DISSERTATION

submitted to the
Combined Faculty of Mathematics,
Engineering and Natural Sciences
at
Ruprecht-Karls-Universität Heidelberg, Germany

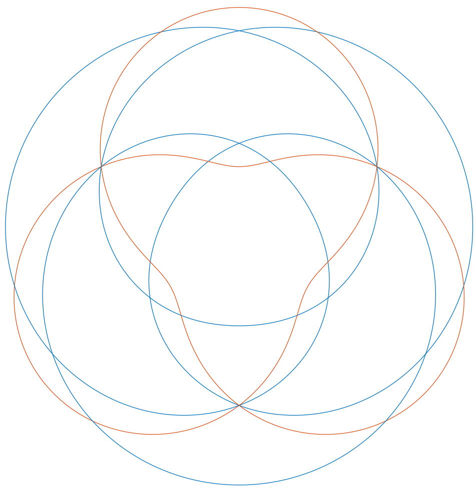
for the degree of
Doctor of Natural Sciences

by
Irene Elisabeth Seifert
born 18.01.1993 in Freiburg im Breisgau, Germany

Date of oral examination:

Polyfold methods for the study of periodic delay orbits

Irene Elisabeth Seifert



Advisor: Prof. Dr. Peter Albers

Abstract

We use methods from symplectic geometry to study periodic solutions of differential delay equations (DDEs, also known as retarded functional differential equations, RFDEs). Using polyfold theory, we prove that near a given non-degenerate 1-periodic orbit of a vector field in $\mathbb{R}^n$, there is a 1-dimensional family of 1-periodic delay orbits smoothly parametrized by delay. Then we generalize this result in several ways. Moreover, we prove an abstract compactness theorem for perturbed non-local unregularized gradient flow lines in $\mathbb{R}^{2n}$, which is one step towards the construction of Floer theory for Hamiltonian delay equations.

Zusammenfassung

Wir benutzen Methoden aus der symplektischen Geometrie, um periodische Lösungen von Verzögerungsgleichungen (*differential delay equations*, DDEs, auch bekannt als retardierte Differentialgleichungen, RFDEs) zu untersuchen. Mithilfe von Polyfold-Theory beweisen wir, dass es nahe einem gegebenen, nicht-degenerierten 1-periodischen Orbit eines Vektorfeldes in $\mathbb{R}^n$ eine 1-dimensionale Familie von 1-periodischen Delay-Orbits gibt, die glatt durch die Verzögerung parametrisiert werden kann. Danach verallgemeinern wir dieses Resultat auf verschiedene Art. Außerdem beweisen wir ein abstraktes Kompaktheits-Theorem für gestörte nicht-lokale deregularisierte Gradientenflusslinien in $\mathbb{R}^{2n}$, das einen Schritt in Richtung der Konstruktion einer Floer-Theorie für Hamiltonsche Verzögerungsgleichungen darstellt.

Acknowledgements

There are times when doing math is a one-person activity – just sitting there, concentrating, writing, and thinking. But most of the time math is done in interaction of several people: questions, answers, corrections, discussions. If I had not had lots of wonderful people around me, I would never have earned any degree in mathematics.

All this started with "Zettelrechnen" for the very first mathematics lectures in Münster. Some people were just trying to understand anything of what was going on, others argued about which of two possible solutions was nicer. Thanks a lot, "Rechnen im F10", for math discussions, picknicks, holidays and everything else! A few years later there was a small group of students who were interested in symplectic geometry and helped each other in surviving a lecture (and exercise classes...) about Gromov's non-squeezing theorem. Thanks, Basti, Anna-Maria, Kevin and Jay!

And then I was lucky to move to Heidelberg for my PhD. Here I definitely found what can be called an active and international research environment! Since 2021 it carries the name "Research Station Geometry and Dynamics", but the ideas of openness and internationality have been lived here long before. Lots of friendly and nice people; lots of activities; lots of ideas; cooperations with other universities or faculties within the scope of RTG 2229, SFB/TRR 191 and the excellence cluster STRUCTURES; enough funding for conferences; et cetera. All this only exists because there are PIs, postdocs and PhD students who care for their projects and for each other. Thanks to everybody! And it could not exist without those who organize everything, settle the accounts and buy coffee and fruit for the workshop buffet. Thanks a lot to all the secretaries!

It was great to meet people working in very different parts of geometry – I learned a lot from you all, and I want to thank you for the time we spent together! Most relevant to my own project were of course the seminars and discussions with the other members of the symplectic group. Thanks a lot for countless instances of topics seminars and BACH, for listening to practice talks and for proofreading, and also for coffee, cake, fondue and raclette! A special thank you goes to my office mates Anna-Maria, Matthias, Arnaud, Merik and Fernando, and also to Anna and Mareike, who were always ready for a little chat about math or non-math.

Last but definitely not least: Thanks, Peter! Thanks for suggesting this nice topic to me; thanks for countless discussions both about vague ideas and about smallest details; thanks for the freedom to follow my own ideas when I had some and thanks for giving me new ideas when I had none. It was great to have you as an advisor!

Two more things are needed when doing math, at least for me: encouragement by non-math people, and distraction. I was given encouragement by my wonderful husband, by my parents, by my sisters, and by other family members and friends. Thank you, Michael, and thanks to all the others! The distraction I got from the same people and from making music, but most of all from my wonderful-wonderful daughters. Of course at times it is difficult to combine family and research, especially when daycare is closed because of a pandemic. On the other hand, the two worlds complement each other in a very healthy way. (If the brain hurts, go to the playground. If a wrong computation destroyes a proof, buy a box of strawberries and read a children's book. And if the head is full of family mental load, do Floer homology.) Fanny and Edda, I am so glad that you are there, and I love you so much!

Contents

Chapter 1

Introduction

The purpose of this thesis is to study periodic delay orbits (that is, periodic solutions of delay differential equations) by using methods from symplectic geometry. The project was suggested by my supervisor Peter Albers, and it is related closely to the articles [AFS19] and [AFS20] by him and his collaborators Urs Frauenfelder and Felix Schlenk. Delay differential equations play a role in many parts of natural sciences; for instance, modelling predator-prey dynamics or the spread of a disease usually involves a delay given by the time between two generations of a species or by incubation time, respectively. Not much is known in general about the existence of periodic solutions to delay differential equations. On the other hand, symplectic geometry provides a plethora of methods for the study of periodic solutions of Hamiltonian differential equations without delay. In this thesis, I use some of these methods – in particular, polyfold theory – to prove results about periodic delay orbits.

Let us start by defining the notion of periodic delay orbit. Suppose that $X : S^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth time-dependent vector field which is periodic in time, where $S^1 = \mathbb{R}/\mathbb{Z}$. We write $X_t(p) := X(t, p)$. A 1-*periodic orbit* of X is a map $x : S^1 \to \mathbb{R}^n$ such that $\partial_t x(t) = X_t(x(t))$ for all $t \in S^1$. (Throughout this thesis, we denote the derivative of x in direction of t by $\partial_t x$. This avoids misleading notation at a later point, and at the same time makes it easy to talk about the differential operator ∂_t taking x to $\partial_t x$.) The corresponding delay equation with constant delay $\tau \in \mathbb{R}$ is

$$\partial_t x(t) = X_t(x(t - \tau)), \tag{1.1}$$

and a 1-*periodic* τ-*delay orbit* is a solution $x : S^1 \to \mathbb{R}^n$ of (1.1).[1]

Delay differential equations (DDEs) are much harder to deal with than usual differential equations. For instance, the initial value problem is non-local. Thus, there could be different solutions of (1.1) passing through the same point at $t = 0$. In particular, the dynamical behavior of equation (1.1) cannot be captured by a flow on $\mathbb{R}^n$. Moreover, note that if we replace $\mathbb{R}^n$ by a manifold M, then (1.1) does not make sense as stated, since the two sides of the equation typically belong to different tangent spaces. However, one can think of many other useful and interesting delay equations on general manifolds. There is a rich literature about delay differential equations, which we briefly review in Chapter 2.

[1] In this thesis, we focus on 1-periodic solutions instead of solutions of another period $T \in \mathbb{R}$ only for ease of presentation; everything works in the exact same manner for T-periodic (delay) orbits.

After two chapters reviewing the background on DDEs and on polyfold theory, this thesis is divided into two parts, which are devoted to the two main results.

Part I concerns the following questions: If we have a non-degenerate 1-periodic orbit x_0 (without delay) of a vector field X, does it persist for small delay τ? Can we find a whole family of delay orbits near x_0 smoothly parametrized by their (small) delay? To the best of our knowledge, these questions have not been addressed in the literature before. They seem very natural, though: When instead of delay we consider a smooth perturbation of the vector field, then the answer is a clear yes, and it can be shown by simply using the implicit function theorem for smooth Fredholm sections in Banach bundles. However, for the case of delay, this strategy breaks down due to a lack of smoothness. This is why we pass on to scale calculus, the analytical framework used in polyfold theory, which was developed for the study of moduli spaces of (perturbed) pseudoholomorphic curves. Indeed, using the M-polyfold implicit function theorem, we can prove the following:

Theorem A (Theorem 4.1)**.** *If x_0 is a non-degenerate 1-periodic orbit of X, then there is $\tau_0 > 0$ such that for every delay τ with $|\tau| \leq \tau_0$ there exists a (locally unique) smooth 1-periodic solution x_τ of the delay equation* (1.1)*. Moreover, the parametrization $\tau \mapsto x_\tau$ is smooth.*

Theorem A is a sample theorem in the sense that the same methods can be used to prove the corresponding statement for a variety of other delay differential equations. Some possible generalizations are Theorems 7.2, 7.13, 7.15, and 7.33.

The topic of Part II is a little more technical. Here, the idea is to understand periodic orbits of Hamiltonian delay equations as critical points of an action functional, which allows using ideas from Floer homology. A crucial question is then compactness of the space of gradient flow lines. Keeping this in mind, we prove an abstract compactness result for *non-local gradient flow lines.* Very roughly, it could be stated as follows:

Theorem B (a generalization of [AFS19, Theorem 2.4])**.** *Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a sequence of maps $w_\nu : I \to \mathrm{H}$ from an interval I to a certain function space $\mathrm{H} \supset \mathcal{C}^\infty(S^1, \mathbb{R}^{2n})$. Assume that each w_ν satisfies a certain non-local gradient flow equation depending on some $R_\nu \in \mathbb{R}_{\geq 0}$, and assume that as $\nu \to \infty$, the numbers R_ν converge to some $R_* \in \mathbb{R}_{\geq 0}$. Further assume that all w_ν are bounded in a certain topology. Then there is a subsequence which converges (in the same topology) to a limit w_*, which is a solution of the corresponding non-local gradient flow equation for R_*.*

Neither delay equations nor polyfold theory appear explicitly in the compactness theorem or in its proof, but they are the motivation behind the whole result.

We expect that many more interesting results can emerge in this way. We hope that this thesis serves as an example of how fruitful it is to bring together ideas from symplectic geometry and from the study of differential delay equations.

Overview of this thesis

This thesis is organized as follows:

Part 0 contains some background on the two fields of research which we bring together, namely delay differential equations (DDEs) and polyfold theory. **Chapter 2** is a very brief overview of DDEs, their periodic orbits, and dependence of solutions on the delay. We give references to both general textbooks about DDEs and articles concerned with specific questions. In **Chapter 3** we explain the fundamentals of scale calculus and polyfold theory.

Part I is about Theorem A and its generalizations. The main result is stated again as Theorem 4.1 and discussed in the beginning of **Chapter 4**. In the remaining part of **Chapter 4**, we prove it in all detail. **Chapter 5** presents an example in $\mathbb{R}^2$ where the families of periodic delay orbits can be computed explicitly. Since the main result excludes autonomous equations, we discuss this case in **Chapter 6**. In **Chapter 7** we prove several generalizations of the main result. The case of time-dependent delay (Section 7.4) involves a detailed analysis of the time-dependent shift map.

Part II is about Theorem B and the application we have in mind. In **Chapter 8**, we briefly explain the context, before we pass on to state and prove the compactness theorem in the abstract setting. **Chapter 9** is then dedicated to proving that the theorem indeed applies to perturbed gradient flow lines of Hamiltonian delay action functionals.

Finally, in **Chapter 10**, we give a small outlook on possible future research in the intersection of delay differential equations and symplectic geometry.

There are two appendices. **Appendix A** contains proofs for statements which are mentioned in Section 4.2 but not needed for the proof of Theorem 4.1. **Appendix B** reviews an argument from my master thesis [Sei17] which hopefully can be generalized to the case of Hamiltonian delay equations with help of the compactness result from Part II.

A note on previously published material

Theorem A was published by my supervisor Peter Albers and myself in the article "Periodic delay orbits and the polyfold implicit function theorem" in Commentarii Mathematici Helvetici [AS20]. Some sections have been copied more or less verbatim from the article to this thesis:

- Section 2.3 corresponds to [AS20, Section 2];
- some definitions and the polyfold IFT from Chapter 3 (all originally due to Hofer–Wysocki–Zehnder [HWZ21]) are the same as in [AS20, Sections 4–7];
- Sections 4.1–4.6 correspond to [AS20, Sections 1, 3–7];
- Section 7.2 corresponds to [AS20, Section 8];
- Appendix A corresponds to the appendix of [AS20].

Moreover, Section 3.1 may have certain similarities with Section 1 of the article [BDS+21], which I published together with Franziska Beckschulte, Ipsita Datta, Anna-Maria Vocke and Katrin Wehrheim.

Part 0

Background

Chapter 2

A short guide to delay differential equations

Delay equations (also known as delay differential equations (DDEs) or retarded functional differential equations (RFDEs)) are differential equations of the form

$$\partial_t x(t) = F\big(t, x(t), x(t - \tau)\big), \tag{2.1}$$

where $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is some function taking as inputs not only the time parameter $t \in \mathbb{R}$ and the current state $x(t) \in \mathbb{R}^n$, but also some past state $x(t - \tau) \in \mathbb{R}^n$ of the system. Here, we understand x to be a function $x : I \to \mathbb{R}^n$ from some interval I to real n-dimensional space. Delay equations arise in a very natural way in biology. The rate of expansion of a population of a given species at time t, for example, usually depends on the size of the population some time before; also the spread of a disease will typically depend on the number of infected individuals with a certain delay given by incubation time.

At first sight, equation (2.1) does not look very different from an ordinary differential equation (ODE). However, the delay makes a big qualitative change on the equation. If we want to impose an initial condition on a solution x of (2.1), it is not enough to specify $x(0) \in \mathbb{R}^n$. Instead, we need to prescribe x on a whole interval of length τ by demanding $x|_{[-\tau,0]} = \phi$ with a so-called *initial history* $\phi \in \mathcal{C}^0([-\tau, 0], \mathbb{R}^n)$.

There is, of course, a rich literature on differential delay equations and their solutions. For an introduction to the beautiful mathematical theory and its many different facets we recommend the books [DvGVLW95] and [Hal77]. Other books as [Smi11], [Rih21], and [Kua93] complement it with their stronger focus on applications. Below in Section 2.3 we mention literature on two different aspects of differential delay equations close to the topic of this thesis, namely the existence of periodic solutions and the regularity of solutions with respect to the delay.

2.1 Solving DDEs by hand or numerically

Let us assume that the Function $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is extraordinarily well behaved, say it splits as $F(t, y_1, y_2) = f(t) + g + h(y_2)$, with $g \in \mathbb{R}^n$ fixed (in particular, independent of y_1). Then, given an initial history $\phi \in \mathcal{C}^0([-\tau, 0], \mathbb{R}^n)$, we can find a solution x of (2.1) on the interval $[0, \tau]$ by simply integrating

$$x(t) = \phi(0) + \int_0^t f(s)\, \mathrm{d}s + t \cdot g + \int_0^t h\big(\phi(s - \tau)\big)\, \mathrm{d}s \qquad t \in [0, \tau].$$

Integrating further, we can extend x to the interval $[\tau, 2\tau]$, then to $[2\tau, 3\tau]$ and so on. This is called the *method of steps*. It shows how existence and uniqueness results about ODEs can be used to say something about DDEs as well, and it shows how solutions of DDEs tend to become more and more differentiable over time.

For more complicated functions F, there are ways to numerically find solutions with a given initial history, for example using the Matlab functions `dde23` (for discrete delay) and `ddesd` (also for time-dependent and state-dependent delay).

2.2 The solution semiflow

For an ordinary differential equation on $\mathbb{R}^n$, initial values are elements of $\mathbb{R}^n$, and – under suitable conditions –, the equation defines a flow on $\mathbb{R}^n$. In a similar manner, the dynamics of a linear autonomous DDE can be seen as semiflow on a function space $\mathcal{C}$, which is the space of possible initial histories. For simplicity, think of $\mathcal{C} := \mathcal{C}^0([-\tau, 0], \mathbb{R}^n)$, although it turns out that for analytical reasons other function spaces may be preferable[2]. The solution semiflow $(S_t : \mathcal{C} \to \mathcal{C})_{t \in \mathbb{R}_{\geq 0}}$ is given by

$$S_t(\phi) := \varphi\big(t, N(\phi)|_{[t-\tau,t]}\big),$$

where $N(\phi)$ is the extension of $\phi \in \mathcal{C}$ to a function $N(\phi) : [-\tau, \infty) \to \mathbb{R}^n$ which on $(0, \infty)$ is a solution of the delay equation (2.1), $N(\phi)|_{[t-\tau,t]}$ is its restriction to the interval $[t-\tau, t]$, and $\varphi(t, \cdot)$ means the shift by t which is needed to transport $N(\phi)|_{[t-\tau,t]}$ back to an element of $\mathcal{C}$. $(S_t)_{t \in \mathbb{R}_{\geq 0}}$ is indeed a semiflow in the sense that $S_0 = \mathrm{id}$ is the identity and $S_{t+t'} = S_t \circ S_{t'}$. Note that we cannot expect the existence of backwards solutions, that is, S_t for $t < 0$, since even for a smooth $\phi \in \mathcal{C}$ there is no reason why it should be a solution of (2.1).

Associated to the solution semiflow $(S_t : \mathcal{C} \to \mathcal{C})_t$ of equation (2.1), there is its *infinitesimal generator* $A : \mathcal{D}(A) \to \mathcal{C}$, the closed linear operator defined on a dense domain $\mathcal{D}(A) \subseteq \mathcal{C}$ by

$$A(\phi) := \lim_{t \to 0} \frac{S_t(\phi) - \phi}{t}.$$

Many dynamical properties of $(S_t)_t$ can be studied by analysing A, for instance via its spectral theory, see [DvGVLW95, Chapter 4], [Hal77, Chapter 7]. In particular, given a constant solution x (that is an equilibrium of $(S_t)_t$), under suitable conditions it is

[2] In the theory of DDEs, people use $\mathcal{C}^{\odot *}$ a lot, where $\odot$ is pronounced as "sun". The space $\mathcal{C}^{\odot *}$ is the dual of the space $\mathcal{C}^{\odot} \subseteq \mathcal{C}^*$, which in turn is the closure of the domain of the dual A^* of the *infinitesimal generator* $A : \mathcal{C} \supset \mathcal{D}(A) \to \mathcal{C}$ of the solution semiflow $(S_t)_t$. Note that this semiflow depends on the exact form of the delay equation at hand, and so does $\mathcal{C}^{\odot *}$. There is an embedding $\mathcal{C} \hookrightarrow \mathcal{C}^{\odot *}$. If $\mathcal{C} = \mathcal{C}^{\odot *}$, then $\mathcal{C}$ is said to be $\odot$*-reflexive* with respect to $(S_t)_t$. See e.g. [DvGVLW95, Chapter II].

possible to split $\mathcal{C}$ into a stable subspace, an unstable subspace and a center subspace, resulting in a stable manifold, an unstable manifold and a center manifold. A similar strategy can be used to study the behavior of the system near a periodic solution, see [Hal77, Chapter 10], [DvGVLW95, Chapter XIV].

For non-autonomous (but still linear) DDEs, the dynamics cannot be captured in a solution semiflow $(S_t : \mathcal{C} \to \mathcal{C})_t$. Indeed, it is not sufficient to specify an initial history ϕ and a time $t \geq 0$ after which we are interested in the solution, because also the initial time matters. This means that one has to consider a two-parameter family $(U_{s,t})_{s,t}$ of operators with certain properties, a so-called *forward evolutionary system* on $\mathcal{C}$. If the dependency on time is periodic, one can do some kind of Floquet theory relating the periodic linear DDE to an autonomous one, see [DvGVLW95, Chapter XIII].

If we are given a 1-parameter family of delay equations, it makes sense to ask about how solutions change with the parameter. Indeed, under suitable assumptions, parts of bifurcation theory carry over from the theory of ODEs to the delay setting, see [DvGVLW95, Chapter X], [Hal77, Section 11.1]. Note that the assumptions include that the family of equations is twice continuously differentiable with respect to the parameter; this makes it impossible in this context to handle varying delay as a bifurcation parameter.

2.3 Results and literature related to this thesis

Let us now mention literature on two aspects of the theory of DDEs which are related to the topic of this thesis. The first aspect is the existence of periodic solutions. The second aspects concerns the regularity of the dependence on the delay of (not necessarily periodic) solutions. These results are all very interesting and in some way related to our work, but to the best of our knowledge none of them implies Theorem 4.1. (We point out that we can only give a very limited glimpse into the existing literature on differential delay equations and that it may be biased by our interest in geometric approaches.) However, the use of the polyfold setting and, in particular, the polyfold implicit function theorem is certainly new in the context of differential delay equations.

2.3.1 Existence of periodic solutions

One class of results concerns differential delay equations with fixed delay and asks for existence of periodic solutions with arbitrary period. Here, Mallet-Paret [MP88] and Nussbaum [Nus73] used global methods to find periodic solutions for some classes of differential delay equations. Kaplan and Yorke [KY74] showed the existence (and some properties) of periodic solutions of a differential delay equation with symmetries and fixed delay by converting it to an ordinary differential equation in twice the dimension. The uniqueness counterpart in the Kaplan–Yorke setting was recently solved by López Nieto [LN20]. Existence results for periodic orbits with small delay were proven by Arino–Hbid [AH90] and Hbid–Qesmi [HQ06] locally near a stable equilibrium of the delay equation by bifurcation arguments. In these results the period is allowed to vary with the delay, and there is no statement about the regularity with respect to the delay. Sieber [Sie12] shows how to locally find families of periodic orbits even for state-dependent delay, but he does not consider varying the delay. He uses the concept of "extendable continuous differentiability" (mentioned before in [HKWW06]) which seems to have a certain similarity with the scale differentiability by Hofer–Wysocki–Zehnder [HWZ21].

2.3.2 Smooth dependence on initial history and delay

In the context of solving differential delay equations with the help of a semi-flow acting on a function space, it is natural to ask whether solutions depend smoothly on the initial history and on the delay. This means analyzing the regularity of the solution map

$$(\phi, \tau) \longmapsto x \tag{2.2}$$

sending an initial history $\phi : [-T, 0] \to \mathbb{R}^n$ and a delay $\tau \leq T$ to the appropriate maximal solution $x : [-T, T_{\phi,\tau}] \to \mathbb{R}^n$ of the considered delay equation. It turns out that the differentiability of this map depends heavily on the choice of the space of initial histories. Hale–Ladeira [HL91] showed that in case of $W^{1,\infty}$ as history space, the dependence is of class $\mathcal{C}^1$. Recently, Nishiguchi [Nis19] showed the same for history spaces of general Sobolev type. Walther [Wal19] discusses different kinds of $\mathcal{C}^1$-differentiability in Fréchet spaces. None of these articles touch upon the question of regularity beyond $\mathcal{C}^1$.

However, dependence of solutions on delay in the sense of the map (2.2) above is different from dependence of solutions on delay in the sense of the map

$$\tau \longmapsto x_\tau \tag{2.3}$$

that appears in Theorem 4.1. On one hand, we do not consider dependence on initial histories at the same time, which circumvents the question of what history space to use. This is why, in our case, $\mathcal{C}^1$-dependence is immediate from classical methods, see the discussion in §4.2 (especially Remark 4.7). On the other hand, the parametrization map (2.3) is not just the restriction of the solution map (2.2) to a fixed initial history. Indeed, there is no reason why the periodic orbits from Theorem 4.1 should all agree on an interval of length τ. Therefore, we do not see any direct connection between our theorem and the articles [HL91, Nis19, Wal19] mentioned above.

Chapter 3

A short guide to polyfold theory

Polyfold theory was developed by Helmut Hofer, Krzysztof Wysocki and Eduard Zehnder [HWZ09, HWZ10, HWZ21] as a general framework for the study of moduli spaces of (perturbed) pseudoholomorphic curves. It builds on a new notion of differentiability, called sc-differentiability[3], for maps between Banach spaces with an additional structure, called *sc-Banach spaces*. The power of the theory lies in the following:

- Polyfold theory gives a smooth structure to objects which are not smooth in a classical sense, namely to ambient spaces of compactified moduli spaces of pseudoholomorphic curves.

- Moreover, it provides an implicit function theorem (IFT) assuring that zero sets of certain maps between these objects – namely the compactified moduli spaces themselves – are finite-dimensional classically smooth manifolds.

All definitions and results about sc-differentiability, polyfold theory and the polyfold IFT can be found in all details in the recently published book [HWZ21] by Hofer–Wysocki–Zehnder. For a motivation of the theory, we recommend the survey [FFGW12] or Section 1 of the article [BDS⁺21]. In this chapter, we sketch the idea of polyfold theory and state the definitions which are relevant in the context of this thesis.

3.1 Motivation: Moduli spaces of pseudoholomorphic curves

Let (Σ, i) be a (not necessarily connected) Riemann surface and (M, J) an almost complex manifold, that is, a smooth manifold M together with an automorphism $J : TM \to TM$ of the tangent bundle (i.e. $J_p : T_pM \to T_pM$ for every $p \in M$) satisfying $J^2 = -\mathrm{id}_{TM}$. A map $u : \Sigma \to M$ is called *pseudoholomorphic* (or *J-holomorphic* if we want to stress the almost complex structure) if

$$\mathrm{d}u \circ i = J(u) \circ \mathrm{d}u, \tag{3.1}$$

[3]Here, "sc-" stands for "scale". The polyfold community sometimes pronounces it as "scale", sometimes as "ess see".

that is, its differential respects the almost complex structure.[4] (3.1) is called *Cauchy–Riemann equation*. Pseudoholomorphic maps share many rigidity properties of holomorphic maps, see for instance [Wen15]. For symplectic geometers, pseudoholomorphic maps are most interesting in the case when the manifold M carries a symplectic form ω and the almost complex structure J is *compatible* with ω, meaning that $g_J := \omega(J\cdot, \cdot)$ defines a Riemannian metric on M. Given a symplectic manifold (M, ω), there are plenty of compatible almost complex structures, and they can be used as a means to study (M, ω) and the possible dynamics on it. There is a lot of literature on pseudoholomorphic curves; we refer particularly to the books [MS12] and [Wen18].

In symplectic and contact topology, moduli spaces of pseudoholomorphic curves are used to define homological invariants like Hamiltonian Floer theory [Sal99, AD14], Rabinowitz Floer homology [CF09, AF], symplectic homology [CO18], Gromov–Witten invariants [FO99, HWZ17] or symplectic field theory [EGH00, FH18a, Wen16]. These moduli spaces have the form

$$\mathcal{M}(J) := \{u : (\Sigma, i) \to (M, J) \mid \mathrm{d}u \circ i = J(u) \circ \mathrm{d}u\} / \sim,$$

where $u \sim v$ iff one is a biholomorphic reparametrization of the other, that is, if there is $\psi : (\Sigma, i) \to (\Sigma, i)$ biholomorphic with $u = v \circ \psi$. Usually, there are some more conditions on the map u, for instance a prescribed homotopy class, an energy estimate $\int_\Sigma u^*\omega < \infty$ or point constraints $u(z_j) = p_j$ (in this case the biholomorphic reparametrizations are expected to fix the points $z_j \in \Sigma$), or the Cauchy–Riemann equation is perturbed to the Floer equation.

Such a moduli space is typically not compact, but under suitable assumptions it has a natural compactification. If even more conditions on Σ, M and J are satisfied, then the compactified moduli space $\overline{\mathcal{M}}(J)$ is expected to have the structure of a finite-dimensional smooth manifold, with dimension given by the Riemann–Roch formula. When proving a result like this, one would like to use an implicit function theorem. For this, it would be crucial to see $\overline{\mathcal{M}}(J)$ as the zero set $\bar{\partial}_J^{-1}(0) = \overline{\mathcal{M}}(J)$ of a transverse Fredholm section $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ in a Banach bundle (i.e. the total space $\mathcal{E}$ and the base space $\mathcal{B}$ are Banach manifolds and each fiber is a Banach space). However, one runs into the following problems:

(i) (varying domains) Due to bubbling and breaking phenomena, the compactification $\overline{\mathcal{M}}(J)$ contains equivalence classes of maps living on different Riemann surfaces. Thus, even if we ignore the quotient by reparametrization for a while, the compactified moduli spaces does not sit inside a function space of maps $u : \Sigma \to M$, but rather inside a bigger ambient space $\mathcal{B}$ which contains maps from different Riemann surfaces to M. The topology on $\overline{\mathcal{M}}(J)$ is defined via so-called pregluing maps. These maps can be used to define a topology on $\mathcal{B}$ as well. In some sense they even suggest what local charts should look like. However, these candidates for local charts are in general not homeomorphisms onto open subsets of Banach spaces. Hence, this strategy does not give $\mathcal{B}$ the structure of a smooth Banach manifold. (See e.g. [FFGW12], especially Example 2.1.4 and Remark 2.1.5 therein.)

