \mathrm{ad}_{\rho_x}(1)) = 1$$

since $D_x = D_x(1)$ one deduces $\dim H^1(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + 2$, and thus $\overline{R}^{\mathrm{univ}}_{\rho_x}$ cannot be regular.

Let us now consider case (ii). To compute the tangent space dimension of $\overline{R}^{\mathrm{univ}}_{D_x}$, we make use of [Bel12, Thm. A]. It provides an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(G_K, \mathrm{ad}_{\rho_{x_1}}) \oplus H^1(G_K, \mathrm{ad}_{\rho_{x_2}}) &\longrightarrow \dim \mathbf{t}_{\operatorname{Spec} \overline{R}^{\mathrm{univ}}_{D_x}, x} \otimes_{\kappa(x)} L \\ \longrightarrow \mathrm{Ext}^1_{G_K}(\rho_{x_1}, \rho_{x_2}) \otimes \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) &\xrightarrow{h} \mathrm{Ext}^2_{G_K}(\rho_{x_1}, \rho_{x_2}) \oplus \mathrm{Ext}^2_{G_K}(\rho_{x_1}, \rho_{x_2}), \end{aligned}$$

[7] See Remark 3.3.2 for the case $n = 2$ and $K = \mathbb{Q}_p$.

where the map h is given by the Yoneda product. We conclude as in the proof of [Che11, Lem. 2.4]: $\dim_L H^1(G_K, \mathrm{ad}_{\rho_{x_i}}) \geq 1 + dn_i^2$, $\dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_i}, \rho_{x_{3-i}}) = dn_1n_2$, and the second extension groups vanish. Hence

$$\mathbf{t}_{\mathrm{Spec}\,\overline{R}^{\mathrm{univ}}_{D_x}} = d(n_1^2 + n_2^2) + 2 + d^2n_1^2n_2^2 \geq dn^2 + 1 + (dn_1n_2 - 1)^2.$$

This dimension is strictly larger than $dn^2 + 1$, unless $dn_1n_2 = 1$, i.e., $n_1 = n_2 = 1$ and $K = \mathbb{Q}_p$. This proves the claim in case (ii). □

The following result will give a lower bound for the codimension of the singular locus X^{sing}.

Lemma 3.3.14. *If either $n > 2$, or $n = 2$ and $d > 1$, then one has*

$$\max\{\dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}, \dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{D=D(1)}\} \leq \dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - 3.$$

Proof. We have $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} \subset \bigcup_{n_1+n_2=n} \iota_{n_1,n_2}(\overline{X}_{n_1} \times_{\mathbb{F}} \overline{X}_{n_2})$. Therefore,

$$\dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} \leq \max_{n_1+n_2=n} \dim \overline{X}_{n_1} + \dim \overline{X}_{n_2} = \max_{n_1+n_2=n} d(n_1^2 + n_2^2) + 2 = d((n-1)^2 + 1) + 2,$$

and therefore

$$\dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - \dim(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} \geq dn^2 + 1 - d(n^2 - 2n + 2) - 2 \;=\; 2d(n-1) - 1,$$

which is at least 3 unless $n = 2$ and $d = 1$, in which case the bound is only 1. To complete the proof we appeal to Theorem 3.3.6. It gives $\frac{dn^2}{m} + 2$ for an upper bound of the closure X of $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{D=D(1)} \smallsetminus (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ where $m = [K(1) : K] \geq 2$ and m divides n. Hence now

$$\dim \overline{X}^{\mathrm{univ}}_{\overline{D}} - \dim X \geq dn^2 + 1 - \Big(\frac{dn^2}{m} + 2\Big) \geq \frac{dn^2}{2} - 1,$$

and again this quantity is at least 3 if either $n > 2$, or if $d > 1$ and $n = 2$. □

Corollary 3.3.15 (Theorem E)**.** *Suppose that $\zeta_p \notin K$ and that either $n > 2$, or that $n = 2$ and $d > 1$. Then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr},D(1)\neq D}$ constitutes the regular locus of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ and it follows that $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ satisfies Serre's condition (R_2).*

Proof. This follows from Lemma 3.3.14, Theorem 3.3.13 (Theorem D) and Theorem B (Theorem 3.3.12). □

Journal of Algebra 444 (2015) 81–123

4. IRREDUCIBILITY OF VERSAL DEFORMATION RINGS IN THE (p,p)-CASE FOR 2-DIMENSIONAL REPRESENTATIONS

Gebhard Böckle[1] , Ann-Kristin Juschka
University of Heidelberg, IWR, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany

ARTICLE INFO

Article history:
Received 15 April 2014
Available online 10 August 2015
Communicated by Eva Bayer-Fluckiger

Keywords:
Galois representations
Local fields
Deformation theory
Crystalline representations

ABSTRACT

Let G_K be the absolute Galois group of a finite extension K of $\mathbb{Q}_p$ and $\bar{\rho}: G_K \to \mathrm{GL}_n(\mathbb{F})$ a continuous residual representation for $\mathbb{F}$ a finite field of characteristic p. We investigate whether the versal deformation space $\mathfrak{X}(\bar{\rho})$ of $\bar{\rho}$ is irreducible. For $n = 2$ and $p > 2$ we obtain a complete answer in the affirmative based on the results of [Böc00, Böc10]. As a consequence we deduce from recent results of Colmez, Kisin and Nakamura [Col08, Kis10b, Nak13] that for $n = 2$ and $p > 2$ crystalline points are Zariski dense in the versal deformation space $\mathfrak{X}(\bar{\rho})$.

4.1 *Introduction and statement of main results*

Deformation rings of Galois representations

Let G_K be the absolute Galois group of a finite extension K of $\mathbb{Q}_p$ and let $\bar{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ be a continuous residual representation for $\mathbb{F}$ a finite field of characteristic p. Let $W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$. We shall always write $\mathcal{O}$ for the ring of integers of a finite totally ramified extension of $W(\mathbb{F})[1/p]$ and denote by $\mathfrak{m}_{\mathcal{O}}$ its maximal ideal and by $\varpi_{\mathcal{O}}$ a uniformizer. To simplify notation, we shall write $\mathcal{O}_i$ for the quotient $\mathcal{O}/\varpi_{\mathcal{O}}^i\mathcal{O}$ for any integer $i \geq 1$. Denote by ad the adjoint representation of $\bar{\rho}$, i.e., the representation on $M_n(\mathbb{F})$ induced from $\bar{\rho}$ by conjugation and by ad^0 the subrepresentation on trace zero matrices.

For $\bar{\rho}$ as above, we consider the deformation functor from the category $\widehat{\mathcal{A}r}_{\mathcal{O}}$ of complete Noetherian local $\mathcal{O}$-algebras $(R, \mathfrak{m}_R)$ to the category of sets defined by

$$D_{\bar{\rho}}(R) := \{\rho\colon G_K \to \mathrm{GL}_n(R) \mid \rho \mod \mathfrak{m}_R = \bar{\rho} \text{ and } \rho \text{ is a cont. repr.}\}/\sim$$

where $\rho \sim \rho'$ if there exists $A \in \ker\big(\mathrm{GL}_n(R) \overset{\mathrm{mod}\ \mathfrak{m}_R}{\longrightarrow} \mathrm{GL}_n(\mathbb{F})\big)$ such that $\rho' = A\rho A^{-1}$. An equivalence class $[\rho]$ of ρ under $\sim$ is called a *deformation of* $\bar{\rho}$. Since G_K satisfies the finiteness condition Φ_p from [Maz89, § 1.1], by [Maz89, Prop. 1] with a slight strengthening by [Ram93] one deduces:

[1] Corresponding author.
E-mail addresses: gebhard.boeckle@iwr.uni-heidelberg.de (G. Böckle), ann-kristin.juschka@iwr.uni-heidelberg.de (A.-K. Juschka).

http://dx.doi.org/10.1016/j.jalgebra.2015.07.009

Theorem 4.1.1. *The functor $D_{\bar\rho}$ always possesses a versal hull $(R_{\bar\rho}, \mathfrak{m}_{\bar\rho})$ which is unique up to isomorphism. If in addition $\mathrm{End}_{G_K}(\bar\rho) = \mathbb{F}$, then $D_{\bar\rho}$ is representable and in particular $(R_{\bar\rho}, \mathfrak{m}_{\bar\rho})$ is unique up to unique isomorphism.*

We denote by $\rho_{\bar\rho}\colon G_K \to \mathrm{GL}_n(R_{\bar\rho})$ a representative of the versal class.

For later use, we recall parts of the obstruction theory related to $D_{\bar\rho}$. Suppose we are given a short exact sequence

$$0 \longrightarrow J \longrightarrow R_1 \longrightarrow R_0 \longrightarrow 0,$$

where the morphism $R_1 \to R_0$ is in $\widehat{\mathcal{A}r}_{\mathcal{O}}$, and $\mathfrak{m}_1 \cdot J = 0$ for $\mathfrak{m}_1$ the maximal ideal of R_1; such a diagram is called a *small extension of* R_0. Suppose further that we are given a deformation of $\bar\rho$ to R_0 represented by $\rho_0 : G_K \to \mathrm{GL}_n(R_0)$. Then Mazur defines a canonical obstruction class

$$\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}) \otimes J$$

that vanishes if and only if ρ_0 can be lifted to a deformation $\rho_1 : G_K \to \mathrm{GL}_n(R_1)$ of $\bar\rho$, see [Maz89, p. 398]. By elementary linear algebra, the obstruction class $\mathcal{O}(\rho_0)$ defines an obstruction homomorphism $\mathrm{obs}\colon \mathrm{Hom}_{\mathbb{F}}(J, \mathbb{F}) \to H^2(G_K, \mathrm{ad})$, and conversely from the latter one can recover $\mathcal{O}(\rho_0)$.

The following result describes the mod $\mathfrak{m}_{\mathcal{O}}$ tangent space of $R_{\bar\rho}$ and a bound on the number of generators of an ideal in a minimal presentation of $R_{\bar\rho}$ by a power series ring over $\mathcal{O}$.

Proposition 4.1.2 ([Maz89])**.** *(a) If $\mathbb{F}[\varepsilon]$ denotes the ring of dual numbers of $\mathbb{F}$ and $\overline{\mathfrak{m}}_{\bar\rho} := \mathfrak{m}_{\bar\rho}/\mathfrak{m}_{\mathcal{O}} R_{\bar\rho}$, then one has canonical isomorphisms between the two tangent spaces*

$$t_{D_{\bar\rho}} := D_{\bar\rho}(\mathbb{F}[\varepsilon]) \cong H^1(G_K, \mathrm{ad}) \cong \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\bar\rho}/\overline{\mathfrak{m}}_{\bar\rho}^2, \mathbb{F}\right) =: t_{R_{\bar\rho}}.$$

(b) Let $h_1 := \dim_{\mathbb{F}} H^1(G_K, \mathrm{ad})$, let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}[\![x_1, \ldots, x_{h_1}]\!]$ and let

$$0 \longrightarrow I \longrightarrow \mathcal{O}[\![x_1, \ldots, x_{h_1}]\!] \xrightarrow{\pi} R_{\bar\rho} \longrightarrow 0$$

be a presentation of $R_{\bar\rho}$. Then the obstruction homomorphism

$$\mathrm{obs}\colon \mathrm{Hom}_{\mathbb{F}}(I/\mathfrak{m}I, \mathbb{F}) \longrightarrow H^2(G_K, \mathrm{ad}), \quad f \longmapsto (1 \otimes f)(\mathcal{O}(\rho_{\bar\rho})),$$

is injective, and thus $\dim_{\mathbb{F}} H^2(G_K, \mathrm{ad})$ bounds the minimal number $\dim_{\mathbb{F}} I/\mathfrak{m}I$ of generators of I.

If in a presentation as in (b) the number of variables is minimal, i.e., if the mod $\mathfrak{m}_{\mathcal{O}}$ tangent space of $\mathcal{O}[\![x_1, \ldots, x_{h_1}]\!]$ is isomorphic to that of $R_{\bar\rho}$, then we call the presentation *minimal*. Now fix a character $\psi\colon G_K \to \mathcal{O}^*$ which reduces to $\det \bar\rho$ and denote by $D_{\bar\rho}^{\psi}$ the subfunctor of $D_{\bar\rho}$ of deformations whose determinant is equal to ψ (under the canonical homomorphism $\mathcal{O} \to R$).

Proposition 4.1.3. *If $p \nmid n$, then the results of Theorem 4.1.1 and Proposition 4.1.2 hold for $D_{\bar\rho}^{\psi}$ as well, if one replaces* ad *by the adjoint representation* ad^0 *on trace zero matrices, the pair $(R_{\bar\rho}, \mathfrak{m}_{\bar\rho})$ by the versal deformation ring $(R_{\bar\rho}^{\psi}, \mathfrak{m}_{\bar\rho}^{\psi})$ and the ideal I by a* relation ideal *I^{ψ} in a minimal presentation*

$$0 \longrightarrow I^{\psi} \longrightarrow \mathcal{R} := \mathcal{O}[\![x_1, \ldots, x_h]\!]^{2} \longrightarrow R_{\bar\rho}^{\psi} \longrightarrow 0 \qquad \text{with } h = \dim_{\mathbb{F}} H^1(G_K, \mathrm{ad}^0). \quad (1)$$

[2] To avoid notation such as $\mathfrak{m}_{\mathcal{R}^{\psi}}$, $\mathfrak{m}_{\mathcal{R}^{\psi}}$, we use the simpler notation $\mathcal{R}$ instead of $\mathcal{R}^{\psi}$ for the frequently used ring $\mathcal{R}$

This article presents three results on the deformation rings $R_{\bar\rho}$ and $R^\psi_{\bar\rho}$ introduced above:

For $n=2$, we improve the ring theoretic results from [Böc00] by showing that the rings $R^\psi_{\bar\rho}$ are integral domains. On the technical side, we clarify that for this result and the main results in [Böc00] the knowledge of a suitably defined *(refined) quadratic part*, see Definition 4.2.4, of the relation in a minimal presentation of $R^\psi_{\bar\rho}$ suffices.

Using the irreducibility of $R^\psi_{\bar\rho}$, we deduce the Zariski density of crystalline points in $\operatorname{Spec} R_{\bar\rho}$ for $n=2$, $p>2$ and any p-adic local field K.

For many n and K we give a cohomological description of the quadratic parts of the relations in a minimal presentation of $R^\psi_{\bar\rho}$ via a cup product and a Bockstein formalism in the context of Galois cohomology of p-adic fields.

We now explain these results in greater detail. From now on we assume that $p>2$.

Ring-theoretic results on local versal deformation rings

To describe some auxiliary ring-theoretic results and some ring-theoretic properties of the versal deformation ring $R^\psi_{\bar\rho}$ for a fixed 2-dimensional residual representation $\bar\rho\colon G_K\to \mathrm{GL}_2(\mathbb{F})$, we fix some further notation.

For a ring R in $\widehat{\mathcal{A}r}_{\mathcal{O}}$ and a proper ideal $\mathfrak{n}$ of R, we denote by $\mathrm{gr}_{\mathfrak{n}}(R)$ the associated graded ring $\bigoplus_{i\ge 0}\mathrm{gr}^i_{\mathfrak{n}}(R)$ with $\mathrm{gr}^i_{\mathfrak{n}}(R)=\mathfrak{n}^i/\mathfrak{n}^{i+1}$. By $\mathrm{in}_{\mathfrak{n}}\colon R\to\mathrm{gr}_{\mathfrak{n}}(R)$, we denote the map that sends $r\in R\smallsetminus\{0\}$ to its initial term in $\mathrm{gr}_{\mathfrak{n}}(R)$, i.e., if i_r is the largest integer $i\ge 0$ such that $r\in\mathfrak{n}^i$, then $\mathrm{in}_{\mathfrak{n}}(r)$ is the image of r in $\mathfrak{n}^{i_r}/\mathfrak{n}^{i_r+1}$. Further, we set $\mathrm{in}_{\mathfrak{n}}(0)=0$ and note that $\bigcap_i\mathfrak{n}^i=\{0\}$ for R in $\widehat{\mathcal{A}r}_{\mathcal{O}}$. If we wish to indicate i_r in the notation, we write $\mathrm{in}^{i_r}_{\mathfrak{n}}(r)$. For an ideal $I\subset R$ one denotes by $\mathrm{in}_{\mathfrak{n}}(I)$ the ideal of $\mathrm{gr}_{\mathfrak{n}}(R)$ generated by $\{\mathrm{in}_{\mathfrak{n}}(r)\mid r\in I\}$. To describe the mod $\mathfrak{m}_{\mathcal{O}}$ reduction of pairs $(R,\mathfrak{m}_R)$ in $\widehat{\mathcal{A}r}_{\mathcal{O}}$, we define $\overline{R}:=R/\mathfrak{m}_{\mathcal{O}}R$ and $\overline{\mathfrak{m}}_R:=\mathfrak{m}_R/\mathfrak{m}_{\mathcal{O}}R$. Similarly, we write $\bar r$ for the image of $r\in R$ in $\overline{R}$.

The following is the key technical result to deduce ring theoretic properties of $R^\psi_{\bar\rho}$:

Theorem 4.1.4. *Suppose $\bar\rho$ is of degree 2 and $p>2$. Fix a minimal presentation of $R^\psi_{\bar\rho}$ as in Proposition 4.1.3. Then there exist an $\mathfrak{m}_{\mathcal{R}}$-primary ideal $\mathfrak{m}_s$ of $\mathcal{R}\cong\mathcal{O}[\![x_1,\ldots,x_h]\!]$ of the form $(\varpi^s_{\mathcal{O}},x_1,\ldots,x_h)$ with $\mathfrak{m}_s^2\supset I^\psi$ and generators $f_1,\ldots,f_r$ of I^ψ such that the following hold:*

- *(a) Let $g_j=\mathrm{in}^2_{\mathfrak{m}_s}(f_j)\in\mathfrak{m}_s^2/\mathfrak{m}_s^3$ for $j=1,\ldots,r$. Then the elements $\bar t_0,\bar g_1,\ldots,\bar g_r$ form a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s}(\mathcal{R})}\cong\mathbb{F}[\bar t_0,\bar t_1,\ldots,\bar t_h]$, where $t_0=\mathrm{in}_{\mathfrak{m}_s}(\varpi^s_{\mathcal{O}})$ and $t_i=\mathrm{in}_{\mathfrak{m}_s}(x_i)$ for $i=1,\ldots,h$.*
- *(b) The quotient ring $\overline{\mathrm{gr}_{\mathfrak{m}_s}(\mathcal{R})}/(\bar g_1,\ldots,\bar g_r)$ is an integral domain and one has $(\bar g_1,\ldots,\bar g_r)=\overline{\mathrm{in}_{\mathfrak{m}_s}(I^\psi)}$.*

Theorem 4.1.4 will be proven after Corollary 4.3.6. A cohomological interpretation of the $\bar g_j$ is given in Theorem 4.1.14.

As a consequence of Theorem 4.1.4 and some purely ring-theoretic results summarized in Proposition 4.2.2, we shall obtain the following main theorem in Section 4.2:

Theorem 4.1.5. *Let the residual representation $\bar\rho$ be of degree 2 and let $p>2$. Then the following hold:*

- *(a) The ring $\overline{R}^\psi_{\bar\rho}$ is a complete intersection.*
- *(b) The ring $R^\psi_{\bar\rho}$ is a complete intersection and it is flat over $\mathcal{O}$.*

(c) *The ring* $R_{\bar{\rho}}^{\psi}$ *is an integral domain and in particular irreducible.*

In Lemma 4.4.1 we show that is suffices to prove Theorem 4.1.5 for any fixed choice of lift ψ, for instance for the Teichmüller lift of $\det \rho$.

Remark 4.1.6. Parts (a) and (b) of Theorem 4.1.5 were obtained already in [Böc00]. In fact, our present proof heavily relies on the results of [Böc00] because we shall simply quote the relations of $R_{\bar{\rho}}^{\psi}$ in a minimal presentation from there. However, the present article allows one to redo much of [Böc00] by working with the simpler ring $R_{\bar{\rho}}^{\psi}/\mathfrak{m}_s^3$, and this would avoid most of the technical difficulties occurring in [Böc00]. An example of this is given by the proof of Lemma 4.3.7.

Remark 4.1.7. It does not seem possible to show irreducibility when $n = 2$, $p = 2$ and $K = \mathbb{Q}_2$ with ideas of the present article, i.e., by using suitable initial terms in an associated graded ring of $R_{\bar{\rho}}^{\psi}$. For instance, if $\bar{\rho}$ is the trivial representation, then it is simple to check that the natural degrees of such initial terms are 2 and 3 and that they form a regular sequence. But the resulting associated graded ring is not an integral domain! However, when K is a proper extension of $\mathbb{Q}_2$, as shown in the Master thesis of M. Kremer, the methods of this article suffice to show that $R_{\bar{\rho}}^{\psi}$ is an integral domain for the trivial representation $\bar{\rho}$. For $n = 2$, $p = 2$ and $K = \mathbb{Q}_2$, see however Remark 4.1.13.

Irreducible components of versal deformation spaces and Zariski density of crystalline points

Denote by $\mathfrak{X}(\bar{\rho})$ the *versal deformation space* of a fixed residual representation $\bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$ that is the generic fiber over $\mathcal{O}[1/p]$ of its versal deformation ring $R_{\bar{\rho}}$ in the sense of Berthelot, see [dJ95, § 7]. The points of $\mathfrak{X}(\bar{\rho})$ are in bijection with those p-adic representations of G_K that have a mod p reduction isomorphic to $\bar{\rho}$. To explain the consequences of the the ring-theoretic results in Theorem 4.1.5 to p-adic Galois representations, we introduce the following notions due to Colmez, Kisin and Nakamura:

Definition 4.1.8. Let V be a potentially crystalline p-adic representation of G_K of degree n.

(i) V is called *regular* if for each embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}_p}$ the Hodge-Tate weights of $V \otimes_{K,\sigma} \overline{\mathbb{Q}_p}$ are pairwise distinct.

(ii) V is called *benign* if V is regular and the Frobenius eigenvalues $\alpha_1, \ldots, \alpha_n$ of (the filtered φ-module corresponding to) V are pairwise distinct and satisfy $\alpha_i/\alpha_j \neq p^f$, for any i, j, with $f = [K_0 : \mathbb{Q}_p]$.

Using the following important structure result on the irreducible components of $\mathfrak{X}(\bar{\rho})$, we show in Lemma 4.4.2 that every component of $\mathfrak{X}(\bar{\rho})$ contains a regular crystalline point.

Theorem 4.1.9. *Suppose* $p > 2$ *and let* $\bar{\rho}$ *be a residual representation of* G_K *of degree* 2*. Consider the canonical map* $D\colon \mathfrak{X}(\bar{\rho}) \to \mathfrak{X}(\det \bar{\rho})$ *induced from mapping a deformation of* $\bar{\rho}$ *to its determinant. Then* D *induces a bijection between the irreducible components of* $\mathfrak{X}(\bar{\rho})$ *and those of* $\mathfrak{X}(\det \bar{\rho})$*. Moreover, for both spaces, irreducible and connected components coincide. Lastly, the connected components of* $\mathfrak{X}(\det \bar{\rho})$ *form a principal homogeneous space over the set* $\mu_{p^\infty}(K)$ *of* p*-power roots contained in* K*.*

The proof follows from Theorem 4.1.5 and Lemma 4.4.1, and is thus postponed to Section 4.4.

Question 4.1.10. We wonder whether the assertions of Theorem 4.1.9 hold for all representations $\bar{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ of any degree n, and any p and any finite extension $K/\mathbb{Q}_p$? We also wonder if Theorem 4.1.5 holds in this generality.

The following theorem is shown by methods similar to [Kis10b]. It generalizes a result of Colmez and Kisin for $K = \mathbb{Q}_p$, cf. [Col08, Kis10b], and makes crucial use of an idea of Chenevier [Nak14, Thm. 2.9].

Theorem 4.1.11 ([Nak13, Theorem 1.4])**.** *Suppose $n = 2$ and that every component of $\mathfrak{X}(\bar{\rho})$ contains a regular crystalline point. Then the Zariski closure of the benign crystalline points in $\mathfrak{X}(\bar{\rho})$ is non-empty and a union of irreducible components of $\mathfrak{X}(\bar{\rho})$.*

We remark that the above result is also proven for arbitrary n. This is due to Chenevier [Che13b] for $K = \mathbb{Q}_p$ and to Nakamura [Nak14] for arbitrary finite extensions $K/\mathbb{Q}_p$.
Using Theorems 4.1.5 and 4.1.9, we show in Section 4.4 that Theorem 4.1.11 implies:

Theorem 4.1.12. *Suppose $n = 2$, $p > 2$, K is a finite extension of $\mathbb{Q}_p$ and $\bar{\rho}\colon G_K \to \mathrm{GL}_2(\mathbb{F})$ is any residual representation. Then the benign crystalline points are Zariski dense in $\mathfrak{X}(\bar{\rho})$.*

In Corollary 4.4.3, we prove analogs of Theorems 4.1.9 and 4.1.12 for pseudo-representations, in the sense of Chenevier [Che14].
In the case $K = \mathbb{Q}_p$ and $n = 2$, Theorem 4.1.12 is an important ingredient in Colmez' proof of the p-adic local Langlands correspondence. In that case it is essentially due to Kisin, cf. [Böc10], and it is used to establish the surjectivity of Colmez' functor V, which relies on an analytic continuation argument and the knowledge of the correspondence in the crystalline case; see [Col10, proof of Thm. II.3.3] or alternatively [Kis10b].

Remark 4.1.13. Suppose $p = 2$ and $K = \mathbb{Q}_2$. The assertions of Theorems 4.1.9 and 4.1.12 for the universal framed deformation space of the trivial representation $1 \oplus 1$ were proved by Colmez, Dospinescu and Paskunas [CDP15, Thms. 1.1 and 1.2]. The assertion of Theorem 4.1.9 was proved by Chenevier in the case $n = 2$ if the residual representation is an extension of two distinct characters, and for arbitrary n if the residual representation is absolutely irreducible [Che11, Cor. 4.2]. In these two cases the assertion of Theorem 4.1.12 is deduced in [CDP15, Rem. 9.8].

