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Boundary Distributions for GL_3 over a Local Field and Symmetric Power Coefficients

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Abstract

In this thesis, we construct a residue map and a Poisson kernel between holomorphic discrete series representations on the Drinfeld period domain and harmonic cocycles with certain non-trivial coefficients on the Bruhat-Tits building for GL_3 over a local field of any characteristic. In order to construct the Poisson kernel, we find a new locally analytic kernel function that can be integrated against general boundary distributions. Assuming the existence of certain boundary distributions attached to bounded harmonic cocycles, we prove that the Poisson kernel is a right inverse of the residue map for bounded harmonic cocycles. Moreover, we show that the existence of the needed boundary distributions follows from a non-criticality statement for a new class of automorphic forms. We prove a control theorem that implies this non-criticality statement for trivial coefficients. Finally, we apply our constructions to relate spaces of Drinfeld cusp forms for certain congruence subgroups of GL_3 and spaces of harmonic cocycles extending work of Teitelbaum to GL_3.

Zusammenfassung

In dieser Arbeit konstruieren wir eine Residuenabbildung und einen Poisson-Kern zwischen holomorphen diskreten Reihendarstellungen auf dem Drinfeldschen Periodenraum und harmonischen Kozykeln mit gewissen nicht-trivialen Koeffizienten auf dem Bruhat-Tits Gebäude für GL_3 über einem lokalen Körper von beliebiger Charakteristik. Um den Poisson-Kern zu konstruieren, finden wir eine neue lokal analytische Kernfunktion, die gegen allgemeine Rand-Distributionen integriert werden kann. Unter Annahme der Existenz von gewissen Rand-Distributionen zu beschränkten harmonischen Kozykeln zeigen wir, dass der Poisson-Kern für beschränkte harmonische Kozykel rechts-invers zur Residuenabbildung ist. Darüber hinaus zeigen wir, dass die Existenz der benötigten Rand-Distributionen aus einer Nicht-Kritikalitäts-Aussage für eine neue Klasse von automorphen Formen folgt. Wir beweisen ein Kontrolltheorem, welches die Nicht-Kritikalität für triviale Koeffizienten impliziert. Schließlich wenden wir unsere Konstruktionen an, um Beziehungen zwischen Räumen von Drindfeldschen Spitzenformen zu gewissen Kongruenzuntergruppen von GL_3 und Räumen von harmonischen Kozykeln zu erhalten, was eine Arbeit von Teitelbaum auf GL_3 verallgemeinert.

Contents

Introduction

At the heart of modern arithmetic geometry lies the Langlands program, a network of conjectures and links between number theory, representation theory and geometry. In its essence it seeks to relate two very distinct classes of mathematical objects: On the one side there are Galois groups from number theory, on the other automorphic forms and representations attached to algebraic groups. The Langlands program generalizes classical class field theory and is one of the biggest ongoing research projects within pure mathematics. One of the most famous instances where a link between Galois groups and automorphic forms has been proved is Wiles' proof of the modularity theorem (formerly the Taniyama–Shimura conjecture).

Much current research goes into variants of the Langlands program such as the local Langlands program for general reductive groups and its modern cousin, the p-adic local Langlands program. A common theme within these is the study of the cohomology of locally symmetric spaces. It turns out that the cohomology of these spaces has an extremely rich structure, it is a source of both automorphic and Galois representations. This has been very successfully exploited to prove several instances of a conjectural Langlands correspondence.

In this thesis, we focus on the automorphic side of the Langlands program. In the classical theory of modular and automorphic forms it has proven to be of great use to realize analytic spaces of such forms in a more combinatorial or algebraic way. A very famous instance of such a description goes back to Birch in [Bir71] and Manin in [Man72] and is given by so called modular symbols. To associate a modular symbol to a classical modular form, one needs to consider certain period integrals associated to the modular form. The precise relation between modular forms and modular symbols is given by the Eichler-Shimura isomorphism. Modular symbols are of great use for doing explicit computations with modular forms. Their role within the Langlands program becomes apparent upon realizing that they "know" special values of classical L-functions of modular forms.

It became a natural question whether this theory could be extended to a p-adic situation, i.e., if there is a p-adic analogue of the space of modular symbols which contains information about p-adic families of modular forms and p-adic L-functions. One possible answer is provided by the theory of so called overconvergent modular symbols, which were first introduced by Stevens in [Ste94]. Applications and generalizations of overconvergent modular symbols play a central role in modern number theory. These include Heegner point computations, overconvergent cohomology and the theory of

eigenvarieties.
It turns out that in various situations, it is natural to consider a second analogue of classical modular symbols in the p-adic setting, the so called harmonic cocycles, which play a central role in this thesis. The starting point for this theory is the so called *Drinfeld period domain* $\mathcal{X}$ for GL_n over a (non-archimedean) local field K. When $n = 2$ it has been first observed by Mumford in [Mum72] and Čerednik in [Čer76] that a large class of algebraic curves over K can by uniformized by quotients of $\mathcal{X}$ under certain groups. More generally, Rapoport and Zink have shown that quotients of $\mathcal{X}$ appear as p-adic uniformizations of certain Shimura varieties of unitary type, see [RZ96]. This is in analogy with the classical complex uniformization of Shimura varieties. Over function fields, Drinfeld realized that $\mathcal{X}$ is *the* central object when studying moduli spaces of Drinfeld modules, see [Dri74]. Moreover, he showed that $\mathcal{X}$ can be obtained as a moduli space of formal groups, see [Dri76], which hints at the important role that $\mathcal{X}$ plays in the local Langlands program. All of this makes it clear that forms on $\mathcal{X}$ are directly linked to various types of modular and automorphic forms both over number fields and function fields.
To obtain a combinatorial description for such forms on $\mathcal{X}$, one exploits the fact that the rigid analytic space $\mathcal{X}$ is deeply connected to the Bruhat-Tits building $\mathcal{T}$ for $\mathrm{GL}_n(K)$ via a reduction map $\mathrm{red}\colon \mathcal{X} \to \mathcal{T}$. Over a local field of characteristic zero, Schneider in [Sch84] (for $n = 2$) and Schneider-Teitelbaum in [ST97] (for general n) have used this link to construct a so called *residue map*

$$\mathrm{Res}_0\colon \Omega_{\mathcal{X}}^{n-1} \to C_{\mathrm{har}}(\mathcal{T}, K),$$

where the space on the right hand side is the space of harmonic cocycles, whose elements are certain K-valued functions on the pointed chambers of $\mathcal{T}$ that satisfy harmonicity conditions. This residue map should be viewed as an analogue of the period integrals used to associate modular symbols to classical modular forms. The applications of this construction are plentiful. For example, the residue map can be used to construct $\mathcal{L}$-invariants attached to modular forms on certain Shimura varieties of unitary type, see [BdS16], which are of great interest in the p-adic local Langlangs program. However, if one is interested in $\mathcal{L}$-invariants of modular forms of higher weight, it becomes necessary to replace $\Omega_{\mathcal{X}}^{n-1}$ by more general geometric $\mathrm{GL}_n(K)$-representations, so called holomorphic discrete series representations. It can also be used to obtain a p-adic Eichler-Shimura isomorphism, see [deS89]. Going in a different direction, it is a natural question whether this construction can be extended to local fields of any characteristic. For $n = 2$, this is due to Teitelbaum, see [Tei91], and has broad applications in the theory of Drinfeld modular forms. For example, it is a central ingredient in the construction of an Eichler-Shimura isomorphism for Drinfeld modular forms, see [Böc02]. Moreover, it can be used to gain a better understanding for the Hecke-module structures of spaces of Drinfeld modular forms, see [BGP19] and for explicit computations, see [BB12].
The aim of this thesis is to provide simultaneous generalizations of the residue map and of its surrounding constructions to more general holomorphic discrete series representations and to local fields of any characteristic for $\mathrm{GL}_3(K)$, the first interesting case

going beyond $\mathrm{GL}_2(K)$. Our method is completely independent of the characteristic of the underlying local field which is a very rare feature within the Langlands program. Before we explain our results, let us mention a further application of the residue map, namely how it fits into the ongoing investigation of the cohomology of $\mathcal{X}$ (and of its étale coverings). By work of de Shalit, see [deS01], the map Res_0 induces an isomorphism

$$H^{n-1}_{\mathrm{dR}}(\mathcal{X}) \to C_{\mathrm{har}}(\mathcal{T}, K).$$

The existence of an abstract isomorphism between the two spaces has been known prior, see [SS91]. In fact, in their pioneering work [SS91], Schneider and Stuhler computed the cohomology of $\mathcal{X}$ for all cohomology theories satisfying a natural set of axioms such as ℓ-adic and de Rham cohomology. One source for the interest in the cohomology of $\mathcal{X}$ and of its étale coverings is a (now proven) conjecture of Drinfeld. It asserts that the ℓ-adic cohomology of the étale coverings of $\mathcal{X}$ realizes the supercuspidal part of the local Langlands correspondence. But also the p-adic cohomology seems to contain a wealth of arithmetic information. Recently Colmez, Dospinescu and Nizioł have computed the p-adic étale cohomology of $\mathcal{X}$ in [CDN20b]. Moreover, in [CDN20a] they have shown that for $n = 2$ the p-adic local Langlands correspondence for de Rham Galois representations of dimension 2 (of weight 0 and 1) can be realized in the p-adic étale cohomology of the étale coverings of $\mathcal{X}$.

Let us now explain the history and our results in more detail. The Drinfeld period domain for $G = \mathrm{GL}_n(K)$ is the rigid space obtained by removing all K-rational hyperplanes from the projective space $\mathbb{P}^{n-1}_K$, i.e.,

$$\mathcal{X} = \mathbb{P}^{n-1}_K \setminus \bigcup_{H \in \mathcal{H}} H,$$

where $\mathcal{H}$ denotes the set of all K-rational hyperplanes in $\mathbb{P}^{n-1}_K$. The holomorphic discrete series representations of weight k is then just the space of rigid analytic functions on $\mathcal{X}$ with a weight k action, denoted by $\mathcal{O}_{\mathcal{X}}(k)$. In particular, via the well-known isomorphism $\mathcal{O}_{\mathcal{X}}(n) \cong \Omega^{n-1}_{\mathcal{X}}$, this includes the special case discussed in the previous paragraph. Correspondingly, one needs to consider harmonic cocycles on $\mathcal{T}$ with values in more general representations, namely the space $C_{\mathrm{har}}(\mathcal{T}, V_k)$, where V_k is the dual of a symmetric power representation $\mathcal{P}_k$ of weight k. While the holomorphic discrete series representation is a purely analytic object, harmonic cocycles are very combinatorial objects. A third central object is related to the boundary G/B of $\mathcal{T}$, where B denotes the Borel subgroup of upper triangular matrices in G. This is the so called locally analytic Steinberg representation $\mathrm{St}^{\mathrm{an}}_n(k)$. Elements of its continuous dual $\mathrm{St}^{\mathrm{an}}_n(k)'$ are often referred to as boundary distributions.

For $n = 2$, the general residue map is due to Schneider in [Sch84]. It is a G-equivariant map

$$\mathrm{Res}_k \colon \mathcal{O}_{\mathcal{X}}(k+2) \to C_{\mathrm{har}}(\mathcal{T}, V_k)$$

constructed by taking the residues of a rigid analytic function along the annuli in $\mathcal{X}$ attached to the edges of $\mathcal{T}$ via the reduction map. In order to understand its properties, a central idea is due to Teitelbaum in [Tei90]: He constructed the so called Poisson kernel, a G-equivariant map

$$I_k\colon \mathrm{St}_2^{\mathrm{an}}(k)' \to \mathcal{O}_{\mathcal{X}}(k+2).$$

It is given by integrating an explicit locally analytic kernel function against the boundary distributions in $\mathrm{St}_2^{\mathrm{an}}(k)'$. There is a natural subspace $C^b_{\mathrm{har}}(\mathcal{T}, V_k)$ of $C_{\mathrm{har}}(\mathcal{T}, V_k)$, consisting of so called bounded harmonic cocycles, which plays a central role in the theory. By a theorem of Amice-Velu and Vishik, there is a third map $C^b_{\mathrm{har}}(\mathcal{T}, V_k) \to \mathrm{St}_2^{\mathrm{an}}(k)'$ which is constructed by extending certain distributions attached to bounded harmonic cocycles to allow integration of a larger class of locally analytic functions. If one sets $\mathcal{O}_{\mathcal{X}}(k+2)^b = \mathrm{Res}_k^{-1}(C^b_{\mathrm{har}}(\mathcal{T}, V_k))$, the composition with I_k becomes a right inverse of Res_k,

$$\begin{tikzcd} \mathcal{O}_{\mathcal{X}}(k+2)^b \arrow[r, shift left, "\mathrm{Res}_k"] & C^b_{\mathrm{har}}(\mathcal{T}, V_k) \arrow[l, shift left, dashed, "I_k"] \arrow[r] & 0. \end{tikzcd}$$

An excellent overview of the above constructions can be found in [DT08]. For local fields of positive characteristic the analogous theorem has been proved by Teitelbaum in [Tei91].
One of the primary applications of these constructions is the following. If one takes invariants under certain arithmetic subgroups $\Gamma \subset \mathrm{GL}_2(K)$, the boundedness condition is automatically satisfied. In favorable situations, Res_k then induces isomorphisms between spaces of rigid analytic forms on $\mathcal{X}$ satisfying invariance properties for the action of Γ and spaces of Γ-invariant harmonic cocycles. This is the key ingredient for constructing $\mathcal{L}$-invariants attached to classical and Hilbert modular forms as in [Tei90], [IS03] and [CMP15] or for the applications in the theory of Drinfeld modular forms mentioned above.

When going beyond $n = 2$, the starting point is the work of Schneider and Stuhler in [SS91], which shows that there is an abstract isomorphism

$$H^{n-1}_{\mathrm{dR}}(\mathcal{X}) \to (\mathrm{St}_n^{\infty})',$$

where St_n^{∞} denotes the locally constant Steinberg representation of G, i.e., the space of locally constant functions on the flag variety G/B, modulo functions factoring over some G/P, where P is a parabolic subgroup strictly containing B. Building on the work of Schneider and Stuhler, in analogy with the case $n = 2$, in [ST97] Schneider and Teitelbaum construct maps

$$\mathrm{Res}_0\colon \mathcal{O}_{\mathcal{X}}(n) \to C_{\mathrm{har}}(\mathcal{T}, K) \quad \text{and} \quad I_0\colon (\mathrm{St}_n^{\infty})'^{,b} \to \mathcal{O}_{\mathcal{X}}(n).$$

One should note the fact that here the locally constant Steinberg representation is considered. This feature is already present in the theory for $n = 2$: A bounded distribution on the locally constant Steinberg representation can easily be extended to integrate even elements of the continuous Steinberg representation $\mathrm{St}_3^{\mathrm{con}}$, i.e., continuous functions on the K-analytic manifold G/B modulo functions that factor over some G/P with P as above. This is a dichotomy between the cases $k = 0$ and $k > 0$ which plays a central role in the theory. Since $C^b_{\mathrm{har}}(\mathcal{T}, K) \cong (\mathrm{St}_n^\infty)'^{,b}$, one obtains

$$\begin{tikzcd} \mathcal{O}_{\mathcal{X}}(n)^b \arrow[r, "\mathrm{Res}_0"] & C^b_{\mathrm{har}}(\mathcal{T}, K) \arrow[l, dashed, bend left, "I_0"] \arrow[r] & 0, \end{tikzcd}$$

as in the case $n = 2$. Another construction of I_0 can be found in [IS01]. For a more detailed analysis of the situation see [ST02b].

While the work of Schneider and Teitelbaum covers GL_n for any $n \geq 2$, it only considers trivial weight $k = 0$ and local fields of characteristic zero. The aim of Part I of this thesis is to provide an extension to higher weights k and to local fields of any characteristic in the case $n = 3$. However, we should point out that, contrary to [ST97], due to some technical issues, we primarily work over a fixed completion of an algebraic closure of K. There are three main difficulties one has to overcome.

(a) Most prominently in [ST02b] and in the extension [Orl08] due to Orlik, the p-adic representation theory developed by Schneider and Teitelbaum plays a central role. At present, there is no analogous theory for representations over local fields of positive characteristic.

(b) As indicated above, for weights $k > 0$ one needs a finer integration theory compared to the case $k = 0$. Namely, to construct the map $C^b_{\mathrm{har}}(\mathcal{T}, V_k) \to \mathrm{St}_3^{\mathrm{an}}(k)'$, a higher dimensional analogue of the theorem of Amice-Velu and Vishik is needed.

(c) The kernel function used to define I_0 in [ST97] is not locally analytic everywhere, but only on the big cell. As soon as one considers weights $k > 0$ this becomes a major obstacle as it is unclear how to integrate this kernel function against elements of $\mathrm{St}_3^{\mathrm{an}}(k)'$. In [ST02b], Schneider and Teitelbaum show that one can work with the class of the kernel function in the space of locally analytic vectors of $\mathrm{St}_3^{\mathrm{con}}$. At least for $K = \mathbb{Q}_p$, this space is just $\mathrm{St}_3^{\mathrm{an}}(0)$. Nevertheless one lacks an explicit locally analytic representative for this class. Even worse, over other base fields, and particularly function fields, the situation is completely unclear.

In order to address (a), we compensate the lack of theoretical framework by a more explicit study of the objects involved. The first central step is to extend the residue map Res_0 of Schneider and Teitelbaum to

$$\mathrm{Res}_k \colon \mathcal{O}_{\mathcal{X}}(k+3) \to C_{\mathrm{har}}(\mathcal{T}, V_k).$$

We are able to construct this extension by analysing the relationship between different holomorphic discrete series representations, see Chapter 3 for details. The next step is to construct the Poisson kernel I_k. Due to (c) this is significantly more involved. We prove the following.

Theorem A (Theorem 4.17, Theorem 4.22 and Theorem 4.26)**.**

(i) There is an explicit locally analytic representative for the class of the kernel function in $\mathrm{St}_3^{\mathrm{con}}$.

(ii) Integrating the representative in (i) induces, for each $k \geq 0$ *with* $3 \mid k$, *a G-equivariant map*

$$I_k \colon \mathrm{St}_3^{\mathrm{an}}(k)' \to \mathcal{O}_{\mathcal{X}}(k+3).$$

For precise statements see Theorem 4.17, Theorem 4.22 and Theorem 4.26. The construction of this new kernel function is based on geometric considerations: It turns out that the singular locus of the kernel function of Schneider-Teitelbaum is a projective line inside the flag variety G/B. We are able to remove these singularities by modifying the kernel function in a very natural way on an open neighbourhood of this projective line. The proof of (ii) uses detailed knowledge of the relations between various holomorphic discrete series and locally analytic principal series representations.

Point (b) has proven to be quite difficult. In analogy with the theorem of Amice-Velu and Vishik for $n = 2$, this boils down to extending bounded distributions on locally polynomial functions to certain locally analytic functions. One would like to construct such an extension in a canonical and organized way. We realized that one can use certain automorphic forms to construct such extensions systematically. This is inspired by [Gre06], [FM14] and [Grä19], where similar automorphic forms for $n = 2$ are used to compute moments of the corresponding distributions after their existence is already known. But only this thesis makes use of them as a way of extending distributions.

Let us describe our setup in more detail. A new and distinctive feature of the automorphic forms we define is that they do not need to satisfy an invariance property under the action of an arithmetic subgroup, but instead we require a boundedness condition. This means that our automorphic forms are purely local objects. However we should note that after taking invariants under such arithmetic subgroups, one recovers global automorphic forms for this arithmetic subgroup, justifying the term automorphic form. The upshot of this new approach is twofold: First of all, one realizes that the control theorems for automorphic forms such as in [Gre06] are completely independent of the arithmetic group under consideration and can be extended to our automorphic forms, see Theorem 5.55. One can then recover the known control theorems by taking invariants. This makes our approach very broadly applicable. Even more importantly for us, it turns out that the space of bounded harmonic cocycles is naturally isomorphic to such a space of automorphic forms $\mathbb{A}(V_k)_b^{\mathrm{new}}$, whose elements are eigenforms for four Hecke operators: $U_{\pi,i}$ and $W_{\pi,i}$ for $i \in \{1,2\}$ with explicit eigenvalues, see Proposition 5.43. Such automorphic forms are called *non-critical* if they admit unique lifts to spaces of (partially) overconvergent automorphic forms. We are able to prove the following theorem which links (b) to a non-criticality statement for automorphic forms.

Theorem B (Theorem 5.48 and Corollary 5.50)**.** *Assume that every automorphic form in* $\mathbb{A}(V_k)_b^{\mathrm{new}}$ *is non-critical. Then there is a G-equivariant map*

$$C_{\mathrm{har}}^b(\mathcal{T}, V_k) \to \mathrm{St}_3^{\mathrm{an}}(k)'$$

with additional nice properties.

For the precise properties see Conjecture 5.49. Unfortunately, the control theorem we are able to prove is not strong enough to verify the assumptions of Theorem B aside from the case $k = 0$, see Theorem 5.52. This is consistent with the bounds in the literature, see for example [BC09] or [Wil18]. Nevertheless we are able to prove a conjectural analogue of the result of Schneider and Teitelbaum.

Theorem C (Theorem 4.31)**.** *Assume that there is a G-equivariant map $C^b_{\mathrm{har}}(\mathcal{T}, V_k) \to \mathrm{St}^{\mathrm{an}}_3(k)'$ as in Theorem B. Then the composition of this map with I_k is a G-equivariant right inverse of* Res_k*. Consequently,*

$$\mathrm{Res}_k \colon \mathcal{O}_{\mathcal{X}}(k+3)^b \to C^b_{\mathrm{har}}(\mathcal{T}, V_k)$$

is surjective.

The precise assumptions can be found in Conjecture 4.29 (or a stronger version in Conjecture 5.49). As indicated above for $k = 0$, by our control theorem, the assumptions of Theorem C are satisfied. Thus, in this case, our results extend the results of Schneider and Teitelbaum in the case $n = 3$ to local fields of any characteristic. While this has been known to the experts, we give the first full proof.

In Part II of this thesis, we consider an application of our theory over function fields, namely to Drinfeld modular forms of rank 3. In recent years, Basson, Breuer and Pink and independently Gekeler have initiated the systematic study of Drinfeld modular forms of higher rank, i.e., of certain arithmetic forms on $\mathcal{X}$ for GL_n over a local field of positive characteristic, see in particular [BBP18a], [BBP18b] and [BBP18c]. In analogy with the work of Teitelbaum, [Tei91], one expects that Drinfeld cusp forms for a congruence subgroup Γ are isomorphic to Γ-invariant harmonic cocycles. Let $A = \mathbb{F}_q[t]$. If we denote the space of Drinfeld cusp forms of weight k and type ℓ for a congruence subgroup $\Gamma \subseteq \mathrm{GL}_3(A)$ by $\mathcal{S}_{k,\ell}(\Gamma)$, we are able to prove the following theorem.

Theorem D (Theorem 8.9 and Corollary 8.12)**.** *Assume that there is a G-equivariant map $C^b_{\mathrm{har}}(\mathcal{T}, V_k) \to \mathrm{St}^{\mathrm{an}}_3(k)'$ as in Theorem B and let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup such that $\Gamma(t) \subseteq \Gamma$. Then* Res_k *induces a Hecke-equivariant isomorphism*

$$\mathcal{S}_{k+3,\ell}(\Gamma) \to C_{\mathrm{har}}(\mathcal{T}, V_{k,\ell})^{\Gamma},$$

where $V_{k,\ell} = V_k \otimes_K \det^{\ell-1-k/3}$.

The proof follows the ideas of Teitelbaum in [Tei91]: We show that Γ-invariant harmonic cocycles are automatically cuspidal and in particular bounded, which makes the theory developed in Part I of this thesis applicable in this situation. A key difference compared to [Tei91] is that in the higher rank situation, one has formulas for the dimension of spaces of Drinfeld modular and cusp forms only in a few special cases, see for example [Pin19]. This is the reason for the restriction on the congruence subgroups we consider. To obtain similar dimension estimates for the space of Γ-invariant harmonic cocycles, we use knowledge of a fundamental domain of the action of the congruence subgroup $\Gamma(t)$ on $\mathcal{T}$. Theorem D lays the groundwork for extending the results of [Böc02] and [BGP19] to Drinfeld cusp forms of rank 3.

There are several natural follow-up questions we would like to mention. The first and most pressing is whether one can prove a stronger control theorem to remove the conjectural assumptions in Theorem C and Theorem D. While there have been lifting results in the critical slope case, see [PS13], our situation seems to be fundamentally different and new techniques are needed. The second natural question is whether our results can be extended to general GL_n. The bottleneck here lies in the construction of our kernel function in Theorem A. Our explicit geometric considerations need to be generalized to flag varieties of higher dimension. Finally, as an application over number fields, we expect that our results can be used to construct $\mathcal{L}$-invariants attached to modular forms of higher weight on certain Shimura varieties of unitary type, similarly to [BdS16].

Outline of the thesis

Part I – Boundary Distributions for GL_3

In Chapter 1, we recall some basic facts on the Drinfeld period domain $\mathcal{X}$ for $G = \mathrm{GL}_3(K)$, where K is a non-archimedean local field of any characteristic.

Chapter 2 is devoted to studying the Bruhat-Tits building $\mathcal{T}$ attached to G. After introducing various basic notions, the focus lies on the reduction map as well as studying the boundary G/B of $\mathcal{T}$, where B denotes the Borel subgroup of upper triangular matrices in G.

In Chapter 3, we first introduce harmonic cocycles on $\mathcal{T}$. The aim of the remainder of this chapter is the construction of the residue map. For this, we introduce holomorphic discrete series representations and construct natural filtrations on them. This enables us to construct the residue map first for $k = 0$ and then extend the construction to general weights.

Chapter 4 is the theoretical heart of this thesis. After introducing locally analytic principal series representations, we construct our new kernel function. We then proceed to construct the Poisson kernel, first for $k = 0$ and then for general weights. Finally, we show that the Poisson kernel is a right inverse of the residue map for bounded harmonic cocycles assuming that an analogue of the theorem of Amice-Velu and Vishik holds.

In Chapter 5, we want to relate the missing map from the previous chapter to a lifting theorem for certain automorphic forms. We begin by introducing various coefficient modules which are built out of the locally analytic principal series representations from

Chapter 4. After defining our automorphic forms and the relevant Hecke operators on them, we proceed to prove the link between these forms and the boundary distributions. Finally, we prove an abstract control theorem, which we then apply in our situation to show that for $k = 0$ we obtain the needed analogue of the theorem of Amice-Velu and Vishik.

In Appendix A we quickly develop some terminology on locally analytic manifolds and representations which has been missing for local fields of positive characteristic.

Part II – Application to Drinfeld modular forms of rank 3

In Chapter 6, we recall the basic constructions needed to define Drinfeld modular forms of rank 3. Moreover, we state various dimension formulas for spaces of Drinfeld modular and cusp forms and introduce Hecke operators acting on them.

In Chapter 7, we study the action of congruence subgroups on the Bruhat-Tits building in more detail. We explicitly compute various stabilizers and find a fundamental domain for the action of the congruence subgroup $\Gamma(t)$ on $\mathcal{T}$.

In the final Chapter 8, we prove an analogue of Teitelbaum's isomorphism between Drinfeld cusp forms and harmonic cocycles in our situation. For this, we first show that harmonic cocycles that are invariant under the action of a congruence subgroup are automatically cuspidal. We proceed to give a dimension estimate for the space of harmonic cocycles invariant under the action of $\Gamma(t)$. This is based on our knowledge of the fundamental domain constructed in Chapter 7. By combining this with the results from Part I we are then able to prove our main theorem.

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PART I

Boundary Distributions for GL_3

CHAPTER 1

The Drinfeld period domain

Throughout this thesis, we denote by K a non-archimedean local field of any characteristic with ring of integers O_K and residue field κ. Let p be the characteristic of the residue field κ and q its (finite) cardinality. Furthermore, let π denote a uniformizing parameter in K, let ν be the normalized valuation on K such that $\nu(\pi) = 1$ and let $|\cdot| = q^{-\nu(\cdot)}$ be the associated absolute value. Moreover, we fix the completion of an algebraic closure of K and denote it by $\mathbb{C}_K$. We denote the extension of ν and $|\cdot|$ to $\mathbb{C}_K$ by the same symbols.

In this chapter, we quickly recall the construction of the Drinfeld period domain as a rigid space over K. We follow [SS91, Section 1]. However, our normalizations are closer to [DT08]. Let $G = \mathrm{GL}_3(K)$ and let V be a fixed 3-dimensional vector space over K, viewed as the space of row vectors $[x_1, x_2, x_3]$ with $x_i \in K$ for $i \in \{1, 2, 3\}$ on which $g \in G$ acts on the left via

$$g([x_1, x_2, x_3]) = [x_1, x_2, x_3]g^{-1}.$$

In the sequel, we denote by $\mathbb{P}^2$ the projective space $\mathbb{P}(V)$ with the induced G-action. We denote elements of $\mathbb{P}^2$ by $z = [z_1 : z_2 : z_3]$.

Let Ξ_1, Ξ_2 and Ξ_3 denote the dual elements in $V^* = \mathrm{Hom}_K(V, K)$ to the standard basis elements $b_1 = [1, 0, 0]$, $b_2 = [0, 1, 0]$ and $b_3 = [0, 0, 1]$ of V. We act with $g \in G$ on $\Xi \in V^*$ by

$$(g_*\Xi)(v) = \Xi(g^{-1}v) \quad \text{for } v \in V.$$

We define a coordinate function $\omega = (\omega_1, \omega_2)$ via

$$\omega_i = \frac{\Xi_i}{\Xi_3}, \quad i = 1, 2.$$

Then, $g = (g_{ij})_{1\leq i,j\leq 3} \in G$ acts on ω by

$$\begin{aligned} g_*(\omega)[x_1, x_2, x_3] &= \omega(g^{-1}([x_1, x_2, x_3])) = \omega([x_1, x_2, x_3]g) \\ &= \omega([g_{11}x_1 + g_{21}x_2 + g_{31}x_3, g_{12}x_1 + g_{22}x_2 + g_{33}x_3, g_{13}x_1 + g_{23}x_2 + g_{33}x_3]) \\ &= \left(\frac{g_{11}\omega_1 + g_{21}\omega_2 + g_{31}}{g_{13}\omega_1 + g_{23}\omega_2 + g_{33}}, \frac{g_{12}\omega_1 + g_{22}\omega_2 + g_{32}}{g_{13}\omega_1 + g_{23}\omega_2 + g_{33}}\right). \end{aligned}$$

In the remainder of this chapter, we regard $\mathbb{P}^2$ as a rigid space over K. The above action by G is by rigid analytic automorphisms of $\mathbb{P}^2$.

1.1. Definition. Let $\mathcal{H}$ denote the set of all K-rational hyperplanes in $\mathbb{P}^2$. The *Drinfeld period domain over* K is (as a set)

$$\mathcal{X} = \mathbb{P}^2(\mathbb{C}_K) \setminus \bigcup_{H \in \mathcal{H}} H.$$

In this chapter, we choose *unimodular* coordinates for points $z \in \mathcal{X}$, i.e., we pick representatives $z = [z_1 : z_2 : z_3]$ such that $\max_{1 \le i \le 3} |z_i| = 1$. Following [SS91], the space $\mathcal{X}$ is an admissible open subset of $\mathbb{P}^2$ and thus an open rigid analytic subvariety. To prove this, Schneider and Stuhler construct an explicit family $(\mathcal{X}_n)_{n \ge 0}$ of open affinoid subvarieties of $\mathbb{P}^2$ with the properties

(i) $\mathcal{X} = \bigcup_{n \ge 0} \mathcal{X}_n$,

(ii) Any K-morphism $f\colon Y \to \mathbb{P}^2$ from a K-affinoid variety with $f(Y) \subseteq \mathcal{X}$ factors through some $\mathcal{X}_n$.

The subvarieties $\mathcal{X}_n$ are realized by removing certain neighbourhoods of K-rational hyperplanes. For a fixed hyperplane $H \in \mathcal{H}$, we choose a unimodular linear form ℓ_H such that

$$H = \left\{ z \in \mathbb{P}^2(\mathbb{C}_K) \;\middle|\; \ell_H(z) = 0 \right\}.$$

Here, unimodular means that ℓ_H has coefficients in $\mathcal{O}_K$ and at least one coefficient is a unit. In particular, ℓ_H is determined up to multiplication by a unit in $\mathcal{O}_K$ and consequently, $|\ell_H(z)|$ is independent of the choice of ℓ_H for $z \in \mathbb{P}^2(\mathbb{C}_K)$.

1.2. Definition. Let $\varepsilon \in \mathbb{Q}_{>0}$. The set

$$H(\varepsilon) = \left\{ z \in \mathbb{P}^2(\mathbb{C}_K) \;\middle|\; |\ell_H(z)| < \varepsilon \right\}$$

is called the *ε-neighbourhood* of the hyperplane $H \in \mathcal{H}$.

Note that this differs from the definition in [SS91]. We consider a strict inequality, which proves to be more useful in our later computation, compare also [DT08, Section 1.2.1] or [ST02b, Section 0].

1.3. Proposition. *The sets*

$$\mathcal{X}_n = \mathbb{P}^2(\mathbb{C}_K) \setminus \bigcup_{H \in \mathcal{H}} H(q^{-n})$$

are admissible open in $\mathbb{P}^2$ and the collection $(\mathcal{X}_n)_{n \ge 0}$ satisfies properties (i) and (ii) above. Consequently, $\mathcal{X}$ is an admissible open subspace of $\mathbb{P}^2$.

Proof. See [SS91, Section 1]. □

1.4. Remark. The key observation in the above proposition is that each $\mathcal{X}_n$ is in fact defined by only finitely many hyperplanes, see [SS91, Section 1, Lemma 2]. An alternative approach to proving the above proposition involves the Bruhat-Tits building of G. We will get back to this approach in Section 2.2. In a third approach, one constructs a formal scheme $\hat{\mathcal{X}}$ over $\mathrm{Spf}(\mathcal{O}_K)$ realizing $\mathcal{X}$ as its rigid analytic generic fibre. For details see [Dri76].

Clearly the action of G on $\mathbb{P}^2$ induces an action of G on $\mathcal{X}$ by rigid analytic automorphisms. Note that for $z \in \mathcal{X}$ by definition the coordinate function $\omega(z)$ has no zeros in the denominator. Hence, we can always represent elements of $\mathcal{X}$ via ω. More precisely, if z is a point in $\mathcal{X}$, we can always renormalize it such that $z = [z_1 : z_2 : 1]$. Then we have $\omega_i(z) = z_i$ for $i \in \{1, 2\}$. Thus, in the sequel we often do not distinguish between ω and $\omega(z)$.

We denote by $\mathcal{O}_\mathcal{X}$ the ring of rigid analytic functions on $\mathcal{X}$ over $\mathbb{C}_K$, i.e., the global sections of the structure sheaf of $\mathcal{X}$ over $\mathbb{C}_K$. By abuse of notation we do not distinguish between $\mathcal{X}$ and its base change to $\mathbb{C}_K$. Note that by our construction of $\mathcal{X}$, we have

$$\mathcal{O}_\mathcal{X} = \varprojlim_n \mathcal{O}_{\mathcal{X}_n},$$

where each $\mathcal{O}_{\mathcal{X}_n}$ is an affinoid algebra. The restriction maps between these affinoid algebras are compact.

1.5. Proposition. *The restriction maps $\mathcal{O}_{\mathcal{X}_{n+1}} \to \mathcal{O}_{\mathcal{X}_n}$ have dense image. Thus, $\mathcal{X}$ is a Stein space.*

Proof. See [SS91, Section 1, Proposition 4]. □

CHAPTER 2

The Bruhat-Tits building for $G = \mathrm{GL}_3(K)$

In this chapter, we recall some standard facts on the Bruhat-Tits building $\mathcal{T}$ for the group $G = \mathrm{GL}_3(K)$. Of particular importance for us are the reduction map relating $\mathcal{T}$ to the Drinfeld period domain constructed in the previous chapter as well as a detailed study of its boundary.

2.1. The Bruhat-Tits building $\mathcal{T}$

We follow [deS01, Section 1] and [ST97]. Let V be as in Chapter 1. Recall that $G = \mathrm{GL}_3(K)$ and let T denote the diagonal torus in G. We fix the Borel subgroup of upper triangular matrices B in G and its unipotent radical U. Similarly, B^- and U^- denote the Borel subgroup of lower triangular matrices and its unipotent radical. The Weyl group W of G is the symmetric group S_3. It is generated by the two elementary reflections s_1 and s_2, which are explicitly given by

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in G.$$

Then $w_0 \coloneqq s_1 s_2 s_1 = s_2 s_1 s_2 \in W$ is the element of maximal length. Furthermore, we denote by $\mathcal{I}$ the Iwahori subgroup of $\mathrm{GL}_3(O_K)$ of matrices which are upper triangular mod π.

2.1. Definition. The *Bruhat-Tits building* $\mathcal{T}$ of G is the simplicial complex given as follows. The *vertices* $\mathcal{T}_0$ consist of homothety classes $[\Lambda]$ of lattices $\Lambda \subset V^*$. The *n-cells* $\mathcal{T}_n$ for $n \in \{1, 2\}$ consist of sets $\{[\Lambda_0], \dots, [\Lambda_n]\}$ where

$$\pi\Lambda_0 \subsetneq \Lambda_n \subsetneq \Lambda_{n-1} \subsetneq \cdots \subsetneq \Lambda_0.$$

A *pointed n-cell* is an n-cell with a distinguished vertex $v_0 = [\Lambda_0]$. Equivalently, a pointed n-cell is given by a tuple $\sigma = ([\Lambda_0], \dots, [\Lambda_n])$ where the Λ_i are as above. We shall write

$$\sigma = (\pi\Lambda_0 \subsetneq \Lambda_n \subsetneq \cdots \subsetneq \Lambda_0)\,,$$

where $[\Lambda_0]$ is the distinguished vertex. The set of pointed n-cells is denoted by $\widehat{\mathcal{T}_n}$. The 1-cells $\mathcal{T}_1$ are also called *edges*. The 2-cells $\mathcal{T}_2$ are also called *chambers*.

Recall that Ξ_1, Ξ_2 and Ξ_3 denote the dual elements in V^* to the standard basis of V and put

$$[i_1, i_2, i_3] = [\langle \pi^{i_1}\Xi_1, \pi^{i_2}\Xi_2, \pi^{i_3}\Xi_3 \rangle_{O_K}] \in \mathcal{T}_0 \quad \text{for } (i_1, i_2, i_3) \in \mathbb{Z}^3.$$

Note that since we consider homothety classes of lattices, we have

$$[i_1, i_2, i_3] = [i_1 + n, i_2 + n, i_3 + n] \quad \text{for } n \in \mathbb{Z}.$$

The *standard apartment* A_0 is the maximal simplicial subcomplex of $\mathcal{T}$ based on the vertices

$$\{[i_1, i_2, i_3] \mid i_1, i_2, i_3 \in \mathbb{Z}\}.$$

It can be viewed as a triangulation of $\mathbb{R}^2$. The *standard chamber* σ_0 is given by

$$\sigma_0 = \{[0, 0, 0], [0, 0, 1], [0, 1, 1]\} \in \mathcal{T}_2.$$

The *standard pointed chamber* is σ_0 with distinguished vertex $v_0 = [0, 0, 0]$. Finally, the *standard sector* S_0 is the sector based in σ_0, i.e., the maximal simplicial subcomplex based on

$$\{[i_1, i_2, i_3] \mid i_1 \leq i_2 \leq i_3\}.$$

Similarly, for any ordered basis $\mathcal{B}$ of V^*, we may define the apartment $A_{\mathcal{B}}$ and sector $S_{\mathcal{B}}$ attached to $\mathcal{B}$ by replacing the ordered basis (Ξ_1, Ξ_2, Ξ_3) with $\mathcal{B}$ in the above construction.

Note that the group G acts on the lattices in V^*, which induces an action on $\mathcal{T}$. Then one has $\mathrm{Stab}_G(v_0) = K^\times \mathrm{GL}_3(O_K)$. This induces a bijection

$$G/K^\times \mathrm{GL}_3(O_K) \cong \mathcal{T}_0.$$

In the following proposition, we collect some important facts on the action on (pointed) chambers.

2.2. Proposition.

(i) We have $\mathrm{Stab}_G(\sigma_0) = K^\times \mathcal{I}$.

(ii) The map $g \mapsto g\sigma_0$ *induces a bijection between* $G/K^\times \mathcal{I}$ *and* $\widehat{\mathcal{T}_2}$.

(iii) Let $\sigma \in \mathcal{T}_2$. *The group* H_σ *stabilizing* σ *modulo the group fixing* σ *pointwise is cyclic of order* 3.

Proof. See [ST97, Lemma 5 and Lemma 7]. □

We also need the standard parahoric subgroups $\mathcal{I}_i := \mathcal{I} \cup \mathcal{I} s_i \mathcal{I} \subset G$ for $i \in \{1, 2\}$. Let

$$e_1 := \{[0, 0, 0], [0, 0, 1]\} \in \widehat{\mathcal{T}_1} \quad \text{and} \quad e_2 := \{[0, 0, 0], [0, 1, 1]\} \in \widehat{\mathcal{T}_1},$$

both with distinguished vertex v_0. Then we have the following.

2.3. Proposition. *We have*

$$\mathrm{Stab}_G(e_i) = K^\times \mathcal{I}_i.$$

Proof. See [deS01, Section 1.4]. □

2.2. The reduction map

The building $\mathcal{T}$ is closely related to $\mathcal{X}$ through the reduction map, whose construction we recall in this section. We follow [ST97], [deS01, Section 6.1] and [DT08, Section 1.3]. In the sequel, we denote by $|\mathcal{T}|$ the *topological realization* of $\mathcal{T}$, i.e., the topological simplical complex associated to $\mathcal{T}$.

2.4. Definition. A *norm* on V^* is a map $\gamma\colon V^* \to \mathbb{R}$ such that

(i) $\gamma(x+y) \leq \max\{\gamma(x), \gamma(y)\}$,

(ii) $\gamma(ax) = |a| \cdot \gamma(x)$ for $a \in K$,

(iii) $\gamma(x) = 0$ if and only if $x = 0$.

We say that two norms γ and γ' on V^* are *homothetic* if there is a constant $C \in \mathbb{R}_{>0}$ such that

$$\gamma(x) = C \cdot \gamma'(x) \quad \text{for all } x \in V^*.$$

There is an explicit bijection between homothety classes of norms on V^* and points of $|\mathcal{T}|$ given as follows: For a lattice $\Lambda \subset V^*$ we set

$$\gamma_\Lambda(x) \coloneqq \inf \left\{ |a| \;\middle|\; a \in K^\times, a^{-1}x \in \Lambda \right\}.$$

More generally, if $\sigma = (\pi\Lambda_0 \subsetneq \Lambda_n \subsetneq \cdots \subsetneq \Lambda_0)$ is a pointed n-cell, the points in $|\sigma|$ can be uniquely written as

$$t = \sum_{i=0}^{n} t_i[\Lambda_i], \quad \text{where } t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1.$$

We set

$$\gamma_t(x) \coloneqq \max \left\{ \gamma_{\Lambda_0}(x), q^{-t_0}\gamma_{\Lambda_1}(x), \ldots, q^{-\sum_{i=0}^{n-1} t_i}\gamma_{\Lambda_n}(x) \right\}.$$

It is easy to check that the homothety classes of these norms are independent of the chosen representing lattice and the choice of the distinguished vertex. By [BT72], this sets up the bijection between homothety classes of norms on V^* and points in $|\mathcal{T}|$. This enables us to define the reduction map. Let $z \in \mathcal{X}$. We define a norm γ_z on V^* via

$$\gamma_z(\ell) \coloneqq |\ell(z)| \quad \text{for } \ell \in V^*,$$

where we use unimodular coordinates for $z \in \mathcal{X}$ as in Chapter 1. The following fact is well known, see [DT08, Lemma 1.3.7] in the $\mathrm{GL}_2(K)$-case.

2.5. Proposition. *The map*

$$\mathrm{red}\colon \mathcal{X} \to |\mathcal{T}|$$

given by $z \mapsto \gamma_z$ is G-equivariant.

We need to study certain fibers of the reduction map, which become important in order to define the residue map in Chapter 3. For this, let

$$R_0 = \left\{ z \in \mathcal{X} \;\middle|\; q^{-1} < |\omega_1(z)| < |\omega_2(z)| < 1 \right\} \subset \mathcal{X}.$$

Then R_0 is an admissible open subset. In fact, it can be realized as the direct limit of the affinoid subdomains $R_{0,N}$, where by definition $z \in \mathcal{X}$ belongs to $R_{0,N}$ if the following conditions are satisfied:

(i) $|\omega_2(z)| \leq q^{-1/N}$,

(ii) $|\omega_1(z)| \geq q^{-1+1/N}$,

(iii) $|\omega_1(z)|q^{1/N} \leq |\omega_2(z)|$,

see [ST97, p. 405] after taking into account the different normalizations.

2.6. Proposition. *Denote by $|\sigma_0|^\circ$ the interior of the chamber $|\sigma_0|$ in $|\mathcal{T}|$. Then we have* $\mathrm{red}^{-1}(|\sigma_0|^\circ) = R_0$. *Consequently, for each $\sigma \in \mathcal{T}_2$ the set $R(\sigma) = \mathrm{red}^{-1}(|\sigma|^\circ)$ is admissible open.*

Proof. See [ST97, Lemma 16]. □

2.3. The boundary of $\mathcal{T}$

The boundary of $\mathcal{T}$ plays a crucial role in the construction of the Poisson kernel in Chapter 4. In this section, we study it in detail and fix various coordinates which will be of great use for explicit computations.

2.3.1. Cell decompositions

Before we study the boundary in more detail, we need some elementary facts about the decomposition of G and the quotient G/B into various cells. All of this is standard and well known. We keep the notation from the previous sections.

2.7. Proposition (Bruhat-Decomposition)**.** *We have*

$$G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} U_w wB,$$

where $U_w = U \cap (wU^- w^{-1})$.

Proof. See [Spr98, Theorem 8.3.8] □

In particular, if we set $n(w) = l(w)(l(w)-1)/2$ for $w \in W$ we have $G/B \cong \bigsqcup_{w \in W} C(w)$, where $C(w) = U_w wB/B$ is an affine space of dimension $n(w)$, see [Spr98, Lemma 8.3.6]. Let $w_0 \in W$ denote the element of maximal length. The affine space $C(w_0)$ is called the *big cell* of G/B. We also set $C^o(w) = w_0 C(w)$ for $w \in W$. Note that $C^o(w_0) = U^- B/B$, the so called *opposite big cell*.

We are also interested in the parabolic subgroups $P_i = B \cup Bs_iB \subset G$ for $i \in \{1,2\}$. By definition, the Weyl group $W_i \subset W$ of P_i is cyclic of order 2 generated by s_i. We set $W^i := W/W_i$. In the sequel, we regard W^i as a subset of W by picking *minimal representatives*: In each class in W^i there is a unique permutation $w \in W$ such that the length of w is minimal among the wv with $v \in W_i$. Then we also have a Bruhat decomposition with respect to P_i.

2.8. Proposition. *We have*

$$G = \bigsqcup_{w \in W^i} BwP_i = \bigsqcup_{w \in W^i} U_w w P_i,$$

with U_w as in Proposition 2.7.

