INTEGRATION WITH RESPECT TO A WEIGHT

MICHAEL LEINERT

Mathematics Department
University of Heidelberg
D-69 Heidelberg
West Germany

Received 5 July 1990

A simple approach to non-commutative integration for weights is described, following the lines of [7] i.e., using a natural upper integral (which is in fact an integral) and interpolation.

If $\mathscr{A}$ is a von Neumann algebra on the Hilbert space H and φ is a faithful normal semifinite weight on $\mathscr{A}$, the space D of all φ-bounded vectors in H is contained in the domain of every closed positive form coming from a positive self-adjoint operator T affiliated to $\mathscr{A}$ with finite upper integral $\bar{\varphi}(T)$. The (classes of) linear combinations of such forms constitute $\mathscr{L}^1$. In an obvious sense, $\mathscr{A}$ consists of forms, too (bounded ones). $\mathscr{L}^p$ is the complex interpolation space $[\mathscr{A}, \mathscr{L}^1]_{1/p}$. It is checked that $\mathscr{L}^p$ is isometrically isomorphic to V_p in [10], so $\mathscr{L}^p$ is what it ought to be.

Preliminaries

Let H be a Hilbert space, $\mathscr{A}$ a von Neumann algebra on H, and φ a faithful semifinite normal weight on $\mathscr{A}$. Let $n = \{T \in \mathscr{A} | \varphi(T^*T) < \infty\}$ and $m = \mathrm{Lin}\{A^*B | A, B \in n\}$ (where Lin denotes the linear span). Let H_φ denote the completion of n with respect to the scalar product $(A|B) = \varphi(B^*A)$ and let α be the inclusion map from n into H_φ. For $A \in \mathscr{A}$ denote by L_A the bounded operator on H_φ which on n is left multiplication by A. Let $J\Delta^{1/2}$ be the polar decomposition of the closure (as an operator on H_φ) of the involution operator $B \mapsto B^*$ on $n \cap n^*$. For $S, T \in n$ and $A \in \mathscr{A}$ define

$$\varphi_{S^*T}(A) = \big(\alpha(T) | JL_A J\alpha(S)\big). \tag{1}$$

Then $S^*T \mapsto \varphi_{S^*T}$ extends to a linear *-isomorphism preserving positivity from m onto its image in the predual $\mathscr{A}_*$. For $T = T^* \in m$ one has

$$\|\varphi_T\| = \inf\{\varphi(A) + \varphi(B) | T = A - B, A, B \in m^+\} \tag{2}$$

(see [3] or [9]), where $\|\varphi_T\|$ is the norm of the functional φ_T.

The Space of φ-bounded Vectors and Sesquilinear Forms on this Space

Let $D = D(H, \varphi)$ be the linear space of φ-bounded vectors $\{\xi \in H | \text{there is } C > 0$ such that $\|A\xi\| \le C\|\alpha(A)\|_{H_\varphi}$ for all $A \in n\}$. The inequality involved is equivalent to $\omega_\xi(A^*A) \le C'\varphi(A^*A)$ i.e., to $\omega_\xi \le C'\varphi$ on m^+ (where $\omega_\xi(B) = (B\xi|\xi)$).

International Journal of Mathematics Vol 2 No 2 (1991) 177–182

(3) **Lemma.** a) *Let $I \subset H$ be such that $\varphi = \sum_{x \in I} \omega_x$ (such I exists by* [3] *and* [2], *p.* 51). *Then $I \subset D$.*

b) *Let $T \in m^+$ and let $I_T \subset H$ be such that $\varphi_T = \sum_{x \in I_T} \omega_x$. Then $I_T \subset D$.*

Proof. a) For $x \in I$ we have $\omega_x \leq \varphi$, so x is in D.

b) Let $x \in I_T$ and $B \in m^+$. Then $\omega_x(B) \leq \varphi_T(B) = \varphi_B(T) \leq \|T\| \varphi(B)$, so $\omega_x \leq \|T\| \cdot \varphi$ on m^+ i.e., $x \in D$.

We shall see below that, for each $x \in D$, $\omega_x = \varphi_T$ for some $T \in m^+$.

As D is invariant under $\mathscr{A}'$, the orthogonal projection p_D onto the closure of D is in $\mathscr{A}'' = \mathscr{A}$. Letting $p_D^\perp = 1 - p_D$, by a) of the Lemma we have $\varphi(p_D^\perp) = 0$, hence $p_D^\perp = 0$, as φ is faithful. So D is dense in H.