Generation of quadratic parts of relation ideals through cohomological operations

One possible source of obstruction classes in $H^2(G_K, \mathrm{ad}^0)$ stems from the cup product in cohomology: Namely, if one composes the Lie bracket $[\cdot,\cdot]\colon \mathrm{ad}^0 \times \mathrm{ad}^0 \to \mathrm{ad}^0, (A, B) \mapsto AB - BA$, with the cup product $H^1(G_K, \mathrm{ad}^0) \times H^1(G_K, \mathrm{ad}^0) \to H^2(G_K, \mathrm{ad}^0 \otimes \mathrm{ad}^0)$, which are both alternating, one obtains a symmetric $\mathbb{F}$-bilinear pairing

$$b\colon H^1(G_K, \mathrm{ad}^0) \times H^1(G_K, \mathrm{ad}^0) \longrightarrow H^2(G_K, \mathrm{ad}^0),$$

often called the *bracket cup product*. As remarked in [Maz89, §1.6], if $p \neq 2$ the pairing b *gives the quadratic relations (up to higher terms) satisfied by a minimal set of formal parameters for* $\overline{R}^{\psi}_{\bar{\rho}}$. We shall prove this and give a precise interpretation in Lemma 4.5.2.
In Section 4.6, we shall explain how further information on the relation ideal I^{ψ} may arise from cohomology, namely from a Bockstein homomorphism $\widetilde{\beta}_{s+1}\colon H^1(G_K, \mathrm{ad}^0) \to H^2(G_K, \mathrm{ad}^0)$. The Bockstein homomorphism can be defined whenever $\bar{\rho}$ admits a lift to $\mathcal{O}_s = \mathcal{O}/\varpi^s_{\mathcal{O}}\mathcal{O}$ for some s. It measures to what extent lifts from the dual number $\mathbb{F}[\varepsilon]$ can be lifted to $\mathcal{O}_s[\varepsilon]$. In Section 4.6 we then combine the bracket cup product with the Bockstein homomorphism, to show that these two cohomological operations (essentially) suffice to describe the refined quadratic relations in a minimal presentation of $R^{\psi}_{\bar{\rho}}$.
The results of Sections 4.5 and 4.6 have the following consequences. First, we complement Theorem 4.1.4:

Theorem 4.1.14. *Let the notation be as in Theorem 4.1.4. Then in addition to the assertions of Theorem 4.1.4, the following hold:*

(a) The elements $\bar{g}_j$ are the images of an $\mathbb{F}$-basis of $H^2(G_K, \mathrm{ad}^0)^\vee$ under the composite of the dual obstruction homomorphism $\mathrm{obs}^\vee\colon H^2(G_K, \mathrm{ad})^\vee \to I^\psi/\mathfrak{m}_R I^\psi$ with the canonical homomorphism $I^\psi/\mathfrak{m}_R I^\psi \to \overline{\mathfrak{m}_s^2/\mathfrak{m}_s^3}$.

(b) The dual of the map $H^2(G_K, \mathrm{ad}^0)^\vee \longrightarrow \overline{\mathfrak{m}_s^2/\mathfrak{m}_s^3}$ from (a) factors via $-\frac{1}{2}b \oplus -\widetilde{\beta}_{s+1}$.

We prove Theorem 4.1.14 at the end of Section 4.6 by verifying the hypotheses needed to apply Theorem 4.6.8. In particular, this shows that cohomological information alone suffices to deduce all parts of Theorem 4.1.5.

Second, we observe in Example 4.2.3 that the bracket cup product alone need not suffice to show that $R^\psi_{\bar\rho}$ is an integral domain. Thus important ring-theoretic information is not visible by the bracket cup product but requires in addition the Bockstein homomorphism.

Third, our results show that for 2-dimensional residual representations of G_K for $p > 2$ the refined quadratic part of I^ψ in a minimal presentation of $R^\psi_{\bar\rho}$ suffices to prove Theorem 4.1.5. Theorem 4.6.8 then explains that essentially the cohomological operations suffice to deduce all ring-theoretic properties we are interested in.

The third point above is particular to the set-up we work in. For general fields K little is known about the pairing b and whether it generates a significant portion of the elements in the relation ideal I^ψ of Proposition 4.1.3. However for K a finite extension of $\mathbb{Q}_p$ and $p > 2$, the universal deformation $\rho_{\bar\rho}\colon G_K \to \mathrm{GL}_2(R_{\bar\rho})$ factors via a profinite group that is an extension of a finite group by the pro-p-completion $\widehat{G}_L$ of the absolute Galois group of a finite extension L of K. The group $\widehat{G}_L$ is either a free pro-p group or a Demushkin group, and topologically finitely generated. In the former case, $R_{\bar\rho}$ will be unobstructed. In the latter case $\widehat{G}_L$ is isomorphic to the pro-p completion of a group on generators $a_1, b_1, \ldots, a_g, b_g$ with a single relation $a_1^q \cdot (a_1, b_1) \cdot \ldots \cdot (a_g, b_g) = 1$, where $g = [L : \mathbb{Q}_p]$ and (x, y) denotes the commutator bracket $x^{-1}y^{-1}xy$; cf. [Lab67] for the classification of Demushkin groups. The Demushkin case should be compared with the deformation results [GM88a, GM88b] by Goldman and Millson, as already suggested in [Maz89]. Goldman-Millson study the deformation theory of representations of fundamental groups of compact Kähler manifolds, and show in this context that all relations in a minimal presentation of their deformation rings are purely quadratic. A typical example is the fundamental group of a compact Riemann surface, which is a group on $2g$ generators $a_1, b_1, \ldots, a_g, b_g$ subject to a single relation $(a_1, b_1) \cdot \ldots \cdot (a_g, b_g) = 1$. The formal similarity of the relation except for the term a_1^q suggests that the deformation rings might be very similar. The term a_1^q might explain the importance of the Bockstein homomorphism when trying to detect the refined quadratic relations from cohomology.

Outline of the article

We briefly explain the organization of the article. In Section 4.2, we adapt some results from commutative algebra in the way we later wish to apply them. In particular, these results give a sufficient criterion for certain rings R in $\widehat{\mathcal{A}r}_{\mathcal{O}}$ to be complete intersections and to be integral domains in terms of homogeneous initial terms of a presentation of R. The main results of Section 4.2 together with Theorem 4.1.4 imply the ring-theoretic properties stated in Theorem 4.1.5. In Section 4.3, we recall the explicit presentations of the versal deformation rings for 2-dimensional representations $\bar\rho$ from [Böc00]. In Lemma 4.3.7, we also give a detailed treatment of some results from [Böc00, §8], whose proofs are somewhat sketchy. At the end of Section 4.3, we give the proof of Theorem 4.1.4.

The short proof of the Zariski-density of crystalline points in local deformation spaces is the content of Section 4.4. We end this article with Sections 4.5 and 4.6 with (presumably well-known) results regarding the bracket cup product and the Bockstein homomorphism. These results might be relevant for tackling higher dimensional cases in future work. The proof of Theorem 4.1.14 ends Section 4.6.

Acknowledgments: The first author thanks M. Blickle for help in finding the reference [Her81] which led to [VV78] as well as G. Chenevier for some stimulating discussions. Both authors thank A. Muller for some useful remarks. Both authors also thank the referee for a very careful reading and many useful suggestions. G.B. received funding from the DFG via the Forschergruppe 1920 and the SPP 1489. A.-K.J. benefited from a PhD grant of the Landesgraduiertenförderung Baden-Württemberg.

4.2 Results from commutative algebra

The aim of this section is to prove some results in commutative algebra in order to deduce from Theorem 4.1.4 the ring-theoretic results stated in Theorem 4.1.5. In particular, we wish to transfer ring-theoretic properties from a certain associated graded ring to the ring itself. Recall that above Theorem 4.1.4 we define an initial term map $\mathrm{in}_{\mathfrak{n}}$ from a ring R in $\widehat{\mathcal{A}r}_{\mathcal{O}}$ to the associated graded ring $\mathrm{gr}_{\mathfrak{n}} R$ with respect to a proper ideal $\mathfrak{n} \subset R$, and that we write $\mathcal{O}_i$ for $\mathcal{O}/\varpi_{\mathcal{O}}^i\mathcal{O}$, $\overline{R}$ for $R/\mathfrak{m}_{\mathcal{O}}R$, $\overline{\mathfrak{m}}_R$ for $\mathfrak{m}_R/\mathfrak{m}_{\mathcal{O}}R$ and $\bar{x}$ for the image of $x \in R$ in $\overline{R}$.

Lemma 4.2.1. *For a ring R in $\widehat{\mathcal{A}r}_{\mathcal{O}}$ and proper ideals $I = (f_1, \ldots, f_r), \mathfrak{n} \subset R$, the following hold:*

(a) If $\mathrm{gr}_{\mathfrak{n}} R$ is an integral domain, then so is R.

(b) If $f_1, \ldots, f_r$ is a regular sequence in R so that R/I is an integral domain, then R is an integral domain.

(c) The natural homomorphism $\mathrm{gr}_{\mathfrak{n}} R \to \mathrm{gr}_{(\mathfrak{n}+I)/I}(R/I)$ induces an isomorphism

$$(\mathrm{gr}_{\mathfrak{n}} R)/\,\mathrm{in}_{\mathfrak{n}}(I) \cong \mathrm{gr}_{(\mathfrak{n}+I)/I}(R/I).$$

(d) If $\mathrm{in}_{\mathfrak{n}}(f_1), \ldots, \mathrm{in}_{\mathfrak{n}}(f_r)$ is a regular sequence in $\mathrm{gr}_{\mathfrak{n}} R$, then $\mathrm{in}_{\mathfrak{n}}(I) = (\mathrm{in}_{\mathfrak{n}}(f_1), \ldots, \mathrm{in}_{\mathfrak{n}}(f_r))$.

(e) If $\mathrm{in}_{\mathfrak{n}}(f_1), \ldots, \mathrm{in}_{\mathfrak{n}}(f_r)$ is a regular sequence in $\mathrm{gr}_{\mathfrak{n}} R$, then $f_1, \ldots, f_r$ is a regular sequence in R.

Proof. Part (a) is [Eis95, Cor. 5.5], and (b) follows by induction on r: For $r = 1$ we have a short exact sequence $0 \to R \xrightarrow{\cdot f_1} R \to R/(f_1) \to 0$ so that $\mathrm{gr}_{(f_1)} R \cong R/(f_1)[t]$, and R is an integral domain by (a). Parts (c), (d) and (e) are [VV78, middle p. 94], [VV78, Prop. 2.1] and [VV78, Cor. 2.7], respectively. ☐

The next result is a refinement of Lemma 4.2.1 suited for our purposes. As a preparation we introduce the following graded ring. Denote by $\mathfrak{m}_s$ the ideal $(\varpi_{\mathcal{O}}^s, x_1, \ldots, x_h)$ of $\mathcal{R} = \mathcal{O}[\![x_1, \ldots, x_h]\!]$[3] for some integer $s \geq 1$. Setting $t_0 := \mathrm{in}_{\mathfrak{m}_s}(\varpi_{\mathcal{O}}^s)$ and $t_i := \mathrm{in}_{\mathfrak{m}_s}(x_i)$ for $i = 1, \ldots, h$, we have $\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R} = \mathcal{O}_s[t_0, \ldots, t_h]$, $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}} = \mathbb{F}[\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_h]$ and $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}} \overline{\mathcal{R}} \cong \mathbb{F}[\bar{t}_1, \ldots, \bar{t}_h]$, where $\bar{t}_i$ is identified with $\mathrm{in}_{\overline{\mathfrak{m}}_{\mathcal{R}}}(\bar{x}_i)$ for $i = 1, \ldots, h$.

[3] In this section, and here only, by $(\mathcal{R}, \mathfrak{m}_{\mathcal{R}})$ we denote a formally smooth ring over $\mathcal{O}$ in $\widehat{\mathcal{A}r}_{\mathcal{O}}$ and not necessarily a ring in a presentation as in Proposition 4.1.3.

Proposition 4.2.2. *Let $\mathcal{R}, \mathfrak{m}_s$ and s be as above, and let $I \subset \mathcal{R}$ be an ideal generated by elements $f_1, \ldots, f_r \in \mathcal{R}$. Then the following hold:*

(a) *If $\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_1), \ldots, \mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_r)$ is a regular sequence in $\mathrm{gr}_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}}$, then so is $\varpi_\mathcal{O}, f_1, \ldots, f_r$ in $\mathcal{R}$. In this case, $\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{I}) = (\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_1), \ldots, \mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_1))$ in $\mathrm{gr}_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}}$ and $\mathcal{R}/I$ is flat over $\mathcal{O}$.*

(b) *If $\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_1), \ldots, \mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_r)$ is a regular sequence in $\mathrm{gr}_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}}$ and $\mathrm{gr}_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}}/\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{I})$ is an integral domain, then also $\mathcal{R}/I$ is an integral domain.*

(c) *If $\overline{\mathrm{in}_{\mathfrak{m}_s}(f_1)}, \ldots, \overline{\mathrm{in}_{\mathfrak{m}_s}(f_r)}$ is a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$, then so is $f_1, \ldots, f_r$ in $\mathcal{R}$. In this case, $\mathrm{in}_{\mathfrak{m}_s}(I) = (\mathrm{in}_{\mathfrak{m}_s}(f_1), \ldots, \mathrm{in}_{\mathfrak{m}_s}(f_r))$ and $\overline{\mathrm{in}_{\mathfrak{m}_s}(I)} = (\overline{\mathrm{in}_{\mathfrak{m}_s}(f_1)}, \ldots, \overline{\mathrm{in}_{\mathfrak{m}_s}(f_r)})$.*

(d) *If $\overline{\mathrm{in}_{\mathfrak{m}_s}(f_1)}, \ldots, \overline{\mathrm{in}_{\mathfrak{m}_s}(f_r)}, \bar{t}_0$ is a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$ and $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}/\overline{\mathrm{in}_{\mathfrak{m}_s}(I)}$ is an integral domain, then also $\mathcal{R}/I$ is an integral domain.*

We postpone the proof of Proposition 4.2.2, and first explain some of its content.

Proof of Theorem 4.1.5. Theorem 4.1.4 together with Proposition 4.2.2 applied to the relation ideal I^ψ imply the assertions of Theorem 4.1.5 on $R_{\bar{\rho}}^\psi \cong \mathcal{R}/I^\psi$. □

The following instructive example shows the benefits of using the graded ring associated with the ideal $\mathfrak{m}_s$ in (c) and (d) instead of the one associated with $\overline{\mathfrak{m}}_\mathcal{R}$ in (a) and (b).

Example 4.2.3. Define $R := \mathcal{R}/I$ for $\mathcal{R} = W(\mathbb{F})[[x_1, x_2, x_3]]$, $I = (f)$ with $f = qx_1 - x_2x_3$, $q = p^s$ and $s \geq 1$ an integer.[4] Then by Proposition 4.2.2(a) $\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{I}) = \big(\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f})\big) = (\bar{t}_2\bar{t}_3)$ in $\mathrm{gr}_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}} = \mathbb{F}[\bar{t}_1, \bar{t}_2, \bar{t}_3]$, and criterion (b) fails to show that R is an integral domain since $\bar{t}_2$ and $\bar{t}_3$ are nonzero zero divisors in $\mathrm{gr}_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}}/\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{I})$. However, if we consider the graded ring of $\mathcal{R}$ with respect to $\mathfrak{m}_s = (q, x_1, x_2, x_3)$, then $\overline{\mathrm{in}_{\mathfrak{m}_s}(f)} = \overline{t_0t_1} - \overline{t_2t_3}$ lies in $\overline{\mathrm{gr}^2_{\mathfrak{m}_s} \mathcal{R}} \subset \overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}} = \mathbb{F}[\bar{t}_0, \bar{t}_1, \bar{t}_2, \bar{t}_3]$ and R is an integral domain by Proposition 4.2.2(d).

In Sections 4.5 and 4.6, we show that one can use cohomological methods to compute the quadratic relations in $\mathrm{gr}^2_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}}$ resp. $\overline{\mathrm{gr}^2_{\mathfrak{m}_s} \mathcal{R}}$ from the above example. To distinguish there between these two quadratic relations, we introduce the following notions:

Definition 4.2.4. Let $\mathcal{R}$, $\mathfrak{n}$, $\mathfrak{m}_s$ and s be as above, and let $f \in \mathcal{R}$.

(a) If $\bar{f} \in \overline{\mathfrak{m}}_\mathcal{R}^2$, then the *quadratic part of* f is $\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}^{(2)}) \in \mathrm{gr}^2_{\overline{\mathfrak{m}}_\mathcal{R}} \overline{\mathcal{R}}$, where $\bar{f}^{(2)}$ is the homogeneous part of $\bar{f}$ of degree 2 with respect to the grading of $\overline{\mathcal{R}}$ defined by $\overline{\mathfrak{m}}_\mathcal{R}$.

(b) If $f \in \mathfrak{m}_s^2$, then the *refined quadratic part of* f is $\overline{\mathrm{in}_{\mathfrak{m}_s}(f^{(2)})} \in \overline{\mathrm{gr}^2_{\mathfrak{m}_s} \mathcal{R}}$, where $f^{(2)}$ is the homogeneous part of f of degree 2 with respect to the grading of $\mathcal{R}$ defined by $\mathfrak{m}_s$.

The *(refined) quadratic part of an ideal* $I \subset \mathcal{R}$ consists of the (refined) quadratic parts of all elements in I.

Proof of Proposition 4.2.2. It follows from the hypothesis of (a) and Lemma 4.2.1(e) that $(\overline{f_j})_{j=1,\ldots,r}$ is a regular sequence in $\overline{\mathcal{R}}$. Since clearly $\varpi_\mathcal{O}$ is a non-zero divisor of $\mathcal{R}$, the first assertion of (a) is proved. From Lemma 4.2.1(d) it follows that $\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{I}) = \big(\mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_1), \ldots, \mathrm{in}_{\overline{\mathfrak{m}}_\mathcal{R}}(\bar{f}_r)\big)$. Finally, since $\mathcal{R}$ is local, the order of the elements in the regular sequence $\varpi_\mathcal{O}, f_1, \ldots, f_r$ is arbitrary.

[4] The relation ideal of the ring $\widetilde{R}$ from [Böc10, Thm. 5] has the shape $6d - bc$ modulo $\mathfrak{m}_1^3$, where $p = 3$. So in a qualitative sense R occurs as a versal deformation ring. At the expense of heavy notation, one could also use $\widetilde{R}$ in the example.

Hence from the definition of a regular sequence it follows that $\varpi_{\mathcal{O}}$ is a non-zero divisor of the $\mathcal{O}$-algebra $\mathcal{R}/(f_1,\dots,f_r)$, which means precisely that the latter algebra is flat over $\mathcal{O}$.

To prove (b), we deduce by Lemma 4.2.1(c) and (a) that $\overline{\mathcal{R}/I}$ is an integral domain. By the last assertion of (a) and Lemma 4.2.1(b) we also have that $\mathcal{R}/I$ is an integral domain.

For the proof of (c), we define $R_i := \operatorname{gr}_{\mathfrak{m}_s}\mathcal{R}/(\operatorname{in}_{\mathfrak{m}_s}(f_1),\dots,\operatorname{in}_{\mathfrak{m}_s}(f_i))$ for $i=0,\dots,r$ and g_i as the image of $\operatorname{in}_{\mathfrak{m}_s}(f_i)$ in R_i. By the remarks preceding the proposition, $\operatorname{gr}_{\mathfrak{m}_s}\mathcal{R}\cong\mathcal{O}_s[t_0,\dots,t_h]$ and clearly this ring is flat over $\mathcal{O}_s$. By our hypothesis, $\bar{g}_i$ is a non-zero divisor of $\overline{R_{i-1}}$ for $i=1,\dots,r$. We claim, and prove this by induction on i, that R_i is flat over $\mathcal{O}_s$ and that g_i is a non-zero divisor of R_{i-1} for each $i=1,\dots,r$. If this is proved, then we have shown that $\operatorname{in}_{\mathfrak{m}_s}(f_1),\dots,\operatorname{in}_{\mathfrak{m}_s}(f_r)$ is a regular sequence in $\operatorname{gr}_{\mathfrak{m}_s}\mathcal{R}$. Then the first assertion of (c) follows from Lemma 4.2.1(e). The first equality of ideals in (c) follows from Lemma 4.2.1(d) and the assertion just proved, the second is immediate by reduction modulo $\varpi_{\mathcal{O}}$.

To prove the claim, we consider for some $j=2,\dots,s$ the following diagram obtained by tensoring the short exact sequence $0\to\mathcal{O}_{j-1}\to\mathcal{O}_j\to\mathbb{F}\to 0$ of $\mathcal{O}$-modules with the right exact sequence $R_{i-1}\to R_{i-1}\to R_i\to 0$ where the map on the left is multiplication by g_i:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{j-1}\otimes R_{i-1} & \longrightarrow & \mathcal{O}_j\otimes R_{i-1} & \longrightarrow & \mathbb{F}\otimes R_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes g_i} & & \downarrow{\scriptstyle \mathrm{id}\otimes g_i} & & \downarrow{\scriptstyle \mathrm{id}\otimes g_i} & & \\
0 & \longrightarrow & \mathcal{O}_{j-1}\otimes R_{i-1} & \longrightarrow & \mathcal{O}_j\otimes R_{i-1} & \longrightarrow & \mathbb{F}\otimes R_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_{j-1}\otimes R_i & \longrightarrow & \mathcal{O}_j\otimes R_i & \longrightarrow & \mathbb{F}\otimes R_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We assume that the claim is proved for $i-1$. Then the two top horizontal sequences are exact since by induction hypothesis the ring R_{i-1} is flat over $\mathcal{O}_s$. The left and middle vertical sequences are exact because the tensor product is right exact. The right vertical sequence is exact, because $\bar{g}_i$ is a non-zero divisor of $\overline{R_{i-1}}$ by hypothesis.

While i is fixed, we proceed by induction on $j=2,\dots,s$ to show that all rows and columns in the above diagram are in fact left exact as well: In each induction step, the left-most column is a short exact sequence by induction hypothesis. This implies the same for the middle column and it follows that all columns are short exact sequences. In this situation, the 9-lemma implies that the lower row is also a short exact sequence, and the induction step is complete. If we consider the central column for $j=s$, then this shows that g_i is a non-zero divisor of R_{i-1}. If we consider the lower row for $j=s$, we see that $\operatorname{Tor}_1^{\mathcal{O}_s}(\mathbb{F},R_i)=0$ and hence that R_i is flat over $\mathcal{O}_s$. This proves the claim.

Finally, we prove (d). By the proof of (c), we know that the ring $\operatorname{gr}_{\mathfrak{m}_s}\mathcal{R}/\operatorname{in}_{\mathfrak{m}_s}(I)$ is flat over $\mathcal{O}_s$ and its reduction modulo $\varpi_{\mathcal{O}}$ is $\overline{\operatorname{gr}_{\mathfrak{m}_s}\mathcal{R}/\operatorname{in}_{\mathfrak{m}_s}(I)}$. Consider elements f,g in $\mathcal{R}\smallsetminus I$. We claim that there exist integers $a,b\in\{0,1,\dots,s-1\}$ such that $f'=\varpi_{\mathcal{O}}^a f$ and $g'=\varpi_{\mathcal{O}}^b g$ have non-zero image in

$$\overline{\operatorname{gr}_{\mathfrak{m}_s}\mathcal{R}/\operatorname{in}_{\mathfrak{m}_s}(I)}\cong\overline{\operatorname{gr}_{(\mathfrak{m}_s+I)/I}(\mathcal{R}/I)}\cong\bigoplus_{i\ge 0}(\mathfrak{m}_s^i+I)/(\mathfrak{m}_s^{i+1}+\varpi_{\mathcal{O}}\mathfrak{m}_s^i+I),$$

where the first isomorphism follows from Lemma 4.2.1(c). If the claim is shown, then this means that there exist $i, j \geq -1$ such that $f' \in (\mathfrak{m}_s^{i+1} + I) \smallsetminus (\mathfrak{m}_s^{i+2} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + I)$ and $g' \in (\mathfrak{m}_s^{j+1} + I) \smallsetminus (\mathfrak{m}_s^{j+2} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{j+1} + I)$. Since by hypothesis $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}/\mathrm{in}_{\mathfrak{m}_s}(I)}$ is an integral domain, it follows that $f'g' \in (\mathfrak{m}_s^{i+j+2} + I) \smallsetminus (\mathfrak{m}_s^{i+j+3} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+j+2} + I)$ and hence that the class of $f'g'$ is non-zero in $\mathcal{R}/I$. But $f'g' = \varpi_{\mathcal{O}}^{a+b} fg$, and we deduce that the class of fg is non-zero in $\mathcal{R}/I$ and thus assertion (d) follows.

To prepare the proof of the claim, we make some technical remarks:

(i) We have $I \subset \mathfrak{m}_s$ since the hypothesis of (d) implies that $\mathrm{in}_{\mathfrak{m}_s}(f_i) \notin \mathcal{R}/\mathfrak{m}_s$ for all $1 \leq i \leq r$. In particular, $(\mathfrak{m}_s + I)/I = \mathfrak{m}_s/I$.

(ii) If $f \in \mathfrak{m}_s^j \smallsetminus \mathfrak{m}_s^{j+1} + I$ for $j \geq 0$, then $\mathrm{in}_{\mathfrak{m}_s}(f) = \mathrm{in}_{\mathfrak{m}_s/I}(f + I)$. In particular, $t_0 = \mathrm{in}_{\mathfrak{m}_s/I}(\varpi_{\mathcal{O}}^s + I)$ since by hypothesis $\bar{t}_0$ is a non-zero divisor in $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}/\mathrm{in}_{\mathfrak{m}_s}(I)}$ so that $\varpi_{\mathcal{O}}^s \in \mathfrak{m}_s \smallsetminus \mathfrak{m}_s^2 + I$.