Proof. See [Bri05, Section 1.1]. □

Note that we have $G/P_i \cong \mathbb{P}^2(K)$ for $i \in \{1,2\}$. In particular, these are K-analytic manifolds in the sense of Appendix A. We also need the Bruhat-Iwahori decomposition.

2.9. Proposition (Bruhat-Iwahori-Decomposition)**.** *We have*

$$G = \bigsqcup_{w \in W} \mathcal{I} wB = \bigsqcup_{w \in W} \mathcal{I}_w wB,$$

where $\mathcal{I}_w = \mathcal{I} \cap (wU^- w^{-1})$.

Proof. See [Sch11, Proof of Lemme 2.13]. □

The above decompositions allow us to define coordinates on the various cells. The coordinates we will use throughout this thesis are given explicitly in Table 1. For later use, we set

$$D(w) := w_0 \mathcal{I}_w wB/B \subset G/B. \tag{1}$$

The sets $D(w)$ are compact open in G/B. Via the coordinates in the last column of Table 1, every set $D(w)$ can be regarded as a compact open subset of O_K^3. We can make the following definition.

2.10. Definition. For $u \in \mathcal{I}_w$ and $r \in \mathbb{R}$ sufficiently big, we denote by $B_w(u,r) \subseteq D(w)$ the open subset corresponding to the closed polydisc $D(u,r)$ in O_K^3, see Appendix A.

This defines the structure of a K-analytic manifold on the quotient G/B.

$w \in W$	U_w	$\mathcal{I}_w$
w_0	$\begin{pmatrix} 1 & x_3 & x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$ $[x_i \in K]$	$\begin{pmatrix} 1 & x_3 & x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$ $[x_i \in \mathcal{O}_K]$
$w_1 := s_1 s_2$	$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $[x_2, x_3 \in K]$	$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix}$ $[x_2, x_3 \in \mathcal{O}_K, x_1 \in \pi\mathcal{O}_K]$
$w_2 := s_2 s_1$	$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$ $[x_1, x_2 \in K]$	$\begin{pmatrix} 1 & 0 & x_1 \\ x_3 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$ $[x_1, x_2 \in \mathcal{O}_K, x_3 \in \pi\mathcal{O}_K]$
s_1	$\begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $[x_1 \in K]$	$\begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ x_3 & x_2 & 1 \end{pmatrix}$ $[x_1 \in \mathcal{O}_K, x_2, x_3 \in \pi\mathcal{O}_K]$
s_2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}$ $[x_3 \in K]$	$\begin{pmatrix} 1 & 0 & 0 \\ x_2 & 1 & x_3 \\ x_1 & 0 & 1 \end{pmatrix}$ $[x_3 \in \mathcal{O}_K, x_1, x_2 \in \pi\mathcal{O}_K]$
id	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}$ $[x_i \in \pi\mathcal{O}_K]$

Table 1. Bruhat- and Bruhat-Iwahori-cells for $G = \mathrm{GL}_3(K)$

2.3.2. The flag variety G/B and the Plücker embedding

The K-analytic manifold G/B can be viewed as the boundary of the building $\mathcal{T}$. In order to understand its geometry, we recall some facts on flag varieties. We follow [Ful97, Chapter 9] and [Bri05]. Let $\mathcal{F}$ denote the *flag variety* over K whose K-points are given by

$$\mathcal{F}(K) = \{0 \subsetneq W_2 \subsetneq W_1 \subsetneq V^* \mid W_1, W_2 \text{ are } K\text{-linear subspaces of } V^*\}\,.$$

Furthermore, for $i \in \{1, 2\}$, let $\mathcal{G}_i(V^*)$ denote the *Grassmannian* of K-linear subspaces of V^* of codimension i. Then the assignments $\langle v_1, v_2\rangle \mapsto \langle v_1 \wedge v_2\rangle$ and $\langle v_1\rangle \mapsto \langle v_1\rangle$ set up isomorphisms

$$\mathcal{G}_i(V^*) \to \mathbb{P}(\textstyle\bigwedge^{3-i} V^*)$$

In particular, $\mathcal{G}_i(V^*)$ is projective. Let $g \in G$ and denote its columns by v_1, v_2, v_3. The maps $G \to \mathcal{G}_i(V^*)$ given by

$$g = (v_1, v_2, v_3) \mapsto \langle v_1, v_2\rangle \quad \text{and} \quad g = (v_1, v_2, v_3) \mapsto \langle v_1\rangle$$

induce isomorphisms $G/P_i \to \mathcal{G}_i(V^*)$.

2.11. Proposition. *The so called Plücker embedding $\mathcal{F} \to \mathcal{G}_2(V^*) \times \mathcal{G}_1(V^*)$ given on K-valued points by*

$$(W_2 \subseteq W_1) \mapsto (W_2, W_1)$$

is a closed immersion. Consequently, $\mathcal{F}$ is projective. It is of dimension 3.

Proof. See [Ful97, Chapter 9]. □

As above, let $g \in G$ and denote its columns by v_1, v_2, v_3. The map $G \to \mathcal{F}(K)$ given by

$$g = (v_1, v_2, v_3) \mapsto (\langle v_1\rangle \subsetneq \langle v_1, v_2\rangle)$$

induces an isomorphism $G/B \to \mathcal{F}(K)$. Let $pl\colon G/B \to \mathcal{G}_2(V^*) \times \mathcal{G}_1(V^*)$ denote the composition of this isomorphism with the Plücker embedding. Then, we can make all of the above maps explicit as follows: For $g \in G$, we define coordinates $\alpha_i(g)$ and $\beta_i(g)$ for $i = 1, 2, 3$. The column vector $(\alpha_1(g), \alpha_2(g), \alpha_3(g))$ is the first column of g. Additionally, let $\beta_i(g)$ be the determinant of the 2×2 submatrix of g consisting of the first two columns and row $4 - i$ removed. We obtain the following.

2.12. Proposition. *The maps $G/P_2 \to \mathbb{P}^2(K)$ and $G/P_1 \to \mathbb{P}^2(K)$ given by*

$$g \mapsto [\alpha_1(g) : \alpha_2(g) : \alpha_3(g)] \quad \textit{and} \quad g \mapsto [\beta_1(g) : \beta_2(g) : \beta_3(g)]$$

are isomorphisms. Moreover, the map $pl\colon G/B \to \mathbb{P}^2(K) \times \mathbb{P}^2(K)$ is given by

$$g \mapsto ([\alpha_1(g) : \alpha_2(g) : \alpha_3(g)], [\beta_1(g) : \beta_2(g) : \beta_3(g)])$$

and is a closed immersion. We have

$$\mathrm{im}(pl) = \left\{(\alpha, \beta) \in \mathbb{P}^2(K) \times \mathbb{P}^2(K) \,\middle|\, \alpha_3\beta_1 - \alpha_2\beta_2 + \alpha_1\beta_3 = 0\right\}.$$

Proof. All assertions are clear except for the description of $\mathrm{im}(pl)$. For this, see [Ful97, Chapter 9, Lemma 2]. □

The images of the cells $D(w)$ under the Plücker embedding are given in Table 2. These explicit descriptions will be very useful in Chapter 4.

$w \in W$	$\mathcal{I}_w$		$pl(D(w))$
w_0	$\begin{pmatrix} 1 & x_3 & x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$	$[x_i \in \mathcal{O}_K]$	$([1 : x_1 : x_2], [1 : x_3 : x_1x_3 - x_2])$
$w_1 = s_1s_2$	$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix}$	$[x_2, x_3 \in \mathcal{O}_K, x_1 \in \pi\mathcal{O}_K]$	$([x_1 : 1 : x_2], [-1 : x_1x_3 - x_2 : x_3])$
$w_2 = s_2s_1$	$\begin{pmatrix} 1 & 0 & x_1 \\ x_3 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$	$[x_1, x_2 \in \mathcal{O}_K, x_3 \in \pi\mathcal{O}_K]$	$([1 : x_2 : x_1], [x_3 : 1 : x_2 - x_1x_3])$
s_1	$\begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ x_3 & x_2 & 1 \end{pmatrix}$	$[x_1 \in \mathcal{O}_K, x_2, x_3 \in \pi\mathcal{O}_K]$	$([x_2 : 1 : x_1], [-x_3 : x_2 - x_1x_3 : 1])$
s_2	$\begin{pmatrix} 1 & 0 & 0 \\ x_2 & 1 & x_3 \\ x_1 & 0 & 1 \end{pmatrix}$	$[x_3 \in \mathcal{O}_K, x_1, x_2 \in \pi\mathcal{O}_K]$	$([x_1 : x_2 : 1], [x_1x_3 - x_2 : -1 : -x_3])$
id	$\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}$	$[x_i \in \pi\mathcal{O}_K]$	$([x_2 : x_1 : 1], [x_2 - x_1x_3 : -x_3 : -1])$

Table 2. Bruhat-Iwahori-cells under the Plücker embedding

2.3.3. Chambers and compact open subsets of G/B

In this subsection, we relate G/B to $\mathcal{T}$; more precisely we want to see how to associate compact open subsets of G/B to pointed chambers in $\mathcal{T}$. This construction is crucial for the construction of the Poisson kernel. We follow [AdS02, Section 1.6].

2.13. Definition. Let $\sigma = (\pi\Lambda_0 \subsetneq \Lambda_2 \subsetneq \Lambda_1 \subsetneq \Lambda_0) \in \widehat{\mathcal{T}_2}$. A flag $W = (W_2 \subsetneq W_1) \in \mathcal{F}(K)$ is called *compatible* with σ if

$$\Lambda_i = \pi\Lambda_0 + (W_i \cap \Lambda_0) \quad \text{for } i = 1, 2.$$

We denote by $U(\sigma) \subset \mathcal{F}(K)$ the set of all flags which are compatible with σ.

2.14. Proposition.

(i) *$U(\sigma) \subset \mathcal{F}(K)$ is compact open. Every compact open set in $\mathcal{F}(K)$ can be written as a finite disjoint union of subsets of the form $U(\sigma)$, $\sigma \in \widehat{\mathcal{T}_2}$.*

(ii) *We have $U(g\sigma) = gU(\sigma)$ for all $g \in G$.*

(iii) *Under the identification $\mathcal{F}(K) \cong G/B$ we have $U(\sigma_0) = \mathcal{I}B/B$.*

Proof. Property (i) follows from (ii) and (iii) by [SS91, Section 4, Proposition 8]. Properties (ii) and (iii) are straightforward from the definitions. □

If we let

$$y := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi^2 \end{pmatrix} \in G,$$

then it is shown in [SS91, Section 4, Proposition 8] that in (i) it suffices to consider sets of the form $gy^nU(\sigma_0)$ with $g \in \mathrm{GL}_3(\mathcal{O}_K)$ for $n \geq 0$ big enough.

2.15. Remark. We want to mention that property (iii) in the above proposition can be interpreted geometrically as follows: We pick the standard sector S_0 which is based at v_0 and contains σ_0. This sector determines a chamber σ_∞ in the spherical building at infinity in the Borel-Serre compactification of $\mathcal{T}$, see [BS76]. Its stabilizer is given by B and thus, $U(\sigma_0)$ is given by the orbit $\mathrm{Stab}_G(\sigma_0)\mathrm{Stab}_G(\sigma_\infty)$ in G/B. This holds for all chambers $\sigma \in \widehat{\mathcal{T}_2}$, see [ST97, Proposition 8].

Let $\mathbb{Z}[\hat{\mathcal{T}}_2]$ denote the free abelian group on the pointed chambers of $\mathcal{T}$. By the above proposition we obtain a G-equivariant surjective homomorphism

$$\begin{aligned} \psi\colon \mathbb{Z}[\hat{\mathcal{T}}_2] &\to C^\infty(G/B, \mathbb{Z}), \\ \sigma &\mapsto \mathbb{1}_{U(\sigma)}, \end{aligned}$$

where the right hand side denotes the locally constant functions on G/B with values in $\mathbb{Z}$. Let

$$y_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} \in G \quad \text{and} \quad y_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix} \in G.$$

Then we have $y = y_1 \cdot y_2 = y_2 \cdot y_1$. We obtain the following description of $\ker(\psi)$.

2.16. Proposition. *Let I be the G-submodule of $\mathbb{Z}[\hat{\mathcal{T}}_2]$ generated by $\mathcal{I}\, y_i \sigma_0 - \sigma_0$ for $i \in \{1, 2\}$. Then $I = \ker(\psi)$.*

Proof. See [SS91, Section 4, Proposition 11]. □

Note that $\mathcal{I}\, y_i \sigma_0$ is a finite sum of chambers for $i \in \{1, 2\}$, so the above is well-defined.

CHAPTER 3

The residue map

In this chapter, we construct the residue map relating rigid analytic functions on $\mathcal{X}$ to harmonic cocycles on $\mathcal{T}$. This is due to Schneider and Teitelbaum in the case of trivial coefficients. Building on Schneider's theory of so called *holomorphic discrete series representations*, we construct a natural analogue for more general coefficients. This is inspired by a construction of Schneider and Stuhler in [SS91] for $\mathrm{GL}_2(K)$.

3.1. Harmonic cocycles

We begin by introducing harmonic cocycles on $\mathcal{T}$ and the coefficients we are primarily interested in. We fix a complete extension L of K inside $\mathbb{C}_K$ and denote its ring of integers by $\mathcal{O}_L$. In the sequel, by an $L[G]$-module we always mean a left $L[G]$-module.

3.1. Definition. Let M be an $L[G]$-module. A function $c\colon \widehat{\mathcal{T}_2} \to M$ is called a *harmonic cocycle with values in* M if the following conditions are satisfied:

(i) Let $\sigma \in \widehat{\mathcal{T}_2}$ and let ρ_σ be a generator of the group H_σ as in Proposition 2.2. Then

$$c(\rho_\sigma \sigma) = c(\sigma).$$

(ii) Let $e \in \mathcal{T}_1$. Then

$$\sum_{\sigma \mapsto e} c(\sigma) = 0,$$

where the sum is over all pointed chambers $\sigma \in \widehat{\mathcal{T}_2}$ sharing the face e, each with distinguished vertex opposite to e.

The space of harmonic cocycles with values in M is denoted by $C_{\mathrm{har}}(\mathcal{T}, M)$. It carries a natural L-vector space structure. Moreover, if we set

$$(g \cdot c)(\sigma) = g \cdot c(g^{-1}\sigma) \quad \text{for } g \in G, \sigma \in \widehat{\mathcal{T}_2}, c \in C_{\mathrm{har}}(\mathcal{T}, M),$$

it itself becomes an $L[G]$-module.

An alternative, more representation-theoretic description of the space of harmonic cocycles can be found in [ST97, Definition 9].

3.2. Remark. We should mention that a funny new phenomenon appears in the definition of harmonic cocycles when working with $\mathrm{GL}_3(K)$ instead of $\mathrm{GL}_2(K)$: Part (i) of the definition implies that one can directly regard harmonic cocycles as functions on the set of chambers $\mathcal{T}_2$. This will be a convenient point of view for various explicit computations in Part II of this thesis.

The coefficients we are primarily interested in are given as follows. Let $\mathcal{P}_k(L)$ denote the $L[G]$-module $(\mathrm{Sym}^k(V^*) \otimes_K \det^{-k/3}) \otimes_K L$ for $k \geq 0$ with $3 \mid k$. In the sequel, it will be very useful to have the following concrete description of this module. Using the basis of V^* from Chapter 1, we have $\mathcal{P}_k(L) \cong L[X_1, X_2, X_3]_{\deg = k}$ with the G-action given by

$$\begin{aligned}&(g_*F)(X_1, X_2, X_3) = \\ &\qquad \det(g)^{-k/3} F(g_{11}X_1 + g_{21}X_2 + g_{31}X_3, g_{12}X_1 + g_{22}X_2 + g_{32}X_3, g_{13}X_1 + g_{23}X_2 + g_{33}X_3).\end{aligned}$$

We denote the L-vector space dual of $\mathcal{P}_k(L)$ by $V_k(L)$ and equip it with the left action $(g \cdot v)(F) = v((g^{-1})_*F)$ for $v \in V_k(L)$, $F \in \mathcal{P}_k(L)$ and $g \in G$. In the sequel, we suppress L from the notation and just write $\mathcal{P}_k$ and V_k instead of $\mathcal{P}_k(L)$ and $V_k(L)$. Both $\mathcal{P}_k$ and V_k are in fact algebraic representations of the algebraic group underlying G.

3.3. Remark. Note that if $\mathrm{char}(K) = 0$ both $\mathcal{P}_k$ and V_k are irreducible algebraic representations. This is not true in general in positive characteristic, see [Jan03, II.2.16].

The following fact is simple combinatorics.

3.4. Lemma. *We have*

$$\dim_L \mathcal{P}_k = \dim_L V_k = \binom{k+2}{2} = \frac{(k+2)(k+1)}{2}.$$

The representation $\mathcal{P}_k$ has highest weight $(2k/3, -k/3, -k/3)$ with respect to B.

In the sequel, we write $C_{\mathrm{har}}(\mathcal{T}, k) := C_{\mathrm{har}}(\mathcal{T}, V_k)$. Next, we want to define the subspace of bounded harmonic cocycles. For this, we need some preparations. Let $c \in C_{\mathrm{har}}(\mathcal{T}, k)$. We define a function $\varphi_c \colon K^\times \backslash G \to V_k$ by

$$\varphi_c(g) = g^{-1} \cdot c(g\sigma_0),$$

where σ_0 is the standard pointed chamber. Since $\mathrm{Stab}_G(\sigma_0) = K^\times \mathcal{I}$, by Proposition 2.2, we have $\varphi_c(gh) = h^{-1} \cdot \varphi_c(g)$ for $g \in G$, $h \in \mathcal{I}$. Now, we set

$$V_k^{\mathrm{int}} := \left\{ v \in V_k \;\middle|\; v(X_1^{i_1} X_2^{i_2} X_3^{i_3}) \in \pi^{i_1 + i_2} \mathcal{O}_L \text{ for } i_1 + i_2 + i_3 = k \right\}.$$

One easily verifies that V_k^{int} is an $\mathcal{I}$-stable $\mathcal{O}_L$-module, see also the proof of Proposition 5.25.

3.5. Definition. A harmonic cocycle $c \in C_{\mathrm{har}}(\mathcal{T}, k)$ is called *bounded* if there exists $\alpha \in \mathcal{O}_L \setminus \{0\}$ such that $\alpha\varphi_c(g) \in V_k^{\mathrm{int}}$ for all $g \in G$.

3.6. Remark. Note that for $k = 0$, we have $V_0 \cong L$ and $V_0^{\mathrm{int}} \cong \mathcal{O}_L$, both with the trivial G-action. Then, a harmonic cocycle $c \in C_{\mathrm{har}}(\mathcal{T}, 0)$ is bounded if and only if its image $\{c(\sigma) \mid \sigma \in \widehat{\mathcal{T}_2}\} \subseteq L$ is bounded.

We denote by $C^b_{\mathrm{har}}(\mathcal{T}, k)$ the set of bounded harmonic cycles in $C_{\mathrm{har}}(\mathcal{T}, k)$.

3.7. Proposition. *The space $C^b_{\mathrm{har}}(\mathcal{T}, k)$ is an $L[G]$-submodule of $C_{\mathrm{har}}(\mathcal{T}, k)$.*

Proof. It is clear from the definition that $C^b_{\mathrm{har}}(\mathcal{T}, k)$ is an L-subvector space. To see that $C^b_{\mathrm{har}}(\mathcal{T}, k)$ is G-stable, let $c \in C^b_{\mathrm{har}}(\mathcal{T}, k)$ and $g \in G$. Then we have

$$\varphi_{g\cdot c}(h) = h^{-1} \cdot (g \cdot c)(h\sigma_0) = (h^{-1}g) \cdot c(g^{-1}h\sigma_0) = \varphi_c(g^{-1}h)$$

for all $h \in G$. In particular, c is bounded if and only if $g \cdot c$ is bounded. □

3.8. Remark. We should point out that the above definition of boundedness looks different than the one considered in [DT08, Section 2.3]: There, one works with an $\mathcal{I}$-equivariant norm on V_k. The two approaches can be linked by observing that the space V_k^{int} defines an $\mathcal{I}$-equivariant norm on V_k via

$$\gamma(v) := \inf\left\{|a| \;\middle|\; a \in L^\times, a^{-1}v \in V_k^{\mathrm{int}}\right\}.$$

as in Section 2.2. Then boundedness with respect to V_k^{int} and with respect to γ are equivalent. We will also give a very natural interpretation of the notion of boundedness when passing from harmonic cocycles to certain automorphic forms in Chapter 5. We should also mention that in many applications one is in fact interested in harmonic cocycles invariant under certain (arithmetic) subgroups $\Gamma \subset \mathrm{GL}_3(K)$. For cocompact groups, boundedness is then automatic.

3.2. Filtrations on $\mathcal{P}_k$ and V_k

In the sequel, we also need to consider both $\mathcal{P}_k$ and V_k as modules for the parabolic subgroup P_1. In particular, we construct a filtration of $L[P_1]$-submodules on both spaces. Let

$$\mathcal{J} := \left\{(i_1, i_2, i_3) \in \mathbb{Z}^3_{\geq 0} \;\middle|\; i_1 + i_2 + i_3 = k\right\}.$$

To shorten the notation, we write $\underline{X}^I = X_1^{i_1}X_2^{i_2}X_3^{i_3}$ for $I = (i_1, i_2, i_3) \in \mathcal{J}$. We set

$$F^j\mathcal{P}_k := \left\langle \underline{X}^I \;\middle|\; I \in \mathcal{J},\ i_3 \leq k - j\right\rangle_L \subseteq \mathcal{P}_k \quad \text{for } j \in \{0, \dots, k\}.$$

We also set $F^{k+1}\mathcal{P}_k = 0$.

3.9. Proposition. *Each $F^j\mathcal{P}_k$ is P_1-stable. We have*

$$\mathcal{P}_k = F^0\mathcal{P}_k \supsetneq F^1\mathcal{P}_k \supsetneq \cdots \supsetneq F^k\mathcal{P}_k = \left\langle X_1^{i_1}X_2^{i_2} \;\middle|\; i_1+i_2=k \right\rangle_L \supsetneq F^{k+1}\mathcal{P}_k = 0,$$

i.e., $(F^j\mathcal{P}_k)_{0\le j\le k+1}$ is a decreasing filtration of $\mathcal{P}_k$ by $L[P_1]$-submodules.

Proof. This is immediate from the definitions. □

3.10. Remark. Note that $F^0\mathcal{P}_k/F^1\mathcal{P}_k$ is one-dimensional, i.e., given by a character of P_1. Explicitly, it is given by

$$\begin{aligned} \chi_k\colon P_1 &\to K^\times \subseteq L^\times, \\ p = \left(\begin{smallmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{smallmatrix}\right) &\mapsto \det(p)^{-k/3}p_{33}^k. \end{aligned}$$

This character plays an important role in Section 4.1.

By duality, we also obtain a filtration on V_k. It is given as follows. Let

$$F^jV_k \coloneqq \ker(V_k \to (F^{k+1-j}\mathcal{P}_k)^*) \quad \text{for } j \in \{0,\dots,k+1\}.$$

We immediately obtain the following corollary.

3.11. Corollary. *As above, $(F^jV_k)_{0\le j\le k+1}$ is a decreasing filtration of V_k by $L[P_1]$-submodules.*

3.3. Holomorphic discrete series representations

Recall that we have defined a coordinate function ω on $\mathcal{X}$ in Chapter 1, whose components ω_i can be regarded as elements of $\mathcal{O}_\mathcal{X}$. We have seen that if z is a point in $\mathcal{X}$, we can always renormalize it such that $z = [z_1 : z_2 : 1]$. Then we have $\omega_i(z) = z_i$ for $i \in \{1,2\}$. For $g \in G$ we set

$$j(g,\omega) \coloneqq g_{13}\omega_1 + g_{23}\omega_2 + g_{33} \in \mathcal{O}_\mathcal{X},$$

the *factor of automorphy*. Clearly, it satisfies the usual cocycle relation

$$j(gh,\omega) = j(g,\omega)\cdot j(h, g_*\omega) \quad \text{for } g,h \in G.$$

The ring $\mathcal{O}_\mathcal{X}$ of global rigid analytic functions on $\mathcal{X}$ carries the structure of a left $\mathbb{C}_K[G]$-module via $g_*f = f(g_*\omega)$ for $f \in \mathcal{O}_\mathcal{X}$ and $g \in G$. We are interested in various other G-actions on this space, which are given as follows.

3.12. Definition. For $k \in \mathbb{Z}_{\ge 0}$ with $3 \mid k$ we denote by $\mathcal{O}_\mathcal{X}(k)$ the ring $\mathcal{O}_\mathcal{X}$ equipped with the following *weight k action*,

$$g_*f = \det(g)^{k/3} j(g,\omega)^{-k} f(g_*\omega),$$

where $g \in G$ and $f \in \mathcal{O}_\mathcal{X}$.

3.13. Remark. Note that $\mathcal{O}_{\mathcal{X}}(0) = \mathcal{O}_{\mathcal{X}}$ and in the case $k = 3$ we have an isomorphism

$$\begin{aligned} \mathcal{O}_{\mathcal{X}}(3) &\to \Omega^2_{\mathcal{X}}, \\ f &\mapsto f(\omega)\mathrm{d}\omega := f(\omega)\mathrm{d}\omega_1 \wedge \mathrm{d}\omega_2, \end{aligned}$$

where $\Omega^2_{\mathcal{X}}$ denotes the global sections of the sheaf of two-forms on $\mathcal{X}$.

Each of the spaces $\mathcal{O}_{\mathcal{X}}(k)$ in fact carries a structure of an $\mathcal{O}_{\mathcal{X}}[G]$-module by left multiplication with elements in $\mathcal{O}_{\mathcal{X}}$.

The following construction is crucial and essentially a special case of [Sch92, Section 3]. However, since we work over a local field of any characteristic and use different conventions, we redevelop all necessary constructions in our situation. See also [SS91, p. 95 ff.] for a similar construction for $\mathrm{GL}_2(K)$. In the remainder of this section, we work with $L = \mathbb{C}_K$. We are interested in the $\mathbb{C}_K$-vector space $\mathcal{O}_{\mathcal{X}} \otimes_{\mathbb{C}_K} V_k$ equipped with the diagonal left G-action, which we denote by

$$g \cdot (f \otimes v) = g_* f \otimes g \cdot v \quad \text{for } f \in \mathcal{O}_{\mathcal{X}}, v \in V_k, g \in G.$$

We will also view this space as the base-change of the algebraic representation V_k from $\mathbb{C}_K$ to $\mathcal{O}_{\mathcal{X}}$. In particular, it also carries a second action by the group $\mathrm{GL}_3(\mathcal{O}_{\mathcal{X}})$, whose restriction to G is different from the diagonal action of G. For $g \in \mathrm{GL}_3(\mathcal{O}_{\mathcal{X}})$ and $m \in \mathcal{O}_{\mathcal{X}} \otimes_{\mathbb{C}_K} V_k$, we write gm for the element obtained by acting with g on m. We set

$$u(\omega) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\omega_1 & -\omega_2 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}_K(\omega_1, \omega_2)) \subset \mathrm{GL}_3(\mathcal{O}_{\mathcal{X}})$$

and define a projection

$$\begin{aligned} \theta \colon \mathrm{GL}_3(\mathcal{O}_{\mathcal{X}}) &\to \mathbb{P}^2(\mathcal{O}_{\mathcal{X}}), \\ g &\mapsto [0:0:1]g^{-1}. \end{aligned}$$

Then θ is equivariant with respect to the action defined in the beginning of Chapter 1. It induces an isomorphism

$$\mathrm{GL}_3(\mathcal{O}_{\mathcal{X}})/P_1(\mathcal{O}_{\mathcal{X}}) \cong \mathbb{P}^2(\mathcal{O}_{\mathcal{X}}).$$

We have

$$\theta(u(\omega)) = [\omega_1 : \omega_2 : 1].$$

In particular, we obtain

$$\theta(g^{-1}u(\omega)) = \theta(u(g_*\omega)).$$

Hence we find $p_g \in P_1(\mathcal{O}_{\mathcal{X}})$ such that $u(g_*\omega) = g^{-1}u(\omega)p_g$. Note that the entries of $\theta(g^{-1}u(\omega))$ and $\theta(u(g_*\omega))$ differ precisely by a renormalization with the factor of automorphy $j(g, \omega)$.

3.14. Lemma. *Let $M \subseteq V_k$ be a $\mathbb{C}_K[P_1]$-submodule. Then*

$$u(\omega)(\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} M) \subseteq \mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k$$

is a $\mathbb{C}_K[G]$-submodule.

Proof. Let $g \in G$. For $f \in \mathcal{O}_\mathcal{X}$ and $m \in M$ we have

$$g \cdot (u(\omega)(f \otimes m)) = (gu(g_*\omega))(g_*f \otimes m) = u(\omega)(p_g(g_*f \otimes m))$$

for some $p_g \in P_1(\mathcal{O}_\mathcal{X})$ by the discussion above. Since M is P_1-stable, this completes the proof. □

Consequently, we obtain a decreasing filtration by $\mathbb{C}_K[G]$-modules on $\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k$ by setting

$$F^j(\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k) := u(\omega)(\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} F^j V_k) \quad \text{for } j \in \{0, \ldots, k+1\}.$$

In the following, we need an explicit description of this filtration. For this purpose, let $v_I \in V_k$ be given by

$$v_I(F) = \text{coefficient of } \underline{X}^I \text{ in } F,$$

for $F \in \mathcal{P}_k$. Then by construction, we have

$$F^j V_k = \langle v_I \mid I \in \mathcal{J},\, i_3 \geq j \rangle_K .$$

Now, let $D_I := u(\omega)v_I \in \mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k$, where, here and in the sequel, we write v_I for the element $1 \otimes v_I \in \mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k$. Then we have

$$F^j(\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k) = \langle D_I \mid I \in \mathcal{J},\, i_3 \geq j \rangle_{\mathcal{O}_\mathcal{X}} .$$

We define the following order on the set $\mathcal{J}$.

3.15. Definition. Let $I, J \in \mathcal{J}$. We say I *is less or equal than* J and write

$$I \leq J \quad \text{if and only if} \quad (i_1 \leq j_1 \text{ and } i_2 \leq j_2).$$

This defines a partial order on $\mathcal{J}$.

3.16. Proposition. *We have*

$$D_I = \sum_{I \leq J} \binom{j_1}{i_1}\binom{j_2}{i_2} \omega_1^{j_1 - i_1} \omega_2^{j_2 - i_2} v_J .$$

Proof. We compute

$$\begin{aligned}(u(\omega)v_I)(\underline{X}^J) &= v_I(u(\omega)^{-1}\underline{X}^J) = v_I((X_1 + \omega_1 X_3)^{j_1}(X_2 + \omega_2 X_3)^{j_2} X_3^{j_3}) \\ &= \begin{cases} \binom{j_1}{i_1}\binom{j_2}{i_2} \omega_1^{j_1 - i_1} \omega_2^{j_2 - i_2}, & \text{if } i_1 \leq j_1, i_2 \leq j_2, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

By linearity, we obtain the result. □

3.17. Remark. The above formula shows that the maps D_I behave similarly to classical hyperderivatives.

We also need the following corollary.

3.18. Corollary. *In $\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k$ we have*

$$v_I = \sum_{I \leq J} \binom{j_1}{i_1}\binom{j_2}{i_2}(-\omega_1)^{j_1-i_1}(-\omega_2)^{j_2-i_2} D_J.$$

Proof. Let $f \in \mathcal{P}_k$. By definition, we have

$$D_I(u(\omega)f)(\omega) = v_I(f).$$

By substituting ω with $-\omega$ and observing that $u(-\omega) = u(\omega)^{-1}$, we obtain

$$v_I(f) = D_I(u(\omega)^{-1}f)(-\omega).$$

Now we apply Proposition 3.16 to obtain

$$v_I(f) = \sum_{I \leq J} \binom{j_1}{i_1}\binom{j_2}{i_2}(-\omega_1)^{j_1-i_1}(-\omega_2)^{j_2-i_2} v_J(u(\omega)^{-1}f),$$

which gives the desired formula. $\square$

Let $m \geq 3$. We are also interested in the $\mathbb{C}_K[G]$-modules $\mathcal{O}_\mathcal{X}(m) \otimes_{\mathbb{C}_K} V_k$. We obtain a filtration of $\mathbb{C}_K[G]$-modules by observing that

$$\mathcal{O}_\mathcal{X}(m) \otimes_{\mathbb{C}_K} V_k = \mathcal{O}_\mathcal{X}(m) \otimes_{\mathcal{O}_\mathcal{X}} (\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k)$$

as $\mathbb{C}_K[G]$-modules and setting

$$F^j(\mathcal{O}_\mathcal{X}(m) \otimes_{\mathbb{C}_K} V_k) := \mathcal{O}_\mathcal{X}(m) \otimes_{\mathcal{O}_\mathcal{X}} F^j(\mathcal{O}_\mathcal{X} \otimes_{\mathbb{C}_K} V_k) \quad \text{for } j \in \{0, \dots, k+1\}.$$

The explicit description of the filtration above Definition 3.15 is directly transferred to this situation. The following result is crucial and the primary reason why we are interested in this filtration.

3.19. Proposition. *The translation map*

$$\begin{aligned} t_{k,m} \colon \mathcal{O}_\mathcal{X}(k+m) &\to \mathcal{O}_\mathcal{X}(m) \otimes_{\mathbb{C}_K} V_k, \\ f &\mapsto f D_{(0,0,k)}, \end{aligned}$$

is G-equivariant and injective with image $F^k(\mathcal{O}_\mathcal{X}(m) \otimes_{\mathbb{C}_K} V_k)$.

Proof. The only statement requiring a proof is the G-equivariance. For this, observe that by Proposition 3.16,

$$D_{(0,0,k)}(F)(\omega) = F(\omega_1, \omega_2, 1) \quad \text{for } F \in \mathcal{P}_k.$$

Now let $g \in G$ and $f \in \mathcal{O}_{\mathcal{X}}(k+m)$. We have

$$\begin{aligned} g \cdot (t_{k,m}(f))(F) &= g \cdot (f(\omega) D_{(0,0,k)}(\omega))(F) \\ &= \det(g)^{m/3} j(g,\omega)^{-m} f(g_*\omega) D_{(0,0,k)}(g_*^{-1}F)(g_*\omega) \\ &= \det(g)^{m/3} j(g,\omega)^{-m} f(g_*\omega)(g_*^{-1}F)((g_*\omega)_1, (g_*\omega)_2, 1) \\ &= \det(g)^{m/3} j(g,\omega)^{-m} f(g_*\omega) \det(g)^{k/3} j(g,\omega)^{-k} F(\omega_1,\omega_2,1) \\ &= t_{k,m}(g_*f)(F), \end{aligned}$$

completing the proof. □

3.20. Remark. The above construction is inspired by the so called *translation principle* in classical Lie algebra representation theory. In the non-archimedean situation, this first appeared in [SS91, p. 95 ff.] for $\mathrm{GL}_2(K)$.

3.4. The residue map

We have now developed all necessary tools to construct the residue map. We first consider the case $k = 0$ and then extend the construction to more general weights using the translation map. Throughout this section, we always work with $L = \mathbb{C}_K$.

3.4.1. The case $k = 0$

We recall the construction of the residue map from [ST97]. A source in the $\mathrm{GL}_2(K)$-case that uses similar normalizations to ours is [DT08, Section 2.2]. We first need to understand the space $\mathcal{O}_{R_0}$ of rigid analytic functions on R_0. For this, recall that $R_0 = \mathrm{red}^{-1}(|\sigma_0|^\circ)$ is the direct limit of the affinoid subdomains $R_{0,N}$ constructed in Section 2.2.

3.21. Lemma. *Let $N \geq 3$. We have*

$$\mathcal{O}_{R_{0,N}} = \left\{ \sum_{\underline{i}=(i_1,i_2)\in\mathbb{Z}^2} a(\underline{i})\omega_1^{i_1}\omega_2^{i_2} \;\middle|\; a(\underline{i}) \in \mathbb{C}_K, |a(\underline{i})| q^{-\ell_N(\underline{i})} \to 0 \text{ for } |i_1| + |i_2| \to \infty \right\},$$

where

$$\ell_N(\underline{i}) = \begin{cases} \dfrac{2}{N} i_1 + \dfrac{1}{N} i_2, & \text{for } i_1 \geq 0, i_1 + i_2 \geq 0, \\ \left(1 - \dfrac{1}{N}\right) i_1 + \left(1 - \dfrac{2}{N}\right) i_2, & \text{for } i_2 \leq 0, i_1 + i_2 \leq 0, \\ \left(1 - \dfrac{1}{N}\right) i_1 + \dfrac{1}{N} i_2, & \text{for } i_1 \leq 0, i_2 \geq 0. \end{cases}$$

Proof. See [ST97, Lemma 17] or, in the $\mathrm{GL}_2(K)$-case, [DT08, Subsection 2.2.1]. □

The rather complicated convergence conditions stem from the inequalities defining the affinoid subdomain $R_{0,N}$, see Section 2.2. As a consequence of Lemma 3.21, we obtain that $\mathcal{O}_{R_0}$ consists of Laurent series of the above form, where the convergence conditions are satisfied for *each* $N \geq 3$. We are now able to make the following definition.

3.22. Definition. Let $\eta \in \Omega^2_{\mathcal{X}}$. Then, by the above, on the annulus R_0 we can expand η as

$$\eta = \sum_{\underline{i}=(i_1,i_2)\in\mathbb{Z}^2} a_\eta(\underline{i})\omega_1^{i_1}\omega_2^{i_2}\mathrm{d}\omega.$$

with the convergence conditions from above. We define the *residue of η at σ_0* to be $\mathrm{res}_{\sigma_0}(\eta) := a_\eta(-1,-1) \in \mathbb{C}_K$.

3.23. Lemma. *Let $\eta \in \Omega^2_{\mathcal{X}}$ and $g \in K^\times\mathcal{I}$. Then*

$$\mathrm{res}_{\sigma_0}(g_*\eta) = \mathrm{res}_{\sigma_0}(\eta).$$

Proof. See [ST97, Lemma 20]. □

The above lemma ensures that the following construction is well-defined.

3.24. Definition. Let $\sigma \in \widehat{\mathcal{T}_2}$ and $\eta \in \Omega^2_{\mathcal{X}}$. Choose $g \in G$ such that $g\sigma = \sigma_0$. The *residue of η at σ* is defined to be

$$\mathrm{res}_\sigma(\eta) := \mathrm{res}_{\sigma_0}(g_*\eta).$$

Note that by Remark 3.13, we have $\mathcal{O}_{\mathcal{X}}(3) \cong \Omega^2_{\mathcal{X}}$. We obtain the following key result.

3.25. Proposition. *The map*

$$\mathrm{Res}_0 \colon \mathcal{O}_{\mathcal{X}}(3) \to C_{\mathrm{har}}(\mathcal{T}, 0)$$

given by $\mathrm{Res}_0(f(\omega)\mathrm{d}\omega)(\sigma) = \mathrm{res}_\sigma(f(\omega)\mathrm{d}\omega)$ *for* $\sigma \in \widehat{\mathcal{T}_2}$ *and* $f \in \mathcal{O}_{\mathcal{X}}(3)$ *is well-defined,* $\mathbb{C}_K$*-linear and G-equivariant.*

Proof. See [ST97, Proposition 22]. □

3.4.2. Extension to general weights

Extending the residue map to allow more general weights is now straightforward using the constructions in Section 3.3. This approach is due to Schneider and Stuhler in the $\mathrm{GL}_2(K)$-case, see [SS91, p. 97]. Recall that we have the G-equivariant translation map $t_{k,3} \colon \mathcal{O}_{\mathcal{X}}(k+3) \to \mathcal{O}_{\mathcal{X}}(3) \otimes_{\mathbb{C}_K} V_k$. We also have a G-equivariant map

$$\begin{aligned} s_k \colon C_{\mathrm{har}}(\mathcal{T}, 0) \otimes_{\mathbb{C}_K} V_k &\to C_{\mathrm{har}}(\mathcal{T}, k), \\ c \otimes v &\mapsto [\sigma \mapsto c(\sigma)v], \end{aligned}$$

where the left hand side is equipped with the diagonal G-action.

3.26. Definition. The *residue map of weight k* is defined as

$$\mathrm{Res}_k := s_k \circ (\mathrm{Res}_0 \otimes \mathrm{id}) \circ t_{k,3} \colon \mathcal{O}_{\mathcal{X}}(k+3) \to C_{\mathrm{har}}(\mathcal{T}, k).$$

By construction, Res_k is G-equivariant.

We need the following explicit description.

3.27. Proposition. *Let $f \in \mathcal{O}_{\mathcal{X}}(k+3)$, $\sigma \in \widehat{\mathcal{T}_2}$ and $F \in \mathcal{P}_k$. Then*

$$\mathrm{Res}_k(f)(\sigma)(F) = \mathrm{res}_\sigma(F(\omega_1, \omega_2, 1) f(\omega) \mathrm{d}\omega).$$

Proof. By Proposition 3.16 we have

$$\begin{aligned}
\mathrm{Res}_k(f) &= (s_k \circ (\mathrm{Res}_0 \otimes \mathrm{id}))(f D_{(0,0,k)}) \\
&= (s_k \circ (\mathrm{Res}_0 \otimes \mathrm{id}))\left(\sum_{J \in \mathcal{J}} (\omega_1^{j_1} \omega_2^{j_2} f \otimes v_J)\right) \\
&= \sum_{J \in \mathcal{J}} \mathrm{Res}_0(\omega_1^{j_1} \omega_2^{j_2} f) v_J.
\end{aligned}$$

Now by definition of Res_0, plugging in σ and F gives the desired result. □

3.28. Remark. One can also take the above formula as the definition of the map Res_k and check the G-equivariance directly. Since we need the translation map $t_{k,3}$ for the construction of an integration map later, we took the approach above.

Now, let $\mathcal{O}_{\mathcal{X}}(k+3)^b := \mathrm{Res}_k^{-1}(C^b_{\mathrm{har}}(\mathcal{T}, k)) \subseteq \mathcal{O}_{\mathcal{X}}(k+3)$, the *space of rigid analytic functions on $\mathcal{X}$ with bounded residues of weight $k+3$*. The aim of the subsequent chapter is to show that (under some additional assumptions) the induced map

$$\mathrm{Res}_k \colon \mathcal{O}_{\mathcal{X}}(k+3)^b \to C^b_{\mathrm{har}}(\mathcal{T}, k)$$

is surjective and has a G-equivariant right inverse. We will construct this right inverse explicitly, it is given by the so called *Poisson kernel*.

3.29. Remark. We should point out that our definition of $\mathcal{O}_{\mathcal{X}}(k+3)^b$ differs from the standard literature, where a similar space is considered for $k = 0$, see [IS01, Definition 4.6] and [BdS16, Subsection 2.1.4]. The relationship between the two notions is clarified in Remark 4.33.

CHAPTER 4

The Poisson kernel

In this chapter, we construct the Poisson kernel, an explicit right inverse of the residue map constructed in the previous chapter. The crucial step in the construction is finding an appropriate *locally analytic* kernel function, which we can then integrate. The kernel function in [ST97] is not locally analytic everywhere, which makes integration very delicate as soon as one moves away from the trivial coefficients in [ST97]. Using our knowledge of the geometry of the boundary G/B of $\mathcal{T}$, we are able to construct such a kernel function. We first integrate it in the case of trivial coefficients and then extend to general coefficients using the holomorphic discrete series representations studied in the previous chapter. For this, we also need certain locally analytic principal series representations.

4.1. Locally analytic principal series representations

This section is inspired by [Sch11]. Let L be a complete extension of K inside $\mathbb{C}_K$. We use the notation and the results from Appendix A. Let $k \in \mathbb{Z}_{\geq 0}$ with $3 \mid k$ and let $\chi_k \colon T \to K^\times$ denote the algebraic character given by

$$t = \left(\begin{smallmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{smallmatrix}\right) \mapsto \det(t)^{-k/3} t_{33}^k.$$

We extend χ_k to a character of B by letting U act trivially. The following representation is a central object throughout this thesis. Let

$$\mathcal{A}_k := \operatorname{Ind}_B^G(\chi_k) = \left\{ f \in C^{\mathrm{an}}(G, L) \;\middle|\; f(gb) = \chi_k(b^{-1}) f(g) \text{ for } g \in G, b \in B \right\}, \qquad (2)$$

a locally analytic G-representation in the sense of Appendix A, where $g \in G$ acts via $(g_* f)(h) = f(g^{-1}h)$ for $g, h \in G$ and $f \in \mathcal{A}_k$. By Proposition A.16, $\mathcal{A}_k$ is of compact type. Our first aim is to relate $\mathcal{A}_k$ to the algebraic representation $\mathcal{P}_k$ studied in the previous chapter.

4.1. Proposition. *We have a G-equivariant embedding $\iota \colon \mathcal{P}_k \to \mathcal{A}_k$ given by*

$$\iota(F)(g) = \det(g)^{k/3} F([0,0,1] g^{-1}) \quad \textit{for } F \in \mathcal{P}_k \textit{ and } g \in G.$$

Proof. Let $g \in G$ and $F \in \mathcal{P}_k$. Then for $b \in B$ one has

$$\begin{aligned}\iota(F)(gb) &= \det(gb)^{k/3}F([0,0,1]b^{-1}g^{-1})\\ &= \det(b)^{k/3}b_{33}^{-k}\iota(F)(g) = \chi_k(b^{-1})\iota(F)(g)\end{aligned}$$

and for $h \in G$ we compute

$$\begin{aligned}\iota(h_*F)(g) &= \det(g)^{k/3}h_*F([0,0,1]g^{-1})\\ &= \det(g)^{k/3}\det(h)^{-k/3}F([0,0,1]g^{-1}h) = \iota(F)(h^{-1}g),\end{aligned}$$

proving the well-definedness and G-equivariance. The injectivity is obvious. □

4.2. Remark. Note that χ_k extends to a character of P_1, see Remark 3.10. In fact, one has that $\mathcal{P}_k \cong \operatorname{Ind}_B^{G,\mathrm{alg}}(\chi_k) \otimes_K L \cong \operatorname{Ind}_{P_1}^{G,\mathrm{alg}}(\chi_k) \otimes_K L$ under the above map, see [Jan03, II.2.16]. Moreover, by [Jan03, I.3.5], we have $\mathcal{P}_k \cong \operatorname{Ind}_{B(L)}^{G(L),\mathrm{alg}}(\chi_k)$, where we regard χ_k as an L-algebraic character.

We also need the following construction. Let

$$F_{P_i,k} := \operatorname{Ind}_B^{P_i,\mathrm{alg}}(\chi_k) \otimes_K L \cong \operatorname{Ind}_{B(L)}^{P_i(L),\mathrm{alg}}(\chi_k).$$

We want to describe these spaces more explicitly. For this purpose, we need some notation. Note that $P_2 = L_2 \cdot U_2$, where L_2 denotes the standard Levi-component and U_2 the unipotent radical of P_2. Explicitly,

$$L_2 = \begin{pmatrix} * & 0 & 0\\ 0 & * & *\\ 0 & * & * \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} 1 & * & *\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

In particular $L_2 \cong \mathrm{GL}_1(K) \times \mathrm{GL}_2(K)$. Since U_2 is normal in P_2, if M is an algebraic representation of L_2, we can view it as a representation of P_2 by letting U_2 act trivially.