[4] The prefix "pseudo" reflects the fact that (M, J) is in general only *almost* complex; that is, J does not necessarily give rise to a holomorphic atlas on M.

(ii) (quotient, transition maps, non-trivial isotropy) The moduli space $\overline{\mathcal{M}}(J)$ consists of maps modulo reparametrization. Thus also the ambient space $\mathcal{B}$ should consist of maps modulo reparametrization. This causes two problems. First, the reparametrization group does not act smoothly on the function space. This implies that even if we know how to define local charts (using a slice of the action), the corresponding transition maps will not be smooth. Second, if the reparametrization group acts on the function space with non-trivial isotropy, this means that we should expect $\mathcal{B}$ to have the structure of an orbifold rather than a manifold.

(iii) (transversality) For using an implicit function theorem, the transversality condition is important. Even for the space of J-holomorphic maps on a fixed domain (i.e. without taking into account the quotient) this requires a detailed analysis of the section $\bar{\partial}_J$ and a careful choice of almost complex structure J. In some situations, however, for geometric reasons it is not even possible to find a suitable J for which $\bar{\partial}_J$ is transverse. (See e.g. [MS12], [Wen15], [Wen18].)

Thus, in general there is no transverse Fredholm section $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ in a Banach bundle with $\bar{\partial}_J^{-1}(0) = \overline{\mathcal{M}}(J)$. Instead, for proving that $\overline{\mathcal{M}}(J)$ is a manifold, one uses the implicit function theorem only for the space of J-holomorphic maps (without quotient) on a fixed domain, and then checks by hand that indeed the manifold structure descends and extends to $\overline{\mathcal{M}}(J)$. Polyfold theory was developed to overcome all these problems in one unified framework. All the steps which usually need to be done for every single moduli space individually should be incorporated into abstract theory. It is clear that such a theory is necessarily quite complicated, but also very powerful.[5]

As mentioned above, polyfold theory builds on the notion of sc-differentiability which we describe in detail in Section 3.2 below. The following is a very rough sketch of how problems (i)-(iii) are approached in polyfold theory:

(i) (varying domains) The candidates for local charts map to open subsets of *sc-retracts in open quadrants of sc-Banach spaces*, and these are exactly the local models for (M-)polyfolds.

(ii) (quotient, transition maps, non-trivial isotropy) The reparametrization groups act sc-smoothly on the function spaces at hand. Thus, the candidates for transition maps between local charts are sc-smooth, as they should be for (M-)polyfolds. Moreover, the definition of polyfolds via groupoids includes orbifold behavior.

(iii) (transversality) Polyfold theory incorporates an abstract perturbation scheme. With that at hand, there is no need to choose the almost complex structure J in a specific way; instead, one works with the preferred J and perturbs the section $\bar{\partial}_J$ abstractly. This way, not $\overline{\mathcal{M}}(J)$ itself is given the structure of a smooth finite dimensional manifold, but an arbitrarily small perturbation of it.

[5] The hope is that at some point people can build their polyfold bundles from existing polyfold models via a ‘building-block system’ or an ‘imprinting method’, thus using the theory mainly as a black box, see [FH18b]. At the moment, this is still difficult, since there are not yet that many polyfold constructions to build on. For examples how polyfold theory can be used to prove famous results, see [FW21a] and [BDS$^+$21].

More detailed discussions can be found in [FFGW12] or [BDS+21, Section 1]. Here we pass on to state the definitions and results which are relevant in our context.

3.2 Sc-differentiability and sc-smoothness

Definition 3.1 ([HWZ21, Definition 1.1.1])**.** A *sc-Hilbert space* E is a Hilbert space E_0 together with a filtration

$$\cdots \subseteq E_{m+1} \subseteq E_m \subseteq \cdots \subseteq E_0$$

by subspaces E_m, $m \in \mathbb{N}_0$, all of which are Hilbert spaces in their own right, in such a way that all inclusions $E_{m+1} \hookrightarrow E_m$ are compact and dense.

The norm on the Hilbert space E_m will be denoted by $\|\cdot\|_{E_m}$. We use the notation

$$\mathrm{E}^1 = \big((\mathrm{E}^1)_m = E_{m+1}\big)_{m\in\mathbb{N}_0}$$

to denote the subspace E_1 with the induced filtration. Elements of the intersection $E_\infty := \cap_{m\in\mathbb{N}_0} E_m$ are called *smooth points*. We observe that every finite dimensional Hilbert space E is a sc-Hilbert space E with the constant filtration $E_m = E$. In the infinite dimensional case, it follows from compactness of the inclusions that $E_{m+1} \neq E_m$ for all $m \in \mathbb{N}_0$. Note that products and sums of sc-Hilbert spaces are sc-Hilbert spaces again.

We now state the definitions of sc-continuity, sc-differentiability and sc-smoothness for maps between sc-Hilbert spaces. In the book [HWZ21], these notions are defined more generally for maps between open subsets of quadrants of sc-Banach spaces.

Definition 3.2 ([HWZ21, Definition 1.1.13])**.** A map $f : \mathrm{E} \to \mathrm{F}$ between sc-Hilbert spaces E and F is *sc-continuous* (sc^0) if it satisfies $f(E_m) \subseteq F_m$ for all $m \in \mathbb{N}_0$ and the induced maps $f : E_m \to F_m$ are continuous.

Definition 3.3 ([HWZ21, Definition 1.1.15])**.** Let $f : \mathrm{E} \to \mathrm{F}$ be a map between sc-Hilbert spaces. It is called *sc-differentiable* (sc^1) if the following holds:

1. f is sc-continuous.
2. For every $x \in E_1$ there exists a bounded linear operator $\mathrm{d}f(x) : E_0 \to F_0$ such that
$$\lim_{h\in E_1, \|h\|_{E_1}\to 0} \frac{\|f(x+h) - f(x) - \mathrm{d}f(x)h\|_{F_0}}{\|h\|_{E_1}} = 0.$$
3. The *tangent map* Tf given by
$$\begin{aligned} Tf : \mathrm{E}^1 \oplus \mathrm{E} &\longrightarrow \mathrm{F}^1 \oplus \mathrm{F} \\ (x,h) &\longmapsto \mathrm{d}f(x)h \end{aligned}$$
is sc-continuous.

Note that this definition does not require the map $E_1 \to \mathcal{L}(E_0, F_0), x \mapsto \mathrm{d}f(x)$ to be continuous with respect to the operator norm. Indeed, in general this will not be the case. In finite dimensions, since the sc-structure is constant, sc^1-maps are differentiable in the usual sense, and by Proposition 3.8 below they are exactly the $\mathcal{C}^1$-maps. Lots of examples of sc^1-maps in infinite dimensions can be found in [HWZ10].

Having the notion of sc-differentiability, we can proceed inductively to define sc-smoothness:

Definition 3.4 ([HWZ21, below Remark 1.1.16])**.** Let $k \geq 2$. A map $f : \mathrm{E} \to \mathrm{F}$ between sc-Hilbert spaces E and F is sc^k if it is sc^{k-1} and its tangent map is sc^{k-1}. It is *sc-smooth* (sc^∞) if it is sc^k for every $k \in \mathbb{N}$.

In practice, when working with differentiability, one usually relies on a chain rule. From the definition of sc-differentiability it is not obvious that there should be a chain rule in sc-calculus: After all, in the definition of sc-differentiability there is a shift in levels, and we might expect these shifts to add up when we concatenate maps. However, Hofer–Wysocki–Zehnder showed that sc-differentiability satisfies a true chain rule without any shift in levels:

Theorem 3.5 (chain rule, [HWZ21, Theorem 1.3.1])**.** *Let $f : \mathrm{E} \to \mathrm{F}$ and $g : \mathrm{F} \to \mathrm{G}$ be sc^1-maps. Then $g \circ f : \mathrm{E} \to \mathrm{G}$ is also sc^1, and the tangent map satisfies $T(g \circ f) = Tg \circ Tf$.*

3.3 Properties and alternative characterizations of sc-differentiability

The following two results are very useful in applications.

Proposition 3.6. *[HWZ21, Proposition 1.2.2] Let $f : \mathrm{E} \to \mathrm{F}$ be sc^k. Then the induced map $f : \mathrm{E}^1 \to \mathrm{F}^1$ is also sc^k.*

Proposition 3.7. *[HWZ21, Proposition 1.2.4] Let $f : \mathrm{E} \to \mathrm{F}$ be a map such that for every $m \geq 0$ and $0 \leq l \leq k$, it induces a map*

$$f : E_{m+l} \to F_m$$

which is of class $\mathcal{C}^{l+1}$. Then f is sc^{k+1}.

There are several alternative characterizations of sc-differentiability which are helpful in recognizing sc^1-maps[6]. The following is due to Frauenfelder–Weber. It is similar (but not equal) to [HWZ21, Proposition 1.2.1].

Proposition 3.8 ([FW21b, Lemma 4.6])**.** *Let $f : \mathrm{E} \to \mathrm{F}$ be sc^0. Then it is sc^1 if and only if the following conditions are satisfied:*

(i) $f : E_1 \to F_0$ is pointwise differentiable in the usual sense.

[6]In particular, comparing the properties of the shift map φ that we collect in Section 4.2 with the conditions in Proposition 3.8 suggests that sc-calculus indeed is a good framework for the proof of Theorem 4.1.

(ii) For every $x \in E_k$, the differential $\mathrm{d}f(x) : E_1 \to F_0$ *has a continuous extension* $\mathrm{d}f(x) : E_0 \to F_0$.

(iii) For all $m \geq 0$ and $x \in E_{m+1}$, the continuous extension $\mathrm{d}f(x) : E_0 \to F_0$ *from (ii) restricts to a continuous map* $\mathrm{d}f(x)|_{E_m} : E_m \to F_m$ *such that*

$$\mathrm{d}f|_{E_{m+1}\oplus E_m} : E_{m+1} \oplus E_m \longrightarrow F_m$$

is continuous.

In the same spirit, Frauenfelder–Weber [FW21b] gave a description of sc^k-maps in terms of the classical kth derivative and its continuity with respect to the compact-open topology. While the original definition of sc^k via tangent maps is very elegant, the alternative characterization is easier to check in applications.

Proposition 3.9 ([FW21b, Proposition 4.10][7])**.** *Let $f : \mathrm{E} \to \mathrm{F}$ be sc^{k-1}. Then it is sc^k if and only if the following conditions are satisfied:*

(i) $f : E_k \to F_0$ is pointwise k times differentiable in the usual sense.

(ii) For every $x \in E_k$, the kth differential $\mathrm{d}^k f(x) : E_k \oplus \cdots \oplus E_k \longrightarrow F_0$ has a continuous extension

$$\mathrm{d}^k f(x) : \underbrace{E_{k-1} \oplus \cdots \oplus E_{k-1}}_{k \text{ times}} \longrightarrow F_0.$$

(iii) For all $m \geq k-1$ and $x \in E_{m+1}$, the continuous extension $\mathrm{d}^k f(x) : E_{k-1} \oplus \cdots \oplus E_{k-1} \longrightarrow F_0$ from (ii) restricts to a continuous k-fold multilinear map

$$\mathrm{d}^k f(x)|_{E_m\oplus\cdots\oplus E_m} : \underbrace{E_m \oplus \cdots \oplus E_m}_{k \text{ times}} \longrightarrow F_{m-(k-1)} = F_{m-k+1}$$

such that

$$\mathrm{d}^k f|_{E_{m+1}\oplus(E_m\oplus\cdots\oplus E_m)} : E_{m+1} \oplus \underbrace{(E_m \oplus \cdots \oplus E_m)}_{k \text{ times}} \longrightarrow F_{m-k+1}$$

is continuous.

3.4 Sc-Hilbert manifolds and strong bundles

An n-dimensional smooth manifold is defined to be a topological space which is locally homeomorphic to $\mathbb{R}^n$, in such a way that the transition maps between the local charts are smooth. Using sc-Hilbert spaces as local models, we can make the following definition:

[7] In comparison to [FW21b, Proposition 4.10], here we switched the roles of k and m, in order to bring notation more in line with notation from [HWZ21].

Definition 3.10. A *sc-Hilbert manifold* $\mathcal{B}$ is a paracompact Hausdorff space which is locally homeomorphic to open subsets of sc-Hilbert spaces, in such a way that the transition maps between the local charts are sc-smooth.
The filtration on the sc-Hilbert spaces induces a filtration $\mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \dots$ on $\mathcal{B}$. In particular, it makes sense to talk about *smooth points* $x \in \mathcal{B}_\infty$ of a sc-Hilbert manifold.

The notion of a bundle over a sc-Hilbert space or manifold should of course respect the sc-structure of base space and fiber. In order to define bundles, we need the following notion.

Definition 3.11. [HWZ21, above Definition 2.6.1] Let E and F be sc-Hilbert spaces. The *non-symmetric product* $\mathrm{E} \triangleright \mathrm{F}$ is the set $\mathrm{E} \times \mathrm{F}$ equipped with the double filtration

$$(\mathrm{E} \triangleright \mathrm{F})_{m,k} := E_m \oplus F_k$$

for $m \in \mathbb{N}_{\geq 0}$, $0 \leq k \leq m+1$. For $i = 0, 1$, we define sc-manifolds $(\mathrm{E} \triangleright \mathrm{F})[i]$ by their filtrations

$$(\mathrm{E} \triangleright \mathrm{F})[i]_m := E_m \oplus F_{m+i}.$$

The projection $\mathrm{E} \triangleright \mathrm{F} \to \mathrm{E}$ together with all this extra structure is called a *trivial strong sc-Hilbert bundle.*

To make this clearer and to define non-trivial bundles, we need to say which properties of $\mathrm{E} \triangleright \mathrm{F}$ should be preserved by a bundle map.

Definition 3.12. [HWZ21, Definition 2.6.1] A map $\Phi : \mathrm{E} \triangleright \mathrm{F} \longrightarrow \mathrm{E}' \triangleright \mathrm{F}'$ is a *strong bundle map* if

- it preserves the double filtration,
- it is of the form $\Phi(x, y) = \big(\phi(x), \Gamma(x, y)\big)$, where the map Γ is linear in y,
- and the maps

  $$\Phi[i] : (\mathrm{E} \triangleright \mathrm{F})[i] \longrightarrow (\mathrm{E}' \triangleright \mathrm{F}')[i]$$

  are sc-smooth for $i = 0, 1$.

A *strong bundle isomorphism* is an invertible strong bundle map whose inverse is also a strong bundle map.

Finally, we are ready to define strong sc-Hilbert bundles over sc-Hilbert manifolds.

Definition 3.13. [HWZ21, special case of Definitions 2.6.4, 2.6.5] Let $\mathcal{B}$ be a sc-Hilbert manifold, and let $P : \mathcal{E} \to \mathcal{B}$ be a surjective map from a paracompact Hausdorff space $\mathcal{E}$ to $\mathcal{B}$, such that for every $x \in \mathcal{B}$ the fiber $P^{-1}(\{x\})$ is a Hilbert space.

- A *strong sc-Hilbert bundle chart* for $P : \mathcal{E} \to \mathcal{B}$ consists of an open subset $V \subseteq \mathcal{B}$ of $\mathcal{B}$, an open subset $O \subset \mathrm{E}$ of a sc-Hilbert space E, a trivial strong sc-Hilbert bundle $\mathrm{E} \triangleright \mathrm{F}$, and a homeomorphism

  $$\Phi : P^{-1}(V) \longrightarrow O \triangleright \mathrm{F}$$

  satisfying the following:

- Φ covers a homeomorphism $\phi : V \to O$.
- For each $x \in V$, the homeomorphism Φ restricts to a bounded linear isomorphism between the Hilbert spaces $P^{-1}(\{x\})$ and $\phi(x) \times F_0 \cong F_0$.

- Two strong sc-Hilbert bundle charts with $V \cap V' \neq \emptyset$ are *compatible* if the transition maps

$$\Phi' \circ \Phi^{-1}[i] : \Phi\big(P^{-1}(V \cap V')\big)[i] \longrightarrow \Phi'\big(P^{-1}(V \cap V')\big)[i]$$

are sc-smooth for $i = 0, 1$.

- A *strong bundle atlas* for $P : \mathcal{E} \to \mathcal{B}$ is a collection of strong bundle charts covering $\mathcal{E}$. Equivalence of atlases is defined in the usual way.

- The surjective map $P : \mathcal{E} \to \mathcal{B}$ together with an equivalence class of strong bundle atlases is called a *strong sc-Hilbert bundle* over $\mathcal{B}$.

3.5 M-polyfolds

All M-polyfolds and bundles considered in this thesis are in fact sc-Hilbert spaces (resp. sc-Hilbert manifolds in Section 7.2) and trivial bundles (resp. strong sc-Hilbert bundles). However, the implicit function theorem in sc-calculus that we will use (see Theorem 3.23 below) is stated in the more general context of M-polyfold bundles. Therefore, we chose to at least sketch the definitions of the notions of M-polyfolds[8] and polyfolds as well. The reader who is not interested in these definitions may pass on directly to Sections 3.7 and 3.8, replacing "M-polyfold" by "sc-Hilbert manifold" everywhere.

So, how do sc-Hilbert manifolds generalize to M-polyfolds? One difference is that from the very beginning, we do not demand a Hilbert space structure any more, just a Banach space. This makes hardly any difference in the definitions and results. Only for the implicit function theorem we need to make sure that there exist sc-smooth bump functions; this is satisfied if everything is modeled on sc-Hilbert spaces, but in the sc-Banach setting it is a non-empty condition. Another difference is that M-polyfolds allow for boundary and corners. This means that all definitions, starting from sc-continuity, sc-differentiability and the tangent, need to be generalized to open subsets of *partial quadrants* in sc-Banach spaces, and that in some results boundary points play a special role. The third and main difference is that M-polyfolds are not modeled on open subsets of (partial quadrants in) sc-Banach spaces, but instead on open subsets of *sc-smooth retracts* in (partial quadrants in) sc-Banach spaces.

Definition 3.14. [HWZ21, Definitions 2.1.1, 2.1.3, 2.1.12] A *sc-smooth retraction* is a sc-smooth map r such that $r \circ r = r$. A subset $O \subset \mathrm{E}$ of a sc-Banach space is a *sc-smooth retract* if there exists a sc-smooth retraction r on a (relatively open subset of a partial quadrant in a) sc-Banach space such that $O = \mathrm{im}(r)$. The tangent of a sc-smooth retract O defined by a retraction r is $TO = \mathrm{im}(Tr)$.

[8] "M-polyfold" stands for "manifold-type polyfold" and means a polyfold without orbifold behavior.

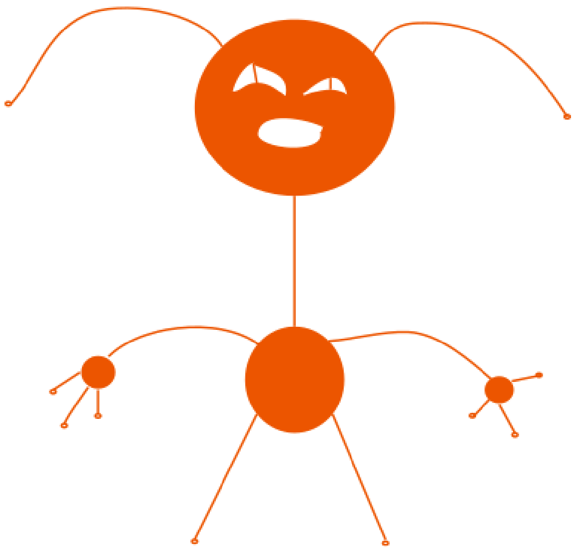


Figure 3.1: An M-polyfold with locally varying finite dimensions. The sketch is taken from the book [HWZ21].

Passing to sc-smooth retracts does not only mean carrying more notation (because the retraction comes into play) and more concerns about well-definedness (because one retract could be defined via different retraction maps, resulting in a priori different sc-smooth structures) – it really makes a qualitative change. While the images of classically smooth retractions are classical submanifolds (see [HWZ21, Proposition 2.1.2], the images of sc-smooth retractions have different behavior: They can have locally varying finite dimensions. The prototype example is [HWZ10, Example 1.22], where the image of sc-smooth retraction is homeomorphic to $\big(\mathbb{R}_{\leq 0} \times \{0\}\big) \cup \big(\mathbb{R}{>}\ 0 \times \mathbb{R}\big) \subset \mathbb{R}^2$. Despite this non-differentiable appearance, Hofer–Wysocki–Zehnder were able to show that the following notion of sc-smoothness for maps on sc-smooth retracts makes sense and satisfies the chain rule.

Definition 3.15. [HWZ21, Definition 2.1.14] Let $f : O \to \mathrm{E}$ be a map from a sc-smooth retract O to a sc-Banach space E, and let r be a sc-smooth retraction for O. Then f is *sc-smooth* if the composition $f \circ r$ is sc-smooth. In this case, the *tangent map* of f is $Tf := T(f \circ r)|_{TO}$.

Given this definition, one can pass on to define charts and atlases in the usual manner. In short, the notion of M-polyfold is defined as follows.

Definition 3.16. [HWZ21, Definition 2.3.4] An *M-polyfold* is a paracompact Hausdorff space which is locally homeomorphic to open subsets sc-smooth retracts, in such a way that the transition maps between the local charts are sc-smooth.

One famous M-polyfold is sketched in Figure 3.1. We suspect, though, that this M-polyfold has not been used in applications. Strong bundles over M-polyfolds are defined similarly as strong bundles over sc-Hilbert manifolds: via strong bundle charts covering M-polyfold charts. We skip the details here because they do not help for a better understanding.

3.6 Polyfolds

Passing from M-polyfolds to polyfolds means adding the orbifold behavior, hence another level of abstraction. Before we give the definition of polyfolds, let us recall that orbifolds can be defined via the notion of Lie groupoid. For this, we recommend the survey [Moe02] and the references therein.

Definition 3.17. [Moe02, Sections 1.1, 1.2, 1.4, 1.5, 2.1, 2.4, 3.1, 3.2, and 3.3]

- A *groupoid* is a small category in which every morphism is an isomorphism. Thus, a groupoid $\mathcal{G}$ consists of a set G_0 of objects, a set G_1 of isomorphisms and structure maps s, t, m, u, and i satisfying various conditions. Here $s, t : G_1 \to G_0$ are the *source* and *target* maps associating to each morphism its source and target;

$$\begin{aligned} m : G_1 \times_{G_0} G_1 := \{(h, g) \in G_1 \times G_1 \mid s(h) = t(g)\} &\longrightarrow G_1 \\ (h, g) &\longmapsto h \circ g \end{aligned}$$

  is the (associative) *composition* or *multiplication* map; $u : G_0 \to G_1, x \mapsto 1_x$ is the *unit* map assigning to each object x the identity morphism $1_x : x \to x$; and $i : G_1 \to G_1, g \mapsto g^{-1}$ is the *inverse* map assigning to each morphism its inverse.

- A *Lie groupoid* is a groupoid $\mathcal{G}$ for which the sets G_0 and G_1 are smooth manifolds, the structure maps s, t, m, u, i are smooth, and $s, t : G_1 \to G_0$ are submersions.

- A *homomorphism between Lie groupoids* is a smooth functor. There is a notion of *equivalence* of Lie groupoids.

- Let $\mathcal{G}$ be a Lie groupoid. Then for every object $x \in G_0$, the set $G_x := \{g \in G_1 \mid s(g) = t(g) = x\}$ of morphisms from x to itself is a Lie group. It is called the *isotropy* or *stabilizer* at x.

- Let $\mathcal{G}$ be a Lie groupoid. For every $x \in G_0$ the set

$$t\big(s^{-1}(x)\big) = \{y \in G_0 \mid \exists g : x \to y\}$$

  of targets of morphisms starting in x is called the *orbit* of x. It is a smooth manifold. The collection of all orbits is called *orbit space* and denoted by $|\mathcal{G}|$; it is a quotient of G_0, and in general it is not a manifold.

- A Lie groupoid $\mathcal{G}$ is called *proper* if the map $(s, t) : G_1 \to G_0 \times G_0$ proper.

- A Lie groupoid $\mathcal{G}$ is called *étale* if s and t are local diffeomorphisms.

- A Lie groupoid $\mathcal{G}$ is called a *foliation groupoid* if each isotropy group G_x is discrete. In particular, every étale groupoid is a foliation groupoid.

- An *orbifold groupoid* is a proper foliation groupoid. If $\mathcal{G}$ is an orbifold groupoid, then its orbit space $|\mathcal{G}|$ is a locally compact Hausdorff space.

- If Z is a locally compact Hausdorff space, then an *orbifold structure* on Z consists of an orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to Z$. There is a notion of *equivalence* of orbifold structures.

- An *orbifold* is a locally compact Hausdorff space equipped with an equivalence class of orbifold structures.

The same strategy is used by Hofer–Wysocki–Zehnder to define polyfolds. However, while the notion of étale proper Lie groupoids has a long history and is well studied, the corresponding notion in polyfold theory ("ep-groupoids") is new. Thus, a big part of the book [HWZ21] is dedicated to establishing the theory of ep-groupoids, and the definition of a polyfold is not stated untill Chapter 16, that is page 707 of the printed book. Therefore, here we only sketch the definition and refer to [HWZ21] for all details.

Definition 3.18. [HWZ21]

- ([HWZ21, Definition 7.1.3]) An *ep-groupoid* is a groupoid $\mathcal{G}$ for which the sets G_0 and G_1 are M-polyfolds, the structure maps s, t, m, u, i are sc-smooth, and
  - (étale) s and t are local sc-diffeomorphisms,
  - (proper)[9] and every object $x \in G_0$ has a neighborhood $V \subseteq G_0$ such that the map

    $$t|_{s^{-1}(\overline{V})} : s^{-1}(\overline{V}) \to G_0$$

    is proper.

- ([HWZ21, Proposition 7.1.12]) Let $\mathcal{G}$ be an ep-groupoid. Then for every object $x \in G_0$ the isotropy G_x is finite.

- ([HWZ21, Theorem 7.3.1]) Let $\mathcal{G}$ be an ep-groupoid. Then the orbit space $|\mathcal{G}|$ is a locally metrizable, regular, Hausdorff space. If in addition $|\mathcal{G}|$ is paracompact, then it is metrizable.

- ([HWZ21, Definitions 16.1.1, 16.1.2]) If Z is a topological space, then a *polyfold structure* on Z consists of an ep-groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to Z$. There is a notion of *equivalence* of polyfold structures.

- ([HWZ21, Definition 16.1.3]) A *polyfold* is a topological space equipped with an equivalence class of polyfold structures.

3.7 The sc-Fredholm property

While the definition of the nonlinear sc-Fredholm property is quite involved (see below), linear sc-Fredholm operators are defined in a straightforward way:

[9] This properness condition is stronger than the one for Lie groupoids. See also [HWZ21, below Definition 7.1.4].

Definition 3.19 ([HWZ21, Definition 1.1.9])**.** A sc-continuous linear operator $T : \mathrm{E} \to \mathrm{F}$ is a *sc-Fredholm operator* if there are splittings $\mathrm{E} = \mathrm{K} \oplus \mathrm{X}$ and $\mathrm{F} = \mathrm{C} \oplus \mathrm{Y}$ respecting the sc-structure such that the following holds:

- K is the kernel of T and finite dimensional.
- Y is the image of T and C is finite dimensional.
- $T : \mathrm{X} \to \mathrm{Y}$ is a sc-isomorphism.

The *Fredholm index* of T is the integer $\mathrm{ind}(T) := \dim \mathrm{K} - \dim \mathrm{C}$.

Another characterization of the linear sc-Fredholm property is the following.

Lemma 3.20 ([Weh12, Lemma 3.6])**.** *A sc-continuous linear operator $T : \mathrm{E} \to \mathrm{F}$ is a sc-Fredholm operator if and only if it is regularizing (that is, if $e \in E_0$ and $T(e) \in F_m$, then $e \in E_m$) and $T : E_0 \to F_0$ is a classical Fredholm operator.*

The linear sc-Fredholm property is invariant under a class of perturbations called *sc^+-perturbations*. Kernel and cokernel of sc-Fredholm operators consist of smooth points.

