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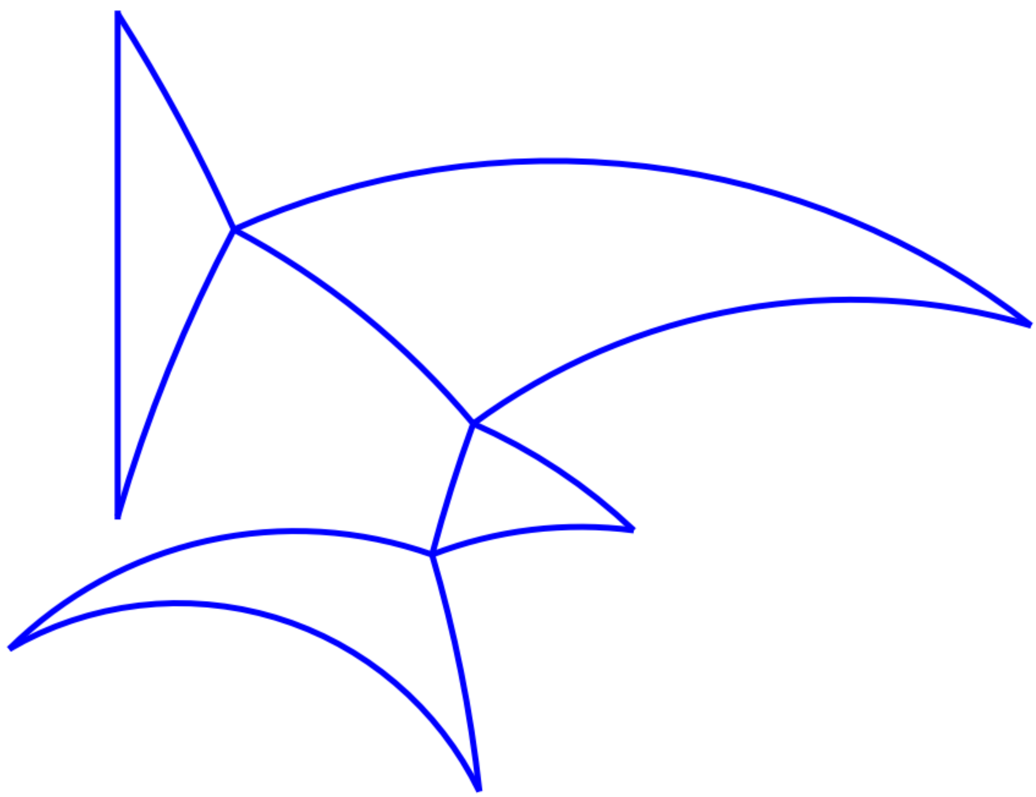

Under supervision of

Prof. Dr. Peter Albers

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À mon plus viel ami, Alain.

KARADOC: *On va commencer avec 12, ça fera un pallier.*
PERCEVAL: *Pis après on fera 13, 14, 15, ... ; enfin, tous les chiffres impairs jusqu'à 22.*

Kaamelott
Livre II, *Unagi II*

Abstract

We study a compact family of totally elliptic representations of the fundamental group of a punctured sphere into $\mathrm{PSL}(2,\mathbb{R})$ discovered by Deroin and Tholozan and named after them. We describe a polygonal model that parametrizes the relative character variety of Deroin–Tholozan representations in terms of chains of triangles in the hyperbolic plane. We extract action-angle coordinates from our polygonal model as geometric quantities associated to chains of triangles. The coordinates give an explicit isomorphism between the space of representations and the complex projective space. We prove that they are almost global Darboux coordinates for the Goldman symplectic form.

This work also investigates the dynamics of the mapping class group action on the relative character variety of Deroin–Tholozan representations. We apply symplectic methods developed by Goldman and Xia to prove that the action is ergodic.

Zusammenfassung

Wir untersuchen eine kompakte Familie von total elliptischen Darstellungen der Fundamentalgruppe einer punktierten Sphäre in $\mathrm{PSL}(2,\mathbb{R})$, die von Deroin und Tholozan entdeckt und nach ihnen benannt wurde. Wir beschreiben ein polygonales Modell, das die relative Charaktervarietät der Deroin–Tholozan-Darstellungen in Form von Ketten von Dreiecken in der hyperbolischen Ebene parametrisiert. Wir extrahieren Wirkungs-Winkelkoordinaten aus unserem polygonalen Modell als geometrische Größen, die mit Dreiecksketten assoziiert sind. Die Koordinaten ergeben einen expliziten Isomorphismus zwischen dem Raum der Darstellungen und dem komplexen projektiven Raum. Wir beweisen, dass sie fast globale Darboux-Koordinaten für die symplektische Goldman-Form sind.

In dieser Arbeit wird auch die Dynamik der Wirkung der Abbildungsklassengruppe auf der relativen Charaktervarietät von Deroin–Tholozan-Darstellungen untersucht. Wir wenden symplektische Methoden an, die von Goldman und Xia entwickelt wurden, um zu beweisen, dass die Wirkung ergodisch ist.

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Kaamelott
Livre II, *La Botte secrète II*

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LOTH: *Odi panem quid meliora. Ça veut rien dire, mais je trouve que ça boucle bien.*

Kaamelott
Livre V, *Corvus corone*

Contents

1. Introduction

A *character variety* is, broadly speaking, a symplectic manifold constructed from a closed oriented surface Σ and a quadrable[1] Lie group G. It is defined as the space of conjugacy classes of representations of the fundamental group of Σ into G. If the surface Σ is n times punctured, the same construction gives a Poisson manifold whose symplectic leaves are called *relative character varieties*. They are defined for a choice of n conjugacy classes inside G and consist of representations whose holonomy around each puncture lies in the prescribed conjugacy class. We elaborate on these concepts in Section 2.4.

This work focuses on the case where Σ is a sphere with at least three punctures and G is $\mathrm{PSL}(2, \mathbb{R})$. The relative character varieties obtained by prescribing elliptic conjugacy classes for the holonomy around each puncture of the sphere has been shown to contain compact connected components [DT19]. We investigate the geometry and the mapping class group dynamics on these compact components which we call *Deroin–Tholozan relative character varieties*, see Chapter 3 for a precise definition. The representations themselves are referred to as *Deroin–Tholozan representations*. Deroin–Tholozan representations have the remarkable property of being *totally elliptic*. This means that the image of any homotopy class of loops containing a simple closed curve is an elliptic element of $\mathrm{PSL}(2, \mathbb{R})$, see Proposition 3.1.10.

1.1. The results

The original content of the present work has been published by the author in two different papers: [Mar21] and [Mar20].

1.1.1. Action-angle coordinates

The first result is a description of action-angle coordinates for the Deroin–Tholozan relative character variety. The notion of *action-angle coordinates* refers to the canonical coordinates

[1] Quadrable Lie groups are defined in Definition 2.1.1. For now, it is sufficient to think about reductive or semisimple Lie groups instead.

of an integrable system in the sense of the Arnold–Liouville Theorem, see e.g. [CdS01, Thm. 18.12]. Our construction mimics the definition of Fenchel–Nielsen coordinates for Teichmüller space. Let Σ_n denote a connected and oriented surface of genus zero with $n \geqslant 3$ punctures and fundamental group $\pi_1(\Sigma_n)$. The action-angle coordinates that we construct depend on the choice of a pants decomposition of Σ_n. We fix a pants decomposition $\mathcal{P}$ and consider a Deroin–Tholozan representation $\phi\colon \pi_1(\Sigma_n) \to \mathrm{PSL}(2,\mathbb{R})$. To each of the $n-2$ pairs of pants $P_0, \dots, P_{n-3}$ in $\mathcal{P}$, we associate a geodesic triangle Δ_i in the upper half-plane whose vertices are the unique fixed points of the images of the three boundary curves of P_i (we use that ϕ is totally elliptic). This produces a chain of $n-2$ geodesic triangles $\Delta_0, \dots, \Delta_{n-3}$ in the upper half-plane as illustrated by Figure 1.1. The term "necklace" was used in [DT19, §0.3] to hint at the construction; we will, however, stick to "chain of triangles".

The $(n-3)$-torus action given by rotation of a chain of triangles around one of the $n-3$ common vertices of two consecutive triangles in the chain defines a maximal Hamiltonian torus action on the Deroin–Tholozan relative character variety. We describe this action in greater details in Section 4.1.3. Let a_i be twice the area of the triangle Δ_i and γ_i be the angle between the triangles Δ_{i-1} and Δ_i measured at their common vertex, see Figure 1.1.

Theorem A. *If we set $\sigma_i := \gamma_1 + \ldots + \gamma_i$, then*

$$\{a_1, \dots, a_{n-3}, \sigma_1, \dots, \sigma_{n-3}\}$$

are action-angle coordinates for the Deroin–Tholozan relative character variety.

Theorem A says that the dynamical system induced by rotating the triangles around their common vertices is integrable with canonical coordinates $\{a_1, \dots, a_{n-3}, \sigma_1, \dots, \sigma_{n-3}\}$. In fact, it corresponds to the maximal Hamiltonian torus action on the Deroin–Tholozan relative character variety described in [DT19, §3] by considering the twist flows à la Goldman along the separating curves defining the pants decomposition $\mathcal{P}$. This equips the Deroin–Tholozan relative character variety with the structure of a symplectic toric manifold. We deduce Theorem A from

Theorem B. *The map from the Deroin–Tholozan relative character variety to $\mathbb{CP}^{n-3}$ defined in homogeneous coordinates by*

$$\left[\sqrt{a_0} : \sqrt{a_1}e^{i\sigma_1} : \ldots : \sqrt{a_{n-3}}e^{i\sigma_{n-3}}\right]$$

is an isomorphism of symplectic toric manifolds.

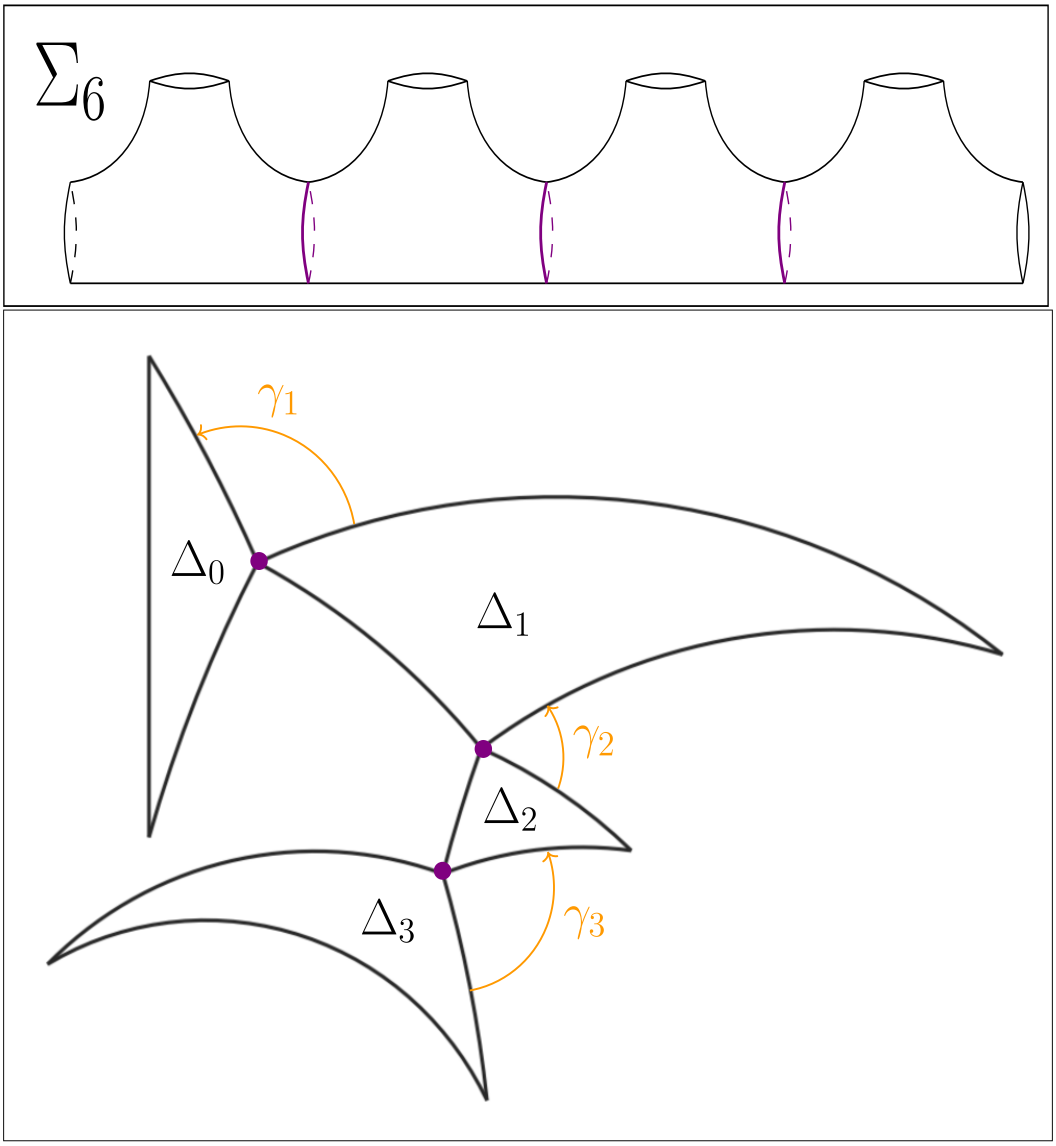


Figure 1.1.: On top: a pants decomposition of a sphere with six punctures into four pairs of pants. On bottom: a corresponding chain of geodesic triangles in the upper half-plane. The angles between consecutive triangles in the chain are denoted by γ_i.

Both spaces involved in the statement of Theorem B are equipped with a natural symplectic structure: the Goldman symplectic form for the character variety and the Fubini-Study form for the complex projective space. It was already proven in [DT19, Thm. 4] that the two spaces are isomorphic using Delzant's classification of symplectic toric manifolds. Theorem B provides an explicit isomorphism.

The main difficulty in the proof of Theorem B lies in checking that the map is differentiable. This requires a careful analysis of all the parameters involved. The primary source of trouble is the erratic behaviour of the parameters σ_i when a triangle in a chain degenerates to a single point and the presence of square roots on the parameters a_i. An immediate consequence of Theorem B is

Theorem C. *On an open and dense subset of the Deroin–Tholozan relative character variety, it holds that*

$$\omega_{\mathcal{G}} = \frac{1}{2}\sum_{i=1}^{n-3} da_i \wedge d\sigma_i,$$

where $\omega_{\mathcal{G}}$ is the Goldman symplectic form. In particular, the 2-form $\sum_{i=1}^{n-3} da_i \wedge d\sigma_i$ is independent of the pants decomposition used to define the coordinates $\{a_1, \ldots, a_{n-3}, \sigma_1, \ldots, \sigma_{n-3}\}$.

Theorem C is the analogue of a famous result of Wolpert known as *Wolpert's magic formula* in the context of Teichmüller space. We briefly explain the analogy. The Teichmüller space of a closed hyperbolic surface of genus g can be identified with $(0, \infty)^{3g-3} \times \mathbb{R}^{3g-3}$ using Fenchel–Nielsen coordinates, see e.g. [FM12, §10.6]. Fenchel–Nielsen coordinates consist of length parameters $l_1, \ldots, l_{3g-3}$ and twist parameters $\theta_1, \ldots, \theta_{3g-3}$. They depend on a choice of pants decomposition of the surface. Wolpert proved in [Wol83] that the length and twist parameters are dual to each other and that the 2-form

$$\frac{1}{2}\sum_{i=1}^{3g-3} dl_i \wedge d\theta_i \tag{1.1.1}$$

is independent of the choice of the pants decomposition. He did so by proving that the 2-form (1.1.1) is equal to the Weil-Petersson form on Teichmüller space. This relation is nowadays known as Wolpert's magic formula. Goldman proved in [Gol84] that the Weil-Petersson form is a multiple of the Goldman symplectic form if one sees Teichmüller space as a component of the character variety of representations of the fundamental group of the surface into $\mathrm{PSL}(2, \mathbb{R})$.

The cornerstone of the construction of the coordinates $\{a_1, \ldots, a_{n-3}, \sigma_1, \ldots, \sigma_{n-3}\}$ is the modelling of Deroin–Tholozan representations in terms of chains of geodesic triangles in the upper half-plane. Formally, we introduce a moduli space of *chains of geodesic triangles* in the upper half-plane and show that it is in one-to-one correspondence with the relative character variety of Deroin–Tholozan representations. We refer to it as the *polygonal model* for Deroin–Tholozan representations. Other character varieties arise as moduli spaces of combinatorial structures. An example is the relation singled out by Kapovich–Millson in [KM96] between the moduli space of polygons in the three-dimensional Euclidean space $\mathbb{E}^3$ and the character variety of representations of the fundamental group of a punctured sphere into the isometries of $\mathbb{E}^3$.

1.1.2. Mapping class group dynamics

The *mapping class group* of an oriented surface Σ is the group of isotopy classes of orientation-preserving homeomorphisms $\Sigma \to \Sigma$. If Σ is punctured, then only homeomorphisms that fix each puncture individually are considered. The mapping class group of Σ is denoted by $\mathrm{Mod}(\Sigma)$ and will be discussed in more details in Section 2.6.2.

There is a natural action of $\mathrm{Mod}(\Sigma)$ on any (relative) character variety of representations of the fundamental group of Σ into a Lie group G. The action is by pre-composition, after identifying $\mathrm{Mod}(\Sigma)$ with a subgroup of the group of outer automorphisms of the fundamental group of Σ. The action always preserves the symplectic structure of the character variety. It is known to be ergodic if G is compact, whereas its dynamical nature remains widely unknown if G is not compact, and specifically if G has higher dimension or rank. It is nevertheless proven to be proper and discontinuous on the Teichmüller components of the $\mathrm{PSL}(2,\mathbb{R})$-character variety of a closed surface and conjectured by Goldman to be ergodic on the remaining components (see Subsection 1.1.3). Our results investigate the mapping class group action on the Deroin–Tholozan relative character varieties.

Theorem D. *The action of* $\mathrm{Mod}(\Sigma_n)$ *on the Deroin–Tholozan relative character variety is ergodic with respect to the Goldman symplectic measure.*

Theorem D is the contribution of the author to a series of results about mapping class group dynamics on character varieties. In Section 1.1.3, we briefly provide the reader with an overview of this field which has been studied extensively in the past decades.

We prove Theorem D by applying methods developed in [GX11] and in [MW16]. The argument has a strong symplectic geometry flavour. The cornerstone of the proof relates the action of a Dehn twist in $\mathrm{Mod}(\Sigma_n)$ to a certain Hamiltonian flow on the Deroin–Tholozan relative character variety, see Proposition 5.1.1 for a precise statement. A coarse sketch of the proof consists of the following steps:

1. Identify sufficiently many Dehn twists in $\mathrm{Mod}(\Sigma_n)$ such that the associated Hamiltonian flows locally act transitively on the Deroin–Tholozan relative character variety.

2. Prove that this implies that any integrable $\mathrm{Mod}(\Sigma_n)$-invariant function of the Deroin–Tholozan relative character variety must be constant almost everywhere.

Theorem D can be refined to a stronger statement. Namely, we also prove

Theorem E. *For $n \geqslant 5$, there exists a proper subgroup $\mathcal{H}$ of $\mathrm{Mod}(\Sigma_n)$ whose action on the Deroin–Tholozan relative character variety is ergodic with respect to the Goldman symplectic measure. Moreover, $\mathcal{H}$ can be chosen to be finitely generated by $2(n-3)$ Dehn twists.*

1.1.3. Some context about dynamics on character varieties

The list below is certainly non-exhaustive and reflects the taste of the author.

Goldman proved in [Gol97] that the mapping class group action is ergodic whenever $\Sigma_{g,n}$ has negative Euler characteristic and G is a Lie group whose simple factors are isomorphic to $\mathrm{SU}(2)$. In [GX11] Goldman–Xia provided a new proof of the ergodicity for $\mathrm{SU}(2)$-character varieties relying on the symplectic geometry of the character variety. Goldman conjectured in [Gol97, Conj. 1.3] that the mapping class group action is ergodic for any compact Lie group. The conjecture was proven by Pickrell–Xia in [PX02, PX03] for all $\Sigma_{g,n}$ with negative Euler characteristic except $\Sigma_{1,1}$. Goldman–Lawton–Xia established ergodicity for $\Sigma_{1,1}$ and $G = \mathrm{SU}(3)$ in [GLX21].

If G is not compact, the dynamics of the mapping class group action exhibit a different behaviour. It is, for instance, long known that the mapping class group acts properly and discontinuously on Teichmüller space which can be realized as a connected component of the $\mathrm{PSL}(2,\mathbb{R})$-character variety of $\Sigma_{g,0}$. More generally, the action is proper on the spaces of maximal and Hitchin representations [Wie06], [Lab08]. Ergodic actions contrast with proper actions by producing chaotic dynamics. Goldman promotes the following dichotomy in [Gol06]. Assume that G is noncompact and semisimple. The action is expected to be "nice" on connected components of the character variety that have a "strong" geometrical meaning (such as Teichmüller space). On the other hand, it is expected to give rise to more "complicated" dynamics on the remaining components. He conjectured, for instance, that the action is ergodic on the non-Teichmüller components of the $\mathrm{PSL}(2,\mathbb{R})$-character variety of a closed surface [Gol06, Conj. 3.1]. Marché –Wolff proved in [MW16, MW19] that the conjecture holds for $\Sigma_{2,0}$ on the connected components of Euler class ± 1 and disproved the conjecture for the component of Euler class zero. They also introduce the subspace $\mathcal{NH}_g^k$ of the character variety that consists of representations with Euler class k which map a simple closed curve to a non-hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ and prove that the action is ergodic on $\mathcal{NH}_g^k$ for $(g,k) \neq (2,0)$, see [MW16, Thm. 1.6]. This shows that Goldman's conjecture is equivalent to $\mathcal{NH}_g^k$ having full measure in the corresponding connected component.

The counterpart of Goldman's conjecture for non-closed surfaces was formulated recently by Yang. He investigated in [Yan16] the mapping class group action on $\mathrm{PSL}(2,\mathbb{R})$-relative character varieties with parabolic holonomy around each puncture. In the case of a 4-punctured sphere, he proved that the action is ergodic on every connected component of non-extremal Euler class, generalizing a result known to Maloni–Palesi–Tan for the components of Euler class ± 1 [MPT15]. He further conjectured that the analogous statement holds for every punctured surface [Yan16, Conj. 1.4].

Several authors have also considered the action of remarkable subgroups of $\mathrm{Mod}(\Sigma_{g,n})$ on character varieties. For instance, the *Johnson group* is the subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ generated by Dehn twists along simple closed curves which are null-homologous in $H_1(\Sigma_{g,n}, \mathbb{Z})$. Goldman–Xia proved in [GX12] that the action of the Johnson group on the $\mathrm{SU}(2)$-relative character variety of $\Sigma_{1,2}$ is ergodic for a generic choice of conjugacy classes for the holonomy around the punctures. This result was extended to all closed surfaces $\Sigma_{g,0}$ with $g \geqslant 2$ by Funar–Marché in [FM13]. Another remarkable subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ is the *Torelli group*. If $n \leqslant 1$, then the Torelli group is the subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ acting trivially on $H_1(\Sigma_{g,n}, \mathbb{Z})$. The Johnson group is a subgroup of the Torelli group, see e.g. [FM12, §6] for more details. Bouilly recently proved in [Bou20] that the action of the Torelli group on each connected component of the character variety of $\Sigma_{g,0}$ is ergodic, for any $g \geqslant 2$ and for any compact connected semisimple Lie group G.

The mapping class group action remains of interest on character varieties on which the Goldman symplectic form cannot be defined, for there are ways to define an alternative natural invariant measure, see e.g. [Pal11] and references therein. The first kind of examples are character varieties of non-orientable surfaces. Palesi proved in [Pal11] that the mapping class group action is ergodic for every non-orientable surfaces with Euler characteristic at most -2, including punctured surfaces, and $G = \mathrm{SU}(2)$. Maloni–Palesi–Yang studied in [MPY21] the mapping class group action on certain representations of the 3-punctured projective plane into $\mathrm{PGL}(2, \mathbb{R})$ that map peripheral loops to parabolic isometries. They proved that the action is ergodic on most of the connected components of non-maximal Euler characteristic. They expect ergodicity to hold on the remaining components as well.

The existence of an invariant symplectic structure may also fail for certain Lie groups. An example is the group $\mathrm{Aff}(\mathbb{C})$ of affine transformations of the complex plane. Ghazouani showed in [Gha16] that the mapping class group action on the $\mathrm{Aff}(\mathbb{C})$-character variety of a closed surface does not preserve any symplectic form. There exists however an invariant measure for which the mapping class group is ergodic [Gha16].

1.2. Organisation of the work

Chapter 2 introduces the notions of representation and character varieties. We cover the basic definitions and expend on the conjugacy action on representation varieties. This leads to the notion of character variety on which we provide several perspectives. We introduce the Goldman symplectic form for character varieties of surface group representations, along with the notion of volume of a representation of a surface group into a Hermitian Lie group. We conclude with an exposition of the mapping class group action on character varieties.

The recent results of Deroin–Tholozan [DT19] and Tholozan–Toulisse [TT21] on compact components of relative character varieties for punctured spheres are recalled in Chapter 3. The case of representations into $\mathrm{PSL}(2,\mathbb{R})$ is treated in detail, whereas the generalization to Hermitian Lie groups of higher rank is covered succinctly. We insist on the total elliptic nature of these representations.

The material presented in Chapters 4 and 5 is the original work of the author. In Chapter 4, we explain the construction of the action-angle coordinates for the Deroin–Tholozan relative character variety. Most of the chapter is dedicated to the proof of Theorem B (see also Theorem 4.2.1), from which we deduce Theorems A and C. The mapping class group dynamics is studied in Chapter 5 where we prove Theorems D and E.

We also provide the reader with two Appendices. Appendix A covers some useful facts about the Lie groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$. These include the classification of the infinite algebraic subgroups and irreducible subgroups of $\mathrm{SL}(2,\mathbb{C})$. We give several formulas related to the action of $\mathrm{SL}(2,\mathbb{R})$ on the upper half-plane. A brief introduction to group (co)homology, focusing on all relevant results and formulae for this work, is provided in Appendix B.

2. A note on character varieties

The material presented is this chapter is classical. The aim is to introduce the notions of *representation* and *character varieties*, taking into account various approaches found in the literature. We cover both the analytic and algebraic perspectives and insist on the symplectic geometry aspects of character varieties at the end of the notes. Most of it is inspired from [Sik12], [Mon16, §2], [Lab13], and [BGPGW07].

2.1. Representation varieties

A representation variety is an analytic, sometimes algebraic, object associated to a finitely generated group Γ and a Lie group G. It consists of the space of group homomorphisms from Γ to G. We start by recalling some generalities about Lie groups, including algebraic groups, and finitely generated groups. Most of the results later in this note require to restrict the groups Γ and G to finer classes. The relevant notions are presented in the next section.

2.1.1. Setting: Lie groups and finitely generated groups

Lie groups

A *Lie group* G is a real smooth manifold with a group structure for which the operations of multiplication and inverse are smooth maps. Lie groups always admit an analytic atlas, unique up to analytic diffeomorphism, such that multiplication and inverse are analytic maps[1]. Lie groups are not necessarily connected. We denote by G° the identity component of G. The *centralizer* of a subset $S \subset G$ is denoted $Z(S) := \{g \in G : gsg^{-1} = s, \forall s \in S\}$. It is a closed subgroup of G and hence a Lie subgroup of G. The standard examples of Lie groups are $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, and all their closed subgroups, called *linear Lie groups*, which include $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SU}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{SO}(n, \mathbb{R})$.

[1] This is a consequence of the Campbell-Hausdorff formula, see e.g. [Ser06, Chap. IV, §7-8]

The Lie algebra of a Lie group G is denoted $\mathfrak{g}$. Most of the time, we will think of $\mathfrak{g}$ as the tangent space to G at the identity. In various places we will make use of the Lie theoretic exponential map $\exp\colon \mathfrak{g} \to G$, which, in the case that G is a linear Lie group, is the matrix exponential map. The adjoint representation of G on $\mathfrak{g}$ is denoted by $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ and is defined by

$$\mathrm{Ad}(g)(\xi) := \left.\frac{d}{dt}\right|_{t=0} g\exp(t\xi)g^{-1}, \quad g \in G,\ \xi \in \mathfrak{g}.$$

A Lie algebra $\mathfrak{g}$ is

- *simple* if it is not abelian and if its only proper ideal is the zero ideal. Since ideals of $\mathfrak{g}$ are in one-to-one correspondence with sub-representations of its adjoint representation, $\mathfrak{g}$ is simple if and only if its adjoint representation is irreducible and $\mathfrak{g}$ is not a one-dimensional abelian Lie algebra.

- *semisimple* if it has no nonzero abelian ideals. Equivalently, a Lie algebra is semisimple if it is a direct sum of simple Lie algebras [Bou98, Chap. I, §6.2, Cor. 1]. By Cartan's criterion, $\mathfrak{g}$ is semisimple if and only if its *Killing form*

$$\begin{aligned} K\colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (\xi_1, \xi_2) &\mapsto \mathrm{Tr}(\mathrm{ad}(\xi_1)\,\mathrm{ad}(\xi_2)) \end{aligned}$$

  is nondegenerate [Bou98, Chap. I, §6.1, Thm. 1].

- *reductive* if it is the direct sum of an abelian and a semisimple Lie algebra. Equivalently, $\mathfrak{g}$ is reductive if and only if its adjoint representation is completely reducible[2], which is further equivalent to $\mathfrak{g}$ admitting a faithful, completely reducible, finite-dimensional representation [Bou98, Chap. I, §6.4, Prop. 5].

We call a connected Lie group *simple, semisimple* or *reductive* if its Lie algebra is simple, semisimple or reductive, respectively. Simple Lie groups are semisimple and semisimple Lie groups are reductive. The groups $\mathrm{SL}(n, \mathbb{R})$ for $n \geqslant 2$, $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{SU}(p, q)$ for $p + q \geqslant 2$ are simple. The group $\mathrm{SO}(n, \mathbb{R})^\circ$ is simple for $n \geqslant 3, n \neq 4$ and semisimple for $n = 4$. In contrast, the group $\mathrm{GL}(n, \mathbb{R})^\circ$ is not semisimple for any $n \geqslant 1$ (its Killing form is degenerate). It is however reductive, because its Lie algebra is the direct sum of the simple Lie algebra of traceless matrices and the abelian Lie algebra of diagonal matrices. It is worth observing that a connected linear Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$ is reductive if and

[2]Recall that a *completely reducible* representation is a representation that decomposes as a direct sum of irreducible representations. Such representations are sometimes called *semisimple*.

only if the trace form

$$\begin{aligned}\mathrm{Tr}\colon \mathfrak{g}\times\mathfrak{g} &\to \mathbb{R}\\ (\xi_1,\xi_2) &\mapsto \mathrm{Tr}(\xi_1\xi_2)\end{aligned}$$

is nondegenerate. This can be seen as a consequence of the classification of semisimple Lie algebras and [Bou98, Chap. I, §6.4, Prop. 5]. The previous statement also holds for connected linear Lie groups $G \subset \mathrm{GL}(n,\mathbb{C})$. If the (in this case, complex-valued) trace form is nondegegenrate, then so is its real part $\Re(\mathrm{Tr})\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ which gives a nondegenerate, symmetric, Ad-invariant, real-valued bilinear form.

A Lie group is called a *complex Lie group* if it has the structure of a complex manifold and the group operations are holomorphic. Standard examples of complex Lie groups include $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$.

Quadrable Lie groups

An important class of Lie groups for the purpose of this work are those that admit a nondegenerate, symmetric and Ad-invariant pairing on their Lie algebra. Such Lie groups carry different names throughout the literature, see [Ova16] for an overview. We opt for the name quadrable.

Definition 2.1.1 (Quadrable Lie groups)**.** A Lie group G is called *quadrable* if there exists a bilinear form (also called pairing)

$$B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$$

which is nondegenerate, symmetric and Ad-invariant.

Quadrable Lie groups are common among the standard Lie groups. For example, all semisimple Lie groups, and more generally all reductive Lie groups, are quadrable. Indeed, a nondegenerate, symmetric and Ad-invariant bilinear form on a reductive Lie algebra can be taken to be the Killing form on the semisimple part and any nondegenerate, symmetric bilinear form on the abelian part. Alternatively, one may consider the trace form associated to a faithful, finite-dimensional representation[3] of $\mathfrak{g}$. We point out that not all quadrable Lie groups are reductive, see [Gol84, Footnote p. 204].

[3]The *trace form of a representation* $\rho\colon \mathfrak{g}\to\mathrm{GL}(n,\mathbb{R})$ is the symmetric bilinear form $\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ given by $(\xi_1,\xi_2)\mapsto\mathrm{Tr}(\rho(\xi_1)\rho(\xi_2))$. For instance, the Killing form is the trace form of the adjoint representation.

Example 2.1.2. For instance, $G = \mathrm{SL}(2, \mathbb{R})$ is quadrable. We usually chose to work with the pairing given by the trace form: $\mathrm{Tr}\colon \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R} \to \mathbb{R}$, $(\xi_1, \xi_2) \mapsto \mathrm{Tr}(\xi_1\xi_2)$. The trace of a matrix is invariant under conjugation, so the trace form is Ad-invariant. In the basis

$$\mathfrak{sl}_2\mathbb{R} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

the trace form is given by the pairing $2x_1x_2 + y_1y_2 + z_1z_2$. It is clearly symmetric and nondegenerate. Actually, in this case, the pairing $\mathrm{Tr}\colon \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$ is also positive-definite.

Example 2.1.3. The Heisenberg group H is an example of a non-quadrable Lie group. Recall that H is defined to be the group of strictly upper triangular 3×3 real matrices:

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{h}$ of H is generated by the three matrices

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple computation shows that Z commutes with any element of H. Further

$$\mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(X) = X - Z, \quad \mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(Y) = Y, \tag{2.1.1}$$

and

$$\mathrm{Ad}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(X) = X, \quad \mathrm{Ad}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(Y) = Y + Z. \tag{2.1.2}$$

So, because of (2.1.1), any symmetric and Ad-invariant bilinear form $B\colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$, must satisfy

$$B(X, Z) = B(X - Z, Z) \text{ and } B(X, Y) = B(X - Z, Y)$$

which implies $B(Z, Z) = 0$ and $B(Y, Z) = 0$. Moreover, because of (2.1.2), it must also satisfy

$$B(X, Y) = B(X, Y + Z)$$

and thus $B(X, Z) = 0$. This shows that B is degenerate.

Algebraic groups

A group G is called an *algebraic group* if it is an algebraic variety[4] and if the operations are regular maps. The Zariski closure of any subgroup of G is an algebraic subgroup [Mil17, Lem. 1.40] and any algebraic subgroup of G is Zariski closed [Mil17, Prop. 1.41]. For instance, the centralizer $Z(S)$ of a subset $S \subset G$ is Zariski closed and hence an algebraic subgroup. All algebraic groups over the fields of real or complex numbers, respectively called *real or complex algebraic groups*, are also Lie groups, see [Mil13, III, §2] and references therein. Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. The group $\mathrm{GL}(n, \mathbb{K})$, and all its Zariski closed subgroups, such as $\mathrm{SL}(n, \mathbb{K})$, $\mathrm{Sp}(2n, \mathbb{K})$ or $\mathrm{SO}(n, \mathbb{K})$, are algebraic groups. They are called *linear algebraic groups*. Algebraic groups, however, are not necessarily linear (for instance, elliptic curves are non-linear algebraic groups). The group $\mathrm{SU}(p, q)$ is a real algebraic group, but is not a complex algebraic variety, see e.g. [SKKT00, Exercise 1.1.2].

Any algebraic group contains a unique maximal normal connected solvable subgroup called the *radical*, see [Mil17, Chap. 6, §h]. A *reductive algebraic group* is a connected algebraic group whose radical over $\mathbb{C}$ is an algebraic torus, i.e. isomorphic to $(\mathbb{C}^*)^n$ for some $n \geqslant 0$. A reductive algebraic group over the fields of real or complex numbers is a reductive Lie group in the previous sense, hence quadrable [Mil13, II, §4].

Connected linear algebraic groups $G \subset \mathrm{GL}(n, \mathbb{C})$ are reductive if and only if the trace form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $(\xi_1, \xi_2) \mapsto \mathrm{Tr}(\xi_1 \xi_2)$ is nondegenerate. In particular, $\mathrm{SL}(n, \mathbb{C})$ for $n \geqslant 2$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ for $n \geqslant 3$ are reductive algebraic groups.

Finitely generated groups

The second ingredient of a representation variety is a finitely generated group Γ. Finitely generated groups are always equipped with the discrete topology. Our guiding example of finitely generated groups are surface groups.

Definition 2.1.4 (Surface group)**.** Let $g \geqslant 0$ and $n \geqslant 0$ be two integers. A group is called a *surface group* if it can presented as

$$\pi_{g,n} := \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \ : \ \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{n} c_j = 1 \right\rangle, \tag{2.1.3}$$

[4]In the context of this work, an *algebraic variety* is understood to be the zero locus of a set of polynomial equations over $\mathbb{R}$ or $\mathbb{C}$ (in other words, algebraic varieties are always affine). We make no assumption about irreducibility and, in particular, we don't distinguish algebraic varieties and algebraic sets. Morphisms of algebraic varieties are restrictions of polynomial maps and are called *regular maps*.

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ denotes the commutator of a_i and b_i. If $n = 0$, then it is called a *closed surface group*.

The closed surface groups $\pi_{g,0}$ are pairwise non-isomorphic (because their cohomology with real coefficients differs in degree 1), non-free for $g \geqslant 1$ and non-abelian for $g \geqslant 2$. If $n \geqslant 1$, then the surface group $\pi_{g,n}$ is isomorphic to the free group on $2g + n - 1$ generators. The name "surface group" is explained by the following lemma.

Lemma 2.1.5. *Let $\Sigma_{g,n}$ denote a connected orientable topological surface of genus $g \geqslant 0$, with $n \geqslant 0$ punctures. The fundamental group of $\Sigma_{g,n}$ is isomorphic to $\pi_{g,n}$.*

Proof. The proof for the case $n = 0$ is explained in [Lab13, Thm. 2.3.15]. Its generalization to punctured surfaces can be understood in two steps. First, observe that a sphere with $n \geqslant 1$ punctures is homotopy equivalent to the wedge of $n-1$ circles. Hence, its fundamental group is the free group on $n-1$ generators. Similarly, a surface of genus g with one puncture is homotopy equivalent to the wedge of $2g$ circles. Thus, its fundamental group is the free group on $2g$ generators. Now, note that $\Sigma_{g,n}$ is the union of two sub-surfaces $\Sigma_{g,1}$ and $\Sigma_{0,n+1}$. The conclusion now follows from Van Kampen's Theorem. □

The generators c_i in (2.1.3) will play a central role later in Section 2.4.2 in the context of relative representation varieties. They should be thought of as homotopy classes of based loops enclosing the ith puncture of $\Sigma_{g,n}$.

2.1.2. Definition

Definition 2.1.6 (Representation variety)**.** The *representation variety* associated to a finitely generated group Γ and a Lie group G is the set of group homomorphisms from Γ to G and is denoted by

$$\mathrm{Hom}(\Gamma, G).$$

The elements $\phi \in \mathrm{Hom}(\Gamma, G)$ are called *representations*.

The topology on the representation variety $\mathrm{Hom}(\Gamma, G)$ is defined to be the subspace topology induced by the compact-open topology on the space G^Γ of all (necessarily continuous) functions $\Gamma \to G$.

Let $(\gamma_1, \dots, \gamma_n)$ be a set of generators of Γ. We introduce the subspace

$$X(\Gamma, G) := \left\{ \left(\phi(\gamma_1), \dots, \phi(\gamma_n)\right) : \phi \in \mathrm{Hom}(\Gamma, G) \right\} \subset G^n.$$

Lemma 2.1.7. *Let G be a Lie group equipped with an analytic atlas. The set $X(\Gamma, G)$ is an analytic subvariety*[5] *of G^n and is homeomorphic to* $\mathrm{Hom}(\Gamma, G)$*. In particular,* $\mathrm{Hom}(\Gamma, G)$ *has a natural structure of analytic variety and the structure does not depend on the choice of generators of* Γ*.*

Proof. Let $R = \{r_i\}$ denote a (maybe infinite) set of relations for the generators $\gamma_1, \ldots, \gamma_n$. Each relation r_i defines an analytic map $r_i \colon G^n \to G$ because multiplication and inverse are assumed to be analytic operations on G. The map r_i is called a *word map*. The set $X(\Gamma, G)$ is the analytic subset of G^n cut out by the relations $r_i(g_1, \ldots, g_n) = 1$ for every i.

Since a group homomorphism $\phi \colon \Gamma \to G$ is determined by the images of a set of generators of Γ, the map

$$\begin{aligned} \Pi \colon \mathrm{Hom}(\Gamma, G) &\to X(\Gamma, G) \\ \phi &\mapsto \big(\phi(\gamma_1), \ldots, \phi(\gamma_n)\big) \end{aligned}$$

is a bijection. We prove that Π is a homeomorphism. Recall that all the sets

$$V(K, U) := \{f \colon \Gamma \to G : K \subset \Gamma \text{ finite}, \, U \subset G \text{ open}, \, f(K) \subset U\}$$

form a sub-basis for the compact-open topology on $\mathrm{Hom}(\Gamma, G)$. To see that Π is a continuous map, observe that, for a collection of open sets $U_1, \ldots, U_n \subset G$,

$$\Pi^{-1}\left(X(\Gamma, G) \,\cap\, U_1 \times \ldots \times U_n\right) = \mathrm{Hom}(\Gamma, G) \cap \bigcap_{i=1}^{n} V(\{\gamma_i\}, U_i).$$

To prove that the inverse map Π^{-1} is also continuous, note that any element $k \in \Gamma$, seen as a word in the generators $\gamma_1, \ldots, \gamma_n$, determines an analytic function $k \colon G^n \to G$. Now, given a finite set $K \subset \Gamma$ and an open set $U \subset G$, we have

$$\Pi\left(\mathrm{Hom}(\Gamma, G) \,\cap\, V(K, U)\right) = X(\Gamma, G) \cap \bigcap_{k \in K} k^{-1}(U).$$

We conclude that both Π and its inverse are continuous. Hence, Π is a homeomorphism.

If $(\gamma_1', \ldots, \gamma_{n'}')$ is another set of generators of Γ and $X'(\Gamma, G)$ is the associated space, then the map from $X(\Gamma, G)$ to $X'(\Gamma, G)$ defined as the composition

$$X(\Gamma, G) \to \mathrm{Hom}(\Gamma, G) \to X'(\Gamma, G)$$

[5] An *analytic variety* is understood to be the zero locus of a set of analytic functions over $\mathbb{R}$ or $\mathbb{C}$.

is an isomorphism of analytic varieties. Indeed, the map sends $\big(\phi(\gamma_1), \dots, \phi(\gamma_n)\big)$ to $\big(\phi(\gamma_1'), \dots, \phi(\gamma_{n'}')\big)$. Now, since γ_i' is a word in the generators $\gamma_1, \dots, \gamma_n$, it follows that $\phi(\gamma_i')$ is a word in $\phi(\gamma_1), \dots, \phi(\gamma_n)$. This shows that the map is analytic because word maps are analytic by assumption on G. ☐

Lemma 2.1.8. *Assume that G has the structure of a real or complex algebraic group, then $X(\Gamma, G)$ is an algebraic subset of G^n. In particular,* $\mathrm{Hom}(\Gamma, G)$ *has a natural structure of real or complex algebraic variety and the structure does not depend on the choice of generators of* Γ*.*

Proof. The argument is analogous to the proof of Lemma 2.1.7. The key observation is that the relations $R = \{r_i\}$ give regular maps $r_i \colon G^n \to G$ by assumption on G. ☐

Remark 2.1.9 (Finitely generated versus finitely presented)*.* Since we assumed Γ to be finitely generated, and not finitely presented, the set of equations that define $X(\Gamma, G)$ might be infinite. However, Hilbert's basis theorem implies that any algebraic variety over a field can be described as the zero locus of finitely many polynomial equations, see e.g. [SKKT00, §2.2].

Remark 2.1.10 (Standard topology versus Zariski topology)*.* If G is a real or complex algebraic group, then it is also a Lie group, as mentioned earlier. This means that the representation variety $\mathrm{Hom}(\Gamma, G)$ has both the structure of an analytic variety and of an algebraic variety. The underlying topology of the analytic structure is called the *standard topology* and that of the algebraic structure the *Zariski topology*. The standard topology on an algebraic variety is always Hausdorff. The Zariski topology is coarser than the standard topology. Indeed, Zariski open sets are open in the standard topology because polynomials are continuous functions. A nonempty Zarsiki open set is also dense in both the standard and the Zariski topology.

Example 2.1.11 (Surface groups)**.** Representations $\pi_{g,n} \to G$ typically arise as *holonomies* (or *monodromies*) of (G, X)-structures on $\Sigma_{g,n}$, see [Gol21] for further details. Not all the representations $\pi_{g,n} \to G$ are holonomies of (G, X)-structures. However, if $n = 0$, then the set of holonomies is an open subset of $\mathrm{Hom}(\pi_{g,0}, G)$ [Gol21, Cor. 7.2.2]. For instance, if $G = \mathrm{PSL}(2, \mathbb{R})$, then the holonomies of hyperbolic structures on the closed surface $\Sigma_{g,0}$, $g \geqslant 2$, are precisely the discrete and faithful representations in $\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(2, \mathbb{R}))$. They form two connected components of the representation variety.

In the vocabulary of category theory, we can say that representation variety is a bifunctor from the product of the category of finitely generated groups and the category of Lie/algebraic groups to the category of analytic/algebraic varieties. This is a consequence of Lemmata 2.1.7 and 2.1.8, and of the following.

Lemma 2.1.12. *Let Γ be a finitely generated group and G be a Lie/algebraic group.*

1. *If $\tau\colon \Gamma_1 \to \Gamma_2$ is a morphism of finitely generated groups, then the induced map $\tau^*\colon \mathrm{Hom}(\Gamma_2, G) \to \mathrm{Hom}(\Gamma_1, G)$ is an analytic/regular map.*
2. *If $r\colon G_1 \to G_2$ is a morphism of Lie groups or of algebraic groups, then the induced map $r_*\colon \mathrm{Hom}(\Gamma, G_1) \to \mathrm{Hom}(\Gamma, G_2)$ is an analytic map or a regular map, respectively.*

Proof. The second assertion is immediate. To prove the first statement, note that if $(\gamma_1^1, \dots, \gamma_n^1)$ is a set of generators for Γ_1 and $(\gamma_1^2, \dots, \gamma_m^2)$ is a set of generators for Γ_2, then $(\tau^*\phi)(\gamma_i^1) = \phi(\tau(\gamma_i^1))$ is a word in $\phi(\gamma_1^2), \dots, \phi(\gamma_m^2)$. Word maps are analytic, respectively regular, and thus so is τ^*. □

2.1.3. Symmetries

The representation variety $\mathrm{Hom}(\Gamma, G)$ has two natural symmetries given by the right action of the group $\mathrm{Aut}(\Gamma)$ of automorphisms of Γ by pre-composition and the left action of $\mathrm{Aut}(G)$ by post-composition:

$$\mathrm{Aut}(G) \circlearrowright \mathrm{Hom}(\Gamma, G) \circlearrowleft \mathrm{Aut}(\Gamma).$$

An immediate consequence of Lemma 2.1.12 is

Corollary 2.1.13. *The actions of the groups $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(G)$ on $\mathrm{Hom}(\Gamma, G)$ preserve its analytic/algebraic structure.*

There is a normal subgroup of $\mathrm{Aut}(G)$ that is of particular interest: namely, the subgroup of inner automorphisms of G, denoted $\mathrm{Inn}(G)$. Recall that an *inner automorphism* of G is an automorphism given by conjugation by a fixed element of G. In particular, $\mathrm{Inn}(G) \cong G/Z(G)$, where $Z(G)$ denotes the centre of G (which is a closed and normal subgroup of G). The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is relevant in many concrete cases. For instance, the holonomy representations mentioned in Example 2.1.11 are really defined up to conjugation by an element of G and so it makes sense to see them as elements of the quotient

$$\mathrm{Hom}(\Gamma, G)/\,\mathrm{Inn}(G). \tag{2.1.4}$$

The quotient (2.1.4) is the prototype of the notion of character variety introduced below.

The action of $\mathrm{Aut}(\Gamma)$ on the representation variety descends to an action of $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ on the quotient (2.1.4). The group $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ is denoted $\mathrm{Out}(\Gamma)$ and is called the group of *outer automorphisms* of Γ.

Example 2.1.14 (Surface groups)**.** The group of outer automorphisms of the surface group $\pi_{g,n}$ has a particular significance. It contains the (pure) mapping class group of the surface $\Sigma_{g,n}$ as a subgroup. This is known as the Dehn–Nielsen Theorem. We develop this observation further in Section 2.6.2.

2.1.4. Zariski tangent spaces

In this section, we would like to determine the Zariski tangent spaces to representation varieties. We start by recalling the classical notion of Zariski tangent spaces for analytic varieties in $\mathbb{R}^n$.

Definition 2.1.15 (Zariski tangent spaces)**.** Let $X \subset \mathbb{R}^n$ is an analytic variety defined as the zero locus of some analytic functions $f_1, \dots, f_m \colon \mathbb{R}^n \to \mathbb{R}$. The *Zariski tangent space* at $x \in X$ is the kernel of the $m \times n$ Jacobi matrix

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j}. \tag{2.1.5}$$

Equivalently, the Zariski tangent space at x consists of all tangent vectors $x'(0)$ tangent to a smooth path $x(t)$ inside $\mathbb{R}^n$ with $x(0) = x$ and that satisfies the relations $f_i = 0$ *up to first order* by which we mean that $f_i(x(0)) = 0$ and $\frac{d}{dt}\big|_{t=0} f_i(x(t)) = 0$.

To specialize to the case of representation varieties, we need a notion of Zariski tangent spaces for analytic varieties in the infinite product G^Γ. We follow the approach of [Kar92] and refer the reader to that paper for more details. The relevant notion here is that of real valued ringed space.

Definition 2.1.16 (Real valued ringed space)**.** A *real valued ringed space* is a topological space with a sheaf of real valued continuous functions.

Examples of real valued ringed spaces include smooth manifolds together with the sheaf of smooth functions, analytic varieties together with the sheaf of analytic functions or algebraic varieties together with the sheaf of rational maps. There is a notion of *Zariski* *tangent space* for real valued ringed spaces that generalizes the notion of tangent spaces for manifolds and that of Zariski tangent spaces for analytic and algebraic varieties.

On the space G^Γ, one can define a notion of smooth functions. A function $F\colon G^\Gamma \to \mathbb{R}$ is called *locally smooth* if it is locally a smooth function of a finite number of coordinates. The space G^Γ, together with the sheaf of locally smooth real-valued functions on G^Γ, is a real valued ringed space. In the case of G^Γ, the Zariski tangent space at any point can be identified with $\mathfrak{g}^\Gamma$ via left translation.

The representation variety $\mathrm{Hom}(\Gamma, G)$ is the subspace of the space G^Γ cut out by the equations

$$\phi(xy)\phi(y)^{-1}\phi(x)^{-1} = 1, \quad \forall x, y \in \Gamma.$$

As such, it has an induced ringed space structure. Previously, in the context of Lemma 2.1.7, we explained that $\mathrm{Hom}(\Gamma, G)$ inherits its structure from the embedding inside G^n that depends on a choice of generators for Γ. In contrast, the embedding $\mathrm{Hom}(\Gamma, G) \subset G^\Gamma$ does not require to fix a set of generators for Γ. The disadvantage is that G^Γ, unlike G^n, is an infinite product.

Lemma 2.1.17 ([Kar92])**.** *Fix a set of n generators of Γ and let F_n be the free group on n generators. The following diagram is a commutative diagram of real valued ringed spaces:*

$$\begin{tikzcd} & G^n \arrow[dr] & \\ \mathrm{Hom}(\Gamma, G) \arrow[ur] \arrow[dr] & & G^{F_n} \\ & G^\Gamma \arrow[ur] & \end{tikzcd}$$

In particular, the structures induced by G^n and G^Γ on $\mathrm{Hom}(\Gamma, G)$ *coincide.*

We refer the reader to [Kar92] for a proof of Lemma 2.1.17.

Working with the embedding $\mathrm{Hom}(\Gamma, G) \subset G^\Gamma$, we can determine the Zariski tangent space to the representation variety without referring to a presentation of Γ. Let $F_{x,y}\colon G^\Gamma \to G$ be defined by $F_{x,y}(f) := f(xy)f(y)^{-1}f(x)^{-1}$. The Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at ϕ is the intersection of the kernels of the linear forms $D_\phi F_{x,y}\colon \mathfrak{g}^\Gamma \to \mathfrak{g}$ for all $x, y \in \Gamma$ (each tangent space to G is naturally identified to $\mathfrak{g}$ via left translation).