(iii) If $\varpi_{\mathcal{O}}^b \,\mathrm{in}_{\mathfrak{m}_s/I}(h) \neq 0$ for $h \in \mathcal{R}$ and $0 < b < s$, then $\varpi_{\mathcal{O}}^b \,\mathrm{in}_{\mathfrak{m}_s/I}(h) = \mathrm{in}_{\mathfrak{m}_s/I}(\varpi_{\mathcal{O}}^b h)$ as follows from the definition of multiplication on $\mathrm{gr}_{\mathfrak{m}_s/I} \mathcal{R}/I$.

(iv) The graded components $M^j = \mathrm{gr}^j_{\mathfrak{m}_s/I}(\mathcal{R}/I) \cong \mathrm{gr}^j_{\mathfrak{m}_s} \mathcal{R}/\,\mathrm{in}_{\mathfrak{m}_s}(I)$ are finite over the ring $\mathcal{O}_s = M^0$. At the beginning of (d) we observed that the M^j are flat over $\mathcal{O}_s$, and hence they are finite and free over $\mathcal{O}_s$. In particular one has $\mathrm{Ker}(\varpi_{\mathcal{O}}^b \colon M^j \to M^j) = \mathrm{Im}(\varpi_{\mathcal{O}}^{s-b} \colon M^j \to M^j)$.

We now verify the claim for f; the proof for g is analogous. Choose $i \geq 0$ such that $f \in (\mathfrak{m}_s^i + I) \smallsetminus (\mathfrak{m}_s^{i+1} + I)$. This is equivalent to $\mathrm{in}_{\mathfrak{m}_s/I}(f + I)$ lying in the i-th graded piece of $\mathrm{gr}_{\mathfrak{m}_s/I}(\mathcal{R}/I)$. If the image of $\mathrm{in}_{\mathfrak{m}_s/I}(f + I)$ in $\overline{\mathrm{gr}_{\mathfrak{m}_s/I}(\mathcal{R}/I)}$ is non-zero, we choose $a = 0$ and are done. Else we have $f \in \varpi_{\mathcal{O}}\mathfrak{m}_s^i + \mathfrak{m}_s^{i+1} + I$, and since $\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}/\,\mathrm{in}_{\mathfrak{m}_s}(I)$ is annihilated by $\varpi_{\mathcal{O}}^s$, we can find $a \in \{1, \ldots, s-1\}$ such that $\varpi_{\mathcal{O}}^{a-1} f \notin (\mathfrak{m}_s^{i+1} + I)$ but $\varpi_{\mathcal{O}}^a f \in (\mathfrak{m}_s^{i+1} + I)$. To prove the claim, it remains to show that $\varpi_{\mathcal{O}}^a f$ does not lie in $\mathfrak{m}_s^{i+2} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + I$.

By (iv), there exists $f_0 \in \mathfrak{m}_s^i \smallsetminus \mathfrak{m}_s^{i+1} + I$ such that $\varpi_{\mathcal{O}}^{s-a} \,\mathrm{in}_{\mathfrak{m}_s/I}(f_0 + I) = \mathrm{in}_{\mathfrak{m}_s/I}(f + I)$ in $\mathrm{gr}_{\mathfrak{m}_s/I}(\mathcal{R}/I)$. In terms of ideals this means $\varpi_{\mathcal{O}}^{s-a} f_0 - f \in \mathfrak{m}_s^{i+1} + I$, using (iii), and $f_0 \notin \underline{\varpi_{\mathcal{O}}\mathfrak{m}_s^i + \mathfrak{m}_s^{i+1}} + I$. By (ii) and the hypothesis in (d), the element $\bar{t}_0$ is a non-zero divisor of $\overline{\mathrm{gr}_{\mathfrak{m}_s}/I(\mathcal{R}/I)}$, and so we have $\varpi_{\mathcal{O}}^s f_0 \notin \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + \mathfrak{m}_s^{i+2} + I$. This implies $\varpi_{\mathcal{O}}^a f \notin \mathfrak{m}_s^{i+2} + \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + I$, because we have

$$\varpi_{\mathcal{O}}^s f_0 - \varpi_{\mathcal{O}}^a f = \varpi_{\mathcal{O}}^a \cdot (\varpi_{\mathcal{O}}^{s-a} f_0 - f) \in \varpi_{\mathcal{O}}^a \mathfrak{m}_s^{i+1} + \varpi_{\mathcal{O}}^a I \subset \varpi_{\mathcal{O}}\mathfrak{m}_s^{i+1} + \mathfrak{m}_s^{i+2} + I. \qquad \square$$

We end this section with a simple result on regular sequences, flatness and integral domains:

Lemma 4.2.5. *Suppose I is an ideal of $\mathcal{R} \cong \mathcal{O}[\![x_1, \ldots, x_h]\!]$ such that I is minimally generated by $m := \dim_{\mathbb{F}} I/\mathfrak{m}_{\mathcal{R}} I$ elements. Suppose $g_1, \ldots, g_l$ are elements of $\mathcal{R}$ and let $J = I + \mathcal{R}g_1 + \ldots + \mathcal{R}g_l$.*

(a) If $\mathcal{R}/J$ is a complete intersection ring of Krull dimension $h + 1 - l - m$, then $\mathcal{R}/I$ is a complete intersection ring and I is generated by a regular $\mathcal{R}$-sequence.

(b) If (a) holds and if $\mathcal{R}/J$ is flat over $\mathcal{O}$, then $\mathcal{R}/I$ is flat over $\mathcal{O}$.

(c) If (a) holds and if $\mathcal{R}/J$ is an integral domain, then $\mathcal{R}/I$ is an integral domain.

Proof. By induction, it suffices to prove the lemma for $l = 1$. Let $f_1, \ldots, f_m$ denote a minimal set of generators of I. The hypothesis of (a) implies that $\mathcal{R}/(f_1, \ldots, f_m, g_1)$ is a complete intersection ring of dimension $h + 1 - m - 1 = h - m$. It follows that $f_1, \ldots, f_m, g_1$ must be a regular sequence, and now (a) is immediate. To see (b), observe that its hypothesis implies that $\mathcal{R}/(J + \varpi_{\mathcal{O}})$ is a complete intersection ring of Krull dimension $h - l - m$. It follows from (a) that $f_1, \ldots, f_m, g_1, \varpi_{\mathcal{O}}$ is a regular sequence. Part (b) is now clear. For (c) note that since $g_1 \pmod{I}$ is a non-zero divisor in $\mathcal{R}/I$ by the proof of (a), we may now apply Lemma 4.2.1(b) to complete (c). □

4.3 Explicit presentations of the versal deformation rings

In order to prove Theorem 4.1.4 using the explicit minimal presentations of versal deformation rings computed in [Böc00], we note that we can work over the ring of Witt vectors $W(\mathbb{F})$ by [CDT99, A.1]. First we need to introduce some notation: Denote by H the image of a fixed residual representation $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ of degree two, and by U a p-Sylow subgroup of H. Since G_K is prosolvable, the group H is solvable. Either $\#H$ is of order prime to p, or U is a normal subgroup of H. By the lemma of Schur-Zassenhaus, we can find a subgroup G of H of order prime to p such that $U \rtimes G = H$. By $\overline{H}, \overline{G}$ we denote the images of H, G in $\mathrm{PGL}_2(\mathbb{F})$. Note that U is isomorphic to its image in $\mathrm{PGL}_2(\mathbb{F})$ because its order is prime to the order of $\mathbb{F}^*$ and hence we may identify U with its image. The following can now be deduced from Dickson's classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{F})$, see [Hup67, II.7]. The group $\overline{G}$ is either cyclic or dihedral and if U is non-trivial, $\overline{G}$ must be cyclic (we assume $p > 2$). We also introduce finite extensions $L \supset F \supset K$ in a fixed algebraic closure of K by the conditions $G_L = \ker(\bar{\rho}) \subset G_F = \bar{\rho}^{-1}(U) \subset G_K$.

For a character $\xi : G_K \to \mathbb{F}^*$ we denote by $\mathbb{F}^\xi$ the one-dimensional vector space $\mathbb{F}$ together with the action via ξ. We let $\mathrm{triv} : G_K \to \mathbb{F}^*$ be the trivial character and $\varepsilon : G_K \to \mathbb{F}^*$ be the mod p cyclotomic character. Observe that $\mathrm{ad} = \mathrm{End}(\bar{\rho}) \cong \bar{\rho} \otimes_{\mathbb{F}} \bar{\rho}^\vee \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathrm{ad}^0$ since $p > 2$ and thus $\mathrm{ad}^0 \cong \mathrm{Hom}_{\mathbb{F}}\left(\mathrm{ad}^0, \mathbb{F}\right)$. Using local Tate duality, one obtains that $H^2(G_K, \mathrm{ad}^0) \cong \left((\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon\right)^G$.

In the remainder of this section, we distinguish the following five cases.

(A) $\overline{G} \neq \{1\}$ is cyclic and U is trivial. Then $\bar{\rho} \sim \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} \otimes \eta$ for some characters $\xi, \eta : G_K \to \mathbb{F}^*$. Moreover,

$$\mathrm{ad} \cong (\mathbb{F}^{\mathrm{triv}})^2 \oplus \mathbb{F}^\xi \oplus \mathbb{F}^{\xi^{-1}} \quad \text{and} \quad (\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^\varepsilon \oplus \mathbb{F}^{\xi\varepsilon} \oplus \mathbb{F}^{\xi^{-1}\varepsilon}.$$

(B) $\overline{G} \neq \{1\}$ is cyclic and U is nontrivial. Then $\bar{\rho} \sim \begin{pmatrix} 1 & \star \\ 0 & \xi \end{pmatrix} \otimes \eta$ for some characters $\xi, \eta : G_K \to \mathbb{F}^*$; here $\star$ denotes a non-trivial extension, i.e., a non-trivial class in $H^1(G_K, \mathbb{F}^\xi)$. Moreover,

$$(\mathrm{ad})^U \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^{\xi^{-1}} \quad \text{and} \quad (\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^{\xi^{-1}\varepsilon}.$$

(C) $\overline{G}$ is dihedral. Then $\overline{H} = \overline{G}$, and $\overline{U}$ is trivial. By [Mul13, Prop. 2.1.1], there exists a character ξ' of a normal cyclic subgroup C_n of $\overline{G}$ of index 2 such that $\bar{\rho} \sim \mathrm{Ind}_{C_n}^{\overline{G}}(\xi')$. Then we have

$$\mathrm{ad} \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^\varphi \oplus \mathrm{Ind}_{C_n}^{\overline{G}} \mathbb{F}^\xi \quad \text{and} \quad \mathrm{ad}^0 \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^{\varphi\varepsilon} \oplus \mathrm{Ind}_{C_n}^{\overline{G}} \mathbb{F}^\xi \otimes \mathbb{F}^\varepsilon,$$

where $\varphi \colon \overline{G}/C_n \to \mathbb{F}^*$ is the unique non-trivial character of order two and $\xi : C_n \to \mathbb{F}^*$ is the character $g \mapsto \xi'(g)^{1-\#k_K}$ for k_K the residue field of K.

(D) $\overline{G}$ and $\overline{U}$ are trivial. Then $\overline{H}$ is trivial, and H is in the scalars of $\mathrm{GL}_2(\mathbb{F})$. Moreover,

$$\mathrm{ad} \cong (\mathbb{F}^{\mathrm{triv}})^4 \quad \text{and} \quad \mathrm{ad}^0 \otimes \mathbb{F}^\varepsilon \cong (\mathbb{F}^\varepsilon)^3.$$

(E) $\overline{G}$ is trivial and $\overline{U}$ is nontrivial. Then $\bar{\rho} \sim \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \otimes \eta$ for some character $\eta : G_K \to \mathbb{F}^*$, where $\star$ denotes a non-trivial extension. Moreover,

$$(\mathrm{ad})^U \cong (\mathbb{F}^{\mathrm{triv}})^2 \quad \text{and} \quad (\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon \cong \mathbb{F}^\varepsilon.$$

Remark 4.3.1. We would like to correct a mistake in [Böc00, Lem. 6.1] when U is nontrivial. As the character ψ defined at the beginning of [Böc00, §5] corresponds to the character ξ^{-1} in the notation used here, in [Böc00, Lem. 6.1] the line $((\mathrm{ad}_{\bar{\rho}})^U \otimes \mu_p(L))^G \cong (k^\chi \oplus k^{\psi^{-1}\chi})^G$ should be replaced by $((\mathrm{ad}_{\bar{\rho}})^U \otimes \mu_p(L))^G \cong (k^\chi \oplus k^{\psi\chi})^G$. Further, the condition in case (ix) should read $\chi = \psi^{-1}$ and not $\chi = \psi$ as written.

We know from [Böc00, Theorem 2.6] that the versal deformation ring is isomorphic to the quotient $W(\mathbb{F})[\![x_1, \ldots, x_h]\!]/I^\psi$, where I^ψ is generated by exactly $h_2 := \dim_{\mathbb{F}} H^2(G_K, \mathrm{ad}^0)$ relations.

Lemma 4.3.2 (Cf. [Böc00, Lem. 6.1])**.** *If $\mu_{p^\infty}(F) = \{1\}$, then $h_2 = 0$. Else, the dimensions h_2 and h take the following values in the cases (A)-(E) introduced above.*

(A) (i) *If $\varepsilon = \mathrm{triv}$, then $h_2 = 1$ and $h = 3[K : \mathbb{Q}_p] + 2$;*
(ii) *If $\varepsilon = \xi$ and the order of ξ is two, then $h_2 = 2$ and $h = 3[K : \mathbb{Q}_p] + 3$;*
(iii) *If $\varepsilon = \xi$ or $\varepsilon = \xi^{-1}$ and $\xi \neq \xi^{-1}$, then $h_2 = 1$ and $h = 3[K : \mathbb{Q}_p] + 2$;*
(iv) *In all other cases $h_2 = 0$ and $h = 3[K : \mathbb{Q}_p] + 1$.*

(B) (i) *If $\varepsilon = \xi^{-1}$ and the order of ξ is two, then $h_2 = 1$ and $h = 3[K : \mathbb{Q}_p] + 1$;*
(ii) *If $\varepsilon = \xi^{-1}$ and $\xi \neq \xi^{-1}$, then $h_2 = 1$ and $h = 3[K : \mathbb{Q}_p] + 1$;*[5]
(iii) *In all other cases $h_2 = 0$ and $h = 3[K : \mathbb{Q}_p]$.*

(C) (i) *If $\varepsilon = \varphi$, then $h_2 = 1$ and $h = 3[K : \mathbb{Q}_p] + 1$;*
(ii) *In all other cases $h_2 = 0$ and $h = 3[K : \mathbb{Q}_p]$.*

(D) (i) *If $\varepsilon = \mathrm{triv}$, then $h_2 = 3$ and $h = 3[K : \mathbb{Q}_p] + 6$;*
(ii) *In all other cases $h_2 = 0$ and $h = 3[K : \mathbb{Q}_p] + 3$.*

(E) (i) *If $\varepsilon = \mathrm{triv}$, then $h_2 = 1$ and $h = 3[K : \mathbb{Q}_p] + 2$;*
(ii) *In all other cases $h_2 = 0$ and $h = 3[K : \mathbb{Q}_p] + 1$.*

Proof. If F contains no p-power roots of unity, then the maximal pro-p quotient $G_F(p)$ of G_F is a free pro-p group and $h_2 = 0$ by [Lab67, §1.4]. Otherwise we use the above decompositions of $\mathrm{ad} \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathrm{ad}^0$ and $(\mathrm{ad}^0)^U \otimes \mathbb{F}^\varepsilon$ in the cases (A)–(E), and obtain the values of h_2 and $h_0 := \dim_{\mathbb{F}} H^0(G_K, \mathrm{ad}^0)$. Recall next that the Euler-Poincaré characteristic of ad^0 is $3[K : \mathbb{Q}_p] = -h_0 + h - h_2$ from which one computes h. □

For the following explicit descriptions of minimal presentations of $R^\psi_{\bar{\rho}}$, we recall the functor E_Π from [Böc00, Proposition 2.3]. It is always representable and its universal ring is a versal hull for $D^\psi_{\bar{\rho}}$. To describe E_Π we need to fix some notation. Since U is a p-group in $\mathrm{GL}_2(\mathbb{F})$ we shall assume that U lies in the set of unipotent upper triangular matrices $U_2(\mathbb{F})$. If U is non-trivial,

[5] The reason for not combining (i) and (ii) in case (B) into a single case is that the cases of Lemma 4.3.2 are used throughout this section, and in later parts the distinction is necessary.

let $\{g_n\}_n$ be a minimal set of topological generators of the maximal pro-p quotient $G_F(p)$ of G_F so that the $\bar{\rho}(g_n) = \mathrm{Mat}\, 1\bar{u}_n 01$ for $\bar{u}_n \in \mathbb{F}$ generate U as a G-module. If U is non-trivial, there is a smallest index i_0 for which $\bar{u}_{i_0}$ is a unit. Then by conjugation by an element of the form $\mathrm{Mat}\, \lambda 001$, $\lambda \in \mathbb{F}^*$, which clearly lifts to $\mathrm{GL}_2(W(\mathbb{F}))$, we will assume from now on that $\bar{u}_{i_0} = 1$. For any ring R in $\widehat{\mathcal{A}r}_W(\mathbb{F})$ we denote by $\widetilde{\Gamma}_2(R)$ the inverse image of $U_2(\mathbb{F})$ under the reduction homomorphism $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathbb{F})$. We set $\bar{\alpha} := \bar{\rho}|_{G_F(p)}$. If $\bar{G} = \{1\}$, then we define the functor $E_\Pi \colon \widehat{\mathcal{A}r}_W(\mathbb{F}) \to \mathit{Sets}$ by sending $(R, \mathfrak{m}_R)$ to the set

$$\left\{ \alpha \in \mathrm{Hom}_G(G_F(p), \tilde{\Gamma}_2(R)) \,\middle|\, \alpha(g_{i_0}) = \begin{pmatrix} 1 & 1 \\ * & * \end{pmatrix} \text{ and } \alpha \equiv \bar{\alpha} \pmod{\mathfrak{m}_R} \right\}$$

if U is non-trivial, and else to $\mathrm{Hom}_G(G_F(p), \tilde{\Gamma}_2(R))$. Observe that $E_\Pi(\mathbb{F}) = \{\bar{\alpha}\}$. As noted above, E_Π is representable and its universal ring, we write $R_{\bar{\alpha}}$, is isomorphic to $R^\psi_{\bar{\rho}}$. The gain is that it is rather elementary to write down explicitly $R_{\bar{\alpha}}$.

Lastly, we define q as the number of p-power roots of unity contained in F and g_q as the polynomial

$$g_q(x) := \sum_{k=0}^{(q-1)/2} \frac{q}{(2k+1)!} \prod_{j=0}^{k-1} (q^2 - (2j+1)^2) x^k.$$

Note that the polynomial g_q lies in fact in $\mathbb{Z}[x]$.

Remark 4.3.3. We take this opportunity to correct another mistake from [Böc00, Rem. 5.5 (i)]: In the formulas for $a_{n,k}$ and $b_{n,k}$, the expressions $(2k)!$ and $(2k+1)!$, respectively, should be in the denominator.

Theorem 4.3.4 (Cf. [Böc00, Thm. 6.2 and Rem. 6.3 (iv)]). *Suppose $\mu_{p^\infty}(F) \neq \{1\}$ and set $m := [K : \mathbb{Q}_p]$. There exists a minimal presentation $0 \to I^\psi \to \mathcal{R} \to R^\psi_{\bar{\rho}} \to 0$ of $R^\psi_{\bar{\rho}}$, where $\mathcal{R}$ and I^ψ are as follows in the respective cases of the previous lemma.*

(A) *(i)* $\mathcal{R} = W(\mathbb{F})[\![\{b_i, c_i\}_{i=1}^m, \{d_j\}_{j=0}^{m+1}]\!]$ *and* $I^\psi = \Big(\sum_{i=1}^m c_i b_{m-i+1} - ((1+d_0)^q - 1)(1+d_0)^{-\frac{q}{2}} \Big)$;

(ii) $\mathcal{R} = W(\mathbb{F})[\![\{b_i, c_i, d_i\}_{i=0}^m]\!]$ *and* $I^\psi = \Big(\sum_{i=0}^m b_i d_{m-i} - b_0 g_q(b_0 c_0), -\sum_{i=0}^m c_i d_{m-i} - c_0 g_q(b_0 c_0) \Big)$;

(iii) *If* $\varepsilon = \psi$, *then* $\mathcal{R} = W(\mathbb{F})[\![\{b_i, d_i\}_{i=0}^m, \{c_j\}_{j=1}^m]\!]$ *and* $I^\psi = \Big(\sum_{i=0}^m d_i b_{m-i} - q b_0 \Big)$;
If $\varepsilon = \psi^{-1}$, *then* $\mathcal{R} = W(\mathbb{F})[\![\{b_i\}_{i=1}^m, \{c_j, d_j\}_{j=0}^m]\!]$ *and* $I^\psi = \Big(\sum_{i=0}^m d_i c_{m-i} - q c_0 \Big)$;

(iv) $\mathcal{R} = W(\mathbb{F})[\![\{b_i, c_i\}_{i=1}^m, \{d_j\}_{j=0}^m]\!]$ *and* $I^\psi = (0)$.

(B) *(i)* $\mathcal{R} = W(\mathbb{F})[\![\{b_i, c_i, d_i\}_{i=0}^m]\!]/(b_{i_0}, d_{m-i_0})$ *and* $I^\psi = \Big(-\sum_{i=0, i\neq i_0}^m c_i d_{m-i} - \delta_{i_0}(c_0 + 2c_{i_0} b_0) \cdot g_q(b_0(c_0 + c_{i_0} b_0)) - 2(1-\delta_{i_0}) c_0 g_q(c_0) \Big)$, *where* $\delta_{i_0} \in \{0, 1\}$ *is 0 if* $i_0 = 0$ *and else 1;*

(ii) $\mathcal{R} = W(\mathbb{F})[\![\{b_i\}_{i=1}^m, \{c_j, d_j\}_{j=0}^m]\!]/(b_{i_0})$ *and* $I^\psi = \Big(\sum_{i=0}^m c_i d_{m-i} - q c_0 \Big)$;

(iii) $\mathcal{R} = W(\mathbb{F})[\![\{b_i, c_i, d_i\}_{i=1}^m]\!]$ *and* $I^\psi = (0)$.

(C) *(i)* $\mathcal{R} = W(\mathbb{F})[\![\{b_i\}_{i=1}^{2m}, \{d_j\}_{j=0}^m]\!]$ *and* $I^\psi = \Big(\sum_{i=1}^m b_i b_{2m-i+1} - \big((1+d_0)^{\frac{q}{2}} - (1+d_0)^{-\frac{q}{2}} \big) \Big)$;

(ii) $\mathcal{R} = W(\mathbb{F})[\![\{b_i\}_{i=1}^{2m}, \{d_j\}_{j=1}^{m}]\!]$ *and* $I^{\psi} = (0)$.

Remark 4.3.5. We point out that in front of the sum $\sum c_i d_{m-i}$ in the second generator of the relation ideals in [Böc00, Theorem 6.2(ii) and Prop. 7.3(ii)] a minus sign is missing. This originates from a sign mistake in [Böc00, Lem. 5.6(B)]: There the matrix $\begin{pmatrix} 0 & b_i \\ c_i & 0 \end{pmatrix}$ should read instead $\begin{pmatrix} 0 & b_i \\ -c_i & 0 \end{pmatrix}$.

Proof. The relation ideal of the versal hull of the deformation functor without fixing the determinant is listed in the respective cases in [Böc00, Theorem 6.2]. We remark that the relation $(1+a_0)^q - 1$ from there is omitted due to our condition on the determinant, and we used a change of variables according to [Böc00, Remark 6.3(iv)] to simplify the expressions for the relations and variables. In order to obtain the right number of indeterminates of the power series ring $\mathcal{R}$, we follow the steps described in the proof of [Böc00, Theorem 2.6].