Let X be the linear space of all sesquilinear forms on H whose domain contains $D \times D$. Equality on $D \times D$ (write $\underset{D}{=}$ for this) is an equivalene relation on X. The space $(X, \underset{D}{=})$ is given the topology of pointwise convergence on $D \times D$. By polarization, $A_\mu \to A$ in X if and only if $A_\mu(x, x) \to A(x, x)$ for all $x \in D$. Clearly $(X, \underset{D}{=})$ is a Hausdorff topological linear space. The canonical map from $\mathscr{A}$ into X is injective and continuous.

Upper Integral and $\mathscr{L}^1$

Let T be a positive self-adjoint operator on H which is affiliated to $\mathscr{A}$ (in symbols: $T \sim \mathscr{A}$). The closed positive form on H corresponding to T is again denoted by T. (It is the form $(x, y) \to (T^{1/2}x | T^{1/2}y)$. By abuse of notation we write $(Tx|y)$ for $(T^{1/2}x|T^{1/2}y)$ whenever $x, y \in D(T^{1/2})$ and set $(Tx|x) = \infty$ for $x \notin D(T^{1/2})$). Define

$$\bar{\varphi}(T) = \inf\left\{\sum_1^\infty \varphi(A_n) | A_n \in m^+, \sum_1^\infty A_n \geq T\right\}$$

where $\sum A_n \geq T$ means $\sum (A_n x|x) \geq (Tx|x)$ for all $x \in H$.

(4) If $\bar{\varphi}(T) < \infty$, then

a) $$D \subset D(T^{1/2})$$

b) There are $B_n \in m^+$ with $T = \sum_1^\infty B_n$ (pointwise as forms) and $\bar{\varphi}(T) = \sum_1^\infty \varphi(B_n)$.

Proof. a) Let $x \in D$ and suppose $T \leq \sum_1^\infty A_n$ with $\sum_1^\infty \varphi(A_n) < \infty$. In order to obtain $x \in D(T^{1/2})$, it suffices to show $\sum_1^\infty (A_n x|x) < \infty$. But this is clear, since $\omega_x \leq C \cdot \varphi$.

b) If $T = \int_0^\infty \lambda de_\lambda$ is the spectral representation of T, let $B_n = \int_{n-1}^n \lambda de_\lambda$. Then $T = \sum_1^\infty B_n$ (as forms), so $\bar{\varphi}(T) \leq \sum_1^\infty \varphi(B_n)$. On the other hand, for $A_n \in \mathscr{A}^+$ with $\sum_1^\infty A_n \geq T = \sum_1^\infty B_n$, using that the weight can be written $\varphi = \sum_{x \in I} \omega_x$, we obtain $\sum_1^\infty \varphi(A_n) \geq \sum_1^\infty \varphi(B_n)$, so $\bar{\varphi}(T) \geq \sum_1^\infty \varphi(B_n)$. Hence equality holds.

In particular we see that $\bar{\varphi}(T) = \sum_{x \in I} (Tx|x)$ and that $\bar{\varphi}(T) = \varphi(T)$ if T is in $\mathscr{A}^+$.

(5) **Definition.** Let $\mathscr{L}^1 = \mathscr{L}^1(\mathscr{A}, \varphi)$ be the space of all (equivalence classes of) complex linear combinations of closed positive forms coming from positive self-adjoint operators $T \sim \mathscr{A}$ with $\bar{\varphi}(T) < \infty$. For $T \in \mathscr{L}_h^1$ (i.e., $T \in \mathscr{L}^1$, T a Hermitian form) we set

$$\|T\|_1 = \inf\{\bar{\varphi}(A) + \bar{\varphi}(B) | A, B \sim \mathscr{A}, A, B \geq 0, T \underset{D}{=} A - B\}.$$

Then $\| \ \|_1$ is a semi-norm on $\mathscr{L}_h^1$, and $\|T\|_1 = 0$ implies $T \underset{D}{=} 0$ (use the definition of φ-bounded vectors and representations $A = \sum A_n$, $B = \sum B_n$ with $A_n, B_n \in m^+$), so $\mathscr{L}_h^1$ with $\| \ \|_1$ is a normed linear space.

Summation of a sequence of positive bounded forms leads to a closed positive form, and affiliation to $\mathscr{A}$ is preserved in this procedure. Using this one obtains that, for $T_n \in \mathscr{L}_h^1, \sum \|T_n\|_1 < \infty$ implies convergence of $\sum T_n$ in $\mathscr{L}_h^1$ (and pointwise convergence, too, on $D \times D$) so $\mathscr{L}_h^1$ is complete. Thus $\mathscr{L}_h^1$ is Banach, and so is $\mathscr{L}^1$ in any norm equivalent to the sum norm (for the precise definition of $\| \ \|_1$ on non-Hermitian elements see below).