4.3. Proposition. *We have*

$$F_{P_i,k} = \begin{cases} \chi_k, & \text{for } i = 1,\\ ((1_{\mathrm{GL}_1(K)} \otimes_K \operatorname{Sym}^k((K^2)^*)) \otimes_K \det^{-k/3}) \otimes_K L, & \text{for } i = 2, \end{cases}$$

where we regard $1_{\mathrm{GL}_1(K)} \otimes_K \operatorname{Sym}^k((K^2)^*)$ as a representation of P_2 as explained above.

Proof. We first observe that by definition it suffices to consider the case $L = K$. Let $i = 1$. Note that χ_k is in fact a character of P_1 by Remark 3.10. We have

$$F_{P_1,k} = \operatorname{Ind}_B^{P_1,\mathrm{alg}}(\chi_k|_B) = \chi_k \otimes_K \operatorname{Ind}_B^{P_1,\mathrm{alg}}(1_B)$$

by the tensor product identity, see [Jan03, I.3.6]. Now by [Jan03, I.5.10] we have

$$\operatorname{Ind}_B^{P_1,\mathrm{alg}}(1_B) = H^0(P_1/B, \mathcal{O}_{P_1/B}) = K.$$

This concludes the case $i = 1$. Now consider the case $i = 2$. By definition, we have

$$F_{P_2,k} = \left\{ f\colon P_2 \to \mathbb{A}^1_K \text{ K-algebraic} \;\middle|\; f(pb) = \chi_k(b^{-1})f(p) \text{ for } p \in P_2, b \in B \right\}.$$

But since $U_2 \subseteq B \cap \ker(\chi_k)$ and $(L_2 \cap B) \cdot U_2 = B$, this can be rewritten as

$$F_{P_2,k} = \left\{ f\colon L_2 \to \mathbb{A}^1_K \text{ K-algebraic} \;\middle|\; f(pb) = \chi_k(b^{-1})f(p) \text{ for } p \in L_2, b \in L_2 \cap B \right\}.$$

Now, since $L_2 \cong \mathrm{GL}_1(K) \times \mathrm{GL}_2(K)$ and $L_2 \cap B \cong \mathrm{GL}_1(K) \times B_2$, where $B_2 \subseteq \mathrm{GL}_2(K)$ denotes the Borel subgroup of upper triangular matrices, we can apply [Jan03, II.2.16] to obtain the desired description. □

4.4. Remark. We can also realize the space $F_{P_2,k}$ as a quotient of $\mathcal{P}_k$ as follows. It is easy to check that

$$\langle \underline{X}^I \mid I \in \mathcal{J}, i_1 \geq 1 \rangle_L \subseteq \mathcal{P}_k$$

is P_2-stable. The quotient by this submodule is isomorphic to $F_{P_2,k}$. In fact this is the top graded piece of a filtration of $\mathcal{P}_k$ by P_2-submodules analogous to the one studied in Section 3.2 for P_1.

Now, we can define the locally analytic representations

$$\mathcal{A}_{P_i,k} \coloneqq \mathrm{Ind}^G_{P_i}(F_{P_i,k}) \quad \text{for } i \in \{1, 2\}. \tag{3}$$

By the transitivity of the induction functor, see Proposition A.19, and since the K-points of P_i lie dense in the algebraic group underlying P_i, these are naturally $L[G]$-submodules of $\mathcal{A}_k$. By the same reasoning, $\mathcal{P}_k$ is naturally a $L[G]$-submodule of $\mathcal{A}_{P_i,k}$ for $i \in \{1, 2\}$. We are now able to define the object we are primarily interested in.

4.5. Definition. The *locally analytic Steinberg representation of G of weight k* is the $L[G]$-module

$$\mathrm{St}^{\mathrm{an}}_3(k) = \mathcal{A}_k / (\mathcal{A}_{P_1,k} + \mathcal{A}_{P_2,k}).$$

4.6. Remark. Note that our definition of the locally analytic Steinberg representation looks slightly more complicated than the usual one, as defined for example in [Sch11, Section 2.4]. There, the spaces $F_{P_i,k}$ are replaced with the irreducible representation of highest weight $(k, 0, 0)$ of the corresponding Levi-components. If $\mathrm{char}(K) = 0$ both definitions agree, however if $\mathrm{char}(K) = p > 0$ this is not the case anymore. We used the so called ∇-modules of weight $(k, 0, 0)$ as the replacement here (and twisted with $\det^{-k/3}$). This is inspired by the $\mathrm{GL}_2(K)$-case studied by Teitelbaum in positive characteristic, where this turns out to be the correct point of view.

The following proposition is an analogue of [Sch11, Lemme 2.23] and will become very important.

4.7. Proposition. *There is a natural commutative diagram of topological $L[G]$-modules*

$$\begin{tikzcd}
(\mathcal{A}_{P_1,0} + \mathcal{A}_{P_2,0}) \otimes_L \mathcal{P}_k \arrow[r] \arrow[d] & \mathcal{A}_{P_1,k} + \mathcal{A}_{P_2,k} \arrow[d] \\
\mathcal{A}_0 \otimes_L \mathcal{P}_k \arrow[r] & \mathcal{A}_k
\end{tikzcd}$$

with surjective horizontal arrows. This induces a G-equivariant continuous surjective map $T_k\colon \mathrm{St}_3^{\mathrm{an}}(0) \otimes_L \mathcal{P}_k \to \mathrm{St}_3^{\mathrm{an}}(k)$. Explicitly, T_k is given by

$$[f] \otimes F \mapsto [g \mapsto \iota(F)(g)f(g) = \det(g)^{-2k/3}F(\beta_3(g), -\beta_2(g), \beta_1(g))f(g)],$$

where β_i for $i \in \{1, 2, 3\}$ are the coordinate functions from Proposition 2.12.

Proof. Let $P \in \{B, P_1, P_2\}$ and write $F_{B,k} := \chi_k$. Then, we need to construct a continuous map

$$\mathrm{Ind}_P^G(1_P) \otimes_L \mathcal{P}_k \to \mathrm{Ind}_P^G(F_{P,k}).$$

But by the tensor product identity for the locally analytic induction we have

$$\mathrm{Ind}_P^G(1_P) \otimes_L \mathcal{P}_k \cong \mathrm{Ind}_P^G(\mathcal{P}_k|_P)$$

The isomorphism is given by $f \otimes F \mapsto [g \mapsto f(g)g^{-1}F]$. We also have a natural P-equivariant surjection $\mathcal{P}_k|_P \to F_{P,k}$ by Remark 3.10 and Remark 4.4. By Proposition A.20 we obtain the desired surjection by composition. Clearly, these constructions are compatible with the vertical arrows in the diagram, making it commutative. By taking quotients, we obtain the map $\mathrm{St}_3^{\mathrm{an}}(0) \otimes_L \mathcal{P}_k \to \mathrm{St}_3^{\mathrm{an}}(k)$. The explicit formula can now be read off by construction; we only need to compute the coefficient of X_3^k in $g^{-1}F$, or in other words, $\det(g)^{k/3}F([0, 0, 1]g^{-1})$, i.e., we need to compute the last row of the matrix g^{-1}. It is a simple computation to see that it is given by

$$\det(g)^{-1}(\beta_3(g), -\beta_2(g), \beta_1(g)),$$

proving the explicit formula. □

4.2. The kernel function

In this section, we define the central object of this chapter, the kernel function. Following [ST97], we first define a continuous kernel function, locally analytic only on the (opposite) big cell. We then describe how this kernel function can be modified in natural way to make it locally analytic everywhere. This uses the geometry of the flag variety G/B and the Plücker embedding: We realize that the singular locus of the kernel function is a projective line inside the flag variety. We then construct an explicit open neighbourhood of this projective line and modify the kernel function on this open neighbourhood. We should note that our normalizations differ from [ST97]. For the remainder of this chapter, we set $L = \mathbb{C}_K$.

Let $\xi \in \mathcal{O}_{\mathcal{X}}$ be given by $\xi(\omega) = \frac{1}{\omega_1 \cdot \omega_2}$.

4.8. Definition. The *(continuous) kernel function* $\theta \colon G/B \times \mathcal{X} \to \mathbb{C}_K$ is defined as follows. For $u \in U^-$ we set

$$\theta(u, \omega) = (u_* \xi)(\omega).$$

Following [ST97], we extend θ to a function on $G/B \times \mathcal{X}$ as follows:

$$\theta(g, \omega) = \begin{cases} 0, & \text{for } g \notin C^o(w_0), \\ \theta(u, \omega), & \text{for } g = ub \in C^o(w_0). \end{cases}$$

By adapting [ST97, Lemma 31] to our choice of Plücker coordinates in Proposition 2.12, we can rewrite this as

$$\theta(g, \omega) = \theta_1(g, \omega) \cdot \theta_2(g, \omega),$$

where

$$\theta_1(g, \omega) = \frac{\alpha_1(g)}{\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g)} \quad \text{and} \quad \theta_2(g, \omega) = \frac{\beta_1(g)}{\beta_1(g)\omega_2 + \beta_2(g)}.$$

In the sequel, an even more explicit description on the opposite big cell will be useful. We work with the coordinates in Table 1. Explicitly, this means we fix the embedding $u \colon K^3 \to G$ given by $\underline{x} = (x_1, x_2, x_3) \mapsto u(\underline{x})$, where

$$u(\underline{x}) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix} \in w_0 U_{w_0} w_0 = U^- \subset G.$$

By definition, the partial kernel functions $\theta_i(\cdot, \omega) \colon G/B \to \mathbb{C}_K$ for $i = 1, 2$ satisfy

$$u(\underline{x})_* \omega = (\theta_1(u(\underline{x}), \omega)^{-1}, \theta_2(u(\underline{x}), \omega)^{-1}).$$

More explicitly,

$$\theta_1(u(\underline{x}), \omega) = \frac{1}{\omega_1 + x_1\omega_2 + x_2} \quad \text{and} \quad \theta_2(u(\underline{x}), \omega) = \frac{1}{\omega_2 + x_3}.$$

This description resembles the kernel function used in the $\mathrm{GL}_2(K)$-case, see [Tei90, Section 2]. We have the following.

4.9. Lemma. *The function $\theta(\cdot, \omega) \colon G/B \to \mathbb{C}_K$ is continuous. Moreover, it is locally analytic on $C^o(w_0)$.*

Proof. See [ST97, Proposition 29 and Proposition 47]. □

We can analyze the singular locus in more detail.

4.10. Lemma. *The singular locus of the kernel function $\theta(\cdot,\omega)\colon G/B \to \mathbb{C}_K$ is*

$$C^o(s_1) \cup C^o(\mathrm{id}) = w_0 P_1/B \subset G/B.$$

Proof. The partial kernel function $\theta_i(\cdot,\omega)$ are quotients of locally analytic (even algebraic) functions. In particular, they are locally analytic except at the zeros of the denominators. Since $\omega \in \mathcal{X}$, such zeros can only occur if the corresponding Plücker coordinates appearing in the denominators simultaneously vanish. More precisely, fix $g \in G$, then, since g is invertible, the functions α_i for $i \in \{1,2,3\}$ do not have a common zero, hence $\theta_1(\cdot,\omega)$ is locally analytic everywhere. The singular locus of $\theta_2(\cdot,\omega)$ is given by

$$\{g \in G \mid \beta_2(g) = \beta_1(g) = 0\} = \{g \in G \mid \beta_2(w_0 g) = \beta_3(w_0 g) = 0\}.$$

Let $S = \{g \in G \mid \beta_2(g) = \beta_3(g) = 0\}$. It is easy to check that S is left- and right-B-invariant, hence S is a union of Bruhat cells. Now, one verifies that $w \in W$ is in S if and only if $w \in W_1 = \{\mathrm{id}, s_1\}$, completing the proof. □

We need the following object, where for a topological space M, we denote by $C(M,\mathbb{C}_K)$ the space of $\mathbb{C}_K$-valued continuous functions on M.

4.11. Definition. The *continuous Steinberg representation* of G is the $\mathbb{C}_K[G]$-module

$$\mathrm{St}_3^{\mathrm{con}} := C(G/B,\mathbb{C}_K)/C_{\mathrm{inv}}(G/B,\mathbb{C}_K),$$

where

$$C_{\mathrm{inv}}(G/B,\mathbb{C}_K) := C(G/P_1,\mathbb{C}_K) + C(G/P_2,\mathbb{C}_K)$$

and G acts via $(g_* f)(h) = f(g^{-1}h)$ for $g, h \in G$ and $f \in C(G/P,\mathbb{C}_K)$ for $P \in \{B, P_1, P_2\}$.

4.12. Remark. Note that we have a natural inclusion $\mathrm{St}_3^{\mathrm{an}}(0) \to \mathrm{St}_3^{\mathrm{con}}$ as locally analytic functions are continuous.

To build a suitable integration theory, we are interested in the class $[\theta(\cdot,\omega)] \in \mathrm{St}_3^{\mathrm{con}}$. We want to show that there exists a representing function for this class which is locally analytic everywhere. By the above, we need to modify the kernel function only on an open neighbourhood of $C^o(s_1) \cup C^o(\mathrm{id})$ in G/B. Recall from Proposition 2.8 that we have

$$G = \bigsqcup_{w \in W^1} U_w w P_1.$$

Note that the minimal representative for the class of w_0 in W^1 is $w_1 = s_1 s_2$. We define a kernel function with respect to P_1 as follows. For $u \in U_{w_1}$, we set

$$\theta_{\mathrm{inv}}(w_0 u w_0, \omega) := ((w_0 u w_0)_* \xi)(\omega) \in \mathbb{C}_K.$$

As above, we extend $\theta_{\mathrm{inv}}(\cdot,\omega)$ to G/P_1 by 0 outside $w_0 U_{w_1} w_0 P_1$. We need the following simple topological lemma.

4.13. Lemma. *Let Y be a topological space and Z be a closed subset of Y. Let $f\colon Y \to \mathbb{C}_K$ be continuous such that f vanishes along Z and $h\colon Y \setminus Z \to \mathbb{C}_K$ be bounded and continuous. Then the product fh defines a continuous function on Y, which vanishes along Z.*

Proof. See [ST97, Lemma 34]. □

4.14. Proposition. *We have*

$$\theta_{\mathrm{inv}}(g,\omega) = \frac{\beta_1(g)}{\beta_1(g)\omega_1 - \beta_3(g)} \cdot \frac{\beta_1(g)}{\beta_1(g)\omega_2 + \beta_2(g)} \quad \textit{for } g \in G.$$

Consequently, $\theta_{\mathrm{inv}}(\cdot,\omega)$ is continuous on G/P_1.

Proof. We claim that for $g = w_0uw_0p \in w_0U_{w_1}w_0P_1$ we have

$$u = \begin{pmatrix} 1 & \beta_2(g)/\beta_1(g) & -\beta_3(g)/\beta_1(g) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

First off, we observe that

$$\frac{\beta_i(g)}{\beta_1(g)} = \frac{\beta_i(w_0uw_0)}{\beta_1(w_0uw_0)} \quad \text{for } i = 2,3.$$

Now, the matrix w_0uw_0 is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}$$

and we compute that

$$\frac{\beta_3(w_0uw_0)}{\beta_1(w_0uw_0)} = -x_2 \quad \text{and} \quad \frac{\beta_2(w_0uw_0)}{\beta_1(w_0uw_0)} = x_3,$$

which proves our claim. Since the opposite big cell with respect to W^1 is the Zariski open set where β_1 is non-zero, the formula in the proposition now follows by applying the definition of $\theta_{\mathrm{inv}}(\cdot,\omega)$. The continuity can be proved analogously to the reasoning in [ST97, Lemma 33]: The two factors entering into $\theta_{\mathrm{inv}}(\cdot,\omega)$ are bounded outside the common zeros of β_1, β_3 and β_1, β_2 respectively. But since the functions $\beta_1, \beta_2, \beta_3$ can not simultaneously vanish, we can apply Lemma 4.13 twice to complete the proof. □

To finish the construction and for Section 5.1, we need the following. Let $i \in \{1,2\}$ and recall that $\mathcal{I}_i = \mathcal{I} \cup \mathcal{I}s_i\mathcal{I}$. Note that by Lemma 2.3 the set $K^\times\mathcal{I}_i$ is the stabilizer of the edge $e_i \in \widehat{\mathcal{T}_1}$. Recall that $U(\sigma_0) = \mathcal{I}B/B$. We set

$$U_i(\sigma_0) := \mathcal{I}_iU(\sigma_0) \subseteq G/B.$$

Note that this is a finite disjoint union of compact open subsets of G/B, hence compact open.

4.15. Lemma. *We have*

$$U_i(\sigma_0) = \mathcal{I} P_i/B.$$

Proof. Recall that by Proposition 2.9, we have

$$U_i(\sigma_0) = \mathcal{I}_{\mathrm{id}} B/B \cup \mathcal{I} s_i \mathcal{I}_{\mathrm{id}} B/B.$$

But since $s_i \mathcal{I}_{\mathrm{id}} s_i \subseteq \mathcal{I}$, we obtain, again by Proposition 2.9,

$$U_i(\sigma_0) = \mathcal{I} B/B \cup \mathcal{I} s_i B/B = \mathcal{I}_{\mathrm{id}} B/B \cup \mathcal{I}_{s_i} s_i B/B.$$

Thus, to show that $\mathcal{I} P_i/B \subseteq U_i(\sigma_0)$, it suffices to show that $\mathcal{I}_{\mathrm{id}} B \cup \mathcal{I}_{s_i} s_i B$ is right P_i-stable. Since $P_i = P_i(\mathcal{O}_K)B$, it further suffices to show that $\mathcal{I}_{\mathrm{id}} B(\mathcal{O}_K) \cup \mathcal{I}_{s_i} s_i B(\mathcal{O}_K)$ is right $P_i(\mathcal{O}_K)$-stable. Let $t \colon G(\mathcal{O}_K) \to G(\kappa)$ be the canonical projection. Let $p \in P_i(\mathcal{O}_K)$ and denote by r_p and $r_{t(p)}$ the right multiplications by p on $G(\mathcal{O}_K)$ and by $t(p)$ on $G(\kappa)$. The following diagram commutes:

$$\begin{tikzcd}
G(\mathcal{O}_K) \arrow[r, "r_p"] \arrow[d, "t"'] & G(\mathcal{O}_K) \arrow[d, "t"] \\
G(\kappa) \arrow[r, "r_{s(p)}"'] & G(\kappa)
\end{tikzcd}$$

Note that by definition $\mathcal{I}_w w B(\mathcal{O}_K) = t^{-1}(U_w(\kappa) w B(\kappa))$ for all $w \in W$. Now, for $w \in W_i$, we obtain

$$\begin{aligned}
r_p(\mathcal{I}_w w B(\mathcal{O}_K)) &= r_p(t^{-1}(U_w(\kappa) w B(\kappa))) \subseteq t^{-1}(r_{t(p)}(U_w(\kappa) w B(\kappa))) \\
&\subseteq t^{-1}(U_{\mathrm{id}}(\kappa) B(\kappa) \cup U_{s_i}(\kappa) s_i B(\kappa)) = \mathcal{I}_{\mathrm{id}} B(\mathcal{O}_K) \cup \mathcal{I}_{s_i} s_i B(\mathcal{O}_K).
\end{aligned}$$

since $P_i(\kappa)$ is a group. For the other inclusion, recall that $P_i = B \cup B s_i B$. Hence we have

$$U_i(\sigma_0) \subseteq \mathcal{I} B/B \cup \mathcal{I} B s_i B/B = \mathcal{I} P_i/B,$$

which completes the proof. $\square$

We set $\mathcal{U} := w_0 U_1(\sigma_0) \subset G/B$, a compact open neighbourhood of $w_0 P_1/B$. Now we can define our modified kernel function.

4.16. Definition. We define the *locally analytic kernel function* $\hat{\theta} \colon G/B \times \mathcal{X} \to \mathbb{C}_K$ by

$$\hat{\theta}(g, \omega) = \theta(g, \omega) - \mathbb{1}_{\mathcal{U}}(g) \cdot \theta_{\mathrm{inv}}(g, \omega) \quad \text{for } g \in G.$$

The following theorem justifies the term locally analytic kernel function.

4.17. Theorem. *The function $\hat{\theta}(\cdot, \omega)$ is locally analytic everywhere. We have*

$$[\hat{\theta}(\cdot, \omega)] = [\theta(\cdot, \omega)]$$

in $\mathrm{St}_3^{\mathrm{con}}$.

Proof. First note that by construction $\hat{\theta}(\cdot,\omega)$ is continuous. Moreover, by Lemma 4.15 and Proposition 4.14, we have $\mathbb{1}_{\mathcal{U}}(g)\cdot\theta_{\mathrm{inv}}(g,\omega)\in C(G/P_1,\mathbb{C}_K)$ proving the second assertion. To prove the local analyticity, note that by Lemma 4.10, we only need to show that $\hat{\theta}(\cdot,\omega)$ is locally analytic on $\mathcal{U}$. For this purpose, let $g\in\mathcal{U}$. Then, by the Plücker relation in Proposition 2.12, we have

$$\begin{aligned}\hat{\theta}(g,\omega) &= \frac{\beta_1(g)}{\beta_1(g)\omega_2+\beta_2(g)}\left[\frac{\alpha_1(g)}{\alpha_1(g)\omega_1+\alpha_2(g)\omega_2+\alpha_3(g)}-\frac{\beta_1(g)}{\beta_1(g)\omega_1-\beta_3(g)}\right]\\ &= \frac{\beta_1(g)}{\beta_1(g)\omega_2+\beta_2(g)}\left[\frac{-\alpha_1(g)\beta_3(g)-\alpha_3(g)\beta_1(g)-\alpha_2(g)\beta_1(g)\omega_2}{(\alpha_1(g)\omega_1+\alpha_2(g)\omega_2+\alpha_3(g))\cdot(\beta_1(g)\omega_1-\beta_3(g))}\right]\\ &= \frac{\beta_1(g)}{\beta_1(g)\omega_2+\beta_2(g)}\left[\frac{-\alpha_2(g)\beta_2(g)-\alpha_2(g)\beta_1(g)\omega_2}{(\alpha_1(g)\omega_1+\alpha_2(g)\omega_2+\alpha_3(g))\cdot(\beta_1(g)\omega_1-\beta_3(g))}\right]\\ &= \frac{\alpha_2(g)}{\alpha_1(g)\omega_1+\alpha_2(g)\omega_2+\alpha_3(g)}\cdot\frac{-\beta_1(g)}{\beta_1(g)\omega_1-\beta_3(g)}.\end{aligned}$$

Thus, we only need to show that β_1 and β_3 do not simultaneously vanish on $\mathcal{U}$. This can be done explicitly. Recall that in the proof of Lemma 4.15 we have seen that $\mathcal{U}=w_0(\mathcal{I}_{\mathrm{id}}B\cup\mathcal{I}_{s_1}s_1B)/B=D(\mathrm{id})\cup D(s_1)$. Therefore, Table 2 implies that we have $\beta_3(g)\neq 0$ for all $g\in\mathcal{U}$. □

From the formula in the above proof we immediately obtain the following corollary.

4.18. Corollary. *The modified kernel function is explicitly given by*

$$\hat{\theta}(g,\omega)=\begin{cases}\dfrac{\alpha_2(g)}{\alpha_1(g)\omega_1+\alpha_2(g)\omega_2+\alpha_3(g)}\cdot\dfrac{-\beta_1(g)}{\beta_1(g)\omega_1-\beta_3(g)} & \text{for } g\in\mathcal{U},\\ \dfrac{\alpha_1(g)}{\alpha_1(g)\omega_1+\alpha_2(g)\omega_2+\alpha_3(g)}\cdot\dfrac{\beta_1(g)}{\beta_1(g)\omega_2+\beta_2(g)} & \text{for } g\in\mathcal{U}^{\mathrm{c}}.\end{cases}$$

We need the following constructions. For $h\in G$ we let

$$\begin{aligned}E_h(g,\omega)&:=\theta(g,h_*\omega)-\det(h)^{-1}j(h,\omega)^3\theta(hg,\omega)\quad\text{for } g\in G,\\ \hat{E}_h(g,\omega)&:=\hat{\theta}(g,h_*\omega)-\det(h)^{-1}j(h,\omega)^3\hat{\theta}(hg,\omega)\quad\text{for } g\in G,\end{aligned}$$

and

$$E_h^{\mathrm{inf}}(g,\omega):=\hat{E}_h(g,\omega)-E_h(g,\omega)\quad\text{for } g\in G.$$

Then we have $\hat{E}_h(\cdot,\omega)\in C^{\mathrm{an}}(G/B,\mathbb{C}_K)$ and $E_h^{\mathrm{inf}}(\cdot,\omega)\in C(G/P_1,\mathbb{C}_K)$ by Proposition 4.14 and Theorem 4.17. Moreover, the cocycle relation of the factor of automorphy implies that

$$\hat{E}_{h_1h_2}(g,\omega)=\hat{E}_{h_2}(g,(h_1)_*\omega)+\det(h_2)^{-1}j(h_2,(h_1)_*\omega)^3\hat{E}_{h_1}(h_2g,\omega).\tag{4}$$

The same relation holds for the functions $E_h(g,\omega)$ and $E_h^{\mathrm{inf}}(g,\omega)$.

4.19. Lemma. *We have the following.*

(i) $E_h(\cdot,\omega) = 0$ *for* $h \in B^-$.

(ii) $E_{s_i}(\cdot,\omega) \in C(G/P_i, \mathbb{C}_K)$ *for* $i \in \{1,2\}$.

Proof. See [ST97, Proposition 29]. □

The following proposition will be very useful.

4.20. Proposition. *Let* $h \in G$. *We have*

$$\hat{E}_h(\cdot,\omega) \in C^{\mathrm{an}}(G/P_1, \mathbb{C}_K) + C^{\mathrm{an}}(G/P_2, \mathbb{C}_K).$$

Proof. First of all, we observe that by (4) and the Bruhat-Decomposition (with respect to B^-) it suffices to show

$$\hat{E}_h(\cdot,\omega) \in C^{\mathrm{an}}(G/P_1, \mathbb{C}_K) + C^{\mathrm{an}}(G/P_2, \mathbb{C}_K) \quad \text{for } h \in B^- \cup \{s_1, s_2\}.$$

We begin with the case $h \in B^-$. Then by Lemma 4.19 (i) we have

$$\hat{E}_h(\cdot,\omega) = E_h^{\mathrm{inf}}(\cdot,\omega) \in C^{\mathrm{an}}(G/B, \mathbb{C}_K) \cap C(G/P_1, \mathbb{C}_K) = C^{\mathrm{an}}(G/P_1, \mathbb{C}_K).$$

Thus, we are left with the considering $h \in \{s_1, s_2\}$. We observe that by Lemma 4.19 (ii) we have

$$\hat{E}_{s_1}(\cdot,\omega) = E_{s_1}(\cdot,\omega) + E_{s_1}^{\mathrm{inf}}(\cdot,\omega) \in C^{\mathrm{an}}(G/B, \mathbb{C}_K) \cap C(G/P_1, \mathbb{C}_K) = C^{\mathrm{an}}(G/P_1, \mathbb{C}_K).$$

Finally, we need to consider the reflection s_2. For this, we need a finer result than Lemma 4.19 (ii). We begin by computing the function $E_{s_2}(\cdot,\omega)$ explicitly. Let $g \in G$. Then it is easy to verify that

$$[\alpha_1(s_2 g), \alpha_2(s_2 g), \alpha_3(s_2 g)] = [\alpha_1(g), \alpha_3(g), \alpha_2(g)]$$

and

$$[\beta_1(s_2 g), \beta_2(s_2 g), \beta_3(s_2 g)] = [\beta_2(g), \beta_1(g), -\beta_3(g)].$$

Thus, we compute

$$\theta(s_2 g, \omega) = \frac{\alpha_1(g)}{\alpha_1(g)\omega_1 + \alpha_3(g)\omega_2 + \alpha_2(g)} \cdot \frac{\beta_2(g)}{\beta_2(g)\omega_2 + \beta_1(g)}.$$

Moreover, we obtain

$$\begin{aligned}\theta(g, (s_2)_*\omega) &= \frac{\alpha_1(g)}{\alpha_2(g)\frac{\omega_1}{\omega_2} + \alpha_2(g)\frac{1}{\omega_2} + \alpha_3(g)} \cdot \frac{\beta_1(g)}{\beta_1(g)\frac{1}{\omega_2} + \beta_2(g)} \\ &= \frac{\alpha_1(g)\omega_2}{\alpha_1(g)\omega_1 + \alpha_3(g)\omega_2 + \alpha_2(g)} \cdot \frac{\beta_1(g)\omega_2}{\beta_2(g)\omega_2 + \beta_1(g)}.\end{aligned}$$

Upon observing that $\det(s_2) = -1$ and $j(s_2, \omega) = \omega_2$, we can put all of this together to obtain

$$E_{s_2}(g, \omega) = \frac{\alpha_1(g)\omega_2^2}{\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g)}.$$

But since the functions α_i for $i \in \{1, 2, 3\}$ do not have a common zero, this means that in fact we have $E_{s_2}(\cdot, \omega) \in C^{\mathrm{an}}(G/P_2, \mathbb{C}_K)$. It follows that

$$E_{s_2}^{\mathrm{inf}}(\cdot, \omega) = \hat{E}_{s_2}(\cdot, \omega) - E_{s_2}(\cdot, \omega) \in C^{\mathrm{an}}(G/B, \mathbb{C}_K) \cap C(G/P_1, \mathbb{C}_K) = C^{\mathrm{an}}(G/P_1, \mathbb{C}_K).$$

Hence we obtain $\hat{E}_{s_2}(\cdot, \omega) \in C^{\mathrm{an}}(G/P_1, \mathbb{C}_K) + C^{\mathrm{an}}(G/P_2, \mathbb{C}_K)$ which completes the proof. □

4.21. Remark. We should explain our motivation for finding a locally analytic kernel function: First of all, the phenomenon of singularities in the kernel function is not present in the theory for $\mathrm{GL}_2(K)$, where the kernel function is naturally locally analytic everywhere. But more importantly, in [ST02b], it is shown that one can work with the class of the kernel function in the space of so called locally analytic vectors of the continuous Steinberg representation. If $K = \mathbb{Q}_p$, the functor "taking locally analytic vectors" is exact by [ST03, Theorem 7.1]. This means that the locally analytic vectors of the continuous representation are just the locally analytic Steinberg representation. Thus, in this situation we already know that there is locally analytic representative, but we have no explicit description. Even though the functor "taking locally analytic vectors" is not exact in general, see [Sch09], and not even defined for local fields of positive characteristic, it seemed natural that the definition of the kernel function should be independent of the base field, as in the $\mathrm{GL}_2(K)$-case.

4.3. The integration map

In this section, we prove that by integrating the locally analytic kernel function against continuous linear forms on the spaces $\mathrm{St}_3^{\mathrm{an}}(k)$, we obtain elements of the spaces $\mathcal{O}_{\mathcal{X}}(k+3)$. We first do this in the case $k = 0$ and then extend the result to general k by using the representation theory we developed in the previous chapter.

4.3.1. The case $k = 0$

Since our kernel function is locally analytic everywhere, we can work with the locally analytic Steinberg representation instead of the continuous one as in [ST97]. The following theorem is the adaptation of [ST97, Theorem 42] to this situation. Since we do not impose any boundedness conditions, the proof becomes more involved. Our proof is inspired by [DT08, Proof of Proposition 2.2.6]. We write

$$\mathrm{St}_3^{\mathrm{an}}(k)' := \mathrm{Hom}_{\mathrm{cont}}(\mathrm{St}_3^{\mathrm{an}}(k), \mathbb{C}_K).$$

In the proof we use the notation from Appendix A.

4.22. Theorem. *Let $\lambda \in \mathrm{St}_3^{\mathrm{an}}(0)'$. Then the function $f_\lambda \colon \mathcal{X} \to \mathbb{C}_K$ given by*

$$f_\lambda(\omega) = \lambda(g \mapsto \hat{\theta}(g, \omega))$$

is in $\mathcal{O}_\mathcal{X}$. The map $I_0 \colon \mathrm{St}_3^{\mathrm{an}}(0)' \to \mathcal{O}_\mathcal{X}(3)$ given by $\lambda \mapsto f_\lambda$ is G-equivariant.

Proof of Theorem 4.22. The main idea of the proof can be summarized as follows: We need to show that the restriction of f_λ to each $\mathcal{X}_n$ is rigid analytic. If we choose a covering of G/B by compact open balls, we realize that the denominators of the function $\hat{\theta}(g, \omega)$ are essentially just given by hyperplane equations. But since we cut out balls around each hyperplane in the definition of $\mathcal{X}_n$, we can ensure that we can expand into a convergent series if we choose the covering fine enough. The resulting series can then integrated term by term after observing that the restriction of λ to each compact open ball is bounded by [Bos14, Appendix B, Lemma 1].

We make the approach above explicit as follows: Fix $n \geq 0$. Since by Proposition 2.9, the Bruhat-Iwahori cells $D(w)$ from (1) cover G/B disjointly, it suffices to show that $\lambda(\hat{\theta}(g, \omega)\mathbb{1}_{D(w)}(g))$ is rigid analytic on $\mathcal{X}_n$ for all $w \in W$. Moreover, by using the coordinates as in Table 1 for $g \in G$, we can simplify the situation even further. Let $u \in \mathcal{I}_w$ with coordinates (u_1, u_2, u_3) as in Table 1 and consider the polydisc $B_w(u, 2n+1)$. Since we can cover $D(w)$ disjointly by finitely many polydiscs of this form, it suffices to show that $\lambda(\hat{\theta}(g, \omega)\mathbb{1}_{B_w(u,2n+1)}(g))$ is rigid analytic. Observe that, for $g \in B_w(u, 2n+1)$ with coordinates as in Table 1, we have $|x_i - u_i| \leq q^{-2n-1}$ for $i \in \{1, 2, 3\}$.

We first consider the case $w = w_0$. By Table 2 we have

$$\hat{\theta}(g, \omega) = \frac{1}{f_1(g, \omega)} \cdot \frac{1}{f_2(g, \omega)},$$

where $f_1(g, \omega) = \omega_1 + x_1\omega_2 + x_2$ and $f_2(g, \omega) = \omega_2 + x_3$. Then, for $g \in B_{w_0}(u, 2n+1)$ we write

$$f_1(g, \omega) = f_1(u, \omega) + (x_2 - u_2) + (x_1 - u_1)\omega_2.$$

Then, by definition of $\mathcal{X}_n$, we have $|f_1(u, \omega)| \geq q^{-n}$. Moreover, we have $|\omega_2| \leq q^n$. Consequently,

$$|(x_2 - u_2) + (x_1 - u_1)\omega_2| \leq q^{-n-1}$$

uniformly on $B_{w_0}(u, 2n+1)$. Thus, we can expand

$$\frac{1}{f_1(g, \omega)} = \frac{1}{f_1(u, \omega)} \left(1 + \frac{(x_2 - u_2) + (x_1 - u_1)\omega_2}{f_1(u, \omega)}\right)^{-1},$$

as a uniformly convergent power series on $B_{w_0}(u, 2n+1)$. In an analogous way, we can expand $f_2(g, \omega)$ as a uniformly convergent power series on $B_{w_0}(u, 2n+1)$. By multiplying these series expansions, we obtain

$$\hat{\theta}(g, \omega) = \sum_I c_I \frac{\omega_2^{i_1}}{f_1(u, \omega)^{i_1+i_2+1} f_2(u, \omega)^{i_3+1}} (\underline{x} - \underline{u})^I$$

with $c_I \in \mathcal{O}_K$. Now, since the restriction of λ to the Banach space $A_{\mathbb{C}_K}(B_{w_0}(u, 2n+1))$, see Appendix A, is bounded, we find a constant $C > 0$ depending only on u and n such that

$$|\lambda((\underline{x} - \underline{u})^I \mathbb{1}_{B_{w_0}(u,2n+1)})| \leq C \cdot \|(\underline{x} - \underline{u})^I \mathbb{1}_{B_{w_0}(u,2n+1)}\|_{B_{w_0}(u,2n+1)} = Cq^{(-2n-1)|I|}$$

Putting all of this together, we see that this exhibits $\lambda(\hat{\theta}(g,\omega)\mathbb{1}_{B_{w_0}(u,2n+1)}(g))$ as an element of $\mathcal{O}_{X_n}$.

Next we consider the case $w = w_1$. Again, by Table 2, we have

$$\hat{\theta}(g,\omega) = \frac{x_1}{f_1(g,\omega)} \cdot \frac{1}{f_2(g,\omega)},$$

where $f_1(g,\omega) = x_1\omega_1 + \omega_2 + x_2$ and $f_2(g,\omega) = \omega_2 + x_2 - x_1x_3$. As above we can expand

$$\frac{1}{f_1(g,\omega)} = \frac{1}{f_1(u,\omega)}\left(1 + \frac{(x_2-u_2)+(x_1-u_1)\omega_1}{f_1(u,\omega)}\right)^{-1},$$

and

$$\frac{1}{f_2(g,\omega)} = \frac{1}{f_2(u,\omega)}\left(1 + \frac{x_2 - x_1x_3 - (u_2 - u_1u_3)}{f_2(u,\omega)}\right)^{-1}.$$

Consequently, we have

$$\hat{\theta}(g,\omega) = \sum_I c_I \frac{\omega_1^{i_1}}{f_1(u,\omega)^{i_1+i_2+1} f_2(u,\omega)^{i_3+1}} x_1(x_1-u_1)^{i_1}(x_2-u_2)^{i_2}(x_2-x_1x_3-(u_2-u_1u_3))^{i_3}$$

with $c_I \in \mathcal{O}_K$. Now, observe that

$$\begin{aligned}
&|\lambda(x_1(x_1-u_1)^{i_1}(x_2-u_2)^{i_2}(x_2-x_1x_3-(u_2-u_1u_3))^{i_3}\mathbb{1}_{B_{w_1}(u,2n+1)})| \\
&\leq C \cdot \|(x_1(x_1-u_1)^{i_1}(x_2-u_2)^{i_2}(x_2-x_1x_3-(u_2-u_1u_3))^{i_3}\mathbb{1}_{B_{w_1}(u,2n+1)})\|_{B_{w_1}(u,2n+1)} \\
&\leq Cq^{(-2n-1)|I|}.
\end{aligned}$$

Thus, as in the previous case, we obtain an element of $\mathcal{O}_{X_n}$.

The remaining cases can be proved analogously after observing that by Table 2 we have

$$\hat{\theta}(g,\omega) = \begin{cases}
\dfrac{1}{\omega_1 + x_2\omega_2 + x_1} \cdot \dfrac{x_3}{x_3\omega_2 + 1}, & w = w_2, \\
\dfrac{1}{x_2\omega_1 + \omega_2 + x_1} \cdot \dfrac{-x_3}{x_3\omega_1 + 1}, & w = s_1, \\
\dfrac{x_1}{x_1\omega_1 + x_2\omega_2 + 1} \cdot \dfrac{x_2 - x_1x_3}{(x_2 - x_1x_3)\omega_2 + 1}, & w = s_2, \\
\dfrac{x_1}{x_2\omega_1 + x_1\omega_2 + 1} \cdot \dfrac{x_1x_3 - x_2}{(x_2 - x_1x_3)\omega_1 + 1}, & w = \mathrm{id}.
\end{cases}$$

The key observation being that contrary to the kernel function in [ST97], the linear forms appearing in the denominators are always unimodular. We still need to show the G-equivariance. But this follows directly from Proposition 4.20. Let $h \in G$. Then we have

$$\begin{aligned} f_{h\cdot\lambda}(\omega) &= (h\cdot\lambda)(\hat{\theta}(g,\omega)) = \lambda(\hat{\theta}(hg,\omega)) \\ &= \det(h)j(h,\omega)^{-3}\lambda(\hat{\theta}(g,h_*\omega)) = h_*f_\lambda(\omega). \end{aligned}$$

This completes the proof. □

4.23. Remark. Note that it follows from combining the work of Orlik and of Schraen, see [Orl08] and [Sch11] that in general I_0 cannot be an isomorphism. This is a stark contrast to the situation for $\mathrm{GL}_2(K)$, where the analogous map in fact turns out to be an isomorphism, see [DT08, Theorem 2.2.1].

4.3.2. Extension to general weights

The aim of this subsection is to construct, for any $k \geq 0$ with $3 \mid k$, a G-equivariant integration map $I_k\colon \mathrm{St}_3^{\mathrm{an}}(k)' \to \mathcal{O}_{\mathcal{X}}(k+3)$ building on the case $k = 0$ studied in the previous subsection. Our approach is inspired by [Sch11, Section 6]. For this purpose, recall that by Proposition 4.7 we have an explicit G-equivariant continuous surjection $T_k\colon \mathrm{St}_3^{\mathrm{an}}(0) \otimes_{\mathbb{C}_K} \mathcal{P}_k \to \mathrm{St}_3^{\mathrm{an}}(k)$. By duality, we obtain a G-equivariant injection

$$\eta_k\colon \mathrm{St}_3^{\mathrm{an}}(k)' \to \mathrm{St}_3^{\mathrm{an}}(0)' \otimes_{\mathbb{C}_K} V_k.$$

The following lemma is just the dual of the formula in Proposition 4.7.

4.24. Lemma. *Let $\lambda \in \mathrm{St}_3^{\mathrm{an}}(k)'$, $f \in \mathrm{St}_3^{\mathrm{an}}(0)$ and $F \in \mathcal{P}_k$. Then*

$$\eta_k(\lambda)(f\otimes F) = \sum_{I\in\mathcal{J}} \lambda\left(g \mapsto \det(g)^{-2k/3}\beta_3(g)^{i_1}(-\beta_2(g))^{i_2}\beta_1(g)^{i_3}f(g)\right)\cdot v_I(F).$$

Recall the filtration $(F^i(\mathcal{O}_{\mathcal{X}}(3)\otimes_{\mathbb{C}_K} V_k))_{0\leq i\leq k+1}$ on $\mathcal{O}_{\mathcal{X}}(3)\otimes_{\mathbb{C}_K} V_k$ from Section 3.3. The following observation is the central ingredient for the construction of I_k.

4.25. Proposition. *We have an inclusion*

$$((I_0\otimes\mathrm{id})\circ\eta_k)\left(\mathrm{St}_3^{\mathrm{an}}(k)'\right) \subseteq F^k(\mathcal{O}_{\mathcal{X}}(3)\otimes_{\mathbb{C}_K} V_k).$$

Proof. Let $\lambda \in \mathrm{St}_3^{\mathrm{an}}(k)'$. By Lemma 4.24 and the definition of I_0, we have

$$((I_0\otimes\mathrm{id})\circ\eta_k)(\lambda) = \sum_{I\in\mathcal{J}} \lambda\left(g \mapsto \det(g)^{-2k/3}\beta_3(g)^{i_1}(-\beta_2(g))^{i_2}\beta_1(g)^{i_3}\hat{\theta}(g,\omega)\right)\otimes v_I.$$

Now, by Corollary 3.18, we have

$$v_I = \sum_{I\leq J}\binom{j_1}{i_1}\binom{j_2}{i_2}(-\omega_1)^{j_1-i_1}(-\omega_2)^{j_2-i_2}D_J.$$

By simple algebraic manipulations, we arrive at

$$((I_0 \otimes \mathrm{id}) \circ \eta_k)(\lambda) = \sum_{J \in \mathcal{J}} \lambda \left(g \mapsto f_J(g, \omega) \right) D_J,$$

where

$$f_J(g, \omega) := \det(g)^{-2k/3} \beta_1(g)^{j_3} (\beta_3(g) - \beta_1(g)\omega_1)^{j_1} (-\beta_2(g) - \beta_1(g)\omega_2)^{j_2} \hat{\theta}(g, \omega).$$

Thus, we are left with showing that

$$f_J(g, \omega) \in \mathcal{A}_{P_1,k} + \mathcal{A}_{P_2,k} \quad \text{for } J \neq (0, 0, k).$$

Assume first that $J = (1, 0, k-1)$. Then,

$$\begin{aligned} f_J(g, \omega) &= \det(g)^{-2k/3} \beta_1(g)^{k-1} (\beta_3(g) - \beta_1(g)\omega_1) \hat{\theta}(g, \omega) \\ &= \begin{cases} \dfrac{\beta_1(g)^k}{\det(g)^{2k/3}} \dfrac{\alpha_2(g)}{\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g)} & \text{for } g \in \mathcal{U}, \\ \dfrac{\beta_1(g)^k}{\det(g)^{2k/3}} \dfrac{\alpha_1(g)(\beta_3(g) - \beta_1(g)\omega_1)}{(\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g))(\beta_1(g)\omega_2 + \beta_2(g))} & \text{for } g \in \mathcal{U}^{\mathrm{c}}. \end{cases} \end{aligned}$$

Now, using the Plücker relation in Proposition 2.12, we may write

$$f_J(g, \omega) = \det(g)^{-2k/3} \beta_1(g)^k (f_{J,1}(g, \omega) + f_{J,2}(g, \omega)),$$

where

$$f_{J,1}(g, \omega) := \begin{cases} \dfrac{-\beta_1(g)}{\beta_1(g)\omega_2 + \beta_2(g)} & \text{for } g \in \mathcal{U}^{\mathrm{c}}, \\ 0 & \text{for } g \in \mathcal{U}, \end{cases}$$

and

$$f_{J,2}(g, \omega) := \frac{\alpha_2(g)}{\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g)}.$$

Again, by Proposition 2.12 and since $\mathcal{U}$ is right P_1-stable, we see that $f_{J,i}(g, \omega) \in \mathcal{A}_{P_i,0}$ for $i \in \{1, 2\}$. Now consider the commutative diagram from Proposition 4.7:

$$\begin{tikzcd} (\mathcal{A}_{P_1,0} + \mathcal{A}_{P_2,0}) \otimes_{\mathbb{C}_K} \mathcal{P}_k \arrow[r] \arrow[d] & \mathcal{A}_{P_1,k} + \mathcal{A}_{P_2,k} \arrow[d] \\ \mathcal{A}_0 \otimes_{\mathbb{C}_K} \mathcal{P}_k \arrow[r] & \mathcal{A}_k \end{tikzcd}$$

Our computation shows that $f_J(g, \omega)$ is in the image of $(\mathcal{A}_{P_1,0} + \mathcal{A}_{P_2,0}) \otimes_{\mathbb{C}_K} \mathcal{P}_k$, hence by commutativity in $\mathcal{A}_{P_1,k} + \mathcal{A}_{P_2,k}$.

Assume now that $J = (0, 1, k-1)$. Then,

$$f_J(g,\omega) = \det(g)^{-2k/3}\beta_1(g)^{k-1}(-\beta_2(g) - \beta_1(g)\omega_2)\hat{\theta}(g,\omega)$$
$$= \begin{cases} \dfrac{\beta_1(g)^k}{\det(g)^{2k/3}} \dfrac{\alpha_2(g)(\beta_2(g) + \beta_1(g)\omega_1)}{(\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g))(\beta_1(g)\omega_1 - \beta_3(g))} & \text{for } g \in \mathcal{U}, \\ \dfrac{\beta_1(g)^k}{\det(g)^{2k/3}} \dfrac{-\alpha_1(g)}{\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g)} & \text{for } g \in \mathcal{U}^c. \end{cases}$$

As above, by using the Plücker relation in Proposition 2.12, we may write

$$f_J(g,\omega) = \det(g)^{-2k/3}\beta_1(g)^k(f_{J,1}(g,\omega) + f_{J,2}(g,\omega)),$$

where

$$f_{J,1}(g,\omega) := \begin{cases} \dfrac{\beta_1(g)}{\beta_1(g)\omega_1 - \beta_3(g)} & \text{for } g \in \mathcal{U}, \\ 0 & \text{for } g \in \mathcal{U}^c, \end{cases}$$

and

$$f_{J,2}(g,\omega) := \frac{-\alpha_1(g)}{\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g)}.$$

We see again that $f_{J,i}(g,\omega) \in \mathcal{A}_{P_i,0}$ for $i \in \{1,2\}$ and we conclude exactly as in the first case.