Since by assumption F contains a p-power root of unity, $G_F(p)$ is a Demuškin group, and its Frattini quotient $\bar{G}_F(p)$ is isomorphic to $\mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^{\varepsilon} \oplus \mathbb{F}_p[G]^m$ as a G-module. By the Burnside basis theorem, there are closed subgroups P_n of $G_F(p)$ such that the Frattini quotients $\bar{P}_n$ of P_n are irreducible and $\overline{G}_F(p) = \oplus_n \bar{P}_n$. Since the tangent space $t_E := E_\Pi(\mathbb{F}[t]/(t^2))$ of E_Π is isomorphic to the tangent space t_D and $\mathrm{ad}^0 \cong \tilde{\Gamma}_2(\mathbb{F}[t]/(t^2))$ as a G-module, we have $h = \dim_{\mathbb{F}} t_D = \dim_{\mathbb{F}} t_E \leq \dim_{\mathbb{F}} \mathrm{Hom}_G(\overline{G}_F(p), \mathrm{ad}^0)$. We can compute the right hand side in terms of those G-submodules $\bar{P}_n$ of $\overline{G}_F(p)$ that occur in decompositions of both $\overline{G}_F(p)$ and ad^0 into irreducible G-modules, because the G-submodules that do not occur in a decomposition of ad^0 have trivial image (prime-to-adjoint principle). As remarked in [Böc00, §6], the multiplicities of the G-submodules occurring in a decomposition of ad^0 are $(\overline{G}_F(p), \mathrm{Ind}_{C_n}^{\bar{G}} \mathbb{F}^{\psi})_G = 2m$ if $\bar{G}$ is dihedral, $(\overline{G}_F(p), \mathbb{F}^{\tau})_G = m$ for any non-trivial character $\tau \neq \varepsilon$, and $(\overline{G}_F(p), \mathbb{F}^{\mathrm{triv}})_G = (\overline{G}_F(p), \mathbb{F}^{\varepsilon})_G = m + 1 + \delta_K$, where $(X, Y)_G := \dim_{\mathbb{F}}(\mathrm{Hom}_G(X, Y))$ for G-modules and δ_K is 1 if ε acts trivially and 0 otherwise. By [Böc00, Lem. 5.3], we can choose $x_n \in P_n$ such that Gx_n topologically generates P_n, and whose image under a homomorphism $\alpha : P_n \to \Gamma_n(R)$ is either the identity if $\bar{P}_n$ does not occur in a decompositions of ad^0 or a matrix of the type

$$S(b,c) := \begin{pmatrix} \sqrt{1+bc} & \frac{b}{\sqrt{1+bc}} \\ c & \end{pmatrix} \qquad \text{or} \qquad D(d) := \begin{pmatrix} \sqrt{1+d} & 0 \\ 0 & \sqrt{1+d}^{-1} \end{pmatrix}$$

for any ring R in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ and $b, c, d \in \mathfrak{m}_R$. If U is non-trivial, we shall take for the g_n in the definition of E_Π the generators x_n. If $\bar{\rho}(x_0) \neq \mathrm{id}$, then we take $g_1 := x_0$, else we shall assume that $g_1 := x_1$ by a suitable permutation of the indices n. In cases (A)–(C), we will consider the power series ring $\mathcal{R}$ over $W(\mathbb{F})$ in the variables b, c, d occurring in the images $S(b,c)$ and $D(d)$ of all generators. Then we will obtain the universal object $(R_{\bar{\alpha}}, \alpha_{\bar{\alpha}})$ representing E_Π, where $R_{\bar{\alpha}}$ is the quotient ring of $\mathcal{R}$ modulo the respective relations in terms of the variables b, c, d from [Böc00, Lemma 5.6 and Theorem 6.2] .

We begin with explicitly describing $\mathcal{R}$ and the relation ideal I^{ψ} in case (A). Then we have that $\mathrm{ad}^0 \cong \mathbb{F}^{\mathrm{triv}} \oplus \mathbb{F}^{\xi} \oplus \mathbb{F}^{\xi^{-1}}$ and $h = \dim_{\mathbb{F}} t_E = (\overline{G}_F(p), \mathbb{F}^{\mathrm{triv}})_G + (\overline{G}_F(p), \mathbb{F}^{\xi})_G + (\overline{G}_F(p), \mathbb{F}^{\xi^{-1}})_G$. The following table displays the respective multiplicities of the subrepresentations $\mathbb{F}^{\mathrm{triv}}$, $\mathbb{F}^{\xi}$ and $\mathbb{F}^{\xi^{-1}}$ in $\overline{G}_F(p)$:

		$k_1 = (\overline{G}_F(p), \mathbb{F}^{\text{triv}})_G$	$k_2 = (\overline{G}_F(p), \mathbb{F}^{\xi})_G$	$k_3 = (\overline{G}_F(p), \mathbb{F}^{\xi^{-1}})_G$
(i)	$\varepsilon = \text{triv}$	m+2	m	m
(ii)	$\varepsilon = \xi$, $\varepsilon = \varepsilon^{-1}$	m+1	m+1	m+1
(iii)	$\varepsilon = \xi$, $\varepsilon \neq \varepsilon^{-1}$	m+1	m+1	m
	$\varepsilon = \xi^{-1}$, $\varepsilon \neq \varepsilon^{-1}$	m+1	m	m+1
(iv)	$\varepsilon \notin \{\text{triv}, \xi, \xi^{-1}\}$	m+1	m	m

By [Böc00, Lem. 5.3(ii)-(iv)], there exist $b_n, c_{n'}, d_{n''} \in \mathfrak{m}_R$ (with $n = (m+1-k_2), \dots, m$, $n' = (m+1-k_3), \dots, m$ and $n'' = 0, \dots, k_1 - 1$) such that a generator x_n of a subgroup P_n gets mapped to either

$$S(b_n, 0), \quad S(0, c_{n'}), \quad S(b_n, c_n), \quad D(d_{n''}) \quad \text{or} \quad D(0)$$

under a G-equivariant homomorphism $P_n \to \text{GL}_2(R)$. Finally, in [Böc00, Lem. 5.6(A)-(D),(F)] the image of the Demuškin relation involving these matrices is completely described. The thereby obtained equations define the respective relation ideal I^ψ (as in [Böc00, Theorem 6.2(i)-(iv)]).

In case (B), we have that $\text{ad}^0 \cong \mathbb{F}^{\text{triv}} \oplus \mathbb{F}^{\xi} \oplus \mathbb{F}^{\xi^{-1}}$ and $h = \dim_{\mathbb{F}} t_D = \dim_{\mathbb{F}} t_E < h' := (\overline{G}_F(p), \mathbb{F}^{\text{triv}})_G + (\overline{G}_F(p), \mathbb{F}^{\xi})_G + (\overline{G}_F(p), \mathbb{F}^{\xi^{-1}})_G$ due to the further conditions that $\alpha \in E_\Pi(\mathbb{F}[t]/t^2)$ has to satisfy if U is non-trivial. As in case (A), there exist $b_n, c_{n'}, d_{n''} \in \mathfrak{m}_R$ (with $n = (m+1-k_2), \dots, m$, $n' = (m+1-k_3), \dots, m$ and $n'' = 0, \dots, k_1 - 1$) such that a generator x_n of a subgroup P_n gets mapped to either

$$S(\bar{u}_n + b_n, 0), \quad S(0, c_{n'}), \quad S(\bar{u}_n + b_n, c_n), \quad D(d_{n''}) \quad \text{or} \quad D(0)$$

under a G-equivariant homomorphism $P_n \to \text{GL}_2(R)$. Due to the condition on the image of x_{i_0}, the variable b_{i_0} occurring in the image of x_{i_0} must vanish. In [Böc00, Lem. 5.6(A)-(D),(F)] the image of the Demuškin relation involving these matrices is completely described. By [Böc00, Theorem 6.2(ii)-(iii)], this gives rise to the following generators of I^ψ:

$$\sum_{i=0}^{m} (\bar{u}_i + b_i) d_{m-i} - (\bar{u}_0 + b_0) g_q((\bar{u}_0 + b_0)c_0) \quad \text{and} \quad -\sum_{i=0}^{m} c_i d_{m-i} - c_0 g_q((\bar{u}_0 + b_0)c_0) \quad \text{in case (i)}$$

and in case (ii) to $\sum_{i=0, i\neq i_0}^{m} c_i d_{m-i} - q c_0$. In (i), we use the first relation $d_{m-i_0} = (\bar{u}_0 + b_0) g_q((\bar{u}_0 + b_0)c_0) - \sum_{i=0, i\neq i_0}^{m} (\bar{u}_i + b_i) d_{m-i}$ to also eliminate d_{m-i_0}. Then the second equation reads

$$-\sum_{i=0}^{m} c_i d_{m-i} - c_0 g_q((\bar{u}_0+b_0)c_0) = -\sum_{i=0, i\neq i_0}^{m} \big(c_i - c_{i_0}(\bar{u}_i + b_i)\big) d_{m-i} - (c_0 + c_{i_0}(\bar{u}_0 + b_0)) g_q((\bar{u}_0 + b_0)c_0).$$

We perform a linear change of coordinates by replacing $c_i + c_{i_0}(\bar{u}_i + b_i)$ by c_i for $i \neq i_0$. Note that $\bar{u}_0 = 0$ if $i_0 > 0$ so that we obtain the respective generators of I^ψ displayed in case (B).

In case (C), we have that $\text{ad}^0 \cong \mathbb{F}^{\varphi} \oplus \text{Ind}_{C_n}^{\overline{G}} \mathbb{F}^{\xi}$ and $h = \dim_{\mathbb{F}} t_E = (\overline{G}_F(p), \mathbb{F}^{\varphi})_G + 2(\overline{G}_F(p), \text{Ind}_{C_n}^{\overline{G}} \mathbb{F}^{\xi})_G$. This means that the multiplicities of the subrepresentations $\text{Ind}_{C_n}^{\overline{G}} \mathbb{F}^{\xi}$ in $\overline{G}_F(p)$ are $2m$, and the ones of the subrepresentations $\mathbb{F}^{\varphi}$ are $m+1$ if $\varepsilon = \varphi$ and m if $\varepsilon \neq \varphi$. By [Böc00, Lem. 5.3(ii),(v)-(vii)], there exist $b_n, d_{n'} \in \mathfrak{m}_R$ (with $n = 1, \dots, 2m$, $n' = 0, \dots, m$ in (i) and $n' = 1, \dots, m$ in (ii)) such that a generator x_n of a subgroup P_n gets mapped to either

$$S(b_n, b_n), \quad S(b_n, -b_n), \quad D(d_{n'}) \quad \text{or} \quad D(0)$$

under a G-equivariant homomorphism $P_n \to \text{GL}_2(R)$. Finally, in [Böc00, Lem. 5.6(E)-(F)] the image of the Demuškin relation involving these matrices is completely described. The thereby obtained equations define the respective relation ideal I^ψ (as in [Böc00, Theorem 6.2(v)-(vii)]). $\square$

We define $\mathfrak{n}$ to be the ideal in $\mathcal{R}$ generated by all the variables $b_i, c_{i'}, d_{i''}$ occurring in the respective definitions of $\mathcal{R}$ in the previous theorem. Further, define the ideal $\mathfrak{m}_s \subset \mathcal{R}$ as $\mathfrak{m}_s := q\mathcal{R} + \mathfrak{n}$. In cases (A)–(C) it is now a simple matter to read off from the previous theorem the initial terms for the graded rings naturally associated to $\mathcal{R}$. Checking that these initial terms form a regular sequence will imply most parts of Theorem 4.1.4 and, when combined with Proposition 4.2.2, the assertions of Theorem 4.1.5 in cases (A)–(C).

Corollary 4.3.6. *In the cases (A)–(C) of the previous lemma, denote the two generators of I^ψ in case (A)(ii) by f_1 and f_2, and in the other cases the generator of I^ψ by f_1.*

(a) Let in *be the initial term map $\overline{\mathcal{R}} \to \mathrm{gr}_{\bar{\mathfrak{n}}} \overline{\mathcal{R}}$. Then the following are the initial terms of the generators of I^ψ in $\mathrm{in}(\overline{I^\psi}) \subset \mathrm{gr}_{\bar{\mathfrak{n}}} \overline{\mathcal{R}}$ in the cases (A)–(C) of Lemma 4.3.2, where we only list those cases in which h_2 is non-zero.*

(A) (i) $\mathrm{in}(\overline{f_1}) = \sum_{i=1}^m \bar{c}_i \bar{b}_{m-i+1}$;

(ii) $\mathrm{in}(\overline{f_1}) = -\sum_{i=0}^m \bar{b}_i \bar{d}_{m-i}$ *and* $\mathrm{in}(\overline{f_2}) = \sum_{i=0}^m \bar{c}_i \bar{d}_{m-i}$;

(iii) If $\varepsilon = \psi$, then $\mathrm{in}(\overline{f_1}) = \sum_{i=0}^m \bar{d}_i \bar{b}_{m-i}$;
If $\varepsilon = \psi^{-1}$, then $\mathrm{in}(\overline{f_1}) = \sum_{i=0}^m \bar{d}_i \bar{c}_{m-i}$;

(B) (i) $\mathrm{in}(\overline{f_1}) = -\sum_{i=0, i\neq i_0}^m \bar{c}_i \bar{d}_{m-i} - \overline{1-\delta_{i_0}} \cdot \frac{q}{3} \cdot \bar{c}_0^2$;[6]

(ii) $\mathrm{in}(\overline{f_1}) = \sum_{i=0}^m \bar{c}_i \bar{d}_{m-i}$;

(C) (i) $\mathrm{in}(\overline{f_1}) = \sum_{i=1}^m \bar{b}_i \bar{b}_{2m-i+1}$.

(b) Let in *be the initial term map $\mathcal{R} \to \mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}$ and set $t_0 := \mathrm{in}(q)$. Then the following are the initial terms of the generators of I^ψ in $\overline{\mathrm{in}(I^\psi)} \subset \overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$ in the cases (A)–(C) of Lemma 4.3.2, where we only list those cases in which h_2 is non-zero.*

(A) (i) $\overline{\mathrm{in}(f_1)} = \sum_{i=1}^m \bar{c}_i \bar{b}_{m-i+1} - \bar{t}_0 \bar{d}_0$;

(ii) $\overline{\mathrm{in}(f_1)} = \sum_{i=0}^m \bar{b}_i \bar{d}_{m-i} - \bar{t}_0 \bar{b}_0$ *and* $\overline{\mathrm{in}(f_2)} = -\sum_{i=0}^m \bar{c}_i \bar{d}_{m-i} - \bar{t}_0 \bar{c}_0$;

(iii) If $\varepsilon = \psi$, then $\overline{\mathrm{in}(f_1)} = \sum_{i=0}^m \bar{d}_i \bar{b}_{m-i} - \bar{t}_0 \bar{b}_0$;
If $\varepsilon = \psi^{-1}$, then $\overline{\mathrm{in}(f_1)} = \sum_{i=0}^m \bar{d}_i \bar{c}_{m-i} - \bar{t}_0 \bar{c}_0$;

(B) (i) $\overline{\mathrm{in}(f_1)} = -\sum_{i=0, i\neq i_0}^m \bar{c}_i \bar{d}_{m-i} - \overline{2-\delta_{i_0}} \cdot \bar{t}_0 \bar{c}_0 - \overline{1-\delta_{i_0}} \cdot \frac{q}{3} \cdot \bar{c}_0^2$;[6]

(ii) $\overline{\mathrm{in}(f_1)} = \sum_{i=0}^m \overline{c_i} \bar{d}_{m-i} - \bar{t}_0 \bar{c}_0$;

(C) (i) $\overline{\mathrm{in}(f_1)} = \sum_{i=1}^m \bar{b}_i \bar{b}_{2m-i+1} - \bar{t}_0 \bar{d}_0$.

Proof of Theorem 4.1.4. First note that we can reduce to the case $\mathcal{O} = W(\mathbb{F})$ as follows: by [Maz97, § 12 Prop.] there is an isomorphism $R_{\bar\rho} \cong R_{\bar\rho, W(\mathbb{F})} \otimes_{W(\mathbb{F})} \mathcal{O}$, where $R_{\bar\rho}$ and $R_{\bar\rho, W(\mathbb{F})}$ are the universal deformation rings of $\bar\rho$ that parametrize all deformations of $\bar\rho$ to coefficient rings in $\widehat{\mathcal{A}r}_{\mathcal{O}}$ and $\widehat{\mathcal{A}r}_W(\mathbb{F})$, respectively. If the fixed character $\psi\colon G_K \to \mathcal{O}^*$ takes values in $W(\mathbb{F})^*$, then the same argument shows that $R_{\bar\rho}^\psi \cong R_{\bar\rho, W(\mathbb{F})}^\psi \otimes_{W(\mathbb{F})} \mathcal{O}$, where $R_{\bar\rho, W(\mathbb{F})}^\psi$ is the universal deformation ring of $\bar\rho$ that parametrizes all deformations of $\bar\rho$ with fixed determinant ψ to coefficient rings in $\widehat{\mathcal{A}r}_W(\mathbb{F})$. If $\psi\colon G_K \to \mathcal{O}^*$ is arbitrary, by Lemma 4.4.1 below it can be twisted so that its image lies in $W(\mathbb{F})^*$.

We next give the proof in cases (A)–(C): In all cases of (A)–(C) with $h_2 \neq 0$, Theorem 4.1.4(a) holds since the initial terms given in Corollary 4.3.6(b) together with $\bar{t}_0$ form regular sequences

[6] Note that the term involving $\bar{c}_0^2$ vanishes unless $q = 3$.

in $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}$ with $t_0 := \mathrm{in}(q)$. Moreover, by Proposition 4.2.2(c) the initial terms from Corollary 4.3.6(b) generate $\overline{\mathrm{in}(I^\psi)}$ in the respective cases and one checks that $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}/\overline{\mathrm{in}(I^\psi)}$ is an integral domain. Thus Theorem 4.1.4(b) follows from Proposition 4.2.2(d) in cases (A)–(C).

Theorem 4.1.4 in the remaining cases (D) and (E) is a direct consequence of the following lemma. □

Lemma 4.3.7. *In the cases (D) and (E) let q denote the minimum of p and the number of p-power roots of unity in K. Then there exists a minimal presentation*

$$0 \longrightarrow I^\psi = (r_1,\dots,r_m) \longrightarrow \mathcal{R} \cong W(\mathbb{F})[\![x_1,\dots,x_h]\!] \longrightarrow R^\psi_{\bar\rho} \longrightarrow 0$$

such that, letting $\mathfrak{m}_s = (q,x_1,\dots,x_h)$, the following hold:

(a) $\mathfrak{m}_s^2 \supset I^\psi$ *and* $\overline{\mathrm{in}(q)},\overline{\mathrm{in}(r_1)},\dots,\overline{\mathrm{in}(r_m)} \in \overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}$ *is a regular sequence in* $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}$*;*

(b) $\overline{\mathrm{gr}_{\mathfrak{m}_s}\mathcal{R}}/\big(\overline{\mathrm{in}(r_1)},\dots,\overline{\mathrm{in}(r_m)}\big)$ *is an integral domain and* $\overline{\mathrm{in}(I^\psi)} = \big(\overline{\mathrm{in}(r_1)},\dots,\overline{\mathrm{in}(r_m)}\big)$*;*

(c) $\mathrm{in}(\bar r_1),\dots,\mathrm{in}(\bar r_m) \in \mathrm{gr}^2_{\overline{\mathfrak{m}}_{\mathcal{R}}}\overline{\mathcal{R}}$ *form a regular sequence in* $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}}\overline{\mathcal{R}}$*;*

(d) $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}}\overline{\mathcal{R}}/\big(\mathrm{in}(\bar r_1),\dots,\mathrm{in}(\bar r_1)\big)$ *is an integral domain and* $\mathrm{in}(\overline{I^\psi}) = \big(\mathrm{in}(\bar r_1),\dots,\mathrm{in}(\bar r_1)\big)$*;*

(e) $m = \dim_{\mathbb{F}} H^2(G_K,\mathrm{ad}^0)$ *and* $\dim_{\mathrm{Krull}} R^\psi_{\bar\rho} = h+1-m$*.*

Proof. The proof proceeds along the lines of the proof of [Böc00, Theorem 2.6], but it is simpler in our case as we shall only determine the initial parts of the g_i and r_j, and since there is no action of a finite group of order prime to p. We recall that $\bar\rho \sim \mathrm{Mat}\,1{\star}01 \otimes \eta$ for some character $\eta : G_K \to \mathbb{F}^*$, where $\star$ denotes an extension. As a preliminary reduction, we may twist $\bar\rho$ by η^{-1} so that the image of $\bar\rho$ is a p-group. Twisting all deformations by the Teichmüller lift of η^{-1} provides an isomorphism to the deformation functor of the twist of $\bar\rho$. In particular both functors are represented by isomorphic versal rings. Since now $\det\bar\rho$ is trivial, we shall also assume that its fixed lift ψ is the trivial character, since again, changing ψ has no effect on the versal deformation ring up to isomorphism. After this reduction, the first case to consider is that when K does not contain a non-trivial p-power root of unity. Then by Lemma 4.3.2 we have $h_2 = 0$. Hence $R^\psi_{\bar\rho}$ is unobstructed and thus formally smooth, and assertions (a)–(e) are obvious.

Suppose from now on that K contains a primitive p-th root of unity ζ_p. Then the maximal pro-p-quotient $G_K(p)$ of G_K is known to be a Demushkin group of rank $2g = [K:\mathbb{Q}_p]+2$, cf. [Lab67, §5]. By the classification of Demushkin groups with $q>2$ [Lab67, Theorem 7], the pro-p group $G_K(p)$ is isomorphic to the pro-p completion Π of the discrete group

$$\langle x_1,\dots,x_{2g} \mid r\rangle$$

for the Demushkin relation $r = x_1^q(x_1,x_2)(x_3,x_4)\dots(x_{2g-1},x_{2g})$ – recall that $(x,y) = x^{-1}y^{-1}xy$. In the following we fix an isomorphism $G_K(p) \cong \Pi$.[7] Note also that $2g \geq 4$, because K has to contain $\mathbb{Q}_p(\zeta_p)$ and $[\mathbb{Q}_p(\zeta_p):\mathbb{Q}_p] = p-1 \geq 2$. If $\mathrm{im}\,(\bar\rho)$ is non-trivial, the functor $E_\Pi\colon \widehat{\mathcal{A}\mathrm{r}}_W(\mathbb{F}) \to Sets$ is given by

$$(R,\mathfrak{m}_R) \mapsto \left\{ \alpha \in \mathrm{Hom}(\Pi,\tilde\Gamma_2(R)) \,\middle|\, \alpha(x_{i_0}) = \begin{pmatrix} 1 & 1 \\ * & * \end{pmatrix},\ \bar\alpha(x_i) \equiv \begin{pmatrix} 1 & \bar u_i \\ 0 & 1 \end{pmatrix} \bmod \ \mathfrak{m}_R \text{ for all } i \right\},$$

[7] By slight abuse of notation we shall therefore regard the topological generators x_i of Π as elements of G_K.

and else by $(R, \mathfrak{m}_R) \mapsto \mathrm{Hom}(\Pi, \tilde{\Gamma}_2(R))$. As the elements $\{g_n\}$ from the bottom of page 102 we take $x_1, \ldots, x_{2g}$. As noted there, E_Π is always representable and its universal ring $R_{\bar{\alpha}}$ is isomorphic to $R^{\psi}_{\bar{\rho}}$.

In order to find an explicit presentation of $R_{\bar{\alpha}}$, we define $\mathcal{S} := W(\mathbb{F})[\![b_i, c_i, d_i : i = 1, \ldots, 2g]\!]$. For each $1 \leq i \leq 2g$ let

$$M_i := 1_2 + \begin{pmatrix} a_i & b_i + u_i \\ c_i & d_i \end{pmatrix} \qquad \text{with } 1_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where we choose a lift $u_i \in W(\mathbb{F})$ of $\bar{u}_i \in \mathbb{F}$, subject to the requirement $u_i = 0$ whenever $\bar{u}_i = 0$, and where $a_i \in \mathcal{S}$ is chosen so that $\det M_i = 1$, i.e., $a_i = ((b_i + u_i)c_i - d_i) \sum_{n \geq 0} (-1)^n d_i^n$. Observe that in case (D) all $u_i = 0$. We define polynomials r_k in $\mathcal{S}$ by

$$1_2 + \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} := M_1^q [M_1, M_2] \ldots [M_{2g-1}, M_{2g}],$$

where $[M_i, M_{i'}]$ is the commutator bracket $M_i^{-1} M_{i'}^{-1} M_i M_{i'}$. Note that $(1+r_1)(1+r_4) - r_2 r_3 = 1$ and that, as we shall explain in a moment, $(r_1, \ldots, r_4) \subset \mathfrak{m}_s = (q, b_i, c_i, d_i : i = 1, \ldots, 2g)$. It is now straightforward to see that the ring

$$R_{\bar{\alpha}} := \begin{cases} W(\mathbb{F})[\![b_i, c_i, d_i : i = 1, \ldots, 2g]\!]/(r_1, r_2, r_3) & \text{in case (D)} \\ W(\mathbb{F})[\![b_i, c_i, d_i : i = 1, \ldots, 2g]\!]/(r_1, r_2, r_3, b_{i_0}, d_{i_0} - c_{i_0}) & \text{in case (E)} \end{cases}$$

together with the homomorphism $\alpha_{\bar{\alpha}} : \Pi \to \mathrm{SL}_2(R_{\bar{\alpha}})$ defined by mapping x_i to M_i – the latter regarded as a matrix over $R_{\bar{\alpha}}$ – is a universal object for E_Π. Note that $\alpha_{\bar{\alpha}}$ is well-defined precisely because we imposed the condition that all r_k vanish. In case (E) we may and shall assume that $i_0 \leq 4$ by permuting the **indices** of the x_i in pairs $(2i'-1, 2i')$ for $i' \in \{2, \ldots, g\}$.

