

## COMPLEX INTERPOLATION AND NON-COMMUTATIVE INTEGRATION

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*This paper is dedicated to Uffe Haagerup*

Communicated by U. Franz

ABSTRACT. We show that under suitable conditions interpolation between a Banach space and its dual yields a Hilbert space at  $\theta = \frac{1}{2}$ . By application of this result to the special case of the non-commutative  $L^p$ -spaces of Leinert [Int. J. Math. **2** (1991), no. 2, 177–182] and Terp [J. Operator Theory **8** (1982), 327–360] we conclude that  $L^2$  is a Hilbert space and that  $L^p$  is isometrically isomorphic to the dual of  $L^q$  without using the isomorphisms of these spaces to  $L^p$ -spaces of Hilsun [J. Funct. Anal. **40** (1981), 151–169.] and Haagerup [Colloq. Internat. CNRS, 274, CNRS, Paris, 1979]. Haagerup and Pisier [Canad. J. Math. **41** (1989), no. 5, 882–906.], Pisier [Mem. Amer. Math. Soc. **122** (1996), no. 585, viii+103 pp] and Watbled [C. R. Acad. Sci. Paris, t. 321, Série I, p. 1437–1440, 1995] gave conditions under which interpolation between a Banach space and its conjugate dual yields a Hilbert space at  $\frac{1}{2}$ . The result mentioned above when put in “conjugate form” extends their results.

### HISTORICAL REMARK

Some 20 years ago, Klaus Werner proved a theorem on interpolation (that under certain conditions the “middle” interpolation space between a Banach space and its dual (or antidual) is a Hilbert space). His result was stronger than the best results of this time (due to Pisier and to Watbled, see [12]) and at the same time solved an open problem in noncommutative integration. He earned his PhD for this in 1996 at Heidelberg University. In 1998, Cobos and

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Schonbek [4] proved an interpolation theorem of similar strength, and similarly Watbled [13] in a second paper in 2000. Since Werner's result was not published in a journal, his achievement was not properly recognised at the time. This is a pity, because he was the first to prove such a strong type of interpolation result. I therefore urged him to have it published in a journal. It seems suitable for this issue dedicated to Uffe Haagerup, since Haagerup and Pisier in [7] also proved a version of the above-mentioned interpolation theorem.

Werner's PhD thesis was written in German. He also prepared a shorter English version, but he did not care too much about publishing, since he was going into industry anyway.

Except for this historical remark and the addition of the just mentioned references [4] and [13], this note is the original English version of twenty years ago.

M. Leinert

## 1. PRELIMINARIES

Details on complex interpolation can be found in [2] and [3]. If  $(A_0, A_1)$  is a compatible couple of Banach spaces we denote the complex interpolation spaces by  $((A_0, A_1)_{[\theta]}, \|\cdot\|_{[\theta]})$  and  $((A_0, A_1)^{[\theta]}, \|\cdot\|^{[\theta]})$ . The corresponding function spaces are denoted by  $\mathcal{F}(A_0, A_1)$  and  $\mathcal{G}(A_0, A_1)$  respectively.

Let  $\mathcal{A}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$  and  $\varphi$  a normal faithful semifinite weight on  $\mathcal{A}$ .

$$\begin{aligned} N_\varphi &:= \{x \in \mathcal{A} \mid \varphi(x^*x) < \infty\} \\ M_\varphi &:= \text{Lin}\{x^*y \mid x, y \in N_\varphi\}. \end{aligned}$$

One has  $M_\varphi \subset N_\varphi$  since  $N_\varphi$  is a left ideal in  $\mathcal{A}$ . Let  $\mathcal{H}_\varphi$  be the completion of  $N_\varphi$  with respect to the scalar product  $\langle x, y \rangle_{\mathcal{H}_\varphi} = \varphi(y^*x)$ . Let  $x_\varphi$  be the image of  $x \in N_\varphi$  under the inclusion  $N_\varphi \hookrightarrow \mathcal{H}_\varphi$ . Let  $J\Delta_\varphi^{\frac{1}{2}}$  be the polar decomposition of the closure of the involution operator on  $N_\varphi \cap N_\varphi^*$  (as an operator in  $\mathcal{H}_\varphi$ ). For  $x \in \mathcal{A}$  denote by  $L_x$  the bounded operator on  $\mathcal{H}_\varphi$  which on  $N_\varphi$  is left multiplication by  $x$ . Let  $L$  be the set of all  $x \in \mathcal{A}$  for which there exists an element  $\varphi_x$  of the predual  $\mathcal{A}_*$  of  $\mathcal{A}$  such that

$$\langle \varphi_x, z^*y \rangle = \langle J(L_x)^* Jy_\varphi, z_\varphi \rangle_{\mathcal{H}_\varphi} \quad \forall y, z \in N_\varphi,$$

with the norm  $\|x\|_L := \max\{\|\varphi_x\|, \|x\|\}$ . The set  $L$  contains  $M_\varphi$ . Using the inclusion map  $x \mapsto \varphi_x$  one may consider  $L$  as a subset of  $\mathcal{A}_*$ .

Some of Terp's results are summarized in the following theorem (cf. [11]):

**Theorem 1.1** ([11]).

- (i)  $L = \{x \in \mathcal{A} \mid \exists \varphi_x \in \mathcal{A}_*, \text{ such that } \langle \varphi_x, y \rangle = \langle \varphi_y, x \rangle \ \forall y \in M_\varphi\}$ .
- (ii)  $L$  is a norm dense subset of  $\mathcal{A}_*$ .
- (iii)  $M_\varphi \subset L$  and for  $x \in L$  there exists a net  $(x_i)_{i \in I}$  in  $M_\varphi$ , such that
 
$$\begin{aligned} \sup\{\|x_i\|_L \mid i \in I\} &< \infty, \\ x_i &\longrightarrow x \quad \sigma\text{-weakly}, \\ \|\varphi_{x_i} - \varphi_x\| &\longrightarrow 0. \end{aligned}$$
- (iv)  $\langle \varphi_x, y \rangle = \langle \varphi_y, x \rangle$  for all  $x, y \in L$ .
- (v) Let  $x \in \mathcal{A}$ . Then
 
$$x \in L \iff \exists C \geq 0, \text{ such that } |\langle \varphi_y, x \rangle| \leq C\|y\| \quad \forall y \in M_\varphi.$$

In the following  $\mathcal{A}_*$  is denoted by  $L^1$ ,  $\|\varphi_x\|$  by  $\|x\|_1$  for  $x \in L$ ,  $\mathcal{A}$  by  $L^\infty$  and its norm by  $\|\cdot\|$ . Put  $L^p := (L^\infty, L^1)_{[\frac{1}{p}]}$  for  $1 < p < \infty$  as defined in [9] and [11].

First, we show that the duality on  $L^1$  and  $L^\infty$  defines a scalar product on  $L$ .

## 2. THE SCALAR PRODUCT ON $L$

**Lemma 2.1.** Let  $x, y \in M_\varphi$ . Then  $\langle \varphi_x, y^* \rangle = \langle \Delta_\varphi^{\frac{1}{2}} x_\varphi, y_\varphi \rangle_{\mathcal{H}_\varphi}$ , hence

$$\langle x, y \rangle_{\mathcal{H}_0} := \langle \varphi_x, y^* \rangle$$

is a scalar product on  $M_\varphi$ . The completion of  $M_\varphi$  with respect to this scalar product is denoted by  $\mathcal{H}_0$ .