Lemma 2.1.18. *It holds that*

$$D_\phi F_{x,y}(v) = v(xy) - v(x) - \mathrm{Ad}(\phi(x))v(y)$$

for $v \in \mathfrak{g}^\Gamma$ and $\phi \in$ $\mathrm{Hom}(\Gamma, G)$*.*

Proof. By definition, we have that

$$\begin{aligned} D_\phi F_{x,y}(v) &= \frac{d}{dt}\Big|_{t=0} F_{x,y}(\exp(tv)\phi) \\ &= \frac{d}{dt}\Big|_{t=0} \exp(tv(xy))\phi(xy)\phi(y)^{-1}\exp(-tv(y))\phi(x)^{-1}\exp(-tv(x)) \\ &= v(xy) - v(x) - \mathrm{Ad}(\phi(x))v(y). \end{aligned}$$

Here $\exp\colon \mathfrak{g} \to G$ denotes the Lie theoretic exponential map. □

We conclude

Corollary 2.1.19 ([Gol84], [Kar92])**.** *The Zariski tangent space to* $\mathrm{Hom}(\Gamma, G)$ *at* ϕ *is*

$$T_\phi \mathrm{Hom}(\Gamma, G) = \left\{ v \in \mathfrak{g}^\Gamma : v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y), \quad \forall x, y \in \Gamma \right\}.$$

Corollary 2.1.19 can be reformulated in terms of group cohomology[6]. A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ equips $\mathfrak{g}$ with the structure of a Γ-module by

$$\Gamma \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}).$$

The resulting Γ-module is denoted by $\mathfrak{g}_\phi$. The set of 1-cochains in the bar complex that computes the cohomology of Γ with coefficients in $\mathfrak{g}_\phi$ is $\mathfrak{g}^\Gamma$, see Appendix B.2 for more details on the bar complex. The space of 1-cocycles is

$$Z^1(\Gamma, \mathfrak{g}_\phi) := \left\{ v \in \mathfrak{g}^\Gamma : v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y), \quad \forall x, y \in \Gamma \right\}$$

and thus identifies with the Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at ϕ. The space of 1-coboundaries, defined by

$$B^1(\Gamma, \mathfrak{g}_\phi) := \left\{ v \in \mathfrak{g}^\Gamma : \exists \xi \in \mathfrak{g}, \quad v(x) = \xi - \mathrm{Ad}(\phi(x))\xi, \quad \forall x \in \Gamma \right\},$$

also plays a role in this context. They can be identified with the Zarisiki tangent space to the $\mathrm{Inn}(G)$-orbit of $\phi \in \mathrm{Hom}(\Gamma, G)$ at ϕ (recall from Section 2.1.3 that $\mathrm{Inn}(G)$ acts on the representation variety by post-composition). We denote this orbit by

$$\mathcal{O}_\phi \subset \mathrm{Hom}(\Gamma, G).$$

[6] We provide an introduction to group (co)homology, containing all the relevant notions for this work, in Appendix B.

Proposition 2.1.20 ([Gol84], [Kar92]). *The Zariski tangent space to $\mathcal{O}_\phi$ at ϕ is*

$$T_\phi \mathcal{O}_\phi = \left\{ v \in \mathfrak{g}^\Gamma : \exists \xi \in \mathfrak{g}, \quad v(x) = \xi - \operatorname{Ad}(\phi(x))\xi, \quad \forall x \in \Gamma \right\}.$$

Proof. The orbit $\mathcal{O}_\phi$ is a smooth manifold isomorphic to the quotient of G by the stabilizer of ϕ for the conjugation action. The stabilizer of ϕ is the centralizer $Z(\phi) := Z(\phi(\Gamma))$ of $\phi(\Gamma)$ inside G, which is a closed subgroup of G. In particular, the Zariski tangent space to $\mathcal{O}_\phi$ at ϕ coincides with the usual notion of tangent space.

A smooth deformation of ϕ inside $\mathcal{O}_\phi$ is of the form $\phi_t = g(t)\phi g(t)^{-1}$, where $g(t)$ is a smooth 1-parameter family inside G with $g(0) = 1$. The tangent vector to ϕ_t at $t = 0$ is the coboundary $v(x) = \xi - \operatorname{Ad}(\phi(x))\xi$ where $\xi \in \mathfrak{g}$ is the tangent vector to $g(t)$ at $t = 0$. Conversely, for any $\xi \in \mathfrak{g}$, the coboundary $v(x) = \xi - \operatorname{Ad}(\phi(x))\xi$ is tangent to $\exp(t\xi)\phi\exp(-t\xi)$ at $t = 0$. ☐

Observe that $B^1(\Gamma, \mathfrak{g}_\phi)$ can be identified with the quotient $\mathfrak{g}/\mathfrak{z}(\phi)$, where $\mathfrak{z}(\phi)$ is the Lie algebra of $Z(\phi)$. In particular, it holds that

$$\dim B^1(\Gamma, \mathfrak{g}_\phi) = \dim \mathcal{O}_\phi = \dim G - \dim Z(\phi). \tag{2.1.6}$$

We mention that the quotient

$$H^1(\Gamma, \mathfrak{g}_\phi) = Z^1(\Gamma, \mathfrak{g}_\phi)/B^1(\Gamma, \mathfrak{g}_\phi)$$

is known as the first cohomology group of the group Γ with coefficients in the Γ-module $\mathfrak{g}_\phi$ introduced in Definition B.2.

Example 2.1.21 (Surface groups). In the special case of a closed surface group, one can obtain the conclusion of Corollary 2.1.19 from the embedding $\operatorname{Hom}(\pi_{g,0}, G) \subset G^{2g}$. Let $\phi \in \operatorname{Hom}(\pi_{g,0}, G)$ and let $A_i := \phi(a_i)$ and $B_i := \phi(b_i)$, where a_i and b_i are the generators of $\pi_{g,0}$ in the presentation (2.1.3). The Zariski tangent space to $\operatorname{Hom}(\pi_{g,0}, G)$ at ϕ is isomorphic to the kernel of the differential of the map

$$\begin{aligned} F \colon G^{2g} &\to G \\ (X_1, \ldots, X_g, Y_1, \ldots, Y_g) &\mapsto \prod_{i=1}^{g} [X_i, Y_i] \end{aligned}$$

at $(A_1, \dots, A_g, B_1, \dots, B_g)$. A simple computation shows that the kernel of $D_{(A_i,B_i)}F$ corresponds to the subset of $\mathfrak{g}^{2g}$ that consists of all those $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ such that

$$\begin{aligned}
&\big(\alpha_1 + \mathrm{Ad}(A_1)\beta_1\big) - \mathrm{Ad}\left([A_1, B_1]\right)\big(\beta_1 + \mathrm{Ad}(B_1)\alpha_1\big) \\
&+ \mathrm{Ad}\left([A_1, B_1]\right)\big(\alpha_2 + \mathrm{Ad}(A_2)\beta_2\big) - \mathrm{Ad}\left([A_1, B_1][A_2, B_2]\right)\big(\beta_2 + \mathrm{Ad}(B_2)\alpha_2\big) \\
&+ \dots \\
&= \sum_{i=1}^{g} \mathrm{Ad}\left(\prod_{j=1}^{i-1}[A_j, B_j]\right)\big(\alpha_i + \mathrm{Ad}(A_i)\beta_i\big) - \mathrm{Ad}\left(\prod_{j=1}^{i}[A_j, B_j]\right)\big(\beta_i + \mathrm{Ad}(B_i)\alpha_i\big)
\end{aligned} \tag{2.1.7}$$

vanishes, compare [Lab13, Prop. 5.3.12]. Once again, we identified $T_{A_i}G \cong \mathfrak{g}$ and $T_{B_i}G \cong \mathfrak{g}$ via left translation.

To see the correspondence between this description of the Zariski tangent space and that of Corollary 2.1.19, we proceed as follows. First, if one defines $v\colon \pi_{g,0} \to \mathfrak{g}$ by $v(a_i) := \alpha_i$ and $v(b_i) := \beta_i$ for $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ that satisfy (2.1.7), and extend to $\pi_{g,0}$ using $v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y)$, then v defines an element of $Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$. Indeed, it is sufficient to check that $v\big(\prod[a_i, b_i]\big) = 0$. If one develops $v\big(\prod[a_i, b_i]\big)$ using $v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y)$ and $v([x, y]) = v(xy) - \mathrm{Ad}(\phi([x, y]))v(yx)$, then one gets that $v\big(\prod[a_i, b_i]\big) = 0$ is equivalent to (2.1.7) vanishing. Conversely, given $v \in Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$, then $(v(a_1), \dots, v(a_g), v(b_1), \dots, v(b_g))$ satisfies 2.1.7 by the same argument as above.

2.1.5. Smooth points

Smooth points of analytic varieties in $\mathbb{R}^n$ are defined as follows.

Definition 2.1.22 (Smooth points)**.** A point x of an analytic variety $X \subset \mathbb{R}^n$ is a *smooth point* if there is an open neighbourhood $U \subset X$ of x such that U is an embedded submanifold of $\mathbb{R}^n$.

Using the Implicit Function Theorem, we can reformulate the condition and say that x is a smooth point of X if and only if the rank of the Jacobi matrix (2.1.5) at x is maximal. By the Rank-Nullity Theorem, this happens if and only if the dimension of the Zariski tangent space to X at x is minimal. If every point of an analytic variety is smooth, then it is an analytic manifold.

In the context of representation varieties, we will use the characterization of smooth points as the ones that minimize the dimension of the Zariski tangent space. For instance, if Γ is a free group, then $\mathrm{Hom}(\Gamma, G)$ is an analytic manifold because of the absence of relations (recall from Lemma 2.1.7 that representation varieties are analytic varieties).

Lemma 2.1.23. *The set of smooth points of* $\mathrm{Hom}(\Gamma, G)$ *is invariant under the* $\mathrm{Inn}(G)$*-action.*

Proof. The action of G on itself by conjugation is analytic. Therefore, it preserves smooth neighbourhoods of points inside $\mathrm{Hom}(\Gamma, G)$. We can give an alternative argument by observing that the Zariski tangent spaces at ϕ and $g\phi g^{-1}$ are isomorphic as Γ-modules, and hence have the same dimension. The isomorphism is given by

$$\begin{aligned} Z^1(\Gamma, \mathfrak{g}_\phi) &\to Z^1(\Gamma, \mathfrak{g}_{g\phi g^{-1}}) \\ v &\mapsto \mathrm{Ad}(g)v. \end{aligned}$$

□

In the case that $\Gamma = \pi_{g,0}$ is a closed surface group and G is quadrable, it is possible to describe the smooth points of the representation variety explicitly.

Proposition 2.1.24 ([Gol84])**.** *Let G be a quadrable Lie group. The smooth points of* $\mathrm{Hom}(\pi_{g,0}, G)$ *are those representations ϕ satisfying*

$$\dim Z(G) = \dim Z(\phi),$$

where $Z(G)$ denotes the centre of G and $Z(\phi)$ is the centralizer of $\phi(\pi_{g,0})$ inside G (the dimensions are to be understood in terms of manifolds here).

Proof. We compute the dimension of the Zariski tangent space to $\mathrm{Hom}(\pi_{g,0}, G)$ at ϕ. We use the identification with $Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$ provided by Corollary 2.1.19. Recall that the group cohomology of $\pi_{g,0}$ with coefficients in $\mathfrak{g}_\phi$ is isomorphic to the de Rham cohomology of the surface $\Sigma_{g,0}$ with coefficients in the flat vector bundle E_ϕ associated to $\mathfrak{g}_\phi$ (i.e. the adjoint bundle of the principal G-bundle $(\widetilde{\Sigma}_{g,0} \times G)/\pi_{g,0}$ built from ϕ, see [Gol84] for more details):

$$H^*(\pi_{g,0}, \mathfrak{g}_\phi) \cong H^*_{dR}(\Sigma_{g,0}, E_\phi).$$

In particular, it vanishes in degrees larger than 2.

Goldman observed that the quantity

$$\dim H^0(\pi_{g,0}, \mathfrak{g}_\phi) - \dim H^1(\pi_{g,0}, \mathfrak{g}_\phi) + \dim H^2(\pi_{g,0}, \mathfrak{g}_\phi) \tag{2.1.8}$$

is independent of ϕ. Indeed, using that the space of cochains $C^*(\Sigma_{g,0}, E_\phi)$ in the de Rham complex is finite-dimensional in every degree, we conclude that (2.1.8) is equal to the alternating sum of the dimensions of the spaces of cochains in the de Rham complex. The latter is independent of ϕ, because the structure of $\pi_{g,0}$-module of $\mathfrak{g}_\phi$ only intervenes in the differential, see the definition of the bar resolution (B.2). If ϕ is the trivial representation, then $\mathfrak{g}_\phi$ is the trivial $\pi_{g,0}$-module and (2.1.8) is equal to the Euler characteristic of $\Sigma_{g,0}$ times the dimension of G. We conclude

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\phi) = (2g-2)\dim G + \dim H^0(\pi_{g,0}, \mathfrak{g}_\phi) + \dim H^2(\pi_{g,0}, \mathfrak{g}_\phi).$$

Poincaré duality (see Appendix B.7) implies $H^2(\pi_{g,0}, \mathfrak{g}_\phi) \cong H^0(\pi_{g,0}, \mathfrak{g}_\phi^*)^*$. The existence of a non-degenerate, Ad-invariant, symmetric, bilinear form on $\mathfrak{g}$ implies that $\mathfrak{g}_\phi \cong \mathfrak{g}_\phi^*$ as $\pi_{g,0}$-modules. Hence, $\dim H^0(\pi_{g,0}, \mathfrak{g}_\phi) = \dim H^2(\pi_{g,0}, \mathfrak{g}_\phi)$. It is easy to see that $H^0(\pi_{g,0}, \mathfrak{g}_\phi)$ is the space of $\mathrm{Ad}(\phi)$-invariant elements of $\mathfrak{g}$, namely $\mathfrak{z}(\phi)$. Hence

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\phi) = (2g-2)\dim G + 2\dim Z(\phi).$$

Recall from (2.1.6) that the dimension of $B^1(\pi_{g,0}, \mathfrak{g}_\phi)$ is equal to $\dim G - \dim Z(\phi)$. Finally, we obtain

$$\dim Z^1(\pi_{g,0}, \mathfrak{g}_\phi) = (2g-1)\dim G + \dim Z(\phi).$$

Since $Z(G) \subset Z(\phi)$, it holds that $\dim Z(G) \leqslant \dim Z(\phi)$, and we conclude that ϕ minimizes the dimension of its Zariski tangent space if and only if $\dim Z(G) = \dim Z(\phi)$. □

Alternative proof. Instead of using group cohomology (and the embedding of the representation variety in G^Γ), one can alternatively compute the dimension of the Zariski tangent space at a representation ϕ from the embedding $\mathrm{Hom}(\pi_{g,0}, G) \subset G^{2g}$, compare [Lab13, Prop. 5.3.12]. The infinitesimal kernel of the unique relation of a closed surface group is described by (2.1.7), where $A_i = \phi(a_i)$ and $B_i = \phi(b_i)$.

Consider the orthogonal complement V in $\mathfrak{g}$, with respect to the Ad-invariant pairing B coming from the quadrability of G, of the image of the map $\mu\colon \mathfrak{g}^{2g} \to \mathfrak{g}$ defined by (2.1.7). A simple computation leads to

$$\begin{aligned}\mu(\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g) = &\sum_{i=1}^{g}\left(\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\right)(\alpha_i - \mathrm{Ad}(A_iB_iA_i^{-1})\alpha_i)\\ &- \sum_{i=1}^{g}\left(\prod_{j\leqslant i}\mathrm{Ad}\left([A_j,B_j]\right)\right)(\beta_i - \mathrm{Ad}(B_iA_iB_i^{-1})\beta_i).\end{aligned}$$

The orthogonal complement of the Lie algebra of the centralizer $Z(g)$ of any element $g \in G$ is equal to the image of the map $\mathfrak{g} \to \mathfrak{g}$ given by $\xi \mapsto \xi - \mathrm{Ad}(g)\xi$. Therefore, using the general fact that $Z(ghg^{-1}) = gZ(h)g^{-1}$ for any $g, h \in G$, we obtain that V must contain the Lie algebra of

$$\begin{aligned}
&\bigcap_{i=1}^{g}\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\big(Z(A_iB_iA_i^{-1})\cap Z(A_iB_iA_iB_i^{-1}A_i^{-1})\big)\\
&=\bigcap_{i=1}^{g}\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\mathrm{Ad}(A_iB_i)\big(Z(B_i)\cap Z(A_i)\big)\\
&=\bigcap_{i=1}^{g}\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\big(Z(B_i)\cap Z(A_i)\big)\\
&=\bigcap_{i=1}^{g}\big(Z(A_i)\cap Z(B_i)\big).
\end{aligned}$$

Hence, $\mathfrak{z}(\phi)\subset V$. The reverse inclusion is obvious. Using the Rank-Nullity Theorem, we conclude, as before, that the dimension of the Zariski tangent space at the representation ϕ is

$$\dim Z^1(\pi_{g,0},\mathfrak{g}_\phi)=\dim\mathrm{Ker}(\mu)=(2g-1)\dim G+\dim Z(\phi).\qquad\square$$

Proposition 2.1.24 applies to closed surface groups. In Proposition 2.4.9 below, we will discuss an analogous description of smooth points for fundamental groups of punctured surfaces.

2.2. The action by conjugation

In this section, we elaborate on the action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma,G)$ by post-composition. We sometimes refer to this action as the *the conjugation action of G on the representation variety.*

2.2.1. Freeness

The action of $\mathrm{Inn}(G)\cong G/Z(G)$ on $\mathrm{Hom}(\Gamma,G)$ is never free, since the trivial representation is always a global fixed point. It is easy to see that the stabilizer of a representation $\phi\in\mathrm{Hom}(\Gamma,G)$ is $Z(\phi)/Z(G)$. In particular

Lemma 2.2.1. *The $\mathrm{Inn}(G)$-action is free on the $\mathrm{Inn}(G)$-invariant subset that consists of all the representations ϕ such that*

$$Z(G)=Z(\phi).$$

There is a neat characterization of the points where the action is locally free. Recall that the action of a topological group on a set X is *locally free at* $x \in X$ if the stabilizer of x is discrete.

Proposition 2.2.2 ([Gol84])**.** *The action of* $\mathrm{Inn}(G)$ *on* $\mathrm{Hom}(\Gamma, G)$ *is locally free at* ϕ *if and only if*

$$\dim Z(G) = \dim Z(\phi).$$

Proof. The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ induces, for any representation ϕ, a surjective linear map $\mathfrak{Inn}(G) \to T_\phi \mathcal{O}_\phi$, where $\mathfrak{Inn}(G)$ denotes the Lie algebra of $\mathrm{Inn}(G)$ and $\mathcal{O}_\phi$ the $\mathrm{Inn}(G)$-orbit of ϕ. The map is given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)(\phi).$$

Observe that the action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is locally free at ϕ if and only if the induced map $\mathfrak{Inn}(G) \to T_\phi \mathcal{O}_\phi$ is injective. Since the map is always surjective, this is equivalent to asking that both spaces $\mathfrak{Inn}(G)$ and $T_\phi \mathcal{O}_\phi$ have the same dimension. The dimension of $\mathfrak{Inn}(G)$ is $\dim G - \dim Z(G)$ and the dimension of $T_\phi \mathcal{O}_\phi$ is $\dim G - \dim Z(\phi)$, as computed in (2.1.6). Hence, the dimensions coincide if and only if $\dim Z(G) = \dim Z(\phi)$. □

Example 2.2.3 (Surface groups)**.** It is striking that the condition of Proposition 2.2.2 coincides with that of Proposition 2.1.24. This means that if $\Gamma = \pi_{g,0}$ is a closed surface group, then the smooth points of $\mathrm{Hom}(\pi_{g,0}, G)$ are precisely those where the action of $\mathrm{Inn}(G)$ is locally free.

Proposition 2.2.2 motivates the following definition.

Definition 2.2.4 (Regular representations)**.** A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is called *regular* if

$$\dim Z(G) = \dim Z(\phi).$$

We denote by $\mathrm{Hom}^{\mathrm{reg}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of regular representations. If it further holds that $Z(G) = Z(\phi)$, we say that ϕ is *very regular*. The $\mathrm{Inn}(G)$-invariant subspace of very regular representations is denoted by $\mathrm{Hom}^{\mathrm{vReg}}(\Gamma, G)$.

We will see later that if G is a reductive algebraic group, then most representations are regular, see Proposition 2.2.14.

Example 2.2.5. In the case $G = \mathrm{PSL}(2,\mathbb{R})$, the representations $\phi\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ that are not regular are of a particular kind. We use the description of centralizers in $\mathrm{PSL}(2,\mathbb{R})$ provided by Lemma A.9. It tells us that a non-regular representation is of one of the following kinds:

1. ϕ is the trivial representation.
2. The elements of $\phi(\Gamma)$ are rotations around the same point of $\mathbb{H}$ and $Z(\phi) \cong \mathrm{PSO}(2,\mathbb{R})$.
3. The elements of $\phi(\Gamma)$ fix a common geodesic in $\mathbb{H}$ and $Z(\phi) \cong \mathbb{R}_{>0}$.
4. The elements of $\phi(\Gamma)$ fix the same point in the boundary of $\mathbb{H}$ and $Z(\phi) \cong \mathbb{R}$.

As soon as the image of $\phi(\Gamma)$ contains, for instance, two elements of different nature (elliptic, hyperbolic or parabolic) or two rotations around different points, then $Z(\phi) = Z(\mathrm{PSL}(2,\mathbb{R}))$ is trivial and ϕ is regular, actually very regular.

2.2.2. Properness

The conjugation action of G on $\mathrm{Hom}(\Gamma, G)$ is in general not proper.

Example 2.2.6. Consider the case where $\Gamma = F_2 = \langle a, b\rangle$ is the free group on two generators and $G = \mathrm{PSL}(2,\mathbb{R})$. Let $\phi_1\colon F_2 \to \mathrm{PSL}(2,\mathbb{R})$ be the representation given by $\phi_1(a) = \mathrm{par}^+$ (see (A.6)) and $\phi_1(b)$ is the identity. Let ϕ_2 denote the trivial representation. Since the closure of the conjugacy class of any parabolic element of $\mathrm{PSL}(2,\mathbb{R})$ contains the identity, we observe that

$$\phi_2 \in \overline{\mathcal{O}_{\phi_1}} \smallsetminus \mathcal{O}_{\phi_1} \quad \text{and} \quad \{\phi_2\} = \mathcal{O}_{\phi_2}.$$

So, the orbits $\mathcal{O}_{\phi_1}$ and $\mathcal{O}_{\phi_2}$ cannot be separated by disjoint open sets in the (topological) quotient $\mathrm{Hom}(F_2, \mathrm{PSL}(2,\mathbb{R}))/\,\mathrm{Inn}(\mathrm{PSL}(2,\mathbb{R}))$. In particular, the quotient is not Hausdorff and the conjugacy action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathrm{Hom}(F_2, \mathrm{PSL}(2,\mathbb{R}))$ is not proper.

Example 2.2.6 hints at the pathological behaviour of representations whose image lies in a parabolic subgroup. This is essentially a worst case scenario, as we explain below.

Definition 2.2.7 (Borel and parabolic subgroups)**.** Let G be an algebraic group. A *Borel subgroup* of G is a maximal, Zariski closed, solvable connected subgroup of G. A Zariski closed subgroup of G that contains a Borel subgroup is called a *parabolic subgroup* of G.

By definition, a Borel subgroup of G is automatically a Borel subgroup of G°. Similarly, P is a parabolic subgroup of G if and only if P° is a parabolic subgroup of G°. If G is connected, then all parabolic subgroups are connected [Mil17, Cor. 17.49].

Example 2.2.8. Let $G = \mathrm{GL}(n, \mathbb{C})$. The subgroup of upper triangular matrices is a Borel subgroup of G. More generally, the Borel subgroups of $\mathrm{GL}(n, \mathbb{C})$ are the ones that preserve a full flag in $\mathbb{C}^n$ and the parabolic subgroups are those that preserve a (partial) flag in $\mathbb{C}^n$ [Bou05, Chap. VIII, §13].

Definition 2.2.9 (Irreducible representations)**.** Let G be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *irreducible* if the image of ϕ does not lie in a proper parabolic subgroup of G. We denote by $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of irreducible representations.

Observe that if $G = \mathrm{GL}(n, \mathbb{C})$, then ϕ being irreducible in the sense of Definiton 2.2.9 is equivalent to $\mathbb{C}^n$ being an irreducible Γ-module (i.e. ϕ is an irreducible representation in the classical sense). This is a consequence of Example 2.2.8.

Example 2.2.10. Let $G = \mathrm{SL}(2, \mathbb{C})$. The irreducible representations into $\mathrm{SL}(2, \mathbb{C})$ can be characterized in terms of traces:

Lemma 2.2.11. *A representation $\phi\colon \Gamma \to G$ is irreducible if and only there exists an element $\gamma \in [\Gamma, \Gamma] \subset \Gamma$ of the commutator subgroup of Γ such that $\mathrm{Tr}(\phi(\gamma)) \neq 2$.*

A proof of Lemma 2.2.11 can be found in [CS83, Lem. 1.2.1]. The argument relies on the following observation: if $A, B \in \mathrm{SL}(2, \mathbb{C})$ are two upper-triangular matrices, then their commutator $[A, B]$ is upper-triangular and has trace 2 (i.e. upper-triangular with ones on the diagonal).

Definition 2.2.12 (Irreducible subgroups)**.** A subgroup of an algebraic group G is called *irreducible* if it is not contained in a proper parabolic subgroup of G.

In particular, a representation $\phi\colon \Gamma \to G$ is irreducible if and only if its image is an irreducible subgroup of G. The centralizer of an irreducible subgroup in a reductive group G is a finite extension of $Z(G)$ [Sik12, Prop. 15] (see also [Sik12, Cor. 17]). Hence

Lemma 2.2.13. *Let G be a reductive algebraic group. Irreducible representations into G are regular:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{reg}}(\Gamma, G).$$

It is important to note the following

Proposition 2.2.14. *Let G be a reductive algebraic group. The subspace of irreducible representations* $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ *is Zariski open in the representation variety* $\mathrm{Hom}(\Gamma, G)$*. Moreover, if* $\Gamma = \pi_{g,n}$ *is a surface group, then* $\mathrm{Hom}^{\mathrm{irr}}(\pi_{g,n}, G)$ *is dense in a nonempty set of irreducible components of* $\mathrm{Hom}(\pi_{g,n}, G)$*.*

We refer the reader to [Sik12, Prop. 27 & 29] for a proof. The main result of this section says that if one restricts to irreducible representations, then the conjugation action of G becomes proper.

Theorem 2.2.15 ([JM87])**.** *Let G be a reductive algebraic group. The* $\mathrm{Inn}(G)$*-action on* $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ *is proper.*

We refer the reader to [JM87, Prop. 1.1] and references therein for a proof of Theorem 2.2.15. Following [JM87], we introduce the notion of good representations.

Definition 2.2.16 (Good representations)**.** Let G be an algebraic group. A representation $\phi \colon \Gamma \to G$ is called *good*[7] if it is irreducible and very regular. We denote by $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of good representations.

Lemma 2.2.1 implies that the $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ is free and by Theorem 2.2.15 it is also proper. It is, however, not clear a priori whether good representations exist. However, one can prove the following

Lemma 2.2.17 ([JM87])**.** *Let G be a reductive algebraic group. The set of good representations* $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ *is Zariski open in the representation variety* $\mathrm{Hom}(\Gamma, G)$*.*

Lemma 2.2.17 is proven in [JM87, Prop 1.3 & Lem. 1.3]. In general, $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ might not be a smooth manifold. However, it is the case for closed surface groups by Proposition 2.1.24. We conclude from Theorem 2.2.15 and Lemma 2.2.1 that

Corollary 2.2.18. *Let G be a reductive algebraic group. Let* $\Gamma = \pi_{g,0}$ *be a closed surface group. The space of good representations* $\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)$ *is an analytic manifold of dimension* $(2g-1)\dim G + \dim Z(G)$*. The* $\mathrm{Inn}(G)$*-action on* $\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)$ *is proper and free, and the quotient*

$$\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)/\,\mathrm{Inn}(G)$$

is an analytic manifold of dimension $(2g-2)\dim G + 2\dim Z(G)$*.*

[7] In [JM87] and [Sik12] a good representation is defined to be a very regular reductive representation (see Definition 2.2.21). If G is reductive, then their definition is equivalent to ours (see Lemma 2.2.23).

Note that the dimension of the quotient in Corollary 2.2.18 is always even. This observation will be relevant later in Section 2.4 when we discuss the symplectic nature of character varieties.

The notion of irreducible representations can be generalized to the notion of reductive representations.

Definition 2.2.19 (Linearly reductive groups)**.** An algebraic group is called *linearly reductive* if all its finite-dimensional representations are completely reducible.

Equivalently, over the fields of real or complex numbers, an algebraic group G is linearly reductive if and only if the algebraic subgroup that consists of the identity component for the Zariski topology is reductive [Mil17, Cor. 22.43].

Definition 2.2.20 (Completely reducible subgroups)**.** A subgroup of an algebraic group is called *completely reducible* if and only if its Zariski closure is linearly reductive.

Definition 2.2.21 (Reductive representations)**.** Let G be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *reductive* (or *completely reducible*) if $\phi(\Gamma) \subset G$ is completely reducible. We denote by $\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of reductive representations.

In particular, a representation $\phi\colon \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is reductive if and only if $\mathbb{C}^n$ is a completely reducible Γ-module (i.e. a direct sum of irreducible Γ-modules).

Lemma 2.2.22. *Let G be a reductive algebraic group. Irreducible representations $\phi\colon \Gamma \to G$ are reductive:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{red}}(\Gamma, G).$$

Proof. The proof relies on the observation that irreducible subgroups of reductive algebraic groups are completely reducible. This is proved in [Sik12, §3] using the notion of Levi subgroups. □

The converse of Lemma 2.2.22 is not true in general. However

Lemma 2.2.23. *Let G be a reductive algebraic group. A reductive representation into G is irreducible if and only if it is regular:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G) \cap \mathrm{Hom}^{\mathrm{reg}}(\Gamma, G).$$

The reader is referred to [Sik12, Cor. 17] for a proof of Lemma 2.2.23. Reductive representations can be characterized as follows:

Proposition 2.2.24. *Let G be a reductive algebraic group. A representation $\phi\colon \Gamma \to G$ is reductive if and only if the the $\operatorname{Inn}(G)$-orbit $\mathcal{O}_\phi$ of ϕ is closed in $\operatorname{Hom}(\Gamma, G)$.*

A proof of Proposition 2.2.24 can be found in [Sik12, Thm. 30], based on an argument of [JM87]. An immediate consequence of Proposition 2.2.24 is that the points of the topological quotient $\operatorname{Hom}^{\mathrm{red}}(\Gamma, G)/\operatorname{Inn}(G)$ are closed, i.e. it is a $\mathcal{T}_1$ space[8].

Proposition 2.2.25 ([RS90])**.** *Let G be a reductive algebraic group. The topological quotient*

$$\operatorname{Hom}^{\mathrm{red}}(\Gamma, G)/\operatorname{Inn}(G)$$

is Hausdorff.

The reader is referred to [RS90, §7.3] and references therein for a proof of Proposition 2.2.25. Some authors favour the notion of Zariski dense representations over irreducible representations, see for instance [Lab13], [Mon16].

Definition 2.2.26 (Zariski dense representations)**.** Let G be an algebraic Lie group. A representation $\phi \in \operatorname{Hom}(\Gamma, G)$ is called *Zariski dense* if $\phi(\Gamma)$ is a Zariski dense subgroup of G. It is called *almost Zariski dense* if the Zariski closure of $\phi(\Gamma)$ contains G°. The $\operatorname{Inn}(G)$-invariant spaces of Zariski dense and almost Zariski dense representations are denoted $\operatorname{Hom}^{\mathrm{Zd}}(\Gamma, G)$ and $\operatorname{Hom}^{\mathrm{aZd}}(\Gamma, G)$, respectively.

Recall that a subgroup H of an algebraic groups G is Zariski dense if and only if any regular function that vanishes on H also vanishes on G.

Lemma 2.2.27. *Let G be an algebraic Lie group. Almost Zariski dense representations are irreducible:*

$$\operatorname{Hom}^{\mathrm{aZd}}(\Gamma, G) \subset \operatorname{Hom}^{\mathrm{irr}}(\Gamma, G).$$

Proof. Let $\phi\colon \Gamma \to G$ be almost Zariski dense. By definition, the Zariski closure of $\phi(\Gamma)$ contains G°. In particular, no proper parabolic subgroups of G° can contain the identity component of the Zariski closure of $\phi(\Gamma)$. Since parabolic subgroups are by definition Zariski closed, no proper parabolic subgroup of G can contain $\phi(\Gamma)$. $\square$

[8] See Section 2.3 for a reminder of some notions of separability.

Example 2.2.28. Let $\alpha_1, \ldots, \alpha_n \in (0, 2\pi)^n$ be angles such that $\alpha_1 + \ldots + \alpha_n = 2k\pi$ for some integer k. Let $F_n = \langle a_1, \ldots, a_n \rangle$ denote the free group on n generators. We consider the representation $\phi\colon F_n \to \mathrm{PSL}(2, \mathbb{R})$ defined by $\phi(a_i) = \mathrm{rot}_{\alpha_i}$ (see (A.2)). The representation ϕ is not Zariski dense because its image lies inside $\mathrm{PSO}(2, \mathbb{R})$ which is Zariski closed in $\mathrm{PSL}(2, \mathbb{R})$. However, ϕ is irreducible as one can check that $\phi(\Gamma)$ has no fixed point in $\mathbb{RP}^1 = \mathbb{R}^2/\mathbb{R}^\times$. Consider now the representation $\overline{\phi}$ defined as the composition of ϕ with the inclusion $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{C})$. Observe that $\overline{\phi}\colon F_n \to \mathrm{PSL}(2, \mathbb{C})$ is reducible since it fixes $[1 : i] \in \mathbb{CP}^1 = \mathbb{C}^2/\mathbb{C}^\times$, but it is still not Zariski dense because its image lies inside $\mathrm{PSO}(2, \mathbb{C})$ which is Zariski closed in $\mathrm{PSL}(2, \mathbb{C})$.

Lemma 2.2.29. *Let G be an algebraic group such that $Z(G) = Z(G^\circ)$. If $\phi \in \mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G)$, then ϕ is very regular, i.e.*

$$Z(G) = Z(\phi).$$

In particular, almost Zariski dense representations are good:

$$\mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{good}}(\Gamma, G).$$

Proof. The argument is taken from [Lab13, §5.3]. Denote by $Z(Z(\phi))$ the centralizer of $Z(\phi) = Z(\phi(\Gamma))$ in G. It is a Zariski closed subgroup of G that contains $\phi(\Gamma)$. Hence, by almost Zariski density of $\phi(\Gamma)$, it holds $G^\circ \subset Z(Z(\phi))$ and thus $Z(\phi) \subset Z(G^\circ)$. Since we assumed $Z(G^\circ) = Z(G)$, we conclude that $Z(G) = Z(\phi)$. It now follows from 2.2.27 that almost Zarsiki dense representations are good. □

It follows from Theorem 2.2.15 and Lemma 2.2.27 that, for a reductive algebraic group G (hence connected) and $\Gamma = \pi_{g,0}$ a closed surface group, the $\mathrm{Inn}(G)$-action on the subspace of Zariski dense representations is free and proper, compare [Lab13, Thm. 5.2.6] and [Mon16, Lem. 2.10]. It is interesting to note that the resulting quotient, at least in the case when $Z(G)$ is finite, has the same dimension as the quotient from Corollary 2.2.18.

By way of conclusion, we provide the reader with a Venn diagram that illustrates the different relations of inclusion between the various notions of representations introduced in this section, see Figure 2.1.

2.2.3. Invariant functions

The real- or complex-valued functions of $\mathrm{Hom}(\Gamma, G)$ that are invariant under the conjugation action of G are called *invariant functions* of the representation variety. We consider the case where G is an algebraic group over $\mathbb{C}$. The algebra of regular functions on the

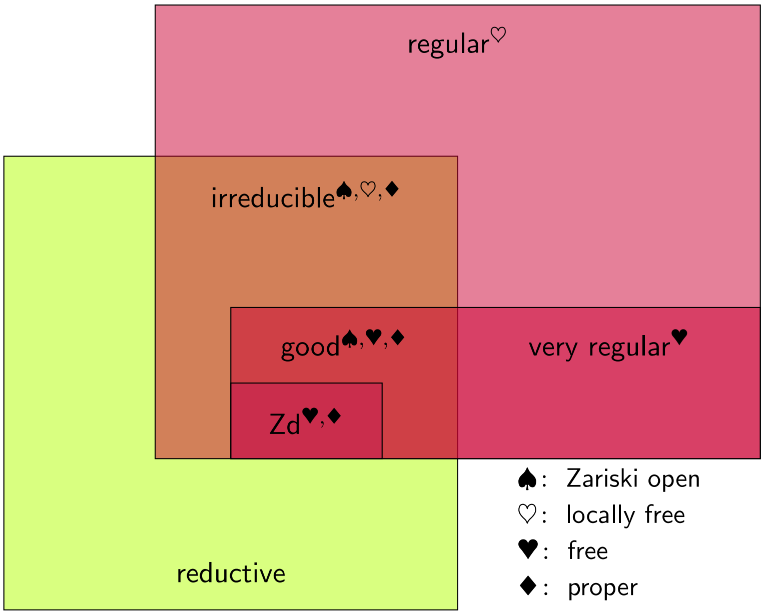


Figure 2.1.: We assume for simplicity that G is a reductive algebraic group (hence connected). The two largest families of representations are the regular and the reductive ones. Their intersection is the set of irreducible representations. A representation that is irreducible and very regular is called good. Zariski dense representations are good.

variety $\mathrm{Hom}(\Gamma, G)$, a.k.a. its coordinate ring, is denoted $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ and the subalgebra of invariant functions is denoted by

$$\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G.$$

In this section, we will only consider the case of a linear algebraic group $G \subset \mathrm{GL}(m, \mathbb{C})$. The main example of invariant functions are the so-called trace functions (recall that $\mathrm{Tr}\colon \mathrm{GL}(m, \mathbb{C}) \to \mathbb{C}$ is a conjugacy invariant).

Definition 2.2.30 (Trace functions)**.** Let $\gamma \in \Gamma$. The function

$$\begin{aligned} \mathrm{Tr}_\gamma \colon \mathrm{Hom}(\Gamma, G) &\to \mathbb{C} \\ \phi &\mapsto \mathrm{Tr}(\phi(\gamma)). \end{aligned}$$

is called the *trace function of* γ. We denote by $\mathcal{T}(\Gamma, G)$ the subalgebra of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ generated by trace functions.

In most cases, as for instance when G is one of the classical complex Lie groups, invariant functions of the representation variety are generated by trace functions. In other words,

$\mathcal{T}(\Gamma, G) = \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. This is a consequence of Procesi's Theorem (see Theorem 2.2.32 below) on invariants of matrices.

Remark 2.2.31. Nagata's Theorem implies that, if G is a reductive algebraic group, then $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated, see for instance [Dol03, Thm. 3.3].

Let $\mathbb{K}$ denote either the field of real or complex numbers. We denote by $M_m(\mathbb{K})$ the algebra of $m \times m$ matrices with coefficients in $\mathbb{K}$. Let $M_m(\mathbb{K})^n = M_m(\mathbb{K}) \times \ldots \times M_m(\mathbb{K})$ and $\mathbb{K}[M_m(\mathbb{K})^n]$ be the algebra of polynomial functions in n matrix variables $\xi_k = (x^k_{i,j})_{i,j=1,\ldots,m}$. The group $\mathrm{GL}(m, \mathbb{K})$ acts diagonally on $M_m(\mathbb{K})^n$ by conjugation. For any subgroup $G \subset \mathrm{GL}(m, \mathbb{K})$, the subalgebra of $\mathbb{K}[M_m(\mathbb{K})^n]$ that consists of G-invariant polynomials is denoted $\mathbb{K}[M_m(\mathbb{K})^n]^G$.

Theorem 2.2.32 ([Pro76])**.** *The following hold:*

- *If* $G \in \{\mathrm{GL}(m, \mathbb{K}), \mathrm{SL}(m, \mathbb{K})\}$*, then* $\mathbb{K}[M_m(\mathbb{K})^n]^G$ *is finitely generated by trace polynomials* $\mathrm{Tr}(W)$*, where* W *is a reduced word in* $\xi_1, \ldots, \xi_n$ *of length at most* $2^m - 1$*.*
- *If* $G \in \{\mathrm{O}(m, \mathbb{K}), \mathrm{SO}(m, \mathbb{K})\}$*, then* $\mathbb{K}[M_m(\mathbb{K})^n]^G$ *is finitely generated by trace polynomials* $\mathrm{Tr}(W)$*, where* W *is a reduced word of length at most* $2^m - 1$ *in* $\xi_1, \ldots, \xi_n$ *and their orthogonal transposes*[9]*.*
- *If* $G = \mathrm{Sp}(2m, \mathbb{K})$*, then* $\mathbb{K}[M_{2m}(\mathbb{K})^n]^G$ *is finitely generated by trace polynomials* $\mathrm{Tr}(W)$*, where* W *is a reduced word of length at most* $2^m - 1$ *in* $\xi_1, \ldots, \xi_n$ *and their symplectic transposes*[10]*.*

The reader is referred to [Pro76] for the proof of Theorem 2.2.32, see also [DCP17].

Back to the context of representation varieties: Assume that Γ admits a generating family $(\gamma_1, \ldots, \gamma_n)$, then the embedding $\imath\colon \mathrm{Hom}(\Gamma, G) \subset G^n$ induces a surjective morphism

$$\imath^*\colon \mathbb{C}[G^n] \twoheadrightarrow \mathbb{C}[\mathrm{Hom}(\Gamma, G)]. \tag{2.2.1}$$

The morphism $\imath^*$ maps invariant functions to invariant functions and thus restricts to a morphism

$$(\imath^*)^G\colon \mathbb{C}[G^n]^G \to \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G. \tag{2.2.2}$$

[9]The *orthogonal transpose* of a matrix is the inverse of its transpose. The orthogonal group $\mathrm{O}(m, \mathbb{K})$ consists precisely of the matrices that are equal to their orthogonal transposes.

[10]The *symplectic transpose* of a matrix $A \in M_{2m}(\mathbb{K})$ is the matrix JA^tJ, where $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ and I_m is the $m \times m$ identity matrix. The symplectic group $\mathrm{Sp}(2m, \mathbb{K})$ consists precisely of the matrices that are equal to their symplectic transposes.

If we further assume G to be reductive, then $(\imath^*)^G$ is surjective. This is a consequence of the existence of Reynolds operators, see [Sik13, Rem. 25] or [Hos15, Cor. 4.23]. The morphism $(\imath^*)^G$ maps trace functions to trace functions in the following sense.

Lemma 2.2.33. *Let W be a reduced word in the matrices variables $\xi_1, \ldots, \xi_n$. It holds that*

$$(\imath^*)^G(\mathrm{Tr}(W)) = \mathrm{Tr}_{W(\gamma_1,\ldots,\gamma_n)}\,.$$

Proof. The word W induces a word map $W\colon G^n \to G$. The trace function $\mathrm{Tr}(W)\colon G^n \to \mathbb{C}$ sends $(g_1, \ldots, g_n)$ to $\mathrm{Tr}(W(g_1, \ldots, g_n))$. The image $(\imath^*)^G(\mathrm{Tr}(W))$ is the invariant function $\mathrm{Hom}(\Gamma, G) \to \mathbb{C}$ given by $\phi \mapsto \mathrm{Tr}(W(\phi(\gamma_1), \ldots, \phi(\gamma_n)))$. Because ϕ is a group homomorphism, it holds that $\mathrm{Tr}(W(\phi(\gamma_1), \ldots, \phi(\gamma_n))) = \mathrm{Tr}(\phi(W(\gamma_1, \ldots, \gamma_n))$, where we now think of W as a function $W\colon \Gamma^n \to \Gamma$. We conclude that $(\imath^*)^G(\mathrm{Tr}(W)) = \mathrm{Tr}_{W(\gamma_1,\ldots,\gamma_n)}$. □

Lemma 2.2.34. *Let $G \subset \mathrm{GL}(m, \mathbb{C})$ be a reductive linear algebraic group. If the algebra $\mathbb{C}[G^n]^G$ is generated by trace functions, then*

$$\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G = \mathcal{T}(\Gamma, G).$$

Proof. If G is reductive, then $(\imath^*)^G$ is surjective and so $(\imath^*)^G(\mathbb{C}[G^n]^G) = \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. Moreover, $(\imath^*)^G$ maps trace functions to trace functions, thus, if $\mathbb{C}[G^n]^G$ is generated by trace functions, then it holds $(\imath^*)^G(\mathbb{C}[G^n]^G) = \mathcal{T}(\Gamma, G)$. □

Lemma 2.2.35. *Let G be one of the reductive groups* $\mathrm{GL}(m, \mathbb{C})$ *or* $\mathrm{SL}(m, \mathbb{C})$ *with $m \geqslant 2$,* $\mathrm{O}(m, \mathbb{C})^\circ$ *or* $\mathrm{SO}(m, \mathbb{C})$ *with $m \geqslant 3$, or* $\mathrm{Sp}(2m, \mathbb{C})$*. Then $\mathbb{C}[G^n]^G$ is generated by trace functions.*

Proof. The inclusion $G \subset M_m(\mathbb{C})$ induces a surjective morphism $\mathbb{C}[M_m(\mathbb{C})^n]^G \twoheadrightarrow \mathbb{C}[G^n]^G$. Theorem 2.2.32 says that $\mathbb{C}[M_m(\mathbb{C})^n]^G$ is generated by trace of words of matrices and their transposes. In particular, a similar argument as in the proof of Lemma 2.2.34 implies that $\mathbb{C}[G^n]^G$ is generated by traces of words. We used here that the inverse transpose and the symplectic transpose of any matrix in $\mathrm{O}(m, \mathbb{C})$ and $\mathrm{Sp}(2m, \mathbb{C})$, respectively, is the matrix itself. □

We conclude

Corollary 2.2.36. *Let G be one of the reductive groups* $\mathrm{GL}(m,\mathbb{C})$ *or* $\mathrm{SL}(m,\mathbb{C})$ *with* $m \geqslant 2$*,* $\mathrm{O}(m,\mathbb{C})^\circ$ *or* $\mathrm{SO}(m,\mathbb{C})$ *with* $m \geqslant 3$*, or* $\mathrm{Sp}(2m,\mathbb{C})$*. Then*

$$\mathbb{C}[\mathrm{Hom}(\Gamma,G)]^G = \mathcal{T}(\Gamma,G).$$

Example 2.2.37. Let $G = \mathrm{SL}(2,\mathbb{C})$. Corollary 2.2.36 says that the algebra of invariant functions $\mathbb{C}[\mathrm{Hom}(\Gamma,\mathrm{SL}(2,\mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})}$ is generated by Tr_γ for $\gamma \in \Gamma$. The trace formula $\mathrm{Tr}(A)\,\mathrm{Tr}(B) = \mathrm{Tr}(AB) + \mathrm{Tr}(AB^{-1})$ for 2×2 matrices gives the relation

$$\mathrm{Tr}_{\gamma_1}\mathrm{Tr}_{\gamma_2} = \mathrm{Tr}_{\gamma_1\gamma_2} + \mathrm{Tr}_{\gamma_1\gamma_2^{-1}}.$$

It is folklore knowledge (see [MS21, §1]) that the trace formula, together with the relation $\mathrm{Tr}_1 = 2$, is a complete set of relations. In other words, there is an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\mathrm{Hom}(\Gamma,\mathrm{SL}(2,\mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})} \cong \mathbb{C}[X_\gamma : \gamma \in \Gamma] \Big/ \left(X_1 - 2, X_{\gamma_1}X_{\gamma_2} - X_{\gamma_1\gamma_2} - X_{\gamma_1\gamma_2^{-1}}\right).$$

2.2.4. Characters

A character is the analogue of a trace function where a representation is now fixed and $\gamma \in \Gamma$ is the variable. We assume again that $G \subset \mathrm{GL}(m,\mathbb{C})$ is a linear algebraic group.

Definition 2.2.38 (Characters)**.** The *character* of a representation $\phi \in \mathrm{Hom}(\Gamma,G)$ is the function

$$\begin{aligned}\chi_\phi \colon \Gamma &\to \mathbb{C}\\ \gamma &\mapsto \mathrm{Tr}(\phi(\gamma)).\end{aligned}$$

In other words, $\chi_\phi(\gamma) = \mathrm{Tr}_\gamma(\phi)$. We denote by $\chi(\Gamma,G) \subset \mathbb{C}^\Gamma$ the set of all characters of representations in $\mathrm{Hom}(\Gamma,G)$ equipped with the subspace topology inherited from the compact-open topology on $\mathbb{C}^\Gamma$.

Note that $\chi(\Gamma,G) \subset \mathbb{C}^\Gamma$ is automatically a Hausdorff space because $\mathbb{C}^\Gamma$ is a Hausdorff space.

Theorem 2.2.39 ([CS83])**.** *The space* $\chi(\Gamma,G) \subset \mathbb{C}^\Gamma$ *is a closed algebraic variety for* $G = \mathrm{SL}(2,\mathbb{C})$*.*

We refer the reader to [CS83, Cor. 1.4.5] for a proof of Theorem 2.2.39. The natural projection

$$\mathrm{Hom}(\Gamma, G) \to \chi(\Gamma, G)$$

factors through the quotient $\mathrm{Hom}(\Gamma, G)/\operatorname{Inn}(G)$. A character does not necessarily determine a unique conjugacy class of representations. For instance, the two representations of Example 2.2.6 are not conjugate but determine the same character. However, the following is true.

Proposition 2.2.40. *Let $G \subset \mathrm{GL}(m, \mathbb{C})$ be a linear algebraic group. Conjugacy classes of irreducible representations are determined by their characters.*

Culler–Shalen provide a proof of Proposition 2.2.40 in [CS83, Prop. 1.5.2] for the case $G = \mathrm{SL}(2, \mathbb{C})$ and claim that the result still holds when $\mathrm{SL}(2, \mathbb{C})$ is replaced by $\mathrm{GL}(m, \mathbb{C})$. The analogous result for almost Zariski dense representations can be found in [Lab13, Cor. 5.3.7].

2.3. Character varieties

The previous sections highlighted the relevance of the quotient space $\mathrm{Hom}(\Gamma, G)/\operatorname{Inn}(G)$. However, it was also explained that there is no reason to expect that this quotient has any nice structure, since the action of G by conjugation on the representation variety is non-free and non-proper in general. The goal of this section is to construct an alternative space, with a nicer structure than the topological quotient $\mathrm{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ and with a projection from $\mathrm{Hom}(\Gamma, G)$ that factors through $\mathrm{Hom}(\Gamma, G)/\operatorname{Inn}(G)$. The specification is to construct the largest possible space, while ensuring some regularity such as being Hausdorff or being a variety or manifold. The resulting space will be called a *character variety* of the finitely generated group Γ and the Lie group G. Several constructions explained below lead to richer structures but require more assumptions on the Lie group G.

We start by recalling the definitions of two separability properties. A topological space X is said to be

- $\mathcal{T}_1$ if for any pair of distinct points in X, each point lies in an open set that does not contain the other, or, equivalently, X is $\mathcal{T}_1$ if the points of X are closed,
- $\mathcal{T}_2$ or *Hausdorff* if for any pair of distinct points in X, there are two disjoint open sets such that each contains one of the two points.

Note that the quotient $\mathrm{Hom}(F_2, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{Inn}(\mathrm{PSL}(2,\mathbb{R}))$ of Example 2.2.6 is not only non-Hausdorff, but is also not $\mathcal{T}_1$. Indeed, the closure of the orbit of ϕ_1 always contains the orbit of ϕ_2.

2.3.1. Hausdorff quotient

The first approach consists in considering the Hausdorffization the topological quotient. The Hausdorffization of a topological space X is basically the largest Hausdorff quotient of X.

Definition 2.3.1 (Hausdorffization)**.** Consider the equivalence relation on X given by $x \sim y$ if and only if $x \approx y$ for all equivalence relations $\approx$ on X such that $X/\approx$ is Hausdorff (such a relation $\approx$ always exists, as one can identify all the points of X). The quotient

$$\mathrm{Haus}(X) := X/\sim$$

is the *Hausdorffization of* X.

Lemma 2.3.2. *The space* $\mathrm{Haus}(X)$ *is a Hausdorff topological space. Moreover, the space* $\mathrm{Haus}(X)$ *has the following universal property: If* Y *is a Hausdorff topological space, then any continuous surjective map* $X \to Y$ *factors uniquely through the projection* $X \to \mathrm{Haus}(X)$*.*

Proof. First we prove that $\mathrm{Haus}(X)$ is a Hausdorff space. Let $x, y \in X$ be two points with $x \nsim y$. By definition, there exists an equivalence relation $\approx$ on X with Hausdorff quotient such that $x \not\approx y$. Since the projections of x and y in $X/\approx$ are separable and the map $X/\sim \to X/\approx$ is continuous, the projections of x and y are also separable in $X/\sim$.

Let now Y be a Hausdorff space and $f\colon X \to Y$ be a continuous surjection. Define an equivalence relation on X by $x \approx y$ if and only if $f(x) = f(y)$. The quotient $X/\approx$ is homeomorphic to the Hausdorff space Y. This implies the existence of a continuous surjective map $\mathrm{Haus}(X) \to Y$ such that f is the composition $X \to \mathrm{Haus}(X) \to Y$. The factoring map is uniquely determined by f. $\square$

Corollary 2.3.3. *If* x *and* y *are two points of* X *such that* $\overline{\{x\}} \cap \overline{\{y\}} \neq \varnothing$*, then* $x \sim y$*.*

Proof. Since $\mathrm{Haus}(X)$ is Hausdorff, its points are closed. In particular, the conjugacy classes for the relation $\sim$ are closed subsets of X. If we assume that $x \nsim y$, then the conjugacy classes of x and y are disjoint closed subsets of X. This implies that the closures of $\{x\}$ and $\{y\}$ are disjoint. $\square$

Definition 2.3.4 (Hausdorff character variety)**.** The *Hausdorff character variety* of a finitely generated group Γ and a Lie group G is the Hausdorffization of the topological quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ and is denoted

$$\mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G) := \mathrm{Haus}\left(\mathrm{Hom}(\Gamma, G) \Big/ \mathrm{Inn}(G)\right).$$

The construction of character varieties by Hausdorff quotients has the advantage to work in a broad sense (it could even be defined for topological groups G). It is the approach favoured in [Mon16], for instance.