For $k = 2, 3$ and $j = 0, 1$ we define

$$G^{k,k-j}(\mathcal{S}) := \left\{ 1_2 + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{S}) : c \in \mathfrak{m}_s^k, a, b, d \in \mathfrak{m}_s^{k-j} \right\}.$$

We set $\delta_{q=3} = 1$ if $q = 3$ and $\delta_{q=3} = 0$ if $q \neq 3$. One can easily check the following facts, where starting from (2) we let $j = 0$ in case (D) and $j = 1$ in case (E):

(1) the sets $G^{k,k-j}(\mathcal{S})$ defined above are subgroups of $\mathrm{SL}_2(\mathcal{S})$, and moreover $G^{3,3-j}(\mathcal{S})$ is a normal subgroup of $G^{2,2-j}(\mathcal{S})$ for $j \in \{0, 1\}$;

(2) the matrices M_1^q and $[M_{2i-1}, M_{2i}]$, for $i = 1, \ldots, g$, lie in $G^{2,2-j}(\mathcal{S})$;

(3) in case (D), computing modulo $G^{3,3}(\mathcal{S})$, for $i = 1, \ldots, g$ one has $M_1 \equiv 1_2 + \mathrm{Mat}\, -qd_1 qb_1 qc_1 qd_1$ and

$$[M_{2i-1}, M_{2i}] \equiv 1_2 + \begin{pmatrix} b_{2i-1}c_{2i} - b_{2i}c_{2i-1} & 2b_{2i-1}d_{2i} - 2b_{2i}d_{2i-1} \\ -2c_{2i-1}d_{2i} + 2c_{2i}d_{2i-1} & -b_{2i-1}c_{2i} + b_{2i}c_{2i-1} \end{pmatrix};$$

(4) in case (E), computing modulo $G^{3,2}(\mathcal{S})$, for $i = 1, \ldots, g$ one has

$$M_1^q \equiv \begin{cases} 1_2 + \begin{pmatrix} 0 & 1 \\ c_1 & 0 \end{pmatrix} \cdot (q + \delta_{q=3} c_1), & \text{if } i_0 = 1, \\ 1_2 + \begin{pmatrix} 0 & 0 \\ qc_1 & 0 \end{pmatrix}, & \text{if } i_0 > 1, \end{cases}$$

$$[M_{2i-1}, M_{2i}] \equiv$$

$$1_2 + \begin{pmatrix} u_{2i-1}c_{2i} - u_{2i}c_{2i-1} & u_{2i-1}^2 c_{2i} - u_{2i}^2 c_{2i-1} + 2u_{2i-1}d_{2i} - 2u_{2i}d_{2i-1} \\ u_{2i}c_{2i-1}^2 - 2(u_{2i-1} - u_{2i})c_{2i-1}c_{2i} - u_{2i-1}c_{2i}^2 - 2c_{2i-1}d_{2i} + 2c_{2i}d_{2i-1} & -u_{2i-1}c_{2i} + u_{2i}c_{2i-1} \end{pmatrix};$$

(5) for $M, M' \in G^{2,2-j}(\mathcal{S})$ one has $MM' \equiv M + M' - 1_2 \pmod{G^{3,3-j}(\mathcal{S})}$.

Using these facts we can explicitly compute the initial terms of the relations r_i since for $j = 0, 1$:

$$1_2 + \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} = M_1^q \cdot \prod_{i=1}^{g} [M_{2i-1}, M_{2i}] \equiv M_1^q + \sum_{i=1}^{g} \big([M_{2i-1}, M_{2i}] - 1_2\big) \bmod G^{3,3-j}(\mathcal{S}).$$

In case (D) we have $r_1, r_2, r_3 \in \mathfrak{m}_s^2$ from (3) and (5), and in case (E) we deduce $r_1, r_2 \in \mathfrak{m}_s$ and $r_3 \in \mathfrak{m}_s^2$ from (4) and (5). Below we make the initial terms of the r_k more explicit. To then analyze properties of $R_{\bar{\alpha}}$, we shall need the following results from commutative algebra, which are simple exercises:

(α) if R is a ring and $a_1, a_2, a_3 \in R$, then using total degrees $w = xy - a_1 x - a_2 y + a_3$ is a non-zero divisor in the polynomial ring $R[x, y]$ over R; if moreover R is an integral domain and $a_3 \neq a_1 a_2$, then $R[x, y]/(w)$ is an integral domain, as can be seen by performing a linear coordinate change with x and y, and then passing to $\mathrm{Frac}(R)[x, y]/(w)$.

(β) if R is an $\mathbb{N}$-graded Noetherian ring and if $f_1, \ldots, f_\omega \in R$ are homogeneous of positive degree, then they form a regular sequence if they do so in any order (see [Mat89, Remark after Thm. 16.3]);

(γ) if R and $f_1, \ldots, f_\omega$ are as in (β), if the f_i form a regular sequence and if $R[\frac{1}{f_{\omega'+1} \cdot \ldots \cdot f_\omega}]/(f_1, \ldots, f_{\omega'})$ is an integral domain for any $1 \leq \omega' \leq \omega$, then $R/(f_1, \ldots, f_{\omega'})$ is an integral domain, as well.

We first show assertions (a)–(e) in case (D). Here we take $\mathcal{R} = \mathcal{S}$. Because $\mathfrak{m}_s^2$ contains (r_1, r_2, r_3), the presentation $0 \to (r_1, r_2, r_3) \to \mathcal{R} \to R_{\bar{\alpha}}^{\psi} \to 0$ is minimal. We shall consider the canonical reduction map $\pi\colon \mathcal{R} \to \mathcal{R}' = \mathcal{R}/(b_i, c_i, d_i, i = 5, \ldots, 2g)$, and we let $\mathfrak{m}_s' = \pi(\mathfrak{m}_s)$ and $r_k' = \pi(r_k)$ for $k = 1, 2, 3$. The ring $\mathcal{R}'$ is a power series ring over $W(\mathbb{F})$ in 12 variables. Thus $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}'}} \overline{\mathcal{R}}'$ and $\overline{\mathrm{gr}_{\mathfrak{m}_s'} \mathcal{R}'}/(\bar{t}_0)$, for $t_0 := \mathrm{in}(q)$, are polynomials rings over $\mathbb{F}$ in 12 variables. The elements $\mathrm{in}^2(\bar{r}_k') \equiv \overline{\mathrm{in}^2(r_k')} \pmod{\bar{t}_0}$ are homogeneous elements of degree 2 for $k = 1, 2, 3$, which by (3) are given by the expressions

$$\bar{b}_1\bar{c}_2 - \bar{b}_2\bar{c}_1 + \bar{b}_3\bar{c}_4 - \bar{b}_4\bar{c}_3, \quad \bar{b}_1\bar{d}_2 - \bar{b}_2\bar{d}_1 + \bar{b}_3\bar{d}_4 - \bar{b}_4\bar{d}_3 \quad \text{and} \quad \bar{c}_1\bar{d}_2 - \bar{c}_2\bar{d}_1 + \bar{c}_3\bar{d}_4 - \bar{c}_4\bar{d}_3.$$

Using (α) and (β), one easily deduces that $\bar{c}_1, \bar{d}_2, \bar{b}_3, \bar{b}_4$ together with the three displayed relations above form a regular sequence in any order in $R = \mathbb{F}[\bar{b}_k, \bar{c}_k, \bar{d}_k : k = 1, \ldots, 4]$. To complete the argument, we wish to apply (γ). If we invert $\bar{b}_4$ in R, then forming the quotient of R by the first two relations is equivalent to eliminating $\bar{c}_3, \bar{d}_3$ in R. This will change $\overline{\mathrm{in}^2(r_3')} \pmod{\bar{t}_0}$ to

$$w := c_1 d_2 - \frac{b_2}{b_4} d_4 c_1 - \frac{b_1}{b_4} c_4 d_2 - c_2 d_1 + \frac{b_2}{b4} c_4 d_1 + \frac{b_1}{b_4} c_2 d_4 \in R' := \mathbb{F}[b_1, b_2, b_3, b_4, c_2, c_4, d_1, d_4, \frac{1}{b_4}][c_1, d_2].$$

Since $(b_2 d_4)(b_1 c_4) \neq b_4^2(-c_2 d_1 + \frac{b_2}{b4} c_4 d_1 + \frac{b_1}{b_4} c_2 d_4)$ in the polynomial ring $\mathbb{F}[b_1, b_2, b_3, b_4, c_2, c_4, d_1, d_4]$, the ring $R'/(w)$ is an integral domain by (α). Therefore by (γ) the ring $\overline{\mathrm{gr}_{\mathfrak{m}_s'} \mathcal{R}'}/(\bar{t}_0, \overline{\mathrm{in}(r_1')}, \overline{\mathrm{in}(r_2')}, \overline{\mathrm{in}(r_3')})$ is an integral domain, as well. This implies that $\bar{t}_0, \overline{\mathrm{in}(r_1)}, \overline{\mathrm{in}(r_2)}, \overline{\mathrm{in}(r_3)}, \bar{b}_5, \bar{c}_5, \bar{d}_5, \ldots, \bar{b}_{2g}, \bar{c}_{2g}, \bar{d}_{2g}$ is a regular sequence in $\overline{\mathrm{gr}_{\mathfrak{m}_s} \mathcal{R}}$ and that the corresponding quotient ring is an integral domain. Invoking Lemma 4.2.1(b) for the domain property, this completes the proof of (a) and (b) in case (D). The proof of (c) and (d) is analogous since the elements $\mathrm{in}(\bar{r}'_k)$ and $\overline{\mathrm{in}(r'_k)} \pmod{\bar{t}_0}$ are formally given by the same expressions for $k = 1, 2, 3$. Part(e) follows from Lemma 4.3.2.

We now turn to case (E). Recall that here we have $u_{i_0} = 1$ by definition of E_{Π}. Let $i_1 \neq i_0$ denote the index in $\{1, 2, 3, 4\}$ such that $\{i_0, i_1\}$ is either $\{1, 2\}$ or $\{3, 4\}$. Using (4) above, one finds that the coefficients of c_{i_1} in r_1 and of d_{i_1} in r_2 are in $\{\pm 1, \pm 2\} \subset W(\mathbb{F})^*$. In particular $\mathrm{in}^1(\bar{r}_k)$, $k = 1, 2$, and $\overline{\mathrm{in}^1(r'_k)} \pmod{\bar{t}_0}$, $k = 1, 2$, are $\mathbb{F}$-linearly independent elements in $\overline{\mathfrak{m}}_{\mathcal{R}'}/\overline{\mathfrak{m}}^2_{\mathcal{R}'}$ and $\overline{\mathfrak{m}'_s/(\mathfrak{m}'_s)^2} \pmod{\bar{t}_0}$, respectively. We define $\mathcal{R} = \mathcal{S}/(r_1, r_2, , b_{i_0}, d_{i_0} - c_{i_0})$. Using r_1 and r_2 as replacement rules to eliminate the variables c_{i_1} and d_{i_1}, we find that the homomorphism

$$W(\mathbb{F})[[c_{i_0}, b_{i_1}, b_k, c_k, d_k : k \in \{1, \ldots, 2g\} \smallsetminus \{i_0, i_1\}]] \to \mathcal{R},$$

which sends each formal variable to the same named variable in $\mathcal{R}$, is an isomorphism. By $\widetilde{r}_3$ we denote the image of r_3 in $\mathcal{R}$. It is clear from (2) that $\widetilde{r}_3$ lies in $\mathfrak{m}_s^2$, where now $\mathfrak{m}_s$ is the image of $(q, b_i, c_i, d_i, i = 1, \ldots, 2g)$ in $\mathcal{R}$. In particular, $0 \to (\widetilde{r}_3) \to \mathcal{R} \to R^{\psi}_{\alpha} \to 0$ is a minimal presentation.

As in the analysis of (D), we consider the reduction map $\pi\colon \mathcal{R} \to \mathcal{R}' = \mathcal{R}/(b_i, c_i, d_i, i = 5, \ldots, 2g)$, we define $\mathfrak{m}'_s = \pi(\mathfrak{m}_s)$ and $r'_3 = \pi(\widetilde{r}_3)$. The ring $\mathcal{R}'$ is now a power series ring over $W(\mathbb{F})$ in 8 variables. A short computation shows

$$w := \mathrm{in}^2(\bar{r}'_3) \equiv \overline{\mathrm{in}^2(r'_3)} \pmod{\bar{t}_0} \equiv \begin{cases} 2\bar{d}_3\bar{c}_4 - 2\bar{d}_4\bar{c}_3 + \text{ other terms}, & \text{if } i_0 \in \{1, 2\}, \\ 2\bar{d}_1\bar{c}_2 - 2\bar{d}_2\bar{c}_1, & \text{if } i_0 \in \{3, 4\}. \end{cases}$$

From $w \neq 0$ we deduce (a) and (c). The proof of (e) follows from Lemma 4.3.2. Arguing as for (D), to prove (b) and (d) it suffices to show that w is a non-zero divisor in

$$\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}'}} \overline{\mathcal{R}}' \cong \overline{\mathrm{gr}_{\mathfrak{m}'_s} \mathcal{R}'}/(\bar{t}_0) \cong \mathbb{F}[[c_{i_0}, b_{i_1}, b_k, c_k, d_k : k \in \{1, 2, 3, 4\} \smallsetminus \{i_0, i_1\}]].$$

We need to show that w is irreducible, i.e., not a product of two linear terms. For this one may consider w as a bilinear from. If w was reducible, the representing Gram matrix would have rank at most 2. However, the displayed coefficients of w imply that this rank is at least 4. □

Remark 4.3.8. (a) In Section 4.2, we showed Theorem 4.1.5 by combining Theorem 4.1.4 with Proposition 4.2.2. Alternatively, in cases (D) and (E) Theorem 4.1.5 follows easily from Lemma 4.3.7(c),(d) combined with Proposition 4.2.2(a),(b).

(b) In cases (D) and (E), Theorem 4.1.4 can also be deduced from [Böc00, §8]. However, we felt that the arguments there are somewhat sketchy. To make them more precise, we would have needed to introduce much notation. Since the above proof follows nicely from the ideas of Section 4.2, we chose this path.

4.4 Crystalline points in components of versal deformation spaces

Let $\mathfrak{X}(\bar{\rho})$ be the versal deformation space of a fixed residual representation $\bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$. The Zariski density of benign crystalline points in $\mathfrak{X}(\bar{\rho})$ for $n = 2$ is an important consequence of the integrality results of the previous sections. The purpose of this section is to prove Theorem 4.1.9 on irreducible components of $\mathfrak{X}(\bar{\rho})$, and Theorem 4.1.12 on the Zariski density of crystalline points by showing that any component of $\mathfrak{X}(\bar{\rho})$ contains a crystalline point.

We fix a character $\psi\colon G_K \to \mathcal{O}^*$ that reduces to $\det \bar{\rho}$. As is well-known, e.g. [Böc08, Prop. 2.1] for results of this type, one has the following result:

Lemma 4.4.1. *Suppose p does not divide n and $\psi'\colon G_K \to \mathcal{O}^*$ is a second lift of $\det \bar{\rho}$. Then*

(a) *$D_{\bar\rho} \to D^{\psi}_{\bar\rho} \times D_{\det\bar\rho}, [\rho] \mapsto ([\rho \otimes (\psi \det \rho^{-1})^{1/n}], \det \rho)$ is an isomorphism of functors with inverse $([\rho'], \varphi') \mapsto [\rho' \otimes (\varphi'\psi^{-1})^{1/n}]$. In particular one has a natural isomorphism $R_{\bar\rho} \cong R^{\psi}_{\bar\rho} \hat\otimes_{\mathcal{O}} R_{\det\bar\rho}$.*

(b) *$D^{\psi}_{\bar\rho} \to D^{\psi'}_{\bar\rho}, [\rho] \mapsto [\rho \otimes \sqrt{\psi^{-1}\psi'}]$ is an isomorphism of functors so that $R^{\psi}_{\bar\rho}$ and $R^{\psi'}_{\bar\rho}$ are isomorphic.*

Lemma 4.4.1 shows that it suffices to prove Theorem 4.1.5 for any fixed choice of lift ψ, for instance for the Teichmüller lift of $\det \rho$. Furthermore, together with Theorem 4.1.5, it implies Theorem 4.1.9:

Proof of Theorem 4.1.9. By Theorem 4.1.5 and part (a) of the previous lemma, the map $D\colon \mathfrak{X}(\bar\rho) \to \mathfrak{X}(\det\bar\rho)$ of Theorem 4.1.9 induces a bijection of irreducible components. Moreover the irreducible components of both spaces will be connected components if this holds for $\mathfrak{X}(\det\bar\rho)$. To prove this and the remaining assertion of Theorem 4.1.9, it will suffice to describe $R_{\det\bar\rho}$ explicitly. This however has been carried out in [Maz89, § 1.4]: Denote by Π the abelianized pro-p completion of G_K, which by class field theory is isomorphic to $(\mathbb{Z}_p, +) \times (1 + \mathfrak{m}_K, \cdot)$. Then $R_{\bar\eta} \cong \mathcal{O}[\![\Pi]\!] \cong \mathcal{O}[\![T_0, \ldots, T_{[K:\mathbb{Q}_p]}]\!][X]/((1+X)^q - 1)$ for any character $\bar\eta\colon G_K \to \mathbb{F}^*$, where $q = \#\mu_{p^\infty}(K)$. The remaining assertions are now immediate. □

Proof of Theorem 4.1.12. By [Mul13, Thm. 0.0.4], we may choose a crystalline p-adic Galois representation $\rho_0\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ which is a lift of $\bar\rho$, i.e., so that $[\rho_0] \in \mathfrak{X}(\bar\rho)$. By the construction in [Mul13], we can assume ρ_0 to be regular. We want to show that any component of $\mathfrak{X}(\bar\rho)$ contains a regular crystalline point so that the hypothesis of Theorem 4.1.11 holds. Denote by ψ the determinant of ρ_0, so that ψ is crystalline, and by $\mathfrak{X}(\bar\rho)^{\psi}$ the rigid analytic space that is the generic fiber of $R^{\psi}_{\bar\rho}$ in the sense of Berthelot. By Lemma 4.4.1, we have the isomorphism

$$\mathfrak{X}(\bar\rho)^{\psi} \times \mathfrak{X}(\det\bar\rho) \xrightarrow{\cong} \mathfrak{X}(\bar\rho),\ ([\rho'], \varphi') \longmapsto [\rho' \otimes (\varphi'\psi^{-1})^{1/2}].$$

By the following lemma, we have a crystalline point φ'_i in any component i of $\mathfrak{X}(\det\bar\rho)$. Now the components form a torsor over $\mu_{p^\infty}(K)$, which is a finite cyclic group of p-power order. Because 2 is prime to p, the characters $(\varphi'_i)^2$ still exhaust all components of $\mathfrak{X}(\det\bar\rho)$, and the same holds for the translates $\psi(\varphi'_i)^2$. Now under the above map we have $([\rho_0], \psi(\varphi'_i)^2) \mapsto [\rho_0 \otimes \varphi'_i]$, and by Theorem 4.1.9 we see that the latter representations give a regular crystalline lift in any component of $\mathfrak{X}(\bar\rho)$. Applying Theorem 4.1.11 completes the proof of Theorem 4.1.12. □

Lemma 4.4.2. *Any component of $\mathfrak{X}(\det\bar\rho)$ contains a crystalline point.*

Proof. By twisting by ψ^{-1} it will suffice to prove the lemma for the trivial character 1 in place of $\det\bar\rho$. The crystalline points in $\mathfrak{X}(1)$ correspond to characters $G_K \to \overline{\mathbb{Q}_p}^*$ with trivial reduction 1. We shall use the classification of one-dimensional crystalline representations to describe the crystalline points. Let $\mathrm{rec}_K\colon \hat{\mathbb{Z}} \times \mathcal{O}_K^* \xrightarrow{\sim} G_K^{\mathrm{ab}}$ be the local Artin map. Consider the induced projection $\mathrm{pr}_2\colon G_K^{\mathrm{ab}} \to \mathcal{O}_K^*$, and let $\mathcal{P}_K$ be the set of embeddings $K \hookrightarrow \overline{\mathbb{Q}_p}$. Then for any $\tau_0 \in \mathcal{P}_K$ one defines a character χ_{τ_0} as the composite

$$\chi_{\tau_0} : G_K \longrightarrow G_K^{\mathrm{ab}} \xrightarrow{\mathrm{pr}_2} \mathcal{O}_K^* \xrightarrow{\tau_0} \overline{\mathbb{Q}_p}^*.$$

One has the following assertions, cf. [Con11, App. B]:

(a) The character χ_{τ_0} is crystalline with labeled Hodge-Tate weights $(a_\tau)_{\tau\in\mathcal{P}_K}$ where $a_{\tau_0}=1$ and $a_\tau=0$ for $\tau\in\mathcal{P}_K\smallsetminus\{\tau_0\}$.[8]

(b) Any crystalline character of G_K is of the form $\nu\prod_{\tau\in\mathcal{P}_K}\chi_\tau^{\ell_\tau}$ for integers ℓ_τ and an unramified character ν. The tuple $(\ell_\tau)_{\tau\in\mathcal{P}_K}$ is its labeled Hodge-Tate weight.

As discussed in the proof of Theorem 4.1.9, $R_1\cong\mathcal{O}[\![\Pi]\!]\cong\mathcal{O}[\![T_0,\dots,T_{[K:\mathbb{Q}_p]}]\!][X]/((1+X)^q-1)$ so that $\mathfrak{X}(1)$ has $q=\#\mu_{p^\infty}(K)$ connected components. In order to find a crystalline point in any component of $\mathfrak{X}(1)$, we introduce a labeling of its connected components by $\mu_{p^\infty}(K)$: Any point in $\mathfrak{X}(1)$ corresponds to a character $G_K\to\overline{\mathbb{Q}_p}^*$ with trivial mod p reduction, which factors via the abelianized pro-p completion Π of G_K, i.e., it induces a character $\eta:\Pi\to\overline{\mathbb{Q}_p}^*$. Via the isomorphism $\mathrm{rec}_{K,p}\colon\mathbb{Z}_p\times(1+\mathfrak{m}_{\mathcal{O}_K})\xrightarrow{\sim}\Pi$ induced from rec_K by pro-p completion, the torsion subgroup $\mu_{p^\infty}(K)$ of $(1+\mathfrak{m}_{\mathcal{O}_K})$ is isomorphic to the torsion subgroup of Π so that we can define the label of η to be $\eta\circ\mathrm{rec}_{K,p}|_{\mu_{p^\infty}(K)}(\zeta)\in\mu_{p^\infty}(K)$ for a chosen generator ζ of $\mu_{p^\infty}(K)$. Equivalently, one can say that the component of $\mathfrak{X}(1)$ that contains η is determined by the restriction $\eta\circ\mathrm{rec}_{K,p}|_{\mu_{p^\infty}(K)}$.

Now we use the above labeling of components to find a crystalline character in each component. Recall that $f=[K_0:\mathbb{Q}_p]$, and denote by $\tau_0\in\mathcal{P}_K$ our usually chosen embedding $K\hookrightarrow\overline{\mathbb{Q}_p}$. By (b) above, for any $\ell\in\mathbb{Z}$ the character $\chi_{\tau_0}^{\ell(q^f-1)}\colon G_K\to\overline{\mathbb{Q}_p}^*$ is crystalline. Because of the factor q^f-1 in the exponent, its image is a pro-p group, and it is straightforward to see that for the induced character $\eta\colon\Pi\to\overline{\mathbb{Q}_p}$ we have $\eta\circ\mathrm{rec}_{K,p}|_{1+\mathfrak{m}_{\mathcal{O}_K}}=\tau_0^{\ell(q^f-1)}|_{1+\mathfrak{m}_{\mathcal{O}_K}}$. Hence $\eta\circ\mathrm{rec}_{K,p}|_{\mu_{p^\infty}(K)}$ is equal to the homomorphism

$$\mu_{p^\infty}(K)\longrightarrow\mu_{p^\infty}(K),\quad \alpha\longmapsto\alpha^{\ell(q^f-1)}=\alpha^{-\ell}.$$

By choosing ℓ suitably, it is clear that η can be made to lie in any connected component of $\mathfrak{X}(1)$. □

For the following result, we assume that the reader is familiar with the theory of determinants as introduced in [Che14]. Following [WE13] we shall call them pseudo-representations. Let R be in $\mathcal{A}r_{\mathcal{O}}$. To any representation $\rho\colon G_K\longrightarrow\mathrm{GL}_n(R)$ one can attach a pseudo-representation of degree n, i.e., a multiplicative R-polynomial law $\tau=\tau_\rho\colon R[G_K]\to R$ homogeneous of degree n. To describe the latter, denote for any R-module M by $\underline{M}$ the functor from R-algebras A to sets that assigns to A the set $M\otimes_R A$. Then τ is the natural transformation $\underline{R[G_K]}\to\underline{R}$ that on any R-algebra A is given by $\tau_A\colon A[G_K]\to A$, $\sum r_ig_i\mapsto\det\big(\sum r_i\rho(g_i)\big)$. In particular, any residual representation $\bar\rho\colon G_K\longrightarrow\mathrm{GL}_n(\mathbb{F})$ has an associated pseudo-representation $\bar\tau$. By [Che14], if τ arises from a representation ρ over R, then the characteristic polynomial $\chi_\rho(g)$ of ρ is equal to $\chi_\tau(g,T):=\tau_{R[T]}(T-g)\in R[T]$ for any $g\in G_K$. The determinant of τ is defined as the representation $\det\tau:=\tau_R=(-1)^n\chi_\tau(\_,0)\colon G_K\to\mathrm{GL}_1(R)$.

In [Che14, § 3.1], Chenevier defines a deformation functor D_τ for a residual pseudo-representations $\bar\tau\colon\mathbb{F}[G_K]\to\mathbb{F}$. By [Che14, Prop. 3.3 and Ex. 3.7], the functor $D_{\bar\tau}$ is representable by a ring $R_{\bar\tau}$ in $\mathcal{A}r_{\mathcal{O}}$. By $\mathfrak{X}^{\mathrm{ps}}(\bar\tau)$ we denote the generic fiber of $\mathrm{Spf}\,R_{\bar\tau}$ in the sense of Berthelot, see [dJ95, § 7]. If $\bar\tau$ is associated to $\bar\rho$ then there are natural functors

$$\mathfrak{X}(\bar\rho)\xrightarrow{\pi_1}\mathfrak{X}^{\mathrm{ps}}(\bar\tau)\xrightarrow{\pi_2}\mathfrak{X}(\det\bar\rho),\tag{2}$$

where π_1 is defined by mapping a deformation to the associated pseudo-representation, and π_2 by mapping a pseudo-representation to its determinant. Note that the composite is defined by the usual determinant of representations.

[8] For the definition of labeled Hodge-Tate weights, see [DS15, Def. 3.2].

Corollary 4.4.3. *Suppose $\bar{\rho}$ is a semisimple 2-dimensional residual representation of G_K and $p > 2$.*

(a) The morphisms of connected components

$$\pi_0(\mathfrak{X}(\bar{\rho})) \xrightarrow{\pi_0(\pi_1)} \pi_0(\mathfrak{X}^{\mathrm{ps}}(\bar{\tau})) \xrightarrow{\pi_0(\pi_2)} \pi_0(\mathfrak{X}(\det \bar{\rho})),$$

induced from (2) are bijective.