*Proof.* Let  $x = \sum_{i=1}^n w_i^* v_i$  with  $v_i, w_i \in N_\varphi$  for  $1 \leq i \leq n$ .

$$\begin{aligned} \langle \varphi_x, y^* \rangle &= \sum_{i=1}^n \langle J L_y J(v_i)_\varphi, (w_i)_\varphi \rangle_{\mathcal{H}_\varphi} = \sum_{i=1}^n \langle L_{v_i} J y_\varphi, (w_i)_\varphi \rangle_{\mathcal{H}_\varphi} \\ &= \sum_{i=1}^n \langle J y_\varphi, (L_{v_i})^*(w_i)_\varphi \rangle_{\mathcal{H}_\varphi} = \sum_{i=1}^n \langle J y_\varphi, (v_i^* w_i)_\varphi \rangle_{\mathcal{H}_\varphi} \\ &= \left\langle J y_\varphi, \left( \sum_{i=1}^n v_i^* w_i \right)_\varphi \right\rangle_{\mathcal{H}_\varphi} = \langle J y_\varphi, x_\varphi^* \rangle_{\mathcal{H}_\varphi} \\ &= \langle J y_\varphi, J \Delta_\varphi^{\frac{1}{2}} x_\varphi \rangle_{\mathcal{H}_\varphi} = \overline{\langle y_\varphi, \Delta_\varphi^{\frac{1}{2}} x_\varphi \rangle_{\mathcal{H}_\varphi}} \\ &= \langle \Delta_\varphi^{\frac{1}{2}} x_\varphi, y_\varphi \rangle_{\mathcal{H}_\varphi}. \end{aligned}$$

□

**Lemma 2.2.**  $\langle x, y \rangle_{\mathcal{H}} := \langle \varphi_x, y^* \rangle$ ,  $x, y \in L$  is a scalar product on  $L$ . The completion  $\mathcal{H}$  of  $L$  with respect to this scalar product is isometrically isomorphic to  $\mathcal{H}_0$  by the extension of the inclusion  $M_\varphi \hookrightarrow L$ .

*Proof.* Obviously  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is a sesquilinear form which coincides with  $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$  on  $M_{\varphi}$ .

Let  $x, y \in L$  and  $(x_i)_{i \in I}$ ,  $(y_j)_{j \in K}$  resp. be nets as in Theorem 1.1, (iii). Since  $(y_k)_{k \in K}$  is bounded in  $\|\cdot\|_L$  and thus also in  $\|\cdot\|$  (the norm of  $L^\infty$ ) and  $x_i \rightarrow x$  in  $\|\cdot\|_1$ , we have

$$\langle \varphi_x - \varphi_{x_i}, y_k^* \rangle \longrightarrow 0$$

uniformly in  $k$ . As  $y_k^* \longrightarrow y^*$   $\sigma$ -weakly, we conclude

$$\langle \varphi_x, y_k^* \rangle \longrightarrow \langle \varphi_x, y^* \rangle.$$

Hence  $\langle x_i, y_k \rangle_{\mathcal{H}}$  converges to  $\langle x, y \rangle_{\mathcal{H}}$ :

$$\begin{aligned} \langle x, y \rangle_{\mathcal{H}} - \langle x_i, y_k \rangle_{\mathcal{H}} &= \langle \varphi_x, y^* \rangle - \langle \varphi_{x_i}, y_k^* \rangle \\ &= \langle \varphi_x - \varphi_{x_i}, y^* \rangle + \langle \varphi_{x_i}, y^* \rangle - \langle \varphi_{x_i}, y_k^* \rangle + \langle \varphi_{x_i}, y_k^* \rangle \\ &\longrightarrow 0 + \langle \varphi_x, y^* \rangle - \langle \varphi_x, y^* \rangle + 0 \\ &= 0. \end{aligned}$$

This convergence yields first that  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is Hermitian

$$\langle x, y \rangle_{\mathcal{H}} \longleftarrow \langle x_i, y_k \rangle_{\mathcal{H}} = \overline{\langle y_k, x_i \rangle_{\mathcal{H}}} \longrightarrow \overline{\langle y, x \rangle_{\mathcal{H}}},$$

and secondly, with  $x = y$ , that  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is positive semidefinite,

$$0 \leq \langle x_i, x_i \rangle_{\mathcal{H}} \longrightarrow \langle x, x \rangle_{\mathcal{H}} \geq 0.$$

It remains to show that  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is positive definite.

Let  $\langle x, x \rangle_{\mathcal{H}} = 0$ . By the Cauchy-Schwarz inequality (for positive semi-definite sesquilinear forms) we get

$$0 = \langle x, y^* \rangle_{\mathcal{H}} = \langle \varphi_x, y \rangle = \langle \varphi_y, x \rangle \quad \forall y \in L.$$

Hence  $x = 0$ , because  $\varphi_y, y \in L$ , are dense in the predual.

So  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is a scalar product on  $L$  which coincides with the scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$  on  $M_{\varphi} \subset L$ .  $M_{\varphi}$  is dense in  $\mathcal{H}$ , as for  $x \in L$ ,  $(x_i)_{i \in I}$  (as above),  $x_i$  converges to  $x$  in  $\|\cdot\|_{\mathcal{H}}$ :

$$\|x - x_i\|_{\mathcal{H}}^2 = \langle \varphi_{(x-x_i)}, (x-x_i)^* \rangle = \langle \varphi_x - \varphi_{x_i}, x^* - x_i^* \rangle \longrightarrow 0,$$

since the set  $\{x^* - x_i^* \mid i \in I\}$  is norm-bounded.

Hence  $\mathcal{H}$  is isometrically isomorphic to  $\mathcal{H}_0$ .  $\square$

**Lemma 2.3.** *On  $(L, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  and on  $(L, \|\cdot\|_1)$  the involution is an isometric mapping.*

*Proof.* Let  $x, y \in L$ .

The first part of the assertion follows from the equation

$$\langle x^*, y^* \rangle_{\mathcal{H}} = \langle \varphi_{x^*}, y \rangle = \langle \varphi_y, x^* \rangle = \langle y, x \rangle_{\mathcal{H}}.$$

The second part is a consequence of the fact that  $(\varphi_x)^* = \varphi_{x^*}$ :

$$\langle \varphi_{x^*}, y \rangle = \langle x^*, y^* \rangle_{\mathcal{H}} = \langle y, x \rangle_{\mathcal{H}} = \overline{\langle x, y \rangle_{\mathcal{H}}} = \overline{\langle \varphi_x, y^* \rangle}$$

for all  $y \in L$ .

Since  $M_\varphi \subset L$  is  $\sigma$ -weakly dense in  $L^\infty$ , it follows that  $\varphi_{x^*} = (\varphi_x)^*$ . Hence

$$\|x^*\|_1 = \|\varphi_{x^*}\| = \|(\varphi_x)^*\| = \|\varphi_x\| = \|x\|_1.$$

□

### 3. INTERPOLATION BETWEEN A BANACH SPACE AND ITS DUAL

By an involution on a Banach space (without multiplicative structure) we mean a conjugate-linear self-inverse mapping. Lemma 3.1 includes the case of an isometric involution.