Now, for the general case, let $J \neq (0,0,k)$. Then, we have $(1,0,k-1) \leq J$ or $(0,1,k-1) \leq J$ with respect to the partial order from Definition 3.15. For simplicity, assume that we are in the first case. We have

$$f_J(g,\omega) = \beta_1(g)^{j_3-k+1}(\beta_3(g) - \beta_1(g)\omega_1)^{j_1-1}(-\beta_2(g) - \beta_1(g)\omega_2)^{j_2} f_{(1,0,k-1)}(g,\omega)$$

and thus again by invoking the above commutative diagram, $f_J(g,\omega) \in \mathcal{A}_{P_1,k} + \mathcal{A}_{P_2,k}$. This completes the proof. □

Consequently, by invoking the translation map constructed in Proposition 3.19, we may set

$$I_k := t_{k,3}^{-1} \circ (I_0 \otimes \mathrm{id}) \circ \eta_k \colon \mathrm{St}_3^{\mathrm{an}}(k)' \to \mathcal{O}_{\mathcal{X}}(k+3).$$

The following theorem is now an easy consequence.

4.26. Theorem. *The map $I_k \colon \mathrm{St}_3^{\mathrm{an}}(k)' \to \mathcal{O}_{\mathcal{X}}(k+3)$ is G-equivariant. Explicitly, it is given by*

$$I_k(\lambda)(\omega) = \lambda\left(g \mapsto \det(g)^{-2k/3}\beta_1(g)^k\hat{\theta}(g,\omega)\right).$$

Proof. The G-equivariance is a consequence of Proposition 3.19. The explicit description can be read off in the proof of Proposition 4.25. □

4.27. Remark. Note that the analogous theorem in the $\mathrm{GL}_2(K)$-case can be easily proved by a direct computation, see for example [DT08, Theorem 2.2.1]. The key difference is that in this case the locally analytic Steinberg representation $\mathrm{St}_2^{\mathrm{an}}(k)$ is much better understood: It can be described explicitly as locally analytic functions on $\mathbb{P}^1(K)$ with a pole of order at most k at ∞ modulo global polynomial functions of degree less or equal to k. This is also the reason why, contrary to [DT08, Theorem 2.2.1], a scaling factor depending on k appears in our formula: We work directly with the locally analytic induction. If we were to pull back to the opposite big cell via an explicit embedding as in [DT08, Section 2.1.2] to obtain functions on G/B with certain singularities, this factor would disappear. Describing the functions one obtains this way explicitly for the spaces $\mathcal{A}_{P_i,k}$ for $i \in \{1,2\}$ has proven to be very complicated, which is why we developed a different approach.

4.28. Remark. We should remark that in fact the constructions of I_0 and I_k for $k > 0$ should descent to K, i.e., one should be able to construct analogous maps between locally analytic Steinberg representations over K and holomorphic discrete series representations over K. In order to do this, one should use the integration map constructed in [ST02a, Theorem 2.2], see also [Eme17, Proposition 2.2.10]. As in the proof of Theorem 4.22 one can show that the map $g \mapsto [\omega \mapsto \hat{\theta}(g,\omega)]$ defines an element of $C^{\mathrm{an}}(G/B, \mathcal{O}_{\mathcal{X}})$ whenever $\mathcal{O}_{\mathcal{X}}$ is endowed with the topology coming from a *fixed* $\mathcal{X}_n$. These maps should glue (for varying n) and give rise to an integration map I_0, defined over K. However, showing that this map is G-equivariant when restricted to the dual of the Steinberg representation and the extension procedure to obtain I_k require further developments. In particular, various compatibilities need to be established.

4.4. The main theorem

In this section, we prove our main theorem, which states that the residue map is surjective on functions with bounded residues and has a G-equivariant right inverse. It relies on a conjecture regarding the existence of an extension of the distribution attached to a bounded harmonic cocycle. In the next chapter, we will see how this conjecture can be interpreted as a non-criticality statement for certain automorphic forms.

4.29. Conjecture. *For each $c \in C^b_{\mathrm{har}}(\mathcal{T}, k)$ there exists $\lambda_c \in \mathrm{St}_3^{\mathrm{an}}(k)'$ with the following properties:*

(i) $\lambda_c([\mathbb{1}_{U(\sigma)}] \otimes F) = c(\sigma)(F)$ for all $F \in \mathcal{P}_k$, $\sigma \in \widehat{\mathcal{T}_2}$, where we regard $[\mathbb{1}_{U(\sigma)}] \otimes F$ as an element of $\mathrm{St}_3^{\mathrm{an}}(k)$ via the map in Proposition 4.7.

(ii) The map $C^b_{\mathrm{har}}(\mathcal{T}, k) \to \mathrm{St}_3^{\mathrm{an}}(k)'$ given by $c \mapsto \lambda_c$ is $\mathbb{C}_K$-linear and G-equivariant.

4.30. Remark. The above conjecture should be viewed as an analogue of a theorem of Amice-Velu-Vishik, see for example [DT08, Theorem 2.3.2] for an analogous statement in the $\mathrm{GL}_2(K)$-case. Note however that we do not require uniqueness here. Moreover, we do not require a strong estimate as in [DT08, Theorem 2.3.2 (3)]. We will see in the next chapter that requiring a similar estimate is very natural and in fact makes the distribution unique. But we do not need this to prove the main theorem. The stronger version including uniqueness will be stated later in Conjecture 5.49.

4.31. Theorem (Main theorem). *Assume that Conjecture 4.29 holds and let $c \in C^b_{\mathrm{har}}(\mathcal{T}, k)$. Then we have*

$$\mathrm{Res}_k(I_k(\lambda_c)) = c.$$

Consequently, the residue map $\mathrm{Res}_k \colon \mathcal{O}_X(k+3)^b \to C^b_{\mathrm{har}}(\mathcal{T}, k)$ is surjective.

In order to prove Theorem 4.31, we need the following proposition, whose proof is a lengthy and technical computation. An analogous statement for $k = 0$ can be found in [ST97, Lemma 49].

4.32. Proposition. *Let $\lambda \in \mathrm{St}^{\mathrm{an}}_3(k)'$. Then we have*

$$\mathrm{Res}_k(I_k(\lambda))(\sigma_0)(F) = \lambda([\mathbb{1}_{U(\sigma_0)}] \otimes F)$$

for all $F \in \mathcal{P}_k$.

Proof. First off, we can assume that $F = \underline{X}^I$, where $i_1 + i_2 + i_3 = k$. Secondly, we may write

$$\mathrm{Res}_k(I_k(\lambda))(\sigma_0)(F) = \mathrm{res}_{\sigma_0}(\omega_1^{i_1}\omega_2^{i_2} I_k(\lambda)(\omega)\mathrm{d}\omega).$$

Now note that by Theorem 4.26, we may write

$$\begin{aligned} I_k(\lambda)(\omega) &= \lambda\left(\det(g)^{-2k/3}\beta_1(g)^k\hat{\theta}(g,\omega)\right) \\ &= \sum_{w \in W} \lambda\left(\det(g)^{-2k/3}\beta_1(g)^k\hat{\theta}(g,\omega)\mathbb{1}_{w_0\mathcal{I}_w wB}(g)\right), \end{aligned}$$

as in the proof of Theorem 4.22. By Proposition A.17, we have an isomorphism of topological vector spaces $\mathcal{A}_k \cong C^{\mathrm{an}}(G/B, \mathbb{C}_K)$ induced by the splitting

$$\iota \colon G/B = \bigsqcup_{w \in W} D(w) \to G$$

given by picking the unique representatives $g = w_0 u w$ with $u \in \mathcal{I}_w$ in each cell, see (1). By composition with the natural surjection $\mathcal{A}_k \to \mathrm{St}^{\mathrm{an}}_3(k)$ and duality, we obtain an injective map

$$\mathrm{St}^{\mathrm{an}}_3(k)' \to C^{\mathrm{an}}(G/B, \mathbb{C}_K)'.$$

This enables us to regard λ as an element of the space on the right hand side, which by abuse of notation we also denote by λ. Under this identification, each summand in the above sum becomes

$$S_w(\omega) := \lambda\left(\beta_1(g)^k\hat{\theta}(g,\omega)\mathbb{1}_{w_0\mathcal{I}_w w}(g)\right)$$

At this point we are in a similar situation as in the proof of Theorem 4.22. We need to find a covering of $w_0\mathcal{I}_w w$ such that we can expand $S_w(\omega)$ into a convergent Laurent series on R_0. Then we can read off the residues. Note that this is simpler than the series expansion in Theorem 4.22, where we needed convergence on $\mathcal{X}_n$ not just on R_0. As in the proof of Theorem 4.22, we will describe the covering explicitly, but will only compute the series expansion in a few cases. The remaining cases can be computed analogously. We use the coordinates $\underline{x} = (x_1, x_2, x_3)$ from Table 1. The coverings are then given in Table 3.

$w \in W$	Covering $\{U_i(w)\}_i$ of $w_0\mathcal{I}_w w$
w_0	$U_1(w_0) = \left\{|x_1| \le q^{-1}, |x_2| \le q^{-1}, |x_3| \le q^{-1}\right\}$
	$U_2(w_0) = \left\{|x_1| \le q^{-1}, |x_2| \le q^{-1}, |x_3| = 1\right\}$
	$U_3(w_0) = \left\{|x_1| = 1, |x_2| \le q^{-1}, |x_3| \le q^{-1}\right\}$
	$U_4(w_0) = \left\{|x_1| = 1, |x_2| \le q^{-1}, |x_3| = 1\right\}$
	$U_5(w_0) = \left\{|x_1| \le 1, |x_2| = 1, |x_3| \le q^{-1}\right\}$
	$U_6(w_0) = \left\{|x_1| \le 1, |x_2| = 1, |x_3| = 1\right\}$
w_1	$U_1(w_1) = \left\{|x_1| \le q^{-1}, |x_2| \le q^{-1}, |x_3| \le 1\right\}$
	$U_2(w_1) = \left\{|x_1| \le q^{-1}, |x_2| = 1, |x_3| \le 1\right\}$
w_2	$U_1(w_2) = \left\{|x_1| \le q^{-1}, |x_2| \le q^{-1}, |x_3| \le q^{-1}\right\}$
	$U_2(w_2) = \left\{|x_1| = 1, |x_2| \le 1, |x_3| \le q^{-1}\right\}$
	$U_3(w_2) = \left\{|x_1| \le q^{-1}, |x_2| = 1, |x_3| \le q^{-1}\right\}$
s_1	$U_1(s_1) = \left\{|x_1| \le q^{-1}, |x_2| \le q^{-1}, |x_3| \le q^{-1}\right\}$
	$U_2(s_1) = \left\{|x_1| = 1, |x_2| \le q^{-1}, |x_3| \le q^{-1}\right\}$
s_2	$U_1(s_2) = \left\{|x_1| \le q^{-1}, |x_2| \le q^{-1}, |x_3| \le 1\right\}$
id	$U_1(\mathrm{id}) = \left\{|x_1| \le q^{-1}, |x_2| \le q^{-1}, |x_3| \le q^{-1}\right\}$

Table 3. Explicit coverings

We first study the covering of $w_0\mathcal{I}_{w_1}w_1$. Note that by Table 2, we have

$$\hat{\theta}(g,\omega) = \frac{x_1}{f_1(g,\omega)} \cdot \frac{1}{f_2(g,\omega)},$$

where $f_1(g,\omega) = x_1\omega_1 + \omega_2 + x_2$ and $f_2(g,\omega) = \omega_2 + x_2 - x_1x_3$.

(i) On $U_1(w_1)$ we may write

$$\frac{x_1}{f_1(g,\omega)} = \sum_{n\ge 0}(-1)^n x_1(x_1\omega_1 + x_2)^n \omega_2^{-n-1} = \sum_{n\ge 0}(-1)^n \sum_{j=0}^{n} \binom{n}{j} x_1^{j+1} x_2^{n-j} \omega_1^j \omega_2^{-n-1},$$

which converges on R_0 as $|x_1\omega_1 + x_2| < |\omega_2|$. Similarly, we have

$$\frac{1}{f_2(g,\omega)} = \sum_{m\ge 0}(x_1x_3 - x_2)^m \omega_2^{-m-1},$$

which converges on R_0 as $|x_1x_3 - x_2| < |\omega_2|$. Thus, after combining all factors and applying $\lambda(\cdot)$, we obtain

$$\begin{aligned}
&\lambda\left(\beta_1(g)^k\hat{\theta}(g,\omega)\mathbb{1}_{U_1(w_1)}(g)\right)\\
&\quad=\sum_{n\geq 0}\sum_{m\geq 0}(-1)^{n+k}\sum_{j=0}^{n}\binom{n}{j}\lambda(x_1^{j+1}x_2^{n-j}(x_1x_3-x_2)^m)\omega_1^j\omega_2^{-n-m-2}\\
&\quad=\sum_{\substack{m_1\geq 0\\ m_2\leq -m_1-2}}\left(\sum_{\substack{n\geq m_1\\ n\leq -m_2-2}}(-1)^{n+k}\binom{n}{m_1}\lambda((x_1^{m_1+1}x_2^{n-m_1}(x_1x_3-x_2)^{-n-2-m_2})\right)\omega_1^{m_1}\omega_2^{m_2}.
\end{aligned}$$

If we denote the coefficients of the above series expansion by $a(m_1, m_2)$, we see that $|a(m_1, m_2)| \leq Cq^{m_2+1}$ for some constant $C > 0$ by the continuity of λ as in the proof of Theorem 4.22. Thus, we have

$$|a(m_1,m_2)|q^{-\ell_N(m_1,m_2)} \leq Cq^{m_2+1-(1-1/N)m_1-(1-2/N)m_2}$$

with the notation as in Lemma 3.21. But since

$$m_2 + 1 - (1-1/N)m_1 - (1-2/N)m_2 \leq 1 - 2/N(|m_1| + |m_2|)$$

for $N \geq 3$, the above expansion converges and defines an element of $\mathcal{O}_{R_0}$.

(ii) On $U_2(w_1)$ we may write

$$\frac{x_1}{f_1(g,\omega)} = \sum_{n\geq 0}(-1)^n x_1(x_1\omega_1+\omega_2)^n x_2^{-n-1} = \sum_{n\geq 0}(-1)^n\sum_{j=0}^{n}x_1^{j+1}x_2^{-n-1}\omega_1^j\omega_2^{n-j},$$

which converges on R_0 as $|x_1\omega_1 + \omega_2| < |x_2|$. Similarly, we have

$$\frac{1}{f_2(g,\omega)} = \sum_{m\geq 0}\omega_2^m(x_1x_3-x_2)^{-m-1},$$

which converges on R_0 as $|\omega_2| < |x_1x_3 - x_2|$. Thus, after combining all factors and applying $\lambda(\cdot)$, we obtain

$$\begin{aligned}
&\lambda\left(\beta_1(g)^k\hat{\theta}(g,\omega)\mathbb{1}_{U_2(w_1)}(g)\right)\\
&\quad=\sum_{n\geq 0}\sum_{m\geq 0}(-1)^{n+k}\sum_{j=0}^{n}\binom{n}{j}\lambda(x_1^{j+1}x_2^{-n-1}(x_1x_3-x_2)^{-m-1})\omega_1^j\omega_2^{m+n-j}\\
&\quad=\sum_{\substack{m_1\geq 0\\ m_2\geq 0}}\left(\sum_{\substack{n\geq m_1\\ n\leq m_1+m_2}}(-1)^{n+k}\lambda(x_1^{m_1+1}x_2^{-n-1}(x_1x_3-x_2)^{-m_2-m_1-1+n})\right)\omega_1^{m_1}\omega_2^{m_2}.
\end{aligned}$$

If we denote the coefficients of the above series expansion by $a(m_1, m_2)$, we see that $|a(m_1, m_2)| \leq Cq^{-m_1-1}$ for some constant $C > 0$ by the continuity of λ. Thus, we have

$$|a(m_1, m_2)|q^{-\ell_N(m_1,m_2)} \leq Cq^{-(m_1+1+2m_1/N+m_2/N)}$$

with the notation as in Lemma 3.21. But since

$$-(m_1 + 1 + 2m_1/N + m_2/N) \leq -1/N(|m_1| + |m_2|)$$

for $N \geq 3$, the above expansion converges and defines an element of $\mathcal{O}_{R_0}$.

Now, we observe that none of the above series expansions involve terms of the form $\omega_1^{n_1}\omega_2^{n_2}$ with $n_1 < 0$ and $n_2 < 0$. By definition of the residue, see Definition 3.22, this means that

$$\mathrm{res}_{\sigma_0}(\omega_1^{i_1}\omega_2^{i_2}S_{w_1}(\omega)\mathrm{d}\omega) = 0.$$

This completes the computation for $w_0\mathcal{I}_{w_1}w_1$. Now, in all other cases, except $w = w_0$, a similar computation yields

$$\mathrm{res}_{\sigma_0}(\omega_1^{i_1}\omega_2^{i_2}S_w(\omega)\mathrm{d}\omega) = 0.$$

Now we consider the case $w = w_0$. The interesting part is the first open ball in the covering, $U_1(w_0)$, which by definition corresponds precisely to $U(\sigma_0)$. By Table 2 we have

$$\hat{\theta}(g, \omega) = \frac{1}{f_1(g, \omega)} \cdot \frac{1}{f_2(g, \omega)},$$

where $f_1(g, \omega) = \omega_1 + x_1\omega_2 + x_2$ and $f_2(g, \omega) = \omega_2 + x_3$. We may write

$$\frac{1}{f_1(g, \omega)} = \sum_{n\geq 0}(-1)^n x_1(x_1\omega_2 + x_2)^n \omega_1^{-n-1} = \sum_{n\geq 0}(-1)^n \sum_{j=0}^{n}\binom{n}{j}x_1^j x_2^{n-j}\omega_1^{-n-1}\omega_2^j,$$

which converges on R_0 as $|x_1\omega_2 + x_2| < |\omega_1|$. Similarly, we have

$$\frac{1}{f_2(g, \omega)} = \sum_{m\geq 0}(-1)^m x_3^m \omega_2^{-m-1},$$

which converges on R_0 as $|x_3| < |\omega_2|$. Thus, after combining the all factors and applying $\lambda(\cdot)$, we obtain

$$\begin{aligned}
&\lambda\left(\beta_1(g)^k\hat{\theta}(g, \omega)\mathbb{1}_{U_1(w_0)}(g)\right)\\
&\quad= \sum_{n\geq 0}\sum_{m\geq 0}(-1)^{n+m}\sum_{j=0}^{n}\binom{n}{j}\lambda(x_1^j x_2^{n-j}x_3^m)\omega_1^{-n-1}\omega_2^{-m-1+j}\\
&\quad= \sum_{\substack{m_1\leq -1\\ m_2\leq -m_1-2}}\left(\sum_{\substack{n\geq 0\\ n\geq m_2+1\\ n\leq -m_1-1}}(-1)^{m_1+m_2-n}\binom{-m_1-1}{n}\lambda(x_1^n x_2^{-m_1-1-n}x_3^{n-1-m_2})\right)\omega_1^{m_1}\omega_2^{m_2}.
\end{aligned}$$

Denote the coefficients of the above series expansion by $a(m_1, m_2)$. We consider two separete cases. Assume first that $m_2 \leq -1$. Then we see that $|a(m_1, m_2))| \leq Cq^{m_1+m_2+2}$ for some constant $C > 0$ by the continuity of λ as in the proof of Theorem 4.22. Thus, we have

$$|a(m_1, m_2)|q^{-\ell_N(m_1,m_2)} \leq Cq^{m_1+m_2+2-(1-1/N)m_1-(1-2/N)m_2}$$

with the notation as in Lemma 3.21. Assume now that $m_2 \geq 0$. Then we obtain $|a(m_1, m_2)| \leq Cq^{m_1+1}$ and we see that

$$|a(m_1, m_2)|q^{-\ell_N(m_1,m_2)} \leq Cq^{m_1+1-(1-1/N)m_1-(1/N)m_2}.$$

Combining both cases, we obtain

$$|a(m_1, m_2)|q^{-\ell_N(m_1,m_2)} \leq Cq^{2-1/N(|m_1|+|m_2|)}$$

which tends to 0 for all $N \geq 3$. Consequently, the above expansion defines an element of $\mathcal{O}_{R_0}$. Now, by analogous computations to the above cases, or, alternatively since these computations happen inside the (opposite) big cell, as in [ST97, Lemma 49], we obtain that

$$\begin{aligned}\mathrm{res}_{\sigma_0}(\omega_1^{i_1}\omega_2^{i_2}S_{w_0}(\omega)\mathrm{d}\omega) &= \lambda\left(\sum_{n=0}^{i_1}(-1)^{i_1+i_2-n}\binom{i_1}{n}x_1^n x_2^{i_1-n}x_3^{i_2+n}\mathbb{1}_{U(\sigma_0)}\right)\\ &= \lambda\left((-x_2+x_1x_3)^{i_1}(-x_3)^{i_2}\mathbb{1}_{U(\sigma_0)}\right),\end{aligned}$$

where we just read off the residue in the above series expansion. Combining all of the above yields

$$\mathrm{Res}_k(I_k(\lambda))(\sigma_0)(F) = \lambda([\mathbb{1}_{U(\sigma_0)}] \otimes F)$$

via the map in Proposition 4.7, which completes the proof. $\square$

Now we can prove Theorem 4.31

Proof of Theorem 4.31. Let $\sigma \in \widehat{\mathcal{T}_2}$ and $F \in \mathcal{P}_k$. Then we find $g \in G$ such that $g\sigma = \sigma_0$. By Proposition 4.32 we have

$$\begin{aligned}\mathrm{Res}_k(I_k(\lambda_c))(\sigma)(F) &= \mathrm{Res}_k(I_k(\lambda_c))(g^{-1}\sigma_0)(g_*^{-1}g_*F)\\ &= (g\cdot\mathrm{Res}_k(I_k(\lambda_c)))(\sigma_0)(g_*F)\\ &= \mathrm{Res}_k(g_*(I_k(\lambda_c)))(\sigma_0)(g_*F)\\ &= \mathrm{Res}_k(I_k(g\cdot\lambda_c))(\sigma_0)(g_*F)\\ &= \mathrm{Res}_k(I_k(\lambda_{g\cdot c}))(\sigma_0)(g_*F)\\ &= (g\cdot c)(\sigma_0)(g_*F) = c(\sigma)(F)\end{aligned}$$

by applying Conjecture 4.29 and the G-equivariance of Res_k and I_k, see Subsection 3.4.2 and Theorem 4.26. $\square$

4.33. Remark. We should clarify why our main theorem looks different than for example [BdS16, Theorem 2.2] where (for $k = 0$) the space of rigid analytic functions with bounded residues of weight $k+3$ is *isomorphic* to $C^b_{\mathrm{har}}(\mathcal{T}, k)$ under the residue map. The point is that the space $\mathcal{O}_\mathcal{X}(k+3)^b$ is defined differently in [IS01, Definition 4.6]. In our notation, the analogue would be

$$\mathcal{O}_\mathcal{X}(k+3)^b_{\mathrm{IS}} \coloneqq \left\{ I_k(\lambda_c) \;\middle|\; c \in C^b_{\mathrm{har}}(\mathcal{T}, k) \right\} \subseteq \mathcal{O}_\mathcal{X}(k+3)^b.$$

By Conjecture 4.29 (ii), this is a G-invariant $\mathbb{C}_K$-subvector space of $\mathcal{O}_\mathcal{X}(k+3)^b$. It follows directly from Theorem 4.31 that Res_k induces an isomorphism

$$\mathcal{O}_\mathcal{X}(k+3)^b_{\mathrm{IS}} \to C^b_{\mathrm{har}}(\mathcal{T}, k).$$

We chose to work with the space $\mathcal{O}_\mathcal{X}(k+3)^b$ instead as this space can be defined without assuming Conjecture 4.29.

We will apply the above theorem in Part II of this thesis to realize certain spaces of Drinfeld cusp forms of rank 3 via harmonic cocycles of the type above.

CHAPTER 5

Overconvergent automorphic forms and distributions

The aim of this chapter is to lay the foundations for studying extensions of distributions as in Conjecture 4.29 from a more conceptual point of view, namely as classicality statements for certain overconvergent automorphic forms. This is inspired by the ideas in [FM14] and [Grä19], where these types of automorphic forms are used to explicitly compute values of these distributions when their existence (and uniqueness) is already established. Our aim is to go in the opposite direction: We show that by lifting certain automorphic forms, we can construct the needed distributions. This reformulation sheds some light on the complexity of Conjecture 4.29, as the question of liftability is quite delicate. We also feel that this is the correct setting to study Conjecture 4.29 in. Most of the notation and framework in this chapter is new and tailored to specifically to address questions such as Conjecture 4.29, even in more generality. Throughout this chapter, L denotes an arbitrary complete extension of K in $\mathbb{C}_K$. Its ring of integers is denoted by $\mathcal{O}_L$.

5.1. Coefficient modules

We begin by introducing the coefficients for our automorphic forms. Roughly speaking, these are local building blocks of the representations studied in Section 4.1, similarly to the $\mathrm{GL}_2(K)$-case, where this role is played by the Tate algebra, see [Gre06]. We define a monoid $\Sigma' \subseteq G$ by

$$\Sigma' \coloneqq \left\{ g \in G \,\middle|\, g^{-1}U(\sigma_0) \subseteq U(\sigma_0) \right\}.$$

Then, since $\mathrm{Stab}_G(\sigma_0) = K^\times \mathcal{I}$, we have $K^\times \mathcal{I} \subseteq \Sigma'$. Moreover, we let

$$\Sigma'_i \coloneqq \left\{ g \in G \,\middle|\, g^{-1}U_i(\sigma_0) \subseteq U_i(\sigma_0) \right\}$$

for $i \in \{1, 2\}$. Then Σ'_i is again a monoid. By definition, we have $K^\times \mathcal{I}_i \subseteq \Sigma'_i$.

5.1. Lemma. *For $i \in \{1, 2\}$ we have $\Sigma' \subseteq \Sigma'_i$.*

Proof. Let $g \in \Sigma'$, then by definition $g^{-1}U(\sigma_0) \subseteq U(\sigma_0)$, i.e., $g^{-1}\mathcal{I}B \subseteq \mathcal{I}B$. Thus, $g^{-1}\mathcal{I}P_i \subseteq \mathcal{I}P_i$ and by Lemma 4.15, $g^{-1}U_i(\sigma_0) \subseteq U_i(\sigma_0)$ □

By Proposition 2.16, we have $y_i^{-1} \in \Sigma'$ for $i \in \{1,2\}$. The following monoids play a central role in this chapter.

5.2. Definition.

(i) We denote by Σ be the submonoid of Σ' generated by y_1^{-1}, y_2^{-1} and $\mathcal{I}$.

(ii) We denote by Σ_i be the submonoid of Σ_i' generated by y_1^{-1}, y_2^{-1} and $\mathcal{I}_i$ for $i \in \{1,2\}$.

By definition, we have $\Sigma \subseteq \Sigma_i$ for $i \in \{1,2\}$.

5.3. Definition.

(i) A *coefficient module* is an L-vector space endowed with a right-action by Σ.

(ii) Let $i \in \{1,2\}$. A *P_i-admissible coefficient module* is an L-vector space endowed with a right-action by Σ_i.

We also need to consider integral structures on our coefficient modules which are defined as follows.

5.4. Definition. Let M be a coefficient module. An *integral structure* on M is a tuple $(M^{\text{int}}, (n_i)_{i\in\{1,2\}})$, where M^{int} is an $\mathcal{O}_L$-submodule of M and $(n_i)_{i\in\{1,2\}}$ is a tuple of elements of $\mathbb{Z}_{\geq 0}$, such that the following conditions are satisfied:

(i) $M^{\text{int}} \otimes_{\mathcal{O}_L} L = M$.

(ii) M^{int} is $\mathcal{I}$-stable.

(iii) For $i \in \{1,2\}$ we have

$$\pi^{n_i}(m \cdot y_i^{-1}) \in M^{\text{int}} \quad \text{for all } m \in M^{\text{int}}.$$

5.5. Remark. Let $(M^{\text{int}}, (n_i)_{i\in\{1,2\}})$ be a tuple that satisfies conditions (ii) and (iii) in the above definition. Then $M^{\text{int}} \otimes_{\mathcal{O}_L} L$ becomes a coefficient module by extending the $\mathcal{I}$-action in (ii) naturally and setting

$$(m \otimes x) \cdot y_i^{-1} := \pi^{n_i}(m \cdot y_i^{-1}) \otimes (x\pi^{-n_i}) \quad \text{for } m \in M^{\text{int}}, x \in L,$$

which is well-defined by (iii).

We also need a stronger version of an integral structure on P_i-admissible coefficient modules.

5.6. Definition. Let M be a P_i-admissible coefficient module. A *P_i-admissible integral structure* on M is a triple $(M^{\text{int}}, (n_j)_{j\in\{1,2\}}, m_i)$, where M^{int} is an $\mathcal{O}_L$-submodule of M, $(n_j)_{j\in\{1,2\}}$ is a tuple of elements of $\mathbb{Z}_{\geq 0}$ and m_i is another element of $\mathbb{Z}_{\geq 0}$, such that the following conditions are satisfied:

(i) $(M^{\text{int}}, (n_j)_{j\in\{1,2\}})$ is an integral structure on M.

(ii) We have
$$\pi^{m_i}(m \cdot s_i) \in M^{\mathrm{int}} \quad \text{for all } m \in M^{\mathrm{int}}.$$

Recall that the space $\mathcal{P}_k$ carries a left G-action. Then the dual V_k naturally becomes a coefficient module by taking the dual right action. It is P_i-admissible for $i = 1, 2$. We also obtain an integral structure on V_k by considering the O_L-module

$$V_k^{\mathrm{int}} = \left\{ v \in V_k \;\middle|\; v(\underline{X}^I) \in \pi^{i_1+i_2} O_L \text{ for all } I \right\}.$$

as in Section 3.1. Because V_k is finite-dimensional, condition (i) in Definition 5.4 is trivially satisfied. The optimal constants n_i in condition (iii) can be explicitly computed as follows. We have

$$(v \cdot y_i^{-1})(\underline{X}^I) = v((y_i^{-1})_* \underline{X}^I) = \begin{cases} \pi^{k/3-i_3} v(\underline{X}^I), & \text{for } i = 1, \\ \pi^{2k/3-i_2-i_3} v(\underline{X}^I), & \text{for } i = 2, \end{cases}$$

for all $v \in V$. Thus, we see that $n_1 = 2k/3$ and $n_2 = k/3$ satisfy (iii). Moreover, V_k^{int} can be turned into a P_i-admissible integral structure for $i \in \{1, 2\}$. We compute the optimal constants via

$$(v \cdot s_i)(\underline{X}^I) = v((s_i)_* \underline{X}^I) = \begin{cases} v(X_1^{i_2} X_2^{i_1} X_3^{i_3}), & \text{for } i = 1, \\ v(X_1^{i_1} X_2^{i_3} X_3^{i_2}), & \text{for } i = 2, \end{cases}$$

which shows that we can choose $m_1 = 0$ and $m_2 = k$.

The other coefficient modules we are primarily interested in are constructed from the G-representations $\mathcal{A}_k$ and $\mathcal{A}_{P_i,k}$ for $i \in \{1, 2\}$, which were introduced in Section 4.1, see (2) and (3).

5.7. Definition. We set

$$\mathcal{A}_k(\sigma_0) := \{ f \in \mathcal{A}_k \mid \operatorname{supp}(f) \subseteq U(\sigma_0) \},$$
$$\mathcal{A}_{P_i,k}(\sigma_0) := \{ f \in \mathcal{A}_{P_i,k} \mid \operatorname{supp}(f) \subseteq U_i(\sigma_0) \},$$

for $i \in \{1, 2\}$.

These spaces are closed subspaces of $\mathcal{A}_k$ respectively $\mathcal{A}_{P_i,k}$, see [Fea99, Subsection 2.3.1].

5.8. Remark. We should remark that the B-equivariance of elements of $\mathcal{A}_k$ implies that the notion of support is well-defined on the quotient G/B. The set $U(\sigma_0) = \mathcal{I}B/B$ is open and closed in G/B. Hence the functions in $\mathcal{A}_k(\sigma_0)$ are just the elements of $\mathcal{A}_k$ that vanish outside $\mathcal{I}B$. Similarly, the functions in $\mathcal{A}_{P_i,k}(\sigma_0)$ are the elements of $\mathcal{A}_{P_i,k}$ that vanish outside $\mathcal{I}_i B = \mathcal{I} P_i$.

We often regard elements of both $\mathcal{A}_k(\sigma_0)$ and $\mathcal{A}_{P_i,k}(\sigma_0)$ for $i \in \{1,2\}$ as functions on $\mathcal{I}B$. Note that by the P_i-equivariance of elements of $\mathcal{A}_{P_i,k}(\sigma_0)$ the restriction to $\mathcal{I}B$ is injective. Observe that by definition of Σ, the left action of G on $\mathcal{A}_k$ then induces a left action of Σ on $\mathcal{A}_k(\sigma_0)$. Similarly, Σ_i acts from the left on $\mathcal{A}_{P_i,k}(\sigma_0)$. We should point out that the action by $s_i \in \mathcal{I}_i$ on $\mathcal{A}_{P_i,k}(\sigma_0)$ is then given by extending the function to $\mathcal{I}_i B$, acting with s_i and then restricting back to $\mathcal{I}B$.

We have Σ_i-equivariant inclusions $\iota_i \colon \mathcal{P}_k \to \mathcal{A}_{P_i,k}(\sigma_0)$ given by

$$\iota_i(F)(g)(p) = \det(gp)^{k/3} F([0,0,1](gp)^{-1}) \quad \text{for } g \in \mathcal{I}B, p \in P_i,$$

as in Proposition 4.1. Furthermore we define maps $\iota^i \colon \mathcal{A}_{P_i,k}(\sigma_0) \to \mathcal{A}_k(\sigma_0)$ by

$$\iota^i(f)(g) = f(g)(\mathrm{id}) \quad \text{for } f \in \mathcal{A}_{P_i,k}(\sigma_0), g \in \mathcal{I}B.$$

5.9. Proposition. *The maps $\iota^i \colon \mathcal{A}_{P_i,k}(\sigma_0) \to \mathcal{A}_k(\sigma_0)$ are Σ-equivariant, injective and continuous for $i \in \{1,2\}$. The diagram*

$$\begin{tikzcd} & \mathcal{A}_{P_1,k}(\sigma_0) \arrow[dr, "\iota^1"] & \\ \mathcal{P}_k \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & \mathcal{A}_k(\sigma_0) \\ & \mathcal{A}_{P_2,k}(\sigma_0) \arrow[ur, "\iota^2"'] & \end{tikzcd}$$

commutes.

Proof. Let $f \in \mathcal{A}_{P_i,k}(\sigma_0)$. Then for $g \in \Sigma$, we have by definition

$$\iota^i(g_* f)(h) = f(g^{-1}h)(\mathrm{id}) = (g_* \iota^i(f))(h).$$

Moreover, if $\iota^i(f) = 0$, we have $f(g)(\mathrm{id}) = 0$ for $g \in \mathcal{I}B$. Let $h \in \mathcal{I}B$ and $p \in \mathcal{I}B \cap P_i$. Then we have

$$f(h)(p) = f(hp)(\mathrm{id}) = 0.$$

Hence, the algebraic morphism $f(h) \colon P_i(L) \to \mathbb{A}^1_L$ vanishes on the dense subset $\mathcal{I}B \cap P_i$, see [Mil17, Proposition 1.11], and is therefore globally zero, i.e., we have $f(h)(p) = 0$ for all $p \in P_i$, which shows that $f = 0$. To show continuity, observe that we can write ι^i as

$$\mathcal{A}_{P_i,k}(\sigma_0) \subseteq \mathcal{A}_{P_i,k} \to \mathcal{A}_k \to \mathcal{A}_k(\sigma_0),$$

where the middle arrow is the natural inclusion and the last arrow is given by $f \mapsto f \mathbb{1}_{\mathcal{I}B}$. It is immediate that this map is continuous proving the continuity of ι^i. We still need to show that the diagram commutes. For this, let $F \in \mathcal{P}_k$. Then we have

$$\iota^i(\iota_i(F))(g) = \det(g)^{k/3} F([0,0,1]g^{-1}) \quad \text{for } g \in \mathcal{I}B,$$

which is independent of i and thus completes the proof. $\square$

5.10. Remark. It is worth pointing out that ι^i is *not* just induced by the natural inclusion $\mathcal{A}_{P_i,k} \to \mathcal{A}_k$. In fact, this observation will become important later on. In the sequel, we write $\iota_0 := \iota^i \circ \iota_i \colon \mathcal{P}_k \to \mathcal{A}_k(\sigma_0)$.

We need a more explicit description of the above spaces. Following Table 1, we identify $U(\sigma_0)$ with $(\pi\mathcal{O}_K)^3$ as follows:

$$\underline{x} = (x_1, x_2, x_3) \mapsto u(\underline{x})^{-1}, \quad \text{where } u(\underline{x}) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix} \in \mathcal{I} \cap U^- = \mathcal{I}_{\mathrm{id}}. \tag{5}$$

Note that in contrast to Table 1, we are taking the inverse of $u(\underline{x})$ here, the reason being that one obtains significantly simpler formulas using this normalization, see also [DT08, Section 2.1.2] and, in the sequel, Remark 5.13. Now, let $g \in \Sigma$. Under the above identification, g^{-1} acts on $(\pi\mathcal{O}_K)^3$. Explictly, we define $u(g^{-1}\underline{x}) \in \mathcal{I}_{\mathrm{id}}$ by

$$g^{-1}u(\underline{x})^{-1} = u(g^{-1}\underline{x})^{-1} b_{g,\underline{x}}, \quad \text{with } b_{g,\underline{x}} \in B \text{ unique.}$$

The computation in the following lemma is analogous to [PP09, Lemma 2.1].

5.11. Lemma. *We denote by m_{ij} the ij-th minor of g. Then we have*

$$u(g^{-1}\underline{x}) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{m_{32}(x_1x_3 - x_2) + m_{22}x_1 + m_{12}}{m_{31}(x_1x_3 - x_2) + m_{21}x_1 + m_{11}} & 1 & 0 \\ \frac{g_{11}x_2 + g_{21}x_3 + g_{31}}{g_{13}x_2 + g_{23}x_3 + g_{33}} & \frac{g_{12}x_2 + g_{22}x_3 + g_{32}}{g_{13}x_2 + g_{23}x_3 + g_{33}} & 1 \end{pmatrix}$$

and $\chi_k(b_{g,\underline{x}}^{-1}) = \det(g)^{-k/3}(g_{13}x_2 + g_{23}x_3 + g_{33})^k$.

Proof. We compute

$$h := u(\underline{x})g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{11}x_1 + g_{21} & g_{12}x_1 + g_{22} & g_{13}x_1 + g_{23} \\ g_{11}x_2 + g_{21}x_3 + g_{31} & g_{12}x_2 + g_{22}x_3 + g_{32} & g_{13}x_2 + g_{23}x_3 + g_{33} \end{pmatrix}.$$

Let $b \in B$. Then,

$$bh = \begin{pmatrix} b_{11}h_{11} + b_{12}h_{21} + b_{13}h_{31} & b_{11}h_{12} + b_{22}h_{21} + b_{13}h_{32} & b_{11}h_{13} + b_{12}h_{23} + b_{13}h_{33} \\ b_{22}h_{21} + b_{23}h_{31} & b_{22}h_{22} + b_{23}h_{32} & b_{22}h_{23} + b_{23}h_{33} \\ b_{33}h_{31} & b_{33}h_{32} & b_{33}h_{33} \end{pmatrix}.$$

Since $b_{g,\underline{x}}$ is the unique $b \in B$ such that $bh \in \mathcal{I}_{\mathrm{id}}$, we obtain $b_{33} = 1/h_{33}$. We can solve the resulting equations for the entries of b and obtain in particular

$$b_{22} = \frac{h_{33}}{h_{22}h_{33} - h_{23}h_{32}}, \quad b_{23} = \frac{-h_{23}}{h_{22}h_{33} - h_{23}h_{32}} \quad \text{and} \quad b_{11} = \frac{h_{22}h_{33} - h_{23}h_{32}}{\det(g)}.$$

Now, it is straightforward to compute

$$b_{22}h_{21} + b_{23}h_{31} = \frac{m_{32}(x_1x_3 - x_2) + m_{22}x_1 + m_{12}}{m_{31}(x_1x_3 - x_2) + m_{21}x_1 + m_{11}},$$

which completes the proof. $\square$

Now, we can pull back elements of $\mathcal{A}_k(\sigma_0)$ to functions on $U(\sigma_0) \cong \mathcal{I}_{\mathrm{id}} \cong (\pi\mathcal{O}_K)^3$ via (5).

5.12. Proposition. *We keep the notation as in Lemma 5.11. Under the above identification, we have*

$$\mathcal{A}_k(\sigma_0) \cong C^{\mathrm{an}}(U(\sigma_0), L)$$

as topological vector spaces. Here, $g \in \Sigma$ acts on the space on the right hand side by

$$\begin{aligned}(g_* f)(\underline{x}) &= \det(g)^{-k/3}(g_{13}x_2 + g_{23}x_3 + g_{33})^k \\ &\cdot f\left(\frac{m_{32}(x_1x_3 - x_2) + m_{22}x_1 + m_{12}}{m_{31}(x_1x_3 - x_2) + m_{21}x_1 + m_{11}}, \frac{g_{11}x_2 + g_{21}x_3 + g_{31}}{g_{13}x_2 + g_{23}x_3 + g_{33}}, \frac{g_{12}x_2 + g_{22}x_3 + g_{32}}{g_{13}x_2 + g_{23}x_3 + g_{33}}\right).\end{aligned}$$

Proof. Since elements of $\mathcal{A}_k(\sigma_0)$ are locally analytic, we obtain locally analytic functions on $U(\sigma_0)$. Moreover, since elements of $\mathcal{A}_k(\sigma_0)$ are completely determined by their values on $\mathcal{I}_{\mathrm{id}}$ and every locally analytic function on $\mathcal{I}_{\mathrm{id}}$ can be extended uniquely to an element of $\mathcal{A}_k(\sigma_0)$, the association is bijective. That we obtain an isomorphism of topological vector spaces can be verified as in [Fea99, Satz 4.3.1] or [Eme07, Lemma 2.3.3]: We have implicitly already constructed an inverse, but we need to verify that it is continuous. For this, note that since $\mathcal{I}_{\mathrm{id}} \cap B = \{\mathrm{id}\}$, we can regard $\mathcal{A}_k(\sigma_0)$ as a subspace of $C^{\mathrm{an}}(\mathcal{I}_{\mathrm{id}} \times B, L)$. The inverse can then be described as the composition of the two maps

$$\begin{aligned} C^{\mathrm{an}}(U(\sigma_0), L) &\to C^{\mathrm{an}}(B, L) \times C^{\mathrm{an}}(\mathcal{I}_{\mathrm{id}}, L), \\ f &\mapsto (\chi_k^{-1}, f), \end{aligned}$$

and

$$\begin{aligned} C^{\mathrm{an}}(B, L) \times C^{\mathrm{an}}(\mathcal{I}_{\mathrm{id}}, L) &\to C^{\mathrm{an}}(\mathcal{I}_{\mathrm{id}} \times B, L), \\ (\chi, f) &\mapsto [(g, b) \mapsto \chi(b) f(g)]. \end{aligned}$$

The continuity of the first map is immediate. That the second map is well-defined and continuous follows from [Fea99, Satz 2.4.3]. The formula for the Σ-action is a direct consequence of Lemma 5.11. □

5.13. Remark. Let $F \in \mathcal{P}_k$. Then, as we have seen in Section 4.1, we can regard F as an element of $\mathcal{A}_k$ by considering the map $g \mapsto F([0,0,1]g^{-1})$. If we pull back the function $g \mapsto F([0,0,1]g^{-1})\mathbb{1}_{\mathcal{I}B}(g)$ to $U(\sigma_0)$, we obtain by construction the function $f \in C^{\mathrm{an}}(U(\sigma_0), L)$ given by

$$f(\underline{x}) = F(x_2, x_3, 1)$$

with the above Σ-action. This is precisely the action one obtains by dehomogenizing with respect to the variable X_3 in the definition of $\mathcal{P}_k$, an operation frequently considered in the $\mathrm{GL}_2(K)$-case, which further justifies our normalization.

Next, we want to consider the spaces $\mathcal{A}_{P_i,k}(\sigma_0)$ for $i \in \{1,2\}$, and derive similar explicit descriptions. For this, observe that we may write

$$u(\underline{x}) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 - x_1x_3 & 0 & 1 \end{pmatrix}. \tag{6}$$

We denote this factorization by $u(\underline{x}) = p_i(\underline{x}) \cdot u_i(\underline{x})$ for $i \in \{1,2\}$, where $p_1(\underline{x})$ and $u_1(\underline{x})$ are the matrices in the first factorization and $p_2(\underline{x})$ and $u_2(\underline{x})$ are the matrices in the second factorization. Note that $p_i(\underline{x}) \in P_i$. If we denote by $\mathcal{I}^i \subset \mathcal{I}_{\mathrm{id}}$ the subgroup of matrices of the same form as $u_i(\underline{x})$, i.e.,

$$\mathcal{I}^1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{I}^2 := \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix},$$

we have $\mathcal{I}^i P_i = \mathcal{I} P_i$ and $\mathcal{I}^i \cap P_i = \{\mathrm{id}\}$. This enables us to obtain the following.

5.14. Proposition. *Let $i \in \{1,2\}$. We have*

$$\mathcal{A}_{P_i,k}(\sigma_0) \cong C^{\mathrm{an}}(\mathcal{I}^i, F_{P_i,k})$$

as topological vector spaces.

Proof. This is completely analogous to the proof of Proposition 5.12. □

We want to define certain *rigid analytic* counterparts of these spaces, which resemble the spaces considered in [PP09, Section 2] and [Wil18, Section 4]. Automorphic forms with coefficients in (the duals of) these spaces are the ones for which we shall first prove certain lifting theorems. We use the notation from Appendix A, in particular, $A_L(U(\sigma_0))$ denotes the space of rigid analytic functions on $U(\sigma_0) = (\pi \mathcal{O}_K)^3$.

5.15. Definition. We set

$$\mathcal{A}_k^{\mathrm{rig}}(\sigma_0) := A_L(U(\sigma_0))$$

and $\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) := A_L(\mathcal{I}^i) \otimes_L F_{P_i,k}$ for $i \in \{1,2\}$.

