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Dipl.-Math. Oliver Thomas
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# On Analytic and Iwasawa Cohomology

Betreuer: Prof. Dr. Otmar Venjakob

**Abstract**

We generalise spectral sequences for Iwasawa adjoints of Jannsen to higher dimensional coefficient rings by systematically employing Matlis, local, Koszul and Tate duality. With the same strategy we achieve a generalisation of Venjakob's local duality theorem for Iwasawa algebras and compute the $\Lambda$-torsion of the first Iwasawa cohomology group, both locally and globally.

Furthermore, we develop a flexible framework to prove standard results of group cohomology for topologised monoids with coefficients in topologised modules, using explicit methods dating back to Hochschild and Serre. This closes a few argumentative gaps in the literature.

We also prove a form of Poincaré duality for Lie groups over arbitrary complete non-archimedean fields of characteristic zero.

Finally, we take tentative steps towards applying these results to $(\varphi, \Gamma)$-modules.

**Zusammenfassung**

Wir verallgemeinern Spektralfolgen für Iwasawa-Adjungierte von Jannsen auf höherdimensionale Koeffizientenringe, indem wir systematisch Matlis-, lokale, Koszul- und Tate-Dualität verwenden. So erreichen wir auch eine Verallgemeinergung von Venjakobs Theorem über lokale Dualität für Iwasawa-Algebren und bestimmen die $\Lambda$-Torsion der ersten Iwasawa-Kohomologie-Gruppen, sowohl lokal als auch global.

Ferner entwickeln wir ein flexibles Framework, um Standard-Resultate der Gruppenkohomologie für topologisierte Monoide mit Koeffizienten in topologisierten Moduln zu zeigen. Hier nutzen wir explizite Methoden, die auf Hochschild und Serre zurückgehen. Dies schließt einige argumentative Lücken in der Literatur.

Wir zeigen eine Form von Poincaré-Dualität für Lie-Gruppen über beliebigen vollständigen nicht-archimedischen Körpern der Charakteristik null.

Schlussendlich gehen wir erste Schritte in Richtung einer Anwendung dieser Resultate auf $(\varphi, \Gamma)$-Moduln.

Diese Arbeit wäre nicht entstanden ohne die vielen interessanten und hilfreichen mathematischen Diskussionen mit meinem Betreuer Otmar Venjakob. Für seine Geduld, sein offenes Ohr und sein Vertrauen in meine Fähigkeiten bedanke ich mich herzlich. Auch gebührt Kay Wingberg besonderer Dank, der nicht nur oft das rechte Wort zur rechten Zeit wusste, sondern auch die Entwicklung dieser Arbeit mit Interesse verfolgt hat.

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# Contents

# Introduction

Cohomology has jokingly been referred to as the fifth operation of elementary arithmetic after addition, substraction, multiplication and division.[1] The development of the cohomological machinery started in the early twentieth century in the area of algebraic topology. Since then it has been an invaluable tool in many areas of mathematics, algebraic number theory chief among them. Since at least the publication of Artin's and Tate's notes on class field theory (cf. [AT68]) it is universally accepted that the cohomological point of view is the most conceptual and insightful one for proving the results of class field theory – a major achievement in algebraic number theory in the twentieth century. It is therefore no wonder that many modern results in algebraic number theory are phrased in a cohomological language, where a rich machinery is available.

One example of this is Iwasawa theory. In his seminal paper [Iwa73], Iwasawa uses an ad hoc construction for something that is nowadays called an *Iwasawa adjoint*, and which can be described as an Ext-group. Norm compatible elements in field extensions can via class field theory also be expressed as corestriction compatible systems of Galois cohomology classes, which are closely linked to Euler systems and play a key role in the construction of $p$-adic $L$-functions, a major area of research in algebraic number theory in general and Iwasawa theory in particular.

The first chapter of this thesis is devoted to studying Iwasawa adjoints and Iwasawa cohomology. We systematically employ Matlis duality, local duality, Koszul duality and Tate duality to derive more general versions of important spectral sequences due to Jannsen (cf. [Jan89; Jan14]): Where he only considered $\mathbb{Z}_p$-coefficients, we allow higher dimensional local rings. This also allows us to prove a quite general local duality theorem for Iwasawa algebras, generalising the results of [Ven02]. We also show that the first Iwasawa cohomology groups are torsion-free over their Iwasawa algebras. This chapter has been accepted for publication as [TV19].

The first chapter mainly handles variants of two cohomology theories: $\operatorname{Ext}^\bullet_R(M, R)$ where $R$ is a commutative ring, and group cohomology $H^\bullet(G, A)$, where $G$ is a locally compact group and $A$ a discrete $G$-module. By cleverly combining these two types of cohomology theories, we also gain access to the cohomology groups $\operatorname{Ext}^\bullet_\Lambda(M, \Lambda)$, where $\Lambda$ is the (non-commutative) Iwasawa algebra. The arguments are not purely algebraic in nature: We often employ continuous duals and have to take care about how we topologise certain modules. However, as the modules in question are finitely generated, this does not pose an actual obstacle.

Everything becomes significantly more complicated if instead of considering discrete $G$-modules $A$, one wants to look at coefficients with more complicated topological data. It gets even worse when $G$ and $A$ are not only topological groups, but also have the structure of e. g. manifolds. We will call these groups and coefficients *topologised* to denote a potentially finer structure than that of topological groups. Developing cohomology theories which make use of this finer structure, one is confronted with a fundamental issue:

[1] We would like to point out that even the addition of natural numbers can benefit from a cohomological point of view, cf. [Grü17].

The classical machinery of homological algebra refuses to function and results are much more difficult to prove.

There seem to be three possibilities to circumvent these problems. The most conceptional one, due to Flach (cf. [Fla08]), embeds topological modules in a strictly larger category. He is able to elegantly recover standard results – however, many of the objects appearing are not representable by actual cohomology groups and only exist in this larger category, which makes explicit determinations rather difficult. A middle ground between explicit calculations and a more abstract point of view can be found for example in the work of Casselman and Wigner (cf. [CW74]), comprehensively in the book of Borel and Wallach (cf. [BW00]) and with applications to locally analytic representations in the work of Kohlhaase (cf. [Koh11]). Here, injective resolutions are replaced with certain acyclic resolutions with "strictly injective" differentials or similarly, whose existence in applications is somewhat unclear. The third possibility is the one originally used by Hochschild and Serre (cf. [HS53]), where everything is explicitly done on the level of cochains.

One result one would like to recover is a version of the Hochschild-Serre spectral sequence for topologised groups. The discrete version of the theorem reads as follows:

**Theorem.** *Let $G$ be a group and $N$ a normal subgroup. Let $A$ be a $G$-module. Then there is a convergent first-quadrant spectral sequence*

$$H^p(G/N, H^q(N, A)) \Longrightarrow H^{p+q}(G, A).$$

Another important result is known as Shapiro's lemma, which in the discrete case can be stated as such:

**Theorem.** *Let $G$ be a group and $H \leq G$ a subgroup. Let $A$ be an $H$-module. Then there is an easily defined induced module $\mathrm{Ind}_G^H(A)$, on which $G$ operates, such that there are natural isomorphisms for all $n$:*

$$H^n(G, \mathrm{Ind}_G^H(A)) \cong H^n(H, A).$$

There are numerous instances in the literature where variants of these two results are used for topologised groups and coefficients – sometimes in a generality we find implausible, often in ways that turn out to be correct, but where we find the arguments insufficient, cf. e. g. [Lec12, lemma 5.26, theorem 5.27], [GR18, proposition 3.3], [Ked16, lemma 3.3], [Pot13, theorem 2.8], [BF17, remarks after definition 2.1.2].

The aim of the second and third chapter is to remedy this oversight. Our philosophy largely follows the original ideas of Hochschild and Serre and especially something they call the *direct method* (for reasons that will become painfully apparent in chapter 3). The reason for following this approach is that we can give insight into cohomology groups with explicit descriptions in terms of cochains (in contrast to the more conceptional one as laid out by Flach) and hold unconditionally (in contrast to the Casselman, Wigner, et. al. approach, where everything depends on the existence of certain resolutions.)

We first introduce a framework for topologised groups, which somewhat axiomatises the idea of *group cohomology by means of cochain complexes* in chapter 2. This is necessary as we want our proofs to hold in the most general version we can imagine, so that no-one else has to go through the tedious calculations of chapter 3 ever again. There we prove the *maximal* versions of aforementioned results we consider plausible. Specific instances of our results on topologised group cohomology are the following.

**Theorem** (theorems 3.3.2, 3.4.2, 3.5.8, 3.6.27 and 3.8.1)**.** *Cohomology theories such as continuous, L-analytic, pro-L-analytic and bounded cohomology yield a long exact sequence of cohomology groups. Shapiro's lemma, the Hochschild-Serre spectral sequence and the classification of torsors via $H^1$ all hold.*

The precise statements of course do not hold in all generality, assumptions on certain subgroups will need to be imposed. However, in practice these assumptions are easy to check. We also prove a stronger version than the Hochschild-Serre spectral sequence itself for direct products in theorem 3.7.6, where we show that the spectral sequence stems from a quasi-isomorphism to a certain double complex.

Of the aforementioned examples of cohomology theories for topological groups, we won't show applications in bounded cohomology (used in differential geometry) and continuous cohomology (with plentiful applications in number theory) – the application we are most interested in is *(pro-)analytic cohomology*. Here an analytic group over a $p$-adic field acts locally analytically on a locally convex space. The first major result might be due to Lazard (cf. [Laz65, V.(2.3.10)]), who showed that $\mathbb{Q}_p$-analytic cohomology is just continuous cohomology. For analyticity over proper extensions of $\mathbb{Q}_p$ these cohomology theories however no longer coincide. Another important result of Lazard is the fact that Lie groups over $\mathbb{Q}_p$ are Poincaré groups, i. e., their continuous cohomology groups are dual to one another (cf. [Laz65, V.(2.5.8)]).

If $G$ is a Lie group over a $p$-adic field $L$, then it is of course also a Lie group over $\mathbb{Q}_p$ and hence also satisfies duality over $\mathbb{Q}_p$. But is there is also a corresponding duality result for the $L$-analytic cohomology groups?

Our main result of chapter 4 is an affirmative answer to this.

**Theorem** (theorem 4.5.3 and corollary 4.5.4)**.** *Let $G$ be a compact Lie group over a non-archimedean field of characteristic zero of dimension $d$. Then Poincaré duality holds, i. e., for an analytic representation $V$ of $G$ with continuous dual $V'$ we have isomorphisms of analytic cohomology groups*

$$H^i(G, V') \longrightarrow H^{d-i}(G, V)',$$

*which are functorial in $V$.*

The precise statement is more complicated than this, as we also want to have duality for infinite-dimensional representations of $G$. But we then run into topological difficulties, as locally convex representations are not particularly well behaved.[2] Above statement is however unconditionally true for finite dimensional representations.

Analytic cohomology naturally turns up in the study of $(\varphi, \Gamma)$-modules. Here we however not only have an analytic group acting on the coefficients, but also a non-invertible operator $\varphi$, which necessitates to prove the results of chapter 3 for *monoids* instead of groups. This poses additional technical difficulties.

Using the results of chapters 3 and 4, we are able to extend an exact sequence due to Berger and Fourquaux in theorem 5.1.5. We end this thesis with a discussion of a possible duality in the Herr complex: Morally speaking, analytic duality should apply to the Herr complex, which would then be self-dual. In section 5.4 we discuss this possibility with a somewhat pessimistic answer: While we can construct a duality morphism, we do not expect it to be a quasi-isomorphism in general.

[2]In [Sci85], Rota is quoted as follows: "Other branches of mathematics are not so clear-cut. Functional analysis of infinite-dimensional vector spaces is never fully convincing; you don't get a feeling of having done an honest day's work."

CHAPTER 1

# Spectral Sequences for Iwasawa Adjoints

A slightly abridged version of this chapter has been published as [TV19]. The results are based on ideas of the second author as written up in [Ven15] and hence resulted in a joint publication.

The second author's contributions are easily seen in [Ven15], the differences between the following chapter and the preprint being due to us. Our contributions include: all results up to proposition 1.5.5, which include answers to questions posed in the preprint; generalising all remaining results to the derived setting (the original statements now being present as remarks); simplifying their proofs at the same time; and completely rewriting the exposition of the material.

## 1.1. Overview

Let $\mathcal{O}$ be a complete discrete valuation ring with uniformising element $\pi$ and finite residue field. Consider the ring of formal power series $R = \mathcal{O}[[X_1, \ldots, X_t]]$ in $t$ variables, which is a complete regular local ring of dimension $d = t + 1$ with maximal ideal $\mathfrak{m}$. While our results hold under more general assumptions, this is the case we are most interested in.

In this setting, we can look at four types of dualities.

First, there is Matlis duality: Denote with $\mathcal{E}$ an injective hull of $R/\mathfrak{m}$ as an $R$-module. Then $T = \operatorname{Hom}_R(-, \mathcal{E})$ induces a contravariant involutive equivalence between Noetherian and Artinian $R$-modules akin to Pontryagin duality.

Second, there is local duality: If $\mathbf{R}\Gamma_{\mathfrak{m}}$ denotes the right derivation of

$$M \longmapsto \varinjlim_k \operatorname{Hom}_R(R/\mathfrak{m}^k, M)$$

in the derived category of $R$-modules, then

$$\mathbf{R}\Gamma_{\mathfrak{m}} \cong [-d] \circ T \circ \mathbf{R}\operatorname{Hom}_R(-, R)$$

on finitely generated $R$-modules.

Third, there is Koszul duality: The complex $\mathbf{R}\Gamma_{\mathfrak{m}}$ can be computed by means of a Koszul complex $K^\bullet$ which is self-dual: $K^\bullet = \operatorname{Hom}_R(K^\bullet, R)[d]$.

Finally, there is Tate duality: Let $G$ be a pro-$p$ duality group of dimension $s$. Then for finite $p$-torsion $G$-modules $A$ we have

$$H^i(G, \operatorname{Hom}_{\mathbb{Z}}(A, I)) \cong H^{s-i}(G, A)^*$$

for a dualising module $I$. Here $-^*$ denotes the abstract dual

$$(-)^* = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}).$$

Consider $\Lambda_R(G) = \varprojlim_U R[G/U]$ where $U$ runs through the open normal subgroups of $G$ for a topological group $G$. It is well known that $\Lambda_R(\mathbb{Z}_p^s) \cong R[[Y_1, \ldots, Y_s]]$ and $R \cong \Lambda_{\mathcal{O}}(\mathbb{Z}_p^r)$. The maximal ideal of $\Lambda_R(\mathbb{Z}_p^s)$ is then generated by the regular sequence

$$(\pi, X_1, \ldots, X_t, Y_1, \ldots, Y_s)$$

and no matter how we split up this regular sequence into two, they will remain regular. The Koszul complex then gives rise to a number of interesting spectral sequences and these should (at least morally) recover the spectral sequences

$$\mathrm{Tor}_n^{\mathbb{Z}_p}(D_m(M^\vee), \mathbb{Q}_p/\mathbb{Z}_p) \Longrightarrow \mathrm{Ext}^{n+m}_{\Lambda_{\mathbb{Z}_p}(G)}(M, \Lambda_{\mathbb{Z}_p}(G))^\vee \tag{1}$$

and

$$\varinjlim_k D_n(\mathrm{Tor}_m^{\mathbb{Z}_p}(\mathbb{Z}_p/p^k, M)^\vee) \Longrightarrow \mathrm{Ext}^{n+m}_{\Lambda_{\mathbb{Z}_p}(G)}(M, \Lambda_{\mathbb{Z}_p}(G))^\vee, \tag{2}$$

which show up in Jannsen's proof of [Jan89, 2.1 and 2.2]. The functors $D_n$ stem from Tate's spectral sequence and are a corner stone in the theory of duality groups. In contrast to the abstract dual $-^*$ above, $-^\vee$ denotes the Pontryagin dual $\mathrm{Hom}_{\mathrm{cts}}(-, \mathbb{R}/\mathbb{Z})$.

This chapter is structured as follows: After briefly fixing a few conventions, we lay the ring theoretic groundwork in section 1.3, including a discussion of local cohomology. Afterwards, we compare Matlis with Pontryagin duality in section 1.4 and observe a relation between Tate cohomology and local cohomology in section 1.5. We will then show in section 1.6 that aforementioned spectral sequences (and many more) are consequences of the four duality principles laid out above. This also allows us to generalise Jannsen's spectral sequences to more general coefficients. For example, generalisations of eq. (1) and eq. (2) are subjects of proposition 1.6.6 and of proposition 1.6.10 respectively. While another spectral sequence for Iwasawa adjoints has already been generalised to more general coefficients (cf. theorem 1.8.1), the generalizations of the aforementioned spectral sequences are missing in the literature. We can even generalise an explicit calculation of Iwasawa adjoints (cf. [Jan89, corollary 2.6], [NSW08, (5.4.14)]) in theorem 1.6.15.

Furthermore, we generalise Venjakob's result on local duality for Iwasawa algebras (cf. [Ven02, theorem 5.6]) to more general coefficients (cf. theorem 1.7.2). As an application we determine the torsion submodule of local Iwasawa cohomology generalising a result of Perrin-Riou in the case $R = \mathbb{Z}_p$ in theorem 1.8.2.

## 1.2. Conventions

In this chapter, a *ring* will always be unitary and associative, but not necessarily commutative. If not explicitly stated otherwise, "module" means left-module, "Noetherian" means left-Noetherian etc.

We will furthermore use the language of derived categories. If $\mathbf{A}$ is an abelian category, we denote with $\mathbf{D}(\mathbf{A})$ the derived category of unbounded complexes, with $\mathbf{D}^+(\mathbf{A})$ the derived category of complexes bounded below, with $\mathbf{D}^-(\mathbf{A})$ the derived category of complexes bounded above and with $\mathbf{D}^b(\mathbf{A})$ the derived category of bounded complexes.

As we simultaneously have to deal with left- and right-exact functors, both covariant and contravariant, recovering spectral sequences from isomorphisms in the derived category is a bit of a hassle regarding the indices. Suppose that $\mathbf{A}$ has enough injectives and projectives and that $M$ is a (suitably bounded) complex of objects of $\mathbf{A}$. Then for a covariant functor $F\colon \mathbf{A} \longrightarrow \mathbf{A}$ we set $\mathbf{R}F(M) = F(Q)$ and $\mathbf{L}F(M) = F(P)$ with $Q$ a complex of injective objects, quasi-isomorphic to $M$ and $P$ a complex of projectives, quasi-isomorphic to $M$. If $F$ is contravariant, we set $\mathbf{L}F(M) = F(Q)$ and $\mathbf{R}F(M) = F(P)$. For indices, this implies the following: Assume that $M$ is concentrated in degree zero. Then for $F$ covariant, $\mathbf{R}F(M)$ has non-vanishing cohomology at most in non-negative degrees and $\mathbf{L}F(M)$ at most in non-positive degrees. For $F$ contravariant, it's exactly the other way around. We set $\mathbf{L}^q F(M) = H^q(\mathbf{L}F(M))$ and $\mathbf{R}^q F(M) = H^q(\mathbf{R}F(M))$. Note that with these conventions

- $\mathbf{R}^p(-)^G(M) = H^p(G, A)$
- $\mathbf{L}^q(- \otimes N)(M) = \mathrm{Tor}_{-q}(M, N)$
- $\mathbf{R}^p\,\mathrm{Hom}(-, N)(M) = \mathrm{Ext}^p(M, N)$

- $\mathbf{L}^q(\varinjlim_U (-^U)^*)(M) = \varinjlim_U H^{-q}(U,M)^*$

If $F\colon \mathbf{A} \longrightarrow \mathbf{A}$ is exact, then $F$ maps quasi-isomorphic complexes to quasi-isomorphic complexes. Its derivation $\mathbf{R}F$ (or $\mathbf{L}F$) is then given by simply applying $F$ and we will make no distinction between $F$ and $\mathbf{R}F\colon \mathbf{D}(\mathbf{A}) \longrightarrow \mathbf{D}(\mathbf{A})$ in this case.

For every integer $d \in \mathbb{Z}$ we have a *shift operator* $[d]$, so that for complexes $C$ and $n \in \mathbb{Z}$ the following holds:

$$([d](C))^n = C^{n+d}.$$

We will at times write $C[d]$ instead of $[d](C)$. Note that although we occasionally cite [Wei94], we deviate from Weibel's conventions in this regard: Our $[d]$ is Weibel's $[-d]$. We furthermore set $\mathrm{Hom}(C^\bullet, D^\bullet)$ to be the complex with entries

$$\mathrm{Hom}(C^\bullet, D^\bullet)^i = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(C^k, D^{k+i}).$$

Sign conventions won't matter in this chapter.

If $R$ is Noetherian, the category of finitely generated $R$-modules $\mathbf{f.g.}$-$R$-$\mathbf{Mod}$ is abelian. Its inclusion into the category of all $R$-modules induces equivalences $\mathbf{D}^*(\mathbf{f.g.}\text{-}R\text{-}\mathbf{Mod}) \cong \mathbf{D}^*_c(R\text{-}\mathbf{Mod})$ for $* \in \{+, b\}$ where subscript “c” means complexes with finitely generated cohomology. (The letter “c” actually stands for “coherent,” which for Noetherian rings amounts to finitely generated.)

## 1.3. A Few Facts on $R$-Modules

**1.3.1. Non-commutative rings.** Let $R$ be a ring. The intersection of all maximal left ideals coincides with the intersection of all maximal right ideals and is called the *Jacobson radical* of $R$ and is hence a two-sided ideal, denoted by $J(R)$. For every $r \in J(R)$ the element $1 - r$ then has both a left and a right inverse and the following form of Nakayama's lemma holds, cf. e. g. [Lam91, (4.22)].

**Lemma 1.3.1** (Azumaya-Krull-Nakayama)**.** *Let $M \in R$-$\mathbf{Mod}$ be a finitely generated $R$-module. If $J(R)M = M$, then $M = 0$.*

Proof. Suppose $M \neq 0$ and $m_1, \dots, m_n$ is a minimal generating system of $M$. By assumption there exist $r_1, \dots, r_n \in J(R)$ with $\sum_i r_i m_i = \sum_i m_i$, i.e.,

$$\sum_i (1 - r_i) m_i = 0.$$

Multiplying this equation with $(1 - r_n)^{-1}$ from the left, we see that $m_n$ lies in the span of $m_1, \dots, m_{n-1}$, a contradiction. ■

Recall that a ring is called local if it has a unique maximal left-ideal. This unique left ideal is then also the ring's unique maximal right-ideal and the group of two-sided units is the complement of this maximal ideal.

The following is well known and gives rise to the notion of “finitely presented” (or “compact”) objects in arbitrary categories.

**Lemma 1.3.2.** *Let $R$ be a ring and $M$ an $R$-module. Then*

$$\mathrm{Hom}_R(M, -)$$

*commutes with all direct limits if and only if $M$ is finitely presented.*

Proof. Suppose that $\mathrm{Hom}_R(M, -)$ commutes with direct limits. Let $M_i$ be a family of $R$-modules with limit $M$. Then by assumption canonically

$$\varinjlim_i \mathrm{Hom}_R(M, M_i) \cong \mathrm{Hom}_R(M, \varinjlim_i M_i) = \mathrm{Hom}_R(M, M).$$

The identity then has a preimage, represented by $\varphi\colon M \longrightarrow M_k$, i. e.,

$$M \xrightarrow{\varphi} M_k \longrightarrow M$$

is the identity and hence $M$ is a direct summand of $M_k$. But it is easy to see that if $M = F(X)/S$ with $F(X)$ the free $R$-module with basis $X$, then

$$M = \varinjlim_{\substack{X' \subseteq X \text{ finite,} \\ S' \subseteq S \cap F(X') \text{ finitely generated}}} F(X')/S'.$$

and is hence the direct limit of finitely presented modules. Therefore, $M$ is a direct summand of a finitely presented module and thus itself finitely presented.

On the other hand it is clear that for finitely generated free modules $M$ the functor $\operatorname{Hom}_R(M, -)$ commutes with all direct limits. For the general case, consider

$$R^r \longrightarrow R^d \longrightarrow M \longrightarrow 0$$

exact and an arbitrary direct system of $R$-modules $(N_i)_i$. Apply $\operatorname{Hom}_R(-, \varinjlim_i N_i)$ to the presentation of $M$. Comparing this with $\varinjlim_i \operatorname{Hom}_R(-, N_i)$ yields the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_i \operatorname{Hom}_R(M, N_i) \arrow[r] \arrow[d] & \varinjlim_i \operatorname{Hom}_R(R^d, N_i) \arrow[r] \arrow[d] & \varinjlim_i \operatorname{Hom}_R(R^r, N_i) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_R(M, \varinjlim_i N_i) \arrow[r] & \operatorname{Hom}_R(R^d, \varinjlim_i N_i) \arrow[r] & \operatorname{Hom}_R(R^r, \varinjlim_i N_i)
\end{tikzcd}$$

A diagram chase (or the five lemma if you're lazy) completes the proof. ∎

If $R$ is Noetherian, this isomorphism extends to higher Ext-groups.

**Proposition 1.3.3.** *Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module, and $(N_i)_i$ a direct system of $R$-modules. Then*

$$\operatorname{Ext}_R^q(M, \varinjlim_i N_i) \cong \varinjlim_i \operatorname{Ext}_R^q(M, N_i).$$

Proof. As $R$ is Noetherian, there exists a resolution of $M$ by finitely generated projective $R$-Modules. As furthermore $\varinjlim$ commutes with homology, lemma 1.3.2 yields the result. ∎

**Remark 1.3.4.** Recall the following subtleties: Let $R, S, T$ be rings, $N$ a $S$-$R$-bimodule and $P$ a $S$-$T$-bimodule. Then $\operatorname{Hom}_S(N, P)$ has the natural structure of an $R$-$T$-bimodule via $(rf)(n) = f(nr)$ and $(ft)(n) = f(n)t$.

Furthermore let $M$ be a $R$-left-module. Then canonically

$$\operatorname{Hom}_R(M, \operatorname{Hom}_S(N, P)) = \operatorname{Hom}_S(N \otimes_R M, P)$$

as $T$-right-modules.

**Lemma 1.3.5.** *If $P$ is an $R$-$R$-bimodule that is flat as an $R$-right-module and $Q$ an injective $R$-left-module, then $\operatorname{Hom}_R(P, Q)$ is again an injective $R$-left-module.*

Proof. $\operatorname{Hom}_R(-, \operatorname{Hom}_R(P, Q)) = \operatorname{Hom}_R(-, Q) \circ (P \otimes_R -)$ is a composition of exact functors and hence exact. ∎

**Lemma 1.3.6.** *Let $N$ be an $R$-$R$-bimodule and $M$ an $R$-left-module. Then*

$$\operatorname{Tor}_q^R(N, M) = 0$$

*if and only if*

$$\operatorname{Ext}_R^q(M, \operatorname{Hom}_R(N, Q)) = 0$$

*for all injective $R$-left-modules $Q$.*

Proof. The isomorphism of functors

$$\operatorname{Hom}_R(-,Q) \circ (N \otimes_R -) \cong \operatorname{Hom}_R(-, \operatorname{Hom}_R(N,Q))$$

yields an isomorphism

$$\operatorname{Hom}_R(-,Q) \circ (N \otimes_R^{\mathbf{L}} -) \cong \mathbf{R}\operatorname{Hom}_R(-, \operatorname{Hom}_R(N,Q))$$

in the derived category, which in turn yields

$$\operatorname{Hom}_R(\operatorname{Tor}_q^R(N,M),Q) \cong \operatorname{Ext}_R^q(M, \operatorname{Hom}_R(N,Q))$$

for all $q$. As the category of $R$-left-modules has sufficiently many injectives, this shows the proposition. ■

**Definition 1.3.7.** Let $R$ be a ring. A sequence $(r_1,\dots,r_d)$ of central elements in $R$ is called *regular*, if for each $i$ the residue class of $r_{i+1}$ in $R/(r_1,\dots,r_i)$ is not a zero-divisor.

**Definition 1.3.8.** For a regular sequence $\underline{r} = (r_1,\dots,r_d)$ we denote by $\underline{r}^{(k)}$ the sequence $(r_1^k,\dots,r_d^k)$, which is again regular (cf. e. g. [Mat86, theorem 16.1]). If $I$ is an ideal generated by a regular sequence $\underline{r}$, we will by abuse of notation refer to the ideal generated by $\underline{r}^{(k)}$ as $I^{(k)}$. Note that $I^{(k)}$ actually depends on the chosen regular sequence, which is either going to be clear from the context or arbitrary as long as chosen consistently.

**Proposition 1.3.9.** *Let $R$ be a ring and $(r_1,\dots,r_k,s_1,\dots,s_l)$ such a regular sequence that the sequence $(s_1,\dots,s_l)$ is itself regular. Let $I = (r_i)_i$ and $J = (s_i)_i$ be the ideals generated by the first and second part of the regular sequence. Then for all $q \geq 1$*

$$\operatorname{Tor}_q^R(R/I, R/J) = 0$$

*and*

$$\varprojlim_n \operatorname{Tor}_q^R(R/I^n, R/J^n) = 0.$$

Proof. Let us first show that $\operatorname{Tor}_1^R(R/I,R/J) = 0$ and then reduce to this case by induction on $l$. Consider the exact sequence of $R$-modules

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

and apply $\operatorname{Tor}_\bullet^R(R/I,-)$. As $R$ is flat,

$$\operatorname{Tor}_1^R(R/I,R/J) = \ker\left(R/I \otimes_R J \longrightarrow R/I\right) = \frac{I \cap J}{IJ}.$$

We argue by induction on $l$ that this is zero: For $l = 1$, $x \in I \cap J$ implies $x = \lambda s_1 = s_1 \lambda \in I$, so $\lambda = 0$ in $R/I$, so $\lambda \in I$ and hence $x \in IJ$. Denote with $J'$ the ideal generated by $s_1,\dots,s_{l-1}$. By induction $I \cap J' = IJ'$. Let $x \in I \cap J = I \cap (J' + s_l R)$, so $x = a + s_l b$ with $a \in J'$ and $b \in R$. Clearly $s_l b = 0$ in $R/(I + J')$, so $b \in I + J'$ by regularity of the sequence and hence

$$I \cap J = I \cap (J' + s_l R) = I \cap (J' + s_l(I + J')) = I \cap (J' + s_l I) = I \cap J' + I \cap s_l I$$

as $s_l I \subseteq I$. Now $I \cap J' + I \cap s_l I = IJ' + s_l I = I(J' + s_l R) = IJ$, and this was to be shown. We now argue by induction on $l$ that $\operatorname{Tor}_q^R(R/I,R/J) = 0$ for all $q > 0$. For $l = 1$ we have the free resolution

$$0 \longrightarrow R \xrightarrow{s_1} R \longrightarrow R/J \longrightarrow 0$$

hence $\operatorname{Tor}_q^R(R/I,R/J) = 0$ for $q > 1$ and for $q = 1$ by what we saw above. Let $J'$ be again the ideal generated by $s_1,\dots,s_{l-1}$. By induction we can assume that all $\operatorname{Tor}_q^R(R/I,R/J')$ vanish. Consider the sequence

$$0 \longrightarrow R/J' \xrightarrow{s_l} R/J' \longrightarrow R/J \longrightarrow 0,$$

which is exact as the subsequence $(s_1, \ldots, s_l)$ is regular. Applying $\mathrm{Tor}^R_\bullet(R/I, -)$ shows that $\mathrm{Tor}^R_q(R/I, R/J) = 0$ for $q > 1$ by induction hypothesis — and by what we saw above also for $q = 1$.

Let $m' = \max\{k, l\}$. Clearly $I^{m'm} \subseteq I^{(m)} \subseteq I^m$ and the same is true for $J$, hence the natural map

$$\mathrm{Tor}^R_q(R/I^{m'm}, R/J^{m'm}) \longrightarrow \mathrm{Tor}^R_q(R/I^m, R/J^m)$$

factors through $\mathrm{Tor}^R_q(R/I^{(m)}, R/J^{(m)})$, which is zero. ■

**1.3.2. The Koszul Complex.** We recall a couple of well-known facts about the Koszul complex (cf. e. g. [Wei94, section 4.5]).

**Definition 1.3.10.** Denote by

$$K_\bullet(x) = 0 \longrightarrow R \xrightarrow{\;x\;} R \longrightarrow 0$$

the chain complex (i. e., the degree decreases to the right) concentrated in degrees one and zero for a ring $R$ and a central element $x \in Z(R)$. For central elements $x_1, \ldots, x_d$ the complex

$$K_\bullet(x_1, \ldots, x_d) = K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_d)$$

is called the *Koszul complex attached to* $x_1, \ldots, x_d$. We will also consider the cochain complex $K^\bullet(x_1, \ldots, x_d)$ with entries $K^p(x_1, \ldots, x_d) = K_{-p}(x_1, \ldots, x_d)$.

**Remark 1.3.11.** While this definition is certainly elegant, a more down to earth description is given as follows: $K_p(x_1, \ldots, x_d)$ is the free $R$-module generated by the symbols $e_{i_1} \wedge \cdots \wedge e_{i_p}$ with $i_1 < \cdots < i_p$ with differential

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}.$$

This description emphasises the importance of using central elements $(x_i)_i$.

**Remark 1.3.12.** The importance of the Koszul complex for our purposes stems from the following fact: If $x_1, \ldots, x_d$ is a regular sequence, then $K_\bullet(x_1, \ldots, x_d)$ is a free resolution of $R/(x_1, \ldots, x_d)$, cf. e. g. [Wei94, corollary 4.5.5].

**Proposition 1.3.13.** *The complex* $K_\bullet = K_\bullet(x_1, \ldots, x_d)$ *is isomorphic to the complex*

$$0 \longrightarrow \mathrm{Hom}_R(K_0, R) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_R(K_d, R) \longrightarrow 0,$$

*where* $\mathrm{Hom}_R(K_0, R)$ *is in degree* $d$ *and* $\mathrm{Hom}_R(K_d, R)$ *in degree zero. Analogously*

$$K^\bullet \cong \mathrm{Hom}_R(K^\bullet, R)[d].$$

Proof. We have to describe isomorphisms $K_p \cong \mathrm{Hom}_R(K_{d-p}, R)$ such that all diagrams

$$\begin{tikzcd} K_p \arrow[r, "d"] \arrow[d, "\cong"] & K_{p-1} \arrow[d, "\cong"] \\ \mathrm{Hom}_R(K_{d-p}, R) \arrow[r, "d^*"] & \mathrm{Hom}_R(K_{d-p+1}, R) \end{tikzcd}$$

commute. Consider the map

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \longmapsto \left(e_{j_1} \wedge \cdots \wedge e_{j_{d-p}} \longmapsto \mathrm{sgn}(\sigma)\right)$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation

$$\sigma(1) = i_1, \ldots, \sigma(p) = i_p, \sigma(p+1) = j_1, \ldots, \sigma(d) = j_{d-p},$$

i. e., in the exterior algebra $\bigwedge^d R^d$ we have

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-p}} = \mathrm{sgn}(\sigma) e_1 \wedge \cdots \wedge e_d.$$

(Note that $\mathrm{sgn}(\sigma) = 0$ if $\sigma$ is not bijective.) It is then easy to verify that the diagram above does indeed commute: Identifying $R$ with $\bigwedge^d R^d$, all but at most one summand vanishes in the ensuing calculation and the difference in sign is precisely the difference in the permutations. ∎

**Proposition 1.3.14.** *Let $R$ be a ring and $x_1, \dots, x_d$ a regular sequence of central elements in $R$. Then in the bounded derived category of $R$-modules*

$$[d] \circ \mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_n), -) \cong R/(x_1, \dots, x_n) \otimes_R^{\mathbf{L}} -.$$

Proof. Denote with $K_\bullet$ the Koszul (chain) complex $K_\bullet(x_1, \dots, x_d)$ (concentrated in degrees $d, d-1, \dots, 0$) and with $K^\bullet$ the Koszul (cochain) complex (concentrated in degrees $-d, -d+1, \dots, 0$).

As $x_1, \dots, x_d$ form a regular sequence, $K^\bullet$ is a free resolution of $R/(x_1, \dots, x_n)$ and hence allows us to calculate the derived functors as follows.

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_n), M)[d] &= \mathrm{Hom}_R(K^\bullet, M)[d] \\ &\cong \mathrm{Hom}_R(K^\bullet[-d], R) \otimes_R M \\ &\cong K^\bullet \otimes_R M \\ &= R/(x_1, \dots, x_d) \otimes_R^{\mathbf{L}} M,\end{aligned}$$

with the crucial isomorphisms being due to the fact that $K^\bullet$ is a complex of free modules and proposition 1.3.13. It is clear that these isomorphisms are functorial in $M$. ∎

**Corollary 1.3.15.** *Let $R$ be a commutative ring, $x_1, \dots, x_d$ a regular sequence in $R$ and $T = \mathrm{Hom}_R(-, Q)$ with $Q$ injective. Then*

$$\mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_d), -) \circ T = T \circ [d] \circ \mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_d), -)$$

*on the derived category of $R$-modules.*

Proof. The functor $T$ is exact and

$$\mathrm{Hom}_R(R/(x_1, \dots, x_d), -) \circ T = T \circ (R/(x_1, \dots, x_d) \otimes_R -)$$

by adjointness. Hence also

$$\mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_d), -) \circ T = T \circ (R/(x_1, \dots, x_d) \otimes_R^{\mathbf{L}} -).$$

By proposition 1.3.14, this is just $T \circ [d] \circ \mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_d), -)$. ∎

**Corollary 1.3.16.** *Let $R$ be a commutative ring and $x_1, \dots, x_d$ a regular sequence in $R$. Let further $M$ be an $R$-module. Then*

$$\mathrm{Ext}_R^{d-p}(R/(x_1, \dots, x_d), M) = \mathrm{Tor}_p^R(R/(x_1, \dots, x_d), M).$$

Proof. This is just proposition 1.3.14, taking extra care of the indices:

$$\begin{aligned}\mathrm{Ext}_R^{d-p}(R/(x_1, \dots, x_d), M) &= \mathbf{R}^{d-p}\,\mathrm{Hom}_R(R/(x_1, \dots, x_d), M) \\ &= H^{-p}\,\mathbf{R}\mathrm{Hom}_R(R/(x_1, \dots, x_d)[-d], M) \\ &= H^{-p}(R/(x_1, \dots, x_d) \otimes_R^{\mathbf{L}} M) \\ &= \mathrm{Tor}_p^R(R/(x_1, \dots, x_d), M).\end{aligned}$$

∎

### 1.3.3. Local Cohomology.

**Definition 1.3.17.** Let $R$ be a ring and $\underline{J} = (J_n)_{n\in\mathbb{N}}$ a decreasing sequence of two-sided ideals. (The classical example is to take a two-sided ideal $J$ and set $\underline{J} = (J^n)_n$.) For an $R$-left-module $M$ set

$$\Gamma_{\underline{J}}(M) = \{m \in M \mid J_n m = 0 \text{ for some } n\}.$$

It is clear that $\Gamma_{\underline{J}}$ is a left-exact functor with values in $R$-**Mod**. Denote its right-derived functor in the derived category $\mathbf{D}^+(R\text{-}\mathbf{Mod})$ by $\mathbf{R}\Gamma_{\underline{J}}$.

**Remark 1.3.18.** $\Gamma_{\underline{J}} = \varinjlim_n \operatorname{Hom}_R(R/J_n, -)$, so

$$\mathbf{R}\Gamma_{\underline{J}} = \varinjlim_n \mathbf{R}\operatorname{Hom}_R(R/J_n, -)$$

and

$$\mathbf{R}^q\Gamma_{\underline{J}} = \varinjlim_n \operatorname{Ext}^q_R(R/J_n, -).$$

**Remark 1.3.19.** Let $I$ be an ideal generated by a regular sequence in a ring $R$. Then by cofinality of the systems

$$\Gamma_{(I^n)_n} = \Gamma_{(I^{(n)})_n}.$$

**Lemma 1.3.20.** *Let $\mathbf{A}, \mathbf{B}$ be abelian categories, with additive functors $L\colon \mathbf{A} \longrightarrow \mathbf{B}$ left adjoint to $R\colon \mathbf{B} \longrightarrow \mathbf{A}$. If $L$ is exact, $R$ preserves injective objects.*

Proof. Let $Q$ be an injective object in $\mathbf{B}$ and let

$$M^\bullet = 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence in $\mathbf{A}$. Apply $\operatorname{Hom}_{\mathbf{A}}(-, R(Q))$. We want to see that the resulting complex $\operatorname{Hom}_{\mathbf{A}}(M^\bullet, R(Q))$ has trivial homology. But by adjointness, this is the complex $\operatorname{Hom}_{\mathbf{B}}(F(M^\bullet), Q)$, which by assumptions on $F$ and $Q$ is exact. ∎

**Remark 1.3.21.** Let $\varphi\colon R \longrightarrow S$ be a homomorphism between unitary rings. Let $\underline{J}$ be decreasing sequence of two-sided ideals in $R$ and denote with $\underline{JS}$ the induced sequence of two-sided ideals in $S$. If $\varphi(R)$ lies in the centre of $S$, then $\Gamma_{\underline{J}} \circ \operatorname{res}_\varphi = \operatorname{res}_\varphi \circ \Gamma_{\underline{JS}}$. If furthermore injective $S$-modules are also injective as $R$-modules, e. g., if $S$ is a flat $R$-module via lemma 1.3.20, then $\mathbf{R}\Gamma_{\underline{J}} \circ \operatorname{res}_\varphi = \operatorname{res}_\varphi \circ \mathbf{R}\Gamma_{\underline{JS}}$. Local cohomology is thus independent of the base ring for flat extensions and we will omit $\operatorname{res}_\varphi$ and the distinction between $\underline{JS}$ and $\underline{J}$ in the future. Note especially that if $R$ is complete, then $R[[G]] = R[G]^\wedge$ is a flat $R$-module.

**Proposition 1.3.22.** *Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field. Let $M$ be a finitely generated $R$-module. Then $\Gamma_{\underline{\mathfrak{m}}}(M)$ is the maximal finite submodule of $M$.*

Proof. Denote with $T$ the maximal finite submodule of $M$ (which exists as $M$ is Noetherian). By Nakayama there exists a $k \in \mathbb{N}$ with $\mathfrak{m}^k T = 0$ and hence $T \subseteq \Gamma_{\underline{\mathfrak{m}}}(M)$. Conversely $R/\mathfrak{m}^k$ is a finite ring for each $k$, hence $Rm$ is a finite module for each $m \in \Gamma_{\underline{\mathfrak{m}}}(M)$ and is thus contained in $T$. ∎

**Proposition 1.3.23.** *If $R$ is a Noetherian ring and $\underline{J}$ a decreasing sequence of ideals, then $\mathbf{R}\Gamma_{\underline{J}}$ and $\mathbf{R}^q\Gamma_{\underline{J}}$ commute with direct limits.*

Proof. This is just proposition 1.3.3, as for Noetherian rings, direct limits of injective modules are again injective. ∎

**Definition 1.3.24.** For $\underline{I}$ and $\underline{J}$ decreasing sequences of two-sided ideals of a ring $R$ set $(\underline{I} + \underline{J})_n = I_n + J_n$.

**Remark 1.3.25.** If $I$ and $J$ are two-sided ideals of a ring $R$, then generally $\underline{I} + \underline{J} \neq \underline{I + J}$. But as these two families are cofinal, $\Gamma_{\underline{I}+\underline{J}} = \Gamma_{\underline{I+J}}$.