**Lemma 3.1.** *Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces whose intersection  $\Delta := A_0 \cap A_1$  is dense in  $A_j$ ,  $j = 0, 1$ .*

*Let  $T : \Delta \rightarrow \Delta$  be a conjugate-linear, surjective mapping that is isometric in both  $\|\cdot\|_0$  and  $\|\cdot\|_1$ . Then  $T$  is also isometric with respect to the norms of  $(A_0, A_1)_{[\theta]}$  and  $(A_0, A_1)^{[\theta]}$ ,  $0 \leq \theta \leq 1$ .*

*Proof.*  $T$  can be extended uniquely to a conjugate-linear isometric surjective mapping on  $A_0, A_1$  and thus also on  $A_0 + A_1$ . This mapping again is denoted by  $T$ . For a function  $f : S \rightarrow A_0 + A_1$  let  $\bar{T}(f) : S \rightarrow A_0 + A_1$  be defined by  $\bar{T}(f)(z) = T(f(\bar{z}))$  and  $\bar{T}^{-1}(f)$  by  $\bar{T}^{-1}(f)(z) = T^{-1}(f(\bar{z}))$ .

Let  $a \in (A_0, A_1)_{[\theta]}$ . It is easy to see that  $\bar{T}$  maps

$$\mathcal{F}_a := \{f : S \rightarrow A_0 + A_1 \mid f \in \mathcal{F}(A_0, A_1), f(\theta) = a\}$$

onto

$$\mathcal{F}_{T(a)} := \{g : S \rightarrow A_0 + A_1 \mid g \in \mathcal{F}(A_0, A_1), g(\theta) = T(a)\}$$

bijjectively and isometrically with respect to the norm of  $\mathcal{F}(A_0, A_1)$ . Hence  $\|a\|_{[\theta]} = \|T(a)\|_{[\theta]}$ .

A similar proof yields the same result for the other interpolation method. □

**Lemma 3.2.** *Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces. If there exists some  $\theta_0 \in [0, 1]$  such that  $(A_0, A_1)_{[\theta_0]}$  is reflexive, then*

(i)  $(A_0, A_1)_{[\theta]}$  is reflexive for  $0 < \theta < 1$ .

(ii)  $(A_0, A_1)_{[\theta]} = (A_0, A_1)^{[\theta]}$  for  $0 < \theta < 1$ .

*Proof.* Let  $0 < \theta < 1$ ,  $\theta \neq \theta_0$ . By the reiteration theorem (see [5])  $(A_0, A_1)_{[\theta]}$  is an interpolation space of  $K := (A_0, A_1)_{[\theta_0]}$  and  $A_j$  where  $j = 0$  if  $\theta < \theta_0$  and  $j = 1$  if  $\theta > \theta_0$ . We may assume  $\theta < \theta_0$ . With  $\eta = \theta/\theta_0$  we have

$$(A_0, A_1)_{[\theta]} = (A_0, K)_{[\eta]}.$$

By assumption  $K$  is reflexive, hence  $(A_0, A_1)_{[\theta]}$  is also reflexive (see [3],[2]) and  $(A_0, K)_{[\eta]} = (A_0, K)^{[\eta]}$ . As  $(A_0, A_1)^{[\theta]}$  is contained in  $(A_0, K)^{[\eta]}$ , we have

$$(A_0, A_1)^{[\theta]} \subset (A_0, K)^{[\eta]} = (A_0, K)_{[\eta]} = (A_0, A_1)_{[\theta]}.$$

The opposite inclusion is always true.

Since all interpolation spaces  $(A_0, A_1)_{[\theta]}$ ,  $0 < \theta < 1$ , are reflexive one can repeat the proof with an arbitrary  $\theta_1 \neq \theta_0$  instead of  $\theta_0$  to get assertion (ii) also for  $\theta_0$  if  $\theta_0 \neq 0, 1$ . □

The next theorem yields sufficient conditions for the middle interpolation space to be a Hilbert space. These conditions are fulfilled by the interpolation couple  $(L^\infty, L^1)$  as will be seen in Theorem 4.2.

**Theorem 3.3.** *Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces such that  $A'_1 = A_0$  and the intersection  $\Delta := A_0 \cap A_1$  is dense in  $A_1$ . Suppose that for  $x \in \Delta$*

$$\|x\|_{A_1} = \sup\{|\langle x, y \rangle_{A_1, A_0}| : y \in \Delta, \|y\|_{A_0} \leq 1\}.$$

*Assume that for any linear functional  $\psi$  on  $\Delta$  which is continuous with respect to the norms of  $A_0$  and  $A_1$  there exists some  $z \in \Delta$  such that*

$$\psi(x) = \langle x, z \rangle_{A_1, A_0} \quad \forall x \in \Delta.$$

*Suppose further that on  $\Delta$  there is an involution  $*$  such that*

$$\langle x, y \rangle_{\mathcal{H}} := \langle x, y^* \rangle_{A_1, A_0}$$

*is a scalar product on  $\Delta$ . Let  $\mathcal{H}$  denote the completion of  $\Delta$  with respect to this scalar product. Assume that  $*$  is isometric with respect to the norms of  $A_0$ ,  $A_1$  and  $\mathcal{H}$ .*

*Then*

$$(A_0, A_1)_{[\frac{1}{2}]} \cong \mathcal{H}.$$

*For  $0 < \theta < 1$  the following holds:*

- (i)  $(A_0, A_1)_{[\theta]}$  is reflexive,
- (ii)  $(A_0, A_1)_{[\theta]} = (A_0, A_1)^{[\theta]}$ ,
- (iii)  $(A_0, A_1)'_{[\theta]} = (A_0, A_1)_{[1-\theta]}$ .

*Remark 3.4.* 1. Strictly speaking the last assertion should be  $(A_0, A_1)'_{[\theta]} \cong (A_0, A_1)_{[1-\theta]}$ . In the proof all spaces are considered as subspaces of the dual of  $\Delta$ . In this space real equality holds.

2. It may be confusing that the Hilbert space  $(A_0, A_1)_{[\frac{1}{2}]}$  is equal to its dual instead of its conjugate dual. That's because the duality on  $\mathcal{H} \times \mathcal{H}$  is an extension of the duality of  $A_1$  and  $A_0$  on  $\Delta \times \Delta$  (see remark after Theorem 5.1).

*Proof.* Put  $B_0 := \overline{\Delta}^{\|\cdot\|_{A_0}}$  (the closure of  $\Delta$  in  $A_0$ ). Then we have  $(A_0, A_1)_{[\theta]} = (B_0, A_1)_{[\theta]}$ .  $\Delta$  is dense in  $A_1$  and  $B_0$ . Hence by the Duality Theorem  $(B_0, A_1)'_{[\theta]} = (B'_0, A'_1)^{[\theta]}$ , where  $(B'_0, A'_1)$  is the dual interpolation couple defined by the canonical inclusions of  $B'_0$  and  $A'_1$  into the dual  $\Delta'$  of  $\Delta$ . Especially  $\Delta \subset A_0 = A'_1 \subset \Delta'$ .