Note that by definition we have continuous inclusions

$$\mathcal{A}_k^{\mathrm{rig}}(\sigma_0) \to \mathcal{A}_k(\sigma_0) \quad \text{and} \quad \mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \to \mathcal{A}_{P_i,k}(\sigma_0)$$

for $i \in \{1,2\}$. Moreover, we have a continuous map

$$\begin{aligned} \mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) &\to \mathcal{A}_k^{\mathrm{rig}}(\sigma_0), \\ f &\mapsto [\underline{x} \mapsto f(u_i(\underline{x})^{-1})(p_i(\underline{x})^{-1})]. \end{aligned}$$

We obtain the following.

5.16. Proposition. *Let $i \in \{1, 2\}$. The diagram*

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \arrow[r] \arrow[d] & \mathcal{A}^{\mathrm{rig}}_{k}(\sigma_0) \arrow[d] \\
\mathcal{A}_{P_i,k}(\sigma_0) \arrow[r, "\iota^i"] & \mathcal{A}_{k}(\sigma_0)
\end{tikzcd}$$

commutes.

Proof. This follows directly from (6) since by definition we have

$$f(u(\underline{x})^{-1})(\mathrm{id}) = f(u_i(\underline{x})^{-1} \cdot p_i(\underline{x})^{-1})(\mathrm{id}) = f(u_i(\underline{x})^{-1})(p_i(\underline{x})^{-1})$$

for $f \in \mathcal{A}_{P_i,k}(\sigma_0)$. □

Hence, the map $\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \to \mathcal{A}^{\mathrm{rig}}_{k}(\sigma_0)$ is injective. In the sequel, we denote this map also by ι^i. We want to study the image of $\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0)$ under ι^i in more detail. For this, we need some preparations.

5.17. Definition.

(i) A function $f \in A_L(U(\sigma_0))$ is called P_1-*analytic* if there is a function $f_1 \in A_L((\pi\mathcal{O}_K)^2)$ with $f(\underline{x}) = f_1(x_2, x_3)$.

(ii) A function $f \in A_L(U(\sigma_0))$ is called P_2-*analytic* if there is a collection of functions $f_{2,j} \in A_L((\pi\mathcal{O}_K)^2)$ for $j \in \{0, \dots, k\}$ with

$$f(\underline{x}) = \sum_{j=0}^{k} f_{2,j}(x_1, x_2 - x_1x_3)x_3^j.$$

For $i \in \{1, 2\}$ we denote the L-vector space of P_i-analytic functions by $A_{L,i}(U(\sigma_0))$.

It is immediate from the definition that the spaces $A_{L,i}(U(\sigma_0))$ for $i \in \{1, 2\}$ are closed subspaces of the Banach space $\mathcal{A}^{\mathrm{rig}}_{k}(\sigma_0)$, hence also Banach spaces.

5.18. Remark. The space $A_{L,1}(U(\sigma_0))$ is very similar to the space considered in [Wil18, Proposition 4.3]. The key difference is that we only consider a more restricted class of weights.

The following proposition is of central importance.

5.19. Proposition. *Let $i \in \{1, 2\}$. Then ι^i induces an isomorphism of topological vector spaces*

$$\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \cong A_{L,i}(U(\sigma_0)).$$

Proof. By definition, we have $(\iota^i(f))(u(\underline{x}^{-1})) = f(u_i(\underline{x})^{-1})(p_i(\underline{x})^{-1})$. By the explicit description of $F_{P_i,k}$ in Proposition 4.3, we see that $\iota^i(f) \in A_{L,i}(U(\sigma_0))$ for $i \in \{1,2\}$. The other inclusion is proved separately for $i = 1$ and $i = 2$. We begin with the case $i = 1$. Let $f \in A_{L,1}(U(\sigma_0))$. Then, since $F_{P_1,k} = \chi_k$, we can directly regard f as a function $f' \colon \mathcal{I}^1 \to F_{P_1,k}$. Clearly, it is rigid analytic. By definition, we have $\iota^1(f') = f$. The case $i = 2$ is slightly more complicated. Let $f \in A_{L,2}(U(\sigma_0))$. Then we may write

$$f(\underline{x}) = \sum_{j=0}^{k} f_{2,j}(x_1, x_2 - x_1x_3)x_3^j.$$

with $f_{2,j} \in A_L((\pi\mathcal{O}_K)^2)$ for $j \in \{0,\dots,k\}$. Note that we can identify $F_{P_2,k} \cong L[y]_{\deg\leq k}$. Then we may define $f' \colon \mathcal{I}^2 \to F_{P_2,k}$ by

$$f' = \sum_{j=0}^{k} f_{2,j}y^j.$$

Since the functions $f_{2,j}$ are rigid analytic, so is f'. By definition, we have $\iota^2(f') = f$. Thus, for both $i \in \{1,2\}$ we have constructed a continuous bijection

$$\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \to A_{L,i}(U(\sigma_0)).$$

But as both spaces are Banach spaces, we can apply the open mapping theorem, see for example [Sch02, Corollary 8.7], to conclude that it is a topological isomorphism. □

Now, we can study the action by Σ and Σ_i for $i \in \{1,2\}$ on these spaces.

5.20. Lemma. *The subspace $\mathcal{A}^{\mathrm{rig}}_k(\sigma_0) \subset \mathcal{A}_k(\sigma_0)$ is Σ-stable for the action in Proposition 5.12.*

Proof. For the elements y_i^{-1} for $i \in \{1,2\}$ this is easily verified by the formula in Proposition 5.12. For $g \in \mathcal{I}$, the only possibly problematic terms in Proposition 5.12 are $(m_{31}(x_1x_3 - x_2) + m_{21}x_1 + m_{11})^{-1}$ and $(g_{13}x_2 + g_{23}x_3 + g_{33})^{-1}$. But since $g_{33} \in \mathcal{O}_K^\times$ and $m_{11} \in \mathcal{O}_K^\times$, the power series expansion of these functions is again rigid analytic on $U(\sigma_0)$. □

5.21. Lemma. *The subspaces $A_{L,i}(U(\sigma_0)) \subset A_L(U(\sigma_0))$ for $i \in \{1,2\}$ are Σ-stable for the action in Proposition 5.12.*

Proof. For $i = 1$, this is immediate from the definition and the description of the action in Proposition 5.12. For $i = 2$, we need to do more computations. Let $g \in \Sigma$ and $f \in A_{L,2}(U(\sigma_0))$. We use the notation as in the proof of Lemma 5.11. Then

$$\begin{aligned}(g_* f)(\underline{x}) &= \det(g)^{-k/3}h_{33}^k f\left(\frac{h_{33}h_{21} - h_{23}h_{31}}{h_{22}h_{33} - h_{23}h_{32}}, \frac{h_{31}}{h_{33}}, \frac{h_{32}}{h_{33}}\right)\\ &= \det(g)^{-k/3}\sum_{j=0}^{k} f_{2,j}\left(\frac{h_{33}h_{21} - h_{23}h_{31}}{h_{22}h_{33} - h_{23}h_{32}}, \frac{h_{33}h_{21} - h_{23}h_{31}}{h_{22}h_{33} - h_{23}h_{32}}\frac{h_{32}}{h_{33}} - \frac{h_{31}}{h_{33}}\right)h_{32}^j h_{33}^{k-j}\end{aligned}$$

We have already seen that the first argument of $f_{2,j}$ involves only the variables x_1 and $x_2 - x_1x_3$. For the second argument, we compute

$$\begin{aligned}\frac{h_{33}h_{21} - h_{23}h_{31}}{h_{22}h_{33} - h_{23}h_{32}}\frac{h_{32}}{h_{33}} - \frac{h_{31}}{h_{33}} &= \frac{h_{32}h_{21} - h_{22}h_{31}}{h_{22}h_{33} - h_{23}h_{32}} \\ &= \frac{m_{33}(x_1x_3 - x_2) + m_{23}x_1 + m_{13}}{m_{31}(x_1x_3 - x_2) + m_{21}x_1 + m_{11}},\end{aligned}$$

which again only involves x_1 and $x_2 - x_1x_3$. Now, we have

$$\begin{aligned}h_{32}^j &= (g_{12}x_2 + g_{22}x_3 + g_{32})^j = (g_{12}(x_2 - x_1x_3) + (g_{22} + g_{12}x_1)x_3 + g_{32})^j \\ h_{33}^{k-j} &= (g_{13}x_2 + g_{23}x_3 + g_{33})^{k-j} = (g_{13}(x_2 - x_1x_3) + (g_{23} + g_{13}x_1)x_3 + g_{33})^{k-j}\end{aligned}$$

and we see that the product can be written as a polynomial in x_3 of degree less than or equal to k with coefficients polynomials in x_1 and $x_2 - x_1x_3$. Putting all of this together completes the proof. □

Note that via Proposition 5.14, similarly to Lemma 5.20, the space $\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0)$ inherits a Σ_i-action from $\mathcal{A}_{P_i,k}(\sigma_0)$. Going back through the definitions, one sees that the inclusion $\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \to \mathcal{A}^{\mathrm{rig}}_{k}(\sigma_0)$ is Σ-equivariant. Moreover, we see that s_i acts on the image $A_{L,i}(U(\sigma_0))$. This action can be made explicit as follows:

5.22. Lemma.

(i) Let $f \in A_{L,1}(U(\sigma_0))$. Then we have

$$((s_1)_*f)(\underline{x}) = f(x_1, x_3, x_2).$$

(ii) Let $f \in A_{L,2}(U(\sigma_0))$ be given by

$$f(\underline{x}) = \sum_{j=0}^{k} f_{2,j}(x_1, x_2 - x_1x_3)x_3^j$$

with $f_{2,j} \in A_L((\pi\mathcal{O}_K)^2)$ for $j \in \{0, \dots, k\}$. Then we have

$$((s_2)_*f)(\underline{x}) = \sum_{j=0}^{k} f_{2,k-j}(x_2 - x_1x_3, x_1)x_3^j.$$

Proof. We observe that

$$s_iu(\underline{x})^{-1} = s_iu_i(\underline{x})^{-1}p_i(\underline{x})^{-1} = (s_iu_i(\underline{x})^{-1}s_i)(s_ip_i(\underline{x})^{-1}).$$

for $i \in \{1, 2\}$. We have $s_iu_i(\underline{x})^{-1}s_i \in \mathcal{I}^i$ and $s_ip_i(\underline{x})^{-1} \in P_i$. Thus, we obtain

$$\iota^i((s_i)_*f)(u(\underline{x})^{-1}) = f(s_iu_i(\underline{x})^{-1}s_i)(s_ip_i(\underline{x})^{-1}).$$

Now, we see that $s_1 u_1(\underline{x})^{-1} s_1 = u(0, x_3, x_2)^{-1}$. Since $F_{P_1,k} = \chi_k$, we obtain (i). For part (ii), we see that $s_2 u_2(\underline{x})^{-1} s_2 = u(x_2 - x_1 x_3, x_1, 0)^{-1}$. Since

$$F_{P_2,k} = ((1_{\mathrm{GL}_1(K)} \otimes_K \mathrm{Sym}^k((K^2)^*)) \otimes_K \det{}^{-k/3}) \otimes_K L,$$

we only need to see how s_2 acts on this space, which can be computed directly by Remark 4.4 and gives the desired formula. $\square$

Our constructions can be summarized as follows. The maps ι_i and ι^i for $i \in \{1, 2\}$ give rise to the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{A}^{\mathrm{rig}}_{P_1,k}(\sigma_0) \arrow[dr, "\iota^1"] & \\
\mathcal{P}_k \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & \mathcal{A}^{\mathrm{rig}}_k(\sigma_0) \\
 & \mathcal{A}^{\mathrm{rig}}_{P_2,k}(\sigma_0) \arrow[ur, "\iota^2"'] &
\end{tikzcd}$$

All of the above maps are topological embeddings. Now we set

$$\mathbb{D}^{\bullet}_{*,k}(\sigma_0) \coloneqq \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}^{\bullet}_{*,k}(\sigma_0), L) \quad \text{for } \bullet \in \{\emptyset, \mathrm{rig}\},\ * \in \{\emptyset, P_1, P_2\}.$$

By the above, all of these spaces are naturally coefficient modules by endowing them with the dual right action by Σ. For $* = P_i$, they are P_i-admissible. We also set $V^{\mathrm{rig}}_k \coloneqq V_k$ and denote the maps induced by ι_i and ι^i by π_i and π^i for $i \in \{1, 2\}$. All natural diagrams of maps between these various coefficient modules commute. In the sequel, we refer to $\mathbb{D}^{\bullet}_k(\sigma_0)$ as *overconvergent* coefficients and to $\mathbb{D}^{\bullet}_{P_i,k}(\sigma_0)$ for $i \in \{1, 2\}$ as *partially overconvergent* coefficients.

5.23. Lemma. *The maps*

$$\pi^i \colon \mathbb{D}^{\mathrm{rig}}_k(\sigma_0) \to \mathbb{D}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \quad \textit{and} \quad \pi_i \colon \mathbb{D}^{\mathrm{rig}}_{P_i,k}(\sigma_0) \to V^{\mathrm{rig}}_k$$

are surjective.

Proof. The Banach spaces $\mathcal{A}^{\mathrm{rig}}_k(\sigma_0)$ and $\mathcal{A}^{\mathrm{rig}}_{P_i,k}(\sigma_0)$ are by definition of countable type, i.e., they contain a countable subset whose linear span is dense. Thus, we can apply the theorem of Hahn-Banach, see [Sch86, Theorem 4.2 and Theorem 4.4], to obtain the result. $\square$

5.24. Remark. A more explicit proof of Lemma 5.23 is given as follows: Each of the (continuous) maps ι_i and ι^i for $i \in \{1, 2\}$ has a continuous splitting given by truncation, i.e., by removing terms from the series expansion to obtain an element of the relevant subspace. Since all spaces are endowed with the topology from $A_L(U(\sigma_0))$, it is clear that these splittings are continuous.

Now, we want to construct integral structures on our rigid analytic coefficient modules. For this, let

$$\mathbb{D}_k^{\mathrm{int}}(\sigma_0) := \left\{ \mu \in \mathbb{D}_k^{\mathrm{rig}}(\sigma_0) \;\middle|\; \mu(\underline{x}^I) \in \pi^{|I|}\mathcal{O}_L \text{ for all } I \right\},$$

in analogy with the definition of V_k^{int}.

5.25. Proposition. *Let $n_1 = 2k/3$ and $n_2 = k/3$. Then $(\mathbb{D}_k^{\mathrm{int}}(\sigma_0), (n_i)_{i\in\{1,2\}})$ is an integral structure on $\mathbb{D}_k^{\mathrm{rig}}(\sigma_0)$.*

Proof. We need to verify the properties in Definition 5.4. We begin with property (i). Let $\mu \in \mathbb{D}_k^{\mathrm{rig}}(\sigma_0)$. Then μ is bounded, i.e., we find a constant $C > 0$ such that

$$|\mu(\underline{x}^I)| \le C\|\underline{x}^I\|_{U(\sigma_0)} = Cq^{-|I|} \quad \text{for all } I.$$

This shows that we find $\alpha \in \mathcal{O}_L \setminus \{0\}$ such that $\alpha\mu \in \mathbb{D}_k^{\mathrm{int}}(\sigma_0)$, which proves (i). To prove (ii), note that we have

$$\mathcal{I} = (\mathcal{I} \cap U^-) \cdot (\mathcal{I} \cap B) = \mathcal{I}_{\mathrm{id}} \cdot (\mathcal{I} \cap B).$$

We consider matrices in the two factors in the product above separately. Let

$$g = \begin{pmatrix} 1 & 0 & 0 \\ g_{21} & 1 & 0 \\ g_{31} & g_{32} & 1 \end{pmatrix} \in \mathcal{I}_{\mathrm{id}}.$$

For $\mu \in \mathbb{D}_k^{\mathrm{int}}(\sigma_0)$ we compute

$$(\mu \cdot g)(\underline{x}^I) = \mu((x_1 + g_{21})^{i_1}(x_2 + g_{21}x_3 + g_{31})^{i_2}(x_2 + g_{32}x_3)^{i_3}).$$

Since we have $g_{21}, g_{31}, g_{32} \in \pi\mathcal{O}_K$ it is easy to see that $(\mu \cdot g)(\underline{x}^I) \in \pi^{|I|}\mathcal{O}_L$. Let now

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{pmatrix} \in \mathcal{I} \cap B.$$

For $\mu \in \mathbb{D}_k^{\mathrm{int}}(\sigma_0)$ we compute

$$(\mu \cdot g)(\underline{x}^I) = \mu(\det(g)^{-k/3} f_g(\underline{x})(g_{11}(g_{23}(x_1x_3 - x_2) + g_{33}x_1))^{i_1}(g_{11}x_2)^{i_2}(g_{12}x_2 + g_{22}x_3)^{i_3}),$$

with

$$f_g(\underline{x}) = ((g_{12}g_{23} - g_{22}g_{13})(x_1x_3 - x_2) + g_{12}g_{33}x_1 + g_{22}g_{33})^{-i_1}(g_{13}x_2 + g_{23}x_3 + g_{33})^{k-i_2-i_3}.$$

Observe that $f_g(\underline{x})$ is a (convergent) power series in $\underline{x}$ since $g_{22}, g_{33} \in \mathcal{O}_K^\times$. This shows that $(\mu \cdot g)(\underline{x}^I) \in \pi^{|I|}\mathcal{O}_L$. Finally, to prove (iii), we compute

$$(\pi^{n_i} y_i^{-1} \cdot \mu)(\underline{x}^I) = \begin{cases} \pi^{i_2+i_3}\mu(\underline{x}^I), & \text{for } i = 1, \\ \pi^{i_1+i_2}\mu(\underline{x}^I), & \text{for } i = 2, \end{cases}$$

which completes the proof. $\square$

5.26. Remark. It is worth noting that we have $\mu \in \mathbb{D}_k^{\mathrm{int}}(\sigma_0)$ if and only if

$$\mu(x_1^{i_1}(x_2 - x_1x_3)^{i_2}x_3^{i_3}) \in \pi^{|I|}\mathcal{O}_L \quad \text{for all } I.$$

This will be useful in the following.

Now we set

$$\mathbb{D}_{P_i,k}^{\mathrm{int}}(\sigma_0) := \pi^i(\mathbb{D}_k^{\mathrm{int}}(\sigma_0)) \subset \mathbb{D}_{P_i,k}^{\mathrm{rig}}(\sigma_0)$$

for $i \in \{1,2\}$. Explicitly, we have

$$\mathbb{D}_{P_1,k}^{\mathrm{int}}(\sigma_0) = \left\{\mu \in \mathbb{D}_{P_1,k}^{\mathrm{rig}}(\sigma_0) \;\middle|\; \mu(\underline{x}^I) \in \pi^{|I|}\mathcal{O}_L \text{ for all } I\right\}$$

and

$$\mathbb{D}_{P_2,k}^{\mathrm{int}}(\sigma_0) = \left\{\mu \in \mathbb{D}_{P_2,k}^{\mathrm{rig}}(\sigma_0) \;\middle|\; \mu(x_1^{i_1}(x_2 - x_1x_3)^{i_2}x_3^{i_3}) \in \pi^{|I|}\mathcal{O}_L \text{ for all } I\right\}$$

by the remark above.

5.27. Proposition. *Let n_1 and n_2 as above and set $m_1 = 0$ and $m_2 = k$. Let $i \in \{1,2\}$. Then the triple $(\mathbb{D}_{P_i,k}^{\mathrm{int}}(\sigma_0), (n_j)_{j\in\{1,2\}}, m_i)$ is a P_i-admissible integral structure on $\mathbb{D}_{P_i,k}^{\mathrm{rig}}(\sigma_0)$.*

Proof. Since the projection maps π^i defined above are surjective, it is immediate that the tuple $(\mathbb{D}_{P_i,k}^{\mathrm{int}}(\sigma_0), (n_j)_{j\in\{1,2\}})$ is an integral structure. The second property can be easily checked by Lemma 5.22 and the explicit description above. □

5.28. Remark. Note that we have $\pi_i(\mathbb{D}_{P_i,k}^{\mathrm{int}}(\sigma_0)) = V_k^{\mathrm{int}}$ for $i \in \{1,2\}$ and that the choices for $(n_i)_{i\in\{1,2\}}$ and $(m_i)_{i\in\{1,2\}}$ are compatible with our choice for V_k^{int}. In the sequel, we always fix these choices when working with any of the above coefficient modules.

5.2. Automorphic forms

With the preparations of the previous section we can now define the spaces of automoprhic forms we are interested in and various operators acting on them.

5.29. Definition. Let M be a coefficient module. An *automorphic form* with coefficients in M is a left $K^\times$-invariant and right $\mathcal{I}$-equivariant map $\varphi\colon G \to M$. The L-vector space of automorphic forms is denoted by $\mathbb{A}(M)$. It becomes an $L[G]$-module via $(g\varphi)(h) = \varphi(g^{-1}h)$ for $g,h \in G$, $\varphi \in \mathbb{A}(M)$.

5.30. Remark. Let $(M^{\mathrm{int}}, (n_i)_{i\in\{1,2\}})$ be an integral structure on M. Then we can define the space $\mathbb{A}(M^{\mathrm{int}})$ by requiring the same conditions. This space is a G-stable $\mathcal{O}_L$-submodule of $\mathbb{A}(M)$.

Note that, contrary to the notion of automorphic forms in [Gre06, Section 4] and [Grä19, Section 3], we do not require invariance under a subgroup $\Gamma \subset G$. This is due to the fact that we want to study general bounded harmonic cocycles, not just Γ-invariant ones. By taking invariants $\mathbb{A}(M)^\Gamma$, one obtains the usual spaces of automorphic forms.

In particular, one can obtain global descriptions for these automorphic forms, see for example [Gre06] for such a description in the $\mathrm{GL}_2(K)$-case. We think that this justifies using the term automorphic form here. One downside of this approach is that our space of automorphic forms is not finite-dimensional, even if the coefficient module M is. To remedy this, we need the notion of *bounded* automorphic forms.

5.31. Definition. Let M be a coefficient module that admits an integral structure $(M^{\mathrm{int}}, (n_i)_{i\in\{1,2\}})$. An automorphic form $\varphi \in \mathbb{A}(M)$ is called *bounded* (with respect to M^{int}) if there exists $\alpha \in \mathcal{O}_L \setminus \{0\}$ such that $\alpha\varphi \in \mathbb{A}(M^{\mathrm{int}})$. Clearly, the bounded automorphic forms form an $L[G]$-submodule of $\mathbb{A}(M)$, which we denote by $\mathbb{A}(M)_b$.

The notion of boundedness depends strongly on the chosen integral structure by definition. The following operators play a central role.

5.32. Definition. We fix a coefficient module M together with an integral structure $(M^{\mathrm{int}}, (n_i)_{i\in\{1,2\}})$. Write

$$\mathcal{I} y_i \mathcal{I} = \bigsqcup_j y_{i,j} \mathcal{I}$$

for $i \in \{1,2\}$. The *$U_{\pi,i}$-operator* on $\mathbb{A}(M)$ is the L-linear map $U_{\pi,i} \colon \mathbb{A}(M) \to \mathbb{A}(M)$ given by

$$(U_{\pi,i}\varphi)(g) = \pi^{n_i} \sum_j \varphi(g y_{i,j}) \cdot y_{i,j}^{-1} \quad \text{for } \varphi \in \mathbb{A}(M), g \in G.$$

It is easy to check that these operators are well-defined and independent of the choice of coset representatives. Note that by definition $U_{\pi,i}$ preserves $\mathbb{A}(M^{\mathrm{int}})$ and $\mathbb{A}(M)_b$.

If the coefficient module is P_i-admissible, we have another operator acting on the space of automorphic forms.

5.33. Definition. Let $i \in \{1,2\}$ and let M be a P_i-admissible coefficient module with a P_i-admissible integral structure $(M^{\mathrm{int}}, (n_j)_{j\in\{1,2\}}, m_i)$. Let $s_i \in \Sigma_i$ as in the previous section. Write

$$\mathcal{I} s_i \mathcal{I} = \bigsqcup_j s_{i,j} \mathcal{I}.$$

The *Atkin-Lehner $W_{\pi,i}$-operator* on $\mathbb{A}(M)$ is the L-linear map $W_{\pi,i} \colon \mathbb{A}(M) \to \mathbb{A}(M)$ given by

$$(W_{\pi,i}\varphi)(g) = \pi^{m_i} \sum_j \varphi(g s_{i,j}) \cdot s_{i,j}^{-1} \quad \text{for } \varphi \in \mathbb{A}(M), g \in G.$$

As above, it is easy to check that these operators are well-defined and independent of the choice of coset representatives. Note that by definition $W_{\pi,i}$ preserves $\mathbb{A}(M^{\mathrm{int}})$ and $\mathbb{A}(M)_b$.

More generally, if M is any coefficient module without a fixed integral structure, one can still define the $U_{\pi,i}$-operators for $i \in \{1,2\}$ as above with arbitrary choices of n_1 and n_2. Of course the operators then depend on these choices. The same holds true for the $W_{\pi,i}$-operator and the choice of m_i for a P_i-admissible coefficient module.

5.34. Remark. We should point out that the above definition of the Atkin-Lehner operators is different than the one considered in [Grä19, Definition 13] in the $\mathrm{GL}_2(K)$-case. Geometrically, the difference can be described as follows: The operator in [Grä19] flips the orientation on a chamber, whereas here we take the sum over all chambers sharing a specified face with the given chamber. In the $\mathrm{GL}_2(K)$-case this just means taking all edges with the same target as the given edge. If one requires harmonicity, these conditions are equivalent. However, in the $\mathrm{GL}_3(K)$-case the present condition seems to be the natural one if one is in interested in the attached distributions following [ST97]. We should also remark that the presence of the scalar factors π^{m_i} makes the name *Atkin-Lehner operator* a bit misleading as these operators are involutions only up to scalar, which follows from $s_i^2 = 1$ for $i \in \{1, 2\}$.

5.35. Remark. It is easy to verify that the number of cosets in the disjoint union used in the definition of $W_{\pi,i}$ is q, whereas for $U_{\pi,i}$ there are q^2 coset representatives. In the latter case, one natural choice for the coset representatives are the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pi a & \pi b & \pi \end{pmatrix} \quad \text{for } i = 1 \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ \pi a & \pi & 0 \\ \pi b & 0 & \pi \end{pmatrix} \quad \text{for } i = 2,$$

where a, b run through a set a representatives in O_K of the q residue classes in κ.

5.36. Lemma. *We have $U_{\pi,1} \circ U_{\pi,2} = U_{\pi,2} \circ U_{\pi,1}$ and $W_{\pi,i} \circ U_{\pi,i} = U_{\pi,i} \circ W_{\pi,i}$ for $i \in \{1, 2\}$.*

Proof. This follows directly from $y_1 \cdot y_2 = y_2 \cdot y_1$ and $y_i s_i = s_i y_i$ for $i \in \{1, 2\}$. □

Now, we turn to the coefficient modules constructed in Section 5.1. By functoriality of the formation of automorphic forms, we have commutative diagrams

$$\begin{tikzcd} & \mathbb{A}(\mathbb{D}^\bullet_{P_1,k}(\sigma_0)) \arrow[dr, "\rho_1"] & \\ \mathbb{A}(\mathbb{D}^\bullet_k(\sigma_0)) \arrow[ur, "\rho^1"] \arrow[dr, "\rho^2"'] & & \mathbb{A}(V^\bullet_k), \\ & \mathbb{A}(\mathbb{D}^\bullet_{P_2,k}(\sigma_0)) \arrow[ur, "\rho_2"'] & \end{tikzcd}$$

for $\bullet \in \{\emptyset, \mathrm{rig}, \mathrm{int}\}$, where the maps are induced by π^i and π_i for $i \in \{1, 2\}$. We refer to the maps in the above diagram as *specialization maps*. All specialization maps are G-equivariant. For all of the rigid coefficient modules we fix the integral structures from the previous section, in particular $n_1 = 2k/3$ and $n_2 = k/3$. We also fix $m_1 = 0$ and $m_2 = k$. Then the specialization maps preserve boundedness. We use the same constants to define the $U_{\pi,i}$-operators and $W_{\pi,i}$-operators on the locally analytic coefficient modules. Then whenever one of the above operators $U_{\pi,i}$ and $W_{\pi,i}$ for $i \in \{1, 2\}$ is defined on the source and target of a specialization map, the respective map is equivariant for the action of this operator. We also set $\rho_0 := \rho_i \circ \rho^i$ (which is independent of $i \in \{1, 2\}$).

Space of automorphic forms	$U_{\pi,1}$	$U_{\pi,2}$	$W_{\pi,1}$	$W_{\pi,2}$
$\mathbb{A}(V_k^\bullet)$	✓	✓	✓	✓
$\mathbb{A}(\mathbb{D}^\bullet_{P_1,k}(\sigma_0))$	✓	✓	✓	✗
$\mathbb{A}(\mathbb{D}^\bullet_{P_2,k}(\sigma_0))$	✓	✓	✗	✓
$\mathbb{A}(\mathbb{D}^\bullet_{k}(\sigma_0))$	✓	✓	✗	✗

Table 4. $U_{\pi,i}$- and $W_{\pi,i}$-operators on automorphic forms

In Table 4, we summarize which of the above operators act on our spaces of automorphic forms.

Our first aim is to compare eigenforms with locally analytic and rigid analytic coefficients. The proof of the following theorem is an extension of [PS13, Lemma 5.3] to the $\mathrm{GL}_3(K)$-case.

5.37. Theorem. *Let $\alpha_1, \alpha_2 \in \mathcal{O}_L \setminus \{0\}$. Then, the natural maps $\mathbb{D}_{*,k}(\sigma_0) \to \mathbb{D}^{\mathrm{rig}}_{*,k}(\sigma_0)$ for $* \in \{\emptyset, P_1, P_2\}$ induce isomorphisms*

$$\Psi\colon \mathbb{A}(\mathbb{D}_k(\sigma_0))^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}} \to \mathbb{A}(\mathbb{D}^{\mathrm{rig}}_k(\sigma_0))^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$$

and

$$\Psi_i\colon \mathbb{A}(\mathbb{D}_{P_i,k}(\sigma_0))^{U_{\pi,i}=\alpha_i} \to \mathbb{A}(\mathbb{D}^{\mathrm{rig}}_{P_i,k}(\sigma_0))^{U_{\pi,i}=\alpha_i}$$

for $i \in \{1,2\}$.

Proof. We begin with the first map and start by proving its injectivity. For this, let $\Phi \in \mathbb{A}(\mathbb{D}_k(\sigma_0))^{(U_{\pi,1}=\alpha_i)_{i\in\{1,2\}}}$ such that the image of Φ in $\mathbb{A}(\mathbb{D}^{\mathrm{rig}}_k(\sigma_0))^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$ is 0. This means that for each $g \in G$, the distribution $\Phi(g) \in \mathbb{D}_k(\sigma_0)$ vanishes on $\mathcal{A}^{\mathrm{rig}}_k(\sigma_0)$. We need to show that it vanishes on the whole space $\mathcal{A}_k(\sigma_0)$. For this, let $f \in \mathcal{A}_k(\sigma_0)$. By examining the action of the matrices $y_{i,j}$ from the definition $U_{\pi,i}$-operators on $U(\sigma_0)$ as in Remark 5.35, we see that we can choose $m \in \mathbb{Z}_{\geq 0}$ big enough such that we have

$$(h^{-1})_* f \in \mathcal{A}^{\mathrm{rig}}_k(\sigma_0)$$

for each element $h \in G$ of the form

$$\left(\prod_{n=1}^{m} y_{1,j_n} y_{2,j'_n}\right)$$

with $j_n, j'_n \in \{0, \ldots, q^2-1\}$ for all $n \in \{1, \ldots, m\}$. But since $\Phi \in \mathbb{A}(\mathbb{D}_k(\sigma_0))^{(U_{\pi,1}=\alpha_i)_{i\in\{1,2\}}}$, if we put $\tilde{U} := \alpha_1^{-1}\alpha_2^{-1}U_{\pi,1}U_{\pi,2}$, we obtain

$$\Phi(g)(f) = (\tilde{U}^m\Phi)(g)(f).$$

By definition of the $U_{\pi,i}$-operators, the right hand side is a linear combination of elements of the form

$$\Phi(gh)((h^{-1})_* f)$$

with h as above, which are all zero since $(h^{-1})_* f \in \mathcal{A}_k^{\mathrm{rig}}(\sigma_0)$. Thus, we have proved the injectivity. We have also derived a strategy to prove the surjectivity: Let $f \in \mathcal{A}_k(\sigma_0)$ and $\varphi \in \mathbb{A}(\mathbb{D}_k^{\mathrm{rig}}(\sigma_0))^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$. We choose m (depending on f) as above. We define $\Phi(g)(f)$ by writing $(\tilde{U}^m\varphi)(g)$ as a sum of elements of the form $\varphi(gh)\cdot h^{-1}$ with h of the above form. Then the value $(\varphi(gh)\cdot h^{-1})(f)$ makes sense. It is now easy to check that this does not depend on the choice of m and defines an element of $\mathbb{A}(\mathbb{D}_k(\sigma_0))^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$. For the second statement, observe that by Proposition 5.14 for $f \in \mathcal{A}_{P_i,k}(\sigma_0)$, it suffices to apply products of the matrices $y_{i,j}$ to f to obtain a rigid function. From here on, the proof is exactly as in the first case by considering just the $U_{\pi,i}$-operator. □

5.38. Remark. In fact, the above proof shows that one can also work directly with an eigenform for the operator $U_\pi := U_{\pi,1} \circ U_{\pi,2}$ in the first case and obtain the analogous result in this situation.

Even though we do not have integral structures on the coefficient modules $\mathbb{D}_{*,k}(\sigma_0)$ for $* \in \{\emptyset, P_1, P_2\}$, if we just consider eigenforms, we can transfer the notion of boundedness via the above theorem.

5.39. Definition. Let $\alpha_1, \alpha_2 \in \mathcal{O}_L \setminus \{0\}$. We define

$$\mathbb{A}(\mathbb{D}_k(\sigma_0))_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}} := \Psi^{-1}(\mathbb{A}(\mathbb{D}_k^{\mathrm{rig}}(\sigma_0))_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}})$$

and

$$\mathbb{A}(\mathbb{D}_{P_i,k}(\sigma_0))_b^{U_{\pi,i}=\alpha_i} := \Psi_i^{-1}(\mathbb{A}(\mathbb{D}_{P_i,k}^{\mathrm{rig}}(\sigma_0))_b^{U_{\pi,i}=\alpha_i})$$

for $i \in \{1,2\}$.

The following notion is of central importance.

5.40. Definition. Let $\alpha_1, \alpha_2 \in \mathcal{O}_L \setminus \{0\}$. We say that the pair (α_1, α_2) is *non-critical* (for the weight k) if the following conditions hold:

(i) The map $\rho_0 \colon \mathbb{A}(\mathbb{D}_k(\sigma_0))_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}} \to \mathbb{A}(V_k)_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$ is an isomorphism.

(ii) The maps $\rho_i \colon \mathbb{A}(\mathbb{D}_{P_i,k}(\sigma_0))_b^{U_{\pi,i}=\alpha_i} \to \mathbb{A}(V_k)_b^{U_{\pi,i}=\alpha_i}$ are isomorphisms for $i \in \{1,2\}$.

Similarly, we make the following definition.

5.41. Definition. Let $\varphi \in \mathbb{A}(V_k)_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$, where $\alpha_i \in \mathcal{O}_L \setminus \{0\}$ for $i \in \{1,2\}$. We say that φ is *non-critical* if there exist unique elements $\Phi \in \mathbb{A}(\mathbb{D}_k(\sigma_0))_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$ and $\Phi_i \in \mathbb{A}(\mathbb{D}_{P_i,k}(\sigma_0))_b^{U_{\pi,i}=\alpha_i}$ which lift φ.

By definition, if the pair (α_1, α_2) is non-critical, then every element of $\mathbb{A}(V_k)_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$ is non-critical.

5.3. Distributions attached to automorphic forms

In this section, we see how the theory of automorphic forms developed in the previous section is related to the distributions studied in Section 4.4. In particular, we show that the non-criticality of a certain class of automorphic forms implies Conjecture 4.29. In the next section we will then investigate the question which forms are non-critical further. We should remark that the relation between automorphic forms and distributions studied here is similar to [Grä19, Section 3.4]. The objective if rather different though: Whereas in [Grä19] the extension of the relevant distributions is already known and automorphic forms provide a good framework for explicit computations, here we want to promote the standpoint that automorphic forms provide a better framework towards proving results for distributions.

Recall that $n_1 = 2k/3$ and $n_2 = k/3$. To shorten the notation we set

$$\mathbb{A}(V_k)^{\mathrm{eig}} := \mathbb{A}(V_k)^{(U_{\pi,i}=\pi^{n_i})_{i\in\{1,2\}}}, \quad \mathbb{A}(\mathbb{D}_k(\sigma_0))^{\mathrm{eig}} := \mathbb{A}(\mathbb{D}_k(\sigma_0))^{(U_{\pi,i}=\pi^{n_i})_{i\in\{1,2\}}}$$

and

$$\mathbb{A}(\mathbb{D}_k(\sigma_0))_b^{\mathrm{eig}} := \mathbb{A}(\mathbb{D}_k(\sigma_0))_b^{(U_{\pi,i}=\pi^{n_i})_{i\in\{1,2\}}}.$$

Moreover, we define

$$\mathbb{A}(V_k)^{\mathrm{new}} := \{\varphi \in \mathbb{A}(V_k)^{\mathrm{eig}} \mid W_{\pi,i}\varphi = -\pi^{m_i}\varphi \text{ for } i \in \{1,2\}\},$$

where $m_1 = 0$ and $m_2 = k$. Our first aim is to connect harmonic cocycles to automorphic forms. The following construction is an analogue of [Grä19, Proposition 10]. Let $c \in C_{\mathrm{har}}(\mathcal{T}, k)$. We define

$$\varphi_c \colon K^\times\backslash G \to V_k$$

by $\varphi_c(g) = g^{-1} \cdot c(g\sigma_0) = c(g\sigma_0) \cdot g$.

5.42. Lemma. *We have $\varphi_c \in \mathbb{A}(V_k)^{\mathrm{new}}$.*

Proof. Let $x \in K^\times$ and $h \in \mathcal{I}$. Then we have

$$\varphi_c(xgh) = c(xgh\sigma_0) \cdot (xgh) = c(g\sigma_0) \cdot (gh) = \varphi_c(g) \cdot h,$$

since $\mathrm{Stab}_G(\sigma_0) = K^\times\mathcal{I}$ and $K^\times$ acts trivially on V_k. This shows that $\varphi_c \in \mathbb{A}(V_k)$. Now, by [ST97, Definition 9 and Lemma 10], we have that c satisfies

$$c(g\sigma_0) = \sum_j c(gy_{i,j}\sigma_0) \quad \text{and} \quad c(g\sigma_0) = -\sum_j c(gs_{i,j}\sigma_0)$$

for $i \in \{1,2\}$ and $g \in G$. These conditions directly translate to the above relations. □

5.43. Proposition. *The map $C_{\mathrm{har}}(\mathcal{T}, k) \to \mathbb{A}(V_k)^{\mathrm{new}}$ given by $c \mapsto \varphi_c$ is a G-equivariant isomorphism. Moreover, $c \in C_{\mathrm{har}}(\mathcal{T}, k)$ is bounded if and only if φ_c is bounded.*

Proof. We just need to show that the map is an isomorphism. The remaining statements are just a rephrasing of Proposition 3.7. Let $\varphi \in \mathbb{A}(V_k)^{\mathrm{new}}$. We define $c_\varphi \colon \widehat{\mathcal{T}_2} \to V_k$ by

$$c(\sigma) = \varphi(g) \cdot g^{-1},$$

where we choose $g \in G$ such that $\sigma = g\sigma_0$. It is straightforward to check that c_φ is well-defined and satisfies

$$c_\varphi(g\sigma_0) = \sum_j c_\varphi(gy_{i,j}\sigma_0) \quad \text{and} \quad c_\varphi(g\sigma_0) = -\sum_j c_\varphi(gs_{i,j}\sigma_0)$$

for $i \in \{1,2\}$ and $g \in G$. But then we have $c_\varphi \in C_{\mathrm{har}}(\mathcal{T},k)$ by [Aït06, Theorem 3.3] (see also [ST97, Proposition 11]). It is immediate from the definitions that $\varphi_{c_\varphi} = \varphi$ and $c_{\varphi_c} = c$ for $c \in C_{\mathrm{har}}(\mathcal{T},k)$ and $\varphi \in \mathbb{A}(V_k)^{\mathrm{new}}$. ◻

Our next aim is to relate elements of $\mathbb{A}(\mathbb{D}_k(\sigma_0))^{\mathrm{eig}}$ to certain distributions. Recall that we denote by $\mathcal{A}_k'$ and $\mathrm{St}_3^{\mathrm{an}}(k)'$ the continuous duals of $\mathcal{A}_k$ and $\mathrm{St}_3^{\mathrm{an}}(k)$. Let $\Phi \in \mathbb{A}(\mathbb{D}_k(\sigma_0))^{\mathrm{eig}}$, $f \in \mathcal{A}_k$ and $g \in G$. We set

$$\lambda_\Phi(f\mathbb{1}_{g\mathcal{I}B}) := \Phi(g)((g^{-1})_*(f\mathbb{1}_{g\mathcal{I}B})) \in L,$$

where we observe that $(g^{-1})_*(f\mathbb{1}_{g\mathcal{I}B}) = ((g^{-1})_*f)\mathbb{1}_{\mathcal{I}B} \in \mathcal{A}_k(\sigma_0)$.

5.44. Proposition. *The function λ_Φ can be uniquely extended to an element of $\mathcal{A}_k'$.*

Proof. We want to define the extension by

$$\lambda_\Phi(f) = \sum_g \lambda_\Phi(f\mathbb{1}_{g\mathcal{I}B}),$$

where the sum is over any finite set of matrices $g \in G$ such that the corresponding open sets $g\mathcal{I}B$ form a disjoint covering of G. Once this well-defined, it is clear that this is the unique extension. In order to see the well-definedness, we first observe that

$$\begin{aligned}\lambda_\Phi(f\mathbb{1}_{gxh\mathcal{I}B}) &= \Phi(gxh)((((gxh)^{-1})_*(f\mathbb{1}_{xgh\mathcal{I}B})) \\ &= (\Phi(gh)\cdot h^{-1})((g^{-1})_*(f\mathbb{1}_{g\mathcal{I}B})) = \lambda(f\mathbb{1}_{g\mathcal{I}B})\end{aligned}$$

for $x \in K$, $h \in \mathcal{I}$, which shows that each summand is independent of the choice of g. Thus, we are left with showing the independence of the chosen covering. For this, we define

$$\mu_{\Phi,f} \colon \mathbb{Z}[\widehat{\mathcal{T}_2}] \to L$$

by $\mu_{\Phi,f}(\sigma) = \lambda_\Phi(f\mathbb{1}_{g\mathcal{I}B})$ for $\sigma \in \widehat{\mathcal{T}_2}$, where $g \in G$ is chosen such that $\sigma = g\sigma_0$. Our previous computation ensures that this is independent of the choice of g. Assume that $\mu_{\Phi,f}$ satisfies

$$\mu_{\Phi,f}(g\mathcal{I}y_i\sigma_0 - g\sigma_0) = 0 \quad \text{for } g \in G, i \in \{1,2\}.$$

Then, by Proposition 2.16, $\mu_{\Phi,f}$ defines a linear functional on $C^\infty(G/B,\mathbb{Z})$. But going back through the definition this just means that

$$\lambda(f\mathbb{1}_{g\mathcal{I}B}) = \sum_{g'} \lambda(f\mathbb{1}_{g'\mathcal{I}B}),$$

whenever the sets $g'\mathcal{I}B$ cover $g\mathcal{I}B$ disjointly, which proves the independence of the chosen covering. Thus, we need to show that $\mu_{\Phi,f}(g\mathcal{I}y_i\sigma_0 - g\sigma_0) = 0$ for $g \in G$, $i \in \{1,2\}$. If we write

$$\mathcal{I}y_i\mathcal{I} = \bigsqcup_j y_{i,j}\mathcal{I}$$

and set $f_g := (g^{-1})_*(f\mathbb{1}_{g\mathcal{I}B}) \in \mathcal{A}_k(\sigma_0)$, we compute

$$\begin{aligned}
\mu_{\Phi,f}(g\mathcal{I}y_i\sigma_0 - g\sigma_0) &= \left(\sum_j \mu_{\Phi,f}(gy_{i,j}\sigma_0)\right) - \mu_{\Phi,f}(g\sigma_0) \\
&= \left(\sum_j \Phi(gy_{i,j})(((gy_{i,j})^{-1})_*(f\mathbb{1}_{gy_{i,j}\mathcal{I}B}))\right) - \Phi(g)(f_g) \\
&= \left(\sum_j (\Phi(gy_{i,j})\cdot y_{i,j}^{-1})(f_g)\right) - \Phi(g)(f_g) \\
&= \pi^{-n_i}U_{\pi,i}\Phi(g)(f_g) - \Phi(g)(f_g) = 0.
\end{aligned}$$

The continuity of λ_Φ follows directly from the fact that $\Phi(g) \in \mathbb{D}_k(\sigma_0)$ is continuous for each $g \in G$. □

5.45. Proposition. *The map $\mathbb{A}(\mathbb{D}_k(\sigma_0))^{\mathrm{eig}} \to \mathcal{A}_k'$ given by $\Phi \mapsto \lambda_\Phi$ is a G-equivariant isomorphism.*

Proof. We first show the G-equivariance. By definition of λ_Φ, it suffices to show that

$$(g\cdot\lambda_\Phi)(f\mathbb{1}_{h\mathcal{I}B}) = \lambda_{g\cdot\Phi}(f\mathbb{1}_{h\mathcal{I}B})$$

for $f \in \mathcal{A}_k$, $g,h \in G$. We compute

$$\begin{aligned}
(g\cdot\lambda_\Phi)(f\mathbb{1}_{h\mathcal{I}B}) &= \lambda_\Phi((g^{-1})_*(f\mathbb{1}_{h\mathcal{I}B})) \\
&= \lambda_\Phi(((g^{-1})_*(f))\mathbb{1}_{g^{-1}h\mathcal{I}B}) \\
&= \Phi(g^{-1}h)((h^{-1}g)_*(((g^{-1})_*(f))\mathbb{1}_{g^{-1}h\mathcal{I}B}) \\
&= (g\cdot\Phi)(h)((h^{-1})_*(f\mathbb{1}_{h\mathcal{I}B})) \\
&= \lambda_{g\cdot\Phi}(f\mathbb{1}_{h\mathcal{I}B}).
\end{aligned}$$

To show that $\Phi \mapsto \lambda_\Phi$ is an isomorphism, we construct an explicit inverse. Let $\lambda \in \mathcal{A}_k'$. We define $\Phi_\lambda\colon K^\times\backslash G \to \mathbb{D}_k(\sigma_0)$ by

$$\Phi_\lambda(g)(f) = \lambda(g_*f) \in L$$

for $g \in G$ and $f \in \mathcal{A}_k(\sigma_0)$, where we regard f as an element of $\mathcal{A}_k$ by extending with zero. A similar but simpler computation as in Proposition 5.44 then shows that $\Phi_\lambda \in \mathbb{A}(\mathbb{D}_k(\sigma_0)^{\mathrm{eig}}$. It is easy to check that the two maps are inverse to each other, completing the proof. $\square$

By requiring stronger conditions on Φ, we can restrict the distribution λ_Φ even further.