**Remark 1.3.26.** Clearly $\Gamma_{\underline{I}+\underline{J}} = \Gamma_{\underline{I}} \circ \Gamma_{\underline{J}}$, but regrettably

$$\mathbf{R}\Gamma_{\underline{I}+\underline{J}} = \mathbf{R}\Gamma_{\underline{I}}\mathbf{R}\Gamma_{\underline{J}}$$

is in general false if the families $\underline{I}$ and $\underline{J}$ are not sufficiently independent from one another: For $R = \mathbb{Z}$, $\underline{I} = \underline{J} = (n_i\mathbb{Z})_i$ any descending sequence of non-trivial ideals and $M = \mathbb{Q}/\mathbb{Z}$, the five-term-sequence in cohomology would start as follows:

$$\begin{array}{ccccc} 0 & \longrightarrow & \varinjlim_{i,j} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \operatorname{Hom}(\mathbb{Z}/n_j, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & \varinjlim_i \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \mathbb{Q}/\mathbb{Z}) \\ & & & & \downarrow \\ & & & & \cdots \end{array}$$

But clearly $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \mathbb{Q}/\mathbb{Z}) = 0$ and

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \operatorname{Hom}(\mathbb{Z}/n_j, \mathbb{Q}/\mathbb{Z})) = \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \mathbb{Z}/n_j) = \mathbb{Z}/(n_i, n_j),$$

hence

$$\varinjlim_{i,j} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n_i, \operatorname{Hom}(\mathbb{Z}/n_j, \mathbb{Q}/\mathbb{Z})) = \varinjlim_i \mathbb{Z}/n_i,$$

so the sequence above cannot possibly be exact.

This argument of course generalises: Were $\mathbf{R}\Gamma_{\underline{I}+\underline{J}} = \mathbf{R}\Gamma_{\underline{I}}\mathbf{R}\Gamma_{\underline{J}}$, then [CE56, chapter XV, theorem 5.12] implied that

$$\varinjlim_i \operatorname{Ext}^p_R(R/I_i, \varinjlim_j \operatorname{Hom}_R(R/J_j, Q)) = 0$$

for all $p > 0$ and $Q$ injective, i. e., if the isomorphism in the derived category holds, then because $\mathbf{R}\Gamma_{\underline{J}}$ mapped injective objects to $\Gamma_{\underline{I}}$-acyclics. Using lemma 1.3.6, a sufficient criterion for that to happen is that the transition maps eventually factor through $\operatorname{Ext}^p_R(R/\widetilde{I}_i, \operatorname{Hom}_R(R/\widetilde{J}_j, Q))$ for some $\widetilde{I}_i, \widetilde{J}_j$ with $\operatorname{Tor}^R_p(R/\widetilde{I}_i, R/\widetilde{J}_j) = 0$ for all $p > 0$ and this criterion appears to be close to optimal. The following proposition is a simple application of this principle.

**Proposition 1.3.27.** *Let $R$ be a commutative ring and*

$$(r_1, \ldots, r_k, s_1, \ldots, s_l)$$

*such a regular sequence, that $(s_1, \ldots, s_l)$ is itself again regular. Then for the ideals $I = (r_i)_i$ and $J = (s_i)_i$ we have*

$$\mathbf{R}\Gamma_{\underline{I}+\underline{J}} = \mathbf{R}\Gamma_{\underline{I}}\mathbf{R}\Gamma_{\underline{J}} = \mathbf{R}\Gamma_{\underline{J}}\mathbf{R}\Gamma_{\underline{I}}.$$

Proof. The transition maps in the system

$$\varinjlim_i \operatorname{Ext}^p_R(R/I^i, \varinjlim_j \operatorname{Hom}_R(R/J^j, Q)) = \varinjlim_{i,j} \operatorname{Ext}^p_R(R/I^i, \operatorname{Hom}_R(R/J^j, Q))$$

eventually factor through $\operatorname{Ext}^p_R(R/I^{(n)}, \operatorname{Hom}_R(R/J^{(n)}, Q))$. But by lemma 1.3.6 and proposition 1.3.9 this vanishes for $p \geq 1$. As $\operatorname{Tor}^R_\bullet(-,-)$ is symmetrical for commutative rings, the same argument also applies for $\mathbf{R}\Gamma_{\underline{J}}\mathbf{R}\Gamma_{\underline{I}}$. ■

## 1.4. (Avoiding) Matlis Duality

First recall Pontryagin duality.

**Theorem 1.4.1** (Pontryagin duality, e. g. [NSW08, (1.1.11)]). *The functor*

$$\Pi = \mathrm{Hom}_{\mathrm{cts}}(-, \mathbb{R}/\mathbb{Z})$$

*induces a contravariant auto-equivalence on the category of locally compact (Hausdorff) abelian groups. It interchanges compact with discrete groups. The isomorphism* $A \longrightarrow \Pi(\Pi(A))$ *is given by* $a \longmapsto (\varphi \longmapsto \varphi(a))$.
*If* $G$ *is pro-*$p$*, then* $\Pi(G) = \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{Q}_p/\mathbb{Z}_p)$*. If* $D$ *is a discrete torsion group or a topologically finitely generated profinite group, then* $\Pi(D) = \mathrm{Hom}_{\mathbb{Z}}(D, \mathbb{Q}/\mathbb{Z})$*.*
*We will write* $-^\vee$ *for* $\Pi$ *if it is notationally more convenient.*

Matlis duality is commonly stated as follows:

**Theorem 1.4.2** (Matlis duality, [BH93, theorem 3.2.13]). *Let* $R$ *be a complete Noetherian commutative local ring with maximal ideal* $\mathfrak{m}$ *and* $\mathcal{E}$ *a fixed injective hull of the* $R$*-module* $R/\mathfrak{m}$*. Then* $\mathrm{Hom}_R(-, \mathcal{E})$ *induces an equivalence between the finitely generated modules and the Artinian modules with inverse* $\mathrm{Hom}_R(-, \mathcal{E})$*.*

**Example 1.4.3.** If $R$ is a discrete valuation ring, then $Q(R)/R$ is an injective hull of its residue field.

Matlis duality – using an *abstract* dualising module instead of a topological one – behaves very nicely in relation to local cohomology. In applications however the Matlis module $\mathcal{E}$ is cumbersome and in general not particularly easy to construct.

**Example 1.4.4.** Consider the rings $R = \mathbb{Z}_p$, $S_1 = \mathbb{Z}_p[\pi]$ and $S_2 = \mathbb{Z}_p[[T]]$ with $\pi$ a uniformiser of $\mathbb{Q}_p(\sqrt{p})$. Clearly the homomorphisms $R \longrightarrow S_i$ are local and flat and their respective residue fields agree. But while $\mathcal{E}_R = \mathbb{Q}_p/\mathbb{Z}_p$, $\mathcal{E}_{S_1} \cong \mathbb{Q}_p/\mathbb{Z}_p^{\oplus 2}$ as an abelian group. Furthermore, $\mathcal{E}_{S_2} \cong \bigoplus_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p$ as an abelian group by proposition 1.4.8.

The best we can hope for in general is the following.

**Proposition 1.4.5** ([Stacks, Tag 08Z5]). *Let* $R \longrightarrow S$ *be a flat and local homomorphism between Noetherian local rings with respective maximal ideals* $\mathfrak{m}$ *and* $\mathfrak{M}$*. Assume that* $R/\mathfrak{m}^n \cong S/\mathfrak{M}^n$ *for all* $n$*. Then an injective hull of* $S/\mathfrak{M}$ *as an* $S$*-module is also an injective hull of* $R/\mathfrak{m}$ *as an* $R$*-module.*

Starting with pro-$p$ local rings, Matlis modules are however intimately connected with Pontryagin duality.

**Lemma 1.4.6.** *Let* $R$ *be a pro-*$p$ *local ring with maximal ideal* $\mathfrak{m}$*. Then there exists an isomorphism of* $R$*-modules* $R/\mathfrak{m} \cong \mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}, \mathbb{Q}_p/\mathbb{Z}_p)$*.*

Proof. As $R/\mathfrak{m}$ is finite and hence a commutative field, both objects are isomorphic as abelian groups. As vector spaces of the same finite dimension over $R/\mathfrak{m}$ they are hence isomorphic as $R/\mathfrak{m}$-modules and thus as $R$-modules. ∎

**Lemma 1.4.7.** *Let* $R$ *be a pro-*$p$ *local ring with maximal ideal* $\mathfrak{m}$ *and* $M$ *a finitely presented or a discrete* $R$*-module. Then* $\Pi(M) = \mathrm{Hom}_R(M, \Pi(R))$*.*

Proof. Let first $M = \varinjlim_i M_i$ be an arbitrary direct limit of finitely presented $R$-modules. Then by lemma 1.3.2

$$\begin{aligned}\mathrm{Hom}_R(\varinjlim_i M_i, \Pi(R)) &= \varprojlim_i \mathrm{Hom}_R(M_i, \varinjlim_k \Pi(R/\mathfrak{m}^k)) \\ &= \varprojlim_i \varinjlim_k \mathrm{Hom}_R(M_i, \mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}^k, \mathbb{Q}_p/\mathbb{Z}_p)) \\ &= \varprojlim_i \varinjlim_k \mathrm{Hom}_{\mathbb{Z}_p}(M_i/\mathfrak{m}^k, \mathbb{Q}_p/\mathbb{Z}_p) \\ &= \varprojlim_i \Pi(M_i).\end{aligned}$$

If $M$ itself is finitely presented, this shows the proposition. If $M$ is discrete, we can take the $M_i$ to be discrete and finitely presented (i. e., finite). The projective limit of their duals exists in the category of compact $R$-modules and it follows that $\varprojlim_i \Pi(M_i) = \Pi(M)$. ∎

**Proposition 1.4.8.** *Let $R$ be a Noetherian pro-$p$ local ring with maximal ideal $\mathfrak{m}$. Then $\Pi(R) = \mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$ is an injective hull of $R/\mathfrak{m}$ as an $R$-module.*

Proof. $\Pi(R)$ is injective as an abstract $R$-module: By Baer's criterion it suffices to show that $\mathrm{Hom}_R(R, \Pi(R)) \longrightarrow \mathrm{Hom}_R(I, \Pi(R))$ is surjective for every (left-)ideal $I$ of $R$. By lemma 1.4.7, this is equivalent to the surjectivity of $\Pi(R) \longrightarrow \Pi(I)$, which is clear. In lieu of lemma 1.4.6 it hence suffices to show that

$$\mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}, \mathbb{Q}_p/\mathbb{Z}_p) \subseteq \mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$$

is an essential extension, so take

$$H \leq \mathrm{Hom}_{\mathrm{cts}}(R, \mathbb{Q}_p/\mathbb{Z}_p)$$

an $R$-submodule and $0 \neq f \in H$. Then by continuity, $f$ descends to

$$f\colon R/\mathfrak{m}^{k+1} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

with $k$ minimal. It follows that there exists an element $r \in \mathfrak{m}^k$ with $f(r) \neq 0$. The map $rf\colon R \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ is consequentially also not zero, lies in $H$ but now descends to

$$rf\colon R/\mathfrak{m} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

i. e., $H \cap \mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}, \mathbb{Q}_p/\mathbb{Z}_p) \neq 0$. ∎

**Corollary 1.4.9.** *Let $R$ be a commutative pro-$p$ Noetherian commutative local ring. Then if $M$ is finitely generated or Artinian, Matlis and Pontryagin duality agree.*

Proof. Immediate from lemma 1.4.7 and proposition 1.4.8. ∎

**Proposition 1.4.10.** *Let $R$ satisfy Matlis duality via*

$$T = \mathrm{Hom}_R(-, \mathcal{E}).$$

*Let $\underline{I}$ be a decreasing family of ideals generated by regular sequences of length $d$. Then*

$$\mathbf{R}\Gamma_{\underline{I}} = \varinjlim_n T \circ [d] \circ \mathbf{R}\mathrm{Hom}(R/I_n, -) \circ T$$

*on $\mathbf{D}^+_c(R\text{-}\mathbf{Mod})$.*

Proof. By corollary 1.3.15 it follows that

$$\mathbf{R}\Gamma_{\underline{I}} = \varinjlim_n \mathbf{R}\mathrm{Hom}(R/I_n, -) \circ T \circ T = \varinjlim_n T \circ [d] \circ \mathbf{R}\mathrm{Hom}(R/I_n, -) \circ T.$$

∎

## 1.5. Tate Duality and Local Cohomology

**Remark 1.5.1.** Working in the derived category makes a number of subtleties more explicit than working only with cohomology groups. Assume that $R$ is a complete local commutative Noetherian ring with finite residue field of characteristic $p$ and $G$ an analytic pro-$p$ group. Then every $\Lambda = R[[G]]$-module has a natural topology via the filtration of augmentation ideals of $\Lambda$. It is furthermore obvious to consider the following two categories:

- $\mathcal{C}(\Lambda)$, the category of compact $\Lambda$-modules (with continuous $G$-action),
- $\mathcal{D}(\Lambda)$, the category of discrete $\Lambda$-modules (with continuous $G$-action).

Pontryagin duality then induces equivalences between $\mathcal{C}(\Lambda)$ and $\mathcal{D}(\Lambda^\circ)$, where $-^\circ$ denotes the opposite ring. It is furthermore well-known that both categories are abelian, that $\mathcal{C}(\Lambda)$ has exact projective limits and enough projectives and analogously that $\mathcal{D}(\Lambda)$ has exact direct limits and enough injectives, cf. e. g. [RZ00, chapter 5]. It is important to note that the notion of continuous $\Lambda$-homomorphisms and abstract ones often coincides: If $M$ is finitely generated with the quotient topology and $N$ is either compact or discrete, every $\Lambda$-homomorphism $M \longrightarrow N$ is continuous, cf. [Lim12, lemma 3.1.4].

In what follows we want to compare Tate cohomology, i. e. $\mathbf{L}D$ as defined below, with other cohomology theories such as local cohomology. Now Tate cohomology is defined on the category of *discrete* $G$-modules and we hence have a contravariant functor

$$\mathbf{L}D \colon \mathbf{D}^+(\mathcal{D}(\Lambda)) \longrightarrow \mathbf{D}^-(\Lambda^\circ\text{-}\mathbf{Mod})$$

Local cohomology on the other hand is defined on $\mathbf{D}^+(\Lambda\text{-}\mathbf{Mod})$ or any subcategory that contains sufficiently many acyclic (e. g. injective) modules. This is not necessarily satisfied for $\mathbf{D}^-(\mathcal{C}(\Lambda))$. A statement such as

$$\mathbf{L}D \circ \Pi = [d] \circ \mathbf{R}\Gamma_{\underline{I}}$$

without further context hence does not make much sense: The implication would be that this would be an isomorphism of functors defined on $\mathbf{D}^b(\mathcal{C}(\Lambda))$, but $\mathbf{R}\Gamma_{\underline{I}}$ doesn't exist on $\mathbf{D}^b(\mathcal{C}(\Lambda))$.

**Definition 1.5.2.** Let $G$ be a profinite group and $A$ a discrete $G$-module. Denote with $D$ the functor

$$D \colon A \longmapsto \varinjlim_U (A^U)^*$$

where $N^* = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$, the limit runs over the open normal subgroups of $G$ with the dual of the corestriction being the transition maps (cf. [NSW08, II.5 and III.4]). $D$ is right exact and contravariant and $D(A)$ has a continuous action of $G$ *from the right.* Denote its left derivation in the derived category of discrete $G$-modules by

$$\mathbf{L}D \colon \mathbf{D}^+(\mathcal{D}(\widehat{\mathbb{Z}}[[G]])) \longrightarrow \mathbf{D}^-(\widehat{\mathbb{Z}}[[G]]^\circ\text{-}\mathbf{Mod})$$

(where $-^\circ$ denotes the opposite ring), so

$$D_i(A) = \mathbf{L}^{-i}D(A) = \varinjlim_U H^i(U, A)^*.$$

If $G$ is a profinite group, $R$ a profinite ring and $A$ a discrete $R[[G]]$-module, then $D(A)$ is again an $R[[G]]$-module, so

$$\mathbf{L}D \colon \mathbf{D}^+(\mathcal{D}(R[[G]])) \longrightarrow \mathbf{D}^-(R[[G]]^\circ\text{-}\mathbf{Mod}),$$

where $\mathcal{D}(R[[G]])$ denotes the category of discrete $R[[G]]$-modules. Furthermore, we can of course also look at the functor

$$\mathbf{L}D \colon \mathbf{D}^+(R[[G]]\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^-(R[[G]]^\circ\text{-}\mathbf{Mod}).$$

Naturally, these functors don't necessarily coincide.

**Proposition 1.5.3.** *Let $R$ be such a profinite ring, that the structure morphism $\widehat{\mathbb{Z}} \longrightarrow R$ gives it the structure of a finitely presented flat $\widehat{\mathbb{Z}}$-module. Let $G$ be a profinite group such that $R[[G]]$ is a Noetherian local ring with finite residue field. (This is the case if $G$ is a $p$-adic analytic group and $R$ is the valuation ring of a finite extension over $\mathbb{Z}_p$.)*
*Then an injective discrete $R[[G]]$-module is an injective discrete $G$-module.*

Proof. By lemma 1.3.20 it suffices to show that $\text{¿}\colon \mathcal{D}(R[[G]]) \longrightarrow \mathcal{D}(\widehat{\mathbb{Z}}[[G]])$ has an exact left adjoint. It is clear that

$$M \longmapsto R[[G]] \otimes_{\widehat{\mathbb{Z}}[[G]]} M$$

is an algebraic exact left adjoint, so it remains to show that $R \otimes_{\widehat{\mathbb{Z}}} M = R[[G]] \otimes_{\widehat{\mathbb{Z}}[[G]]} M$ is a discrete $R[[G]]$-module. Now $M$ is the direct limit of finite modules, hence so is $R \otimes_{\widehat{\mathbb{Z}}} M$. But for a finite $R[[G]]$-module $N$ this is clear as then $\mathfrak{m}^k M_i = 0$ for some $k$ with $\mathfrak{m}$ the maximal ideal of $R[[G]]$ by Nakayama. ∎

**Corollary 1.5.4.** *The following diagram commutes if $R$ is a finitely presented flat $\widehat{\mathbb{Z}}$-module with $R[[G]]$ Noetherian and local with finite residue field:*

$$\begin{tikzcd}
\mathbf{D}^+(\mathcal{D}(R[[G]])) \arrow[r, "¿"] \arrow[d, "\mathbf{L}D"] & \mathbf{D}^+(\mathcal{D}(\widehat{\mathbb{Z}}[[G]])) \arrow[d, "\mathbf{L}D"] \\
\mathbf{D}^-(R[[G]]^\circ\text{-}\mathbf{Mod}) \arrow[r, "¿"] & \mathbf{D}^-(\widehat{\mathbb{Z}}[[G]]^\circ\text{-}\mathbf{Mod})
\end{tikzcd}$$

Proof. Clearly the forgetful functors and $D$ all commute on the level of categories of modules. The result then follows from proposition 1.5.3. ∎

**Proposition 1.5.5** ([Lim12, corollary 3.1.6, proposition 3.1.8])**.** *Let $M, N$ be $\Lambda$-modules.*

1. *If $M$ is Artinian, then $\Lambda \times M \longrightarrow M$ is continuous if we give $M$ the discrete topology.*
2. *If $N$ is Noetherian, then $N$ is compact if we give it the topology induced by $\Lambda$.*
3. *The functor*
$$\mathbf{f.g.}\text{-}\Lambda\text{-}\mathbf{Mod} \longrightarrow C(\Lambda)$$
*maps projective objects to projectives.*

**Proposition 1.5.6.** *Let $\mathcal{O}$ be a pro-$p$ discrete valuation ring, $R = \mathcal{O}[[X_1, \dots, X_t]]$ with maximal ideal $\mathfrak{m}$, $G = \mathbb{Z}_p^d$ and $\Lambda = \varprojlim_i R[G/G^{p^i}]$ with generalised augmentation ideals $I_i = \ker \Lambda \longrightarrow R[G/G^{p^i}]$. Then the following holds in $\mathbf{D}_c^b(\Lambda\text{-}\mathbf{Mod})$:*

$$\mathbf{L}D \circ T \cong [d] \circ \mathbf{R}\Gamma_{\underline{I}},$$

*Especially the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{D}_c^b(\Lambda\text{-}\mathbf{Mod}) \arrow[d, hook] & \mathbf{D}^b(\mathbf{f.g.}\text{-}\Lambda\text{-}\mathbf{Mod}) \arrow[l, "\cong"'] \arrow[r] & \mathbf{D}^b(C(\Lambda)) \arrow[r, "T"] & \mathbf{D}^b(\mathcal{D}(\Lambda^\circ)) \arrow[d, "\mathbf{L}D"] \\
\mathbf{D}^b(\Lambda\text{-}\mathbf{Mod}) \arrow[rrr, "{[d]\circ\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}}"] & & & \mathbf{D}^b(\Lambda\text{-}\mathbf{Mod})
\end{tikzcd}$$

Proof. $\Lambda$ is a regular local ring with maximal ideal generated by $(\pi, X_1, \dots, X_t, \gamma_1 - 1, \dots, \gamma_d - 1)$ for any set of topological generators $(\gamma_i)_i$ of $G$ and uniformiser $\pi$ of $\mathcal{O}$. One immediately verifies that the sequences $\gamma_1^{p^i} - 1, \dots, \gamma_d^{p^i} - 1$ are again regular and generate the ideals $I_i$.
By proposition 1.4.10,

$$[d] \circ \mathbf{R}\Gamma_{\underline{I}} = \varinjlim_n T \circ \mathbf{R}\mathrm{Hom}_\Lambda(\Lambda/I_n, -) \circ T.$$

Take a bounded complex $M$ of finitely generated $R$-modules that is quasi-isomorphic to a bounded complex $P$ of finitely generated projective modules. The resulting complex $T(P)$ is then not only a bounded complex of injective discrete modules by Pontryagin duality and proposition 1.5.5, but also a bounded complex of injective abstract $\Lambda$-modules by lemma 1.3.5. In all relevant derived categories $T(M) \cong T(P)$ holds. As $\mathrm{Hom}_\Lambda(\Lambda/I_n, -) = (-)^{G^{p^n}}$ by construction, we can compute $[d] \circ \mathbf{R}\Gamma_{\underline{I}}(M)$ as follows (keeping corollary 1.4.9 in mind):

$$\begin{aligned}
[d] \circ \mathbf{R}\Gamma_{\underline{I}}(M) &= [d] \circ \mathbf{R}\Gamma_{\underline{I}}(P) \\
&= \varinjlim_n T \circ \mathbf{R}\mathrm{Hom}_\Lambda(\Lambda/I_n, -) \circ T(P) \\
&= \varinjlim_n T(\mathrm{Hom}_\Lambda(\Lambda/I_n, T(P))) \\
&= \varinjlim_n T(T(P)^{G^{p^n}}) \\
&= \varinjlim_n \Pi(T(P)^{G^{p^n}}) \\
&= D(T(P)) = \mathbf{L}D \circ T(M).
\end{aligned}$$

∎

**Lemma 1.5.7.** *Let $R$ be a commutative Noetherian ring with unit and $R \longrightarrow S$ a flat ring extension with $R$ contained in the centre of $S$ and $S$ again (left-)Noetherian.*
*Then*

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^-(R\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^+(R\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^+(R\text{-}\mathbf{Mod})$$

*extends to*

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^-(R\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^+(S\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^+(S\text{-}\mathbf{Mod}),$$

*which in turn restricts to*

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^b_c(R\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^b_c(S\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^b_c(S\text{-}\mathbf{Mod}),$$

Proof. First note that if $M$ is an $R$-left-module and $N$ an $S$-left-module, then the abelian group $\mathrm{Hom}_R(M, N)$ carries the structure of an $S$-left-module via $(sf)(m) = sf(m)$. Then $\mathrm{Hom}_R(R, S) \cong S$ as $S$-left-modules and the following diagram commutes:

$$\begin{tikzcd}
S\text{-}\mathbf{Mod} \arrow[r, "\mathrm{Hom}_R(M,-)"] \arrow[d, "¿"] & S\text{-}\mathbf{Mod} \arrow[d, "¿"] \\
R\text{-}\mathbf{Mod} \arrow[r, "\mathrm{Hom}_R(M,-)"] & R\text{-}\mathbf{Mod}
\end{tikzcd}$$

As $S$ is a flat $R$-module, ¿ preserves injectives by lemma 1.3.20 and we can compute $\mathbf{R}\mathrm{Hom}_R(M, -)$ in either category.

If furthermore $M$ is a finitely generated $R$-module and $N$ a finitely generated $S$-module, then $\mathrm{Hom}_R(M, N)$ is again a finitely generated $S$-module, as $S$ is left-Noetherian. If $M$ is a bounded complex of finitely generated $R$-modules, then it is quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules. The result then follows at once. ∎

**Remark 1.5.8.** Note however that $\mathbf{R}\mathrm{Hom}_R$ does *not* extend to a functor

$$\mathbf{R}\mathrm{Hom}_R \colon \mathbf{D}^-(S\text{-}\mathbf{Mod})^{\mathrm{opp}} \times \mathbf{D}^+(R\text{-}\mathbf{Mod}) \longrightarrow \mathbf{D}^+(S\text{-}\mathbf{Mod}).$$

Even in those cases where we can give $\mathrm{Hom}_R(M, A)$ the structure of an $S$-module (e. g. when $S$ has a Hopf structure with antipode $s \longmapsto \bar{s}$ via $(sf)(m) = f(\bar{s}m)$), projective $S$-modules in general are not projective. This is specially true for $R[[G]]$, which is a flat, but generally not a projective $R$-module.

An essential ingredient in the proof of this section's main theorem is Grothendieck local duality. It is commonly stated as follows:

**Theorem 1.5.9** (Local duality, [Har66, theorems V.6.2, V.9.1])**.** *Let $R$ be a commutative regular local ring of dimension $d$ with maximal ideal $\mathfrak{m}$, and $\mathcal{E}$ a fixed injective hull of the $R$-module $R/\mathfrak{m}$. Denote with $R[d]$ the complex concentrated in degree $-d$ with entry $R$. Then*

$$\mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \cong T \circ \mathbf{R}\mathrm{Hom}_R(-, R[d]) = [-d] \circ T \circ \mathbf{R}\mathrm{Hom}_R(-, R)$$

*on $\mathbf{D}^b_c(R\text{-}\mathbf{Mod})$.*

The regularity assumption on $R$ can be weakened if one is willing to deal with a dualising complex that is not concentrated in just one degree (loc. cit.). Relaxing commutativity however is more subtle and will be the focus of section 1.7.

**Theorem 1.5.10.** *Let $\mathcal{O}$ be a pro-$p$ discrete valuation ring, $R = \mathcal{O}[[X_1, \ldots, X_t]]$ with maximal ideal $\mathfrak{m}$, $G = \mathbb{Z}_p^s$ and $\Lambda = R[[G]]$. Then*

$$T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda) \cong [t+1] \circ \mathbf{R}\Gamma_{\mathfrak{m}} \circ \mathbf{L}D \circ T$$

*on $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$. The right hand side can furthermore be expressed as*

$$\mathbf{R}\Gamma_{\mathfrak{m}} \circ \mathbf{L}D \circ T \cong \varinjlim_k \mathbf{L}D \circ T \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -).$$

Proof. $\Lambda$ is again a regular local ring, now of dimension $t+s+1$. Denote its maximal ideal by $\mathfrak{M}$. By theorem 1.5.9

$$\mathbf{R}\Gamma_{\underline{\mathfrak{M}}} \cong T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda[s+t+1]) = [-s-t-1] \circ T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda).$$

Now $\mathfrak{M} = \mathfrak{m} + (\gamma_1 - 1, \ldots, \gamma_s - 1)$ and if $x_1, \ldots, x_{t+1}$ is a regular sequence in $R$, then $x_1, \ldots, x_{t+1}, \gamma_1 - 1, \ldots, \gamma_s - 1$ is a regular sequence in $\Lambda$. Furthermore, the sequence $\gamma_1 - 1, \ldots, \gamma_s - 1$ is of course itself regular in $\Lambda$. Let it generate the ideal $I$. Then we can apply proposition 1.3.27, i. e.,

$$\mathbf{R}\Gamma_{\mathfrak{M}} \cong \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ \mathbf{R}\Gamma_{\underline{I}}.$$

By proposition 1.5.6, we have $\mathbf{R}\Gamma_{\underline{I}} = [-s] \circ \mathbf{L}D \circ T$, which shows the first isomorphism.

Consider furthermore the functor $\varinjlim_k \mathbf{L}D \circ T \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -)$. By lemma 1.5.7 we can compute it on $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$ as

$$\begin{aligned}
\varinjlim_k \mathbf{L}D \circ T \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -) &\cong \varinjlim_k [s] \circ \mathbf{R}\Gamma_{\underline{I}} \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -) \\
&\cong [s] \circ \mathbf{R}\Gamma_{\underline{I}} \circ \varinjlim_k \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^k, -) \\
&= [s] \circ \mathbf{R}\Gamma_{\underline{I}} \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \\
&\cong [s] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ \mathbf{R}\Gamma_{\underline{I}} \\
&\cong [s] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ [-s] \circ \mathbf{L}D \circ T \\
&= \mathbf{R}\Gamma_{\underline{\mathfrak{m}}} \circ \mathbf{L}D \circ T,
\end{aligned}$$

as by proposition 1.3.23, local cohomology commutes with direct limits, $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$ and $\mathbf{R}\Gamma_{\underline{I}}$ commute by proposition 1.3.27, and by two applications of proposition 1.5.6. ∎

**Remark 1.5.11.** Proposition 1.5.6 and theorem 1.5.10 should together be compared with the duality diagram [Nek06, (0.4.4)].

**Remark 1.5.12.** If we express theorem 1.5.10 in terms of a spectral sequence, it looks like this:

$$\varinjlim_k \mathbf{L}^p D(T(\mathrm{Ext}^q_R(R/\mathfrak{m}^k, M))) \Longrightarrow T(\mathrm{Ext}^{t+1-p-q}_\Lambda(M, \Lambda)).$$

Writing $\mathrm{E}^\bullet_\Lambda$ for $\mathrm{Ext}^\bullet_\Lambda(-,\Lambda)$, flipping the sign of $p$ and shifting $q \longmapsto t+1-q$ then yields

$$\varinjlim_k D_p(\mathrm{Ext}_R^{t+1-q}(R/\mathfrak{m}^k, M)^\vee) \Longrightarrow \mathrm{E}_\Lambda^{p+q}(M)^\vee$$

and the following exact five term sequence:

$$\begin{tikzcd} \mathrm{E}^2_\Lambda(M)^\vee \arrow[r] & \varinjlim_k D_2(\mathrm{Ext}_R^{t+1}(R/\mathfrak{m}^k, M)^\vee) \arrow[r] & \varinjlim_k D(\mathrm{Ext}_R^{t}(R/\mathfrak{m}^k, M)^\vee) \arrow[dll, out=-10, in=170] \\ \mathrm{E}^1_\Lambda(M)^\vee \arrow[r] & \varinjlim_k D_1(\mathrm{Ext}_R^{t+1}(R/\mathfrak{m}^k, M)^\vee) \arrow[r] & 0 \end{tikzcd}$$

## 1.6. Iwasawa Adjoints

In this section let $R$ be a pro-$p$ commutative local ring with maximal ideal $\mathfrak{m}$ and residue field of characteristic $p$. Let $G$ be a compact $p$-adic Lie group and

$$\Lambda = \Lambda(G) = \varprojlim_U R[[G/U]],$$

where $U$ ranges over the open normal subgroups of $G$. As is customary, we set again $\mathrm{E}^\bullet_\Lambda(M) = \mathrm{Ext}^\bullet_\Lambda(M, \Lambda)$.

**Remark 1.6.1.** Note that if $M$ is a left $\Lambda$-module, it also has an operation of $\Lambda$ from the right given by $mg = g^{-1}m$. This of course does not give $M$ the structure of a $\Lambda$-bimodule, as the actions are not compatible. We can however still give $\mathrm{Hom}_\Lambda(M, \Lambda)$ the structure of a left $\Lambda$-module by $(g.\varphi)(m) = \varphi(m)g^{-1}$.

The following lemma is based on an observation in the proof of [Jan89, theorem 2.1].

**Lemma 1.6.2.** $\mathrm{E}^0_\Lambda(M) = \varprojlim_U \mathrm{Hom}_R(M_U, R)$ *for finitely generated $\Lambda$-modules $M$, where the transition map for a pair of open normal subgroups $U \le V$ are given by the dual of the trace map*

$$M_V \longrightarrow M_U, m \longmapsto \sum_{g\in V/U} gm.$$

Proof. Note first that as $\mathrm{Hom}_\Lambda(M,-)$ commutes with projective limits,

$$\mathrm{Hom}_\Lambda(M,\Lambda) = \varprojlim_U \mathrm{Hom}_\Lambda(M, R[G/U]) = \varprojlim_U \mathrm{Hom}_{R[G/U]}(M_U, R[G/U]).$$

For $U$ an open normal subgroups of $G$, consider the trace map

$$\begin{aligned} \mathrm{Hom}_R(M_U, R) &\longrightarrow \mathrm{Hom}_{R[G/U]}(M_U, R[G/U]) \\ \varphi &\longmapsto \left( m \longmapsto \sum_{g\in G/U} \varphi(g^{-1}m)\cdot g \right), \end{aligned}$$

which is clearly an isomorphism of $R$-modules and induces the required isomorphism to the projective system mentioned in the proposition. ∎

**Proposition 1.6.3.** *On* $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$ *we have*

$$\Pi \circ \mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda) \cong \varinjlim_U \Pi \circ \mathbf{R}\mathrm{Hom}_R(-,R) \circ \mathbf{L}(-)_U.$$

Proof. Immediate by lemma 1.6.2, as $(-)_U$ clearly maps finitely generated free $\Lambda$-modules to finitely generated free $R$-modules. ∎

**Remark 1.6.4.** The spectral sequence attached to proposition 1.6.3 looks like this:

$$\varinjlim_U \operatorname{Ext}^p_R(H_q(U,M),R)^\vee \Longrightarrow \mathrm{E}^{p+q}_\Lambda(M)^\vee$$

Its five term exact sequence is given by

$$\begin{tikzcd} \mathrm{E}^2_\Lambda(M)^\vee \arrow[r] & \varinjlim_U \operatorname{Ext}^2_R(M_U,R)^\vee \arrow[r] & \varinjlim_U \operatorname{Hom}_R(H_1(U,M),R)^\vee \arrow[dll, out=-10, in=170] \\ \mathrm{E}^1_\Lambda(M)^\vee \arrow[r] & \varinjlim_U \operatorname{Ext}^1_R(M_U,R)^\vee \arrow[r] & 0 \end{tikzcd}$$

The following lemma is also based on an observation in the proof of [Jan89, theorem 2.1].

**Lemma 1.6.5.** $\operatorname{Hom}_\Lambda(M,\Lambda)^\vee \cong \varinjlim_U R^\vee \otimes_R M_U$ *for finitely generated* $\Lambda$*-modules* $M$.

PROOF.

$$\begin{aligned} \operatorname{Hom}_\Lambda(M,\Lambda)^\vee &\cong \varinjlim_U \Pi(\operatorname{Hom}_R(M_U,R)) \\ &\cong \varinjlim_U \Pi \operatorname{Hom}_R(\Pi(R),\Pi(M_U)) \\ &\cong \Pi\circ\Pi(\varinjlim_U M_U \otimes_R \Pi(R)) \\ &\cong \varinjlim M_U \otimes_R R^\vee. \end{aligned}$$

■

**Proposition 1.6.6.** $\Pi\circ\mathbf{R}\operatorname{Hom}_\Lambda(-,\Lambda) \cong \left(R^\vee \otimes^{\mathbf{L}}_R -\right)\circ\mathbf{L}D\circ\Pi$ *on* $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$.

PROOF. Using lemmas 1.6.2 and 1.6.5, usual Pontryagin duality, and the fact that tensor products commute with direct limits, we get:

$$\begin{aligned} \operatorname{Hom}_\Lambda(M,\Lambda)^\vee &\cong \varinjlim_U R^\vee \otimes_R M_U \\ &\cong R^\vee \otimes_R \varinjlim \Pi(\Pi(M)^U) \\ &= R^\vee \otimes_R D(\Pi(M)). \end{aligned}$$

It hence remains to show that $(D\circ\Pi)$ maps projective objects to $R^\vee\otimes_R -$-acyclics and it actually suffices to check this for the module $\Lambda$. But $D(\Pi(\Lambda)) = \varinjlim_U R[G/U]$ is clearly $R^\vee \otimes_R -$-acyclic. ■

**Remark 1.6.7.** The spectral sequence attached to proposition 1.6.6 looks like this:

$$\operatorname{Tor}^R_p(R^\vee, D_q(M^\vee)) \Longrightarrow \mathrm{E}^{p+q}_\Lambda(M)^\vee,$$

which yields the following five term exact sequence:

$$\begin{tikzcd} \mathrm{E}^2_\Lambda(M)^\vee \arrow[r] & \operatorname{Tor}^R_2(R^\vee, D(M^\vee)) \arrow[r] & R^\vee\otimes_R D_1(M^\vee) \arrow[dll, out=-10, in=170] \\ \mathrm{E}^1_\Lambda(M)^\vee \arrow[r] & \operatorname{Tor}^R_1(R^\vee, D(M^\vee)) \arrow[r] & 0 \end{tikzcd}$$

This also proves that $\mathrm{E}^p_\Lambda(M) = 0$ if $p > \dim G + \dim R$. If $\dim R = 1$, the spectral sequence degenerates and we can compute $\mathrm{E}^p_\Lambda(\operatorname{tor}_R M)$ and $\mathrm{E}^p_\Lambda(M/\operatorname{tor}_R M)$ akin to [NSW08, (5.4.13)]. The spectral sequence for $R = \mathbb{Z}_p$ first appeared in [Jan89, theorem 2.1].

**Lemma 1.6.8.** $R^\vee \cong \varinjlim_k R/\mathfrak{m}^{(k)}$ *if* $R$ *is regular.*

Proof. $R$ satisfies local duality by assumption, hence by corollaries 1.3.16 and 1.4.9 $R^\vee = T(R) \cong \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}(R) = \varinjlim_k \mathrm{Ext}^d_R(R/\mathfrak{m}^{(k)}, R) \cong \varinjlim_k R/\mathfrak{m}^{(k)}$. ∎

**Lemma 1.6.9.** $\mathrm{Hom}_\Lambda(M,\Lambda)^\vee \cong \varinjlim_{U,k}(M/\mathfrak{m}^{(k)})_U$ *for finitely generated $\Lambda$-modules $M$ and regular $R$.*

Proof.

$$\begin{aligned}\mathrm{Hom}_\Lambda(M,\Lambda)^\vee &\cong R^\vee \otimes_R \varinjlim_U M_U\\ &\cong \varinjlim_U \varinjlim_k R/\mathfrak{m}^{(k)} \otimes_R M_U\\ &\cong \varinjlim_U \varinjlim_k (M/\mathfrak{m}^{(k)}M)_U\end{aligned}$$

using lemmas 1.6.5 and 1.6.8. ∎

**Proposition 1.6.10.** *If $R$ is regular, $\Pi \circ \mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda) \cong \varinjlim_k \mathbf{L}D \circ \Pi \circ \left(R/\mathfrak{m}^{(k)} \otimes^{\mathbf{L}}_R -\right) \cong \varinjlim_k \mathbf{L}D \circ \Pi \circ [d] \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^{(k)}, -)$ on $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$.*

Proof. By lemma 1.6.9

$$\begin{aligned}\mathrm{Hom}_\Lambda(M,\Lambda)^\vee &\cong \varinjlim_U \varinjlim_k (M/\mathfrak{m}^{(k)}M)_U\\ &\cong \varinjlim_{U,k} \Pi(\Pi(M/\mathfrak{m}^{(k)}M)^U)\\ &= \varinjlim_k D(\Pi(R/\mathfrak{m}^{(k)} \otimes_R M)).\end{aligned}$$

By proposition 1.3.14 it suffices to show that $(\Lambda/\mathfrak{m}^{(k)})^\vee$ is $D$-acyclic. But

$$\mathbf{L}^{-i}D((\Lambda/\mathfrak{m}^{(k)})^\vee) = \varinjlim_U H^i(U, R/\mathfrak{m}^{(k)}[[G]]^\vee)^* = \varinjlim_U H_i(U, R/\mathfrak{m}^{(k)}[[G]]),$$

which is zero for $i > 0$ by Shapiro's Lemma.

The other isomorphism now follows from proposition 1.3.14. ∎

**Remark 1.6.11.** Write $D_p$ for $\mathbf{L}^{-p}D$. The spectral sequences attached to proposition 1.6.10 look like this:

$$\varinjlim_k D_p(\mathrm{Tor}^R_q(R/\mathfrak{m}^{(k)}, M)^\vee) \Longrightarrow \mathrm{E}^{p+q}_\Lambda(M)^\vee$$

and

$$\varinjlim_k D_p(\mathrm{Ext}^{d-q}_R(R/\mathfrak{m}^{(k)}, M)^\vee) \Longrightarrow \mathrm{E}^{p+q}_\Lambda(M)^\vee$$

with exact five term sequences

$$\mathrm{E}^2_\Lambda(M)^\vee \longrightarrow \varinjlim_k D_2((M/\mathfrak{m}^{(k)}M)^\vee) \longrightarrow \varinjlim_k D(\mathrm{Tor}^R_1(R/\mathfrak{m}^{(k)}, M)^\vee) \longrightarrow \mathrm{E}^1_\Lambda(M)^\vee \longrightarrow \varinjlim_k D_1((M/\mathfrak{m}^{(k)}M)^\vee) \longrightarrow 0$$

and

$$\mathrm{E}^2_\Lambda(M)^\vee \longrightarrow \varinjlim_k D_2(\mathrm{Ext}^d_R(R/\mathfrak{m}^{(k)}, M)^\vee) \longrightarrow \varinjlim_k D(\mathrm{Ext}^{d-1}_R(R/\mathfrak{m}^{(k)}, M)^\vee) \longrightarrow \mathrm{E}^1_\Lambda(M)^\vee \longrightarrow \varinjlim_k D_1(\mathrm{Ext}^d_R(R/\mathfrak{m}^{(k)}, M)^\vee) \longrightarrow 0$$

respectively. For $R = \mathbb{Z}_p$, these appear in the proof of [Jan89, theorem 2.1].

**Lemma 1.6.12** ([Lim12, proposition 3.1.7]). *Let $M$ be a finitely generated $\Lambda$-module. Then $M \cong \varprojlim_k M/\mathfrak{M}^k M$ algebraically and topologically.*

**Lemma 1.6.13.** $\Pi \circ \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^{(k)}, -)$ *maps bounded complexes of $\Lambda$-modules with finitely generated cohomology to bounded complexes whose cohomology modules are discrete $p$-torsion $G$-modules. If $M$ is a complex in $\varinjlim_U \mathbf{D}^b_c(R[G/U]\text{-}\mathbf{Mod})$, then all cohomology groups of*

$$\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}^{(k)}, M)^\vee$$

*are furthermore finite.*

Proof. The groups $\mathrm{Ext}^q_R(R/\mathfrak{m}^{(k)}, M)$ for $M$ finitely generated over $\Lambda$ are clearly $p$-torsion and finitely generated as $\Lambda$-modules, hence compact by lemma 1.6.12, and consequentially topologically profinite and pro-$p$. Their Pontryagin duals are thus discrete $p$-torsion $G$-modules.