(b) The benign crystalline points are Zariski dense in $\mathfrak{X}^{\mathrm{ps}}(\bar{\rho})$.

Proof. To prove (a) observe that by Theorem 4.1.9, the composite $\pi_0(\pi_2) \circ \pi_0(\pi_1)$ is a bijection. Moreover the map $\pi_0(\pi_1)$ is surjective: For this it suffices to show that any pseudo-representation τ over $\overline{\mathbb{Q}_p}$, i.e. any closed point in $\mathfrak{X}^{\mathrm{ps}}(\bar{\tau})$, arises from a representation ρ, i.e. a closed point in $\mathfrak{X}(\bar{\rho})$. By [Che14, Thm. 2.12], it is known that τ is the pseudo-representation for a semisimple representation $G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$. The latter can be realized over a finite extension E of $\mathbb{Q}_p$ and then, in turn by a representation $\rho' \colon G_K \to \mathrm{GL}_2(\mathcal{O}_E)$ for $\mathcal{O}_E$ the valuation ring of E. Moreover, by possibly enlarging E and choosing a suitable lattice, one can also assume that the reduction $\bar{\rho}'$ of ρ' modulo $\mathfrak{m}_{\mathcal{O}_E}$ is semisimple. Now on the one hand, we have $\chi_{\bar{\tau}} = \chi_{\bar{\rho}}$. On the other hand $\pi_1(\rho') = \tau$ yields $\chi_\tau = \chi_{\rho'}$, and reducing mod $\mathfrak{m}_{\mathcal{O}_E}$ we deduce $\chi_{\bar{\rho}'} = \chi_{\bar{\tau}} = \chi_{\bar{\rho}}$. By the semisimplicity of $\bar{\rho}$ and $\bar{\rho}'$, the theorem of Brauer-Nesbitt now implies $\bar{\rho} \cong \bar{\rho}'$. But then ρ' represents an element of $\mathfrak{X}(\bar{\rho})$ that maps to τ, completing the proof of (a).

To prove (b), observe that, by what we just proved, the map π_1 is surjective on (closed) points. Moreover for rigid spaces all Zariski closed subsets are the Zariski closures of their closed points. But then the image under π_1 of a Zariski dense subset is Zariski dense. It follows from Theorem 4.1.12 that the set of benign crystalline points in $\mathfrak{X}^{\mathrm{ps}}(\bar{\tau})$, which is the image of the set of benign crystalline points in $\mathfrak{X}(\bar{\rho})$, is Zariski dense in $\mathfrak{X}^{\mathrm{ps}}(\bar{\tau})$. □

4.5 The cup product and quadratic obstructions

In the remainder of the article, we consider a residual representation $\bar{\rho} \colon G_K \longrightarrow \mathrm{GL}_n(\mathbb{F})$ for $n \in \mathbb{N}$ arbitrary. Let $0 \to I^\psi \to \mathcal{R} \xrightarrow{\pi} R^\psi_{\bar{\rho}} \to 0$ be a minimal presentation of $R^\psi_{\bar{\rho}}$ as in (1) of Proposition 4.1.3. In this section, we show that the bracket cup product $b \colon \mathrm{Sym}^2(H^1(G_K, \mathrm{ad}^0)) \to H^2(G_K, \mathrm{ad}^0)$ determines the quadratic part of the relation ideal $\bar{I}^\psi$ in the sense of Definition 4.2.4.

As recalled in Proposition 4.1.2 and 4.1.3, Mazur attaches to any small extension $0 \to J \to R_1 \to R_0 \to 0$ in $\widehat{\mathcal{A}r}_{\mathcal{O}}$ and deformation $\rho_0 \colon G_K \to \mathrm{GL}_n(R_0)$ with determinant ψ an obstruction class $\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}^0) \otimes J$ for lifting ρ_0 to a deformation to R_1. First one chooses a continuous set-theoretic lift $\rho_1 \colon G_K \to \mathrm{GL}_n(R_1)$ of ρ_0 which still satisfies $\det \circ \rho_1 = \psi$.[9] Then $\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}^0 \otimes J)$ is given by the 2-cocycle

$$(g,h) \longmapsto \rho_1(gh)\rho_1(h)^{-1}\rho_1(g)^{-1} - 1. \tag{3}$$

Similarly, $\mathcal{O}(\rho_0)$ can be described by the obstruction homomorphism $\mathrm{obs} \colon \mathrm{Hom}_{\mathbb{F}}(J, \mathbb{F}) \to H^2(G_K, \mathrm{ad}^0)$. The latter is defined as follows: For any $f \in \mathrm{Hom}_{\mathbb{F}}(J, \mathbb{F})$, form the pushout on the left of the given small extension and denote the result by $0 \to \mathbb{F} \to R_f \to R_0 \to 0$. If

[9] Such a map always exists: For instance choose a continuous set-theoretic splitting $R_0 \to R_1$ of the given homomorphism $R_1 \to R_0$. Observe that since the R_i are local, it induces a continuous set-theoretic splitting of $\mathrm{GL}_n(R_1) \to \mathrm{GL}_n(R_0)$. Finally, fix the determinant similar to Lemma 4.4.1.

$\rho_f\colon G_K \to \mathrm{GL}_n(R_f)$ is a continuous set-theoretic lift of ρ_0 satisfying $\det\circ\rho_f = \psi$, then we set $\mathrm{obs}(f) := \big(\mathcal{O}(\rho_0), f\big) := (\mathrm{id}\otimes f)\big(\mathcal{O}(\rho_0)\big) \in H^2(G_K, \mathrm{ad}^0)$, i.e., $\mathrm{obs}(f)$ is given by the 2-cocycle $(g,h) \longmapsto \rho_f(gh)\rho_f(h)^{-1}\rho_f(g)^{-1} - 1$.

The following lemma shows that the obstruction class is independent of a chosen small extension. Its simple proof is left as an exercise.

Lemma 4.5.1. *Consider a morphism of small extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & R_1 & \longrightarrow & R_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} & & \\ 0 & \longrightarrow & J' & \longrightarrow & R_1' & \longrightarrow & R_0' & \longrightarrow & 0, \end{array}$$

i.e., a commuting diagram with both rows a small extension and the right hand square in $\widehat{\mathcal{A}\mathrm{r}}_{\mathcal{O}}$. Let $\mathcal{O}(\rho_0) \in H^2(G_K, \mathrm{ad}^0 \otimes J)$ be the obstruction of a deformation $\rho_0 : G_K \to \mathrm{GL}(R_0)$ of $\bar\rho$. Then

$$(\mathrm{id}\otimes\pi)\big(\mathcal{O}(\rho_0)\big) = \mathcal{O}(\pi_0\circ\rho_0) \in H^2(G_K, \mathrm{ad}^0\otimes J') \cong H^2(G_K, \mathrm{ad}^0)\otimes J'.$$

Recall that $^{-}$ means that we pass to rings mod $\mathfrak{m}_{\mathcal{O}}$, and minimality of the presentation of $R^{\psi}_{\bar\rho}$ implies that π induces an isomorphism $\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 \cong \overline{\mathfrak{m}}^{\psi}_{\bar\rho}/(\overline{\mathfrak{m}}^{\psi}_{\bar\rho})^2$. In particular, $\bar I^{\psi} \subset \overline{\mathfrak{m}}_{\mathcal{R}}^2$. In this section, we consider the filtration $\{\overline{\mathfrak{m}}_{\mathcal{R}}^i\}_{i\ge 0}$ on $\overline{\mathcal{R}}$, and let in denote the initial term map $\overline{\mathcal{R}} \to \mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}}\overline{\mathcal{R}}$. The following basic result relates the bracket cup product and the quadratic part of I^{ψ}:

Lemma 4.5.2. *We assume $p > 2$. Then the following diagram is commutative:*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{F}}\left(H^2(G_K,\mathrm{ad}^0),\mathbb{F}\right) & \xrightarrow{\ \mathrm{obs}^\vee\ } & \bar I^{\psi}/\overline{\mathfrak{m}}_{\mathcal{R}}\bar I^{\psi} & \longrightarrow & \left(\bar I^{\psi}+\overline{\mathfrak{m}}_{\mathcal{R}}^3\right)/\overline{\mathfrak{m}}_{\mathcal{R}}^3 \\ \downarrow{\scriptstyle -\frac12 b^\vee} & & & & \downarrow \\ \mathrm{Sym}^2\left(\mathrm{Hom}_{\mathbb{F}}\left(H^1(G_K,\mathrm{ad}^0),\mathbb{F}\right)\right) & \xrightarrow{\ \sim\ } & \mathrm{Sym}^2\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2\right) & \xrightarrow{\ \sim\ } & \overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3, \end{array}$$

where $b^\vee$ is induced by the dual of the bracket cup product, and $\mathrm{obs}^\vee$ is dual to the obstruction homomorphism. In particular, the quadratic part $\mathrm{in}^2(\bar I^{\psi})$ of $\bar I^{\psi}$ in $\overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3$ agrees with the image of $b^\vee$.

Proof. Let $\bar J := (\bar I^{\psi} + \overline{\mathfrak{m}}_{\mathcal{R}}^3)/\overline{\mathfrak{m}}_{\mathcal{R}}^3$. We prove that the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Sym}^2\left(H^1(G_K,\mathrm{ad}^0)\right) & \xrightarrow{\ \sim\ } & \mathrm{Sym}^2\left(\mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2,\mathbb{F}\right)\right) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3,\mathbb{F}\right) \\ \downarrow{\scriptstyle -\frac12 b} & & & & \downarrow \\ H^2(G_K,\mathrm{ad}^0) & \xleftarrow[\ \mathrm{obs}\ ]{} & \mathrm{Hom}_{\mathbb{F}}\left(\bar I^{\psi}/\overline{\mathfrak{m}}_{\mathcal{R}}\bar I^{\psi},\mathbb{F}\right) & \longleftarrow & \mathrm{Hom}_{\mathbb{F}}\left(\bar J,\mathbb{F}\right). \end{array}$$

The first isomorphism in the upper row is the canonical isomorphism from Proposition 4.1.2(a). We shall show that the image of any $c_1 \in H^1(G_K, \mathrm{ad}^0)$ in $H^2(G_K, \mathrm{ad}^0)$ is independent of whether we apply $-\frac12 b$ or the clockwise composite morphism that passes via obs. Since both maps are $\mathbb{F}$-linear and elements of the form c_1^2 generate $\mathrm{Sym}^2\left(H^1(G_K,\mathrm{ad}^0)\right)$ as an $\mathbb{F}$-vector space, this will prove commutativity. Before we embark on the lengthy computation of the composite morphism, we observe that the bracket cup product of c_1 with itself is represented by the explicit 2-cocycle $(g,h) \mapsto [c_1(g), \mathrm{Ad}\,\bar\rho(g)c_1(h)]$, see [Was97, §2] – we write $\mathrm{Ad}\,\bar\rho$ for the adjoint action of G_K on ad^0 to have clear notation.

We now compute the clockwise composite morphism that passes via obs. First we extend c_1 to a basis $\{c_1, \dots, c_h\}$ of $H^1(G_K, \mathrm{ad}^0)$. Via the isomorphisms $H^1(G_K, \mathrm{ad}^0) \cong \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2, \mathbb{F}\right)$, we obtain a basis of $\mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2, \mathbb{F}\right)$, which by slight abuse of notation, we also denote $\{c_1, \dots, c_h\}$. For the corresponding dual basis of $\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2$ we write $\{\bar{x}_1, \dots, \bar{x}_h\}$ so that $c_i(\bar{x}_j)$ is the Kronecker symbol δ_{ij}. We lift the latter elements to a system of parameters $\{x_1, \dots, x_h\}$ of $\overline{\mathfrak{m}}_{\mathcal{R}}$; this defines an isomorphism $\overline{\mathcal{R}} \cong \mathbb{F}[\![x_1, \dots, x_h]\!]$. With this notation, the image of c_1^2 in $\mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3, \mathbb{F}\right)$ is characterized by $c_1^2(\bar{x}_i\bar{x}_j) = 0$ if one of i, j is at least 2 and $c_1^2(\bar{x}_1^2) = 1$. The image of c_1^2 in $\mathrm{Hom}_{\mathbb{F}}\left(\bar{J}, \mathbb{F}\right)$ is the restriction $c_1^2|_{\bar{J}}$ to the subspace $\bar{J} \subset \overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3$. Finally, the composition of the canonical homomorphism $\bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi \to \bar{J}$ and $c_1^2|_{\bar{J}}$ defines an element f in $\mathrm{Hom}_{\mathbb{F}}\left(\bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi, \mathbb{F}\right)$. To evaluate $\mathrm{obs}(f) = (\mathcal{O}(\rho_{\bar{\rho}}), f)$, we consider the following diagram which displays three morphisms of small extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{I}^\psi/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi & \longrightarrow & \bar{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}\bar{I}^\psi & \longrightarrow & \bar{\mathcal{R}}/\bar{I}^\psi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{J} & \longrightarrow & \bar{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^3 & \longrightarrow & \bar{\mathcal{R}}/(\bar{I}^\psi + \overline{\mathfrak{m}}_{\mathcal{R}}^3) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3 & \longrightarrow & \bar{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^3 & \longrightarrow & \bar{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_1^2} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{F} & \longrightarrow & \bar{\mathcal{R}}/\ker(c_1^2) & \longrightarrow & \bar{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 & \longrightarrow & 0,
\end{array}$$

where the last row is obtained by pushout along c_1^2 and where we denote by $\ker(c_1^2)$ the ideal of $\overline{\mathcal{R}}$ that is the preimage under $\overline{\mathcal{R}} \to \overline{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^3$ of the kernel of $c_1^2\colon \overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3 \to \mathbb{F}$. Note that since $\overline{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 \cong \overline{R}_{\bar{\rho}}^\psi/(\overline{\mathfrak{m}}_{\bar{\rho}}^\psi)^2$, the right column is the morphism defining the deformation $\rho_{\bar{\rho}}^\psi \pmod{(\overline{\mathfrak{m}}_{\bar{\rho}}^\psi)^2}$ to $\overline{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2$.

By Lemma 4.5.1, we can use the last row to compute $\mathrm{obs}(f)$. For this, we need a suitable set-theoretic lift of $\rho_{\bar{\rho}}^\psi \pmod{(\overline{\mathfrak{m}}_{\bar{\rho}}^\psi)^2}$ to $\bar{\mathcal{R}}/\ker(c_1^2)$. We begin with a cohomological description of $\rho_{\bar{\rho}}^\psi \pmod{(\overline{\mathfrak{m}}_{\bar{\rho}}^\psi)^2}$: using vector space duality, the canonical isomorphism $H^1(G_K, \mathrm{ad}^0) \cong \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2, \mathbb{F}\right)$ can be described equivalently by the 1-cocycle $\sum_{i=1}^h c_i \otimes \bar{x}_i$ in $Z^1(G_K, \mathrm{ad}^0 \otimes \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2)$. Therefore, $\rho_{\bar{\rho}}^\psi \pmod{(\overline{\mathfrak{m}}_{\bar{\rho}}^\psi)^2}$ is given by the formula

$$g \longmapsto \Big(1 + \sum_{i=1}^h c_i(g) \otimes \bar{x}_i\Big)\bar{\rho}(g).$$

We want to obtain a formula for a set-theoretic lift to $\bar{\mathcal{R}}/\ker(c_1^2)$. It will be convenient to use the exponential map $\exp_2(x) = 1 + x + \frac{1}{2}x^2$ to level 2, which is well-defined as the rings $(R, \mathfrak{m}_R)$ in $\widehat{\mathcal{A}\mathrm{r}}_{\mathcal{O}}$ have characteristics different from 2. Moreover, $\exp_2$ can be applied to matrices $A \in M_n(\mathfrak{m}_R)$. If in addition $\mathfrak{m}_R^3 = 0$, then one can also verify that $\det(\exp_2(A)) = \exp_2(\mathrm{Tr}(A))$. In particular, $\exp_2(A)$ has determinant equal to 1 if A is of trace zero. Now we take as our set-theoretic lift

$$\rho_0'\colon G_K \longrightarrow \mathrm{GL}_n(\bar{\mathcal{R}}/\ker(c_1^2)), \quad g \longmapsto \exp_2\Big(\sum_{i=1}^h c_i(g) \otimes x_i\Big)\bar{\rho}(g) \pmod{\ker(c_1^2)}.$$

By the remark above on $\exp_2$, we have $\det(\rho_0'(g)) = \det(\bar{\rho}(g)) = \psi(g) \pmod{\mathfrak{m}_{\mathcal{O}}}$ for all $g \in G_K$. In $\mathcal{R}/\ker(c_1^2)$, we have $x_ix_j = 0$ whenever $i > 1$ or $j > 1$. Hence, the expressions

$\exp_2(c_i(g) \otimes x_i)$ commute for all i and we have $\exp_2\left(\sum_{i=1}^h c_i(g) \otimes x_i\right) = \prod_{i=1}^h \exp_2(c_i(g)x_i)$. Using these properties, the class $\mathrm{obs}(f)$ is represented by the 2-cocycle

$$(g,h) \longmapsto \rho_0'(gh)\rho_0'(h)^{-1}\rho_0'(g)^{-1} - 1 = \rho_1'(gh)\rho_1'(h)^{-1}\rho_1'(g)^{-1} - 1,$$

where ρ_1' is the lift $G_K \to \mathrm{GL}_n(\bar{\mathcal{R}}/\ker(c_1^2))$, $g \mapsto \exp_2\left(c_1(g) \otimes x_1\right)\bar\rho(g) \pmod{\ker(c_1^2)}$, of $\bar\rho$. At this point, it is a simple if lengthy computation to verify that the right hand side of the previous expression is the 2-cocycle $(g,h) \mapsto -\frac{1}{2}[c_1(g), \mathrm{Ad}\,\bar\rho(g)c_1(h)] \otimes x_1^2$. Now x_1^2 is our chosen $\mathbb{F}$-basis of the lower left term in the above diagram and via c_1^2 it is mapped to 1. Hence, $\mathrm{obs}(f)$ agrees with the expression for $-\frac{1}{2}b(c_1, c_1)$ given above. □

Remark 4.5.3. The use of the exponential map in the above proof seems standard, e.g. [Gol84, 1.3].

Corollary 4.5.4. *Suppose $\bar\rho$ is of degree 2 and $p > 2$. Then the homomorphism*

$$b\colon \mathrm{Sym}^2 H^1(G_K, \mathrm{ad}^0) \longrightarrow H^2(G_K, \mathrm{ad}^0)$$

induced from the bracket cup product is surjective.

Proof. Consider a minimal presentation $0 \to I^\psi \to \mathcal{R} \to R_{\bar\rho} \to 0$ of $R_{\bar\rho}$. By Lemma 4.5.2, it suffices to show that the images of the quadratic parts of generators of I^ψ span a subspace of dimension equal to $\dim_{\mathbb{F}} H^2(G_K, \mathrm{ad}^0)$. This follows from Corollary 4.3.6(a) in cases (A)–(C), Lemma 4.3.7(c)–(e) in cases (D)–(E) and Lemma 4.3.2 by direct inspection in the respective cases of Section 4.3. □

4.6 Further quadratic obstructions from the Bockstein homomorphism

Let $\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ be a residual representation and $0 \to I^\psi \to \mathcal{R} \xrightarrow{\pi} R^\psi_{\bar\rho} \to 0$ be a fixed minimal presentation as in Proposition 4.1.3. In the previous section we gave a description of the contribution of the bracket cup product $b\colon \mathrm{Sym}^2(H^1(G_K, \mathrm{ad}^0)) \to H^2(G_K, \mathrm{ad}^0)$ to the relation ideal I^ψ. By Lemma 4.5.2, knowing b is equivalent to knowing the quadratic part of $\bar I^\psi$. In Example 4.2.3 we saw that knowing the refined quadratic part may have stronger ring-theoretic implications than knowing the quadratic part only. The theme of this section is the Bockstein homomorphism and its additional contribution to the relation ideal I^ψ. The upshot is a cohomological description of the refined quadratic part of I^ψ in cohomological terms in Lemma 4.6.6 and Theorem 4.6.8.

We suppose that there is a representation $\rho_{s+1}\colon G_K \to \mathrm{GL}_n(\mathcal{O}_{s+1})$ lifting $\bar\rho$ for some integer s.[10] Observe that ρ_{s+1} defines a homomorphism $\alpha_{s+1}\colon R^\psi_{\bar\rho}/(\varpi_{\mathcal{O}}^{s+1}) \to \mathcal{O}_{s+1}$. For the following discussion it will be convenient to choose a regular sequence of parameters of $\mathcal{R}$ that is compatible with α_{s+1} in the following sense: Since the morphism $\alpha_{s+1} \circ (\pi \pmod{\varpi_{\mathcal{O}}^{s+1}})\colon \mathcal{R}/(\varpi_{\mathcal{O}}^{s+1}) \to \mathcal{O}_{s+1}$ is a surjective homomorphism of formally smooth $\mathcal{O}_{s+1}$-algebras, it possesses an $\mathcal{O}_{s+1}$-splitting. Thus we may choose $x_1, \ldots, x_h$ of $\mathcal{R}$ with $h = \dim_{\mathbb{F}} H^1(G_K, \mathrm{ad}^0)$ such that $\mathcal{R}/\varpi_{\mathcal{O}}^{s+1} \cong \mathcal{O}_{s+1}[\![x_1, \ldots, x_h]\!]$ and such that under this identification the homomorphism $\alpha_{s+1} \circ (\pi \pmod{\varpi_{\mathcal{O}}^{s+1}})$ sends all x_i to zero. For $1 \le i \le s$, let $\alpha_i := \alpha_{s+1} \pmod{\varpi_{\mathcal{O}}^i}\colon R^\psi_{\bar\rho}/(\varpi_{\mathcal{O}}^i) \to \mathcal{O}_i$ and $\rho_i := \rho_{s+1} \pmod{\varpi_{\mathcal{O}}^i}\colon G_K \to \mathrm{GL}_n(\mathcal{O}_i)$. Further, for $1 \le i \le s+1$, the adjoint representation $\mathrm{Ad}\,\rho_i : G_K \to \mathrm{ad}_i^0$ of G_K on trace zero matrices $\mathrm{ad}_i^0 := \mathrm{Mat}_n^0(\mathcal{O}_i)$ is given by conjugation with ρ_i so that $\mathrm{ad}_1^0 = \mathrm{ad}^0$. Then for all $2 \le i \le s+1$ there is a short exact sequence of G_K-modules

$$0 \longrightarrow \mathrm{ad}_{i-1}^0 \xrightarrow{\cdot\varpi_{\mathcal{O}}} \mathrm{ad}_i^0 \xrightarrow{\mathrm{pr}_i} \mathrm{ad}^0 \longrightarrow 0.$$

[10] The delicate matter of the correct choice of $q = p^s$ is discussed in Lemma 4.6.9 and Remark 4.6.10.

Definition 4.6.1. For $2 \le i \le s+1$, the *i-th Bockstein operator* or the *p^i-Bockstein homomorphism* is the connecting homomorphism β_i in the induced long exact cohomology sequence

$$\begin{array}{l} \ldots \longrightarrow H^1(G_K, \mathrm{ad}^0_{i-1}) \xrightarrow{\cdot\varpi_{\mathcal{O}}} H^1(G_K, \mathrm{ad}^0_i) \xrightarrow{\mathrm{pr}_i^*} H^1(G_K, \mathrm{ad}^0) \\ \xrightarrow{\beta_i} H^2(G_K, \mathrm{ad}^0_{i-1}) \xrightarrow{\cdot\varpi_{\mathcal{O}}} H^2(G_K, \mathrm{ad}^0_i) \xrightarrow{\mathrm{pr}_i^*} H^2(G_K, \mathrm{ad}^0) \longrightarrow \ldots \end{array} \tag{4}$$

Now we give an explicit description of β_i that will be useful later.

Lemma 4.6.2. *Let $2 \le i \le s+1$, let $c \in Z^1(G_K, \mathrm{ad}^0)$, and let $\widetilde{\ }$ denote a set-theoretic splitting of $\mathrm{ad}^0_i \to \mathrm{ad}^0$. The i-th Bockstein operator is given explicitly by*

$$\beta_i([c]) = \Big((g,h) \longmapsto \varpi_{\mathcal{O}}^{-1} \cdot \big(\mathrm{Ad}\,\rho_i(g)\widetilde{c}(h) - \widetilde{c}(gh) + \widetilde{c}(g)\big)\Big) \pmod{B^2(G_K, \mathrm{ad}^0_{i-1})}. \tag{5}$$

Proof. The connecting homomorphism β_i is defined by applying the snake lemma to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & \bar{C}^1(G_K, \mathrm{ad}^0_{i-1}) & \xrightarrow{\cdot\varpi_{\mathcal{O}}} & \bar{C}^1(G_K, \mathrm{ad}^0_i) & \xrightarrow{\mathrm{pr}_i^*} & \bar{C}^1(G_K, \mathrm{ad}^0) \longrightarrow 0 \\ & & \downarrow \partial_{i-1} & & \downarrow \partial_i & & \downarrow \partial_1 \\ 0 \longrightarrow & & Z^2(G_K, \mathrm{ad}^0_{i-1}) & \xrightarrow[\cdot\varpi_{\mathcal{O}}]{} & Z^2(G_K, \mathrm{ad}^0_i) & \xrightarrow[\mathrm{pr}_i^*]{} & Z^2(G_K, \mathrm{ad}^0), \end{array}$$

where we let $\bar{C}^1(G_K, \mathrm{ad}^0_j) := C^1(G_K, \mathrm{ad}^0_j)/B^1(G_K, \mathrm{ad}^0_j)$ and ∂_j is induced by the coboundary map

$$C^1(G_K, \mathrm{ad}^0_j) \longrightarrow C^2(G_K, \mathrm{ad}^0_j), \quad b \longmapsto \big((g,h) \mapsto \big(\mathrm{Ad}\,\rho_j(g)b(h) - b(gh) + b(g)\big)\big).$$

for any $1 \le j \le s+1$. We lift the given 1-cocycle $c \in Z^1(G_K, \mathrm{ad}^0)$ to the 1-cochain $b_0 := \big(g \mapsto \widetilde{c}(g)\big) : G_K \to \mathrm{ad}^0_i$, and denote the image of c and b_0 in $\bar{C}^1(G_K, \mathrm{ad}^0_i)$ by $\bar{c}$ and $\bar{b}_0$, respectively. Since by assumption $\partial_1(\bar{c})$ vanishes and the right hands side of the diagram is commutative, we conclude that $\partial_i(\bar{b}_0) \in \ker(\mathrm{pr}_i^*)$. Using the exactness of the lower row, we may define $\beta_i([c]) := \varpi_{\mathcal{O}}^{-1} \cdot \partial_i(\bar{b}_0) \pmod{B^2(G_K, \mathrm{ad}^0_{i-1})}$ so that the desired formula (5) follows from the definition of ∂_i. □

The meaning of the Bockstein operator for obstructions is given by the following straightforward result.