2.3.2. $\mathcal{T}_1$ quotient

An alternative to the Hausdorff quotient is the $\mathcal{T}_1$ quotient used in [RS90, §7]. Let G be a topological group acting on a space X. For any $x \in X$, we denote the G-orbit of x by $\mathcal{O}_x$. We make the following crucial assumption:

$$\forall x \in X, \quad \overline{\mathcal{O}_x} \subset X \text{ contains a unique closed } G\text{-orbit.} \tag{2.3.1}$$

Let $X /\!/ G$ denote the set of closed orbits for the action of G on X and define

$$\pi \colon X \to X /\!/ G$$

to be the map that sends x to the unique closed orbit contained in $\overline{\mathcal{O}_x}$. A topology on $X /\!/ G$ is defined by declaring π to be a quotient map, i.e $Z \subset X /\!/ G$ is closed if and only if $\pi^{-1}(Z) \subset X$ is closed. Define a relation on X by

$$x \approx y \quad \Leftrightarrow \quad \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \varnothing.$$

Lemma 2.3.5. *Under the assumption* (2.3.1)*, the relation $\approx$ is an equivalence relation and $X /\!/ G$ is homeomorphic to the quotient $X/\approx$.*

Proof. The relation $\approx$ is obviously symmetric and reflexive. We prove that it is also transitive. Assume that $x \approx y$ and $y \approx z$. In particular, $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ is nonempty and thus contains an element w. Since $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ is closed and G-invariant, it holds $\overline{O_w} \subset \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$. We conclude that $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ contains a unique closed orbit which is the one contained in $\overline{O_w}$. Similarly, $\overline{\mathcal{O}_y} \cap \overline{\mathcal{O}_z}$ contains a unique closed orbit. By uniqueness of the closed orbit contained in $\overline{\mathcal{O}_y}$, the two must coincide. Hence, $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \cap \overline{\mathcal{O}_z}$ contains $\overline{\mathcal{O}_w}$ and is therefore nonempty. This shows that $x \approx z$.

To see that $X /\!\!/ G \cong X/\!\approx$, observe that, by the above argument, $\pi(x) = \pi(y)$ if and only if $x \approx y$. Both are quotients of X and therefore homeomorphic. $\square$

Lemma 2.3.6. *The space $X /\!\!/ G$ has the following universal property: For every $\mathcal{T}_1$ space Y, any continuous map $X \to Y$ that is constant on G-orbits factors uniquely through $\pi\colon X \to X /\!\!/ G$.*

Proof. Let Y be $\mathcal{T}_1$ with a continuous map $f\colon X \to Y$ that is constant on G-orbits. Let $x \in X$. We want to prove that f is constant on $\overline{\mathcal{O}_x}$. Let $y = f(x)$. Since Y is $\mathcal{T}_1$, the singleton $\{y\} \subset Y$ is closed and so is $f^{-1}(y)$. Therefore, $\overline{\mathcal{O}_x} \subset f^{-1}(y)$ and f is constant on $\overline{\mathcal{O}_x}$. This shows that $f\colon X \to Y$ factors through $X /\!\!/ G$. The factoring map $\overline{f}\colon X /\!\!/ G \to Y$ is continuous and uniquely determined by f. $\square$

In the case that $X /\!\!/ G$ is a $\mathcal{T}_1$ space says that $X /\!\!/ G$ is the largest $\mathcal{T}_1$ quotient of X. There is a relation between $X /\!\!/ G$ and the Hausdorffization of the topological quotient X/G. Namely

Lemma 2.3.7. *There is a natural surjective continuous map*

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, two heads, "\pi"] & X/G \arrow[d, two heads] \\ X /\!\!/ G \arrow[r, dashed, two heads, "\exists"] & \mathrm{Haus}(X/G) \end{tikzcd}$$

Proof. Let x and y be two points of X. Lemma 2.3.5 says that if $\pi(x) = \pi(y)$, then $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \varnothing$. This means the closures of $\mathcal{O}_x$ and $\mathcal{O}_y$, seen as singletons in X/G, have a nonempty intersection. By Corollary 2.3.3, we conclude that x and y project to the same point in $\mathrm{Haus}(X/G)$. $\square$

Corollary 2.3.8. *If $X /\!\!/ G$ is Hausdorff, then it is homeomorphic to the Hausdorffization of X/G.*

Definition 2.3.9 ($\mathcal{T}_1$ character variety)**.** If the conjugation action of G on the representation variety $\mathrm{Hom}(\Gamma, G)$ satisfies property (2.3.1), we define the $\mathcal{T}_1$ *character variety* of Γ and G to be

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) := \mathrm{Hom}(\Gamma, G) /\!\!/ \mathrm{Inn}(G).$$

Note that the $\mathcal{T}_1$ character variety of Γ and G might not be a $\mathcal{T}_1$ space, but always lies over any $\mathcal{T}_1$ quotient of $\mathrm{Hom}(\Gamma, G)$ by Lemma 2.3.6. In particular, by Lemma 2.3.7, there is a surjection

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \twoheadrightarrow \mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G)$$

which is a homeomorphism when $\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G)$ is Hausdorff.

2.3.3. GIT quotient

In this section, we sketch a construction of character variety in the case that G is a complex reductive algebraic group. It is based on geometric invariant theory (GIT). The reader may consult [Sik12], [Dre04, §2] or [Lou15, §B.5] for more details.

If G is a complex algebraic group then the representation variety $\mathrm{Hom}(\Gamma, G)$ is an algebraic variety by Lemma 2.1.8. Recall that the algebra of regular functions of $\mathrm{Hom}(\Gamma, G)$ is denoted $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ and the subalgebra of G-invariant functions is denoted $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. Nagata's theorem implies that $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated, see Remark 2.2.31. In particular, there is an algebraic variety denoted $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ whose algebra of polynomial functions is $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. This variety is also known as the *GIT quotient* of $\mathrm{Hom}(\Gamma, G)$.

Definition 2.3.10 (GIT character variety)**.** The *GIT character variety* of a finitely generated group Γ and a complex reductive algebraic group G is defined to be

$$\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G) := \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G).$$

The GIT character variety has by definition the structure of an algebraic variety and is, in particular, a Hausdorff topological space with the standard topology. The inclusion $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ induces a surjective morphism of algebraic varieties

$$p\colon\ \mathrm{Hom}(\Gamma, G) \twoheadrightarrow \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G).$$

We recall here some general properties of GIT quotients and refer the reader to [Dre04, §2] and [Lou15, §B.5], and references therein for proofs.

Lemma 2.3.11. *The GIT quotient* $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ *has the following universal property: for every algebraic variety Y, any morphism* $\mathrm{Hom}(\Gamma, G) \to Y$ *that is constant on G-orbits factors uniquely through* $p\colon\ \mathrm{Hom}(\Gamma, G) \to \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$.

Lemma 2.3.12. *The GIT quotient* $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ *satisfies the following properties:*

1. *For two representations $\phi_1, \phi_2 \in \operatorname{Hom}(\Gamma, G)$, it holds that*

$$p(\phi_1) = p(\phi_2) \quad \Leftrightarrow \quad \overline{\mathcal{O}_{\phi_1}} \cap \overline{\mathcal{O}_{\phi_2}} \neq \varnothing.$$

2. *Any fibre of p contains a unique closed orbit (compare (2.3.1)).*

Lemma 2.3.12, combined with Lemma 2.3.5, implies that the underlying topological structure of the GIT character variety of Γ and G coincides with the $\mathcal{T}_1$ character variety. Since the GIT character variety is a Hausdorff space, it further coincides with the Hausdorff character variety by Corollary 2.3.8:

$$\operatorname{Rep}^{\mathrm{GIT}}(\Gamma, G) \cong \operatorname{Rep}^{\mathcal{T}_1}(\Gamma, G) \cong \operatorname{Rep}^{\mathcal{T}_2}(\Gamma, G).$$

2.3.4. Analytic quotient

If one is interested in constructing a character variety that is an analytic manifold, one can restrict to good representations defined in Definition 2.2.16. If $\operatorname{Hom}^{\mathrm{good}}(\Gamma, G)$ is a nonempty analytic manifold (recall from Corollary 2.2.18 that it is the case if $\Gamma = \pi_{g,0}$ is a closed surface group and G is a reductive algebraic group), then the quotient $\operatorname{Hom}^{\mathrm{good}}(\Gamma, G)/\operatorname{Inn}(G)$ is an analytic manifold.

Definition 2.3.13 (Analytic character variety)**.** The *analytic character variety* of a closed surface group $\Gamma = \pi_{g,0}$ and a reductive algebraic group G is defined to be

$$\operatorname{Rep}^{\infty}(\pi_{g,0}, G) := \operatorname{Hom}^{\mathrm{good}}(\pi_{g,0}, G)/G.$$

The topology of an analytic character variety is a Hausdorff. Hence, by Lemma 2.3.2, there is a projection from the Hausdorff character variety (which does not need to be a homeomorphism)

$$\operatorname{Rep}^{\mathcal{T}_2}(\pi_{g,0}, G) \twoheadrightarrow \operatorname{Rep}^{\infty}(\pi_{g,0}, G).$$

2.3.5. Variant of the GIT and analytic quotients

The GIT character variety can be described more concretely as follows.

Definition 2.3.14 (Stability of representations)**.** Let G be an algebraic group. A representation $\phi \colon \Gamma \to G$ is

- *polystable* if $\mathcal{O}_\phi$ is closed.
- *stable* if ϕ is polystable and regular.

The $\mathrm{Inn}(G)$-invariant subspace of polystable representations is denoted $\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)$ and the subspace of stable representations is denoted $\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$.

These notions are redundant if G is a reductive complex algebraic group because of the following.

Proposition 2.3.15. *Let G be a reductive complex algebraic group. Let $\phi \in \mathrm{Hom}(\Gamma, G)$ be a representation. Then*

1. *ϕ is reductive if and only if ϕ is polystable,*
2. *ϕ is irreducible if and only if ϕ is stable.*

The first assertion of Proposition 2.3.15 was already stated in Proposition 2.2.24. The second assertion is a consequence of Lemma 2.2.23.

Theorem 2.3.16. *Let G be a reductive complex algebraic group. The topological quotient*

$$\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)/\,\mathrm{Inn}(G) = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\,\mathrm{Inn}(G)$$

is homeomorphic to $\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G)$. It contains, as an open subset, the topological quotient

$$\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)/\,\mathrm{Inn}(G) = \mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)/\,\mathrm{Inn}(G)$$

which is an orbifold whenever $Z(G)$ is finite.

Proof. Polystable representations have a closed orbit under the $\mathrm{Inn}(G)$-action by definition. So, the first statement of Lemma 2.3.12 implies that the projection $p\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ factors through an injective map

$$\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)/\,\mathrm{Inn}(G) \to \mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G).$$

We can use the second statement of Lemma 2.3.12 to see that this map is also surjective.

Recall now from Proposition 2.2.14 that $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) = \mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$. To prove the orbifold statement, we use that an algebraic variety over the real or the complex numbers has a finite number of connected components in the usual topology, see e.g. [DK81, Thm. 4.1]. So, if $Z(G)$ is finite, then a polystable representation $\phi\colon \Gamma \to G$ is stable if and only if $Z(\phi)$ is finite. Equivalently, ϕ is stable if and only if it has a finite stabilizer for the $\mathrm{Inn}(G)$-action. This shows that the quotient is an orbifold since the $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$ is proper by Theorem 2.2.15. □

Theorem 2.3.16 says that there is a natural structure of algebraic variety on the quotient of the space of reductive representations by the $\mathrm{Inn}(G)$-action, given that G is a reductive complex algebraic group. In the case that G is a real algebraic group, we have the following

Theorem 2.3.17 ([RS90]). *Let G be a real algebraic group. The quotient*

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\,\mathrm{Inn}(G)$$

is a real semialgebraic[11] variety.

Theorem 2.3.17 is proved in [RS90, Thm. 7.6].

2.4. Symplectic structure of character varieties

Throughout this section we assume that G is a quadrable Lie group. We also fix a non-degenerate, symmetric, Ad-invariant bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Goldman described in [Gol84] a natural symplectic structure on the character variety of representations of a closed surface group into a quadrable group. We remind the reader of the construction.

Assume for now that Γ is any finitely generated group. We explained in Corollary 2.1.19 that the Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at a representation ϕ can be identified with $Z^1(\Gamma, \mathfrak{g}_\phi) \subset \mathfrak{g}^\Gamma$. To define a 2-form on the representation variety $\mathrm{Hom}(\Gamma, G)$ we use the cup product in group cohomology (B.11). Combined with the pairing B, this gives a map

$$\omega\colon Z^1(\Gamma, \mathfrak{g}_\phi) \times Z^1(\Gamma, \mathfrak{g}_\phi) \xrightarrow{\smile} Z^2(\Gamma, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi) \xrightarrow{B_*} Z^2(\Gamma, \mathbb{R}). \tag{2.4.1}$$

The map ω is bilinear and anti-symmetric because the cup product is anti-symmetric in degree 1 (Lemma B.11) and B is symmetric.

Theorem 2.4.1 ([Kar92]). *Let $\varphi\colon Z^2(\Gamma, \mathbb{R}) \to \mathbb{R}$ be any continuous linear function that vanishes on $B^2(\Gamma, \mathbb{R})$. Then, $\varphi \circ \omega$ is a closed 2-form on* $\mathrm{Hom}(\Gamma, G)$.

[11] A *semialgebraic variety* is defined to be a set of points satisfying polynomial equalities and inequalities.

The main conclusion of Theorem 2.4.1 is the statement that the form $\varphi \circ \omega$ is closed. Karshon gives an elementary proof of the closeness via direct computations in group cohomology.

The cup product of coboundaries in $B^1(\Gamma, \mathfrak{g}_\phi)$ is itself a coboundary inside $B^2(\Gamma, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi)$. This shows that the 2-form $\varphi \circ \omega$ is degenerate. Recall from Proposition 2.1.20 that the tangent space at ϕ to the G-orbit $\mathcal{O}_\phi \subset \operatorname{Hom}(\Gamma, G)$ can be identified with the 1-coboundaries $B^1(\Gamma, \mathfrak{g}_\phi) \subset \mathfrak{g}^\Gamma$. So, $\varphi \circ \omega$ is degenerate at least along the tangent directions to the G-orbit of ϕ. In general, the kernel of $\varphi \circ \omega$ might contain more degenerate directions than those which arise from $\mathcal{O}_\phi$.

Definition 2.4.2 (Goldman symplectic form)**.** In the case that the G-orbits are the only directions of degeneracy of $\varphi \circ \omega$, we denote by $\omega_{\mathcal{G}}$ the induced nondegenrate closed form on cohomology:

$$(\omega_{\mathcal{G}})_\phi \colon H^1(\Gamma, \mathfrak{g}_\phi) \times H^1(\Gamma, \mathfrak{g}_\phi) \to \mathbb{R}.$$

We say that $\omega_{\mathcal{G}}$ is the *the Goldman symplectic form* on $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$.

The index $\mathcal{G}$ refers to Goldman. We are abusing the terminology "symplectic form" here. The topological quotient $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ does not need to be a variety in general and it is abusive to say that the "Zariski tangent space" at $[\phi] \in \operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ is the quotient space $H^1(\Gamma, \mathfrak{g}_\phi) = Z^1(\Gamma, \mathfrak{g}_\phi)/B^1(\Gamma, \mathfrak{g}_\phi)$. What $\omega_{\mathcal{G}}$ really is, is a 2-form on $\operatorname{Hom}(\Gamma, G)$ that is degenerate precisely along the orbits of the $\operatorname{Inn}(G)$-action.

2.4.1. Closed surface groups

Let $\Gamma = \pi_{g,0}$ be a closed surface group. Let $[\pi_{g,0}]$ be a generator of $H_2(\pi_{g,0}, \mathbb{Z}) \cong \mathbb{Z}$ (where $\mathbb{Z}$ is the trivial $\pi_{g,0}$-module). In other words, $[\pi_{g,0}]$ corresponds to an orientation of the surface $\Sigma_{g,0}$ under the isomorphism $H_2(\pi_{g,0}, \mathbb{Z}) \cong H_2(\Sigma_{g,0}, \mathbb{Z})$ of Theorem B.8. Integration against $[\pi_{g,0}]$ gives an isomorphism

$$[\pi_{g,0}] \frown : H^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}.$$

Let $\varphi \colon Z^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}$ be given by the composition of the quotient map $Z^2(\pi_{g,0}, \mathbb{R}) \to H^2(\pi_{g,0}, \mathbb{R})$ and the integration against $[\pi_{g,0}]$. Clearly, φ vanishes on $B^2(\pi_{g,0}, \mathbb{R})$.

Lemma 2.4.3. *Let $\Gamma = \pi_{g,0}$ be a closed surface group. The composition of $\varphi \colon Z^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}$ with the form ω of (2.4.1) defines a 2-form on $\operatorname{Hom}(\pi_{g,0}, G)$ whose kernel is $B^1(\pi_{g,0}, \mathbb{R})$.*

Proof. The proof relies on Poincaré duality in group cohomology for the group $\pi_{g,0}$. It implies that the cup product

$$H^1(\pi_{g,0}, \mathbb{R}) \times H^1(\pi_{g,0}, \mathbb{R}) \xrightarrow{\smile} H^2(\pi_{g,0}, \mathbb{R})$$

is a nondegenerate pairing. This means that the form $\varphi \circ \omega$ is degenerate on $B^1(\pi_{g,0}, \mathbb{R})$ only. □

The induced nondegenerate closed form $(\omega_{\mathcal{G}})_\phi \colon H^1(\pi_{g,0}, \mathfrak{g}_\phi) \times H^1(\pi_{g,0}, \mathfrak{g}_\phi) \to \mathbb{R}$ is the celebrated Goldman symplectic form for character varieties of closed surface groups representations. The original argument of Goldman in [Gol84] to prove that the $\omega_{\mathcal{G}}$ is closed is inspired by the treatment of the case when G is compact in [AB83]. The proof involves an infinite dimensional symplectic reduction from the affine space of connections on some vector bundle, see [Gol84] and [Lab13, §6] for more details.

Remark 2.4.4. The Goldman symplectic form depends on the pairing B on the Lie algebra of G. Different choices of pairing for the same Lie group G may lead to different symplectic structures. Abusing once again of the term “symplectic manifold”, one can say that Goldman’s construction is a functor form the product category of the category of closed connected oriented surfaces $\Sigma_{g,0}$ with the category of quadrable Lie groups G with a choice of a form pairing B to the category of “symplectic manifold”

$$\big(\Sigma_{g,0}, (G, B)\big) \rightsquigarrow \big(\operatorname{Hom}(\pi_1(\Sigma_{g,0}), G)/\operatorname{Inn}(G), \omega_{\mathcal{G}}\big).$$

We point out that the quotients $\operatorname{Hom}(\pi_1(\Sigma_{g,0}), G)/\operatorname{Inn}(G)$ obtained for different choices of basepoints in $\Sigma_{g,0}$ are naturally isomorphic (the isomorphism does *not* depend on the choice of path connecting different basepoints).

2.4.2. General surface groups

Let $\Gamma = \pi_{g,n}$ be a surface group. We will assume in this section that $n > 0$. As mentioned earlier, in that case $\pi_{g,n}$ is a free group and the representation variety $\operatorname{Hom}(\pi_{g,n}, G)$ is isomorphic to the product G^{2g+n-1}. It can be written as the disjoint union of so-called relative representation varieties.

Definition 2.4.5 (Relative representation variety)**.** Let $\mathcal{C} = (C_1, \ldots, C_n)$ be an ordered collection of n conjugacy classes in G. The *relative representation variety* associated to $(\pi_{g,n}, G, \mathcal{C})$ is the subspace of $\operatorname{Hom}(\pi_{g,n}, G)$ given by

$$\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G) := \{\phi \in \operatorname{Hom}(\pi_{g,n}, G) : \phi(c_i) \in C_i, \ \forall i = 1, \ldots, n\},$$

where $c_1, \ldots, c_n$ refer to the generators of $\pi_{g,n}$ in the presentation (2.1.3).

If G/G denotes the set of conjugacy classes in G, then

$$\operatorname{Hom}(\pi_{g,n}, G) = \bigsqcup_{\mathcal{C} \in (G/G)^n} \operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G).$$

Relative character varieties are really associated to the particular presentation of $\pi_{g,n}$ that we fixed in (2.1.3). The conjugation action of G on $\operatorname{Hom}(\pi_{g,n}, G)$ restricts to $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$.

Lemma 2.4.6. *Let G be a Lie group equipped with an analytic atlas. The relative representation variety $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is naturally an analytic subvariety of G^{2g+n}. If G is a complex algebraic group, then $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is an algebraic subvariety of $\operatorname{Hom}(\pi_{g,n}, G)$. If G is a real algebraic group, then $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is a semialgebraic subvariety of $\operatorname{Hom}(\pi_{g,n}, G)$.*

Proof. The proof is analogous to the proof of Lemma 2.1.7. A conjugacy class $C \in G/G$ is a smooth submanifold of G isomorphic to $G/Z(c)$, where c is any element of C (recall that $Z(c)$ is a closed subgroup of G). It has a unique structure of real analytic manifold that makes the projection map $G \to G/Z(c)$ an analytic submersion. The relative representation variety $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is naturally identified with the subspace of $G^{2g} \times C_1 \times \ldots \times C_n$ cut out by the single relation of the surface group $\pi_{g,n}$ (see (2.1.3)). This shows that $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is an analytic subvariety of G^{2g+n}. Observe now that, if G is a complex algebraic group, then conjugacy classes in G are algebraic subvarieties of G. This can be seen as a consequence of Chevalley's Theorem. Moreover, if G is a real algebraic group, then conjugacy classes in G are semialgebraic subvarieties of G[12]. This, in turn, is a consequence of Tarski–Seidenberg Theorem. □

We would like to determine the Zariski tangent space to relative character varieties. We follow the approach of [GHJW97, §4]. Let $\phi \in \operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$. The Zariski tangent space to $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ at ϕ is the space of all tangent vectors in $Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ tangent to a smooth deformation ϕ_t of ϕ inside $\operatorname{Hom}(\pi_{g,n}, G)$ that satisfies $\phi_t(c_i) \in C_i$ up to first order. Observe that the condition $\phi_t(c_i) \in C_i$ is equivalent to the existence of a smooth 1-parameter family $g_i(t) \in G$, with $g_i(0) = 1$, and

$$\phi_t(c_i) = g_i(t)\phi(c_i)g_i(t)^{-1}. \tag{2.4.2}$$

[12] An example of conjugacy classes that are a semialgebraic subvarieties, but not algebraic subvarieties, are parabolic conjugacy classes inside $\mathrm{SL}(2, \mathbb{R})$.

Lemma 2.4.7. *A vector $v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ tangent to ϕ_t at $t = 0$ satisfies (2.4.2) up to first order if and only if*

$$v(c_i) = \dot{g}_i - \mathrm{Ad}(\phi(c_i))\dot{g}_i,$$

where $\dot{g}_i \in \mathfrak{g}$ is the tangent vector to $g_i(t)$ at $t = 0$.

Proof. We use $\frac{d}{dt}\big|_{t=0} \phi_t(c_i)\phi(c_i)^{-1} = v(c_i)$ and derive the relation (2.4.2). □

Corollary 2.4.8 ([GHJW97])**.** *The Zariski tangent space to $\mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ at ϕ is*

$$T_\phi \mathrm{Hom}_{\mathcal{C}}(\Gamma, G) = \left\{ v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi) \ : \ \forall i = 1, \ldots, n, \ \exists \xi_i \in \mathfrak{g}, \ v(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i \right\}.$$

The cocycles $v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ that satisfy the property stated in the conclusion of Corollary 2.4.8 are called *parabolic 1-cocycles*, see Appendix B.8. The subspace of parabolic cocycles is denoted

$$Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \subset Z^1(\pi_{g,n}, \mathfrak{g}_\phi).$$

The tangent space to the G-orbit $\mathcal{O}_\phi$ of $\phi \in \mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ still identifies with $B^1(\pi_{g,n}, \mathfrak{g}_\phi)$. The quotient of parabolic 1-cocycles by 1-coboundaries is the first parabolic group cohomology group of $\pi_{g,n}$ with coefficients in the $\pi_{g,n}$-module $\mathfrak{g}_\phi$:

$$H^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) = Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi)/B^1(\pi_{g,n}, \mathfrak{g}_\phi).$$

Proposition 2.4.9. *Let G be a quadrable Lie group. The dimension of the Zariski tangent space to $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ at ϕ is*

$$(2g - 1) \dim G + \sum_{i=1}^{n} \dim C_i + \dim Z(\phi).$$

In particular, the smooth points of $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ are the representations ϕ such that

$$\dim Z(G) = \dim Z(\phi).$$

Proof. We proceed as in the alternative proof of Proposition 2.1.24. Let $A_i = \phi(a_i)$, $B_i = \phi(b_i)$ and $R_i = \phi(c_i)$, where a_i, b_i, c_i refer to the presentation (2.1.3). Consider the map $\mu\colon \mathfrak{g}^{2g+n} \to \mathfrak{g}$ obtained by differentiating the unique surface group relation:

$$\begin{aligned}\mu(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n) = &\sum_{i=1}^{g} \left(\prod_{j<i} \mathrm{Ad}\left([A_j, B_j]\right) \right) (\alpha_i - \mathrm{Ad}(A_i B_i A_i^{-1})\alpha_i) \\ &- \sum_{i=1}^{g} \left(\prod_{j\leqslant i} \mathrm{Ad}\left([A_j, B_j]\right) \right) (\beta_i - \mathrm{Ad}(B_i A_i B_i^{-1})\beta_i) \\ &+ \prod_{k=1}^{g} \mathrm{Ad}\left([A_k, B_k]\right) \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} \mathrm{Ad}\left(R_j\right) \right) (\gamma_i - \mathrm{Ad}(R_i)\gamma_i).\end{aligned}$$

Let V be the orthogonal complement of the image of μ with respect to the pairing B. Similarly as in the alternative proof of Proposition 2.1.24, we conclude that $V = \mathfrak{z}(\phi)$. The Rank-Nullity Theorem gives

$$\dim T_\phi \operatorname{Hom}_{\mathcal{C}}(\Gamma, G) = \dim \operatorname{Ker}(\mu) = (2g-1)\dim G + \sum_{i=1}^{n} \dim C_i + \dim Z(\phi). \qquad \square$$

Remark 2.4.10. We make a little digression on the dimension of conjugacy orbits inside Lie groups. Recall that any conjugacy class $\mathcal{C} \in G/G$ is a smooth submanifold of G diffeomorphic to the quotient $G/Z(g)$. If G is quadrable, the pairing B on $\mathfrak{g}$ can be used to identify coadjoint orbits in $\mathfrak{g}^*$ to adjoint orbits in $\mathfrak{g}$. Coadjoint orbits are naturally symplectic, see e.g. [CdS01, Homework 17]. The exponential map maps the adjoint orbit of $\xi \in \mathfrak{g}$ to the conjugacy orbit of $\exp(\xi)$ in G. Recall however that the Lie theoretic exponential map needs not be a local diffeomorphism at ξ. If it were, it would imply that the conjugacy orbit of $\exp(\xi)$ in G has even dimension. M. Riestenberg pointed out to the author a class of examples of Lie groups that contain conjugacy classes of odd dimension. They consist of the group of all isometries of an odd-dimensional symmetric space X. In that case, the conjugacy class of the orientation-reversing isometry s_p that reflects through a point p is the set of all the orientation-reversing isometries s_q for $q \in X$ and is therefore isomorphic to X.

Question 2.4.11. When does a conjugacy orbit in a quadrable Lie group G have even dimension? Is it necessarily the case if it lies in the image of the exponential map?

We close the digression and go back to relative representation varieties. We would like to obtain an analogue of the Goldman symplectic form for general surface groups. We denote by $\partial_i \pi_{g,n}$ the subgroup of $\pi_{g,n}$ generated by c_i. We write $\partial\pi_{g,n}$ for the collection of subgroups $\{\partial_i \pi_{g,n}\}$. Observe that the cup product in group cohomology restricts to the product (B.15) in parabolic group cohomology. It gives an anti-symmetric bilinear form

$$\omega\colon Z^1_{par}(\pi_{g,n},\mathfrak{g}_\phi)\times Z^1_{par}(\pi_{g,n},\mathfrak{g}_\phi)\xrightarrow{\smile} Z^2(\pi_{g,n},\partial\pi_{g,n},\mathfrak{g}_\phi\otimes\mathfrak{g}_\phi)\xrightarrow{B_*} Z^2(\pi_{g,n},\partial\pi_{g,n},\mathbb{R}).$$

Let $[\pi_{g,n}]$ be a generator of $H_2(\pi_{g,n},\partial\pi_{g,n},\mathbb{Z})\cong\mathbb{Z}$, that corresponds to a choice of orientation for the surface $\Sigma_{g,n}$. Integrating against the fundamental class $[\pi_{g,n}]$ gives an isomorphism $H^2(\pi_{g,n},\partial\pi_{g,n},\mathbb{R})\xrightarrow{\cong}\mathbb{R}$. Let $\varphi\colon Z^2(\pi_{g,n},\partial\pi_{g,n},\mathbb{R})\to\mathbb{R}$ be the composition of the quotient map $Z^2(\pi_{g,n},\partial\pi_{g,n},\mathbb{R})\to H^2(\pi_{g,n},\partial\pi_{g,n},\mathbb{R})$ with the integration against $[\pi_{g,n}]$. Similarly as in the closed case, it was proven in [GHJW97, §3] that the 2-form $\varphi\circ\omega$ is degenerate precisely on $B^1(\pi_{g,n},\mathfrak{g}_\phi)$ and is furthermore closed [GHJW97, Thm. 7.1] (see also [Law09]). We obtain

Theorem 2.4.12 ([GHJW97])**.** *Let $\Gamma=\pi_{g,n}$ be a surface group. The composition of*

$$\omega\colon Z^1_{par}(\pi_{g,n},\mathfrak{g}_\phi)\times Z^1_{par}(\pi_{g,n},\mathfrak{g}_\phi)\to Z^2(\pi_{g,n},\partial\pi_{g,n},\mathbb{R})$$

with $\varphi\colon Z^2(\pi_{g,n},\partial\pi_{g,n},\mathbb{R})\to\mathbb{R}$ gives a nondegenerate closed 2-form

$$(\omega_{\mathcal{G}})_\phi\colon H^1_{par}(\pi_{g,n},\mathfrak{g}_\phi)\times H^1_{par}(\pi_{g,n},\mathfrak{g}_\phi)\to\mathbb{R}$$

Definition 2.4.13 (Relative character varieties)**.** The Hausdorffization of the topological quotient

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n},G)/\,\mathrm{Inn}(G)$$

is called the *relative character variety* associated to $(\pi_{g,n},G,\mathcal{C})$. The nondegenerate closed 2-form $\omega_{\mathcal{G}}$ is the *the Goldman symplectic form* on $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n},G)/\,\mathrm{Inn}(G)$.

Depending on the properties of the group G, the definition of relative character variety can be refined in order to get a better control of its structure similarly as in Section 2.3.

Remark 2.4.14 (Poisson structure)*.* The representation variety $\mathrm{Hom}(\pi_{g,n},G)$ is the disjoint union of all the relative representation varieties $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n},G)$ over all possible choices for $\mathcal{C}\in(G/G)^n$. The quotient of each relative representation variety by the $\mathrm{Inn}(G)$-action has a symplectic structure in the sense of Theorem 2.4.12. It turns out that these quotients are the symplectic leaves of a Poisson structure on the quotient of the representation variety by the $\mathrm{Inn}(G)$-action. The reader is referred to [BJ21] for a precise statement, a proof, and references to prior proofs.

Definition 2.4.15 (Goldman symplectic measure)**.** Both in the case of character varieties for closed surfaces and in the case of relative character varieties for punctured surfaces, the measure obtained from the Goldman symplectic form is denoted $\nu_{\mathcal{G}}$ and called the *Goldman symplectic measure.*

The Goldman symplectic measure is a strictly positive Borel measure. It means that open sets are measurable and always have positive measure if they are nonempty.

Case of a punctured sphere

In the case that $\Gamma = \pi_{0,n}$ is the fundamental group of a punctured sphere, then one can obtain fairly explicit formulae for the Goldman symplectic form on $\mathrm{Hom}_{\mathcal{C}}(\pi_{0,n}, G)$. We abbreviate $\pi_n := \pi_{0,n}$ in this section. We first need to compute a fundamental class $[\pi_n]$ explicitly. All computations are lead in the bar complex for group cohomology introduced in Appendix B.2.

Lemma 2.4.16. *Let $e \in \mathbb{Z}[\pi_n \times \pi_n]$ be given by*

$$e := (c_1, c_2) + (c_1c_2, c_3) + \ldots + (c_1c_2 \cdot \ldots \cdot c_{n-1}, c_n) + (1, 1). \tag{2.4.3}$$

Then $(e, c_1, \ldots, c_n) \in Z^2(\pi_n, \partial\pi_n, \mathbb{Z})$, i.e. the 2-chain $(e, c_1, \ldots, c_n)$ is closed. Moreover, $[(e, c_1, \ldots, c_n)]$ is a generator of $H_2(\pi_n, \partial\pi_n, \mathbb{Z})$.

Proof. Let $\imath_i \colon \partial_i\pi_n \hookrightarrow \pi_n$ denote the inclusion of the subgroup $\partial_i\pi_n$ (generated by c_i) into π_n. The long exact sequence (B.9) in group homology for the pair $(\pi_n, \partial\pi_n)$ contains

$$\ldots \to H_2(\pi_n, \mathbb{Z}) \longrightarrow H_2(\pi_n, \partial\pi_n, \mathbb{Z}) \xrightarrow{\delta} H_1(\partial\pi_n, \mathbb{Z}) \xrightarrow{\oplus \imath_i} H_1(\pi_n, \mathbb{Z}) \to \ldots.$$

Since $H_2(\pi_n, \mathbb{Z}) = 0$, the connecting morphism δ is an isomorphism onto its image. Hence $H_2(\pi_n, \partial\pi_n, \mathbb{Z}) \cong \mathrm{Ker}(\bigoplus \imath_i)$. Recall that $H_2(\pi_n, \partial\pi_n, \mathbb{Z}) \cong \mathbb{Z}$, and so $\mathrm{Ker}(\bigoplus \imath_i) \cong \mathbb{Z}$. The strategy to find a fundamental class is to first find an isomorphism $\psi\colon \mathrm{Ker}(\bigoplus \imath_i) \to \mathbb{Z}$, then compute $\psi^{-1}(1) \in H_1(\partial\pi_n, \mathbb{Z})$ and finally compute its preimage under δ.

Recall that the bar chain complex that computes the homology of the group π_n with coefficients in the trivial π_n-module $\mathbb{Z}$ is defined by $C_k(\pi_n, \mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_n^k] \cong \mathbb{Z}[\pi_n^k]$, where $\pi_n^k = \pi_n \times \ldots \times \pi_n$. The differentials in degrees 1 and 2 are

$$\begin{array}{ccccc} C_2(\pi_n\mathbb{Z}) & \xrightarrow{\partial} & C_1(\pi_n, \mathbb{Z}) & \xrightarrow{\partial} & C_0(\pi_n, \mathbb{Z}) \\ & & g & \longmapsto & 0 \\ (g, h) & \longmapsto & g + h - gh. & & \end{array}$$

In particular, the first homology group is

$$H_1(\pi_n, \mathbb{Z}) = \mathbb{Z}[\pi_n]/(g + h - gh), \tag{2.4.4}$$

Since $c_n = \prod_{i=1}^{n-1} c_i^{-1}$ by construction, it holds that $c_n = \sum_{i=1}^{n-1} -c_i$ and $c_i^k = k \cdot c_i$ inside $\mathbb{Z}[\pi_n]/(g+h-gh)$. This gives an isomorphism $\mathbb{Z}[\pi_n]/(g+h-gh) \cong \mathbb{Z} \cdot c_1 \oplus \ldots \oplus \mathbb{Z} \cdot c_{n-1}$. For the same reason,

$$H_1(\partial_i \pi_n, \mathbb{Z}) = \mathbb{Z}[\partial_i \pi_n]/(g + h - gh) \cong \mathbb{Z} \cdot c_i.$$

We are interested in the morphism $\varphi \colon \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ induced by $\oplus \imath_i$ in the following diagram

$$\begin{tikzcd}
H_1(\partial \pi_n, \mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "\oplus \imath_i"] & \mathbb{Z} \cdot c_1 \oplus \ldots \oplus \mathbb{Z} \cdot c_n \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}^n \arrow[d, dashed, "\varphi"] \\
H_1(\pi_n, \mathbb{Z}) \arrow[r, "\cong"] & \mathbb{Z} \cdot c_1 \oplus \ldots \oplus \mathbb{Z} \cdot c_{n-1} \arrow[r, "\cong"] & \mathbb{Z}^{n-1}
\end{tikzcd}$$

The previous identifications implies that φ is the morphism

$$\varphi(m_1, \ldots, m_n) = (m_1 - m_n, \ldots, m_{n-1} - m_n).$$

Therefore, the kernel of φ consists of vectors having identical entries and thus $\mathrm{Ker}(\oplus \imath_i)$ is generated by $[(c_1, \ldots, c_n)] \in H_1(\partial \pi_n, \mathbb{Z})$.

It remains to compute $\delta^{-1}([(c_1, \ldots, c_n)])$. Since δ is induced from the projection $\mathbb{Z}[\pi_n^2] \oplus \mathbb{Z}[\partial \pi_n] \to \mathbb{Z}[\partial \pi_n]$, it is enough to find a chain $e \in \mathbb{Z}[\pi_n^2]$ such that $(e, c_1, \ldots, c_n)$ is closed. This is the case for e given by (2.4.3) because $\partial_2 e = -c_1 - \ldots - c_n$ and hence $\partial_2(e, c_1, \ldots, c_n) = 0$. $\square$

The fundamental class $[\pi_n]$ was already computed in [GHJW97, Section 2] using different methods. We now give explicit formulae for the Goldman symplectic form.

Let $u, v \in Z^1_{par}(\pi_n, \mathfrak{g}_\phi)$. By definition of parabolic cocycles, there exist $\xi_i, \zeta_i \in \mathfrak{g}$ such that

$$u(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i, \quad v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i, \quad i = 1, \ldots, n.$$

The first step consists in computing a preimage of u inside $Z^1(\pi_n, \partial \pi_n, \mathfrak{g}_\phi)$. Note that

$$\partial \xi_i(c_i) = \mathrm{Ad}(\phi(c_i))\xi_i - \xi_i = -u(c_i).$$

Hence, the 1-cochain $(u, -\xi_1, \ldots, -\xi_n)$ is closed and is a preimage of u.

To compute $\omega_{\mathcal{G}}(u, v)$, we proceed as follows:

1. Apply the cup product to $(u, -\xi_1, \dots, -\xi_n)$ and v.

2. Apply the pairing B-

3. Take the cap product with the fundamental form $[\pi_n]$ computed in Lemma 2.4.16 (here we use Lemma B.15).

This gives

$$\omega_G(u, v) = B_*(u \smile v)(e) + \sum_{i=1}^{n} B_*(\xi_i \smile v)(c_i). \tag{2.4.5}$$

We develop each cup product according to (B.11) and plug in the value of e computed in Lemma 2.4.16. The right-hand side of (2.4.5) becomes

$$\sum_{i=2}^{n} B(u(c_1 \cdot \dots \cdot c_{i-1}) \cdot \mathrm{Ad}(\phi(c_1 \cdot \dots \cdot c_{i-1}))v(c_i)) + \sum_{i=1}^{n} B(\xi_i \cdot v(c_i)). \tag{2.4.6}$$

We can further simplify (2.4.6) using to the Ad-invariance of B and the formula $u(x^{-1}) = -\mathrm{Ad}(\phi(x^{-1}))u(x)$. It is useful to introduce the notation $b_{i-2} := c_{i-1}^{-1} \cdots c_1^{-1}$. In particular, $b_0 = c_1^{-1}$ and $b_{n-1} = 1$. We obtain

$$\omega_{\mathcal{G}}(u, v) = -\sum_{i=2}^{n} B(u(b_{i-2}) \cdot v(c_i)) + \sum_{i=1}^{n} B(\xi_i \cdot v(c_i)). \tag{2.4.7}$$

Using that ω_G and the cup product are anti-symmetric, we get the following equivalent form of (2.4.7)

$$\omega_{\mathcal{G}}(u, v) = -\sum_{i=2}^{n} B(u(b_{i-2}) \cdot v(c_i)) - \sum_{i=1}^{n} B(\zeta_i \cdot u(c_i)). \tag{2.4.8}$$

Formulae (2.4.5), (2.4.8), and (2.4.7), were already obtained in the proof of [GHJW97, Key Lemma 8.4]. We go one step further.

Lemma 2.4.17. *It holds that*

$$\omega_{\mathcal{G}}(u, v) = \sum_{i=1}^{n-2} B((\zeta_{i+1} - \zeta_{i+2}) \cdot u(b_i)). \tag{2.4.9}$$

Proof. Using $v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i$ and the Ad-invariance of B, we get

$$B(u(b_{i-2}) \cdot v(c_i)) = B(\zeta_i \cdot u(b_{i-2})) - B(\mathrm{Ad}(\phi(c_i^{-1}))u(b_{i-2}) \cdot \zeta_i)$$

By construction, $b_{i-1} = c_i^{-1} b_{i-2}$ and thus $u(b_{i-1}) = u(c_i^{-1}) + \mathrm{Ad}(\phi(c_i^{-1}))u(b_{i-2})$. So,

$$B(u(b_{i-2}) \cdot v(c_i)) = B(\zeta_i \cdot u(b_{i-2})) - B(\zeta_i \cdot u(b_{i-1})) + B(\zeta_i \cdot u(c_i^{-1})).$$

Therefore, (2.4.8) becomes

$$\begin{aligned}
\omega_{\mathcal{G}}(u,v) = &\sum_{i=2}^{n} B(\zeta_i \cdot u(b_{i-1})) - B(\zeta_i \cdot u(b_{i-2})) \\
&- B(\zeta_1 \cdot u(c_1)) - \sum_{i=2}^{n} B(\zeta_i \cdot (u(c_i^{-1}) + u(c_i)) \\
=& B(\zeta_2 \cdot u(b_1)) + \sum_{i=3}^{n} B(\zeta_i \cdot u(b_{i-1})) - B(\zeta_i \cdot u(b_{i-2})) \\
&- \sum_{i=1}^{n} B\big(\zeta_i \cdot (u(c_i^{-1}) + u(c_i))\big) \\
=& \sum_{i=1}^{n-2} B\big((\zeta_{i+1} - \zeta_{i+2}) \cdot u(b_i)\big) - \underbrace{\sum_{i=1}^{n} B\big(\zeta_i \cdot (u(c_i^{-1}) + u(c_i))\big)}_{=:\Omega},
\end{aligned}$$

where in the second equality we used $b_0 = c_1^{-1}$ and in the third equality that $u(b_{n-1}) = u(1) = 0$. It remains to prove that $\Omega = 0$. Using $u(x^{-1}) = -\mathrm{Ad}(\phi(x^{-1}))u(x)$, we get

$$B(\zeta_i \cdot u(c_i^{-1})) = -B(\mathrm{Ad}(\phi(c_i))\zeta_i \cdot u(c_i)).$$

Therefore, using $v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i$, we conclude

$$\Omega = \sum_{i=1}^{n} B(u(c_i) \cdot v(c_i)).$$

By construction, $B(u(\cdot) \cdot v(\cdot))$ defines a 1-cocycle in $Z^1(\pi_n, \mathbb{R})$. Closeness can also be computed directly using (B.2), similarly as in the proof of Lemma B.11. Therefore, Ω is equal to the evaluation of the 1-cocycle $B(u(\cdot) \cdot v(\cdot))$ on the 1-cycle $c_1 + \ldots + c_n$. The identification (2.4.4) shows that the 1-cycle $\sum_{i=1}^{n} c_i$ vanishes in homology (this is a consequence of the fact that $\prod_{i=1}^{n} c_i = 1$). Hence, $\Omega = B(u(1) \cdot v(1)) = 0$ as desired. ∎

2.5. Volume of a representation

The topology of a representation variety is notably known to be complicated. The enumeration of its connected components is a non-trivial task. The volume of a representation is an invariant that lets us approach this problem. We recall its definition below and recommend [BIW10] for more details.

2.5.1. Definition

The volume is defined in [BIW10] for representations of surface groups $\Gamma = \pi_{g,n}$ into Hermitian Lie groups G. Recall that a *Hermitian Lie group* G is a semisimple Lie group, with finite center and no compact factors, such that its associated symmetric space X is a Hermitian manifold. The Kähler form obtained from the unique G-invariant Hermitian metric of constant sectional curvature -1 on X is denoted ω_X. The classical examples of Hermitian Lie groups include $\mathrm{SU}(p,q)$ and $\mathrm{Sp}(2n,\mathbb{R})$.

Example 2.5.1. The guiding example in this section is the group $G = \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$. It is a simple Lie group, without compact factor and with center $Z(\mathrm{SL}(2,\mathbb{R})) = \{\pm I\}$. It is of Hermitian type. It is sometimes more convenient to consider the center-free quotient $\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ instead, which is also of Hermitian type. The associated symmetric space is the upper half-plane $X = \mathbb{H}$ on which $\mathrm{SL}(2,\mathbb{R})$ acts by Möbius transformations, see Appendix A for more considerations on the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$. The group of orientation-preserving isometries of $\mathbb{H}$ is $\mathrm{PSL}(2,\mathbb{R})$. The associated Kähler form is $\omega_{\mathbb{H}} = (dx \wedge dy)/y^2$.

Let G be a Hermitian Lie group with symmetric space X. Given three points z_1, z_2, z_3 in X, we denote by $\Delta(z_1, z_2, z_3)$ the oriented geodesic triangle in X with vertices z_1, z_2, z_3. Its signed area, computed with the area form associated to ω_X, is denoted by

$$[\Delta(z_1, z_2, z_3)] := \int_{\Delta(z_1,z_2,z_3)} \omega_X.$$

Fix a basepoint $z \in X$ and consider the function

$$\begin{aligned} c\colon G \times G &\to \mathbb{R} \\ (g_1, g_2) &\to \big[\Delta\big(z, g_1 z, g_1 g_2 z\big)\big]. \end{aligned} \tag{2.5.1}$$

Lemma 2.5.2. *The function c satisfies the cocycle condition*

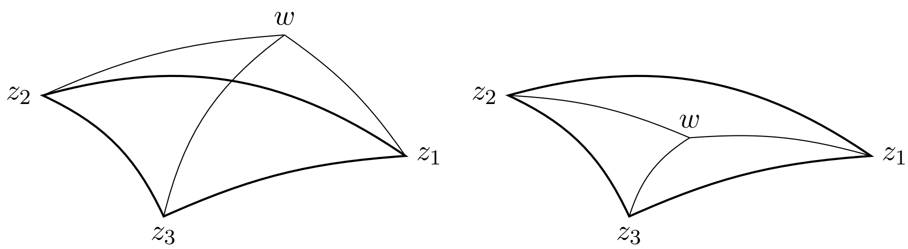
$$c(g_2, g_3) - c(g_1g_2, g_3) + c(g_1, g_2g_3) - c(g_1, g_2) = 0 \tag{2.5.2}$$

for every $g_1, g_2, g_3 \in G$, compare (B.2).

Proof. We need the following identity: if z_1, z_2, z_3 are any three points in X, then, for any fourth point $w \in X$,

$$[\Delta(z_1, z_2, z_3)] = [\Delta(z_1, z_2, w)] + [\Delta(z_2, z_3, w)] + [\Delta(z_3, z_1, w)]. \tag{2.5.3}$$

The following picture should convince the reader of (2.5.3).



In terms of triangle areas, the cocycle condition (2.5.2) is equivalent to

$$[\Delta(z, g_2z, g_2g_3z)] + [\Delta(z, g_1z, g_1g_2g_3z)]$$

being equal to

$$[\Delta(z, g_1g_2z, g_1g_2g_3z)] + [\Delta(z, g_1z, g_1g_2z)].$$

Since $g_1 \in G$ acts by isometry on X and preserves the orientation, the latter is equivalent to

$$[\Delta(g_1z, g_1g_2z, g_1g_2g_3z)] + [\Delta(z, g_1z, g_1g_2g_3z)]$$

being equal to

$$[\Delta(z, g_1g_2z, g_1g_2g_3z)] + [\Delta(z, g_1z, g_1g_2z)].$$

This is precisely formula (2.5.3) applied to $z_1 = z$, $z_2 = g_1z$, $z_3 = g_1g_2z$ and $w = g_1g_2g_3z$. □

Lemma 2.5.2 implies that c defines a cohomology class $\kappa := [c]$ inside $H^2(G, \mathbb{R})$. The function c is bounded because the area of a geodesic triangle in X is bounded. This means that the cohomology class κ gives a class $\kappa \in H_b^2(G, \mathbb{R})$ in the second bounded cohomology group of G. We recommend [Löh10] for an introduction to bounded group cohomology.

Lemma 2.5.3. *The cohomology class κ is independent of the choice of the basepoint z involved in the definition of c (whereas c does depend on the point z).*

Proof. For the purpose of this proof, we will write c_z instead of c for the cocycle (2.5.1) to emphasize the dependence on the basepoint z. Given another basepoint $x \in X$, we prove that $c_z - c_x$ is a coboundary.

First, we develop $c_z(g_1, g_2) = [\Delta(z, g_1 z, g_1 g_2 z)]$ using (2.5.3) with $w = g_1 x$. We obtain

$$\begin{aligned} c_z(g_1, g_2) &= [\Delta(z, g_1 z, g_1 x)] + [\Delta(g_1 z, g_1 g_2 z, g_1 x)] + [\Delta(g_1 g_2 z, z, g_1 x)] \\ &= -[\Delta(x, z, g_1^{-1} z)] + [\Delta(x, z, g_2 z)] + [\Delta(g_1 g_2 z, z, g_1 x)]. \end{aligned}$$

Now, we develop $[\Delta(g_1 g_2 z, z, g_1 x)]$ using (2.5.3) with $w = x$. This gives

$$\begin{aligned} [\Delta(g_1 g_2 z, z, g_1 x)] &= [\Delta(g_1 g_2 z, z, x)] + [\Delta(z, g_1 x, x)] + [\Delta(g_1 x, g_1 g_2 z, x)] \\ &= -[\Delta(x, z, g_1 g_2 z)] - [\Delta(z, x, g_1 x)] + [\Delta(g_1 x, g_1 g_2 z, x)]. \end{aligned}$$

Finally, we develop $[\Delta(g_1 x, g_1 g_2 z, x)]$ using (2.5.3) with $w = g_1 g_2 x$. We have

$$\begin{aligned} [\Delta(g_1 x, g_1 g_2 z, x)] &= [\Delta(g_1 x, g_1 g_2 z, g_1 g_2 x)] + [\Delta(g_1 g_2 z, x, g_1 g_2 x)] + [\Delta(x, g_1 x, g_1 g_2 x)] \\ &= [\Delta(z, x, g_2^{-1} x)] - [\Delta(z, x, g_2^{-1} g_1^{-1} x)] + c_x(g_1, g_2). \end{aligned}$$

Consider the 1-cochain $v_{x,z}(g) := [\Delta(x, z, gz)]$. It holds that

$$\partial v_{x,z}(g_1, g_2) = [\Delta(x, z, g_1 z)] + [\Delta(x, z, g_2 z)] - [\Delta(x, z, g_1 g_2 z)].$$

In particular, $\partial v_{x,z}(g, g^{-1}) = [\Delta(x, z, gz)] + [\Delta(x, z, g^{-1} z)]$. The previous computations show that

$$c_z(g_1, g_2) - c_x(g_1, g_2) = \partial v_{x,z}(g_1, g_2) - \partial v_{x,z}(g_1, g_1^{-1}) + \partial v_{z,x}(g_2^{-1}, g_1^{-1}) - \partial v_{z,x}(g_1, g_1^{-1}).$$

We conclude as predicted that $c_z - c_x$ is a coboundary. $\square$

Given a representation $\phi\colon \pi_{g,n} \to G$, we can pull back κ to the class $\phi^* \kappa$ inside $H_b^2(\pi_{g,n}, \mathbb{R})$. An important property of the bounded cohomology of the group $\pi_{g,n}$ is that the map

$$j\colon H_b^2(\pi_{g,n}, \partial \pi_{g,n}, \mathbb{R}) \to H_b^2(\pi_{g,n}, \mathbb{R}) \tag{2.5.4}$$

from the long exact sequence in cohomology for the pair $(\pi_{g,n}, \partial \pi_{g,n})$ is an isomorphism, see [Löh10, Thm. 2.6.14]. Recall finally that integrating along a fundamental class $[\pi_{g,n}]$ gives an isomorphism $H^2(\pi_{g,n}, \partial \pi_{g,n}, \mathbb{R}) \cong \mathbb{R}$.

Definition 2.5.4 (Volume of a representation, [BIW10])**.** Let G be a Hermitian Lie group. The *volume* of a representation[13] $\phi\colon \pi_{g,n} \to G$ is the real number defined by

$$\mathrm{vol}(\phi) := j^{-1}(\phi^* \kappa) \frown [\pi_{g,n}].$$

The volume is a generalization of the Euler number of a representation of a closed surface group into $\mathrm{PSL}(2,\mathbb{R})$. The latter is equal to the Euler number of the flat $\mathbb{RP}^1$-bundle $(\widetilde{\Sigma}_{g,0} \times \mathbb{RP}^1)/\pi_{g,0} \to \Sigma_{g,0}$ associated to a representation $\pi_{g,0} \to \mathrm{PSL}(2,\mathbb{R})$.

2.5.2. Properties

Lemma 2.5.5. *The volume is invariant under the conjugation action of G on $\mathrm{Hom}(\pi_{g,n}, G)$ and thus descends to a function*

$$\mathrm{vol}\colon\ \mathrm{Hom}(\pi_{g,n}, G)/\,\mathrm{Inn}(G) \to \mathbb{R}.$$

Proof. Consider the cocycle c defined in (2.5.1). The diagonal conjugation action of an element $g \in G$ on $G \times G$ amounts to a change of basepoint in the definition of c. Indeed, if c_z denotes the cocycle (2.5.1) defined using the basepoint $z \in X$, then it holds that $c_z(gg_1g^{-1}, gg_2g^{-1}) = c_{g^{-1}z}(g_1, g_2)$. Since, by Lemma 2.5.3, the cohomology class κ is independent of the choice of the basepoint defining c, we conclude that the volume is an invariant of conjugation. □

The main properties of the volume are the following. We denote by $\chi(\Sigma_{g,n})$ the Euler characteristic of $\Sigma_{g,n}$.