The inclusion $\mathscr{L}^1 \subset X$ is continuous, since for $x \in D$ and $T \geq 0$ one has "$\omega_x(T)$" $= (Tx|x) \leq C \cdot \bar{\varphi}(T)$ (use $T = \sum A_n$ with $A_n \in m^+$).

$\mathscr{L}^1$ as Predual of $\mathscr{A}$

(6) **Proposition.** *Let $A_n, A \in m^+, \sum_1^\infty A_n \geq A$. Then $\sum \varphi_{A_n} \geq \varphi_A$ pointwise on m^+. If in addition $\sum \varphi_{A_n}$ converges in norm, the last inequality holds on all of $\mathscr{A}^+$.*

Proof. Let $B \in m^+$. Since φ_B is normal, we have $\sum \varphi_{A_n}(B) = \sum \varphi_B(A_n) \geq \varphi_B(A) = \varphi_A(B)$. If $\sum \varphi_{A_n}$ converges in norm, to $\psi \in \mathscr{A}_*$ say, we obtain $\psi \geq \varphi_A$ on $\mathscr{A}^+$, since m^+ is ultraweakly dense in $\mathscr{A}^+$.

(7) **Remark.** a) The above Proposition holds for increasing nets (in the place of the sequence of partial sums) too.

b) In the Proposition we may use the weaker assumption $\sum A_n \underset{D}{\geq} A$ instead of $\sum A_n \geq A$, since by b) of Lemma (3) we have $\varphi_B = \sum_{x \in I_B} \omega_x$ with $I_B \subset D$, so $\sum_n \varphi_B(A_n) = \sum_{n,x} \omega_x(A_n) \geq \sum_x \omega_x(A) = \varphi_B(A)$.

(8) **Corollary.** *If $A_n, B_n \in \mathscr{A}^+$ with $\sum A_n \underset{D}{=} \sum B_n$ and $\sum \varphi(A_n) (= \sum \varphi(B_n)) < \infty$, then $\sum \varphi_{A_n} = \sum \varphi_{B_n} (\in \mathscr{A}_*^+)$ and the map $A = \sum A_n \mapsto \varphi_A = \sum \varphi_{A_n}$ is positive linear. So the map*

$$A - B = \sum A_n - \sum B_n \mapsto \varphi_{A-B} = \sum \varphi_{A_n} - \sum \varphi_{B_n} \tag{9}$$

from $\mathscr{L}_h^1$ to $(\mathscr{A}_*)_h$ is well defined and real linear. It is norm-decreasing and is isometric on elements $A = \sum A_n$ with $A_n \in \mathscr{A}^+, \sum \varphi(A_n) < \infty$, as $\bar{\varphi}(A) = \sum \varphi(A_n) = \sum \|\varphi_{A_n}\| = \|\sum \varphi_{A_n}\|$.

The map $\varphi_T \mapsto T \in \mathscr{L}_h^1$ (for $T \in m_h$) by (2) and the definition of $\| \ \|_1$ is norm-decreasing. It can be extended, since $\mathscr{L}_h^1$ is complete, to a norm-decreasing map from $(\mathscr{A}_*)_h$ to $\mathscr{L}_h^1$ and on $\{\varphi_T | T \in m_h\}$ it is the inverse of the map defined in (9). So (9) defines an isometric isomorphism from $\mathscr{L}_h^1$ onto $(\mathscr{A}_*)_h$. Extending C-linearly to $\mathscr{L}^1$ we obtain an isomorphism from $\mathscr{L}^1$ onto $\mathscr{A}_*$. Defining $\| \ \|_1$ for non-Hermitian elements of $\mathscr{L}^1$ according to this isomorphism we obtain

(10) **Theorem.** *$\mathscr{L}^1 \cong \mathscr{A}_*$ isometrically by the isomorphism which on $\mathscr{L}_h^1$ is defined in* (9).

Definition of $\mathscr{L}^p$ and Proof that it is the Usual Space

$\mathscr{L}^1$ and $\mathscr{A}$ are continuously embedded in X, so interpolation theory applies. For $1 < p < \infty$ we define $\mathscr{L}^p$ to be the complex interpolation space $[\mathscr{A}, \mathscr{L}^1]_{1/p}$. This is the usual $\mathscr{L}^p$ space as we shall show. In order to establish an isometric isomorphism to Terp's $\mathscr{L}^p$ space [9], we need some preparation.