In classical calculus, a map is defined to be Fredholm if its linearization at any point is a Fredholm operator. This implies the existence of a contraction normal form which can be used to prove the implicit function theorem for Fredholm maps (see [Weh12, Remark 4.2]). Hence, one might try to define sc-Fredholm maps as sc-smooth maps with differentials that are linear sc-Fredholm operators. Again, this implies the existence of a normal form. However, in sc-calculus this normal form does not necessarily involve a contraction on the whole space, rather a contraction from one level to another (see [Weh12, Remark 4.2]). In particular, there is no implicit function theorem for this class of maps. Counterexamples and a detailed discussion of these problems can be found in [FZW19]. To obtain an implicit function theorem, one has to restrict to sc-smooth maps satisfying some extra condition. This *nonlinear sc-Fredholm property* was defined by Hofer–Wysocki–Zehnder in terms of a special form (*basic germ*, [HWZ21, Definition 3.1.7]) that the map needs to take after *sc^+*-perturbation and a sc-smooth coordinate change (see [HWZ21, Definitions 3.1.11, 3.1.16]). The proof of the M-polyfold implicit function theorem (see Theorem 3.23 for the statement and [HWZ21] for the proof) and the counterexamples and discussion in [FZW19] suggest that this sc-Fredholm property is exactly what is needed to make an implicit function theorem possible. However, in applications the right sc-smooth coordinate change may be hard to find. Katrin Wehrheim suggested the following alternative definition of a *sc-Fredholm property* (at a point) *with respect to a splitting*:

Definition 3.21 ([Weh12, Definition 4.3])**.** Let $f : \mathrm{E} \to \mathrm{F}$ be a *sc*-smooth map. Then f is *sc-Fredholm at* 0 *with respect to the splitting* $\mathrm{E} = \mathbb{R}^d \oplus \mathrm{E}'$ if the following holds:

(i) f is regularizing as germ, that is for every $m \in \mathbb{N}_0$ there exists $\varepsilon_m > 0$ such that $f(e) \in F_{m+1}$ and $\|e\|_{E_m} \leq \varepsilon_m$ implies $e \in E_{m+1}$.

(ii) $\mathrm{E} = \mathbb{R}^d \oplus \mathrm{E}'$ is a sc-isomorphism and for every $m \in \mathbb{N}_0$ there exists $\varepsilon_m > 0$ such that $f(r, \cdot) : B^{E'_m}_{\varepsilon_m} \to F_m$ is differentiable for all $\|r\|_{\mathbb{R}^d} < \varepsilon_m$. Moreover, for fixed $m \in \mathbb{N}_0$, the differential $\mathrm{d}_{\mathrm{E}'} f(r_0, e_0) : E'_m \to F_m$ in direction of E' has the following continuity properties:

(a) For $r \in B^{\mathbb{R}^d}_{\varepsilon_m}$ the differential operator

$$\begin{aligned} B^{E'_m}_{\varepsilon_m} &\longrightarrow \mathcal{L}(E'_m, F_m) \\ e &\longmapsto \mathrm{d}_{\mathrm{E}'} f(r, e) \end{aligned}$$

is continuous, and the continuity is uniform in a neighborhood of $(r, e) = (0, 0)$.

(b) For sequences $\mathbb{R}^d \ni r_\nu \to 0$ and $e_\nu \in B^{E'_m}_1$ with $\|\mathrm{d}_{\mathrm{E}'} f(r_\nu, 0)e_\nu\|_{F_m} \to 0$, $\nu \to \infty$, there exists a subsequence such that $\|\mathrm{d}_{\mathrm{E}'} f(0, 0)e_\nu\|_{F_m} \to 0$.

(iii) The differential $\mathrm{d}_{\mathrm{E}'} f(0, 0) : \mathrm{E}' \to \mathrm{F}$ is a sc-Fredholm operator. Moreover, $\mathrm{d}_{\mathrm{E}'} f(r, 0) : E'_0 \to F_0$ is classically Fredholm for all $\|r\|_{\mathbb{R}^d} < \varepsilon_0$, with Fredholm index equal to that for $r = 0$, and weakly regularizing, meaning that $\ker \mathrm{d}_{\mathrm{E}'} f(r, 0) \subseteq E'_1$.

As in [Weh12, Definition 4.3], above the sc-Fredholm property is defined only at the origin $(\tau, x) = (0, 0)$. At a smooth point $(\tau^*, x^*) \in \mathbb{R} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n) = \mathbb{R} \times \bigcap_m H_m$ the appropriate conditions are obtained by conjugation with the sc-smooth map $(\tau, x) \mapsto (\tau - \tau^*, x - x^*)$. This definition of the sc-Fredholm property (*with respect to a splitting*) is not equivalent to the original one ([HWZ21, Definitions 3.1.11, 3.1.16]). However, Wehrheim proved the following:

Theorem 3.22 ([Weh12, Theorem 4.5])**.** *Let* $f : \mathrm{E} \to \mathrm{F}$ *be a sc-smooth map that is sc-Fredholm at* 0 *with respect to a splitting* $\mathrm{E} = \mathbb{R}^d \oplus \mathrm{E}'$*. Then* $f|_{E_1} : \mathrm{E}^1 \to \mathrm{F}^1$ *is sc-Fredholm at* 0*.*

In the implicit function theorem, in the end one is interested only in the zero set $\{f = 0\}$ of a given sc-Fredholm map f, and this zero set is then automatically contained in the set E_∞ of smooth points. Therefore the shift in scales occuring in Theorem 3.22 is irrelevant for the conclusions of the implicit function theorem. This means that, although the two definitions are not strictly equivalent, in practice one can choose which one to work with.

3.8 The M-polyfold implicit function theorem

We now state the implicit function theorem in the context of strong M-polyfold bundles. Some notions used in the theorem are not defined above because in the setting of this thesis they are not important; see Remarks 3.24 and 3.25 below.

Theorem 3.23 (M-polyfold Implicit Function Theorem [HWZ21, Theorem 3.6.8])**.** *Let* f *be a sc-Fredholm section of a tame strong M-polyfold bundle* $\mathrm{Y} \to \mathrm{X}$*. If* $f(x) = 0$*, and if the sc-Fredholm germ* (f, x) *is in good position, then there exists an open neighborhood* V *of* $x \in \mathrm{X}$ *such that the solution set* $\mathcal{S} = \{y \in V \mid f(y) = 0\}$ *in* V *has the following properties.*

- *At every point* $y \in \mathcal{S}$*, the sc-Fredholm germ* (f, y) *is in good position.*
- $\mathcal{S}$ *is a sub-M-polyfold of* X *and the induced M-polyfold structure is equivalent to a smooth manifold structure with boundary with corners.*

Remark 3.24. Let us make a few comments on Theorem 3.23.

- The regularizing property of a sc-Fredholm section implies that the solution set $\mathcal{S}$ is contained in $\mathrm{X}_\infty = \bigcap_{m\in\mathbb{N}} \mathrm{X}_m$, the set of smooth points of X.
- If the M-polyfold X in Theorem 3.23 does not have boundary or corners, then the solution space $\mathcal{S}$ is a smooth, finite-dimensional manifold without boundary or corners.
- Being in good position consists of two conditions. The first one is surjectivity of $\mathrm{d}f$ at the point $x \in \mathrm{X}$. The second condition is concerned with the case that X has boundary or corners and is thus not relevant in our context, see Remark 3.25.
- As in the classical implicit function theorem, the tangent space of $\mathcal{S}$ at a point $y \in \mathcal{S}$ is given by

  $$T_y\mathcal{S} = \ker \mathrm{d}f(y) \subseteq T_y\mathrm{X}$$

  (see [HWZ21, Theorem 3.1.22]). In particular, the local dimension of the solution space $\mathcal{S}$ equals the Fredholm index of the linearized section.

Remark 3.25. As mentioned before, all M-polyfolds considered in this thesis are in fact sc-Hilbert spaces or sc-Hilbert manifolds. This leads to significant technical simplifications. For instance, neither retraction maps nor boundaries need to be considered.

Part I

Families of periodic delay orbits

Chapter 4

The main theorem

4.1 Idea and statement

It is folklore knowledge that, under a non-degeneracy assumption, a periodic orbit of a vector field persists under smooth perturbations of the vector field. The reason is that periodic orbits correspond to zeroes of a suitable Fredholm section in a Banach space bundle. This section is transverse to the zero section by non-degeneracy. Therefore the implicit function theorem provides a smooth family of periodic orbits of dimension equal to the Fredholm index. Here, instead of perturbing the vector field, we perturb the equation $\partial_t x(t) = X_t(x(t))$ by adding a delay and considering it as the perturbation parameter. The result (see Theorem 4.1 below) is similar to the one for perturbations of the vector field, but the methods to prove it are much more involved.

Let us be a bit more precise and suppose that we are given a smooth time-dependent vector field $X : S^1 \times \mathbb{R}^n \to \mathbb{R}^n$, where $S^1 = \mathbb{R}/\mathbb{Z}$. For a constant delay parameter $\tau \in \mathbb{R}$, we want to study maps $x : S^1 \to \mathbb{R}^n$ which solve the corresponding delay equation

$$\partial_t x(t) = X_t(x(t - \tau)) \qquad \text{for all } t \in S^1. \tag{4.1}$$

For $\tau = 0$, this recovers the case without delay, and solutions of (4.1) are exactly the 1-periodic orbits of X. For small $\tau \neq 0$, we think of (4.1) as a slight perturbation of the ODE $\partial_t x(t) = X_t(x(t))$, and so we might expect that non-degenerate periodic orbits of X give rise to families of periodic delay orbits.

However, the corresponding Banach section is merely of class $\mathcal{C}^1$, see the discussion below. Therefore, the classical implicit function theorem does not provide a smooth family of periodic delay orbits. Instead, we use the implicit function theorem for M-polyfold bundles ([HWZ21], stated here as Theorem 3.23), which was developed in the context of symplectic field theory [EGH00], see below for details.

Recall that a 1-periodic orbit of a vector field X is called *non-degenerate* if the linearized time-1-map of the flow of X at any point of the orbit does not have 1 as an eigenvalue, see also Definition 4.15. We prove the following:

Theorem 4.1. *If x_0 is a non-degenerate 1-periodic orbit of X, then there is $\tau_0 > 0$ such that for every delay τ with $|\tau| \leq \tau_0$ there exists a (locally unique) smooth 1-periodic solution x_τ of the delay equation* (4.1)*. Moreover, the parametrization $\tau \mapsto x_\tau$ is smooth.*

The proof of Theorem 4.1 is given in Section 4.6. The idea behind the proof can be used to cover more general delay equations (for instance, with more delay parameters or on manifolds) as well. This is demonstrated in Chapter 7.

Note that we do not consider delay in the time-dependence of the vector field, that is, equations of the form $\partial_t x(t) = X_{t-\tau}(x(t))$, since this is merely a deformation of the vector field X_t and can therefore be treated by the classical implicit function theorem.

Let us now formulate the functional analytic setup. We denote by

$$\begin{aligned} \varphi : \mathbb{R} \times L^2(S^1, \mathbb{R}^n) &\longrightarrow L^2(S^1, \mathbb{R}^n) \\ (\tau, x) &\longmapsto x(\cdot - \tau). \end{aligned} \tag{4.2}$$

the shift map, and define a map

$$\begin{aligned} s : \mathbb{R} \times W^{1,2}(S^1, \mathbb{R}^n) &\longrightarrow L^2(S^1, \mathbb{R}^n) \\ (\tau, x) &\longmapsto \partial_t x - X(\varphi(\tau, x)). \end{aligned} \tag{4.3}$$

Then the set of solutions of the delay equation (4.1) for all delays $\tau \in \mathbb{R}$ at once corresponds to the zero set of s. Besides, every solution x_0 of $\partial_t x(t) = X_t(x(t))$ satisfies $s(0, x_0) = 0$. Thus it seems plausible to use an implicit function theorem to show that, under a suitable non-degeneracy assumption on x_0, the zero set of s carries the structure of a smooth submanifold of $\mathbb{R} \times W^{1,2}(S^1, \mathbb{R}^n)$ having dimension equal to the Fredholm index of s. This Fredholm index is expected to be 1, because $\partial_t : W^{1,2}(S^1, \mathbb{R}^n) \to L^2(S^1, \mathbb{R}^n)$ has index 0.[10] So the implicit function theorem would prove existence of solutions of (4.1) and also give a parametrization. However, the map s as defined in (4.3) is in general not smooth; we will see that it is, in general, only $\mathcal{C}^1$. The reason is that the shift map φ is not smooth in τ, as will be explained in more detail in Section 4.2. The lack of regularity of s implies that also the parametrization which we get from a classical implicit function theorem can only be of regularity $\mathcal{C}^1$.

Analyzing the properties of this shift map in detail, we see that it is very natural to pass from classical to sc-calculus. Recall from Chapter 3 that sc-calculus provides a new notion of smoothness for maps between Banach spaces equipped with a scale structure, and that it was developed by Hofer–Wysocki–Zehnder mainly for the study of moduli spaces of J-holomorphic curves. In that context, non-smoothness of reparametrization actions is one of the main problems, and sc-calculus was made to deal with this. Indeed, Frauenfelder and Weber [FW21b] showed that the shift map φ defined above in (4.2) is sc-smooth between appropriate sc-spaces. Thus, sc-calculus provides a natural way to deal with the problem described above as follows. Using the definition of the sc-Fredholm property in [Weh12], we show that the map s is a sc-smooth sc-Fredholm section in a sc-Hilbert space bundle. Sc-Hilbert space bundles are the easiest examples of tame strong M-polyfold bundles defined in [HWZ21]. Thus, to prove Theorem 4.1, we can apply the implicit function theorem from sc-calculus [HWZ09, HWZ21] (stated here as Theorem 3.23).

[10] The kernel consists of constant loops, the cokernel is generated by loops of non-zero integral, hence both are isomorphic to $\mathbb{R}^n$.

4.2 Classical differentiability

From now on, for an integer $m \geq 0$ we denote by

$$H_m := W^{m,2} := W^{m,2}(S^1, \mathbb{R}^n) \tag{4.4}$$

the Hilbert space of periodic maps of Sobolev class $(m, 2)$ with values in $\mathbb{R}^n$. In particular, $H_0 = L^2 = L^2(S^1, \mathbb{R}^n)$. Consider the following shift map:

$$\begin{aligned} \varphi : \mathbb{R} \times H_m &\longrightarrow H_m \\ (\tau, x) &\longmapsto x(\cdot - \tau) \end{aligned}$$

It is easy to see that φ is continuous after evaluation, but it is not continuous in the operator topology. This can be remedied by choosing a higher regularity level of the domain while keeping the one on the target. In this section we collect these facts. Proofs following Frauenfelder–Weber [FW21b] can be found in Appendix A. We use the notation $\mathcal{L}(\cdot, \cdot)$ for spaces of linear maps.

Lemma 4.2 ([FW21b, Lemma 2.1]). *For every $m \in \mathbb{N}_0$, the map*

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{L}\big(H_m, H_m\big) \\ \tau &\longmapsto \varphi(\tau, \cdot) \end{aligned}$$

is continuous with respect to the compact-open topology on $\mathcal{L}(H_m, H_m)$.

Lemma 4.3 ([FW21b, Lemma 2.2]). *The shift map φ is not continuous as a map*

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow \mathcal{L}\big(H_0, H_0\big) \\ \tau &\longmapsto \varphi(\tau, \cdot), \end{aligned}$$

where the target space carries the operator norm topology.

Proof. For every small τ construct a function $x_\tau \in H_0$ of norm 1 such that $\|\varphi(\tau, x_\tau) - \varphi(0, x_\tau)\|_{H_0} = c > 0$. This implies $\|\varphi(\tau, \cdot) - \varphi(0, \cdot)\|_{\mathcal{L}} \geq c$. Note that by Lemma 4.2, such a family $(x_\tau)_{\tau>0}$ cannot converge in $H_0 = L^2$. An easy construction of x_τ with $c = \sqrt{2}$ is contained in [FW21b]. □

Now let us consider the shift map φ as a map from $\mathbb{R} \times H_1$ to H_0.

Lemma 4.4. *The shift map*

$$\begin{aligned} \varphi : \mathbb{R} \times H_1 &\longrightarrow H_0 \\ (\tau, x) &\longmapsto x(\cdot - \tau) \end{aligned}$$

is differentiable with derivative at a point (τ, x) given by

$$\begin{aligned} \mathrm{d}\varphi(\tau, x) : \mathbb{R} \times H_1 &\longrightarrow H_0 \\ (T, \hat{x}) &\longmapsto \varphi(\tau, \hat{x}) - T \cdot \varphi(\tau, \partial_t x). \end{aligned} \tag{4.5}$$

The statement of this Lemma 4.4 follows from [FW21b, Theorem 6.1] together with [HWZ21, Proposition 1.2.3]. For convenience of the reader, we include a direct proof in Appendix A.

Remark 4.5. In fact, one can even show that the derivative is continuous as a map $\mathrm{d}\varphi : \mathbb{R} \times H_1 \to \mathcal{L}(\mathbb{R} \times H_1, H_0)$, that is, $\varphi : \mathbb{R} \times H_1 \to H_0$ is $\mathcal{C}^1$.

In the same way, for each $m \in \mathbb{N}$ we can consider φ as a map

$$\begin{aligned} \varphi : \mathbb{R} \times H_{m+1} &\longrightarrow H_m \\ (\tau, x) &\longmapsto x(\cdot - \tau) \end{aligned}$$

and see that it is $\mathcal{C}^1$. This is easiest to prove if one works with the norm $\|x\|_m := \|x\|_{L^2} + \|\partial_t^m x\|_{L^2}$ which is equivalent to the usual Sobolev norm $\|\cdot\|_{H_m} = \|\cdot\|_{W^{m,2}}$. In order to gain regularity, one might consider φ as a map $\varphi : \mathbb{R} \times H_m \longrightarrow H_0$ to gain regularity.

Lemma 4.6. *For $m \in \mathbb{N}$ the map*

$$\begin{aligned} \varphi : \mathbb{R} \times H_m &\longrightarrow H_0 \\ (\tau, x) &\longmapsto x(\cdot - \tau) \end{aligned}$$

is of class $\mathcal{C}^m$.

Remark 4.7. We recall from the introduction, see (4.3), the map $s : \mathbb{R} \times H_m \to H_0$ defined by

$$\begin{aligned} s : \mathbb{R} \times H_m &\longrightarrow H_0 \\ (\tau, x) &\longmapsto \partial_t x - X(\varphi(\tau, x)) \end{aligned} \tag{4.6}$$

where $X : S^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is some time-dependent smooth vector field. Since s is $\mathcal{C}^1$, the classical implicit function theorem implies (s is indeed Fredholm) the existence of zeroes of s (i.e. solutions of (4.1) for small $\tau \in \mathbb{R}$) under a suitable non-degeneracy assumption. The implicit function theorem will guarantee the parametrization of these solutions to be $\mathcal{C}^1$ in τ. A priori this parametrization will not be of higher regularity, though. In order to gain better regularity one might be tempted to pass to the C^2-map $s : \mathbb{R} \times H_2 \to H_0$. However, since $s : \mathbb{R} \times H_2 \to H_0$ factors through the compact embedding $H_1 \hookrightarrow H_0$ it fails to be Fredholm. In addition, its linearization is never surjective.

One aim of this thesis is to employ the natural framework of scale calculus and the corresponding scale implicit function theorem in order to directly prove the existence of a $\mathcal{C}^\infty$-family of solutions to (4.1) leading to Theorem 4.1.

4.3 Sc-Smoothness

The goal of this section is to show that the map s (which was defined in equation (4.3) and cuts out the solution space) is sc-smooth between appropriate sc-Hilbert spaces.

Consider the sc-Hilbert space defined by

$$\mathrm{H} := \left(H_m = W^{m,2}(S^1, \mathbb{R}^n)\right)_{m \in \mathbb{N}_0},$$

that is the Hilbert space $H_0 = L^2(S^1, \mathbb{R}^n)$ with filtration given by the numbers of weak derivatives. Moreover, recall from Chapter 3 that $\mathbb{R}$ is a sc-Hilbert space with the constant filtration $\mathbb{R}_m \equiv \mathbb{R}$. Then the shift map φ is a map between these sc-spaces, that is

$$\begin{aligned} \varphi : \mathbb{R} \times \mathrm{H} &\longrightarrow \mathrm{H} \\ (\tau, x) &\longmapsto x(\cdot - \tau). \end{aligned}$$

When we compare the properties of φ established in Section 4.2 to the properties of sc-differentiability mentioned in Section 3.3, it makes sense to expect the following result by Frauenfelder–Weber:

Theorem 4.8 ([FW21b, Theorem 6.1])**.** *$\varphi : \mathbb{R} \times \mathrm{H} \to \mathrm{H}$ is sc-smooth.*

Now consider the sc-Hilbert space H^1, that is H_1 with the induced filtration. Then equation (4.3) defines a map

$$\begin{aligned} s : \mathbb{R} \times \mathrm{H}^1 &\longrightarrow \mathrm{H} \\ (\tau, x) &\longmapsto \partial_t x - X(\varphi(\tau, x)) \end{aligned} \tag{4.7}$$

between sc-Hilbert spaces. Here, $X : S^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is a time-dependent smooth vector field on $\mathbb{R}^n$. The implicit function theorem (Theorem 3.23) is formulated in the language of sections in strong M-polyfold bundles. To translate s into this language we define

$$\begin{aligned} S : \mathbb{R} \times \mathrm{H}^1 &\longrightarrow \mathbb{R} \times \mathrm{H}^1 \triangleright \mathrm{H} \\ (\tau, x) &\longmapsto (\tau, x, s(\tau, x)). \end{aligned} \tag{4.8}$$

S is a section in the trivial strong sc-Hilbert space bundle $\mathbb{R} \times \mathrm{H}^1 \triangleright \mathrm{H} \longrightarrow \mathbb{R} \times \mathrm{H}^1$.

Remark 4.9. As pointed out above, sc-Hilbert spaces are trivially M-polyfolds. In fact, they admit global charts and do not require retractions. Moreover, the trivial bundle $\mathbb{R} \times \mathrm{H}^1 \triangleright \mathrm{H} \longrightarrow \mathbb{R} \times \mathrm{H}^1$ is a *strong* bundle in the sense of Definition 3.13 and [HWZ21, Definitions 2.6.4, 2.6.5]. The map s is the *principal part* of S, see [HWZ21, Definition 2.6.3]. Finally, since we do not need to consider boundary nor retractions, the *tameness* condition defined in [HWZ21, Definitions 2.5.2, 2.5.7] is trivially satisfied. Thus, the bundle $\mathbb{R} \times \mathrm{H}^1 \triangleright \mathrm{H} \longrightarrow \mathbb{R} \times \mathrm{H}^1$ is a tame strong M-polyfold bundle. Since everything is modeled on sc-Hilbert spaces, these M-polyfolds automatically admit sc-smooth bump functions.

Proposition 4.10. *The section S is sc-smooth. Its vertical sc-differential at the point $(\tau, x) \in (\mathbb{R} \times \mathrm{H}^1)_1 = \mathbb{R} \times H_2$ is*

$$\begin{aligned} \mathrm{d}s(\tau, x) : \mathbb{R} \times H_1 &\longrightarrow H_0 \\ (T, \hat{x}) &\longmapsto \partial_t \hat{x} - \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x}) + T \cdot \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \partial_t x). \end{aligned} \tag{4.9}$$

In particular, at $(0, x) \in (\mathbb{R} \times \mathrm{H}^1)_1$ this simplifies to

$$\mathrm{d}s(0, x)(T, \hat{x}) = \partial_t \hat{x} - \mathrm{d}X(x) \cdot \hat{x} + T \cdot \mathrm{d}X(x) \cdot \partial_t x. \tag{4.10}$$

Remark 4.11. Note that by $\mathrm{d}X(y)$ we mean the map

$$\begin{aligned} \mathrm{d}X(y) : S^1 &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \mathrm{d}X_t(y(t)). \end{aligned}$$

Since S^1 is compact and X is smooth, $\mathrm{d}X(y)$ has the same Sobolev regularity as y.

Proof of Proposition 4.10. First, we observe that the operator $\partial_t : \mathrm{H}^1 \to \mathrm{H}$ is sc-smooth. Indeed, for every m, the operator $\partial_t : H_{m+1} \to H_m$ is a bounded linear map, in particular it is classically smooth. Thus, by Proposition 3.7, ∂_t is sc-smooth.

Next, note that Theorem 4.8 implies that φ is sc-smooth also as a map $\varphi : \mathbb{R} \times \mathrm{H}^1 \to \mathrm{H}^1 \hookrightarrow \mathrm{H}$ since the inclusion $\mathrm{H}^1 \hookrightarrow \mathrm{H}$ is level-wise compact.

Since the vector field X is smooth, the map $\mathrm{H} \ni x \mapsto X(x) \in \mathrm{H}$ is sc-smooth. Now the chain rule from scale calculus implies that S is sc-smooth, and it also gives the formula for the derivative. Here we may use the fact that φ is classically $\mathcal{C}^1$ and therefore sc^1 and its sc-differential agrees with the classical differential given by equation (4.5). ☐

4.4 The sc-Fredholm property

Next we establish the sc-Fredholm property. We show that the section S is sc-Fredholm in the sense of Definition 3.21, keeping in mind that this implies that – at least after restricting S to a map $\mathbb{R} \times \mathrm{H}^2 \to \mathbb{R} \times \mathrm{H}^2 \triangleright \mathrm{H}^1$ – it is also sc-Fredholm in the sense of Hofer–Wysocki–Zehnder.

Theorem 4.12. *S is a sc-Fredholm section.*

Proof. We first show that S is sc-Fredholm at $(\tau, x) = (0, 0)$ with respect to a splitting by checking conditions (i), (ii), and (iii) of Definition 3.21. After this we revisit the case of a general smooth point.

As a splitting, in the sense of Wehrheim, of the domain $\mathbb{R} \times \mathrm{H}^1$ we take the one induced by the Cartesian product. In particular, we have $d = 1$.

(i) First we show that s is regularizing. Take $(\tau, x) \in (\mathbb{R} \times \mathrm{H}^1)_m = \mathbb{R} \times H_{m+1}$ with

$$s(\tau, x) = \partial_t x - X(\varphi(\tau, x)) \in H_{m+1}.$$

Since $x \in H_{m+1}$, we have $\varphi(\tau, x) \in H_{m+1}$ and thus $X(\varphi(\tau, x)) \in H_{m+1}$. This means that $\partial_t x = s(\tau, x) + X(\varphi(\tau, x))$ lies in H_{m+1} and so $x \in H_{m+2}$, thus

$$(\tau, x) \in (\mathbb{R} \times \mathrm{H}^1)_{m+1}$$

as desired.

(ii) For fixed $\tau \in \mathbb{R}$ and $m \in \mathbb{N}$, the map

$$\begin{aligned} s_{\tau,m} := s(\tau, \cdot) : H_{m+1} &\longrightarrow H_m \\ x &\longmapsto \partial_t x - X(\varphi(\tau, x)) \end{aligned}$$

is clearly classically smooth with differential

$$\begin{aligned} \mathrm{d}s_{\tau,m}(x) : H_{m+1} &\longrightarrow H_m \\ \hat{x} &\longmapsto \partial_t \hat{x} - \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x}). \end{aligned} \tag{4.11}$$

(a) For fixed m and small τ,

$$\mathrm{d}s_{\tau,m} : H_{m+1} \longrightarrow \mathcal{L}(H_{m+1}, H_m)$$

needs to be uniformly continuous in x near $x = 0$ (note that non-uniform continuity follows from classical smoothness).

In more detail, we need to show that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $\|x\|_{H_{m+1}} < \delta$, for all $x' \in H_{m+1}$ with $\|x - x'\|_{H_{m+1}} < \delta$, and for all $\hat{x} \in H_{m+1}$

$$\|\mathrm{d}s_{\tau,m}(x)\hat{x} - \mathrm{d}s_{\tau,m}(x')\hat{x}\|_{H_m} \leq \varepsilon \cdot \|\hat{x}\|_{H_{m+1}}$$

holds. Indeed, from equation (4.11) we get the following:

$$\begin{aligned}
&\|\mathrm{d}s_{\tau,m}(x)\hat{x} - \mathrm{d}s_{\tau,m}(x')\hat{x}\|_{H_m} \\
&\qquad = \|\left(\mathrm{d}X(\varphi(\tau, x')) - \mathrm{d}X(\varphi(\tau, x))\right) \cdot \varphi(\tau, \hat{x})\|_{H_m} \\
&\qquad \leq \|\mathrm{d}X(\varphi(\tau, x')) - \mathrm{d}X(\varphi(\tau, x))\|_{\mathcal{C}^m(S^1, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))} \cdot \underbrace{\|\varphi(\tau, \hat{x})\|_{H_{m+1}}}_{=\|\hat{x}\|_{H_{m+1}}}
\end{aligned}$$

The last estimate follows from the operator norm inequality for fixed $t \in S^1$, $\tau \in \mathbb{R}$ and linear maps on $\mathbb{R}^n$.

For δ small enough the first factor in this estimate is smaller than ε since $\mathrm{d}X$ is continuous and $\|x - x'\|_{H_{m+1}} < \delta$ implies $\|\varphi(\tau, x) - \varphi(\tau, x')\|_{\mathcal{C}^m} < \mathit{const} \cdot \|\varphi(\tau, x) - \varphi(\tau, x')\|_{H_{m+1}} < \mathit{const} \cdot \delta$ (recall that φ is an isometry in its second argument).