Theorem 2.5.6 ([BIW10])**.** *The volume, seen as a function* $\mathrm{vol}\colon \mathrm{Hom}(\pi_{g,n}, G) \to \mathbb{R}$*, has the following properties:*

1. vol *is a continuous function.*
2. vol *is locally constant on each relative representation variety.*
3. (Milnor–Wood inequality) vol *is bounded:*

$$|\,\mathrm{vol}\,| \leqslant 2\pi \cdot |\chi(\Sigma_{g,n})| \cdot \mathrm{rank}(G),$$

[13] Up to a constant, the volume of a representation ϕ is sometimes called the *Toledo number* of the representation and is, in that case, denoted $\mathrm{Tol}(\phi)$. The two notions are related by the identity $\mathrm{vol}(\phi) = 2\pi\,\mathrm{Tol}(\phi)$.

moreover, if $n > 0$, then vol *is a surjective function onto the interval*

$$[-2\pi \cdot |\chi(\Sigma_{g,n})| \cdot \operatorname{rank}(G),\, 2\pi \cdot |\chi(\Sigma_{g,n})| \cdot \operatorname{rank}(G)]\,.$$

4. vol *is additive: if $\Sigma_{g,n}$ is separated by a simple closed curve into two surfaces S_1 and S_2, then, for every $\phi \in \operatorname{Hom}(\pi_{g,n}, G)$,*

$$\operatorname{vol}(\phi) = \operatorname{vol}(\phi\restriction_{\pi_1(S_1)}) + \operatorname{vol}(\phi\restriction_{\pi_1(S_2)}).$$

The first and second statement in Theorem 2.5.6 imply that the set of representations of a given volume forms a collection of connected components of each relative character variety. Recall that in the case of a closed surface group and $G = \operatorname{PSL}(2,\mathbb{R})$, the Euler number completely distinguishes the connected components of the character variety [Gol88].

The volume has an interesting symmetry that comes from reversing the orientation of X. By definition, for each $z \in X$, there exists an orientation-reversing isometry s_z of X that fixes z. This gives an involutive automorphism $\sigma \colon G \to G$ defined by $\sigma(g) := s_z \circ g \circ s_z$. Indeed, if $g \in G$ is an orientation-preserving isometry of X, then $s_z \circ g \circ s_z$ is again an orientation-preserving isometry of X, and hence belongs to G. Using the functoriality of representation varieties (see Lemma 2.1.12), the involution σ descends to an analytic involution

$$\sigma \colon \operatorname{Hom}(\pi_{g,n}, G) \to \operatorname{Hom}(\pi_{g,n}, G).$$

Lemma 2.5.7. *The involution σ satisfies the following properties:*

1. *σ preserves conjugacy classes of representations, and therefore descends to an involution*

$$\overline{\sigma} \colon \operatorname{Hom}(\pi_{g,n}, G)/\operatorname{Inn}(G) \to \operatorname{Hom}(\pi_{g,n}, G)/\operatorname{Inn}(G).$$

2. *σ depends on the choice of $z \in X$ only up to conjugation, in particular, $\overline{\sigma}$ is independent of the choice of $z \in X$.*

3. *For any representation $\phi \in \operatorname{Hom}(\pi_{g,n}, G)$ it holds that*

$$\operatorname{vol}(\sigma(\phi)) = -\operatorname{vol}(\phi).$$

Proof. The first assertion follows from $\sigma(g\phi g^{-1}) = (s_z \circ g \circ s_z)\sigma(\phi)(s_z \circ g^{-1} \circ s_z)$ and the observation that $s_z \circ g \circ s_z$ is orientation-preserving. If $z' \in X$ is a second point, then it holds that $s_{z'} \circ g \circ s_{z'} = (s_{z'} \circ s_z)(s_z \circ g \circ s_z)(s_z \circ s_{z'})$, which proves the second assertion because

$s_{z'} \circ s_z$ is orientation-preserving. Finally, note that $(\sigma(\phi))^*\kappa = \phi^*(\sigma^*\kappa)$ and $\sigma^*\kappa = -\kappa$ because s_z reverses the orientation of X. $\square$

Example 2.5.8. Consider the case $G = \mathrm{SL}(2, \mathbb{R})$. An example of orientation-reversing isometry of the upper half-plane is given by $z \mapsto -\bar{z}$. It fixes the imaginary axis. The associated involutive automorphism σ of $\mathrm{SL}(2, \mathbb{R})$ is given by conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of determinant -1.

The involution $\sigma\colon \mathrm{Hom}(\pi_{g,n}, G) \to \mathrm{Hom}(\pi_{g,n}, G)$ maps the relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ to the relative representation variety $\mathrm{Hom}_{\sigma(\mathcal{C})}(\pi_{g,n}, G)$. Since G is of Hermitian type, it is by definition semisimple and hence quadrable. The Goldman symplectic form built from the Killing form on $\mathfrak{g}$ is invariant under σ. This is a consequence of the fact that the Killing form is invariant under automorphisms of $\mathfrak{g}$. In this case, the involution $\sigma\colon G \to G$ induces an automorphism $D\sigma\colon \mathfrak{g} \to \mathfrak{g}$.

2.5.3. Alternative definition

A downside of Definition 2.5.4 is the lack of computability. Given a representation $\phi\colon \pi_{g,n} \to G$, computing $j^{-1}(\phi^*\kappa)$ means finding a primitive in $H^1(\partial_i \pi_{g,n}, \mathbb{R})$ for each restriction $\phi^*\kappa\restriction_{\partial_i \pi_{g,n}}$. This is a non-trivial task in general. There is an alternative definition of the volume of a representation that makes it easier to compute. It is based on a notion of rotation number that generalizes the classical notion of rotation number for homeomorphisms of the circle, see [Ghy01] for an exposition of the classical theory of rotation numbers. The *rotation number* in our context is a function $\rho\colon G \to \mathbb{R}/2\pi\mathbb{Z}$ that lifts to a quasimorphism $\widetilde{\rho}\colon \widetilde{G} \to \mathbb{R}$ of the universal cover of G. We explain the construction in the case $G = \mathrm{PSL}(2, \mathbb{R})$ and refer the reader to [BIW10, §7] for the general case. The main result is

Theorem 2.5.9 ([BIW10]). *Let $\widetilde{\phi}\colon \pi_{g,n} \to \widetilde{G}$ be a group homomorphism that covers ϕ. Then*

$$\mathrm{vol}(\phi) = -\sum_{i=1}^{n} \widetilde{\rho}\left(\widetilde{\phi}(c_i)\right),$$

where c_i are the generators of $\pi_{g,n}$ of presentation (2.1.3).

Example 2.5.10. Let's study the case $G = \mathrm{PSL}(2, \mathbb{R})$. We fix a topological group structure on $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$ by fixing a unit e in the fibre over the identity. The action of $\mathrm{PSL}(2, \mathbb{R})$ on the circle $\mathbb{R}/2\pi\mathbb{Z}$ (see Lemma A.4) gives a group homomorphism $f\colon \mathrm{PSL}(2, \mathbb{R}) \to$

$\mathrm{Homeo}^+(\mathbb{R}/2\pi\mathbb{Z})$. This action lifts to a faithful action of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ on the universal cover $\mathbb{R}/2\pi\mathbb{Z}$. The classical rotation number is a function $\mathrm{rot}\colon \mathrm{Homeo}^+(\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}$, see [Ghy01]. The quasimorphism $\widetilde{\rho}\colon \widetilde{\mathrm{PSL}(2,\mathbb{R})} \to \mathbb{R}$ is the unique lift of $\rho := \mathrm{rot} \circ f$ satisfying $\widetilde{\rho}(e) = 0$.

We can describe ρ more explicitly by considering conjugacy classes in $\mathrm{PSL}(2,\mathbb{R})$. Recall that, if $\mathcal{E}$ denotes the set of elliptic conjugacy classes in $\mathrm{PSL}(2,\mathbb{R})$, then there is a well-defined angle function $\vartheta\colon \mathcal{E} \to (0,2\pi)$, see Lemma A.7. It extends to an upper semi-continuous function $\overline{\vartheta}\colon \mathrm{PSL}(2,\mathbb{R}) \to [0,2\pi]$ by

$$\overline{\vartheta}(A) := \begin{cases} \vartheta(A), & \text{if } A \text{ is elliptic,} \\ 0, & \text{if } A \text{ is hyperbolic or positively parabolic,} \\ 2\pi, & \text{if } A \text{ is the identity or negatively parabolic.} \end{cases} \tag{2.5.5}$$

The notions of positively and negatively parabolic refer to the two conjugacy classes of parabolic elements in $\mathrm{PSL}(2,\mathbb{R})$ represented by (A.6). The definition of the function $\overline{\vartheta}$ is ad hoc, however it satisfies $\overline{\vartheta} = \rho$ modulo 2π. In particular, the correction term

$$k(\phi) := \frac{1}{2\pi}\left(\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) - \sum_{i=1}^{n} \widetilde{\rho}\left(\widetilde{\phi}(c_i)\right)\right) \tag{2.5.6}$$

is an integer called the *relative Euler class* of ϕ. The definition of the relative Euler class very much depends on the choice of the extension $\overline{\vartheta}$ of ϑ. Theorem 2.5.9 implies

$$k(\phi) = \frac{1}{2\pi}\left(\mathrm{vol}(\phi) + \sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i))\right).$$

The range of the relative Euler class over $\mathrm{Hom}(\pi_{g,n}, G)$ was studied in [DT19]. The authors proved that

Proposition 2.5.11 ([DT19]). *Let $\phi\colon \pi_{g,n} \to \mathrm{PSL}(2,\mathbb{R})$ be a representation. Then*

$$k(\phi) \leqslant \max\left\{|\chi(\Sigma_{g,n})|, \frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i))\right\}.$$

Remark 2.5.12. Observe that, as soon as $g \geqslant 1$, then $|\chi(\Sigma_{g,n})| \geqslant n \geqslant \frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i))$ and thus the inequality $k(\phi) \leqslant |\chi(\Sigma_{g,n})|$ prevails. In the case $g = 0$, it is however possible that $\frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) > |\chi(\Sigma_{0,n})|$.

2.6. Mapping class group dynamics

We expand on some results and remarks from Section 2.1.3. Let G be a Lie group and Γ be a finitely generated group. Recall that the $\mathrm{Aut}(\Gamma)$-action on the representation variety $\mathrm{Hom}(\Gamma, G)$ descends to an action of the outer automorphisms group $\mathrm{Out}(\Gamma)$ on the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. This action preserves the analytic/algebraic structure of $\mathrm{Hom}(\Gamma, G)$ by Corollary 2.1.13. When $\Gamma = \pi_{g,n}$ is a surface group, then $\mathrm{Out}(\pi_{g,n})$ contains the mapping class group of the surface $\Sigma_{g,n}$ as a subgroup, compare Example 2.1.14. The induced action is the so-called *mapping class group action on character varieties*.

We start with some general considerations on the $\mathrm{Aut}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)$ and then specialize to the case of a surface group.

2.6.1. Remarks on the $\mathrm{Aut}(\Gamma)$-action

Lemma 2.6.1. *The* $\mathrm{Aut}(\Gamma)$*-action on* $\mathrm{Hom}(\Gamma, G)$ *preserves the subspaces of (very) regular, reductive, irreducible, good and (almost) Zariski dense representations.*

Proof. All these particular notions of representations are defined in terms of the image of the representation. However, for any $\tau \in \mathrm{Aut}(\Gamma)$ and $\phi \in \mathrm{Hom}(\Gamma, G)$, it holds that $\phi(\Gamma) = (\phi \circ \tau)(\Gamma)$. □

A consequence of Lemma 2.6.1 is that the $\mathrm{Out}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ restricts to an action of $\mathrm{Out}(\Gamma)$ on the GIT character variety $\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G)$ (by Theorem 2.3.16, assuming G is a reductive complex algebraic group) and on the analytic character variety $\mathrm{Rep}^{\infty}(\pi_{g,0}, G)$.

Lemma 2.6.2. *The* $\mathrm{Aut}(\Gamma)$*-action on* $\mathrm{Hom}(\Gamma, G)$ *preserves closed orbits.*

Proof. This is an immediate consequence of Corollary 2.1.13. □

In particular, Lemma 2.6.2 implies that the $\mathrm{Aut}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)$ induces an $\mathrm{Out}(\Gamma)$-action on the $\mathcal{T}_1$ character variety $\mathrm{Rep}^{\mathcal{T}_1}(\pi_{g,0}, G)$. It is not clear to the author whether there is an induced action of $\mathrm{Out}(\Gamma)$ on the Hausdorff character variety in general.

2.6.2. Generalities about mapping class groups

The *mapping class group* of a closed and oriented surface $\Sigma_{g,0}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,0}$. In the case of a punctured oriented surface $\Sigma_{g,n}$, the mapping class group is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ that fix each puncture individually[14]. The mapping class group is denoted by $\mathrm{Mod}(\Sigma_{g,n})$ and the isotopy class of an orientation-preserving homeomorphism $f\colon \Sigma_{g,n} \to \Sigma_{g,n}$ is denoted $[f] \in \mathrm{Mod}(\Sigma_{g,n})$. The group law is given by composition and the identity element correspond to the identity homeomorphism.

Theorem 2.6.3. *The mapping class group is finitely presented. Generators can be chosen to be Dehn twists along simple closed curves on $\Sigma_{g,n}$.*

More details about Theorem 2.6.3, including proof and explicit generating family, can be found in [FM12, §4]. In [GW17], the question of the minimal number of generators of $\mathrm{Mod}(\Sigma_{0,n})$ is treated, see also Remark 5.2.9.

A homeomorphism f of $\Sigma_{g,n}$ induces a group isomorphism $\pi_1(\Sigma_{g,n}, x) \to \pi_1(\Sigma_{g,n}, f(x))$. After choosing a continuous path from x to $f(x)$, we get an induced automorphism of the fundamental group of $\Sigma_{g,n}$ (that depends up to conjugation on the choice of the path). This gives a group homomorphism

$$\mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Out}(\pi_{g,n}).$$

The Dehn–Nielsen Theorem says that it is injective and provides a description of its image.

Theorem 2.6.4 (Dehn–Nielsen Theorem)**.** *The mapping class group $\mathrm{Mod}(\Sigma_{g,0})$ is an index two subgroup of $\mathrm{Out}(\pi_{g,0})$ for $g \geqslant 1$ (and is trivial for $g = 0$). Moreover, if $\Sigma_{g,n}$ has negative Euler characteristic, then the mapping class group $\mathrm{Mod}(\Sigma_{g,n})$ is an index two subgroup of $\mathrm{Out}^\star(\pi_{g,n})$, where $\mathrm{Out}^\star(\pi_{g,n})$ is the subgroup of $\mathrm{Out}(\pi_{g,n})$ that consists of the outer automorphisms that act by conjugation on the generators c_i of $\pi_{g,n}$ (in the presentation* (2.1.3)*).*

We refer the reader to [FM12, §8] for more considerations on the Dehn–Nielsen Theorem. Theorem 2.6.4 implies that the $\mathrm{Aut}(\pi_{g,0})$-action on the representation variety $\mathrm{Hom}(\pi_{g,0}, G)$

[14] In the terminology of [FM12], if punctures are fixed individually, then the group is called the *pure mapping class group*. It contrasts with the mapping class group where punctures can be permuted.

induces an action

$$\mathrm{Mod}(\Sigma_{g,0}) \circlearrowright \mathrm{Hom}(\pi_{g,0}, G)/\operatorname{Inn}(G).$$

The action is analytic/algebraic on the regular part of the quotient by Corollary 2.1.13. In the case of a punctured surface, the action of $\mathrm{Aut}(\pi_{g,n})$ on $\mathrm{Hom}(\pi_{g,n}, G)$ restricts to an action of $\mathrm{Aut}^\star(\pi_{g,n})$ on any relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$. This gives, by Theorem 2.6.4, an action

$$\mathrm{Mod}(\Sigma_{g,n}) \circlearrowright \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)/\operatorname{Inn}(G),$$

for any choice of conjugacy classes $\mathcal{C} \in (G/G)^n$. These two actions are what we call the *mapping class group action on character varieties.*

2.6.3. Properties of the mapping class group action

The first property is that the mapping class group action preserves the Goldman symplectic form. We start with the case of a closed surface. Let $[f] \in \mathrm{Mod}(\Sigma_{g,0})$ and take any $\tau \in \mathrm{Aut}(\pi_{g,0})$ that lies over the image of $[f]$ inside $\mathrm{Out}(\pi_{g,0})$. We choose the generator $[\pi_{g,0}]$ of $H_2(\pi_{g,0}, \mathbb{Z})$ that corresponds to the orientation of the surface $\Sigma_{g,0}$. Since f is orientation-preserving, it holds that $\tau_*[\pi_{g,0}] = [\pi_{g,0}]$. For any $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$, the automorphism τ induces a map $(d\tau)_\phi \colon Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \to Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau})$, $v \mapsto v \circ \tau$, on the Zariski tangent spaces to the representation variety.

Lemma 2.6.5. *If $\omega_{\mathcal{G}}$ denotes the Goldman symplectic form from Definition 2.4.2, then, for any $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$, the following diagram commutes*

$$\begin{tikzcd}
Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \times Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \arrow[r, "(\omega_{\mathcal{G}})_\phi"] \arrow[d, "(d\tau)_\phi \times (d\tau)_\phi"'] & \mathbb{R} \\
Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau}) \times Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau}) \arrow[ur, bend right, "(\omega_{\mathcal{G}})_{\phi\circ\tau}"'] &
\end{tikzcd}$$

In other words, it holds that

$$\tau^*\omega_{\mathcal{G}} = \omega_{\mathcal{G}}.$$

Proof. Let $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the pairing used in the definition of $\omega_{\mathcal{G}}$. For any $v, w \in Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$, we have

$$\begin{aligned}
(\omega_{\mathcal{G}})_{\phi\circ\tau}(v \circ \tau, w \circ \tau) &= B(v \circ \tau, w \circ \tau) \frown [\pi_{g,0}] \\
&= B(v, w) \frown \tau_*[\pi_{g,0}].
\end{aligned}$$

Since $\tau_*[\pi_{g,0}] = [\pi_{g,0}]$, we conclude $(\omega_{\mathcal{G}})_{\phi\circ\tau}(v\circ\tau, w\circ\tau) = (\omega_{\mathcal{G}})_\phi(v,w)$. □

As a consequence of Lemma 2.6.5, we obtain that the $\mathrm{Mod}(\Sigma_{g,0})$-action on the quotient $\mathrm{Hom}(\pi_{g,0}, G)/\mathrm{Inn}(G)$ preserves the Goldman symplectic measure $\nu_{\mathcal{G}}$ from Definition 2.4.15.

The situation is similar for punctured surfaces. Let $[f] \in \mathrm{Mod}(\Sigma_{g,n})$ and take any $\tau \in \mathrm{Aut}^\star(\pi_{g,n})$ that lies over the image of $[f]$ inside $\mathrm{Out}^\star(\pi_{g,n})$. The generator $[\pi_{g,n}]$ of $H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z})$ is again chosen to correspond to the orientation of the surface $\Sigma_{g,n}$. Similarly as before, $\tau_*[\pi_{g,n}] = [\pi_{g,n}]$. Moreover, the map $(d\tau)_\phi$ restricts to to a map $(d\tau)_\phi \colon Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \to Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau})$. Indeed, note that if $v(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i$ and $\tau(c_i) = g_i c_i g_i^{-1}$, then

$$(v\circ\tau)(c_i) = \big(v(g_i) + \mathrm{Ad}(\phi(g_i))\xi_i\big) - \mathrm{Ad}\,\big((\phi\circ\tau)(c_i)\big)\big(v(g_i) + \mathrm{Ad}(\phi(g_i))\xi_i\big).$$

Lemma 2.6.6. *If $\omega_{\mathcal{G}}$ denotes the Goldman symplectic form from Definition 2.4.13, then, for any $\phi \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$, the following diagram commutes*

$$\begin{tikzcd}
Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \arrow[r, "(\omega_{\mathcal{G}})_\phi"] \arrow[d, "(d\tau)_\phi \times (d\tau)_\phi"'] & \mathbb{R} \\
Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau}) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau}) \arrow[ur, bend right, "(\omega_{\mathcal{G}})_{\phi\circ\tau}"'] &
\end{tikzcd}$$

In other words, it holds that

$$\tau^*\omega_{\mathcal{G}} = \omega_{\mathcal{G}}.$$

The proof is analogous to the proof of Lemma 2.6.5.

The second property is that the mapping class group action also preserves the volume of a representation. As before, let $[f] \in \mathrm{Mod}(\Sigma_{g,n})$ and take any $\tau \in \mathrm{Aut}^\star(\pi_{g,n})$ that lies over the image of $[f]$ inside $\mathrm{Out}^\star(\pi_{g,n})$. Again, $\tau_*[\pi_{g,n}] = [\pi_{g,n}]$.

Lemma 2.6.7. *Let G be a Hermitian Lie group. For any $\phi \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$, it holds that*

$$\mathrm{vol}(\phi\circ\tau) = \mathrm{vol}(\phi).$$

Proof. We compute directly from Definition 2.5.4 that

$$\begin{aligned}\mathrm{vol}(\phi \circ \tau) &= j^{-1}\big((\phi \circ \tau)^* \kappa\big) \frown [\pi_{g,n}] \\ &= j^{-1}\big(\tau^* \phi^* \kappa\big) \frown [\pi_{g,n}] \\ &= j^{-1}\big(\phi^* \kappa\big) \frown \tau_*[\pi_{g,n}].\end{aligned}$$

We conclude by using $\tau_*[\pi_{g,n}] = [\pi_{g,n}]$. $\square$

3. Compact components in genus zero

In a series of recent work, compact components of relative character varieties of representations of the fundamental group of a punctured sphere into Hermitian Lie groups have been identified, see [DT19] and [TT21]. These constructions generalize an older result of [BG99] in the case of a 4-punctured sphere. See also [Gol22] for a recent survey.

3.1. Deroin–Tholozan representations

Let $\Gamma = \pi_n$ be the group

$$\pi_n := \langle c_1, \ldots, c_n : \prod_{i=1}^{n} c_i = 1 \rangle.$$

Recall from Example 2.1.11 that $\pi_n = \pi_{0,n}$ is isomorphic to the fundamental group of an oriented and connected surface Σ_n of genus 0 with $n \geqslant 3$ labelled punctures. We consider, for now, the case $G = \mathrm{PSL}(2,\mathbb{R})$. Let $\phi\colon \pi_n \to \mathrm{PSL}(2,\mathbb{R})$ be a representation and $k(\phi)$ be its relative Euler class, as defined in (2.5.6). Proposition 2.5.11 says

$$k(\phi) \leqslant \max\left\{ n-2,\, \frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) \right\}.$$

From the definition of the function $\overline{\vartheta}\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}$ provided in (2.5.5), it is immediate that $\frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) \leqslant n$ and $\frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) = n$ if and only if ϕ is the trivial representation. Moreover, if $\phi(c_i)$ is elliptic for every i, and if $\vartheta(\phi(c_1)) + \ldots + \vartheta(\phi(c_n)) > 2\pi(n-1)$, then Proposition 2.5.11 becomes

$$k(\phi) \leqslant n-1.$$

Definition 3.1.1 (Deroin–Tholozan representations)**.** A representation $\phi\colon \pi_n \to \mathrm{PSL}(2,\mathbb{R})$ for which $\phi(c_i)$ is elliptic for every i and such that $k(\phi) = n-1$ is said to be a *Deroin–Tholozan representation.*

It is proved in [DT19] that Deroin–Tholozan representations exist and that they form a compact connected component of the corresponding relative character variety, see Theorem 3.1.6 below. To state a precise result, we first need to introduce some notation.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0, 2\pi)^n$ be angles such that

$$\alpha_1 + \ldots + \alpha_n > 2\pi(n-1). \tag{3.1.1}$$

Each angle α_i determines a unique elliptic conjugacy class in $\mathrm{PSL}(2,\mathbb{R})$ that consists of all the elements $g \in \mathrm{PSL}(2,\mathbb{R})$ such that $\overline{\vartheta}(g) = \alpha_i$. We consider the relative representation variety (see Definition 2.4.5)

$$\mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R})) := \{\phi \colon \pi_n \to \mathrm{PSL}(2,\mathbb{R}) : \overline{\vartheta}(\phi(c_i)) = \alpha_i\}.$$

Lemma 3.1.2. *Assuming $\alpha_1 + \ldots + \alpha_n > 2\pi(n-1)$, the relative representation variety $\mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ is a smooth manifold of dimension $2n-3$.*

Proof. We prove that any $\phi \in \mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ is a smooth point of the relative representation variety. Since $\alpha_1 + \ldots + \alpha_n$ is not an integer multiple of 2π, the elliptic elements $\phi(c_1), \ldots, \phi(c_n)$ cannot have the same fixed point in the upper half-plane. So, by Example 2.2.5, we deduce that ϕ is regular (actually $Z(\phi)$ is trivial). In particular, it is a smooth point of $\mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ by Proposition 2.4.9. Moreover, since $\mathrm{PSL}(2,\mathbb{R})$ is 3-dimensional and any elliptic conjugacy class in $\mathrm{PSL}(2,\mathbb{R})$ is 2-dimensional, we use Proposition 2.4.9 again to deduce that $\dim \mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R})) = 2n-3$. □

Remark 3.1.3. Deroin–Tholozan representations were originally called *supra-maximal* because they maximize the relative Euler class. However, these representations do not have maximal volume and are thus not *maximal* in the sense of [BIW10]. They even tend to minimize the volume in absolute value. Indeed, by (2.5.6), if $\phi \in \mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ satisfies $k(\phi) = n-1$, then

$$\mathrm{vol}(\phi) = 2\pi(n-1) - \alpha_1 - \ldots - \alpha_n \in (-2\pi, 0).$$

The range of the volume over $\mathrm{Hom}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$, according to the Milnor–Wood inequality stated in Theorem 2.5.6, is $[-2\pi(n-2), 2\pi(n-2)]$. To avoid any further confusion we prefer the terminology of *Deroin–Tholozan representations* instead of that of supra-maximal representations.

Definition 3.1.4 (Scaling factor)**.** The real number

$$\lambda := \alpha_1 + \ldots + \alpha_n - 2\pi(n-1)$$

is called the *scaling factor*. Note that $\lambda < 2\pi$. The condition (3.1.1), or equivalently the condition $\lambda > 0$, is referred to as the *angles condition* on α.

Observe that $\phi \in \operatorname{Hom}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ satisfies $k(\phi) = n-1$ if and only if it satisfies $\operatorname{vol}(\phi) = -\lambda$ because of (2.5.6). The *Deroin–Tholozan relative representation variety* is defined to be the set of Deroin–Tholozan representations inside the relative character variety $\operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$:

$$\operatorname{Hom}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R})) := \operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R})) \cap \operatorname{vol}^{-1}(-\lambda).$$

We know from Theorem 2.5.6 that the volume is locally constant on relative representation varieties. This implies that $\operatorname{Hom}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ is a collection of connected components of $\operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ and, thus, also a smooth manifold.

Lemma 3.1.5. *The* $\operatorname{Inn}(\mathrm{PSL}(2,\mathbb{R}))$*-action on* $\operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ *is free and proper. In particular, the topological quotient*

$$\operatorname{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R})) := \operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))/\operatorname{Inn}(\mathrm{PSL}(2,\mathbb{R}))$$

is naturally a smooth symplectic manifold of dimension $2(n-3)$*.*

Proof. It was already explained in the proof of Lemma 3.1.2 that $Z(\phi)$ is trivial for any $\phi \in \operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$. This shows that the action is free on $\operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ by Lemma 2.2.1. Using the criterion of Lemma 2.2.11, for instance, we see that any $\phi \in \operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ is irreducible. So, by Theorem 2.2.15, the $\operatorname{Inn}(\mathrm{PSL}(2,\mathbb{R}))$-action on $\operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ is also proper. We conclude that the relative character variety $\operatorname{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ is a smooth manifold of dimension $2(n-3)$. We equip it with the Goldman symplectic form $\omega_{\mathcal{G}}$ built from the trace form:

$$\begin{aligned} \operatorname{Tr}\colon \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) &\to \mathbb{R} \\ (\xi_1, \xi_2) &\mapsto \operatorname{Tr}(\xi_1 \xi_2). \end{aligned}$$ ∎

The *Deroin–Tholozan character variety* is the submanifold of $\operatorname{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ obtained by restricting to Deroin–Tholozan representations

$$\operatorname{Rep}_\alpha^{\mathrm{DT}} := \operatorname{Rep}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R})) := \operatorname{Hom}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))/\operatorname{Inn}(\mathrm{PSL}(2,\mathbb{R})).$$

As usual, the conjugacy class of a representation $\phi \in \operatorname{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ is denoted by $[\phi] \in \operatorname{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$.

Theorem 3.1.6 ([DT19])**.** *The Deroin–Tholozan relative character variety is a nonempty and compact connected component of the relative character variety. It is moreover sym-*

plectomorphic to the complex projective space of complex dimension $n-3$*:*

$$\left(\mathrm{Rep}_\alpha^{\mathrm{DT}}, \omega_{\mathcal{G}}\right) \cong \left(\mathbb{CP}^{n-3}, \lambda \cdot \omega_{\mathcal{FS}}\right),$$

where $\omega_{\mathcal{FS}}$ *is the Fubini-Study symplectic form on* $\mathbb{CP}^{n-3}$ *with volume* $\pi^{n-3}/(n-3)!$.

Remark 3.1.7. These compact connected components were already discovered by Benedetto–Goldman in the case $n = 4$ [BG99].

Remark 3.1.8. The involution σ of Lemma 2.5.7, specified in Example 2.5.8 for $G = \mathrm{PSL}(2,\mathbb{R})$, maps $\mathrm{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ to $\mathrm{Rep}_{2\pi-\alpha}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$. It maps the connected component of Deroin–Tholozan representations in $\mathrm{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ to a compact connected component inside $\mathrm{Rep}_{2\pi-\alpha}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$. It consists of representations ϕ for which $\mathrm{vol}(\phi) = \lambda \in (0, 2\pi)$ and $k(\phi) = 1$. In the terminology of [DT19], these representations could be called *infra-minimal*.

Deroin–Tholozan representations have an important property called total ellipticity.

Definition 3.1.9 (Totally elliptic)**.** A representation $\phi\colon \pi_n \to \mathrm{PSL}(2,\mathbb{R})$ is called *totally elliptic* if it maps any simple closed curve on Σ_n to an elliptic element.

Total ellipticity for Deroin–Tholozan representations was originally proved in [DT19] for a particular collection of simple closed curves. The argument generalizes immediately to any simple closed curve.

Proposition 3.1.10. *Let* $a \in \pi_1(\Sigma_n) \cong \pi_n$ *denote the homotopy class of a simple closed curve on* Σ_n*. Then* $\phi(a) \in \mathrm{PSL}(2,\mathbb{R})$ *is elliptic for any* $\phi \in \mathrm{Hom}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$*.*

Proof. In a slight abuse of notation we denote by a both the homotopy class and the associated simple closed curve on Σ_n that represents the class a (which is unique up to free homotopy).

If a is homotopic to a puncture, then $\phi(a)$ is elliptic by definition of the relative character variety. Otherwise, a separates Σ_n into two surfaces $S_1 \sqcup_a S_2 = \Sigma_n$ of negative Euler characteristic. Let ϕ_1 and ϕ_2 denote the restrictions of ϕ to $\pi_1(S_1)$ and $\pi_1(S_2)$. The curve a also determines a partition of the set $\{1, \ldots, n\}$ into two subsets J_1 and J_2 of respective cardinality m_1 and m_2. Theorem 2.5.6 implies

$$\mathrm{vol}(\phi_i) = 2\pi k(\phi_i) - \sum_{j \in J_i} \alpha_j - \overline{\vartheta}(\phi_i(a)), \quad i = 1, 2.$$

Since ϕ is Deroin–Tholozan,

$$\mathrm{vol}(\phi) = 2\pi(n-1) - \sum_{i=1}^{n} \alpha_i.$$

By additivity of the volume (Theorem 2.5.6), $\mathrm{vol}(\phi) = \mathrm{vol}(\phi_1) + \mathrm{vol}(\phi_2)$ and thus

$$2\pi(n-1) + \overline{\vartheta}(\phi_1(a)) + \overline{\vartheta}(\phi_2(a)) = 2\pi\big(k(\phi_1) + k(\phi_2)\big). \tag{3.1.2}$$

Because of Proposition 2.5.11, it holds $k(\phi_i) \leqslant m_i$ for $i = 1, 2$. So, recalling that $m_1 + m_2 = n$, we deduce from (3.1.2) that

$$\overline{\vartheta}(\phi_1(a)) + \overline{\vartheta}(\phi_2(a)) \leqslant 2\pi.$$

By construction $\phi_1(a) = \phi_2(a)^{-1}$. Thus, the sum $\overline{\vartheta}(\phi_1(a)) + \overline{\vartheta}(\phi_2(a))$, being at most 2π, is either 0 or 2π.

Assume first that $\overline{\vartheta}(\phi_1(a)) + \overline{\vartheta}(\phi_2(a)) = 0$. Then both $\overline{\vartheta}(\phi_1(a))$ and $\overline{\vartheta}(\phi_2(a))$ vanish. With this extra information, our application of Proposition 2.5.11 to ϕ_i can be refined and now gives $k(\phi_i) \leqslant m_i - 1$ for $i = 1, 2$. This contradicts (3.1.2).

Assume now that $\overline{\vartheta}(\phi_1(a)) + \overline{\vartheta}(\phi_2(a)) = 2\pi$. Then (3.1.2), together with the inequalities $k(\phi_i) \leqslant m_i$ for $i = 1, 2$, imply that $k(\phi_i) = m_i$ for $i = 1, 2$. For Proposition 2.5.11 to hold for ϕ_1 and ϕ_2, one must necessarily have $\overline{\vartheta}(\phi_1(a)) > 0$ and $\overline{\vartheta}(\phi_2(a)) > 0$. Therefore, $\overline{\vartheta}(\phi_i(a)) \in (0, 2\pi)$ for $i = 1, 2$ and we conclude that $\phi(a)$ is elliptic. $\square$

Remark 3.1.11 (Totally elliptic versus discrete and faithful). There is an active domain of research called *Higher Teichmüller Theory* that studies discrete and faithful representations of finitely generated groups into Lie groups, see [Wie18] for an overview. Totally elliptic representations into $\mathrm{PSL}(2, \mathbb{R})$ are, in nature, the opposite of a discrete and faithful representation. Indeed, if the image of a representation into $\mathrm{PSL}(2, \mathbb{R})$ contains an elliptic element, then either the angle of rotation is rational and the representation is not faithful, or the angle of rotation is irrational and the representation is not discrete.

Question 3.1.12. Does the converse of Proposition 3.1.10 hold? Namely, if α satisfies the angles condition (3.1.1) and $\phi \in \mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2, \mathbb{R}))$ is totally elliptic, is ϕ necessarily a Deroin–Tholozan representation, i.e. does it hold $\mathrm{vol}(\phi) = -\lambda$?

If $n = 3$ or $n = 4$, then the answer to Question 3.1.12 is yes. This relies on the trichotomy for the case $n = 3$ provided in Lemma 4.1.2. If $n = 3$ and $\phi \in \mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2, \mathbb{R}))$, then one of the following holds:

- $\alpha_1 + \alpha_2 + \alpha_3 \in (0, 2\pi]$ and $k(\phi) = 1$, or
- $\alpha_1 + \alpha_2 + \alpha_3 \in [4\pi, 6\pi)$ and $k(\phi) = 2$.

In particular, if we assume $\alpha_1 + \alpha_2 + \alpha_3 > 4\pi$, then $k(\phi) = 2$ and hence ϕ is Deroin–Tholozan. If $n = 4$, consider the pants decomposition $\Sigma_4 = S_1 \sqcup_{b_1} S_2$, where b_1 is a simple closed curve in the free homotopy class of $c_2^{-1}c_1^{-1}$ (see Figure 5.2). Let $\phi \in \mathrm{Hom}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ and denote by ϕ_i the restriction of ϕ to $\pi_1(S_i)$. Because of the above dichotomy, it must hold $k(\phi_i) = 2$ for $i = 1, 2$, otherwise $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 6\pi$, contradicting the angles conditions. Hence ϕ is Deroin–Tholozan. The same argument does not apply if $n \geqslant 5$ and the question whether totally elliptic representations are Deroin–Tholozan remains open.

The Deroin–Tholozan relative character variety admits a natural maximal and effective Hamiltonian torus action[1]. Recall that a torus action on a symplectic manifold is called *maximal* if the dimension of the torus is half the dimension of the manifold and it is called *effective* if only the identity element acts trivially. The action is constructed following the work of Goldman in [Gol86] on invariant functions. By Proposition 3.1.10, any simple closed curve a on Σ_n gives a Hamiltonian function

$$\begin{aligned} \vartheta_a \colon \mathrm{Rep}_\alpha^{\mathrm{DT}} &\to (0, 2\pi) \\ [\phi] &\mapsto \vartheta(\phi(a)). \end{aligned}$$

The associated Hamiltonian flow Φ_a has period π for any curve a, see [DT19]. We refer to this flow as the *twist flow* along the curve a. Goldman proved in [Gol86] that two twist flows Φ_{a_1} and Φ_{a_2} commute if the curves a_1 and a_2 are disjoint. Recall that a maximal collection of disjoint and non-homotopic simple closed curves on Σ_n has cardinality $n - 3$. Each such collection of curves therefore defines a Hamiltonian action of the torus $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$ on $\mathrm{Rep}_\alpha^{\mathrm{DT}}(\Sigma_n, G)$ via the associated twist flows. Since $\mathrm{Rep}_\alpha^{\mathrm{DT}}(\Sigma_n, G)$ has dimension $2(n-3)$, this action is maximal and equips $\mathrm{Rep}_\alpha^{\mathrm{DT}}(\Sigma_n, G)$ with the structure of a symplectic toric manifold.

Theorem 3.1.6 is proved in [DT19] using Delzant's classification of symplectic toric manifolds, see e.g. [CdS01] for a neat presentation of Delzant's classification. To any symplectic toric manifold you can associate a polytope called the *moment polytope*. Delzant's classification says that the moment polytopes of two symplectic toric manifolds agree if and only if the two symplectic toric manifolds are isomorphic. Here isomorphism means an equivariant symplectomorphism. It was observed in [DT19] that the moment polytope for

[1]The reader can find in [CdS01] the definition of all relevant concepts from symplectic geometry needed for this work.

$(\mathrm{Rep}_\alpha^{\mathrm{DT}}, \omega_{\mathcal{G}})$ and the torus action above is the standard simplex in $\mathbb{R}^{n-3}$, which coincides with the moment polytope of $(\mathbb{CP}^{n-3}, \lambda \cdot \omega_{\mathcal{FS}})$ for the standard torus action.

An interesting open question is whether there exist other kind of compact connected components inside $\mathrm{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ in addition to the Deroin–Tholozan one.

Question 3.1.13. Is the Deroin–Tholozan relative character variety the only compact connected component of $\mathrm{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$?

The numeric simulations conducted in [BG99] seem to indicate a positive answer, at least in the case of a 4-punctured sphere, see Figure 3.1. One must also mention the results of [Mon16] on the topology of relative character varieties of representations of general surface groups into $\mathrm{PSL}(2,\mathbb{R})$, and especially the characterization of compactness given in [Mon16, Cor. 4.17].

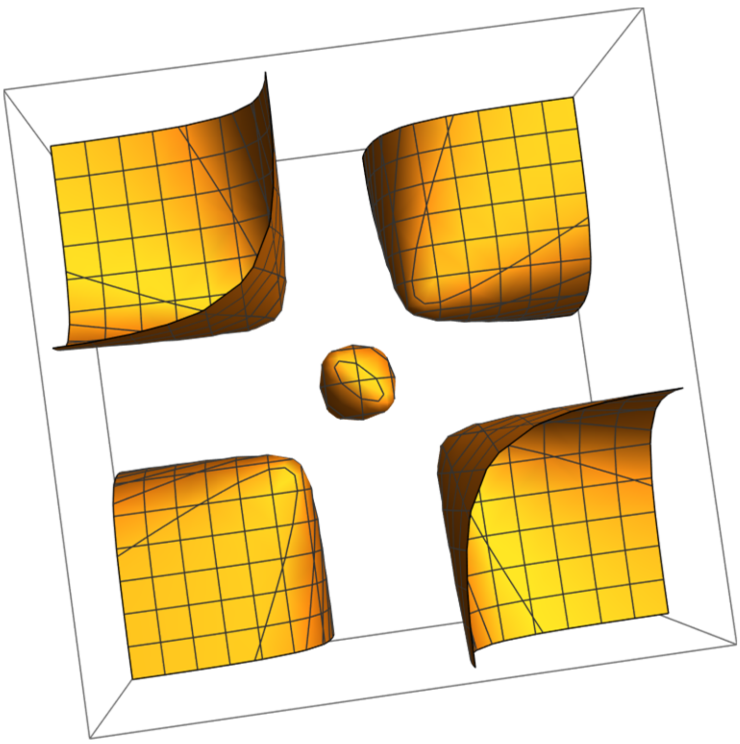


Figure 3.1.: Illustration of $\mathrm{Rep}_\alpha(\pi_4, \mathrm{PSL}(2,\mathbb{R}))$ for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 7\pi/4$. The relative character variety is the algebraic variety inside $\mathbb{R}^3$ cut out by the equation $x^2+y^2+z^2-xyz-s=0$, where $s=4-4a^2+a^4$ and $a=2\cos(7\pi/8)$, see [Gol21, §3]. The figure indicates five connected components among which one is compact and corresponds to the Deroin–Tholozan relative character variety. The image was produced with Wolfram Mathematica.

3.2. Generalization to Hermitian Lie groups

The treatment of the topology of relative character varieties of representations into $\mathrm{PSL}(2,\mathbb{R})$ for punctured surfaces provided in [Mon16] inspired the generalization of the results of [DT19] in [TT21]. The construction uses the non-abelian Hodge correspondence that relates relative character varieties to moduli spaces of parabolic Higgs bundles. These methods were successfully used in [TT21] to identify compact components of relative character varieties of representations into Hermitian Lie groups. We explain now the main results of [TT21].

Let again $\Gamma = \pi_n$ and consider the case $G = \mathrm{SU}(p,q)$. This is a generalization of the previous case in the sense that $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$. An $\mathrm{SU}(p,q)$-*multiweight* is an ordered collection of real numbers $(\alpha,\beta) \in (\mathbb{R}^p)^n \times (\mathbb{R}^q)^n$ such that for all $i = 1,\dots,n$

$$0 \leqslant \alpha_1^i \leqslant \ldots \leqslant \alpha_p^i < 4\pi, \quad 0 \leqslant \beta_1^i \leqslant \ldots \leqslant \beta_q^i < 4\pi, \quad \sum_{j=1}^{p} \alpha_j^i + \sum_{j=1}^{q} \beta_j^i \in 4\pi\mathbb{Z}.$$

The set of all $\mathrm{SU}(p,q)$-multiweights is denoted

$$W(n,p,q) \subset (\mathbb{R}^p)^n \times (\mathbb{R}^q)^n.$$

For any $(\alpha,\beta) \in W(n,p,q)$, we denote by $\mathrm{Hom}_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$ the relative representation variety of all representations $\phi\colon \pi_n \to \mathrm{SU}(p,q)$ such that $\phi(c_i)$ is conjugate to the diagonal matrix in $\mathrm{SU}(p,q)$ with entries $e^{i\alpha_1^i},\dots,e^{i\alpha_p^i},e^{i\beta_1^i},\dots,e^{i\beta_q^i}$. We denote by

$$\mathrm{Rep}_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$$

the corresponding Hausdorff relative character variety, see Definition 2.3.4. For $(\alpha,\beta) \in (\mathbb{R}^p)^n \times (\mathbb{R}^q)^n$, we write $\|\alpha\| := \sum_{i=1}^n \sum_{j=1}^p \alpha_j^i$ and $\|\beta\| := \sum_{i=1}^n \sum_{j=1}^q \beta_j^i$. We also introduce the quantity

$$\varepsilon(\alpha,\beta) := \sum_{i=1}^{n} \beta_q^i - \alpha_1^i.$$

We say that $(\alpha,\beta) \in W(n,p,q)$ satisfies the *compactness criterion* if

$$\alpha_p^i < \beta_1^i,\ \forall i = 1,\dots,n \quad \text{and} \quad 0 < \varepsilon(\alpha,\beta) < 8\pi. \tag{3.2.1}$$

In that, case we write $J_{(\alpha,\beta)}$ for the open interval of $\mathbb{R}$ defined by

$$J_{(\alpha,\beta)} := \big(\|\beta\| - \|\alpha\|, \|\beta\| - \|\alpha\| + 8\pi - \varepsilon(\alpha,\beta)\big).$$

Proposition 3.2.1 ([TT21]). *If $(\alpha, \beta) \in W(n,p,q)$ satisfies the compactness criterion (3.2.1) and $d \in J_{(\alpha,\beta)} \cap 4\pi\mathbb{Z}$, then the subspace of $\mathrm{Rep}_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$ consisting of all $[\phi]$ with*

$$\mathrm{vol}(\phi) = \frac{1}{2}(\|\beta\| - \|\alpha\| - d)$$

is compact.

We denote by

$$\mathrm{Rep}^d_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$$

the compact subspace of $\mathrm{Rep}_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$ defined by $\mathrm{vol}(\phi) = \frac{1}{2}(\|\beta\| - \|\alpha\| - d)$. We emphasize that $\mathrm{Rep}^d_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$ might well be empty at this stage. Theorem 2.5.6 implies that $\mathrm{Rep}^d_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$ is a union of connected components of $\mathrm{Rep}_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$. Note that for any $[\phi] \in \mathrm{Rep}^d_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(p,q))$ it holds that

$$\mathrm{vol}(\phi) \in (-4\pi, 0),$$

compare Remark 3.1.3.

An $\mathrm{SU}(p,q)$-mutliweight $(\alpha, \beta) \in W(n,p,q)$ is said to be a *constant* $\mathrm{SU}(p,q)$*-multiweight* if $\alpha^i := \alpha^i_1 = \ldots = \alpha^i_p$ and $\beta^i := \beta^i_1 = \ldots = \beta^i_q$ for every $i = 1, \ldots, n$. In particular, it holds that $p\alpha^i + q\beta^i \in 4\pi\mathbb{Z}$ for every i. It is proved in [TT21] that there exists a constant $\mathrm{SU}(p,q)$-multiweight (α, β) such that

$$\alpha^i < \beta^i, \ \forall i = 1, \ldots, n \quad \text{and} \quad 0 < \varepsilon(\alpha, \beta) < 4\pi. \tag{3.2.2}$$

The condition (3.2.2) is called the *nonemptiness criterion*. A constant $\mathrm{SU}(p,q)$-mutliweight that satisfies the nonemptiness criterion also satisfies the compactness criterion (3.2.1), but the converse is not true.

Theorem 3.2.2 ([TT21]). *Assume that $n > 2 + p/q + q/p$ and that $(\alpha, \beta) \in W(n,p,q)$ is a constant $\mathrm{SU}(p,q)$-multiweight that satisfies the nonemptiness criterion (3.2.2). There exists an open neighbourhood $W(\alpha, \beta)$ of (α, β) inside $W(n,p,q)$, such that for every $(\alpha', \beta') \in W(\alpha, \beta)$ and $d' \in J_{(\alpha',\beta')} \cap 4\pi\mathbb{Z}$,*

$$\mathrm{Rep}^{d'}_{(\alpha',\beta')}(\pi_n, \mathrm{SU}(p,q))$$

is compact and nonempty.

It is explained in [TT21, §6.2] that an analogue of Theorem 3.2.2 is true if $\mathrm{SU}(p,q)$ is replaced by $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{SO}^*(2n)$. We point out that the representations inside these components satisfy, like any Deroin–Tholozan representation, a property of total ellipticity. Namely, the image of any simple closed curve on Σ_n has only complex eigenvalues of modulus 1, see [TT21, Thm. 2].

Example 3.2.3. We explain how Theorem 3.2.2 generalizes the notion of Deroin–Tholozan relative character variety. Assume that $p = q = 1$. Any $\mathrm{SU}(1,1)$-multiweight (α, β) is constant. If (α, β) satisfies the nonemptiness criterion (3.2.2), then it holds $\beta^i = 4\pi - \alpha^i$. This is because $\alpha^i + \beta^i \in 4\pi\mathbb{Z}$ and, by assumption, $0 \leqslant \alpha^i < \beta^i < 4\pi$. In particular, since $\alpha^i < \beta^i$, it holds that $\alpha^i \in (0, 2\pi)$. It is interesting to observe that

$$\varepsilon(\alpha, \beta) < 4\pi \quad \Leftrightarrow \quad \alpha^1 + \ldots + \alpha^n > 2\pi(n-1).$$

We recognize here the angles condition (3.1.1) that is part of the hypotheses of Theorem 3.1.6. Moreover, $J_{(\alpha,\beta)} = (\varepsilon(\alpha, \beta), 8\pi)$ and so $J_{(\alpha,\beta)} \cap 4\pi\mathbb{Z} = \{4\pi\}$. We conclude that the compact relative character variety $\mathrm{Rep}^{4\pi}_{(\alpha,\beta)}(\pi_n, \mathrm{SU}(1,1))$ is mapped to $\mathrm{Rep}^{\mathrm{DT}}_{\alpha}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ under the projection induced by the quotient map $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$.

4. Action-angle coordinates for Deroin–Tholozan representations

4.1. A polygonal model

The coordinates for the Deroin–Tholozan relative character variety we are about to construct depend on the choice of a pants decomposition of Σ_n. Each choice of pants decomposition of Σ_n leads to action-angle coordinates by the same construction.

We fix a maximal collection of disjoint and non-homotopic simple closed curves $b_1, \ldots, b_{n-3}$ on Σ_n. It is convenient to work with the curves

$$b_i := c_{i+1}^{-1} c_i^{-1} \cdot \ldots \cdot c_1^{-1} \in \pi_n$$

for $i = 1, \ldots, n-3$, where the curves c_i refer to the presentation of π_n fixed in (2.1.3). The curves b_i are illustrated on Figure 4.1. We set $b_0 := c_1^{-1}$ and $b_{n-2} := c_n$ for convenience. Below, we fix a maximal Hamiltonian torus action on the Deroin–Tholozan relative character variety $\mathrm{Rep}_\alpha^{\mathrm{DT}} = \mathrm{Rep}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ using a combination of the twist flows along the disjoint curves $b_1, \ldots, b_{n-3}$, see Section 4.1.3. To describe angle coordinates for this torus action, we introduce a polygonal model for Deroin–Tholozan representations.

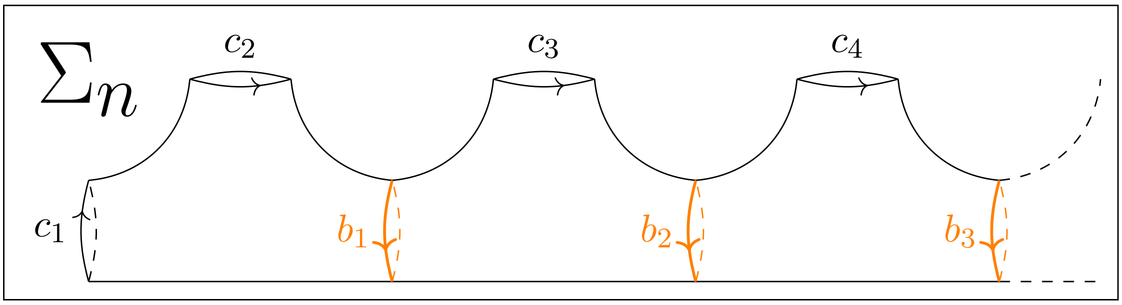


Figure 4.1.: The simple closed curves $b_1, \ldots, b_{n-3}$ and the peripheral curves $c_1, \ldots, c_n$. This illustration is modelled on [DT19, Fig. 2].

Let $[\phi]$ denote the conjugacy class of a Deroin–Tholozan representation $\phi\colon \pi_n \to \mathrm{PSL}(2,\mathbb{R})$. By definition of the Deroin–Tholozan relative character variety, $\phi(c_i)$ is elliptic and satisfies

$\vartheta(\phi(c_i)) = \alpha_i$ for every $i = 1, \ldots, n$. Let

$$C_1(\phi), \ldots, C_n(\phi) \in \mathbb{H}$$

be the fixed points of $\phi(c_1), \ldots, \phi(c_n)$, respectively. Proposition 3.1.10 says that $\phi(b_i)$ is elliptic for all $i = 1, \ldots, n-3$. Let

$$B_1(\phi), \ldots, B_{n-3}(\phi) \in \mathbb{H}$$

be the fixed points of $\phi(b_1), \ldots, \phi(b_{n-3})$, respectively. We emphasize that those fixed points are associated to the representation ϕ and not to its conjugacy class $[\phi]$. A different representative of the class $[\phi]$ leads to a different set of fixed points. However, for $A \in \mathrm{PSL}(2,\mathbb{R})$, it holds that $C_i(A\phi A^{-1}) = A \cdot C_i(\phi)$ and $B_i(A\phi A^{-1}) = A \cdot B_i(\phi)$. This observation motivates the following. Let $\mathbb{H}^n = \mathbb{H} \times \ldots \times \mathbb{H}$. We introduce the topological quotient $(\mathbb{H}^n \times \mathbb{H}^{n-3}) / \mathrm{PSL}(2,\mathbb{R})$ where $\mathrm{PSL}(2,\mathbb{R})$ acts diagonally on $\mathbb{H}^n \times \mathbb{H}^{n-3}$. We refer to it as *the moduli space of point configurations* in $\mathbb{H}$. It allows for the definition of a map

$$\mathfrak{P}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \longrightarrow (\mathbb{H}^n \times \mathbb{H}^{n-3}) / \mathrm{PSL}(2,\mathbb{R})$$

that sends $[\phi]$ to the equivalence class of the points $(C_1(\phi), \ldots, C_n(\phi), B_1(\phi), \ldots, B_{n-3}(\phi))$ in the moduli space of point configurations. The map $\mathfrak{P}$ is injective because a Deroin–Tholozan representation ϕ is entirely determined by the fixed points of $\phi(c_1), \ldots, \phi(c_n)$ (recall that the angles of rotation $\alpha_1, \ldots, \alpha_n$ are fixed parameters). Let

$$\mathrm{ChTri}_\alpha \subset (\mathbb{H}^n \times \mathbb{H}^{n-3}) / \mathrm{PSL}(2,\mathbb{R})$$

denote the image of the map $\mathfrak{P}$. The inverse map

$$\mathfrak{P}^{-1}\colon \mathrm{ChTri}_\alpha \longrightarrow \mathrm{Rep}_\alpha^{\mathrm{DT}}$$

maps an equivalence class of points $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$ to the conjugacy class of the representation $\phi\colon \pi_n \to \mathrm{PSL}(2,\mathbb{R})$ that sends each generator c_i of π_n to the rotation of angle α_i around C_i.