$A_1$  can be regarded as a subspace of  $B'_0 \subset \Delta'$  because each  $x \in A_1$  defines a functional  $i_{A_1}(x)$  on  $B_0$  by  $y \mapsto \langle x, y \rangle_{A_1, A_0}$ ,  $y \in B_0$ . Since

$$\|i_{A_1}(x)\|_{B'_0} = \sup\{|\langle x, y \rangle_{A_1, A_0}| : y \in \Delta, \|y\|_{B_0} \leq 1\} = \|x\|_{A_1} \quad \forall x \in \Delta$$

we infer that the embedding  $i_{A_1} : A_1 \hookrightarrow B'_0$  is isometric on  $\Delta$  and thus on all of  $A_1$ . The last equality holds by assumption because the norms of  $B_0$  and  $A_0$

coincide on  $\Delta$ .

The involution is isometric in the norm of  $\mathcal{H}$ . For the scalar product this means:

$$\begin{aligned}
 \langle x, y \rangle_{\mathcal{H}} &= \frac{1}{4} \left( \sum_{n=0}^3 i^n \langle x + i^n y, x + i^n y \rangle_{\mathcal{H}} \right) \\
 &= \frac{1}{4} \left( \sum_{n=0}^3 i^n \langle (x + i^n y)^*, (x + i^n y)^* \rangle_{\mathcal{H}} \right) \\
 &= \frac{1}{4} \left( \sum_{n=0}^3 i^n \langle x^* + i^{-n} y^*, x^* + i^{-n} y^* \rangle_{\mathcal{H}} \right) \\
 &= \frac{1}{4} \left( \sum_{n=0}^3 i^{-n} \langle x^* + i^{-n} y^*, x^* + i^{-n} y^* \rangle_{\mathcal{H}} \right) \\
 &= \overline{\langle x^*, y^* \rangle_{\mathcal{H}}}
 \end{aligned}$$

for all  $x, y \in \Delta$ .

For  $x \in \Delta$  the values of the functionals  $i_{A_1}(x) \in B'_0$  and  $x \in A'_1$  are the same on  $\Delta$  since

$$\begin{aligned}
 \langle y, i_{A_1}(x) \rangle_{B_0, B'_0} &= \langle x, y \rangle_{A_1, A_0} = \langle x, y^* \rangle_{\mathcal{H}} \\
 &= \overline{\langle y^*, x \rangle_{\mathcal{H}}} = \langle y, x^* \rangle_{\mathcal{H}} \\
 &= \langle y, x \rangle_{A_1, A_0} \quad \forall x, y \in \Delta.
 \end{aligned}$$

Hence  $i_{A_1}(x) = x$  as elements of  $\Delta'$  for all  $x \in \Delta \subset A_0 = A'_1$ . The intersection of  $A'_1$  and  $B'_0$  also equals  $\Delta$  as on the one hand  $\Delta \subset A'_1 \cap B'_0$ . On the other hand let  $y \in A'_1 \cap B'_0$ . Then by assumption there exists some  $z \in \Delta$  such that

$$\langle x, y \rangle_{B_0, B'_0} = \langle x, y \rangle_{A_1, A'_1} = \langle x, z \rangle_{A_1, A'_1} \quad \forall x \in \Delta.$$

The last equation implies that  $z$  and  $y$  are equal as elements of  $\Delta'$ . Hence we also get  $A'_1 \cap B'_0 \subset \Delta$ . For the closure of the intersection in  $B'_0$  we have

$$\overline{\Delta}^{\|\cdot\|_{B'_0}} = \overline{\Delta}^{\|\cdot\|_{A'_0}} = A_1,$$

since the norms of  $B'_0$ ,  $A'_0$  and  $A_1$  coincide on  $\Delta$ . Altogether the above implies

$$(A_0, A_1)'_{[\theta]} = (B_0, A_1)'_{[\theta]} = (B'_0, A'_1)^{[\theta]} = (A_1, A_0)^{[\theta]} = (A_0, A_1)^{[1-\theta]}. \quad (3.1)$$

Especially for  $\theta = \frac{1}{2}$ :  $(A_0, A_1)'_{[\frac{1}{2}]} = (A_0, A_1)^{[\frac{1}{2}]}$ .

By Bergh's Theorem in [1]  $K := (A_0, A_1)_{[\frac{1}{2}]}$  is contained in  $(A_0, A_1)^{[\frac{1}{2}]} = K'$  and the norms coincide, i. e.  $\|x\|_K = \|x\|_{K'}$  for  $x \in K$ .

Let  $x \in \Delta$ . Lemma 3.1 implies  $\|x^*\|_K = \|x\|_K$ .

$$\begin{aligned}
 \|x\|_{\mathcal{H}}^2 &= \langle x, x \rangle_{\mathcal{H}} \\
 &= \langle x, x^* \rangle_{A_1, A_0} \\
 &= \langle x, x^* \rangle_{K, K'} \\
 &\leq \|x\|_K \|x^*\|_{K'} \\
 &= \|x\|_K \|x^*\|_K \\
 &= \|x\|_K^2
 \end{aligned}$$

Thus  $\|x\|_H \leq \|x\|_K$  for all  $x \in \Delta$ . The reverse inequality also holds since

$$\begin{aligned} \|x\|_K = \|x\|_{K'} &= \sup\{|\langle y, x \rangle_{K, K'}| : \|y\|_K \leq 1, y \in \Delta\} \\ &\leq \sup\{|\langle y, x \rangle_{K, K'}| : \|y\|_H \leq 1, y \in \Delta\} \\ &= \sup\{|\langle y, x \rangle_{A_1, A_0}| : \|y\|_H \leq 1, y \in \Delta\} \\ &= \sup\{|\langle y, x^* \rangle_{\mathcal{H}}| : \|y\|_H \leq 1, y \in \Delta\} \\ &= \|x^*\|_{\mathcal{H}} \\ &= \|x\|_{\mathcal{H}} \end{aligned}$$

for all  $x \in \Delta$ .

Since  $\Delta$  is dense in both  $K$  and  $\mathcal{H}$  it follows that  $K \cong \mathcal{H}$ . Consequently  $K$  is reflexive. By Lemma 3.2 and equality (3.1) we infer

$$(A_0, A_1)_{[\theta]} = (A_0, A_1)^{[\theta]}$$

and

$$(A_0, A_1)'_{[\theta]} = (A_0, A_1)^{[1-\theta]} = (A_0, A_1)_{[1-\theta]}$$

for  $0 < \theta < 1$ . □

*Remark 3.5.* The isomorphism between  $(A_0, A_1)_{[\frac{1}{2}]}$  and  $\mathcal{H}$  in the proof is the extension of the identity mapping on the intersection  $\Delta$ . If we assume that  $\mathcal{H}$  is contained in  $A_0 + A_1$  we even get

$$(A_0, A_1)_{[\frac{1}{2}]} = \mathcal{H}.$$

#### 4. APPLICATION TO THE COUPLE $(L^\infty, L^1)$

Let  $L$  and  $L^p$ ,  $1 \leq p \leq \infty$  be defined as in the beginning. Theorem 3.3 is now applied to the interpolation couple  $(L^\infty, L^1)$  in order to show that  $L^2$  is a Hilbert space and that  $L^q$  is the dual of  $L^p$ .