If $M$ is finitely generated over some $R[G/U]$, it is also finitely generated over $R$ and $\mathrm{Ext}^q_R(R/\mathfrak{m}^{(k)}, M)$ finitely generated over $R/\mathfrak{m}^{(k)}$, hence finite. ∎

**Proposition 1.6.14.** *Assume that $R$ is regular. Let $G$ be a duality group (cf. [NSW08, (3.4.6)]) of dimension $s$ at $p$. Then*

$$\begin{aligned}\mathrm{E}^m_\Lambda(M)^\vee &\cong \varinjlim_k \mathbf{L}^{-s} D\,\mathrm{Ext}^{d-(m-s)}_R(R/\mathfrak{m}^{(k)}, M)^\vee\\ &\cong \varinjlim_k \mathbf{L}^{-s} D\,\mathrm{Tor}^R_{m-s}(R/\mathfrak{m}^{(k)}, M)^\vee.\end{aligned}$$

*for finitely generated $R[G/U]$-modules $M$. Especially $\Pi \circ \mathrm{E}^s_\Lambda$ is then right-exact.*

Proof. As $G$ is a duality group of dimension $s$ at $p$, the complex $\mathbf{L}D(M')$ has trivial cohomology outside of degree $-s$ if $M'$ is a finite discrete $p$-torsion $G$-module. Together with lemma 1.6.13 this implies that the spectral sequence attached to proposition 1.6.10 degenerates and gives

$$\mathrm{E}^m_\Lambda(M)^\vee \cong \varinjlim_k \mathbf{L}^{-s} D\,\mathrm{Ext}^{d-(m-s)}_R(R/\mathfrak{m}^{(k)}, M)^\vee.$$

The other isomorphism follows with exactly the same argument. Note furthermore that as $\dim R = d$, $\mathrm{Ext}^{d-(m-s)}_R(R/\mathfrak{m}^{(k)}, M) = 0$ if $m - s < 0$, hence $\mathrm{E}^m_\Lambda(M)^\vee = 0$ if $m < s$. ∎

**Theorem 1.6.15.** *Assume that $R$ is regular and that $G$ is a Poincaré group at $p$ of dimension $s$ with dualising character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ (i. e., $\varinjlim_\nu D_s(\mathbb{Z}/p^\nu) \cong \mathbb{Q}_p/\mathbb{Z}_p(\chi)$, cf. [NSW08, (3.7.1)]), which gives rise to the “twisting functor” $\boldsymbol{\chi}\colon M \longmapsto M(\chi)$. Assume that $R$ is a commutative complete Noetherian ring of global dimension $d$ with maximal ideal $\mathfrak{m}$. Then*

$$T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda) = \boldsymbol{\chi} \circ [d + s] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$$

*on $\varinjlim_U \mathbf{D}^b_c(R[G/U]\text{-}\mathbf{Mod})$.*

Proof. Let $I = \mathbb{Q}_p/\mathbb{Z}_p(\chi) = \boldsymbol{\chi}(\mathbb{Q}_p/\mathbb{Z}_p)$ be the dualising module of $G$. For any $p$-torsion $G$-module $A$ we have by [NSW08, (3.7)] that $\mathbf{L}^{-s}D(A) = \varinjlim_U H^s(U, A)^* \cong \varinjlim_U \mathrm{Hom}_{\mathbb{Z}_p}(A, I)^U = \mathrm{Hom}_{\mathbb{Z}_p}(A, I)$, as $I$ is also a dualising module for every open subgroup of $G$.

Note that

$$H^0(\boldsymbol{\chi} \circ [d] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}}) = \boldsymbol{\chi} \circ \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}$$

and that $\chi \circ \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}$ is hence right-exact. Note furthermore that

$$\begin{aligned} H^0([-s] \circ T \circ \mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda)) &= \mathrm{E}^s(-)^\vee \\ &\cong \varinjlim_k \mathbf{L}^{-s} D \operatorname{Ext}^d_R(R/\mathfrak{m}^{(k)}, -)^\vee \\ &\cong \varinjlim_k \mathrm{Hom}_{\mathbb{Z}_p}(\operatorname{Ext}^d_R(R/\mathfrak{m}^{(k)}, -)^\vee, I) \\ &\cong \chi \circ \varinjlim_k \operatorname{Ext}^d_R(R/\mathfrak{m}^{(k)}, -) \\ &= \chi \circ \mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}} \end{aligned}$$

using proposition 1.6.14 and Pontryagin duality.

By [Har66, proposition I.7.4] the left derivation of $\mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}$ is $[d] \circ \mathbf{R}\Gamma_{\underline{\mathfrak{m}}}$: The complex $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}(R)$ is concentrated in degree $d$ and hence every module is a quotient of a module with this property, as local cohomology commutes with arbitrary direct limits. ∎

Note that even though this theorem suspiciously looks like local duality, the local cohomology on the right hand side is with respect to the maximal ideal of the coefficient ring, not the whole Iwasawa algebra. The local duality result is subject of the next section.

We end this section with a generalisation of [Jan89, corollary 2.6], where $R = \mathbb{Z}_p$ was considered.

**Corollary 1.6.16.** *In the setup of theorem 1.6.15 assume that $M$ is a finitely generated $R[G/U]$-module. Then the following hold:*

*(1) If $M$ is free over $R$, then*

$$\mathrm{E}^q_\Lambda(M)^\vee \cong \begin{cases} M \otimes_R R^\vee(\chi) & \text{if } q = s \\ 0 & \text{else} \end{cases}$$

*(2) If $M$ is $R$-torsion, then $\mathrm{E}^q_\Lambda(M) = 0$ for all $q \leq s$.*

*(3) If $M$ is finite, then*

$$\mathrm{E}^q_\Lambda(M)^\vee \cong \begin{cases} M(\chi) & \text{if } q = d + s \\ 0 & \text{else} \end{cases}$$

Proof. By theorem 1.6.15, we have

$$\mathrm{E}^q_\Lambda(M)^\vee \cong \mathbf{R}^{d+s-q}\Gamma_{\underline{\mathfrak{m}}}(M)(\chi)$$

in any case.

In the first case, this is just $M \otimes_R R^\vee$ for $q = s$ and zero else. In the second case, local duality yields $\mathbf{R}^d\Gamma_{\underline{\mathfrak{m}}}(M) \cong \mathrm{Hom}_R(M, R)^\vee = 0$. In the third case, we note that $M$ has an injective resolution by modules that are the direct limit of finite modules. Proposition 1.3.23 together with proposition 1.3.22 then imply the result. ∎

### 1.7. Local Duality for Iwasawa Algebras

This section gives a streamlined proof of a local duality result for Iwasawa algebras as first published in [Ven02] and generalises the result to more general coefficient rings. Let $G$ be a pro-$p$ Poincaré group of dimension $s$ with dualising character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ and $R$ a commutative Noetherian pro-$p$ regular local ring with maximal ideal $\mathfrak{m}$ of global dimension $d$. Set $\Lambda = R[[G]]$, which is of global dimension $r = d + s$.

**Proposition 1.7.1.** $\mathbf{R}\Gamma_{\underline{\mathfrak{M}}}(\Lambda) \cong \Lambda^\vee[-d-s]$ *and* $\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{M}^l, \Lambda) \cong \mathbf{R}\Gamma^{d+s-i}_{\underline{\mathfrak{m}}}(\Lambda/\mathfrak{M}^l)(\chi)$.

Proof. By proposition 1.6.14,

$$\begin{aligned}\mathbf{R}^i\Gamma_{\underline{\mathfrak{M}}}(\Lambda) &= \varinjlim_l \mathrm{E}^i(\Lambda/\mathfrak{M}^l)\\ &\cong \varinjlim_l \left(\varinjlim_k \mathbf{L}^{-s}D(\mathrm{Ext}_R^{d-(i-s)}(R/\mathfrak{m}^{(k)},\Lambda/\mathfrak{M}^l)^\vee)\right)^\vee.\end{aligned}$$

As in the proof of theorem 1.6.15, we can express

$$\mathbf{L}^{-s}D(\mathrm{Ext}_R^{d-(i-s)}(R/\mathfrak{m}^{(k)},\Lambda/\mathfrak{M}^l)^\vee)$$

as

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ext}_R^{d-(i-s)}(R/\mathfrak{m}^{(k)},\Lambda/\mathfrak{M}^l)^\vee,\mathbb{Q}_p/\mathbb{Z}_p(\chi)),$$

which we again see is isomorphic to

$$\mathrm{Ext}_R^{d-(i-s)}(R/\mathfrak{m}^{(k)},\Lambda/\mathfrak{M}^l)(\chi).$$

In the direct limit over $k$ this becomes $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}^{d-(i-s)}(\Lambda/\mathfrak{M}^l)(\chi)$.

$\Gamma_{\underline{\mathfrak{m}}}$ restricted to the subcategory of finite $\Lambda$- (or $R$-)modules is the identity. As $\Gamma_{\underline{\mathfrak{m}}}$ commutes with arbitrary direct limits, this is also true for the category of discrete $\Lambda$-modules. As the latter category contains sufficiently many injectives, $\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}(N) = N$ if $N$ is a complex of discrete $\Lambda$-modules.

Now $\Lambda/\mathfrak{M}^l$ is such a finite module, hence

$$\mathbf{R}\Gamma_{\underline{\mathfrak{M}}}(\Lambda) = [-d-s]\circ \varinjlim_l(\mathbf{R}\Gamma_{\underline{\mathfrak{m}}}(\Lambda/\mathfrak{M}^l)(\chi))^\vee = \Lambda(\chi)^\vee[-r].$$

The proposition now follows at once if we observe that $\Lambda \cong \Lambda(\chi)$ as a $\Lambda$-module via $g \longmapsto \chi(g)g$. ∎

**Theorem 1.7.2** (Local duality for Iwasawa algebras). 

$$\mathbf{R}\Gamma_{\underline{\mathfrak{M}}} \cong [-r]\circ\Pi\circ\mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda)$$

*on* $\mathbf{D}_c^b(\Lambda)$.

Proof. Because of proposition 1.7.1, this follows verbatim as in [Har67, theorem 6.3]: The functors $\mathbf{R}^r\Gamma_{\underline{\mathfrak{M}}}$ and $\mathrm{Hom}_\Lambda(-,\Lambda)^\vee$ are related by a pairing of Ext-groups, are both covariant and right-exact and agree on $\Lambda$, hence also agree on finitely generated modules. As the complex $\mathbf{R}\Gamma_{\underline{\mathfrak{M}}}(\Lambda)$ is concentrated in degree $r$, the same argument as in theorem 1.6.15 shows that the left derivation of $\mathbf{R}^r\Gamma_{\underline{\mathfrak{M}}}$ is just $[r]\circ\mathbf{R}\Gamma_{\underline{\mathfrak{M}}}$ and the result follows. ∎

## 1.8. Torsion in Iwasawa Cohomology

There are notions of both local and global Iwasawa cohomology. Our result about their torsion below holds in both cases and we will first deal with the local case.

In both subsections, $R$ is a commutative Noetherian pro-$p$ local ring of residue characteristic $p$.

### 1.8.1. Torsion in Local Iwasawa Cohomology.

Let $K$ be a finite extension of $\mathbb{Q}_p$ and $K_\infty|K$ a Galois extension with an analytic pro-$p$ Galois group $G$ without elements of finite order. Let $T$ be a finitely generated $\Lambda = R[[G]]$-module and set $A = T\otimes_R R^\vee$. Due to Lim and Sharifi we have the following spectral sequence (stemming from an isomorphism of complexes in the derived category). Write $H^i_{\mathrm{Iw}}(K_\infty,T) = \varprojlim_{K'} H^i(G_{K'},T)$ where the limit is taken with respect to the corestriction maps over all finite field extensions $K'|K$ contained in $K_\infty$, and where we denote by $G_L$ the absolute Galois group of a field $L$.

**Theorem 1.8.1.** *There is a convergent spectral sequence*

$$\mathrm{E}^i_\Lambda(H^j(G_{K_\infty}, A)^\vee) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K, T).$$

Proof. This is [LS13, theorem 4.2.2, remark 4.2.3], which generalises a local version of the main result of [Jan14] to more general coefficients. ∎

**Theorem 1.8.2.** *If $G$ is a pro-$p$ Poincaré group of dimension $s \geq 2$ with dualising character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ and if $R$ is regular, then*

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) = 0.$$

*If $s = 1$, then*

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) \cong \operatorname{Hom}_R((T^*)_G, R)(\chi^{-1}),$$

*where $T^* = \operatorname{Hom}_R(T, R)$.*

Proof. The exact five-term sequence attached to theorem 1.8.1 starts like this:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{E}^1_\Lambda(H^0(G_{K_\infty}, A)^\vee) \arrow[r] & H^1_{\mathrm{Iw}}(K_\infty, T) \arrow[r] & \mathrm{E}^0_\Lambda(H^1(G_{K_\infty}, A)^\vee) \arrow[d] \\ & & & \mathrm{E}^2_\Lambda(H^0(G_{K_\infty}, A)^\vee) \end{tikzcd}$$

Note that $\mathrm{E}^2_\Lambda(M)$ is pseudo-null and hence $\Lambda$-torsion for every finitely generated module $M$, which follows from the spectral sequence attached to the isomorphism $\mathbf{R}\operatorname{Hom}_\Lambda(-, \Lambda) \circ \mathbf{R}\operatorname{Hom}_\Lambda(-, \Lambda) \cong \mathrm{id}$ on $\mathbf{D}^b_c(\Lambda\text{-}\mathbf{Mod})$. Furthermore $\mathrm{E}^0_\Lambda(M)$ is always $\Lambda$-torsion free, as $\Lambda$ is integral. It follows that $\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) \subseteq \mathrm{E}^1_\Lambda(H^0(G_{K_\infty}, A)^\vee)$. As the latter is $\Lambda$-torsion,

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) = \mathrm{E}^1_\Lambda(H^0(G_{K_\infty}, A)^\vee).$$

The result now follows immediately from corollary 1.6.16. ∎

**1.8.2. Torsion in Global Iwasawa Cohomology.** Let $K$ be a finite extension of $\mathbb{Q}$ and $S$ a finite set of places of $K$. Let $K_S$ be the maximal extension of $K$ which is unramified outside $S$ and $K_\infty|K$ a Galois extension contained in $K_S$. Suppose that $G = G(K_\infty|K)$ is an analytic pro-$p$ group without elements of finite order. Let $T$ be a finitely generated $\Lambda = R[[G]]$-module and set $A = T \otimes_R R^\vee$. Due to Lim and Sharifi we have the following spectral sequence (stemming from an isomorphism of complexes in the derived category). Write $H^i_{\mathrm{Iw}}(K_\infty, T) = \varprojlim_{K'} H^i(G(K_S|K'), T)$ where the limit is taken with respect to the corestriction maps over all finite field extensions $K'|K$ contained in $K_\infty$.

**Theorem 1.8.3.** *There is a convergent spectral sequence*

$$\mathrm{E}^i_\Lambda(H^j(G(K_S|K_\infty), A)^\vee) \Longrightarrow H^{i+j}_{\mathrm{Iw}}(K_\infty, T).$$

Proof. This follows from [LS13, theorem 4.5.1], which generalises the main result of [Jan14] to more general coefficients. ∎

From this we derive the following.

**Theorem 1.8.4.** *If $G$ is a pro-$p$ Poincaré group of dimension $s \geq 2$ with dualising character $\chi\colon G \longrightarrow \mathbb{Z}_p^\times$ and if $R$ is regular, then*

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) = 0.$$

*If $s = 1$, then*

$$\operatorname{tor}_\Lambda H^1_{\mathrm{Iw}}(K_\infty, T) \cong \operatorname{Hom}_R((T^*)_G, R)(\chi^{-1}),$$

*where $T^* = \operatorname{Hom}_R(T, R)$.*

Proof. Replace "$G_{K_\infty}$" with "$G(K_S|K_\infty)$" in the proof of theorem 1.8.2. ∎

CHAPTER 2

# A Framework for Topologised Groups

There are numerous variants of group cohomology defined via cochains: They can be assumed to be continuous, analytic, bounded etc. The results we will prove will be valid for all of these topologised group cohomology theories, as all proofs will be done with the *direct method* of Hochschild and Serre on the level of cochains without appealing to homological arguments.

While the direct method is ridiculously flexible, it is quite hard to present one streamlined proof that shows off many of its features. For this reason, we will introduce a simple axiomatic framework that allows us to deal with all of these variants in one go.

One application we have in mind is analytic cohomology of $(\varphi, \Gamma)$-modules in the sense of [Col14]. For this reason, we are strictly speaking not setting up a framework for topologised groups, but for topologised monoids. Regrettably, this does complicate a few arguments.

## 2.1. Topological Categories

There are a number of notions of topological categories in the literature, none of which is standard. For our purposes it is sufficient to have a good notion of discrete spaces.

**Definition 2.1.1.** A *concrete category* is a faithful functor $¿ : \mathbf{C} \longrightarrow \mathbf{Set}$. One often only says that a category $\mathbf{C}$ is a concrete category, even though the forgetful functor $¿$ is an essential part of the datum.

**Definition 2.1.2.** A concrete category $¿ : \mathbf{C} \longrightarrow \mathbf{Set}$ is called a *category admitting discrete objects*, if $\mathbf{C}$ has finite limits and $¿$ admits a fully faithful left adjoint $\mathbf{F}$.

We will denote by

$$\mathbf{C}^{\delta} = \{X \in \mathbf{C} \mid ¿(\mathrm{Hom}_{\mathbf{C}}(X, -)) = \mathrm{Hom}_{\mathbf{Set}}(¿X, ¿(-))\}$$

the *discrete objects* in $\mathbf{C}$. This terminology is justified as all objects in $\mathbf{Set}$ give rise to discrete objects, cf. proposition 2.1.7. By abuse of notation, we will often only say that a category admits discrete objects without specifying the forgetful functor.

We will always denote by $\bullet$ a singleton set.

**Remark 2.1.3.** Note that for a category admitting discrete objects, the forgetful functor is represented by $\mathbf{F}\bullet$.

**Proposition 2.1.4.** *Let $\mathbf{C}$ be a category admitting discrete objects. Then $¿ \circ \mathbf{F} \simeq \mathrm{id}_{\mathbf{Set}}$.*

Proof. The isomorphism $\mathrm{Hom}_{\mathbf{Set}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(\mathbf{F}X, \mathbf{F}Y) = \mathrm{Hom}_{\mathbf{Set}}(X, ¿\mathbf{F}Y)$ shows that $Y \cong ¿\mathbf{F}Y$. ∎

**Proposition 2.1.5.** *Let $\mathbf{C}$ be a category admitting discrete objects. Then $¿$ maps monomorphisms to monomorphisms.*

Proof. Let $X \longrightarrow Y$ be a monomorphism in $\mathbf{C}$. It suffices to show that for $\alpha \neq \beta \in \mathrm{Hom}_{\mathbf{Set}}(\bullet, ¿X)$ also their induced maps in $\mathrm{Hom}_{\mathbf{Set}}(\bullet, ¿Y)$ differ. Assume they did not.

$\alpha$ and $\beta$ correspond to $\alpha', \beta' \in \mathrm{Hom}_{\mathbf{C}}(\mathbf{F}\bullet, X)$. As $¿\mathbf{F} \simeq \mathrm{id}_{\mathbf{Set}}$ by proposition 2.1.4, this implies that

$$¿\left(\mathbf{F}\bullet \xrightarrow{\alpha'} X \longrightarrow Y\right) = ¿\left(\mathbf{F}\bullet \xrightarrow{\beta'} X \longrightarrow Y\right),$$

so $\alpha' = \beta'$ as $X \longrightarrow Y$ was assumed to be mono. ∎

**Proposition 2.1.6.** *Subobjects of discrete objects are discrete.*

Proof. Let $X$ be a discrete object in a category admitting discrete objects $\mathbf{C}$, $D$ a subobject and $Y$ an arbitrary object in $\mathbf{C}$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(X, Y) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathbf{Set}}(¿X, ¿Y) \arrow[d] \\
\mathrm{Hom}_{\mathbf{C}}(D, Y) \arrow[r, hook, "¿"] & \mathrm{Hom}_{\mathbf{Set}}(¿D, ¿Y)
\end{tikzcd}$$

As $¿D \longrightarrow ¿X$ is again a mono by proposition 2.1.5, the map on the right is surjective and hence so is the bottom one, i. e., $D$ is discrete. ∎

**Proposition 2.1.7.** *Let $\mathbf{C}$ be a category admitting discrete objects. Then $\mathbf{F}S$ is discrete for every set $S$. $\mathbf{F}$ is essentially surjective onto the discrete objects, so $\mathbf{F}\colon \mathbf{Set} \longrightarrow \mathbf{C}^{\delta}$ is an equivalence of categories.*

Proof. Let $S$ be a set. Then

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{F}S, Y) \xrightarrow{¿} \mathrm{Hom}_{\mathbf{Set}}(¿\mathbf{F}S, ¿Y) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{F}¿\mathbf{F}S, Y) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{F}S, Y),$$

and the identifications imply that the first map is an isomorphism.
On the other hand, if $X$ is discrete, i. e., if

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(¿X, ¿Y) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{F}¿X, Y) \text{ for all } Y,$$

then Yoneda implies $X \cong \mathbf{F}¿X$, so $\mathbf{F}$ is essentially surjective onto discrete objects. ∎

**Example 2.1.8.** Categories admitting discrete objects don't quite behave like topological spaces, as for example singleton objects need not be isomorphic. Consider the following category $\mathbf{C}$ where the objects are tuples $(A, \tau_A)$ with $A$ any set and $\tau_A$ *any* subset of the power set $2^A$ of $A$. A morphism $(A, \tau_A) \longrightarrow (B, \tau_B)$ in $\mathbf{C}$ is defined as a map $f\colon A \longrightarrow B$ subject to the condition that for $b \in \tau_B$, $f^{-1}(b) \in \tau_A$.
The forgetful functor has the obvious left adjoint $A \longmapsto (A, 2^A)$, but singleton sets need not be isomorphic: $(\bullet, \varnothing) \not\cong (\bullet, 2^{\bullet})$ and only the latter object is discrete while only $(\bullet, \varnothing)$ is final in $\mathbf{C}$.

**Definition 2.1.9.** We say that a category $\mathbf{C}$ admitting discrete objects is *topological*, if the functor $\mathbf{F}\colon \mathbf{S} \longrightarrow \mathbf{C}$ commutes with finite limits and if for every discrete object $D$ and all objects $X, Y$ the natural map

$$\mathrm{Hom}_{\mathbf{C}}(D \times X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(¿D, \mathrm{Hom}_{\mathbf{C}}(X, Y))$$

is a bijection.

**Remark 2.1.10.** If $\mathbf{F}$ commutes with finite limits, it especially maps a final object to a final object.

**Remark 2.1.11.** The isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(D \times X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(¿D, \mathrm{Hom}_{\mathbf{C}}(X, Y))$$

replaces some kind of internal Hom-functor: In the category of compactly generated weakly Hausdorff spaces admits we endow for spaces $X, Y$ the set $\mathrm{Hom}_{\mathbf{CGWH}}(X, Y)$ with the compact open topology and call the resulting object $[X,Y]$. This results in a pair of adjoint functors:

$$\mathrm{Hom}_{\mathbf{CGWH}}(Z\times X, Y)\cong \mathrm{Hom}_{\mathbf{CGWH}}(Z,[X,Y]).$$

If $Z$ is furthermore discrete, this reads as

$$\mathrm{Hom}_{\mathbf{CGWH}}(Z\times X, Y)\cong \mathrm{Hom}_{\mathbf{CGWH}}(Z,[X,Y]) = \mathrm{Hom}_{\mathbf{Set}}(¿Z, \mathrm{Hom}_{\mathbf{CGWH}}(X,Y)),$$

which is precisely the second requirement we posed for a category admitting discrete objects to be topological.

**Example 2.1.12.** Examples of topological categories are: the category of topological spaces, of Hausdorff topological spaces, of metric spaces – all with continuous maps. The category of analytic manifolds (over some arbitrary base) is also topological. In all cases, ¿ is the obvious forgetful functor to **Set** and **F** maps a set to the same set with the discrete topology. Here we regard discrete sets as zero-dimensional manifolds.

**Proposition 2.1.13.** *In a topological category, every constant map of sets lifts to a morphism.*

Proof. A constant map of sets factors as

$$¿X \longrightarrow \bullet \longrightarrow ¿Y.$$

As $\mathbf{F}\bullet$ is terminal in a topological category, this factorisation lifts to morphisms in the topological category. ∎

## 2.2. Topologised Groups

**Definition 2.2.1.** A *topologised group* is a group object in a topological category. Similarly, a *topologised monoid* will mean a monoid object in a topological category. A morphism of topologised groups is a morphism $\phi\colon G\longrightarrow H$ in the ambient category such that the diagram

$$\begin{tikzcd} G\times G \arrow[r, "\mathrm{mul}"] \arrow[d, "(\phi,\phi)"] & G \arrow[d, "\phi"] \\ H\times H \arrow[r, "\mathrm{mul}"] & H \end{tikzcd}$$

commutes. For a morphism of topologised monoids we furthermore require the commutativity of the following diagram:

$$\begin{tikzcd} G \arrow[rr, "\phi"] & & H \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

Here $1$ is the trivial group structure on a final object of the ambient category and the morphisms $1\longrightarrow H, 1\longrightarrow G$ are the inclusion of identity elements.

**Remark 2.2.2.** Note that if $\mathbf{C}$ is a topological category, ¿ maps topologised groups to groups and $\mathbf{F}$ maps groups to (discrete) group objects in $\mathbf{C}$. We again have an equivalence between the category of (abstract) groups and discrete topologised groups via $\mathbf{F}$.

**Remark 2.2.3.** If this topological category is the category of (Hausdorff) topological spaces, our notions of topologised groups and monoids coincide with the standard ones of topological groups and monoids. Other important examples are the categories of $L$-analytic manifolds where $L$ is a local field.

**Definition 2.2.4.** A morphism $N \longrightarrow G$ of topologised groups is called *normal*, if its cokernel exists in the category of topologised groups with kernel exactly $N \longrightarrow G$. The cokernel $G \longrightarrow C$ will also be called the *quotient* of $G$ by $N$ and we will simply write $C \cong G/N$. A morphism $U \longrightarrow G$ is called an open normal subgroup, if it is normal and $G/U$ is discrete.

**Remark 2.2.5.** Note that this definition allows us to avoid the notion of *strictness*, which is rather cumbersome, cf. remark 2.2.10. Indeed, consider the bijective morphism $\mathbb{R}^\delta \longrightarrow \mathbb{R}$ in the category of locally compact groups, where $\mathbb{R}^\delta$ carries the discrete topology and $\mathbb{R}$ the usual one. It is easy to see that the cokernel of this morphism is the trivial morphism $\mathbb{R} \longrightarrow 1$, which has kernel $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$, so $\mathbb{R}^\delta \longrightarrow \mathbb{R}$ is not normal.

**Proposition 2.2.6.** *Let $U \longrightarrow G$ be an open normal subgroup of topologised groups. Then*

$$¿(G/U) \cong ¿G/¿U.$$

Proof. Note that ¿ commutes with arbitrary limits and especially with taking kernels. Therefore

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Grp}}(¿(G/U), H) &= ¿\,\mathrm{Hom}_{\mathbf{Grp_C}}(G/U, \mathbf{F}H) \\
&= ¿\left\{f \in \mathrm{Hom}_{\mathbf{Grp_C}}(G, \mathbf{F}H) \mid \ker f \supseteq U\right\} \\
&\subseteq \left\{f \in \mathrm{Hom}_{\mathbf{Grp}}(¿G, H) \mid \ker f \supseteq ¿U\right\} \\
&= \mathrm{Hom}_{\mathbf{Grp}}(¿G/¿U, H),
\end{aligned}$$

which yields a surjection

$$¿G/¿U \twoheadrightarrow ¿(G/U),$$

as epimorphisms in the category of groups are exactly the surjective group homomorphisms. On the other hand we also have a natural injection $¿G/¿U \hookrightarrow ¿(G/U)$ as ¿ preserves kernels. As both maps clearly coincide, this proves the proposition. ∎

**Proposition 2.2.7.** *Let $G$ be a topologised group with open normal subgroup $U$. Then $G \longrightarrow G/U$ admits a section in $\mathbf{C}$.*

Proof. Consider any section $¿G/¿U \longrightarrow ¿G$, which by proposition 2.2.6 is a section $¿(G/U) \longrightarrow ¿G$. As $G/U$ is discrete, this lifts to a section in $\mathbf{C}$. ∎

**Proposition 2.2.8.** *Let $G$ be a topologised group with normal subgroup $N$. If $G \longrightarrow G/N$ admits a section in $\mathbf{C}$, then $¿(G/N) \cong ¿G/¿N$.*

Proof. As ¿ preserves kernel, we always have an injection $¿G/¿N \longrightarrow ¿(G/N)$. The existence of a section implies that it is also surjective and hence an isomorphism of groups. ∎

**Proposition 2.2.9.** *Let $G'$ be a topologised group, $M$ a discrete topologised monoid and $U$ an open normal subgroup of $G'$. Then $U \times 1 \longrightarrow G' \times M$ is the kernel of $G' \times M \longrightarrow G'/U \times M$ and the latter map is the cokernel of the former map in the category of topologised monoids.*

Proof. That $U \times 1 \longrightarrow G' \times M$ is the kernel is clear, as kernels are stable under taking products. For $G' \times M \longrightarrow G'/U \times M$ being its cokernel, note that in the diagram

$$\begin{tikzcd}
U \times 1 \arrow[r] & G' \times M \arrow[r] \arrow[dr] & G'/U \times M \arrow[d, dashed] \\
& & D
\end{tikzcd}$$

the object $G'/U \times M$ is discrete as $\mathbf{F}$ commutes with finite limits, so the corresponding proposition in the category of (abstract) monoids yields the proposition by proposition 2.2.6. ∎

**Remark 2.2.10.** The following subtlety in the definition of strictness caused the author much grief. A morphism is called *strict*, if its image and coimage coincide.

Consider the following notion, which we will call the *classical image*, which is often simply called the image of a morphism, cf. [Mit65, p. I.10]: The classical image of a morphism $f\colon X \longrightarrow Y$ is a monomorphism $CI \hookrightarrow Y$ and a morphism $X \longrightarrow CI$ such that $f = X \longrightarrow CI \hookrightarrow Y$ and for every other factorisation $f = X \longrightarrow D \hookrightarrow Y$ there is a unique morphism $CI \longrightarrow D$ such that the obvious diagrams commute. We can analogously define the *classical coimage* of a morphism.

Note that it is easy to see that in the category of topological spaces, the classical image is the set theoretic image with the quotient topology (i. e., $V \subseteq f(X)$ is open if and only if $f^{-1}(V)$ is), while the classical coimage is the set theoretic image with the subspace topology of the codomain.

These notions have to be strictly differentiated from the notions of *regular images* and *regular coimages*, which are often simply called the image and coimage of a morphism, cf. [KS06, definition 5.1.1]: The regular image of a morphism $f\colon X \longrightarrow Y$ is defined as the equaliser $\lim Y \rightrightarrows Y \sqcup_X Y$ and its regular coimage as the coequaliser $\operatorname{colim} X \times_Y X \rightrightarrows X$.

It is again easy to see that in the category of topological spaces, the regular image of a morphism is the set theoretic image with the subspace topology, and that the regular coimage is given by the set theoretic image with the quotient topology, i. e., in the category of topological spaces the classical image is the regular coimage and the classical coimage is the regular image!

Indeed, a number of sources simply call regular coimages *images* to make the confusion complete.

## 2.3. Rigidified $G$-Modules

Let $\mathbf{C}$ be a topological category and $G$ a topologised monoid in $\mathbf{C}$. Then we can define a $G$-module as an abelian group object $A$ in $\mathbf{C}$ together with a morphism $G \times A \longrightarrow A$ subject to the usual diagrams. Regrettably this definition is too restrictive for our applications.

**Definition 2.3.1.** Let $\mathbf{C}$ be a topological category and $\mathbf{D}$ a concrete category. A rigidification from $\mathbf{C}$ to $\mathbf{D}$ is a bifunctor

$$h\colon \mathbf{C}^\circ \times \mathbf{D} \longrightarrow \mathbf{Set}$$

such that functorially in $X$ and $Y$,

$$h(X, Y) \subseteq \operatorname{Hom}_{\mathbf{Set}}(¿X, ¿Y)$$

and

$$h(\mathbf{F}\bullet, -) \cong ¿.$$

**Example 2.3.2.** Even though we haven't yet defined the notion of $h$-pliant objects, we want to give an overview of the most important examples of rigidifications.

| $\mathbf{C}$ | $\mathbf{D}$ | $h(X, Y)$ | $h$-pliant objects |
|---|---|---|---|
| any topological category | $\mathbf{C}$ | $\operatorname{Hom}_{\mathbf{C}}(X, Y)$ | all discrete objects |
| analytic manifolds | LF-spaces | locally analytic maps in the sense of [Col16, section 5] | all discrete spaces (considered as zero-dimensional manifolds) |
| topological spaces | metric spaces | bounded continuous maps | finite discrete spaces |

But even this notion of rigidifications is in some cases to restrictive.

**Remark 2.3.3.** LF-spaces and induced modules are the main reason we have to consider rigidified objects and not just rigidifications: Assume a group $G$ with normal subgroup $N$ was to act on an LF-space $A$ in a suitable sense. Then $A^N$, being a kernel, need not be an LF-space itself, cf. [Gro54]. But we still have an object with $\mathbf{C}$-rigidification in the sense of definition 2.3.4.

**Definition 2.3.4.** Let $\mathbf{C}$ be a topological category. A *set with* $\mathbf{C}$*-rigidification* is a set $Y$ and a contravariant functor $h_Y : \mathbf{C}^\circ \longrightarrow \mathbf{Set}$ such that functorially in $X \in \mathbf{C}$,

$$h_Y(X) \subseteq \mathrm{Hom}_{\mathbf{Set}}(¿X, Y).$$

We furthermore require that $h_Y(\mathbf{F}\bullet) = Y$.

For $f \in h_Y(X)$ we will also write $f : X \rightsquigarrow Y$. If for discrete $D$ and all $X$ we have an equality $h_Y(D \times X) = \mathrm{Hom}_{\mathbf{Set}}(¿D, h_Y(X))$, we say that $D$ is $Y$-pliant. It follows that if $D$ is $Y$-pliant, then $h_Y(D) = \mathrm{Hom}_{\mathbf{Set}}(¿D, Y)$.

**Remark 2.3.5.** There is an obvious notion of the category of objects with $\mathbf{C}$-rigidification where objects are tuples $(Y, h_Y)$, and a rigidification could then be defined a functor from $\mathbf{D}$ to this category.

**Remark 2.3.6.** Let $h$ be a rigidification from $\mathbf{C}$ to $\mathbf{D}$. Any object in $\mathbf{D}$ then gives rise to an object with $\mathbf{C}$-rigidification via $Y \longmapsto (¿Y, h(-, Y))$.

**Definition 2.3.7.** Let $h$ be a rigidification from $\mathbf{C}$ to $\mathbf{D}$. A discrete object $D$ in $\mathbf{C}$ is called $h$-pliant, if for all $Y \in \mathbf{D}$, $D$ is $(Y, h(-, Y))$-pliant.

**Definition 2.3.8.** Let $\mathbf{C}$ be a topological category and $G$ a topologised monoid in $\mathbf{C}$. A $G$-module with $\mathbf{C}$-rigidification is a set with $\mathbf{C}$-rigidification $(A, h_A)$ together with a $¿G$-module structure on $A$ such that functorially in $X \in \mathbf{C}$

- $h_A(X)$ is a subgroup of $\mathrm{Hom}_{\mathbf{Set}}(¿X, A)$
- for $f \in h_A(X)$ the induced map

$$¿G \times ¿X \xrightarrow{(\mathrm{id}, f)} ¿G \times A \xrightarrow{\mu} A$$

  lies in $h_A(G \times X)$.

A morphism of $G$-modules with $\mathbf{C}$-rigidification $(A, h_A) \longrightarrow (B, h_B)$ is a morphism of functors $h_A \longrightarrow h_B$ such that the induced map $A = h_A(\mathbf{F}\bullet) \longrightarrow h_B(\mathbf{F}\bullet) = B$ is a morphism of $¿G$-modules. A sequence

$$(A, h_A) \longrightarrow (B, h_B) \longrightarrow (C, h_C)$$

is called a short exact sequence of $G$-modules with $\mathbf{C}$-rigidification, if for all $X \in \mathbf{C}$ the sequence of abelian groups

$$0 \longrightarrow h_A(X) \longrightarrow h_B(X) \longrightarrow h_C(X) \longrightarrow 0$$

is exact.

**Remark 2.3.9.** Let $G$ be a topologised group in a topological category $\mathbf{C}$ and $A$ a $G$-module, i. e., an abelian group object in $\mathbf{C}$ with a morphism $G \times A \longrightarrow A$ subject to the usual diagrams. Then $(¿A, \mathrm{Hom}_{\mathbf{C}}(-, A))$ is a $G$-module with $\mathbf{C}$-rigidification.

If conversely $A$ is an object in $\mathbf{C}$ and $(¿A, h_A)$ a $G$-module with $\mathbf{C}$-rigidification, then we can in general only recover a morphism $G \times A \longrightarrow A$ in $\mathbf{C}$ if $h_A = ¿(\mathrm{Hom}_{\mathbf{C}}(-, A))$: In this case, the map

$$¿G \times ¿A \xrightarrow{(\mathrm{id}, ¿\,\mathrm{id})} ¿G \times ¿A \xrightarrow{\mu} ¿A$$

lies in $¿(\mathrm{Hom}_{\mathbf{C}}(G \times A, A))$.

**Remark 2.3.10.** The definition of an exact sequence of rigidified $G$-modules has concrete interpretations in practice, as the following proposition shows. Indeed they boil down to the usual requirements as for example in [NSW08, (2.7.2)].

The same arguments also show that in a topological category, a sequence of $G$-modules is exact if it is strict (cf. remark 2.2.10) and the last morphism admits a section in $\mathbf{C}$.

**Proposition 2.3.11.** *Let $\mathbf{C}$ be the category of compactly generated weakly Hausdorff spaces and $G$ a group object in $\mathbf{C}$. We fix $G$-modules $A, B, C$ with corresponding rigidifications $h_A$, $h_B$, and $h_C$. Then the short exact sequences*

$$(A, h_A) \longrightarrow (B, h_B) \longrightarrow (C, h_C)$$

*are in one-to-one correspondence with exact sequences*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

*where all maps are continuous, $A$ carries the subspace topology of $B$ and there is a continuous section $C \longrightarrow B$.*

Proof. Let us start with a short exact sequences

$$(A, h_A) \longrightarrow (B, h_B) \longrightarrow (C, h_C).$$

By Yoneda, morphisms $h_A \longrightarrow h_B$ are given by morphisms $A \longrightarrow B$ etc. Evaluating at $\mathbf{F}\bullet$ hence gives a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with continuous maps. As by assumption

$$\mathrm{Hom}_{\mathbf{C}}(C, B) = h_B(C) \longrightarrow h_C(C) = \mathrm{Hom}_{\mathbf{C}}(C, C)$$

is surjective and the latter includes the identity, this yields a section.

We also clearly have a continuous bijective map $\iota\colon A \longrightarrow \iota(A)$, where the latter carries the subspace topology. The inclusion $\iota(A) \subseteq A$ clearly get mapped to zero in $h_C(\iota(A))$, so has to come from an element in $h_A(\iota(A))$, which is the (continuous) inverse to $\iota$.

Conversely, if we start with a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with all maps continuous, $B \longrightarrow C$ admitting a section and $A$ carrying the subspace topology, it is easy to see that indeed all sequences

$$0 \longrightarrow h_A(X) \longrightarrow h_B(X) \longrightarrow h_C(X) \longrightarrow 0$$

are exact. ∎

**Definition 2.3.12.** Let $\mathbf{C}$ be a topological category, $G$ a topologised monoid in $\mathbf{C}$ and $(A, h_A)$ a $G$-module with $\mathbf{C}$-rigidification. For a normal subgroup $N \leq G$ we define $A^N = (A^{¿N}, h_{A^N})$ by

$$h_{A^N}(X) = \left\{ f \in h_A(X) \mid f(¿X) \subseteq A^{¿N} \right\}.$$

We immediately see that this is again a $G$-module with $\mathbf{C}$-rigidification.

**Remark 2.3.13.** Let $\mathbf{C}$ be a topological category and $G$ a topologised group with normal subgroup $N$. Let $A$ be a $G$-module in the sense that $A$ is an abelian group object in $\mathbf{C}$ together with a morphism $G \times A \longrightarrow A$ subject to the usual diagrams.

Then there is a slightly more natural notion of the invariants $A^N$: Let $g \in ¿G$. Then proposition 2.1.13 yields a morphism

$$m_g\colon A \longrightarrow \mathbf{F}\bullet \times A \longrightarrow G \times A \longrightarrow A$$

that we call multiplication by $g$. We will also denote $m_g - \mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ by $g - 1$.

For a finite set $R \subseteq ¿G$, we denote by

$$A^R = \ker \; A \xrightarrow{(g-1)_{g\in R}} \prod_{g\in R} A.$$

and clearly

$$¿(A^R) = ¿(A)^R.$$

If $\mathbf{C}$ admits arbitrary limits, we can analogously define $A^G$. If there is a finite set $R \subseteq ¿G$ such that $¿(A)^{¿G} = ¿(A)^R$, then we will also call $A^G = A^R$ and it is an easy exercise that both definitions of $A^G$ (when applicable) coincide. In this case, the universal property of the kernel yields an action of $G$ on $A^N$. In the presence of a section $G/N \longrightarrow G$ in $\mathbf{C}$, we also get a morphism $G/N \times A^N \longrightarrow A^N$ and we can check on the level of sets that this gives $A^N$ the structure of a $G/N$-module.
It is easy to check that both definitions of invariants coincide, i. e.,

$$(¿(A^N), \mathrm{Hom}_{\mathbf{C}}(-, A^N)) = ((¿A)^{¿N}, h_{A^N}),$$

where $h_{A^N}$ is defined as in definition 2.3.12.

## 2.4. The Induced Module

Let $\mathbf{C}$ be a topologised category, $G$ a topologised monoid in $\mathbf{C}$ and $H$ a submonoid of $G$. Let $(A, h_A)$ be an $H$-module with $\mathbf{C}$-rigidification.

**Definition 2.4.1.**

$$\mathrm{Ind}_G^H(A)\colon \mathbf{C}^\circ \longrightarrow \mathbf{Set}$$

$$X \longmapsto \{f \in h_A(X \times G) \mid f(x, hg) = h.f(x, g) \text{ for all } x \in ¿X, h \in ¿H, g \in ¿G\}$$

is called the induced module of $A$ from $H$ to $G$.

**Proposition 2.4.2.** *Set $I = \mathrm{Ind}_G^H(A)$, then $I$ is a $G$-module with $\mathbf{C}$-rigidification, if we give the set $I(\mathbf{F}\bullet) \subseteq h_A(G)$ the $¿G$-module action of right translation:*

$$(gf)(\sigma) = f(\sigma g) \text{ for } f \in I(\mathbf{F}\bullet), g, \sigma \in ¿G.$$

Proof. The only difficulty lies in the formalism.
Note first that for $g \in ¿G$ and $f \in I(\mathbf{F}\bullet)$, $gf$ indeed lies in $h_A(G)$, as right-multiplication by $g$ is a morphism in $\mathbf{C}$. It then follows immediately that $gf \in I(\mathbf{F}\bullet)$.
We have to show that for $f \in I(X) \subseteq \mathrm{Hom}_{\mathbf{Set}}(¿X, I(\mathbf{F}\bullet)) \subseteq \mathrm{Hom}_{\mathbf{Set}}(¿X, h_A(G))$ the induced map

$$¿G \times ¿X \xrightarrow{(\mathrm{id}, f)} ¿G \times I(\mathbf{F}\bullet) \xrightarrow{\mu} I(\mathbf{F}\bullet)$$

lies in $I(G \times X) \subseteq h_A(G \times X \times G)$.
For this note that there is a morphism $G \times X \times G \longrightarrow X \times G$ in $\mathbf{C}$, which on the level of elements is given by $(g, x, g') \longmapsto (x, g'g)$. Precomposing with this morphism yields a map

$$h_A(X \times G) \longrightarrow h_A(G \times X \times G)$$

and it is evident that under this map, the subset $I(X)$ gets sent into $I(G \times X)$, which is precisely the map we need. ∎

**Remark 2.4.3.** If $\mathbf{C}$ is the category of analytic manifolds over a non-archimedean field, there is also a less sheafy view on the subject of induced modules: Let $G$ be a group object in $\mathbf{C}$, $H \leq G$ a closed subgroup, and $A$ an analytic representation of $H$, cf. definition 4.2.2. Then there is a natural topology on the induced module $\mathrm{Ind}_G^H(A)$, such that the action of $G$ on $\mathrm{Ind}_G^H(A)$ is itself analytic, cf. [Féa99, Kapitel 4].

CHAPTER 3

# Cohomology of Topologised Monoids

We have now set the stage to define the cohomology of topologised monoids with coefficients in a rigidified module. Our aim for this chapter is to prove some standard results of group cohomology in this setting. Namely, we compare cohomology of topologised monoids with their discrete counterparts in proposition 3.3.1, show the existence of a long exact sequence in theorem 3.3.2, prove two versions of Shapiro's lemma in theorems 3.5.8 and 3.8.1, and show variants of the classical Hochschild-Serre spectral sequence in theorems 3.6.27 and 3.7.6.