The notation ChTri_α for the image of $\mathfrak{P}$ is an abbreviation of *chain of triangles* and is motivated by the following construction. Let $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$ be a configuration of points in $\mathbb{H}^n \times \mathbb{H}^{n-3}$ whose isometry class lies in ChTri_α. For convenience, we let $B_0 := C_1$ and $B_{n-2} := C_n$. For every $i = 0, \ldots, n-3$, we consider the oriented geodesic triangle

$$\Delta_i := \Delta(B_i, C_{i+2}, B_{i+1})$$

in the upper half-plane, see Figure 4.2. The triangles Δ_i and Δ_{i+1} share the common vertex B_i. The geometric quantities associated to the triangles Δ_i, such as their area or interior angles, are invariant of the isometry class of $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$. We refer to $(\Delta_0, \ldots, \Delta_{n-3})$ as a *chain of triangles*. Chain of triangles constitute the polygonal model for the Deroin–Tholozan relative character variety.

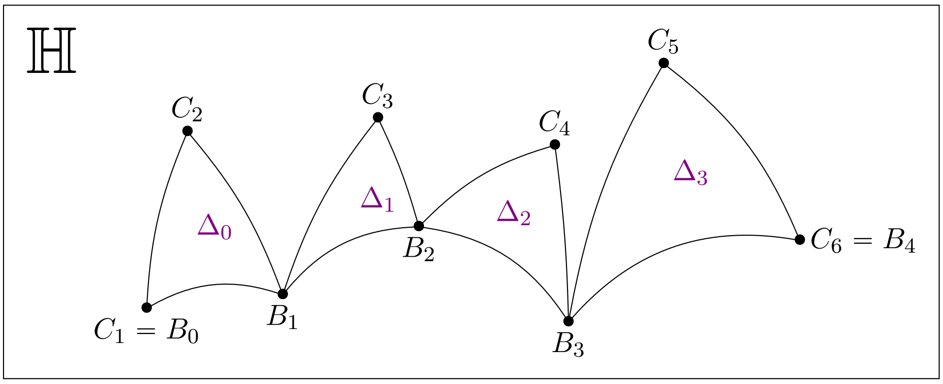


Figure 4.2.: Example of a configuration of the fixed points and the associated chain of triangles in the case $n = 6$.

We advertise two results to convince the reader about the pertinence of the polygonal model for $\mathrm{Rep}_\alpha^{\mathrm{DT}}$. The first concerns angle coordinates which can be read directly from the chain of triangles. We prove below in Section 4.3 that the angles between the geodesic rays $\overrightarrow{B_iC_{i+2}}$ and $\overrightarrow{B_iC_{i+1}}$ are angle coordinates for the Hamiltonian torus action on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$, see Figure 4.6.

The second example concerns the action coordinates which also appear as geometric quantities in the chain of triangles. For $i = 1, \ldots, n-3$, we write

$$\beta_i(\phi) := \vartheta_{b_i}(\phi) = \vartheta(\phi(b_i)) \tag{4.1.1}$$

for the angle of rotation of the elliptic element $\phi(b_i) \in G$. Let further, in accordance to our previous conventions, $\beta_0(\phi) := 2\pi - \alpha_1$ and $\beta_{n-2}(\phi) := \alpha_n$. The functions β_i are the components of the moment map for the torus action defined by the twist flows along the curves b_i. We prove the following below in Subsection 4.1.2, see Figure 4.4.

Lemma 4.1.1. *Let Δ_i be a non-degenerate triangle in the chain built from $\mathfrak{P}([\phi])$ for some $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. The following holds: The triangle Δ_i is clockwise oriented and the interior angle of Δ_i at B_i equals $\beta_i(\phi)/2$, the interior angle at C_{i+2} equals $\pi - \alpha_{i+2}/2$ and the interior angle at B_{i+1} equals $\pi - \beta_{i+1}(\phi)/2$.*

The remainder of this section is dedicated to the study of the possible configurations of points inside ChTri_α. We want to find sufficient geometrical conditions for a chain of triangles to be a configuration of fixed points associated to a Deroin–Tholozan representation. We start with the case $n = 3$ and then explain how the cases $n \geqslant 4$ are built from the case $n = 3$.

4.1.1. The case of the thrice-punctured sphere

Assume that $n = 3$ and let Σ_3 be an oriented and connected sphere with three labelled punctures. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (0, 2\pi)^3$ be a triple of angles. At this stage, we make no particular assumption concerning a lower bound for $\alpha_1 + \alpha_2 + \alpha_3$. Let $[\phi] \in \mathrm{Rep}_\alpha(\pi_3, \mathrm{PSL}(2, \mathbb{R}))$. The following lemma describes the possible configurations of the fixed points C_1, C_2, C_3 of $\phi(c_1)$, $\phi(c_2)$, $\phi(c_3)$. The lemma is transcribed from [DT19] and the proof is included for completeness.

Lemma 4.1.2 ([DT19])**.** *The points C_1, C_2, $C_3 \in \mathbb{H}$ are arranged in one of the following three configurations:*

1. *All three points coincide and $\alpha_1 + \alpha_2 + \alpha_3 \in \{2\pi, 4\pi\}$.*

2. *The points form a non-degenerate triangle $\Delta(C_1, C_2, C_3)$ which is oriented clockwise and has interior angles $\pi - \alpha_i/2$ at C_i for $i = 1, 2, 3$. Moreover, $\alpha_1 + \alpha_2 + \alpha_3 > 4\pi$.*

3. *The points form a non-degenerate triangle $\Delta(C_1, C_2, C_3)$ which is oriented anti-clockwise and has interior angles $\alpha_i/2$ at C_i for $i = 1, 2, 3$. Moreover, $\alpha_1 + \alpha_2 + \alpha_3 < 2\pi$.*

Proof. Assume that $C_i = C_j$ for some $i \neq j$. Let $k \in \{1, 2, 3\}$ be the third index. Up to permutation of i and j, it holds that $\phi(c_k) = \phi(c_i)^{-1}\phi(c_j)^{-1}$ because $c_1c_2c_3 = 1$ by assumption. So, $\phi(c_k)$ fixes both C_k and $C_i = C_j$. Therefore, all three points must coincide because $\phi(c_k)$ is elliptic. It means that $\phi(c_1), \phi(c_2)$ and $\phi(c_3)$ are rotations about the same point. Since their product is the identity, $\alpha_1 + \alpha_2 + \alpha_3$ is an integer multiple of 2π.

Assume now that C_1, C_2 and C_3 are distinct. Let ζ_3 be the geodesic through C_1 and C_2. Let ζ_2 be the image of ζ_3 by a clockwise rotation of $\pi - \alpha_1/2$ around C_1. Let ζ_1 be the image of ζ_3 by an anti-clockwise rotation of $\pi - \alpha_2/2$ around C_2, see Figure 4.3. We denote by $\tau_i \colon \mathbb{H} \to \mathbb{H}$ the reflection through the geodesic ζ_i. By construction, $\phi(c_1) = \tau_2\tau_3$ and $\phi(c_2) = \tau_3\tau_1$. Hence, $\phi(c_3) = \phi(c_1)^{-1}\phi(c_2)^{-1} = \tau_1\tau_2$. Since $\phi(c_3)$ fixes C_3, the geodesics ζ_1 and ζ_2 must intersect at C_3.

We distinguish two cases according to the orientation of $\Delta(C_1, C_2, C_3)$.

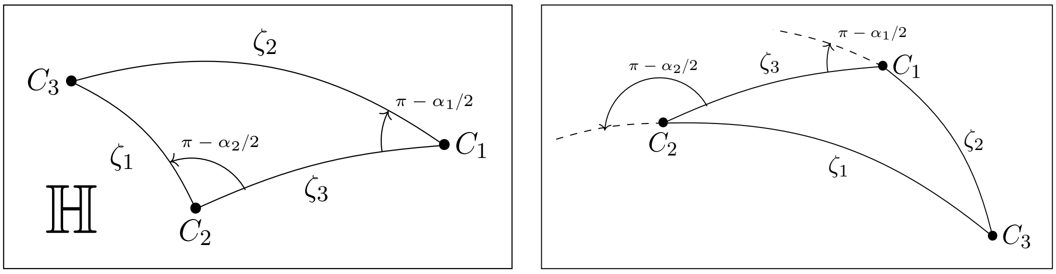


Figure 4.3.: The two non-degenerate configurations of fixed points. Above: the configuration where $\Delta(C_1, C_2, C_3)$ is clockwise oriented and the interior angles are $\pi - \alpha_i/2$. Below: the configuration where $\Delta(C_1, C_2, C_3)$ is anti-clockwise oriented and the interior angles are $\alpha_i/2$.

- First, assume that the triangle is clockwise oriented. It that case, $\tau_2\tau_3$ is a clockwise rotation around C_1 of twice the interior angle at C_1. Since $\phi(c_1)$ is by definition an anti-clockwise rotation of angle α_1 around C_1 and $\phi(c_1) = \tau_2\tau_3$, the interior angle at C_1 must be $\pi - \alpha_1/2$. For the same reason, the interior angles at C_2 and C_3 are $\pi - \alpha_2/2$ and $\pi - \alpha_3/2$, respectively. The positive area of the triangle $\Delta(C_1, C_2, C_3)$ is equal to the angle defect:

$$\pi - \sum_{i=1}^{3}(\pi - \alpha_i/2) = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 - 4\pi).$$

  We conclude that $\alpha_1 + \alpha_2 + \alpha_3 > 4\pi$.

- Conversely, if the triangle is anti-clockwise oriented, then the same argument shows that the interior angle at C_i is $\alpha_i/2$. In this case, the positive area of the triangle $\Delta(C_1, C_2, C_3)$ is equal to

$$\pi - \sum_{i=1}^{3}\alpha_i/2 = \frac{1}{2}(2\pi - \alpha_1 - \alpha_2 - \alpha_3).$$

  We conclude that $\alpha_1 + \alpha_2 + \alpha_3 < 2\pi$. □

A consequence of Lemma 4.1.2 is that $\mathrm{Rep}_\alpha(\pi_3, \mathrm{PSL}(2,\mathbb{R}))$ is empty whenever $\alpha_1+\alpha_2+\alpha_3 \in (2\pi, 4\pi)$. The next lemma shows that the volume of $[\phi]$ is directly proportional to the signed area of the triangle $\Delta(C_1, C_2, C_3)$.

Lemma 4.1.3. *Let* $[\phi] \in \mathrm{Rep}_\alpha(\pi_3, \mathrm{PSL}(2,\mathbb{R}))$. *Then*

$$\mathrm{vol}(\phi) = -2 \cdot [\Delta(C_1, C_2, C_3)].$$

Proof. The proof is an explicit computation of $\mathrm{vol}(\phi)$ using Definition 2.5.4. The computations are conducted in the bar resolution for group cohomology and use the explicit form of the fundamental class $[\pi_3]$ computed in Lemma 2.4.16.

Let z be a base point in $\mathbb{H}$. We start by computing the preimage of the cocycle $\phi^*\kappa \in H_b^2(\pi_3;\mathbb{R})$ under the isomorphism $j\colon H_b^2(\pi_3,\partial\pi_3;\mathbb{R}) \to H_b^2(\pi_3;\mathbb{R})$, see (2.5.4). This means finding primitives for $\phi^*c\colon \pi_3 \times \pi_3 \to \mathbb{R}$ restricted to the subgroup $\langle c_i\rangle$ of π_3, where c is the cocycle defined in (2.5.1). For $i = 1, 2, 3$, consider the functions $k_i\colon \langle c_i\rangle \to \mathbb{R}$ defined by

$$k_i(c_i) := [\Delta(C_i, z, \phi(c_i)z)].$$

We claim that the functions k_i are the desired primitives. By definiton of the bar complex, k_i is a primitive for ϕ^*c restricted to $\langle c_i\rangle$ if for any two integers a and b, it holds that $k_i(c_i^a) + k_i(c_i^b) - k_i(c_i^{a+b}) = c(\phi(c_i)^a, \phi(c_i)^b)$. We compute $k_i(c_i^a) + k_i(c_i^b) - k_i(c_i^{a+b})$. This is, by definition of k_i, equal to

$$[\Delta(C_i, z, \phi(c_i)^a z)] + [\Delta(C_i, z, \phi(c_i)^b z)] - [\Delta(C_i, z, \phi(c_i)^{a+b} z)].$$

Since $\phi(c_i)^a$ is an orientation-preserving isometry of the upper half-plane that fixes C_i, it holds that

$$[\Delta(C_i, z, \phi(c_i)^b z)] = [\Delta(C_i, \phi(c_i)^a z, \phi(c_i)^{a+b} z)].$$

Recall from (2.5.3) that for any A, B, C, D in $\mathbb{H}$, it holds that

$$[\Delta(A, B, C)] + [\Delta(C, D, A)] = [\Delta(B, C, D)] + [\Delta(B, D, A)]. \tag{4.1.2}$$

Thus, with $A = C_i$, $B = z$, $C = \phi(c_i)^a z$ and $D = \phi(c_i)^{a+b} z$, we deduce

$$\begin{aligned} k_i(c_i^a) + k_i(c_i^b) - k_i(c_i^{a+b}) &= [\Delta(z, \phi(c_i)^a z, \phi(c_i)^{a+b} z)] \\ &= c(\phi(c_i)^a, \phi(c_i)^b). \end{aligned}$$

This proves the claim. Hence

$$j^{-1}(\phi^*\kappa) = [(\phi^*c, k_1, k_2, k_3)].$$

Definition 2.5.4 says that

$$\mathrm{vol}(\phi) = [(\phi^*c, k_1, k_2, k_3)] \frown [\pi_3].$$

The fundamental class $[\pi_3]$ is the homology class of the 2-chain (e, c_1, c_2, c_3) where e is given by (2.4.3). Using the explicit expression of the cap product in the bar complex provided by Lemma B.15, we obtain

$$
\begin{aligned}
\operatorname{vol}([\phi]) &= (\phi^*c)(e) - k_1(c_1) - k_2(c_2) - k_3(c_3) \\
&= [\Delta(z, \phi(c_1)z, \phi(c_1c_2)z)] + [\Delta(z, \phi(c_1c_2)z, \phi(c_1c_2c_3)z)] + [\Delta(z, z, z)] \\
&\quad - [\Delta(C_1, z, \phi(c_1)z)] - [\Delta(C_2, z, \phi(c_2)z)] - [\Delta(C_3, z, \phi(c_3)z)] \\
&= [\Delta(z, \phi(c_1)z, \phi(c_1c_2)z)] - \sum_{i=1}^{3}[\Delta(C_i, z, \phi(c_i)z)]. \qquad (4.1.3)
\end{aligned}
$$

The volume is independent of the choice of the base point z, so we may as well assume $z = C_1$. After obvious cancellations, (4.1.3) becomes

$$\operatorname{vol}(\phi) = [\Delta(C_1, C_2, \phi(c_2)C_1)] + [\Delta(C_1, C_3, \phi(c_3)C_1)].$$

Using $\phi(c_3)C_1 = \phi(c_2)^{-1}C_1$, we further compute

$$\operatorname{vol}([\phi]) = [\Delta(C_1, C_2, \phi(c_2)C_1)] + [\Delta(\phi(c_2)C_1, \phi(c_2)C_3, C_1)]. \qquad (4.1.4)$$

We make use of (4.1.2) again. Letting $A = C_1$, $B = C_2$, $C = \phi(c_2)C_1$ and $D = \phi(c_2)C_3$, the relation (4.1.4) becomes

$$
\begin{aligned}
\operatorname{vol}(\phi) &= [\Delta(C_2, \phi(c_2)C_1, \phi(c_2)C_3)] + [\Delta(C_2, \phi(c_2)C_3, C_1)] \\
&= -[\Delta(C_1, C_2, C_3)] + [\Delta(C_1, C_2, \phi(c_2)C_3)]. \qquad (4.1.5)
\end{aligned}
$$

If $C_1 = C_2 = C_3$ then $\operatorname{vol}(\phi) = 0$ by (4.1.5), and so $\operatorname{vol}(\phi) = -2[\Delta(C_1, C_2, C_3)]$ as desired. Otherwise, we know from the proof of Lemma 4.1.2 that all three points are distinct and $\phi(c_2) = \tau_3\tau_1$. Observe that the triangle $\Delta(C_1, C_2, \phi(c_2)C_3)$ is the image under τ_3 of the triangle $\Delta(C_1, C_2, C_3)$ because τ_3 fixes C_1 and C_2 and τ_1 fixes C_3. Hence, for τ_3 is orientation-reversing,

$$[\Delta(C_1, C_2, \phi(c_2)C_3)] = -[\Delta(C_1, C_2, C_3)].$$

and (4.1.5) becomes $\operatorname{vol}(\phi) = -2[\Delta(C_1, C_2, C_3)]$. This finishes the proof of the lemma. □

We can compile the conclusions of Lemma 4.1.2 and Lemma 4.1.3 into the following summary table, see Table 4.1.

So far, we discussed the properties of the elements of $\operatorname{Rep}_\alpha(\pi_3, \mathrm{PSL}(2,\mathbb{R}))$. Now, we address the question of existence and uniqueness of such elements. If $\alpha_1 + \alpha_2 + \alpha_3 > 4\pi$, then there exists a unique clockwise oriented triangle Δ_α in $\mathbb{H}$, up to orientation-preserving isometries, with interior angles $\pi - \alpha_i/2$. The composition of the reflections through the

angles	volume	relative Euler class	configuration of $\Delta(C_1, C_2, C_3)$
$\sum \alpha_i \in \{2\pi, 4\pi\}$	0	$k = 1$ if $\sum \alpha_i = 2\pi$, $k = 2$ if $\sum \alpha_i = 4\pi$	$C_1 = C_2 = C_3$
$\sum \alpha_i > 4\pi$	$4\pi - \sum \alpha_i$	$k = 2$	clockwise oriented, interior angles $\pi - \alpha_i/2$
$\sum \alpha_i < 2\pi$	$2\pi - \sum \alpha_i$	$k = 1$	anti-clockwise oriented, interior angles $\alpha_i/2$

Table 4.1.: Summary of the different configurations of fixed points in the case $n = 3$.

sides of Δ_α, as in the proof of Lemma 4.1.2, defines an element of $\mathrm{Rep}_\alpha(\pi_3, \mathrm{PSL}(2, \mathbb{R}))$. This element is unique because Δ_α is unique up to isometry. If $\alpha_1 + \alpha_2 + \alpha_3 = 4\pi$, then Δ_α is degenerate to a point. The rotations of angle α_i around that point define an element of $\mathrm{Rep}_\alpha(\pi_3, \mathrm{PSL}(2, \mathbb{R}))$. This element is unique because G acts transitively on the upper half-plane. The case $\alpha_1 + \alpha_2 + \alpha_3 \leqslant 2\pi$ is similar. In conclusion, we obtain

Lemma 4.1.4. *If $\alpha_1 + \alpha_2 + \alpha_3 \in (0, 2\pi] \cup [4\pi, 6\pi)$, then $\mathrm{Rep}_\alpha(\pi_3, \mathrm{PSL}(2, \mathbb{R}))$ is a singleton and ChTri_α consists of only the isometry class of Δ_α. If $\alpha_1 + \alpha_2 + \alpha_3 \in (2\pi, 4\pi)$, then $\mathrm{Rep}_\alpha(\pi_3, \mathrm{PSL}(2, \mathbb{R}))$ and ChTri_α are empty.*

4.1.2. The general case

Let us first prove that the chain of triangles associated to a Deroin–Tholozan representation has the geometric properties stated in Lemma 4.1.1. The curves $b_1, \ldots, b_{n-3}$ illustrated in Figure 4.1 define a pants decomposition of Σ_n into $n - 2$ pair of pants $P_0, \ldots, P_{n-3}$. The pair of pants P_i has boundary curves b_i^{-1}, c_{i+2} and b_{i+1} (with the convention that $b_0 = c_1^{-1}$ and $b_{n-2} = c_n$). Let $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. The conjugacy class $[\phi\restriction_{P_i}]$ of the restriction of ϕ to P_i lies in the relative character variety $\mathrm{Rep}_{\varpi_i}(P_i, G)$ where ϖ_i is the vector of angles $(2\pi - \beta_i(\phi), \alpha_{i+2}, \beta_{i+1}(\phi))$. Indeed, the functions β_i, introduced in (4.1.1), measure the angle of rotation of the evaluation on the curve b_i. Deroin–Tholozan observed in [DT19] that the relative Euler classes of all the $\phi\restriction_{P_i}$ are automatically maximal. The argument is simple. Since the volume of a representation is additive, it holds that $\mathrm{vol}(\phi) = \mathrm{vol}(\phi\restriction_{P_0}) + \ldots + \mathrm{vol}(\phi\restriction_{P_{n-3}})$ or equivalently

$$
\begin{aligned}
2\pi(n-1) - \sum_{i=1}^{n} \alpha_i &= \sum_{i=0}^{n-3} \left(2\pi k(\phi\restriction_{P_i}) - (2\pi - \beta_i(\phi) + \alpha_{i+2} + \beta_{i+1}(\phi))\right) \qquad (4.1.6)\\
&= 2\pi \sum_{i=0}^{n-3} k(\phi\restriction_{P_i}) - 2\pi(n-3) - \sum_{i=1}^{n} \alpha_i.
\end{aligned}
$$

So, we conclude $k(\phi\restriction_{P_0}) + \ldots + k(\phi\restriction_{P_{n-3}}) = 2(n-2)$. Table 4.1 says that $k(\phi\restriction_{P_i}) \in \{1,2\}$ for every i. Therefore, it must hold $k(\phi\restriction_{P_i}) = 2$ for every $i = 0, \ldots, n-3$ and the relative Euler class of each $\phi\restriction_{P_i}$ is indeed maximal.

We can apply the case distinction of Table 4.1 to the triangles $\Delta_0, \ldots, \Delta_{n-3}$ built from $\mathfrak{P}([\phi])$. Let Δ_i be any of these triangles. For $k(\phi\restriction_{P_i}) = 2$, we have $2\pi - \beta_i(\phi) + \alpha_{i+2} + \beta_{i+1}(\phi) \geqslant 4\pi$ or equivalently

$$\alpha_{i+2} + \beta_{i+1}(\phi) - \beta_i(\phi) \geqslant 2\pi.$$

If $\alpha_{i+2} + \beta_{i+1}(\phi) - \beta_i(\phi) > 2\pi$, then Δ_i is a non-degenerate, clockwise oriented, triangle with interior angles $\beta_i(\phi)/2$, $\pi - \alpha_{i+1}/2$ and $\pi - \beta_{i+1}(\phi)/2$, such as stated in Lemma 4.1.1. If $\alpha_{i+2} + \beta_{i+1}(\phi) - \beta_i(\phi) = 2\pi$, then Δ_i is degenerate to a point. In both cases,

$$\mathrm{vol}(\phi\restriction_{P_i}) = -2[\Delta_i] = -(\alpha_{i+2} + \beta_{i+1}(\phi) - \beta_i(\phi) - 2\pi).$$

Observe that, thanks to the clockwise orientation of Δ_i, its area is always nonnegative. Table 4.2 summarizes the above discussion.

angles	$\mathrm{vol}(\phi\restriction_{P_i})$	configuration of $\Delta_i = \Delta(B_i, C_{i+2}, B_{i+1})$
$\alpha_{i+2} + \beta_{i+1} - \beta_i > 2\pi$	$-(\alpha_{i+2} + \beta_{i+1} - \beta_i - 2\pi)$	clockwise oriented, interior angles $\beta_i/2$, $\pi - \alpha_{i+1}/2$ and $\pi - \beta_{i+1}/2$
$\alpha_{i+2} + \beta_{i+1} - \beta_i = 2\pi$	0	degenerate, $B_i = C_{i+2} = B_{i+1}$.

Table 4.2.: The two different natures of $[\phi\restriction_{P_i}]$.

It turns out that Lemma 4.1.1 completely determines ChTri_α in the case the triangles are non-degenerate. This allows for a purely geometric description of the subset ChTri_α of the moduli space of point configurations in $\mathbb{H}$. This is the purpose of Lemma 4.1.5. In the case none of the triangles are degenerate, there is a cleaner formulation of the sufficient conditions for a chain of triangles to lie in ChTri_α. We state it as Corollary 4.1.6.

Lemma 4.1.5. *Let $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$ be a configuration of points in the upper half-plane and let $(\Delta_0, \ldots, \Delta_{n-3})$ be the chain of triangles defined by $\Delta_i = \Delta(B_i, C_{i+2}, B_{i+1})$, with the usual convention that $B_0 = C_1$ and $B_{n-2} = C_n$. Further, for $i = 0, \ldots, n-4$, let*

$$\beta_{i+1} := \sum_{j=0}^{i} 2[\Delta_j] - \sum_{j=1}^{i+2} \alpha_j + 2(i+2)\pi.$$

The isometry class of $(C_1, \dots, C_n, B_1, \dots, B_{n-3})$ *lies in* ChTri_α *if and only if the following conditions on* $\Delta_0, \dots, \Delta_{n-3}$ *are fulfilled.*

1. *If* $[\Delta_i] > 0$*, then* Δ_i *is clockwise oriented and has interior angle* $\beta_i/2$ *at* B_i*,* $\pi - \alpha_{i+2}/2$ *at* C_{i+2} *and* $\pi - \beta_{i+1}/2$ *at* B_{i+1}*. Moreover, if* $i = 0$*, then* Δ_0 *has interior angle* $\pi - \alpha_1/2$ *at* C_1 *and if* $i = n-3$*, then* Δ_{n-3} *has interior angle* $\pi - \alpha_n/2$ *at* C_n*.*

2. *If* $[\Delta_i] = 0$*, then* $B_i = C_{i+2} = B_{i+1}$*.*

Proof. The forward implication follows from the discussion that lead to Table 4.2. To prove the backward implication, start with a configuration of points $(C_1, \dots, C_n, B_1, \dots, B_{n-3})$ in the upper half-plane that satisfy the properties *(1)* and *(2)*. We construct a Deroin–Tholozan representation $[\phi]$ such that $\mathfrak{P}([\phi])$ is the isometry class of $(C_1, \dots, C_n, B_1, \dots, B_{n-3})$. Define $\phi(c_i)$ to be the rotation of angle α_i with fixed point C_i. We first claim that ϕ is a representation π_n into $\mathrm{PSL}(2, \mathbb{R})$, i.e. $\phi(c_1) \cdot \ldots \cdot \phi(c_n) = 1$. Indeed, arguing as in the proof of Lemma 4.1.2, we observe that $\phi(c_2)^{-1}\phi(c_1)^{-1}$ is a rotation of angle

$$2[\Delta_0] - \alpha_1 - \alpha_2 + 4\pi$$

around B_1. This angle is by definition equal to β_1. Similarly, $\phi(c_{n-2})^{-1} \cdot \ldots \cdot \phi(c_1)^{-1}$ is a rotation of angle

$$2[\Delta_{n-4}] - \alpha_{n-2} - (2\pi - \beta_{n-4}) + 4\pi$$

around B_{n-3}. Again, observe that this angle is by definition equal to β_{n-3}. Moreover, the same argument shows that $\phi(c_{n-1})\phi(c_n)$ is also a rotation of angle β_{n-3} around B_{n-3}. Hence $\phi(c_{n-2})^{-1} \cdot \ldots \cdot \phi(c_1)^{-1} = \phi(c_{n-1})\phi(c_n)$. This proves that ϕ is a representation of π_n into G. It is immediate from the definition of ϕ that $[\phi] \in \mathrm{Rep}_\alpha(\Sigma_n, G)$. We now prove that $\mathrm{vol}(\phi) = -\lambda$. In fact, using both the additivity of the volume and Lemma 4.1.3, we obtain

$$\mathrm{vol}(\phi) = -2 \sum_{i=0}^{n-3} [\Delta_i].$$

We express $[\Delta_{n-3}]$ in therms of the interior angles of Δ_{n-3}:

$$\begin{aligned} -2[\Delta_{n-3}] &= -2\pi + (2\pi - \alpha_n) + (2\pi - \alpha_{n-1}) + \beta_{n-3} \\ &= 2\pi - \alpha_n - \alpha_{n-1} + \beta_{n-3}. \end{aligned}$$

By definition of β_{n-3} it holds

$$-2 \sum_{i=0}^{n-4} [\Delta_i] = -\beta_{n-3} - \sum_{i=1}^{n-2} \alpha_i + 2(n-2)\pi.$$

We conclude that $\mathrm{vol}(\phi) = 2\pi(n-1) - \sum_{i=1}^{n} \alpha_i = -\lambda$ and thus $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. By construction, the chain of triangles built from $\mathfrak{P}([\phi])$ is $(\Delta_0, \ldots, \Delta_{n-3})$. We conclude that the isometry class of $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$ lies in ChTri_α as desired. $\square$

If all the triangles are non-degenerate, then Lemma 4.1.5 admits a cleaner formulation which we state as a corollary.

Corollary 4.1.6. *Let $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$ be a configuration of points in the upper half-plane and let $(\Delta_0, \ldots, \Delta_{n-3})$ be the chain of triangles it defines. Assume that none of the triangles Δ_i are degenerate. The isometry class of $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$ lies in* ChTri_α *if and only if the following conditions on $\Delta_0, \ldots, \Delta_{n-3}$ are fulfilled.*

1. *The triangle Δ_i is clockwise oriented and has interior angle $\pi - \alpha_{i+2}/2$ at C_{i+2}. Moreover, if $i = 0$, then Δ_0 has interior angle $\pi - \alpha_1/2$ at C_1 and if $i = n-3$, then Δ_{n-3} has interior angle $\pi - \alpha_n/2$ at C_n.*

2. *The interior angles of Δ_i and Δ_{i+1} at their common vertex B_{i+1} are supplementary.*

The conditions of Corollary 4.1.6 are illustrated on Figure 4.4.

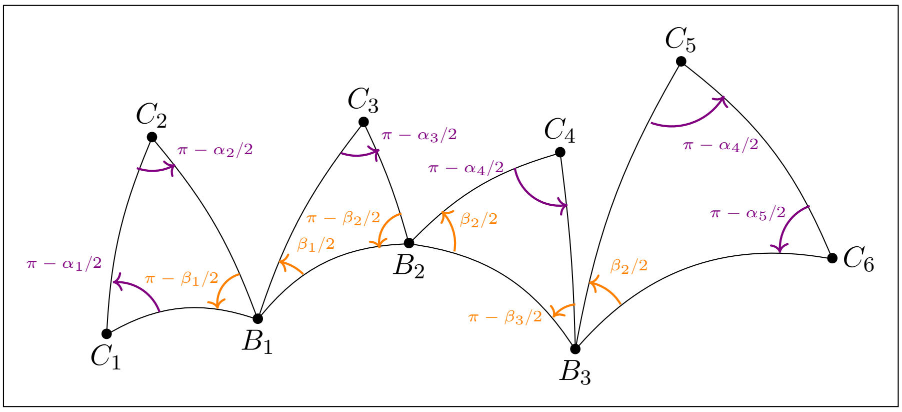


Figure 4.4.: Example of a configuration of points whose isometry class lies in ChTri_α in the case $n = 6$.

4.1.3. The torus action revisited

We explained how to use Proposition 3.1.10 to associate to a maximal collection of simple closed curves on Σ_n a maximal torus action on the Deroin–Tholozan relative character variety. In this section we first fix a parametrization of the maximal torus action associated to the curves $b_1, \ldots, b_{n-3}$ we intend to work with. We should emphasize that our choice of parametrization is different from that of Deroin–Tholozan in [DT19]. Deroin–Tholozan work with the torus action given by the Hamiltonian flows of the functions

$\beta_1, \ldots, \beta_{n-3} \colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to (0, 2\pi)$ defined in (4.1.1). We choose to consider the Hamiltonian flows of the functions $1/2(\beta_{i+1} - \beta_i)$ instead. They define an effective action

$$\mathbb{T}^{n-3} := (\mathbb{R}/2\pi\mathbb{Z})^{n-3} \circlearrowright \mathrm{Rep}_\alpha^{\mathrm{DT}}. \tag{4.1.7}$$

The reason for considering $1/2(\beta_{i+1} - \beta_i)$ instead of β_i is that the expression $1/2(\beta_{i+1} - \beta_i)$ is up to constant equal to the area of the triangle Δ_i, see Table 4.2.

Following [Gol86] we can write down explicitly how the Hamiltonian flows of the functions $1/2(\beta_{i+1} - \beta_i)$ act on representations. For $\theta = (\theta_1, \ldots, \theta_{n-3}) \in \mathbb{T}^{n-3}$ we introduce the notation

$$\overline{\theta}_i := \theta_i - \theta_{i-1}, \quad i = 1, \ldots, n-3,$$

where it is understood that $\theta_0 = 0$. The unique elliptic element of $\mathrm{PSL}(2, \mathbb{R})$ that fixes $z \in \mathbb{H}$ with angle of rotation $\vartheta \in (0, 2\pi)$ is denoted

$$\mathrm{rot}_\vartheta(z).$$

Let $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ and let $B_i \in \mathbb{H}$ be the fixed point of $\phi(b_i)$, with the convention that $B_{n-2} = C_n$ is the fixed point of $\phi(c_n)$. Under the action (4.1.7) the image of $\theta \in \mathbb{T}^{n-3}$ acting on $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ is the conjugacy class of the representation $\theta \cdot \phi$ given by

$$(\theta \cdot \phi)(c_i) = \left(\prod_{j=1}^{i-2} \mathrm{rot}_{\overline{\theta}_j}(B_j) \right) \cdot \phi(c_i) \cdot \left(\prod_{j=1}^{i-2} \mathrm{rot}_{\overline{\theta}_j}(B_j) \right)^{-1}.$$

Or more explicitly

$$\begin{cases} (\theta \cdot \phi)(c_1) = \phi(c_1), \\ (\theta \cdot \phi)(c_2) = \phi(c_2), \\ (\theta \cdot \phi)(c_3) = \mathrm{rot}_{\overline{\theta}_1}(B_1) \cdot \phi(c_3) \cdot \mathrm{rot}_{\overline{\theta}_1}(B_1)^{-1}, \\ (\theta \cdot \phi)(c_4) = \mathrm{rot}_{\overline{\theta}_1}(B_1)\, \mathrm{rot}_{\overline{\theta}_2}(B_2) \cdot \phi(c_4) \cdot \mathrm{rot}_{\overline{\theta}_2}(B_2)^{-1}\, \mathrm{rot}_{\overline{\theta}_1}(B_1)^{-1}, \\ \vdots \\ (\theta \cdot \phi)(c_{n-1}) = \left(\prod_{i=1}^{n-3} \mathrm{rot}_{\overline{\theta}_i}(B_i) \right) \cdot \phi(c_{n-1}) \cdot \left(\prod_{i=1}^{n-3} \mathrm{rot}_{\overline{\theta}_i}(B_i) \right)^{-1}, \\ (\theta \cdot \phi)(c_n) = \left(\prod_{i=1}^{n-3} \mathrm{rot}_{\overline{\theta}_i}(B_i) \right) \cdot \phi(c_n) \cdot \left(\prod_{i=1}^{n-3} \mathrm{rot}_{\overline{\theta}_i}(B_i) \right)^{-1}. \end{cases} \tag{4.1.8}$$

Observe that both $\phi(c_{n-1})$ and $\phi(c_n)$ are conjugated by the same element because they correspond to the same triangle in the chain built from $\mathfrak{P}([\phi])$. The reader is referred to

[Gol86] for explanations on how the explicit action (4.1.8) corresponds to the torus action (4.1.7) given by the Hamiltonian flows of the functions $1/2(\beta_{i+1} - \beta_i)$.

The action (4.1.7) is a Hamiltonian torus action on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ equipped with the symplectic form $1/\lambda \cdot \omega_{\mathcal{G}}$ with moment map $\mu\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{R}^{n-3}$ defined by

$$\mu_i([\phi]) := \frac{1}{2\lambda}(\alpha_{i+2} + \beta_{i+1}(\phi) - \beta_i(\phi) - 2\pi). \tag{4.1.9}$$

Recall that λ is the scaling factor introduced in Definition 3.1.4. Comparing Table 4.2 one observes that

$$\mu_i([\phi]) = \frac{1}{\lambda}[\Delta_i].$$

The image of μ inside $\mathbb{R}^{n-3}$ is the moment polytope for the action of $\mathbb{T}^{n-3}$ on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$. The area of the triangles in a chain corresponding to an element of $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ are nonnegative numbers that sum up to $\lambda/2$:

$$[\Delta_i] \in [0, \lambda/2] \subset [0, \pi), \quad [\Delta_0] + \ldots + [\Delta_{n-3}] = \lambda/2.$$

This is a consequence of the additivity of the volume and Lemma 4.1.3; the computation is similar to (4.1.6). Hence

$$\mu_i \in [0, 1/2], \quad \mu_1 + \ldots + \mu_{n-3} \leqslant 1/2. \tag{4.1.10}$$

This shows that the moment polytope is the $(n-3)$-simplex in $\mathbb{R}^{n-3}$ with side length 1/2. If we compare Lemma 4.1.5 and the range of $[\Delta_i]$ we deduce

$$\beta_i \in \left[2(i+1)\pi - \sum_{j=1}^{i+1} \alpha_j, \sum_{j=i+2}^{n} \alpha_j - 2\pi(n-i-2)\right] \subset (0, 2\pi). \tag{4.1.11}$$

Observe that the length of the range of the function β_i is equal to λ and that the range of the function β_{i+1} is obtained from that of β_i by a translation of $2\pi - \alpha_{i+2}$. The moment polytope equations (4.1.10) translated in terms of β_i read

$$\begin{cases} \beta_1 \geqslant 4\pi - \alpha_1 - \alpha_2, \\ \beta_i - \beta_{i+1} \leqslant \alpha_{i+2} - 2\pi, \quad i = 1, \ldots, n-4 \\ \beta_{n-3} \leqslant \alpha_{n-1} + \alpha_n - 2\pi. \end{cases}$$

Lemma 4.1.7. *The fibre of the moment map μ over a point of the moment polytope is an embedded torus of dimension $k \in \{0, \ldots, n-3\}$ in $\mathrm{Rep}_\alpha^{\mathrm{DT}}$, where $(n-3)-k$ is the number of degenerate triangles in the chain associated to any element of the fibre.*

Lemma 4.1.7 is a standard fact about symplectic toric manifolds. The toric fibres of maximal dimension $n-3$ form an open dense subset of $\mathrm{Rep}_\alpha^{\mathrm{DT}}$. They are called *regular fibres* of the moment map. Their union is the preimage under μ of the interior of the moment polytope. We denote this subspace by

$$\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R})) \subset \mathrm{Rep}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R})).$$

We will abbreviate $\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}} := \mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$. It is a full measure subset that consists exactly of the points where $\mathbb{T}^{n-3}$ acts freely.

The torus action (4.1.7) explicitly described by (4.1.8) may look, in the words of a retired analyst, baroque. It can be easily visualized if we translate it to our polygon model. This is yet another pleasant feature of the polygonal model for the Deroin–Tholozan relative character variety. For this purpose, we declare the bijection $\mathfrak{P}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathrm{ChTri}_\alpha$ to be equivariant and define therewith an action of $\mathbb{T}^{n-3}$ on ChTri_α. Let $\theta \in \mathbb{T}^{n-3}$. We denote the fixed points of $(\theta\cdot\phi)(c_i)$ and $(\theta\cdot\phi)(b_i)$ by C_i^θ and B_i^θ, respectively. From (4.1.8), we obtain that

$$C_1^\theta = C_1, \quad C_2^\theta = C_2, \quad C_3^\theta = \mathrm{rot}_{\overline{\theta}_1}(B_1)\cdot C_3, \quad \ldots, \quad C_n^\theta = \prod_{i=1}^{n-3} \mathrm{rot}_{\overline{\theta}_i}(B_i)\cdot C_n, \qquad (4.1.12)$$

and

$$B_1^\theta = B_1, \quad B_2^\theta = \mathrm{rot}_{\overline{\theta}_1}(B_1)\cdot B_2, \quad \ldots, \quad B_{n-3}^\theta = \prod_{i=1}^{n-4} \mathrm{rot}_{\overline{\theta}_i}(B_i)\cdot B_{n-3}. \qquad (4.1.13)$$

This means that $\theta \in \mathbb{T}^{n-3}$ acts on a chain of triangles in ChTri_α by successive rotations of the sub-chain of triangles $\Delta_i, \ldots, \Delta_{n-3}$ by an angle $\overline{\theta}_i$ around B_i, see Figure 4.5.

4.2. Complex projective coordinates

In this section, we construct an explicit equivariant symplectomorphism from $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ to $\mathbb{CP}^{n-3}$. It is based on the polygonal model developed in the previous section.

4.2.1. Definition of the map

Let $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. We associate to $[\phi]$ a collection of parameters defined using the chain of triangles $\Delta_0, \ldots, \Delta_{n-3}$ built from $\mathfrak{P}([\phi]) \in \mathrm{ChTri}_\alpha$. The first collection of parameters

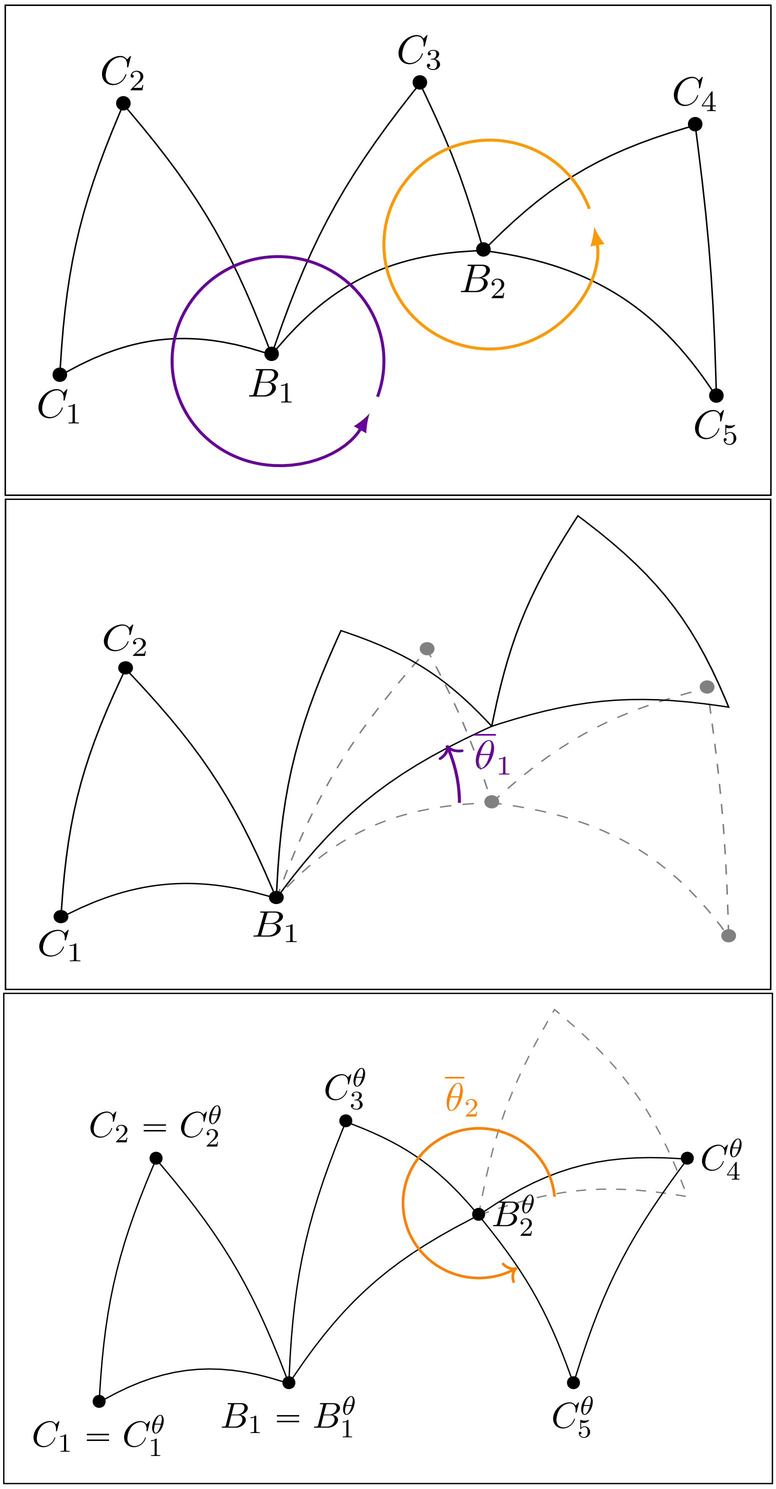


Figure 4.5.: The action of $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$ in the case $n = 5$. The initial configuration is drawn on top. The intermediate configuration is obtained after rotating the triangles Δ_1 and Δ_2 together by an angle $\overline{\theta}_1$ around B_1. The triangle Δ_0 is not moved during this step. The final configuration is obtained from the intermediate configuration after rotating the triangle Δ_2 by an angle $\overline{\theta}_2$ around B_2. The triangles Δ_0 and Δ_1 are not moved during this step.

$a_0, \dots, a_{n-3} \colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to [0, \infty)$ are called *area parameters* and are defined to be twice the area of the triangle Δ_i:

$$a_i([\phi]) := 2[\Delta_i], \quad i = 0, \dots, n-3.$$

Lemma 4.1.1 implies that

$$a_i([\phi]) = \alpha_{i+2} + \beta_{i+1}([\phi]) - \beta_i([\phi]) - 2\pi \geqslant 0. \tag{4.2.1}$$

Each area parameter takes value in $[0, \lambda]$ and their sum is constant and equal to the scaling factor $\lambda > 0$. This was already observed earlier when we computed the moment polytope for the moment map (4.1.9). In particular, at least one area parameter is nonzero. Since the functions β_i are analytic functions of $\mathrm{Rep}_\alpha^{\mathrm{DT}}$, the area parameters are analytic functions as well. Observe that, because of (4.1.9), it holds that

$$a_i([\phi]) = 2\lambda \cdot \mu_i([\phi]). \tag{4.2.2}$$

The second set of parameters $\sigma_1, \dots, \sigma_{n-3} \colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{R}/2\pi\mathbb{Z}$ are called *angle parameters*. Their definiton is more subtle as one needs to consider the case where some triangles of the chain are degenerate to a point. First, assume that $a_i([\phi]) \neq 0$ for every $i = 0, \dots, n-3$ or equivalently that $[\phi]$ lies in a regular fibre of the moment map. This ensures that the fixed points $B_i(\phi)$, $C_{i+1}(\phi)$, $C_{i+2}(\phi)$, abbreviated B_i, C_{i+1}, C_{i+2} below, are distinct points for every i. In this case, we define, for $i = 1, \dots, n-3$, the angle $\gamma_i([\phi]) \in \mathbb{R}/2\pi\mathbb{Z}$ to be the oriented angle between the geodesic rays $\overrightarrow{B_iC_{i+2}}$ and $\overrightarrow{B_iC_{i+1}}$ (see Figure 4.6):

$$\gamma_i([\phi]) := \angle(\overrightarrow{B_iC_{i+2}}, \overrightarrow{B_iC_{i+1}}).$$

In less rigorous words, γ_i is the angle between the triangle Δ_{i+1} and the triangle Δ_i. In the case that some of the area parameters vanish, we define $\gamma_i([\phi]) \in \mathbb{R}/2\pi\mathbb{Z}$ to be

$$\gamma_i([\phi]) := \begin{cases} 0, & \text{if } a_j([\phi]) = 0, \forall j < i, \\ \pi - \alpha_{i+2}/2, & \text{if } a_i([\phi]) = 0 \text{ and } \exists j < i,\ a_j([\phi]) > 0, \\ \angle(\overrightarrow{B_iC_{i+2}}, \overrightarrow{B_iC_{m(i)+2}}), & \text{if } a_i([\phi]) > 0 \text{ and } \exists j < i,\ a_j([\phi]) > 0, \end{cases}$$

where $m(i)$ is the largest index smaller than i such that $a_{m(i)}([\phi]) > 0$, see Figure 4.6. Whenever $[\phi]$ lies in a regular fibre of the moment map, then $m(i) = i - 1$ for every i, showing that the definition of γ_i is consistent. Note that the parameters $\gamma_i([\phi])$ are well-defined in the sense that if $a_i([\phi]) > 0$ then $B_i \neq C_{i+2}$ and $B_i \neq C_{m(i)+2}$. We finally define the angle parameters $\sigma_i([\phi]) \in \mathbb{R}/2\pi\mathbb{Z}$ for $i = 1, \ldots, n-3$ by

$$\sigma_i([\phi]) := \sum_{j=1}^{i} \gamma_j([\phi]).$$

Below, we will refer to both sets of parameters $\{\gamma_1, \ldots, \gamma_{n-3}\}$ and $\{\sigma_1, \ldots, \sigma_{n-3}\}$ as angle parameters, without distinction. The angle parameters γ_i and σ_i are analytic functions on $\mathring{\mathrm{Rep}}^{\mathrm{DT}}_\alpha$ and may have points of discontinuity on the complement of $\mathring{\mathrm{Rep}}^{\mathrm{DT}}_\alpha$.

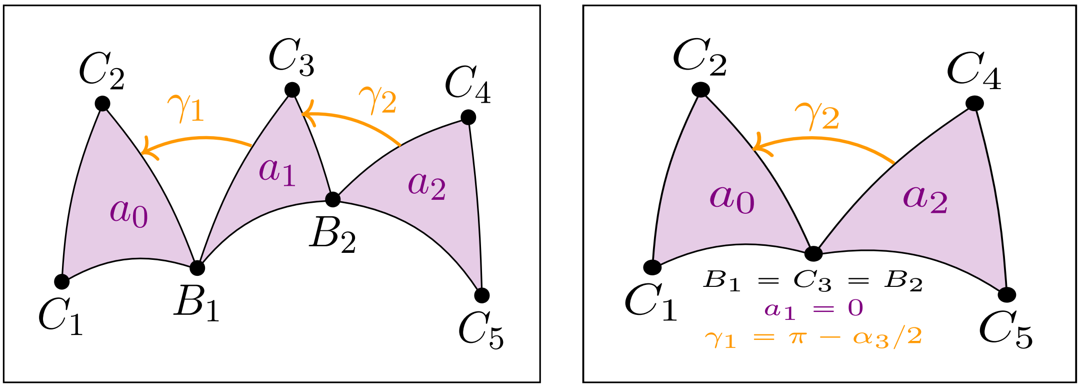


Figure 4.6.: The angles γ_i for two configurations of fixed points in the case $n = 5$. The left picture corresponds to a representation in a regular fiber of the moment map. The right picture corresponds to a representation for which a_1 vanishes.

Area and angle parameters completely characterize Deroin–Tholozan representations. To see this, we introduce the map

$$\begin{aligned} \mathfrak{C} \colon \mathrm{Rep}^{\mathrm{DT}}_\alpha &\to \mathbb{CP}^{n-3} \\ [\phi] &\mapsto \left[\sqrt{a_0([\phi])} : \sqrt{a_1([\phi])}e^{i\sigma_1([\phi])} : \ldots : \sqrt{a_{n-3}([\phi])}e^{i\sigma_{n-3}([\phi])}\right]. \end{aligned} \tag{4.2.3}$$

Recall that the area parameters are nonnegative and cannot vanish all at once. Moreover, recall that both the area and angle parameters are geometric invariants of $\mathfrak{P}([\phi]) \in \mathrm{ChTri}_\alpha$. We thus see that the map $\mathfrak{C} \colon \mathrm{Rep}^{\mathrm{DT}}_\alpha \to \mathbb{CP}^{n-3}$ is well-defined.

Recall that the Deroin–Tholozan relative character variety has the structure of a symplectic toric manifold with symplectic form $1/\lambda \cdot \omega_{\mathcal{G}}$ and the torus action (4.1.7). We equip $\mathbb{CP}^{n-3}$ with the Fubini-Study symplectic form $\omega_{\mathcal{FS}}$ of volume $\pi^{n-3}/(n-3)!$, see e.g. [CdS01] for more details on the symplectic nature of the complex projective space. We further equip $\mathbb{CP}^{n-3}$ with the $\mathbb{T}^{n-3}$-action defined in homogeneous coordinates by

$$\theta \cdot [z_0 : z_1 : \ldots : z_{n-3}] := [z_0 : e^{-\theta_1} z_1 : \ldots : e^{-\theta_{n-3}} z_{n-3}], \quad \theta \in \mathbb{T}^{n-3}. \tag{4.2.4}$$

This action is a maximal effective Hamiltonian torus action with moment map

$$\nu([z_0 : z_1 : \ldots : z_{n-3}]) := \left(\frac{|z_1|^2}{2|z|}, \ldots, \frac{|z_{n-3}|^2}{2|z|}\right) \in \mathbb{R}^{n-3}, \tag{4.2.5}$$

where $|z|^2 := |z_0|^2 + |z_1|^2 + \ldots + |z_{n-3}|^2$. Now that all the notation as been introduced, we can re-state Theorem B in a more formal fashion.

Theorem 4.2.1 (Theorem B). *The map* $\mathfrak{C}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ *defined in* (4.2.3) *is an isomorphism of symplectic toric manifolds. In other words,* $\mathfrak{C}$ *is an equivariant diffeomorphism such that*

$$\mu = \nu \circ \mathfrak{C} \quad \textit{and} \quad \mathfrak{C}^*\omega_{\mathcal{FS}} = 1/\lambda \cdot \omega_{\mathcal{G}}.$$

The proof of Theorem 4.2.1 is unfolded, step by step, below. The main difficulty in the proof is showing that the map $\mathfrak{C}$ is differentiable at the points in the irregular fibres of the moment map – that is, on the complement of $\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}$. On these fibres the area parameters can vanish causing the angles parameters γ_i to be discontinuous.

A direct consequence of Theorem 4.2.1, already pointed out in [DT19], says that the symplectic volume of the Deroin–Tholozan relative character variety is equal to

$$\frac{(\lambda\pi)^{n-3}}{(n-3)!}.$$

4.2.2. A Wolpert-type formula

Theorem 4.2.1 implies Theorem A which says that the coordinates

$$\{a_1, \ldots, a_{n-3}, \sigma_1, \ldots, \sigma_{n-3}\}$$

are action-angle coordinates for the Deroin–Tholozan relative character variety. In particular, as a corollary of Theorem 4.2.1, we prove that the coordinates are Darboux coordinates for the Goldman symplectic form.