Lemma 4.6.3. *Let $i \in \{2, \ldots, s+1\}$ and consider a deformation $\bar{\rho}_c = (1 + c\varepsilon) \cdot \bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$ of $\bar{\rho}$ for some $c \in Z^1(G_K, \mathrm{ad}^0)$. Then ρ_i has a deformation to $\mathcal{O}_i[\varepsilon]$ that lifts $\bar{\rho}_c$ if and only if $\beta_i([c]) = 0$.*

Proof. As in the mod $\varpi_{\mathcal{O}}$ case we can write any deformation to $\mathcal{O}_i[\varepsilon]$ of ρ_i as

$$\rho_{i,c_i} = (1 + c_i\varepsilon) \cdot \rho_i : G_K \to \mathrm{GL}_n(\mathcal{O}_i[\varepsilon])$$

for some $c_i \in Z^1(G_K, \mathrm{ad}^0_i)$. Using the functorial homomorphism $\mathrm{pr}_i^* : C^1(G_K, \mathrm{ad}^0_i) \to C^1(G_K, \mathrm{ad}^0)$, we find that the image of ρ_{i,c_i} under reduction mod $\varpi_{\mathcal{O}}$ is given by

$$(1 + \mathrm{pr}_i^*(c_i)\varepsilon) \cdot \bar{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon]).$$

Hence, such a deformation $\rho_{i,c_i} : G_K \to \mathrm{GL}_n(\mathcal{O}_i[\varepsilon])$ of ρ_i that lifts $\bar{\rho}_c$ exists if and only if $\mathrm{pr}_i^*(c_i) = c$. The long exact sequence of group cohomology (4) implies that the latter holds if and only if $[c]$ lies in the kernel of β_i. □

Corollary 4.6.4. *Let i be in $\{2,\dots,s+1\}$ and consider the presentation*

$$0 \longrightarrow I_i \longrightarrow \mathcal{R}_i := \mathcal{O}_i[x_1,\dots,x_h]/(x_1,\dots,x_h)^2 \xrightarrow{\pi_i} R_i := R^{\psi}_{\bar\rho}/\pi((x_1,\dots,x_h)^2+\varpi^i_{\mathcal{O}}\mathcal{R}) \longrightarrow 0 \tag{6}$$

induced from (1) in Proposition 4.1.3. Then $\beta_i = 0$ if and only if $I_i = 0$, i.e., if and only if π_i is an isomorphism. In particular, if $\beta_s = 0$, then $\beta_j = 0$ for all $j = 2,\dots,s$.

Proof. Suppose that I_i is non-zero and let $f \neq 0$ be an element of I_i. By multiplying f by a suitable power of $\varpi_{\mathcal{O}}$, we may assume that f lies in $\varpi^{i-1}_{\mathcal{O}}\mathcal{R}_i$, i.e., that f is of the form $\varpi^{i-1}_{\mathcal{O}}(\sum_{j=1}^h \lambda_j x_j)$ for suitable $\lambda_j \in \mathcal{O}_i$ such that at least one λ_j lies in $\mathcal{O}_i^*$. Let $\bar\alpha_\varepsilon\colon R_i \to \mathbb{F}[\varepsilon]$ be an $\mathcal{O}$-algebra homomorphism such that $\bar\alpha_\varepsilon\big(\sum_{j=1}^h \lambda_j x_j\big)$ is non-zero. Since $\beta_i = 0$, there exists an $\mathcal{O}$-algebra homomorphism

$$\alpha_{i,\varepsilon}\colon R_i \to \mathcal{O}_i[\varepsilon]$$

such that $\alpha_{i,\varepsilon} \equiv \bar\alpha_\varepsilon \pmod{\varpi_{\mathcal{O}}}\colon R_i \to \mathbb{F}[\varepsilon]$. We deduce

$$0 \overset{\pi_i(I_i)=0}{=} (\alpha_{i,\varepsilon}\circ\pi_i)\Big(\varpi^{i-1}_{\mathcal{O}}\big(\sum_{j=1}^h \lambda_j x_j\big)\Big) \overset{\mathcal{O}\text{-hom.}}{=} \varpi^{i-1}_{\mathcal{O}}(\alpha_{i,\varepsilon}\circ\pi_i)\Big(\sum_{j=1}^h \lambda_j x_j\Big) \in \mathcal{O}_i[\varepsilon],$$

and it follows that $(\alpha_{i,\varepsilon}\circ\pi_i)\big(\sum_{j=1}^h \lambda_j x_j\big)$ lies in $\varpi_{\mathcal{O}}\mathcal{O}_i[\varepsilon]$, or, in other words, that $\bar\alpha_\varepsilon\big(\sum_{j=1}^h \lambda_j x_j\big) = 0$. This is a contradiction. □

Lemma 4.6.5. *Suppose that $\beta_s = 0$, so that also $\beta_2 = \ldots = \beta_{s-1} = 0$. Then the following hold:*

(a) For $i = 2,\dots,s$, the short exact sequence

$$0 \longrightarrow \mathrm{ad}^0 \xrightarrow{\cdot\varpi^{i-1}_{\mathcal{O}}} \mathrm{ad}^0_i \xrightarrow{\gamma_i} \mathrm{ad}^0_{i-1} \longrightarrow 0.$$

yields a short exact sequence $0 \longrightarrow H^2(G_K,\mathrm{ad}^0) \xrightarrow{\cdot\varpi^{i-1}_{\mathcal{O}}} H^2(G_K,\mathrm{ad}^0_i) \xrightarrow{\gamma_i^*} H^2(G_K,\mathrm{ad}^0_{i-1}) \longrightarrow 0$.

(b) The Bockstein homomorphism $\beta_{s+1}\colon H^1(G_K,\mathrm{ad}^0) \to H^2(G_K,\mathrm{ad}^0_s)$ induces a homomorphism

$$\widetilde\beta_{s+1}\colon H^1(G_K,\mathrm{ad}^0) \longrightarrow H^2(G_K,\mathrm{ad}^0)$$

with $\beta_{s+1} = \varpi^{s-1}_{\mathcal{O}}\widetilde\beta_{s+1}$ and the following property: A deformation $\rho_c = (1+c\varepsilon)\bar\rho : G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$ of $\bar\rho$ given by $c \in Z^1(G_K,\mathrm{ad}^0)$ lifts to a deformation of ρ_{s+1} to $\mathcal{O}_{s+1}[\varepsilon]$ if and only if $\widetilde\beta_{s+1}([c]) = 0$.

(c) If $\widetilde c \in Z^1(G_K,\mathrm{ad}^0_{s+1})$ denotes a set-theoretic lift of $c \in Z^1(G_K,\mathrm{ad}^0)$, then one has the explicit formula

$$\tilde\beta_{s+1}([c]) = \Big((g,h) \longmapsto \varpi^{-s}_{\mathcal{O}}\big(\mathrm{Ad}\,\rho_{s+1}(g)\widetilde c(h) - \widetilde c(gh) + \widetilde c(g)\big)\Big) \pmod{B^2(G_K,\mathrm{ad}^0)}.$$

Proof. For (a), recall that one has $\mathrm{scd}\,G_K = 2$ for the strict cohomological dimension of K. Thus from $\beta_s = 0$ and from (4) we obtain the short exact sequence

$$0 \longrightarrow H^2(G_K,\mathrm{ad}^0_{s-1}) \xrightarrow{\cdot\varpi_{\mathcal{O}}} H^2(G_K,\mathrm{ad}^0_s) \xrightarrow{\mathrm{pr}_i^*} H^2(G_K,\mathrm{ad}^0) \longrightarrow 0.$$

The groups $H^2(G_K, \mathrm{ad}_i^0)$ are finite, and we deduce

$$\#H^2(G_K, \mathrm{ad}_s^0) = \#H^2(G_K, \mathrm{ad}_{s-1}^0) \cdot \#H^2(G_K, \mathrm{ad}^0). \tag{7}$$

The sequence in (a) of second cohomology groups is part of a long exact cohomology sequence. Its right exactness thus follows from $\operatorname{scd} G_K = 2$, and then its left exactness is immediate from (7).

For (b) and (c) we consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & H^1(G_K, \mathrm{ad}^0) & & & & \\
 & & \swarrow^{\widetilde{\beta}_{s+1}} & & \downarrow{\scriptstyle \beta_{s+1}} & \searrow^{\beta_s} & & & \\
0 & \longrightarrow & H^2(G_K, \mathrm{ad}^0) & \xrightarrow{\cdot \varpi_{\mathcal{O}}^{s-1}} & H^2(G_K, \mathrm{ad}_s^0) & \xrightarrow{\gamma_s^*} & H^2(G_K, \mathrm{ad}_{s-1}^0) & \longrightarrow & 0
\end{array}$$

with exact second row. Because $\beta_s = 0$, the dashed arrow $\widetilde{\beta}_{s+1}$ exists, and this proves (b). Finally the formula for $\widetilde{\beta}_{s+1}$ in (c) follows from multiplying the formula (5) for β_{s+1} by $\varpi_{\mathcal{O}}^{-(s-1)}$. □

The next result gives the meaning of the Bockstein operator for the relation ideal I^ψ.

Lemma 4.6.6. *For $i = 1, \ldots, s+1$, let $\mathfrak{m}_i$ be the kernel of the composition morphism $\mathcal{R} \xrightarrow{\pi} R_{\bar\rho}^\psi \twoheadrightarrow R_{\bar\rho}^\psi/(\varpi_{\mathcal{O}}^i) \xrightarrow{\alpha_i} \mathcal{O}_i$, i.e., $\mathfrak{m}_i = (\varpi_{\mathcal{O}}^i, x_1, \ldots, x_h)$. Let I_{s+1} be the relation ideal in (6) and denote by $I^\psi \to I_{s+1}$ the canonical homomorphism. Suppose $\beta_s = 0$. Then one has the following commutative diagram:*

$$\begin{array}{ccccc}
H^2(G_K, \mathrm{ad}^0)^\vee & \xrightarrow{\mathrm{obs}^\vee} & I^\psi/\mathfrak{m}_{\mathcal{R}} I^\psi & \longrightarrow & I_{s+1} \\
\downarrow{\scriptstyle -\widetilde{\beta}_{s+1}^\vee} & & & & \downarrow \\
H^1(G_K, \mathrm{ad}^0)^\vee & \xrightarrow{\sim} & \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 & \xrightarrow[\varpi_{\mathcal{O}}^s \cdot]{\sim} & V,
\end{array}$$

where $V := \left(\mathfrak{m}_s^2 + \varpi_{\mathcal{O}}^{s+1}\mathcal{R}\right)/\left(\mathfrak{m}_{s+1}^2 + \varpi_{\mathcal{O}}^{s+1}\mathcal{R}\right)$ is an $\mathbb{F}$-vector space with basis $\{\varpi_{\mathcal{O}}^s x_j\}_{j=1,\ldots,h}$.

Proof. As in the proof of Lemma 4.5.2, we prove commutativity of the dual diagram

$$\begin{array}{ccccc}
H^2(G_K, \mathrm{ad}^0) & \xleftarrow{\mathrm{obs}} & \mathrm{Hom}_{\mathbb{F}}\left(I^\psi/\mathfrak{m}_{\mathcal{R}} I^\psi, \mathbb{F}\right) & \longleftarrow & \mathrm{Hom}_{\mathbb{F}}\left(I_{s+1}, \mathbb{F}\right) \\
\uparrow{\scriptstyle -\widetilde{\beta}_{s+1}} & & & & \uparrow \\
H^1(G_K, \mathrm{ad}^0) & \xleftarrow{\sim} & \mathrm{Hom}_{\mathbb{F}}\left(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2, \mathbb{F}\right) & \xleftarrow[\varpi_{\mathcal{O}}^{-s} \cdot]{\sim} & \mathrm{Hom}_{\mathbb{F}}\left(V, \mathbb{F}\right).
\end{array}$$

We start by computing $\mathrm{obs}(\bar f) \in H^2(G_K, \mathrm{ad}^0)$, where $\bar f$ is the image in $\mathrm{Hom}_{\mathbb{F}}\left(I^\psi/\mathfrak{m}_{\mathcal{R}} I^\psi, \mathbb{F}\right)$ of a homomorphism $f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. For this, we use f to construct certain deformations of $\bar\rho$ and corresponding 1-cocycles that at the end of the proof also determine the image of f in $H^2(G_K, \mathrm{ad}^0)$ under the other composite morphism passing through $H^1(G_K, \mathrm{ad}^0)$.

In order to compute $\mathrm{obs}(\bar f)$ with the help of Lemma 4.5.1, let $\widetilde{f}_{s+1}\colon V = \oplus_{j=1}^h \mathbb{F}\varpi_{\mathcal{O}}^s x_j \to \mathcal{O}_{s+1}$ be a set-theoretic lift of f, and define $f_{s+1}\colon \mathcal{R}_{s+1} \to \mathcal{O}_{s+1}[\varepsilon]$ by mapping x_i to $\widetilde{f}_{s+1}(\varpi_{\mathcal{O}}^s x_i) \cdot \varepsilon$. Then we consider the quotient $R_{s+} := \mathcal{R}_{s+1}/\left(\mathfrak{m}_s^2 + \varpi_{\mathcal{O}}^{s+1}\mathcal{R}\right)$ of $\mathcal{R}_{s+1}$. Note that R_{s+} is the ring fiber product

$$\begin{array}{ccc}
R_{s+} & \longrightarrow & \mathcal{R}_s \\
\downarrow & \square & \downarrow \\
\mathcal{O}_{s+1} & \longrightarrow & \mathcal{O}_s.
\end{array}$$

The deformation ρ_{s+1} defines a homomorphism $R_{\bar\rho}^{\psi} \to \mathcal{O}_{s+1}$, and since $\beta_s = 0$ there is a surjection $R_{\bar\rho}^{\psi} \twoheadrightarrow \mathcal{R}_s$ by the previous lemma. By universality of the fiber product R_{s+}, there exists a homomorphism $g : R_{\bar\rho}^{\psi} \to R_{s+}$ that corresponds to a deformation $\rho_{s+}\colon G_K \to \mathrm{GL}_n(R_{s+})$ of $\bar\rho$. Moreover, the homomorphism $\big(f_{s+1} \pmod{\varpi_{\mathcal{O}}^s \mathcal{O}_{s+1}}\big) \circ g\colon R_{\bar\rho}^{\psi} \to R_{s+} \to \mathcal{O}_{s+1}[\varepsilon]/(\varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1})$ defines a deformation $\bar\rho_{s+}\colon G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_{s+1}[\varepsilon]/(\varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1}))$. Finally, we form the pushout R_f of $V \hookrightarrow \mathcal{R}_{s+1}$ and f so that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^{\psi}/\mathfrak{m}_{\mathcal{R}} I^{\psi} & \longrightarrow & \mathcal{R}/\mathfrak{m}_{\mathcal{R}} I^{\psi} & \longrightarrow & R_{\bar\rho}^{\psi} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \ \overset{g}{\searrow} & & \\
0 & \longrightarrow & I_{s+1} & \longrightarrow & \mathcal{R}_{s+1} & \longrightarrow & R_{s+1} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & \mathcal{R}_{s+1} & \longrightarrow & R_{s+} & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow \ \overset{f_{s+1}}{\searrow} & & \| & & \\
0 & \longrightarrow & \mathbb{F} & \longrightarrow & R_f & \longrightarrow & R_{s+} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} & \longrightarrow & \mathcal{O}_{s+1}[\varepsilon] & \longrightarrow & \mathcal{O}_{s+1}[\varepsilon]/\varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} & \longrightarrow & 0
\end{array}$$

whose rows are small extensions in $\widehat{\mathcal{A}r}_{\mathcal{O}}$. Using Lemma 4.5.1, we obtain

$$\mathrm{obs}(\bar f) \otimes \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} = \big(\mathcal{O}(\rho_{s+}), f\big) \otimes \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1} = \mathcal{O}(\bar\rho_{s+}) \in H^2\big(G_K, \mathrm{ad}^0 \otimes_{\mathbb{F}} \varpi_{\mathcal{O}}^s \varepsilon \mathcal{O}_{s+1}\big). \qquad (8)$$

Now we follow the steps explained above Lemma 4.5.1: Namely, we first define a suitable set-theoretic lift $G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_{s+1}[\varepsilon])$ of $\bar\rho_{s+}$ and then compute the obstruction class (8) by applying formula (3). Composing the surjection $R_{\bar\rho}^{\psi} \to R_s$ and $f_{s+1} \pmod{\varpi_{\mathcal{O}}^s}\colon R_s \to \mathcal{O}_s[\varepsilon]$ determines a deformation $\rho_{s,\varepsilon} = (1 + \varepsilon c_s)\rho_s\colon G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_s[\varepsilon])$ for some 1-cocylcle $c_s \in H^1(G_K, \mathrm{ad}_s)$. Let $\widetilde{c_s} \in Z^1(G_K, \mathrm{ad}_{s+1}^0)$ be a set-theoretic lift of c_s that by construction defines a set-theoretic lift

$$\widetilde{\rho}_{s,\varepsilon} := (1 + \varepsilon \widetilde{c_s})\rho_{s+1}\colon G_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_{s+1}[\varepsilon]) \qquad (9)$$

of $\bar\rho_{s+}$. Using formula (3), we calculate a representative in $Z^2\big(G_K, \mathrm{ad}^0 \otimes (\varpi_{\mathcal{O}}^s \varepsilon)\big)$ for (8) by evaluating

$$\begin{aligned}
(h,k) &\longmapsto \widetilde{\rho}_{s,\varepsilon}(hk)\widetilde{\rho}_{s,\varepsilon}(k)^{-1}\widetilde{\rho}_{s,\varepsilon}(h)^{-1} - 1 \\
&\overset{(9)}{=} \big(1 + \varepsilon \widetilde{c_s}(hk)\big)\rho_{s+1}(hk)\rho_{s+1}(k)^{-1}\big(1 - \varepsilon \widetilde{c_s}(k)\big)\rho_{s+1}(h)^{-1}\big(1 - \varepsilon \widetilde{c_s}(h)\big) - 1 \\
&= \varpi_{\mathcal{O}}^{-s}\big(\widetilde{c_s}(hk) - \mathrm{Ad}_{\rho_{s+1}(h)}\,\widetilde{c_s}(k) - \widetilde{c_s}(h)\big) \cdot \varpi_{\mathcal{O}}^s \varepsilon.
\end{aligned}$$

Hence, the class $\mathrm{obs}(\bar f) \in H^2(G_K, \mathrm{ad}^0)$ is obtained from dividing by $\varepsilon \varpi_{\mathcal{O}}^s$.

It remains to compute the image of the homomorphism $f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ under the composite morphism passing through $H^1(G_K, \mathrm{ad}^0)$. First note that the map $f_{s+1} \pmod{\varpi_{\mathcal{O}}}\colon \mathbb{F}[x_1, \ldots, x_h]/(x_1, \ldots, x_h)^2 \to \mathbb{F}[\varepsilon]$ induces a homomorphism $f_1 \in \mathrm{Hom}_{\mathbb{F}}\big(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2, \mathbb{F}\big)$, which under multiplication by $\varpi_{\mathcal{O}}^s$ is mapped to $f \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. We want to compute $\widetilde{\beta}_{s+1}([c])$, where $c \in Z^1(G_K, \mathrm{ad}^0)$ is a representative of the image of f_1 under the canonical isomorphism $\mathrm{Hom}_{\mathbb{F}}\big(\overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2, \mathbb{F}\big) \xrightarrow{\sim} H^1(G_K, \mathrm{ad}^0)$. Since by construction $\rho_{s,\varepsilon} = (1 + \varepsilon c_s)\rho_s\colon G_K \to \mathrm{GL}_n(\mathcal{O}_s[\varepsilon])$ lifts $(1 + \varepsilon c)\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$, it is clear that $\widetilde{c_s} \in Z^1(G_K, \mathrm{ad}_{s+1}^0)$ is a set-theoretic lift of c. By Lemma 4.6.5(c), it thus provides us with the representative

$$(h,k) \longmapsto \varpi_{\mathcal{O}}^{-s}\left(\mathrm{Ad}_{\rho_{s+1}(h)}\,\widetilde{c_s}(k) - \widetilde{c_s}(hk) + \widetilde{c_s}(h)\right) \quad \in Z^2(G_K, \mathrm{ad}^0)$$

for $\widetilde{\beta}_{s+1}([c])$. This shows that $\widetilde{\beta}_{s+1}([c]) = -\mathrm{obs}(\bar f)$, proving the lemma. □

If $\bar\rho$ has a lift to $\mathcal{O}_{2s}$, then there is a natural refinement of the above with regards to the filtration of $\mathcal{R}$ given by $\mathfrak{m}_s$. Denoting by in the initial term map with respect to this filtration, one has isomorphisms

$$\overline{\mathfrak{m}_s^2/(\mathfrak{m}_s^3+\mathfrak{m}_{2s}^2)} \cong \mathbb{F}\cdot \mathrm{in}(\varpi_{\mathcal{O}}^s)^2 \oplus \bigoplus_{i=1}^h \mathbb{F}\cdot \mathrm{in}(\varpi_{\mathcal{O}}^s)\cdot \mathrm{in}(x_i) \cong \mathbb{F}\cdot \mathrm{in}(\varpi_{\mathcal{O}}^s)^2 \oplus V$$

and

$$\ker\left(\overline{\mathfrak{m}_s^2/\mathfrak{m}_s^3} \to \overline{\mathfrak{m}_s^2/(\mathfrak{m}_s^3+\mathfrak{m}_{2s}^2)}\right) \cong \bigoplus_{1\le i\le j\le n} \mathbb{F}\cdot \mathrm{in}(x_i)\cdot \mathrm{in}(x_j) \cong \overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3.$$

In other words, we have a natural 2-step filtration of $\overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}$ whose first subquotient is isomorphic to $\overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3$ and whose second subquotient is isomorphic to $\mathbb{F}\cdot\mathrm{in}(\varpi_{\mathcal{O}}^s)^2\oplus V$ with V as above. A variant of the above lemma is the following whose proof we leave to the reader:

Lemma 4.6.7. *Let I_{s+1} be the relation ideal in (6) and let $I^\psi \to I_{s+1}$ be the canonical homomorphism. Suppose that ρ_{s+1} possesses a lift to $\mathcal{O}_{2s}$ and that $\beta_s = 0$. Then one has the following commutative diagram:*

$$\begin{array}{ccccc} H^2(G_K,\mathrm{ad}^0)^\vee & \longrightarrow & I^\psi/\mathfrak{m}_{\mathcal{R}}I^\psi & \longrightarrow & I_{s+1} \\ \downarrow{\scriptstyle -\widetilde\beta_{s+1}^\vee} & & & & \downarrow \\ H^1(G_K,\mathrm{ad}^0)^\vee & \xrightarrow{\sim} & \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2 & \xhookrightarrow{\varpi_{\mathcal{O}}^s\cdot} & \mathbb{F}\cdot\mathrm{in}(\varpi_{\mathcal{O}}^s)^2\oplus V. \end{array}$$

The use of the above 2-step filtration of $\overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}$ allows one to apply our results on the Bockstein operator on one piece and that of the bracket cup product on the other. This gives precise information on the *refined quadratic parts* in $\overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}$ which arise from $H^2(G_K,\mathrm{ad}^0)$ – with the possible exception of the quotient $\mathrm{in}(\varpi_{\mathcal{O}}^s)^2\cdot\mathbb{F}$. Namely, we have the following result:

Theorem 4.6.8. *Suppose $\bar\rho$ has a lift to $\mathcal{O}_{s+1}$ and that $\beta_s = 0$. Then I_ψ is contained in $\mathfrak{m}_s^2+\varpi_{\mathcal{O}}^{s+1}\mathcal{R}$ and the following diagram is commutative, where all homomorphisms are the natural ones, as given either in Lemma 4.5.2 or Lemma 4.6.6:*

$$\begin{array}{ccccc} H^2(G_K,\mathrm{ad}^0)^\vee & \longrightarrow & I^\psi/\mathfrak{m}_{\mathcal{R}}I^\psi & \longrightarrow & \overline{I^\psi/\left(I^\psi\cap(\mathfrak{m}_s^3+\varpi_{\mathcal{O}}^{s+1}\mathcal{R})\right)} \\ \downarrow{\scriptstyle -\widetilde\beta_{s+1}^\vee\oplus-\frac12 b^\vee} & & & & \downarrow \\ H^1(G_K,\mathrm{ad}^0)^\vee\oplus\mathrm{Sym}^2H^1(G_K,\mathrm{ad}^0)^\vee & \xrightarrow{\sim} & \overline{\mathfrak{m}}_{\mathcal{R}}/\overline{\mathfrak{m}}_{\mathcal{R}}^2\oplus\overline{\mathfrak{m}}_{\mathcal{R}}^2/\overline{\mathfrak{m}}_{\mathcal{R}}^3 & \xhookrightarrow{(\varpi_{\mathcal{O}}^s\cdot\_)\oplus\mathrm{incl.}} & \overline{\mathrm{gr}^2_{\mathfrak{m}_s}\mathcal{R}}/(\mathbb{F}\cdot\mathrm{in}(\varpi_{\mathcal{O}}^s)^2). \end{array}$$

If in addition $\bar\rho$ has a lift to $\mathcal{O}_{2s}$, then the above diagram still commutes if one removes the symbols '$+\varpi_{\mathcal{O}}^{s+1}\mathcal{R}$' in the top right and $\mathbb{F}\cdot\mathrm{in}(\varpi_{\mathcal{O}}^s)^2$' in the lower right corner.