## 3.1. Abstract Monoid Cohomology

Note first that the cohomology of monoids is trickier than one might expect.

**Definition 3.1.1.** Let $M$ be an abstract monoid. Then we define the standard resolution as follows: Denote by $F_n$ the free $\mathbb{Z}[M]$-module with basis $M^n$ and define the coboundary operator via

$$\begin{aligned} \partial\colon F_{n+1} &\longrightarrow F_n, \\ (x_1,\dots,x_{n+1}) &\longmapsto x_1\cdot(x_2,\dots,x_{n+1}) + (-1)^{n+1}(x_1,\dots,x_n) \\ &\qquad + \sum_{i=1}^{n}(-1)^i(x_1,\dots,x_{i-1},x_ix_{i+1},x_{i+1},\dots,x_{n+1}). \end{aligned}$$

**Proposition 3.1.2.**

$$\dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

*is a free resolution of the integers.*

Proof. The map $F_0 = \mathbb{Z}[M] \longrightarrow \mathbb{Z}$ is given by the augmentation

$$\sum_{m\in M} a_m m \longmapsto \sum_{m\in M} a_m.$$

Proofs that $\partial\partial = 0$ if $M$ is a group are readily available in the literature and as they at no point use inverse elements or the cancellation property, we will not repeat them here.
To see that the complex is exact, we will construct maps

$$h\colon F_n \longrightarrow F_{n+1}$$

such that $\partial h + h\partial = \mathrm{id}_{F_n}$.
Set

- $h\colon \mathbb{Z} \longrightarrow \mathbb{Z}[M], a \longmapsto a,$
- $h\colon \mathbb{Z}[M] \longrightarrow F_2, a\cdot m \longmapsto a\cdot(m),$
- $h\colon F_n \longrightarrow F_{n+1}, m\cdot(x_1,\dots,x_n) \longmapsto (m,x_1,\dots,x_n)$ continued $\mathbb{Z}$-linearly.

We leave the exactness in low degrees to the unusually enthusiastic reader. For $n \geq 2$, consider an element $m \cdot (x_1, \dots, x_n)$ in the obvious $\mathbb{Z}$-basis of $F_n$. Then on the one hand,

$$\begin{aligned}\partial h(m \cdot (x_1, \dots, x_n)) &= \partial(m, x_1, \dots, x_n) \\ &= m \cdot (x_1, \dots, x_n) + (-1)^{n+1}(m, x_1, \dots, x_{n-1}) - (mx_1, x_2, \dots, x_n) \\ &\quad + \sum_{i=2}^{n} (-1)^i (m, x_1, \dots, x_{i-2}, x_{i-1}x_i, x_{i+1}, \dots, x_n),\end{aligned}$$

and on the other hand

$$\begin{aligned}h\partial(m \cdot (x_1, \dots, x_n)) &= h\left(m \cdot \partial(x_1, \dots, x_n)\right) \\ &= h\Bigg(m \cdot \big(x_1(x_2, \dots, x_n) + (-1)^n (x_1, \dots, x_{n-1}) \\ &\qquad + \sum_{i=1}^{n-1} (-1)^i (x_1, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_n)\big)\Bigg) \\ &= (mx_1, x_2, \dots, x_n) + (-1)^n (m, x_1, \dots, x_{n-1}) \\ &\qquad + \sum_{i=1}^{n-1} (-1)^i (m, x_1, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_n),\end{aligned}$$

so clearly $h\partial + \partial h = \mathrm{id}$. ∎

**Proposition 3.1.3.** *Let $M$ be an abstract monoid and $A$ an $M$-module. Then the inhomogeneous cochain complex computes the cohomology of $A \longmapsto A^M$.*

Proof. Immediate from proposition 3.1.2. ∎

**Remark 3.1.4.** The homogeneous cochains do not necessarily form a free resolution. Indeed, set $M = (\mathbb{Z}/2, \cdot)$. Then the homogeneous complex is given by $F'_n = \mathbb{Z}[M^{n+1}]$ with diagonal action and the usual differential. However, it is *not* a free resolution of the integers: It is evident that $F'_1$ is not cyclic. But every two elements $e_1, e_2 \in F'_1$ admit a non-trivial combination of zero: Multiplied by the monoid element $(0)$, both are contained in $(0,0)\mathbb{Z} \subseteq \mathbb{Z}[M^2] = F'_1$, say, $(0) \cdot e_1 = \alpha \cdot (0,0)$ and $(0) \cdot e_2 = \beta \cdot (0,0)$. If $\alpha$ or $\beta$ is zero, this is a non-trivial combination of zero, otherwise $\beta \cdot (0) \cdot e_1 - \alpha \cdot (0) \cdot e_2$ will do. Nonetheless, $F'_1$ is still a *projective* $\mathbb{Z}[M]$-module. Consider the $\mathbb{Z}$-linear homomorphisms $F'_1 \longrightarrow \mathbb{Z}[M]$

$$\begin{aligned} A_1 &: (1,1) \longmapsto (1); (0,1), (1,0), (0,0) \longmapsto (0) \\ A_2 &: (1,0) \longmapsto (1) - (0); (1,1), (0,1), (0,0) \longmapsto 0 \\ A_3 &: (0,1) \longmapsto (1) - (0); (1,1), (1,0), (0,0) \longmapsto 0. \end{aligned}$$

It is easy to verify that these maps are actually $\mathbb{Z}[M]$-linear and that

$$x \longmapsto (1,1)A_1(x) + (1,0)A_2(x) + (0,1)A_3(x)$$

is the identity on $F'_1$, which by the dual basis theorem is therefore projective. One can analogously show that $F'_\bullet$ is still a projective resolution of $\mathbb{Z}$ in this case.

## 3.2. Setup

For the remainder of this chapter, we fix a topological category $\mathbf{C} = (\mathbf{C}, ¿, \mathbf{F})$ in which everything takes place, a topologised group $G'$ with an open normal subgroup $U'$ and abelian topologised monoids $M_1, \dots, M_r$. Set

$$U = U' \times M_1^{e_1} \times \cdots \times M_r^{e_r}$$

with all $e_i \in \{0,1\}$ and $e_i = 1$ if $M_i$ is not discrete. We also set $G = G' \times \prod_i M_i$ and see that $U \longrightarrow G$ has a cokernel $G/U \cong G' \times \prod_i M_i^{1-e_1}$, whose kernel is $U$ and which is discrete, cf. proposition 2.2.9. We will furthermore use the shorthand $M = \prod_i M_i$.

We let further $N'$ be a normal subgroup of $G'$ and $N = N' \times \prod_i M_i^{e'_i}$ with $e'_i \in \{0,1\}$. It is again evident that $N \longrightarrow G$ has a cokernel (which we denote by $G/N$) and that the kernel of this cokernel is precisely $N$. We furthermore require the existence of a section $s\colon G/N \longrightarrow G$ in $\mathbf{C}$ whose image on the level of sets contains the neutral element. If $N = U$, this exists automatically by proposition 2.2.7 and everything that we prove for the $N$ will also automatically be true for $U$, but the converse does not hold. The section is of vital importance; without it, statements such as $¿(G/N) \cong ¿G/¿N$ need not be true, cf. proposition 2.2.8.

The projections $G \longrightarrow G/N$ and $G \longrightarrow G/U$ will both be denoted by $\pi$. It will always be clear from the context which map is meant.

The section gives rise to two important morphisms: On the one hand, the *choice of a representative* morphism $(-)^*\colon G \longrightarrow G$, which is the composition of the projection onto $G/N$ followed by the section $s\colon G/N \longrightarrow G$, and on the other hand the morphism $(-)_N\colon G \longrightarrow G$ which on $¿G$ is given by $x \longmapsto (x^*)^{-1}x$ with the obvious interpretation of this on the monoid parts (either the identity or constant 1). It is clear that $(-)_N$ factors through $N \longrightarrow G$, and that the composition

$$N \longrightarrow G \xrightarrow{(-)_N} N$$

is the identity on $N$. Evidently we can factor the identity on $G$ as

$$G \xrightarrow{\Delta} G \times G \xrightarrow{((-)^*,(-)_N)} G \times G \xrightarrow{\mu} G.$$

It is important that these maps exist in $\mathbf{C}$, which is why we spell out these details.

We also fix a $G$-module with $\mathbf{C}$-rigidification $A = (A, h_A)$ and assume that $G/U$ is $A$-pliant, cf. example 2.3.2 for examples of what this means in practice.

For $n \geq 0$ set

- $X^n = X^n(G,A) = h_A(G^n) \subseteq \mathrm{Hom}_{\mathbf{Set}}(¿(G)^n, A)$, the inhomogeneous cochains, with the convention that $G^0 \cong \mathbf{F}\bullet$ and hence $X^0 = A$,
- $C^n = C^n(G,A)$ those maps $f \in X^n(G,A)$ such that $f(x_1,\ldots,x_n) = 0$ if at least one of the $x_i$ is 1, the normalised cochains,
- $I^jC^n$ those maps in $C^n$ that come from morphisms in $h_A(G^{n-j} \times (G/N)^j)$ (i. e., $I^jC^n = C^n \cap h_A(G^{n-j} \times (G/N)^j)$, the intersection taking place in $X^n$). We set $I^jC^n = 0$ for $j > n$.

We can characterise the filtration as follows:

**Lemma 3.2.1.** *Let $f \in C^n$. Then $f \in I^jC^n$ if and only if the last $j$ arguments are $¿N$-invariant, i. e.,*

$$f(x_1,\ldots,x_{n-j},x_{n-j+1}\sigma_1,\ldots,x_n\sigma_j) = f(x_1,\ldots,x_n) \text{ for all } x_i \in ¿G, \sigma_i \in ¿N.$$

Proof. The "only if" part of the proposition is clear.

For the "if" part, we only show the case of $j = n = 1$, as the other cases follow completely analogously.

As we have a section $s\colon G/N \longrightarrow G$, proposition 2.2.8 shows that $¿G/¿N \cong ¿(G/N)$. Consider the following commutative diagram:

$$\begin{tikzcd}
h_A(G/N) \arrow[r, hook] \arrow[d, "h_A(\pi)"] \arrow[dd, bend right=60, "\mathrm{id}"'] & \mathrm{Hom}_{\mathbf{Set}}(¿G/¿N, A) \arrow[d, "\mathrm{Hom}_{\mathbf{Set}}(¿\pi, A)"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
h_A(G) \arrow[r, hook] \arrow[d, "h_A(s)"] & \mathrm{Hom}_{\mathbf{Set}}(¿G, A) \arrow[d, "\mathrm{Hom}_{\mathbf{Set}}(¿s, A)"] \\
h_A(G/N) \arrow[r, hook] & \mathrm{Hom}_{\mathbf{Set}}(¿G/¿N, A)
\end{tikzcd}$$

Starting with an $¿N$-invariant $f \in h_A(G)$, we know from the assumption that it comes from an element in $\mathrm{Hom}_{\mathbf{Set}}(¿G/¿N, A)$. But the diagram implies that this element is necessarily identical to $h_A(s)(f)$. ∎

Note that for $f \in X^n$, the induced face maps

$$s_k(f)\colon (x_i) \longmapsto f(x_1, \ldots, x_k, 1, x_{k+1}, \ldots, x_{n-1})$$

lie in $X^{n-1}$.

We will often omit the forgetful functor $¿$, e. g., instead of $x \in ¿N$ we will simply write $x \in N$.

**Proposition 3.2.2.** *The assignment*

$$\begin{aligned}\partial f(x_1, \ldots, x_{n+1}) = x_1.f(x_2, \ldots, x_{n+1}) &+ (-1)^{n+1} f(x_1, \ldots, x_n) \\ &+ \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{n+1}),\end{aligned}$$

*induced by definition 3.1.1, gives rise to well-defines maps*

$$\begin{aligned}\partial\colon X^n &\longrightarrow X^{n+1}, \\ \partial\colon C^n &\longrightarrow C^{n+1}, \\ \partial\colon I^j C^j &\longrightarrow I^j C^{n+1}.\end{aligned}$$

Proof. To see that $\partial f \in X^{n+1}$ for $f \in X^n$, it suffices to check this for each of the summands, as $X^{n+1}$ is an abelian group per definition. All but the first summand stem from a composition

$$G^{n+1} \longrightarrow G^n \overset{f}{\rightsquigarrow} A$$

and hence lie in $X^{n+1}$. That the first summand lies in $X^{n+1}$ is precisely the second requirement in definition 2.3.8.

If $f$ is normalised, then in the expansion of $\partial f(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_n)$ there are exactly two terms that are not trivially zero. But these terms are identical except for an opposing sign and hence cancel.

If $f \in I^j C^n$, we want to show that the last $j$ arguments of $\partial f$ are $¿N$-invariant. But the last $j$ arguments of $\partial f$ only contribute to the last $j$ arguments of every individual summand in the coboundary expansion of $\partial f$, which are $¿N$-invariant. ∎

**Remark 3.2.3.** It might seem artificial to consider cochains whose *last* $j$ arguments are $¿N$-invariant instead of the *first* $j$, which will also lead to somewhat counter-intuitive definitions later on. But the equation

$$\partial f(x, y) = x.f(y) - f(xy) + f(x)$$

shows that if $f$ is $¿N$-invariant, only the second argument of $\partial f$ is $¿N$-invariant and not necessarily the first.

**Remark 3.2.4.** For $G'$ we can also form the complex $\widetilde{X}^\bullet(G', A)$ given by

$$\widetilde{X}^n(G', A) = h_A((G')^{n+1})$$

with differential

$$\widetilde{\partial}(f)(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}).$$

By the usual arguments (cf. e. g. [NSW08, p. 14]), we get an isomorphism of complexes

$$X^\bullet(G', A) \cong \widetilde{X}^\bullet(G', A).$$

However, the usual morphism

$$\phi\colon X^n(G', A) \longrightarrow \widetilde{X}^n(G', A),$$

which is given by

$$\phi(f)(x_0, \ldots, x_n) = x_0 f(x_0^{-1}x_1, x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x_n)$$

can only be defined for the group object $G'$ and not for the monoid object $G$. The example in remark 3.1.4 shows that both complexes cannot be isomorphic in general. They might still be quasi-isomorphic, however, we were unable to show this.

### 3.3. Cohomology of Topologised Monoids

We will call $H^\bullet(G, A) = H^\bullet(X^\bullet, \partial)$ the ($\mathbf{C}$-)cohomology of $G$ with coefficients in $A$. Note that if $G$ is $A$-pliant, the $\mathbf{C}$-cohomology of $G$ with coefficients in $A$ is just the (abstract) monoid cohomology of $¿G$ with coefficients in $A$. Generally, comparing topological cohomology with abstract cohomology only works well in low degrees.

**Proposition 3.3.1.** *For every $n$ there is a natural morphism*

$$H^n(G, A) \longrightarrow H^n(¿G, A),$$

*which is an isomorphism for $n = 0$ and injective for $n = 1$.*

Proof. Clearly the following diagram commutes

$$\begin{tikzcd}
X^{n-1}(G, A) \arrow[r, "\partial"] \arrow[d, hook] & X^n(G, A) \arrow[r, "\partial"] \arrow[d, hook] & X^{n+1}(G, A) \arrow[d, hook] \\
X^{n-1}(¿G, A) \arrow[r, "\partial"] & X^n(¿G, A) \arrow[r, "\partial"] & X^{n+1}(¿G, A),
\end{tikzcd}$$

which yields the required comparison morphisms. By definition,

$$X^0(G, A) = X^0(¿G, A),$$

so the morphism is indeed an isomorphism for $n = 0$ and injective for $n = 1$. ∎

Our definition of an exact sequence of rigidified $G$-modules is custom tailored to admit a long exact sequence of cohomology groups.

**Theorem 3.3.2.** *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*be an exact sequence of rigidified $G$-modules. Then there is a long exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & A^{¿G} \arrow[r] & B^{¿G} \arrow[r] & C^{¿G} \arrow[r] & \dots \arrow[dlll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
 & H^n(G,A) \arrow[r] & H^n(G,B) \arrow[r] & H^n(G,C) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
 & H^{n+1}(G,A) \arrow[r] & \dots & &
\end{tikzcd}$$

Proof. By definition of exactness of a sequence of rigidified $G$-modules, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
0 \arrow[r] & X^n(G,A) \arrow[r] \arrow[d, "\partial"] & X^n(G,B) \arrow[r] \arrow[d, "\partial"] & X^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & X^{n+1}(G,A) \arrow[r] \arrow[d, "\partial"] & X^{n+1}(G,B) \arrow[r] \arrow[d, "\partial"] & X^{n+1}(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

As usual, the snake lemma implies the existence of the long exact sequence as required. ∎

As in the classical case, the normalised cochains compute the same cohomology as all inhomogeneous cochains.

**Proposition 3.3.3.** *The inclusion $C^\bullet \longrightarrow X^\bullet$ is a quasi-isomorphism.*

Proof. The original proof in [EM47, § 6] works without issues, as for every $f \in X^n$ the map $s_k f\colon (x_1,\dots,x_{n-1}) \longmapsto f(x_1,\dots,x_{k-1},1,x_k,\dots,x_{n-1})$ is again in $X^{n-1}$. ∎

## 3.4. Cohomology and Extensions

For discrete coefficients, the groups $H^i(G,A)$ have concrete interpretations for $i \leq 2$. We will give two concrete interpretations for $H^1$ also in the topological case. For this matter, we fix in this section another topological category $\mathbf{D}$ and a rigidification

$$h\colon \mathbf{C}^\circ \times \mathbf{D} \longrightarrow \mathbf{Set}.$$

In this section, we assume that $A$ is actually an abelian group object in $\mathbf{D}$ and that $h_A = h(-,A)$.

**Definition 3.4.1.** An $A$-torsor is an object $X$ in $\mathbf{D}$ with a right action from $A$ (i. e., an arrow $\mu\colon X\times A \longrightarrow X$ in $\mathbf{D}$ subject to the usual conditions), such that the induced map

$$m\colon X\times A \xrightarrow{(\mathrm{id},\mu)} X\times X$$

is an isomorphism. The composition $\pi_A \circ m^{-1}\colon X\times X \longrightarrow A$ will be denoted by $\backslash$.

An $A$-torsor with $G$-rigidification is an $A$-torsor $X$, such that $(¿X, h(-,X))$ is a $G$-set with $\mathbf{C}$-rigidification, and on the level of sets for all $g \in ¿G$, $x \in ¿X$, and $a \in ¿A$ the following holds:

$$g\mu(x,a) = \mu(gx, ga).$$

An isomorphism of $A$-torsors with $G$-rigidification $j\colon X \longrightarrow Y$ is an isomorphism in $D$ such that on the level of sets, $j$ commutes with both the $A$- and $G$-action.

**Theorem 3.4.2.** *$H^1(G, A)$ stands in one-to-one correspondence with isomorphism classes of $A$-torsors with $G$-rigidification.*

Proof. Given a torsor $X$, we construct an element in $H^1(G, A)$ as follows: First choose $x \in ¿X$. The definition of a $G$-set, together with the existence of constant maps in topological categories, imply the existence of $\cdot x \in h(G, X)$, which on the level of sets is just given by $g \longmapsto g \cdot x$. As $h$ is a bifunctor, we can compose it as follows:

$$\begin{tikzcd}
G \arrow[r, squiggly, "\cdot x"] \arrow[ddr, squiggly, "c_X"'] & X \arrow[d, "{(x,\mathrm{id})}"] \\
& X \times X \arrow[d, "\backslash"] \\
& X
\end{tikzcd}$$

Note that on the level of sets, $c_X(g)$ is the unique element such that $g \cdot x = xc_X(g)$. The verification that $c_X$ is a well-defined cocycle independent of $x \in ¿X$ is standard.

For the other direction, take a cocycle in $H^1(G, A)$, represented by $c\colon G \rightsquigarrow A$. Define $X = A$ and $\mu\colon X \times A \longrightarrow X$ as the addition in $A$. We define the $¿G$-action on $¿X$ via

$$g.x = c(g) + g \cdot x,$$

where $g \cdot x$ is the given action of $G$ on $A$. It is easy to check that this gives a well defined $G$-module with $\mathbf{C}$-rigidification. The verifications that this construction is (up to isomorphism) independent of the class of $c$ and both left and right inverse to the previous construction is again standard. ∎

Let $R$ be a ring object in $\mathbf{D}$, which is furthermore commutative and unitary. For the remainder of this section we assume that $A$ is an $R$-module, i. e., we additionally require the existence of a morphism $R \times A \longrightarrow A$ subject to the usual conditions. We also assume that $G$ operates on $R$ via ring endomorphisms in $\mathbf{D}$ and that on the level of sets, the action on $A$ is $R$-semi-linear, i. e.,

$$g \cdot (r \cdot a) = (g \cdot r) \cdot (g \cdot a) \text{ for all } g \in ¿G, r \in ¿R, a \in ¿A.$$

We will call such an object a semi-linear $G$-module over $R$.

An exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of semi-linear $G$-modules is an exact sequence of $G$-modules with $\mathbf{C}$-rigidification in the sense of definition 2.3.8, where we additionally require the morphisms

$$0 \longrightarrow h_{M'}(X) \longrightarrow h_M(X) \longrightarrow h_{M''}(X) \longrightarrow 0$$

to be $R$-linear. In the above exact sequence, $M$ will be called an extension of $M''$ by $M'$.

An equivalence of extensions of $M''$ by $M'$, which will be denoted by $M \approx \widetilde{M}$, is an isomorphism of functors $h_M \xrightarrow{\cong} h_{\widetilde{M}}$ such that for all $X$ the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & h_{M'}(X) \arrow[r] \arrow[d, equal] & h_M(X) \arrow[r] \arrow[d, "\cong"] & h_{M''}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & h_{M'}(X) \arrow[r] & h_{\widetilde{M}}(X) \arrow[r] & h_{M''}(X) \arrow[r] & 0.
\end{tikzcd}$$

The equivalence classes of extensions of $M''$ by $M'$ will be denoted by $\mathrm{Ext}(M'', M')$.

**Theorem 3.4.3.** $H^1(G, A) \cong \mathrm{Ext}(R, A)$.

Proof. Because of the similarity to theorem 3.4.2, we only sketch the construction. Let

$$0 \longrightarrow A \longrightarrow E \xrightarrow{p} R \longrightarrow 0$$

be an extension and denote by $1 \in ¿R$ the unit in $R$. We can construct a cochain via

$$g \longmapsto g \cdot e - e,$$

where $e \in ¿E$ is any preimage of 1.

On the other hand, for a cochain $c\colon G \rightsquigarrow A$ define a $G$-action on $A \times R$ via

$$g \cdot (a, r) = ((g \cdot r) \cdot c(g) + g \cdot a, g \cdot r),$$

which is a well-defined semi-linear $G$-module over $R$. The universal property of the product yields the exactness. ∎

**Remark 3.4.4.** An alternative to the construction of theorem 3.4.3 goes as follows: As $\mathbf{D}$ is a topological category and all limits exist, we get an object and a morphism $X = p^{-1}(1) \longrightarrow E$. It follows that we have an action $X \times A \longrightarrow X$ given by addition in $E$ and that the composition

$$X \times A \xrightarrow{(\mathrm{id},+)} X \times X \xrightarrow{(\mathrm{id}, x_2 - x_1)} X \times E$$

factors through a morphism $X \times A \longrightarrow X \times A$ and induces the identity, i. e., $X$ is a $A$-torsor. It is also evident that $X$ inherits a $G$-rigidification from $E$.

Directly constructing an extension from a torsor $X$ is regrettably not straight-forward, as it is very cumbersome to define the correct $R$-module structure on $X \times R$.

## 3.5. Shapiro's Lemma for Topologised Groups

We will now prove Shapiro's lemma for the induction of subgroups, i. e., we will assume in this section that $M = 1$. Later on we will also prove Shapiro's lemma for actual monoids, cf. section 3.8.

Let $H \leq G$ be a subgroup of $G$ and assume the existence of a map

$$_H(-)\colon G \longrightarrow H$$

in $\mathbf{C}$ with the following properties:

- $_H(1) = 1$,
- $_H(hg) = h \cdot {_H(g)}$ for all $h \in H, g \in G$

**Remark 3.5.1.** Instead of requiring the existence of such a morphism in $\mathbf{C}$, one could also construct it as follows: Assume that the push-out $H\backslash G$ of the following diagram exists:

$$\begin{tikzcd} H \times G \arrow[r, "\pi_2"] \arrow[d, "\mu"] & G \arrow[d, "p"] \\ G \arrow[r] & H\backslash G \end{tikzcd}$$

$H\backslash G$ is then the space of right cosets of $H$ in $G$. Assume the existence of a section $s\colon H\backslash G \longrightarrow G$ in $\mathbf{C}$ with $p \circ s = \mathrm{id}_{H\backslash G}$ and with the neutral element contained in the image of $s$. We would then define

$$_H(g) = gs(p(g))^{-1}.$$

Note the similarity with $(-)_N$, which was defined as $(g)_N = s(\pi(g))^{-1}g$. As we wanted to use the same convention for the action of $G$ on $\mathrm{Ind}_G^H(A)$ as in the literature, we have to use a different convention here. However, we would then need to check many basic properties of this construction, which wouldn't shed any additional light on what actually happens.

**Example 3.5.2.** These requirements are always satisfied if $G$ is an analytic group and $H$ a closed subgroup, cf. [Bou89, section III.1.6].

**Definition 3.5.3.** Consider the following maps:

(1) $\alpha_n : C^n(G, \mathrm{Ind}_G^H(A)) \longrightarrow C^n(H, A)$ given by
$$\alpha_n(f)(h_1, \dots, h_n) = f(h_1, \dots, h_n, 1).$$
Note that our formalism ensures that this is a well-defined map.

(2) $\beta_n : C^n(H, A) \longrightarrow C^n(G, \mathrm{Ind}_G^H(A))$ by
$$\begin{aligned}\beta_n(f)(g_1, \dots, g_n, x) =& {}_H(x) f({}_H(x)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots, \\ & {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n)).\end{aligned}$$
As we can express $\beta_n(f)$ in a (very large) diagram in $\mathbf{C}$, it again gives a well-defined element in $h_A(G^n \times G)$, and we immediately verify that it indeed lies in $C^n(G, \mathrm{Ind}_G^H(A))$.

(3) $\kappa_{n+1} : C^{n+1}(G, \mathrm{Ind}_G^H(A)) \longrightarrow C^n(G, \mathrm{Ind}_G^H(A))$ given by
$$\begin{aligned}&\kappa_{n+1} f(g_1, \dots, g_n, x) \\ &\quad = f(x^{-1}{}_H(x), {}_H(x)^{-1}{}_H(xg_1), \\ &\qquad\quad {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots, {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n), x) \\ &\qquad + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i, (xg_1 \dots g_i)^{-1}{}_H(xg_1 \dots g_i), \\ &\qquad\qquad\qquad {}_H(xg_1 \dots g_i)^{-1}{}_H(xg_1 \dots g_{i+1}), \dots, \\ &\qquad\qquad\qquad {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n), x).\end{aligned}$$
The summands of $\kappa_{n+1}$ are given by morphisms in $\mathbf{C}$ followed by $f$, so indeed $\kappa_{n+1}(f) \in X^n(G, \mathrm{Ind}_G^H(A))$ and also in $C^n(G, \mathrm{Ind}_G^H(A))$.

**Remark 3.5.4.** While we can adapt the definition of $\beta$ to also work in a monoid setting by explaining what the map is supposed to do on the monoid part, this is no longer true for $\kappa$. We will prove later in section 3.8 that Shapiro's lemma still holds in the monoid setting.

The proof of Shapiro's lemma now consists of the following few lemmata which show that $\alpha_\bullet$ and $\beta_\bullet$ are quasi-isomorphisms. Their proofs are routine, excruciatingly unenlightening, and given only for sake of completeness.

**Lemma 3.5.5.** *$\alpha_\bullet$ and $\beta_\bullet$ are maps of chain complexes, i. e., they commute with $\partial$.*

Proof. First,
$$\begin{aligned}&\partial\alpha_n f(h_1, \dots, h_{n+1}) \\ &\quad = h_1\alpha_n f(h_2, \dots, h_{n+1}) + (-1)^{n+1}\alpha_n f(h_1, \dots, h_n) \\ &\qquad + \sum_{i=1}^{n} (-1)^i \alpha_n f(h_1, \dots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \dots, h_{n+1}) \\ &\quad = h_1 f(h_2, \dots, h_{n+1}, 1) + (-1)^{n+1} f(h_1, \dots, h_n, 1) \\ &\qquad + \sum_{i=1}^{n} (-1)^i f(h_1, \dots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \dots, h_{n+1}, 1) \\ &\quad = f(h_2, \dots, h_{n+1}, 1 \cdot h_1) + (-1)^{n+1} f(h_1, \dots, h_n, 1) \\ &\qquad + \sum_{i=1}^{n} (-1)^i f(h_1, \dots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \dots, h_{n+1}, 1)\end{aligned}$$

$$\begin{aligned} &= \partial f(h_1, \dots, h_{n+1}, 1) \\ &= \alpha_{n+1}\partial f(h_1, \dots, h_{n+1}). \end{aligned}$$

Secondly,

$$\begin{aligned} &\partial\beta_n f(g_1, \dots, g_{n+1}, x) \\ &\quad= \beta_n f(g_2, \dots, g_{n+1}, xg_1) + (-1)^{n+1}\beta_n f(g_1, \dots, g_n, x) \\ &\qquad+ \sum_{i=1}^{n}(-1)^i \beta_n f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}, x) \\ &\quad= {}_H(xg_1) f\big({}_H(xg_1)^{-1}{}_H(xg_1g_2), {}_H(xg_1g_2)^{-1}{}_H(xg_1g_2g_3), \dots, \\ &\qquad\qquad {}_H(xg_1\dots g_n)^{-1}{}_H(xg_1\dots g_{n+1})\big) \\ &\qquad+ \sum_{i=1}^{n}(-1)^i {}_H(x) f\big({}_H(x)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots, \\ &\qquad\qquad\qquad {}_H(xg_1\dots g_{i-1})^{-1}{}_H(xg_1\dots g_{i+1}), \\ &\qquad\qquad\qquad {}_H(xg_1\dots g_{i+1})^{-1}{}_H(xg_1\dots g_{i+2}), \dots, \\ &\qquad\qquad\qquad {}_H(xg_1\dots g_n)^{-1}{}_H(xg_1\dots g_{n+1}))\big) \\ &\qquad+ (-1)^{n+1}{}_H(x) f({}_H(x)^{-1}{}_H(xg_1), \dots, {}_H(xg_1\dots g_{n-1})^{-1}{}_H(xg_1\dots g_n)) \\ &\quad= {}_H(x)\left({}_H(x)^{-1}{}_H(xg_1) f(\dots) + \sum_{i=1}^{n}(-1)^i f(\dots) + (-1)^{n+1} f(\dots)\right) \\ &\quad= {}_H(x)\partial f\big({}_H(x)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots, \\ &\qquad\qquad {}_H(xg_1\dots g_n)^{-1}{}_H(xg_1\dots g_{n+1})\big) \\ &\quad= \beta_{n+1}\partial f(g_1, \dots, g_{n+1}, x). \end{aligned}$$

■

**Lemma 3.5.6.** $\alpha_\bullet \circ \beta_\bullet = \mathrm{id}_{C^\bullet(H,A)}$.

Proof. Note that ${}_H(-)$ restricted to $H$ is the identity. We hence have

$$\begin{aligned} \alpha_n\beta_n f(h_1, \dots, h_n) &= \beta_n f(h_1, \dots, h_n, 1) \\ &= {}_H(1) f\big({}_H(1)^{-1}{}_H(1\cdot h_1), {}_H(1\cdot h_1)^{-1}{}_H(1\cdot h_1h_2), \dots, \\ &\qquad {}_H(1\cdot h_1\dots h_{n-1})^{-1}{}_H(1\cdot h_1\dots h_n)\big) \\ &= f(h_1, \dots, h_n). \end{aligned}$$

■

**Lemma 3.5.7.** $\partial \circ \kappa_n + \kappa_{n+1} \circ \partial = \beta_n \circ \alpha_n - \mathrm{id}_{C^n(G,\mathrm{Ind}_G^H(A))}$, *i. e., $\kappa_\bullet$ is a chain homotopy from $\beta_\bullet \circ \alpha_\bullet$ to the identity.*

Proof. This is going to be as bad as it looks. Let us first compute $\partial \circ \kappa_n$ and $\kappa_{n+1} \circ \partial$, subtract $(\beta_n \circ \alpha_n - \mathrm{id})$ from this and show that the sum of the remaining terms is zero. We first compute $\partial \circ \kappa_n$:

$$\begin{aligned} &(\partial \circ \kappa_n) f(g_1, \dots, g_n, x) \\ &\qquad= \kappa_n(f)(g_2, \dots, g_n, xg_1) + (-1)^n \kappa_n(f)(g_1, \dots, g_{n-1}, x) \\ &\qquad\quad+ \sum_{i=1}^{n-1}(-1)^i \kappa_n(f)(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n, x). \end{aligned}$$

We can also expand

$$
\begin{aligned}
(\blacksquare)\qquad & \kappa_n f(g_2, \dots, g_n, xg_1) \\
&= f((xg_1)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \\
&\qquad {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots, {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n), xg_1) \\
&\quad + \sum_{j=1}^{n} (-1)^j f(g_2, \dots, g_{j+1}, (xg_1 \dots g_{j+1})^{-1}{}_H(xg_1 \dots g_{j+1}), \\
&\qquad\qquad {}_H(xg_1 \dots g_{j+1})^{-1}{}_H(xg_1 \dots g_{j+2}), \dots, \\
&\qquad\qquad {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n), xg_1),
\end{aligned}
$$

$$
\begin{aligned}
& (-1)^n \kappa_n f(g_1, \dots, g_{n-1}, x) \\
(\Diamond)\qquad &= (-1)^n f(x^{-1}{}_H(x), {}_H(x)^{-1}{}_H(xg_1), \\
&\qquad {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots, {}_H(xg_1 \dots g_{n-2})^{-1}{}_H(xg_1 \dots g_{n-1}), x) \\
(\star)\qquad &\quad + \sum_{j=1}^{n-1} (-1)^{j+n} f(g_1, \dots, g_j, (xg_1 \dots g_j)^{-1}{}_H(xg_1 \dots g_j), \\
&\qquad\qquad {}_H(xg_1 \dots g_j)^{-1}{}_H(xg_1 \dots g_{j+1}) \dots, \\
&\qquad\qquad {}_H(xg_1 \dots g_{n-2})^{-1}{}_H(xg_1 \dots g_{n-1}), x)
\end{aligned}
$$

and

$$
\begin{aligned}
& (-1)^i \kappa_n(f)(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n, x) \\
(\maltese.i)\qquad &= (-1)^i f(x^{-1}{}_H(x), {}_H(x)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots, \\
&\qquad {}_H(xg_1 \dots g_{i-2})^{-1}{}_H(xg_1 \dots g_{i-1}), \\
&\qquad {}_H(xg_1 \dots g_{i-1})^{-1}{}_H(xg_1 \dots g_{i+1}), \\
&\qquad {}_H(xg_1 \dots g_{i+1})^{-1}{}_H(xg_1 \dots g_{i+2}), \dots, \\
&\qquad {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n), x) \\
(◖.i)\qquad &\quad + \sum_{j=1}^{i-1} (-1)^{i+j} f(g_1, \dots, g_j, (xg_1 \dots g_j)^{-1}{}_H(xg_1 \dots g_{j+1}), \\
&\qquad\qquad {}_H(xg_1 \dots g_{j+1})^{-1}{}_H(xg_1 \dots g_{j+2}), \dots, \\
&\qquad\qquad {}_H(xg_1 \dots g_{i-2})^{-1}{}_H(xg_1 \dots g_{i-1}), \\
&\qquad\qquad {}_H(xg_1 \dots g_{i-1})^{-1}{}_H(xg_1 \dots g_{i+1}), \\
&\qquad\qquad {}_H(xg_1 \dots g_{i+1})^{-1}{}_H(xg_1 \dots g_{i+2})^{-1}, \dots, \\
&\qquad\qquad {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n), x) \\
(\blacktriangleleft.i)\qquad &\quad + \sum_{j=i}^{n-1} (-1)^{i+j} f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{j+1}, \\
&\qquad\qquad (xg_1 \dots g_{j+1})^{-1}{}_H(xg_1 \dots g_{j+1}), \\
&\qquad\qquad {}_H(xg_1 \dots g_{j+1})^{-1}{}_H(xg_1 \dots g_{j+2}), \dots, \\
&\qquad\qquad {}_H(xg_1 \dots g_{n-1})^{-1}{}_H(xg_1 \dots g_n), x).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(\kappa_{n+1} \circ \partial) f(g_1, \ldots, g_n, x) \\
&\qquad = \partial(f)(x^{-1}{}_H(x), {}_H(x)^{-1}{}_H(xg_1), \\
&\qquad\qquad {}_H(xg_1)^{-1}{}_H(xg_1g_2), \ldots, {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), x) \\
&\qquad\quad + \sum_{j=1}^{n} (-1)^j \partial(f)(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_j), \\
&\qquad\qquad\qquad {}_H(xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \ldots, \\
&\qquad\qquad\qquad {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), x)
\end{aligned}
$$

and

$$
\begin{aligned}
&(-1)^j \partial(f)(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_j), \\
&\qquad {}_H(xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), x)
\end{aligned}
$$

(■.$j$)
$$
\begin{aligned}
&= (-1)^j f(g_2, \ldots, g_j, (xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_j), \\
&\qquad {}_H(xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), xg_1)
\end{aligned}
$$

(► .$j$)
$$
\begin{aligned}
&+ \sum_{i=1}^{j-1} (-1)^{i+j} f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_j, \\
&\qquad (xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_j), \\
&\qquad {}_H(xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), x)
\end{aligned}
$$

(☾ .$j$)
$$
\begin{aligned}
&+ (-1)^{j+j} f(g_1, \ldots, g_{j-1}, (xg_1 \ldots g_{j-1})^{-1}{}_H(xg_1 \ldots g_j), \\
&\qquad {}_H(xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), x)
\end{aligned}
$$

(☾ $'$.$j$)
$$
\begin{aligned}
&+ (-1)^{j+j+1} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \\
&\qquad {}_H(xg_1 \ldots g_{j+1})^{-1}{}_H(xg_1 \ldots g_{j+2}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), x)
\end{aligned}
$$

( ◗.$j$)
$$
\begin{aligned}
&+ \sum_{i=j+2}^{n} (-1)^{i+j} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_j), \\
&\qquad {}_H(xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{i-3})^{-1}{}_H(xg_1 \ldots g_{i-2}), \\
&\qquad {}_H(xg_1 \ldots g_{i-2})^{-1}{}_H(xg_1 \ldots g_i), \\
&\qquad {}_H(xg_1 \ldots g_i)^{-1}{}_H(xg_1 \ldots g_{i+1}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{n-1})^{-1}{}_H(xg_1 \ldots g_n), x)
\end{aligned}
$$

(★.$j$)
$$
\begin{aligned}
&+ (-1)^{j+n+1} f(g_1, \ldots, g_j, (xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_j), \\
&\qquad {}_H(xg_1 \ldots g_j)^{-1}{}_H(xg_1 \ldots g_{j+1}), \ldots, \\
&\qquad {}_H(xg_1 \ldots g_{n-2})^{-1}{}_H(xg_1 \ldots g_{n-1}), x).
\end{aligned}
$$

We furthermore expand

$$
\begin{aligned}
&\partial(f)(x^{-1}{}_H(x), {}_H(x)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots,\\
&\qquad {}_H(xg_1\dots g_{n-1})^{-1}{}_H(xg_1\dots g_n), x)\\
(\circ)\quad &\qquad = f({}_H(x)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots,\\
&\qquad\qquad {}_H(xg_1\dots g_{n-1})^{-1}{}_H(xg_1\dots g_n), {}_H(x))\\
(\diamond')\quad &\qquad + (-1)^{n+1} f(x^{-1}{}_H(x), {}_H(x)^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots,\\
&\qquad\qquad {}_H(xg_1\dots g_{n-2})^{-1}{}_H(xg_1\dots g_{n-1}), x)\\
(\P)\quad &\qquad - f(x^{-1}{}_H(xg_1), {}_H(xg_1)^{-1}{}_H(xg_1g_2), \dots,\\
&\qquad\qquad {}_H(xg_1\dots g_{n-1})^{-1}{}_H(xg_1\dots g_n), x)\\
(\maltese)\quad &\qquad + \sum_{i=2}^{n}(-1)^i f(x^{-1}{}_H(x), {}_H(x)^{-1}{}_H(xg_1), \dots,\\
&\qquad\qquad {}_H(xg_1\dots g_{i-3})^{-1}{}_H(xg_1\dots g_{i-2}),\\
&\qquad\qquad {}_H(xg_1\dots g_{i-2})^{-1}{}_H(xg_1\dots g_i),\\
&\qquad\qquad {}_H(xg_1\dots g_i)^{-1}{}_H(xg_1\dots g_{i+1}), \dots,\\
&\qquad\qquad {}_H(xg_1\dots g_{n-1})^{-1}{}_H(xg_1\dots g_n), x)
\end{aligned}
$$

Clearly $(\circ) = \beta_n \circ \alpha_n(f)(g_1, \dots, g_n, x)$ and $(\star.n) = -f(g_1, \dots, g_n, x)$, so it remains to show that the other summands amount to zero.

Note first that

$$
\begin{aligned}
(☾ .1) &= -(\P),\\
(☾ '.j) &= -(☾ .j+1) \qquad\qquad \text{for } j = 1, \dots, n-1,\\
(☾ '.n) &= 0,
\end{aligned}
$$

so in $\kappa_{n+1} \circ \partial$ all $(☾ ), (☾ ')$, and $(\P)$-terms cancel. Furthermore it is immediately evident that

$$
\begin{aligned}
(\diamond) &= -(\diamond'),\\
\sum_{j=1}^{n}(\blacksquare.j) &= -(\blacksquare),\\
\sum_{i=1}^{n-1}(\maltese'.i) &= -(\maltese), \text{ and}\\
\sum_{i=1}^{n-1}(\star.j) &= -(\star).
\end{aligned}
$$

It remains to show that

$$\sum_{i=1}^{n}(◖ .i) = -\sum_{j=1}^{n}( ◗.j)$$

and

$$\sum_{i=1}^{n}(\blacktriangleleft .i) = -\sum_{j=1}^{n}(\blacktriangleright .j).$$

Write

$$\begin{aligned}F(i,j) = f(&g_1,\dots,g_j,(xg_1\dots g_j)^{-1}{}_H(xg_1\dots g_{j+1}),\\&{}_H(xg_1\dots g_{j+1})^{-1}{}_H(xg_1\dots g_{j+2}),\dots,\\&{}_H(xg_1\dots g_{i-2})^{-1}{}_H(xg_1\dots g_{i-1}),\\&{}_H(xg_1\dots g_{i-1})^{-1}{}_H(xg_1\dots g_{i+1}),\\&{}_H(xg_1\dots g_{i+1})^{-1}{}_H(xg_1\dots g_{i+2})^{-1},\dots,\\&{}_H(xg_1\dots g_{n-1})^{-1}{}_H(xg_1\dots g_n),x),\\G(i,j) = f(&g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_{j+1},\\&(xg_1\dots g_{j+1})^{-1}{}_H(xg_1\dots g_{j+1}),\\&{}_H(xg_1\dots g_{j+1})^{-1}{}_H(xg_1\dots g_{j+2}),\dots,\\&{}_H(xg_1\dots g_{n-1})^{-1}{}_H(xg_1\dots g_n),x),\end{aligned}$$

then

$$\begin{aligned}\sum_{i=1}^{n}(◖.i) &= \sum_{i=1}^{n}\sum_{j=1}^{i-1}(-1)^{i+j}F(i,j) = \sum_{1\le j<i\le n}(-1)^{i+j}F(i,j),\\ \sum_{j=1}^{n}(◗.j) &= \sum_{j=1}^{n}\sum_{i=j+2}^{n}(-1)^{i+j}F(i-1,j) = \sum_{1\le j<i\le n}(-1)^{i+j+1}F(i,j),\\ \sum_{i=1}^{n}(◀.i) &= \sum_{i=1}^{n}\sum_{j=i}^{n-1}(-1)^{i+j}G(i,j) = \sum_{1\le i\le j\le n-1}(-1)^{i+j}G(i,j),\ \text{and}\\ \sum_{j=1}^{n}(▶.j) &= \sum_{j=1}^{n}\sum_{i=1}^{j-1}(-1)^{i+j}G(i,j-1) = \sum_{1\le i\le j\le n-1}(-1)^{i+j-1}G(i,j),\end{aligned}$$

so indeed their sum amounts to zero. ∎

We immediately deduce Shapiro's lemma.