Corollary 4.2.2 (Theorem C). *The restriction of the Goldman form on* $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ *to* $\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}$ *can be written as*

$$\omega_{\mathcal{G}} = \frac{1}{2}\sum_{i=1}^{n-3} da_i \wedge d\sigma_i = \frac{1}{2}\sum_{i=1}^{n-3} d\gamma_i \wedge d\beta_i.$$

Proof. At any point $[z_0 : z_1 : \ldots : z_{n-3}] \in \mathbb{CP}^{n-3}$ for which $z_i \neq 0$ for all $i = 0, \ldots, n-3$, the Fubini-Study form can be written as

$$\omega_{\mathcal{FS}} = \sum_{i=1}^{n-3} d\nu_i \wedge d\theta_i,$$

where $(\nu_1, \ldots, \nu_{n-3})$ are the components of the moment map (4.2.5) and θ_i is the complex argument of z_i (defined up to a constant). The coordinates $\{\nu_1, \ldots, \nu_{n-3}, \theta_1, \ldots, \theta_{n-3}\}$ are action-angle coordinates for the integrable dynamics on $\mathbb{CP}^{n-3}$ defined by (4.2.4). Theorem 4.2.1 says that $\omega_{\mathcal{G}} = \lambda \cdot \mathfrak{C}^* \omega_{\mathcal{FS}}$. It also implies $\mathfrak{C}^* d\nu_i = d\mu_i = da_i/(2\lambda)$ (where we used (4.2.2)) and $\mathfrak{C}^* d\theta_i = d\sigma_i$. Hence, on $\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}$, it holds that

$$\omega_{\mathcal{G}} = \lambda \cdot \mathfrak{C}^* \omega_{\mathcal{FS}} = \lambda \sum_{i=1}^{n-3} \mathfrak{C}^* d\nu_i \wedge \mathfrak{C}^* d\theta_i = \frac{1}{2} \sum_{i=1}^{n-3} da_i \wedge d\sigma_i.$$

Using $da_i = d\beta_{i+1} - d\beta_i$, with $d\beta_0 = d\beta_{n-2} = 0$, and $d\sigma_{i+1} - d\sigma_i = d\gamma_{i+1}$, it follows that

$$\sum_{i=1}^{n-3} da_i \wedge d\sigma_i = \sum_{i=1}^{n-3} d\gamma_i \wedge d\beta_i. \qquad \square$$

Corollary 4.2.2 implies that, even if the definition of the coordinates $\{a_1, \ldots, a_{n-3}, \sigma_1, \ldots, \sigma_{n-3}\}$ depends on the choice of a pants decomposition of Σ_n, the 2-form $\sum_{i=1}^{n-3} da_1 \wedge d\sigma_i$ does not. This is because the Goldman symplectic form on the Deroin–Tholozan relative character variety is defined without any reference to a pants decomposition.

4.3. Proof of Theorem B

In this section we prove Theorem 4.2.1, i.e. we show that the map $\mathfrak{C} \colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ defined in (4.2.3) is an equivariant symplectomorphism.

4.3.1. Homeomorphism property

We start by proving

Proposition 4.3.1. *The map $\mathfrak{C} \colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ is a homeomorphism.*

To prove Proposition 4.3.1, we show that $\mathfrak{C}$ is a continuous bijection. This is done in Lemmata 4.3.2, 4.3.3 and 4.3.4 below. Since $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ is compact by Theorem 3.1.6 and $\mathbb{CP}^{n-3}$ is a Hausdorff space, it follows that $\mathfrak{C}$ is a homeomorphism.

Lemma 4.3.2. *The map* $\mathfrak{C}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ *is surjective.*

Proof. Let $[z_0 : \ldots : z_{n-3}] \in \mathbb{CP}^{n-3}$. We may assume that $|z_0|^2 + \ldots + |z_{n-3}|^2 = \lambda$ and that the first nonzero z_i is a positive real number. The goal is to build a representation $\phi\colon \pi_n \to \mathrm{PSL}(2,\mathbb{R})$ such that $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ and $\mathfrak{C}([\phi]) = [z_0 : \ldots : z_{n-3}]$. To do so, we build a chain of triangles satisfying the properties of Lemma 4.1.5 such that the corresponding Deroin–Tholozan representation has the desired image under $\mathfrak{C}$. The triangles are constructed in $n-2$ steps starting with Δ_0.

0. *Step 0.* Let C_1 be any point in $\mathbb{H}$. If $z_0 = 0$, then we let $C_2 := B_1 := C_1$. Now, assume $z_0 \neq 0$. By assumption, z_0 is a positive real number. First, observe that $|z_0|^2/2 = z_0^2/2 \leqslant \lambda/2 < \pi$. Further, let $\beta_1 := z_0^2 - \alpha_1 - \alpha_2 + 4\pi$. Note that, since $4\pi > \alpha_1 + \alpha_2$ and $\lambda - \alpha_1 - \alpha_2 < -2\pi$, it holds $\beta_1 \in (0, 2\pi)$. In particular, there exists a clockwise oriented hyperbolic triangle $\Delta_0 = \Delta(C_1, B_1, C_2)$ such that

   - Δ_0 has area $z_0^2/2$,
   - Δ_0 has interior angles $\pi - \alpha_1/2$ at C_1 and $\pi - \alpha_2/2$ at C_2.

   The triangle Δ_0 is not uniquely determined as it can be arbitrarily rotated around C_1. We fix one such triangle Δ_0. By construction, Δ_0 has interior angle $\pi - \beta_1/2$ at B_1.

1. *Step 1.* If $z_1 = 0$, then we let $C_3 = B_2 = B_1$. Now, assume $z_1 \neq 0$. Again, observe that $|z_1|^2/2 \leqslant \lambda/2 < \pi$ and $\beta_2 := |z_1|^2 - \alpha_3 + \beta_1 + 2\pi \in (0, 2\pi)$, because $-\alpha_3 + \beta_1 + 2\pi \geqslant 6\pi - \alpha_1 - \alpha_2 - \alpha_3 > 0$ and $\lambda - \alpha_1 - \alpha_2 - \alpha_3 < -4\pi$. So, there exists a clockwise oriented hyperbolic triangle $\Delta_1 = \Delta(B_1, C_3, B_2)$ such that

   - Δ_1 has area $|z_1|^2/2$,
   - Δ_1 has interior angles $\pi - \alpha_3/2$ at C_3 and $\beta_1/2$ at B_1.

   If $z_0 = 0$, then as before Δ_1 can be arbitrarily rotated around B_1. If $z_0 \neq 0$, then Δ_1 is uniquely determined if we further impose

   - the angle $\angle(\overrightarrow{B_1C_3}, \overrightarrow{B_1C_2})$ is equal to the complex argument of z_1.

   If Δ_1 is non-degenerate, then by construction it has interior angle $\pi - \beta_2/2$ at B_2.

2. *Step 2.* If $z_2 = 0$, then we let $C_4 = B_3 = B_2$. Now, assume $z_2 \neq 0$. It holds $|z_2|^2/2 \leqslant \lambda/2 < \pi$ and $\beta_3 := |z_2|^2 - \alpha_4 + \beta_2 + 2\pi \in (0, 2\pi)$. There exists a clockwise oriented hyperbolic triangle $\Delta_2 = \Delta(B_2, C_4, B_3)$ such that

   - Δ_2 has area $|z_2|^2/2$,

- Δ_2 has interior angles $\pi - \alpha_4/2$ at C_4 and $\beta_2/2$ at B_2.

If $z_0 = 0$ and $z_1 = 0$, then Δ_2 can be arbitrarily rotated around B_2. If $z_0 \neq 0$ and $z_1 = 0$, then Δ_2 is uniquely determined if we impose

- the angle $\angle(\overrightarrow{B_2C_4}, \overrightarrow{B_2C_2})$ is equal to the complex argument of z_2.

If $z_1 \neq 0$, then Δ_2 is uniquely determined if we impose

- the angle $\angle(\overrightarrow{B_2C_4}, \overrightarrow{B_2C_3})$ is equal to the complex argument of z_2 minus the complex argument of z_1.

If Δ_2 is non-degenerate, then by construction it has interior angle $\pi - \beta_3/2$ at B_3.

This process can be repeated $n-5$ times until the point $C_n = B_{n-2}$ has been constructed. The last triangle in the chain, namely $\Delta_{n-3} = \Delta(B_{n-3}, C_{n-1}, C_n)$, has area $|z_{n-3}|^2/2$ and interior angles $\pi - \alpha_{n-1}/2$ at C_{n-1} and $\beta_{n-3}/2$ at B_{n-3}, assuming $z_{n-3} \neq 0$. Since

$$|z_{n-3}|^2 = \lambda - |z_0|^2 - \ldots - |z_{n-4}|^2 = \alpha_n + \alpha_{n-1} - \beta_{n-3} - 2\pi,$$

it follows that the interior angle of Δ_{n-3} at C_n is $\pi - \alpha_n/2$. Therefore, the configuration of points $(C_1, \ldots, C_n, B_1, \ldots, B_{n-3})$ we just built satisfies the properties of Lemma 4.1.5. Its preimage under $\mathfrak{P}$ is the conjugacy class of a Deroin–Tholozan representation $[\phi]$. It follows from the construction that $\mathfrak{C}([\phi]) = [z_0 : \ldots : z_{n-3}]$. $\square$

Lemma 4.3.3. *The map* $\mathfrak{C}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ *is injective.*

Proof. Let $[\phi]$ and $[\phi']$ be two elements of $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ such that $\mathfrak{C}([\phi]) = \mathfrak{C}([\phi'])$. We want to prove that $[\phi] = [\phi']$. To achieve this, it is sufficient to check that the chain of triangles built from $\mathfrak{P}([\phi])$ and $\mathfrak{P}([\phi'])$ are isometric because $\mathfrak{P}$ is injective.

Let $a_i = \alpha_{i+2} + \beta_{i+1} - \beta_i - 2\pi$ and $a_i' = \alpha_{i+2} + \beta_{i+1}' - \beta_i' - 2\pi$ be the area parameters associated to $[\phi]$ and $[\phi']$, respectively. Similarly, let γ_i, σ_i and γ_i', σ_i' be their respective angle parameters. Recall that $a_0 + \ldots + a_{n-3} = a_0' + \ldots + a_{n-3}' = \lambda$. By definition of $\mathfrak{C}$ (see (4.2.3)), since we assume $\mathfrak{C}([\phi]) = \mathfrak{C}([\phi'])$, it follows that $a_i = a_i'$ for every $i = 0, \ldots, n-3$. Moreover, it also implies $\sigma_i = \sigma_i' + \sigma$ for every $i = 1, \ldots, n-3$, where σ is some constant. Note that, if $a_0 = a_0' > 0$, then $\sigma = 0$.

From $a_i = a_i'$, it follows $\beta_i = \beta_i'$ for every i. Thus, by Lemma 4.1.1, the oriented triangles Δ_i and Δ_i' inside $\mathbb{H}$ have the same interior angles and are therefore isometric for every i. To conclude that the two chains are isometric, it suffices to check that the angles between consecutive non-degenerate triangles in each chain are equal. Since $\sigma_i = \sigma_i' + \sigma$, we have $\gamma_1 = \gamma_1' + \sigma$ and $\gamma_i = \gamma_i'$ for every $i \geqslant 2$. Since $\sigma = 0$ whenever $a_0 = a_0' > 0$, this shows that the angles between the corresponding pairs of consecutive non-degenerate triangles in each chain are equal. We conclude that $\mathfrak{P}([\phi]) = \mathfrak{P}([\phi'])$ and thus $[\phi] = [\phi']$. □

Lemma 4.3.4. *The map* $\mathfrak{C}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ *is continuous.*

Proof. The continuity of $\mathfrak{C}$ is immediate at any point in a regular fibre of the moment map. The task is more subtle when some area parameters vanish because of the discontinuity of the angle parameters γ_i.

Let $[\phi_0] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. We prove that $\mathfrak{C}$ is continuous at $[\phi_0]$. Let $i \geqslant 0$ be the smallest index such that $a_i([\phi_0]) > 0$. We work in the chart $\{z_i \neq 0\}$ of $\mathbb{CP}^{n-3}$. Continuity is guaranteed for every index j such that $a_j([\phi_0]) = 0$. It thus suffices to prove that $\sigma_j([\phi]) - \sigma_i([\phi])$ is continuous around $[\phi_0]$ for every index $j > i$ such that $a_j([\phi]) > 0$. Let $i = i_1 < i_2 < \ldots < i_d$ denote the indices such that $a_{i_l}([\phi]) > 0$. Because of telescopic cancellations, it is sufficient to prove that $\sigma_{i_{l+1}}([\phi]) - \sigma_{i_l}([\phi])$ is continuous around $[\phi_0]$ for every $l = 1, \ldots, d-1$.

We treat the case $l = 1$. Let $i = i_1 < i_2 = j$. We first consider the case $j - i = 1$ first. In this case,

$$\sigma_j([\phi]) - \sigma_i([\phi]) = \gamma_{i+1}([\phi]).$$

Since $a_{i+1}([\phi_0]) > 0$ and $a_i([\phi_0]) > 0$ by assumption, the angle parameter $\gamma_{i+1}([\phi])$ is a continuous function around $[\phi_0]$.

Now, we consider the general case $j - i \geqslant 2$. Recall that it corresponds the situation where $a_i([\phi_0]) > 0$, $a_j([\phi_0]) > 0$ and $a_l([\phi_0]) = 0$ for all $i < l < j$. For clarity, we let $[\phi_k]$ be a sequence that converges to $[\phi_0]$. We will assume that $a_\ell([\phi_k]) > 0$ for every k and every $i \leqslant \ell \leqslant j$. The argument below can be adapted to the case where, for some $i < \ell < j$, $a_\ell([\phi_k]) = 0$ for infinitely many k. Since we assume $a_j([\phi_k]) > 0$ and $a_i([\phi_k]) > 0$, it holds $B_j(\phi_k) \neq C_{j+2}(\phi_k)$ and $B_{i+1}(\phi_k) \neq C_{i+2}(\phi_k)$. For k large enough, we may assume that the geodesics $\overrightarrow{B_j(\phi_k)C_{j+2}(\phi_k)}$ and $\overrightarrow{B_{i+1}(\phi_k)C_{i+2}(\phi_k)}$ intersect, because they do so at the limit. Recall that, by definition, $\gamma_j = \angle\big(\overrightarrow{B_jC_{j+2}}, \overrightarrow{B_{i+1}C_{i+2}}\big)$ (see Figure 4.6) and so

$$\gamma_j([\phi_0]) = \lim_{k\to\infty} \angle\big(\overrightarrow{B_j(\phi_k)C_{j+2}(\phi_k)}, \overrightarrow{B_{i+1}(\phi_k)C_{i+2}(\phi_k)}\big). \tag{4.3.1}$$

The angle $\angle\big(\overrightarrow{B_jC_{j+2}}, \overrightarrow{B_{i+1}C_{i+2}}\big)$ can be decomposed as follows:

$$\angle\big(\overrightarrow{B_jC_{j+2}}, \overrightarrow{B_jC_{j+1}}\big) + \angle\big(\overrightarrow{B_jC_{j+1}}, \overrightarrow{B_{j-1}C_{j+1}}\big) + \angle\big(\overrightarrow{B_{j-1}C_{j+1}}, \overrightarrow{B_{j-1}C_j}\big) + \ldots + \angle\big(\overrightarrow{B_{i+1}C_{i+3}}, \overrightarrow{B_{i+1}C_{i+2}}\big).$$

Using

$$\angle\big(\overrightarrow{B_m(\phi_k)C_{m+2}(\phi_k)}, \overrightarrow{B_m(\phi_k)C_{m+1}(\phi_k)}\big) = \gamma_m([\phi_k])$$

and

$$\angle\left(\overrightarrow{B_m(\phi_k)C_{m+1}(\phi_k)}, \overrightarrow{B_{m-1}(\phi_k)C_{m+1}(\phi_k)}\right) = \pi - \frac{\alpha_{m+1}}{2},$$

and recalling that

$$\gamma_m([\phi_0]) = \pi - \frac{\alpha_{m+2}}{2}, \quad m = i+1, \ldots, j-1,$$

we conclude

$$\angle\left(\overrightarrow{B_j(\phi_k)C_{j+2}(\phi_k)}, \overrightarrow{B_{i+1}(\phi_k)C_{i+2}(\phi_k)}\right) = \sigma_j([\phi_k]) - \sigma_i([\phi_k]) - \gamma_{j-1}([\phi_0]) - \ldots - \gamma_{i+1}([\phi_0]).$$

Because of (5.2.3) we conclude that $\sigma_j([\phi_k]) - \sigma_i([\phi_k])$ converges to $\gamma_j([\phi_]) + \ldots + \gamma_{i+1}([\phi_0]) = \sigma_j([\phi_0]) - \sigma_i([\phi_0])$. □

4.3.2. Equivariance property

We prove

Proposition 4.3.5. *The map $\mathfrak{C}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ is equivariant with respect to the torus actions* (4.1.8) *and* (4.2.4)*. Moreover,*

$$\mu = \nu \circ \mathfrak{C},$$

where μ and ν are the moment maps defined in (4.1.9) *and* (4.2.5)*.*

Proof. Both torus actions and both moment maps are continuous. The map $\mathfrak{C}$ is continuous by Lemma 4.3.4. It thus suffices to check the conclusion of the proposition on the dense open subset given by the regular fibres of the moment map μ. Let $[\phi]$ an element in a regular fibre and let $\theta \in \mathbb{T}^{n-3}$. The relations (4.1.12) and (4.1.13) (see also Figure 4.5) show that, for any $i = 0, \ldots, n-3$ and $j = 1, \ldots, n-3$,

$$a_i(\theta \cdot [\phi]) = a_i([\phi]) \quad \text{and} \quad \gamma_j(\theta \cdot [\phi]) = \gamma_j([\phi]) - \overline{\theta}_j.$$

Hence $\sigma_j(\theta \cdot [\phi]) = \sigma_j([\phi]) - \theta_j$. This implies $\mathfrak{C}(\theta \cdot [\phi]) = \theta \cdot \mathfrak{C}([\phi])$. Observe further that, for every $i = 1, \ldots, n-3$, it holds that

$$\nu_i \circ \mathfrak{C}([\phi]) = \frac{a_i([\phi])}{2\lambda} = \mu_i([\phi]),$$

where we used that the sum of the area parameters is equal to λ. □

4.3.3. Differentiablity property

In this section, we prove that

Proposition 4.3.6. *The map* $\mathfrak{C}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ *is continuously differentiable.*

The map $\mathfrak{C}$ restricted to $\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}$ is analytic because both the area and angle parameters are analytic functions of $\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}$. As mentioned earlier, two factors lead to complications when trying to prove differentiability on the complement of $\mathring{\mathrm{Rep}}_\alpha^{\mathrm{DT}}$. The first one is the presence of square roots on the area parameters. The second one is the discontinuity of the angle parameters whenever triangles are degenerate.

The proof that $\mathfrak{C}$ is a continuous function (Lemma 4.3.4) showed the importance of considering consecutive indices for which the corresponding area parameters vanish. This leads to the notion of *chain of degeneracy* for $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ by which we mean a *maximal* collection of consecutive degenerate triangles in the chain built from $\mathfrak{P}([\phi])$. A chain of degeneracy is said to be *of type* (j,k) if the maximal collection of consecutive degenerate triangles is $\Delta_j, \ldots, \Delta_{j+k-1}$. The number k is the length of the chain. The maximality assumption means that the triangles Δ_{j-1} and Δ_{j+k}, if they exist, are non-degenerate.

To conclude the proof of Proposition 4.3.6 it remains to check that $\mathfrak{C}$ is continuously differentiable at every $[\phi]$ with at least one chain of degeneracy. For simplicity, we only cover the case where $a_{n-3}([\phi]) > 0$. The case $a_{n-3}([\phi]) = 0$ can be treated in similar manner.

Let $[\phi_0] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ such that $a_{n-3}([\phi_0]) > 0$. Assume that $[\phi_0]$ has exactly $d \geqslant 1$ chains of degeneracy of types (j_1,k_1), …, (j_d,k_d) with $j_1 < \ldots < j_d$. This means that $a_{j_l+k_l}([\phi_0]) > 0$ for every $l = 1, \ldots, d$ (the case $l = d$ follows from the assumption $a_{n-3}([\phi_0]) > 0$). This implies that the angle parameters γ_i are analytic in a neighbourhood of $[\phi_0]$ for every index i in the complement of

$$\{j_1, \ldots, j_1 + k_1\} \cup \ldots \cup \{j_d, \ldots, j_d + k_d\}.$$

More precautions must be taken to deal with the case where $j_1 = 0$, i.e. when $a_0([\phi_0]) = 0$. To prove that $\mathfrak{C}$ is continuously differentiable at $[\phi_0]$ we claim that it is sufficient to prove

Lemma 4.3.7. *The following functions are continuously differentiable in a neighbourhood of* $[\phi_0]$*:*

1. $[\phi] \mapsto \exp\left(i \cdot (\gamma_{j_1}([\phi]) + \ldots + \gamma_{j_1+k_1}([\phi]))\right)$ *if* $j_1 \neq 0$,
2. $[\phi] \mapsto \exp\left(i \cdot (\gamma_{j_l}([\phi]) + \ldots + \gamma_{j_l+k_l}([\phi]))\right)$ *for every* $l = 2, \ldots, d$,

3. $[\phi] \mapsto \sqrt{a_i([\phi])} \exp\left(i \cdot (-\gamma_{i+1}([\phi]) - \ldots - \gamma_{j_l+k_l}([\phi]))\right)$ *for every* $i = j_l, \ldots, j_l + k_l - 1$ *and* $l = 1, \ldots, d$.

We now explain how Proposition 4.3.6 follows from Lemma 4.3.7.

Proof of Proposition 4.3.6. We prove that $\mathfrak{C}$ is continuously differentiable at $[\phi_0]$. First assume $j_1 \neq 0$. The first two statements of Lemma 4.3.7, together with the observation made just before stating Lemma 4.3.7, imply that $\exp\left(i \cdot \sigma_i([\phi])\right)$ is continuously differentiable in a neighbourhood of $[\phi_0]$ for every index i in the complement of

$$\{j_1, \ldots, j_1 + k_1 - 1\} \cup \ldots \cup \{j_d, \ldots, j_d + k_d - 1\}.$$

These are precisely the indices i for which $a_i([\phi_0]) > 0$. Denote the collection of these indices $\mathcal{I}_{reg}$. If $j_1 = 0$, then we may only conclude that $\exp\left(i \cdot (\sigma_i([\phi]) - \sigma_{j_1+k_1}([\phi])\right)$ is continuously differentiable in a neighbourhood of $[\phi_0]$ for every index i in $\mathcal{I}_{reg}$. So, in both cases we know that

$$\exp\left(i \cdot (\sigma_i([\phi]) - \sigma_{j_1+k_1}([\phi])\right) \tag{4.3.2}$$

is continuously differentiable in a neighbourhood of $[\phi_0]$ for every index i in $\mathcal{I}_{reg}$.

Recall that if $a_i([\phi_0]) > 0$, then $\sqrt{a_i([\phi])}$ is differentiable in a neighbourhood of $[\phi_0]$. We decide to work in the chart $\{z_{j_1+k_1} \neq 0\}$ of $\mathbb{CP}^{n-3}$. So, proving that $\mathfrak{C}$ is continuously differentiable at $[\phi_0]$ amounts to prove that all the functions

$$\sqrt{a_i([\phi])} \exp\left(i \cdot (\sigma_i([\phi]) - \sigma_{j_1+k_1}([\phi])\right) \tag{4.3.3}$$

are continuously differentiable in a neighbourhood of $[\phi_0]$ for every $i \neq j_1 + k_1$. This is immediate for $i \in \mathcal{I}_{reg}$. For all the indices i such that $a_i([\phi_0]) = 0$, we proceed as follows. Recall from (4.3.2) that the functions $\exp\left(i \cdot (\sigma_i([\phi]) - \sigma_{j_1+k_1}([\phi])\right)$ are continuously differentiable for $i = j_l + k_l$ with $l = 2, \ldots, d$. So, proving that the functions of the type (4.3.3) are continuously differentiable for $i \notin \mathcal{I}_{reg}$ is equivalent to proving that all the functions

$$\sqrt{a_i([\phi])} \exp\left(i \cdot (\sigma_i([\phi]) - \sigma_{j_l+k_l}([\phi])\right)$$

are differentiable in a neighbourhood of $[\phi_0]$ for all $i = j_l, \ldots, j_l + k_l - 1$ and all $l = 1, \ldots, d$. This is exactly the third statement of Lemma 4.3.7. $\square$

The rest of this section is devoted to prove Lemma 4.3.7. The idea is to express the area and angle parameters as functions of the coordinates of the points $C_i = x_{C_i} + i \cdot y_{C_i}$ and $B_i = x_{B_i} + i \cdot y_{B_i}$. We start with the area parameters.

Lemma 4.3.8. *Let* $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. *For any* $i = 0, \dots, n-3$, *it holds*

$$a_i([\phi]) = 4 \arcsin \left(\frac{\sin\left(\frac{\alpha_{i+2}}{2}\right) \sin\left(\frac{\beta_i}{2}\right)}{2 \sin\left(\frac{\alpha_{i+2}+2\pi-\beta_{i+1}-\beta_i}{4}\right)} \cdot y_{C_{i+2}}^{-1} y_{B_i}^{-1} \left((x_{C_{i+2}} - x_{B_i})^2 + (y_{C_{i+2}} - y_{B_i})^2\right) \right),$$

where we abbreviated $\beta_i = \beta_i([\phi])$, $\beta_{i+1} = \beta_{i+1}([\phi])$, $C_{i+2} = C_{i+2}(\phi)$ *and* $B_i = B_i(\phi)$.

Proof. The formula is true if the triangle Δ_i is degenerate because then $B_i = C_{i+2}$. Recall that the hyperbolic distance $d(C_{i+2}, B_i)$ in the upper half-plane is given by

$$\cosh(d(C_{i+2}, B_i)) = 1 + \frac{(x_{C_{i+2}} - x_{B_i})^2 + (y_{C_{i+2}} - y_{B_i})^2}{y_{C_{i+2}} y_{B_i}}. \tag{4.3.4}$$

The hyperbolic law of cosines applied to the triangle $\Delta_i = \Delta(B_i, C_{i+2}, B_{i+1})$ gives

$$\cos\left(\pi - \frac{\beta_{i+1}}{2}\right) = -\cos\left(\pi - \frac{\alpha_{i+2}}{2}\right) \cos\left(\frac{\beta_i}{2}\right) + \sin\left(\pi - \frac{\alpha_{i+2}}{2}\right) \sin\left(\frac{\beta_i}{2}\right) \cosh(d(C_{i+2}, B_i)).$$

For geometric reasons, it makes sense to keep using $2\pi - \beta_{i+1}$ and not simplify the corresponding trigonometric terms. Using the angle sum identity for the cosine, this can be rewritten as

$$\begin{aligned} \cos\left(\pi - \frac{\beta_{i+1}}{2}\right) &= \cos\left(\frac{\alpha_{i+2}}{2}\right) \cos\left(\frac{\beta_i}{2}\right) + \sin\left(\frac{\alpha_{i+2}}{2}\right) \sin\left(\frac{\beta_i}{2}\right) \cosh(d(C_{i+2}, B_i)) \\ &= \cos\left(\frac{\alpha_{i+2} - \beta_i}{2}\right) + \sin\left(\frac{\alpha_{i+2}}{2}\right) \sin\left(\frac{\beta_i}{2}\right) (\cosh(d(C_{i+2}, B_i)) - 1). \end{aligned} \tag{4.3.5}$$

We use the trigonometric identity $\cos(x) + \cos(y) = -2\sin((x-y)/2)\sin((x+y)/2)$ to write

$$\cos\left(\frac{2\pi - \beta_{i+1}}{2}\right) - \cos\left(\frac{\alpha_{i+2} - \beta_i}{2}\right) = -2 \sin\left(\frac{2\pi - \beta_{i+1} - \alpha_{i+2} + \beta_i}{4}\right) \sin\left(\frac{2\pi - \beta_{i+1} + \alpha_{i+2} - \beta_i}{4}\right).$$

Using (4.2.1) we obtain

$$\cos\left(\frac{2\pi - \beta_{i+1}}{2}\right) - \cos\left(\frac{\alpha_{i+2} - \beta_i}{2}\right) = 2 \sin\left(\frac{a_i([\phi])}{4}\right) \sin\left(\frac{\alpha_{i+2} + 2\pi - \beta_i - \beta_{i+1}}{4}\right). \tag{4.3.6}$$

The conclusion follows from (4.3.4), (4.3.5) and (4.3.6). $\square$

The formula of Lemma 4.3.8 for the area parameters is relevant for the following reasons. Recall that the ranges of the functions β_i over $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ are compact subsets of $(0, 2\pi)$

explicitly written down in (4.1.11). So, the range of the ratio

$$\frac{\sin\left(\frac{\alpha_{i+2}}{2}\right)\sin\left(\frac{\beta_i}{2}\right)}{2\sin\left(\frac{\alpha_{i+2}+2\pi-\beta_i-\beta_{i+1}}{4}\right)}$$

over $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ is a compact interval inside the positive real numbers. On the other hand, the expression

$$y_{C_{i+2}}^{-1}y_{B_i}^{-1}\left((x_{C_{i+2}}-x_{B_i})^2+(y_{C_{i+2}}-y_{B_i})^2\right)$$

is zero whenever the triangle Δ_i is degenerate. This means that the function

$$\frac{a_i}{y_{C_{i+2}}^{-1}y_{B_i}^{-1}\left((x_{C_{i+2}}-x_{B_i})^2+(y_{C_{i+2}}-y_{B_i})^2\right)}$$

extends analytically to any $[\phi_0]$ such that $a_i([\phi_0])=0$. Moreover, its value at $[\phi_0]$ is the positive number

$$\frac{2\sin\left(\frac{\alpha_{i+2}}{2}\right)\sin\left(\frac{\beta_i([\phi_0])}{2}\right)}{\sin\left(\frac{\beta_{i+1}([\phi_0])}{2}\right)},$$

which remains uniformly bounded away from zero by the above remark for every such $[\phi_0]$. We conclude that the function

$$\sqrt{\frac{a_i}{y_{C_{i+2}}^{-1}y_{B_i}^{-1}\left((x_{C_{i+2}}-x_{B_i})^2+(y_{C_{i+2}}-y_{B_i})^2\right)}} \tag{4.3.7}$$

also extends analytically to any $[\phi_0]$ such that $a_i([\phi_0])=0$. We proved

Lemma 4.3.9. *The function defined by* (4.3.7) *on the subspace of* $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ *of all* $[\phi]$ *for which* $a_i([\phi])>0$ *extends analytically to* $\mathrm{Rep}_\alpha^{\mathrm{DT}}$.

We now proceed with a computation of the angle parameters. We start by introducing the function $\Gamma\colon \mathbb{H}\setminus\{i\}\to\mathbb{R}/2\pi\mathbb{Z}$ defined as

$$\Gamma(x+iy):=\begin{cases}0, & x=0 \text{ and } y>1,\\ \pi, & x=0 \text{ and } y<1,\\ 3\pi/2, & x^2+y^2=1 \text{ and } x>0,\\ \pi/2, & x^2+y^2=1 \text{ and } x<0,\\ \pi-\arctan\left(\frac{2x}{x^2+y^2-1}\right), & x^2+y^2<1,\\ -\arctan\left(\frac{2x}{x^2+y^2-1}\right), & x^2+y^2>1.\end{cases}$$

The different domains involved in the definition of Γ are illustrated on Figure 4.7.

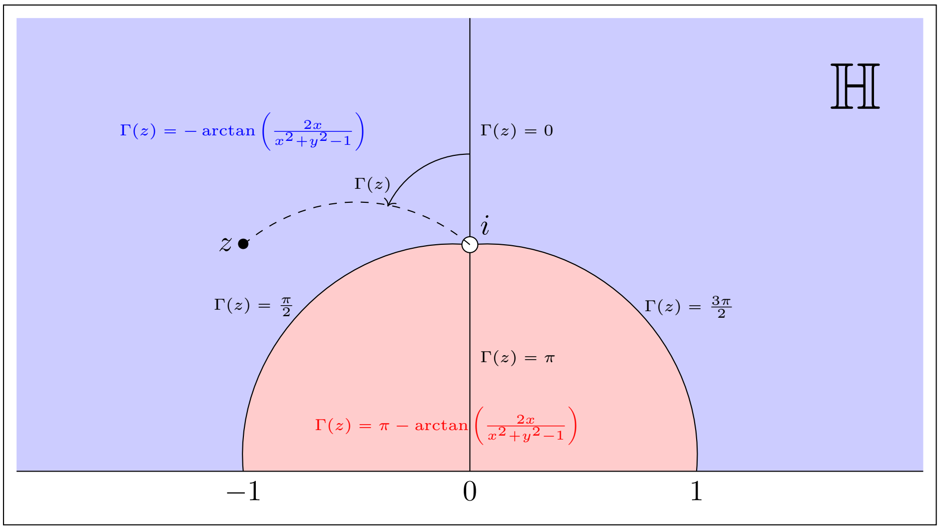


Figure 4.7.: Illustration of the different domains involved in the definition of the function Γ and the value of Γ in each of these regions.

The function Γ has a geometric interpretation. It measures the oriented angle between the vertical geodesic ray leaving from i and the geodesic ray leaving from i and going through $x + iy$. This can easily be seen after noticing that the ratio

$$\frac{x^2 + y^2 - 1}{2x}$$

is the point on the boundary of the upper half-plane which is the center of the semi-circle supporting the geodesic through i and $x + iy$.

Lemma 4.3.10. *The function* $\Gamma\colon \mathbb{H} \smallsetminus \{i\} \to \mathbb{R}/2\pi\mathbb{Z}$ *is continuously differentiable.*

Proof. We refer to Figure 4.7. The function Γ is continuously differentiable in the blue and red regions. These regions are open subdomains of $\mathbb{H}$. If one carefully studies the limit behaviour of Γ at the boundary of the blue and red regions, one sees that Γ is a continuous function. The partial derivatives inside the blue and red regions are

$$\frac{\partial}{\partial x}\left(-\arctan\left(\frac{2x}{x^2+y^2-1}\right)\right) = \frac{2(x^2-y^2+1)}{4x^2+(x^2+y^2-1)^2}$$

and

$$\frac{\partial}{\partial y}\left(-\arctan\left(\frac{2x}{x^2+y^2-1}\right)\right) = \frac{4xy}{4x^2+(x^2+y^2-1)^2}.$$

These partial derivatives extend continuously to $\mathbb{H}\smallsetminus\{i\}$. We conclude that Γ is continuously differentiable. $\square$

Lemma 4.3.11. *It holds*

$$\exp(i \cdot \Gamma(x+iy)) = \frac{x^2+y^2-1-i\cdot 2x}{\sqrt{4x^2+(x^2+y^2-1)^2}}.$$

Proof. We use the following identity:

$$e^{i\cdot\arctan(x)} = \frac{1+ix}{\sqrt{x^2+1}}.$$

It implies

$$\exp(i\cdot\Gamma(x+iy)) = \begin{cases} -\exp\left(i\arctan\left(\frac{2x}{x^2+y^2-1}\right)\right), & x^2+y^2<1,\\ \exp\left(i\arctan\left(\frac{2x}{x^2+y^2-1}\right)\right), & x^2+y^2>1.\end{cases}$$

Observe that

$$\sqrt{\frac{4x^2}{(x^2+y^2-1)^2}+1} = \frac{\sqrt{4x^2+(x^2+y^2-1)^2}}{|x^2+y^2-1|}.$$

Hence

$$\begin{aligned}\exp(i\cdot\Gamma(x+iy)) &= \frac{x^2+y^2-1}{\sqrt{4x^2+(x^2+y^2-1)^2}} - i\cdot\frac{2x}{x^2+y^2-1}\cdot\frac{x^2+y^2-1}{\sqrt{4x^2+(x^2+y^2-1)^2}}\\ &= \frac{x^2+y^2-1-i\cdot 2x}{\sqrt{4x^2+(x^2+y^2-1)^2}}.\end{aligned}$$ □

Let $p = x_p + iy_p$ be a point in $\mathbb{H}$. We introduce the function $\Gamma_p \colon \mathbb{H}\setminus\{p\} \to \mathbb{R}/2\pi\mathbb{Z}$ defined by

$$\Gamma_p(z) := \Gamma(y_p^{-1}(z-x_p)).$$

Note that the function Γ_p is defined to be the composition of the function Γ with the orientation-preserving isometry

$$\pm y_p^{1/2}\begin{pmatrix} y_p^{-1} & -x_p y_p^{-1}\\ 0 & 1\end{pmatrix}$$

of the upper half-plane that sends p to i. This isometry sends vertical geodesics to vertical geodesics. In other words, Γ_p measures the oriented angle between the vertical geodesic ray leaving from p and the geodesic ray leaving from p and going through z. The analogue of Lemma 4.3.11 for the function Γ_p reads

$$\exp(i\cdot\Gamma_p(x+iy)) = \frac{(x-x_p)^2+y^2-y_p^2-i\cdot 2y_p(x-x_p)}{\sqrt{4(x-x_p)^2+((x-x_p)^2+y^2-y_p^2)^2}}. \tag{4.3.8}$$

Lemma 4.3.12. *The function that maps a pair of distinct points (p, z) in $\mathbb{H} \times \mathbb{H}$ to*

$$\exp\big(i \cdot (\Gamma_z(p) - \Gamma_p(z))\big)$$

extends to a continuously differentiable function of $\mathbb{H} \times \mathbb{H}$.

Proof. Let $p = x_p + iy_p$ and $z = x_z + iy_z$. We use (4.3.8) to compute, with the help of Wolfram Mathematica[1],

$$\begin{aligned}\frac{\exp\big(i \cdot \Gamma_z(p)\big)}{\exp\big(i \cdot \Gamma_p(z)\big)} &= \frac{(x_p - x_z)^2 + y_p^2 - y_z^2 - i \cdot 2y_z(x_p - x_z)}{\sqrt{4(x_p - x_z)^2 + ((x_p - x_z)^2 + y_p^2 - y_z^2)^2}} \cdot \frac{\sqrt{4(x_z - x_p)^2 + ((x_z - x_p)^2 + y_z^2 - y_p^2)^2}}{(x_z - x_p)^2 + y_z^2 - y_p^2 - i \cdot 2y_p(x_z - x_p)} \\ &= \frac{(x_p - x_z) - i \cdot (y_p + y_z)}{(x_p - x_z) + i(y_p + y_z)}.\end{aligned}$$

The last expression is a continuously differentiable function of $\mathbb{H} \times \mathbb{H}$. □

The relation between the function Γ and the angle parameters is immediate. Let $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ be such that $a_i([\phi]) > 0$ and $a_{i-1}([\phi]) > 0$. Let $\ell_i(\phi)$ be the vertical geodesic ray leaving from $B_i(\phi)$. Using the definition of γ_i we obtain

$$\begin{aligned}\gamma_i([\phi]) &= \angle(\overrightarrow{B_i(\phi)C_{i+2}(\phi)}, \overrightarrow{B_i(\phi)C_{i+1}(\phi)}) \\ &= \angle(\ell_i(\phi), \overrightarrow{B_i(\phi)C_{i+1}(\phi)}) - \angle(\ell_i(\phi), \overrightarrow{B_i(\phi)C_{i+2}(\phi)}) \\ &= \Gamma_{B_i(\phi)}(C_{i+1}(\phi)) - \Gamma_{B_i(\phi)}(C_{i+2}(\phi)).\end{aligned} \tag{4.3.9}$$

The second conclusion of Corollary 4.1.6 says that

$$\angle(\overrightarrow{B_i(\phi)B_{i+1}(\phi)}, \overrightarrow{B_i(\phi)C_{i+2}(\phi)}) + \angle(\overrightarrow{B_i(\phi)C_{i+1}(\phi)}, \overrightarrow{B_i(\phi)B_{i-1}(\phi)}) = \beta_i/2 + \pi - \beta_i/2 = \pi.$$

This implies

$$\begin{aligned}\gamma_i([\phi]) &= \pi - \angle(\overrightarrow{B_i(\phi)B_{i-1}(\phi)}, \overrightarrow{B_i(\phi)B_{i+1}(\phi)}) \\ &= \pi - \big(\angle(\ell_i(\phi), \overrightarrow{B_i(\phi)B_{i+1}(\phi)}) - \angle(\ell_i(\phi), \overrightarrow{B_i(\phi)B_{i-1}(\phi)})\big) \\ &= \pi - \big(\Gamma_{B_i(\phi)}(B_{i+1}(\phi)) - \Gamma_{B_i(\phi)}(B_{i-1}(\phi))\big).\end{aligned} \tag{4.3.10}$$

Lemma 4.3.13. *Let $j < k$ and $[\phi_0] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ be such that $a_{j-1}([\phi_0]) > 0$, $a_{j+k}([\phi_0]) > 0$ and $a_i([\phi_0]) = 0$ for every $l = j, \ldots, j+k-1$. Then the function*

$$\exp\big(i \cdot (\gamma_j([\phi]) + \ldots + \gamma_{j+k}([\phi])\big)$$

[1] version 12.2.0.0

is continuously differentiable in a neighbourhood of $[\phi_0]$.

Proof. We know from the proof of Lemma 4.3.4 that the function is continuous. Using (4.3.10) we write (dropping the dependence on ϕ)

$$\begin{aligned}\gamma_j + \ldots + \gamma_{j+k} &= (k+1)\pi - \sum_{i=j}^{j+k} \left(\Gamma_{B_i}(B_{i+1}) - \Gamma_{B_i}(B_{i-1})\right) \\ &= (k+1)\pi + \Gamma_{B_j}(B_{j-1}) - \Gamma_{B_{j+k}}(B_{j+k+1}) - \sum_{i=j}^{j+k-1} \left(\Gamma_{B_i}(B_{i+1}) - \Gamma_{B_{i+1}}(B_i)\right).\end{aligned}$$

In a neighbourhood of $[\phi_0]$ we may assume that a_{j-1} and a_{j+k} are nonzero, so that $B_j \neq B_{j-1}$ and $B_{j+k} \neq B_{j+k+1}$. This means that the functions $\Gamma_{B_j}(B_{j-1})$ and $\Gamma_{B_{j+k}}(B_{j+k+1})$ are continuously differentiable around $[\phi_0]$. Lemma 4.3.12 implies that every summand in the remaining sum extends to a continuously differentiable function around $[\phi_0]$. This concludes the proof of the lemma. $\square$

The next step consists in expressing $\exp\left(i \cdot \Gamma_{B_{i+1}}(C_{i+2})\right)$ in terms of the coordinates of the points B_i and C_{i+1}. We first need to compute the coordinates of the point B_{i+1} in terms of that of B_i and C_{i+2}. Recall that B_{i+1} is the fixed point of $\phi(c_{i+2})^{-1}\phi(b_i)$. Using Lemma A.8 we can write down explicitly $\phi(c_{i+2})$ and $\phi(b_i)$ in terms of the coordinates of C_{i+2} and B_i, and the angles α_{i+2} and β_i. We can then compute the product $\phi(c_{i+2})^{-1}\phi(b_i)$ and deduce the coordinates of B_{i+1} using formula (A.1). With the help of Wolfram Mathematica, we obtain

$$x_{B_{i+1}} = \frac{2\left(x_{C_{i+2}} y_{B_i} \cos\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right) + x_{B_i} y_{C_{i+2}} \cos\left(\frac{\alpha_{i+2}}{2}\right)\sin\left(\frac{2\pi-\beta_i}{2}\right)\right) + \sin\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right)(x_{B_i}^2 - x_{C_{i+2}}^2 + y_{B_i}^2 - y_{C_{i+2}}^2)}{2\left(y_{B_i}\cos\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right) + y_{C_{i+2}}\cos\left(\frac{\alpha_{i+2}}{2}\right)\sin\left(\frac{2\pi-\beta_i}{2}\right) + \sin\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right)(x_{B_i} - x_{C_{i+2}})\right)}$$

and

$$x_{C_{i+2}} - x_{B_{i+1}} = \frac{-\sin\left(\frac{2\pi-\beta_i}{2}\right)\left(2y_{C_{i+2}}(x_{B_i} - x_{C_{i+2}})\cos\left(\frac{\alpha_{i+2}}{2}\right) + \sin\left(\frac{\alpha_{i+2}}{2}\right)\left((x_{B_i} - x_{C_{i+2}})^2 + y_{B_i}^2 - y_{C_{i+2}}^2\right)\right)}{2\left(y_{B_i}\cos\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right) + y_{C_{i+2}}\cos\left(\frac{\alpha_{i+2}}{2}\right)\sin\left(\frac{2\pi-\beta_i}{2}\right) + \sin\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right)(x_{B_i} - x_{C_{i+2}})\right)}.$$

We also have

$$y_{C_{i+2}}^2 - y_{B_{i+1}}^2 = y_{C_{i+2}}^2 - \frac{4y_{B_i}^2 y_{C_{i+2}}^2 - \left(-2y_{B_i} y_{C_{i+2}}\cos\left(\frac{2\pi-\beta_i}{2}\right)\cos\left(\frac{\alpha_{i+2}}{2}\right) + \sin\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right)\left((x_{B_i} - x_{C_{i+2}})^2 + y_{B_i}^2 + y_{C_{i+2}}^2\right)\right)^2}{4\left(y_{B_i}\cos\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right) + y_{C_{i+2}}\cos\left(\frac{\alpha_{i+2}}{2}\right)\sin\left(\frac{2\pi-\beta_i}{2}\right) + \sin\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right)(x_{B_i} - x_{C_{i+2}})\right)^2}.$$

We apply (4.3.8) to get

$$\begin{aligned} e^{i\cdot\Gamma_{B_{i+1}}(C_{i+2})} = \; & \frac{(x_{C_{i+2}} - x_{B_{i+1}})^2 + y_{C_{i+2}}^2 - y_{B_{i+1}}^2 - i\cdot 2y_{B_{i+1}}(x_{C_{i+2}} - x_{B_{i+1}})}{\sqrt{\left((x_{B_i} - x_{C_{i+2}})^2 + (y_{B_i} - y_{C_{i+2}})^2\right)\left((x_{B_i} - x_{C_{i+2}})^2 + (y_{B_i} + y_{C_{i+2}})^2\right)}} \\ & \cdot \frac{y_{B_i}\cos\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right) + y_{C_{i+1}}\cos\left(\frac{\alpha_{i+2}}{2}\right)\sin\left(\frac{2\pi-\beta_i}{2}\right) + \sin\left(\frac{2\pi-\beta_i}{2}\right)\sin\left(\frac{\alpha_{i+2}}{2}\right)(x_{B_i} - x_{C_{i+2}})}{-y_{C_{i+2}}\sin\left(\frac{2\pi-\beta_i}{2}\right)}.\end{aligned}$$

The crucial observation is that the irregularity of the function $\exp\left(i \cdot \Gamma_{B_{i+1}}(C_{i+2})\right)$ comes from the presence of the term $\sqrt{(x_{B_i} - x_{C_{i+2}})^2 + (y_{B_i} - y_{C_{i+2}})^2}$ in the denominator. If we compare this observation with Lemma 4.3.9 we conclude

Lemma 4.3.14. *The function*

$$[\phi] \mapsto \sqrt{a_i([\phi])} \exp\left(i \cdot \Gamma_{B_{i+1}(\phi)}(C_{i+2})(\phi)\right)$$

is continuously differentiable on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$.

We are now ready to prove Lemma 4.3.7.

Proof of Lemma 4.3.7. The first and second assertions of Lemma 4.3.7 follow from Lemma 4.3.13. To prove the third assertion, first observe that (as in the proof of Lemma 4.3.13) $-\gamma_{i+1}([\phi]) - \ldots - \gamma_{j_l+k_l}([\phi])$ can be written as

$$-(j_l + k_l - i)\pi - \Gamma_{B_{i+1}}(B_i) + \Gamma_{B_{j_l+k_l}}(B_{j_l+k_l+1}) + \sum_{m=i+1}^{j_l+k_l-1} \left(\Gamma_{B_m}(B_{m+1}) - \Gamma_{B_{m+1}}(B_m)\right).$$

The functions $\exp\left(i\cdot(\Gamma_{B_m}(B_{m+1}) - \Gamma_{B_{m+1}}(B_m))\right)$ are continuously differentiable by Lemma 4.3.12. Since $a_{j_l+k_l}([\phi_0]) > 0$ by hypothesis, the points $B_{j_l+k_l}(\phi)$ and $B_{j_l+k_l+1}(\phi)$ are distinct for $[\phi]$ is a neighbourhood of $[\phi_0]$. Thus the function $\exp\left(i \cdot \Gamma_{B_{j_l+k_l}}(B_{j_l+k_l+1})\right)$ is continuously differentiable in the same neighbourhood of $[\phi_0]$. Now observe that

$$\Gamma_{B_{i+1}}(B_i) = \Gamma_{B_{i+1}}(C_{i+2}) + \beta_{i+1}/2.$$

For the function β_{i+1} is analytic on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$, it is sufficient to prove that

$$\sqrt{a_i} \exp\left(i \cdot \Gamma_{B_{i+1}}(C_{i+2})\right)$$

is continuously differentiable in a neighbourhood of $[\phi_0]$ to conclude the third assertion of Lemma 4.3.7. This is precisely the statement of Lemma 4.3.14. □

This finishes the proof of Proposition 4.3.6.

4.3.4. Diffeomorphism property

We now prove

Proposition 4.3.15. *The map* $\mathfrak{C}\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ *is a diffeomorphism.*

Proof. Thanks to Proposition 4.3.1 and Proposition 4.3.6 we know that $\mathfrak{C}$ is a continuously differentiable bijection. To prove that $\mathfrak{C}$ is a diffeomorphism it is thus sufficient to prove that the differential of $\mathfrak{C}$ is regular at every point.

Let $[\phi_0] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ and assume for simplicity that $a_0([\phi_0]) > 0$. We decide to work in the chart $\{z_0 \neq 0\}$ of $\mathbb{CP}^{n-3}$. In a slight abuse of notation, we write $\mathfrak{C} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_{n-3})$ for the map

$$[\phi] \mapsto \left(\sqrt{\frac{a_1([\phi])}{a_0([\phi])}} e^{i\sigma_1([\phi])}, \ldots, \sqrt{\frac{a_{n-3}([\phi])}{a_0([\phi])}} e^{i\sigma_{n-3}([\phi])} \right) \in \mathbb{C}^{n-3}$$

defined for every $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ such that $a_0([\phi]) > 0$. We prove that $(d\mathfrak{C})_{[\phi_0]}$ is surjective. We distinguish two cases according to whether some area parameters of $[\phi_0]$ vanish.

First assume that $a_i([\phi_0]) > 0$ for every i. We consider the decomposition of the tangent space to $\mathbb{CP}^{n-3}$ at $\mathfrak{C}([\phi_0])$ as the direct sum of the kernel of the differential of the moment map ν defined in (4.2.5) and its complement V:

$$T_{\mathfrak{C}([\phi_0])}\mathbb{CP}^{n-3} = \mathrm{Ker}\left((d\nu)_{\mathfrak{C}([\phi_0])}\right) \oplus V.$$

Both subspaces have dimension $n-3$ because $\mathfrak{C}([\phi_0])$ lies in a regular fibre of ν. Since we are assuming $a_i([\phi_0]) > 0$ for every i, the exterior derivative of $\mathfrak{C}$ at $[\phi_0]$ is given in components by

$$(d\mathfrak{C}_i)_{[\phi_0]} = e^{i\sigma_i([\phi_0])} \left(i \cdot (d\sigma_i)_{[\phi_0]} \sqrt{\frac{a_i([\phi_0])}{a_0([\phi_0])}} + \frac{a_0([\phi_0])(da_i)_{[\phi_0]} - a_i([\phi_0])(da_0)_{[\phi_0]}}{2a_0([\phi_0])\sqrt{a_i([\phi_0])a_0([\phi_0])}} \right). \tag{4.3.11}$$

Note that since the torus action on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ is by diffeomorphisms, we can neglect the term $e^{i\sigma_i([\phi_0])}$ appearing in (4.3.11). Said differently, $d\mathfrak{C}$ is surjective at $[\phi_0]$ if and only if it is surjective at $[\phi_0']$, where $[\phi_0']$ and $[\phi_0]$ lie in the same fibre of the moment and $\sigma_i([\phi_0']) = 0$ for every i.

We consider a first family of smooth deformations of $[\phi_0]$ along the orbits corresponding to a fixed component of the torus action. Assume that we deform along the orbit corresponding to the ith component of the torus. Along that deformation all the area parameters a_j are constant and the angle parameters σ_j are constant for $j \neq i$. The image under the differential of $\mathfrak{C}$ of that deformation is generated by the complex direction of the ith component of $\mathbb{C}^{n-3}$ according to (4.3.11). Moreover, the images of each such deformation lie in the kernel of $(d\nu)_{\mathfrak{C}([\phi_0])}$ by Proposition 4.3.5. Comparing dimensions, we conclude that the image of $(d\mathfrak{C})_{[\phi_0]}$ contains the kernel of $(d\nu)_{\mathfrak{C}([\phi_0])}$.

Next we consider a second family of smooth deformations of $[\phi_0]$ corresponding to the complement W of the kernel of the differential of the moment map μ defined in (4.1.9):

$$T_{[\phi_0]} \operatorname{Rep}_\alpha^{\mathrm{DT}} = \operatorname{Ker}\left((d\mu)_{[\phi_0]}\right) \oplus W.$$

Proposition 4.3.5 says that $\mu = \nu \circ \mathfrak{C}$ which implies $(d\mathfrak{C})_{[\phi_0]}(W) \subset V$. Because both $(d\mu)_{[\phi_0]}$ and $(d\nu)_{\mathfrak{C}([\phi_0])}$ have maximal rank, we conclude that $(d\mathfrak{C})_{[\phi_0]}$ maps W surjectively onto V. This shows that the image of $(d\mathfrak{C})_{[\phi_0]}$ contains V. We conclude that $(d\mathfrak{C})_{[\phi_0]}$ is surjective.