We now discuss various issues about the Bockstein homomorphism that were left open so far, for instance the existence of lifts ρ_{s+1} and the choice of s.

Lemma 4.6.9. *Let $p>2$ and $\bar\rho\colon G_K\to\mathrm{GL}_n(\mathbb{F})$ be a representation. Let s be an integer.*[11] *We fix a minimal presentation of $R_{\bar\rho}^\psi$ as in Proposition 4.1.3 and an isomorphism $\mathcal{R}\cong\mathcal{O}[\![x_1,\ldots,x_h]\!]$, and set $\mathfrak{m}_s := (\varpi_{\mathcal{O}}^s, x_1,\ldots,x_h)$. Then the following hold:*

[11] In different items, s may take different values.

(a) *If* $n = 2$, *then* $\bar{\rho}$ *has a lift to* $\mathcal{O}$.

(b) *For general* n, *if* $\bar{\rho}(G_K)$ *is a* p*-group and if* $p^s = \#\mu_{p^\infty}(K) > 1$, *then* $\bar{\rho}$ *has a lift to the ring* $\mathcal{O}/\varpi_{\mathcal{O}}^{s+1}\mathcal{O}$.

(c) *If the relation ideal* I^ψ *lies in* $\mathfrak{m}_s^2$, *then* $\bar{\rho}$ *has a lift to* $\mathcal{O}_{2s}$.

(d) *If* $\bar{\rho}$ *has a lift to* $\mathcal{O}_{2s}$ *and if* $\beta_s = 0$, *then any choice* $y_i \in x_i + \varpi_{\mathcal{O}}^s\mathcal{R}$, $i = 1, \ldots, h$, *induces a change of coordinates isomorphism* $\mathcal{R} \cong \mathcal{O}[\![x_1, \ldots, x_h]\!] \cong \mathcal{O}[\![y_1, \ldots, y_h]\!]$ *such that* $\mathfrak{m}_s$ *is independent of whether we use the* x_i *or the* y_i *to define it.*

(e) *If* $R_{\bar{\rho}}^\psi$ *is flat over* $\mathcal{O}$, *then there exists a finite totally ramified extension of* $\mathcal{O}[1/p]$ *with ring of integers* $\mathcal{O}'$ *and a homomorphism* $R_{\bar{\rho}}^\psi \to \mathcal{O}'$ *in* $\widehat{\mathcal{A}\mathrm{r}}_{\mathcal{O}}$, *i.e.,* $\bar{\rho}$ *has a lift to characteristic zero, and in particular lifts to* $\mathcal{O}'/(\varpi'_{\mathcal{O}})^s$ *for every integer* s.

Regarding (e) note that A. Muller [Mul13] has constructed crystalline lifts of a large class of mod p Galois representations $\bar{\rho}$ for any n. Whether such a lift always exists is still an open question.

Proof. For $\mathcal{O} = W(\mathbb{F})$, part (a) can be obtained from a simple adaption of [Kha97, Theorem 2] – Khare's proof using Kummer theory works for all field of characteristic zero – and part (b) is [Böc03, Prop. 2.1]. For general $\mathcal{O}$, one can apply Lemma 4.4.1 to replace the fixed character $\psi\colon G_K \to \mathcal{O}^*$ by a twist of ψ whose image lies in $W(\mathbb{F})^*$. Part (c) is rather trivial: the hypothesis implies that $R_{\bar{\rho}}^\psi \cong \mathcal{R}/I^\psi$ surjects onto $\mathcal{R}/(p^{2s}, x_1, \ldots, x_h) \cong \mathcal{O}_{2s}$. Part (d) is also obvious. For (e) observe that by flatness the ring $R_{\bar{\rho}}^\psi[1/p]$ is non-zero. Hence, its generic fiber $\mathfrak{X}(\rho)^\psi$ is a non-empty rigid analytic space over $\mathcal{O}[1/p]$. Thus it has points over some finite extension of $\mathcal{O}[1/p]$. These points are the desired lifts. ∎

Remark 4.6.10. The definition of the Bockstein operators β_i depends on a choice of a *base point*, i.e., a lift ρ_{s+1} of $\bar{\rho}$ to $\mathcal{O}_{s+1}$. We do not know in general in what sense the vanishing of β_s and the non-vanishing of β_{s+1} could be independent of such a lift. A change of base point as described in Lemma 4.6.9(d), clearly does not change the integer s for which $\beta_s = 0$ and $\beta_{s+1} \neq 0$, assuming the existence of ρ_{s+1}. We also do not know, what an optimal choice of s, independently of a choice of the lift ρ_{s+1} means, although Lemma 4.6.9 provides some reasonable guesses. If one does have an explicit choice of ρ_{s+1}, and a situation where one can then determine its infinitesimal deformations, then one can determine whether $\beta_s = 0$ and $\beta_{s+1} \neq 0$. Such an approach is sketched in the proof of Proposition 4.6.11.

Before giving the proof of Theorem 4.1.14, we discuss the existence of such a base point in cases (D) and (E) of Section 4.3. For the remainder, suppose that $q = \#\mu_{p^\infty}(K) > 1$ and set $s := \log_p q$. Suppose also that the image of $\bar{\rho}$ is a p-group and that the fixed lift ψ of $\det \bar{\rho}$ is the trivial character – both can be assumed without loss of generality by twisting; cf. the proof of Lemma 4.3.7.

Proposition 4.6.11. *In cases (D) and (E) of Section 4.3 there exists a deformation* ρ_∞ *in* $D_{\bar{\rho}}^\psi(W(\mathbb{F}))$ *such that* $\beta_s = 0$ *and* $\beta_{s+1} \neq 0$.

Proof. We ask the reader to have the notation and concepts used in the proof of Lemma 4.3.7 at hand. We define

$$M_i := \begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix} \text{ for } i = 1, 3, \ldots, 2g \quad \text{and} \quad M_2 := \begin{pmatrix} \sqrt{1-q}^{\,-1} & u_2 \\ 0 & \sqrt{1-q} \end{pmatrix},$$

where the u_i are all zero in case (D). Then it is easy to verify that the M_i satisfy the Demushkin relation $M_1^q[M_1,M_2]\dots[M_{2g-1},M_{2g}]=1$. Hence, the map $\Pi\to \mathrm{GL}_2(W(\mathbb{F}))$ defined by mapping x_i to M_i yields the desired lift ρ_∞.

We use this base point to determine the Bockstein relations, and thus to determine the correct value of s such that $\beta_s=0$ and $\beta_{s+1}\neq 0$, by computing explicitly infinitesimal deformations of ρ_∞. Namely, we define $N_i:=M_i(1+\varepsilon A_i)\in \mathrm{GL}_2(W(\mathbb{F})[\varepsilon])$ for matrices $A_i=\begin{pmatrix} a_i & b_i\\ c_i & -a_i\end{pmatrix}$. Computing the Demushkin relation $N_1^q[N_1,N_2]\dots[N_{2g-1},N_{2g}]=1$, we obtain a linear relation whose coefficients lie in $qW(\mathbb{F})$ but not in $pqW(\mathbb{F})$. The assertion follows. □

Remark 4.6.12. We note that the base point lift chosen in the proof of the previous proposition is obtained as a specialization of the variables in the proof of Lemma 4.3.7 within $qW(\mathbb{F})$. Hence, by Lemma 4.6.9(d), the trivial specialization that sends all variables to zero gives a lift to $W(\mathbb{F})/q^2W(\mathbb{F})$ (in fact to $W(\mathbb{F})$) so that $\beta_s=0$ and $\beta_{s+1}\neq 0$.

Proof of Theorem 4.1.14. By the same reduction as in the proof of Theorem 4.1.4, given after Corollary 4.3.6, it suffices to treat the case $\mathcal{O}=W(\mathbb{F})$. By Theorem 4.3.4, we have in cases (A)–(C) of Section 4.3 that a lift $\rho_{2s}\colon G_K\to \mathrm{GL}_2(W(\mathbb{F})/p^{2s}W(\mathbb{F}))$ exists for $s=\log_p q$ if we specialize all variables to zero. Then all the specialized relations will vanish modulo q^2. Moreover for this choice, we have $\beta_s=0$ and $\beta_{s+1}\neq 0$ because the linear terms of the relations vanish modulo q but not modulo pq. By Corollary 4.3.6(b), the images of the quadratic parts of generators of I^ψ span a subspace of dimension equal to $h_2=\dim_{\mathbb{F}} H^2(G_K,\mathrm{ad}^0)$. Thus Theorem 4.1.14 follows from Theorem 4.6.8.

It remains to consider cases (D) and (E). We take the specialization from Remark 4.6.12 as our lift to $W(\mathbb{F})/q^2W(\mathbb{F})$ so that $\beta_s=0$ and $\beta_{s+1}\neq 0$. By Lemma 4.3.7(c),(e), there exists a presentation

$$0\longrightarrow (r_1,\dots,r_m)\longrightarrow \mathcal{R}\longrightarrow R^\psi_{\bar\rho}\longrightarrow 0$$

such that $\mathrm{in}(\bar r_1),\dots,\mathrm{in}(\bar r_m)\in \mathrm{gr}^2_{\overline{\mathfrak{m}}_{\mathcal{R}}}\overline{\mathcal{R}}$ form a regular sequence in $\mathrm{gr}_{\overline{\mathfrak{m}}_{\mathcal{R}}}\overline{\mathcal{R}}$ and $m=\dim_{\mathbb{F}} H^2(G_K,\mathrm{ad}^0)$. We complete the proof of Theorem 4.1.14 by a further appeal to Theorem 4.6.8. □

BIBLIOGRAPHY

[BC09] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, (324):xii+314, 2009.

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939, 2001. URL: https://doi.org/10.1090/S0894-0347-01-00370-8.

[Bel12] Joël Bellaïche. Pseudodeformations. *Math. Z.*, 270(3-4):1163–1180, 2012. URL: http://dx.doi.org/10.1007/s00209-011-0846-2, doi:10.1007/s00209-011-0846-2.

[Ber10] Laurent Berger. On some modular representations of the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. *Compos. Math.*, 146(1):58–80, 2010. URL: http://dx.doi.org/10.1112/S0010437X09004345, doi:10.1112/S0010437X09004345.

[BJ15] Gebhard Böckle and Ann-Kristin Juschka. Irreducibility of versal deformation rings in the (p,p)-case for 2-dimensional representations. *J. Algebra*, 444:81–123, 2015. URL: http://dx.doi.org/10.1016/j.jalgebra.2015.07.009, doi:10.1016/j.jalgebra.2015.07.009.

[Böc00] Gebhard Böckle. Demuškin groups with group actions and applications to deformations of Galois representations. *Compositio Math.*, 121(2):109–154, 2000. URL: http://dx.doi.org/10.1023/A:1001746207573, doi:10.1023/A:1001746207573.

[Böc03] Gebhard Böckle. Lifting mod p representations to characteristics p^2. *J. Number Theory*, 101(2):310–337, 2003. URL: http://dx.doi.org/10.1016/S0022-314X(03)00058-1, doi:10.1016/S0022-314X(03)00058-1.

[Böc08] Gebhard Böckle. Deformations and the rigidity method. *J. Algebra*, 320(10):3613–3658, 2008. URL: http://dx.doi.org/10.1016/j.jalgebra.2008.07.023, doi:10.1016/j.jalgebra.2008.07.023.

[Böc10] Gebhard Böckle. Deformation rings for some mod 3 Galois representations of the absolute Galois group of $\mathbf{Q}_3$. *Astérisque*, (330):529–542, 2010.

[CDP14] Pierre Colmez, Gabriel Dospinescu, and Vytautas Paškūnas. The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. *Camb. J. Math.*, 2(1):1–47, 2014. URL: https://doi.org/10.4310/CJM.2014.v2.n1.a1.

[CDP15] Pierre Colmez, Gabriel Dospinescu, and Vytautas Paškūnas. Irreducible components of deformation spaces: wild 2-adic exercises. *Int. Math. Res. Not. IMRN*, (14):5333–5356, 2015. URL: http://dx.doi.org/10.1093/imrn/rnu089, doi:10.1093/imrn/rnu089.

[CDT99] Brian Conrad, Fred Diamond, and Richard Taylor. Modularity of certain potentially Barsotti-Tate Galois representations. *J. Amer. Math. Soc.*, 12(2):521–567, 1999. URL: http://dx.doi.org/10.1090/S0894-0347-99-00287-8, doi:10.1090/S0894-0347-99-00287-8.

[Che11] Gaëtan Chenevier. Sur la variété des caractères p-adique du groupe de galois absolu de $\mathbb{Q}_p$. *Preprint*, 2011. URL: http://gaetan.chenevier.perso.math.cnrs.fr/pub.html.

[Che13a] Gaëtan Chenevier. Repr\'esentations galoisiennes automorphes et cons\'equences arithm\'etiques des conjectures de langlands et arthur. *Habilitation, Parix XI*, 2013. URL: http://gaetan.chenevier.perso.math.cnrs.fr/pub.html.

[Che13b] Gaëtan Chenevier. Sur la densité des représentations cristallines de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. *Math. Ann.*, 355(4):1469–1525, 2013. URL: http://dx.doi.org/10.1007/s00208-012-0815-z, doi:10.1007/s00208-012-0815-z.

[Che14] Gaëtan Chenevier. The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings. In *Automorphic forms and Galois representations*, volume 94 of *Proceedings of the LMS Durham Symposium*, 2014. To appear. URL: http://gaetan.chenevier.perso.math.cnrs.fr/pub.html, arXiv:arXiv:0809.0415.

[Coh73] Gerard Elie Cohen. Profinite modules. *Canad. Math. Bull.*, 16:405–415, 1973. URL: https://doi.org/10.4153/CMB-1973-064-5.

[Col08] Pierre Colmez. Représentations triangulines de dimension 2. *Astérisque*, (319):213–258, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (φ, Γ)-modules.

[Col10] Pierre Colmez. Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (φ, Γ)-modules. *Astérisque*, (330):281–509, 2010.

[Con99] Brian Conrad. Ramified deformation problems. *Duke Math. J.*, 97(3):439–513, 1999. URL: http://dx.doi.org/10.1215/S0012-7094-99-09718-1, doi:10.1215/S0012-7094-99-09718-1.

[Con11] Brian Conrad. Lifting global representations with local properties. *Preprint*, 2011. URL: http://math.stanford.edu/$\sim$conrad/papers/locchar.pdf.

[CR62] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[CR81] Charles W. Curtis and Irving Reiner. *Methods of representation theory. Vol. I*. John Wiley & Sons, Inc., New York, 1981. With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication.

[Dem61] S. P. Demuškin. The group of a maximal p-extension of a local field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:329–346, 1961.

[Dem63] S. P. Demuškin. On 2-extensions of a local field. *Sibirsk. Mat. Ž.*, 4:951–955, 1963.

[DGPS18] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann. Singular 4-1-1 — A computer algebra system for polynomial computations, 2018. URL: http://www.singular.uni-kl.de.

[dJ95] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.*, (82):5–96 (1996), 1995. URL: http://www.numdam.org/item?id=PMIHES_1995__82__5_0.

[DS74] Pierre Deligne and Jean-Pierre Serre. Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup. (4)*, 7:507–530 (1975), 1974.

[DS15] Fred Diamond and David Savitt. Serre weights for locally reducible two-dimensional Galois representations. *J. Inst. Math. Jussieu*, 14(3):639–672, 2015. URL: http://dx.doi.org/10.1017/S1474748014000164, doi:10.1017/S1474748014000164.

[Ebe14] Rouven Eberhardt. Irreduzibilität eines versellen Deformationsringes im $(2, 2)$-Fall. Bachelor thesis, Heidelberg University, 2014.

[EGA I] Alexander Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas (rédigés avec la collaboration de Jean Dieudonné). *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[EGA IV_1] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I (rédigés avec la collaboration de Jean Dieudonné). *Inst. Hautes Études Sci. Publ. Math.*, (20):259, 1964.

[EGA IV_2] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II (rédigés avec la collaboration de Jean Dieudonné). *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[EGA IV_3] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III (rédigés avec la collaboration de Jean Dieudonné). *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry. URL: http://dx.doi.org/10.1007/978-1-4612-5350-1, doi:10.1007/978-1-4612-5350-1.

[Eme06] Matthew Emerton. A local-global compatibility conjecture in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$. *Pure Appl. Math. Q.*, 2(2, Special Issue: In honor of John H. Coates. Part 2):279–393, 2006. URL: https://doi.org/10.4310/PAMQ.2006.v2.n2.a1.

[Ger10] David James Geraghty. *Modularity lifting theorems for ordinary Galois representations*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Harvard University. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:3414729.

[GM88a] William M. Goldman and John J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (67):43–96, 1988. URL: http://www.numdam.org/item?id=PMIHES_1988__67__43_0.

[GM88b] William M. Goldman and John J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 18(2):153–158, 1988. URL: http://dx.doi.org/10.1090/S0273-0979-1988-15631-5, doi:10.1090/S0273-0979-1988-15631-5.

[Gol84] William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984. URL: http://dx.doi.org/10.1016/0001-8708(84)90040-9, doi:10.1016/0001-8708(84)90040-9.

[Gou01] Fernando Q. Gouvêa. Deformations of Galois representations. In *Arithmetic algebraic geometry (Park City, UT, 1999)*, volume 9 of *IAS/Park City Math. Ser.*, pages 233–406. Amer. Math. Soc., Providence, RI, 2001. Appendix 1 by Mark Dickinson, Appendix 2 by Tom Weston and Appendix 3 by Matthew Emerton.

[Gro68] Alexander Grothendieck. Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 46–66. North-Holland, Amsterdam, 1968.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Her81] J. Herzog. When is a regular sequence super regular? *Nagoya Math. J.*, 83:183–195, 1981. URL: http://projecteuclid.org/euclid.nmj/1118786483.

[Hoc07] Melvin Hochster. F-finite rings. *Lecture Notes, Math 711*, 2007. URL: http://math.lsa.umich.edu/~hochster/711F07/L09.21.pdf.

[Hup67] B. Huppert. *Endliche Gruppen. I.* Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, 1967.

[Kha97] Chandrashekhar Khare. Base change, lifting, and Serre's conjecture. *J. Number Theory*, 63(2):387–395, 1997. URL: http://dx.doi.org/10.1006/jnth.1997.2093, doi:10.1006/jnth.1997.2093.

[Kis03] Mark Kisin. Overconvergent modular forms and the Fontaine-Mazur conjecture. *Invent. Math.*, 153(2):373–454, 2003. URL: http://dx.doi.org/10.1007/s00222-003-0293-8, doi:10.1007/s00222-003-0293-8.

[Kis09] Mark Kisin. The Fontaine-Mazur conjecture for GL_2. *J. Amer. Math. Soc.*, 22(3):641–690, 2009. URL: https://doi.org/10.1090/S0894-0347-09-00628-6.

[Kis10a] Mark Kisin. Deformations of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ representations. *Astérisque*, (330):511–528, 2010.

[Kis10b] Mark Kisin. Deformations of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ representations. *Astérisque*, (330):511–528, 2010. appendix to [Col10].

[KO74] Max-Albert Knus and Manuel Ojanguren. *Théorie de la descente et algèbres d'Azumaya.* Lecture Notes in Mathematics, Vol. 389. Springer-Verlag, Berlin-New York, 1974.

[Koc00] Helmut Koch. *Number theory*, volume 24 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2000. Algebraic numbers and functions, Translated from the 1997 German original by David Kramer.

[Kre13] Maximilian Kremer. Cup-Produkt und Nullteilerfreiheit eines darstellenden Rings im Rahmen der Deformationstheorie. Master thesis, Heidelberg University, 2013.

[KW09a] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture. I. *Invent. Math.*, 178(3):485–504, 2009. URL: https://doi.org/10.1007/s00222-009-0205-7.

[KW09b] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture. II. *Invent. Math.*, 178(3):505–586, 2009. URL: https://doi.org/10.1007/s00222-009-0206-6.

[Lab67] John P. Labute. Classification of Demushkin groups. *Canad. J. Math.*, 19:106–132, 1967.

[Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Maz89] B. Mazur. Deforming Galois representations. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989. URL: http://dx.doi.org/10.1007/978-1-4613-9649-9_7, doi:10.1007/978-1-4613-9649-9_7.

[Maz97] Barry Mazur. An introduction to the deformation theory of Galois representations. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 243–311. Springer, New York, 1997.

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[Mul13] Alain Muller. *Relèvements cristallins de représentations galoisiennes*. PhD thesis, Université de Strasbourg, 2013.

[Nak00] Kazunori Nakamoto. Representation varieties and character varieties. *Publ. Res. Inst. Math. Sci.*, 36(2):159–189, 2000. URL: http://dx.doi.org/10.2977/prims/1195143100, doi:10.2977/prims/1195143100.

[Nak13] Kentaro Nakamura. Deformations of trianguline B-pairs and Zariski density of two dimensional crystalline representations. *J. Math. Sci. Univ. Tokyo*, 20(4):461–568, 2013.

[Nak14] Kentaro Nakamura. Zariski density of crystalline representations for any p-adic field. *J. Math. Sci. Univ. Tokyo*, 21(1):79–127, 2014.

[Nek06] Jan Nekovář. Selmer complexes. *Astérisque*, (310):viii+559, 2006.

[Neu99] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[NSW00] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.

[Paš17] Vytautas Paškūnas. On 2-adic deformations. *Math. Z.*, 286(3-4):801–819, 2017. URL: https://doi.org/10.1007/s00209-016-1785-8.

[Pro87] Claudio Procesi. A formal inverse to the Cayley-Hamilton theorem. *J. Algebra*, 107(1):63–74, 1987. URL: http://dx.doi.org/10.1016/0021-8693(87)90073-1, doi:10.1016/0021-8693(87)90073-1.

[Ram93] Ravi Ramakrishna. On a variation of Mazur's deformation functor. *Compositio Math.*, 87(3):269–286, 1993. URL: http://www.numdam.org/item?id=CM_1993__87_3_269_0.

[Rob63] Norbert Roby. Lois polynomes et lois formelles en théorie des modules. *Ann. Sci. École Norm. Sup. (3)*, 80:213–348, 1963.

[Rob80] Norbert Roby. Lois polynômes multiplicatives universelles. *C. R. Acad. Sci. Paris Sér. A-B*, 290(19):A869–A871, 1980.

[Rou96] Raphaël Rouquier. Caractérisation des caractères et pseudo-caractères. *J. Algebra*, 180(2):571–586, 1996. URL: http://dx.doi.org/10.1006/jabr.1996.0083, doi:10.1006/jabr.1996.0083.

[Sag14] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 6.4.1)*, 2014. URL: http://www.sagemath.org.

[Sch68] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[Ser95] Jean-Pierre Serre. Structure de certains pro-p-groupes (d'après Demuškin). In *Séminaire Bourbaki, Vol. 8*, pages Exp. No. 252, 145–155. Soc. Math. France, Paris, 1995.

[Sim94] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. II. *Inst. Hautes Études Sci. Publ. Math.*, (80):5–79 (1995), 1994. URL: http://www.numdam.org/item?id=PMIHES_1994__80__5_0.

[Sta18] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu, 2018.

[Tay91] Richard Taylor. Galois representations associated to Siegel modular forms of low weight. *Duke Math. J.*, 63(2):281–332, 1991. URL: http://dx.doi.org/10.1215/S0012-7094-91-06312-X, doi:10.1215/S0012-7094-91-06312-X.

[TW95] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995. URL: https://doi.org/10.2307/2118560.

[Vac09] Francesco Vaccarino. Homogeneous multiplicative polynomial laws are determinants. *J. Pure Appl. Algebra*, 213(7):1283–1289, 2009. URL: http://dx.doi.org/10.1016/j.jpaa.2008.11.036, doi:10.1016/j.jpaa.2008.11.036.

[VV78] Paolo Valabrega and Giuseppe Valla. Form rings and regular sequences. *Nagoya Math. J.*, 72:93–101, 1978. URL: http://projecteuclid.org/euclid.nmj/1118785674.

[Was97] Lawrence C. Washington. Galois cohomology. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 101–120. Springer, New York, 1997.

[WE13] Carl William Wang Erickson. *Moduli of Galois Representations*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Harvard University. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:3567116.

[WE17] Carl William Wang Erickson. Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings. To appear in Mathematische Annalen., 2017. URL: https://link.springer.com/article/10.1007/s00208-017-1557-8.

[Web16] Peter Webb. *A course in finite group representation theory*, volume 161 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016. URL: http://dx.doi.org/10.1017/CBO9781316677216, doi:10.1017/CBO9781316677216.

[Wil88] A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988. URL: https://doi.org/10.1007/BF01394275.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995. URL: http://dx.doi.org/10.2307/2118559, doi:10.2307/2118559.

[Zip86] Dieter Ziplies. A characterization of the norm of an Azumaya algebra of constant rank through the divided powers algebra of an algebra. *Beiträge Algebra Geom.*, (22):53–70, 1986.

[Zip87] Dieter Ziplies. Generators for the divided powers algebra of an algebra and trace identities. *Beiträge Algebra Geom.*, (24):9–27, 1987.