**Theorem 3.5.8** (Shapiro's lemma)**.** *In the derived category of abelian groups,*

$$C^\bullet(G,\mathrm{Ind}_G^H(A)) \cong C^\bullet(H,A).$$

*Especially*

$$H^n(G,\mathrm{Ind}_G^H(A)) \cong H^n(H,A)\ \textit{for all}\ n.$$

## 3.6. A Hochschild-Serre Spectral Sequence

We devote this section to proving a Hochschild-Serre spectral sequence in a rather general fashion. Constructing cochains of course happens in $\mathrm{Hom}_{\mathbf{Set}}(¿G^n, A)$ and is done by the usual tedious calculations. As $G$ is only a monoid, extra care is required. Showing that these cochains stem from (then necessarily unique) elements in $X^\bullet$ poses an additional difficulty.

The spectral sequence will indeed follow from the following spectral sequence attached to the filtered complex $I^\bullet C^\bullet$.

**Definition 3.6.1.** As for all $n$ the filtration $I^\bullet C^n$ is a finite filtration and all $I^jC^\bullet$ form a subcomplex, there is a $E_1$-spectral sequence

$$ss(I^\bullet C^\bullet)_1^{p,q} \Longrightarrow H^{p+q}(G,A),$$

where the $E_1$-terms are defined as

$$ss(I^\bullet C^\bullet)_1^{p,q} = \frac{\ker\left(I^pC^{p+q}/I^{p+1}C^{p+q} \xrightarrow{\partial} I^pC^{p+q+1}/I^{p+1}C^{p+q+1}\right)}{\operatorname{im}\left(I^pC^{p+q-1}/I^{p+1}C^{p+q-1} \xrightarrow{\partial} I^pC^{p+q}/I^{p+1}C^{p+q}\right)},$$

cf. e. g. [NSW08, (2.2.1)].

**Definition 3.6.2.** To simplify reading, we will use the following notational convention: The first time a variable is used, a superscript will denote the set it belongs to. For example, instead of "Let $\underline{x} \in G^p$. Then define $f(\underline{x}) = \dots$" we will simply write "Define $f(\overset{G^p}{\underline{x}}) = \dots$"

### 3.6.1. The Meaning of $C^i(G/N, C^j(N, A))$.

In the previous section, we fixed a concrete category $\mathbf{C}$ to encapsulate our topological data. Our setup allowed us to give meaning to $C^j(N, A)$ for $G$-modules $A$. It is however very unclear how we can give the abstract module $C^j(N, A)$ again the structure of an object in $\mathbf{C}$.

If $\mathbf{C}$ is the category of Hausdorff topological spaces, one can topologise $C^j(N, A)$ with the compact-open topology, which if we further restrict to compactly generated spaces, has somewhat nice properties and can be called canonical. But computing cohomology, we are presented with the issue that images of differentials need not be closed and one subsequently loses the Hausdorff property, cf. [CW74] for a thorough discussion of these issues.

If $\mathbf{C}$ is a bit more exotic, e. g., analytic $\mathbb{Q}_p$-manifolds, then there is no obvious way to give $C^j(N, A)$ the structure of an analytic $\mathbb{Q}_p$-manifold compatible with the additional structure — and especially none that also correctly topologises the cohomology groups.

If the quotient is discrete, we can identify $C^q(U, A)$ with $\mathbf{F}C^q(U, A)$ and define

$$C^p(G/U, C^q(U, A)) = C^p(G/U, \mathbf{F}C^q(U, A)),$$

but note that while $C^q(U, A)$ carries the structure of a $G$-module, in general it does *not* carry the structure of a $G/U$-module (the action is only ¿$U$-invariant after passing to cohomology).

**Definition 3.6.3.** For $N = U$, let $f \in I^pC^{p+q}(G, A)$ and define its $p$-restriction $r_p(f) \in C^p(G/U, C^q(U, A))(= C^p(G/U, \mathbf{F}C^q(U, A)))$ via

$$r_p(f)(\overset{G/U}{x_1}, \dots, \overset{G/U}{x_p})(\overset{U}{y_1}, \dots, \overset{U}{y_q}) = f(y_1, \dots, y_q, s(x_1), \dots, s(x_p)).$$

This is well-defined: The induced map

$$r_p(f)(\overset{(G/U)^p}{\underline{x}}) \colon U^q \rightsquigarrow A$$

stems from the composition

$$U^q \xrightarrow{(\mathrm{id}, s(x_1), \dots, s(x_p))} U^q \times G^p \longrightarrow G^q \times G^p \overset{f}{\rightsquigarrow} A,$$

where we use the existence of constant maps (cf. proposition 2.1.13). As $G/U$ is discrete, $\underline{x} \longmapsto r_p(f)(\underline{x})$ is indeed in $C^p(G/U, C^q(U, A))$.

**Lemma 3.6.4.** *If $p' < p$ and $f \in I^pC^{p+q}$ then $r_{p'}(f) = 0$.*

Proof.

$$\begin{aligned} r_{p'}(f)(\overset{(G/U)^{p'}}{\underline{x}})(\overset{U}{y_1}, \dots, \overset{U}{y_{p+q-p'}}) &= f(y_1, \dots, y_q, y_{q+1}, \dots, y_{p+q-p'}, s(\underline{x})) \\ &= f(y_1, \dots, y_q, 1, \dots, 1, s(\underline{x})) \\ &= 0, \end{aligned}$$

as $f$ is normalised by assumption. ∎

**3.6.2. Extensions of cochains.** Comparing $C^{p+q}(G, A)$ with $C^p(G/U, C^q(U, A))$, we will have to extend maps $U^q \times (G/U)^p \longrightarrow A$ to maps $G^q \times (G/U)^p \longrightarrow A$. This extension process also works if we work with $N$ instead of $U$.

**Definition 3.6.5.** Let $g\colon G^{k-1} \times N^{q-k} \times G^p \rightsquigarrow A$ be a normalised map (meaning its value being zero if one of the arguments is 1) with $k \geq 2$. Then for normalised $f\colon G^k \times N^{q-k} \times G^p \rightsquigarrow A$ we define

$$\operatorname{ext}_f(g)\colon G^{k-2} \times G \times G \times N^{q-k-1} \times G^p \rightsquigarrow A$$

$$(\overset{G^{k-2}}{\underline{y}}, \overset{G}{w}, \overset{G}{x}, \overset{N^{q-k-1}}{\underline{\sigma}}, \overset{G^p}{\underline{z}}) \longmapsto g(\underline{y}, wx^*, x_N, \underline{\sigma}, \underline{z}) + (-1)^k f(\underline{y}, w, x^*, x_N, \underline{\sigma}, \underline{z})$$

It is called the extension of $g$ along $f$ and is again normalised. Note that it actually lies in $h_A$, as the modification of the arguments is done via morphisms in $\mathbf{C}$.

Calling it an extension is due to the following fact which is immediately verified:

**Lemma 3.6.6.** *In the setting of definition 3.6.5 the following diagram commutes:*

$$\begin{tikzcd}
G^{k-1} \times N \times N^{q-k-1} \times G^p \arrow[d, hook] \arrow[dr, squiggly, "g"] & \\
& A \\
G^{k-1} \times G \times N^{q-k-1} \times G^p \arrow[ur, squiggly, "\operatorname{ext}_f(g)"] &
\end{tikzcd}$$

Proof. Vectors in the image of the inclusion have $x^* = 1$ and $x_N = x$. ∎

**Lemma 3.6.7.** *In the setting of definition 3.6.5, the usual coboundary formula gives meaning to the function*

$$\partial(\operatorname{ext}_f(g))\colon G^k \times N^{q-k} \times G^p \rightsquigarrow A$$

*and the following diagram commutes:*

$$\begin{tikzcd}
G^{k-1} \times N \times N^{q-k} \times G^p \arrow[d, hook] \arrow[dr, squiggly, "\partial(g)"] & \\
& A \\
G^{k-1} \times G \times N^{q-k} \times G^p \arrow[ur, squiggly, "\partial(\operatorname{ext}_f(g))"] &
\end{tikzcd}$$

Proof. Immediate from lemma 3.6.6. ∎

**Remark 3.6.8.** In the setting of definition 3.6.5, $g$ is only defined on $G^{k-1} \times N \times N^{q-k-1} \times G^p$. For the coboundary formula to make sense, all terms $(x_1, \ldots, x_i x_{i+1}, \ldots, x_{p+q})$ must lie in $G^{k-1} \times N \times N^{q-k-1} \times G^p$. This is the reason $\partial g$ is only defined on $G^{k-1} \times N \times N^{q-k} \times G^p$.

**Proposition 3.6.9.** *Assume $N = U$. Let $q \geq 2$ and $f \in I^p C^{p+q}$ with $\partial f \in I^{p+1} C^{p+q+1}$. Take $u \in C^p(G/U, C^{q-1}(U, A))$. Define an element $g = g(u, f)$ as follows:*

- $g_0(\overset{U^{q-1}}{\underline{\sigma}}, \overset{G^p}{\underline{y}}) = u(\underline{y})(\underline{\sigma})$,
- $g_1(\overset{G}{x}, \overset{U^{q-2}}{\underline{\sigma}}, \overset{G^p}{\underline{y}}) = x^*.g_0(x_U, \underline{\sigma}, \underline{y}) - f(x^*, x_U, \underline{\sigma}, \underline{y})$,
- $g_k = \operatorname{ext}_f(g_{k-1}) \in h_A(G^k \times U^{q-1-k} \times G^p)$ *for* $2 \leq k \leq q-1$.
- $g = g_{q-1}$.

*Then the following hold:*

(1) *$g \in I^p C^{p+q-1}$,*
(2) *$r_p(g) = u$,*
(3) *if $r_p(f)(\overset{(G/U)^p}{\underline{x}}) = \partial(u(\underline{x}))$ for all $\underline{x} \in (G/U)^p$, then $f - \partial g \in I^{p+1}C^{p+q}$,*

Proof. The first assertion follows immediately from the definitions, as none of the manipulations touches the last $p$ arguments.

The second assertion follows inductively: $r_p(g_0) = u$ is obvious and noting that $x^* = 1$ and $x_U = x$ for $x \in U$ implies that $r_p(g_1) = g$. Using lemma 3.6.6, we find that $r_p(g_k) = u$ for all $k$.

Let us now prove the third assertion. For $1 \leq l \leq k$, it is

$$g_l(\overset{G^{l-1}}{\underline{y}}, \overset{s(G/U)}{x^*}, \overset{U^{q-l-1}}{\underline{\sigma}}, \overset{G^p}{\underline{z}}) = g_k(\underline{y}, x^*, \underline{\sigma}, \underline{z}) = 0,$$

where the first equality is due to lemma 3.6.6 and the second is due to the definition of the extension, as $(x^*)_U = 1$ and all $g_i$ are normalised.

By definition

$$\partial g_l \in h_A(G^l \times U^{q-l} \times G^p),$$

but restricting to $G^{l-1} \times s(G/U) \times U^{q-l} \times G^p$ this shows that the coboundary expansion of

$$\partial(g_l)(\overset{G^{l-1}}{\underline{y}}, \overset{s(G/U)}{x^*}, \overset{U}{\tau}, \overset{U^{q-l-1}}{\underline{\sigma}}, \overset{G^p}{\underline{z}})$$

consists of only two terms and can be further computed as follows by lemma 3.6.6 and the definition of the extension: For $l \geq 2$ we have

$$\begin{aligned}\partial(g_l)(\overset{G^{l-2}}{\underline{y}}, \overset{G}{w}, \overset{s(G/U)}{x^*}, \overset{U}{\tau}, \overset{U^{q-l-1}}{\underline{\sigma}}, \overset{G^p}{\underline{z}}) &= (-1)^{l-1} g_l(\underline{y}, wx^*, \tau, \underline{\sigma}, \underline{z}) + (-1)^l g_l(\underline{y}, w, x^*\tau, \underline{\sigma}, \underline{z}) \\ &= (-1)^l \left(\mathrm{ext}_f(g_{l-1})(\underline{y}, w, x^*\tau, \underline{\sigma}, \underline{z}) - g_{l-1}(\underline{y}, wx^*, \tau, \underline{\sigma}, \underline{z})\right) \\ &= f(\underline{y}, w, x^*, \tau, \underline{\sigma}, \underline{x}).\end{aligned}$$

We also have

$$\partial g_1(\overset{s(G/U)}{x^*}, \overset{U}{\tau}, \overset{U^{q-1}}{\underline{\sigma}}, \overset{G^p}{\underline{z}}) = x^*.g_1(\tau, \underline{\sigma}, \underline{z}) - g_1(x^*\tau, \underline{\sigma}, \underline{z}) = f(x^*, \tau, \underline{\sigma}, \underline{z}),$$

so altogether we have

$$(\star) \qquad (f - \partial(g_l)) = 0 \text{ on } G^{l-1} \times s(G/U) \times U^{q-l} \times G^p$$

We want to show that this holds on all of $G^l \times U^{q-l} \times G^p$ by induction on $k$. Note first that $f - \partial g_0$ is zero on $U^q \times G^p$: Indeed the first $q$ terms in the coboundary expansion of $\partial(g_0)(\overset{U}{\tau}, \overset{U^{q-1}}{\underline{\sigma}}, \overset{G^p}{\underline{z}})$ are by definition the first $q$ terms in the coboundary expansion of $\partial(u(\underline{z}))(\tau, \underline{\sigma})$. The $q+1$-th term is

$$\begin{aligned}(-1)^q g_0(\tau, \sigma_1, \ldots, \sigma_{q-2}, \sigma_{q-1}z_1, \ldots, z_p) &= (-1)^q u(\sigma_{q-1}z_1, z_2, \ldots, z_p)(\tau, \sigma_1, \ldots, \sigma_{q-2}) \\ &= (-1)^q u(\underline{z})(\tau, \sigma_1, \ldots, \sigma_{q-2}),\end{aligned}$$

as $U$ is normal in $G$, i. e., the first $q+1$ terms in the coboundary expansion of $\partial(g_0)(\tau, \underline{\sigma}, \underline{z})$ are equal to $\partial(u(\underline{z}))(\tau, \underline{\sigma})$. All the other summands are of the form

$$\pm u(\sigma_{q-1}, \ldots)(\ldots)$$

and hence zero as $u$ was supposed to be normalised. As $r_p(f)(\underline{z}) = \partial(u(\underline{z}))$, this shows this first assertion.

Assume we had already proved that

$$f - \partial g_l = 0 \text{ on } G^l \times U \times U^{q-l-1} \times G^p.$$

As $\partial f \in I^{p+1}$ and $\partial\partial g_{l+1} = 0$,

$$\partial(f - \partial g_{l+1}) = 0 \text{ on } G^l \times s(G/U) \times U \times U^{q-l-1} \times G^p,$$

so the coboundary expansion of

$$\partial(f - \partial g_{l+1})(\overset{G^l}{\underline{y}}, \overset{s(G/U)}{x^*}, \overset{U}{\tau}, \overset{U^{q-l-1}}{\underline{\sigma}}, \overset{G^p}{\underline{z}}) = 0$$

yields that

$$(f - \partial g_{l+1})(\underline{y}, x^*\tau, \underline{\sigma}, \underline{z})$$

is the sum of terms of the form $\pm(f - \partial g_{l+1})(\dots)$ where the $l+1$-th argument is either $x^*$ or $\tau$. If the $l+1$-th argument is $x^*$, it is zero by eq. ($\star$). If it is $\tau \in U$, then it is $(f - \partial g_l)(\dots)$ by lemma 3.6.7 and hence zero by induction hypothesis, so we have finally shown that

$$(f - \partial(g_{l+1})) = 0 \text{ on } G^{l+1} \times U^{q-l-1} \times G^p$$

and hence that for all $k \leq q-1$

$$(\star\star) \qquad (f - \partial(g_k)) = 0 \text{ on } G^k \times U^{q-k} \times G^p.$$

Consider at last the map $f - \partial g_{q-1} \colon G^{q-1} \times G \times G^p \rightsquigarrow A$. We want to show that the last $p+1$ arguments are $U$-invariant. But just as in the previous step, the coboundary expansion of

$$\partial(f - \partial g_{q-1})(\overset{G^{q-1}}{\underline{y}}, \overset{s(G/U)}{x^*}, \overset{U}{x_U}, \overset{G^p}{\underline{z}}) = 0$$

expresses $(f - \partial g_{q-1})(\overset{G^{q-1}}{\underline{y}}, \overset{G}{x}, \overset{G^p}{\underline{z}})$ as a sum of terms of which a lot are zero by eq. ($\star\star$), lots of others are zero because the $(q+1)$th argument is $x_U \in U$, and the only non-zero summand is

$$(f - \partial g_{q-1})(\underline{y}, x^*, x_U z_1, z_2, \dots, z_p).$$

But as $f - \partial g_{q-1}$ lies in $I^p C^{p+q}$, this is independent of $x_U$, so actually $f - \partial g \in I^{p+1}C^{p+q}$. ∎

**Remark 3.6.10.** The cochain $g = g(u, f)$ of proposition 3.6.9 has a rather unwieldy definition. However, if $f = 0$ and $U$ is a direct factor of $G$, then $g$ has an explicit description, cf. proposition 3.7.12.

### 3.6.3. Comparison of the first page.

**Proposition 3.6.11.**

$$ss(I^\bullet C^\bullet)_1^{p,0} \cong C^p(G/N, A^N)$$

*for all p.*

Proof. By definition, $ss(I^\bullet C^\bullet)_1^{p,0} = \ker I^p C^p \xrightarrow{\partial} I^p C^{p+1}/I^{p+1}C^{p+1}$ and furthermore, $f \in I^p C^p$ comes from a (necessarily unique) morphism $(G/N)^p \rightsquigarrow A$, which yields an element in $C^p(G/N, A)$. We first need to show that the image of $f$ is contained in $A^N$, i. e.,

$$(\overset{N}{n} - 1) f(\overset{G^p}{\underline{x}}) = 0.$$

As $f$ is in the aforementioned kernel, $\partial f(n, \underline{x}) = \partial f(1, \underline{x}) = 0$, and as $f \in I^p C^p$ by assumption, the difference between the coboundary expansions of $\partial f(n, \underline{x})$ and $\partial f(1, \underline{x})$ is exactly $(n-1)f(\underline{x})$ and hence also zero. Injectivity of the map is then clear.

On the other hand, for $f \in C^p(G/N, A^N)$ coming from $\widetilde{f} \in h_{A^N}((G/N)^p)$, consider the induced morphism

$$\widetilde{g} \colon G^p \longrightarrow (G/N)^p \overset{\widetilde{f}}{\rightsquigarrow} A^N \longrightarrow A.$$

It is clear that $g \in I^pC^p$ and gets mapped to $f$. As the image of $g$ lies in $A^N$, $\partial g$ is ¿$N$-invariant for all of its $p+1$ arguments, so lemma 3.2.1 implies that indeed $g \in ss(I^\bullet C^\bullet)_1^{p,0}$.

∎

**Remark 3.6.12.** By abuse of notation, even for not necessarily open $N$ we will refer to the map $ss(I^\bullet C^\bullet)_1^{p,0} \longrightarrow C^p(G/N, A^N)$ of proposition 3.6.11 as a map $r_p\colon ss(I^\bullet C^\bullet)_1^{p,0} \longrightarrow C^p(G/N, H^0(N,A))$. This is clearly compatible with the previous definition of $r_p$.

**Proposition 3.6.13.** *Suppose that $N = U$. Then $r_p\colon I^pC^{p+q} \longrightarrow C^p(G/U, C^q(U,A))$ induces an isomorphism between the $E_1$-terms:*

$$ss(I^\bullet C^\bullet)_1^{p,q} \cong C^p(G/U, H^q(U,A)).$$

Proof. Recall that by definition,

$$\begin{aligned} ss(I^\bullet C^\bullet)_1^{p,q} &= \frac{\ker\left(I^pC^{p+q}/I^{p+1}C^{p+q} \xrightarrow{\ \partial\ } I^pC^{p+q+1}/I^{p+1}C^{p+q+1}\right)}{\operatorname{im}\left(I^pC^{p+q-1}/I^{p+1}C^{p+q-1} \xrightarrow{\ \partial\ } I^pC^{p+q}/I^{p+1}C^{p+q}\right)} \\ &= \frac{\ker\left(I^pC^{p+q} \xrightarrow{\ \partial\ } C^{p+q+1}/I^{p+1}C^{p+q+1}\right)}{\partial(I^pC^{p+q-1}) + I^{p+1}C^{p+q}}. \end{aligned}$$

We will first prove injectivity. Therefore, take $f \in I^pC^{p+q}$ with $\partial f \in I^{p+1}C^{p+q+1}$. Assume that $f$ is zero in $C^p(G/U, H^q(U,A))$, i. e.,

$$r_p(f)(\overset{(G/U)^p}{\underline{x}}) = \partial(u(\underline{x}))$$

for some $u \in C^p(G/U, C^{q-1}(U,A))$. We want to find an $h \in I^pC^{p+q-1}$ with $f - \partial(h) \in I^{p+1}C^{p+q}$.

The case of $q = 0$ was already dealt with in proposition 3.6.11.

If $q = 1$, then define $h \in I^pC^{p+1-1}$ as the normalised cocycle corresponding to $u \in C^p(G/U, C^0(U,A)) = C^p(G/U, A)$. Note that by assumption $u$ has the property

$$f(\overset{G}{x}, \overset{G^p}{\underline{y}}) = r_p(f)(\underline{y})(x) = \partial(u(\underline{y}))(x) = x.u(\underline{y}) - u(\underline{y}).$$

We want to show that $f - \partial(h) \in I^{p+1}C^{p+1}$. For that matter we need to show that

$$(f - \partial(h))(\overset{G}{x} \cdot \overset{U}{\sigma}, \overset{G^p}{\underline{y}})$$

is independent of $\sigma$. As $\partial f \in I^{p+1}C^{p+2}$, we see that $\partial(f)(\overset{G}{x}, \overset{U}{\sigma}, \overset{G^p}{\underline{y}}) = 0$. Expansion of the coboundary operator hence yields

$$f(x\sigma, \underline{y}) = x.f(\sigma, \underline{y}) + f(x, \underline{y}),$$

as $f \in I^pC^{p+q}$, and also

$$\partial(h)(x\sigma, \underline{y}) = x.\sigma.h(\underline{y}) + \text{ terms independent of } \sigma,$$

as $h \in I^pC^p$. We hence get

$$\begin{aligned} (f - \partial(h))(x\sigma, \underline{y}) &= f(x\sigma, \underline{y}) - x.\sigma.h(\underline{y}) + \text{ terms independent of } \sigma \\ &= x.f(\sigma, \underline{y}) + f(x, \underline{y}) - x.\sigma.h(\underline{y}) + \text{ terms independent of } \sigma \\ &= x.(\sigma.u(\underline{y}) - u(\underline{y})) - x.\sigma.u(\underline{y}) + \text{ terms independent of } \sigma, \end{aligned}$$

which is independent of $\sigma$.

For $q > 1$ we are in the situation of proposition 3.6.9, which deals with exactly this case.

For surjectivity, take $u \in C^p(G/U, C^q(U,A))$ such that $\partial(u(\overset{(G/U)^p}{\underline{x}})) = 0$ for all $\underline{x}$. For $q = 0$ finding a preimage is trivial, for $q \geq 1$ proposition 3.6.9 yields a preimage $g \in I^pC^{p+q}$ with $\partial g \in I^{p+1}C^{p+q+1}$ (take $f = 0$ in the proposition).

∎

**3.6.4. The shuffling mechanism.** To compare the differential in the spectral sequence attached to $I^\bullet C^\bullet$, Hochschild and Serre use a process they call *shuffling*. However, in the proof of proposition 3.6.22 their argument misses the terms of type (M3) and leaves the constructions of the various $\psi$ to the reader, which is why we spell out the details.

**Lemma 3.6.14.** *Every $x \in ¿G$ induces a conjugation action $G \longrightarrow G$ that on $M$ is trivial and on $G'$ is the usual conjugation by the $G'$-part of $x$.*

Proof. On $G'$ conjugation is defined via the composition

$$G' \cong \mathbf{F} \bullet \times G' \times \mathbf{F} \bullet \xrightarrow{(x^{-1}, \mathrm{id}, x)} G' \times G' \times G' \xrightarrow{\mathrm{mult} \circ \mathrm{mult}} G',$$

where we use the existence of constant maps from proposition 2.1.13. ∎

**Remark 3.6.15.** We will write formulas like $x^{-1}yx$ even though $x$ need not be invertible in $G$. As $M$ is central in $G$, the usual identities such as $y(y^{-1}xy) = xy$ still hold.

**Definition 3.6.16.** We will make use of the ordered sets

$$\lceil n \rceil = \{1, \dots, n\}$$

for $n \in \mathbb{N}$. For every injective morphism of ordered sets

$$\phi \colon \lceil p \rceil \longrightarrow \lceil p+q \rceil$$

there exists a unique (injective) morphism

$$\phi^* \colon \lceil q \rceil \longrightarrow \lceil p+q \rceil$$

such that

$$\lceil p+q \rceil = \operatorname{im} \phi \cup \operatorname{im} \phi^*.$$

We furthermore define

$$\operatorname{sgn} \phi = (-1)^{\sum_{i=1}^{q} \phi^*(i) - i}.$$

**Lemma 3.6.17.** *Let $\phi \colon \lceil p \rceil \longrightarrow \lceil p+q \rceil$ be an injective morphism of ordered sets. Then* $\operatorname{sgn}(\phi) \cdot \operatorname{sgn}(\phi^*) = (-1)^{p \cdot q}$.

Proof.

$$\begin{aligned} \sum_{i=1}^{q} \phi^*(i) - i + \sum_{i=1}^{p} \phi(i) - i - p \cdot q &= \frac{(p+q)(p+q+1) - q(q+1) - p(p+1)}{2} - pq \\ &= \frac{2pq}{2} - pq = 0. \end{aligned}$$

∎

**Definition 3.6.18.** Denote by $F_{p+q}$ the free $\mathbb{Z}[¿G]$-module with basis $¿G^{p+q}$.

For $\phi \colon \lceil p \rceil \longrightarrow \lceil p+q \rceil$ an injective morphism of ordered sets define

$$(x_1, \dots, x_q, y_1, \dots, y_p)^\phi = (\gamma_1, \dots, \gamma_{p+q})$$

with

$$\gamma_{\phi(i)} = y_i$$

and

$$\gamma_{\phi^*(i)} = (y_1 \cdots y_{\phi^*(i)-i})^{-1} x_i (y_1 \cdots y_{\phi^*(i)-i}).$$

If we are considering multiple morphisms $\phi$, we will also write $\gamma(\phi, k)$ instead of $\gamma_k$.

Define now

$$\mathrm{shuffle}_p^{p+q}(\overset{¿G^{p+q}}{\underline{z}}) = \sum_{\phi} \operatorname{sgn}(\phi) \underline{z}^\phi \in F_{p+q},$$

where here and in the following an unspecified sum over $\phi$ denotes the sum over all injective morphisms of ordered sets with $p$ and $p+q$ clear from the context.

Every $g \in C^{p+q}$ gives rise to $g^{\phi} \in C^{p+q}$ via

$$g^{\phi}(\underline{z}) = g(\underline{z}^{\phi}).$$

Indeed $g^{\phi} \in X^{p+q}$ as both conjugation and reordering come from morphisms in $\mathbf{C}$. We can also define $\mathrm{shuffle}_p^{p+q}\, g \in C^{p+q}$ via

$$\mathrm{shuffle}_p^{p+q}(g)(\underline{z}) = \sum_{\phi} \mathrm{sgn}(\phi) g(\underline{z}^{\phi}).$$

We will use the convention that $\mathrm{shuffle}_0^n = \mathrm{id}$.

**Proposition 3.6.19.** *Let $\phi\colon \lceil p \rceil \longrightarrow \lceil p+q \rceil$ be the unique injective morphism of ordered sets with $\phi(1) = q+1$. Then $\underline{z} = \underline{z}^{\phi}$ for all $z \in G^{p+q}$. If $g \in I^p C^{p+q}$, then*

$$\mathrm{shuffle}_p^{p+q}\, g = g \text{ on } N^q \times G^p.$$

Proof. The first assertion is clear from the definitions, as then $\phi^*(i) = i$ for all $1 \le i \le q$.
For the second assertion we will show that for all other $\varphi$, $g^{\varphi} = 0$ on $N^q \times G^p$. In this case, there exists $q+1 \le i \le p+q$ with $i = \varphi^*(k)$ for some $k$ and $\gamma_i$ is then equal to a conjugate of $\alpha_k$, which lies by assumption again in $N$. As $g$ was supposed to be normalised and $N$-invariant in the last $p$ components, this implies that $g^{\varphi} = 0$ on $N^q \times G^p$. ∎

**Definition 3.6.20.** For $p, q \ge 1$ we define the following two partial coboundary operators $F_{p+q} \longrightarrow F_{p+q-1}$:

$$\begin{aligned}\partial_q(x_1, \ldots, x_q, y_1, \ldots, y_p) = x_1.(x_2, \ldots, x_q, \underline{y}) + (-1)^q(x_1, \ldots, x_{q-1}, \underline{y})\\ + \sum_{i=1}^{q-1}(-1)^i(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_q, \underline{y})\end{aligned}$$

and

$$\begin{aligned}\delta_p(x_1, \ldots, x_q, y_1, \ldots, y_p) = y_1.(y_1^{-1}\underline{x}y_1, y_2, \ldots, y_p) + (-1)^p(\underline{x}, y_1, \ldots, y_{p-1})\\ + \sum_{i=1}^{p-1}(-1)^k(\underline{x}, y_1, \ldots, y_{i-1}, y_i y_{i+1}, y_{i+2}, \ldots, y_p),\end{aligned}$$

where $\underline{x} = (x_1, \ldots, x_q), \underline{y} = (y_1, \ldots, y_p)$ and $y_1^{-1}\underline{x}y_1 = (y_1^{-1}x_1y_1, \ldots, y_1^{-1}x_qy_1)$. These formulas also give rise to partial coboundary operators $\partial_q, \delta_p\colon C^{p+q-1} \longrightarrow C^{p+q}$ by the same arguments as in proposition 3.2.2.

**Proposition 3.6.21.** *Let $\Phi$ denote the set of injective morphisms of ordered sets $\lceil p \rceil \longrightarrow \lceil p+q \rceil$. For each $1 \le k \le p+q-1$ there is a bijection*

$$\{\phi \in \Phi \mid k \in \operatorname{im}\phi^* \text{ and } k+1 \in \operatorname{im}\phi\} \longleftrightarrow \{\psi \in \Phi \mid k \in \operatorname{im}\psi \text{ and } k+1 \in \operatorname{im}\psi^*\}$$

*with the following property: If $\phi$ corresponds to $\psi$ then $\gamma(\phi, k)\gamma(\phi, k+1) = \gamma(\psi, k)\gamma(\psi, k+1)$ and $\gamma(\phi, i) = \gamma(\psi, i)$ for all $i \ne k, k+1$. Furthermore, $\mathrm{sgn}(\phi) = -\mathrm{sgn}(\psi)$.*

Proof. Construct $\psi$ as follows: Let $k+1 = \phi(a)$. Then

$$\begin{gathered}\psi(1) = \phi(1), \ldots, \psi(a-1) = \phi(a-1),\\ \psi(a) = k,\\ \psi(a+1) = \phi(a+1), \ldots, \psi(p) = \phi(p)\end{gathered}$$

and conversely for given $\psi$ with $\psi(b) = k$ construct $\phi$ via

$$\phi(1) = \psi(1), \dots, \phi(b-1) = \psi(b-1),$$
$$\phi(b) = k+1,$$
$$\phi(b+1) = \psi(b+1), \dots, \phi(p) = \psi(p).$$

It is clear that both constructions are mutually inverse to one another and satisfy above requirements. ■

**Proposition 3.6.22.** *For $p, q \geq 1$ and $\underline{z} \in F_{p+q}$ we have*

$$\partial \operatorname{shuffle}_p^{p+q}(\underline{z}) = (\operatorname{shuffle}_p^{p+q-1} \partial_q \underline{z}) + (-1)^q (\operatorname{shuffle}_{p-1}^{p+q-1} \delta_p \underline{z}).$$

*Consequently, for $f \in C^{p+q-1}$ the following identity holds:*

$$\operatorname{shuffle}_p^{p+q}(\partial f) = \partial_q(\operatorname{shuffle}_p^{p+q-1}(f)) + (-1)^q \delta_p(\operatorname{shuffle}_{p-1}^{p+q-1}(f)).$$

The proof of this is of course a combinatorial nightmare. Before giving it, we want to discuss an example which gives an overview to how we will group summands together.

**Example 3.6.23.** For $p = 2, q = 4$ consider $\phi \colon \lceil 2 \rceil \longrightarrow \lceil 6 \rceil$ given by

$$\phi = \overset{1}{3} < \overset{2}{4},$$

so

$$\phi^* = \overset{1}{1} < \overset{2}{2} < \overset{3}{5} < \overset{4}{6},$$

where $\phi = \dots < \overset{b}{a} < \dots$ means that $\phi(b) = a$ etc. It follows that

$$(x_1, \dots, x_4, y_1, y_2)^\phi = \big(x_1, x_2, y_1, y_2, (y_1 y_2)^{-1} x_3 (y_1 y_2), (y_1 y_2)^{-1} x_4 (y_1 y_2)\big)$$

and

$$\operatorname{sgn} \phi = 1.$$

In the expansion of $\partial \operatorname{shuffle}_2^6(x_1, \dots, x_4, y_1, y_2)$ the terms stemming from $\phi$ are hence as follows, the types being explained in the proof of proposition 3.6.22 below.

| Sign | Term | Type |
|---|---|---|
| $+$ | $x_1 \cdot \big(x_2, y_1, y_2, (y_1y_2)^{-1}x_3(y_1y_2), (y_1y_2)^{-1}x_4(y_1y_2)\big)$ | all of type (S1) to (S3) |
| $(-1)^{2+4}$ | $\big(x_1, x_2, y_1, y_2, (y_1y_2)^{-1}x_3(y_1y_2)\big)$ | all of type (S1) to (S3) |
| $(-1)^1$ | $\big(x_1x_2, y_1, y_2, (y_1y_2)^{-1}x_3(y_1y_2), (y_1y_2)^{-1}x_4(y_1y_2)\big)$ | $x_1x_2$ of type (M1) |
| $(-1)^2$ | $\big(x_1, x_2y_1, y_2, (y_1y_2)^{-1}x_3(y_1y_2), (y_1y_2)^{-1}x_4(y_1y_2)\big)$ | $x_2y_1$ degenerate term of second kind |
| $(-1)^3$ | $\big(x_1, x_2, y_1y_2, (y_1y_2)^{-1}x_3(y_1y_2), (y_1y_2)^{-1}x_4(y_1y_2)\big)$ | $y_1y_2$ of type (M2) |
| $(-1)^4$ | $\big(x_1, x_2, y_1, y_2(y_1y_2)^{-1}x_3(y_1y_2), (y_1y_2)^{-1}x_4(y_1y_2)\big)$ | $y_2(y_1y_2)^{-1}x_3(y_1y_2)$ degenerate term of first kind |
| $(-1)^5$ | $\big(x_1, x_2, y_1, y_2, (y_1y_2)^{-1}x_3x_4(y_1y_2)\big)$ | $(y_1y_2)^{-1}x_3x_4(y_1y_2)$ of type (M3) |

Proof of proposition 3.6.22. Write $\underline{z} = (x_1, \dots, x_q, y_1, \dots, y_p)$. For $\phi \colon \lceil p \rceil \longrightarrow \lceil p+q \rceil$ consider the expansion of

$$\partial((x_1, \dots, x_q, y_1, \dots, y_p)^\phi).$$

In this expansion, consider the terms of the form

$$(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{p+q}),$$

where $i = \phi(j)$ and $i+1 = \phi^*(k)$ or $i = \phi^*(j)$ and $i+1 = \phi(k)$. In the first case,

$$\gamma_i\gamma_{i+1} = y_j(y_1 \cdots y_{i+1-k})^{-1} x_k (y_1 \cdots y_{i+1-k}),$$

which we will call a *degenerate term of the first kind.* In the second case

$$\gamma_i\gamma_{i+1} = (y_1\cdots y_{i-j})^{-1}x_j(y_1\cdots y_{i-j})y_k,$$

which we will call a *degenerate term of the second kind.*

This is the only way in which in the expansion of $\partial\,\mathrm{shuffle}_p^{p+q}(\underline{z})$ arguments of these degenerate two kinds can occur and by proposition 3.6.21 they all cancel, i. e., we are left with sums of ($G$ operating on) basis elements with each component taking one of the following forms:

(S1) $x_i$,
(S2) $y_i$,
(S3) $(y_1\cdots y_{\phi^*(i)-i})^{-1}x_i(y_1\cdots y_{\phi^*(i)-i})$,

except for at most one, which will then be of the form

(M1) $x_ix_{i+1}$,
(M2) $y_iy_{i+1}$, or
(M3) $(y_1\cdots y_{\phi^*(i)-i})^{-1}x_ix_{i+1}(y_1\cdots y_{\phi^*(i)-i})$.

Let us formally compare

$$(\star)\qquad \partial\circ\mathrm{shuffle}_p^{p+q}(x_1,\dots,x_q,y_1,\dots,y_p)$$

with

$$(\blacklozenge)\qquad \left(\mathrm{shuffle}_p^{p+q-1}\circ\partial_q+(-1)^q\,\mathrm{shuffle}_{p-1}^{p+q-1}\circ\delta_p\right)(x_1,\dots,x_q,y_1,\dots,y_p).$$

We will first look at terms in ($\star$) where all terms are of the form (S1) to (S3), which occur in our convention as the first and second summands in the coboundary expansion, so

$$(\star')\qquad \begin{aligned}\partial\circ\mathrm{shuffle}_p^{p+q}(\underline{z}) &= \sum_\phi \mathrm{sgn}(\phi)\cdot\gamma(\phi,1)\cdot(\gamma(\phi,2),\dots,\gamma(\phi,p+q))\\ &\quad+\sum_\phi \mathrm{sgn}(\phi)(-1)^{p+q}(\gamma(\phi,1),\dots,\gamma(\phi,p+q-1))\\ &\quad+\ \text{terms with one component of form (M1) to (M3)}\,,\end{aligned}$$

$$(\blacktriangle)\qquad \begin{aligned}\mathrm{shuffle}_p^{p+q-1}\circ\partial_q(\underline{z}) &= x_1\cdot(\mathrm{shuffle}_p^{p+q-1}(x_2,\dots,x_q,\underline{y}))\\ &\quad+(-1)^q\,\mathrm{shuffle}_p^{p+q-1}(x_1,\dots,x_{q-1},\underline{y})\\ &\quad+\ \text{terms with one component of form (M1) to (M3)}\,,\end{aligned}$$

$$(\blacktriangledown)\qquad \begin{aligned}\mathrm{shuffle}_{p-1}^{p+q-1}\circ\delta_p(\underline{z}) &= y_1\cdot(\mathrm{shuffle}_{p-1}^{p+q-1}(y_1^{-1}\underline{x}y_1,y_2,\dots,y_p))\\ &\quad+(-1)^p\,\mathrm{shuffle}_{p-1}^{p+q-1}(\underline{x},y_1,\dots,y_{p-1})\\ &\quad+\ \text{terms with one component of form (M1) to (M3)}\,.\end{aligned}$$

Let us first take care of the first group of summands in ($\star'$).

Note that independent of $\phi$, $\gamma(\phi,1)$ is either $y_1$ (in which case we wish to find the term in ($\star$) in the $\delta_p$-part of ($\blacklozenge$)) or $x_1$ (in which case we wish to find it in the $\partial_q$-part.)

If $\phi^*(1)=1$ and hence $\gamma(\phi,1)=x_1$, define $\psi\colon\lceil p\rceil\longrightarrow\lceil p+q-1\rceil$ via $\psi(i)=\phi(i)-1$. Then $\psi^*(i-1)=\phi^*(i)-1$ and $\mathrm{sgn}(\phi)=\mathrm{sgn}(\psi)$, so the term

$$\mathrm{sgn}(\phi)\cdot\gamma(\phi,1)\cdot(\gamma(\phi,2),\dots,\gamma(\phi,p+q))$$

in ($\star'$) is equal to the summand stemming from $\psi$ in the expansion of

$$x_1\cdot(\mathrm{shuffle}_p^{p+q-1}(x_2,\dots,x_q,\underline{y}))$$

in (▲). If $\phi(1) = 1$ and hence $\gamma(\phi, 1) = y_1$, define $\psi\colon \lceil p-1 \rceil \longrightarrow \lceil p+q-1 \rceil$ via $\psi(i) = \phi(i+1) - 1$. Then $\psi^*(i) = \phi^*(i) - 1$ and $\mathrm{sgn}(\psi) = (-1)^q \operatorname{sgn}\phi$, so the term

$$\mathrm{sgn}(\phi) \cdot \gamma(\phi, 1) \cdot (\gamma(\phi, 2), \ldots, \gamma(\phi, p+q))$$

in (★′) is equal to $(-1)^q$-times the summand stemming from $\psi$ in the expansion of

$$y_1 \cdot (\mathrm{shuffle}_{p-1}^{p+q-1}(y_1^{-1}\underline{x}y_1, y_2, \ldots, y_p))$$

in (▼).

Let us now take care of the second group of summands in (★′). Similarly to before, $\gamma(\phi, p+q)$ is either $y_p$ or $(y_1 \cdots y_p)^{-1} x_q (y_1 \cdots y_p)$.

If $\phi^*(q) = p+q$ and hence $\gamma(\phi, p+q) = (y_1 \cdots y_p)^{-1} x_q (y_1 \cdots y_p)$, $\phi$ restricts to a morphism $\psi\colon \lceil p \rceil \longrightarrow \lceil p+q-1 \rceil$. $\psi^*$ is then the restriction $\phi^*\colon \lceil q-1 \rceil \longrightarrow \lceil p+q-1 \rceil$ and $\mathrm{sgn}(\psi) = (-1)^p \operatorname{sgn}(\phi)$, so the term in the second group of (★′) corresponding to $\phi$ is the term in the second group of (▲) corresponding to $\psi$.

Analogously, if $\phi(p) = p+q$, then we can restrict $\phi$ to $\psi\colon \lceil p-1 \rceil \longrightarrow \lceil p+q-1 \rceil$. In this case, $\mathrm{sgn}(\psi) = \mathrm{sgn}(\phi)$ and we again identify the term in the second group of (★′) and $(-1)^q$-times the term in (▼).

It remains to consider the summands where one component is of the form (M1) to (M3).

Ad (M1): Assume the (M1)-component is $x_k x_{k+1}$, so especially $\phi^*(k+1) = \phi^*(k) + 1$. Including signs it looks like this in the expansion of (★):

$$(-1)^k \cdot \mathrm{sgn}(\phi) \cdot (x_1, \ldots, x_{k-1}, x_k x_{k+1}, \gamma(\phi, k+2), \ldots, \gamma(\phi, p+q)).$$

Setting $\psi(i) = \phi(i) - 1$, we see that this is equal to

$$(-1)^k \cdot \mathrm{sgn}(\psi) \cdot (x_1, \ldots, x_{k-1}, x_k x_{k+1}, x_{k+2}, \ldots, x_q, \underline{y})^{\psi},$$

a term occurring in (▲).