(b) Suppose we are given a sequence $(\tau_\nu, \hat{x}_\nu)_\nu \subseteq (\mathbb{R} \times \mathrm{H}^1)_m$ such that $\tau_\nu \to 0$ and $\|\hat{x}_\nu\|_{H_{m+1}} < 1$ such that

$$\|\mathrm{d}s_{\tau_\nu}(0)\hat{x}_\nu\|_{H_m} \longrightarrow 0.$$

Then we need to find a subsequence of $(\hat{x}_\nu)_\nu$ (still denoted by the same symbol) such that

$$\|\mathrm{d}s_0(0)\hat{x}_\nu\|_{H_m} \longrightarrow 0.$$

We compute

$$\begin{aligned}
\|\mathrm{d}s_0(0)\hat{x}_\nu\|_{H_m} &= \|\partial_t \hat{x}_\nu - \mathrm{d}X(0) \cdot \varphi(0, \hat{x}_\nu)\|_{H_m} \\
&\leq \|\partial_t \hat{x}_\nu - \mathrm{d}X(0) \cdot \varphi(\tau_\nu, \hat{x}_\nu)\|_{H_m} \\
&\qquad + \|\mathrm{d}X(0) \cdot \left(\varphi(\tau_\nu, \hat{x}_\nu) - \varphi(0, \hat{x}_\nu)\right)\|_{H_m}.
\end{aligned}$$

The first summand converges to zero by assumption. For the second summand we recall that $\mathrm{d}X(0)$ is still t-dependent: For every $t \in S^1$ it denotes the linear map $\mathrm{d}X_t(0) : \mathbb{R}^n \to \mathbb{R}^n$. Since $\|\hat{x}_\nu\|_{H_{m+1}} < 1$ and the inclusion $H_{m+1} \hookrightarrow H_m$ is compact there exists a subsequence (still denoted by $(\hat{x}_\nu)_\nu$) with $(\hat{x}_\nu)_\nu \to \hat{x}$ in H_m. Taking the corresponding subsequence of $(\tau_\nu)_\nu$, we get by Lemma 4.2 that $\varphi(\tau_\nu, \hat{x}_\nu) - \varphi(0, \hat{x}_\nu) \to 0$ in H_m. Finally, since $\mathrm{d}X(0)$ is continuous it follows that

$$\|\mathrm{d}X(0) \cdot \left(\varphi(\tau_\nu, \hat{x}_\nu) - \varphi(0, \hat{x}_\nu)\right)\|_{H_m} \longrightarrow 0.$$

(iii) The third condition again consists of several parts.

(a) Since $0 \in \bigcap_{m \geq 0} H^1_m$, by condition (ii) we have maps $\mathrm{d}s_{0,m}(0) : H^1_m \to H_m$ for all $m \in \mathbb{N}$. Together they define a sc^0-map

$$\mathrm{d}s_0(0) : \mathrm{H}^1 \longrightarrow \mathrm{H}.$$

We have to show that $\mathrm{d}s_0(0)$ is a linear sc-Fredholm operator (Definition 3.19, Lemma 3.20), meaning that it is regularizing and classically linear Fredholm at the 0-level.

The regularizing property follows exactly as in (i). It remains to show that the operator

$$\begin{aligned} W^{1,2}(S^1, \mathbb{R}^n) = H_1 &\longrightarrow H_0 = L^2(S^1, \mathbb{R}^n) \\ \hat{x} &\longmapsto \partial_t \hat{x} - \mathrm{d}X(0) \cdot \hat{x} \end{aligned} \tag{4.12}$$

has closed image and finite dimensional kernel and cokernel. The operator ∂_t is Fredholm between these spaces. Indeed, its kernel is the space of constant maps while its image consists of all periodic maps with mean zero. Thus, kernel and cokernel are isomorphic to $\mathbb{R}^n$. Since $H_1 \hookrightarrow H_0$ is a compact embedding, the second term in (4.12) represents a compact operator and thus does not change the Fredholm property.

(b) The final condition is that for fixed τ near 0, the operator on 0-level

$$\begin{aligned} \mathrm{d}s_\tau(0) : (\mathrm{H}^1)_0 = H_1 &\longrightarrow H_0 \\ \hat{x} &\longmapsto \partial_t \hat{x} - \mathrm{d}X(0) \cdot \varphi(\tau, \hat{x}) \end{aligned}$$

is classically linear Fredholm with the same index as $\mathrm{d}s_0(0)$, and that it is weakly regularizing, meaning

$$\ker \mathrm{d}s_\tau(0) \subseteq (\mathrm{H}^1)_1 = H_2$$

(as opposed to just $\ker \mathrm{d}s_\tau(0) \subseteq H_1$ which holds by definition).

To verify these properties, note that the first term of $\mathrm{d}s_\tau(0)$ is the same in $\mathrm{d}s_0(0)$ and the second one is still compact. In particular, $\mathrm{d}s_\tau(0)$ is Fredholm of the same index as $\mathrm{d}s_0(0)$. Now take $\hat{x} \in \ker \mathrm{d}s_\tau(0) \subseteq H_1$, then

$$\partial_t \hat{x} = \mathrm{d}X(0) \cdot \varphi(\tau, \hat{x}).$$

Since the shift does not change regularity, the right hand side lies in H_1, so $\partial_t \hat{x} \in H_1$ and thus $\hat{x} \in H_2$.

This finishes the proof that S is sc-Fredholm at $(\tau, x) = (0, 0)$. Now we review conditions (i)-(iii) from above and see what needs to be changed for the sc-Fredholm property at a general smooth point $(\tau, x) \in \mathbb{R} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n)$. We recall that these conditions are obtained from a conjugation, as mentioned above.

(i) The proof of the regularization property for $(0, 0)$ can be repeated verbatim at any smooth point $(\tau, x) \in \mathbb{R} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n)$.

(ii) The proof that $s_{\tau,m}$ is classically differentiable for every m did not use $\tau = 0$ and continues to hold at any $(\tau, x) \in \mathbb{R} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n)$.

(a) In the proof of the uniform continuity of $\mathrm{d}s_{\tau,m}$ near $x = 0$ we neither used that τ is small nor that $\|x\|_{H_{m+1}}$ is small. Again, the same proof continues to work.

(b) Here we need to consider more generally sequences $(\tau_\nu, \hat{x}_\nu)_\nu \subseteq (\mathbb{R} \times \mathrm{H}^1)_m$ with $(\tau_\nu)_\nu \to \tau$ and $\|\hat{x}_\nu\|_{H_{m+1}} \leq 1$ such that

$$\|\mathrm{d}s_{\tau_\nu}(x)\hat{x}_\nu\|_{H_m} \to 0$$

and we need to find a subsequence of $(\hat{x}_\nu)_\nu$ (still denoted the same way) such that

$$\|\mathrm{d}s_\tau(x)\hat{x}_\nu\|_{H_m} \to 0.$$

The following is a small modification of our previous argument. Again by compactness of the embedding $H_{m+1} \hookrightarrow H_m$ we pick a subsequence $(\hat{x}_\nu)_\nu$ converging in H_m to some $\hat{x}$, and the corresponding subsequence $(\tau_\nu)_\nu$. Again add zero and use the triangle inequality as follows:

$$\begin{aligned}
\|\mathrm{d}s_\tau(x)\hat{x}_\nu\|_{H_m} &= \|\partial_t \hat{x}_\nu - \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x}_\nu)\|_{H_m} \\
&= \|\partial_t \hat{x}_\nu - \mathrm{d}X(\varphi(\tau_\nu, x)) \cdot \varphi(\tau_\nu, \hat{x}_\nu) \\
&\quad + \mathrm{d}X(\varphi(\tau_\nu, x)) \cdot \varphi(\tau_\nu, \hat{x}_\nu) - \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x}_\nu)\|_{H_m} \\
&\leq \underbrace{\|\partial_t \hat{x}_\nu - \mathrm{d}X(\varphi(\tau_\nu, x)) \cdot \varphi(\tau_\nu, \hat{x}_\nu)\|_{H_m}}_{\to 0 \text{ by assumption}} \\
&\quad + \|\mathrm{d}X(\varphi(\tau_\nu, x)) \cdot \varphi(\tau_\nu, \hat{x}_\nu) - \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x}_\nu)\|_{H_m}
\end{aligned}$$

By Lemma 4.2 we have $\varphi(\tau_\nu, x) \to \varphi(\tau, x)$ in H_{m+1} (hence also in $\mathcal{C}^m$) as well as $\varphi(\tau_\nu, \hat{x}_\nu) \to \varphi(\tau, \hat{x})$ and $\varphi(\tau, \hat{x}_\nu) \to \varphi(\tau, \hat{x})$ in H_m. By continuity of $\mathrm{d}X$ it follows that

$$\|\mathrm{d}X(\varphi(\tau_\nu, x)) \cdot \varphi(\tau_\nu, \hat{x}_\nu) - \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x}_\nu)\|_{H_m} \longrightarrow 0.$$

(iii) Since $x \in \mathcal{C}^\infty$ is a smooth point, by (ii) there are linear maps $\mathrm{d}s_{\tau,m}(x) : H_{m+1} \to H_m$ for all $m \geq 0$. We have to show that these define a linear sc-Fredholm map

$$\mathrm{d}s_\tau(x) : \mathrm{H}^1 \longrightarrow \mathrm{H}$$

with Fredholm index not changing under small changes of τ.

We have

$$\mathrm{d}s_\tau(x)\hat{x} = \partial_t \hat{x} - \mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x})$$

and so we see that $\mathrm{d}s_\tau(x)$ is of class sc^0 and regularizing. For the Fredholm property at the 0-level and the index we use that the second term is still compact. That is, we use that the dependence on τ is only through compact operators.

This concludes the proof of Theorem 4.12. □

We now compute the Fredholm index of ds at some point $(\tau = 0, x)$, where $x \in H_2$. The Fredholm index in sc-calculus is by definition the same as the classical Fredholm index at the 0-level. The following computation applies in particular to the solution x_0 from Theorem 4.1.

Proposition 4.13. *The Fredholm index of* $\mathrm{d}s(0, x)$ *is equal to* 1.

Proof. The expression

$$\mathrm{d}s(0, x)(T, \hat{x}) = \partial_t \hat{x} - \mathrm{d}X(x) \cdot \hat{x} + T \cdot \mathrm{d}X(x) \cdot \partial_t x.$$

was derived in (4.10). The first term is the operator

$$\begin{aligned} (\mathrm{H}^1)_0 = H_1 &\longrightarrow H_0 \\ \hat{x} &\longmapsto \partial_t \hat{x}. \end{aligned}$$

It is Fredholm of index 0, which was explained above in the proof of Theorem 4.12, precisely condition (iiia) below equation (4.12).

The second term of $\mathrm{d}s(0, x)$, the operator $H_1 \ni \hat{x} \mapsto -\mathrm{d}X(x) \cdot \hat{x} \in H_0$, is compact (by compactness of $H_1 \hookrightarrow H_0$) and thus does not change the Fredholm index. It remains to see that adding the third term in $\mathrm{d}s(0, x)$ does not change the Fredholm property and raises the index by 1. This follows from Lemma 4.14 below. □

We prove the following obvious statement here for completeness.

Lemma 4.14. *Assume that* $f : U \to V$ *is a linear Fredholm operator, and choose some* $v \in V$. *Then the operator* $F : \mathbb{R} \times U \longrightarrow V$, $(T, u) \mapsto f(u) + T \cdot v$ *is Fredholm of index* $\mathrm{ind}\, F = \mathrm{ind}\, f + 1$.

Proof. We consider two cases. If $v = f(u) \in \mathrm{im}\, f$, then $\mathrm{im}\, F = \mathrm{im}\, f$ is still closed of the same codimension and $\ker F = (\{0\} \oplus \ker f) \oplus (\mathbb{R} \cdot (-1, u))$, thus $\dim(\ker F) = \dim(\ker f) + 1$. In the other case, $v \notin \mathrm{im}\, f$, the kernel $\ker F = \{0\} \oplus \ker f$ is isomorphic to $\ker f$ and $\mathrm{im}\, F = \mathrm{im}\, f \oplus \langle v \rangle$, therefore $\dim(\mathrm{coker}\, F) = \dim(\mathrm{coker}\, f) - 1$. □

4.5 Transversality

In order to apply an implicit function theorem, we need transversality of the section S to the zero-section at our given solution, that is surjectivity of the vertical differential $\mathrm{d}s(0, x_0)$ of S at $(0, x_0)$ with $s(0, x_0) = 0$. We now analyze what this condition means for x_0. For that, we recall the notion of non-degeneracy of a periodic orbit of a vector field.

Definition 4.15. Denote the flow of X by Φ_X^t. A 1-periodic orbit $x : S^1 \to \mathbb{R}^n$ of X is called *non-degenerate* if the linearized time-1-map $\mathrm{d}\Phi_X^1(x(0))$ does not have 1 as an eigenvalue.

Remark 4.16. We do not assume that X is complete. The existence of a 1-periodic orbit x implies that in an open neighborhood of $x(S^1)$ in $\mathbb{R}^n$ the flow Φ_X^t is defined for $t \in [0, 1]$. In particular, the notion of non-degeneracy is well-defined.

Remark 4.17. If the vector field X is autonomous, that is, if $X_t(\cdot) = X(\cdot)$ does not depend on $t \in S^1$, then there are no non-constant, non-degenerate periodic orbits. Indeed, if $x : S^1 \to \mathbb{R}^n$ is a periodic orbit of X, then for every $\tau \in \mathbb{R}$ and $t \in S^1$ we have

$$\partial_t x(t-\tau) = X(x(t-\tau)),$$

so every reparametrization $\varphi(\tau, x)$ of x is also a periodic orbit of X. We compute, using again that X is autonomous,

$$\begin{aligned} \mathrm{d}\Phi^1_X\big(x(0)\big)\big(\partial_t x(0)\big) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi^1_X(x(t)) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x(t) \\ &= \partial_t x \end{aligned}$$

and conclude that $\partial_t x$ is an eigenvector of $\mathrm{d}\Phi^1_X(x(0))$ with eigenvalue 1.

Our main goal in this section is to show the following:

Proposition 4.18. *The linear map $\mathrm{d}s_0(x_0) = \mathrm{d}s(0, x_0)(0, \cdot) : H_1 \to H_0$ is surjective if and only if x_0 is non-degenerate.*

This has an immediate corollary:

Corollary 4.19. *If x_0 is non-degenerate, then $\mathrm{d}s(0, x_0) : \mathbb{R} \times H_1 \to H_0$ is surjective.*

The eigenvalues of $\mathrm{d}\Phi^1_X(x(0))$ can be computed in terms of $\mathrm{d}X(x)$. This gives the following well-known alternative characterization of non-degeneracy.

Lemma 4.20. *Let $x : S^1 \to \mathbb{R}^n$ be a 1-periodic orbit of X. Set*

$$A(t) := -\mathrm{d}X_t(x(t))^T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

for every $t \in S^1$ and let $Y : \mathbb{R} \to \mathbb{R}^{n\times n}$ be the fundamental system for $A : S^1 \to \mathbb{R}^{n\times n}$, that is, the solution of

$$\begin{cases} \tfrac{\mathrm{d}}{\mathrm{d}t} Y(t) = A(t) \cdot Y(t) \\ Y(0) = \mathbb{1}. \end{cases} \tag{4.13}$$

Then

$$\mathrm{d}\Phi^1_X(x(0)) = \big(Y(1)^T\big)^{-1}$$

In particular, x is non-degenerate if and only if $Y(1)$ does not have 1 as an eigenvalue.

Proof. We use the flow Φ^t_X of X to define $Z(t) := \mathrm{d}\Phi^t_X(x(0))$. Then $Z(0) = \mathbb{1}$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Z(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{d}\Phi^t_X\big(x(0)\big)\Big) \\ &= \mathrm{d}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t_X\big(x(0)\big)\Big) \\ &= \mathrm{d}\Big(X_t\big(\Phi^t_X(x(0))\big)\Big) \\ &= \mathrm{d}X_t\big(\underbrace{\Phi^t_X(x(0))}_{=x(t)}\big) \cdot \mathrm{d}\Phi^t_X(x(0)) \\ &= -A(t)^T \cdot Z(t) \end{aligned}$$

That is, $Z(t)$ satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} Z(t) = -A(t)^T \cdot Z(t) \\ Z(0) = \mathbb{1}, \end{cases}$$

meaning that Z is a fundamental system of the so-called adjoint system of A. One can easily compute, using the two initial value problems, that

$$Z(t)^T \cdot Y(t) = \mathbb{1} \quad \forall t \in \mathbb{R}.$$

Therefore, we have

$$Y(t) = \left(Z(t)^T\right)^{-1} \quad \forall t \in \mathbb{R},$$

in particular,

$$Y(1) = \left(Z(1)^T\right)^{-1} = \left(\mathrm{d}\Phi_X^1(x(0))^T\right)^{-1}.$$

This proves the lemma. □

Remark 4.21. In case that $X = X_H$ is a *Hamiltonian* vector field (meaning that $\mathrm{d}H_t = \omega(X_t, \cdot)$ for some time-dependent function H_t and a symplectic form ω), Lemma 4.20 simplifies slightly due to the fact that the matrix $A(t) = \mathrm{d}X_t(x(0))$ is skew-symmetric. In particular, Y and Z solve the same initial value problem and are thus identical and, in addition, symmetric matrices.

Another simplification occurs in the case of a fixed point of an autonomous vector field. For instance, assume that X is autonomous and $X(0) = 0$. The constant orbit $x_0(t) := 0$ is then also a delay orbit of any delay, thus the existence of smoothly parametrized delay orbits is immediate. However, Theorem 4.1 may still be applied to show local uniqueness. We claim that in this situation non-degeneracy of the 1-periodic orbit x_0 is equivalent to $\mathrm{d}X(0)$ being invertible. Indeed, using the notation of Lemma 4.20 we see that $A(t) = -\mathrm{d}X(0)^T$ is constant. Therefore, the fundamental system is given by $Y(t) = \exp(tA)$. By considering a vector $v \in \mathbb{R}^n$ and the ODE that is satisfied by $v(t) := \exp(tA)v$, one easily sees that the matrix A has an eigenvalue a if and only if $\exp(tA)$ has an eigenvalue e^{ta}. In particular, $Y(1) = \exp(A)$ has an eigenvalue 1 if and only if A has an eigenvalue 0. The latter is, of course, equivalent to $\mathrm{d}X(0)$ having a non-trivial kernel.

In preparation for the proof of Proposition 4.18, we recall the following theorem from Floquet theory.

Theorem 4.22 ([Wal72])**.** *Let $A : S^1 \to \mathbb{R}^{n\times n}$ be a smooth 1-periodic matrix valued function and let $Y : \mathbb{R} \to \mathbb{R}^{n\times n}$ be the fundamental system for A defined by* (4.13)*. Then $\partial_t \eta(t) = A(t)\eta(t)$ has a non-trivial 1-periodic solution if and only if* 1 *is an eigenvalue of $Y(1)$. In this case, the solution is of the form $\eta(t) = Y(t) \cdot \eta(0)$ for all t and $\eta(0)$ being some eigenvector of $Y(1)$ for the eigenvalue* 1*.*

Finally, we are ready to prove Proposition 4.18.

Proof of Proposition 4.18. In the proof of Theorem 4.12 we have shown that

$$\begin{aligned} \mathrm{d}s_0(x_0) = \mathrm{d}s(0,x_0)(0,\cdot) : H_1 &\longrightarrow H_0 \\ \hat{x} &\longmapsto \partial_t \hat{x} - \mathrm{d}X(x_0)\cdot\hat{x} \end{aligned}$$

is classically Fredholm. In particular, $\mathrm{d}s_0(x_0)$ has closed image. Thus, $\mathrm{im}(\mathrm{d}s_0(x_0)) = H_0$ if and only if $(\mathrm{im}(\mathrm{d}s_0(x_0)))^\perp = \{0\}$ in $H_0 = L^2$. Therefore, failure of surjectivity of $\mathrm{d}s_0(x_0)$ is equivalent to the existence of $0 \neq \eta \in H_0$ satisfying

$$\forall \hat{x} \in H_1 : \quad \langle \mathrm{d}s_0(x_0)\hat{x}, \eta\rangle_{H_0} = 0.$$

Using the explicit formula above, we see that this is equivalent to

$$\forall \hat{x} \in H_1 : \quad \langle \partial_t\hat{x} - \mathrm{d}X(x_0)\hat{x}, \eta\rangle_{H_0} = 0.$$

This condition asserts that the weak derivative of η exists and equals

$$\partial_t \eta = -\mathrm{d}X(x_0)^T\eta. \tag{4.14}$$

In particular, bootstrapping shows that $\eta \in \mathcal{C}^\infty$. If we set

$$A(t) := -\mathrm{d}X_t(x_0(t))^T : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \tag{4.15}$$

then (4.14) becomes

$$\partial_t \eta(t) = A(t)\eta(t). \tag{4.16}$$

Now Theorem 4.22 and Lemma 4.20 imply that such η exists if and only if $\mathrm{d}\Phi_X^1(x_0(0))$ has an eigenvalue 1, that is, if x_0 is a degenerate periodic orbit of X.

□

4.6 The proof of Theorem 4.1

We finally use the M-polyfold implicit function theorem proved by Hofer, Wysocki and Zehnder. The full theorem from [HWZ21] for a sc-Fredholm section f of a tame strong M-polyfold bundle $Y \to X$ was stated above as Theorem 3.23.

In our situation, we have $X = \mathbb{R}\times\mathrm{H}^1$ and $Y = \mathbb{R}\times\mathrm{H}^1 \triangleright \mathrm{H}$, thus Y is the trivial bundle and hence it is a tame strong M-polyfold bundle, as mentioned before in Remark 4.9. The sc-Fredholm section f is given by S, see formula (4.8) and Theorem 4.12. The solution set $\mathcal{S}$ consists of pairs (τ, x_τ) nearby $(0, x_0)$, where x_τ is a τ-delay orbit of the vector field X, see equation (4.1), as in Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.10, S is sc-smooth and by Theorem 4.12 it is sc-Fredholm. Moreover, according to Proposition 4.18 and Corollary 4.19, non-degeneracy of x_0 implies that $\mathrm{d}s(0,x_0) : \mathbb{R}\times H_1 \to H_0$ is surjective. Since S is sc-Fredholm, its vertical differential $\mathrm{d}s(0,x_0) : \mathbb{R}\times\mathrm{H}^1 \to \mathrm{H}$ at $(0,x_0)$ is a linear sc-Fredholm operator. In particular, $\mathrm{d}s(0,x_0)$ is surjective on all levels (cp. Definition 3.19), and thus the germ of S at $(0,x_0)$ is in good position. Therefore, we can apply the M-polyfold implicit function theorem, Theorem 3.23, and conclude that the solution set

$$\mathcal{S} = \{(\tau, x) \in \mathbb{R} \times \mathrm{H}^1 \mid s(\tau, x) = 0\}$$

is, near $(0, x_0)$, a finite-dimensional smooth manifold. The dimension of $\mathcal{S}$ equals the Fredholm index, which is $\dim \mathcal{S} = 1$ by Proposition 4.13.

We have seen in Proposition 4.18 that non-degeneracy of x_0 implies that $\mathrm{d}s_0(x_0) = \mathrm{d}s(0, x_0)\big|_{\{0\}\times H_1} : H_1 \to H_0$ is surjective. Moreover, $\mathrm{d}s_0(x_0)$ is a Fredholm operator of index 0, thus an isomorphism. In particular, $\ker \mathrm{d}s(0, x_0)$ is not contained in $\{0\} \times H^1$. Therefore, near $(0, x_0)$ the manifold $\mathcal{S} \subset \mathbb{R} \times H^1$ is a graph over $\mathbb{R}$, that is, near $(0, x_0)$, we can smoothly parametrize $\mathcal{S}$ as $\tau(\tau \mapsto x_\tau)$. □

Remark 4.23. If it was possible to apply the M-polyfold implicit function theorem near every pair $(\tau, x) \in \mathcal{S}$ in the solution set $\mathcal{S}$, then all of $\mathcal{S}$ would carry the structure of a 1-dimensional manifold. However, for $\tau \neq 0$ the linearization $\mathrm{d}s(\tau, x_\tau)$ is significantly more complicated than $\mathrm{d}s(0, x_0)$. It is unclear to us how to formulate a criterion for surjectivity of this map in terms of the vector field.

Chapter 5

An illustrative example

Let us consider the plane $\mathbb{R}^2$ and identify it with $\mathbb{C}$ via $z = x_1 + i \cdot x_2$. We want to define a time-dependent vector field X, examine its degenerate or non-degenerate 1-periodic orbits and analyze the 1-periodic delay orbits near them.

5.1 Two functions

Let $f : S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be any smooth function. For a real parameter $\delta \neq 0$ we define a function $g_\delta : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ by

$$g_\delta(r) = \delta \cdot r \cdot (r-1)$$

and extend it to a continuous function $g_\delta : \mathbb{C} \longrightarrow \mathbb{R}$ by

$$g_\delta(z) := g_\delta(\|z\|) = \delta \cdot \|z\| \cdot (\|z\| - 1)\,.$$

The function g_δ is zero on the origin and on the circle of radius 1. It is negative on the open ball of radius 1 without the origin and positive outside the disk of radius 1. Note that g_δ is smooth everywhere except in the origin.

5.2 The vector field

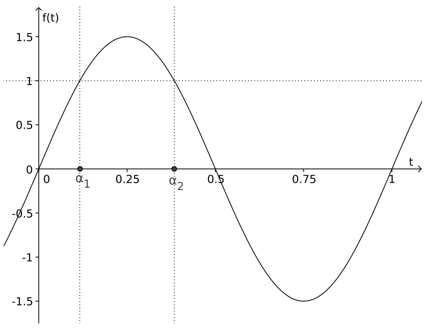
Now we can define a time-dependent vector field $X : S^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ as follows.

$$X_t(z) := g_\delta(z) \cdot \frac{z}{\|z\|} + f\left(\frac{\arg(z)}{2\pi} - t\right) \cdot 2\pi i \cdot z$$

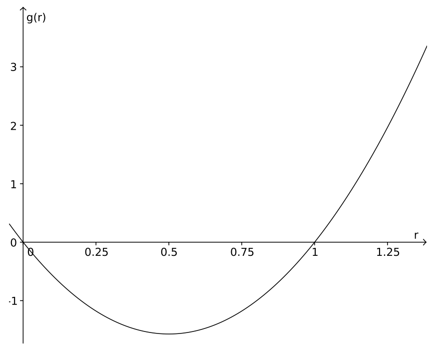
Here, $\arg(z) \in [0, 2\pi)$ denotes the argument of the complex number $z \in \mathbb{R}^2 \cong \mathbb{C}$.

Remark 5.1. Since $g_\delta(z) = \delta \cdot \|z\| \cdot (\|z\| - 1)$, the vector field X_t is well-defined and continuous in 0.

Remark 5.2. The first summand in the definition of X_t is the radial part, the second is the angular part. Because of $X_t(0) \equiv 0$, the origin is a constant orbit of X. On the ball of radius 1, the radial part is negative, so everything here is attracted by the constant orbit. Outside the disk of radius 1, the radial part is positive, so everything there is running off to infinity. Therefore, periodic orbits other than the constant one in the origin necessarily have to lie on the circle of radius one.



(a) The function $f : S^1 \to (-2, 2)$ given by $f(t) = 1.5 \cdot \sin(2\pi t)$. Here, $Z = \{\alpha_1, \alpha_2\}$.



(b) The function $g_{2\pi} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $g_{2\pi}(r) = 2\pi \cdot r \cdot (r - 1)$.

Figure 5.1: Examples of functions f and g_δ.

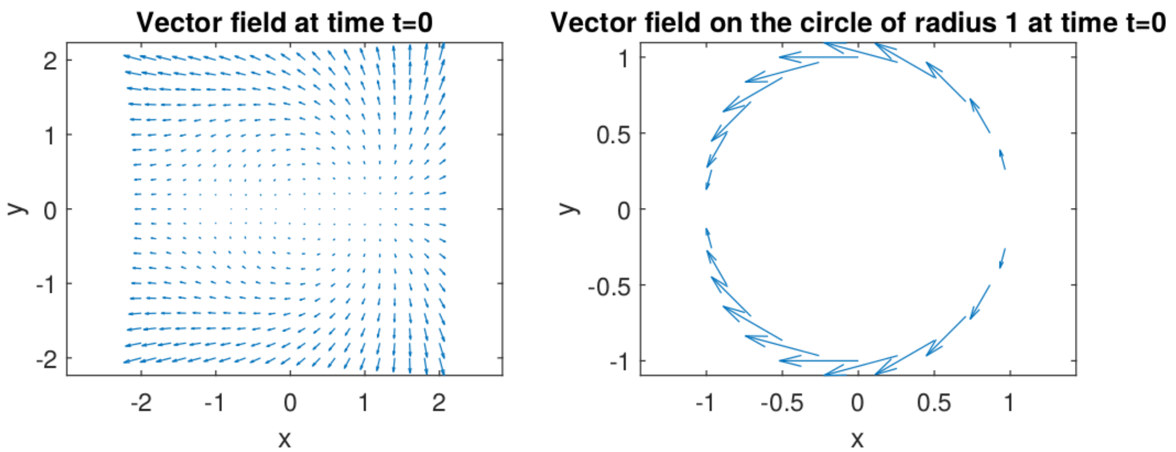


Figure 5.2: The vector field X_t at time $t = 0$. Here we chose $\delta = 2\pi$ and $f(t) = 1.5 \cdot \sin(2\pi t)$.