**Lemma 4.1.** *Let  $\psi$  be a linear functional on  $L$  which is continuous with respect to the norms of  $L^1$  and  $L^\infty$ . Then there exists an element  $z$  of  $L$  such that*

$$\langle x, \psi \rangle = \langle \varphi_x, z \rangle \quad \forall x \in L.$$

*Proof.*  $L$  is dense in  $L^1$ . Hence there exists a unique extension of  $\psi$  to a continuous linear functional on  $L^1$ . Consequently there is a  $z \in L^\infty$  such that

$$\langle x, \psi \rangle = \langle \varphi_x, z \rangle \quad \forall x \in L.$$

Since  $\psi$  is continuous with respect to the norm of  $L^\infty$  there exists a  $C \geq 0$  such that  $|\langle \varphi_x, z \rangle| = |\langle x, \psi \rangle| \leq C\|x\| \quad \forall x \in L$ . Theorem 1.1,(v) implies  $z \in L$ . □

**Theorem 4.2.**  $L^2 = (L^\infty, L^1)_{[\frac{1}{2}]}$  is a Hilbert space and

$$(L^p)' = L^q$$

for  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p, q < \infty$ .



*Proof.* Put  $A_0 = L^\infty$  and  $A_1 = L^1$ . Then the conditions of Theorem 3.3 are fulfilled:

$M_\varphi \subset \Delta = L$  is dense in  $L^1$ .

Let  $x \in \Delta = L$ .

$$\begin{aligned} \|x\|_{A_1} = \|\varphi_x\| &= \sup\{ |\langle \varphi_x, y \rangle| : y \in \mathcal{A}, \|y\| \leq 1 \} \\ &= \sup\{ |\langle \varphi_x, y \rangle| : y \in M_\varphi, \|y\| \leq 1 \} \\ &= \sup\{ |\langle \varphi_x, y \rangle| : y \in L, \|y\| \leq 1 \}, \end{aligned}$$

as  $M_\varphi \subset L$  and  $M_\varphi$  is  $\sigma$ -weakly dense in  $\mathcal{A}$ .

By Lemma 4.1, for a linear functional  $\psi$  on  $\Delta$  which is continuous with respect to the norms of  $A_0$  and  $A_1$  there exists an element  $z$  of  $\Delta$  such that

$$\psi(x) = \langle x, z \rangle_{A_1, A_0} \quad \forall x \in \Delta.$$

The involution  $*$  on  $\Delta$  yields a scalar product

$$\langle x, y \rangle_{\mathcal{H}} := \langle x, y^* \rangle_{A_1, A_0}, \quad x, y \in \Delta$$

by Lemma 2.2 and  $*$  is isometric in  $\|\cdot\|_{A_1}$  and  $\|\cdot\|_{\mathcal{H}}$  by Lemma 2.3. Obviously the involution is also isometric on  $A_0$ .  $\square$

*Remark 4.3.* By Lemma 2.2 and the remark after Theorem 3.3,  $L^2$  is the completion of  $M_\varphi$  (within  $L^\infty + L^1$ ) with respect to the scalar product

$$\langle x, y \rangle = \langle \Delta_\varphi^{\frac{1}{2}} x_\varphi, y_\varphi \rangle_{\mathcal{H}_\varphi}, \quad x, y \in M_\varphi.$$

## 5. INTERPOLATION WITH THE CONJUGATE DUAL

Let  $\overline{X}$  be the conjugate space of a complex vector space  $X$ , i.e. the space consisting of the same elements as  $X$  with conjugate scalar multiplication. Let  $\overline{x} \in \overline{X}$  denote the element corresponding to  $x \in X$ .

For a topological complex vector space  $X$  we have  $\overline{X'} = \overline{X'}$  by

$$\langle \overline{x}, \overline{y} \rangle_{\overline{X}, \overline{X'}} := \overline{\langle x, y \rangle}_{X, X'}.$$

Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces. Then

$$\begin{aligned} \overline{(A_0, A_1)_{[\theta]}} &= (\overline{A_0}, \overline{A_1})_{[\theta]}, \\ \overline{(A_0, A_1)^{[\theta]}} &= (\overline{A_0}, \overline{A_1})^{[\theta]}. \end{aligned}$$

We get the analogous result to Theorem 3.3 for interpolation between a Banach space and its conjugate dual. The proof is similar to the proof of Theorem 3.3 and can easily be done by the reader.

**Theorem 5.1.** *Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces such that  $\overline{A_1} = A_0$  and the intersection  $\Delta := A_0 \cap A_1$  is dense in  $A_1$ . Suppose that for  $x \in \Delta$*

$$\|x\|_{A_1} = \sup\{ |\langle x, \overline{y} \rangle_{A_1, \overline{A_0}}| : y \in \Delta, \|y\|_{A_0} \leq 1 \}.$$

Assume that for any linear functional  $\psi$  on  $\overline{\Delta}$  which is continuous with respect to the norms of  $\overline{A_0}$  and  $\overline{A_1}$  there exists some  $z \in \Delta$  such that

$$\psi(\overline{x}) = \langle \overline{x}, z \rangle_{\overline{A_1}, \overline{A_0}} \quad \forall \overline{x} \in \overline{\Delta}.$$

Suppose further that

$$\langle x, y \rangle_{\mathcal{H}} := \langle x, \overline{y} \rangle_{\overline{A_1}, \overline{A_0}}$$

is a scalar product on  $\Delta$ . Let  $\mathcal{H}$  denote the completion of  $\Delta$  with respect to this scalar product.

Then

$$(A_0, A_1)_{[\frac{1}{2}]} \cong \mathcal{H}.$$

For  $0 < \theta < 1$  the following holds:

- (i)  $(A_0, A_1)_{[\theta]}$  is reflexive,
- (ii)  $(A_0, A_1)_{[\theta]} = (A_0, A_1)^{[\theta]}$ ,
- (iii)  $\overline{(A_0, A_1)'_{[\theta]}} = (A_0, A_1)_{[1-\theta]}$ .

*Remark 5.2.* 1. The isomorphism between  $(A_0, A_1)_{[\frac{1}{2}]}$  and  $\mathcal{H}$  is the canonical extension of the identity map on  $\Delta$ . Whereas  $\mathcal{H}$  is an abstract completion of  $\Delta$ ,  $(A_0, A_1)_{[\frac{1}{2}]}$  is a very concrete one. If we assume  $\mathcal{H} \subset A_0 + A_1$ , we have  $(A_0, A_1)_{[\frac{1}{2}]} = \mathcal{H}$ .

2. The reader may have noticed that no involution is needed for Theorem 5.1.

Consider the following case of interpolation between a Banach space and its conjugate dual:

Let  $\mathcal{H}$  be a Hilbert space and  $v : \mathcal{H} \rightarrow X$  a continuous embedding of  $\mathcal{H}$  into a Banach space  $X$  with dense image. Then the transposed mapping of  $v$  gives an embedding  $\overline{v}' : \overline{X'} \rightarrow \overline{\mathcal{H}'} = \mathcal{H}$ . The composition

$$v \circ \overline{v}' : \overline{X'} \rightarrow X.$$

defines an interpolation couple  $(\overline{X'}, X)$  the intersection of which is  $\overline{X'}$ . Then  $(\overline{X'}, X)_{[\frac{1}{2}]} = \mathcal{H}$  as shown by Pisier [10].