Ad (M2): Assume the (M2)-component is $y_k y_{k+1}$, so especially $\phi(k+1) = \phi(k) + 1$. Including signs it looks like this in the expansion of (★):

$$(-1)^{\phi(k)} \cdot \mathrm{sgn}(\phi) \cdot (\gamma(\phi, 1), \ldots, \gamma(\phi, \phi(k) - 1), y_k y_{k+1}, \gamma(\phi, \phi(k) + 2), \ldots, \gamma(\phi, p+q)).$$

Define

$$\psi(i)\colon \lceil p-1 \rceil \longrightarrow \lceil p+q-1 \rceil$$
$$\psi = \begin{cases} \phi(i) & \text{for } i \le k \\ \phi(i+1) - 1 & \text{for } i \ge k+1, \end{cases}$$

hence

$$(\underline{x}, y_1, \ldots, y_{k-1}, y_k y_{k+1}, y_{k+2}, \ldots, y_p)^{\psi}$$

is up to signs the term we are looking for.

Clearly

$$\psi^*(i) = \begin{cases} \phi^*(i) & \text{if } \phi^*(i) \le k \\ \phi^*(i) - 1 & \text{else}, \end{cases}.$$

There are $\phi(k) - k$ numbers $i$ such that $\phi^*(i) \le k$, so $\operatorname{sgn}\psi = \operatorname{sgn}\phi \cdot (-1)^{q - (\phi(k) - k)}$. The sign in the $\delta_p$-part of (♦) is hence

$$(-1)^q \cdot (-1)^k \cdot \mathrm{sgn}(\phi) \cdot (-1)^{q - (\phi(k) - k)} = \mathrm{sgn}(\phi) \cdot (-1)^{\phi(k)},$$

which is exactly the sign we were looking for.

Ad (M3): Assume the (M3)-component is $(y_1 \cdots y_{\phi^*(k)-k})^{-1} x_k x_{k+1} (y_1 \cdots y_{\phi^*(k)-k})$, which is then the $\phi^*(k)$th component. Note that then $\phi^*(k+1) = \phi(k)^* + 1$. Its sign is given by $(-1)^{\phi^*(k)} \cdot \operatorname{sgn}(\phi)$. Define $\psi$ via

$$\psi(i) = \begin{cases} \phi(i) & \text{if } \phi(i) \leq k \\ \phi(i) - 1 & \text{else,} \end{cases},$$

then

$$\psi^* = \begin{cases} \phi^*(i) & \text{for } i \leq k \\ \phi^*(i+1) - 1 & \text{for } i \geq k+1 \end{cases}$$

and hence

$$\operatorname{sgn}(\psi) = \operatorname{sgn}(\phi) \cdot (-1)^{\phi^*(k)-k}.$$

The sign in the $\partial_q$-part of (♦) corresponding to $(x_1, \ldots, x_{k-1}, x_k x_{k+1}, x_{k+2}, \ldots, x_q, \underline{y})^\psi$ is hence

$$(-1)^k \cdot \operatorname{sgn}(\psi) = (-1)^{\phi^*(k)} \cdot \operatorname{sgn}(\phi),$$

which is the sign we were looking for.

Checking the constructions, we see that we identified each term in the expansion of (⋆) with exactly one term in the expansion of (♦). It is furthermore evident form the construction of the various $\psi$ that every term in the expansion of (♦) has a corresponding term in the expansion of (⋆). ∎

**3.6.5. Comparison of the second page.** So far, we only considered the *groups* $C^p(G/U, H^q(U,A))$ with the $E_1$-terms corresponding to the spectral sequence attached to $I^\bullet C^\bullet$. Now we need to give $C^\bullet(G/U, H^q(U,A))$ the structure of a *complex*. For this it suffices to give $H^q(U,A)$ the structure of a ¿$(G/U)$-module, as then $C^\bullet(G/U, H^q(U,A)) = C^\bullet(G/U, \mathbf{F}H^q(U,A))$ is a complex by section 3.2. The module-structure also exists for non-open subgroups $N$.

We also need compatibility between our partial coboundary operators $\partial_q, \delta_p$ and the operators

$$\delta \colon C^p(G/U, \mathbf{F}C^q(U,A)) \longrightarrow C^{p+1}(G/U, \mathbf{F}C^q(U,A))$$

and

$$\partial \colon C^p(G/U, \mathbf{F}C^q(U,A)) \longrightarrow C^p(G/U, \mathbf{F}C^{q+1}(U,A)).$$

**Proposition 3.6.24.** *$C^q(N,A)$ and $H^q(N,A)$ carry the structure of a ¿$G$-module by the usual conjugation action.*

Proof. Recall that every element in $y \in$ ¿$G$ induces a conjugation morphism on $G$ (lemma 3.6.14), which restricts to a morphism on $N$. Defining

$$(y.(\overset{C^q(N,A)}{f}))(\overset{N^q}{\underline{x}}) = y.(f(y^{-1}\underline{x}y))$$

yields an element in $C^q(N,A)$ because of definition 2.3.8, so altogether we get a $G$-action on $C^q(N,A)$. As in the classical case, this also gives an action on the cohomology groups. ∎

**Lemma 3.6.25.** *The diagrams*

$$\begin{array}{ccc} C^{p+q}(G,A) & \xrightarrow{r_p} & C^p(G/U, C^q(U,A)) \\ \downarrow{\scriptstyle \delta_{p+1}} & & \downarrow{\scriptstyle \delta} \\ C^{p+q+1}(G,A) & \xrightarrow{r_{p+1}} & C^{p+1}(G/U, C^q(U,A)) \end{array}$$

*and*

$$\begin{tikzcd} C^{p+q}(G,A) \arrow[r, "r_p"] \arrow[d, "\partial_{q+1}"] & C^p(G/U, C^q(U,A)) \arrow[d, "\partial"] \\ C^{p+q+1}(G,A) \arrow[r, "r_p"] & C^p(G/U, C^{q+1}(U,A)) \end{tikzcd}$$

*are commutative. As before, $\delta_{p+1}$ and $\partial_{q+1}$ are the partial coboundary operators from definition 3.6.20 and $\delta$ and $\partial$ are the respective coboundary operators of $C^\bullet(G/U,-)$ and $C^\bullet(U,-)$.*

Proof. Immediate from the definitions. ∎

**Proposition 3.6.26.** *¿N operates trivially on $H^q(N,A)$.*

Proof. We use proposition 3.6.22 for the topologised monoid $N$: For $p = 1$ and $f \in C^q(N,A) \cap \ker \partial$ this reads

$$0 = \operatorname{shuffle}_q^{1+q}(\partial f) = \partial_q(\operatorname{shuffle}_1^q(f)) + (-1)^q \delta_1(\operatorname{shuffle}_0^q(f))$$

and hence

$$\delta_1(f) \in \operatorname{im} \partial_q.$$

But $\delta_1(f)$ is explicitly given by

$$(\delta_1 f)(\overset{N^q}{\underline{x}}, \overset{N}{y}) = y.f(y^{-1}\underline{x}y) - f(\underline{x}) = (y.f)(\underline{x}) - f(\underline{x}),$$

so $y.f$ and $f$ are cohomologous, as $\partial_q$ is the differential on $C^\bullet(N,A)$, analogously to lemma 3.6.25. ∎

**Theorem 3.6.27.** *There is a convergent $E_2$-spectral sequence*

$$H^p(G/U, H^q(U,A)) \Longrightarrow H^{p+q}(G,A).$$

*Even if $N$ is not necessarily open, we have the classical five term exact sequence:*

$$\begin{tikzcd} 0 \arrow[r] & H^1(G/N, A^N) \arrow[r] & H^1(G,A) \arrow[r] & H^1(N,A)^{¿(G/N)} \arrow[dll, out=-10, in=170] \\ & H^2(G/N, A^N) \arrow[r] & H^2(G,A). \end{tikzcd}$$

Proof. Consider first the case of $N = U$. By proposition 3.6.13 it suffices to show that the following diagram commutes:

$$(\star)\qquad \begin{tikzcd} ss(I^\bullet C^\bullet)_1^{p,q} \arrow[r, "\partial"] \arrow[d, "r_p"] & ss(I^\bullet C^\bullet)_1^{p+1,q} \arrow[d, "r_{p+1}"] \\ C^p(G/U, H^q(U,A)) \arrow[r, "(-1)^q\delta"] & C^{p+1}(G/U, H^q(U,A)) \end{tikzcd}$$

Here $\delta$ denotes the coboundary operator on $C^p$, not on lifted maps $U^q \times G^p \rightsquigarrow A$. By lemma 3.6.25

$$\delta \circ r_p = r_{p+1} \circ \delta_{p+1}.$$

For $q = 0$, the commutativity follows immediately from the definitions, so assume $q \geq 1$. Take $f \in I^p C^{p+q}$ with $\partial f \in I^{p+1} C^{p+q+1}$ Then by proposition 3.6.22 and multiple applications of proposition 3.6.19,

$$\begin{aligned} r_{p+1}(\partial f) &= r_{p+1}(\operatorname{shuffle}_{p+1}^{p+q+1}(\partial f)) \\ &= r_{p+1}\left(\partial_q(\operatorname{shuffle}_{p+1}^{p+q} f) + (-1)^q \delta_{p+1}(\operatorname{shuffle}_p^{p+q} f)\right) \\ &= r_{p+1}(\partial_q(\operatorname{shuffle}_{p+1}^{p+q} f)) + (-1)^q \delta(r_p(\operatorname{shuffle}_p^{p+q} f)) \\ &= r_{p+1}(\partial_q(\operatorname{shuffle}_{p+1}^{p+q} f)) + (-1)^q \delta(r_p(f)). \end{aligned}$$

As clearly $r_{p+1}(\partial_q(\text{shuffle}^{p+q}_{p+1} f)) = 0$ in $C^{p+1}(G/U, H^q(U, A))$ by lemma 3.6.25, this finishes the proof for open subgroups.

For normal subgroups $N$, the spectral sequence

$$ss(I^\bullet C^\bullet)_2^{p,q} \Longrightarrow H^{p+q}(G, A)$$

still yields a five term exact sequence and we are left with showing that the groups $ss(I^\bullet C^\bullet)_2^{1,0}$, $ss(I^\bullet C^\bullet)_2^{2,0}$ and $ss(I^\bullet C^\bullet)_2^{0,1}$ are precisely the cohomology groups we were looking for. For $q = 0$, the same as above argument works, using proposition 3.6.11 instead of proposition 3.6.13. For the case of $p = 0, q = 1$, we cannot use diagram ($\star$). But the same argument as above yields a commutative diagram

$$\begin{tikzcd}
ss(I^\bullet C^\bullet)_1^{0,1} \arrow[rr, "\partial"] \arrow[d, "r_0"] & & ss(I^\bullet C^\bullet)_1^{1,1} \arrow[d, hook] \\
H^1(N, A) \arrow[r, "="] & C^0(¿(G/N), H^1(N, A)) \arrow[r, "(-1)\delta"] & C^1(¿(G/N), H^1(N, A))
\end{tikzcd}$$

where the map on the left is induced by the restriction of $f \in C^1(G, A)$ to $N$. The map on the right, defined analogously to before, is however only injective.

The map on the left is however still bijective, adapting the proof of proposition 3.6.13: Represent an element of $ss(I^\bullet C^\bullet)_1^{0,1}$ by $f \in C^1(G, A)$. Assume its restriction to $N$ is given by $f(n) = n.a - a$ for some $a \in A$. Consider $h = f - (x \longmapsto x.a - a)$, which is the same as $f$ in

$$ss(I^\bullet C^\bullet)_1^{0,1} = \frac{\ker C^1 \longrightarrow C^2/I^1C^2}{\partial(C^0) + I^1C^1}.$$

We will show that indeed $h \in I^1C^1$ and that hence $h$ and therefore $f$ is zero in $ss(I^\bullet C^\bullet)_1^{0,1}$. By assumption, $\partial f(x, n) = 0$ for all $x \in G, n \in N$, so actually

$$f(xn) = x.f(n) + f(x)$$

We immediately find that

$$h(xn) = f(xn) - xn.a + a = x.(n.a - a) + f(x) - xn.a + a = f(x) - (x.a - a) = h(x),$$

so $h \in I^1C^1$ by lemma 3.2.1.

For surjectivity, choose a representative $\widetilde{f} \in C^1(N, A)$ and simply define $f$ via $\widetilde{f} \circ (-)_N$. Therefore,

$$ss(I^\bullet C^\bullet)_2^{0,1} \cong \ker \delta = \{f \in H^1(N, A) \mid g.f - f = 0 \text{ for all } g \in G\},$$

which is precisely $H^1(N, A)^{¿(G/N)}$, as $N$ already operates trivially by proposition 3.6.26. ∎

**Remark 3.6.28.** With all this effort, we still cannot recover the Hochschild-Serre spectral sequence for Hausdorff compactly generated topological groups $G$ with closed normal subgroup $N$ and discrete coefficients $A$. From the point of view presented above, the spectral sequence

$$H^p(G/N, H^q(N, A)) \Longrightarrow H^{p+q}(G, A)$$

is actually an anomaly: The category $\mathbf{C}$ would be the category of compactly generated weakly Hausdorff spaces, which is cartesian closed, where the exponential objects are given by $\mathrm{Hom}_{\mathbf{C}}(X, Y)$, endowed with the compact-open topology (cf. remark 2.1.11). The analogue of proposition 3.6.13, which shows an isomorphism of the $E_1$-page, is then generally only a bijection – but for discrete $A$, $C^q(N, A)$ (and hence also $H^q(N, A)$) is again discrete and bijectivity then suffices for showing the isomorphism.

In any case, it is much more convenient to derive said spectral sequence from homological algebra and reserve the *direct method* for cases where the homological arguments fail.

## 3.7. A Double Complex

This section is devoted to making a precise statement of the following sort and proving it afterwards:

**Prototheorem 3.7.1.** *Let $G$ be a topologised monoid, $D$ an abelian discrete monoid and $A$ a topologised $D \times G$-module. Then in the derived category of abelian groups the following holds:*

$$C^\bullet(D \times G, A) \cong \operatorname{tot} C^\bullet(D, C^\bullet(G, A)),$$

This is very much related to the previous results: There, we filtered the complex on the left hand side. If we are not looking at a direct product $D \times G$, there is no double complex on the right hand side – but a hypothetical double complex would have $C^\bullet(D, H^\bullet(G, A))$ as the cohomology in one direction. We compared this cohomology with the $E_1$-page of the filtered complex and showed that indeed they coincide.

### 3.7.1. Setup and Precise Statement.

For the whole section, we fix:

- a topological category $\mathbf{C}$,
- a topologised monoid $G$ in $\mathbf{C}$ as in section 3.2,
- a discrete abelian monoid $D$,
- $-^*\colon D \times G \longrightarrow D \times G$, the morphism of topologised monoids which on the level of sets is given by $(d, g) \longmapsto (d, 1)$,
- $-_G\colon D \times G \longrightarrow D \times G$, the morphism of topologised monoids which on the level of sets is given by $(d, g) \longmapsto (1, g)$,
- the canonical projections $\pi_D\colon D \times G \longrightarrow D$, $\pi_G\colon D \times G \longrightarrow G$, and
- such a $D \times G$-module with $\mathbf{C}$-rigidification $A$, that $D$ is $A$-pliant.

As before, $C^n(G, A)$ denotes the set of normalised (inhomogeneous) cochains $G^n \rightsquigarrow A$. We write $C^\bullet$ for $C^\bullet(D \times G, A)$ and denote the boundary operator of section 3.2 by $\partial$. The filtration $I^\bullet C^n$ will be taken with respect to the submonoid $G$ of $D \times G$.

**Lemma 3.7.2.** *The assignment*

$$(\overset{D}{d} f)(\overset{G^n}{\underline{x}}) = d f(\underline{x}) \left(= d f(d^{-1}\underline{x} d)\right)$$

*gives $C^n(G, A)$ the structure of a $D$-module.*

Proof. Clear from proposition 3.6.24. ∎

**Remark 3.7.3.** This only works because of the direct product structure of $D \times G$, cf. section 3.6.1.

**Definition 3.7.4.** Denote by $C^{\bullet,\bullet}$ the commutative double complex $C^p(D, C^q(G, A))$ with differentials

$$\delta\colon C^p(D, C^q(G, A)) \longrightarrow C^{p+1}(D, C^q(G, A))$$
$$\partial\colon C^p(D, C^q(G, A)) \longrightarrow C^p(D, C^{q+1}(G, A))$$

explicitly given by

$$\delta(f)(\overset{D^{p+1}}{\underline{y}})(\overset{G^q}{\underline{x}}) = y_1 f(y_2, \ldots, y_{p+1})(y_1^{-1}\underline{x} y_1) + (-1)^{p+1} f(y_1, \ldots, y_p)(\underline{x})$$
$$+ \sum_{i=1}^{p} f(y_1, \ldots, y_{i-1}, y_i y_{i+1}, y_{i+2}, \ldots, y_{p+1})(\underline{x})$$

and

$$\partial(f)(\overset{D^p}{\underline{y}})(\overset{G^{q+1}}{\underline{x}}) = x_1 f(\underline{y})(x_2, \dots, x_{q+1}) + (-1)^{q+1} f(\underline{y})(x_1, \dots, x_q)$$
$$+ \sum_{i=1}^{q} f(\underline{y})(x_1, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_{p+1}).$$

That this is indeed a double complex follows from lemma 3.7.2 and the previous discussion in section 3.2. We form the total complex

$$(\operatorname{tot} C^{\bullet,\bullet})^n = \bigoplus_{p+q=n} C^{p,q}$$

with total differential

$$\Delta \colon C^{p,q} \longrightarrow C^{p+1,q} \oplus C^{p,q+1}$$
$$\Delta = \partial + (-1)^q \delta$$

**Remark 3.7.5.** If $D \cong \mathbb{N}_0$ (or $D \cong \mathbb{Z}$) operates via a single operator $\varphi$, then because of proposition 3.1.2 and the fact that

$$0 \longrightarrow \mathbb{Z}[\varphi] \xrightarrow{\varphi - 1} \mathbb{Z}[\varphi] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is also a free resolution of the integers, we see that (by abstract nonsense)

$$\operatorname{tot} C^{\bullet,\bullet} \cong \operatorname{tot}\left(C^\bullet(G,A) \xrightarrow{\varphi - 1} C^\bullet(G,A)\right)$$

in the derived category of abelian groups. This immediately generalises to monoids $D \cong \mathbb{N}_0^r \times \mathbb{Z}^s$ by induction.

Our main result of this section can now be stated as follows:

**Theorem 3.7.6.** *There is a quasi-isomorphism of complexes*

$$C^\bullet \longrightarrow \operatorname{tot} C^{\bullet,\bullet}.$$

The preparations of its proof will span the next couple of pages, which itself is given on page 63.

### 3.7.2. The morphism ...

**Definition 3.7.7.** For $f \in C^{p+q}$ denote by $r_p(f) \in C^{p,q}$ the map

$$r_p(f)(\overset{D^p}{\underline{y}})(\overset{G^q}{\underline{x}}) = f((1,x_1), \dots, (1,x_q), (y_1,1), \dots, (y_p,1))$$

and by $\alpha$ the map

$$\alpha \colon C^n \longrightarrow \bigoplus_{p+q=n} C^{p,q}$$
$$\alpha(f) = \bigoplus_{p+q=n} r_p(\operatorname{shuffle}_p^{p+q}(f)).$$

**Proposition 3.7.8.** *$\alpha$ is a morphism of complexes $C^\bullet \longrightarrow \operatorname{tot} C^{\bullet,\bullet}$.*

Proof. Consider the diagram

$$\begin{tikzcd} C^{n-1} \arrow[r] \arrow[d, "\partial"] & \bigoplus_{p+q=n-1} C^{p,q} \arrow[d, "\Delta"] \\ C^n \arrow[r] & \bigoplus_{p+q=n-1} \left(C^{p+1,q} \oplus C^{p,q+1}\right). \end{tikzcd}$$

To show that it commutes, let $p' + q' = n$. We will compare

$$r_{p'}(\operatorname{shuffle}_{p'}^n(\partial f))$$

with the entry of $\Delta(\alpha(f))$ in $C^{p',q'}$. By definition, this entry is equal to

$$\partial(r_{p'}(\mathrm{shuffle}_{p'}^{p'+q'-1}(f))) + (-1)^{q'}\delta(r_{p'-1}(\mathrm{shuffle}_{p'-1}^{p'-1+q'}(f))).$$

By lemma 3.6.25,

$$\partial \circ r_{p'} = r_{p'} \circ \partial_{q'}$$

and

$$\delta \circ r_{p'-1} = r_{p'} \circ \delta_{p'},$$

where $\partial_{q'}$ and $\delta_{p'}$ are the maps from definition 3.6.20. The claim then follows immediately from the additivity of $r_{p'}$ and proposition 3.6.22. ∎

### 3.7.3. ... and its quasi-inverse.

**Lemma 3.7.9.** *Consider the map*

$$(-)^\sharp\colon C^{p,q} \longrightarrow C^{p+q},$$

$$f^\sharp(\overset{(D\times G)^{p+q}}{\underline{z}}) = f(\pi_D(z_{q+1}),\dots,\pi_D(z_{p+q}))(\pi_G(z_1),\dots,\pi_G(z_q)).$$

*Its image lies in $I^pC^{p+q}$ and the composition*

$$C^{p,q} \xrightarrow{-^\sharp} I^pC^{p+q} \xrightarrow{r_p} C^{p,q}$$

*is the identity.*

Proof. Clear from the definitions. ∎

**Proposition 3.7.10.** $(-)^\sharp$ *induces a map*

$$(-)^\sharp\colon (\operatorname{tot} C^{\bullet,\bullet})^n \longrightarrow C^n$$

*with*

$$\alpha \circ (-)^\sharp = \mathrm{id}_{(\operatorname{tot} C^{\bullet,\bullet})^n}.$$

Proof. Let $f \in C^{p,q}$. Lemma 3.7.9 and proposition 3.6.19 (with $N = G$) imply that

$$r_p(\mathrm{shuffle}_p^{p+q}(f^\sharp)) = f.$$

It now remains to show that for $p' \neq p$, $r_{p'}(\mathrm{shuffle}_{p'}^{p+q}(f^\sharp)) = 0$.

Let $\varphi\colon \lceil p'\rceil \longrightarrow \lceil p+q\rceil$ be an injective map of ordered sets, so

$$\begin{aligned} r_{p'}((f^\sharp)^\phi)(\overset{D^{p'}}{\underline{y}})(\overset{G^{p+q-p'}}{\underline{x}}) &= (f^\sharp)^\phi\left((1,x_1),\dots,(1,x_{p+q-p'}),(y_1,1),\dots,(y_{p'},1)\right)\\ &= f^\sharp(\gamma_1,\dots,\gamma_{p+q}),\end{aligned}$$

with $\gamma_k$ a conjugate of one of the $(1,x_i)$ or one of the $(y_i,1)$. Therefore, at least $p'$ of the $\gamma_k$ have $\pi_G(\gamma_k) = 1$ and at least $p+q-p'$ of the $\gamma_k$ have $\pi_D(\gamma_k) = 1$. Now

$$f^\sharp(\gamma_1,\dots,\gamma_{p+q}) = f(\pi_D(\gamma_{q+1}),\dots,\pi_D(\gamma_{p+q}))(\pi_G(\gamma_1),\dots,\pi_G(\gamma_q)),$$

and all cocycles are normalised, so this can only be non-zero if all $\gamma_k$ with $\pi_D(\gamma_k) = 1$ are among the first $q$, so

$$p+q-p' \le q$$

and if all $\gamma_k$ with $\pi_G(\gamma_k) = 1$ are among the last $p$, so

$$p' \le p.$$

But this is impossible if $p' \neq p$. Therefore $r_{p'}((f^\sharp)^\phi) = 0$ and hence also

$$r_{p'}(\mathrm{shuffle}_{p'}^{p+q}(f^\sharp)) = 0.$$

∎

**Remark 3.7.11.** The map $(-)^\sharp$ of proposition 3.7.10 is not a map of complexes, so while it is easy to construct preimages in the direct product case, these are not particularly useful. Showing that $\alpha$ is a quasi-isomorphism hence again uses the calculations of section 3.6.2.

**Proposition 3.7.12.** *Let $u \in C^{p,q}$ and $g = g(u, 0) \in I^pC^{p+q}$ its extension along $0$ from proposition 3.6.9. Then*

$$g(\overset{D\times G}{x_1}, \dots, \overset{D\times G}{x_q}, \overset{(D\times G)^p}{\underline{y}}) = x_1^* \dots x_q^*.u^\sharp(x_1, \dots, x_q, \underline{y}).$$

Proof. Note first that by definition of $-^\sharp$,

$$u^\sharp(z_1, \dots, z_q, z_1', \dots, z_p') = u^\sharp((z_1)_G, \dots, (z_q)_G, z_1'^*, \dots, z_p'^*).$$

Define as in proposition 3.6.9

$$g_1(\overset{D\times G}{x_1}, \overset{G^{q-2}}{\underline{\sigma}}, \overset{(D\times G)^p}{\underline{y}}) = x_1^*.u^\sharp((x_1)_G, \underline{\sigma}, \underline{y}).$$

and $g_k = \operatorname{ext}_0(g_{k-1})$ for $2 \le k \le q$, so that $g = g_q$. We will inductively show that

$$g_k(\overset{D\times G}{x_1}, \dots, \overset{D\times G}{x_k}, \overset{G^{q-k}}{\underline{\sigma}}, \overset{(D\times G)^p}{\underline{y}}) = x_1^* \dots x_k^*.u^\sharp(x_1, \dots, x_k, \underline{\sigma}, \underline{y}),$$

which is trivial for $k = 1$. By definition of the extension,

$$g_{k+1}(\overset{D\times G}{x_1}, \dots, \overset{D\times G}{x_k}, \overset{D\times G}{x_{k+1}}, \overset{G^{q-k-1}}{\underline{\sigma}}, \overset{(D\times G)^p}{\underline{y}}) = g_k(x_1, \dots, x_{k-1}, x_kx_{k+1}^*, (x_{k+1})_G, \underline{\sigma}, \underline{y}),$$

which by induction hypothesis is exactly

$$x_1^* \dots x_{k-1}^* \cdot (x_kx_{k+1}^*)^*.u^\sharp((x_1)_G, \dots, (x_{k-1})_G, (x_kx_{k+1}^*)_G, (x_{k+1})_G, \underline{\sigma}, \underline{y})$$

As in our case $-^*$ and $-_G$ are homomorphisms with $-_G \circ -^* \equiv 1$, this shows the proposition. ∎

**Corollary 3.7.13.** *Let $u \in C^{p,q}$ and define $g = g(u, 0)$ as in proposition 3.6.9. Then*

$$g(\overset{(D\times G)^q}{\underline{x}}, \overset{(D\times G)^p}{\underline{y}}) = 0$$

*if one of the first $q$ arguments lies in $D$.*

Proof. Clear from proposition 3.7.12 and the definition of $-^\sharp$. ∎

**Proposition 3.7.14.** *Let $u \in C^{p,q}$ and $g = g(u, 0) \in I^pC^{p+q}$ its extension along $0$ from proposition 3.6.9. Then $\alpha(g) = (0, \dots, 0, \overset{C^{p,q}}{u}, 0, \dots, 0)$.*

Proof. As $g \in I^pC^{p+q}$, we have for all $p' < p$

$$\alpha(g)^{p',p+q-p'} = r_{p'}(\operatorname{shuffle}_{p'}^{p+q}(g)) = r_{p'}(g) = 0$$

by proposition 3.6.19 and lemma 3.6.4. By propositions 3.6.9 and 3.6.19, $\alpha(g)^{p,q} = u$, so it remains to show that $\alpha(g)^{p',p+q-p'} = 0$ for $p' > p$, i. e., that

$$\operatorname{shuffle}_{p'}^{p+q} g(\overset{D^{p'}}{\underline{d}}, \overset{G^{p+q-p'}}{\underline{y}}) = 0.$$

But the definition of the shuffle operator implies that this is the sum of values of the form

$$\pm g(\overset{(D\times G)^{p+q}}{\underline{\gamma}})$$

where at least $p'$ arguments lie in $D$. As $p' > p$, one of these arguments that lie in $D$ is in one of the first $q$ positions, so $g(\underline{\gamma}) = 0$ by corollary 3.7.13. ∎

**Proposition 3.7.15.** *Extension along zero is a morphism of complexes* $\operatorname{tot} C^{\bullet,\bullet} \longrightarrow C^\bullet$.

Proof. We need to show the following: Let $u \in C^{p,q}$ with $\Delta u = v+w$ with $v \in C^{p+1,q}$ and $w \in C^{p,q+1}$. Call their respective extensions along zero from proposition 3.6.9

$$\begin{aligned} g &= g(u,0) \in I^p C^{p+q}, \\ h &= g(v,0) \in I^{p+1} C^{p+q+1}, \text{ and} \\ h' &= g(w,0) \in I^p C^{p+q+1}. \end{aligned}$$

Then

$$\partial g = h + h'.$$

Using proposition 3.7.12, this is now a straight forward (albeit lengthy) calculation. First of all,

$$\begin{aligned} \partial g(\overset{(D\times G)^{p+q+1}}{\underline{x}}) &= x_1.g(x_2,\ldots,x_{p+q+1}) + (-1)^{p+q+1} g(x_1,\ldots,x_{p+q}) \\ &\quad + \sum_{i=1}^{p+q} (-1)^i g(x_1,\ldots,x_{i-1},x_i x_{i+1},x_{i+2},\ldots,x_{p+q+1}) \\ &= x_1 \cdot x_2^* \ldots x_{q+1}^* . u^\sharp(x_2,\ldots,x_{q+1},x_{q+2},\ldots,x_{p+q+1}) \\ &\quad + (-1)^{p+q+1}(x_1 \ldots x_q)^* . u^\sharp(x_1,\ldots,x_q,x_{q+1},\ldots x_{p+q}) \\ &\quad + \sum_{i=1}^{q} (-1)^i (x_1 \ldots x_{q+1})^* u^\sharp(x_1,\ldots,x_i x_{i+1},\ldots,x_{p+q+1}) && (\Sigma.1) \\ &\quad + \sum_{i=q+1}^{p+q} (-1)^i (x_1 \ldots x_q)^* u^\sharp(x_1,\ldots,x_{i-1},x_i x_{i+1},x_{i+2},\ldots,x_{p+q+1}) && (\Sigma.2) \end{aligned}$$

Expanding $h$ we first get

$$\begin{aligned} h(\overset{(D\times G)^{p+q+1}}{\underline{x}}) &= (x_1 \ldots x_q)^* v^\sharp(x_1,\ldots,x_q,x_{q+1},\ldots,x_{p+q+1}) \\ &= (x_1 \ldots x_q)^* v(x_{q+1}^*,\ldots,x_{p+q+1}^*)((x_1)_G,\ldots,(x_q)_G). \end{aligned}$$

We can furthermore express $(-1)^q v(x_{q+1}^*,\ldots,x_{p+q+1}^*)((x_1)_G,\ldots,(x_q)_G)$ as follows:

$$\begin{aligned} &(-1)^q v(x_{q+1}^*,\ldots,x_{p+q+1}^*)((x_1)_G,\ldots,(x_q)_G) \\ &= x_{q+1}^* . u(x_{q+2}^*,\ldots,x_{p+q+1}^*)((x_1)_G,\ldots,(x_q)_G) \\ &\quad + (-1)^{p+1} u(x_{q+1}^*,\ldots,x_{p+q}^*)((x_1)_G,\ldots,(x_q)_G) \\ &\quad + \sum_{i=1}^{p} (-1)^i u(x_{q+1}^*,\ldots,x_{q+i-1}^*,(x_{q+1}x_{q+i+1})^*,x_{q+i+2}^*,\ldots,x_{p+q+1}^*)((x_1)_G,\ldots,(x_q)_G) \\ &= x_{q+1}^* . u^\sharp(x_1,\ldots,x_q,x_{q+2},\ldots,x_{p+q+1}) \\ &\quad + (-1)^{p+1} u^\sharp(x_1,\ldots,x_q,x_{q+1},\ldots,x_{p+q}) \\ &\quad + \sum_{i=1}^{p} (-1)^i u^\sharp(x_1,\ldots,x_{q+i-1},x_{q+i}x_{q+i+1},x_{q+i+2},\ldots,x_{p+q+1}). \end{aligned}$$

On the other hand,

$$\begin{aligned} h'(\overset{(D\times G)^{p+q+1}}{\underline{x}}) &= (x_1 \ldots x_{q+1})^* w^\sharp(x_1,\ldots,x_{q+1},x_{q+2},\ldots,x_{p+q+1}) \\ &= (x_1 \ldots x_{q+1})^* w(x_{q+2}^*,\ldots,x_{p+q+1}^*)((x_1)_G,\ldots,(x_{q+1})_G), \end{aligned}$$

and we can express $w(x^*_{q+2},\dots,x^*_{p+q+1})((x_1)_G,\dots,(x_{q+1})_G)$ as follows:

$$\begin{aligned}
&w(x^*_{q+2},\dots,x^*_{p+q+1})((x_1)_G,\dots,(x_{q+1})_G)\\
&= (x_1)_G.u(x^*_{q+2},\dots,x^*_{p+q+1})((x_2)_G,\dots,(x_{q+1})_G)\\
&\qquad + (-1)^{q+1}u(x^*_{q+2},\dots,x^*_{p+q+1})((x_1)_G,\dots,(x_q)_G)\\
&\qquad + \sum_{i=1}^{q}(-1)^i u(x^*_{q+2},\dots,x^*_{p+q+1})((x_1)_G,\dots,(x_{i-1})_G,(x_i x_{i+1})_G,x_{i+2},\dots,(x_{q+1})_G)\\
&= (x_1)_G.u^\sharp(x_2,\dots,x_{q+1},x_{q+2},\dots,x_{p+q+1})\\
&\qquad + (-1)^{q+1}u^\sharp(x_1,\dots,x_q,x_{q+2},\dots,x_{p+q+1})\\
&\qquad + \sum_{i=1}^{q}(-1)^i u^\sharp(x_1,\dots,x_{i-1},x_i x_{i+1},x_{i+2},\dots,x_{q+1},x_{q+2},\dots,x_{p+q+1}).
\end{aligned}$$

We see at once that ($\Sigma$.1) appears in our expansion of $h'$ and that ($\Sigma$.2) appears in our expansion of $h$. The remaining terms are as follows:

$$\begin{aligned}
&(\partial g - h - h')(\overset{(D\times G)^{p+q+1}}{\underline{x}})\\
(\star.1)\quad &= x_1\cdot x_2^*\dots x^*_{q+1}.u^\sharp(x_2,\dots,x_{p+q+1})\\
(\star.2)\quad &\qquad + (-1)^{p+q+1}(x_1\dots x_q)^*.u^\sharp(x_1,\dots,x_{p+q})\\
(\star.3)\quad &\qquad - (-1)^q(x_1\dots x_q)^* x^*_{q+1}u^\sharp(x_1,\dots,x_q,x_{q+2},\dots,x_{p+q+1})\\
(\star.4)\quad &\qquad - (-1)^{p+q+1}(x_1\dots x_q)^* u^\sharp(x_1,\dots,x_{p+q})\\
(\star.5)\quad &\qquad - (x_1\dots x_{q+1})^*(x_1)_G u^\sharp(x_2,\dots,x_{p+q+1})\\
(\star.6)\quad &\qquad - (-1)^{q+1}(x_1\dots x_{q+1})^* u^\sharp(x_1,\dots,x_q,x_{q+2},\dots,x_{p+q+1}).
\end{aligned}$$

By construction,

$$x_1\cdot x_2^*\dots x^*_{q+1} = (x_1)_G\cdot x_1^*\dots x^*_{q+1} = (x_1\dots x_{q+1})^*(x_1)_G,$$

so ($\star$.1) and ($\star$.5) cancel. Also ($\star$.2) and ($\star$.4) cancel, as do ($\star$.3) and ($\star$.6). ∎

Armed with this, we are now ready to prove the main result of this section.

Proof of theorem 3.7.6. Indeed $\alpha$ is the required quasi-isomorphism

$$\alpha\colon C^\bullet \xrightarrow{\approx} \operatorname{tot} C^{\bullet,\bullet}.$$

Surjectivity on the level of cohomology follows immediately from propositions 3.7.14 and 3.7.15.

It remains to see that $\alpha$ is injective on cohomology. For this matter, take $f\in C^n$ with $\partial f = 0$ and $\alpha(f) = \Delta(u)$ for some $u\in(\operatorname{tot} C^{\bullet,\bullet})^{n-1}$. Write $\alpha(f) = (f^{p,q})_{p,q}\in\bigoplus_{p+q=n} C^{p,q}$. We will now modify $f$ step by step by elements of $\partial(C^{n-1})$ such that it lies in higher and higher $I^pC^n$ until it lies in $I^{n+1}C^n = 0$, i. e., $f$ is cohomologous to zero.

Let $\widetilde{f}\in C^n$ and $\widetilde{u}\in(\operatorname{tot} C^{\bullet,\bullet})^{n-1}$. We call the tuple $(\widetilde{f},\widetilde{u})$ better than $(f,u)$ at $p$ if the following hold:

(1) $f-\widetilde{f}\in\partial(C^{n-1})$,
(2) $\widetilde{f}\in I^pC^n$,
(3) $\alpha(\widetilde{f}) = \Delta(\widetilde{u})$, and
(4) $\widetilde{u}^{k,n-1-k} = 0$ for $k<p$.

We will inductively construct an $\widetilde{f}\in C^n$, such that $(\widetilde{f},0)$ is better than $(f,u)$ at $n$. We will afterwards show that this $\widetilde{f}$ is already zero and hence $f\in\partial(C^{n-1})$.

Obviously $(f,u)$ itself is better than $(f,u)$ at 0. If $(\overline{f},\overline{u})$ is better than $(f,u)$ at $p$, we construct a tuple $(\widetilde{f},\widetilde{u})$ which is better than $(f,u)$ at $p+1$ as follows:

Note that analogously to proposition 3.7.14, by proposition 3.6.19 and lemma 3.6.4, $\overline{f}^{p',q} = 0$ for all $p' < p$ and $\overline{f}^{p,n-p} = r_p(\overline{f})$. By assumption,

$$r_p(\overline{f}) = \partial\overline{u}^{p,n-1-p}.$$

If $p \leq n-2$, we can do the following: By proposition 3.6.9 (with $u = \overline{u}^{p,n-p-1}, f = \overline{f}, p = p, q = n-p$) we find $g \in I^pC^{n-1}$ with the following properties:

(1) $\alpha(g)^{p',n-1-p'} = \overline{u}^{p',n-1-p'}$ for all $p' \leq p$,
(2) $\overline{f} - \partial(g) \in I^{p+1}C^n$.

Note that for proposition 3.6.9 to be applicable, we need the assumption that $p \leq n-2$.
Now set $\widetilde{f} = \overline{f} - \partial(g)$ and $\widetilde{u} = \overline{u} - \alpha(g)$. To show that $(\widetilde{f},\widetilde{u})$ is better than $(f,u)$ at $p+1$, we only have to show that

$$\alpha(\widetilde{f}) = \Delta(\widetilde{u}),$$

but this is straight forward:

$$\alpha(\widetilde{f}) = \alpha(\overline{f}) - \alpha(\partial g) = \Delta(\overline{u}) - \Delta(\alpha(g)) = \Delta(\widetilde{u}).$$

Repeating this process, we get a tuple $(\overline{f},\overline{u})$, which is better than $(f,u)$ at $n-1$, so

$$\overline{u} = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \overline{u}^{n-1,0}$$

and

$$\alpha(\overline{f}) = 0 \oplus \cdots \oplus 0 \oplus \overline{f}^{n-1,1} \oplus \overline{f}^{n,0}.$$

Now set $g = (\overline{u}^{n-1,0})^\sharp \in I^{n-1}C^{n-1}(D\times G, A), \widetilde{f} = \overline{f} - \partial(g), \widetilde{u} = \overline{u} - \alpha(g)$. (Note that by construction, $\widetilde{u} = 0$.)
It is immediately clear that

$$\alpha(\widetilde{f}) = \Delta(\widetilde{u}) = 0.$$

To see that $(\widetilde{f},0)$ is better than $(f,u)$ at $n$, it only remains to show that $\widetilde{f} \in I^nC^n$. As it is clear from the construction that $\widetilde{f} \in I^{n-1}C^n$, we only need to show that

$$\widetilde{f}\left((\overset{D}{d},\overset{G}{\sigma}),\overset{(D\times G)^{n-1}}{\underline{x}}\right) = \widetilde{f}((d,1),\underline{x}).$$

The equality $\overline{f}^{n-1,1} = \partial\overline{u}^{n-1,0}$ (together with $\overline{f} \in I^{n-1}C^n$) implies

$$\begin{aligned}
\overline{f}((1,\overset{G}{\sigma}),\underline{x}) &= r_{n-1}(\overline{f})(\pi_D(\underline{x}))(\sigma)\\
&= \overline{f}^{n-1,1}(\pi_D(\underline{x}))(\sigma)\\
&= \partial(\overline{u}^{n-1,0}(\pi_D(\underline{x})))(\sigma)\\
&= \sigma\overline{u}^{n-1,0}(\pi_D(\underline{x})) - \overline{u}^{n-1,0}(\pi_D(\underline{x}))\\
&= (1,\sigma)(\overline{u}^{n-1,0})^\sharp(\underline{x}) - (\overline{u}^{n-1,0})^\sharp(\underline{x}).
\end{aligned}$$

As $\partial(\overline{f}) = 0$ and $\overline{f} \in I^{n-1}C^n$, the coboundary expansion of $\partial\overline{f}((d,1),(1,\sigma),\underline{x}) = 0$ yields

$$\begin{aligned}
\overline{f}((d,\sigma),\underline{x}) &= (d,1)\overline{f}((1,\sigma),\underline{x}) + \overline{f}((d,1),\underline{x})\\
&= (d,1)\overline{f}((1,\sigma),\underline{x}) + \text{ terms independent of } \sigma
\end{aligned}$$

and analogously

$$\partial(g)((d,\sigma),\underline{x}) = (d,\sigma)g(\underline{x}) + \text{ terms independent of } \sigma.$$

We can hence compute:

$$\begin{aligned}\widetilde{f}((d,\sigma),\underline{x}) &= (\overline{f}-\partial(g))((d,\sigma),\underline{x})\\ &= (d,1)\overline{f}((1,\sigma),\underline{x}) - (d,\sigma)g(\underline{x}) + \text{ terms independent of } \sigma\\ &= (d,\sigma)(\overline{u}^{n-1,0})^\sharp(\underline{x}) - (d,1)(\overline{u}^{n-1,0})^\sharp(\underline{x}) - (d,\sigma)(\overline{u}^{n-1,0})^\sharp(\underline{x})\\ &\qquad + \text{ terms independent of } \sigma,\end{aligned}$$

which is independent of $\sigma$, hence $\widetilde{f} \in I^n C^n$.
We conclude the proof by showing that if $\widetilde{f} \in I^n C^n \cap \ker\alpha \cap \ker\partial$, then $\widetilde{f} \in \partial C^{n-1}$. But indeed such an $\widetilde{f} \in I^n C^n \cap \ker\alpha$ is already zero:

$$\begin{aligned}\widetilde{f}((\overset{D}{d_1},\overset{G}{x_1}),\dots,(\overset{D}{d_n},\overset{G}{x_n})) &= \widetilde{f}((d_1,1),\dots,(d_n,1))\\ &= r_n(\widetilde{f})(d_1,\dots,d_n)\\ &= \alpha(\widetilde{f})^{n,0}(d_1,\dots,d_n) = 0.\end{aligned}$$

∎

#### 3.7.4. On a theorem of Jannsen.

The main result of [Jan90] also has a variant in the topological setting.
We first recall the following result:

**Proposition 3.7.16** ([NSW08, (2.3.4)])**.** *Let $C^\bullet, D^\bullet$ be complexes of modules over a Dedekind domain $R$. Assume that both complexes are bounded in the same direction or that one of them is bounded above and below. If $C^\bullet$ consists of flat $R$-modules, then there is a non-canonical splitting*

$$H^n(\operatorname{tot} C^\bullet \otimes_R D^\bullet) \cong \bigoplus_{p+q=r} ss(C^\bullet \otimes_R D^\bullet)_2^{p,q},$$

*where $ss(C^\bullet \otimes_R D^\bullet)_2^{p,q}$ denotes the $E_2$-terms of the spectral sequence attached to the double complex (cf. e. g. [NSW08, (2.2.3)] for details).*

**Proposition 3.7.17.** *If ¿D is finite and acts trivially on A, then*

$$H^n(D\times G, A) \cong \bigoplus_{p+q=n} H^p(D, H^q(G,A)).$$

Proof. By theorem 3.7.6 it suffices to show that

$$C^{\bullet,\bullet} \cong C^\bullet(D,\mathbb{Z}) \otimes C^\bullet(G,A)$$

as double complexes, as we can then employ proposition 3.7.16 to get the desired result. As $D$ is finite, it is clear that

$$\begin{aligned} C^p(D,\mathbb{Z}) \otimes C^q(G,A) &\longrightarrow C^p(D, C^q(G,A))\\ (f,g) &\longmapsto \left(\overset{D^p}{\underline{d}} \longmapsto \left(\overset{G^q}{\underline{x}} \longmapsto f(\underline{d})g(\underline{x})\right)\right)\end{aligned}$$

is bijective and it is easily verified that it commutes with differentials. ∎

The assumption of finite $D$ is regrettably crucial in the proof. In [Jan90] the case of *compact* (but not necessarily discrete) $D$ and discrete $A$ is considered. Every morphism $D \rightsquigarrow A$ then has finite image, which induces the isomorphism above.
However, for the easiest monoids we also have the following:

**Proposition 3.7.18.** *If $D \cong \mathbb{N}_0^r$ (or $D \cong \mathbb{Z}^r$) acts trivially on A, then*

$$H^n(D\times G, A) \cong \bigoplus_{k=0}^{r} H^{n-k}(G,A)^{\oplus\binom{r}{k}}.$$

Proof. It suffices to show the proposition for $r = 1$, as the general case then follows by induction. By remark 3.7.5 and theorem 3.7.6

$$C^\bullet(D \times G, A) \cong \operatorname{tot}\left(C^\bullet(G, A) \xrightarrow{0} C^\bullet(G, A)\right) \cong C^\bullet(G, A) \oplus C^{\bullet-1}(G, A),$$

so

$$H^n(D \times G, A) \cong H^n(G, A) \oplus H^{n-1}(G, A).$$

■

## 3.8. Shapiro's Lemma for Topologised Monoids

The results of the previous section allow us to extend Shapiro's lemma to monoids.