5.3 1-periodic orbits

Denote $Z = f^{-1}(\{1\})$; this set could possibly be empty or infinite. For $\alpha \in Z$ we define

$$\begin{aligned} z_\alpha &: S^1 \longrightarrow \mathbb{C} \\ z_\alpha(t) &:= e^{2\pi i(\alpha+t)}. \end{aligned}$$

Lemma 5.3. *For every $\alpha \in Z$ the curve z_α is a 1-periodic orbit of X.*

Proof.

$$\begin{aligned} X_t(z_\alpha(t)) &= X_t(e^{2\pi i(\alpha+t)}) \\ &= 0 \cdot e^{2\pi i(\alpha+t)} + f(\alpha + t - t) \cdot 2\pi i \cdot e^{2\pi i(\alpha+t)} \\ &= 2\pi i \cdot e^{2\pi i(\alpha+t)} \\ &= \partial_t z_\alpha(t) \end{aligned}$$

□

Lemma 5.4. *Let $z : S^1 \to \mathbb{C}$ be any 1-periodic orbit of X. Then either $z(t) \equiv 0$, or there is $\alpha \in Z$ such that $z = z_\alpha$.*

Proof. If x is not the constant orbit in the origin, then by Remark 5.2 it has to be located on the unit circle. Let us write $z(t) = e^{2\pi i\theta(t)}$ with $\theta : S^1 \to S^1$. In particular,

$$\partial_t z(t_0) = \dot\theta(t_0) \cdot 2\pi i \cdot e^{2\pi i\theta(t_0)}. \tag{5.1}$$

Moreover note that on the unit circle, our vector field is of the form

$$X_t(e^{2\pi i\theta}) = f(\theta - t) \cdot 2\pi i \cdot e^{2\pi i\theta}. \tag{5.2}$$

Let us first assume that there is $t_0 \in S^1$ with $\dot\theta(t_0) = 1$. Since z is an orbit of X, by comparing (5.1) and (5.2) we find that $f(\theta(t_0) - t_0) = 1$, in particular $\alpha := \theta(t_0) - t_0 \in Z$. The corresponding 1-periodic orbit z_α satisfies

$$z_\alpha(t_0) = e^{2\pi i(\alpha+t_0)} = e^{2\pi i\theta(t_0)} = z(t_0).$$

Thus, z and z_α are orbits of the same vector field, and at time t_0 they take the same value. This implies $z = z_\alpha$.

Now assume that there is no such t_0, that is $\dot\theta(t) \neq 1$ for all $t \in S^1$. Again comparing (5.1) and (5.2), we see that this means

$$\text{either} \quad f(\theta(t) - t) < 1 \qquad \text{or} \quad f(\theta(t) - t) > 1 \qquad \text{for all } t \in S^1.$$

This implies that the map $t \mapsto e^{2\pi i(\theta(t)-(\alpha+t))}$ has rotation number either ≤ -1 or ≥ 1. In particular, there must be $t_0 \in S^1$ with $e^{2\pi i(\theta(t_0)-(\alpha+t_0))} = 1$, that is $\theta(t_0) = \alpha + t_0$, hence $z(t_0) = e^{2\pi i\theta(t_0)} = e^{2\pi i(\alpha+t_0)} = z_\alpha(t_0)$. But again, both are orbits of the same vector field, so $z = z_\alpha$. This contradicts our assumption. □

5.4 Non-degeneracy

Lemma 5.5. *The 1-periodic orbit z_α is non-degenerate if and only if $f'(\alpha) \neq 0$.*

Proof. The non-degeneracy condition can most easily be examined if we pass to polar coordinates $(r, \theta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}/2\pi$ on $\mathbb{R}^2 \setminus \{0\}$. Note that this is allowed since non-degeneracy – in contrast to our notion of delay orbits – is a coordinate invariant property. In polar coordinates, the vector field X_t is given by

$$X_t(r, \theta) = \left(g_\delta(r),\ 2\pi \cdot f\left(\frac{\theta}{2\pi} - t\right)\right),$$

where $g_\delta(r) = \delta \cdot r \cdot (r - 1)$. The periodic orbit z_α is given in polar coordinates by

$$z_\alpha(t) = \big(1, 2\pi(\alpha + t)\big)$$

Let us use some notation and the alternative description of non-degeneracy given in [AS20, Lemma 6.6]. First we compute that

$$\mathrm{d}X_t(r, \theta) = \begin{pmatrix} g_\delta'(r) & 0 \\ 0 & f'\left(\frac{\theta}{2\pi} - t\right) \end{pmatrix}$$

In particular the matrix

$$A(t) := -\mathrm{d}X_t(z_\alpha(t))^T = \begin{pmatrix} -\delta & 0 \\ 0 & -f'(\alpha) \end{pmatrix}$$

occuring in [AS20, Lemma 6.6] is a time-independent diagonal matrix. Thus the fundamental system Y defined by [AS20, Equation (15)] is explicitly given by $Y(t) = e^{At}$, and therefore

$$Y(1) = e^A = \begin{pmatrix} e^{-\delta} & 0 \\ 0 & e^{-f'(\alpha)} \end{pmatrix}.$$

By [AS20, Lemma 6.6], z_α is non-degenerate if and only if $Y(1)$ does not have 1 as an eigenvalue. This is equivalent to $f'(\alpha) \neq 0$. □

5.5 1-periodic delay orbits with small delay

Fix $\alpha \in Z$ with $f'(\alpha) \neq 0$ (which, by Lemma 5.5, is equivalent to the orbit z_α being non-degenerate). For $\tau \in S^1$ we set

$$\varepsilon(\tau, \delta) := \frac{2\pi}{\delta} \sin(2\pi\tau). \tag{5.3}$$

Note that $\varepsilon(\tau, \delta)$ can be negative; it is $\varepsilon(0, \delta) = \varepsilon(\frac{1}{2}) = 0$, $\varepsilon(\frac{1}{4}, \delta) = \frac{2\pi}{\delta}$, and $\varepsilon(-\frac{1}{4}, \delta) = -\frac{2\pi}{\delta}$. Given a delay $\tau \in S^1$, for some reason that will become clear below we are interested in finding a time $t_{\alpha,\tau} \in S^1$ near $\alpha \in S^1$ which satisfies

$$f(t_{\alpha,\tau}) = \cos(2\pi\tau). \tag{5.4}$$

Depending on f, α and τ, such $t_{\alpha,\tau}$ may or may not exist. Since $f(\alpha) = 1$ and $f'(\alpha) \neq 0$, the function f has a local inverse f_α^{-1} from a neighborhood of 1 to a neighborhood of α. Hence, for small enough τ we can use this inverse to define a unique $t_{\alpha,\tau}$ by

$$t_{\alpha,\tau} := f_\alpha^{-1}\left(\cos(2\pi\tau)\right). \tag{5.5}$$

For $\tau = 0$ this is $t_0 = \alpha$; for τ near 0, $t_{\alpha,\tau}$ is a number near α. Also note that $t_{\alpha,-\tau} = t_{\alpha,\tau}$.

Remark 5.6. (i) Above we talked about τ being "small". Since τ is an element of $S^1 = \mathbb{R}/\mathbb{Z}$, by "small" we mean close to 0 in the metric of $\mathbb{R}/\mathbb{Z}$. For example, $\frac{3}{4} \equiv -\frac{1}{4}$ is smaller than $\frac{1}{2} \equiv -\frac{1}{2}$. The distance between $\tau \in S^1$ and 0 will be denoted by $|\tau|$, e.g. $\left|-\frac{1}{2}\right| = \frac{1}{2}$, $\left|\frac{3}{4}\right| = \frac{1}{4}$.

(ii) How small is sufficient for our purpose depends on δ and f. This is discussed in more detail in Section 5.6 below.

Remark 5.7. If we define $t_{\alpha,\tau}$ by (5.5), it makes sense to think about continuity and smoothness of the map $\tau \mapsto t_{\alpha,\tau}$.

- Since f is smooth and we assumed $f'(\alpha) \neq 0$, the inverse function theorem implies that the local inverse f_α^{-1} is smooth in a neighborhood of 1. Thus, $\tau \mapsto t_{\alpha,\tau}$ is smooth in a neighborhood of 0.

- The same argument works for any other $\tau \in S^1$ if for some reason we know that $f'(t_{\alpha,\tau}) \neq 0$.

Given some $\tau \in S^1$ and a choice of $t_{\alpha,\tau}$ satisfying (5.4), let us now define a curve $z_{\alpha,\tau}$ as follows:

$$\begin{aligned} &z_{\alpha,\tau} : S^1 \longrightarrow \mathbb{C} \\ &z_{\alpha,\tau}(t) := \big(1 - \varepsilon(\tau,\delta)\big) \cdot e^{2\pi i(t_{\alpha,\tau}+\tau+t)} \end{aligned} \tag{5.6}$$

Note that for $\tau = 0$ this recovers the orbit $z_\alpha = z_{\alpha,0}$.

Lemma 5.8. *If $\varepsilon(\tau,\delta) \leq 1$, then the curve $z_{\alpha,\tau}$ is a 1-periodic τ-delay orbit of X.*

Proof. From equation (5.6) we see that

$$\partial_t z_{\alpha,\tau}(t) = \big(1 - \varepsilon(\tau,\delta)\big) \cdot 2\pi i \cdot e^{2\pi i(t_{\alpha,\tau}+\tau+t)}.$$

On the other hand, we compute

$$\begin{aligned}
X_t\left(z_{\alpha,\tau}(t-\tau)\right) &= X_t\left(\big(1-\varepsilon(\tau,\delta)\big)\cdot e^{2\pi i(t_{\alpha,\tau}+\tau+t-\tau)}\right)\\
&= X_t\left(\big(1-\varepsilon(\tau,\delta)\big)\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\right)\\
&= g_\delta\big(1-\varepsilon(\tau,\delta)\big)\cdot \frac{1-\varepsilon(\tau,\delta)}{|1-\varepsilon(\tau,\delta)|}\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\\
&\qquad + f\left(t_{\alpha,\tau}+t-t\right)\cdot 2\pi i\cdot\big(1-\varepsilon(\tau,\delta)\big)\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\\
&= \delta\cdot(1-\varepsilon(\tau,\delta))\cdot(-\varepsilon(\tau,\delta))\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\\
&\qquad + \cos(2\pi\tau)\cdot 2\pi i\cdot\big(1-\varepsilon(\tau,\delta)\big)\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\\
&= \big(1-\varepsilon(\tau,\delta)\big)\cdot 2\pi i\cdot\left(\frac{-\delta\cdot\varepsilon(\tau,\delta)}{2\pi i}+\cos(2\pi\tau)\right)\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\\
&= \big(1-\varepsilon(\tau,\delta)\big)\cdot 2\pi i\cdot(\sin(2\pi\tau)\cdot i+\cos(2\pi\tau))\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\\
&= \big(1-\varepsilon(\tau,\delta)\big)\cdot 2\pi i\cdot e^{2\pi i\tau}\cdot e^{2\pi i(t_{\alpha,\tau}+t)}\\
&= \big(1-\varepsilon(\tau,\delta)\big)\cdot 2\pi i\cdot e^{2\pi i(t_{\alpha,\tau}+\tau+t)}.
\end{aligned}$$

Therefore $z_{\alpha,\tau}$ is a τ-delay orbit of X. □

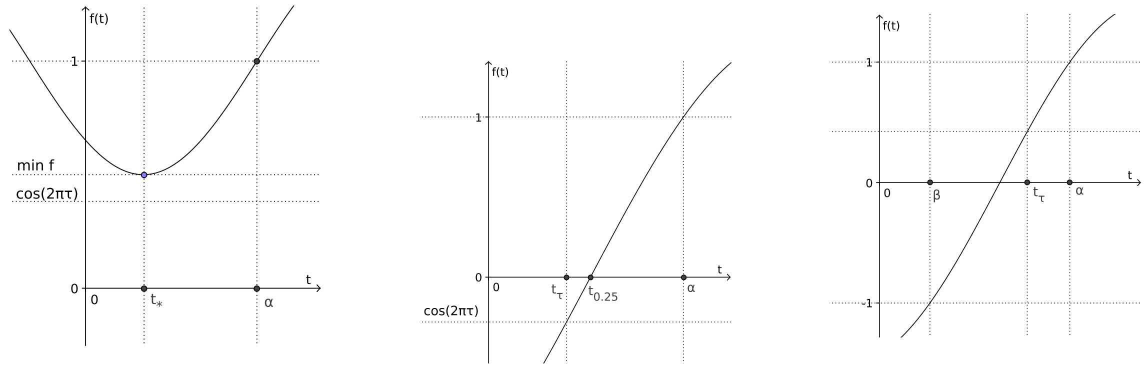
Remark 5.9 (smoothness of the family $(z_{\alpha,\tau})_\tau$)**.** If $\tau\mapsto t_{\alpha,\tau}$ is smooth, then it is clear from the definition in equation (5.6) that the family $(z_{\alpha,\tau})_\tau$ is smooth. By Remark 5.7 this is the case in a neighborhood of $\tau=0$ since we assumed $f'(\alpha)\neq 0$.

Remark 5.10. The delay orbit $z_{\alpha,\tau}$ stays on the circle of radius $1-\varepsilon(\tau,\delta)$ and has constant speed. With that in mind, the whole orbit $z_{\alpha,\tau}\in\mathcal{C}^\infty(S^1,\mathbb{C})$ can be recovered from $z_{\alpha,\tau}(0)$, its position at time $t=0$. Hence, for a better understanding of the family $(z_{\alpha,\tau})_\tau\subset\mathcal{C}^\infty(S^1,\mathbb{C})$, it makes sense to analyse the smooth curve $\tau\mapsto z_{\alpha,\tau}(0)\in\mathbb{C}$. It has a self-intersection if there are $\tau_0\neq\tau_1$ with $\sin(2\pi\tau_0)=\sin(2\pi\tau_0)$ and $t_{\tau_0}+\tau_0=t_{\tau_1}+\tau_1$. In this case, $z_{\alpha,\tau_0}=z_{\alpha,\tau_1}$ is a delay orbit for two different delays. Additionally, the image of the curve can have a cusp, see Remark 5.12.

Remark 5.11 (the degenerate situation)**.** Our main theorem [AS20, Theorem 1.1] only applies under the assumption of non-degeneracy. Here, in this example, we can see what may happen if the original orbit z_α is degenerate, which, by Lemma 5.5, is equivalent to $f'(\alpha)=0$.

- As long as α is just a saddle point of f, the construction of the delay orbits $z_{\tau,\alpha}$ works exactly the same as for $f'(\alpha)\neq 0$, but the map $\tau\mapsto t_{\alpha,\tau}$ may not be smooth in 0 any more.
- If f attains a local minimum in α, then for $|\tau|>0$ there is no $t_{\alpha,\tau}$ near α satisfying (5.4), so we cannot find a τ-delay orbit near z_α.
- If f attains a strict local maximum in α, then the definition of $t_{\alpha,\tau}$ involves a choice between the solutions "to the left" and "to the right" of α. Thus we find two 1-dimensional families of delay orbits near z_α parametrized continuously by delay, and they intersect in $\tau=0$.

For a more detailed illustration of the case when f attains a maximum in α, see the concrete examples in Sections 5.7.3 and 5.7.4 below.



(a) Here, $t_{\alpha,\tau}$ runs into a minimum $t_* = t_{\alpha,\tau_*}$ with $f(t_*) > -1$. For $|\tau| > |\tau_*|$, $t_{\alpha,\tau}$ cannot be defined.

(b) For $\tau = \pm\frac{1}{4}$, $t_{\alpha,\tau}$ needs to pass through a zero of f.

(c) Here, $t_{\alpha,\tau}$ can be defined for all $t \in S^1$, resulting in a smooth S^1-family of periodic delay orbits.

Figure 5.3: What might happen in the definition of $t_{\alpha,\tau}$ when τ grows.

5.6 Families of delay orbits parametrized by delay

In Section 5.5 we constructed 1-periodic τ-delay orbits $z_{\alpha,\tau}$ near z_α whenever δ, f and $\tau \in S^1$ are such that $\varepsilon(\tau, \delta) \leq 1$ and $t_{\alpha,\tau}$ is defined. In particular, if we start with non-degenerate z_α, this gives a smooth 1-dimensional family of 1-periodic τ-delay orbits of X near z_α with small delay τ.

Now we may ask what happens for big τ: What problems occur in the construction? How can they possibly be solved? In addition, we may have a closer look at the degenerate case (see Remark 5.11). Then we can put together our insights to find concrete instances of δ and f where we can see interesting (global) families of delay orbits.

In the construction of the τ-delay orbit $z_{\alpha,\tau}$, two problems can occur:

(I) In the proof that $z_{\alpha,\tau}$ is indeed a τ-delay orbit, we needed the fact that it lies on the circle of radius $1 - \varepsilon(\tau, \delta)$. This obviously only holds as long as $\varepsilon(\tau, \delta) \leq 1$.

(II) The time $t_{\alpha,\tau} \in S^1$ was defined by $f(t_{\alpha,\tau}) = \cos(2\pi\tau)$; for τ small enough we used $t_{\alpha,\tau} = f_\alpha^{-1}(\cos(2\pi\tau))$, where f_α^{-1} is a local inverse of f from a neighborhood of 1 to a neighborhood of $\alpha \in S^1$. This local inverse is smooth near 1 and it gives us a smooth family of times $(t_{\alpha,\tau})_\tau$ near α (see figures 5.3b 5.3c). As τ gets bigger, $\cos(2\pi\tau)$ may leave the neighborhood where f_α^{-1} is defined.

Since we defined $\varepsilon(\tau, \delta) = \frac{2\pi}{\delta}\sin(2\pi\tau)$, problem (I) can be solved by choosing $\delta \geq 2\pi$. Problem (II) is more interesting, so let us have a closer look at what may happen.

- If there is $t_* \in S^1$ with $f'(t_*) = 0$ but f is monotone around t_*, the local inverse f_α^{-1} is not differentiable any more, but it still exists and is continuous. It can be used to define $t_{\alpha,\tau}$ near t_*. Hence the family of delay orbits persists, but it may not be differentiable in the delays $\pm\tau$ with $\cos(2\pi\tau) = f(t_*)$.

- If f attains a (local) minimum at some $t_* \in S^1$ with $f(t_*) > -1$, and $t_* = t_{\tau_*}$ for some $\tau \in S^1$, then for $|\tau| > |\tau_*|$ there is no solution of $f(t_{\alpha,\tau}) = \cos(2\pi\tau)$ near α any more. See Figure 5.3a.

- If there is some $\beta \in S^1$ with $f(\beta) = -1$ and f is monotone between[11] α and β, then $t_{\alpha,\tau}$ is defined for all $\tau \in S^1$ and $t_{\pm\frac{1}{2}} = \beta$. See Figure 5.3c. Thus, we find a continuous S^1-family of delay orbits parametrized by delay. If f' is nonzero between α and β, then the given family is smooth. If $f'(\beta) = 0$, that is if f attains a local minimum in β, then there is another family of delay orbits parametrized by delay, and the two families intersect for $\tau = \pm\frac{1}{2}$.

5.7 Concrete examples

In this section we want to analyse some very concrete examples of f where everything can be computed.

5.7.1 A linear non-degenerate example

As a first example, consider $\delta = 2\pi$ and a smooth function f given by

$$f(t) = \begin{cases} 4t - 2 & \text{if } t \in \left[\frac{1}{8}, \frac{7}{8}\right] \mod 1 \\ \text{some smooth extension} & \text{else} \end{cases}$$

It is $f\left(\frac{3}{4}\right) = 1$, so in the notation from before we have $\frac{3}{4} =: \alpha \in Z$, and the curve $z_\alpha(t) := e^{2\pi i(\frac{3}{4}+t)}$ is a non-degenerate 1-periodic orbit of X. Note that as long as we are only interested in the family of delay orbits through z_α it is really not necessary to specify the smooth extension of f outside $[\frac{1}{8}, \frac{7}{8}]$; the family does not depend on this extension at all. A local inverse of f is given by

$$\begin{aligned} f_\alpha^{-1} &: \left[-\frac{3}{2}, \frac{3}{2}\right] \to \left[-\frac{1}{8}, \frac{7}{8}\right] \\ f_\alpha^{-1}(x) &= \frac{x+2}{4}, \end{aligned}$$

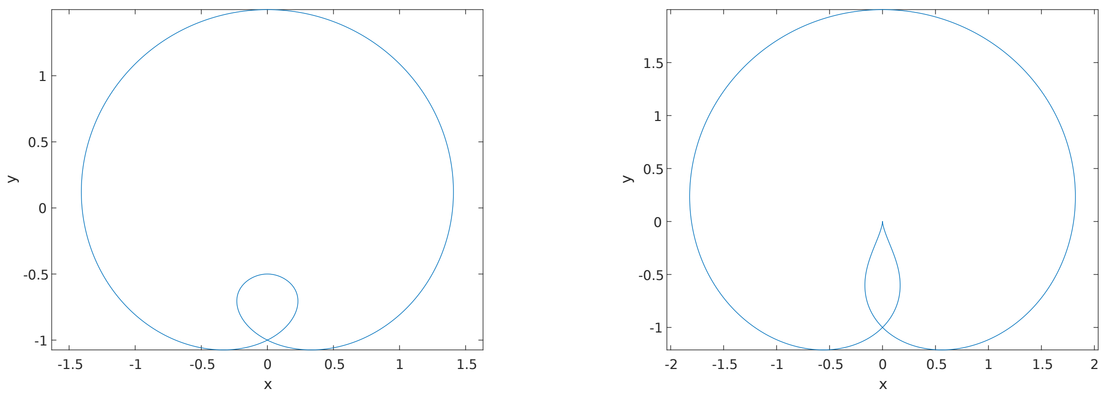
so we compute

$$t_{\alpha,\tau} = f_\alpha^{-1}(\cos(2\pi\tau)) = \frac{\cos(2\pi\tau) + 2}{4}.$$

This is defined for alle $\tau \in S^1$, so we get an S^1-family of 1-periodic delay orbits

$$\begin{aligned} z_{\alpha,\tau}(t) &= \big(1 - \varepsilon(\tau, \delta)\big) \cdot e^{2\pi i(t_{\alpha,\tau}+\tau+t)} \\ &= \big(1 - \sin(2\pi\tau)\big) \cdot e^{2\pi i(\frac{\cos(2\pi\tau)+2}{4}+\tau+t)}. \end{aligned}$$

[11] By "between α and β" we mean the part of S^1 given by $\{t_{\alpha,\tau} | \tau \in S^1\}$.



(a) $\delta = 4\pi$: The image of the curve is non-smooth, but its parametrization is smooth. The double point corresponds to $\tau = 0, \frac{1}{2}$. It exists because the orbit z_α is at the same time a $\frac{1}{2}$-delay orbit.

(b) $\delta = 2\pi$: The double point corresponding to $\tau = 0, \frac{1}{2}$ is still there. Additionally, a cusp appeared, corresponding to delay $\tau = \frac{1}{4}$, when the family runs into the constant orbit in the origin.

Figure 5.4: The curve $S^1 \to \mathbb{C}, \tau \mapsto z_{\alpha,\tau}(0)$, in the situation of Subsection 5.7.1.

For understanding this family, it is h to plot the curve

$$\begin{aligned}\tau \longmapsto z_{\alpha,\tau}(0) &= \big(1 - \varepsilon(\tau, \delta)\big) \cdot e^{2\pi i (t_\tau + \tau)} \\ &= \left(1 - \frac{2\pi}{\delta}\sin(2\pi\tau)\right) \cdot e^{2\pi i \left(\frac{\cos(2\pi\tau)+2}{4} + \tau\right)}\end{aligned}$$

of positions at time $t = 0$, see Figures 5.4a, 5.4b and Remark 5.10. Note that for $\delta = 2\pi$ the image of the curve has a singularity in 0, corresponding to $\tau = \frac{1}{4}$, although it is smoothly parametrized. This singularity does not appear for $\delta > 2\pi$. See also Remark 5.12.

Remark 5.12. In general, for which choice of δ and f and non-degenerate orbit z_α does the image of the curve $\tau \mapsto z_{\alpha,\tau}(0)$ have a cusp (as in Figures 5.4b, 5.5b)? A cusp can appear only if $\frac{\mathrm{d}}{\mathrm{d}t} z_{\alpha,\tau}(0) = 0$. Computing $\frac{\mathrm{d}}{\mathrm{d}\tau} t_{\alpha,\tau} = \frac{-2\pi\sin(2\pi\tau)}{f'(t_{\alpha,\tau})}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} z_{\alpha,\tau}(0) = \left(\frac{4\pi^2}{\delta}\cos(2\pi\tau) + 2\pi i \cdot \left(1 - \frac{2\pi}{\delta}\sin(2\pi\tau)\right) \cdot \left(\left(\frac{\mathrm{d}}{\mathrm{d}\tau} t_{\alpha,\tau}\right) + 1\right)\right) \cdot e^{2\pi i (t_\tau + \tau)},$$

we find that this happens if and only if $\delta = 2\pi$ and $\tau = \frac{1}{4}$. For a specific function f, one could determine whether there really is a cusp by computing

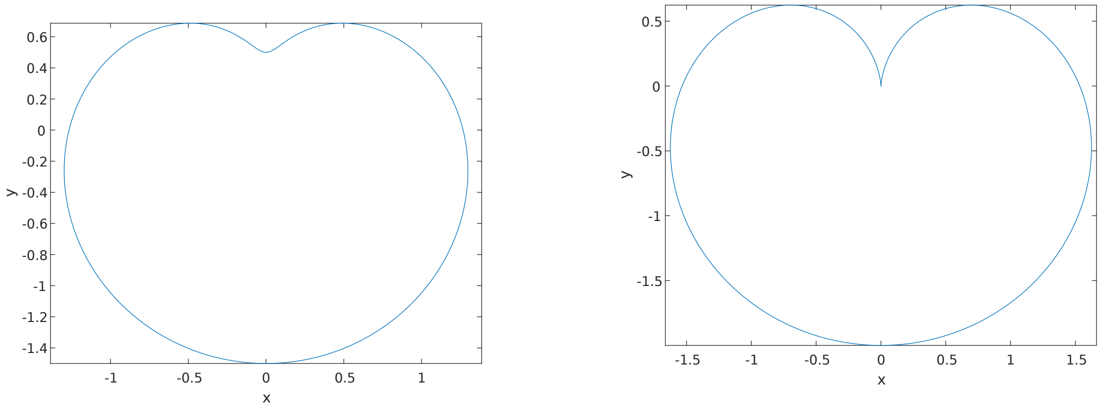
$$\frac{\frac{\mathrm{d}}{\mathrm{d}t} z_{\alpha,\tau}(0)}{\left\|\frac{\mathrm{d}}{\mathrm{d}t} z_{\alpha,\tau}(0)\right\|} \in \{z \in \mathbb{C} \mid \|z\| = 1\}$$

for $\tau \neq \frac{1}{4}$ and analyzing whether it can be continued smoothly to $\tau = \frac{1}{4}$.

5.7.2 Another non-degenerate example

Let us now analyse the situation for the function f defined by

$$f(t) = 1.5\sin(2\pi t).$$



(a) $\delta = 4\pi$: A smooth family of delay orbits, this time without double point or cusp.

(b) $\delta = 2\pi$: Again, a cusp appeares for $\tau = \frac{1}{4}$ when the family runs into the constant orbit at the origin.

Figure 5.5: The curve $S^1 \to \mathbb{C}, \tau \mapsto z_{\alpha_1,\tau}(0)$, in the situation of Subsection 5.7.2.

This time, $Z = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = \frac{1}{2\pi} \arcsin(\frac{2}{3}) \approx 0.116$ and $\alpha_2 = \frac{1}{2} - \alpha_1 \approx 0.384$. The corresponding vector field was depicted in Figure 5.2. There are two 1-periodic orbits of X, both are non-degenerate. A local inverse of f with image around α_1 is given by

$$\begin{aligned} f_{\alpha_1}^{-1} &: [-1.5, 1.5] \longrightarrow \left[-\frac{1}{4}, \frac{1}{4}\right] \\ f_{\alpha_1}^{-1}(x) &= \frac{1}{2\pi} \arcsin\left(\frac{2}{3}x\right), \end{aligned}$$

so for every $\tau \in S^1$ we can define

$$t_{\alpha_1,\tau} = f_{\alpha_1}^{-1}(\cos(2\pi\tau)) = \frac{1}{2\pi} \arcsin\left(\frac{2}{3}\cos(2\pi\tau)\right).$$

For every choice of $\delta \geq 2\pi$, this gives a smooth S^1-family $(z_{\alpha_1,\tau})_{\tau \in S^1}$ of periodic delay orbits; it can be analyzed by plotting the curve

$$\begin{aligned} S^1 &\longrightarrow \mathbb{C} \\ \tau &\longmapsto z_{\alpha_1,\tau}(0) = \left(1 - \frac{2\pi}{\delta}\sin(2\pi\tau)\right) \cdot e^{2\pi i\left(\frac{1}{2\pi}\arcsin\left(\frac{2}{3}\cos(2\pi\tau)\right) + \tau\right)}, \end{aligned}$$

see Figures 5.5a, 5.5b. A similar family through z_{α_2} can be found using a local inverse of f with image around α_2.