As in [9], let $L = \{T \in \mathscr{A} \mid \text{there is } \psi \in \mathscr{A}_* \text{ with } \psi(y) = \varphi_y(T) \text{ for all } y \in m\}$. The map $F: T \mapsto \psi$ is positive and a *-isomorphism from L onto its image in $\mathscr{A}_*$. For $T \in m$, one has $T \in L$ and $\psi = \varphi_T$. For $\psi, \psi' \in \mathscr{A}_*$ corresponding to $T, T' \in L$ the formula

$$\psi(T') = \psi'(T) \tag{11}$$

holds (see [9] p. 332).

(12) **Lemma.** *For any $x \in D$, there is $T \in L^+$ with $F(T) = \omega_x$.*

Proof. Let $x \in D$. For $A, B \in m^+$ we have $((A - B)x|x) \leq (Ax|x) + (Bx|x) \leq C\bar{\varphi}(A) + C\bar{\varphi}(B)$, so $|\omega_x(A - B)| \leq C\|A - B\|_1$. Hence $\omega_x \in (\mathscr{L}^1)^*$, so by the theorem (10) there is $T \in \mathscr{A}$ with $\omega_x(A - B) = \varphi_{A-B}(T)$. Hence $\omega_x(y) = \varphi_y(T)$ for all $y \in m$, i.e., $T \in L$ and $F(T) = \omega_x$. Letting $y \geq 0$ we see that T is positive.

(13) **Corollary.** *Let $A_n, B_n \in \mathscr{A}^+$, $\sum \varphi(A_n) < \infty$, $\sum \varphi(B_n) < \infty$ and $A = \sum A_n$, $B = \sum B_n$. Then $\varphi_{A-B} \geq 0$ if and only if $A - B \underset{D}{\geq} 0$.*

Proof. a) Let $\varphi_{A-B} \geq 0$ and $x \in D$. Let $T \in L^+$ with $\omega_x = F(T)$. We have

$$((A - B)x|x) = \sum_n \omega_x(A_n - B_n) = \sum_n \varphi_{A_n - B_n}(T) = \varphi_{A-B}(T) \geq 0.$$

So $A - B \underset{D}{\geq} 0$.

b) Let $A - B \underset{D}{\geq} 0$ and $T \in m^+$. By b) of Lemma (3) we have

$$\varphi_T = \sum_{x \in J} \omega_x$$

with $J \subset D$. So $\varphi_{A-B}(T) = \sum_n \varphi_{A_n - B_n}(T) = \sum_n \varphi_T(A_n - B_n) = \sum_n \sum_{x \in J} \omega_x(A_n - B_n) = \sum_{x \in J} ((A - B)x|x) \geq 0$. So $\varphi_{A-B} \geq 0$. (The interchange of summations is permissible as the corresponding sum for $A + B$ is an absolutely converging majorant.)

Let us now establish the connection to Terp's $\mathscr{L}^p$ spaces. We roughly have to show that the intersection $\mathscr{L}^1 \cap \mathscr{L}^\infty$ is the same in both pictures. More precisely, using the isomorphisms $\mathscr{L}^1 \to \mathscr{A}_*$ of the Theorem (10) and id : $\mathscr{A} \to \mathscr{A}$, an element S of $\mathscr{L}^1$ is equivalent ($\underset{D}{=}$) to $T \in \mathscr{A}$ if and only if the corresponding element $\psi = \varphi_S \in \mathscr{A}_*$ is identified in the sense of [9] with T (i.e., $F(T) = \psi$).

(14) Let $T \in L$ and $\psi \in A_*$ with $\psi(y) = \varphi_y(T)$ for all $y \in m$. Let $\psi = \varphi_{A-B}$ (we restrict to the Hermitian case, which we may). Then $T \underset{D}{=} A - B$.

Proof. Let $x \in D$. By Lemma (12) there is $S \in L^+$ with $F(S) = \omega_x$. By (11) we have $\omega_x(T) = \psi(S) = \varphi_{A-B}(S) = \sum_n \varphi_{A_n - B_n}(S) = \sum_n \omega_x(A_n - B_n) = \omega_x(A - B)$. So $T \underset{D}{=} A - B$.

By the way, we now see that T in Lemma (12) is really in m^+: Using (14) we have $\varphi(T) = \sum_{x \in I} \omega_x(T) = \sum_{x \in I} \omega_x(A - B) = \bar{\varphi}(A) - \bar{\varphi}(B) < \infty$.