Now, we deal with the case where $a_i([\phi_0]) = 0$ for some index i. The argument here relies on the existence of a smooth deformation $[\phi_t]$ of $[\phi_0]$ such that $a_i([\phi_t]) > 0$ for $t \neq 0$. The existence of such a deformation is a general property of symplectic toric manifolds (recall that a_i is a multiple of the ith component of the moment map μ). Let us abbreviate $a_i(t) := a_i([\phi_t])$. Note that, by assumption, $a_i(0) = a_i'(0) = 0$. We can choose the deformation $[\phi_t]$ to ensure that $a_i''(0) > 0$. Proposition 4.3.6 implies that the function

$$[\phi] \mapsto \sqrt{\frac{a_1([\phi])}{a_0([\phi])}} e^{i\sigma_1([\phi])}$$

is continuously differentiable. We claim that its derivative at $[\phi_0]$ along the deformation $[\phi_t]$ is nonzero. This is the case because we assumed $a_i''(0) > 0$. This means that the image of $(d\mathfrak{C}_i)_{[\phi_0]}$ inside $\mathbb{C}$ has real dimension at least one. However, whenever $a_i([\phi_0]) = 0$, there is an effective action of the ith component of the torus on $T_{[\phi_0]} \operatorname{Rep}_\alpha^{\mathrm{DT}}$. Since the torus action is by diffeomorphisms and $\mathfrak{C}$ is equivariant, we conclude that $(d\mathfrak{C}_i)_{[\phi_0]}$ is surjective. We can repeat this argument for every index i such that $a_i([\phi_0]) = 0$. Combined with the previous case, this shows that $(d\mathfrak{C})_{[\phi_0]}$ is surjective even when some area parameters vanish. □

4.3.5. Symplectomorphism property

Finally, we prove that

Proposition 4.3.16. *The map $\mathfrak{C}\colon \operatorname{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{CP}^{n-3}$ is a symplectomorphism, i.e.*

$$\lambda \cdot \mathfrak{C}^* \omega_{\mathcal{FS}} = \omega_{\mathcal{G}}.$$

To prove Proposition 4.3.16, we need the following result about symplectic toric manifolds. The result is folklore; a proof is included for completeness.

Lemma 4.3.17. *Let M be a compact connected smooth manifold of dimension $2m$. Assume that M is equipped with an effective action of an m-dimensional torus. Let ω_1 and ω_2 be* *two symplectic forms on M for which the torus action is Hamiltonian with respect to the same moment map $\mu\colon M \to \mathbb{R}^m$. Then $\omega_1 = \omega_2$.*

Proof. Let $\mathring{M}$ denote the preimage of the interior of the moment polytope. This is an open and dense subset of M. It is thus sufficient to check that $\omega_1 = \omega_2$ on $\mathring{M}$. The Arnold–Liouville Theorem states (see e.g. [CdS01]) the existence of angle coordinates $(\varphi_1, \dots, \varphi_m)$ and $(\psi_1, \dots, \psi_m)$ defined on $\mathring{M}$ such that

$$\omega_1\restriction_{\mathring{M}} = \sum_{i=1}^{m} d\mu_i \wedge d\varphi_i, \quad \omega_2\restriction_{\mathring{M}} = \sum_{i=1}^{m} d\mu_i \wedge d\psi_i,$$

where $\mu = (\mu_1, \dots, \mu_m)$ is the moment map.

We are assuming that the torus action on M is Hamiltonian with moment map μ for both ω_1 and ω_2. So, for any $\theta \in \mathbb{R}^m$, if Θ denotes the vector field on M defined by the infinitesimal action of θ, then $\omega_1(\Theta, \cdot) = d\langle\mu, \theta\rangle = \omega_2(\Theta, \cdot)$. By letting θ range over the standard basis of $\mathbb{R}^m$, we observe that $d\varphi_i = d\psi_i$ must hold for every i. Hence $\omega_1\restriction_{\mathring{M}} = \omega_2\restriction_{\mathring{M}}$. This concludes the proof of the lemma. $\square$

Proof of Proposition 4.3.16. We want to apply Lemma 4.3.17 for the torus action (4.1.7) on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$. This action is Hamiltonian with respect to the symplectic form $\omega_{\mathcal{G}}$ and the moment map μ defined in (4.1.9). Propostition 4.3.15 says that $\mathfrak{C}$ is a diffeomorphism. So, $\lambda \cdot \mathfrak{C}^*\omega_{\mathcal{FS}}$ is another symplectic form on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$. Proposition 4.3.5 implies that the torus action (4.1.7) on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ is Hamiltonian with respect to the symplectic form $\lambda \cdot \mathfrak{C}^*\omega_{\mathcal{FS}}$ and the moment map μ. Hence, by Lemma 4.3.17, $\lambda \cdot \mathfrak{C}^*\omega_{\mathcal{FS}} = \omega_{\mathcal{G}}$. $\square$

Propositions 4.3.5 and 4.3.16 together prove Theorem 4.2.1.

5. Dynamics on the Deroin-Tholozan relative character variety

5.1. Preliminaries

The mapping class group action on the relative character variety $\mathrm{Rep}_\alpha(\pi_n, \mathrm{PSL}(2,\mathbb{R}))$ restricts to an action on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ because the volume of a representation is $\mathrm{Mod}(\Sigma_n)$-invariant by Lemma 2.6.7. Denote by $\nu_{\mathcal{G}}$ the Goldman symplectic measure on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ normalized so that $\nu_{\mathcal{G}}(\mathrm{Rep}_\alpha^{\mathrm{DT}}) = 1$.

5.1.1. Relation to symplectic geometry

To prove that the $\mathrm{Mod}(\Sigma_n)$-action on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ is ergodic we follow a method developed in [GX11] and [MW16]. It relies essentially on the observation that a Dehn twist τ_a along a non-trivial simple closed curve a on Σ_n is closely related to some Hamiltonian flow. This crucial observation is explained in this section.

Recall that we introduced a function ϑ that maps smoothly elliptic elements in $\mathrm{PSL}(2,\mathbb{R})$ to their rotation angle in $(0, 2\pi)$. Proposition 3.1.10 says that for any non-trivial homotopy class $a \in \pi_n$ freely homotopic to a simple closed curve and any Deroin–Tholozan representation $\phi\colon \pi_n \to \mathrm{PSL}(2,\mathbb{R})$, the image $\phi(a)$ is elliptic. Consider the following function

$$\begin{aligned} \vartheta_a\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} &\longrightarrow (0, 2\pi) \\ [\phi] &\mapsto \vartheta(\phi(a)). \end{aligned} \tag{5.1.1}$$

Let $\Phi_a^t\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathrm{Rep}_\alpha^{\mathrm{DT}}$ denote the Hamiltonian flow of ϑ_a at time $t \in \mathbb{R}$. The flow Φ_a^t is called the *twist flow* of (ϑ, a), in tradition with [Gol84].

Recall that $a \in \pi_n$ determines a unique (up to free homotopy) simple closed curve which we also denote by a. Cutting Σ_n along a determines two surfaces $S_1 \sqcup_a S_2 = \Sigma_n$. The computations conducted in [DT19, Prop. 3.3] from the original definition of twist flows by Goldman show that

$$\Phi_a^{\vartheta_a([\phi])/2}([\phi]) \colon c_i \mapsto \begin{cases} \phi(c_i) & \text{if } c_i \in \pi_1(S_1), \\ \phi(a)\phi(c_i)\phi(a)^{-1} & \text{if } c_i \in \pi_1(S_2). \end{cases} \tag{5.1.2}$$

Goldman–Xia observed in [GX11] that the representation (5.1.2) corresponds precisely to the representation obtained by letting the Dehn twist $\tau_a \in \mathrm{Mod}(\Sigma_n)$ along the curve a act on $[\phi]$. This is the crucial observation mentioned in introduction that connects the symplectic geometry of $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ to the action of $\mathrm{Mod}(\Sigma_n)$. Formally, the following holds.

Proposition 5.1.1. *Let $a \in \pi_n$ be a non-trivial homotopy class of loops freely homotopic to a simple closed curve on Σ_n. Then*

$$\tau_a[\phi] = \Phi_a^{\vartheta_a([\phi])/2}([\phi]), \quad \forall [\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}.$$

Proposition 5.1.1 is used as such in [MW16, Prop. 6.5]. The analogue of Proposition 5.1.1 for SU(2)-character varieties can be found in [GX11, Prop. 5.1].

5.1.2. Ergodic actions

A measure preserving action of a group G on a probability measure space (X, μ) is *ergodic* if for all measurable sets $U \subset X$

$$gU = U, \quad \forall g \in G \quad \Longrightarrow \quad \mu(U) \in \{0, 1\}.$$

Ergodicity means that the dynamical system induced by the G-action on X admits no non-trivial subsystems. Ergodic systems exhibit a certain level of chaos through their dynamics: mixing systems are ergodic and ergodic systems have almost only dense orbits (provided that the measure is Borel). The standard example of ergodic actions are irrational rotations of the circle, see e.g. [EW11, Prop. 2.16].

Ergodicity can be characterized in terms of invariant functions. The regularity class of those functions can be restricted as long as it contains the indicator functions of all measurable sets. For the purpose of this note, and in view of Lemma 5.2.5, we choose to characterize ergodicity in terms of integrable functions.

Lemma 5.1.2. *A measure preserving action of a group G on a probability measure space (X, μ) is ergodic if and only if every G-invariant integrable function $f \colon X \to \mathbb{R}$ is constant almost everywhere.*

We refer the reader to [EW11] for the proof of Lemma 5.1.2 and for further consideration on ergodic actions.

Checking that a function is constant almost everywhere can be done locally. This strategy was employed by Marché –Wolff in [MW16]. The statement is the following. Assume that X is a topological space and μ is a strictly positive Borel measure on X, i.e. $\mu(U) > 0$ for every nonempty open set $U \subset X$.

Lemma 5.1.3. *Let $f\colon X \to \mathbb{R}$ be an integrable function. Assume that there exists an open set $\Omega \subset X$ such that*

1. *Ω is connected,*
2. *$\mu(\Omega) = 1$,*
3. *for all $x \in \Omega$, there exists an open set $U_x \subset \Omega$ containing x such that f is constant almost everywhere on U_x.*

Then f is constant almost everywhere.

Proof. Define the function $F\colon \Omega \to \mathbb{R}$ by

$$F(x) := \frac{1}{\mu(U_x)} \int_{U_x} f \, d\mu.$$

Informally, $F(x)$ is the constant value reached by f almost everywhere on U_x. For every $y \in U_x$, the set $U_x \cap U_y$ is nonempty and thus has positive measure by assumption. Moreover

$$\frac{1}{\mu(U_x)} \int_{U_x} f \, d\mu = \frac{1}{\mu(U_x \cap U_y)} \int_{U_x \cap U_y} f \, d\mu = \frac{1}{\mu(U_y)} \int_{U_y} f \, d\mu.$$

So, $F(x) = F(y)$. This means that F is locally constant on Ω (and not only almost everywhere). For Ω is connected, F is thus constant on Ω.

Now, because $F\restriction_{U_x}$ is constant, f and F coincide almost everywhere on U_x for every $x \in \Omega$. Hence $f = F$ almost everywhere on Ω. Since F is constant on Ω and $\mu(\Omega) = 1$, we conclude that f is constant almost everywhere. $\square$

5.2. The skeleton of the proof

According to Lemma 5.1.2, it is sufficient to show that every $\mathrm{Mod}(\Sigma_n)$-invariant integrable function $f\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{R}$ is constant almost everywhere in order to prove Theorem D. The tool for this is Lemma 5.1.3. We apply the latter by constructing an open set Ω that satisfies the required hypotheses for any $\mathrm{Mod}(\Sigma_n)$-invariant integrable function $f\colon \mathrm{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{R}$.

In this section, we first define the open set Ω. We then state two technical lemmata, namely Lemma 5.2.3 and Lemma 5.2.5. Their proofs are postponed to Sections 5.3 and 5.4. In a third and last part, we prove that Ω satisfies all three conditions of Lemma 5.1.3, assuming that the two lemmata mentioned above hold.

5.2.1. The set Ω

We consider the following $2(n-3)$ elements of π_n: For every $i = 1, \dots, n-3$, let

$$\begin{aligned} b_i &:= c_{i+1}^{-1} c_i^{-1} \cdot \ldots \cdot c_1^{-1}, \\ d_i &:= c_{i+2}^{-1} c_{i+1}^{-1}. \end{aligned}$$

The curves c_i refer to the presentation (2.1.3). The free homotopy classes of loops corresponding to c_i, b_i, d_i can be represented by oriented simple closed curves, also denoted c_i, b_i, d_i, as illustrated on Figure 5.1.

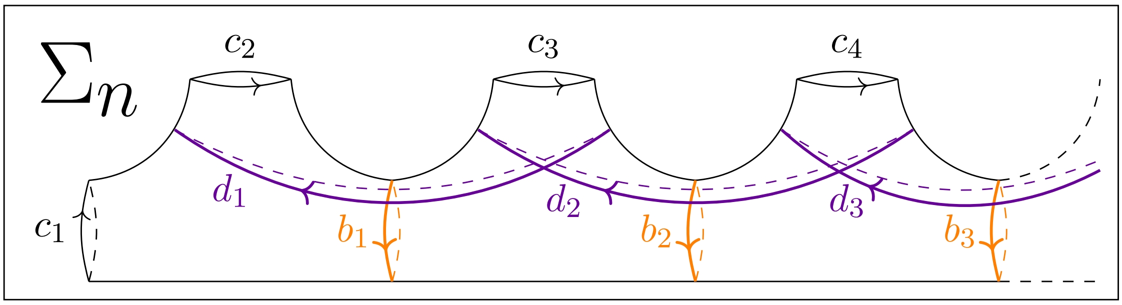


Figure 5.1.: The simple closed curves $b_1, \dots, b_{n-3}$ and $d_1, \dots, d_{n-3}$, and the peripheral curves $c_1, \dots, c_n$. This illustration is modelled on [DT19, Fig. 2].

Deroin–Tholozan proved in [DT19, Prop. 3.3] that the Hamiltonian flows of $\vartheta_{b_1}, \dots, \vartheta_{b_{n-3}}$ are π-periodic and define a symplectic toric manifold structure on $(\mathrm{Rep}_\alpha^{\mathrm{DT}}, \omega_{\mathcal{G}})$. The associated moment map $\mu := (\vartheta_{b_1}, \dots, \vartheta_{b_{n-3}})$ maps $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ to a convex polytope Δ inside $\mathbb{R}^{n-3}$. We denote by $\mathring{\Delta}$ the interior of Δ. The subspace $\mu^{-1}(\mathring{\Delta}) \subset \mathrm{Rep}_\alpha^{\mathrm{DT}}$ is open and dense. The fibres of μ over $\mathring{\Delta}$ are Lagrangian tori.

Because of the symplectic toric structure on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$, for any $i = 1, \dots, n-3$, the Hamiltonian flow Φ_{b_i} has the following orbit structure. Its orbits are either fixed points or circles of length π. Since any of the curves $d_1, \dots, d_{n-3}$ can be mapped to b_1 by a cyclic permutation of the punctures, the Hamiltonian flows $\Phi_{d_1}, \dots, \Phi_{d_{n-3}}$ have the same orbit structure as Φ_{b_1}.

Definition 5.2.1. We call the orbit of $[\phi] \in \operatorname{Rep}_\alpha^{\mathrm{DT}}$ under the combined Hamiltonian flows $\Phi_{b_1}, \ldots, \Phi_{b_{n-3}}$ *regular* if it is homeomorphic to an $(n-3)$-torus, equivalently if $\mu([\phi]) \in \mathring{\Delta}$. It is called *irrational* if it is regular and $\vartheta_{b_i}([\phi]) \in \mathbb{R} \smallsetminus \pi\mathbb{Q}$ for every $i = 1, \ldots, n-3$, equivalently $\mu([\phi]) \in \mathring{\Delta} \cap (\mathbb{R} \smallsetminus \pi\mathbb{Q})^{n-3}$.

As for any symplectic manifold, there is a Poisson bracket on $\operatorname{Rep}_\alpha^{\mathrm{DT}}$ associated to $\omega_{\mathcal{G}}$:

$$\{\cdot,\cdot\}\colon C^\infty\left(\operatorname{Rep}_\alpha^{\mathrm{DT}}\right) \times C^\infty\left(\operatorname{Rep}_\alpha^{\mathrm{DT}}\right) \to C^\infty\left(\operatorname{Rep}_\alpha^{\mathrm{DT}}\right).$$

It is defined as follows: for two smooth functions $\zeta_1, \zeta_2\colon \operatorname{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{R}$ with Hamiltonian vector fields X_{ζ_1}, X_{ζ_2}, let

$$\{\zeta_1, \zeta_2\} := \omega_{\mathcal{G}}(X_{\zeta_1}, X_{\zeta_2}) = d\zeta_2(X_{\zeta_1}). \tag{5.2.1}$$

We denote by $(\tau_{b_i})^{m_i} d_i$ the simple closed curve obtained from d_i by applying m_i iterations of the Dehn twist τ_{b_i}. Inspired by the work of Marché –Wolff [MW16], we introduce

$$\Omega := \left\{[\phi] \in \mu^{-1}(\mathring{\Delta}) : \forall i = 1, \ldots, n-3,\ \exists m_i \in \mathbb{Z},\ \{\vartheta_{b_i}, \vartheta_{(\tau_{b_i})^{m_i} d_i}\}\big([\phi]\big) \neq 0\right\}.$$

Note that $\Omega \subset \operatorname{Rep}_\alpha^{\mathrm{DT}}$ is open and measurable since

$$\Omega = \mu^{-1}(\mathring{\Delta}) \cap \bigcap_{i=1}^{n-3} \bigcup_{m_i \in \mathbb{Z}} \{\vartheta_{b_i}, \vartheta_{(\tau_{b_i})^{m_i} d_i}\}^{-1}\big(\mathbb{R} \smallsetminus \{0\}\big).$$

We claim that Ω satisfies all the hypotheses of Lemma 5.1.3. This follows from Lemma 5.2.7 and Lemma 5.2.8 below.

Remark 5.2.2. It is worth pointing out that the definition of the set Ω does not depend on the function $f\colon \operatorname{Rep}_\alpha^{\mathrm{DT}} \to \mathbb{R}$ that we fixed previously. One may wonder if Ω is actually distinct from $\mu^{-1}(\mathring{\Delta})$. The answer in general remains unknown to the author; however there exist special symmetric cases where the answer is yes.

Assume for simplicity that $n = 4$. Recall that for $n = 4$ the character variety of Deroin–Tholozan representations is symplectomorphic to the 2-sphere. Assume further that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. In this case, the Hamiltonian flows Φ_{b_1} and Φ_{d_1} are rotations around two perpendicular axes of the 2-sphere. We can think of the two fixed points of Φ_{b_1} as the poles of the sphere and the two fixed points of Φ_{d_1} as two diametrically opposite points on the equator (see Figure 5.2). Denote the fixed points of Φ_{d_1} by $[\phi_1]$ and $[\phi_2]$. The equator is the Φ_{b_1}-orbit characterized by $(\vartheta_{b_1})^{-1}(\pi)$. Hence it holds $\tau_{b_1}[\phi_1] = [\phi_2]$ and $\tau_{b_1}[\phi_2] = [\phi_1]$ by Proposition 5.1.1. In particular, because the Hamiltonian vector field of ϑ_{d_1} vanishes at $[\phi_1]$ and $[\phi_2]$, it holds

$$\{\vartheta_{b_1}, \vartheta_{d_1}\}\big((\tau_{b_1})^m[\phi_1]\big) = 0, \quad \forall m \in \mathbb{Z}.$$

Anticipating Lemma 5.2.6, this implies

$$\{\vartheta_{b_1}, \vartheta_{(\tau_{b_1})^m d_1}\}([\phi_1]) = \{\vartheta_{b_1}, \vartheta_{(\tau_{b_1})^m d_1}\}([\phi_2]) = 0, \quad \forall m \in \mathbb{Z}.$$

Therefore $[\phi_1], [\phi_2] \in \mu^{-1}(\mathring{\Delta}) \smallsetminus \Omega$.

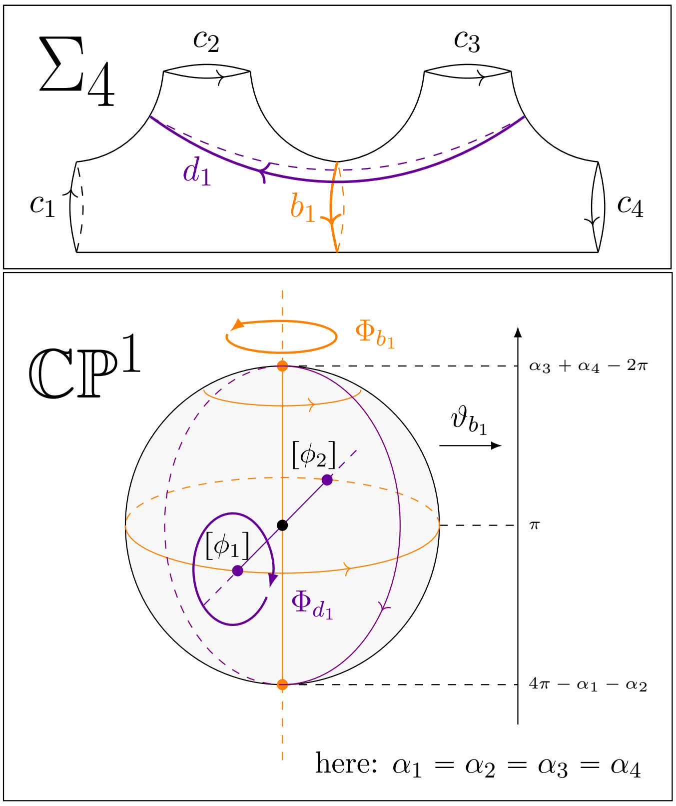


Figure 5.2.: On top: the 4-punctured sphere and the curves b_1, d_1. On the bottom: the flows of Φ_{b_1} and Φ_{d_1} seen as rotations around two perpendicular axes of the 2-sphere when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$.

5.2.2. Two technical lemmata

The proof that Ω is connected and has full measure relies on the following Key Lemma and its corollary. The proof of the Key Lemma 5.2.3 is postponed to Section 5.4 and that of its corollary to Subsection 5.2.3 below.

Lemma 5.2.3 (Key Lemma)**.** *For every $i = 1, \dots, n-3$, every orbit of the Hamiltonian flow Φ_{b_i} contained inside $\mu^{-1}(\mathring{\Delta})$ contains at most two points at which $\{\vartheta_{b_i}, \vartheta_{d_i}\}$ vanishes.*

Moreover, if two such points exist, then they are diametrically opposite, i.e. they are images of each other under $\Phi_{b_i}^{t_0}$ where t_0 is half the minimal period of the corresponding orbit (here $t_0 = \pi/2$).

A particular case where two diametrically opposite points as in the conclusion of the Key Lemma 5.2.3 exist is described in Remark 5.2.2 and illustrated on Figure 5.2. The Key Lemma 5.2.3 has the following implication on the structure of Ω.

Corollary 5.2.4. *The set Ω contains all irrational orbits of the Hamiltonian flows $\Phi_{b_1}, \dots, \Phi_{b_{n-3}}$.*

The third hypothesis of Lemma 5.1.3, namely that f is locally constant almost everywhere on Ω, is a consequence of the ergodicity of irrational circle rotations and of the following result. Consider the unit hypercube $[0,1]^n \subset \mathbb{R}^n$. For $i = 1, \dots, n$, denote by $\pi_i\colon [0,1]^n \to [0,1]^{n-1}$ the projection map defined by forgetting the ith component.

Lemma 5.2.5 (Rectangle trick)**.** *Let $\varphi \in L^1([0,1]^n)$. Assume that there exist full-measure sets $E_1, \dots, E_n \subset [0,1]^{n-1}$ such that for all $i = 1, \dots, n$ and for all $x \in E_i$, $\varphi\restriction_{\pi_i^{-1}(x)}$ is constant almost everywhere. Then φ is constant almost everywhere.*

The case $n = 2$ of Lemma 5.2.5 reads as follows: any integrable function which is constant almost everywhere on almost every vertical and horizontal line in a rectangle is constant almost everywhere on the rectangle. Lemma 5.2.5 is certainly known to experts. However, there is a lack of concrete references in the existing literature and therefore we provide a proof of Lemma 5.2.5 in Section 5.3. We now prove that Ω satisfies the three hypotheses of Lemma 5.1.3.

5.2.3. First and second hypotheses

We start with a useful formula.

Lemma 5.2.6. *Let a, b be two simple closed curves on Σ_n. Then, for any integer m, it holds*

$$\{\vartheta_a, \vartheta_{(\tau_a)^m b}\}([\phi]) = \{\vartheta_a, \vartheta_b\}\big((\tau_a)^m[\phi]\big), \quad \forall [\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}.$$

Proof. Let $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. It suffices to check that

$$\{\vartheta_a, \vartheta_{\tau_a b}\}([\phi]) = \{\vartheta_a, \vartheta_b\}(\tau_a[\phi]).$$

The general formula follows by induction. We compute

$$\begin{aligned}
\vartheta_{\tau_a b}([\phi]) &= \vartheta\big(\phi(\tau_a b)\big) \\
&= \vartheta\big((\tau_a \phi)(b)\big) \\
&= \vartheta_b(\tau_a[\phi]).
\end{aligned}$$

The first and third equalities are an application of the definition of the functions $\vartheta_{\tau_a b}$ and ϑ_b (see (5.1.1)). For the second equality, recall that $\mathrm{Mod}(\Sigma_n)$ acts on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ by precomposition. Using Proposition 5.1.1, we conclude that

$$\vartheta_{\tau_a b}([\phi]) = \vartheta_b \circ \Phi_a^{\vartheta_a([\phi])/2}([\phi]). \tag{5.2.2}$$

Let X_a denote the Hamiltonian vector field of ϑ_a. For every time t it holds

$$X_a\big(\Phi_a^t([\phi])\big) = \big(d\Phi_a^t\big)_{[\phi]}\big(X_a([\phi])\big).$$

In particular for $t = \vartheta_a([\phi])/2$ we get

$$X_a(\tau_a[\phi]) = X_a\left(\Phi_a^{\vartheta_a([\phi])/2}([\phi])\right) = \left(d\Phi_a^{\vartheta_a([\phi])/2}\right)_{[\phi]}\big(X_a([\phi])\big). \tag{5.2.3}$$

Thus

$$\begin{aligned}
\{\vartheta_a, \vartheta_b\}(\tau_a[\phi]) &\overset{(5.2.1)}{=} (d\vartheta_b)_{\tau_a[\phi]}\big(X_a(\tau_a[\phi])\big) \\
&\overset{(5.2.3)}{=} (d\vartheta_b)_{\tau_a[\phi]} \circ \left(d\Phi_a^{\vartheta_a([\phi])/2}\right)_{[\phi]}\big(X_a([\phi])\big) \\
&= d\left(\vartheta_b \circ \Phi_a^{\vartheta_a([\phi])/2}\right)_{[\phi]}\big(X_a([\phi])\big) \\
&\overset{(5.2.2)}{=} (d\vartheta_{\tau_a b})_{[\phi]}\big(X_a([\phi])\big) \\
&\overset{(5.2.1)}{=} \{\vartheta_a, \vartheta_{\tau_a b}\}([\phi]).
\end{aligned}$$

The middle equality is an application of the chain rule. This concludes the proof of the lemma. $\square$

We now proceed with the proof of Corollary 5.2.4 assuming that the Key Lemma 5.2.3 holds.

Proof of Corollary 5.2.4. Let $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ be a point on some irrational orbit of the Hamiltonian flows $\Phi_{b_1}, \ldots, \Phi_{b_{n-3}}$. We want to prove that $[\phi] \in \Omega$.

Assume *ab absurdo* that $[\phi] \notin \Omega$, i.e. there exists $i \in \{1, \ldots, n-3\}$ such that

$$\{\vartheta_{b_i}, \vartheta_{(\tau_{b_i})^m d_i}\}([\phi]) = 0, \quad \forall m \in \mathbb{Z}.$$

Proposition 5.1.1 implies that $\vartheta_{b_i}(\tau_{b_i}[\phi]) = \vartheta_{b_i}([\phi])$ and hence

$$(\tau_{b_i})^m[\phi] = \Phi_{b_i}^{m\vartheta_{b_i}([\phi])/2}([\phi]), \quad \forall m \in \mathbb{Z}.$$

So, by Lemma 5.2.6 we obtain

$$\{\vartheta_{b_i}, \vartheta_{d_i}\}\left(\Phi_{b_i}^{m\vartheta_{b_i}([\phi])/2}([\phi])\right) = 0, \quad \forall m \in \mathbb{Z}.$$

Since by assumption $\vartheta_{b_i}([\phi]) \in \mathbb{R} \smallsetminus \pi\mathbb{Q}$, all the points $\Phi_{b_i}^{m\vartheta_{b_i}([\phi])/2}([\phi])$ for $m \in \mathbb{Z}$ form a dense subset of the Φ_{b_i}-orbit of $[\phi]$. Hence, by continuity, the function $\{\vartheta_{b_i}, \vartheta_{d_i}\}$ vanishes on the whole Φ_{b_i}-orbit of $[\phi]$. This is a contradiction to the Key Lemma 5.2.3. So, we conclude as expected that $[\phi] \in \Omega$. $\square$

Lemma 5.2.7. *The set Ω is connected and satisfies $\nu_{\mathcal{G}}(\Omega) = 1$.*

Proof. The toric manifold structure on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ implies that $\mu^{-1}(\mathring{\Delta}) \subset \mathrm{Rep}_\alpha^{\mathrm{DT}}$ is symplectomorphic to the product of $\mathring{\Delta}$ with the standard $(n-3)$-torus. So, Corollary 5.2.4 immediately implies that $\nu_{\mathcal{G}}(\Omega) = 1$ because $\mathring{\Delta} \cap (\mathbb{R} \smallsetminus \pi\mathbb{Q})^{n-3}$ has full measure in $\mathring{\Delta}$ and $\nu_{\mathcal{G}}(\mu^{-1}(\mathring{\Delta})) = 1$.

We now prove that Ω is connected. The proof essentially uses that $\mathring{\Delta}$ is connected. Assume that $\Omega = A \cup B$ where $A, B \subset \Omega$ are open and disjoint. We prove that under these assumptions either A or B is empty.

By construction $\mu(\Omega) \subset \mathring{\Delta}$. Corollary 5.2.4 says that any irrational orbit is entirely contained in Ω. Moreover, the Key Lemma 5.2.3 also implies that any orbit $\mu^{-1}(x)$ for $x \in \mathring{\Delta} \smallsetminus (\mathbb{R} \smallsetminus \pi\mathbb{Q})^{n-3}$ must intersect Ω. So,

$$\mu(A) \cup \mu(B) = \mathring{\Delta}.$$

Tori being connected, each irrational orbit is contained either in A or in B. Because $\mathring{\Delta} \smallsetminus (\mathbb{R} \smallsetminus \pi\mathbb{Q})^{n-3}$ is dense in $\mathring{\Delta}$ and both A and B are open, $\mu^{-1}(x) \cap \Omega$ must be contained either in A or in B for every $x \in \mathring{\Delta} \smallsetminus (\mathbb{R} \smallsetminus \pi\mathbb{Q})^{n-3}$. Hence,

$$\mu(A) \cap \mu(B) = \varnothing.$$

Recall that moment maps are open maps. So, both $\mu(A)$ and $\mu(B)$ are open subsets of $\mathring{\Delta}$. Because $\mathring{\Delta}$ is connected, it follows that either $\mu(A)$ or $\mu(B)$ is empty, and consequently that either A or B is empty. This concludes the proof of the lemma. $\square$

5.2.4. The third hypothesis

We use the Rectangle Trick (Lemma 5.2.5) to prove

Lemma 5.2.8. *For every $[\phi] \in \Omega$, there exists an open neighbourhood $U_{[\phi]} \subset \Omega$ of $[\phi]$ such that f is constant almost everywhere on $U_{[\phi]}$.*

Proof. Let $[\phi] \in \Omega$. By definition of Ω there exists for every $i = 1, \dots, n-3$ an integer m_i such that

$$\{\vartheta_{b_i}, \vartheta_{(\tau_{b_i})^{m_i} d_i}\}([\phi]) \neq 0.$$

This means that the tangent spaces to $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ in a neighbourhood of $[\phi]$ are generated by the $2(n-3)$ Hamiltonian vector fields

$$X_{b_i}, X_{(\tau_{b_i})^{m_i} d_i}, \quad i = 1, \dots, n-3.$$

Therefore, $[\phi]$ admits a rectangular neighbourhood $\mathcal{R}$ such that $\mathcal{R}$ is isometric to $[0,1]^{2(n-3)}$ and $\mathcal{R}$ is fibred perpendicularly to its faces by the flow lines of Φ_{b_i} and $\Phi_{(\tau_{b_i})^{m_i} d_i}$. Since Ω is open, we can assume $\mathcal{R} \subset \Omega$.

On almost all circle orbits of the Hamiltonian flows Φ_{b_i} and $\Phi_{(\tau_{b_i})^{m_i} d_i}$ crossing $\mathcal{R}$, the corresponding $2(n-3)$ Dehn twists

$$\tau_{b_i}, \tau_{(\tau_{b_i})^{m_i} d_i}, \quad i = 1, \dots, n-3.$$

act by irrational rotation. Indeed, this follows from Proposition 5.1.1 and from full-measureness of irrational numbers. Since irrational rotations are ergodic and f is by assumption $\mathrm{Mod}(\Sigma_n)$-invariant, it is a consequence of Lemma 5.1.2 that f is constant almost everywhere on almost every orbit of the flows crossing $\mathcal{R}$. The Rectangle Trick (Lemma 5.2.5) implies that f is constant almost everywhere on $\mathcal{R}$. This concludes the proof of the lemma. $\square$

Remark 5.2.9. For any $i = 1, \ldots, n-3$, it holds

$$\tau_{(\tau_{b_i})^{m_i} d_i} = (\tau_{b_i})^{m_i} \tau_{d_i} (\tau_{b_i})^{-m_i} \in \mathrm{Mod}(\Sigma_n).$$

This is a general fact about Dehn twists, see e.g. [FM12, §3]. Therefore, we actually proved that the action of the subgroup $\mathcal{H}$ of $\mathrm{Mod}(\Sigma_n)$ generated by the Dehn twists $\tau_{b_1}, \ldots, \tau_{b_{n-3}}, \tau_{d_1}, \ldots, \tau_{d_{n-3}}$ on $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ is ergodic. Now, note the following. Lemma 4.1 in [GW17] (see also [FM12, §9.3]) implies that the minimum number of (Dehn twist) generators of $\mathrm{Mod}(\Sigma_n)$ is $\binom{n-1}{2} - 1$ for $n \geqslant 3$ (recall that $\mathrm{Mod}(\Sigma_n)$ is trivial for $n = 0, 1, 2$). Hence, for $n \geqslant 5$, $\mathcal{H}$ is a proper subgroup of $\mathrm{Mod}(\Sigma_n)$ because $\binom{n-1}{2} - 1 > 2(n-3)$. This proves Theorem E.

5.3. Proof of the Rectangle Trick

This section is dedicated to the proof of Lemma 5.2.5. For clarity, we only give a proof for the case $n = 2$. The proof immediately generalizes to higher dimensional rectangles by induction.

The proof uses the following density result. Let $C_0^\infty([0,1])$ denote the space of smooth functions of the interval with zero integral and let $L_0^1([0,1])$ denote the space of integrable functions of the interval with zero integral.

Lemma 5.3.1. *The space $C_0^\infty([0,1])$ is dense inside $L_0^1([0,1])$.*

Proof. It is a well known fact that $C^\infty([0,1])$ is dense inside $L^1([0,1])$. Let $\varphi \in L_0^1([0,1]) \subset L^1([0,1])$. We want to approximate φ with a sequence of smooth functions with zero integral.

Because of the density of $C^\infty([0,1])$ in $L^1([0,1])$, we can approximate φ with a sequence of smooth functions $\varphi_i \in C^\infty([0,1])$. Consider the sequence of smooth functions

$$\widetilde{\varphi}_i := \varphi_i - \int \varphi_i.$$

By construction $\widetilde{\varphi}_i \in C_0^\infty([0,1])$. Since φ is assumed to be integrable, the sequence of integrals $\int \varphi_i$ converges to $\int \varphi = 0$. So, the sequence $\widetilde{\varphi}_i \in C_0^\infty([0,1])$ converges to $\varphi \in L_0^1([0,1])$. □

Proof of Lemma 5.2.5. Let $\varphi\colon [0,1] \times [0,1] \to \mathbb{R}$ be an integrable function. We assume that φ is constant almost everywhere on almost every vertical and horizontal segment. In other words, we assume that there exist Lebesgue measurable sets $E_h, E_v \subset [0,1]$ such that

- E_h and E_v have measure 1,
- $\varphi\restriction_{\{x\}\times[0,1]}$ is constant almost everywhere for every $x \in E_h$,
- $\varphi\restriction_{[0,1]\times\{y\}}$ is constant almost everywhere for every $y \in E_v$.

We prove that under these assumptions φ is constant almost everywhere.

Consider the functions $c^v\colon E_h \to \mathbb{R}$ and $c^h\colon E_v \to \mathbb{R}$ defined by

$$c^v(x) := \int_0^1 \varphi(x,y)\,dy, \quad c^h(y) := \int_0^1 \varphi(x,y)\,dx.$$

In other words, $c^v(x)$ is the value of the constant reached almost everywhere by the function φ on the vertical segment $\{x\} \times [0,1]$, i.e. $\varphi(x,y) = c^v(x)$ for every $x \in E_h$ and for almost every $y \in [0,1]$. The analogous statement holds for the function c^h. Fubini's Theorem implies that both functions c^h and c^v are measurable and of class L^1. It is sufficient to prove that $c^h\colon E_v \to \mathbb{R}$ is constant almost everywhere to deduce that $\varphi\colon [0,1]\times[0,1] \to \mathbb{R}$ is constant almost everywhere.

For the purpose of showing that c^h is constant almost everywhere, we introduce a test function $\zeta \in C_0^\infty([0,1])$. Using Fubini's Theorem we compute

$$\begin{aligned}\int_0^1\int_0^1 \varphi(x,y)\zeta(y)\,dx\,dy &= \int_0^1 \zeta(y)\int_0^1 \varphi(x,y)\,dx\,dy \\ &= \int_{E_v} \zeta(y)c^h(y)\,dy\end{aligned}$$

Fubini's Theorem also gives

$$\begin{aligned}\int_0^1\int_0^1 \varphi(x,y)\zeta(y)\,dx\,dy &= \int_{E_h}\int_{E_v} \varphi(x,y)\zeta(y)\,dy\,dx \\ &= \int_{E_v} \zeta(y)\,dy \int_{E_h} c^v(x)\,dx.\end{aligned}$$

The last expression vanishes because ζ was chosen to have zero integral. Hence

$$\int_{E_v} \zeta(y)c^h(y)\,dy = 0 \tag{5.3.1}$$

for every test function $\zeta \in C_0^\infty([0,1])$.

By Lemma 5.3.1 we can approximate the function $c^h - \int c^h \in L^1_0([0,1])$ with a sequence of functions $\zeta_i \in C^\infty_0([0,1])$. Because of (5.3.1) we have

$$\int_{E_v} \zeta_i(y) \left(c^h(y) - \int c^h \right) dy = 0$$

for every i. Therefore $c^h - \int c^h$ is the zero function in $L^1([0,1])$. This means that c^h is constant almost everywhere and thus that φ is constant almost everywhere. □

5.4. Proof of the Key Lemma

This section is dedicated to the proof of Lemma 5.2.3. The proof is technical and requires to make explicit computations of the Hamiltonian vector fields $X_{b_1}, \dots, X_{b_{n-3}}$ and of the exterior derivatives of $\vartheta_{d_1}, \dots, \vartheta_{d_{n-3}}$. To that end we start with a short recap of the local structure of relative character varieties.

5.4.1. Tangent spaces to relative character varieties

Recall from Section 2.1.4 that the tangent space to $\mathrm{Rep}_\alpha(\Sigma_n, \mathrm{PSL}(2,\mathbb{R}))$ at $[\phi]$ is given by the first parabolic group cohomology of π_n with coefficients in $(\mathfrak{sl}_2\mathbb{R})_\phi$:

$$T_{[\phi]} \mathrm{Rep}_\alpha(\Sigma_n, \mathrm{PSL}(2,\mathbb{R})) \cong H^1_{par}\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big). \tag{5.4.1}$$

Here, $(\mathfrak{sl}_2\mathbb{R})_\phi$ stands for the π_n-module defined by

$$\pi_n \xrightarrow{\phi} \mathrm{PSL}(2,\mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{sl}_2\mathbb{R}).$$

The identification (5.4.1) depends on the choice of a preferred representative ϕ of the class $[\phi]$ that gives $\mathfrak{sl}_2\mathbb{R}$ the structure of a π_n-module.

For convenience, we recall that the first parabolic group cohomology of π_n can be defined as the quotient

$$H^1_{par}\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big) = \frac{Z^1_{par}\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big)}{B^1\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big)}, \tag{5.4.2}$$

where

- $Z^1_{par}\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big)$ is the set of maps $v\colon \pi_n \to (\mathfrak{sl}_2\mathbb{R})_\phi$ satisfying the cocycle condition

$$v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y), \quad \forall x, y \in \pi_n, \tag{5.4.3}$$

and the coboundary conditions

$$\exists \xi_i \in (\mathfrak{sl}_2\mathbb{R})_\phi, \quad v(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i, \quad \forall i = 1, \ldots, n,$$

- $B^1\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big)$ is the set of maps $v\colon \pi_n \to (\mathfrak{sl}_2\mathbb{R})_\phi$ satisfying the coboundary condition

$$\exists \xi \in (\mathfrak{sl}_2\mathbb{R})_\phi, \quad v(x) = \xi - \mathrm{Ad}(\phi(x))\xi, \quad \forall x \in \pi_n.$$

The reader is referred to Appendix B.8 for more consideration on parabolic group cohomology.

Since $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ is a full dimensional connected component of the relative character variety $\mathrm{Rep}_\alpha(\Sigma_n, \mathrm{PSL}(2,\mathbb{R}))$, the tangent space of $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ at $[\phi]$ is also identified with the first parabolic group cohomology of π_n:

$$T_{[\phi]} \mathrm{Rep}_\alpha^{\mathrm{DT}} \cong H^1_{par}\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big).$$

Accordingly to the quotient (5.4.2), we denote arbitrary element of $T_{[\phi]} \mathrm{Rep}_\alpha^{\mathrm{DT}}$ by the equivalence class $[v]$ of a cocycle $v \in Z^1_{par}\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big)$,

5.4.2. Some preliminary computations

Our first computations concern the zeros of the exterior derivatives of the functions $\vartheta_{d_1}, \ldots, \vartheta_{d_{n-3}}$. Similar computations were already conducted in [DT19]; we include them here for the sake of completeness.

Lemma 5.4.1. *Let $a \in \pi_n$ be a non-trivial homotopy class of loops freely homotopic to a simple closed curve. Let $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$ with preferred representative ϕ and $[v] \in H^1_{par}\big(\pi_n, (\mathfrak{sl}_2\mathbb{R})_\phi\big)$ a tangent vector at $[\phi]$. Then*

$$(d\vartheta_a)_{[\phi]}([v]) = 0 \quad \Longleftrightarrow \quad \mathrm{Tr}(\phi(a)v(a)) = 0.$$

Proof. Consider a smooth path $[\phi_t]$ inside $\mathrm{Rep}_\alpha^{\mathrm{DT}}$ with $[\phi_0] = [\phi]$ and whose tangent vector at $t = 0$ is $[v]$. Let $\vartheta_a(t) := \vartheta_a([\phi_t])$. By definition of the exterior derivative:

$$(d\vartheta_a)_{[\phi]}([v]) = \vartheta_a'(0).$$

We choose smooth lifts in $\mathrm{SL}(2,\mathbb{R})$ of $\phi_t(a) \in \mathrm{PSL}(2,\mathbb{R})$ which we also denote by $\phi_t(a)$. Since the trace is conjugacy invariant and $\phi_t(a)$ is conjugate to $\mathrm{rot}_{\vartheta_a(t)}$, by definition of the function ϑ_a (see (5.1.1)), it follows that

$$2\cos(\vartheta_a(t)/2) = \pm \operatorname{Tr}(\phi_t(a)).$$

Applying a derivative at $t = 0$ we get

$$-2\vartheta_a'(0)\sin(\vartheta_a(0)/2) = \pm \operatorname{Tr}(v(a)\phi(a)).$$

Since $\vartheta_a(0) \in (0, 2\pi)$ by definition of ϑ_a, it follows that $\sin(\vartheta_a(0)/2) \neq 0$ and thus

$$\vartheta_a'(0) = 0 \quad \Longleftrightarrow \quad \operatorname{Tr}(\phi(a)v(a)) = 0. \qquad \square$$

The next computation concerns the Hamiltonian vector fields $X_{b_1}, \ldots, X_{b_{n-3}}$. It is convenient to introduce the following convention. Let us first fix an index $i \in \{1, \ldots, n-3\}$ with the understanding that we are working towards the proof of Lemma 5.2.3.

Convention 5.4.2. Anytime we write $[\phi] \in \operatorname{Rep}_\alpha^{\mathrm{DT}}$ below, we assume that ϕ is a representative of $[\phi]$ such that the unique fixed point of $\phi(b_i)$ in the upper half-plane is the complex unit. Such a representative always exists because $\operatorname{PSL}(2, \mathbb{R})$ acts transitively on the upper half-plane.

For convenience, we introduce the following notation

$$\Xi := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}_2\mathbb{R}.$$

Note that Ξ also belongs to $\operatorname{SL}(2, \mathbb{R})$ and projects to rot_π inside $\operatorname{PSL}(2, \mathbb{R})$. Recall moreover that

$$\operatorname{rot}_t = \pm \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix} = \pm \exp\left(t/2\, \Xi\right).$$

Lemma 5.4.3. *The Hamiltonian vector field X_{b_i} at $[\phi] \in \operatorname{Rep}_\alpha^{\mathrm{DT}}$ is represented by the parabolic cocycle*

$$X_{b_i}([\phi]) \colon c_j \mapsto \begin{cases} 0, & j = 1, \ldots, i+1, \\ \Xi - \operatorname{Ad}(\phi(c_j))\Xi, & j = i+2, \ldots, n. \end{cases}$$

Proof. The action of twist flow Φ_{b_i} on $[\phi]$ was computed in [DT19, Prop. 3.3]:

$$\Phi_{b_i}^t([\phi]) \colon c_j \mapsto \begin{cases} \phi(c_j), & j = 1, \ldots, i+1, \\ \operatorname{rot}_{2t}\phi(c_j)\operatorname{rot}_{2t}^{-1}, & j = i+2, \ldots, n. \end{cases} \tag{5.4.4}$$

Observe that (5.4.4) is a generalization of (5.1.2) which is the special case $t = \vartheta_{b_i}([\phi])/2$. The Hamiltonian flow Φ_{b_i} and the vector field X_{b_i} are related by

$$\Phi^t_{b_i}([\phi])(c_j) = \exp\left(tX_{b_i}([\phi])(c_j)\right)\phi(c_j).$$

So, $X_{b_i}([\phi])(c_j) = 0$ for $j = 1, \dots, i+1$, and for $j = i+2, \dots, n$ we compute

$$\begin{aligned} X_{b_i}([\phi])(c_j) &= \left.\frac{d}{dt}\right|_{t=0} \Phi^t_{b_i}([\phi])(c_j) \cdot \phi(c_j)^{-1} \\ &= \Xi - \mathrm{Ad}(\phi(c_j))\Xi. \end{aligned}$$

For the last equality we used (5.4.4) and $\frac{d}{dt}\big|_{t=0} \mathrm{rot}_{2t} = \Xi$. □

We combine Lemma 5.4.3 and the cocycle formula (5.4.3) to evaluate the parabolic cocycle $X_{b_i}([\phi])$ at $d_i = c_{i+2}^{-1}c_{i+1}^{-1}$:

$$\begin{aligned} X_{b_i}([\phi])(d_i) &= X_{b_i}([\phi])(c_{i+2}^{-1}) + \mathrm{Ad}(\phi(c_{i+2}^{-1}))\underbrace{X_{b_i}([\phi])(c_{i+1}^{-1})}_{=0} \\ &= \Xi - \mathrm{Ad}(\phi(c_{i+2}^{-1}))\Xi. \end{aligned} \tag{5.4.5}$$

5.4.3. A reformulation of the Key Lemma

We make use of the previous computations to reformulate what it means for the Poisson bracket of ϑ_{b_i} and ϑ_{d_i} to vanish.

Lemma 5.4.4. *The Poisson bracket $\{\vartheta_{b_i}, \vartheta_{d_i}\}$ vanishes at $[\phi] \in \mathrm{Rep}^{\mathrm{DT}}_\alpha$ if and only if*

$$\mathrm{Tr}\left(\Xi \cdot \phi(c_{i+2}^{-1})\phi(c_{i+1}^{-1})\right) = \mathrm{Tr}\left(\Xi \cdot \phi(c_{i+1}^{-1})\phi(c_{i+2}^{-1})\right).$$

Proof. Combining Lemma 5.4.1 and (5.4.5) it follows that $\{\vartheta_{b_i}, \vartheta_{d_i}\}([\phi]) = 0$ if and only if

$$\mathrm{Tr}\left(\phi(d_i)(\Xi - \mathrm{Ad}(\phi(c_{i+2}^{-1}))\Xi)\right) = 0.$$

Because the trace is invariant under conjugation, and since $\mathrm{Ad}(\phi(c_{i+2}))\phi(d_i) = \phi(c_{i+1}^{-1})\phi(c_{i+2}^{-1})$, the latter is equivalent to

$$\mathrm{Tr}\left(\Xi \cdot \phi(d_i)\right) = \mathrm{Tr}\left(\Xi \cdot \phi(c_{i+1}^{-1})\phi(c_{i+2}^{-1})\right)$$

which proves the lemma. □

Consider an arbitrary point $[\phi_t] := \Phi_{b_i}^t([\phi])$ on the Φ_{b_i}-orbit of $[\phi] \in \mathrm{Rep}_\alpha^{\mathrm{DT}}$. Thanks to (5.4.4), Lemma 5.4.4 implies that $\{\vartheta_{b_i}, \vartheta_{d_i}\}([\phi_t]) = 0$ if and only if

$$\begin{aligned}&\operatorname{Tr}\left(\Xi \cdot \mathrm{rot}_{2t}\,\phi(c_{i+2}^{-1})\,\mathrm{rot}_{2t}^{-1}\,\phi(c_{i+1}^{-1})\right)\\ =&\operatorname{Tr}\left(\Xi \cdot \phi(c_{i+1}^{-1})\,\mathrm{rot}_{2t}\,\phi(c_{i+2}^{-1})\,\mathrm{rot}_{2t}^{-1}\right).\end{aligned} \tag{5.4.6}$$

What Lemma 5.2.3 claims is that (5.4.6) is satisfied for at most two different values of $t \in [0, \pi)$, provided that $[\phi]$ belongs to $\mu^{-1}(\mathring{\Delta})$.

We now intend to compute (5.4.6) further in terms of the representation ϕ. Let us introduce the following notation

$$\phi(c_{i+2}^{-1}) =: \pm\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \quad \phi(c_{i+1}^{-1}) =: \pm\begin{pmatrix} x & y \\ z & w\end{pmatrix}.$$

Lemma 5.4.5. *The relation* (5.4.6) *holds if and only if*

$$\begin{aligned}&\cos(2t)\left((a-d)(y+z)-(b+c)(x-w)\right)\\ =&\sin(2t)\left((x-w)(d-a)-(b+c)(y+z)\right).\end{aligned} \tag{5.4.7}$$

The proof of Lemma 5.4.5 is a foolish computation and is postponed to end of this section. For now, we prove Lemma 5.2.3 under the assumption that Lemma 5.4.5 holds.

Proof of Lemma 5.2.3. The function $\tan(2t)$ is two-to-one for $t \in [0, \pi)$. So, if (5.4.7) holds for at least three different values of t in $[0, \pi)$, then one must have

$$\begin{cases} (a-d)(y+z) &= (b+c)(x-w), \text{ and} \\ (x-w)(d-a) &= (b+c)(y+z). \end{cases} \tag{5.4.8}$$

We claim that the system (5.4.8) only has trivial solutions over the real numbers, namely

$$\begin{cases} a = d & \text{and} \quad b = -c, \text{ or} \\ x = w & \text{and} \quad y = -z. \end{cases}$$

Indeed, if $a = d$, then $b = -c$, or $x = w$ and $y = -z$. Similarly, if $x = w$, then $y = -z$, or $a = d$ and $b = -c$. The case $y = -z$ leads to the analogous conclusion. If $a \neq d$, $x \neq w$ and $y \neq -z$, then

$$\frac{y+z}{x-w} = \frac{b+c}{a-d} = -\frac{x-w}{y+z}$$

and so $(x-w)^2 + (y+z)^2 = 0$. This is a contradiction.

In the first case, when $a = d$ and $b = -c$, $\phi(c_{i+2}^{-1})$ commutes with Ξ, and thus it commutes with rot_θ for all θ. Hence, if $i \neq n-3$, then (5.4.4) implies $\Phi^\theta_{b_i}([\phi]) = \Phi^\theta_{b_{i+1}}([\phi])$ for every θ , and if $i = n-3$, then (5.4.4) implies $\Phi^\theta_{b_{n-3}}([\phi]) = [\phi]$ for every θ. Both conclusions are in contradiction with the assumption that $[\phi] \in \mu^{-1}(\mathring{\Delta})$.

In the second case, when $x = w$ and $y = -z$, $\phi(c_{i+1}^{-1})$ commutes with Ξ. An analogous argument to the previous case leads to a contradiction.

Therefore, there are at most two different $t_1, t_2 \in [0, \pi)$ that satisfy (5.4.7). Moreover, if they exist, then $|t_2 - t_1| = \pi/2$ and the corresponding points on the Φ_{b_i}-orbit are diametrically opposite. This concludes the proof of Lemma 5.2.3. $\square$

5.4.4. A last computation

It remains to prove Lemma 5.4.5 to conclude the proof of Theorem D.