5.7.3 A degenerate example: The sine function

Let us now have a closer look at a degenerate example. For this, choose the function

$$f(t) = \sin(2\pi t).$$

Here, $Z = \{\alpha\}$ has only one element, namely $\alpha = \frac{1}{4}$. Hence, the curve $z_\alpha(t) = e^{2\pi i\left(\frac{1}{4}+t\right)}$ is the only 1-periodic orbit of X, and it is degenerate. The minimum of f is -1 and it is obtained only in $\beta = \frac{3}{4}$; this means that there is exactly one delay orbit for the delay $\tau = \pm\frac{1}{2}$, corresponding to $t_{\frac{1}{2}} = \beta$; let us denote it by $z_\beta := z_{\alpha,\frac{1}{2}}$. However, we also see that $\alpha + 0 = \beta + \frac{1}{2}$, so $z_\alpha = z_\beta$.

Let us now analyse the delay orbits for delay $\tau \neq 0, \pm\frac{1}{2}$. There are two different local inverses of f (none of them defined on an open neighborhood of 1, and both not differentiable in $1, -1$):

$$\begin{aligned} f_{\alpha,l}^{-1} : [-1,1] &\longrightarrow \left[-\frac{1}{4}, \frac{1}{4}\right] = [\beta, \alpha] \subset S^1 \\ x &\longmapsto \frac{\arcsin(x)}{2\pi} \\ f_{\alpha,r}^{-1} : [-1,1] &\longrightarrow \left[\frac{1}{4}, \frac{3}{4}\right] = [\alpha, \beta] \subset S^1 \\ x &\longmapsto \frac{1}{2} - \frac{\arcsin(x)}{2\pi} \end{aligned}$$

Here, the subscripts l and r indicate that we consider the part of f that is to the left or to the right of α respectively. We use both local inverses to define families of times $(t_{\alpha,\tau}^{lr})_\tau$ and $(t_{\alpha,\tau}^{rl})_\tau$ in the following way:

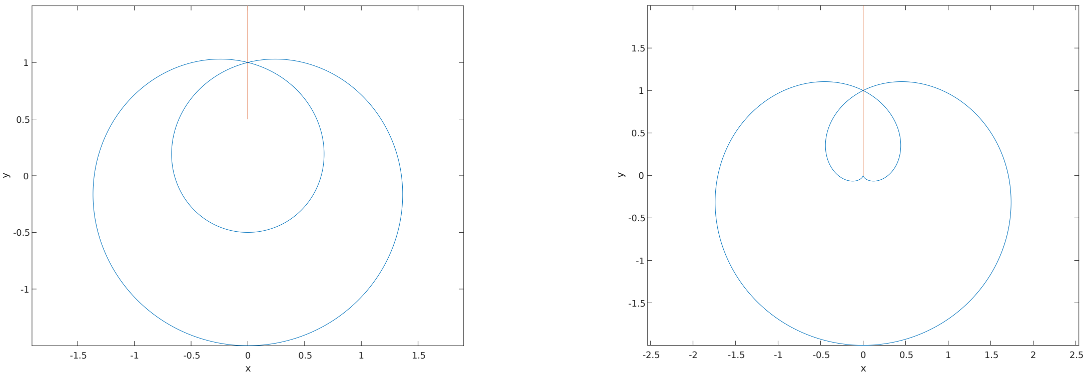
$$\begin{aligned} t_{\alpha,\tau}^{lr} &:= \begin{cases} f_{\alpha,l}^{-1}(\cos(2\pi\tau)) & \text{if } \tau \in \left[-\frac{1}{2}, 0\right] \subset S^1 \\ f_{\alpha,r}^{-1}(\cos(2\pi\tau)) & \text{if } \tau \in \left[0, \frac{1}{2}\right] \subset S^1 \end{cases} \\ &= \tau + \frac{1}{4} \in S^1 \\ t_{\alpha,\tau}^{rl} &:= \begin{cases} f_{\alpha,r}^{-1}(\cos(2\pi\tau)) & \text{if } \tau \in \left[-\frac{1}{2}, 0\right] \subset S^1 \\ f_{\alpha,l}^{-1}(\cos(2\pi\tau)) & \text{if } \tau \in \left[0, \frac{1}{2}\right] \subset S^1 \end{cases} \\ &= -\tau + \frac{1}{4} \in S^1 \end{aligned}$$

Here, the superscripts lr and rl indicate that in one family, we change from $f_{\alpha,l}^{-1}$ to $f_{\alpha,r}^{-1}$ in $\tau = 0$ (and back to $f_{\alpha,l}^{-1}$ in $\tau = \frac{1}{2} = -\frac{1}{2} \in S^1$) and in the other family the other way round. Indeed, both definitions give rise to continuous maps $S^1 \to S^1, \tau \mapsto t_{\alpha,\tau}^{lr}, t_{\alpha,\tau}^{rl}$, and we see that moreover they are smooth.

Remark 5.13. We chose to switch from one local inverse to the other in $\tau = 0, \pm\frac{1}{2}$ because it makes the families $(t_{\alpha,\tau}^{lr})_\tau, (t_{\alpha,\tau}^{rl})_\tau$ smooth. In more general degenerate examples, the situation is similar, and it also makes sense to switch from one local inverse to the other in $\tau = 0$. See Subsection 5.7.4 below.

Using the definition above, for every $\delta \geq 2\pi$ and corresponding $\varepsilon(\tau, \delta)$ we find two smooth S^1-families of 1-periodic delay orbits:

$$\begin{aligned} z_{\alpha,\tau}^{lr}(t) &= \big(1 - \varepsilon(\tau, \delta)\big) \cdot e^{2\pi i(t_{\alpha,\tau}^{lr}+\tau+t)} \\ z_{\alpha,\tau}^{rl}(t) &= \big(1 - \varepsilon(\tau, \delta)\big) \cdot e^{2\pi i(t_{\alpha,\tau}^{rl}+\tau+t)} \end{aligned}$$



(a) $\delta = 4\pi$: The two families intersect in z_α twice, because each family has a self-intersection there for $\tau = 0, \pm\frac{1}{2}$.

(b) $\delta = 2\pi$: There is an additional intersection in the constant orbit for $\tau = \frac{1}{4}$. Both families hav a cusp there.

Figure 5.6: The two curves $S^1 \to \mathbb{C}$, $\tau \mapsto z^{lr}_{\alpha,\tau}(0)$ (blue) and $\tau \mapsto z^{rl}_{\alpha,\tau}(0)$ (orange) in the situation of Subsection 5.7.3.

The two families intersect in $z^{lr}_{\alpha,0} = z^{rl}_{\alpha,0} = z_\alpha$ for $\tau = 0$ and again in $z^{lr}_{\alpha,\frac{1}{2}} = z^{rl}_{\alpha,\frac{1}{2}} = z_\alpha$ for $\tau = \pm\frac{1}{2}$, because $t^{lr}_0 = t^{rl}_0 = \frac{1}{4} = \alpha$ and $t^{lr}_{\frac{1}{2}} = t^{rl}_{\frac{1}{2}} = -\frac{1}{4}$. If $\delta = 2\pi$, there is an additional intersection for $\tau = \frac{1}{4}$ because $z^{lr}_{\alpha,\frac{1}{4}} \equiv 0 \equiv z^{rl}_{\alpha,\frac{1}{4}}(0)$. Again, for understanding the families it makes sense to plot the curves

$$\tau \longmapsto z^{lr}_{\alpha,\tau}(0) = \left(1 - \frac{2\pi}{\delta}\sin(2\pi\tau)\right) \cdot e^{2\pi i\left(\tau + \frac{1}{4} + \tau\right)}$$
$$\tau \longmapsto z^{rl}_{\alpha,\tau}(0) = \left(1 - \frac{2\pi}{\delta}\sin(2\pi\tau)\right) \cdot e^{2\pi i\left(-\tau + \frac{1}{4} + \tau\right)}$$

of points where the τ-delay orbits are at time $t = 0$. The plot for $z^{rl}_{\alpha,\tau}$ is a straight line because $t^{rl}_{\alpha_1,\tau} + \tau \equiv \frac{1}{4}$. See Figures 5.6a, 5.6b.

5.7.4 Another degenerate example

In the previous example, the vector field X had only one 1-periodic orbit. We generalize it by

$$f(t) = \sin(2\pi k t), \ k \in \mathbb{N}_{\geq 1}.$$

Now, $Z = \{\alpha_0, \ldots, \alpha_{k-1}\}$, where $\alpha_j = \frac{4j+1}{4k}$. Similarly, there is a set $\{\beta_0, \ldots, \beta_{k-1}\}$ of times with $f(\beta_j) = -1$ given by $\beta_j = \frac{4j+3}{4k}$. The vector field X has exactly k periodic orbits, namely z_{α_j}, $j = 0, \ldots, k-1$, and they are all degenerate. As in the previous

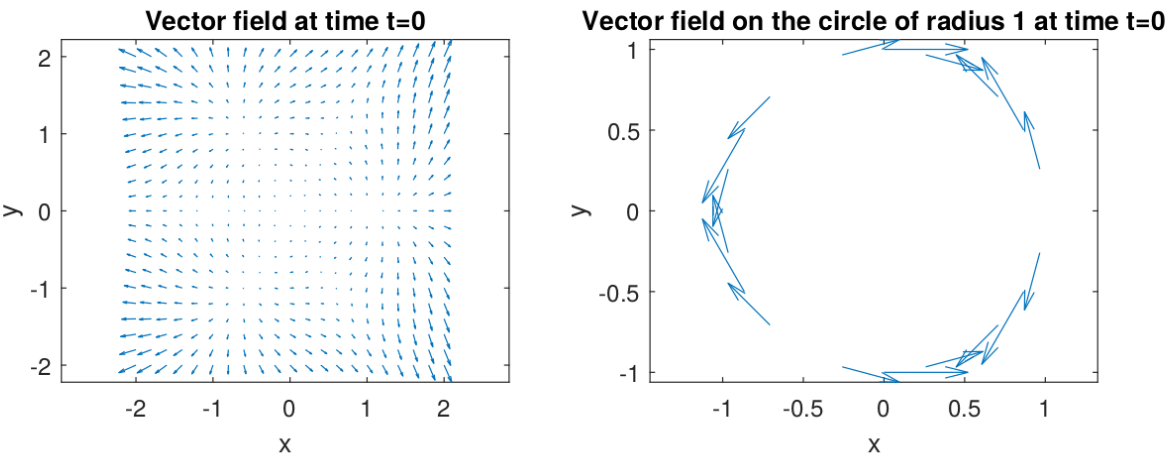


Figure 5.7: The vector field X_t corresponding to $k = 3$ in Subsection 5.7.4 at time $t = 0$. The function is $f(t) = \sin(6\pi t)$ and we chose $\delta = 2\pi$.

example, near each α_j, $j = 0, \dots, k-1$ we find two different local inverses:

$$f^{-1}_{\alpha_j,l} : [-1,1] \longrightarrow [\beta_{j-1}, \alpha_j], \qquad x \longmapsto \frac{\arcsin(x)}{2\pi k} + \frac{j}{k}$$
$$f^{-1}_{\alpha_j,r} : [-1,1] \longrightarrow [\alpha_j, \beta_j], \qquad x \longmapsto \frac{1}{2k} - \frac{\arcsin(x)}{2\pi k} + \frac{j}{k}$$

They are both smooth except in $\{\pm 1\}$ where their derivative explodes. As before, we can define two different smooth families

$$\begin{aligned} t^{lr}_{\alpha_j,\tau} &:= \begin{cases} f^{-1}_{\alpha_j,l}(\cos(2\pi\tau)) & \text{if } \tau \in \left[-\frac{1}{2}, 0\right] \\ f^{-1}_{\alpha_j,r}(\cos(2\pi\tau)) & \text{if } \tau \in \left[0, \frac{1}{2}\right] \end{cases} \\ &= \frac{1}{k}\left(\tau + \frac{1}{4}\right) + \frac{j}{k} \in S^1 \\ t^{rl}_{\alpha_j,\tau} &:= \begin{cases} f^{-1}_{\alpha_j,r}(\cos(2\pi\tau)) & \text{if } \tau \in \left[-\frac{1}{2}, 0\right] \\ f^{-1}_{\alpha_j,l}(\cos(2\pi\tau)) & \text{if } \tau \in \left[0, \frac{1}{2}\right] \end{cases} \\ &= \frac{1}{2k} - \frac{1}{k}\left(\tau + \frac{1}{4}\right) + \frac{j}{k} \in S^1 \end{aligned}$$

and use these to find two families $(z^{lr}_{\alpha_j,\tau})_\tau$, $(z^{rl}_{\alpha_j,\tau})_\tau$ of τ-delay orbits. However, if $k \geq 2$, we should now think of $\tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ as a real parameter, not as an element of S^1 any more, since $t^{lr}_{\alpha_j,-\frac{1}{2}} \neq t^{lr}_{\alpha_j,\frac{1}{2}}$ and $t^{rl}_{\alpha_j,-\frac{1}{2}} \neq t^{rl}_{\alpha_j,\frac{1}{2}}$. Instead, because of

$$\begin{aligned} t^{lr}_{\alpha_j,\frac{1}{2}} &= \beta_j = t^{lr}_{\alpha_{j+1},-\frac{1}{2}} \\ t^{rl}_{\alpha_j,-\frac{1}{2}} &= \beta_j = t^{rl}_{\alpha_{j+1},\frac{1}{2}} \end{aligned} \tag{5.7}$$

we may glue the $\left[-\frac{1}{2}, \frac{1}{2}\right]$-families for all $j = 0, \ldots, k-1$ together and obtain two smooth $\left[-\frac{1}{2}, k-1+\frac{1}{2}\right]$-families as follows:

$$t_\tau^{lr} := t_{\alpha_j,\tau-j}^{lr} \qquad \text{for } \tau \in \left[j - \frac{1}{2}, j + \frac{1}{2}\right]$$
$$t_\tau^{rl} := t_{\alpha_{k-j},\tau-j}^{rl} \qquad \text{for } \tau \in \left[j - \frac{1}{2}, j + \frac{1}{2}\right]$$

Here, of course we denote $\alpha_k = \alpha_0$.

Again using equation (5.7) we see that the maps

$$\begin{aligned} \left[-\frac{1}{2}, k-1+\frac{1}{2}\right] &\longrightarrow S^1 \\ \tau &\longmapsto t_\tau^{lr} \\ \tau &\longmapsto t_\tau^{rl} \end{aligned}$$

can be continued k-periodically to all of $\mathbb{R}$. They correspond to two k-periodic families

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{C}^\infty(S^1, \mathbb{R}^2) \\ \tau &\longmapsto \left[z_\tau^{lr}(t) = \left(1 - \frac{2\pi}{\delta}\sin(2\pi\tau)\right) \cdot e^{2\pi i\left(t_{\alpha,\tau}^{lr}+\tau+t\right)}\right] \\ \tau &\longmapsto \left[z_\tau^{rl}(t) = \left(1 - \frac{2\pi}{\delta}\sin(2\pi\tau)\right) \cdot e^{2\pi i\left(t_{\alpha,\tau}^{rl}+\tau+t\right)}\right] \end{aligned}$$

of 1-periodic delay orbits of X. These two families intersect themselves and each other quite often. See Figures 5.8a, 5.8b for a plot in the case $k = 3$. For $k = 3$ we computed the complete list of intersections:

- original orbits as multi-intersections:

$$z_{\alpha_j} = z_j^{lr} = z_{j+\frac{3}{2}}^{lr} = z_{-j}^{rl} = z_{-j+\frac{3}{2}}^{rl} \qquad j = 0, 1, 2$$

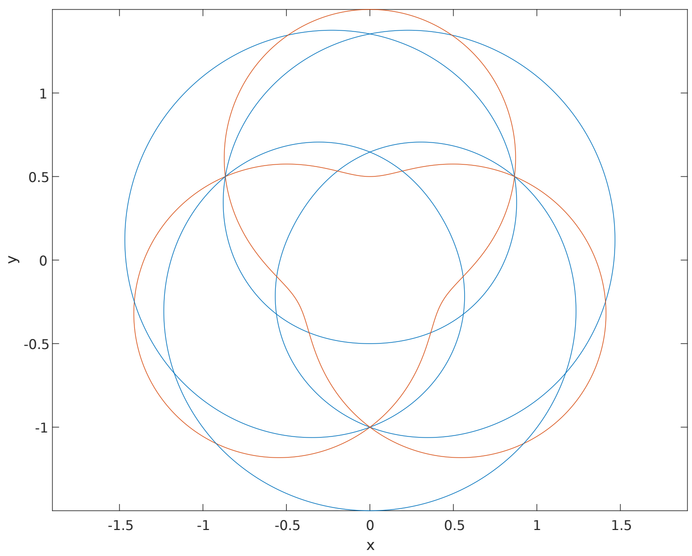
- additional self-intersections of the first family:

$$z_{j+\frac{1}{8}}^{lr} = z_{j+2+\frac{3}{8}}^{lr} \qquad j = 0, 1, 2$$
$$z_{j+\frac{5}{8}}^{lr} = z_{j+2+\frac{7}{8}}^{lr} \qquad j = 0, 1, 2$$

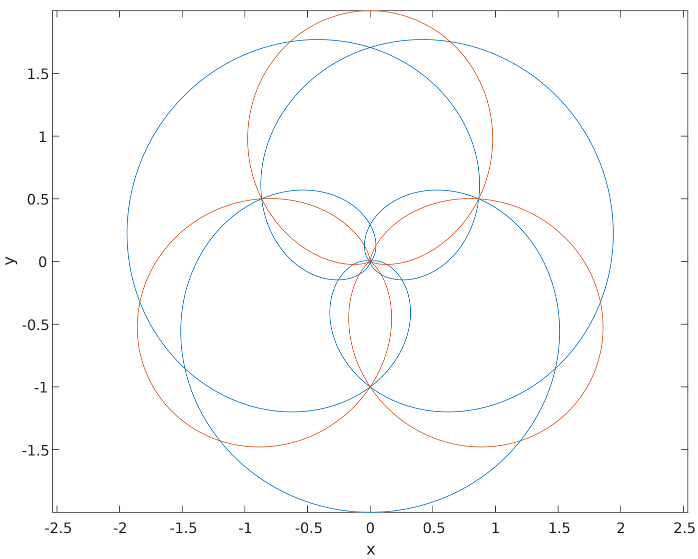
- additional intersections of the two families:

$$z_{j+\frac{1}{6}}^{lr} = z_{j+\frac{1}{3}}^{rl} \qquad j = 0, 1, 2$$
$$z_{j+2+\frac{1}{3}}^{lr} = z_{j+\frac{1}{6}}^{rl} \qquad j = 0, 1, 2$$
$$z_{j+\frac{2}{3}}^{lr} = z_{j+\frac{5}{6}}^{rl} \qquad j = 0, 1, 2$$
$$z_{j+\frac{5}{6}}^{lr} = z_{j+1+\frac{2}{3}}^{rl} \qquad j = 0, 1, 2$$

- If $\delta = 2\pi$, then there is an additional multi-intersection and multi-self-intersection of the two families in the constant orbit, $z_{j+\frac{1}{4}}^{lr} \equiv 0 \equiv z_{j+\frac{1}{4}}^{rl}$, $j = 0, 1, 2$.



(a) $\delta = 4\pi$: The two families intersect themselves and each other many times.



(b) $\delta = 2\pi$: There is an additional multi-intersection and self-intersection of the two families in the constant orbit, namely for $\tau = \frac{1}{4} \bmod \mathbb{Z}$. Everything else is similar as for $\delta = 4\pi$.

Figure 5.8: The 3-periodic curves $\mathbb{R} \to \mathbb{C}$, $\tau \mapsto z_\tau^{lr}(0)$ (blue) and $\tau \mapsto z_\tau^{rl}(0)$ (orange) in the situation of Subsection 5.7.4, $k = 3$.

Remark 5.14. We analysed the case of the very concrete example function $f(t) = \sin(2\pi kt)$, $k \in \mathbb{N}_{\geq 1}$. This made it possible to plot the results. However, the qualitative behavior of the families of delay orbits does not change if we consider a more general function f: Let us assume that

- $f : S^1 \to [-1, 1]$ is smooth,
- there are $\alpha_0, \dots, \alpha_{k-1}$ with $f(\alpha_j) = 1$,
- there are $\beta_0, \dots, \beta_{k-1}$ with $f(\beta_j) = -1$,
- and f is strictly monotone in between.

Then everything works exactly as for $f(t) = \sin(2\pi kt)$; the only difference is that we were not yet able to prove (and are not sure whether to expect) smoothness of the families in $\tau = \frac{m}{2}$, $m \in \mathbb{Z}$.

Chapter 6

The autonomous case

In this chapter we work with an autonomous vector field X, that is, $X_t(\cdot) = X(\cdot)$ does not depend on $t \in S^1$. In this situation, orbits and delay orbits come in S^1-families. Indeed, if $x_\tau : S^1 \to \mathbb{R}^n$ is a 1-periodic τ-delay orbit of X and $t_0 \in S^1$, then

$$\partial_t \varphi(t_0, x_\tau)(t) = \partial_t x_\tau(t - t_0) = X\big(x_\tau(t - t_0 - \tau)\big) = X\big(\varphi(t_0, x_\tau)(t - \tau)\big)$$

and hence $\varphi(t_0, x_\tau)$ is again a τ-delay orbit of X. This is the reason why in the autonomous situation every periodic orbit is degenerate (see also Remark 4.17). So the best that we can hope for is a maximally non-degenerate periodic orbit.

Definition 6.1. A 1-periodic orbit x of an autonomous vector field X is called *maximally non-degenerate* if the eigenspace of the eigenvalue 1 of the linearized time-1-flow $\mathrm{d}\varphi^1_X(x(0))$ is 1-dimensional.

In order to find only one delay orbit from each such family, we may choose to only work with loops x that satisfy a certain condition at $t = 0$. Hence, below we introduce transversal constraints and adapt our setting to looking for 1-dimensional families of delay-orbits x_τ parametrized by τ which all satisfy a constraint at $t = 0$. First, we define splittings of the domain and target of the map s as direct sums. It is convenient to use a shift in scales to make the splitting of the target well-defined.

Fix a maximally non-degenerate 1-periodic orbit x_0 of X. Without loss of generality assume that $x_0(0) = 0 \in \mathbb{R}^n$. Denote $L := \mathbb{R} \cdot X(0)$ and let $N := L^\perp$ be the orthogonal complement of L in $\mathbb{R}^n$. Let $\pi_L : \mathbb{R}^n \to L$ be the orthogonal projection. Now define

$$V := \{x \in H_2 \mid x(0) \in N\}.$$

Then the vector space H_2 splits as

$$\begin{aligned} H_2 &\cong V \oplus L \\ x &\mapsto \big(x - \pi_L(x(0)), \pi_L(x(0))\big) \end{aligned}$$

where we identify each $p \in L$ with the constant loop $x(t) \equiv p$. If H_2 is interpreted as a sc-Hilbert space $\mathrm{H}^2 = (H_{2+m})_{m \geq 0}$ and $\mathrm{V} = (V_m)_{m \geq 0} = (V \cap H_{2+m})_{m \geq 0}$ carries the induced sc-structure, then the splitting above is a sc-Hilbert splitting $\mathrm{H}^2 = \mathrm{V} \oplus L$. As for the inner products, every $\langle \cdot, \cdot \rangle_m$ splits as $\langle \cdot, \cdot \rangle_m = \langle \cdot, \cdot \rangle_m|_V + \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$.

By maximal non-degeneracy, the matrix $\mathrm{d}\Phi_X^1(0)$ has a simple eigenvalue 1. Thus also $C = (\mathrm{d}\Phi_X^1(0)^T)^{-1}$ has a simple eigenvalue 1. This matrix appeared already in the proof of Proposition 4.18; note that in the Hamiltonian case we have $C = \mathrm{d}\Phi_X^1(0)$. Let L' be the eigenspace for the simple eigenvalue 1 of the matrix C, and set $N' := L'^{\perp}$. In the Hamiltonian case we get $L = L'$ and $N = N'$.

Now let us use the embedding $H_1 = W^{1,2}(S^1, \mathbb{R}^n) \hookrightarrow \mathcal{C}^0(S^1, \mathbb{R}^n)$ to define

$$W := \{\xi \in H_1 \mid \xi(0) \in N'\} \tag{6.1}$$

with induced sc-structure $\mathrm{W} = (W_m)_{m\geq 0} = (W \cap H_{1+m})_{m\geq 0}$. This gives a sc-Hilbert-splitting $\mathrm{H}^1 \cong \mathrm{W} \oplus L'$. Let $\pi_W : \mathrm{H}^1 \to \mathrm{W}$ denote the projection given by $\pi_W(\xi) = \xi - \pi_{L'}(\xi(0))$. Instead of

$$\begin{aligned} s : \mathbb{R} \times \mathrm{H}^1 &\longrightarrow \mathrm{H} \\ (\tau, x) &\longmapsto \partial_t x - X(\varphi(\tau, x)) \end{aligned}$$

we now work with

$$\begin{aligned} \widetilde{s} : \mathbb{R} \times \mathrm{V} &\longrightarrow \mathrm{W} \\ (\tau, x) &\longmapsto \pi_W\big(s(\tau, x)\big) = \pi_W\big(\partial_t x - X(\varphi(\tau, x))\big) \end{aligned}$$

or, to formulate it as a section in a trivial bundle,

$$\begin{aligned} \widetilde{S} : \mathbb{R} \times \mathrm{V} &\longrightarrow \mathbb{R} \times \mathrm{V} \times \mathrm{W} \\ (\tau, x) &\longmapsto (\tau, x, \widetilde{s}(\tau, x)). \end{aligned}$$

Unfortunately, $\widetilde{s}(\tau, x_\tau) = 0$ only implies that x_τ solves (4.1) up to a constant error in L'. This is why Theorem 6.5 below is only a weak analogue of Theorem 4.1.

We will show that $\widetilde{S}$ is a sc-Fredholm section of index 1 and see how x_0 being maximally non-degenerate translates into surjectivity of $\mathrm{d}\widetilde{s}(0, x_0)$. This means that we have to prove analogues of Proposition 4.10, Theorem 4.12, Proposition 4.13, Proposition 4.18, and Corollary 4.19.

Proposition 6.2 (cf. Proposition 4.10)**.** *The map $\widetilde{S}$ is a sc-smooth section in a strong M-polyfold bundle.*

Proof. By Proposition 3.6, we can shift the scales in the domain and target and the shifted map

$$s : \mathbb{R} \times \mathrm{H}^2 \longrightarrow \mathrm{H}^1$$

is still sc-smooth. Now restricting it to the sc-subspace $\mathbb{R} \times \mathrm{V} \subset \mathbb{R} \times \mathrm{H}^2$ does certainly not affect sc-smoothness. The projection π_W is sc-smooth since $\mathrm{H}^1 = \mathrm{W} \oplus L'$ is a sc-splitting. So the chain rule for sc-differentiability gives that $\widetilde{s}$ is sc-smooth. □

Note that the linearization of $\widetilde{s}$ at (τ, x) is

$$\begin{aligned} \mathrm{d}\widetilde{s}(\tau, x) : \mathbb{R} \times V &\longrightarrow W \\ (T, \hat{x}) &\longmapsto \pi_W \circ \mathrm{d}s(\tau, x)(T, \hat{x}) \\ &\quad = \pi_W\big(\partial_t \hat{x}\big) - \pi_W\big(\mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x})\big) \\ &\qquad + T \cdot \pi_W\big(\mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \partial_t x)\big). \end{aligned} \tag{6.2}$$

Theorem 6.3 (cf. Theorem 4.12 and Proposition 4.13)**.** *The section $\widetilde{S}$ is sc-Fredholm of index* 1*.*

Proof. We adapt the proofs of Theorem 4.12 and Proposition 4.13. The splitting of the domain is still the obvious one, that is $\mathbb{R} \times \mathrm{V}$, so $d = 1$ in Definition 3.21.

(i) Recall that $\pi_W(\xi) = \xi - \pi_{L'}(\xi(0))$. This implies that for any $\xi \in \mathrm{H}^1$,

$$\pi_W(\xi) \in W_{m+1} = H_{m+2} \cap W \iff \xi \in H_{m+2}.$$

So $\widetilde{s}$ is regularizing by the same argument that was used for s in the proof of Theorem 4.12.

(ii) All differentiability and continuity properties that we showed in part (ii) of the proof of Theorem 4.12 were for fixed scale $m \in \mathbb{N}$, so the shift in scales has no significance for this. Restriction to V does not cause problems for differentiability, and the projection π_W is classically differentiable. This implies that the classical differentiability of $\widetilde{s}$ in direction of the second factor and the continuity properties (a) and (b) of this differential follow directly from the corresponding properties of s.