5.46. Proposition. *Let $\Phi \in \mathbb{A}(\mathbb{D}_k(\sigma_0))^{\mathrm{eig}}$ such that*

$$W_{\pi,i}\rho^i(\Phi) = -\pi^{m_i}\rho^i(\Phi) \quad \textit{for } i \in \{1,2\}.$$

Then we have $\lambda_\Phi \in \mathrm{St}_3^{\mathrm{an}}(k)'$.

Proof. We need stow that λ_Φ vanishes on $\mathcal{A}_{P_i,k}$ for $i \in \{1,2\}$. Fix $i \in \{1,2\}$ and $f \in \mathcal{A}_{P_i,k}$. Observe that we can cover G disjointly by sets of the form $g\mathcal{I}_iB = g\mathcal{I}P_i$, see [SS91, Section 4, Lemma 14 and Proposition 8′]. Let $f_g := (g^{-1})_*(f\mathbb{1}_{g\mathcal{I}B})$ and note that by definition we have $f_g \in \mathcal{A}_{P_i,k}(\sigma_0)$. Then we may write

$$\begin{aligned}\lambda_\Phi(f) &= \sum_g \lambda_\Phi(f\mathbb{1}_{g\mathcal{I}_iB}) = \sum_g\left(\lambda_\Phi(f\mathbb{1}_{g\mathcal{I}B}) + \sum_j \lambda_\Phi(f\mathbb{1}_{gs_{i,j}\mathcal{I}B})\right)\\ &= \sum_g\left(\Phi(g)(f_g) + \sum_j (\Phi(gs_{i,j})\cdot s_{i,j}^{-1})(f_g)\right),\end{aligned}$$

where $\mathcal{I}s_i\mathcal{I} = \bigsqcup_j s_{i,j}\mathcal{I}$ as in the proof of Proposition 5.44. By the assumption and the definition of the $W_{\pi,i}$-operator, we have

$$(\rho^i(\Phi))(g) + \sum_j (\rho^i(\Phi))(gs_{i,j})\cdot s_{i,j}^{-1} = 0.$$

But since $f_g \in \mathcal{A}_{P_i,k}(\sigma_0)$, this means that each summand in the above sum is zero, hence $\lambda_\Phi(f) = 0$. $\square$

5.47. Remark. In the above proof we show more: If $W_{\pi,i}\rho^i(\Phi) = -\pi^{m_i}\rho^i(\Phi)$ for *one* $i \in \{1,2\}$, then λ_Φ vanishes on $\mathcal{A}_{P_i,k}$.

We are now able to prove the following theorem.

5.48. Theorem. *Let $L = \mathbb{C}_K$ and assume that every automorphic form in $\mathbb{A}(V_k)_b^{\mathrm{new}}$ is non-critical. Then Conjecture 4.29 holds.*

Proof. Let $c \in C_{\mathrm{har}}^b(\mathcal{T},k)$ and consider the automorphic form $\varphi_c \in \mathbb{A}(V_k)_b^{\mathrm{new}}$ as in Proposition 5.43. Then by assumption φ_c is non-critical, i.e., there exist unique automorphic forms $\Phi_c \in \mathbb{A}(\mathbb{D}_k(\sigma_0))_b^{\mathrm{eig}}$ and $\Phi_{c,i} \in \mathbb{A}(\mathbb{D}_{P_i,k}(\sigma_0))_b^{U_{\pi,i}=\pi^{n_i}}$ with $\rho(\Phi_c) = \varphi_c$ and $\rho_i(\Phi_{c,i}) = \varphi_c$ for $i \in \{1,2\}$. By uniqueness, we then have

$$\rho^i(\Phi_c) = \Phi_{c,i} \quad \text{for } i \in \{1,2\}.$$

Moreover, we obtain

$$\rho_i(\pi^{-m_i} W_{\pi,i}\Phi_{c,i} + \Phi_{c,i}) = \pi^{-m_i} W_{\pi,i}\varphi_c + \varphi_c = 0.$$

But since $W_{\pi,i} \circ U_{\pi,i} = U_{\pi,i} \circ W_{\pi,i}$, we have $\pi^{-m_i} W_{\pi,i}\Phi_{c,i} \in \mathbb{A}(\mathbb{D}_{P_i,k}(\sigma_0))_b^{U_{\pi,i}=\pi^{n_i}}$. Hence by uniqueness we obtain $W_{\pi,i}\Phi_{c,i} = -\pi^{m_i}\Phi_{c,i}$. This shows that Φ_c satisfies the assumptions of Proposition 5.46. We claim that $\lambda_{\Phi_c} \in \mathrm{St}_3^{\mathrm{an}}(k)'$ satisfies the properties in Conjecture 4.29. For this, let $F \in \mathcal{P}_k$ and $\sigma \in \widehat{\mathcal{T}_2}$. Write $\sigma = g\sigma_0$. Then we have

$$\begin{aligned}
\lambda_{\Phi_c}([\mathbb{1}_{U(\sigma)}] \otimes F) &= \lambda_{\Phi_c}(\iota(F)\mathbb{1}_{g\mathcal{I}B}) = \Phi_c(g)((g^{-1})_*(\iota(F)\mathbb{1}_{g\mathcal{I}B})) \\
&= \Phi_c(g)(\iota_0((g^{-1})_*F)) = \varphi_c(g)((g^{-1})_*(F)) \\
&= (\varphi_c(g) \cdot g^{-1})(F) = c(\sigma)(F)
\end{aligned}$$

by Proposition 4.7 and Proposition 5.43. This shows property (i) in Conjecture 4.29. Property (ii) is immediate as all maps involved are L-linear and G-equivariant. □

Note that we have not explicitly used the boundedness of the lift Φ_c in the above proof. In fact, by incorporating this boundedness we can strengthen Conjecture 4.29 as follows. Note also that we can generalize everything to arbitrary L, i.e., we do *not* need to assume $L = \mathbb{C}_K$.

5.49. Conjecture. *For each $c \in C_{\mathrm{har}}^b(\mathcal{T}, k)$ there exists a unique $\lambda_c \in \mathrm{St}_3^{\mathrm{an}}(k)'$ with the following properties:*

- *(i) $\lambda_c([\mathbb{1}_{U(\sigma)}] \otimes F) = c(\sigma)(F)$ for all $F \in \mathcal{P}_k$, $\sigma \in \widehat{\mathcal{T}_2}$, where we regard $[\mathbb{1}_{U(\sigma)}] \otimes F$ as an element of $\mathrm{St}_3^{\mathrm{an}}(k)$ via the map in Proposition 4.7.*
- *(ii) The map $C_{\mathrm{har}}^b(\mathcal{T}, k) \to \mathrm{St}_3^{\mathrm{an}}(k)'$ given by $c \mapsto \lambda_c$ is L-linear and G-equivariant.*
- *(iii) There exists a constant $C > 0$ such that we have $|\lambda(g_*(\underline{x}^I))| \leq Cq^{-|I|}$ for all I, where we regard $\underline{x}^I \in \mathcal{A}_k^{\mathrm{rig}}(\sigma_0)$ as an element of $\mathcal{A}_k$ by extending with zero.*

We obtain the following immediate consequence.

5.50. Corollary. *Assume that every automorphic form in $\mathbb{A}(V_k)_b^{\mathrm{new}}$ is non-critical. Then Conjecture 5.49 holds.*

Proof. We keep the notation from the proof of Theorem 5.48. We first show how the boundedness of Φ translates to the estimate in (iii). For this, note that by construction we have

$$\lambda_\Phi(g_*(\underline{x}^I)) = \Phi(g)(\underline{x}^I).$$

But since $\underline{x}^I \in \mathcal{A}_k^{\mathrm{rig}}(\sigma_0)$, the right hand side is just $\Psi(\Phi)(g)(\underline{x}^I)$, see Theorem 5.37. The definition of boundedness implies that we find an element $\alpha \in \mathcal{O}_L \setminus \{0\}$ such that $\alpha\Psi(\Phi)(g) \in \mathbb{D}_k^{\mathrm{int}}(\sigma_0)$ for all $g \in G$. Let $C := |\alpha^{-1}|$. Putting all of this together, we obtain

$$|\lambda_\Phi(g_*(\underline{x}^I))| \leq C\pi^{-|I|}.$$

We are left with showing uniqueness. For this, let $\lambda \in \mathrm{St}_3^{\mathrm{an}}(k)'$ be any distribution satisfying properties (i), (ii) and (iii). By Proposition 5.45, we find a corresponding automorphic form $\Phi_\lambda \in \mathbb{A}(\mathbb{D}_k(\sigma_0))^{\mathrm{eig}}$. But property (i) then translates to the fact that Φ_λ lifts φ_c and as above property (iii) translates to the fact that Φ_λ is bounded. This shows that $\Phi_\lambda = \Phi_c$ by non-criticality. □

5.4. Non-critical slopes

The aim of this section is to prove non-criticality for a certain class of $U_{\pi,i}$-eigenvalues. More precisely, we prove that $U_{\pi,i}$-eigenvalues whose valuations are bounded by certain explicit constants are non-critical. While these bounds are not good enough to obtain the non-criticality of the forms needed to prove Conjecture 4.29 in general, we still obtain the needed bounds for $k = 0$. We follow [Wil18].

5.51. Definition. Let $\alpha_1, \alpha_2 \in \mathcal{O}_L \setminus \{0\}$. We say that the pair (α_1, α_2) *has small slope* if

$$v(\alpha_i) \leq v_i^{\mathrm{crit}} \quad \text{where} \quad v_i^{\mathrm{crit}} = \begin{cases} k, & i = 1, \\ 0, & i = 2, \end{cases}$$

for $i \in \{1, 2\}$.

We will prove the following theorem.

5.52. Theorem. *Let $\alpha_1, \alpha_2 \in \mathcal{O}_L \setminus \{0\}$ be such that the pair (α_1, α_2) has small slope. Then (α_1, α_2) is non-critical in the sense of Definition 5.40.*

We obtain the following corollary.

5.53. Corollary. *Let $k = 0$. Then Conjecture 4.29 holds.*

Proof. For $k = 0$ the forms in $\mathbb{A}(V_k)^{\mathrm{new}}$ are $U_{\pi,i}$-eigenforms with eigenvalue 1 for both $i \in \{1, 2\}$. Hence these eigenvalues are small in the sense of the above definition. The result then follows from the above theorem and Theorem 5.48. □

5.54. Remark. We should remark that the bound in Theorem 5.52 is consistent with the standard literature, see for example [Wil18, Theorem 5.13], [BW20, Theorem 4.4] or [BC09, Proposition 7.3.5]. We find that it is a very interesting observation that, in stark contrast to the $\mathrm{GL}_2(K)$-case, the above bounds are not strong enough to prove Conjecture 4.29 aside from the trivial case $k = 0$. In fact we think that this leads to a rather deep underlying question, even for $K = \mathbb{Q}_p$: It seems to indicate that, when considering groups of higher rank, a large class of forms whom one would naturally expect to be non-critical, namely the space of forms new at the prime under consideration, lies outside the proven range of non-criticality. This means that the geometry of the corresponding eigenvarieties is much less understood. This raises many questions, for example on how to construct corresponding p-adic L-functions attached to these forms and other natural objects of deep arithmetic interest. In fact we view the link to distributions proved in the previous section as evidence towards the fact these forms should in fact be non-critical.

The remainder of this section is devoted to proving Theorem 5.52. This will be done in several steps. The first step is proving an abstract control theorem.

5.4.1. An abstract control theorem

The aim of this subsection is to prove a general control theorem, which will be applied to our situation in the subsequent subsections. Our theorem is an adaptation of the ideas in [Wil18, Section 1] to the setting of automorphic forms. All of this is based on [Gre06] and [Gre07]. Let M be a coefficient module with integral structure $(M^{\mathrm{int}}, (n_i)_{i=1,2})$. Assume that there is a decreasing $\mathcal{I}$-stable filtration $(F^n M^{\mathrm{int}})_{n\geq 0}$ on M^{int},

$$M^{\mathrm{int}} \supset F^0 M^{\mathrm{int}} \supset F^1 M^{\mathrm{int}} \supset \dots$$

with $\bigcap_{n\geq 0} F^n M^{\mathrm{int}} = 0$ and $M^{\mathrm{int}} = \varprojlim_n A^n M^{\mathrm{int}}$, where $A^n M^{\mathrm{int}} := M^{\mathrm{int}}/F^n M^{\mathrm{int}}$. We denote the natural projection $M^{\mathrm{int}} \to A^n M^{\mathrm{int}}$ by pr_n.

5.55. Theorem (Control theorem). *Let M be a coefficient module and let $(M^{\mathrm{int}}, (n_i)_{i=1,2})$ and $(F^n M^{\mathrm{int}})_{n\geq 0}$ as above. Fix $\alpha \in O_L \setminus \{0\}$ and $i \in \{1,2\}$. Assume that for all $n \geq 0$ and $\mu \in F^n M^{\mathrm{int}}$ we have*

$$\pi^{n_i} \mu \cdot y_i^{-1} \in \alpha F^{n+1} M^{\mathrm{int}}. \tag{7}$$

Then pr_0 induces an isomorphism

$$\rho\colon \mathbb{A}(M)_b^{U_{\pi,i}=\alpha} \to \mathbb{A}(A^0 M^{\mathrm{int}} \otimes_{O_L} L)_b^{U_{\pi,i}=\alpha}.$$

5.56. Remark. We need to explain what we mean by the space on the right hand side. Note that since we have

$$\pi^{n_i} \mu \cdot y_i^{-1} \in \alpha F^1 M^{\mathrm{int}} \subseteq F^0 M^{\mathrm{int}} \quad \text{for } \mu \in F^0 M^{\mathrm{int}}.$$

by taking quotients, the same condition holds in $A^0 M^{\mathrm{int}}$. This means that $A^0 M^{\mathrm{int}} \otimes_{O_L} L$ carries a natural structure of a coefficient module by Remark 5.5. The natural projection $M \to A^0 M^{\mathrm{int}} \otimes_{O_L} L$ is then G-equivariant and compatible with the chosen integral structures.

Proof. We observe first that since $y_{i,j} \in \mathcal{I} y_i \mathcal{I}$, we can replace y_i with any $y_{i,j}$ in (7). We begin by proving the injectivity. Let $\Phi \in \mathbb{A}(M)_b^{U_{\pi,i}=\alpha}$ with $\rho(\Phi) = 0$. Since Φ is bounded, after rescaling we may assume that $\Phi \in \mathbb{A}(M^{\mathrm{int}})^{U_{\pi,i}=\alpha}$. But then $\rho(\Phi) = 0$ translates to $\Phi \in \mathbb{A}(F^0 M^{\mathrm{int}})$. Recall that

$$(U_{\pi,i}\Phi)(g) = \pi^{n_i} \sum_j \Phi(g y_{i,j}) \cdot y_{i,j}^{-1}.$$

Then (7) implies $U_{\pi,i}\Phi \in \alpha F^1 M^{\mathrm{int}}$. Since $\Phi \in \mathbb{A}(M^{\mathrm{int}})^{U_{\pi,i}=\alpha}$ and M^{int} is torsion-free, it follows that $\Phi \in \mathbb{A}(F^1 M^{\mathrm{int}})$ and inductively, we obtain $\Phi \in \mathbb{A}(\bigcap_{n\geq 0} F^n M^{\mathrm{int}})$. But since

$\bigcap_{n\geq 0} F^n M^{\mathrm{int}} = 0$, we have $\Phi = 0$, hence we have proved the injectivity.

To prove the surjectivity, let $\varphi \in \mathbb{A}(A^0(M^{\mathrm{int}}) \otimes_{\mathcal{O}_K} K)_b^{U_{\pi,i}=\alpha}$. Since φ is bounded, after suitable rescaling, we may assume that $\varphi \in \mathbb{A}(A^0(M^{\mathrm{int}}))^{U_{\pi,i}=\alpha}$. Now we set

$$M_\alpha^{\mathrm{int}} \coloneqq \alpha M^{\mathrm{int}} + F^0 M^{\mathrm{int}}.$$

This is an $\mathcal{I}$-stable submodule of M^{int}. Moreover, by (7) we have

$$\pi^{n_i} \mu \cdot y_i^{-1} \in \alpha M^{\mathrm{int}} \quad \text{for } \mu \in M_\alpha^{\mathrm{int}}. \tag{8}$$

We also have $M_\alpha^{\mathrm{int}} = \varprojlim_n A^n M_\alpha^{\mathrm{int}}$, where $A^n M_\alpha^{\mathrm{int}} \coloneqq M_\alpha^{\mathrm{int}} / F^n M^{\mathrm{int}}$. By definition of M_α^{int}, we have

$$\alpha A^0 M^{\mathrm{int}} = \mathrm{pr}_0(\alpha M^{\mathrm{int}}) = \mathrm{pr}_0(M_\alpha^{\mathrm{int}}) = A^0 M_\alpha^{\mathrm{int}}.$$

Hence, we can even assume that $\varphi \in \mathbb{A}(A^0 M_\alpha^{\mathrm{int}})^{U_{\pi,i}=\alpha}$. Now, assume that we have already constructed a lift $\Phi^n \in \mathbb{A}(A^n M_\alpha^{\mathrm{int}})^{U_{\pi,i}=\alpha}$ of φ. We want to show that then there is a lift $\Phi^{n+1} \in \mathbb{A}(A^{n+1} M_\alpha^{\mathrm{int}})^{U_{\pi,i}=\alpha}$ compatible with Φ^n. For this, we fix coset representatives $y_{i,j}$ as in the definition of the $U_{\pi,i}$-operator. For each j, we pick a lift of Φ^n to a map

$$\tilde{\Phi}_j \colon K^\times \backslash G \to M_\alpha^{\mathrm{int}}.$$

Then we may define another map $\tilde{\Phi} \colon K^\times \backslash G \to M_\alpha^{\mathrm{int}}$ by

$$\tilde{\Phi}(g) \coloneqq \pi^{n_i} \sum_j \tilde{\Phi}_j(g y_{i,j}) \cdot y_{i,j}^{-1}.$$

Now (8) implies that $\tilde{\Phi}(g) \in \alpha M^{\mathrm{int}}$. Hence, we may define

$$\Phi^{n+1} \colon K^\times \setminus G \to A^{n+1} M^{\mathrm{int}}$$

by $\Phi^{n+1}(g) = \mathrm{pr}_{n+1}(\alpha^{-1}\tilde{\Phi}(g))$. By definition $\alpha^{-1}\tilde{\Phi}$ and Φ^{n+1} lift Φ^n, which implies that they take values in M_α^{int} and $A^{n+1} M_\alpha^{\mathrm{int}}$. We claim that Φ^{n+1} is independent of the choices of $(\tilde{\Phi}_j)_j$ in the construction. For this, let $(\tilde{\Phi}'_j)_j$ be another set of lifts. Then by construction

$$\tilde{\Phi}_j(g) - \tilde{\Phi}'_j(g) \in F^n M^{\mathrm{int}}.$$

Hence, the claim follows directly by (7). Next, we claim that in fact $\Phi^{n+1} \in \mathbb{A}(A^{n+1} M_\alpha^{\mathrm{int}})$. For this, let $h \in \mathcal{I}$. Then

$$\begin{aligned}
\Phi^{n+1}(gh^{-1}) \cdot h &= \mathrm{pr}_{n+1}(\alpha^{-1}\tilde{\Phi}(gh^{-1}) \cdot h) \\
&= \mathrm{pr}_{n+1}(\alpha^{-1}\pi^{n_i} \sum_j \tilde{\Phi}_j(gh^{-1}y_{i,j}) \cdot (y_{i,j}^{-1}h)) \\
&= \mathrm{pr}_{n+1}(\alpha^{-1}\pi^{n_i} \sum_j \tilde{\Phi}_j(g(h^{-1}y_{i,j})) \cdot (h^{-1}y_{i,j})^{-1}).
\end{aligned}$$

Since $h \in \mathcal{I}$, by using the double coset decomposition in the definition of the $U_{\pi,i}$-operator, we find elements $h_j \in \mathcal{I}$ and a permutation τ such that

$$\sum_j \tilde{\Phi}_j(g(h^{-1}y_{i,j})) \cdot (h^{-1}y_{i,j})^{-1} = \sum_j \tilde{\Phi}_{\tau(j)}(gy_{i,j}h_j)) \cdot h_j^{-1}y_{i,j}^{-1}$$

Now, since we assume that $\Phi^n \in \mathbb{A}(A^n M_\alpha^{\mathrm{int}})$, every $\tilde{\Phi}_{\tau(j)}(\cdot h_j) \cdot h_j^{-1}$ is again a lift of Φ^n. Hence, by our first claim, we see that

$$\Phi^{n+1}(gh^{-1}) \cdot h = \mathrm{pr}_{n+1}(\alpha^{-1}\pi^{n_i} \sum_j \tilde{\Phi}_j(gy_{i,j}) \cdot y_{i,j}^{-1}) = \Phi^{n+1}(g).$$

Now, we want to show that $\Phi^{n+1} \in \mathbb{A}(A^{n+1}M_\alpha^{\mathrm{int}})^{U_{\pi,i}=\alpha}$. But this follows directly from our first claim since $(\alpha^{-1}\tilde{\Phi})_j$ is another set of lifts of Φ^n. By induction, we obtain a compatible system $(\Phi^n)_{n\geq 0}$ of eigenforms $\Phi^n \in \mathbb{A}(A^n M_\alpha^{\mathrm{int}})^{U_{\pi,i}=\alpha}$ lifting φ. Since $M_\alpha^{\mathrm{int}} = \varprojlim_n A^n M_\alpha^{\mathrm{int}}$, these glue to $\Phi \in \mathbb{A}(M_\alpha^{\mathrm{int}})^{U_{\pi,i}=\alpha}$, proving the surjectivity. □

5.4.2. From algebraic to partially overconvergent coefficients

In this section, we apply Theorem 5.55 to our situation. We want to prove the following theorem, which implies part (ii) of the definition of non-criticality.

5.57. Theorem. *Let $i \in \{1,2\}$ and $\alpha \in \mathcal{O}_L \setminus \{0\}$ such that $v(\alpha) \leq v_i^{\mathrm{crit}}$. Then*

$$\rho_i\colon \mathbb{A}(\mathbb{D}^{\mathrm{rig}}_{P_i,k}(\sigma_0))_b^{U_{\pi,i}=\alpha} \to \mathbb{A}(V_k^{\mathrm{rig}})_b^{U_{\pi,i}=\alpha}$$

is an isomorphism.

We prove the theorem by verifying the assumptions of Theorem 5.55. This has to be done separately for $i = 1$ and $i = 2$. In both cases, by Proposition 5.25, we can work with the integral structure $\mathbb{D}^{\mathrm{int}}_{P_i,k}(\sigma_0)$. Thus, in order to apply Theorem 5.55, we need to construct appropriate filtrations of $\mathbb{D}^{\mathrm{int}}_{P_i,k}(\sigma_0)$ for $i \in \{1,2\}$. We begin with the case $i = 1$, which is also studied in [Wil18, Section 5.1]. For this, for $n \in \mathbb{Z}_{\geq 0}$ let

$$F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) := \left\{\mu \in \mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) \;\middle|\; \mu(\underline{x}^I) \in \pi^n\mathcal{O}_L \text{ for all } I\right\} \cap \ker(\pi_1).$$

Clearly, we have $F^{n+1}\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) \subseteq F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ and $\bigcap_{n\geq 0} F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) = 0$. The following lemma is based on [PP09, Proposition 4.4] and [Wil18, Proposition 5.4 and Lemma 5.7].

5.58. Lemma.

(i) The filtration $(F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0))_{n\geq 0}$ is $\mathcal{I}$-stable.

(ii) For $\mu \in F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ we have $\pi^{n_1}\mu \cdot y_1^{-1} \in \pi^k F^{n+1}\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$.

(iii) We have $\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) = \varprojlim_n A^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ and $A^0\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) = V_k^{\mathrm{int}}$.

Proof. For property (i), note that since π_1 is $\mathcal{I}$-equivariant, $\ker(\pi_1)$ is $\mathcal{I}$-stable. That the other condition is $\mathcal{I}$-stable clear from the definition. To show (ii), we compute

$$(\pi^{n_1}\mu \cdot y_1^{-1})(x_2^{i_2}x_3^{i_3}) = \pi^{i_2+i_3}\mu(x_2^{i_2}x_3^{i_3})$$

for $\mu \in F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$. But since $\mu \in \ker(\pi_1)$, we have $\mu(x_2^{i_2}x_3^{i_3}) = 0$ for $i_2 + i_3 \le k$. Hence, it follows that

$$(\pi^{n_1}\mu \cdot y_1^{-1})(x_2^{i_2}x_3^{i_3}) \in \pi^{k+1}\pi^n\mathcal{O}_L = \pi^k\pi^{n+1}\mathcal{O}_L,$$

which completes the proof of (ii). For (iii), observe that Lemma 5.23 implies

$$V_k^{\mathrm{int}} = \mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)/\ker(\pi_1) = A^0\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0).$$

We need to show that $\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) = \varprojlim_n A^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$. The injectivity follows directly from $\bigcap_{n\ge 0} F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0) = 0$. To prove the surjectivity, let $(\mu_n)_{n\ge 0} \in \varprojlim_n A^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ and fix lifts $\tilde{\mu}_n \in \mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ of μ_n for each n. By definition, we have

$$\tilde{\mu}_n - \tilde{\mu}_m \in F^m\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$$

for all $n \ge m$. This means the the sequence $(\tilde{\mu}(\underline{x}^I))_{n\ge 0} \subset \pi^{|I|}\mathcal{O}_L$ is Cauchy for each I, say converging to some $c_I \in \pi^{|I|}\mathcal{O}_L$. Let $\mu \in \mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ be the unique distribution that satisfies $\mu(\underline{x}^I) \coloneqq c_I$ for each I. Then μ projects to μ_n for each $n \ge 0$. □

5.59. Remark. We would like to point out that the condition $\mu(\underline{x}^I) \in \pi^n\mathcal{O}_L$ in the definition of $F^n\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ is in fact only a restriction on the moments $\mu(\underline{x}^I)$ with $|I| \le n$ as elements of $\mathbb{D}^{\mathrm{int}}_{P_1,k}(\sigma_0)$ satisfy $\mu(\underline{x}^I) \in \pi^{|I|}\mathcal{O}_L$ by definition. This makes it clear that this filtration is the exact analogue of the filtration considered in [Wil18, Section 5.1]. The difference being that we consider spaces of functions on $(\pi\mathcal{O}_K)^3$ whereas in [Wil18] the functions are defined on $\mathcal{O}_K^3$.

The lemma above shows that the assumptions of Theorem 5.55 are satisfied which proves Theorem 5.57 in the case $i = 1$.

Next, we consider the case $i = 2$. We follow the same strategy as in the case $i = 1$ and set

$$F^n\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0) \coloneqq \left\{\mu \in \mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0) \;\middle|\; \mu(x_1^{i_1}(x_2 - x_1x_3)^{i_2}x_3^{i_3}) \in \pi^n\mathcal{O}_L \text{ for all } I\right\} \cap \ker(\pi_2)$$

for $n \in \mathbb{Z}_{\ge 0}$. Clearly, we have $F^{n+1}\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0) \subseteq F^n\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0)$ and $\bigcap_{n\ge 0} F^n\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0) = 0$.

5.60. Lemma.

(i) The filtration $(F^n\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0))_{n\ge 0}$ is $\mathcal{I}$-stable.

(ii) For $\mu \in F^n\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0)$ we have $\pi^{n_2}\mu \cdot y_2^{-1} \in F^{n+1}\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0)$.

(iii) We have $\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0) = \varprojlim_n A^n\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0)$ and $A^0\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0) = V_k^{\mathrm{int}}$.

Proof. Properties (i) and (iii) can be checked in complete analogy with Lemma 5.58. For (ii), we compute

$$(\pi^{n_2}\mu \cdot y_2^{-1})(x_1^{i_1}(x_2 - x_1x_3)^{i_2}x_3^{i_3}) = \pi^{i_1+i_2}\mu(x_1^{i_1}(x_2 - x_1x_3)^{i_2}x_3^{i_3})$$

for $\mu \in F^n\mathbb{D}^{\mathrm{int}}_{P_2,k}(\sigma_0)$. But since $\mu \in \ker(\pi_2)$, we have $\mu(x_1^{i_1}(x_2 - x_1x_3)^{i_2}x_3^{i_3}) = 0$ for $i_1 = i_2 = 0$. Hence, it follows that

$$(\pi^{n_2}\mu \cdot y_2^{-1})(x_1^{i_1}(x_2 - x_1x_3)^{i_2}x_3^{i_3}) \in \pi\pi^n\mathcal{O}_L = \pi^{n+1}\mathcal{O}_L,$$

which completes the proof of (ii). □

Again we see that all assumptions in Theorem 5.55 are satisfied proving Theorem 5.57 in the case $i = 2$.

5.4.3. From partially overconvergent to overconvergent coefficients

To prove Theorem 5.52, we need to lift from partially overconvergent to overconvergent coefficients. It suffices to do this in the case $i = 1$. We want to prove the following theorem by applying Theorem 5.55.

5.61. Theorem. *Let $\alpha \in \mathcal{O}_L \setminus \{0\}$ such that $v(\alpha) = 0$. Then*

$$\rho^1 \colon \mathbb{A}(\mathbb{D}^{\mathrm{rig}}_k(\sigma_0))_b^{U_{\pi,2}=\alpha} \to \mathbb{A}(\mathbb{D}^{\mathrm{rig}}_{P_1,k}(\sigma_0))_b^{U_{\pi,2}=\alpha}$$

is an isomorphism.

5.62. Remark. Similarly, one can prove that

$$\rho^2 \colon \mathbb{A}(\mathbb{D}^{\mathrm{rig}}_k(\sigma_0))_b^{U_{\pi,1}=\alpha} \to \mathbb{A}(\mathbb{D}^{\mathrm{rig}}_{P_2,k}(\sigma_0))_b^{U_{\pi,1}=\alpha}$$

is an isomorphism for $\alpha \in \mathcal{O}_L \setminus \{0\}$ such that $v(\alpha) \leq k$. As this is not needed for our application, we omit the proof.

For the proof, by Proposition 5.25, we can work with the integral structure $\mathbb{D}^{\mathrm{int}}_k(\sigma_0)$. Thus, in order to apply Theorem 5.55, we need to construct an appropriate filtration of $\mathbb{D}^{\mathrm{int}}_k(\sigma_0)$. This is similar to [Wil18, Section 5.2]. For $n \in \mathbb{Z}_{\geq 0}$ let

$$F^n\mathbb{D}^{\mathrm{int}}_k(\sigma_0) := \left\{\mu \in \mathbb{D}^{\mathrm{int}}_k(\sigma_0) \;\middle|\; \mu(\underline{x}^I) \in \pi^n\mathcal{O}_L\right\} \cap \ker(\pi^1).$$

Clearly, we have $F^{n+1}\mathbb{D}^{\mathrm{int}}_k(\sigma_0) \subseteq F^n\mathbb{D}^{\mathrm{int}}_k(\sigma_0)$ and $\bigcap_{n\geq 0} F^n\mathbb{D}^{\mathrm{int}}_k(\sigma_0) = 0$.

5.63. Lemma.

(i) The filtration $(F^n\mathbb{D}^{\mathrm{int}}_k(\sigma_0))_{n\geq 0}$ is $\mathcal{I}$-stable.

(ii) For $\mu \in F^n\mathbb{D}^{\mathrm{int}}_k(\sigma_0)$ we have $\pi^{n_2}\mu \cdot y_2^{-1} \in F^{n+1}\mathbb{D}^{\mathrm{int}}_k(\sigma_0)$.

(iii) We have $\mathbb{D}_k^{\mathrm{int}}(\sigma_0) = \varprojlim_n A^n\mathbb{D}_k^{\mathrm{int}}(\sigma_0)$ *and* $A^0\mathbb{D}_k^{\mathrm{int}}(\sigma_0) = \mathbb{D}_{P_1,k}^{\mathrm{int}}(\sigma_0)$.

Proof. Properties (i) and (iii) can be checked in complete analogy with Lemma 5.58. For (ii), we compute

$$(\pi^{n_2}\mu \cdot y_2^{-1})(\underline{x}^I) = \pi^{i_1+i_2}\mu(\underline{x}^I)$$

for $\mu \in F^n\mathbb{D}_k^{\mathrm{int}}(\sigma_0)$. But since $\mu \in \ker(\pi^1)$, we have $\mu(\underline{x}^I) = 0$ for $i_1 = 0$. Hence, it follows that

$$(\pi^{n_2}\mu \cdot y_2^{-1})(\underline{x}^I) \in \pi\pi^n\mathcal{O}_L = \pi^{n+1}\mathcal{O}_L,$$

which completes the proof of (ii). □

We see that the assumptions of Theorem 5.55 are satisfied, which proves Theorem 5.61.

Now, we have all the tools needed to prove Theorem 5.52.

Proof of Theorem 5.52. First, we observe that by definition we can work with forms with rigid analytic coefficients. Then by Theorem 5.57, part (ii) of Definition 5.40 is satisfied. We need to show part (i). For this, observe that by Theorem 5.61 we have that

$$\rho^1 \colon \mathbb{A}(\mathbb{D}_k^{\mathrm{rig}}(\sigma_0))_b^{U_{\pi,2}=\alpha_2} \to \mathbb{A}(\mathbb{D}_{P_1,k}^{\mathrm{rig}}(\sigma_0))_b^{U_{\pi,2}=\alpha_2}$$

is an isomorphism. But since $U_{\pi,1} \circ U_{\pi,2} = U_{\pi,2} \circ U_{\pi,1}$, this induces an isomorphism

$$\rho^1 \colon \mathbb{A}(\mathbb{D}_k^{\mathrm{rig}}(\sigma_0))_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}} \to \mathbb{A}(\mathbb{D}_{P_1,k}^{\mathrm{rig}}(\sigma_0))_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}.$$

In the same way we obtain the isomorphism

$$\rho_1 \colon \mathbb{A}(\mathbb{D}_{P_1,k}^{\mathrm{rig}}(\sigma_0))_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}} \to \mathbb{A}(V_k)_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}.$$

Composing these two isomorphisms and noting that $\rho_0 = \rho_1 \circ \rho^1$ shows that part (i) is satisfied, completing the proof. □

5.64. Remark. We should remark that the proof of Theorem 5.55, applied to the specific filtrations above, provides an algorithm for computing values of the (unique) lifts in practice: One needs to take an arbitrary initial lift and the iterate the (rescaled) $U_{\pi,i}$-operator to compute the values of the lift to higher and higher precision. This method has proven to be very effective for computations for $\mathrm{GL}_2(K)$, see [Gre06] and [FM14].

APPENDIX A

Locally analytic manifolds and representations

In this appendix, we provide the all notions needed in order to define locally analytic representations and the locally analytic induction functor. Since most of the standard literature only considers only the case where K is a finite extension of $\mathbb{Q}_p$, we quickly develop the necessary tools in more generality. We do not discuss duality and the operation of the Lie algebra here as we expect these constructions to behave very differently over local fields of positive characteristic. However, the notion of a locally analytic representations can be adapted without any difficulty. All of this is based on [Fea99], [ST02a] and [Eme17].

A.1. Locally convex vector spaces

We follow [ST02a, Section 1] and [Fea99, Section 1], see also [Sch02]. Let L be a complete extension of K inside $\mathbb{C}_K$.

A.1. Definition. Let V be a (not necessarily finite-dimensional) topological L-vector space.

(i) V is called *locally convex* if it has a fundamental system of open neighbourhoods consisting of $\mathcal{O}_L$-modules.

(ii) An $\mathcal{O}_L$-submodule of V is called a *lattice* if it L-linearly generates V.

Note that in a locally convex L-vector space any open $\mathcal{O}_L$-submodule is a lattice.

A.2. Definition. Let V be a Hausdorff topological L-vector space. An L-subvector space U of V is called an *FH-space* if there exists a Fréchet-topology τ on U such that the inclusion $(U, \tau) \to V$ is continuous. It is called a *BH-space* if the topology τ is a Banach topology.

A.3. Definition. Let V be a locally convex Hausdorff L-vector space.

(i) V is called *barrelled* if each closed lattice in V is open.

(ii) A subset $B \subseteq V$ is called *compactoid* if for any open lattice $\Lambda \subset V$ there are finitely many vectors $v_1, \dots, v_n$ such that $B \subset \Lambda + \mathcal{O}_L v_1 + \dots + \mathcal{O}_L v_n$.

(iii) A bounded O_L-submodule $B \subseteq V$ is called *c-compact* if it is compactoid and complete.

(iv) Let W be another locally convex Hausdorff L-vector space and let $f: V \to W$ be a continuous linear map. We call f *compact* if there is an open lattice $\Lambda \subset V$ such that the closure of $f(\Lambda)$ in W is c-compact.

(v) V is called *of compact type* if it is the locally convex direct limit of a sequence

$$V_1 \xrightarrow{\iota_1} V_2 \xrightarrow{\iota_2} V_3 \xrightarrow{\iota_3} \cdots$$

of locally convex Hausdorff L-vector spaces V_n for $n \in \mathbb{N}$ with injective compact linear maps ι_n.

A.4. Remark. In part (v) of the above definition one can require the stronger condition that the spaces V_n are Banach spaces, see [Eme17, Section 1.1]. Note that every vector space of compact type is complete, see [Fea99, Theorem 1.2.8].

A.2. Locally analytic manifolds

With the preparations in the previous section we are now able to define locally analytic manifolds and locally analytic functions on them. We follow [DT08, Section 2.1], [Bou67, Section 5.1] and [ST02a, Section 2]. We keep the notation from the previous section. Fix $n \geq 1$. Let $\underline{u} = (u_1, \ldots, u_n) \in K^n$ and $\underline{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$. Let $D(\underline{u}, \underline{r})$ be the closed polydisc in K^n given by

$$D(\underline{u}, \underline{r}) = \left\{ \underline{x} \in K^n \;\middle|\; |x_i - u_i| \leq q^{-r_i} \text{ for } i \in \{1, \ldots, n\} \right\}.$$

For $r \in \mathbb{R}$, we also write $D(\underline{u}, r) = D(\underline{u}, (r, \ldots, r))$. Observe that the above polydiscs form a basis for the topology of K^n. A *K-analytic function on* $D(\underline{u}, \underline{r})$ is given by a convergent power series

$$f(\underline{x}) = \sum_I c_I (\underline{x} - \underline{u})^I,$$

where the sum is over n-tuples $I = (i_1, \ldots, i_n)$ with $i_j \geq 0$, $c_I \in L$ and

$$(\underline{x} - \underline{u})^I := \prod_{j=1}^{n} (x_j - u_j)^{i_j}.$$

The convergence condition is

$$|c_I| \cdot q^{-\sum_{j=1}^n r_j i_j} \to 0 \quad \text{as} \quad |I| = \sum_{j=0}^{n} i_j \to \infty.$$

We denote the L-vector space of such analytic functions by $A_L(D(\underline{u}, \underline{r}))$. It becomes a Banach algebra with respect to the norm

$$\|f\|_{D(\underline{u}, \underline{r})} = \sup_I \left\{ |c_I| \cdot q^{-\sum_{j=1}^n r_j i_j} \right\}.$$

A.5. Definition. Let $U, V \subseteq K^n$ open. A map $f: U \to V$ is called *K-analytic* if for each $\underline{u} \in U$ we find $\underline{r} \in \mathbb{R}^n$ such that $D(\underline{u}, \underline{r}) \subseteq U$ and such that each component of f is K-analytic on $D(\underline{u}, \underline{r})$.

A.6. Definition. Let M be a paracompact topological space.

(i) A *K-analytic chart* (M_i, φ_i) for M is a tuple consisting of an open subset $M_i \subset M$ and a homeomorphism

$$\varphi_i : M_i \to D_i = D(0, \underline{r}) \subset K^n$$

for some $\underline{r} \in \mathbb{R}^n$.

(ii) Two K-analytic charts (M_i, φ_i) and (M_j, φ_j) for M are called *compatible* if the map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(M_i \cap M_j) \to \varphi_i(M_i \cap M_j)$$

is K-analytic.

(iii) A collection of compatible K-analytic charts (M_i, φ_i) for M such that $\bigcup_i M_i = M$ is called an *atlas* for M.

(iv) An atlas $\mathcal{A}$ for M is called *maximal* if there is no atlas $\mathcal{B}$ for M such that $\mathcal{A} \subsetneq \mathcal{B}$.

(v) The pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a maximal atlas for M, is called a *(locally) K-analytic manifold*.

Note that one can naturally define the notion of a K-analytic map between two K-analytic manifolds as in the classical theory.

A.7. Example. The following spaces can be naturally viewed as K-analytic manifolds: The group $\mathrm{GL}_n(K)$ and the projective space $\mathbb{P}^n(K)$ for any $n \geq 1$.

Via the atlas $\mathcal{A}$ for M, we can identify the analytic functions on the set M_i with those on D_i for each chart (M_i, φ_i). We write $A_L(M_i, \varphi_i)$ for this space of functions. Note that every covering of a K-analytic manifold can be refined to a disjoint covering.

A.8. Definition. Let M be a K-analytic manifold. The *locally analytic functions on M* (with values in L) are defined as follows. For each covering of M by disjoint charts (M_i, φ_i) we form the locally convex direct product

$$C^{\mathrm{an}}(\{M_i, \varphi_i\}) := \prod_i A_L(M_i, \varphi_i).$$

We set

$$C^{\mathrm{an}}(M, L) := \varinjlim C^{\mathrm{an}}(\{M_i, \varphi_i\}),$$

where the limit is over finer and finer coverings. This space is equipped with the locally convex direct limit topology.

A.9. Proposition. *Suppose that M is compact. Then the above limit realizes $C^{\mathrm{an}}(M, L)$ as a vector space of compact type in the sense of Definition A.16.*

Proof. This follows from [Fea99, Satz 2.3.2]. □

We also need locally analytic functions with more general coefficients. Let V be a locally convex Hausdorff L-vector space. A *V-index $\mathcal{I}$* on M is a family of triples $(M_i, \varphi_i, V_i)_{i\in I}$, where $(M_i, \varphi_i)_{i\in I}$ is an atlas for M and $V_i \to V$ are BH-spaces. We set

$$\mathcal{F}_{V_i}(M_i, \varphi_i) := A_L(M_i, \varphi_i)\hat{\otimes}_L V_i$$

and form the locally convex direct product

$$\mathcal{F}_V(\mathcal{I}) := \prod_i \mathcal{F}_{V_i}(M_i, \varphi_i).$$

A.10. Definition. Let M be a K-analytic manifold. The *locally analytic functions on M with values in V* are the elements of

$$C^{\mathrm{an}}(M, V) := \varinjlim_{\mathcal{I}} \mathcal{F}_V(\mathcal{I}),$$

where we note that the V-indices on M form a directed set on which $\mathcal{F}_V(\mathcal{I})$ is a directed system. We equip the space with the locally convex direct limit topology.

This definition is compatible with the above definition for $V = L$. We can extend Proposition A.9 as follows.

A.11. Proposition. *Suppose that M is compact and that V is of compact type. Then $C^{\mathrm{an}}(M, V)$ is a vector space of compact type.*

Proof. See [Eme17, Proposition 2.1.18]. □

A.3. Locally analytic representations

We are now able to define locally analytic representations. In this section, we follow [Fea99, Section 4] and [ST02a, Section 3]. Let G be a locally analytic K-group, i.e., G is a group that carries a structure as a K-analytic manifold such that all group operations are analytic.

A.12. Definition. Let V be a barrelled locally convex Hausdorff L-vector space on which G acts via continuous linear endomorphisms. Then V is called a *locally analytic G-representation* if, for each $v \in V$, the orbit map $\rho_v(g) := gv$ is a V-valued locally analytic function on G.

The basic example is given as follows. Let V be a barrelled locally convex Hausdorff L-vector space. We let G act on $C^{\mathrm{an}}(G, V)$ via

$$(g_* f)(h) = f(g^{-1}h) \quad \text{for } g, h \in G, f \in C^{\mathrm{an}}(G, V).$$

Then we obtain the following.

A.13. Proposition. *Let G be compact. Then $C^{\mathrm{an}}(G, V)$ is a locally analytic G-representation.*

Proof. See [Fea99, Satz 3.3.4]. □

Now we can define the locally analytic induction functor. Let $H \subseteq G$ be a closed locally analytic subgroup and $\rho\colon H \to \mathrm{Aut}(V)$ be a locally analytic H-representation. We define

$$\mathrm{Ind}_H^G(\rho) := \left\{ f \in C^{\mathrm{an}}(G, V) \;\middle|\; f(gh) = \rho(h^{-1}) f(g) \text{ for } g \in G, h \in H \right\}.$$

This is a G-stable closed subspace of $C^{\mathrm{an}}(G, V)$.

A.14. Proposition. *The quotient G/H carries a natural structure of a K-analytic manifold. The quotient map $G \to G/H$ splits. Each splitting induces an isomorphism of K-analytic manifolds $G/H \times H \to G$.*

Proof. See [Fea99, Satz 4.1.1]. □

A.15. Proposition. *Assume that G/H is compact. Then $\mathrm{Ind}_H^G(\rho)$ is a locally analytic G-representation.*

Proof. See [Fea99, Satz 4.1.5]. □

A.16. Proposition. *Assume that V is of compact type and that there is a compact open subgroup $\mathcal{K}$ of G such that $G = \mathcal{K}H$. Then $\mathrm{Ind}_H^G(\rho)$ is of compact type.*

Proof. This follows from Proposition A.11 by the same arguments as in [Eme07, Section 2.1]. □

We need the following result.

A.17. Proposition. *Assume that V is a Banach space. Then every splitting ι of the natural projection $G \to G/H$ induces an isomorphism (of topological vector spaces)*

$$\iota^*\colon \mathrm{Ind}_H^G(\rho) \to C^{\mathrm{an}}(G/H, V).$$

Proof. See [Fea99, Satz 4.3.1]. □

The locally analytic induction satisfies a version of Frobenius reciprocity.

A.18. Theorem (Frobenius reciprocity)**.** *Let W be a locally analytic G-representation. We have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{cont},G}(W, \mathrm{Ind}_H^G(\rho)) \cong \mathrm{Hom}_{\mathrm{cont},H}(\mathrm{Res}_H^G(W), \rho),$$

where $\mathrm{Res}_H^G(\cdot)$ is the usual restriction functor on representations.

Proof. See [Fea99, Theorem 4.2.6] and [OS10, Section 2.4]. □

As an immediate consequence we obtain that the functor $\mathrm{Ind}_H^G(\cdot)$ is left-exact. Moreover, we have the following.

A.19. Proposition. *Let $H \subseteq H' \subseteq G$ such that the quotients are compact. Then we have a natural isomorphism*

$$\mathrm{Ind}_{H'}^G(\mathrm{Ind}_H^{H'}(\rho)) \cong \mathrm{Ind}_H^G(\rho).$$

We also need the following.

A.20. Proposition. *Assume that G/H is compact and let $0 \to U \to V \to W \to 0$ be an exact sequence of finite-dimensional locally analytic H-representations. Then the induced sequence*

$$0 \to \mathrm{Ind}_H^G(U) \to \mathrm{Ind}_H^G(V) \to \mathrm{Ind}_H^G(W) \to 0$$

is exact.