Proof of Lemma 5.4.5. To simplify the notation we will abbreviate $\mathfrak{c} = \cos(t)$ and $\mathfrak{s} = \sin(t)$. We first compute the left-hand side of (5.4.6), namely

$$\operatorname{Tr}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{c} & \mathfrak{s} \\ -\mathfrak{s} & \mathfrak{c} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathfrak{c} & -\mathfrak{s} \\ \mathfrak{s} & \mathfrak{c} \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right). \tag{5.4.9}$$

First, note that

$$\begin{pmatrix} \mathfrak{c} & \mathfrak{s} \\ -\mathfrak{s} & \mathfrak{c} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\mathfrak{c} + c\mathfrak{s} & b\mathfrak{c} + d\mathfrak{s} \\ -a\mathfrak{s} + c\mathfrak{c} & -b\mathfrak{s} + d\mathfrak{c} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathfrak{c} & -\mathfrak{s} \\ \mathfrak{s} & \mathfrak{c} \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x\mathfrak{c} - z\mathfrak{s} & y\mathfrak{c} - w\mathfrak{s} \\ x\mathfrak{s} + z\mathfrak{c} & y\mathfrak{s} + w\mathfrak{c} \end{pmatrix}.$$

Hence we have

$$\begin{pmatrix} \mathfrak{c} & \mathfrak{s} \\ -\mathfrak{s} & \mathfrak{c} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathfrak{c} & -\mathfrak{s} \\ \mathfrak{s} & \mathfrak{c} \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \star & l_1 \\ l_2 & \star \end{pmatrix},$$

where

$$\begin{cases} l_1 = ay\mathfrak{c}^2 - aw\mathfrak{c}\mathfrak{s} + cy\mathfrak{c}\mathfrak{s} - cw\mathfrak{s}^2 + by\mathfrak{c}\mathfrak{s} + bw\mathfrak{c}^2 + dy\mathfrak{s}^2 + dw\mathfrak{c}\mathfrak{s}, \\ l_2 = -ax\mathfrak{c}\mathfrak{s} + az\mathfrak{s}^2 + cx\mathfrak{c}^2 - cz\mathfrak{c}\mathfrak{s} - bx\mathfrak{s}^2 - bz\mathfrak{c}\mathfrak{s} + dx\mathfrak{c}\mathfrak{s} + dz\mathfrak{c}^2. \end{cases}$$

So, (5.4.9) is equal to $l_2 - l_1$. We now compute the right-hand side of (5.4.6), namely

$$\operatorname{Tr}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} \mathfrak{c} & \mathfrak{s} \\ -\mathfrak{s} & \mathfrak{c} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathfrak{c} & -\mathfrak{s} \\ \mathfrak{s} & \mathfrak{c} \end{pmatrix}\right). \tag{5.4.10}$$

Because the trace is conjugacy invariant, (5.4.10) is equal to

$$\mathrm{Tr}\left(\begin{pmatrix}\mathfrak{c} & -\mathfrak{s}\\ \mathfrak{s} & \mathfrak{c}\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x & y\\ z & w\end{pmatrix}\begin{pmatrix}\mathfrak{c} & \mathfrak{s}\\ -\mathfrak{s} & \mathfrak{c}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right).$$

Since Ξ and rot_{2t} commute, (5.4.10) is further equal to

$$\mathrm{Tr}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mathfrak{c} & -\mathfrak{s}\\ \mathfrak{s} & \mathfrak{c}\end{pmatrix}\begin{pmatrix}x & y\\ z & w\end{pmatrix}\begin{pmatrix}\mathfrak{c} & \mathfrak{s}\\ -\mathfrak{s} & \mathfrak{c}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right).$$

Now, we can use the previous computations to get

$$\begin{pmatrix}\mathfrak{c} & -\mathfrak{s}\\ \mathfrak{s} & \mathfrak{c}\end{pmatrix}\begin{pmatrix}x & y\\ z & w\end{pmatrix}\begin{pmatrix}\mathfrak{c} & \mathfrak{s}\\ -\mathfrak{s} & \mathfrak{c}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\star & r_1\\ r_2 & \star\end{pmatrix},$$

where

$$\begin{cases} r_1 = bx\mathfrak{c}^2 + dx\mathfrak{c}\mathfrak{s} - bz\mathfrak{c}\mathfrak{s} - dz\mathfrak{s}^2 - by\mathfrak{c}\mathfrak{s} + dy\mathfrak{c}^2 + bw\mathfrak{s}^2 - dw\mathfrak{c}\mathfrak{s},\\ r_2 = ax\mathfrak{c}\mathfrak{s} + cx\mathfrak{s}^2 + az\mathfrak{c}^2 + cz\mathfrak{c}\mathfrak{s} - ay\mathfrak{s}^2 + cy\mathfrak{c}\mathfrak{s} - aw\mathfrak{c}\mathfrak{s} + cw\mathfrak{c}^2.\end{cases}$$

So, (5.4.10) is equal to $r_2 - r_1$.

Therefore, (5.4.6) holds if and only if $l_2 - l_1 = r_2 - r_1$. It holds $l_2 - l_1 = r_2 - r_1$ if and only if

$$\begin{aligned}
&-ax\mathfrak{c}\mathfrak{s} + az\mathfrak{s}^2 + cx\mathfrak{c}^2 - cz\mathfrak{c}\mathfrak{s} - bx\mathfrak{s}^2 - bz\mathfrak{c}\mathfrak{s} + dx\mathfrak{c}\mathfrak{s} + dz\mathfrak{c}^2\\
&-ay\mathfrak{c}^2 + aw\mathfrak{c}\mathfrak{s} - cy\mathfrak{c}\mathfrak{s} + cw\mathfrak{s}^2 - by\mathfrak{c}\mathfrak{s} - bw\mathfrak{c}^2 - dy\mathfrak{s}^2 - dw\mathfrak{c}\mathfrak{s}\\
=&ax\mathfrak{c}\mathfrak{s} + cx\mathfrak{s}^2 + az\mathfrak{c}^2 + cz\mathfrak{c}\mathfrak{s} - ay\mathfrak{s}^2 + cy\mathfrak{c}\mathfrak{s} - aw\mathfrak{c}\mathfrak{s} + cw\mathfrak{c}^2\\
&-bx\mathfrak{c}^2 - dx\mathfrak{c}\mathfrak{s} + bz\mathfrak{c}\mathfrak{s} + dz\mathfrak{s}^2 + by\mathfrak{c}\mathfrak{s} - dy\mathfrak{c}^2 - bw\mathfrak{s}^2 + dw\mathfrak{c}\mathfrak{s}.
\end{aligned}$$

We group all the terms containing $\mathfrak{c}\mathfrak{s}$ on the left-hand side and all the terms containing $\mathfrak{c}^2$ and $\mathfrak{s}^2$ on the other side:

$$\begin{aligned}
&2\mathfrak{c}\mathfrak{s}(-ax - cz - bz + dx + aw - cy - by - dw)\\
=&(\mathfrak{c}^2 - \mathfrak{s}^2)(-cx + az + ay + cw - bx - dz - dy + bw).
\end{aligned}$$

We factorize and use that $\mathfrak{c}^2 - \mathfrak{s}^2 = \cos(2t)$ and $2\mathfrak{c}\mathfrak{s} = \sin(2t)$:

$$\begin{aligned}
&\sin(2t)\big((x-w)(d-a) - (b+c)(y+z)\big)\\
=&\cos(2t)\big((a-d)(y+z) - (b+c)(x-w)\big).
\end{aligned}$$

This finishes the proof of the lemma. $\square$

A. The groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$

This appendix is a reminder of the basic properties of the Lie groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$ and of some relevant results.

A.1. The group $\mathrm{SL}(2,\mathbb{C})$

The group $\mathrm{SL}(2,\mathbb{C})$ is the group of complex 2×2 matrices with determinant 1. It is a complex algebraic group of complex dimension 3. It is also a non-compact simple Lie group. Its center is $Z(\mathrm{SL}(2,\mathbb{C}))=\{\pm I\}$. The only proper parabolic subgroup of $\mathrm{SL}(2,\mathbb{C})$, up to conjugation, is the subgroup of upper triangular matrices. We are interested in the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$ and its irreducible subgroups in the sense of Definition 2.2.12.

Theorem A.1 ([Sit75]). *Let G be an infinite algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then one of the following holds:*

1. $\dim G = 3$ *and* $G=\mathrm{SL}(2,\mathbb{C})$,
2. $\dim G = 2$ *and G is conjugate to the parabolic subgroup of upper triangular matrices,*
3. $\dim G = 1$, *in which case there are three possibilities*

   a) G is conjugate to
   $$\left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a^n = 1,\ a,b\in\mathbb{C}\right\},$$
   and G° is unipotent,

   b) G is conjugate to
   $$\left\{\begin{pmatrix} a & \lambda c \\ c & a\end{pmatrix} : a^2-\lambda c^2 = 1,\ a,c\in\mathbb{C}\right\},$$
   for some $\lambda\in\mathbb{C}^\times$, and G is connected and diagonalizable,

   c) G is conjugate to
   $$\mathrm{SO}^\lambda := \left\{\begin{pmatrix} a & \lambda c \\ c & a\end{pmatrix} : a^2-\lambda c^2 = 1,\ a,c\in\mathbb{C}\right\}\cup\left\{\begin{pmatrix} a & -\lambda c \\ c & -a\end{pmatrix} : -a^2+\lambda c^2 = 1,\ a,c\in\mathbb{C}\right\},$$
   for some $\lambda\in\mathbb{C}^\times$, and G° is diagonalizable.

Recall that the algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$ of dimension 0 are necessarily finite (because algebraic varieties have finitely many connected components in the usual topology, as pointed out earlier). They are well-understood, see e.g. [Sit75, Prop. 1.2]. Also observe that $\mathrm{SO}(2,\mathbb{C}) = \mathrm{SO}^{-1}$ in the notation above. The irreducible subgroups of $\mathrm{SL}(2,\mathbb{C})$ fall into three categories.

Theorem A.2 ([YCo]). *Let G be an irreducible subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then one of the following holds:*

1. *G is Zariski dense in $\mathrm{SL}(2,\mathbb{C})$,*
2. *G is finite and non-abelian,*
3. *the Zariski closure of G is conjugate to*

$$\Delta := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times \right\} \cup \left\{ \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} : a \in \mathbb{C}^\times \right\}.$$

Observe that the matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ conjugates Δ to SO^1 in the notation of Theorem A.1. In particular, Δ is Zariski closed. It is also disconnected and Δ° is the subgroup of diagonal matrices. The anti-diagonal matrices in Δ have order 4.

Remark A.3. It was established in Lemma 2.2.27 that Zariski dense representations into any algebraic group are irreducible. The converse statement for $\mathrm{SL}(2,\mathbb{C})$ can sometimes be found in the literature, see e.g. [Mon16, Rem. 2.13]. It is not true. For instance, given a finite non-abelian subgroup G of $\mathrm{SL}(2,\mathbb{C})$ of order g, then there is a surjective group homomorphism $F_g \to G$, where $F_g = \langle \gamma_1, \ldots, \gamma_g \rangle$ is the free group on g generators. The fundamental group of a closed surface of genus g maps surjectively to F_g by $a_i, b_i \mapsto \gamma_i$, where a_i, b_i refer to the presentation (2.1.3). This gives two irreducible representations $\pi_{g,0} \to \mathrm{SL}(2,\mathbb{C})$ and $F_g \to \mathrm{SL}(2,\mathbb{C})$ that are irreducible but not Zariski dense. It is also possible to build an irreducible representation of a closed surface group with image inside Δ.

A.2. The groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$

The group $\mathrm{SL}(2,\mathbb{R})$ is the subgroup of $\mathrm{SL}(2,\mathbb{C})$ consisting of real matrices. It is a real algebraic group of real dimension 3 that has the topology of an open solid torus. It is a non-compact simple Lie group with center $Z(\mathrm{SL}(2,\mathbb{R})) = \{\pm I\}$. The center-free quotient $\mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ is denoted $\mathrm{PSL}(2,\mathbb{R})$. The group $\mathrm{SL}(2,\mathbb{R})$ is Zariski dense inside $\mathrm{SL}(2,\mathbb{C})$

(actually, even the group $\mathrm{SL}(2,\mathbb{Z})$ is Zariski dense in $\mathrm{SL}(2,\mathbb{C})$). The maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ is $\mathrm{SO}(2,\mathbb{R})$. Note that $\mathrm{SO}(2,\mathbb{R})$ is Zariski closed inside $\mathrm{SL}(2,\mathbb{R})$, but the Zariski closure of $\mathrm{SO}(2,\mathbb{R})$ inside $\mathrm{SL}(2,\mathbb{C})$ is $\mathrm{SO}(2,\mathbb{C})$. The group $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to $\mathrm{SU}(1,1)$. The group $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to the matrix group $\mathrm{SO}(2,1)^\circ$ of special linear transformations of $\mathbb{R}^3$ preserving the Hermitian form $y^2 - xz$ via the map

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

The group $\mathrm{PSL}(2,\mathbb{R})$ can be identified with the group of orientation-preserving isometries of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. It acts on $\mathbb{H}$ by Möbius transformations

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

The action extends to the boundary $\partial\mathbb{H}$ of the upper half-plane.

Lemma A.4. *The action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\partial\mathbb{H}$ *is isomorphic to the action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\mathbb{RP}^1 = \mathbb{R}^2/\mathbb{R}^\times$.

Proof. Identifying $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, one can define a homeomorphism $f\colon \partial\mathbb{H} \to \mathbb{RP}^1$ by $x \mapsto [1:x]$ and $\infty \mapsto [0:1]$. We claim that f conjugates the two actions of $\mathrm{PSL}(2,\mathbb{R})$. Indeed, it is sufficient to compare stabilizers and it is easy to see that the stabilizer of $[1:0] \in \mathbb{RP}^1$ and that of $0 \in \partial\mathbb{H}$ coincide with the subgroup of upper triangular matrices in $\mathrm{PSL}(2,\mathbb{R})$. $\square$

The open subspace of $\mathrm{PSL}(2,\mathbb{R})$ consisting of elements whose trace in absolute value is smaller than 2 is called the subspace of *elliptic* elements of $\mathrm{PSL}(2,\mathbb{R})$. It is denoted $\mathcal{E} \subset \mathrm{PSL}(2,\mathbb{R})$. Equivalently, an element of $\mathrm{PSL}(2,\mathbb{R})$ is elliptic if and only if it has a unique fixed point in $\mathbb{H}$.

Lemma A.5. *If* $A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is elliptic, then* $b \neq 0$ *and* $c \neq 0$.

Proof. If $b = 0$ or $c = 0$, then $\det(A) = ad = 1$. So, $\mathrm{Tr}(A)^2 = (a+d)^2 \geqslant 4ad = 4$ and A is not elliptic. $\square$

Let $A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic element of $\mathrm{PSL}(2,\mathbb{R})$. We denote the unique fixed point of A in $\mathbb{H}$ by $\mathrm{fix}(A)$. It defines a map $\mathrm{fix}\colon \mathcal{E} \to \mathbb{H}$.

Lemma A.6. *The unique fixed point of A is*

$$\mathrm{fix}(A) = \frac{a-d}{2c} + i \cdot \frac{\sqrt{4-(a+d)^2}}{2|c|}, \tag{A.1}$$

and the map $\mathrm{fix}\colon \mathcal{E} \to \mathbb{H}$ is analytic.

Proof. The first assertion is a straightforward computation. Since $c \neq 0$ by Lemma A.5, the map $\mathrm{fix}\colon \mathcal{E} \to \mathbb{H}$ is analytic. □

The elliptic elements of $\mathrm{PSL}(2,\mathbb{R})$ that fix the complex unit $i \in \mathbb{H}$ are of the form

$$\mathrm{rot}_\vartheta := \pm\begin{pmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \tag{A.2}$$

for $\vartheta \in (0, 2\pi)$. Every $A \in \mathcal{E}$ is conjugate to a unique $\mathrm{rot}_{\vartheta(A)}$. This defines a function $\vartheta\colon \mathcal{E} \to (0,2\pi)$. The number $\vartheta(A) \in (0,2\pi)$ is called the *angle of rotation* of A.

Lemma A.7. *The angle of rotation of A is*

$$\vartheta(A) = \arctan\left(\frac{-c}{|c|} \cdot \frac{a+d}{(a+d)^2-2}\sqrt{4-(a+d)^2}\right) + \varepsilon(A), \tag{A.3}$$

where

$$\varepsilon(A) := \begin{cases} 0, & \text{if } (a+d)^2 > 2 \text{ and } (a+d)\frac{-c}{|c|} > 0, \\ \pi, & \text{if } (a+d)^2 < 2, \\ 2\pi, & \text{if } (a+d)^2 > 2 \text{ and } (a+d)\frac{-c}{|c|} < 0. \end{cases}$$

Moreover, the function $\vartheta\colon \mathcal{E} \to (0,2\pi)$ is analytic.

Proof. The number $\vartheta(A)$ can be computed as the complex argument of the complex number

$$\left.\frac{dA}{dz}\right|_{z=\mathrm{fix}\,A} = \left(\frac{(a+d)^2}{2} - 1\right) - i \cdot (a+d)\frac{c}{|c|}\frac{\sqrt{4-(a+d)^2}}{2}. \tag{A.4}$$

Observe that the imaginary part of (A.4) vanishes if and only if $a+d=0$, in which case its real part is equal to -1. This means that the complex number defined by (A.4) takes values inside $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0}$. If we think of the complex argument of a number inside $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0}$

as a function $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0} \to (0, 2\pi)$, then it is analytic. This shows that $\vartheta\colon \mathcal{E} \to (0, 2\pi)$ is an analytic function. $\square$

Lemma A.8. *The map*

$$(\mathrm{fix}, \vartheta)\colon \mathcal{E} \to \mathbb{H} \times (0, 2\pi)$$

is an analytic diffeomorphism that identifies the subset of elliptic elements in $\mathrm{PSL}(2,\mathbb{R})$ *with an open ball.*

Proof. We explained above that the map $(\mathrm{fix}, \vartheta)$ is analytic. The inverse map sends a point $z = x + i \cdot y \in \mathbb{H}$ and an angle $\vartheta \in (0, 2\pi)$ to the elliptic element

$$\mathrm{rot}_\vartheta(z) = \pm \begin{pmatrix} \cos(\vartheta/2) - xy^{-1}\sin(\vartheta/2) & (x^2y^{-1} + y)\sin(\vartheta/2) \\ -y^{-1}\sin(\vartheta/2) & \cos(\vartheta/2) + xy^{-1}\sin(\vartheta/2) \end{pmatrix}. \tag{A.5}$$

Indeed, an immediate computation gives

$$\begin{aligned} \mathrm{fix}(\mathrm{rot}_\vartheta(z)) &= \frac{-2xy^{-1}\sin(\vartheta/2)}{-2y^{-1}\sin(\vartheta/2)} + i \cdot \frac{2\sin(\vartheta/2)}{2y^{-1}\sin(\vartheta/2)} \\ &= x + iy, \end{aligned}$$

and

$$\begin{aligned} \vartheta(\mathrm{rot}_\vartheta(z)) &= \arg\left(\left(\frac{4\cos(\vartheta/2)^2}{2} - 1\right) - i \cdot (2\cos(\vartheta/2)) \cdot (-1) \cdot \frac{2\sin(\vartheta/2)}{2}\right) \\ &= \arg(\cos(\vartheta) + i\sin(\vartheta)) \\ &= \vartheta. \end{aligned}$$

$\square$

The elements of $\mathrm{PSL}(2,\mathbb{R})$ whose trace in absolute value is equal to 2 are called *parabolic*. Parabolic elements are those that have a unique fixed point of the boundary of $\mathbb{H}$. There are two conjugacy classes of parabolic elements represented by

$$\mathrm{par}^+ := \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{par}^- := \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{A.6}$$

The elements conjugate to par^+ are called *positively parabolic* and those conjugate to par^- *negatively parabolic*. Each conjugacy class of parabolic elements is an open annulus whose closures intersect at the identity.

The elements of $\mathrm{PSL}(2,\mathbb{R})$ with a trace larger than 2 in absolute value are called *hyperbolic*. Hyperbolic elements have precisely two fixed points on the boundary of $\mathbb{H}$. Any hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ is conjugate to

$$\mathrm{hyp}_\lambda := \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

for a unique $\lambda > 0$. Hyperbolic conjugacy classes are open annuli.

Elliptic, parabolic and hyperbolic conjugacy classes foliate $\mathrm{PSL}(2,\mathbb{R})$ in a way that is illustrated on Figure 5.3.

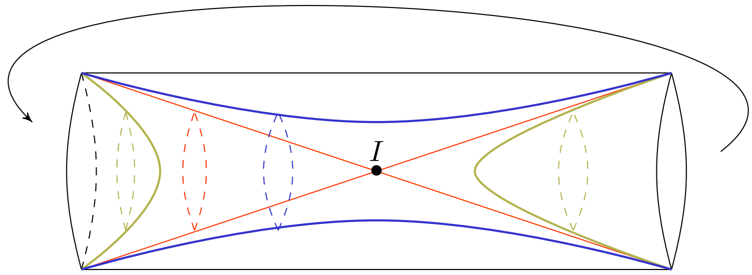


Figure 5.3.: The elliptic conjugacy classes are drawn in green. They foliate an open ball into disks. The open ball is bounded by the two parabolic conjugacy classes which have the shape of two red cones joined at the identity. The hyperbolic conjugacy classes foliate an open solid torus, bounded by the red cones, into blue annuli.

The next lemma describes the centralizers of elements of $\mathrm{PSL}(2,\mathbb{R})$ according to their conjugacy class.

Lemma A.9. *The centralizers of* rot_ϑ, hyp_λ *and* par^+ *are given by*

1. $Z(\mathrm{rot}_\vartheta) = \{\mathrm{rot}_\theta : \theta \in [0,2\pi)\} \cong \mathrm{PSO}(2,\mathbb{R})$,
2. $Z(\mathrm{hyp}_\lambda) = \{\mathrm{hyp}_t : t > 0\} \cong \mathbb{R}_{>0}$,
3. $Z(\mathrm{par}^+) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \cong \mathbb{R}$.

It is worth noticing that the centralizer of an element of $\mathrm{PSL}(2,\mathbb{R})$ always consists of the identity element and of elements of the same nature (i.e. elliptic, parabolic and hyperbolic). In particular, two elements of $\mathrm{PSL}(2,\mathbb{R})$ different from the identity commute if and only if they have the same set of fixed points in $\mathbb{H} \cup \partial\mathbb{H}$.

B. Group (co)homology

This appendix is a short introduction to the topics of group (co)homology and relative group (co)homology. These notions are important because group cohomology is the natural language to describe the Zariski tangent spaces to representation varieties. This note is a short summary of classical literature such as [Nos17, §7], [Löh10] and [BE78].

B.1. Definiton

We begin by recalling the definitions of group (co)homology. Group (co)homology is a functor from the category of discrete groups G with a left G–module M to the category of graded abelian groups:

$$H^*, H_*\colon \left(\begin{array}{c} \text{pairs of a discrete group} \\ \text{and a left module} \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{graded abelian} \\ \text{groups} \end{array} \right).$$

By requiring G to be discrete, we obtain a topological interpretation of group (co)homology. Recall that the natural topology on the fundamental group of a space that admits a universal cover is the discrete topology, because it is the coarser topology that makes the universal cover a principal bundle for the deck transformation action. Discrete groups have the following property.

Theorem B.1 (Classifying Space Theorem)**.** *If G is a discrete group, then there is a unique connected space BG, up to canonical homotopy, called the* classifying space[1] *of G, such that*

$$\pi_1(BG) \cong G, \quad \pi_i(BG) = 0, \quad \forall i \geqslant 2.$$

A possible definition of the (co)homology of the pair (G, M), where G is a discrete group and M is a left G–module, would be to say that it is the singular (co)homology of BG with coefficients in M. We favour however a more intrinsic approach.

Let $\mathbb{Z}[G]$ be the *integral group ring* of G, i.e. the free $\mathbb{Z}$–module generated by the elements of G. Note that a G–module structure is by definition the same as a $\mathbb{Z}[G]$–module structure. Let $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ be the *augmentation map* defined by $g \mapsto 1$, $g \in G$, and extended $\mathbb{Z}$–linearly to $\mathbb{Z}[G]$. We denote by Δ the kernel of the augmentation map.

[1] The names *Eilenberg-MacLane space* or $K(G,1)$*–space* are also common.

Definition B.2 (Group (co)homology)**.** The *group (co)homology* of the discrete group G with coefficients in the left G–module M is

$$H_*(G, M) := \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M), \quad H^k(G, M) := \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Definition B.2 uses the derived functors Tor and Ext. What this really means is that group (co)homology can be computed with projective resolutions of $\mathbb{Z}[G]$–modules. Recall that a module P is *projective* if it satisfies the following lifting property

$$\begin{tikzcd} & A \arrow[d, two heads, "\forall"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & B, \end{tikzcd}$$

by which we mean that every morphism $P \to B$ factors through every surjective morphism $A \to B$. Equivalently, P is projective if every short exact sequence of modules

$$0 \longrightarrow A' \longrightarrow B' \xrightarrow{f} P \longrightarrow 0$$

splits, i.e. there exists a morphism of modules $h \colon P \to B'$, called *section map*, such that $f \circ h$ is the identity on P, see [Bou89, Chap. 2, §2, Prop. 4]. A *projective resolution* $\mathcal{P}$ of a module C (not necessarily projective) is an exact sequence of projective modules ending in $C \to 0$:

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} C \longrightarrow 0 \quad \text{(exact)}.$$

A projective resolution is denoted $\mathcal{P} \twoheadrightarrow C$. The fundamental property of projective resolutions is

Lemma B.3. *Any two projective resolutions of the same module are chain homotopic.*

The derived functors in Definition B.2 mean that if $\mathcal{P} \twoheadrightarrow \Delta = \mathrm{Ker}(\varepsilon)$ is the projective resolution of $\mathbb{Z}[G]$–modules

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

then

$$H_*(G; M) = H_*(\mathcal{P} \otimes_G M), \quad H^*(G; M) = H^*\big(\mathrm{Hom}_G(\mathcal{P}; M)\big).$$

In particular, $H_0(G; M) = \Delta \otimes_G M$ and the negative-degree cohomology modules vanish. Similarly, $H^0(G; M) = \mathrm{Hom}_G(\Delta; M)$. Since any two projective resolutions of Δ are chain homotopic, group (co)homology is independent of the choice of the projective resolution $\mathcal{P} \twoheadrightarrow \Delta$.

Example B.4. We compute the homology of free groups with coefficients in a trivial module M. Let $F_n = \langle a_1, \dots, a_n \rangle$ be the free group on n elements. We claim that Δ is the free $\mathbb{Z}[F_n]$–module given by $\Delta = \langle a_1 - 1, \dots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$. The show the inclusion $\Delta \subset \langle a_1 - 1, \dots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$, argument as follows. If $x \in \Delta$, then $x = \sum n_i h_i$ where $h_i \in F_n$ and the n_i are integers whose sum is zero. An induction on the length of h_i shows that $(h_i - 1) \in \langle a_1 - 1, \dots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$. Now, since $x = \sum n_i h_i = \sum n_i (h_i - 1)$, we conclude that $x \in \langle a_1 - 1, \dots, a_n - 1 \rangle_{\mathbb{Z}[F_n]}$. Since Δ is a free $\mathbb{Z}[F_n]$–module, then

$$0 \longrightarrow \Delta \longrightarrow \mathbb{Z}[F_n] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free, hence projective, resolution of Δ. In particular

$$H_k(F_n, M) = \begin{cases} M, & k = 0 \\ M^n, & k = 1 \\ 0, & k \geqslant 2 \end{cases}.$$

Note that this corresponds to the homology of a sphere with $n + 1$ punctures.

B.2. The bar resolution for (co)homology

Our favourite choice of projective resolution of Δ is the so-called *bar resolution*. It is defined by $P_k := \mathbb{Z}[G^{k+1}]$ for $k \geqslant 1$. Using the canonical isomorphism $M \otimes_G \mathbb{Z}[G^{k+1}] \cong M \otimes_{\mathbb{Z}} \mathbb{Z}[G^k]$, we obtain that the group homology of G with coefficients in M can be computed as the homology of the chain complex

$$C_k(G, M) := M \otimes_{\mathbb{Z}} \mathbb{Z}[G^k], \quad k \geqslant 0.$$

It is called the *bar chain complex* of G and M. The differential $\partial_k \colon C_k(G, M) \to C_{k-1}(G, M)$ is defined by

$$\begin{aligned} \partial_k(a \otimes (g_1, \dots, g_k)) :=& g_1 \cdot a \otimes (g_2, \dots, g_k) \\ &+ \sum_{i=1}^{k-1} (-1)^i a \otimes (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_k) \\ &+ (-1)^k a \otimes (g_1, \dots, g_{k-1}), \end{aligned} \tag{B.1}$$

where $a \in M$ and $(g_1, \dots, g_k) \in G^k$.

The *bar cochain complex* is given by

$$C^k(G, M) := \mathrm{Map}(G^k, M),$$

where $\mathrm{Map}(G^k; M)$ is the G–module of set-theoretic functions from G^k to M. The differential $\partial^k : C^{k-1}(G; M) \to C^k(G; M)$ is defined by

$$\begin{aligned}(\partial^k u)(g_1, \ldots, g_k) := & g_1 \cdot u(g_2, \ldots, g_k) \\ & + \sum_{i=1}^{k-1} (-1)^i u(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_k) \\ & + (-1)^k u(g_1, \ldots, g_{k-1}),\end{aligned} \tag{B.2}$$

where $u \in \mathrm{Map}(G^{k-1}; M)$. One can easily check that the squares of the differentials ∂_k and ∂^k vanish.

There is an obvious relation between the differentials (B.1) and (B.2) given by

$$(\partial^k u)(g_1, \ldots, g_k) = \tilde{u}\big(\partial_k(1 \otimes (g_1, \ldots, g_k))\big), \tag{B.3}$$

where $\tilde{u}: M \otimes_{\mathbb{Z}} \mathbb{Z}[G^{k-1}] \to M$ is the unique lift of the $\mathbb{Z}$–linear map $M \times \mathbb{Z}[G^{k-1}] \to M$, $(a, (g_1, \ldots, g_k)) \mapsto a \cdot u(g_1, \ldots, g_k)$.

The sets of k-cocycles and k-coboundaries of the bar complex are denoted $Z^k(G, M)$ and $B^k(G, M)$, respectively. In particular, the 1-cocycles are

$$Z^1(G, M) := \{u: G \to M : u(g_1 g_2) = u(g_1) + g_1 \cdot u(g_2), \quad \forall g_1, g_2 \in G\}$$

and the 1-coboundaries are

$$B^1(G, M) := \{u: G \to M : \exists a \in M, \quad u(g) = g \cdot a - a, \quad \forall g \in G\}.$$

B.3. Relative group (co)homology

Let $\mathcal{K} = \{K_i : i \in I\}$ be a family of subgroups of G stable under conjugation. We define the group (co)homology of G relative to $\mathcal{K}$ with coefficients in M. Let $\mathbb{Z}[G/\mathcal{K}] := \bigoplus_{i \in I} \mathbb{Z}[G/K_i]$ be the direct sum of the free groups generated by the left cosets of K_i in G. We denote by Δ the kernel of the augmentation map $\varepsilon: \mathbb{Z}[G/\mathcal{K}] \to \mathbb{Z}$.

Definition B.5 (Relative group (co)homology)**.** The *relative (co)homology groups* of G relative to $\mathcal{K}$ with coefficients in the G–module M are defined by

$$H_*(G, \mathcal{K}, M) := \mathrm{Tor}_{*-1}^{\mathbb{Z}[G]}(\mathbb{Z}, \Delta \otimes_G M),$$

$$H^*(G, \mathcal{K}, M) := \mathrm{Ext}_{\mathbb{Z}[G]}^{*+1}(\mathbb{Z}, \mathrm{Hom}_G(\Delta; M)).$$

Observe that

$$H_*(G, \mathcal{K}, M) = H_{*-1}(G, \Delta \otimes_G M), \tag{B.4}$$

$$H^*(G, \mathcal{K}, M) = H^{*-1}(G, \mathrm{Hom}_G(\Delta; M)). \tag{B.5}$$

In particular, $H_0(G, \mathcal{K}, M) = H^0(G, \mathcal{K}, M) = 0$, $H_1(G, \mathcal{K}, M) = \Delta \otimes_G M$ and $H^1(G, \mathcal{K}, M) = \mathrm{Hom}_G(\Delta; M)$.

Remark B.6. Definition B.5 makes perfect sense even if $\mathcal{K}$ is not assumed to be closed under conjugation. This gives a notion of group (co)homology relative to any family of subgroups. However, this notion is equivalent to the former in the following sense. If $\overline{\mathcal{K}}$ denote the *conjugation closure* of $\mathcal{K}$:

$$\overline{\mathcal{K}} := \{gKg^{-1} : g \in G, K \in \mathcal{K}\},$$

then there are canonical isomorphisms

$$H_*(G, \mathcal{K}, M) \cong H_*(G, \overline{\mathcal{K}}, M), \quad H^*(G, \mathcal{K}, M) \cong H^*(G, \overline{\mathcal{K}}, M). \tag{B.6}$$

Indeed, choose a set of coset representatives $\mathcal{X}$ for $G/\mathcal{K}$. This gives an identification $\mathbb{Z}[G/\mathcal{K}] \cong \mathbb{Z}[G/\overline{\mathcal{K}}]$ which induces the desired isomorphisms. The resulting isomorphisms (B.6) are independent of the choice of $\mathcal{X}$, see [BE78, Proposition 7.5].

B.4. Bar resolution for relative (co)homology

The bar resolution for relative group (co)homology is obtained from the bar resolution for group (co)homology using the cone construction. Recall that if A and B are chain complexes and $f : B \to A$ is a morphism of chain complexes, then the *cone* of f is the chain complex $C(f)$ with differential d given by

$$C(f)_k := A_k \oplus B_{k-1}, \quad d(\alpha, \beta) := (-d\alpha + f(\beta), d\beta).$$

This construction produces an exact triangle of complexes $B \to A \to C(f) \to B[-1]$ where $B[-1]$ is the shifted complex obtained from B, also called the *suspension* of B. The exact triangle induces a long exact sequence in (co)homology.

We adopt the shorthand notation

$$C_k(\mathcal{K}, M) := \bigoplus_{i \in I} C_k(K_i, M), \quad C^k(\mathcal{K}, M) := \prod_{i \in I} C^k(K_i, M).$$

The *relative bar chain complex* is given by the cone of the inclusion $K_i \subset G$, i.e.

$$\begin{aligned} C_k(G, \mathcal{K}, M) :&= C_k(G, M) \oplus C_{k-1}(\mathcal{K}, M), \\ &\cong M \otimes_G \big(\mathbb{Z}[G^k] \oplus \mathbb{Z}[\mathcal{K}^{k-1}]\big). \end{aligned}$$

with differential $\partial_k \colon C_k(G, \mathcal{K}, M) \to C_{k-1}(G, \mathcal{K}, M)$ defined by

$$\partial_k(g, h) := \Big(-\partial_k g + \sum_{i \in I} \imath_i h_i,\, \partial_{k-1} h \Big), \tag{B.7}$$

where $g \in C_k(G; M)$ and $h = (h_i)_{i \in I} \in C_{k-1}(\mathcal{K}; M)$. Recall that at most finitely many h_i are nonzero so that the sum in (B.7) makes sense. The *relative bar cochain complex* is defined by

$$\begin{aligned} C^k(G, \mathcal{K}, M) :&= C^k(G, M) \oplus C^{k-1}(\mathcal{K}, M), \\ &\cong \mathrm{Map}\big(\mathbb{Z}[G^k] \oplus \mathbb{Z}[\mathcal{K}^{k-1}], M\big). \end{aligned}$$

The differential $\partial^k \colon C^k(G, \mathcal{K}, M) \to C^{k+1}(G, \mathcal{K}, M)$ is given by

$$\begin{aligned} \partial^k(u, f) :&= \big(\partial^k u,\, u\,\imath_i - \partial^{k-1} f_i\big) \\ &= \big(u\,\partial_{k+1},\, u\,\imath_i - f_i \partial_k\big), \end{aligned} \tag{B.8}$$

where $u \in C^k(G, M)$ and $f = (f_i)_{i \in I} \in C^{k-1}(\mathcal{K}, M)$. The second equality in (B.8) follows from the relation (B.3) which implies $u\partial_{k+1} = \partial^k u$ and $f\partial_k = \partial^{k-1} f$.

There are long exact sequences in group homology and cohomology that read

$$\ldots \longrightarrow H_k(\mathcal{K}, M) \xrightarrow{\oplus(\imath_i)_\star} H_k(G, M) \xrightarrow{j} H_k(G, \mathcal{K}, M) \xrightarrow{r} H_{k-1}(\mathcal{K}, M) \longrightarrow \ldots \tag{B.9}$$

$$\ldots \longrightarrow H^{k-1}(\mathcal{K}, M) \xrightarrow{r} H^k(G, \mathcal{K}, M) \xrightarrow{j} H^k(G, M) \xrightarrow{\times(\imath_i)^\star} H^k(\mathcal{K}, M) \longrightarrow \ldots \tag{B.10}$$

We used the shorthand notations $H_k(\mathcal{K}, M) := \bigoplus_{i \in I} H_k(K_i, M)$ and $H^k(\mathcal{K}, M) := \prod_{i \in I} H^k(K_i, M)$. The morphisms j and r are induced from the inclusion and restriction on the (co)chain complex level. The long exact sequences are obtained by applying the derived functors $\mathrm{Ext}^*_{\mathbb{Z}[G]}(\cdot, M)$ and $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\cdot, M)$ to the short exact sequence

$$0 \longrightarrow \Delta \longrightarrow \mathbb{Z}[G/\mathcal{K}] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

B.5. Relation to singular (co)homology

The purpose of this section is to explain how the singular (co)homology of a space relates to the group (co)homology of its fundamental group.

Definition B.7 (Eilenberg-MacLane pair)**.** A pair of topological spaces (X, Y), $Y \subset X$, is an *Eilenberg-MacLane pair* of type $K(G, \mathcal{K}, 1)$, if X is a $K(G, 1)$–CW-complex and if $Y = \sqcup Y_i$ where each Y_i is a $K(K_i, 1)$–subcomplex of X.

Equivalently, (X, Y) is an Eilenberg-MacLane pair if each inclusion $Y_i \hookrightarrow X$ induces an injective homomorphism $\pi_1(Y_i, y_i) \hookrightarrow \pi_1(X, y_i)$ and if there exists an isomorphism $\varphi\colon \pi_1(X, y_i) \to G$ induced by a suitable choice of path connecting base points such that $\varphi(\pi_1(Y_i, y_i)) = K_i$

$$\begin{tikzcd} \pi_1(Y_i, y_i) \arrow[r, hook] \arrow[d, "\varphi"] & \pi_1(X, y_i) \arrow[d, "\varphi"] \\ K_i \arrow[r, hook, "\imath_i"] & G. \end{tikzcd}$$

The standard examples of Eilenberg-MacLane pairs are pairs (X, Y) where X is a $K(G, 1)$-space and Y is the boundary of X.

Theorem B.8 ([BE78])**.** *Let (X, Y) be an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$. Then there exist isomorphisms in (co)homology in every degree that relates the long exact sequences of the pairs (X, Y) and $(G, \mathcal{K})$ such that the following diagram commutes (up to a minus sign for the middle square)*

$$\begin{tikzcd} H^{k-1}(\mathcal{K}, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(G, \mathcal{K}, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(G, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(\mathcal{K}, M) \arrow[d, dashed, "\cong"] \\ H^{k-1}(Y, M) \arrow[r] & H^k(X, Y, M) \arrow[r] & H^k(X, M) \arrow[r] & H^k(Y, M). \end{tikzcd}$$

Remark B.9. Observe that if (X, Y) is an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$, then it is also an Eilenberg-MacLane pair of type $K(G, \mathcal{K}', 1)$ where $\mathcal{K}'$ is obtained from $\mathcal{K}$ by individually conjugating its elements. So, as a byproduct of Theorem B.8, we get a natural isomorphism between the (co)homology of the pairs $(G, \mathcal{K})$ and $(G, \mathcal{K}')$. This isomorphism corresponds to the one induced by (B.6). In addition there are natural isomorphisms

$$H_\star(X, Y, M) \cong H_\star(G, \overline{\mathcal{K}}, M), \quad H^\star(X, Y, M) \cong H^\star(G, \overline{\mathcal{K}}, M),$$

where $\overline{\mathcal{K}}$ denotes the conjugation closure of $\mathcal{K}$ introduced in Remark B.6.

We refer the reader to [BE78, Thm. 1.3] for a proof of Theorem B.8.

B.6. Cup product

We introduce the cup product in group cohomology using the bar cochain complex as in [Nos17, §7]. Let G be a group and M, M' be two G–modules. Let $u \in C^k(G, M)$ and $v \in C^l(G, M')$. The *cup product* of u and v is defined as the cochain $u \smile v \in C^{k+l}(G, M \otimes_G M')$ defined by

$$u \smile v(g_1, \ldots, g_{k+l}) := u(g_1, \ldots, g_k) \otimes g_1 \ldots g_k \cdot v(g_{k+1}, \ldots, g_l). \tag{B.11}$$

Lemma B.10. *The cup product satisfies the Leibniz rule:*

$$\partial^{k+l+1}(u \smile v) = \partial^{k+1} u \smile v + (-1)^k u \smile \partial^{l+1} v.$$

The Leibniz rule implies that the cup product descends to a well-defined G-invariant product on cohomology:

$$\smile : H^k(G, M) \otimes_G H^l(G, M') \to H^{k+l}(G, M \otimes_G M').$$

Lemma B.11. *Up to the natural identification $M \otimes_G M' \cong M' \otimes_G M$, it holds that*

$$[u \smile v] = (-1)^{kl}[v \smile u], \quad \forall u \in Z^k(G, M), \forall v \in Z^l(G, M').$$

Proof. We treat the case $k = l = 1$. The other cases are similar. We start by computing the differential of $u \otimes v$ using (B.2)

$$\begin{aligned} -\partial^2(u \otimes v)(x, y) &= -u(x) \otimes v(x) + u(xy) \otimes v(xy) - x \cdot (u(y) \otimes v(y)) \\ &= u(x) \otimes x \cdot u(y) + x \cdot u(y) \otimes v(x) \\ &= u \smile v(x, y) + v \smile u(x, y), \end{aligned}$$

where in the second equality we used the cocycle property $u(xy) = u(x) + x \cdot u(y)$. This shows that $u \smile v + v \smile u$ is a coboundary. $\square$

The cup product can be defined on relative cohomology as follows. Let $u \in C^k(G, M)$ and $f \in C^{k-1}(\mathcal{K}, M)$, and $v \in C^l(G, M')$. Define the cup product of (u, f) with v to be the cochain

$$(u \smile v, f \smile v) \in C^{k+l}(G, \mathcal{K}, M \otimes_G M').$$

It induces a cup product in relative cohomology

$$\smile\colon H^k(G,\mathcal{K},M)\otimes_G H^l(G,M') \to H^{k+l}(G,\mathcal{K},M\otimes_G M'). \tag{B.12}$$

B.7. Cap product and Poincaré duality

The purpose of [BE78] was to describe a notion of Poincaré duality for group pairs. This can be done as follows.

Let $\mathcal{P} \twoheadrightarrow \mathbb{Z}$ be a projective resolution of G–modules. Then $\mathcal{P}\otimes_G\mathcal{P}$ is a projective resolution of $\mathbb{Z}$ for the diagonal G–action on $\mathcal{P}\otimes_G\mathcal{P}$. Let $g = p\otimes q\otimes a \in (\mathcal{P}\otimes_G\mathcal{P})\otimes_G M$ and $u \in \mathrm{Hom}_G(\mathcal{P},M')$. The *cap product* of g and u is defined to be

$$g \frown u := q\otimes(a\otimes u(p)) \in P\otimes_G(M\otimes_G M').$$

Lemma B.12. *The cap product is a well-defined operation on complexes and satisfies the Leibniz rule*

$$\partial_k(g\frown u) = (-1)^l\partial_{k+l}\, g\frown u + g\frown \partial^l u.$$

The induced cap product on (co)homology is

$$\frown\colon H_{k+l}(G,M)\otimes_G H^k(G,M') \to H_l(G,M\otimes_G M')$$

The definition of the cap product in relative (co)homology uses the pairing

$$\begin{aligned} B\colon (\Delta\otimes_G M)\otimes_G \mathrm{Hom}_G(\Delta,M') &\to M\otimes_G M' \\ (g\otimes a)\otimes u &\mapsto a\otimes u(g). \end{aligned} \tag{B.13}$$

The *cap product on relative group (co)homology* is the dashed arrow that makes the following diagram commute

$$\begin{tikzcd} H_{k+l}(G,\mathcal{K},M)\otimes_G H^l(G,\mathcal{K},M') \arrow[r, dashed, "\frown"] \arrow[d, equal] & H_l(G,M\otimes_G M') \\ H_{k+l-1}(G,\Delta\otimes_G M)\otimes_G H^{k-1}(G,\mathrm{Hom}_G(\Delta,M')) \arrow[ur, bend right, "B_*\frown"'] & \end{tikzcd}$$

The equality in the first column is an application of (B.4) and (B.5).

Using a modified version of the pairing (B.13), one can define a second variant of the cap product

$$\frown : H_{k+l}(G, \mathcal{K}, M) \otimes_G H^k(G, M') \to H_l(G, \mathcal{K}, M \otimes_G M').$$

The two versions of the cup product are natural operations in group (co)homology, see [BE78] for more details.

The cap product maps the long exact sequence in cohomology for the pair $(G, \mathcal{K})$ to its long exact sequence in homology. This commutes with the corresponding map in singular homology under the isomorphism of Theorem B.8. Indeed, let (X, Y) denote an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$. For any $e \in H_n(G, \mathcal{K}, M)$, let $\overline{e} \in H_n(X, Y; M)$ be the image of e under the isomorphism of Theorem B.8. The following diagram commutes for $k = 0, \ldots, n$ (up to some minus signs depending on the degree of the two lower squares, see [BE78] for complete details)

$$\begin{tikzcd}
H_{n-k}(G, \mathcal{K}, M \otimes_G M') \arrow[r] & H_{n-k-1}(\mathcal{K}, M \otimes_G M') \arrow[r] & H_{n-k-1}(G, M \otimes_G M') \\
H^k(G, M') \arrow[r] \arrow[u, "e\frown"] \arrow[d, "\cong"] & H^k(\mathcal{K}, M') \arrow[r] \arrow[u, "r(e)\frown"] \arrow[d, "\cong"] & H^{k+1}(G, \mathcal{K}, M') \arrow[u, "e\frown"] \arrow[d, "\cong"] \\
H^k(X, M') \arrow[r] \arrow[d, "\overline{e}\frown"] & H^k(Y, M') \arrow[r] \arrow[d, "r(\overline{e})\frown"] & H^{k+1}(X, Y, M') \arrow[d, "\overline{e}\frown"] \\
H_{n-k}(X, Y, M \otimes_G M') \arrow[r] & H_{n-k-1}(Y, M \otimes_G M') \arrow[r] & H_{n-k-1}(X, M \otimes_G M').
\end{tikzcd}$$

Here, r denotes the connecting morphism of the long exact sequence (B.9). In particular, the following square commutes

$$\begin{tikzcd}
H^k(X, Y, M') \arrow[d, "\overline{e}\frown"] & H^k(G, \mathcal{K}, M') \arrow[l, "\cong"'] \arrow[d, "e\frown"] \\
H_{n-k}(X, M \otimes_G M') & H_{n-k}(G, M \otimes_G M'). \arrow[l, "\cong"']
\end{tikzcd}$$

Poincaré duality for de Rham cohomology says that if X is a smooth, compact, connected manifold of dimension n, and $[X]$ is a generator of $H_n(X; \mathbb{Z}) \cong \mathbb{Z}$, then the cap product with $[X]$ is an isomorphism

$$[X] \frown : H^k_{dR}(X, \mathbb{R}) \xrightarrow{\cong} H_{n-k}(X, \mathbb{R}), \quad k = 0, \ldots, n.$$

In the context of group (co)homology, one introduces the notion of Poincaré duality pairs.

Definition B.13 ((Poincaré) duality pairs)**.** The pair $(G, \mathcal{K})$ is called a *duality pair of dimension n*, in short a *D^n-pair*, if there exists a G–module N and an element $e \in H_n(G, \mathcal{K}, N)$ such that both

- $e \frown : H^k(G, M) \to H_{n-k}(G, \mathcal{K}, N \otimes_G M)$,
- $e \frown : H^k(G, \mathcal{K}, M) \to H_{n-k}(G, N \otimes_G M)$

are isomorphisms for every $k = 0, \ldots, n$ and for every G–module M. Moreover, if N can be chosen to be isomorphic to $\mathbb{Z}$ as a group, then $(G, \mathcal{K})$ is called a *Poincaré duality pair of dimension n*, in short a *PD^n-pair*.

If $(G, \mathcal{K})$ is a D^n-pair, then by letting $M = \mathbb{Z}[G]$ and $k = n$, we obtain $H^n(G, \mathcal{K}, \mathbb{Z}[G]) \cong H_0(G, N \otimes_G \mathbb{Z}[G]) \cong N$. Therefore, a duality pair determines a unique *dualizing module N* up to isomorphism. For a PD^n-pair we call each of the two generators of $H_n(G, \mathcal{K}, N) \cong \mathbb{Z}$ a *fundamental class* of $(G, \mathcal{K})$.

Example B.14. Let X be a smooth, compact, connected, manifold of dimension n with non-empty boundary ∂X. Let $[X, \partial X] \in H_n(X, \partial X, \mathbb{Z})$ be a fundamental class. Assume that $(X, \partial X)$ an Eilenberg-MacLane pair of type $K(G, \mathcal{K}, 1)$. Then $(G, \mathcal{K})$ is a PD^n-pair with fundamental class $[G, \mathcal{K}]$ given by the image of $[X, \partial X]$ under the isomorphism of Theorem B.8. In particular, the following diagram commutes

$$\begin{tikzcd}
H^n_{dR}(X, \partial X, \mathbb{R}) \arrow[r, "{[X,\partial X]\frown}"] & H_0(X, \mathbb{R}) \\
H^n(G, \mathcal{K}, \mathbb{R}) \arrow[u, "\cong"] \arrow[r, "{[G,\mathcal{K}]\frown}"] & H_0(G, \mathbb{R}). \arrow[u, "\cong"']
\end{tikzcd}$$

Here, $\mathbb{R}$ is the trivial G–module.

Observe that if $(G, \mathcal{K})$ is a D^n-pair, then there exists an induced isomorphism

$$r(e) \frown : \prod_{i \in I} H^k(K_i; M') \to \bigoplus_{i \in I} H_{n-k-1}(K_i; M \otimes_G M')$$

in every degree k and for every G–modules M, M'. Therefore, $\mathcal{K}$ must be a *finite* collection of subgroups.

Lemma B.15. *Let $(G, \mathcal{K})$ be a PD^n-pair and $\mathbb{R}$ be the trivial G–module. The cap product in degree n for the bar resolution is*

$$\begin{aligned} \frown \colon H_n(G, \mathcal{K}, \mathbb{R}) \otimes_G H^n(G, \mathcal{K}, \mathbb{R}) &\to \mathbb{R} \\ [(g, h_1, \ldots, h_m)] \otimes [(u, f_1, \ldots, f_m)] &\mapsto u(g) - \sum_{i=1}^{m} f_i(h_i), \end{aligned} \tag{B.14}$$

where $u\colon G^n \to \mathbb{R}$ and $f_i\colon K_i^{n-1} \to \mathbb{R}$ have been extended $\mathbb{Z}$-linearly to $\mathbb{Z}[G^n]$, respectively $\mathbb{Z}[K_i^{n-1}]$.

Proof. We only check that (B.14) vanishes if $(g, h_1, \ldots, h_m)$ is exact. A complete proof is given in [KM96, Proposition 5.8].

The condition $\partial^n(u, f_1, \ldots, f_m) = 0$ as defined in (B.8) means that $\partial^n u = 0$ and $u\restriction_{K_i} - \partial^{n-1} f_i = 0$ for all i. Since $(g, h_1, \ldots, h_m)$ is assumed to be exact, there exist $(g', h'_1, \ldots, h'_m) \in C_{n+1}(G, \mathcal{K}, \mathbb{R})$ such that

$$\begin{aligned} (g, h_1, \ldots, h_m) &= \partial_{n+1}(g', h'_1, \ldots, h'_m) \\ &= \left(\sum_{i=1}^{m} h'_i - \partial_{n+1} g',\, \partial_n h'_1, \ldots, \partial_n h'_m \right). \end{aligned}$$

We compute

$$\begin{aligned} u(g) - \sum_{i=1}^{m} f_i(h_i) &= \sum_{i=1}^{m} u\restriction_{K_i}(h'_i) - u(\partial_{n+1} g') - \sum_{i=1}^{m} f_i(\partial_n h'_i) \\ &= \sum_{i=1}^{m} u\restriction_{K_i}(h'_i) - \partial^n u(g') - \sum_{i=1}^{m} \partial^{n-1} f_i(h'_i), \end{aligned}$$

where in the second equality we applied the relation (B.3). The last expression vanishes because $(u, f_1, \ldots, f_m)$ is closed. □

B.8. Parabolic group cohomology

Parabolic group cohomology was introduced in the sixties by André Weil. We give a succinct introduction inspired from [GHJW97].

Let G be a discrete group and $\mathcal{K} = \{K_i : i \in I\}$ be a family of subgroups of G. Let M be a G–module and $k \geqslant 0$ an integer. Define the set of *parabolic cocycles* in the bar complex to be the set k-cocycle $f\colon G^k \to M$ such that all the restrictions $f\restriction_{K_i}$ are exact, i.e. belong to $B^k(K_i, M)$. The set of parabolic cocycles in degree k is denoted

$$Z^k_{par}(G, M) \subset Z^k(G, M).$$

Parabolic cocycles are thus cocycles that are exact on the boundary.

Definition B.16 (Parabolic group cohomology)**.** The *parabolic group cohomology* of G with coefficients in the G–module M is defined to be

$$H^*_{par}(G, M) := Z^*_{par}(G, M)/B^*(G, M) \subset H^*(G; M).$$

It follows from Definition B.16 that parabolic group cohomology is related to relative group cohomology as follows.

Lemma B.17. *Let $j\colon H^k(G, \mathcal{K}, M) \to H^k(G, M)$ be the morphism of the long exact sequence* (B.10) *for the pair $(G, \mathcal{K})$. Then,*

$$H^k_{par}(G, M) = j\big(H^k(G, \mathcal{K}, M)\big) \cong H^k(G, \mathcal{K}, M)/\operatorname{Ker}(j).$$

The Leibniz rule of Lemma B.10 implies that the kernel and the image of j are orthogonal for the cup product (B.12). In particular, there is a non-degenerate induced product

$$\smile\colon H^k_{par}(G, M) \otimes_G H^l_{par}(G, M') \to H^{k+l}(G, \mathcal{K}, M \otimes_G M'). \tag{B.15}$$

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