**Theorem 3.8.1.** *Let $\mathbf{C}$ be a topological category, $G$ a topologised group in $\mathbf{C}$ and $D$ a discrete monoid. Let $H \leq G$ be a subgroup as in section 3.5 and $A$ a rigidified $D \times H$-module with $D$ being $A$-pliant. Then*

$$C^\bullet(D \times G, \operatorname{Ind}_G^H(A)) \cong C^\bullet(D \times H, A)$$

*in the derived category of abelian groups.*

Proof. Let us first note that $D$ is also $\operatorname{Ind}_G^H(A)$-pliant: We need to show that for every $X \in \mathbf{C}$ we have an equality

$$\operatorname{Ind}_G^H(A)(D \times X) = \operatorname{Hom}_{\mathbf{Set}}(¿D, \operatorname{Ind}_G^H(A)(X)).$$

As $D$ is $A$-pliant, $\operatorname{Ind}_G^H(A)(D \times X)$ are those maps in $\operatorname{Hom}_{\mathbf{Set}}(¿D, h_A(X \times G))$ which are $H$-linear in the $G$-argument. But that is exactly $\operatorname{Hom}_{\mathbf{Set}}(¿D, \operatorname{Ind}_G^H(A)(X))$.
We can hence use theorem 3.7.6 to see that

$$C^\bullet(D \times G, \operatorname{Ind}_G^H(A)) \cong \operatorname{tot} C^\bullet(D, C^\bullet(G, \operatorname{Ind}_G^H(A))).$$

By proposition 3.3.3 we have a quasi-isomorphism

$$\operatorname{tot} C^\bullet(D, C^\bullet(G, \operatorname{Ind}_G^H(A))) \cong \operatorname{tot} X^\bullet(D, C^\bullet(G, \operatorname{Ind}_G^H(A))).$$

As $D$ is discrete, $X^\bullet(D, -) = \operatorname{Hom}_{\mathbb{Z}[D]}(F_\bullet, -)$, where $F_\bullet$ is a complex of free $\mathbb{Z}[D]$-modules, cf. proposition 3.1.2. Thus $X^\bullet(D, -)$ preserves quasi-isomorphisms. Using these arguments again, together with theorem 3.5.8, we arrive at quasi-isomorphisms

$$\operatorname{tot} X^\bullet(D, C^\bullet(G, \operatorname{Ind}_G^H(A))) \cong \operatorname{tot} X^\bullet(D, C^\bullet(H, A)) \cong \operatorname{tot} C^\bullet(D, C^\bullet(H, A)) \cong C^\bullet(D \times H, A).$$

■

CHAPTER 4

# Duality for Analytic Cohomology

The aim of this chapter is to prove a duality result for analytic cohomology of Lie groups acting on locally convex vector spaces. The statement of this duality result as given in theorem 4.5.3 is a bit cumbersome, as the functional analysis required to deal with locally convex vector spaces introduces topological difficulties. For finite dimensional vector spaces we however get a very clean result, cf. corollary 4.5.4.

## 4.1. Some Functional Analysis

We want to briefly recall some notions of non-archimedean functional analysis. We refer the reader to [Bou67; Sch02; Eme17] for details. An excellent overview can also be found in [Cre98]. In this section, we fix a complete non-archimedean field $K$ with valuation ring $\mathcal{O}_K$.

### 4.1.1. Foundations.

**Definition 4.1.1.** $K$ is called spherically complete, if every decreasing sequence of closed balls has a non-empty intersection.

**Example 4.1.2.** Every locally compact field is spherically complete. $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$, is not spherically complete.

**Definition 4.1.3.** A lattice $L$ in a $K$-vector space $V$ is an $\mathcal{O}_K$-submodule of $V$, which satisfies

$$V = \bigcup_{\lambda \in K} \lambda L.$$

**Definition 4.1.4.** We call a topological $K$-vector space *locally convex* (or an LCVS), if it has a neighbourhood basis of lattices.

**Remark 4.1.5.** Note that a subset $M$ of a $K$-vector space is an $\mathcal{O}_K$-module if and only if for all $m, m' \in M$ and all $\lambda, \mu$ with $|\lambda|, |\mu| \leq 1$ also $\lambda m + \mu m' \in M$. This is the analogy to the usual notion of convexity. Requiring $\lambda m + (1-\lambda)m' \in M$ regrettably does not suffice.

**Remark 4.1.6.** Let $V$ be a $K$-vector space. For every lattice $L$ in $V$, there is an attached seminorm $p_L$ defined by

$$p_L(v) = \inf_{\lambda \in K, a \in \lambda L} |\lambda|.$$

Conversely, for a seminorm $p \colon V \longrightarrow \mathbb{R}$ and $\varepsilon > 0$ we can define a lattice

$$V_p(\varepsilon) = \{v \in V \mid p(v) < \varepsilon\}.$$

These constructions are inverse to one another in the following sense: For a family of seminorms $(p_i)_i$, the coarsest topology on $V$ such that all $p_i$ are continuous is the locally convex topology generated by the lattices $(V_{p_i}(\varepsilon))_{i,\varepsilon}$. Conversely, if $V$ is locally convex, the topology on $V$ is the coarsest topology, such that all $(p_L)_L$ are continuous, where $L$ ranges over the open lattices in $V$. We refer to [Sch02, section I.4] for details.

**Definition 4.1.7.** A subset $B$ of an LCVS $V$ is called bounded, if for any open lattice $L$ in $V$ there is a $\lambda \in K$ such that $B \subset \lambda L$.

**Proposition 4.1.8.** *Every quasi-compact subset $C$ of an LCVS $V$ is bounded.*

Proof. Let $L$ be an open lattice. By assumption, $V = \bigcup_{\lambda \in K} \lambda L$, so finitely many $\lambda_1 L, \ldots, \lambda_n L$ cover $C$. We can assume that none of the $\lambda_i$ lie in $O_K$. Then $C \subseteq \lambda_1 \cdots \lambda_n L$. ∎

**Remark 4.1.9.** If $K$ is not locally compact, an LCVS over $K$ does not have non-trivial compact $O_K$-submodules.

**Definition 4.1.10.** Let $V$ be an LCVS. We call $V$ bornological, if a $K$-linear map $V \longrightarrow W$ of LCVS is continuous if and only if it respects bounded subsets. $V$ is called barrelled, if every closed lattice is open.

### 4.1.2. Dual spaces.

**Definition 4.1.11.** Let $V, W$ be LCVS. We denote the set of continuous $K$-linear maps from $V$ to $W$ by $\mathscr{L}(V, W)$. For bounded subsets $B \subseteq V$ and open subsets $U \subseteq W$ we denote by $L(B, U) \subseteq \mathscr{L}(V, W)$ those continuous linear maps which map $B$ into $U$. The families

$$\{L(S, U) \mid S \subseteq V \text{ a single point}, U \subseteq W \text{ open}\}$$
$$\{L(C, U) \mid C \subseteq V \text{ compact}, U \subseteq W \text{ open}\}$$
$$\{L(B, U) \mid B \subseteq V \text{ bounded}, U \subseteq W \text{ open}\}$$

generate locally convex topologies on the space $\mathcal{L}(V, W)$ of continuous linear maps from $V$ to $W$, which are called the *weak*, *compact-open*, and *strong* topology respectively. The corresponding LCVS will be denoted by $\mathscr{L}_s(V, W), \mathscr{L}_c(V, W)$, and $\mathscr{L}_b(V, W)$.

**Remark 4.1.12.** The weak topology is coarser than the compact-open topology, which in turn is coarser than the strong topology.

**Remark 4.1.13.** At this point we have to expand on our previous remark 2.1.11 on the compact-open topology. Denote by $\mathbf{T}$ the category of Hausdorff topological spaces. (A variant of this remark also holds in the non-Hausdorff case.) For topological spaces $X, Y$ we denote by $[X, Y]$ the set $\mathrm{Hom}_{\mathbf{T}}(X, Y)$ endowed with the compact-open topology. It is an easy exercise to check that for topological spaces $X, Y, Z$ there is a well-defined map

$$\mathrm{Hom}_{\mathbf{T}}(X \times Y, Z) \longrightarrow \mathrm{Hom}_{\mathbf{T}}(X, [Y, Z])$$

sending $f$ to

$$x \longmapsto (y \longmapsto f(x, y)) .$$

However, the obvious candidate for an inverse

$$\mathrm{Hom}_{\mathbf{T}}(X, [Y, Z]) \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(X \times Y, Z),$$

sending $f$ to

$$(x, y) \longmapsto f(x)(y),$$

in general does not yield continuous maps! Formally speaking, not every topological space is exponentiable. In our setting, we would have a bijection if $Y$ was locally compact, and locally compact spaces are the largest class for which this holds for all spaces $X$ and $Z$. As LCVS are only locally compact if they are finite dimensional, we cannot use the adjointness properties of the compact-open topology. In fact, there is mostly no reason to look at the compact-open topology at all. Considering linear maps, the strong topology plays the same role, but better.

**Proposition 4.1.14** (Hahn-Banach)**.** *If $K$ is spherically complete, $V$ a LCVS and $W$ a linear subspace of $V$ endowed with the subspace topology. Then every continuous linear map $W \longrightarrow K$ extends to a continuous linear map $V \longrightarrow K$.*

Proof. [Sch02, proposition 9.2, corollary 9.4]. ∎

There is also the following version of the Hahn-Banach theorem for LCVS of countable type.

**Definition 4.1.15.** An LCVS $V$ is said to be of countable type, if for every continuous seminorm $p$ on $V$ its completion $V_p$ at $p$ has a dense subspace of countable algebraic dimension.

**Proposition 4.1.16.** *Let $V$ be an LCVS of countable type and $W$ a sub-vector space endowed with the subspace topology. Then every continuous linear map $W \longrightarrow K$ extends to a continuous linear map $V \longrightarrow K$.*

Proof. [PS10, corollary 4.2.6] ∎

**Definition 4.1.17.** We say that Hahn-Banach holds for an LCVS $V$, if $K$ is spherically complete or $V$ is of countable type.

**Remark 4.1.18.** Spaces of countable type are stable under forming subspaces, linear images, projective limits, and countable inductive limits, cf. [PS10, theorem 4.2.13].

**Proposition 4.1.19.** *Let $f: V \times W \longrightarrow X$ be a (jointly) continuous bilinear map of LCVS. Then it induces a continuous map $f: V \longrightarrow \mathscr{L}_b(W, X)$.*

Proof. As a jointly continuous map is also separately continuous, we have a well-defined map $f: V \longrightarrow \mathscr{L}(W, X)$. We only need to show that it is continuous with respect to the strong topology. For this, let $B \subseteq W$ be bounded and $M \subseteq X$ an open lattice. We need to show that the set $T$ of those $w \in W$ such that $f(w, B) \subseteq M$ is open. Let $w \in T$ and $b \in B$. By separate continuity we get open lattices $w \in L, b \in L'$ such that $f(L \times L') \subseteq M$. As $M$ is bounded, there exists $\lambda \in K$ with $B \subseteq \lambda L'$. Then

$$f(w + \lambda^{-1}L, B) = f(w, B) + f(L, \lambda^{-1}B) \subseteq f(w, B) + f(L, L') \subseteq M.$$

∎

**Proposition 4.1.20** (Banach-Steinhaus)**.** *Let $V, W$ be LCVS. If $V$ is barrelled, then every bounded subset $H \subseteq \mathscr{L}_s(V, W)$ is equicontinuous, i. e., for every open lattice $L' \subseteq W$ there exists an open lattice $L \subseteq V$ such that $f(L) \subset L'$ for every $f \in H$.*

Proof. [Sch02, proposition 6.15] ∎

**Proposition 4.1.21.** *Let $G$ be a locally compact topological group and $V$ a barrelled LCVS. Assume that $G$ acts via linear maps on $V$. Then*

$$G \times V \longrightarrow V$$

*is continuous if and only if it is separately continuous.*

Proof. It is clear that a continuous group action is separately continuous.
Let $U \subset V$ be an open lattice and $g \in G, v \in V, gv \in U$. Let $H$ be a compact neighbourhood of $g$ with $Hv \in U$, which exists by local compactness of $G$ and separate continuity of the group action.
Consider the set $M = \{h \cdot - \mid h \in H\}$ of continuous linear maps $V \longrightarrow V$. We want to show that it is bounded in the topology of pointwise convergence on $\mathrm{Hom}_{\mathrm{cts}}(V, V)$. For this matter, take $w \in V, S \subset V$ an open lattice, and denote by $L$ those continuous linear maps $V \longrightarrow V$ which map $w$ into $S$. We need to show that there exists $\lambda \in K$ with $M \subseteq \lambda L$. As $H$ is compact, so is $Hw \subseteq V$, hence there exists $\lambda \in K$ such that $Hw \in \lambda^{-1}S$, i. e., $M \subseteq \lambda L$.
Proposition 4.1.20 now shows the existence of an open lattice $L'$ such that $HL' \subseteq U$, or in other words, $H \times L' \subseteq \mathrm{mult}^{-1}(U)$. ∎

**Definition 4.1.22.** The dual space of an LCVS $V$ is the vector space of continuous $K$-linear functions $V \longrightarrow K$ and will be denoted by $V'$.

We denote by $V'_s$ the dual space equipped with the weak topology, which is the topology of pointwise convergence.

$V'_c$ will denote the dual space equipped with the compact-open topology, which is also the topology of uniform convergence on compact subsets.

The strong dual will be denoted by $V'_b$ and is defined as the topology of uniform convergence on bounded subsets of $V$.

**Remark 4.1.23.** Note that for both the weak and strong duals, the dual of a direct sum of LCVS is the product of its duals. However, only for the strong dual is the dual of a product of LCVS the sum of its duals.

Note that by [Sch02, lemma 6.4], $V'_c$ can be defined as the coarsest topology on $V'$ such that for every quasi-compact $K \subset V$ the map

$$\begin{aligned} V' &\longrightarrow \mathbb{R} \\ v' &\longmapsto \sup_{v \in C} |v'(v)|_K \end{aligned}$$

is continuous.

### 4.1.3. Analyticity.

**Definition 4.1.24.** Let $E$ be a normed $K$-vector space and $V$ a LCVS. A formal sum

$$f = \sum_{n \in \mathbb{N}_0} f_n$$

of continuous functions $f_n \colon E \longrightarrow V$ which are homogeneous of degree $n$ (i. e., $f_n(\lambda x) = \lambda^n f_n(x)$ for all $n \in \mathbb{N}_0, \lambda \in K, x \in E$) is called a convergent power series, if there exists an $R > 0$ such that for every continuous seminorm $p \colon V \longrightarrow K$ the following holds:

$$\|(f_n)_n\|_{p,R} = \sup_{n \in \mathbb{N}_0} \sup_{x \in E, \|x\| \leq 1} R^n p(f_n(x)) < \infty.$$

The supremum over all $R$ such that for every continuous seminorm $p$ we have $\|(f_n)_n\|_{p,R} < \infty$ is called the radius of convergence of $f$.

A map $\widetilde{f} \colon E \longrightarrow V$ is called analytic in $x \in E$, if there exists a convergent power series $f_x$ such that for all $h \in E$ close enough to zero, we have an equality

$$\widetilde{f}(x + h) = f_x(h).$$

It is called analytic, if it is analytic at every point.

Let $M$ be an analytic Banach manifold over $K$ (i. e., $M$ is locally isomorphic to $K$-Banach spaces with analytic transition maps). A map $\widetilde{f} \colon M \longrightarrow V$ with values in $V$ is called locally analytic, if it is analytic in charts. The radius of convergence of $\widetilde{f}$ at $x$ is the radius of the power series development at a local chart. It might be larger than the chart itself.

**Lemma 4.1.25.** *Let $f \colon E \longrightarrow V$ be an analytic map from a normed vector space to a Hausdorff LCVS $V$. The map $r_f \colon E \longrightarrow \mathbb{R}_{>0} \cup \{\infty\}$, mapping a point to the radius of convergence of the power series development of $f$ at that point, is lower semi-continuous, i. e., for every $x \in E$ we have*

$$\liminf_{x' \to x} r_f(x') \geq r_f(x).$$

*Consequently, if $C \subseteq E$ is compact, then*

$$\inf_{x \in C} r_f(x) > 0.$$

Proof. A power series is analytic within its ball of convergence. It follows that the radius of convergence can at most increase. As lower semi-continuous maps attain their infimum in compact sets, the claim follows. ∎

**Proposition 4.1.26.** *The development as a power series is unique, i. e., if $\widetilde{f}\colon E \longrightarrow V$ is an analytic map between a normed $K$-vector space $E$ and a Hausdorff LCVS $V$ and if*

$$\widetilde{f}(x+h) = \sum_{n\in\mathbb{N}_0} f^{(1)}_{x,n}(h) = \sum_{n\in\mathbb{N}_0} f^{(2)}_{x,n}(h)$$

*for sufficiently small $h$ with $f^{(i)}_{x,n}$ continuous and homogeneous of degree $n$, then $f^{(1)}_{x,n} = f^{(2)}_{x,n}$ for all $n$.*

Proof. It suffices to show that if $\sum_n f_n$ is the zero function with $f_n$ continuous and homogeneous of degree $n$ and $(f_n)_n$ convergent close to zero, then all $f_n = 0$. Assume that $f_k \neq 0$. We can assume that $k$ is minimal with this property. Let $p$ be a continuous seminorm on $V$ with $p(f_k(x)) > 0$. By replacing $x$ with $\lambda x$ for some $\lambda$ close to zero, we can assume that $p(f_k(x)) > p(f_{k+n}(x))$ for all $n > 0$: By convergence of the power series, $\{p(f_{k+n}(x)) \mid n\} \subseteq \mathbb{R}$ is bounded from above by some $R \in \mathbb{R}$. Choose now $\lambda \in K$ with $|\lambda| < \max\{1, p(f_k(x))/R\}$, then it is easy to see that indeed $p(f_k(\lambda x)) \neq 0$, $k$ is minimal with this property, and $p(f_k(\lambda x)) > p(f_{k+n}(\lambda x))$ for all $n > 0$.
But then $p(\sum_n f_n(x)) = p(f_k(x)) > 0$, so $\sum_n f_n$ is not the zero function. ∎

**4.1.4. Strictness.** In chapter 2 we went to great lengths to circumvent the notion of *strict morphisms*, cf. remarks 2.2.10 and 2.3.10. At this point, we cannot avoid it any longer.

**Definition 4.1.27.** A linear map $V \longrightarrow W$ of LCVS is called *strict*, if the induced map

$$V/\ker f \longrightarrow \operatorname{im} f$$

with the quotient topology on $V/\ker f$ and the subspace topology on $\operatorname{im} f$ is an isomorphism.

**Remark 4.1.28.** Open linear maps are clearly strict, but strictness is remarkably bad behaved in general: Neither the sum nor the composition of strict maps needs to be strict again.

**Definition 4.1.29.** A sequence of LCVS

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called exact if it is exact as a sequence of vector spaces and if the involved maps are all strict.

**Proposition 4.1.30.** *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*be an exact sequence of LCVS. If Hahn-Banach holds for $B$, then the induced sequence of abelian groups*

$$0 \longrightarrow C' \longrightarrow B' \longrightarrow A' \longrightarrow 0$$

*is also exact.*

Proof. Let

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$$

be exact. It is clear that

$$0 \longrightarrow C' \xrightarrow{\pi^*} B'$$

is exact. Let $f\colon B \longrightarrow K$ be in the kernel of $\iota^*$, i. e., $\iota A \subseteq \ker f$. This induces a map $B/\iota A \longrightarrow K$, which by strictness is a map $C \longrightarrow K$.

It remains to show surjectivity of $\iota^*$, i. e., the existence of a map $\widetilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} \iota A \arrow[r, "\cong"] \arrow[d, phantom, "\cap" rotate=-90] & A \arrow[r, "f"] & K \\ B \arrow[urr, dashed, "\widetilde{f}"'] & & \end{tikzcd}$$

But this extension exists by proposition 4.1.14 or proposition 4.1.16. ∎

**Lemma 4.1.31.** *Let $V$ be an LCVS and $A\colon K^n \longrightarrow K^m$ a linear map. Then the induced map*

$$A \otimes_K V\colon V^{\oplus n} \longrightarrow V^{\oplus m}$$

*is strict.*

Proof. Note that finite direct sums of LCVS coincide with their product. It is then clear that every component map $(A \otimes_K V)_i\colon V^{\oplus n} \longrightarrow V$ is open, so $A \otimes_k V$ is open as well. ∎

**Definition 4.1.32.** A Fréchet space is an LCVS which is isomorphic to the projective limit of Banach spaces.

**Remark 4.1.33.** A space is Fréchet if and only if it is a complete LCVS whose topology is induced by a translation-invariant metric if and only if it is a Hausdorff topological $K$-vector space whose topology is induced by a countable family of semi-norms for which every Cauchy sequence converges.

**Proposition 4.1.34** (Open-mapping theorem)**.** *Let $f\colon V \longrightarrow W$ be a continuous surjective linear map from a Fréchet space to a barrelled Hausdorff LCVS. Then $f$ is open.*

Proof. [Sch02, proposition 8.6] ∎

**Definition 4.1.35.** An LCVS is called an LF-space, if it is the direct limit of a countable family of Fréchet spaces, the limit being formed in the category of locally convex vector spaces.

**Remark 4.1.36.** Let $(A_i)_i$ be a directed system of Fréchet spaces. Then their inductive limit $A = \varinjlim_i A_i$ is the set $\varinjlim_i A_i$ with the finest topology which is locally convex, such that all $A_k \longrightarrow \varinjlim_i A_i$ are continuous. References for LF-spaces in the $p$-adic case are hard to come by with [Eme17] being the most comprehensive one. Arguments such as the one above are best found in the original work of Dieudonné and Schwartz, cf. [DS49], which however strictly speaking only covers the archimedean case.

**Remark 4.1.37.** LF-spaces are Hausdorff.

**Proposition 4.1.38** (Open-mapping theorem for LF-spaces)**.** *Every continuous surjective linear map between LF-spaces is open.*

Proof. [Sch02, proposition 8.8] ∎

**Proposition 4.1.39.** *If a continuous linear map $f\colon V \longrightarrow W$ between LF spaces has finite-dimensional cokernel, it is strict.*

Proof. Note that this does not follow immediately from proposition 4.1.38, as we do not know that $\operatorname{im} f$ is again LF.

Take finitely many independent vectors whose projection to the cokernel form a basis of the cokernel. Their span in $W$ will be called $X$. As $X$ is finite dimensional, it is especially also LF and hence so is $V \oplus X$. The map

$$V \oplus X \xrightarrow{f \oplus \mathrm{id}} W$$

is then bijective, linear and continuous; thus it is an isomorphism by proposition 4.1.34 and $f$ hence an isomorphism onto $f(V)$. ∎

## 4.2. Analytic Actions of Lie Groups

We continue with a fixed non-archimedean field $K$.

**Definition 4.2.1.** A group object in the category of (finite-dimensional analytic) $K$-manifolds is called a Lie group over $K$.

**Definition 4.2.2.** Let $G$ be a Lie group over $K$ and $V$ a separated LCVS. A continuous action $G \times V \longrightarrow V$ by continuous linear maps is called analytic, if every orbit map $g \longmapsto gv$ is analytic. It is called equi-analytic, if it is analytic and the contragradient action on the dual space $G \times V' \longrightarrow V'$ is analytic with respect to the strong topology on $V'$.

**Proposition 4.2.3.** *If a Lie group $G$ acts continuously on an LCVS $V$ and if the evaluation map $V'_b \times V \longrightarrow K$ is continuous, then the contragradient action $G \times V'_b \longrightarrow V'_b$ is also continuous.*

Proof. Consider the following maps:

$$G \times V \times V'_b \xrightarrow{((-)^{-1},\mathrm{id},\mathrm{id})} G \times V \times V'_b \xrightarrow{(\mathrm{mult},\mathrm{id})} V \times V'_b \longrightarrow K$$

The last map is just the evaluation function. The composite is now clearly continuous and by proposition 4.1.19, so is the induced map

$$G \times V'_b \longrightarrow V'_b,$$

which is the contragradient action. ∎

We will spell out the following proposition in more detail than necessary to show where the name *equi-analytic* stems from.

**Proposition 4.2.4.** *An analytic action $G \times V \longrightarrow V$ is equi-analytic, if $V$ is of finite dimension.*

Proof. By proposition 4.2.3 we only need to show that every orbit map

$$g \longmapsto v'(g^{-1} \cdot -)$$

is analytic. Fix $v' \in V'$.
Considering a chart $\mathrm{coord}\colon U \longrightarrow K^d$ of a neighbourhood $U$ of $g$ and $h$ close to the neutral element,

$$(gh)^{-1}v = \sum_{n \in \mathbb{N}_0} F_{g,v,n}(\mathrm{coord}(h))$$

with $F_{g,v,n}\colon K^d \longrightarrow V$ continuous and homogeneous of degree $n$. Define

$$F'_{g,v',n}(x)(v) = v'(F_{g,v,n}(x)).$$

It suffices to show that

$$F'_{g,v',n}\colon U \longrightarrow V'_c$$

is well-defined, continuous, homogeneous of degree $n$, and gives rise to a convergent power series, as then

$$\begin{aligned}\sum_n F'_{g,v',n}(\mathrm{coord}(h))(v) &= \sum_n v'(F_{g,v,n}(\mathrm{coord}(h))) \\ &= v'(\sum_n F_{g,v,n}(\mathrm{coord}(h))) \\ &= v'((gh)^{-1}v) \\ &= ((gh)v')(v).\end{aligned}$$

Note first that by linearity of $v'$, indeed $F'_{g,v',n}$ is homogeneous of degree $n$. Using proposition 4.1.26, we see that $F'_{g,v',n}(h)$ is $K$-linear. The same argument that resulted in proposition 4.2.3 also shows that $F'_{g,v',n}$ is continuous. It remains to show that $(F'_{g,v',n})_n$ is convergent with respect to the strong topology, i. e., we need to show that there exists an $R > 0$ such that for every bounded set $B \subset V$ we have that

$$\sup_{n\in\mathbb{N}_0} \sup_{x\in K^d, \|x\|\leq 1} \sup_{v\in B} R^n \left|F'_{g,v',n}(x)(v)\right| < \infty,$$

which by definition of $F'_{g,v',n}$ is equivalent to

$$(\star) \qquad \sup_{n\in\mathbb{N}_0} \sup_{x\in K^d, \|x\|\leq 1} \sup_{v\in B} R^n \left|v'(F_{g,v,n}(x))\right| < \infty.$$

Analyticity of the group action on the other hand yields that for fixed $g \in G, v \in V$ we have an $R_{g,v} > 0$ such that

$$\sup_{n\in\mathbb{N}_0} \sup_{x\in K^d, \|x\|\leq 1} \sup_{v''\in V'} R^n_{g,v} \left|v''(F_{g,v,n}(x))\right| < \infty.$$

If $B \subseteq V$ is compact, lemma 4.1.25 yields the existence of $R_{g,B} > 0$ such that

$$(\blacklozenge) \qquad \sup_{n\in\mathbb{N}_0} \sup_{x\in K^d, \|x\|\leq 1} \sup_{v\in B} \sup_{v''\in V'} R^n_{g,C} \left|v''(F_{g,v,n}(x))\right| < \infty.$$

We cannot directly deduce ($\star$) from ($\blacklozenge$), as we have no means of controlling the radius of convergence across different compact (or bounded) subsets. This homogeneity is what *equi-analytic* alludes to.

By proposition 4.1.26, for any $u, w \in V$ and $\lambda \in K$

$$F_{g,u+\lambda w,n}(x) = F_{g,u,n}(x) + \lambda F_{g,w,n}(x).$$

It follows that if $u \in V$ is in the linear subspace generated by $w_1, \ldots, w_k \in V$, then for the radii of convergence of the orbit maps $- \cdot u$ and $- \cdot w_i$ we have the following estimate:

$$r_{-\cdot u}(g) \geq \min_i r_{-\cdot w_i}(g).$$

If $V$ is generated by $v_1, \ldots, v_n$ and $R = \min_i r_{-\cdot v_i}(g)$, then $R > 0$ and

$$\sup_{n\in\mathbb{N}_0} \sup_{x\in K^d, \|x\|\leq 1} \sup_{v\in V} \sup_{v'\in V} R^n \left|v'(F_{g,v,n}(x))\right| < \infty,$$

which is more than enough to show ($\star$). ∎

**Lemma 4.2.5.** *Let $\varphi$ be a continuous endomorphism of $V$. Then it induces a continuous map $\varphi' \colon V'_b \longrightarrow V'_b$.*

Proof. We need to show that if $B \subseteq V$ is bounded and $U \subseteq K$ is open, then also $(\varphi')^{-1}(L(B,U)) = L(\varphi(B), U)$ is open. But a continuous map clearly maps bounded sets to bounded sets. ∎

**Proposition 4.2.6.** *Let $M$ be a Banach manifold and $V, W$ separated LCVS. If $f \colon M \longrightarrow V$ is analytic and $\varphi \colon V \longrightarrow W$ continuous and linear, than $\varphi \circ f \colon M \longrightarrow W$ is also analytic.*

PROOF. We can assume that $M$ is a Banach space. Let $x \in M$ be arbitrary and let $f_{x,n}\colon M \longrightarrow V$ be continuous maps, homogeneous of degree $n$, such the family $(f_{x,n})_n$ is a convergent power series and that for all $h$ sufficiently close to zero we have an equality

$$f(x+h) = \sum_n f_{x,n}(h).$$

It suffices to show that the family $(\varphi \circ f_{x,n})_n$ is a convergent power series. By continuity of $\varphi$, for every continuous seminorm $p$ on $W$ we can find a $\lambda_p \in \mathbb{R}$ such that

$$\rho(\varphi(y)) \leq \lambda_p \|y\|.$$

Let $R$ be the radius of convergence of $(f_{x,n})_n$ and $p$ a continuous seminorm on $W$. Then

$$\sup_{n\in\mathbb{N}_0} \sup_{h\in E, \|h\|\leq 1} R^n p(\varphi(f_{x,n}(h))) \leq \lambda_p \sup_{n\in\mathbb{N}_0} \sup_{h\in E, \|h\|\leq 1} R^n \left\|f_{x,n}(h)\right\| < \infty$$

∎

## 4.3. Duality for Lie Algebras

For the general theory of Lie algebras and Lie groups we refer to [Ser92; Bou89]. In this section, we fix a complete non-archimedean field $K$ of characteristic zero and a Lie group $G$ over $K$. We also consider its attached Lie algebra $\mathfrak{g}$ with Lie bracket $[-,-]$. The adjoint action of $G$ on $\mathfrak{g}$ by differentiating conjugation maps will be denoted by $\mathrm{Ad}(-)$, the adjoint action of $\mathfrak{g}$ on itself given by $x \longmapsto [x,-]$ will be denoted by $\mathrm{ad}(-)$.

**Definition 4.3.1.** For a $\mathfrak{g}$-module $M$ we define the Chevalley-Eilenberg complex

$$\mathfrak{C}^\bullet(\mathfrak{g}, V) = \mathrm{Hom}(\textstyle\bigwedge^\bullet \mathfrak{g}, V)$$

concentrated in non-negative degrees by considering the differential

$$d\colon\ \mathfrak{C}^n(\mathfrak{g}, V) \longrightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, V)$$

given by

$$\begin{aligned} df(x_1 \wedge \cdots \wedge x_{n+1}) = &\sum_i (-1)^{i+1} x_i f(x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x+n+1) \\ &+ \sum_{i<j} (-1)^{i+j} f([x_i,x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_{n+1}). \end{aligned}$$

As usual, $\widehat{x_i}$ means omitting $x_i$ etc.

**Remark 4.3.2.** In section 1.3.2 we introduced the Koszul complex $K^\bullet(x_1,\ldots,x_d)$, which for a commutative ring $R$ and a sequence of regular elements $(t_1,\ldots,t_d)$ yielded an $R$-free resolution of $R/(t_1,\ldots,t_d)$. Proposition 1.3.13 said that

$$K^\bullet(x_1,\ldots,x_d) \cong \mathrm{Hom}_R(K^\bullet(x_1,\ldots,x_d), R)[d].$$

In our convention (cf. section 1.2), the complex $K^\bullet(x_1,\ldots,x_d)$ is concentrated in degrees $-d,\ldots,0$ and $\mathrm{Hom}_R(K^\bullet(x_1,\ldots,x_d),R)$ in degrees $0,\ldots,d$. Choosing a basis $(x_i)_i$ of $\mathfrak{g}$, we see that

$$\mathrm{Hom}_K(K^\bullet(x_1,\ldots,x_d), V) = \mathfrak{C}^\bullet(\mathfrak{g}, V)$$

whenever $\mathfrak{g}$ is abelian. The duality of the Koszul complex generalises to the Chevalley-Eilenberg complex as follows.

**Definition 4.3.3.** Let $V$ be a $\mathfrak{g}$-module. Define $V^{\mathrm{tw}}$ as the vector space $V$ with a $\mathfrak{g}$-action given by

$$x \cdot^{\mathrm{tw}} v = xv - \mathrm{Tr}(\mathrm{ad}(x))v,$$

where Tr is the trace map.

**Proposition 4.3.4.** *For $V \neq 0$, $V^{\mathrm{tw}} = V$ if and only if $H^{\dim \mathfrak{g}}(\mathfrak{g}, K) \neq 0$. If $\mathfrak{g}$ is abelian or nilpotent, $V^{\mathrm{tw}} = V$.*

Proof. [Haz70, corollary 2] ∎

In applications, this is very often the case.

**Proposition 4.3.5.** *If $G$ is compact, then $V^{\mathrm{tw}} = V$.*

Proof. As for a compact group $G$, the left and right Haar measures coincide, [Bou89, section III.3.16] implies that

$$\det \operatorname{Ad} g = 1$$

for all $g \in G$. By [Bou89, section III.4.5], we see that for all $x$ in a neighbourhood of zero of $\mathfrak{g}$

$$\operatorname{Ad}(\phi(x)) = \exp(\operatorname{ad} x),$$

where $\phi$ is a local exponential map from this neighbourhood into $G$. Here, $\exp$ is the usual exponential map of $K$ extended to matrices. Applying the determinant, we see that

$$1 = \det \exp(\operatorname{ad} x) = \exp(\operatorname{Tr} \operatorname{ad} x)$$

so $\operatorname{Tr} \operatorname{ad} x = 0$ in a neighbourhood of the identity. Choosing a basis of $\mathfrak{g}$ in this neighbourhood, we see that indeed

$$\operatorname{Tr}(\operatorname{ad} x) = 0$$

for all $x \in G$ and hence $V^{\mathrm{tw}} = V$. ∎

**Remark 4.3.6.** The argument of proposition 4.3.5 shows that if $\operatorname{Tr} \operatorname{ad} x = 0$ for all $x \in \mathfrak{g}$, then $\det \operatorname{Ad}(g) = 1$ for all $g$ in a neighbourhood of the identity. If $G$ is a connected Lie group over $\mathbb{R}$ or $\mathbb{C}$, then $\det \operatorname{Ad} g = 1$ for all $g \in G$. [Bou89, section III.3.16] then implies that the left and right Haar measures of $G$ coincide.

**Proposition 4.3.7.** *Let $M$ be a finite dimensional vector space with basis $e_1, \ldots, e_d$. For an injective morphism of ordered sets $\phi\colon \lceil k \rceil \longrightarrow \lceil d \rceil$ (cf. definition 3.6.16) define*

$$e_\phi = e_{\phi(1)} \wedge \cdots \wedge e_{\phi(k)} \in \bigwedge\nolimits^k M.$$

*Also define the $K$-linear isomorphism*

$$\star\colon \bigwedge\nolimits^k M \longrightarrow \bigwedge\nolimits^{d-k} M$$

*given by*

$$\star e_\phi = \operatorname{sgn}(\phi^*) e_{\phi^*}.$$

*Then for any invertible endomorphism $A$ of $M$ the following holds:*

$$\det A \cdot ((A^{-1})^t \circ \star) = \star \circ A.$$

Proof. This is a straight-forward piece of linear algebra, but we could not find a reference for $k \neq 1$.

Let $\phi, \psi\colon \lceil k \rceil \longrightarrow \lceil d \rceil$ be injective maps of ordered sets. For a matrix $A$ denote by $A_{\phi,\psi}$ the matrix with entries $(a_{\phi(i),\psi(j)})_{i,j \in \lceil k \rceil}$. Now a straight forward calculation (or [Bou98, proposition 9 in III.8.5]) shows that for fixed $\psi$, we have

$$A e_\psi = \sum_\phi (\det A_{\phi,\psi}) e_\phi, \tag{$\spadesuit$}$$

where $\phi$ ranges over the injective maps of ordered sets $\lceil k \rceil \longrightarrow \lceil d \rceil$. We hence also get

$$(A^{-1})^t \star e_\psi = \operatorname{sgn}(\psi^*) \sum_{\phi^*} \det((A^{-1})^t_{\phi^*,\psi^*}) e_{\phi^*},$$

where $\phi^*$ ranges over the injective maps of ordered sets $\lceil d-k \rceil \longrightarrow \lceil d \rceil$. Applying $\star$ to ($\spadesuit$), we are reduced to showing

$$\operatorname{sgn}(\psi^*) \cdot \det(A) \cdot \det(A^{-1})^t_{\phi^*,\psi^*} = \operatorname{sgn}(\phi^*) \cdot \det A_{\phi,\psi}.$$

For $k = 1$, this is precisely Cramer's rule.

Generally, for a matrix $B$, the submatrix $B_{\phi,\psi}$ can be considered as a linear map from the span of $e_{\phi(1)}, \dots, e_{\phi(k)}$ to the span of $e_{\psi(1)}, \dots, e_{\psi(k)}$. Denote this linear map by $B^{\mathrm{res}}_{\phi,\psi}$. Define $B^{\mathrm{ext}}_{\phi,\psi}$ via

$$\begin{aligned} B^{\mathrm{ext}}_{\phi,\psi} e_{\phi(i)} &= B^{\mathrm{res}}_{\phi,\psi} e_{\phi(i)}, \\ B^{\mathrm{ext}}_{\phi,\psi} e_{\phi^*(i)} &= e_{\psi^*(i)}. \end{aligned}$$

It is clear that $\det B^{\mathrm{ext}}_{\phi,\psi} = \varepsilon \det B_{\phi,\psi}$, with $\varepsilon = (-1)^{\sum_i \phi^*(i)+\psi^*(i)}$ so

$$\begin{aligned} \varepsilon \det(B_{\phi,\psi}) \cdot e_1 \wedge \cdots \wedge e_n &= B^{\mathrm{ext}}_{\phi,\psi} e_1 \wedge \cdots \wedge B^{\mathrm{ext}}_{\phi,\psi} e_n \\ &= \operatorname{sgn}(\phi^*) \cdot B^{\mathrm{res}}_{\phi,\psi} e_\phi \wedge e_{\psi^*} \end{aligned}$$

By anticommutativity of the exterior algebra, we see that indeed

$$B^{\mathrm{res}}_{\phi,\psi} e_\phi \wedge e_{\psi^*} = B e_\phi \wedge e_{\psi^*},$$

so

$$(\clubsuit) \qquad \varepsilon \cdot \det(B_{\phi,\psi}) \cdot e_1 \wedge \cdots \wedge e_n = \operatorname{sgn}(\phi^*) \cdot B e_\phi \wedge e_{\psi^*}.$$

If $B$ is invertible, we can apply $B^{-1}$ and get

$$\begin{aligned} \varepsilon \det(B_{\phi,\psi}) \cdot \det(B^{-1}) \cdot e_1 \wedge \cdots \wedge e_n &= \operatorname{sgn}(\phi^*) \cdot e_\phi \wedge B^{-1} e_{\psi^*} \\ &= \operatorname{sgn}(\phi^*)(-1)^{k(d-k)} B^{-1} e_{\psi^*} \wedge e_\phi. \end{aligned}$$

Applying ($\clubsuit$) to $(B^{-1})_{\psi^*,\phi^*}$, we find that

$$B^{-1} e_{\psi^*} \wedge e_\phi = \varepsilon' \cdot \operatorname{sgn}(\psi) \cdot \det((B^{-1})_{\psi^*,\phi^*}) \cdot e_1 \wedge \cdots \wedge e_n$$

with $\varepsilon' = (-1)^{\sum_i \phi(i)+\psi(i)}$. As clearly $\varepsilon\varepsilon' = 1$, we get

$$\det(B_{\phi,\psi}) \cdot \det(B^{-1}) = \operatorname{sgn}(\phi^*) \cdot (-1)^{k(d-k)} \cdot \operatorname{sgn}(\psi) \cdot \det((B^{-1})_{\psi^*,\phi^*}),$$

and using lemma 3.6.17, this becomes

$$\operatorname{sgn}(\phi^*) \cdot \det(B_{\phi,\psi}) \cdot \det(B^{-1}) = \operatorname{sgn}(\psi^*) \cdot \det((B^{-1})_{\psi^*,\phi^*}),$$

which is exactly what we needed to show. ∎

**Theorem 4.3.8.** *Let $V$ be a LCVS with a continuous action by $\mathfrak{g}$. If $\mathfrak{g}$ if of dimension $d$, and if $\operatorname{Tr}(\operatorname{ad}(x)) = 0$ for all $x \in \mathfrak{g}$, then there is a $G$-equivariant functorial isomorphism of complexes*

$$\mathfrak{C}^\bullet(\mathfrak{g}, V') \cong \mathfrak{C}^\bullet(\mathfrak{g}, V)'[-d].$$

Proof. In [Haz70], Hazewinkel shows (without the restriction $\operatorname{Tr} \circ \operatorname{ad} = 0$) that as abstract vector spaces

$$\mathfrak{C}^\bullet(\mathfrak{g}, (V^{\mathrm{tw}})^*) \cong \mathfrak{C}^\bullet(\mathfrak{g}, V)^*[-d],$$

where $(-)^* = \operatorname{Hom}_K(-, K)$. While it is easy to check that the isomorphism respects continuous maps, it is not immediate at all that it is $G$-equivariant. The proof itself is a brutal calculation.