(iii) We have to show that for every $(\tau, x) \in \mathbb{R} \times V_\infty$, the derivative $\mathrm{d}\widetilde{s}_\tau(x) : \mathrm{V} \to \mathrm{W}$ is a linear sc-Fredholm map, and that the index is invariant under small changes of τ. We have

$$\begin{aligned} \mathrm{d}\widetilde{s}_\tau(x) : \mathrm{V} &\longrightarrow \mathrm{W} \\ \hat{x} &\longmapsto \pi_W\big(\partial_t \hat{x}\big) - \pi_W\big(\mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x})\big) \end{aligned}$$

and so it is clear that this is of class sc^0. As in (i), the regularizing property is not affected by π_W. It remains to show that at the 0-level, the map is classically Fredholm with index not depending on τ. Moreover, we want its index to be equal to 0. For this, we start by analysing the first summand. This means that we want to show that

$$\pi_W \circ \partial_t|_V : V \longrightarrow W$$

is Fredholm of index 0. Indeed, we know that $\partial_t : H_2 \to H_1$ is Fredholm of index 0. Note that

$$\{0\} \oplus L \subset \ker \partial_t$$

and

$$\mathrm{im}(\partial_t) \cap (\{0\} \oplus L') = \{0\}.$$

From this we deduce that

$$\dim \ker(\pi_W \circ \partial_t|_V) = \dim \ker(\partial_t) - \dim L = \dim \ker(\partial_t) - 1,$$

and that the image of $\pi_W \circ \partial_t|_V$ is isomorphic to the image of ∂_t. Thus we have

$$\dim \operatorname{coker}(\pi_W \circ \partial_t|_V) = \dim \operatorname{coker}(\partial_t) - 1.$$

So the Fredholm property of $\pi_W \circ \partial_t|_V$ follows from the one of $\partial_t : H_2 \to H_1$ and the index remains 0. In Section 4.4, we used that the summand

$$-\mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x})$$

was compact because it factored through the compact embedding $H_1 \hookrightarrow H_0$. Compactness certainly still holds after applying the projection π_W. So

$$-\pi_W\big(\mathrm{d}X(\varphi(\tau, x)) \cdot \varphi(\tau, \hat{x})\big)$$

is compact and thus does not change the Fredholm property nor the index.

It remains to compute the index of the section $\widetilde{s}$. The sc-differential $\mathrm{d}\widetilde{s}(\tau, x)$ in (6.2) is obtained from $\mathrm{d}\widetilde{s}_\tau(x)$ by adding the $\mathbb{R}$-factor in the domain and adding the third summand. We can now use Lemma 4.14 again to see that this procedure raises the index by 1. ☐

Recall that to make the definitions in the beginning of this section, we fixed a maximally non-degenerate orbit x_0.

Proposition 6.4 (cf. Proposition 4.18 and Corollary 4.19)**.** *The vertical differential* $\mathrm{d}\widetilde{s}_0(x_0) : V \to W$ *is surjective. In particular,* $\mathrm{d}\widetilde{s}(0, x_0) : \mathbb{R} \times V \to W$ *is surjective as well.*

Proof. As in the proof of Proposition 4.18, we see that non-surjectivity of $\mathrm{d}\widetilde{s}_0(x_0)$: $V \to W$ would imply the existence of $0 \neq \eta \in W$ satisfying

$$\forall \hat{x} \in V_1 : \quad \langle \partial_t \hat{x} - \mathrm{d}X(x_0)\hat{x}, \eta \rangle = 0. \tag{6.3}$$

As before, we find that such an η is automatically smooth and that it is a 1-periodic solution of

$$\partial_t \eta(t) = A(t)\eta(t), \tag{6.4}$$

where $A : S^1 \to \mathbb{R}^{n \times n}$ is defined by (4.15). By Theorem 4.22, any solution of equation (6.4) is of the form

$$\eta(t) = Y(t) \cdot \eta(0),$$

where $Y : \mathbb{R} \to \mathbb{R}^{n \times n}$ is the fundamental system given by (4.13) and $\eta(0)$ is some eigenvector of $Y(1)$ for the eigenvalue 1. This means $\eta(0) \in L' \setminus \{0\}$, by definition of the space L' above equation (6.1). So

$$\eta(0) \notin N' = L'^{\perp}.$$

By definition of W in (6.1) on page 56, this contradicts the fact that $\eta \in W$. So there is no non-trivial $\eta \in W$ satisfying (6.3). This proves that $\mathrm{d}\widetilde{s}_0(x_0) : V \to W$ is surjective. ☐

By now we have collected all the ingredients for applying the M-polyfold IFT (Theorem 3.23) to the case of a maximally non-degenerate periodic orbit of an autonomous vector field. We get the following weak analogue of Theorem4.1.

Theorem 6.5 (cf. Theorem 4.1)**.** *If x_0 is a maximally non-degenerate periodic orbit of X, then there is $\tau_0 > 0$ such that for every delay τ with $|\tau| \leq \tau_0$ there is a locally unique smooth $x_\tau \in V$ satisfying*

$$\partial_t x_\tau(t) = X(x_\tau(t - \tau)) + l_{\tau,X}, \qquad t \in S^1, \tag{6.5}$$

where $l_{\tau,X} := \pi_{L'}\big(\partial_t x_\tau(0) - X(x_\tau(-\tau))\big) \in L'$. The parametrization $\tau \mapsto x_\tau$ is smooth.

Proof. We combine Proposition 6.2, Theorem 6.3 and Proposition 6.4 with the M-polyfold IFT (Theorem 3.23) to see that the zero set of $\widetilde{S}$ is, near $(0, x_0)$, a smooth 1-dimensional submanifold of $\mathbb{R} \times V$ that projects to a neighborhood of 0 in $\mathbb{R}$. □

Remark 6.6. Every x_τ in Theorem 6.5 solves the delay equation (4.1) with delay τ in direction of the codimension 1 subspace $N' = L'^{\perp} \subset \mathbb{R}^n$. The number $l_{\tau,X} \in L' \cong \mathbb{R}$ describes how far it is from satisfying the equation also in direction of L'. If we could show that $l_{\tau,X} = 0$, we would have found a honest solution of (4.1). Conversely, if there is any solution of (4.1) near x_0, then it also satisfies (6.5) and so, by local uniqueness of the solution in Theorem 6.5, it coincides with x_τ and so we have $l_{\tau,X} = 0$. Thus, $l_{\tau,X} = 0$ is equivalent to existence of a solution of (4.1) near x_0.

Using (6.5) and that x_τ is 1-periodic, we see that

$$0 = x_\tau(1) - x_\tau(0) = \int_0^1 \partial_t x_\tau(t)\, \mathrm{d}t = \int_0^1 X(x_\tau(t - \tau))\, \mathrm{d}t + l_{\tau,X}$$

and so

$$l_{\tau,X} = -\int_0^1 X(x_\tau(t))\, \mathrm{d}t. \tag{6.6}$$

This is a nice description of $l_{\tau,X}$, but we see no direct way to show that $l_{\tau,X} = 0$.

Chapter 7

Generalizations

So far, we have been working with the delay equation

$$\partial_t x(t) = X_t\big(x(t - \tau)\big) \qquad \text{for all } t \in S^1, \tag{7.1}$$

where $x : S^1 \to \mathbb{R}^n$ is a loop in $\mathbb{R}^n$ and $\tau \in \mathbb{R}$ a real number. The main result was Theorem 4.1 about existence and uniqueness of a 1-dimensional smooth family of solutions for small delay τ near a given solution x_0 for vanishing delay $\tau = 0$. In this chapter, we elaborate on several different generalizations of the above equation. This results in theorems that generalize Theorem 4.1 in different directions, see Theorems 7.2, 7.13, 7.15 and 7.33 below.

7.1 Finitely many discrete delays

Instead of a vector field $X : S^1 \times \mathbb{R}^n \to \mathbb{R}^n$ let us now consider a map $X : S^1 \times (\mathbb{R}^n)^k \to \mathbb{R}^n$. An interesting delay equation is given by

$$\partial_t x(t) = X_t\big(x(t - \tau_1), \dots, x(t - \tau_k)\big) \qquad \text{for all } t \in S^1, \tag{7.2}$$

where now $\tau_1, \dots \tau_k \in \mathbb{R}$ are finitely many discrete delays. It seems that the behavior of equation (7.2) should be similar to the one of equation (7.1). Before we can make this statement precise in Theorem 7.2, we need another definition.

Definition 7.1. Let $X : S^1 \times (\mathbb{R}^n)^k \to \mathbb{R}^n$ be a map. Then

$$\begin{aligned} X^V &: S^1 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ X^V_t(x) &:= X_t(x, \dots, x) \end{aligned}$$

is called the *induced diagonal vector field.*

Note that a map $x_0 : S^1 \to \mathbb{R}^n$ solves (7.2) with $(\tau_1, \dots, \tau_k) = (0, \dots, 0)$ if and only if it is an orbit of the induced diagonal vector field X^V.

Theorem 7.2 (cp. Theorem 4.1)**.** *Let $X : S^1 \times (\mathbb{R}^n)^k \to \mathbb{R}^n$ be smooth, and let x_0 be a non-degenerate 1-periodic orbit of the induced diagonal vector field X^V. Then there is $\tau_0 > 0$ such that for every k-tuple $\tau = (\tau_1, \dots, \tau_k) \in \mathbb{R}^k$ with $\|\tau\|_{\mathbb{R}^k} \le \tau_0$ there exists a (locally unique) smooth 1-periodic solution x_τ of the delay equation* (7.2)*. Moreover, the parametrization $(\tau_1, \dots, \tau_k) = \tau \mapsto x_\tau$ is smooth.*

Recall the shift map φ defined in (4.2). The solutions of (7.2) are cut out by the map

$$\begin{aligned} s : \mathbb{R}^k \times W^{1,2}(S^1, \mathbb{R}^n) &\longrightarrow L^2(S^1, \mathbb{R}^n) \\ \big((\tau_1, \dots, \tau_k), x\big) &\longmapsto \partial_t x - X\big(\varphi(\tau_1, x), \dots, \varphi(\tau_k, x)\big) \end{aligned}$$

which, as the map s from Chapter 4, defines a section

$$\begin{aligned} S : \mathbb{R}^k \times \mathrm{H}^1 &\longrightarrow \mathbb{R} \times \mathrm{H}^1 \triangleright \mathrm{H} \\ (\tau, x) &\longmapsto (\tau, x, s(\tau, x)) \end{aligned}$$

in the trivial sc-Hilbert space bundle $\mathbb{R}^k \times \mathrm{H}^1 \triangleright \mathrm{H} \longrightarrow \mathbb{R}^k \times \mathrm{H}^1$. The zero set

$$\{(\tau, x) \in \mathbb{R}^k \times \mathrm{H}^1 \mid s(\tau, x) = 0\}$$

is the set of 1-periodic solutions of equation (7.2). The statements from Sections 4.3–4.6 carry over to the current set-up with minor modifications, see below. For convenience we refer to the corresponding analogous statements in Chapter 4.

Proposition 7.3 (cf. Proposition 4.10)**.** *The section S is sc-smooth. Its vertical sc-differential* $\mathrm{d}s(\tau, x)$ *at the point* $(\tau, x) \in (\mathbb{R}^k \times \mathrm{H}^1)_1 = \mathbb{R} \times H_2$ *is given by*

$$\begin{aligned} \mathrm{d}s(\tau, x) : \mathbb{R}^k \times H_1 &\longrightarrow H_0 \\ \big((T_1, \dots, T_k), \hat{x}\big) &\longmapsto \partial_t \hat{x} - \mathrm{d}X\big(\varphi(\tau_1, x), \dots, \varphi(\tau_k, x)\big)\big(\varphi(\tau_1, \hat{x}), \dots, \varphi(\tau_k, \hat{x})\big) \\ &\qquad + \mathrm{d}X\big(\varphi(\tau_1, x), \dots, \varphi(\tau_k, x)\big)\big(T_1\varphi(\tau_1, \partial_t x), \dots, T_k\varphi(\tau_k, \partial_t x)\big). \end{aligned}$$

In particular, at $(0, x)$ this simplifies to

$$\mathrm{d}s(0, x) : \mathbb{R}^k \times H_1 \longrightarrow H_0 \tag{7.3}$$

$$\begin{aligned} \big((T_1, \dots, T_k), \hat{x}\big) &\longmapsto \partial_t \hat{x} - \mathrm{d}X\big(x, \dots, x\big)\big(\hat{x}, \dots, \hat{x}\big) \\ &\qquad + \mathrm{d}X\big(x, \dots, x\big)\big(T_1\partial_t x, \dots, T_k\partial_t x\big). \end{aligned} \tag{7.4}$$

Proof. Sc-smoothness follows by chain rule from sc-smoothness of the shift map and classical smoothness of f and X, together with sc-smoothness of ∂_t, exactly as in the proof of Proposition 4.10. The explicit formula for the vertical differential follows by chain rule. □

Theorem 7.4 (cf. Theorem 4.12 and Proposition 4.13)**.** *S is a sc-Fredholm section of Fredholm index k.*

Proof. The proof of Theorem 4.12 carries over with minor adaptations. The suitable splitting of the domain $\mathbb{R} \times \mathrm{H}^1$ is again the one induced by the cartesian product, so this time $d = k$ in Wehrheims definition of the sc-Fredholm property (Definition 3.21). We skip the details here.

Let us compute the index of $\mathrm{d}s(0, x)$. Again, the operator $\partial_t : H_1 \to H_0$ is Fredholm of index 1, and

$$\begin{aligned} H_1 &\longrightarrow H_0 \\ \hat{x} &\longmapsto \mathrm{d}X(x, \dots, x)(\hat{x}, \dots, \hat{x}) \end{aligned}$$

is compact by compactness of the embedding $H_1 \hookrightarrow H_0$. It remains to argue why adding the third term in $\mathrm{d}s(0, x)$ does not change the Fredholm property and raises the index by k. This follows from Lemma 7.5 below. □

The statement of the following lemma (which was used above) is quite obvious; for completeness we include a proof.

Lemma 7.5 (cf. Lemma 4.14). *Assume that* $f : U \to V$ *is a linear Fredholm operator, and that* $g : \mathbb{R}^k \to V$ *is linear. Then* $f + g : \mathbb{R}^k \times U \to V$ *is Fredholm of index* $\mathrm{ind}(f + g) = \mathrm{ind}(f) + k$.

Proof. Let $m = \dim(\mathrm{im}(f) \cap \mathrm{im}(g))$. Then

$$\begin{aligned}\mathrm{ind}(f+g) &= \dim\ker(f+g) - \dim\mathrm{coker}(f+g)\\ &= \big(\dim\ker(f) + (k - \mathrm{rank}(g)) + m\big) - \big(\dim\mathrm{coker}(f) - (\mathrm{rank}(g) - m)\big)\\ &= \dim\ker(f) - \dim\mathrm{coker}(f) + k\\ &= \mathrm{ind}(f) + k.\end{aligned}$$

□

The last step before we can prove Theorem 7.2 is to see how non-degeneracy of x_0 as periodic orbit of the induced diagonal vector field X^V implies transversality of s in $(0, x_0)$.

Proposition 7.6 (cf. Proposition 4.18). *Let* x_0 *be a 1-periodic orbit of the induced diagonal vector field* X^V. *The linear map* $\mathrm{d}s_0(x_0) = \mathrm{d}s(0, x_0)(0, \cdot) : H_1 \longrightarrow H_0$ *is surjective if and only if* x_0 *is non-degenerate.*

Corollary 7.7 (cf. Corollary 4.19). *If* x_0 *is a non-degenerate periodic orbit of* X^V, *then* $\mathrm{d}s(0, x_0) : \mathbb{R}^k \times H_1 \longrightarrow H_0$ *is surjective.*

Proof of Proposition 7.11. From equation (7.4) we see that

$$\begin{aligned}&\mathrm{d}s_0(x_0) : H_1 \longrightarrow H_0\\ &\mathrm{d}s_0(x_0)(\hat{x}) = \partial_t\hat{x} - \mathrm{d}X\big(x, \ldots, x\big)\big(\hat{x}, \ldots, \hat{x}\big).\end{aligned}$$

On the other hand, by definition of X^V, we have

$$\mathrm{d}X^V(x)\hat{x} = \mathrm{d}X(x, \ldots, x)(\hat{x}, \ldots, \hat{x}).$$

Hence, we are really in the situation of Section 4.5 for the vector field X^V, and thus our statement follows immediately from Proposition 4.18. □

The results of this section together with the M-polyfold implicit function theorem imply Theorem 7.2.

7.2 Delay equations on manifolds

As noticed in the introduction, if we pass from $\mathbb{R}^n$ to a manifold M, equation (7.1) does not make sense anymore. Still, of course there are interesting equations on manifolds that involve a delay, for instance Lotka–Volterra equations with delay. For this and further examples see [AFS20, Section 3].

Here, we focus on two different equations. In Subsection 7.2.1, the delay enters only in a function which scales a vector field, so it makes sense on any manifold. In Subsection 7.2.2 we consider Riemannian manifolds and use parallel transport along the loops to map vectors to the right tangent space.

7.2.1 A delay equation with delay entering by scaling

In the following, we want to focus on 1-periodic solutions $x : S^1 \to M$ of equations of the form

$$\partial_t x(t) = f_t(x(t - \tau)) \cdot X_t(x(t)) \qquad \text{for all } t \in S^1,$$

where X is some vector field and f a function on M, both depending smoothly on time. This set-up can be generalized further.

We set $\mathcal{B} := W^{1,2}(S^1, M)$ and equip $\mathcal{B}$ with the scale structure

$$\mathcal{B}_m := W^{1+m,2}(S^1, M).$$

Choosing a Riemannian metric $\langle \cdot, \cdot \rangle_M$ on M turns $\mathcal{B}$ into a sc-Hilbert-manifold. For each $(\tau, x) \in \mathbb{R} \times \mathcal{B}$ denote by $\mathcal{E}_{(\tau,x)} := L^2(S^1, x^*TM)$ the Hilbert space of L^2-vector fields along x with scale structure $\mathcal{E}_{(\tau,x),k} = W^{k,2}(S^1, x^*TM)$. These form a bundle $p : \mathcal{E} \to \mathbb{R} \times \mathcal{B}$ with fiber $\mathcal{E}_{(\tau,x)}$ over (τ, x). The double filtration

$$\mathcal{E}_{m,k} := \big\{ \big((\tau, x), \eta\big) \,\big|\, (\tau, x) \in \mathbb{R} \times \mathcal{B}_m, \eta \in \mathcal{E}_{(\tau,x),k} \big\} \qquad \text{for } 0 \leq k \leq m + 1$$

gives $p : \mathcal{E} \to \mathbb{R} \times \mathcal{B}$ the structure of a tame strong M-polyfold bundle. Still, everything is modeled on sc-Hilbert spaces. We define a section by

$$\begin{aligned} \sigma : \mathbb{R} \times \mathcal{B} &\longrightarrow \mathcal{E} \\ (\tau, x) &\longmapsto \partial_t x - f(\varphi(\tau, x)) \cdot X(x). \end{aligned} \tag{7.5}$$

Then the zero set

$$\{(\tau, x) \in \mathbb{R} \times \mathcal{B} \mid \sigma(\tau, x) = 0\}$$

is the set of 1-periodic solutions of equation (7.5). The statements from Sections 4.3–4.6 carry over to the current set-up with minor modifications, see below. For convenience we refer to the corresponding analogous statements in the previous sections.

Proposition 7.8 (cf. Proposition 4.10)**.** *The section σ is sc-smooth. Its vertical sc-differential $\mathrm{d}^v\sigma(\tau, x)$ at the point $(\tau, x) \in \mathbb{R} \times \mathcal{B}_1$ is given by*

$$\begin{aligned} \mathrm{d}^v\sigma(\tau, x) : \mathbb{R} \times T_x\mathcal{B} &\longrightarrow \mathcal{E}_{(\tau,x)} \\ (T, \hat{x}) &\longmapsto \nabla_{\partial_t x}\hat{x} - f(\varphi(\tau, x)) \cdot \nabla_{\hat{x}} X(x) \\ &\qquad - \mathrm{d}f(\varphi(\tau, x))\big(\varphi(\tau, \hat{x}) - T \cdot \varphi(\tau, \partial_t x)\big) \cdot X(x), \end{aligned}$$

where ∇ is the Levi-Civita connection on M with respect to $\langle \cdot, \cdot \rangle_M$.

Proof. Sc-smoothness follows by chain rule from sc-smoothness of the shift map and classical smoothness of f and X, together with sc-smoothness of ∂_t, exactly as in the proof of Proposition 4.10. To get the explicit formula for the vertical differential, we use the product rule to compute

$$\mathrm{d}^v\sigma(\tau, x)(T, \hat{x}) = \nabla_{\partial_t x}\hat{x} - \Big(\mathrm{d}(f \circ \varphi)(\tau, x)(T, \hat{x}) \cdot X(x) + f(\varphi(\tau, x)) \cdot \nabla_{\hat{x}} X(x)\Big)$$

and, with the chain rule,

$$\begin{aligned} \mathrm{d}(f \circ \varphi)(\tau, x)(T, \hat{x}) &= \mathrm{d}f(\varphi(\tau, x))\big(\mathrm{d}\varphi(\tau, x)(T, \hat{x})\big) \\ &= \mathrm{d}f(\varphi(\tau, x))\big(\varphi(\tau, \hat{x}) - T \cdot \varphi(\tau, \partial_t x)\big). \end{aligned} \qquad \square$$

Theorem 7.9 (cf. Theorem 4.12 and Proposition 4.13). *σ is a sc-Fredholm section of Fredholm index 1.*

Since the sc-Fredholm property and the index computation is local, the proofs of Theorem 4.12 and Proposition 4.13 work with minor adaptations. We skip the details here.

Assume now that we have a solution $x_0 : S^1 \to M$ of equation (7.5) for $\tau = 0$. For simplicity we assume in the following that $x_0^*TM \to S^1$ is trivial. This is, for instance, the case, if M is orientable. The general case can be treated after suitable modifications.

Definition 7.10. Denote the flow of fX by Φ_{fX}^t. Let $x : S^1 \to M$ be a 1-periodic orbit of fX with the property that $x_0^*TM \to S^1$ is trivial. We call x *non-degenerate* if the linearized time-1-map $\mathrm{d}\Phi_X^1(x(0))$ does not have 1 as an eigenvalue.

We want to prove a statement about the existence of solutions of equation (7.5) with small delay $\tau \neq 0$, similar to Theorem 4.1. In order to apply the M-polyfold implicit function theorem it only remains to infer surjectivity of $\mathrm{d}^v\sigma(0, x_0)$ from non-degeneracy of x_0.

Proposition 7.11 (cf. Proposition 4.18). *Assume that $x_0^*TM \to S^1$ is the trivial bundle. Then the linear map $\mathrm{d}^v\sigma_0(x_0) = \mathrm{d}^v\sigma(0, x_0)(0, \cdot) : T_{x_0}\mathcal{B} \longrightarrow \mathcal{E}_{(0,x_0)}$ is surjective if and only if x_0 is non-degenerate as a 1-periodic orbit of the vector field fX.*

As before the following is an immediate corollary.

Corollary 7.12 (cf. Corollary 4.19). *If x_0 is a non-degenerate periodic orbit of the vector field fX and the pullback bundle $x_0^*TM \to S^1$ is trivial, then $\mathrm{d}^v\sigma(0, x_0) : \mathbb{R} \times T_{x_0}\mathcal{B} \longrightarrow \mathcal{E}_{(0,x_0)}$ is surjective.*

Proof of Proposition 7.11. Since the bundle $x_0^*TM \to S^1$ is trivial there is a neighborhood $U \subset M$ of $x_0(S^1)$ which is diffeomorphic to an open set $V \subset \mathbb{R}^n$ by a diffeomorphism $\psi : U \to V$. Then $\psi_*(fX)$ has $\psi(x_0)$ as 1-periodic orbit and x_0 is non-degenerate if and only if $\psi(x_0)$ is non-degenerate. Moreover, the section $\sigma|_{\mathbb{R}\times\mathcal{B}_U} : \mathbb{R}\times\mathcal{B}_U \to \mathcal{E}|_{\mathbb{R}\times\mathcal{B}_U}$, with $B_U := W^{1,2}(S^1, U)$, is conjugated via ψ and $d\psi$ to a section $\widetilde{\sigma}$ of the form (4.8). Finally, $\mathrm{d}^v\sigma(0, x_0)$ is surjective if and only if $\mathrm{d}^v\widetilde{\sigma}(0, \psi(x_0))$ is surjective. This means that we reduced the situation to the case of $\mathbb{R}^n$ and the assertion follows from Proposition 4.18. □

Combining all these results and using the M-polyfold implicit function theorem, we get the following generalization of our main theorem.

Theorem 7.13 (cf. Theorem 4.1). *We consider a vector field X and a function f, both smooth and 1-periodic, on a manifold M. Let x_0 be a non-degenerate 1-periodic orbit of the vector field fX. (In particular, we assume that $x_0^*TM \to S^1$ is trivial.) Then there is $\tau_0 > 0$ such that for every delay τ with $|\tau| \leq \tau_0$ there exists a (locally unique) smooth 1-periodic solution x_τ of the delay equation* (7.5). *Moreover, the parametrization $\tau \mapsto x_\tau$ is smooth.*

Remark 7.14. In the case that x_0^*TM is not the trivial bundle, a straightforward idea is to consider the double cover y_0 of x_0 and work on the space of 2-periodic functions instead. Then, assuming that y_0 is non-degenerate as a 2-periodic orbit of fX, the M-polyfold implicit function theorem will provide a smooth family of 2-periodic delay orbits y_τ near y_0. In this situation non-degeneracy of y_0 is equivalent to the condition that $\mathrm{d}\Phi^1_{fX}(x_0(0))$ has neither 1 nor -1 as an eigenvalue.

7.2.2 A delay equation using parallel transport

In Subsection 7.2.1, we used a Riemannian metric $\langle\cdot,\cdot\rangle_M$ on M just as an auxiliary means needed to make the analysis work; the choice of metric was random and did not have any impact on the equation. If, in contrast, we are given a Riemannian metric $\langle\cdot,\cdot\rangle_M$ on the manifold M from the very beginning, then we might choose to work with another delay equation – after all there is an obvious way of transporting tangent vectors at the point $x(t - \tau)$ to tangent vectors at the point $x(t)$, namely via parallel transport along the curve x. This means considering the delay equation

$$\partial_t x(t) = \Gamma(x)^t_{t-\tau}\Big(X_t\big(x(t - \tau)\big)\Big), \tag{7.6}$$

where $x : S^1 \to M$ is a loop and $\Gamma(x)^t_{t-\tau} : T_{x(t-\tau)}M \to T_{x(t)}M$ denotes parallel transport.

To find solutions of (7.6) near a given solution x_0 for $\tau = 0$, one can use the very same strategy as before. The notion of sc-smoothness is robust enough that it certainly won't be destroyed; the sc-Fredholm property is less robust, but after all parallel transport is an isometry; and for transversality one works with $\tau = 0$ anyway.

7.3 Higher order equations

Another direction of generalizing Theorem 4.1 is working with delay equations of higher order. Let us, for example, consider the second order equation

$$\partial_t^2 x(t) = X_t\big(x(t - \tau)\big) \tag{7.7}$$

where $x : S^1 \to \mathbb{R}^n$ and $X : S^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector field as in the original setting of Chapter 4. One way to deal with equations of higher order is to translate them to systems of equations of order 1, making it possible to apply the results established before. Another way is to directly adjust the framework from Chapter 4. Below we explain both approaches in detail for the second order equation (7.7); both strategies can be adjusted for equations of order 3 or higher.

7.3.1 Reduction to a system of first order equations

Consider the following system of first order equations:

$$\begin{cases} \partial_t x(t) = y(t) \\ \partial_t y(t) = X_t\big(x(t - \tau)\big) \end{cases} \tag{7.8}$$

If $x : S^1 \to \mathbb{R}^n$ solves (7.7), then $(x, \partial_t x) : S^1 \to \mathbb{R}^n \times \mathbb{R}^n$ solves (7.8). Conversely, if $(x, y) : S^1 \to \mathbb{R}^n \times \mathbb{R}^n$ is a solution of (7.8), then

$$\partial_t^2 x(t) = \partial_t x(t) = X_t\big(x(t - \tau)\big),$$

so $x : S^1 \to \mathbb{R}^n$ is a solution of (7.7). Thus we have translated the second order equation on $\mathbb{R}^n$ to a system of two first order equations on $\mathbb{R}^n$, that is one first order equation on $\mathbb{R}^{2n}$.