This result can be obtained as a consequence of Theorem 5.1:

**Corollary 5.3.** For  $v : \mathcal{H} \rightarrow X$ , with the above definitions and notations we have

$$(\overline{X'}, X)_{[\frac{1}{2}]} = \mathcal{H}.$$

For  $0 < \theta < 1$  we have

$$\overline{(\overline{X'}, X)'_{[\theta]}} = (\overline{X'}, X)_{[1-\theta]}.$$

*Proof.*  $\overline{X'} \cap X = \overline{X'}$  is dense in  $X$  since it is dense in  $\mathcal{H}$ .

To see the density in  $\mathcal{H}$ , let  $y \in \mathcal{H}$  with  $\langle x, y \rangle_{\mathcal{H}} = 0$  for all  $x \in \overline{X'}$ . Then  $\langle y, \overline{x} \rangle_{X, X'} = \langle y, x \rangle_{\mathcal{H}} = 0$  for all  $\overline{x} \in X'$  hence  $y = 0$ .

With  $A_1 = X$  and  $A_0 = \overline{X'}$  all the conditions of Theorem 5.1 are fulfilled. The

isometric isomorphism  $(\overline{X'}, X)_{[\frac{1}{2}]} \cong \mathcal{H}$  is in fact an equality, since  $\mathcal{H}$  is contained in  $X$ .  $\square$

Conversely, let  $v$  be an embedding of  $X$  into  $\mathcal{H}$  with dense image. Then we get an embedding

$$\overline{v'} \circ v : X \rightarrow \overline{X'}$$

and thus an interpolation couple  $(\overline{X'}, X)$  with intersection  $X$ . Again the interpolation space at  $\frac{1}{2}$  is  $\mathcal{H}$  as shown by Watbled [12]. This result, too, can be derived from Theorem 5.1:

**Corollary 5.4.** *For  $v : X \rightarrow \mathcal{H}$ , with the above definitions and notations we have*

$$(\overline{X'}, X)_{[\frac{1}{2}]} = \mathcal{H}.$$

For  $0 < \theta < 1$  we have

$$\overline{(\overline{X'}, X)'_{[\theta]}} = (\overline{X'}, X)_{[1-\theta]}.$$

*Proof.* For the convenience of the reader we give a detailed proof but the fast reader might be pleased to know that the first assertion already follows from the first eight lines of Watbled's proof ([12]) and Lemma 3.2.

Let  $B$  be the closure of  $X = \overline{X'} \cap X$  in  $\overline{X'}$  and  $\iota : B \hookrightarrow \overline{X'}$  the corresponding inclusion. The image of  $\mathcal{H} = \overline{\mathcal{H}'}$  under  $\overline{v'}$  is contained in  $B$ , because  $v(X)$  is dense in  $\mathcal{H}$  and  $\overline{v'} \circ v(X) \subset B$  by definition of  $B$ . Since  $(\overline{v'})^{-1}(B)$  is closed and contains  $v(X)$  it contains  $\mathcal{H}$ .

Let  $w : \mathcal{H} \rightarrow B$ , such that

$$\iota \circ w = \overline{v'}.$$

By the duality theorem we get

$$(\overline{X'}, X)'_{[\frac{1}{2}]} = (B, X)'_{[\frac{1}{2}]} = (B', X')_{[\frac{1}{2}]}'. \quad (5.1)$$

The interpolation couple  $(B', X')$  is defined via the transpose  $v' \circ w' : B' \rightarrow X'$  of the embedding  $w \circ v : X \rightarrow B$ .

When applying Corollary 5.3 to  $w : \mathcal{H} \rightarrow B$  we use the embedding  $w \circ \overline{w'} : \overline{B'} \rightarrow B$  and because of

$$\overline{v'} \circ \overline{w'} = \iota \circ w \circ \overline{w'}$$

we get the same interpolation couple as by the duality theorem except for conjugation and taking the closure of the intersection. Therefore the interpolation spaces of one couple are the conjugate spaces of the other.

By Corollary 5.3 and Lemma 3.2 we get

$$(\overline{B'}, \overline{X'})_{[\frac{1}{2}]} = (\overline{B'}, \overline{X'})_{[\frac{1}{2}]}^{\prime\prime} = \mathcal{H}.$$

This and equality (5.1) imply that the conjugate dual space of  $(\overline{X'}, X)_{[\frac{1}{2}]}$  is  $\mathcal{H}$ , hence  $(\overline{X'}, X)_{[\frac{1}{2}]}$  is canonically isomorphic to  $\mathcal{H}$ .

For  $0 < \theta < 1$  we have

$$\begin{aligned}
\overline{(\overline{X'}, X)'}_{[\theta]} &= \overline{(B', X')^{[\theta]}} \\
&= \overline{(B', X')_{[\theta]}} \\
&= (B', X')'_{[1-\theta]} \\
&= ((B', X')^{[1-\theta]})' \\
&= (\overline{X'}, X)''_{[1-\theta]}. \tag{5.2}
\end{aligned}$$

As in Theorem 3.3 and Theorem 5.1 equality is to be understood in  $B'' = (B' \cap X')'$ , the dual of the intersection of the dual interpolation couple.

$\overline{(\overline{X'}, X)'}_{[\theta]} = (\overline{X'}, X)''_{[1-\theta]}$  is canonically isomorphic to  $(\overline{X'}, X)_{[1-\theta]}$ , since this space is reflexive.

They are even equal. To see this it suffices in this case to show that one of the spaces is contained in the other (because all identifications are based on canonical embeddings):

$X \subset \mathcal{H} \subset \overline{X'}$  is contained in  $\overline{B'}$  ( $\subset \mathcal{H} \subset \overline{X'}$ ) as for  $x \in X$  the embedding  $\overline{v'} \circ \overline{w'} : \overline{B'} \rightarrow \overline{X'}$  maps the element  $\overline{z} \in \overline{B'}$  defined by

$$\langle \overline{y}, \overline{z} \rangle_{\overline{B}, \overline{B'}} := \langle x, \overline{v'}(\overline{y}) \rangle_{X, X'}, \quad \overline{y} \in \overline{B}$$

to the image of  $x$  under the embedding  $\overline{v'} \circ v : X \rightarrow \overline{X'}$ :

$$\begin{aligned}
\langle y, \overline{w'}(\overline{z}) \rangle_{\mathcal{H}} &= \langle y, w'(z) \rangle_{\mathcal{H}, \mathcal{H}'} \\
&= \langle w(y), z \rangle_{B, B'} \\
&= \overline{\langle x, \overline{v'} \circ \overline{w'}(\overline{y}) \rangle}_{X, X'} \\
&= \overline{\langle x, v'(\overline{y}) \rangle}_{X, X'} \\
&= \overline{\langle v(x), \overline{y} \rangle}_{\mathcal{H}, \mathcal{H}'} \\
&= \overline{\langle v(x), y \rangle}_{\mathcal{H}} \\
&= \langle y, v(x) \rangle_{\mathcal{H}}
\end{aligned}$$

for all  $y \in \mathcal{H}$ .