Choose a basis $(e_i)_i$ of $\mathfrak{g}$ and define the star operator as in proposition 4.3.7. Hazewinkel's isomorphism stems from the following pairing:

$$\begin{aligned} \langle -,- \rangle \colon \operatorname{Hom}_K(\textstyle\bigwedge^k \mathfrak{g}, V') \times \operatorname{Hom}_K(\textstyle\bigwedge^{d-k} \mathfrak{g}, V) &\longrightarrow K \\ (a,b) &\longmapsto \langle a, b\rangle = \sum_\phi a(e_\phi)(b(\star e_\phi)) \end{aligned}$$

We need to show that

$$\langle ga, b\rangle = \langle a, g^{-1} b\rangle$$

for all $g \in G$. Write $A$ for $\mathrm{Ad}(g)$. Then

$$\langle gx, y\rangle = \sum_\phi (g.x)(e_\phi)(y(\star e_\phi)) = \sum_\phi (g(x(A^{-1}e_\phi)))(y(\star e_\phi)) = \sum_\phi x(A^{-1}e_\phi)(g^{-1}y(\star e_\phi))$$

and

$$\langle x, g^{-1}y\rangle = \sum_\phi x(e_\phi)(g^{-1}y(A \star e_\phi)).$$

In both cases, $\phi$ runs over the injective increasing maps $\lceil k\rceil \longrightarrow \lceil d\rceil$.

By considering the finite dimensional subspace of $V$ generated by all $y(\star e_\phi)$ and their images under $g^{-1}$, we can consider a finite dimensional vector space instead, i. e.,

$$\langle gx, y\rangle = \sum_i (A^{-1}e'_i)^t X^t G^{-1} Y \star e'_i$$

and

$$\langle x, g^{-1}y\rangle = \sum_i e'^t_i X^t G^{-1} Y A \star e'_i$$

for appropriate matrices $X, G^{-1}, Y, \star$ and $(e'_i)_i$ the canonical basis of $K^{\binom{d}{k}}$. (The matrix $G^{-1}$ will not be invertible in general, even though the notation does suggest this.) As $A\star = \star(A^{-1})^t$ by propositions 4.3.5 and 4.3.7, equality follows, as the trace is invariant under cyclic permutations. ∎

## 4.4. Tamme's Comparison Results

We will summarise the results from [Tam15] which we need as follows:

**Theorem 4.4.1.** *Let $K$ be a complete non-archimedean field of characteristic zero. Let $G$ be a Lie group over $K$ and $V$ a barrelled LCVS with an analytic action of $G$. Then there is a functorial morphism*

$$\widetilde{X}^\bullet(G, V) \longrightarrow \mathfrak{C}^\bullet(\mathfrak{g}, V)$$

*from the analytic homogeneous cochains of $G$ with coefficients in $V$ to the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ of $G$ with coefficients in $V$.*

*For an open subgroup $U \le G$, we denote its Lie algebra by $\mathfrak{g}(U)$. Above morphism induces for all $n$ isomorphisms*

$$\varinjlim_{U \le_o G} H^n(U, V) \cong \varinjlim_U H^n(\mathfrak{g}(U), V) = H^n(\mathfrak{g}, V).$$

*The adjoint action of $G$ on $\mathfrak{g}$ together with the action of $G$ on $V$ induce an action of $G$ on the Chevalley-Eilenberg complex and on the Lie algebra cohomology groups. If $G$ is compact, then above morphism of complexes induces an isomorphism*

$$H^n(G, V) \cong H^n(\mathfrak{g}, V)^G$$

*for all $n$.*

Proof. [Tam15, sections 3-5] ∎

## 4.5. The Duality Theorem

**Lemma 4.5.1.** *Let $G$ be a finite group acting linearly on an $L$-vector space $V$. If the order of $G$ is invertible in $L$, the composition of the canonical inclusion and projection*

$$V^G \longrightarrow V \longrightarrow V_G$$

*is an isomorphism.*

Proof. By Maschke's theorem, $L[G]$ is a semisimple ring. Therefore there exists an $L[G]$-submodule $W$ of $V$ with $V = V^G \oplus W$. Without loss of generality we can assume that $W$ is irreducible. Denote by $I$ the augmentation ideal in $L[G]$. Then

$$V_G = V^G/IV^G \oplus W/IW = V^G \oplus W/IW$$

and as $W$ is irreducible, $IW$ is either 0 or $W$. If $IW = 0$, then $W \subseteq V^G$ and hence $W = 0$ by assumption, so $W/IW = 0$ in any case. ∎

Fix now a complete non-archimedean field $K$ of characteristic zero and a Lie group $G$ over $K$, which acts equi-analytically on an LCVS $V$.

**Lemma 4.5.2.** *Let $R$ be a $K$-algebra. Assume that $V$ carries the structure of an $R$-module, such that the operation of $G$ on $V$ is $R$-linear. If $H^i(\mathfrak{g}, V)$ is finitely generated over $R$, then there is an open subgroup of $G$ which acts trivially on $H^i(\mathfrak{g}, V)$.*

Proof. By theorem 4.4.1,

$$\varinjlim_{U \leq_o G, \mathrm{res}} H^i(U, V) = H^i(\mathfrak{g}, V),$$

which is $R$-linear by our assumptions. Taking preimages of the finitely many generators in $H^i(\mathfrak{g}, V)$, we see that there is an open subgroup $U \leq G$ such that $H^i(U, V) \longrightarrow H^i(\mathfrak{g}, V)$ is surjective. This $U$ then operates trivially on $H^i(\mathfrak{g}, V)$. ∎

**Theorem 4.5.3.** *If $G$ is compact and $V, V'_b$ barrelled, we get a functorial (in $V$) morphism of complexes*

$$C^\bullet(G, V'_b) \longrightarrow \mathrm{Hom}_K(C^\bullet(G, V), K)[-d].$$

*If one of the following two conditions is satisfied:*

- *An open subgroup of $G$ operates trivially on the Lie algebra cohomology, the differentials in the Chevalley-Eilenberg complex are strict and Hahn-Banach holds for $V$, or*
- *$V$ is finite-dimensional,*

*then this morphism induces isomorphisms*

$$H^i(G, V'_b) \cong H^{d-i}(G, V)'.$$

Proof. Note first that by lemma 4.5.2, an open subgroup of $G$ operates trivially on the Lie algebra cohomology, no matter the case.

By theorem 4.4.1 and remark 3.2.4 we have morphisms

$$C^\bullet(G, V'_b) \longrightarrow \mathfrak{C}^\bullet(\mathfrak{g}, V'_b)$$

and

$$\mathrm{Hom}_K(\mathfrak{C}^\bullet(\mathfrak{g}, V), K) \longrightarrow \mathrm{Hom}_K(C^\bullet(G, V), K).$$

As $G$ is compact, we can employ theorem 4.3.8 without having to twist the Lie algebra action (cf. proposition 4.3.5). We therefore get a $G$-equivariant isomorphism

$$\mathfrak{C}^\bullet(\mathfrak{g}, V'_b) \cong \mathrm{Hom}_{K,\mathrm{cts}}(\mathfrak{C}^\bullet(\mathfrak{g}, V), K)[-d].$$

Composition with the inclusion

$$\mathrm{Hom}_{K,\mathrm{cts}}(\mathfrak{C}^\bullet(\mathfrak{g}, V), K) \subseteq \mathrm{Hom}_K(\mathfrak{C}^\bullet(\mathfrak{g}, V), K)$$

then yields the comparison morphism, which is clearly functorial in $V$.

If the differentials in the complex $\mathfrak{C}(\mathfrak{g}, V)$ are strict and Hahn-Banach holds for $V$, we get a $G$-equivariant isomorphism on the level of cohomology:

$$H^i(\mathfrak{g}, V'_b) \longrightarrow \mathrm{Hom}_{K,\mathrm{cts}}(H^{d-i}(\mathfrak{g}, V), K).$$

Especially we get the following commutative diagram:

$$\begin{tikzcd}
 & & (H^{d-i}(\mathfrak{g},V)^G)' \arrow[dr, dashed, "\cong"] & \\
H^i(G,V'_b) \arrow[r] \arrow[dr, dashed, "\cong"] & H^i(\mathfrak{g},V'_b) \arrow[r, "\cong"] & H^{d-i}(\mathfrak{g},V)' \arrow[r] \arrow[u, two heads] & H^{d-i}(G,V)' \\
 & H^i(\mathfrak{g},V'_b)^G \arrow[r, "\cong"] \arrow[u, hook] & (H^{d-i}(\mathfrak{g},V)_G)' \arrow[u] &
\end{tikzcd}$$

The dashed isomorphisms are again instances of theorem 4.4.1. If an open subgroup of $G$ acts trivially on the Lie algebra cohomology, then the composition

$$(H^{d-i}(\mathfrak{g},V)_G)' \longrightarrow H^{d-i}(\mathfrak{g},V) \longrightarrow (H^{d-i}(\mathfrak{g},V)^G)'$$

is an isomorphism by lemma 4.5.1 and the claim follows. ∎

**Corollary 4.5.4.** *Let $G$ be a compact Lie group of dimension $d$ acting analytically on a finite dimensional $K$-vector space $V$. Then we have a functorial quasi-isomorphism*

$$C^\bullet(G,V^*) \xrightarrow{\cong} C^\bullet(G,V)^*[-d].$$

Proof. By proposition 4.2.4, we are in the setting of theorem 4.5.3. If $V$ is finite-dimensional, we see that $\mathfrak{C}(\mathfrak{g},V')$ is a complex of finite dimensional vector spaces and the analytic cohomology groups are therefore finite-dimensional as well by theorem 4.4.1. For all cohomology groups involved, their abstract duals hence coincide with their continuous duals and the result follows. ∎

**Remark 4.5.5.** Functoriality in theorem 4.5.3 means the following: Let $V, W$ be LCVS with equi-analytic actions of $G$ on them. Assume that $V, W, V'_b, W'_b$ are all barrelled. Given a $G$-equivariant continuous linear map $\varphi\colon V \longrightarrow W$, we get a commutative diagram:

$$\begin{tikzcd}
C^\bullet(G,W'_b) \arrow[r] \arrow[d, "C(G,\varphi')"] & \mathrm{Hom}_K(C^\bullet(G,W),K)[-d] \arrow[d, "\mathrm{Hom}_K(C(G,\varphi),K)"] \\
C^\bullet(G,V'_b) \arrow[r] & \mathrm{Hom}_K(C^\bullet(G,V),K)[-d]
\end{tikzcd}$$

That the maps involved are well-defined follows from lemma 4.2.5 and proposition 4.2.6.

**Remark 4.5.6.** Of course a quasi-isomorphism would be a nicer result in the setting of theorem 4.5.3. The obvious strategy would be topologise $C^\bullet(G,V)$ in such a manner that the differentials are strict and that the cohomology groups are topologically identical to the topology on the Lie algebra cohomology. The same argument as above would then (under the additional hypotheses on the Chevalley-Eilenberg complex and the Lie algebra cohomology) yield a quasi-isomorphism

$$C^\bullet(G,V') \longrightarrow \mathrm{Hom}_{K,\mathrm{cts}}(C^\bullet(G,V),K).$$

We consider this endeavour to be rather futile, which is the reason we axiomatised and capsuled topological considerations in chapter 3 in the first place.

**Remark 4.5.7.** If $K = \mathbb{Q}_p$ and $V$ is a finite-dimensional $\mathbb{Q}_p$-vector space, then by [Laz65, V.(2.3.10)] analytic cohomology is just continuous cohomology. Theorem 4.5.3 is then a possible way to phrase Poincaré duality, which however does *not* coincide with Poincaré duality due to Lazard (cf. [Laz65, V.(2.5.8)]). Poincaré duality there is an *integral* phenomenon and the dual is given by $\mathrm{Hom}_{\mathbb{Z}_p,\mathrm{cts}}(V,\mathbb{Q}_p/\mathbb{Z}_p)$.

**Example 4.5.8.** Let $V$ be any barrelled LCVS and $G$ a compact abelian Lie group over $K$ of dimension $d$. The trivial action of $G$ on $V$ is of course equi-analytic. The Lie algebra $\mathfrak{g}$ of $G$ then operates by zero on $V$. The differentials in the Chevalley-Eilenberg complex $\mathfrak{C}^\bullet(\mathfrak{g}, V)$ are all zero. Theorem 4.4.1 then yields that $H^i(G, V) \cong H^i(\mathfrak{g}, V) = \mathrm{Hom}_K(\bigwedge^i \mathfrak{g}, V)$ and the isomorphism $H^i(G, V'_b) \cong H^{d-i}(G, V)'$ of theorem 4.5.3 stems from the pairing

$$\wedge\colon\ \bigwedge^i \mathfrak{g} \times \bigwedge^{d-i} \mathfrak{g} \longrightarrow K.$$

CHAPTER 5

# Two Applications to $(\varphi, \Gamma)$-modules

In this chapter we apply the results of the previous chapters to $(\varphi, \Gamma)$-modules. Analytic cohomology as it appears in the theory of $(\varphi, \Gamma)$-modules has mostly had a very strong *ad hoc* flavour. Arguments often used crucially that $\Gamma$ is a one-dimensional Lie group and $\varphi$ a single operator. However, our framework of chapter 2 and our results of chapters 3 and 4 are much more flexible and easily apply themselves to higher-dimensional $\Gamma$ and multiple operators $\varphi_1, \dots, \varphi_d$. The natural objects to look at are thus *multivariable* $(\varphi, \Gamma)$-modules. There is, however, no unified notion of multivariable $(\varphi, \Gamma)$-modules and to our knowledge, no definition of multivariable Lubin-Tate $(\varphi, \Gamma)$-modules has been published. Consequently many results which are well known in the univariate case are unknown to hold in the multivariable case. Our arguments, not relying on the ad hoc constructions of analytic cohomology, should easily carry over to the multivariable case as soon as the necessary category equivalences are shown. An important step towards this has recently been done by [PZ19].

## 5.1. An Exact Sequence of Berger and Fourquaux

We start by improving a result of Berger and Fourquaux, cf. [BF17, theorem 2.2.4], which can be stated without precisely defining $(\varphi, \Gamma)$-modules.

Let $F|\mathbb{Q}_p$ be a local field and consider the category $\mathbf{M}$ of analytic $F$-manifolds.

We fix an LF-space $A \cong \varinjlim_r \varprojlim_s A^{[r,s]}$ with Banach spaces $A^{[r,s]}$ for the remainder of this section (cf. definition 4.1.35). The notation $A^{[r,s]}$ will become apparent in the next section.

**Definition 5.1.1.** For $X \in \mathbf{M}$ let $f : X \longrightarrow A$ be a continuous map. We call $f$ pro-$F$-analytic, if there exists an $r$ and a factorisation

$$f : X \longrightarrow \varprojlim_s A^{[r,s]} \longrightarrow A,$$

such that all induced maps

$$\begin{tikzcd} X \arrow[r] & \varprojlim A^{[r,s]} \arrow[d] \\ & A^{[r,s']} \end{tikzcd}$$

are locally $F$-analytic. We denote the set of pro-$F$-analytic maps from $X$ to $A$ by $h(X, A)$, i. e.,

$$h(X, A) = \varinjlim_r \varprojlim_s h_{\mathrm{an}}(X, A^{[r,s]}),$$

where $h_{\mathrm{an}}$ denote the analytic maps in the sense of definition 4.1.24.

**Remark 5.1.2.** From the definitions it is clear that $h$ is actually a rigidification from $\mathbf{M}$ to $\mathbf{LF}$.

**Proposition 5.1.3.** *An $F$-analytic map into a Fréchet space in the sense of definition 4.1.24 is pro-$F$-analytic.*

Proof. Let $B = \varprojlim_n B_n$ be a Fréchet space with all $B_n$ Banach. We need to show that if $f : M \longrightarrow B$ is analytic, then so is $f : M \longrightarrow B \longrightarrow B_n$ for each $n$. But this is precisely the content of proposition 4.2.6.

∎

**Remark 5.1.4.** The argument in proposition 4.2.6 shows why not every pro-analytic map needs to be analytic: For a pro-analytic map $M \longrightarrow \varprojlim_n B_n$ (and around a fixed point in $M$), we have a positive radius of convergence $R_n$ of the power series development for every $B_n$. But there is no need for $\inf_n R_n$ to be positive, which is the natural estimate for the radius of convergence for $B$.

Let $\Gamma$ be a group object in $\mathbf{M}$, $G = \mathbb{N}_0 \times \Gamma$ and $A$ an LF-space over $F$ which carries the structure of a $G$-module with $h$-rigidification – in other words, let $\Gamma$ be an analytic group over $F$ and $A$ an LF-space an action from $\Gamma$ by pro-$F$-analytic maps and a continuous $L$-linear endomorphism (which we will call $\psi$) of $A$.

We now prove the following stronger version of [BF17, theorem 2.2.4], where only the case of one-dimensional $\Gamma$ is considered. For the one-dimensional case, we can also show that the last map appearing in the exact sequence of Berger and Fourquaux is surjective.

**Theorem 5.1.5.** *There is an exact sequence of cohomology groups with respect to the pro-analytic rigidification h:*

$$0 \longrightarrow H^1(\Gamma, A^{\psi=1}) \longrightarrow H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \longrightarrow (A/(\psi-1)A)^\Gamma \longrightarrow H^2(\Gamma, A^{\psi=1}) \longrightarrow H^2(\psi^{\mathbb{N}_0} \times \Gamma, A)$$

*The second to last group can be replaced by zero if $\Gamma$ is compact, has dimension one over L, and also operates analytically on each $\varprojlim_s A^{[r,s]}$.*

Proof. In lieu of theorem 3.6.27 it only remains to show the surjectivity onto $(A/(\psi - 1)A)^\Gamma$ for compact one-dimensional $\Gamma$. Note that we can't directly use theorem 4.4.1 to compare this cohomology group to Lie algebra cohomology, as pro-analytic maps and analytic maps don't necessarily coincide.

By definition 5.1.1 and the exactness of direct limits we can assume that $A$ is Fréchet. As every analytic map is also pro-analytic, the same proof as for proposition 3.3.1 together with theorem 3.6.27 shows that we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\Gamma, A^{\psi=1}) \arrow[r] & H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \arrow[r] & (A/(\psi-1)A)^\Gamma \arrow[r] & H^2(\Gamma, A^{\psi=1}) \\
0 \arrow[r] & H^1_{\mathrm{an}}(\Gamma, A^{\psi=1}) \arrow[r] \arrow[u, hook] & H^1_{\mathrm{an}}(\psi^{\mathbb{N}_0} \times \Gamma, A) \arrow[r] \arrow[u, hook] & (A/(\psi-1)A)^\Gamma \arrow[r] \arrow[u, equal] & H^2_{\mathrm{an}}(\Gamma, A^{\psi=1}) \arrow[u]
\end{tikzcd}$$

Here, $H^\bullet_{\mathrm{an}}$ denotes cohomology with respect to analytic maps in the sense of definition 4.1.24, which also induces a rigidification. For these cohomology groups, we can apply theorem 4.4.1 to see that

$$H^2_{\mathrm{an}}(\Gamma, A^{\psi=1}) = 0.$$

The exact sequence

$$0 \longrightarrow H^1(\Gamma, A^{\psi=1}) \longrightarrow H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \longrightarrow (A/(\psi-1)A)^\Gamma \longrightarrow 0$$

follows at once. ∎

**Remark 5.1.6.** Assume that $\Gamma$ is compact and of dimension one and that $A = \varprojlim_n A_n$ is Fréchet. Then even if the operation on $A$ is only pro-analytic, the same argument as in theorem 5.1.5 yields exact sequences

$$0 \longrightarrow H^1_{\mathrm{an}}(\Gamma, A_n^{\psi=1}) \longrightarrow H^1_{\mathrm{an}}(\psi^{\mathbb{N}_0} \times \Gamma, A_n) \longrightarrow (A_n/(\psi-1)A_n) \longrightarrow 0$$

for every $n$. Assume furthermore that the image of $A_m$ in $A_n$ is dense for every $m \geq n$. We then have isomorphisms

$$H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \cong \varprojlim_n H^1_{\mathrm{an}}(\psi^{\mathbb{N}_0} \times \Gamma, A_n)$$

by [BF17, proposition 2.1.1]. As taking invariants commutes with projective limits, we therefore get the following exact sequence:

$$0 \longrightarrow H^1(\Gamma, A^{\psi=1}) \longrightarrow H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \longrightarrow (A/(\psi-1)A)^{\Gamma} \longrightarrow \varprojlim_n{}^1 H^1_{\mathrm{an}}(\Gamma, A_n^{\psi=1})$$

Using again theorem 4.4.1, we see that $H^1_{\mathrm{an}}(\Gamma, A_n^{\psi=1}) \cong H^1(\mathfrak{g}, A_n^{\psi=1})^{\Gamma}$, where $\mathfrak{g}$ is the Lie algebra of $\Gamma$. The action of $\Gamma$ on $\mathfrak{g}$ is trivial and $\mathfrak{g} \cong L$, so $H^1(\mathfrak{g}, A_n^{\psi=1})^{\Gamma}$ has a comparatively simple description as the $\Gamma$-invariants of certain quotients of $A_n^{\psi=1}$, which depend on the precise group action, cf. e. g. [Wei94, theorem 7.4.7]. For these it might in certain examples be possible to show the (topological) Mittag-Leffler condition, cf. e. g. [Gro61, (0.13.2.4)], and hence show that

$$0 \longrightarrow H^1(\Gamma, A^{\psi=1}) \longrightarrow H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \longrightarrow (A/(\psi-1)A)^{\Gamma} \longrightarrow 0$$

is exact.

**Remark 5.1.7.** Considering remark 5.1.6, it is natural to ask for the relationship between $H^k(\Gamma, A)$ and $\varprojlim_n H^k(\Gamma, A_n)$, where $A = \varprojlim_n A_n$ is again assumed to be Fréchet. Consider the exact sequence

$$0 \longrightarrow C^\bullet(\Gamma, A) \longrightarrow \prod_n C^\bullet_{\mathrm{an}}(\Gamma, A_n) \xrightarrow{1-u} \prod_n C^\bullet_{\mathrm{an}}(\Gamma, A_n) \longrightarrow \varprojlim_n{}^1 C^\bullet_{\mathrm{an}}(\Gamma, A_n) \longrightarrow 0,$$

whose middle map is given by

$$1-u\colon (f_n)_n \longmapsto (f_n - (A_{n+1} \to A_n) \circ f_{n+1})_n.$$

Its existence follows from very general arguments, cf. [NSW08, (2.4.7)] for a correct statement and proof.

If one could show that $1-u$ is indeed surjective, then the long exact sequence of cohomology would yield the following short exact sequences for every $k$:

$$0 \longrightarrow \varprojlim_n{}^1 H^{k-1}_{\mathrm{an}}(\Gamma, A_n) \longrightarrow H^k(\Gamma, A) \longrightarrow \varprojlim_n H^k_{\mathrm{an}}(\Gamma, A_n) \longrightarrow 0$$

Write $d = \dim_F \Gamma$. Then these short exact sequences would show that $H^k(\Gamma, A) = 0$ for every $k > d+1$ and that $H^{d+1}(\Gamma, A) \cong \varprojlim_n{}^1 H^d_{\mathrm{an}}(\Gamma, A_n)$. Especially, the considerations in theorem 5.1.5 and remark 5.1.6 would coincide for $d = 1$.

## 5.2. So Many Rings

Fix a complete field $L \subseteq \mathbb{C}_p$. We mostly follow the notation of [BF17].

**Definition 5.2.1.** Consider the abelian group $\mathrm{Map}(\mathbb{Z}, L) = L[[X, X^{-1}]]$. In this set, we can define the following rings. Let $I \subset [0,1]$ be an interval. Set

$$\mathbf{B}_L^I = \left\{ \sum_{i \in \mathbb{Z}} a_i X^i \in L[[X, X^{-1}]] \mid \text{convergent for } z \in \mathbb{C}_p, |z| \in I \right\}.$$

For $I = [r,s]$, this is a Banach space over $L$ with norm $\|-\|_{\mathbf{B}_L^{[r,s]}} = \max\{\|-\|_r, \|-\|_s\}$, where $\|-\|_r$ and $\|-\|_s$ are defined via

$$\left\| \sum_{i\in\mathbb{Z}} a_i X^i \right\|_t = \sup_{i\in\mathbb{Z}} |a_i| t^i.$$

We also define

$$\mathbf{B}^{\dagger,r}_{\mathrm{rig},L} = \varprojlim_{r<s<1,s\to 1} \mathbf{B}_L^{[r,s]} \text{ for } 0<r<1, \text{ and}$$

$$\mathbf{B}^{\dagger}_{\mathrm{rig},L} = \varinjlim_{0<r<1,r\to 1} \mathbf{B}^{\dagger,r}_{\mathrm{rig},L}.$$

For $L$ finite over $\mathbb{Q}_p$, we can also define the complete discrete valuation ring

$$\mathbf{A}_L = O_L[[X]][X^{-1}]^\wedge$$

with quotient field

$$\mathbf{B}_L = \mathbf{A}_L[p^{-1}],$$

where $-^\wedge$ denotes $p$-adic completion. We also define

$$\mathbf{B}_L^\dagger = \left\{ f \in \mathbf{B}_L \mid f \text{ has a non-empty domain of convergence} \right\}.$$

If $M|L$ is a finite extension, the theory of the field of norms provides a certain ring extension $\mathbf{A}_{M|L}$ over $\mathbf{A}_L$. Its quotient field $\mathbf{B}_{M|L}$ is an unramified extension of $\mathbf{B}_L$. We define the following complete discrete valuation ring and its quotient field:

$$\mathbf{A} = (\bigcup_{M|L \text{ finite}} \mathbf{A}_{M|L})^\wedge,$$

$$\mathbf{B} = (\bigcup_{M|L \text{ finite}} \mathbf{B}_{M|L})^\wedge.$$

**Remark 5.2.2.** Some authors denote the rings $\mathbf{A}_{M|L}$ and $\mathbf{B}_{M|L}$ simply by $\mathbf{A}_M$ and $\mathbf{B}_M$, obfuscating the fact that these rings are *relative* notions: In our notation, we always have $\mathbf{A}_{M|M} = \mathbf{A}_M$ but generally $\mathbf{A}_{M|L} \neq \mathbf{A}_M$. We consider this abuse of notation in the literature truly abusive.

## 5.3. Lubin-Tate $(\varphi,\Gamma)$-modules

Fix a finite Galois extension $L|\mathbb{Q}_p$ for the remainder of this chapter. We denote the ring of integers of $L$ by $O_L$ and its residue field of cardinality $q$ by $\kappa_L$. We also fix a uniformiser $\pi$ of $O_L$.

**5.3.1. The Lubin-Tate Case.** We assume familiarity with the theory of formal multiplication in local fields, cf. e. g. [Ser67, section 3].

Denote by $LT$ the Lubin-Tate formal $O_L$-module attached to $\pi$, i. e., as a set $LT$ is the maximal ideal of the integral closure of $O_L$ in an algebraic closure of $L$ and the addition is defined via the unique formal group law corresponding to the endomorphism

$$[\pi](T) = T^q + \pi T.$$

Lubin-Tate theory then yields commuting power series $[a](T) \in O[[T]]$ for all $a \in O_L$, which give rise to a $O_L$-module structure on $LT$. It also yields a homomorphism

$$\chi_{LT} : G_L \longrightarrow O_L^\times,$$

which induces an isomorphism

$$\chi_{LT} : \Gamma_L = G(L_\infty|L) \xrightarrow{\cong} O_L^\times,$$

where $L_\infty$ is the extension of $L$ generated by all $\pi^\infty$-torsion points of $LT$.

For $f(T)$ in any of the rings $\mathbf{B}^\dagger_{\mathrm{rig},L}, \mathbf{B}^\dagger_L, \mathbf{A}_L, \mathbf{B}_L$ we have well-defined elements

$$\varphi(f)(T) = f([\pi](T)),$$
$$(gf)(T) = f([\chi_{LT}(g)](T)),\ g \in \Gamma_L.$$

Denote the monoid $\varphi^{\mathbb{N}^0}$ by $\Phi$. Then by construction above formula induce a continuous action of $\Phi \times \Gamma_L$ by ring homomorphisms on each of the above rings with their respective topologies, which for $\mathbf{B}^\dagger_{\mathrm{rig},L}$ is even pro-$L$-analytic cf. e. g. [Ber16, theorem 8.1].

**Definition 5.3.1.** Let $R$ be either of $\mathbf{B}^\dagger_{\mathrm{rig},L}, \mathbf{B}^\dagger_L, \mathbf{A}_L, \mathbf{B}_L$. A $(\varphi, \Gamma_L)$-module $M$ over $R$ is a free $R$-module of finite rank with a semi-linear continuous action of $\Phi \times \Gamma_L$. It is called étale if $\varphi(M)$ generates $M$.

**5.3.2. Relation to Iwasawa Cohomology.** Recall the following result due to Kisin and Ren.

**Theorem 5.3.2** ([KR09, theorem 1.6]). *The functor*

$$V \longmapsto \mathrm{D}_{\mathcal{O}}(V) = (\mathbf{A} \otimes_{\mathcal{O}_L} V)^{G(\overline{\mathbb{Q}}_p|L_\infty)}$$

*establishes an equivalence between the categories of $\mathcal{O}_L$-linear representations of $G_L$ and étale $(\varphi, \Gamma_L)$-modules over $\mathbf{A}_L$.*

For any étale $(\varphi, \Gamma_L)$-module $D$ over $\mathbf{A}_L$ there is an $\mathcal{O}_L$-linear endomorphism

$$\psi\colon D \longrightarrow D$$

satisfying

$$\psi \circ \varphi = \frac{q}{\pi}\,\mathrm{id}_D,$$

cf. e. g. [SV16, p. 416].

There is the following relationship between $(\varphi, \Gamma_L)$-modules and Iwasawa cohomology (cf. section 1.8). While $(\varphi, \Gamma_L)$-modules have plentiful applications, this is our main reason for studying them.

**Theorem 5.3.3** ([SV16, theorem 5.13]).

$$H^1_{\mathrm{Iw}}(L_\infty, V(\chi_{LT}^{-1}\chi_{cyc})) = \mathrm{D}_{\mathcal{O}}(V)^{\psi=1},$$

*where $\chi_{cyc}$ denotes the cyclotomic character.*

**Corollary 5.3.4.** *If $L \neq \mathbb{Q}_p$, we have $D^{\psi=1,\Gamma=1} = 0$ for any étale $(\varphi, \Gamma_L)$-module $D$ over $\mathbf{A}_L$.*

Proof. Together with these two aforementioned results, this follows immediately from theorem 1.8.2, as elements in $D^{\psi=1,\Gamma=1}$ are torsion over the Iwasawa algebra. ∎

**5.3.3. Overconvergence.** For the remainder of this chapter, we also fix a finite extension $F|L$.

**Definition 5.3.5.** For an $L$-linear representation $V$ of $G_F$ set

$$\mathrm{D}(V) = \left(\mathbf{B}^\dagger \otimes_L V\right)^{G(\overline{\mathbb{Q}}_p|FL_\infty)}.$$

$\Gamma = G(FL_\infty|F)$ is an open subgroup of $\Gamma_L$ and by the Lubin-Tate character hence isomorphic to an open subgroup of $\mathcal{O}_L^\times$. $\mathrm{D}(V)$ is an étale $(\varphi, \Gamma)$-module over $\mathbf{B}_L$.

**Definition 5.3.6.** Let $D$ be a $(\varphi, \Gamma)$-module over $\mathbf{B}_L$. If there is a basis of $D$ such that all endomorphisms in $\Phi\times\Gamma$ have representation matrices in $\mathbf{B}^\dagger_L$, we call $D$ overconvergent. This basis generates a $(\varphi, \Gamma)$-module over $\mathbf{B}^\dagger_L$, which we will call $\mathrm{D}^\dagger$. A Galois representation $V$ is called overconvergent if $\mathrm{D}(V)$ is. We will then write $\mathrm{D}^\dagger(V)$ instead of $\mathrm{D}(V)^\dagger$.

**Definition 5.3.7.** Let $V$ be an overconvergent Galois representation. Set

$$\mathrm{D}^{\dagger}_{\mathrm{rig}}(V) = \mathbf{B}^{\dagger}_{\mathrm{rig},F} \otimes_{\mathbf{B}^{\dagger}_{F}} \mathrm{D}^{\dagger}(V).$$

**Definition 5.3.8.** A finite dimensional $L$-linear representation $V$ of $G_F$ is called $L$-analytic, if for all embeddings $\tau\colon L \longrightarrow \bar{\mathbb{Q}}_p$ different from the fixed one,

$$\mathbb{C}_p \otimes^{\tau}_{L} V$$

is a trivial semilinear $\mathbb{C}_p$-representation, i. e., as a Galois module it is isomorphic to $\mathbb{C}_p \otimes_L \widetilde{V}$ for an $L$-vector space $\widetilde{V}$ with trivial Galois operation.

**Lemma 5.3.9.** *A finite dimensional $L$-linear representation $V$ is $L$-analytic if and only if $V^* = \mathrm{Hom}_L(V, L)$ is.*

Proof. Note that a representation is trivial if and only if its dual is. The statement then follows from the isomorphisms

$$(\mathbb{C}_p \otimes^{\tau}_{F} V)^{\vee} \cong \mathbb{C}_p^{\vee} \otimes^{\tau}_{F} V^* \cong \mathbb{C}_p \otimes^{\tau}_{F} V^*,$$

where $-^{\vee} = \mathrm{Hom}_{\mathbb{C}_p}(-, \mathbb{C}_p)$. ∎

**Proposition 5.3.10.** *The action of $\Gamma$ on $\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)$ is pro-$L$ analytic.*

Proof. This follows for example from [Ber16, theorem 8.1]. ∎

**Proposition 5.3.11** ([FX13, p. 2554])**.** *Let $D$ be an étale $(\varphi, \Gamma)$-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},F}$, which be finite generation can be written as $D = \mathbf{B}^{\dagger}_{\mathrm{rig},F} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},F}} D^r$ for some $r$ and some $(\varphi, \Gamma)$-module $D^r$ over $\mathbf{B}^{\dagger,r}_{\mathrm{rig},F}$. Then the series*

$$\log g = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (g-1)$$

*converges for $g$ small enough to an operator on $D^r$. By $\mathbb{Z}_p$-linear extension, this gives rise to a well-defined action of the Lie algebra $\mathfrak{g} \cong L$ via*

$$(x, d) \longmapsto (\log(\exp x))(d).$$

**Definition 5.3.12.** A $(\varphi, \Gamma)$-module $D$ over $\mathbf{B}^{\dagger}_{\mathrm{rig},F}$ is called $L$-analytic, if the action of the Lie algebra of $\Gamma$ on $D$ is $L$-linear.

Berger then shows the following refinement of the category equivalence.

**Theorem 5.3.13** ([Ber16, theorem D])**.** *$V \longmapsto \mathrm{D}^{\dagger}_{\mathrm{rig}}(V)$ is an equivalence of categories between $L$-analytic representations of $G_F$ and étale $L$-analytic $(\varphi, \Gamma)$-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},F}$.*

## 5.4. Towards Duality in the Herr Complex

We continue to use the notation from section 5.3.

Let $V$ be an $L$-analytic representation of $G_F$. Then the Herr complex is given by the double complex

$$C^{\bullet}(\Gamma, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V)) \xrightarrow{\varphi - 1} C^{\bullet}(\Gamma, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V)),$$

whose attached double complex is quasi-isomorphic to

$$C^{\bullet}(\Phi \times \Gamma, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V))$$

by theorem 3.7.6. Here $C^{\bullet}$ denotes the pro-$F$-analytic cochains.

It seems natural to try to apply our duality result theorem 4.5.3 to this setting, however, this is not immediately possible.

Starting with $\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)$, there are (at least) three natural ways to dualise this object: We can consider the $(\varphi,\Gamma)$-module attached to the dual representation, the module theoretic dual over the Robba ring $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$, or the topological dual $\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)'_b$. We first need to investigate how they relate to one another.

**Lemma 5.4.1.** $\mathrm{D}^{\dagger}_{\mathrm{rig}}(V^*) = \mathrm{Hom}_{\mathbf{B}^{\dagger}_{\mathrm{rig},L}}(\mathrm{D}^{\dagger}_{\mathrm{rig}}(V), \mathbf{B}^{\dagger}_{\mathrm{rig},L})$.

Proof. It is well known that the category equivalence over $\mathbf{B}_L$ is a functor of closed monoidal categories and hence respects taking duals. As the duals of analytic representations are again analytic by lemma 5.3.9, the finer category equivalence theorem 5.3.13 also has to respect duals. ∎

Let $\Omega \in \mathbb{C}_p$ be the period of $LT$, cf. [Col16, §1.1.3]. If $L \neq \mathbb{Q}_p$, it is transcendent over $\mathbb{Q}_p$. Let $K$ be the complete subfield of $\mathbb{C}_p$ generated by $L$ and $\Omega$. For a $(\varphi,\Gamma)$-module $D$ over $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ write $D_K$ for the respective $(\varphi,\Gamma)$-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ after extension of scalars. Serre duality implies the following result:

**Proposition 5.4.2.** $\mathrm{Hom}_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}}(\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)_K, \mathbf{B}^{\dagger}_{\mathrm{rig},K})(\chi_{LT}) = (\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)_K)'_b$, *where the dual is taken over* $K$.

Proof. [SV19, lemma 2.37] ∎

Note that as the extension $K|\mathbb{Q}_p$ is infinite, we cannot assume that $K$ is spherically complete. However, we at least have the following.

**Proposition 5.4.3.** *Étale* $(\varphi,\Gamma)$*-modules over* $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ *are of countable type.*

Proof. It suffices to show that $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is of countable type. The completions of $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ at the various continuous seminorms are exactly the rings $\mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},K}$. We will show that the countable set $\widetilde{L}(\Omega)[X,X^{-1}]$ is dense in every $\mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},K}$, where $\widetilde{L}$ is a number field which is dense in $L$.
Let $f = \sum_n a_n X^n \in \mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},K}$ and $\varepsilon > 0$. Convergence of $f$ on the closed annulus of inner radius $r$ and outer radius $s$ implies

$$\sup_{n<k} |a_n| r^n \longrightarrow 0 \quad (k \to -\infty)$$

and

$$\sup_{n>k} |a_n| s^n \longrightarrow 0 \quad (k \to \infty).$$

We can therefore choose $k$ with $\|\sum_{n<-k} a_n X^n\|_r, \|\sum_{n>k} a_n X^n\|_s < \varepsilon$. As $\widetilde{L}(\Omega)$ is dense in $K$, we can also choose $\alpha_{-k}, \dots, \alpha_k \in \widetilde{L}(\Omega)$ such that

$$\max_i |a_i - \alpha_i| < \varepsilon r^k.$$

It follows that

$$\left\| f - \sum_{i=-k}^{k} \alpha_i X^i \right\|_{\mathbf{B}^{\dagger,[r,s]}_{\mathrm{rig},K}} < \varepsilon.$$

∎

**Proposition 5.4.4.** *There are natural morphisms of complexes*

$$C^\bullet(\Psi \times \Gamma, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \longrightarrow \mathrm{Hom}_K(C^\bullet(\Phi \times \Gamma, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V)_K), K)[-2]$$

*and*

$$C^\bullet(\Phi \times \Gamma, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \longrightarrow \mathrm{Hom}_K(C^\bullet(\Psi \times \Gamma, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V)_K), K)[-2]$$

*stemming from a comparison of Lie algebra cohomology.*

PROOF. $\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})$ is an étale $(\varphi,\Gamma)$-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and has the structure of an LF-space over $K$, so

$$\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT}) = \varinjlim_r \varprojlim_s D^{*,[r,s]},$$

where $D^{*,[r,s]}$ are Banach spaces over $K$.

We see that

$$C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})) = \varinjlim_r \varprojlim_s C^\bullet_{\mathrm{an}}(\Gamma, D^{*,[r,s]}).$$

By theorem 4.4.1 we get a morphism

$$C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \longrightarrow \varinjlim_r \varprojlim_s \mathfrak{C}^\bullet(\mathfrak{g}, D^{*,[r,s]}).$$

Now

$$\varinjlim_r \varprojlim_s \mathfrak{C}^\bullet(\mathfrak{g}, D^{*,[r,s]}) = \mathfrak{C}^\bullet(\mathfrak{g},\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT}))$$

by lemma 1.3.2. Analogously we also get a morphism

$$C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K) \longrightarrow \mathfrak{C}^\bullet(\mathfrak{g},\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K).$$

By lemma 5.4.1 and proposition 5.4.2, we can identify $\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})$ with $(\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K)'_b = \mathrm{Hom}_{K,\mathrm{cts}}(\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K, K)_b$, where the strong dual is now taken over $K$. By theorem 4.3.8 we get a $\Gamma$-equivariant $K$-linear morphism

$$\mathfrak{C}^\bullet(\mathfrak{g},\mathrm{Hom}_{K,\mathrm{cts}}(\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K, K)) \longrightarrow \mathrm{Hom}_{K,\mathrm{cts}}(\mathfrak{C}^\bullet(\mathfrak{g},\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K), K)[-1].$$

Composing all these morphism, we get a functorial morphism of complexes

$$C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \longrightarrow \mathrm{Hom}_K(C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K), K)[-1],$$

which we can extend to a double complex as follows:

$$(\star)\qquad \begin{tikzcd} C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \arrow[r] \arrow[d, "C(G,\varphi'-1)"] & \mathrm{Hom}_K(C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K), K)[-1] \arrow[d, "\mathrm{Hom}_K(C(\Gamma,\varphi-1),K)"] \\ C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \arrow[r] & \mathrm{Hom}_K(C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V)), K)[-1] \end{tikzcd}$$

Here, $\varphi$ denotes the intrinsic $\varphi$-operator on $\mathrm{D}^\dagger(V)_K$ and $\varphi'$ its vector space dual. Note that the dualised $\varphi$ operator on $\mathrm{Hom}_{\mathbf{B}^\dagger_{\mathrm{rig},L}}(\mathrm{D}^\dagger_{\mathrm{rig}}(V),\mathbf{B}^\dagger_{\mathrm{rig},L})$ is the intrinsic $\psi$-operator on $\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)$ and vice versa, cf. [SV16, remarks 4.7 and 5.6]. The diagram can hence also be written as

$$\begin{tikzcd} C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \arrow[r] \arrow[d, "C(G,\psi-1)"] & \mathrm{Hom}_K(C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V)_K), K)[-1] \arrow[d, "\mathrm{Hom}_K(C(\Gamma,\varphi-1),K)"] \\ C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})) \arrow[r] & \mathrm{Hom}_K(C^\bullet(\Gamma,\mathrm{D}^\dagger_{\mathrm{rig}}(V)), K)[-1], \end{tikzcd}$$

where $\psi$ is the intrinsic $\psi$-operator of $\mathrm{D}^\dagger_{\mathrm{rig}}(V^*)_K(\chi_{LT})$.

By theorem 3.7.6, this induces the first morphism of complexes as required. The second morphism can be constructed completely analogously: Instead of using the intrinsic $\varphi$-operator of $\mathrm{D}^\dagger(V)_K$ in diagram $(\star)$ on the right hand side, start with its intrinsic $\psi$-operator. Then we get the vector space dual $\psi'$ on the left hand side, which is the intrinsic $\varphi$-operator of $\mathrm{D}^\dagger(V^*)_K$. ∎

**Remark 5.4.5.** The comparison morphism

$$C^\bullet(\Gamma, \mathrm{D}^\dagger_{\mathrm{rig}}(V)_K) \longrightarrow \mathfrak{C}^\bullet(\mathfrak{g}, \mathrm{D}^\dagger_{\mathrm{rig}}(V)_K)$$

does probably not induce an isomorphism on cohomology after taking $G$-invariant on the right hand side. We expect $\varprojlim^1$-terms to appear. Note however that for the first cohomology group, a Mittag-Leffler argument makes a comparison possible, cf. [BF17, proposition 2.1.1].

Apart from this and under the assumptions of theorem 4.5.3, i. e., strict differentials in the Chevalley-Eilenberg complex and an open subgroup of $\Gamma$ operating trivially on the Lie algebra cohomology, we can follow the same argument to compare cohomology groups, as by propositions 4.1.30 and 5.4.3 taking duals is exact.

**Remark 5.4.6.** In degrees zero and one, $\varphi$ and $\psi$ yield the same cohomology groups, cf. [BF17, corollary 2.2.3].

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