(15) Let $T \in \mathscr{A}$, $T \underset{D}{=} A - B$, $\bar{\varphi}(A) < \infty$, $\bar{\varphi}(B) < \infty$. Then $\varphi_{A-B}(y) = \varphi_y(T)$ for all $y \in m$.

Proof. It suffices to prove the assertion for $y \in m^+$. Then $\varphi_y = \sum_{x \in J} \omega_x$ with $J \subset D$ by b) of Lemma (3). We have

$$\varphi_{A-B}(y) = \sum_n \varphi_{A_n - B_n}(y) = \sum_n \varphi_y(A_n - B_n) = \sum_n \sum_x \omega_x(A_n - B_n)$$

$$= \sum_x \omega_x(A - B) = \sum_x \omega_x(T) = \varphi_y(T).$$

The interchange of summations is permissible as the corresponding series for $A + B$ instead of $A - B$ is a dominating absolutely converging series.

(14) and (15) together prove our assertion, so $\mathscr{L}^p$ is isometrically isomorphic to V_p in [9].

Remark. $\mathscr{L}^2$ is a Hilbert space (because of [9], [5], and [4]) and this together with (10) by complex interpolation implies $(\mathscr{L}^p)^* = \mathscr{L}^q$. Denoting the norm in $\mathscr{L}^2 = [\mathscr{A}, \mathscr{L}^1]_{1/2}$ by $\| \ \|_2$, for $T \in n \cap n^*$ we have

$$\|T\|_2 = \|\Delta^{1/4}\alpha(T)\|_{H_\varphi}. \tag{16}$$

In order to show that $\mathscr{L}^2$ is a Hilbert space without referring to any other approach, it would be desirable to obtain (16) at least for T in a sufficiently large subspace of $n \cap n^*$ in a short direct way. The main point for this is the inequality $\|T\|_2 \leq \|\Delta^{1/4}\alpha(T)\|_{H_\varphi}$, so one should write down a suitable analytic function on the strip $0 \leq \operatorname{Re} z \leq 1$, like $f(z) = u|T|^{2z}$ in the trace case (see [7]). For instance, using $\alpha(\sigma_z(T)) = \Delta^{iz}\alpha(T)$ (see [8] p. 32), if $\|\Delta^{1/4}\alpha(T)\|_{H_\varphi} = 1$, the function $f(z) = |\sigma_{(2z-1)/4i}(T^*)|^z \cdot u|\sigma_{(2z-1)/4i}(T)|^z$ satisfies $f(\frac{1}{2}) = T$ and has the right estimates, namely 1, on the line $\operatorname{Re} z = 0$ (estimate in operator norm) and on the line $\operatorname{Re} z = 1$ (estimate in functional norm: $\|\varphi_{f(z)}\|$, see (1)), but it is not analytic. On the other hand, quite a few analytic functions which one might try do not seem to admit the desired estimates. From [9], p. 347 onwards, one can see that the function $f(z) = d^{-z/2}u|d^{1/4}Td^{1/4}|^{2z}d^{-z/2}$ (where d is the spatial derivative of φ with respect to a semifinite faithful normal weight ψ on $\mathscr{A}'$ (in the sense of [1], p. 158) and u is the partial isometry in the polar decomposition of $d^{1/4}Td^{1/4}$) in principal does the job, however it seems difficult to see this directly. Like in the trace case there should be a function which is easily recognized as suitable for the purpose. I have tried myself and also asked a few experts. Nevertheless it may be easy once one looks at things in the right way.

References

1. A. Connes, *On the spatial theory of von Neumann algebras*, J. Funct. Analysis **35** (1980), 153–164.
2. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, deuxième édition, Gauthier-Villars, 1969.
3. U. Haagerup, *Normal weights on W*-algebras*, J. Funct. Analysis **19** (1975), 302–318.
4. U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra*, Colloques internationaux du CNRS 274, Éditions du CNRS, Paris 1979, pp. 175–184.
5. M. Hilsum, *Les espaces L^p d'une algèbre de von Neumann définis par la dérivée spatiale*, J. Funct. Anal. **40** (1981), 151–169.
6. T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, 1984.
7. M. Leinert, *On integration with respect to a trace*, in Aspects of Positivity in Functional Analysis, ed. Nagel et al., North Holland Mathematics Studies, vol. **122**, 1986, pp. 231–239.
8. S. Stratila, *Modular Theory in Operator Algebras*, Abacus Press, 1981.
9. M. Terp, *Interpolation spaces between a von Neumann algebra and its predual*, J. Operator Theory **8** (1982), 327–360.