So  $\overline{w'}(\overline{z}) = v(x)$  in  $\mathcal{H}$  and thus  $\overline{v'} \circ \overline{w'}(\overline{z}) = \overline{v'} \circ v(x)$  in  $\overline{X'}$ .

The inclusion of  $X$  into  $\overline{B'}$  is norm decreasing:

$$\begin{aligned}
\|x\|_X &= \sup\{ |\langle x, y \rangle_{X, X'}| : y \in X', \|y\|_{X'} \leq 1 \} \\
&\geq \sup\{ |\langle x, \overline{v'}(\overline{y}) \rangle_{X, X'}| : y \in \overline{B}, \|y\|_{\overline{B}} \leq 1 \} \\
&= \sup\{ |\langle y, \overline{z} \rangle_{\overline{B}, \overline{B'}}| : y \in \overline{B}, \|y\|_{\overline{B}} \leq 1 \} \\
&= \|\overline{z}\|_{\overline{B'}}.
\end{aligned}$$

Therefore  $\mathcal{F}(\overline{X'}, X) \subset \mathcal{F}(\overline{X'}, \overline{B'})$ . We conclude

$$\overline{(\overline{X'}, X)'}_{[\theta]} \subset (\overline{X'}, \overline{B'})_{[\theta]} = (\overline{B'}, \overline{X'})_{[1-\theta]} \stackrel{(3)}{=} (\overline{X'}, X)''_{[\theta]}.$$

□

## 6. SYMMETRIC VERSION OF THEOREM 5.1

In Theorem 5.1  $A_1$  and  $B_0$  (the closure of  $\Delta$  in  $A_0$ ) play symmetric roles as both spaces are subspaces of the conjugate dual of the other. The density assumption can be replaced by (symmetric) norm conditions for both spaces:

**Theorem 6.1.** *Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces with intersection  $\Delta$ . Let  $\mathcal{H}$  be a Hilbert space containing  $\Delta$  as a dense subset. Suppose that for  $x \in \Delta$*

$$\begin{aligned} \|x\|_{A_1} &= \sup\{|\langle x, y \rangle_{\mathcal{H}}| : y \in \Delta, \|y\|_{A_0} \leq 1\} \\ \|x\|_{A_0} &= \sup\{|\langle x, y \rangle_{\mathcal{H}}| : y \in \Delta, \|y\|_{A_1} \leq 1\} \end{aligned}$$

Assume that for any linear functional  $\psi$  on  $\Delta$  which is continuous with respect to the norms of  $A_0$  and  $A_1$  there exists some  $z \in \Delta$  such that

$$\psi(x) = \langle x, z \rangle_{\mathcal{H}} \quad \forall x \in \Delta.$$

Then

$$(A_0, A_1)_{[\frac{1}{2}]} \cong \mathcal{H}.$$

For  $0 < \theta < 1$  the following holds:

$$\overline{(A_0, A_1)'_{[\theta]}} = (A_0, A_1)_{[1-\theta]}.$$

*Proof.* Let  $B_j$  be the closure of  $\Delta$  in  $A_j$ . Embed  $B_0$  (isometrically) into  $\overline{B_1'}$  by the extension of

$$\langle x, \bar{y} \rangle_{B_1, B_1'} := \langle x, y \rangle_{\mathcal{H}} \quad \forall x, y \in \Delta$$

and apply Theorem 5.1. □

*Remark 6.2.* Theorem 6.1 implies Theorem 3.3. Applying Theorem 6.1 to the situation of Theorem 3.3 one implicitly uses the fact that there is a (linear!) isometric isomorphism  $I : B_0 \rightarrow \overline{B_0}$ ,  $y \mapsto \bar{y}^*$ ,  $y \in \Delta$ .

An interesting question is whether it is possible to get rid of the condition for the functionals that are continuous in both norms. The idea used in Corollary 5.4 to show that the dual space and thus also the original space is a Hilbert space can be of help.

**Theorem 6.3.** *Let  $(A_0, A_1)$  be an interpolation couple of Banach spaces with intersection  $\Delta$ . Let  $B_j$  be the closure of  $\Delta$  in  $A_j$ ,  $j = 0, 1$ . Let  $\mathcal{H} \subset B_0 + B_1$  (continuous embedding) be a Hilbert space containing  $\Delta$  as a dense subset. Suppose that for  $x \in \Delta$*

$$\begin{aligned} \|x\|_{A_1} &= \sup\{|\langle x, y \rangle_{\mathcal{H}}| : y \in \Delta, \|y\|_{A_0} \leq 1\} \\ \|x\|_{A_0} &= \sup\{|\langle x, y \rangle_{\mathcal{H}}| : y \in \Delta, \|y\|_{A_1} \leq 1\} \end{aligned}$$

The intersection  $\Delta_d$  of the dual interpolation couple is contained in  $\mathcal{H}' = \overline{\mathcal{H}}$  (see proof).

Assume that

$$|\langle x, y \rangle_{\overline{\mathcal{H}}}| \leq \|x\|_{B_0'} \|y\|_{B_1'} \quad \forall x, y \in \Delta_d. \quad (6.1)$$

Then

$$(A_0, A_1)_{[\frac{1}{2}]} \cong \mathcal{H}.$$

For  $0 < \theta < 1$  the following holds:

$$\overline{(A_0, A_1)'_{[\theta]}} \cong (A_0, A_1)_{[1-\theta]}.$$

*Remark 6.4.*  $\Delta \subset \mathcal{H} \subset B_0 + B_1$  is a necessary condition for the first assertion.

*Proof.*  $\mathcal{H}$  is dense in  $B_0 + B_1$  as it contains  $\Delta$ . By transposition and the usual identifications we get

$$\Delta_d = (B_0 + B_1)' \subset \mathcal{H}' = \overline{\mathcal{H}} \subset \overline{B_0} + \overline{B_1} \quad (6.2)$$

since the first equality always holds (see [2], Theorem 2.7.1)

For the scalar product we have

$$\langle x, y \rangle_{\overline{\mathcal{H}}} = \langle x, \bar{y} \rangle_{\overline{B_0 + B_1}, \overline{\Delta_d}}$$

for all  $x, y \in \Delta_d$ .

Because of the norm conditions  $\overline{\Delta}$  is isometrically contained in  $B'_0$  and  $B'_1$  with respect to the norms of  $\overline{B_1}$  and  $\overline{B_0}$  respectively. So  $\overline{\Delta}$  is contained in the intersection  $\Delta_d$ .