Proof. It suffices to check exactness on the right. For this, note that as the map in Proposition A.17 is functorial, it suffices to show that $C^{\mathrm{an}}(G/H, V) \to C^{\mathrm{an}}(G/H, W)$ is surjective. But since G/H is compact and the coefficients are finite-dimensional, we have $C^{\mathrm{an}}(G/H, V) \cong C^{\mathrm{an}}(G/H, L) \otimes_L V$ and $C^{\mathrm{an}}(G/H, W) \cong C^{\mathrm{an}}(G/H, L) \otimes_L W$ and the statement is clear. □

A.21. Remark. We should note that the above Proposition holds in much more generality, at least if L is spherically complete and of characteristic zero, see for example [Sch11, Corollaire 4.14]. However, the proof relies on the duality theory of Schneider-Teitelbaum which is not applicable in our situation.

PART II

Application to Drinfeld modular forms of rank 3

CHAPTER 6

Drinfeld modular forms

The aim of the second part of this thesis is to apply the theory developed in Part I to Drinfeld modular forms of arbitrary weight for certain arithmetic subgroups of $\mathrm{GL}_3(K)$. We fix the following notation. Let $A = \mathbb{F}_q[t]$, $F = \mathbb{F}_q(t)$ and denote by ν the discrete valuation on F given by $\nu(f/g) = \deg(g) - \deg(f)$. Then the completion of F with respect to ν is given by $K = \mathbb{F}_q(\!(1/t)\!)$. This is a local field of the type considered in Part I. We keep the remaining notation from Part I. In particular, let $O_K = \mathbb{F}_q[\![1/t]\!]$, the ring of integers in K with residue field $\kappa = \mathbb{F}_q$. Let $\mathbb{C}_K$ be the completion of an algebraic closure of K.

6.1. Basic definitions

We begin with basic constructions needed to define Drinfeld modular forms (of rank 3). We follow [BBP18a]. We should however note that we keep the normalizations from Part I. In particular, our coordinates on $\mathcal{X}$ differ from the ones chosen in [BBP18a]. We will point out the differences that arise at the relevant places throughout this section.

In the sequel, let $\Gamma \subseteq \mathrm{GL}_3(F)$ denote an *arithmetic subgroup*, i.e., a subgroup that is commensurable with $\mathrm{GL}_3(A)$. If furthermore Γ is contained in $\mathrm{GL}_3(A)$ and contains $\Gamma(N) := \ker(\mathrm{GL}_3(A) \to \mathrm{GL}_3(A/N))$ for some non-zero ideal $N \subseteq A$ it is called a *congruence subgroup*. As in the classical case, standard examples of congruence subgroups are

$$\Gamma_1(N) := \left\{ g \in \mathrm{GL}_3(A) \,\middle|\, g \equiv \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\},$$
$$\Gamma_0(N) := \left\{ g \in \mathrm{GL}_3(A) \,\middle|\, g \equiv \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} (\mathrm{mod}\ N) \right\}.$$

Note that usually the congruence conditions in the above definitions are transposed. The reason for this change lies in the fact that we regard $O_{\mathcal{X}}(k)$ as a left G-module, whereas for example in [BBP18a] it is considered as a right G-module, see also Remark 6.3 below.

6.1. Lemma. *Let $\Gamma \subseteq \mathrm{GL}_3(A)$ a congruence subgroup and $g \in \mathrm{GL}_3(A)$. Then the subgroup $g\Gamma g^{-1} \subseteq \mathrm{GL}_3(A)$ is again a congruence subgroup.*

Proof. See [Bas14, Lemma 4.1.1]. □

The following definition is of central importance.

6.2. Definition. Let $k, \ell \in \mathbb{Z}$ such that $k \geq 0$ and $3 \mid k$. A *weak Drinfeld modular form* of weight k and type ℓ for Γ is an element of

$$\mathcal{W}_{k,\ell}(\Gamma) := \left(\mathcal{O}_{\mathcal{X}}(k) \otimes_{\mathbb{C}_K} \det^{\ell-k/3}\right)^{\Gamma}.$$

We also abbreviate $\mathcal{W}_k(\Gamma) := \mathcal{W}_{k,k/3}(\Gamma)$.

6.3. Remark. There are several differences to the analogous definition in [BBP18a, Definition 1.9], which we would like to address: First of all, the coordinates on $\mathcal{X}$ in [BBP18a] are chosen such that the last entry is an unspecified (arithmetic) constant $\xi \in \mathbb{C}_K$. In our situation, we chose $\xi = 1$. Secondly, we regard $\mathcal{O}_{\mathcal{X}}(k)$ as a left G-module, whereas in [BBP18a], it is equipped with a right G-action denoted by $\cdot|_{k,\ell}$. The relation between the two actions is given as follows. Let $f \in \mathcal{O}_{\mathcal{X}}(k)$. Then we have

$$g_* f = f|_{k,\ell} g^t \quad \text{for } g \in G \text{ with } \ell = \tfrac{k}{3}.$$

Now we turn to the expansions at infinity. This needs some preparations. We follow [BBP18a, Section 4] closely. Recall that if z is a point in $\mathcal{X}$, we can always renormalize such that $z = [z_1 : z_2 : 1]$. Then we have $\omega_i(z) = z_i$ for $i \in \{1, 2\}$. By abuse of notation, we simply write ω for points that are normalized in this manner. Let H denote the subgroup of $\mathrm{GL}_3(F)$ of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$$

For any arithmetic subgroup $\Gamma \subseteq \mathrm{GL}_3(F)$ we set $\Gamma_H := \Gamma \cap H$. We have a natural isomorphism $\iota \colon F^2 \to \Gamma_H$ given by $[x_1, x_2] \mapsto \left(\begin{smallmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{smallmatrix}\right)$. By definition, every weak modular form for Γ is a Γ_H-invariant function. Since Γ is arithmetic, the subgroup

$$\Lambda := \iota^{-1}(\Gamma_H) \subset F^2$$

is commensurable with A^2. Let $\mathcal{Y} = \mathbb{P}^1_K \setminus \mathbb{P}^1(K)$, the Drinfeld period domain for $\mathrm{GL}_2(K)$, which is a rigid space over K by exactly the same arguments as in Chapter 1. We have a natural inclusion of rigid spaces

$$\begin{aligned} \mathcal{X} &\to \mathbb{A}^1_K \times \mathcal{Y}, \\ \omega &\mapsto (\omega_1, \tilde{\omega}), \end{aligned}$$

where $\tilde{\omega} = [\omega_2 : 1]$. Observe that we have

$$\iota(\lambda)_*\omega = (\omega_1 + \lambda\tilde{\omega}, \tilde{\omega}) \quad \text{for } \lambda \in \Lambda,$$

where $\lambda\tilde{\omega}$ denotes the matrix product. This extends to an action of Γ_H on $\mathbb{A}^1_K \times \mathcal{Y}$. We define

$$\begin{aligned} e\colon \mathbb{A}^1_K \times \mathcal{Y} &\to \mathbb{A}^1_K, \\ (\omega_1, \tilde{\omega}) &\mapsto \prod_{\lambda \in \Lambda\setminus\{0\}} \left(1 - \frac{\omega_1}{\lambda\tilde{\omega}}\right). \end{aligned}$$

We obtain the following.

6.4. Proposition. *The function $e\colon \mathbb{A}^1_K \times \mathcal{Y} \to \mathbb{A}^1_K$ is well-defined and rigid analytic.*

Proof. See [BBP18a, Proposition 4.7]. □

We define

$$\begin{aligned} \mathcal{E}\colon \mathbb{A}^1_K \times \mathcal{Y} &\to \mathbb{A}^1_K \times \mathcal{Y}, \\ (\omega_1, \tilde{\omega}) &\mapsto (e(\omega_1, \tilde{\omega}), \tilde{\omega}). \end{aligned}$$

The following proposition is the key step towards the expansion at infinity.

6.5. Proposition. *The action of Γ_H on $\mathbb{A}^1_K \times \mathcal{Y}$ is free and discontinuous and the quotient $\Gamma_H\backslash(\mathbb{A}^1_K \times \mathcal{Y})$ exists as a rigid space. The function $\mathcal{E}$ induces an isomorphism*

$$\mathcal{E}\colon \Gamma_H\backslash(\mathbb{A}^1_K \times \mathcal{Y}) \to \mathbb{A}^1_K \times \mathcal{Y},$$

Proof. It follows from [BBP18a, Proposition 4.10] that the quotient exists and that $\mathcal{E}$ is rigid analytic. Moreover, we obtain that the base change of $\mathcal{E}$ to $\mathbb{C}_K$ which we denote by $\mathcal{E}_{\mathbb{C}_K}$, is an isomorphism of rigid spaces over $\mathbb{C}_K$. To conclude that $\mathcal{E}$ is already an isomorphism over K, we can apply [Con06, Theorem 4.2.3] since $K \to \mathbb{C}_K$ is faithfully flat and quasi-compact. □

The following definitions are analogous to [BBP18a, Definition 4.12 and Definition 4.13].

6.6. Definition.

(i) For $n \in \mathbb{Z}_{>0}$ and $R_n > 0$ let

$$I(n, R_n) := \left\{\omega \in \mathcal{X} \;\middle|\; \tilde{\omega} \in \mathcal{Y}_n, d(\omega_1, K^2\tilde{\omega}) \geq R_n\right\},$$

where $d(\cdot,\cdot)$ denotes the usual distance function on $\mathbb{C}_K$ and the sets $\mathcal{Y}_n$ are defined in analogy with $\mathcal{X}_n$, see [BBP18a, Section 3]. Then $I(n, R_n)$ is Γ_H-invariant. A Γ_H-invariant admissible open subset $\mathcal{N} \subseteq \mathcal{X}$ such that for each $n > 0$ there exists an $R_n > 0$ with $I(n, R_n) \subseteq \mathcal{N}$ is called a *neighbourhood of infinity*.

(ii) A subset of $\mathbb{A}^1_K \times \mathcal{Y}$ of the form

$$T = \bigcup_{n \geq 1} D(0, r_n) \times \mathcal{Y}_n$$

with $r_n \in \mathbb{Z}$ is called a *tubular neighbourhood of* $\{0\} \times \mathcal{Y}$, or just a *tubular neighbourhood*. The intersection of a tubular neighbourhood with $(\mathbb{A}^1_K \setminus \{0\}) \times \mathcal{Y}$ is called a *pierced tubular neighbourhood*.

Note that both tubular and pierced tubular neighbourhoods are admissible open. Observe that for $\omega \in \mathcal{X}$ the function e has no zeros. Hence, we may form the inverse

$$u(\omega) := \frac{1}{e(\omega_1, \tilde{\omega})}.$$

We may set

$$\begin{aligned} \vartheta \colon \mathcal{X} &\to (\mathbb{A}^1_K \setminus \{0\}) \times \mathcal{Y}, \\ \omega &\mapsto (u(\omega), \tilde{\omega}), \end{aligned}$$

and obtain the following.

6.7. Theorem.

(i) The map ϑ induces an isomorphism of rigid analytic spaces from $\Gamma_H \backslash \mathcal{X}$ to an admissible open subset of $(\mathbb{A}^1_K \setminus \{0\}) \times \mathcal{Y}$.

(ii) For any neighbourhood of infinity $\mathcal{N} \subseteq \mathcal{X}$, the image $\vartheta(\Gamma_H \backslash \mathcal{N})$ contains a pierced tubular neighbourhood.

(iii) For any pierced tubular neighbourhood $T \subseteq (\mathbb{A}^1_K \setminus \{0\}) \times \mathcal{Y}$ contained in the image of ϑ there is a neighbourhood of infinity $\mathcal{N} \subseteq \mathcal{X}$ such that ϑ induces an isomorphism

$$\vartheta \colon \Gamma_H \backslash \mathcal{N} \to T.$$

Proof. This is proved exactly as [BBP18a, Theorem 4.16] using Proposition 6.5. □

Now we have all preparations needed to define the expansion at infinity. In the sequel, whenever we refer to a rigid analytic function, we regard the underlying rigid space as a rigid space over $\mathbb{C}_K$ as we did for $\mathcal{X}$ in Chapter 1.

6.8. Lemma. *Let $T \subseteq (\mathbb{A}^1_K \setminus \{0\}) \times \mathcal{Y}$ be a pierced tubular neighbourhood. Then any rigid analytic function $f \in \mathcal{O}_T$ has a unique Laurent series expansion*

$$f(\omega_1, \tilde{\omega}) = \sum_{n \in \mathbb{Z}} f_n(\tilde{\omega}) \omega_1^n$$

with $f_n \in \mathcal{O}_{\mathcal{Y}}$ which converges uniformly on every affinoid subset of T.

Proof. See [BBP18a, Lemma 5.3]. □

This lemma and Theorem 6.7 have the following consequence.

6.9. Proposition. *Let $f \in \mathcal{O}_{\mathcal{X}}^{\Gamma_H}$. Then there exist unique rigid analytic functions $f_n \in \mathcal{O}_{\mathcal{Y}}$ such that*

$$f(\omega) = \sum_{n \in \mathbb{Z}} f_n(\tilde{\omega}) u(\omega)^n$$

on some neighbourhood of infinity. The convergence is uniform on each affinoid subset.

Proof. See [BBP18a, Proposition 5.4]. □

Let now L_H denote the subgroup of $\mathrm{GL}_3(F)$ of matrices of the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

so that L_H is the Levi subgroup of the parbolic subgroup $P := L_H \cdot H$ of $\mathrm{GL}_3(F)$. We set

$$\tilde{\Gamma} := \left\{ g \in \mathrm{GL}_2(F) \,\middle|\, \left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right) \in \Gamma \cap L_H \right\}.$$

6.10. Theorem. *Let $f \in \mathcal{W}_{k,\ell}(\Gamma)$ and denote by f_n the coefficients of the expansion in Proposition 6.9. Then f_n is a weak modular form of weight $k - n$ and type ℓ for $\tilde{\Gamma} \subseteq \mathrm{GL}_2(F)$ for each $n \in \mathbb{Z}$, defined for example in [BBP18a, Definition 1.9].*

Proof. See [BBP18a, Theorem 5.9]. □

We can now define the order at infinity as follows.

6.11. Definition. Let $f \in \mathcal{O}_{\mathcal{X}}^{\Gamma_H}$. Then the *order at infinity of f* is

$$\mathrm{ord}_{\Gamma_H}(f) := \inf \{ n \in \mathbb{Z} \mid f_n \neq 0 \},$$

where f_n denote the coefficients from Proposition 6.9. We say that f is *holomorphic at infinity* if $\mathrm{ord}_{\Gamma_H}(f) \geq 0$ and we say that f *vanishes at infinity* if $\mathrm{ord}_{\Gamma_H}(f) \geq 1$.

We will use the following criterion to show that a Γ_H-invariant function vanishes at infinity. This criterion is due to Basson, Breuer and Pink and the proof has been communicated to us. Since it is not publicly available yet, we quickly reprove their criterion.

6.12. Proposition. *Let $f \in \mathcal{O}_{\mathcal{X}}^{\Gamma_H}$. The following conditions are equivalent:*

(i) f vanishes at infinity.

(ii) For all $\varepsilon > 0$ and $n > 0$ there exists $R_{n,\varepsilon} > 0$ such that if $\omega \in I(n, R_{n,\varepsilon})$ we have $|f(\omega)| < \varepsilon$.

(iii) *For all* $\varepsilon > 0$ *and* $\tilde{\omega} \in \mathcal{Y}$ *there exists* $R_{\tilde{\omega},\varepsilon} > 0$ *such that if* $d(\omega_1, K^2\tilde{\omega}) > R_{\tilde{\omega},\varepsilon}$ *we have* $|f(\omega)| < \varepsilon$.

Proof. By Proposition 6.9 we know that f has an expansion of the form

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n(\tilde{\omega})u(\omega)^n \tag{9}$$

on some neighbourhood of infinity. By Theorem 6.7 (ii) this means that we find $r_m \in \mathbb{Z}$ such that (9) converges to a rigid analytic function on each

$$U_m := \{\omega \in \mathcal{X} \mid (u(\omega), \tilde{\omega}) \in D(0, r_m) \times \mathcal{Y}_m\}.$$

We first prove that (i) implies (ii). If f vanishes at infinity, we obtain

$$f(\omega) = \sum_{n\geq 1} f_n(\tilde{\omega})u(\omega)^n$$

Fix $m \geq 0$ and pick $R_m \geq q^{r_m}$. Then we have $|u(\omega)| < 1/R_m \leq q^{-r_n}$ for $\omega \in I(m, R_m)$ by [BBP18a, Proposition 4.7] and consequently $(u(\omega), \tilde{\omega}) \in U_m$. Moreover, by [BBP18a, Lemma 5.1] we have

$$\limsup_{n\to\infty} \|f_n\|_m^{1/n} < q^{r_m} < \infty.$$

Here $\|\cdot\|_m$ denotes the supremum norm of the affinoid algebra $\mathcal{O}_{\mathcal{Y}_m}$. Consequently, we have $N_m := \sup_{n\geq 1} \|f_n\|_m^{1/n} < \infty$. Note that N_m depends on m but is independent of the auxiliary choice of R_m. Together this implies that

$$|f_n(\tilde{\omega})u(\omega)^n| < (N_m/R_m)^n,$$

which shows that for $R_m > N_m$, we obtain

$$|f(\omega)| < N_m/R_m.$$

Thus, by choosing R_m large enough, we can make $|f(\omega)|$ arbitrarily small and we obtain (ii). Note that (ii) implies (iii) since very $\tilde{\omega} \in \mathcal{Y}$ is contained in some $\mathcal{Y}_m$. Thus, we need to show that (iii) implies (i). We do this by contraposition. Assume that we find $N \leq 0$ and $\tilde{\omega} \in \mathcal{Y}$ such that $f_N(\tilde{\omega}) \neq 0$. We choose m and R_m as above. Consider the Newton polygon of the series expansion of f, i.e., the lower convex hull of the set of points

$$(n, v(f_n(\tilde{\omega})))_{n\in\mathbb{Z}}$$

in the Euclidean plane. By [BBP18a, Lemma 5.1] we have $\lim_{n\to-\infty} \|f_n\|_m^{-1/n} = 0$ and consequently $\lim_{n\to-\infty} |f_n(\tilde{\omega})|^{-1/n} = 0$, which shows that the slopes of the Newton polygon tend to $-\infty$ for $n \to -\infty$. This means that the series either has a finite tail, or infinitely many points lie on the Newton polygon for negative n. We consider the line $y = -v(u(\omega))x + c$ tangent to the Newton polygon. After possibly perturbing ω_1, we may assume that this line touches the Newton polygon in exactly one point, say $(n, v(f_n(\tilde{\omega})))$. But then we obtain

$$c = v(f_n(\tilde{\omega})u(\omega)^n) = v(f(\omega))$$

and $v(f_N(\tilde{\omega})u(\omega)^N) \geq c$, which shows that $|f(\omega)| \geq |f_N(\tilde{\omega})| > 0$. Since this is independent of R_m, this shows that (i) does not hold, completing the proof. □

6.13. Remark. There is an analogous criterion for showing that a Γ_H-invariant function is holomorphic at infinity. In the special case $\Gamma = \mathrm{GL}_3(A)$ another such criterion is due to Gekeler, see [Gek17, Proposition 1.8].

Now we can define modular and cusp forms in analogy with the classical case.

6.14. Definition. Let $f \in \mathcal{W}_{k,\ell}(\Gamma)$ be a weak modular form of weight k and type ℓ for an arithmetic group Γ.

(i) We say that f is a *modular form* if

$$\mathrm{ord}_{(g\Gamma g^{-1})_H}(g_* f) \geq 0 \quad \text{for all } g \in \mathrm{GL}_3(F).$$

(ii) We say that f is a *cusp form* if

$$\mathrm{ord}_{(g\Gamma g^{-1})_H}(g_* f) \geq 1 \quad \text{for all } g \in \mathrm{GL}_3(F).$$

The $\mathbb{C}_K$-vector spaces of modular forms and cusp forms are denoted by $\mathcal{M}_{k,\ell}(\Gamma)$ and $\mathcal{S}_{k,\ell}(\Gamma)$. Again we abbreviate $\mathcal{M}_k(\Gamma) := \mathcal{M}_{k,k/3}(\Gamma)$ and $\mathcal{S}_k(\Gamma) := \mathcal{S}_{k,k/3}(\Gamma)$.

6.15. Remark. We should remark that, as the map in Theorem 6.7 is defined over K, one can define spaces of Drinfeld modular and cusp forms as above in complete analogy even over K.

We need the following proposition in which part (iii) is due to the fact that A has class number one.

6.16. Proposition.

(i) We keep the notation from Definition 6.14. We have $\mathrm{ord}_{(g\Gamma g^{-1})_H}(g_* f) = \mathrm{ord}_{(h\Gamma h^{-1})_H}(h_* f)$ *for $h \in \Gamma g P$.*

(ii) The double coset space $\Gamma \backslash \mathrm{GL}_3(F)/P$ is finite.

(ii) If $\Gamma \subseteq \mathrm{GL}_3(A)$ is a congruence subgroup, the double cosets in $\Gamma \backslash \mathrm{GL}_3(F)/P$ can be represented by elements of $\mathrm{GL}_3(A)$.

Proof. See [BBP18a, Proposition 6.2 and Proposition 6.3]. □

Finally, we observe that if Γ_1 is a normal subgroup of Γ, one has

$$\mathcal{M}_{k,\ell}(\Gamma_1)^\Gamma = \mathcal{M}_{k,\ell}(\Gamma) \quad \text{and} \quad \mathcal{S}_{k,\ell}(\Gamma_1)^\Gamma = \mathcal{S}_{k,\ell}(\Gamma).$$

In the case of modular forms this is [BBP18a, (6.7)]. The proof for cusp forms is analogous.

6.2. Dimension formulas

In order to link Drinfeld cusp forms and harmonic cocycles, it will be important to investigate the dimensions of spaces of Drinfeld cusp forms in special cases. We begin with the following general result.

6.17. Theorem. *Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup.*

(i) $\dim_{\mathbb{C}_K} \mathcal{M}_{k,\ell}(\Gamma) < \infty$ *for all integers k and ℓ.*

(ii) $\mathcal{M}_{k,\ell}(\Gamma) = \{0\}$ *for $k < 0$.*

Proof. See [BBP18b, Theorem 11.1]. □

6.18. Remark. We should point out that the proof of the theorem above is deep. It requires the theory of *algebraic Drinfeld modular forms* developed in [Pin13] and [BBP18b], which rely on Satake compactifications of the (algebraic) moduli spaces of Drinfeld modules with level structures. The analytifications of these moduli spaces are uniformized by quotients of $\mathcal{X}$.

The following formulas rely on [PS14].

6.19. Theorem. *We have the following dimension formulas for all $k \geq 0$.*

(i) $\dim_{\mathbb{C}_K} \mathcal{M}_k(\Gamma(t)) = \sum_{i_1, i_2 \in \{0,1\}} q^{\sum_n n \cdot i_n} \binom{k}{\sum_n i_n}$.

(ii) $\dim_{\mathbb{C}_K} \mathcal{M}_k(\Gamma_1(t)) = \binom{k+2}{2}$.

Proof. See [BBP18c, Theorem 17.11] upon noting that there is a small typo in formula (d) in this theorem. □

The following theorem will play a crucial role.

6.20. Theorem. *Let $n \geq 1$ and let $\Gamma \subset \mathrm{GL}_3(K)$ be a subgroup such that $\Gamma(t^n) \subseteq \Gamma \subseteq \Gamma_1(t)$. Then we have*

$$\dim_{\mathbb{C}_K} \mathcal{S}_k(\Gamma) = [\Gamma_1(t) \colon \Gamma] \cdot \binom{k-1}{2}.$$

Proof. See [Pin19, Theorem 3.5.6]. □

6.21. Remark. The above theorems all rely on the explicit knowledge of the relevant Satake compactifications, in particular for $\Gamma = \Gamma(t)$. This is why at present, to our knowledge, there are no proven formulas that work for general congruence subgroups.

6.3. Hecke operators

In this section, we define (analytic) Hecke operators on Drinfeld modular forms. We follow [Bas14, Chapter 4], [BBP18b, Section 12] and [Böc02, Chapter 6]. We need the following proposition.

6.22. Proposition. *Let* $\Gamma \subseteq \mathrm{GL}_3(A)$ *be a congruence subgroup and let* $\delta \in \mathrm{GL}_3(F)$*. Then the double coset* $\Gamma\delta\Gamma$ *can be written as a finite disjoint union of left cosets*

$$\Gamma\delta\Gamma = \bigsqcup_i \delta_i\Gamma.$$

More precisely, the number of left cosets is $[\Gamma : (\Gamma \cap \delta^{-1}\Gamma\delta)]$.

Proof. See [Bas14, Proposition 4.1.2]. □

Let $S_\Gamma \subseteq \mathrm{GL}_3(F)$ be a semigroup containing Γ. We set

$$\mathbb{T}(\Gamma, S_\Gamma) := \mathbb{F}_p[\Gamma\delta\Gamma \mid \delta \in S_\Gamma], \tag{10}$$

the *Hecke algebra* attached to the pair (Γ, S_Γ). A priori, $\mathbb{T}(\Gamma, S_\Gamma)$ is just an $\mathbb{F}_p$-module. The algebra structure is defined as follows. Write

$$\Gamma\alpha\Gamma = \bigsqcup_i \alpha_i\Gamma \quad \text{and} \quad \Gamma\beta\Gamma = \bigsqcup_j \beta_j\Gamma.$$

We define

$$(\Gamma\alpha\Gamma)\cdot(\Gamma\beta\Gamma) := \bigsqcup_\delta a_\delta(\alpha,\beta)\Gamma\delta\Gamma,$$

where the sum is over all double cosets $\Gamma\delta\Gamma$ such that $\bigsqcup_\delta \Gamma\delta\Gamma = \Gamma\alpha\Gamma\beta\Gamma$ and

$$a_\delta(\alpha,\beta) := |\{(i,j) \mid \alpha_i\beta_j\Gamma = \delta\Gamma\}| \pmod p.$$

This turns $\mathbb{T}(\Gamma, S_\Gamma)$ into an algebra. In the sequel, we write

$$T_\delta := \Gamma\delta\Gamma \in \mathbb{T}(\Gamma, S_\Gamma).$$

For a detailed study of the Hecke algebra in the important special case $\Gamma = \mathrm{GL}_3(A)$ and $S_\Gamma = \mathrm{GL}_3(F) \cap \mathrm{M}_3(A)$ see [Bas14, Section 4.2]. The action of the Hecke algebra on modular and cusp forms is defined as follows.

6.23. Definition. Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup and let $\delta \in \mathrm{GL}_3(F)$. The *Hecke operator* T_δ on $\mathcal{M}_{k,\ell}(\Gamma)$ is the $\mathbb{C}_K$-linear operator $T_\delta\colon \mathcal{M}_{k,\ell}(\Gamma) \to \mathcal{M}_{k,\ell}(\Gamma)$ given by

$$T_\delta f = \sum_i (\delta_i)_* f,$$

where we write $\Gamma\delta\Gamma = \bigsqcup_i \delta_i\Gamma$.

Note that since $f \in \mathcal{M}_{k,\ell}(\Gamma)$, this is independent of the choice of the δ_i and well-defined. Moreover, T_δ induces a map

$$T_\delta \colon \mathcal{S}_{k,\ell}(\Gamma) \to \mathcal{S}_{k,\ell}(\Gamma).$$

This defines a structure of a Hecke module for the Hecke algebra $\mathbb{T}(\Gamma, S_\Gamma)$ on $\mathcal{S}_{k,\ell}(\Gamma)$ and $\mathcal{M}_{k,\ell}(\Gamma)$.

6.24. Remark. We should remark that in practice one chooses S_Γ depending on Γ such that the resulting Hecke algebra becomes arithmetically interesting and commutative. This will however not be relevant for us as we will be able to prove the Hecke-equivariance of the map that interests us in complete generality. For $\mathrm{GL}_2(K)$, a more systematic study of the Hecke algebras can be found in [Böc02, Chapter 6].

CHAPTER 7

The action of congruence subgroups on $\mathcal{T}$

In this chapter, we investigate the action of congruence subgroups on $\mathcal{T}$. Our primary focus lies on the group $\Gamma(t)$. Understanding its action on $\mathcal{T}$ in great detail will turn out to be crucial in the sequel.

7.1. The structure of $\mathcal{T}$

Before we can study the action of congruence subgroups on $\mathcal{T}$, we need to understand the structure of $\mathcal{T}$ itself better. For this, we follow [Mül14] and [Geb96]. We start by recalling the following standard facts on the structure of $\mathcal{T}$.

7.1. Lemma. *We have the following.*

(i) Each vertex of $\mathcal{T}$ is a face of exactly $2(q^2+q+1)$ edges.

(ii) Each vertex of $\mathcal{T}$ is a face of exactly $(q+1)(q^2+q+1)$ chambers.

(iii) Each edge of $\mathcal{T}$ is a face of exactly $q+1$ chambers.

Proof. See [Mül14, Lemma 1.33]. □

We also need to investigate the action of $\mathrm{GL}_3(K)$ on $\mathcal{T}$ in more detail. We need the following notion.

7.2. Definition. Let (b_1, b_2, b_3) be a basis of V^* and let $v = [\langle b_1, b_2, b_3\rangle_{O_K}]$. We define the *type* of v to be

$$\mathrm{type}(v) := \nu(\det(b_1, b_2, b_3)) \pmod 3.$$

It is easy to see that this is well-defined (i.e., independent of the representing lattice and the chosen basis) and invariant under the operation of

$$G^+ := \ker(\nu \circ \det\colon G \to \mathbb{Z}) \subset G.$$

We have the following.

7.3. Lemma. *Every chamber of $\mathcal{T}$ contains exactly one vertex of each type.*

Proof. This follows directly from the elementary divisor theorem. □

7.4. Proposition. *Let $g \in G^+$ and $\sigma = \{v_1, \ldots, v_n\} \in \mathcal{T}_n$. Then g stabilizes σ if and only if $gv_i = v_i$ for $i \in \{1, \ldots, n\}$.*

Proof. We have $g\sigma = \sigma$ if and only if $\{gv_1, \ldots, gv_n\} = \{v_1, \ldots, v_n\}$. But by the previous lemma this means that $gv_i = v_i$ for all $i \in \{1, 2, 3\}$ since g preserves the types. □

We begin by computing the stabilizers of vertices in the standard apartment A_0. Recall that A_0 is the maximal simplicial subcomplex of $\mathcal{T}$ based on the vertices

$$\{[i_1, i_2, i_3] \mid i_1, i_2, i_3 \in \mathbb{Z}\},$$

see Section 2.1.

7.5. Lemma. *We have*

$$\mathrm{Stab}_G([i_1, i_2, i_3]) = K^\times \cdot \left\{ \left(\pi^{i_m - i_n} g_{mn}\right)_{1 \leq m,n \leq 3} \;\middle|\; (g_{mn})_{1 \leq m,n \leq 3} \in \mathrm{GL}_3(\mathcal{O}_K) \right\}.$$

Proof. Let $h = \begin{pmatrix} \pi^{i_1} & 0 & 0 \\ 0 & \pi^{i_2} & 0 \\ 0 & 0 & \pi^{i_3} \end{pmatrix} \in G$. Then we have $[i_1, i_2, i_3] = h[0, 0, 0]$. Thus, $g \in G$ stabilizes $[i_1, i_2, i_3]$ if and only if

$$g \in h\mathrm{Stab}_G([0, 0, 0])h^{-1} = K^\times h \mathrm{GL}_3(\mathcal{O}_K) h^{-1}.$$

The result follows directly. □

The following notion is standard and can be found in [AB08, Definition 1.53].

7.6. Definition. A *gallery* in $\mathcal{T}$ is a sequence of chambers $(\sigma_1, \ldots, \sigma_n)$ such that for each $i \in \{2, \ldots, n\}$ the chambers σ_{i-1} and σ_i are adjacent. The integer n is called the *length* of the gallery. We say that the above gallery *connects* σ_1 and σ_n. The minimal length of a gallery connecting two chambers σ and τ is called the *gallery distance* of σ and τ and is denoted by $d(\sigma, \tau)$. Similarly, an *infinite gallery* is an infinite sequence of chambers $(\sigma_1, \sigma_2, \ldots)$ such that for each $i \in \{2, \ldots, n\}$ the chambers σ_{i-1} and σ_i are adjacent.

The following facts are well known, see for example [Geb96, Satz 4.48 and Korollar 4.49].

7.7. Proposition. *Let σ and τ be two chambers in $\mathcal{T}$.*

(i) There exists an apartment containing both σ and τ.

(ii) There exists a gallery connecting σ and τ.

7.2. The action of $\mathrm{GL}_3(A)$

The action of $\mathrm{GL}_3(A)$ on $\mathcal{T}$ can be described very explicitly. The following theorem follows from [AB08, Subsection 11.8.6], see also [Geb96, Satz 3.18].

7.8. Theorem. *The standard sector S_0 is a fundamental domain for the action of $\mathrm{GL}_3(A)$ on $\mathcal{T}$, i.e., for $n \in \{0,1,2\}$ we have*

(i) Each n-cell in $\mathcal{T}$ is $\mathrm{GL}_3(A)$-equivalent to an n-cell in S_0.

(ii) Two distinct n-cells in S_0 are not $\mathrm{GL}_3(A)$-equivalent.

Our aim is to study the growth of certain stabilizers when *tending to the boundary* inside the standard sector S_0. We first need to make precise what this means. Note that the geometric realization of the standard apartment is just a triangulation of $\mathbb{R}^2$. Recall that the standard chamber σ_0 in S_0 is given by

$$\sigma_0 = \{[0,0,0],[0,0,1],[0,1,1]\} \in \mathcal{T}_2.$$

7.9. Definition. Let σ be chamber in S_0.

(i) We say that σ has *positive sign* if there exists a vertex v of σ such that the unique sector in A_0 based in v containing σ is contained in S_0. Otherwise we say that σ has *negative sign*.

(ii) Let $\tau \in S_0$ be a chamber adjacent to σ. We say that τ is *closer to the boundary* than σ if $d(\sigma_0, \tau) > d(\sigma_0, \sigma)$.

To keep the notation short, in the sequel we write $v_{i,j} := [0,i,j]$, where $0 \leq i \leq j$. Then each vertex $v_{i,j}$ belongs to S_0. Figure 1 explains the geometric idea behind the definitions above. The two different colors indicate the chambers of positive and negative sign: Light colored chambers have negative sign, dark colored ones have positive sign. The arrows show how one can move to an adjacent chamber that is closer to the boundary. Note that Figure 1 is not completely accurate as the displayed triangles should be equilateral.
The following lemma formalizes the observations in Figure 1.

7.10. Lemma. *Let σ be chamber in S_0.*

(i) The chamber σ has positive sign precisely if it is of the form

$$\sigma = \left\{v_{i,j}, v_{i,j+1}, v_{i+1,j+1}\right\}$$

where $0 \leq i \leq j$. In this case, σ has exactly one adjacent chamber τ in S_0 that is closer to the boundary. It has negative sign and is given by

$$\tau = \left\{v_{i,j+1}, v_{i+1,j+1}, v_{i+1,j+2}\right\}.$$

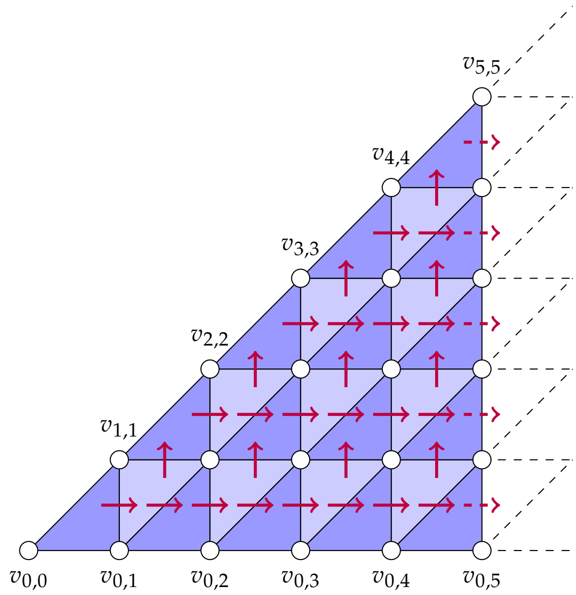

Figure 1. The standard sector S_0

(ii) The chamber σ has negative sign precisely if it is of the form

$$\sigma = \left\{ v_{i,j}, v_{i+1,j}, v_{i+1,j+1} \right\},$$

where $0 \leq i < j$. In this case, σ has exactly two adjacent chambers τ_1 and τ_2 in S_0 that are closer to the boundary. They have positive sign and are given by

$$\tau_1 = \left\{ v_{i+1,j}, v_{i+1,j+1}, v_{i+2,j+1} \right\} \quad \textit{and} \quad \tau_2 = \left\{ v_{i,j}, v_{i,j+1}, v_{i+1,j} \right\}.$$

Proof. Any chamber in S_0 is of the form

$$\sigma = \left\{ v_{i^{(1)},j^{(1)}}, v_{i^{(2)},j^{(2)}}, v_{i^{(3)},j^{(3)}} \right\}.$$

where $0 \leq i^{(n)} \leq j^{(n)}$ for $n \in \{1, 2, 3\}$ and after possible reordering we have

$$i^{(1)} \leq i^{(2)} \leq i^{(3)} \leq i^{(1)} + 1 \quad \text{and} \quad j^{(1)} \leq j^{(2)} \leq j^{(3)} \leq j^{(1)} + 1.$$

Together these conditions leave only two possible forms for σ, exactly the two forms in (i) and (ii). It is also clear that chambers of the form in (i) have positive sign, whereas those in (ii) have negative sign. Thus, we are left with proving the desired descriptions of the adjacent chambers closer to the boundary. But this is an easy exercise by computing the coordinates of the vertices of all adjacent chambers in A_0 and observing which of these lie in S_0 and are closer to the boundary. $\square$

Now we turn to the computation of the stabilizers. This is based on the ideas in [Geb96], [Mül14] and [Hof15]. We can use Lemma 7.5 to compute the stabilizers in $\mathrm{GL}_3(A)$ of cells in the standard apartment. Let $h \in \mathrm{M}_3(\mathbb{Z})$ be a matrix. We define

$$S(h) := \{g \in \mathrm{GL}_3(A) \mid \deg(g_{mn}) \leq h_{mn} \text{ for all } 1 \leq m, n \leq 3\}\,.$$

7.11. Lemma. *We have*

$$\mathrm{Stab}_{\mathrm{GL}_3(A)}([i_1, i_2, i_3]) = S(h_{[i_1,i_2,i_3]}),$$

where $h_{[i_1,i_2,i_3]} = (i_n - i_m)_{1 \leq m,n \leq 3}$.

Proof. We have

$$\begin{aligned}\mathrm{Stab}_{\mathrm{GL}_3(A)}([i_1, i_2, i_3]) &= \mathrm{Stab}_G([i_1, i_2, i_3]) \cap \mathrm{GL}_3(A) \\ &= (K^\times \{g \in G \mid \nu(g_{mn}) \geq i_m - i_n\}) \cap \mathrm{GL}_3(A)\end{aligned}$$

by Lemma 7.5. The claim then follows from

$$\{x \in \mathcal{O}_K \mid \nu(x) \geq N\} \cap A = \{a \in A \mid \deg(a) \leq -N\},$$

which is obvious from the definitions. □

We can now extend this easily to edges and chambers as follows.

7.12. Proposition. *Let* $\sigma = \left\{[i_1^{(1)}, i_2^{(1)}, i_3^{(1)}], \ldots, [i_1^{(d)}, i_2^{(d)}, i_3^{(d)}]\right\}$ *with* $d \leq 3$. *Then we have*

$$\mathrm{Stab}_{\mathrm{GL}_3(A)}(\sigma) = S(h_\sigma),$$

where $h_\sigma = \left(\min_{1 \leq j \leq d} \left\{i_n^{(j)} - i_m^{(j)}\right\}\right)_{1 \leq m,n \leq 3}$.

Proof. Since $\mathrm{GL}_3(A) \subset G^+$, this follows from Proposition 7.4 and Lemma 7.11. □

Now we turn our attention to the standard sector S_0. The following lemma will be needed later.

7.13. Lemma. *Let e be an edge on the boundary of S_0. Then* $\mathrm{Stab}_{\mathrm{GL}_3(A)}(e)$ *permutes the $q+1$ chambers having e as a face transitively.*

Proof. See [Mül14, Lemma 1.36 (ii)]. □

To keep the notation short, for a congruence subgroup $\Gamma \subseteq \mathrm{GL}_3(A)$ we set $\Gamma_\sigma := \mathrm{Stab}_\Gamma(\sigma)$. The following proposition is in the spirit of [Mül14, Lemma 1.36 (i)], but we need a stronger result, i.e., a statement for general congruence subgroups.

7.14. Proposition. *All but finitely many chambers σ of S_0 satisfy the following: If $\tau \in S_0$ is a chamber adjacent to σ such that σ is closer to the boundary than τ, we have $|\Gamma_\sigma/\Gamma_\tau| = q$. The set $(\gamma\tau)_{\gamma \in \Gamma_\sigma/\Gamma_\tau}$ consists precisely of the q chambers other than σ sharing a face with σ and τ.*

Proof. We first consider the case $\Gamma = \Gamma(N)$ for some non-zero monic polynomial $N \in A$. Assume first that τ has positive sign. Then by Lemma 7.10, we have the following explicit descriptions:

$$\tau = \left\{v_{i,j}, v_{i,j+1}, v_{i+1,j+1}\right\} \quad \text{and} \quad \sigma = \left\{v_{i,j+1}, v_{i+1,j+1}, v_{i+1,j+2}\right\}.$$

By Proposition 7.12, it is straightforward to compute the stabilizers of σ and τ in $\mathrm{GL}_3(A)$. We obtain

$$\mathrm{Stab}_{\mathrm{GL}_3(A)}(\tau) = S(h_\tau) \quad \text{and} \quad \mathrm{Stab}_{\mathrm{GL}_3(A)}(\sigma) = S(h_\sigma),$$

where

$$h_\tau = \begin{pmatrix} 0 & i & j \\ -i-1 & 0 & j-i \\ -j-1 & i-j-1 & 0 \end{pmatrix} \quad \text{and} \quad h_\sigma = \begin{pmatrix} 0 & i & j+1 \\ -i-1 & 0 & j-i \\ -j-2 & i-j-1 & 0 \end{pmatrix}.$$

Thus, we see directly that the quotient $\mathrm{GL}_3(A)_\sigma / \mathrm{GL}_3(A)_\tau$ has order q. Assume that $j \geq \deg(N)$. Then $\Gamma_\sigma/\Gamma_\tau$ also has order q. The orbit-stabilizer theorem implies that

$$|\Gamma_\sigma| = |\Gamma_\sigma \circ \tau| \cdot |\Gamma_\tau|,$$

hence we obtain $|\Gamma_\sigma \circ \tau| = q$, which shows that the set $(\gamma\tau)_{\gamma \in \Gamma_\sigma/\Gamma_\tau}$ consists precisely of the q chambers other than σ sharing a face with σ and τ. This completes the proof for τ of positive sign. For τ of negative sign, the argument is completely analogous using the explicit descriptions in Lemma 7.10. We still need to consider general congruence subgroups Γ. Note that by definition we find $N \in A \setminus \{0\}$ such that $\Gamma(N) \subseteq \Gamma$. But then we have

$$\Gamma(N)_\sigma / \Gamma(N)_\tau \subseteq \Gamma_\sigma/\Gamma_\tau \subseteq \mathrm{GL}_3(A)_\sigma / \mathrm{GL}_3(A)_\tau$$

Hence the result for general Γ follows. □

7.15. Remark. In the proof we have seen that we can give an explicit bound after which we can guarantee that the stabilizers behave as in Proposition 7.14. Namely, if $\Gamma(N) \subseteq \Gamma$ and $n = \deg(N)$, we have that the finite number of exceptions is contained in the subcomplex on the vertices

$$\left\{v_{i,j} \mid 0 \leq i \leq j \leq i+n-1 \text{ and } i \leq n-1\right\}.$$

This subcomplex will play an important role later.

7.3. Quotient buildings

Since the quotient of a simplicial complex by a group acting on it is general not a simplicial complex anymore, we need the following notion in order to make sense of the quotient $\Gamma\backslash\mathcal{T}$ for a congruence subgroup $\Gamma \subseteq \mathrm{GL}_3(A)$. We follow [Hof15, Section 2.3].

7.16. Definition. A Δ-*set* is a sequence of sets $(\mathcal{D}_n)_{n\geq 0}$ together with so called *face maps*

$$\delta_n^i\colon \mathcal{D}_n \to \mathcal{D}_{n-1}, \quad \text{where } n > 0, i \in \{0,\dots,n\},$$

such that $\delta_{n-1}^i \circ \delta_n^j = \delta_{n-1}^{j-1} \circ \delta_n^i$ for $n > 1$ and $i < j$.

There is a natural notion of a morphism of Δ-sets, namely a sequence of maps of sets that is compatible with all face maps. The notion of a Δ-set is a natural generalization of a simplicial complex. As the notation suggests, the sets Δ_n are the analogue of the n-cells and the face maps associate to each n-cell its faces of dimension $n-1$. We can view $\mathcal{T}$ as a Δ-set in the following way. We set

$$\begin{aligned}\mathcal{D}_0(\mathcal{T}) &:= \{[\Lambda] \mid \Lambda \text{ is a lattice in } V^*\}\,,\\ \mathcal{D}_n(\mathcal{T}) &:= \{\{[\Lambda_0],\dots,[\Lambda_n]\} \mid \pi\Lambda_0 \subsetneq \Lambda_n \subsetneq \cdots \subsetneq \Lambda_0\}\,,\end{aligned}$$

where the lattice classes are ordered by their type (regarded as a number in $\{0,1,2\}$). We define the face maps δ_n^i by removing the homothety class $[\Lambda_i]$ in $\{[\Lambda_0],\dots,[\Lambda_n]\}$.

7.17. Definition. Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup. The *quotient building* $\Gamma\backslash\mathcal{T}$ is the Δ-set given by

$$\mathcal{D}_n(\Gamma\backslash\mathcal{T}) := \{\Gamma\sigma \mid \sigma \in \mathcal{D}_n(\mathcal{T})\}$$

together with the natural face maps as defined for $\mathcal{T}$.

By definition we have natural morphism of Δ-sets $\mathcal{T} \to \Gamma\backslash\mathcal{T}$. The following proposition is now just a reformulation of Theorem 7.8.

7.18. Proposition. *We have* $\mathrm{GL}_3(A)\backslash\mathcal{T} \cong S_0$.

7.4. The quotient $\Gamma(t)\backslash\mathcal{T}$

In this section, we investigate the quotient $\Gamma(t)\backslash\mathcal{T}$. This is based on [Geb96, Section 4.3] and [Hof15, Section 2.3]. Since S_0 is a fundamental domain for the action of $\mathrm{GL}_3(A)$, we may write $\mathcal{T} = \bigcup_{g\in\mathrm{GL}_3(A)} gS_0$. Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup and let $\{g_i \mid i \in I\} \subseteq \mathrm{GL}_3(A)$ be a (finite) set of representatives for $\Gamma\backslash\mathrm{GL}_3(A)$. We obtain

$$\mathcal{T} = \bigcup_{i\in I} \Gamma g_i S_0.$$

7.19. Definition. Let σ be an n-cell in S_0. We say that an n-cell τ is of *level* σ if $\tau \in \mathrm{GL}_3(A)\sigma$. We denote the set of n-cells of level σ by $\mathcal{L}(\sigma)$. Then we have

$$\mathcal{T} = \bigsqcup_{\sigma\in S_0} \mathcal{L}(\sigma).$$

For a congruence subgroup $\Gamma \subseteq \mathrm{GL}_3(A)$, the morphism of Δ-sets

$$\mathcal{L}\colon \mathcal{T} \to S_0 \cong \mathrm{GL}_3(A)\backslash\mathcal{T}$$

factors over $\Gamma\backslash\mathcal{T}$. Thus, we may write

$$\Gamma\backslash\mathcal{T} = \bigsqcup_{\sigma\in S_0} \mathcal{L}_\Gamma(\sigma),$$

where $\mathcal{L}_\Gamma(\sigma)$ denotes the set of cells of $\Gamma\backslash\mathcal{T}$ that are of level σ. Observe that we have a bijection

$$\begin{aligned} \Gamma\backslash\mathrm{GL}_3(A)/\mathrm{GL}_3(A)_\sigma &\to \mathcal{L}_\Gamma(\sigma), \\ \Gamma g \mathrm{GL}_3(A)_\sigma &\mapsto \Gamma g\sigma. \end{aligned}$$

7.20. Proposition. *Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup and choose a monic polynomial $N \in A$ such that $\Gamma(N) \subseteq \Gamma$. Let $n = \max\{2, \deg(N)\}$ and denote by $S_0(n)$ the maximal subcomplex of S_0 on the vertices*

$$\{[i,j] \mid 0 \leq i \leq j \leq i+n-1 \textit{ and } i \leq n-1\}.$$

Then the quotient $\Gamma\backslash\mathcal{T}$ is completely determined by the fibre of $S_0(n)$. There are no new identifications in the fibres of cells in $S_0 \setminus S_0(n)$.

Proof. See [Hof15, Lemma 2.32] or [Mül14, Satz 1.30]. □

With these preparations, we now specialize to the case $\Gamma = \Gamma(t)$. This case has already been studied in [Geb96, Section 4.3], but we need a slightly finer result. By Proposition 7.20, we are interested in the simplices in $S_0(2)$. Observe first that $\Gamma(t)$ is normal in $\mathrm{GL}_3(A)$. We have $\Gamma(t)\backslash\mathrm{GL}_3(A) \cong \mathrm{GL}_3(\mathbb{F}_q)$. It follows that

$$\mathcal{L}_{\Gamma(t)}(\sigma) \cong \mathrm{GL}_3(\mathbb{F}_q)/((\Gamma(t)\mathrm{GL}_3(A)_\sigma) \cap \mathrm{GL}_3(\mathbb{F}_q)).$$

7.21. Lemma. *We have*

$$\begin{aligned} (\Gamma(t)\mathrm{GL}_3(A)_{v_{1,2}}) \cap \mathrm{GL}_3(\mathbb{F}_q) &= B(\mathbb{F}_q), \\ (\Gamma(t)\mathrm{GL}_3(A)_{v_{1,1}}) \cap \mathrm{GL}_3(\mathbb{F}_q) &= P_2(\mathbb{F}_q), \\ (\Gamma(t)\mathrm{GL}_3(A)_{v_{0,1}}) \cap \mathrm{GL}_3(\mathbb{F}_q) &= P_1(\mathbb{F}_q), \\ (\Gamma(t)\mathrm{GL}_3(A)_{v_{0,0}}) \cap \mathrm{GL}_3(\mathbb{F}_q) &= \mathrm{GL}_3(\mathbb{F}_q). \end{aligned}$$

Proof. This follows easily from Lemma 7.11. □

Upon observing that this implies that

$$|\mathcal{L}_{\Gamma(t)}(v_{0,1})| = |\mathcal{L}_{\Gamma(t)}(v_{1,1})| = q^2+q+1 \quad \text{and} \quad |\mathcal{L}_{\Gamma(t)}(v_{1,2})| = (q+1)(q^2+q+1),$$

we can describe the quotient $\Gamma(t)\backslash\mathcal{T}$ as follows: There are $(q+1)(q^2+q+1)$ copies of the standard sector S_0 parametrized by $\mathrm{GL}_3(\mathbb{F}_q)/B(\mathbb{F}_q)$, each originating in the vertex

$v_{0,0}$. Along the boundary line originating in $\{v_{0,0}, v_{1,1}\}$ two such sectors are glued if and only if they have the same image in $\mathrm{GL}_3(\mathbb{F}_q)/P_2(\mathbb{F}_q)$. Similarly, along the boundary line originating in $\{v_{0,0}, v_{0,1}\}$ two such sectors are glued if and only if they have the same image in $\mathrm{GL}_3(\mathbb{F}_q)/P_1(\mathbb{F}_q)$. That means along each boundary line exactly $q+1$ sectors are glued together. Since $P_1(\mathbb{F}_q) \cap P_2(\mathbb{F}_q) = B(\mathbb{F}_q)$, two sectors that would be glued along two boundary lines are already equal.

The above description of the quotient $\Gamma(t)\backslash\mathcal{T}$ shows that we can in fact realize it *inside* $\mathcal{T}$, i.e., obtain a fundamental domain for the action of $\Gamma(t)$ on $\mathcal{T}$. We make this precise as follows. We have $\mathrm{Stab}_G(S_0) = K^\times B(\mathcal{O}_K)$, see [Geb96, Satz 2.26]. Thus, we obtain

$$\mathrm{Stab}_G(v_0)/\mathrm{Stab}_G(S_0) \cong \mathrm{GL}_3(\mathcal{O}_K)/B(\mathcal{O}_K) \cong \mathrm{GL}_3(\mathbb{F}_q)/B(\mathbb{F}_q).$$

Hence any set of representatives $g_1, \dots, g_{n_q}$ for $\mathrm{GL}_3(\mathbb{F}_q)/B(\mathbb{F}_q)$ regarded as elements of $\mathrm{Stab}_G(v_0)$ stabilizes v_0 and maps S_0 to a sector based in v_0. We have $n_q = (q+1)(q^2+q+1)$. We arrive at the following.

7.22. Theorem. *The set* $\mathcal{F}_{\Gamma(t)} := \bigcup_{i=1}^{n_q} g_i S_0$ *is a fundamental domain for the action of* $\Gamma(t)$ *on* $\mathcal{T}$.

Proof. By construction $\mathcal{F}_{\Gamma(t)}$ maps isomorphically to $\Gamma(t)\backslash\mathcal{T}$ under the natural map $\mathcal{T} \to \Gamma(t)\backslash\mathcal{T}$. □

CHAPTER 8

Drinfeld cusp forms and harmonic cocycles

In this chapter, we want to investigate the relationship between Drinfeld cusp forms and harmonic cocycles. More precisely, we will show that (under various assumptions) the residue map Res_k induces an isomorphism between the space of Drinfeld cusp forms of weight $k+3$ for a congruence subgroup Γ and the space of Γ-invariant harmonic cocycles on $\mathcal{T}$ with coefficients in V_k. This is the natural analogue of the main theorem of [Tei91] in rank 3. Contrary to [Tei91], our theorem requires the congruence subgroup Γ to satisfy $\Gamma(t) \subseteq \Gamma$. Moreover, we need to assume that Conjecture 5.49 holds.

8.1. Cuspidality

The aim of this section is to show that for any congruence subgroup $\Gamma \subseteq \mathrm{GL}_3(A)$, every Γ-invariant harmonic cocycle is Γ-cuspidal in the sense of Definition 8.1. This is analogous to a result of Teitelbaum for $\mathrm{GL}_2(K)$, see [Tei91, Theorem 3], and ensures that the theory developed in Part I of this thesis is applicable in this situation. In the sequel, we write

$$C_{\mathrm{har}}(\Gamma, M) := C_{\mathrm{har}}(\mathcal{T}, M)^{\Gamma},$$

where M is a $\mathbb{C}_K[G]$-module and the group action is as in Section 3.1. From now on we regard harmonic cocycles as functions on the chambers $\mathcal{T}_2$ (and *not* on the pointed chambers $\widehat{\mathcal{T}_2}$) as explained in Remark 3.2.

8.1. Definition. Let M be a $\mathbb{C}_K[G]$-module. A harmonic cocycle $c \in C_{\mathrm{har}}(\mathcal{T}, M)$ is called Γ-*cuspidal* if there exists a finite set S of chambers of $\Gamma\backslash\mathcal{T}$ such that $c(\sigma) \neq 0$ only for $\sigma \in \mathrm{pr}_\Gamma^{-1}(S)$, where $\mathrm{pr}_\Gamma \colon \mathcal{T} \to \Gamma\backslash\mathcal{T}$ denotes the canonical projection.

The aim of this section is to prove the following theorem, which is the natural extension of [Tei91, Theorem 3] to our situation.

8.2. Theorem. *Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup and let M be a $\mathbb{C}_K[G]$-module, finite-dimensional over $\mathbb{C}_K$. Then there exists a finite set S of chambers of $\Gamma\backslash\mathcal{T}$ such that for all $c \in C_{\mathrm{har}}(\Gamma, M)$ we have $c(\sigma) \neq 0$ only for $\sigma \in \mathrm{pr}_\Gamma^{-1}(S)$. Consequently, every $c \in C_{\mathrm{har}}(\Gamma, M)$ is Γ-cuspidal.*

The importance of this theorem becomes evident in the following corollary, which ensures that the theory developed in Part I is applicable in this situation.

8.3. Corollary. *Every harmonic cocycle $c \in C_{\mathrm{har}}(\Gamma, k)$ is bounded.*

Proof. Recall that c is bounded if the automorphic form $\varphi_c \colon K^\times \backslash G \to V_k$ given by $\varphi_c(g) = g^{-1} \cdot c(g\sigma_0)$ is bounded. The fact that c is Γ-invariant means that φ_c factors over $K^\times \Gamma \setminus G$. Now, by Theorem 8.2 (and since each chamber has only three possible orientations), we find $g_1, \dots, g_n \in G$ such that c is supported on $\Gamma g_1 \sigma_0, \dots, \Gamma g_n \sigma_0$. Choose $\alpha \in \mathcal{O}_{\mathbb{C}_K} \setminus \{0\}$ so that $\alpha \varphi_c(g_i) \in V_k^{\mathrm{int}}$ for all $i \in \{1, \dots, n\}$ (which exists by definition of an integral structure). Then, since φ_c is Γ-invariant, $\mathcal{I}$-equivariant and V_k^{int} is $\mathcal{I}$-stable, it follows that $\alpha \varphi_c(g) \in V_k^{\mathrm{int}}$ for all $g \in G$, which completes the proof. □

Now we turn to the proof of Theorem 8.2, which relies on the results from the previous chapter.

Proof of Theorem 8.2. We claim that it suffices to show that there is a finite set S_Γ of chambers in S_0 such that every $c \in C_{\mathrm{har}}(\Gamma, M)$ vanishes on all chambers in S_0 outside S_Γ. To see this, note that because Γ is a congruence subgroup we may write

$$\mathrm{GL}_3(A) = \bigsqcup_{i \in I} \Gamma g_i$$

for some finite set I. Hence by Theorem 7.8, each chamber is Γ-equivalent to a chamber in $g_i S_0$ for some $i \in I$. Thus, it suffices to show that for each $i \in I$, there is a finite set of chambers in $g_i S_0$ such that every $c \in C_{\mathrm{har}}(\Gamma, M)$ vanishes on all other chambers in $g_i S_0$. But the cocycle $c_i := g_i \cdot c$ is invariant under the group $\Gamma_i := g_i \Gamma g_i^{-1}$, which is again a congruence subgroup by Lemma 6.1. Hence, if c_i vanishes on all chambers in S_0 outside S_{Γ_i}, then c vanishes on all chambers in $g_i S_0$ outside $g_i S_{\Gamma_i}$.
Thus, we need to show that there is a finite set S_Γ of chambers in S_0 such that every $c \in C_{\mathrm{har}}(\Gamma, M)$ vanishes on all chambers in S_0 outside S_Γ. For this, we define a partial order on the set of chambers of S_0 by

$$\sigma \leq \tau \quad \text{if and only if} \quad M^\tau \subseteq M^\sigma,$$

where $M^\sigma = M^{\mathrm{Stab}_\Gamma(\sigma)}$. Then, since M is finite-dimensional, each chain with respect to $\leq$ becomes stationary. Thus, we find a maximal element $\sigma_{\max}$ in S_0 with respect to $\leq$, i.e, we have $M^{\sigma_{\max}} \subseteq M^\sigma$ for all σ in S_0. In particular, we have $M^{\sigma_{\max}} = M^\sigma$ for σ adjacent to $\sigma_{\max}$ and closer to the boundary. Thus, we may assume that $\sigma_{\max}$ has positive sign. Then we denote by $S_{\max}$ the subsector of S_0 based in $\sigma_{\max}$. Our first aim is to show that there is a finite set of chambers in $S_{\max}$ such that each c vanishes on all chambers in $S_{\max}$ outside this finite set. By Proposition 7.14 and the harmonicity of c, we obtain for σ and τ in $S_{\max}$ and as in Proposition 7.14 that

$$c(\sigma) = -\sum_{h \in \Gamma_\sigma / \Gamma_\tau} c(h\tau) = -q c(\tau) = 0,$$

since $c(\tau) \in M^\tau = M^\sigma$ by Γ-invariance. To conclude the proof, we still need to consider the chambers in $S_0 \backslash S_{\max}$. But we observe that the collection of chambers of $S_0 \backslash S_{\max}$ is just a finite union of infinite galleries, where in each gallery, when going from one chamber to the next, one moves closer to the boundary, see Figure 2. Thus, after dropping finitely many chambers in each gallery, we can assume that M^σ is independent of the chamber σ in the gallery, and again we can use Proposition 7.14 to conclude by same argument as above. ◻

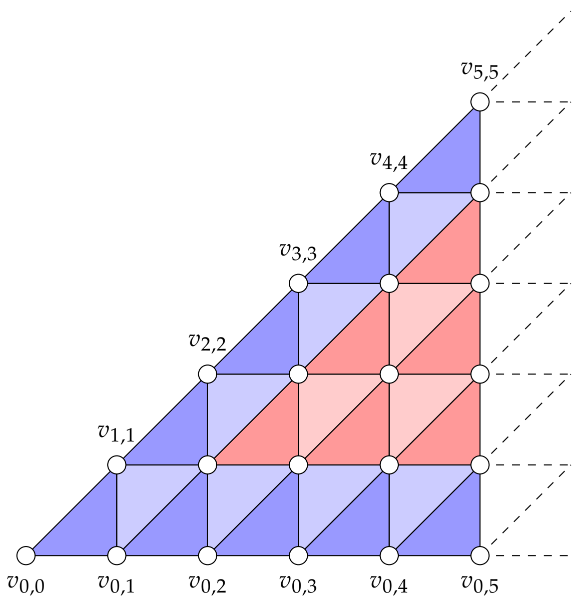

Figure 2. The decomposition of S_0 into $S_{\max}$ and $S_0 \setminus S_{\max}$

We obtain the following corollary.

8.4. Corollary. *Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup and let M be a $\mathbb{C}_K[G]$-module, finite-dimensional over $\mathbb{C}_K$. Then we have*

$$\dim_{\mathbb{C}_K} \mathrm{C}_{\mathrm{har}}(\Gamma, M) < \infty.$$

In the next section, we want to inspect the dimension of $C_{\mathrm{har}}(\Gamma, M)$ in more detail in the special case $\Gamma = \Gamma(t)$.

8.2. Dimension estimates

The aim of this section is to prove the following theorem. The proof relies on our knowledge of the quotient building $\Gamma(t)\backslash\mathcal{T}$ from Section 7.4.

8.5. Theorem. *Let M be a $\mathbb{C}_K[G]$-module, finite-dimensional over $\mathbb{C}_K$. We have*

$$\dim_{\mathbb{C}_K} C_{\mathrm{har}}(\Gamma(t), M) \geq q^3 \dim_{\mathbb{C}_K} M.$$

Our proof is inspired by the proof of [Tei91, Theorem 16] due to Teitelbaum. The idea being that a $\Gamma(t)$-invariant harmonic cocycle is completely determined by its values on the so called *stable part* in a fundamental domain for the action of $\Gamma(t)$ on $\mathcal{T}$. We work out a similar approach in our situation. We should note that Teitelbaum proves this result in much more generality, i.e., for general congruence subgroups. We expect that our method can be expanded in similar fashion, but this requires more knowledge of the boundary of $\Gamma\backslash\mathcal{T}$ for general congruence subgroups Γ. The main reason for restricting to the congruence subgroup $\Gamma(t)$ lies in the fact that we later need to use Theorem 6.20, which has similar restrictions.

8.6. Definition. Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup. We say an n-cell σ of $\mathcal{T}$ is Γ-*stable* if $\Gamma_\sigma = \{\mathrm{id}\}$. Otherwise, we say that σ is Γ-*unstable*.

Now, we can compute the set of $\Gamma(t)$-stable chambers in the fundamental domain $\mathcal{F}_{\Gamma(t)}$. Recall that $\mathcal{F}_{\Gamma(t)} := \bigcup_{i=1}^{n_q} g_i S_0$, where $g_1, \dots, g_{n_q}$ is a set of representatives for $\mathrm{GL}_3(\mathbb{F}_q)/B(\mathbb{F}_q)$.

8.7. Proposition. *The $\Gamma(t)$-stable chambers in $\mathcal{F}_{\Gamma(t)}$ are precisely $\left\{ g_i\sigma_0 \;\middle|\; i \in \{1, \dots, n_q\} \right\}$.*

Proof. Since $\Gamma(t) \subset \mathrm{GL}_3(A)$ is normal, we have $\Gamma(t)_{g_i\sigma} = g_i\Gamma(t)_\sigma g_i^{-1}$. Thus, it suffices to consider the stabilizers of chambers in S_0. But these have already been computed in Proposition 7.14, which proves the claim. □

Proof of Theorem 8.5. Let $m_1, \dots, m_d$ be a basis of M. Let $I \in \{m_1, \dots, m_d\}^{q^3}$. We will show that there are cocycles

$$c_I \in C_{\mathrm{har}}(\Gamma(t), M),$$

such that the family $(c_I)_I$ is linearly independent. We first define the values $c_I(g_i\sigma_0)$. By the discussion above, $(g_i\sigma_0)_i$ are precisely the $(q+1)(q^2+q+1)$ chambers in $\mathcal{T}$ containing v_0. Let e be any edge with face v_0. Then the fact that c_I is supposed to be harmonic at e relates $q+1$ of the values $(c_I(g_i\sigma_0))_i$. Since there are $2(q^2+q+1)$ edges having v_0 as a face this leaves us

$$(q+1)(q^2+q+1) - 2(q^2+q+1) + 1 = q^3$$

choices, where the last 1 comes from the observation that, after going though all but the last edge, at the last edge does not impose any new conditions. Thus, we may define the values $(c_I(g_i\sigma_0))_i$ from I so that c_I is harmonic at each edge with face v_0. Our next aim is to extend c_I to the chambers in $\mathcal{F}_{\Gamma(t)}$. For this, we set

$$c_I(g_i\sigma) := \sum_{g \in \Gamma(t)_{g_i\sigma}} \mathrm{sgn}(\sigma)(g \cdot c_I(g_i\sigma_0)),$$

where

$$\operatorname{sgn}(\sigma) := \begin{cases} -1, & \text{if } \sigma \text{ has negative sign,} \\ 1, & \text{if } \sigma \text{ has positive sign.} \end{cases}$$

This is well-defined because the chambers $g_i\sigma_0$ are $\Gamma(t)$-stable. Finally, we extend c_I to all chambers by setting $c_I(\sigma) = \gamma \cdot c_I(\tau)$, where $\sigma = \gamma\tau$ with τ in $\mathcal{F}_{\Gamma(t)}$. This is well-defined by Theorem 7.22. Thus, we have defined a Γ-invariant function $c_I\colon \mathcal{T}_2 \to M$. Moreover, it is clear that the functions $(c_I)_I$ are linearly independent. We are left with showing that c_I is harmonic. It is clear that is suffices to check harmonicity only at edges e in $\mathcal{F}_{\Gamma(t)}$. We consider two separate cases.

(i) e is in the interior of some g_iS_0.

(ii) e is on the boundary of some g_iS_0.

We start with (i). There are exactly two chambers in S_0 that contain $g_i^{-1}e$. Denote by σ the chamber in S_0 containing $g_i^{-1}e$ that is closer to the boundary than the another chamber τ in S_0 sharing the face $g_i^{-1}e$. In particular, we have $\sigma \neq \sigma_0$. Then we can apply Proposition 7.14 to obtain that $\Gamma(t)_\sigma/\Gamma(t)_\tau$ has order q and permutes the chambers containing $g_i^{-1}e$ other than σ transitively. We obtain

$$\begin{aligned} c_I(g_i\sigma) &= \sum_{g\in\Gamma(t)_{g_i\sigma}} \operatorname{sgn}(\sigma)(g\cdot c_I(g_i\sigma_0)) \\ &= \sum_{g\in\Gamma(t)_\sigma} \operatorname{sgn}(\sigma)((g_igg_i^{-1})\cdot c_I(g_i\sigma_0)) \\ &= \sum_{\gamma\in\Gamma(t)_\sigma/\Gamma(t)_\tau}\ \sum_{g\in\Gamma(t)_\tau} \operatorname{sgn}(\sigma)((g_i\gamma gg_i^{-1})\cdot c_I(g_i\sigma_0)) \\ &= -\sum_{\gamma\in\Gamma(t)_\sigma/\Gamma(t)_\tau} (g_i\gamma g_i^{-1})\cdot c_I(g_i\tau) = -\sum_{\gamma\in\Gamma(t)_\sigma/\Gamma(t)_\tau} c_I(g_i\gamma\tau), \end{aligned}$$

which proves the harmonicity at e. To prove harmonicity in case (ii) we need more preparations. Namely, we need to compute the stabilizers of the chambers with face e on the boundary of g_iS_0. We claim the following. Let σ be any chamber with face e (note that σ is then in some g_iS_0, since $(g_iS_0)_i$ consists of *all* sectors based in chambers containing v_0). Then we have $\Gamma(t)_\sigma = \Gamma(t)_e$. Since $\Gamma(t)$ is normal in $\mathrm{GL}_3(A)$, it suffices to prove this for e being on the boundary of S_0. Moreover, by Lemma 7.13, we may also assume that σ is in S_0. There are two separate cases for the two boundary lines of S_0. We either have

$$e = \{v_{i,i}, v_{i+1,i+1}\} \quad \text{and} \quad \sigma = \{v_{i,i}, v_{i,i+1}, v_{i+1,i+1}\}$$

or

$$e = \{v_{0,i}, v_{0,i+1}\} \quad \text{and} \quad \sigma = \{v_{0,i}, v_{0,i+1}, v_{1,i+1}\}.$$

By Proposition 7.12, we have

$$\operatorname{Stab}_{\mathrm{GL}_3(A)}(\sigma) = S(h_\sigma) \quad \text{and} \quad \operatorname{Stab}_{\mathrm{GL}_3(A)}(e) = S(h_e),$$

where

$$h_\sigma = \begin{pmatrix} 0 & i & i \\ -i-1 & 0 & 0 \\ -i-1 & 0 & 0 \end{pmatrix}, \quad h_e = \begin{pmatrix} 0 & i & i \\ -i-1 & 0 & 0 \\ -i-1 & -1 & 0 \end{pmatrix}$$

in the first case and

$$h_\sigma = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & i \\ -i-1 & -i-1 & 0 \end{pmatrix}, \quad h_e = \begin{pmatrix} 0 & 0 & i \\ -1 & 0 & i \\ -i-1 & -i-1 & 0 \end{pmatrix}$$

in the second case. By taking the intersection with $\Gamma(t)$, we directly obtain the result. With these preparations, we can now show the harmonicity of c_I at e as in case (ii). After reordering, we may assume that the chambers σ_i with face e are in the sectors $g_i S_0$, $i \in \{1, \dots, q+1\}$. It follows that

$$\begin{aligned} \sum_{\sigma \mapsto e} c_I(\sigma) = \sum_{i=1}^{q+1} c_I(\sigma_i) &= \sum_{i=1}^{q+1} \left(\sum_{g \in \Gamma(t)_{\sigma_i}} g \cdot c_I(g_i \sigma_0) \right) \\ &= \sum_{g \in \Gamma(t)_e} g \cdot \left(\sum_{i=1}^{q+1} c_I(g_i \sigma_0) \right) = 0, \end{aligned}$$

since c_I is harmonic at the edges with face v_0. Here, we used that all boundary chambers in S_0 have positive sign. This completes the proof. □

8.3. Hecke operators

Let M be a $\mathbb{C}_K[G]$-module, finite-dimensional over $\mathbb{C}_K$ and $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup. We want to define an action of the Hecke algebra $\mathbb{T}(\Gamma, S_\Gamma)$ on $C_{\mathrm{har}}(\Gamma, M)$. This is analogous to [Böc02, Section 6.3].

8.8. Definition. Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be a congruence subgroup and let $\delta \in \mathrm{GL}_3(F)$. The *Hecke operator* T_δ on $C_{\mathrm{har}}(\Gamma, M)$ is the $\mathbb{C}_K$-linear operator $T_\delta \colon C_{\mathrm{har}}(\Gamma, M) \to C_{\mathrm{har}}(\Gamma, M)$ given by

$$T_\delta c = \sum_i \delta_i \cdot c,$$

where we write $\Gamma\delta\Gamma = \bigsqcup_i \delta_i \Gamma$.

Note that we have

$$(\delta_i g) \cdot c = \delta_i \cdot c \quad \text{for } g \in \Gamma,$$

since $c \in C_{\mathrm{har}}(\Gamma, M)$, which shows that T_δ independent of the choice of the δ_i. Moreover, we have

$$g \cdot (T_\delta c) = g \cdot \left(\sum_i \delta_i \cdot c \right) = \sum_i (g\delta_i) \cdot c = \sum_i \delta_i \cdot c = T_\delta c \quad \text{for } g \in \Gamma,$$

since $(g\delta_i)$ is a set of left coset representatives for $\Gamma\delta\Gamma$, which shows that T_δ is well-defined. This defines a structure of a Hecke module for the Hecke algebra $\mathbb{T}(\Gamma, S_\Gamma)$ on $C_{\mathrm{har}}(\Gamma, M)$, see (10). Now we have developed all the tools we need in order to formulate the main theorem of Part II of this thesis.

8.4. An isomorphism à la Teitelbaum

The aim of this section is to prove the following theorem.

8.9. Theorem. *Let $\Gamma \subseteq \mathrm{GL}_3(A)$ be congruence subgroup such that $\Gamma(t) \subseteq \Gamma$. Assume that Conjecture 5.49 holds. Then the map Res_k induces a Hecke-equivariant isomorphism*

$$\mathcal{S}_{k+3}(\Gamma) \to C_{\mathrm{har}}(\Gamma, k).$$

8.10. Remark. We should note that we expect this result to hold in much more generality, i.e., the assumption $\Gamma(t) \subseteq \Gamma$ should be unnecessary. This would require dimension formulas similar to Theorem 6.20 for more general groups. Going even further, we expect the analogous theorem to hold for more general base rings A in analogy with [Tei91, Theorem 16].

Let us explain the setup of the proof. We ignore the Hecke-equivariance for now. By Theorem 4.31, we have the following diagram.

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{X}}(k+3)^b \arrow[r, "\mathrm{Res}_k"] \arrow[d, "\subseteq"] & C^b_{\mathrm{har}}(\mathcal{T}, k) \arrow[r] \arrow[l, dashed, bend left, "I_k"] \arrow[d, "\subseteq"] & 0 \\
\mathcal{O}_{\mathcal{X}}(k+3) \arrow[r, "\mathrm{Res}_k"] & C_{\mathrm{har}}(\mathcal{T}, k) &
\end{tikzcd}$$

Since the splitting is G-equivariant, the map in the top row stays surjective after taking Γ-invariants. Moreover, by Theorem 8.2, we have $C^b_{\mathrm{har}}(\Gamma, k) = C_{\mathrm{har}}(\Gamma, k)$, and consequently $(\mathcal{O}_{\mathcal{X}}(k+3)^b)^\Gamma = \mathcal{O}_{\mathcal{X}}(k+3)^\Gamma$. We arrive at the following situation.

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{X}}(k+3)^\Gamma \arrow[r, "\mathrm{Res}_k"] & C_{\mathrm{har}}(\Gamma, k) \arrow[r] \arrow[l, dashed, bend left] & 0 \\
\mathcal{S}_{k+3}(\Gamma) \arrow[u, "\subseteq"] \arrow[ur, "\mathrm{Res}_k"'] & &
\end{tikzcd}$$

The key step is now to show that the image of the spitting is in fact in $\mathcal{S}_{k+3}(\Gamma)$. This implies that $\mathcal{S}_{k+3}(\Gamma) \to C_{\mathrm{har}}(\Gamma, k)$ is surjective. Assume for now that we have already shown this. Then specializing to $\Gamma = \Gamma(t)$ by Theorem 6.20, Theorem 8.5 and Lemma 3.4 we have

$$\dim_{\mathbb{C}_K} \mathcal{S}_{k+3}(\Gamma(t)) = [\Gamma_1(t) : \Gamma(t)] \cdot \binom{k+2}{2} = q^3 \dim_{\mathbb{C}_K} V_k \le \dim_{\mathbb{C}_K} C_{\mathrm{har}}(\Gamma(t), k),$$

which proves the injectivity for $\Gamma = \Gamma(t)$. For general Γ, since $\Gamma(t)$ is normal in Γ, we can consider the commutative diagram

$$\begin{tikzcd}
\mathcal{S}_{k+3}(\Gamma) \arrow[r, "\mathrm{Res}_k"] \arrow[d, "\cong"] & C_{\mathrm{har}}(\Gamma, k) \arrow[d, "\cong"] \\
\mathcal{S}_{k+3}(\Gamma(t))^{\Gamma/\Gamma(t)} \arrow[r, "\mathrm{Res}_k"] & C_{\mathrm{har}}(\Gamma(t), k)^{\Gamma/\Gamma(t)},
\end{tikzcd}$$

where the left vertical arrow is an isomorphism by the discussion at the end of Section 6.1. For the right vertical arrow it is immediate from the definition. Since the bottom horizontal arrow is an isomorphism by the above, we obtain the result for general Γ. Thus, in order to complete the proof of Theorem 8.9, we need to prove the following proposition. It boils down to another lengthy computation with the kernel function similar to Theorem 4.22 and Proposition 4.32. The analogous result for $\mathrm{GL}_2(K)$ is [Tei91, Lemma 12]. We should stress that this proposition requires the strong bounds in Conjecture 5.49. The weaker version stated in Conjecture 4.29 is *not* sufficient to prove the proposition.

8.11. Proposition. *Assume that Conjecture 5.49 holds and let $c \in C_{\mathrm{har}}(\Gamma, k)$. Then we have*

$$I_k(\lambda_c) \in \mathcal{S}_{k+3}(\Gamma).$$

Before we begin the proof, we want to explain the main ideas and the new difficulties compared to [Tei91, Lemma 12]. Our proof can be summarized as follows: First of all it suffices to show that $I_k(\lambda_c)$ vanishes at infinity. For this, we use the criterion from Proposition 6.12 (ii). Thus, we need to estimate $|I_k(\lambda_c)(\omega)|$ on certain subsets of $\mathcal{X}$. In order to obtain the desired estimate, we follow the line of thought of the proof of Theorem 4.22: We want to choose a covering of G such that, for the specific ω under consideration, we obtain a "nice" convergent series expansion. Then we want use the *uniform* estimate from Conjecture 4.29 (iii) to estimate the series expansion term by term. So far, this is in complete analogy with [Tei91, Lemma 12].
In the $\mathrm{GL}_2(K)$-case, there are two cells to consider, the big cell and the cell *at infinity*. It turns out that in this situation one can use the following recipe to construct the covering: One removes a small ball around infinity depending on the size of $|\omega|$. The complement is then covered by balls of large radius. This is where the conditions on ω enter: Even on these large balls one obtains a convergent series expansion with a good estimate. Then one realizes that the bound one obtains on the ball around infinity can be chosen independently of $|\omega|$, hence one obtains the desired estimate.
When transferring this strategy to $\mathrm{GL}_3(K)$, the first observation is that one now has six cells instead of two. The locally analytic kernel function is not defined uniformly on all cells, but is modified precisely on two such cells. Over the course of the proof we will see that in fact it suffices to modify the continuous kernel function on a smaller neighbourhood of its singular locus; the resulting integral remains unchanged. It turns

that when constructing the covering, one needs to consider finer conditions than just the size of $|\omega|$. The role of the cell at infinity is played by the cells corresponding to $\{\mathrm{id}, s_1\} \subset W$, exactly the cells where the kernel function is modified.

Proof of Proposition 8.11. Let $f_c := I_k(\lambda_c) \in \mathcal{O}_{\mathcal{X}}(k+3)^{\Gamma}$. First off, we observe that it suffices to check that $\mathrm{ord}_{\Gamma_H}(f_c) \geq 1$. This is because

$$\mathrm{ord}_{(g\Gamma g^{-1})_H}(g_* f_c) = \mathrm{ord}_{(g\Gamma g^{-1})_H}(f_{g\cdot c})$$

and $g \cdot c \in C_{\mathrm{har}}(g\Gamma g^{-1}, k)$ for $g \in \mathrm{GL}_3(A)$, which is sufficient by Proposition 6.16. To check that $\mathrm{ord}_{\Gamma_H}(f_c) \geq 1$, we use the criterion given in Proposition 6.12 (ii). We need some preparations. As explained above, the key idea is to use the estimate in Conjecture 5.49 (iii) to obtain the desired estimate for f_c. For this, we need to use a different covering of G compared to the one used in the proof of Theorem 4.22. We set

$$t(a_1, a_2) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi^{a_1} & 0 \\ 0 & 0 & \pi^{a_2} \end{pmatrix} \quad \text{for } (a_1, a_2) \in \mathbb{Z}^2.$$

Now, for $w \in W$ and $u \in U^-$ we let

$$B^w(u; a_1, a_2) := w_0 w u t(a_1, a_2) \mathcal{I}_{\mathrm{id}} B \subset w_0 w U^- B.$$

Observe that for $(b_1, b_2) \in \mathbb{Z}^2$ such that $b_2 - a_2 \geq b_1 - a_1 \geq 0$ we have

$$B^w(u; a_1, a_2) = \bigsqcup_v B^w(uv; b_1, b_2), \tag{11}$$

where v runs through the left cosets of $t(b_1, b_2)\mathcal{I}_{\mathrm{id}} t(b_1, b_2)^{-1}$ in $t(a_1, a_2)\mathcal{I}_{\mathrm{id}} t(a_1, a_2)^{-1}$. Moreover, note that since

$$w\mathcal{I}_{\mathrm{id}} \subseteq \mathcal{I}_w w \quad \text{for all } w \in W,$$

we have $B^w(u; a_1, a_2) \subseteq w_0 \mathcal{I}_w w B$ for $a_2 \geq a_1 \geq 0$. In fact, the coordinates on $D(w)$ as in (1) and $U(\sigma_0)$ from Table 1 are compatible with this inclusion. We extend the coordinates $\underline{x} = (x_1, x_2, x_3)$ from $D(w)$ to $w_0 w U^- B/B$ so that the inclusion $D(w) \subset w_0 w U^- B/B$ is just the natural inclusion of (a subset of) $\mathcal{O}_K^3$ into K^3. We want to use Conjecture 5.49 (iii) to bound certain values of λ_c. For this, let $f_I := \underline{x}^I$ regarded as an element of $\mathcal{A}_k$ by extending with 0 outside $\mathcal{I}B$. Let $h = w_0 w u t(a_1, a_2) \in G$. Then Conjecture 5.49 (iii) implies that we find $C > 0$, independent of h, such that

$$|\lambda_c(h_* f_I)| \leq C q^{-|I|}.$$

Let $g \in w_0 w U^- \cap B^w(u; a_1, a_2)$. If we denote by $g_{\underline{x}} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{smallmatrix}\right)$ the unique matrix in U^- such that $g = w_0 w g_{\underline{x}}$, we have

$$\begin{aligned} (h_* f_I)(gb) &= f_I(t(a_1, a_2)^{-1} u^{-1} g_{\underline{x}} b) \\ &= \chi_k(b^{-1}) \chi_k(t(a_1, a_2)) f_I(t(a_1, a_2)^{-1} u^{-1} g_{\underline{x}} t(a_1, a_2)) \quad \text{for } b \in B. \end{aligned}$$

Let
$$f_{I,u,(a_1,a_2)}(\underline{x}) := \pi^{a_1(i_1-i_3)+a_2(i_2+i_3)} f_I(t(a_1,a_2)^{-1}u^{-1}g_{\underline{x}}t(a_1,a_2)).$$
Then, since $t(a_1,a_2)^{-1}u^{-1}g_{\underline{x}}t(a_1,a_2) \in \mathcal{I}_{\mathrm{id}}$, we can compute this function explicitly. Note however that the coordinates chosen in (5) are obtained by taking the inverse of the coordinate matrix in Table 1. We obtain
$$\begin{aligned} f_{I,u,(a_1,a_2)}(\underline{x}) &= \pi^{a_1(i_1-i_3)+a_2(i_2+i_3)} f_I\left(\begin{pmatrix} 1 & 0 & 0 \\ \pi^{-a_1}(x_1-u_1) & 1 & 0 \\ \pi^{-a_2}(x_2-u_2-u_3(x_1-u_1)) & \pi^{a_1-a_2}(x_3-u_3) & 1 \end{pmatrix}\right) \\ &= (u_1-x_1)^{i_1}(u_2-x_2-x_3(u_1-x_1))^{i_2}(u_3-x_3)^{i_3}. \end{aligned}$$
Upon observing that $\chi_k(t(a_1,a_2)) = \pi^{a_1(-k/3)+a_2(2k/3)}$, we have
$$\left|\lambda_c\left(\chi_k(b^{-1})f_{I,u,(a_1,a_2)}(\underline{x})\mathbb{1}_{B^w(u;a_1,a_2)}(gb)\right)\right| \leq Cq^{-|I|+a_1(i_3-i_1-k/3)+a_2(-i_2-i_3+2k/3)}. \tag{12}$$
The constant $C > 0$ is independent of w, u and (a_1,a_2). The estimate (12) is the first central ingredient for the proof. The second is given as follows. Let $a_2 \geq a_1 \geq 0$ and put
$$\mathcal{U}(a_1,a_2) := w_0t(a_1,a_2)U_1(\sigma_0).$$
Then by Lemma 4.15 we have $\mathcal{U}(a_1,a_2) = w_0t(a_1,a_2)\mathcal{I}P_1/B \subseteq \mathcal{U}$. We may set
$$\hat{\theta}_{(a_1,a_2)}(g,\omega) := \hat{\theta}(g,\omega) + \mathbb{1}_{\mathcal{U}\setminus\mathcal{U}(a_1,a_2)}(g)\theta_{\mathrm{inv}}(g,\omega).$$
Then as in Proposition 4.17, one can show that $\hat{\theta}_{(a_1,a_2)}(g,\omega)$ is locally analytic on G/B. Even more is true: By construction one can replace the kernel function $\hat{\theta}(g,\omega)$ by $\hat{\theta}_{(a_1,a_2)}(g,\omega)$ in the construction of I_k. The resulting function f_c remains unchanged.
With these preparations, we can now prove that f_c vanishes at infinity. For this, we fix $n \geq 0$ and $N \geq 0$. Let $\omega \in \mathcal{X}$ such that $\tilde{\omega} \in \mathcal{Y}_n$ and $d(\omega_1, K^2\tilde{\omega}) \geq q^{4n+N}$. We will prove that for such ω we have
$$|f_c(\omega)| \leq Cq^{-N},$$
where the constant $C > 0$ is independent of ω, n and N. Then Proposition 6.12 (ii) implies that f_c vanishes at infinity. Note that the above conditions imply in particular that $|\omega_1| \geq q^{4n+N}$. We assume for now that additionally we have $\omega \in \mathcal{X}_m$ for some $m \geq 4n+N$. We choose a covering of G depending on m and n as follows: Write
$$G = \bigsqcup_{w\in W} D_{n,m}(w),$$
where $D_{n,m}(w) := w_0t(n,m)\mathcal{I}_w wB$. This is just a rescaling of the Bruhat-Iwahori decomposition. Note that we have
$$\mathcal{U}(n,m) = D_{n,m}(\mathrm{id}) \cup D_{n,m}(s_1).$$
To keep the notation short, we put
$$S_{w,u,(a_1,a_2)}(\omega) := \lambda_c\left(\det(gb)^{-2k/3}\beta_1(gb)^k\hat{\theta}_{(n,m)}(\omega,gb)\mathbb{1}_{B^w(u;a_1,a_2)}(gb)\right).$$

While the following computations are quite technical, the idea is relatively simple: We will cover each cell $D_{n,m}(w)$ by appropriately chosen balls $B^w(u;a_1,a_2)$ such that we can expand $S_{w,u,(a_1,a_2)}(\omega)$ into a convergent series expansion. Then we can use (12) and the various bounds on ω to obtain the needed estimate. We do this only in a few cases; the remaining cells can be treated in a similar fashion.
We first consider the cell $D_{n,m}(\mathrm{id}) = w_0 t(n,m)\mathcal{I}B = B^{\mathrm{id}}(\mathrm{id};n,m)$. As in the proof of Theorem 4.22, we may write

$$S_{\mathrm{id},\mathrm{id},(n,m)}(\omega) = \sum_I c_I \lambda_c \left(\chi_k(b^{-1}) x_1^{i_1+1} x_2^{i_2} (x_2 - x_1 x_3)^{i_3+k+1} \mathbb{1}_{B^{\mathrm{id}}(\mathrm{id};n,m)}(gb)\right) \omega_1^{i_2+i_3}\omega_2^{i_1}$$

with $c_I \in O_K$. Then a simple computation using (12) reveals that we have

$$|S_{\mathrm{id},\mathrm{id},(n,m)}(\omega)| \leq C \sup_I \left(q^{-|I|+n(-i_1-1-k/3)+m(-i_2-i_3-k-1+2k/3)} q^{ni_1+m(i_2+i_3)}\right) \leq Cq^{-N}.$$

Next, we consider the cell $D_{n,m}(s_1)$. First off, we observe that under the coordinates on U^- chosen above, $D_{n,m}(s_1)$ corresponds to the compact open subset

$$(\pi^{-n}O_K)\times(\pi^{m-n+1}O_K)\times(\pi^{m+1}O_K)\subset K^3.$$

Thus, by (11) we can choose a disjoint covering of $D_{n,m}(s_1)$ by finitely many sets $B^{s_1}(u;m,2m)$ where the coordinates of the matrices u can be chosen so that $u_3 = 0$. We compute

$$S_{s_1,u,(m,2m)}(\omega) = \sum_I c_I \lambda_c \left(\chi_k(b^{-1})(x_1-u_1)^{i_1}(x_2-u_2)^{i_2} x_3^{i_3+k+1}\mathbb{1}_{B^{s_1}(u;m,2m)}(gb)\right)\frac{\omega_1^{i_2+i_3}}{f(u,\omega)^{i_1+i_2+1}},$$

where $c_I \in O_K$ and $f(u,\omega) = u_2\omega_1 + \omega_2 + u_1$. Again, by using (12) and the fact that $\omega \in \mathcal{X}_m$ we obtain

$$\begin{aligned}|S_{s_1,u,(m,2m)}(\omega)| &\leq C\sup_I \left(q^{-|I|+m(i_3+k+1-i_1-k/3)+2m(-i_2-i_3-k-1+2k/3)} q^{m(i_2+i_3)+(m-4n-N)(i_1+i_2+1)}\right)\\ &\leq Cq^{-N}.\end{aligned}$$

Finally, we consider the cell $D_{n,m}(w_0)$. First off, we observe that under the coordinates on U^- chosen above, $D_{n,m}(w_0)$ corresponds to the compact open subset

$$(\pi^{n-m}O_K)\times(\pi^{-m}O_K)\times(\pi^{-n}O_K)\subset K^3.$$

Thus, we can choose a disjoint covering of $D_{n,m}(w_0)$ by finitely many sets $B^{w_0}(u;-2n,-n)$. As in the proof of Theorem 4.22, we have

$$S_{w_0,u,(-2n,-n)}(\omega) = \sum_I c_I\lambda_c\left(\chi_k(b^{-1})(\underline{x}-\underline{u})^I \mathbb{1}_{B^{w_0}(u;-2n,-n)}(gb)\right)\frac{\omega_2^{i_1}}{f_1(u,\omega)^{i_1+i_2+1} f_2(u,\omega)^{i_3+1}}$$

with $c_I \in O_K$, $f_1(u,\omega) = \omega_1 + u_1\omega_2 + u_2$ and $f_2(u,\omega) = \omega_2 + u_3$. Again, a computation using (12) reveals that we have

$$|S_{w_0,u,(-2n,-n)}(\omega)| \leq Cq^{-N}.$$

Similar computations show that each of the remaining cells can be covered in an analogous fashion to obtain the same estimate. Combining all of this shows that $|f_c(\omega)| \leq Cq^{-N}$, independently of the auxiliary choice of m. This completes the proof. □

Now, the final step in the proof of Theorem 8.9 is to check the Hecke-equivariance of the map Res_k.

Proof of Theorem 8.9. The only remaining point is the Hecke-equivariance, which is straightforward from the definitions. Let $\delta \in \mathrm{GL}_3(F)$ and write $\Gamma\delta\Gamma = \bigsqcup_i \delta_i\Gamma$. We have

$$\begin{aligned}\mathrm{Res}_k(T_\delta f) &= \mathrm{Res}_k\left(\sum_i (\delta_i)_* f\right) \\ &= \sum_i \delta_i \cdot \mathrm{Res}_k(f) = T_\delta(\mathrm{Res}_k(f)),\end{aligned}$$

since Res_k is G-equivariant as we have seen in Subsection 3.4.2. □

We can slightly extend Theorem 8.9 as follows. Let $V_{k,\ell} := V_k \otimes_{\mathbb{C}_K} \det^{\ell-1-k/3}$.

8.12. Corollary. *We keep the assumptions from Theorem 8.9. Then there is a Hecke-equivariant isomorphism*

$$\mathcal{S}_{k+3,\ell}(\Gamma) \to C_{\mathrm{har}}(\Gamma, V_{k,\ell}).$$

Proof. By Theorem 8.9 we have the isomorphism $\mathcal{S}_{k+3}(\Gamma(t)) \to C_{\mathrm{har}}(\Gamma(t), k)$. Since $\Gamma(t) \subset \mathrm{SL}_3(A)$, we obtain an isomorphism

$$\mathcal{S}_{k+3,\ell}(\Gamma(t)) \cong \mathcal{S}_{k+3}(\Gamma(t)) \otimes_{\mathbb{C}_K} \det^{\ell-1-k/3} \to C_{\mathrm{har}}(\Gamma(t), k) \otimes_{\mathbb{C}_K} \det^{\ell-1-k/3}$$

of $\Gamma/\Gamma(t)$-modules. After observing that one has a natural isomorphism

$$C_{\mathrm{har}}(\Gamma(t), k) \otimes_{\mathbb{C}_K} \det^{\ell-1-k/3} \cong C_{\mathrm{har}}(\Gamma(t), V_{k,\ell}),$$

we obtain the desired isomorphism by taking $\Gamma/\Gamma(t)$-invariants. Checking the Hecke-equivariance is again straightforward. □

8.13. Remark. By construction, the Hecke-equivariant isomorphism above is given by the same formula as the map Res_k, see Proposition 3.27.

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Introduction

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