Now we show that the dual couple  $(B'_0, B'_1)$  fulfills the conditions of Theorem 6.1.  $\overline{\Delta}$  is dense in  $\overline{B_0}$ , hence

$$\begin{aligned} \|x\|_{B'_0} &= \|\bar{x}\|_{\overline{B'_0}} \\ &= \sup\{ |\langle y, \bar{x} \rangle_{\overline{B_0}, \overline{B'_0}}| : y \in \overline{\Delta}, \|y\|_{\overline{B_0}} \leq 1 \} \\ &= \sup\{ |\langle y, \bar{x} \rangle_{\overline{B_0 + B_1}, \overline{\Delta_d}}| : y \in \overline{\Delta}, \|y\|_{\overline{B_0}} \leq 1 \} \\ &= \sup\{ |\langle y, x \rangle_{\overline{\mathcal{H}}}| : y \in \overline{\Delta}, \|y\|_{\overline{B_0}} \leq 1 \} \\ &= \sup\{ |\langle y, x \rangle_{\overline{\mathcal{H}}}| : y \in \overline{\Delta}, \|y\|_{B'_1} \leq 1 \} \\ &\leq \sup\{ |\langle y, x \rangle_{\overline{\mathcal{H}}}| : y \in \Delta_d, \|y\|_{B'_1} \leq 1 \} \end{aligned}$$

for all  $x \in \Delta_d$ .

Analogously

$$\|x\|_{B'_1} \leq \sup\{ |\langle y, x \rangle_{\overline{\mathcal{H}}}| : y \in \Delta_d, \|y\|_{B'_0} \leq 1 \} \quad \forall x \in \Delta_d.$$

By (6.1) we get the reverse inequality and thus

$$\begin{aligned} \|x\|_{B'_0} &= \sup\{ |\langle y, x \rangle_{\overline{\mathcal{H}}}| : y \in \Delta_d, \|y\|_{B'_1} \leq 1 \} \\ \|x\|_{B'_1} &= \sup\{ |\langle y, x \rangle_{\overline{\mathcal{H}}}| : y \in \Delta_d, \|y\|_{B'_0} \leq 1 \} \end{aligned}$$

for all  $x \in \Delta_d$ .

Let  $\psi$  be a linear functional on  $\Delta_d$  which is continuous with respect to the norms of  $B'_0$  and  $B'_1$ . Then  $\psi$  (restricted to  $\overline{\Delta}$ ) is also continuous with respect to the norms of  $\overline{B_1}$  and  $\overline{B_0}$  because  $\overline{\Delta} \subset \Delta_d$  isometrically. By definition of the dual interpolation couple there is a  $z \in \Delta_d$  such that

$$\psi(x) = \langle x, \bar{z} \rangle_{\overline{B_0 + B_1}, \overline{\Delta_d}} \quad \forall x \in \overline{\Delta}.$$

This equality also holds for  $x \in \Delta_d$  as  $\overline{\Delta}$  is dense in  $\Delta_d$  with respect to the norm of  $\overline{B_0} + \overline{B_1}$  and  $\psi$  is continuous with respect to this norm:

$$\begin{aligned} |\psi(x)| &\leq |\psi(x_0)| + |\psi(x_1)| \\ &\leq C_0 \|x_0\|_{\overline{B_0}} + C_1 \|x_1\|_{\overline{B_1}} \\ &\leq \max(C_0, C_1) (\|x_0\|_{\overline{B_0}} + \|x_1\|_{\overline{B_1}}) \end{aligned}$$

for  $x \in \Delta_d$  and  $x_j \in \overline{B_j}$ ,  $j = 0, 1$ , with  $x = x_0 + x_1$ , where  $C_j$  are norm bounds for  $\psi$ .

Therefore for all  $x \in \Delta_d$  we have

$$\begin{aligned} \psi(x) &= \langle x, \overline{z} \rangle_{\overline{B_0+B_1}, \overline{\Delta_d}} \\ &= \langle x, z \rangle_{\overline{\mathcal{H}}}. \end{aligned}$$

So  $(B'_0, B'_1)$  fulfills all conditions of Theorem 6.1. Consequently

$$(B'_0, B'_1)_{[\frac{1}{2}]} = (B'_0, B'_1)_{[\frac{1}{2}]} \cong \overline{\mathcal{H}}.$$

Hence the dual space of  $(A_0, A_1)_{[\frac{1}{2}]}$  is a Hilbert space isomorphic to  $\overline{\mathcal{H}}$ . We conclude

$$(A_0, A_1)_{[\frac{1}{2}]} \cong \mathcal{H}.$$

We even have equality as  $\mathcal{H} \subset B_0 + B_1 \subset A_0 + A_1$  and all isomorphisms map the intersection  $\Delta$  identically onto itself.

The second assertion of the theorem follows from

$$\begin{aligned} \overline{(A_0, A_1)'_{[\theta]}} &= \overline{(B'_0, B'_1)^{[\theta]}} \\ &= \overline{(B'_0, B'_1)_{[\theta]}} \\ &= (B'_0, B'_1)'_{[1-\theta]} \\ &= ((B'_0, B'_1)^{[1-\theta]})' \\ &= (A_0, A_1)''_{[1-\theta]} \\ &\cong (A_0, A_1)_{[1-\theta]}, \end{aligned}$$

for  $0 < \theta < 1$  since  $(A_0, A_1)_{[1-\theta]}$  is reflexive. □

In Theorem 5.1 and Theorem 6.1  $\Delta_d$  the intersection of the dual couple is equal to  $\overline{\Delta}$ . In the last theorem  $\Delta_d$  can be bigger than  $\overline{\Delta}$  as we can see by the example  $(C_0(\mathbb{R}), L^1(\mathbb{R}))$ . Here  $\Delta$  obviously consists of continuous functions whereas  $\Delta_d$  consists of all essentially bounded  $L^1$ -functions.

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## REFERENCES

1. J. Bergh, *On the relation between the two complex methods of interpolation*, Indiana Univ. Math. J. **28** (1979), no. 5, 775–778.
2. J. Bergh and J. Löfström, *An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.
3. A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.
4. F. Cobos and T. Schonbek, *On a theorem by Lions and Peetre about interpolation between a Banach space and its dual*, Houston J. Math. **24** (1998), 325–344.
5. M. Cwikel, *Complex interpolation spaces, a discrete definition and reiteration*, Indiana Univ. Math. J. **27** (1978), no. 6, 1005–1009.
6. U. Haagerup,  *$L^p$ -spaces associated with an arbitrary von Neumann algebra*, Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977), pp. 175–184, Colloq. Internat. CNRS, 274, CNRS, Paris, 1979.
7. U. Haagerup and G. Pisier, *Factorization of analytic functions with values in noncommutative  $L_1$ -spaces and applications*, Canad. J. Math. **41** (1989), no. 5, 882–906.
8. M. Hilsun, *Les espaces  $L^p$  d'une algèbre de von Neumann définis par la dérivée spatiale*, J. Funct. Anal. **40** (1981), 151–169.
9. M. Leinert, *Integration with respect to a weight*, Int. J. Math. **2** (1991), no. 2, 177–182.
10. G. Pisier, *The operator Hilbert space OH, complex interpolation and tensor norms*, Mem. Amer. Math. Soc. **122** (1996), no. 585, viii+103 pp.
11. M. Terp, *Interpolation spaces between a von Neumann-algebra and its predual*, J. Operator Theory **8** (1982), 327–360.
12. F. Watbled, *Interpolation complexe d'un espace de Banach et de son antidual*, C. R. Acad. Sci. Paris, t. 321, Série I, p. 1437–1440, 1995.
13. F. Watbled, *Complex interpolation of a Banach space with its dual*, Math. Scand. **87** (2000), no. 2, 200–210.

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