This first order equation on $\mathbb{R}^{2n}$ does not exactly fit into the framework of Theorem 4.1, because the right hand side does not only involve the shifted version of x, but also the unshifted version of y. However, it fits into the more general framework of Theorem 7.2, with $k = 2$, $\tau_1 = 0$ and $\tau_2 = \tau$. Indeed, if we define

$$\begin{aligned} \widetilde{X} : S^1 \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n} \\ \widetilde{X}_t\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) &:= \begin{pmatrix} y_1 \\ X_t(x_2) \end{pmatrix}, \end{aligned}$$

then $\widetilde{X}$ is a map as in Section 7.1 for $k = 2$, and the first order system (7.8) translates to

$$\partial_t \begin{pmatrix} x \\ y \end{pmatrix}(t) = \widetilde{X}_t\left(\begin{pmatrix} x \\ y \end{pmatrix}(t), \begin{pmatrix} x \\ y \end{pmatrix}(t - \tau)\right),$$

which is exactly equation (7.2) for $\widetilde{X}$ and $\tau_1 = 0, \tau_2 = \tau$. Let us also recall from Section 7.1 the diagonal vector field $\widetilde{X}^V$ given by

$$\widetilde{X}_t^V \begin{pmatrix} x \\ y \end{pmatrix} = \widetilde{X}_t\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ X_t(x) \end{pmatrix}.$$

Moreover, note that $\|(\tau_1, \tau_2)\|_{\mathbb{R}^2} \leq \tau_0$ implies $\tau_2 \leq \tau_0$. Hence, Theorem 7.2 implies the following:

Theorem 7.15 (cf. Theorems 4.1)**.** *Let $x_0 : S^1 \to \mathbb{R}^n$ be a solution of the second order equation* (7.7) *for $\tau = 0$, and assume that $\begin{pmatrix} x_0 \\ \partial_t x_0 \end{pmatrix} : S^1 \to \mathbb{R}^{2n}$ is non-degenerate as a 1-periodic orbit of the induced diagonal vector field $\widetilde{X}^V$ on $\mathbb{R}^{2n}$. Then there is $\tau_0 > 0$ such that for every τ with $|\tau| \leq \tau_0$ there exists a (locally unique) smooth 1-periodic solution x_τ of the delay equation* (7.2)*. Moreover, the parametrization $\tau \mapsto x_\tau$ is smooth.*

Remark 7.16. The condition that $\begin{pmatrix} x_0 \\ \partial_t x_0 \end{pmatrix} : S^1 \to \mathbb{R}^{2n}$ is non-degenerate as a 1-periodic orbit of the induced diagonal vector field $\widetilde{X}^V$ is not very descriptive. Using Lemma 4.20, we can translate the condition as follows: Define

$$A(t) := -\mathrm{d}\widetilde{X}_t^V \begin{pmatrix} x_0(t) \\ \partial_t x_0(t) \end{pmatrix}^T = \begin{pmatrix} 0 & -\mathrm{d}X_t\big(x_0(t)\big)^T \\ -\mathbb{1}_n & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$$

and let $Y : \mathbb{R} \to \mathbb{R}^{2n \times 2n}$ be the fundamental system for A, that is, the solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} Y(t) = A(t) \cdot Y(t) \\ Y(0) = \mathbb{1}_{2n}. \end{cases}$$

Then $\begin{pmatrix} x_0 \\ \partial_t x_0 \end{pmatrix} : S^1 \to \mathbb{R}^{2n}$ is non-degenerate as a 1-periodic orbit of $\widetilde{X}^V$ if and only if $Y(1)$ does not have 1 as an eigenvalue. In general, this is still hard to check, but in concrete examples it might take an easier form.

Remark 7.17. When translating the second order equation (7.7) into a system of first order equations, one might try with

$$\begin{cases} \partial_t x(t) = y(t - \frac{\tau}{2}) \\ \partial_t y(t) = X_t\big(x(t - \frac{\tau}{2})\big) \end{cases}$$

instead of (7.8). However, if (x, y) solves the system above, then

$$\partial_t^2 x(t) = \partial_t y\left(t - \frac{\tau}{2}\right) = X_{t-\frac{\tau}{2}}\big(x(t - \tau)\big),$$

so x does not solve (7.7). This problem occurs as soon as the first equation in the system involves a delay. It can be resolved by working with

$$\begin{cases} \partial_t x(t) = y(t - \frac{\tau}{2}) \\ \partial_t y(t) = X_{t+\frac{\tau}{2}}\big(x(t - \frac{\tau}{2})\big) \end{cases}$$

instead. However, now the system involves the delay parameter in an explicit way. This is not a fundamental obstacle, but it it prevents us from directly applying Theorem 4.1 or Theorem 7.2.

7.3.2 Directly adjusting the framework

Define

$$\begin{aligned} s : \mathbb{R} \times \mathrm{H}^2 &\longrightarrow \mathrm{H} \\ (\tau, x) &\longmapsto \partial_t^2 x - X(\varphi(\tau, x)). \end{aligned} \tag{7.9}$$

It is clear from the previous discussions that this section is sc-smooth and that its second summand is still compact, because it factors through the compact embedding $H_2 \hookrightarrow H_1 \hookrightarrow H_0$. The first summand, $\partial_t^2 : H_2 \to H$, is classically Fredholm of index 1. Therefore, the sc-Fredholm property follows exactly as in Section 4.4.

The interesting part is, again, finding a suitable condition that ensures transversality. Here, non-surjectivity of $\mathrm{d}s(0, x_0)$ is equivalent to the existence of $\eta \in \mathcal{C}^\infty$ with $\partial_t^2 \eta - \mathrm{d}X(x_0)^T \eta = 0$. This equation is without delay, so it can really be translated to a system of equations of order 1. However, we do not see a geometric meaning of this condition.

Remark 7.18. Here it is obvious that it is possible to add terms of lower order to equation (7.7) without destroying sc-differentiability or the sc-Fredholm property. The condition for transversality, though, will change with the equation.

7.4 Time-dependent delay

In this section, instead of a discrete delay $\tau \in \mathbb{R}$ we want to consider time-dependent delay $\tau \in \mathbb{C}^\infty(S^1, \mathbb{R})$. With this, equation (7.1) generalizes to

$$\partial_t x(t) = X_t\big(x\big(t - \tau(t)\big)\big) \qquad \text{for all } t \in S^1. \tag{7.10}$$

If we vary τ in the infinite-dimensional space $\mathcal{C}^\infty(S^1, \mathbb{R})$, there is little hope of establishing a Fredholm property. Therefore we vary τ only inside a finite-dimensional subspace $\mathcal{T}$ of $\mathcal{C}^\infty(S^1, \mathbb{R})$, and without loss of generality we can assume that $\mathcal{T}$ is 1-dimensional. On $\mathcal{T}$ all norms are equivalent; for convenience we work with the $\mathcal{C}^\infty$-norm $\|\cdot\|_{\mathcal{C}^\infty}$.

Following the strategy from before, we are tempted to define a time-dependent shift map by

$$\varphi(\tau, x)(t) := x\big(t - \tau(t)\big) \tag{7.11}$$

for any map $x : S^1 \to \mathbb{R}^n$. Concerning this "definition", we may ask the following questions:

(1) If x is square-integrable, does it follow from (7.11) that also $\varphi(\tau, x)$ is square-integrable?

(2) If $x = x'$ in L^2, does (7.11) imply that $\varphi(\tau, x) = \varphi(\tau, x')$?

(3) If $x \in H_k$, $k \geq 0$, can we expect $\varphi(\tau, x) \in H_k$?

(4) Recall that we considered $L^2(S^1, \mathbb{R}^n)$ as a sc-Hilbert space H with the filtration

$$\mathrm{H} = \Big(H_0 = L^2(S^1, \mathbb{R}^n) \supset H_1 = W^{1,2}(S^1, \mathbb{R}^n) \supset \dots\Big).$$

If we define $\varphi : \mathcal{T} \times \mathrm{H} \to \mathrm{H}$ as in (7.11), can we expect φ to be sc-smooth?

The answer to these questions is no, no, no and no! At least for general τ, see Example 7.19 below. However, if we pose additional assumptions on τ, the definition in (7.11) does make sense after all. This is what we do in in Subsections 7.4.1-7.4.4. After these initial difficulties, the strategy from before can be applied without major modification to prove an analogon to Theorem 4.1, see Theorem 7.33 below.

Example 7.19. *For question (2) above, consider a smooth map $\tau : S^1 \to \mathbb{R}$ which satisfies $\tau(t) = t$ for all $t \in [0, \frac{1}{2}]$. Then for every map $x : S^1 \to \mathbb{R}^n$ we have*

$$x\big(t - \tau(t)\big) \equiv x(0)\big) \qquad \textit{for all } t \in [0, \tfrac{1}{2}]. \tag{7.12}$$

Take maps $x, x' : S^1 \to \mathbb{R}^n$ which differ only in the value at 0. Then $x = x'$ in L^2. However, with the "definition" in (7.11) together with equation (7.12) we get $\varphi(\tau, x) = \varphi(\tau, x')$. Note that this problem occurs as soon as the map $t \mapsto t - \tau(t)$ is constant on a small interval.

7.4.1 Well-definedness of the shift map

Let us define the following subset of $\mathcal{T}$:

$$\mathcal{O} := \Big\{ \tau \in \mathcal{T} \;\Big|\; \exists\, d(\tau) \in \mathbb{N} \text{ such that } \dot{\tau}(t) \in \big(-d(\tau), 1 \big) \text{ for all } t \in S^1 \Big\} \tag{7.13}$$

The set $\mathcal{O}$ is convex and open in $\mathcal{T}$, and it contains 0. However, it is not symmetric in the sense that $\tau \in \mathcal{O}$ would imply $-\tau \in \mathcal{O}$; this only holds if $d(\tau)$ can be chosen to be 1. From now on, for $\tau \in \mathcal{O}$ we denote by $d(\tau)$ the minimal positive integer that can be used in (7.13).

Remark 7.20. The integer $d(\tau) \in \mathbb{N}$ satisfies the following:

- For any $\tau \in \mathcal{O}$ and $r \in [0,1]$, it is $d(r \cdot \tau) \leq d(\tau)$.
- For $\tau \in \mathcal{O}$ small enough, $d(\tau) = 1$.

Every delay function τ determines a map

$$\begin{aligned} \sigma : S^1 &\longrightarrow S^1 \\ t &\longmapsto t - \tau(t) \end{aligned} \tag{7.14}$$

which, following Nishiguchi [Nis19], we call the *delayed argument function* corresponding to τ. Note that σ and τ determine each other uniquely; equation (7.11) translates to $\varphi(\tau, x)(t) = x(\sigma(t))$. For $\tau \in \mathcal{O}$, the map σ satisfies

$$\dot{\sigma}(t) = 1 - \dot{\tau}(t) \in (0, d(\tau) + 1), \tag{7.15}$$

in particular it is 'monotone' of degree $d(\tau)$. If $d(\tau) = 0$, it follows that σ is a diffeomorphism of S^1, meaning that $\varphi(\tau, x)$ is just a reparametrization of x by a diffeomorphism.

Remark 7.21. In [Har16], Ferenc Hartung analyzes differentiability of solutions of differential equations with time-dependent delay with respect to the delay function. His condition on the delay function is the following, see [Har16, equation (4.1), $\alpha = 1$]:

$$\exists\, 0 \leq \kappa(\tau) < 1 \text{ such that } |\dot{\tau}(t)| \leq \kappa(\tau) \text{ for a.e. } t \in S^1 \tag{7.16}$$

In our situation, we assumed τ to be smooth, so the "almost every $t \in S^1$" is the same as "every $t \in S^1$". Hartung's condition (7.16) implies (7.15) for $d(\tau) = 1$. This is why in [Nis19] it is called "some strict monotonicity assumption".

For general $d(\tau) \in \mathbb{N}$, the delayed argument function σ may not be injective; but we can split it up into several maps as follows. Without loss of generality we may assume that $\sigma(0) = 0$. Denote $t_0 = t_d = 0$. Let $t_1 < \ldots < t_{d(\tau)-1} \in (0,1)$ be the other $d(\tau) - 1$ real numbers with $\sigma(t_i) = 0$. Then the restrictions $\sigma|_{[t_{i-1},t_i]}$ define $d(\tau)$ bijective functions

$$\sigma_i := \sigma|_{[t_{i-1},t_i]} : [t_{i-1}, t_i]/_{t_{i-1} \sim t_i} \cong S^1 \longrightarrow S^1, \qquad 1 \leq i \leq d(\tau)$$

which are diffeomorphisms – except for possibly at the point where the ends of the interval where glued together and the left and right derivatives of σ_i will in general not coincide.

Lemma 7.22. *If* $\tau \in \mathcal{O}$*, then*

$$\begin{aligned}\varphi(\tau,\cdot): H_0 &\longrightarrow H_0 \\ x &\longmapsto \Big[t \mapsto x\big(t - \tau(t)\big)\Big]\end{aligned}$$

is a well-defined bounded linear map.

Proof. Linearity is clear.

Boundedness: We need to show that $\|\varphi(\tau,x)\|_{L^2}$ is bounded in terms of $\|x\|_{L^2}$. For that purpose, recall the $d(\tau)$ maps σ_i defined above. Using integration by substitution for σ_i $(u = \sigma_i(t), t = \sigma_i^{-1}(u), \mathrm{d}t = (\sigma_i^{-1})'(u)\mathrm{d}u)$, we find

$$\begin{aligned}\|\varphi(\tau,x)\|_{L^2}^2 &= \int_0^1 \|x(\sigma(t))\|_{\mathbb{R}^n}^2 \,\mathrm{d}t \\ &= \sum_{i=1}^{d} \int_{t_{i-1}}^{t_i} \|x(\sigma_i(t))\|_{\mathbb{R}^n}^2 \,\mathrm{d}t \\ &= \sum_{i=1}^{d} \int_0^1 \|x(u)\|_{\mathbb{R}^n}^2 \cdot (\sigma_i^{-1})'(u) \,\mathrm{d}u \\ &\leq d(\tau) \cdot c(\tau) \cdot \|x\|_{L^2}^2, \end{aligned} \tag{7.17}$$

since all $|(\sigma_i^{-1})'(u)|$ are bounded by some constant $c(\tau) > 0$ depending on τ. (See Remark 7.23 for a more concrete description of $c(\tau)$.) Note that σ_i' may not be well-defined at the gluing point $t_{i-1} \sim t_i$ and thus $\sigma_i^{-1\prime}$ may not be well-defined at the point $0 \sim 1 \in S^1$, but a point has measure 0 and so this does not matter for the integral.

Well-definedness: Let $x, x' : S^1 \to \mathbb{R}^n$ be maps that are the same in L^2, that is $\|x - x'\|_{L^2} = 0$. We need to show that $\varphi(\tau,x)$ and $\varphi(\tau,x')$ are the same in L^2, that is $\|\varphi(\tau,x) - \varphi(\tau,x')\|_{L^2} = 0$. Using linearity and boundedness we see that

$$\|\varphi(\tau,x) - \varphi(\tau,x')\|_{L^2} = \|\varphi(\tau,x - x')\|_{L^2} \leq c(\tau) \cdot \|x - x'\|_{L^2} = 0.$$

□

Remark 7.23. What can we say about the constant $c(\tau)$ which occured in (7.17), and how does it depend on τ? Above it was only important that it satisfies $|(\sigma_i^{-1})'(u)| \leq c(\tau)$ for all $u \in S^1$; for further use it makes sense to define it more concretely. For $\tau \in \mathcal{O}$, we set

$$\dot{\tau}_{\max} := \max\left\{\dot{\tau}(t) \mid t \in S^1\right\} < 1 \quad \text{and} \quad c(\tau) := \max\left\{\frac{1}{1 - \dot{\tau}_{\max}}, 1\right\}.$$

From $\sigma_i(t) = t - \tau(t)$ we can conclude $\sigma_i^{-1}(u) = u + \tau(\sigma_i^{-1}(u))$. Applying the chain rule, we see that

$$(\sigma_i^{-1})'(u) = \frac{1}{1 - \dot{\tau}(\sigma_i^{-1}(u))} \leq \frac{1}{1 - \dot{\tau}_{\max}} \leq c(\tau), \tag{7.18}$$

hence $c(\tau)$ meets the main requirement. Moreover, it satisfies the following:

- For any $\tau \in \mathcal{O}$ and $r \in [0,1]$, it is $c(r \cdot \tau) \leq c(\tau)$.
- For $\tau \in \mathcal{O}$ small enough, $c(\tau)$ is uniformly bounded by 2.

So far, we analyzed the shift by a fixed delay function τ. Next we want to vary τ.

Lemma 7.24. *The shift map*

$$\begin{aligned} \varphi : \mathcal{O} \times H_0 &\longrightarrow H_0 \\ (\tau, x) &\longmapsto \Big[t \mapsto x\big(t - \tau(t)\big)\Big] \end{aligned}$$

is well-defined and continuous.

Proof. The shift of x by τ is a well-defined element of H_0 by Lemma 7.22. It remains to show continuity, and for this we follow the proof of [AS20, Lemma 3.1]. Let $(\tau_i)_{i\in\mathbb{N}} \subset \mathcal{O}$ and $(x_i)_{i\in\mathbb{N}} \subset H_0$ be sequences such that $\tau_i \to \tau$ in $\mathcal{C}^\infty$ and $x_i \to x$ in H_0. We need to show that $\|\varphi(\tau, x) - \varphi(\tau_i, x_i)\|_{H_0} \to 0$.
Step 0: We claim that without loss of generality we can assume $\tau = 0$. Indeed, it is

$$\begin{aligned} \|\varphi(\tau, x) - \varphi(\tau_i, x_i)\|_{H_0} &= \big\|\varphi\big(\tau, x) - \varphi(\tau_i - \tau, x_i)\big)\big\|_{H_0} \\ &\leq \sqrt{d(\tau)} \cdot \sqrt{c(\tau)} \cdot \|x - \varphi(\tau_i - \tau, x_i)\|_{H_0} \end{aligned}$$

by equation (7.17) from the proof of Lemma 7.22. Here, $d(\tau) \in \mathbb{N}$ is the degree of the delayed argument function σ corresponding to the delay τ, and $c(\tau)$ is a constant depending on τ. Note that to even write down $\varphi(\tau_i - \tau, x_i)$ on the right hand side we need to know that $\tau_i - \tau \in \mathcal{O}$; this is no problem since $\tau_i \to \tau$ and $\mathcal{O} \subset \mathcal{T}$ is open and contains 0.
Step 1: Let us show $\|x_i - \varphi(\tau_i, x_i)\|_{H_0} \to 0$ for the special case when $x_i \equiv x \in H_0$ is a constant sequence. The map x may not be smooth, but it can be approximated in H_0 by smooth elements. Fix $\varepsilon > 0$ and choose $\bar{x} \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$ with

$$\|\bar{x} - x\|_{H_0} \leq \frac{\varepsilon}{8}.$$

Now $\bar{x}$ is uniformly continuous and $\tau_i \to 0$ in $\mathcal{C}^\infty$, so for i sufficiently large we have

$$\|\bar{x}(t) - \bar{x}(t - \tau_i(t))\|_{\mathbb{R}^n} \leq \frac{\varepsilon}{8} \quad \text{for all } t \in S^1$$

and hence

$$\|\bar{x} - \varphi(\tau_i, \bar{x})\|_{H_0} \leq \frac{\varepsilon}{8}.$$

Therefore

$$\begin{aligned} \|x - \varphi(\tau_i, x)\|_{H_0} &\leq \|x - \bar{x}\|_{H_0} + \|\bar{x} - \varphi(\tau_i, \bar{x})\|_{H_0} + \|\varphi(\tau_i, \bar{x}) - \varphi(\tau_i, x)\|_{H_0} \\ &\leq \|x - \bar{x}\|_{H_0} + \|\bar{x} - \varphi(\tau_i, \bar{x})\|_{H_0} + \sqrt{d(\tau_i)} \cdot \sqrt{c(\tau_i)} \cdot \|\bar{x} - x\|_{H_0} \\ &\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \sqrt{2} \cdot \frac{\varepsilon}{8} \\ &< \frac{\varepsilon}{2} \end{aligned}$$

for i sufficiently large. Here, in the second step we used (7.17). In the third step we used that for small τ_i we have $d(\tau_i) = 1$ by Remark 7.20 and $c(\tau_i) \leq 2$ by Remark 7.23.
Step 2: Now we pass on to any converging sequence $x_i \to x \in H_0$. In step 1 we have shown that

$$\|x - \varphi(\tau_i, x)\|_{H_0} \leq \frac{\varepsilon}{2}$$

for i large enough. After increasing i even further we may assume that $\|x - x_i\|_{H_0} \leq \frac{\varepsilon}{4}$. All in all we get

$$\begin{aligned}
\|x - \varphi(\tau_i, x_i)\|_{H_m} &\leq \|x - \varphi(\tau_i, x)\|_{H_m} + \|\varphi(\tau_i, x) - \varphi(\tau_i, x_i)\|_{H_m} \\
&= \|x - \varphi(\tau_i, x)\|_{H_m} + \sqrt{d(\tau_i)} \cdot \sqrt{c(\tau_i)} \cdot \|x - x_i\|_{H_m} \\
&\leq \frac{\varepsilon}{2} + \sqrt{2} \cdot \frac{\varepsilon}{4} \\
&< \varepsilon
\end{aligned}$$

as desired. □

7.4.2 Sc-continuity of the shift map

Lemma 7.25. *The shift map $\varphi : \mathcal{T} \times \mathrm{H} \to \mathrm{H}$ is sc-continuous (sc^0), meaning that for each $m \in \mathbb{N}$, it restricts to a continuous map*

$$\begin{aligned}
\varphi : \mathcal{O} \times H_m &\longrightarrow H_m \\
x &\longmapsto \Big[t \mapsto x\big(t - \tau(t)\big)\Big].
\end{aligned}$$

Proof. Let us first prove that if $x \in H_m$, then $\varphi(\tau, x) \in H_m$. We need to show that there is some $\partial_t^m \varphi(\tau, x) \in H_0$ which qualifies as m-th derivative of $\varphi(\tau, x)$.

The chain rule implies that the first derivative of $\varphi(\tau, x)$ – if it exists – should be given by

$$\partial_t \varphi(\tau, x) = (1 - \partial_t \tau) \cdot \varphi(\tau, \partial_t x). \tag{7.19}$$

If $x \in H_m$, $m \geq 1$, then from $\partial_t x \in H_{m-1}$ and $\tau \in \mathcal{C}^\infty(S^1, \mathbb{R})$ it follows that equation (7.19) really defines an element of H_{m-1}. A simple computation gives that the second derivative of $\varphi(\tau, x)$ – if it exists – should be given by

$$\partial_t^2 \varphi(\tau, x) = (1 - \partial_t \tau)^2 \cdot \varphi(\tau, \partial_t^2 x) - \partial_t^2 \tau \cdot \varphi(\tau, \partial_t x). \tag{7.20}$$

For $x \in H_m$, $m \geq 2$, we see that this is indeed an element of H_{m-2}. Computing derivatives inductively, we find that for every $m \geq 1$, the derivative $\partial_t^m \varphi(\tau, x)$ is a summand of products involving up to m derivatives of τ and shifts by τ of up to m derivatives of x. Thus, for $x \in H_m$ it is indeed an element of H_0. So far we have shown that

$$\varphi(\mathcal{O} \times H_m) \subseteq H_m$$

for every m. Continuity of the restricted map $\varphi : \mathcal{O} \times H_m \to H_m$ now follows from continuity of $\varphi : \mathcal{O} \times H_0 \to H_0$ together with continuity of taking sums and products of functions over S^1. □

7.4.3 Differentiability of the shift map

Next we want to show that $\varphi : \mathcal{O} \times H_1 \to H_0$ is (classically) differentiable. Let us first compute the expected derivative.

$$\begin{aligned}
\mathrm{d}\varphi(\tau, x)(T, \hat{x}) &= \mathrm{d}\varphi(\tau, x)(T, 0) + \mathrm{d}\varphi(\tau, x)(0, \hat{x}) \\
\mathrm{d}\varphi(\tau, x)(0, \hat{x}) &= \varphi(\tau, \hat{x}) \quad \text{as before and by linearity in direction of } x \\
\mathrm{d}\varphi(\tau, x)(T, 0)(t) &= \tfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \varphi(\tau + s \cdot T, x)(t) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} x\big(t - \tau(t) - s \cdot T(t)\big) \\
&= -T(t) \cdot \varphi(\tau, \partial_t x)(t) \qquad (7.21)
\end{aligned}$$

This looks exactly the same as for time-independent delay! In particular, there is only a shift by τ, not by T or anything else. This is good news, since there is no reason why T or $\dot{\tau}$ should be contained in $\mathcal{O}$ (or rather, there is always $T \in T_\tau \mathcal{O} = \mathcal{T}$ which does not belong to $\mathcal{O}$).

From (7.21) we see that for every $(\tau, x) \in \mathcal{O} \times H_1$ there is a bounded linear operator

$$\begin{aligned}
&\mathrm{d}\varphi(\tau, x) : \mathcal{T} \times H_0 \longrightarrow H_0 \\
&\mathrm{d}\varphi(\tau, x)(T, \hat{x}) = \varphi(\tau, \hat{x}) - T \cdot \varphi(\tau, \partial_t x)
\end{aligned}$$

which looks as if it is a derivative of φ. To prove that it is indeed the differential of φ, we need an analogue of Lemma A.2 for time-dependent T.

Lemma 7.26 (cf. Lemma A.2)**.** *Let $x \in H_1$. Then the following holds:*

(i) $\frac{\|\varphi(T,x)-x\|_{H_0}}{\|T\|_{C^\infty}} \leq \sqrt{d(T) \cdot c(T)} \cdot \|\partial_t x\|_{H_0}$ *for every* $T \in \mathcal{O} \setminus \{0\}$

(ii) $\lim_{\|T\| \to 0} \frac{1}{\|T\|_{C^\infty}} \|\varphi(T, x) - x + T \cdot \partial_t x\|_{H_0} = 0$

Recall that $\mathcal{O} \subset \mathcal{T}$ is an open subset containing 0. In particular, for every small enough T the shifted map $\varphi(T, x)$ is indeed a well-defined element of H_0, thus it makes sense to write $\|T\| \to 0$ in *(ii)*.

Proof. We start as in the proof of Lemma A.2 but directly shift by T instead of $-T$ (because from $T \in \mathcal{O}$ it does not follow that $-T \in \mathcal{O}$). Since $x \in H_1$ is weakly differentiable, we get

$$\|x(t) - x(t - h)\| \leq \int_0^1 \|\partial_t x(t - h + sh)\| |h| \, \mathrm{d}s$$

for every $t \in S^1$, $h \in \mathbb{R}$. In particular, if we fix $T \in \mathcal{O} \setminus \{0\}$, this holds for $h = T(t)$. Squaring this and using the Cauchy-Schwarz inequality leads to

$$\begin{aligned}
\|x(t) - x(t - T(t))\|^2 &\leq |T(t)|^2 \int_0^1 \big\|\partial_t x\big(t - T(t) + sT(t)\big)\big\|^2 \, \mathrm{d}s \\
&\leq \max |T|^2 \int_0^1 \big\|\partial_t x\big(t - T(t) + sT(t)\big)\big\|^2 \, \mathrm{d}s.
\end{aligned}$$

As opposed to the proof of Lemma A.2, we now do not divide by $|T(t)|^2$ (because it may be 0) but leave it on the right hand side. We integrate over $t \in S^1$ and get

$$\begin{aligned}
\|x - \varphi(T,x)\|_{H_0}^2 &\leq \int_0^1 \max|T|^2 \int_0^1 \left\|\partial_t x\big(t - (1-s)\cdot T(t)\big)\right\|^2 \,\mathrm{d}s\,\mathrm{d}t \\
&\leq \|T\|_{\mathcal{C}^\infty}^2 \cdot \int_0^1 \int_0^1 \left\|\partial_t x\big(t - (1-s)\cdot T(t)\big)\right\|^2 \,\mathrm{d}t\,\mathrm{d}s \\
&= \|T\|_{\mathcal{C}^\infty}^2 \cdot \int_0^1 \left\|\varphi\big((1-s)\cdot T, \partial_t x\big)\right\|_{H_0}^2 \,\mathrm{d}s \\
&\leq \|T\|_{\mathcal{C}^\infty}^2 \cdot \int_0^1 c\big((1-s)\cdot T\big)\cdot d\big((1-s)\cdot T\big)\,\|\partial_t x\|_{H_0}^2 \,\mathrm{d}s \\
&\leq \|T\|_{\mathcal{C}^\infty}^2 \cdot \int_0^1 c(T)\cdot d(T)\cdot \|\partial_t x\|_{H_0}^2 \,\mathrm{d}s \\
&\leq \|T\|_{\mathcal{C}^\infty}^2 \cdot c(T)\cdot d(T)\cdot \|\partial_t x\|_{H_0}^2.
\end{aligned}$$

Here we used Remarks 7.20 and 7.23 to estimate $c\big((1-s)\cdot T\big)$ and $d\big((1-s)\cdot T\big)$. The calculation shows (i) for every $T \in \mathcal{O}\setminus\{0\}$.

To show (ii), we approximate x in H_1 by smooth functions $x_k \in \mathcal{C}^\infty(S^1,\mathbb{R}^n)$. Using the triangle inequality, for every $k \in \mathbb{N}$ we compute

$$\begin{aligned}
\frac{1}{\|T\|_{\mathcal{C}^\infty}}\|\varphi(T,x) - x + T\cdot\partial_t x\|_{H_0} \leq{}& \frac{1}{\|T\|_{\mathcal{C}^\infty}}\|\varphi(T,x) - x - \varphi(T,x_k) + x_k\|_{H_0} \\
&+ \frac{1}{\|T\|_{\mathcal{C}^\infty}}\|\varphi(T,x_k) - x_k - T\cdot\partial_t x_k\|_{H_0} \\
&+ \frac{1}{\|T\|_{\mathcal{C}^\infty}}\|T\cdot\partial_t x_k - T\cdot\partial_t x\|_{H_0}.
\end{aligned}$$

By (i) (and using $\varphi(T,x) - \varphi(T,x_k) = \varphi(T, x - x_k)$), the first term is bounded by $\sqrt{d(T)\cdot c(T)}\cdot\|\partial_t x - \partial_t x_k\|_{H_0}$. This can be made arbitrarily small by choosing k high enough, since $\sqrt{d(T)\cdot c(T)} \to \sqrt{2}$ as $\|T\|_{\mathcal{C}^\infty} \to 0$ by Remarks 7.20 and 7.23. The third term can also be estimated by $\|\partial_t x_k - \partial_t x\|_{H_0}$ (using Hölder's inequality). To see that the second term tends to 0 as $\|T\|_{\mathcal{C}^\infty} \to 0$, we use $|T(t)| \leq \|T\|_{\mathcal{C}^\infty}$ and the definition of the derivative of the smooth map x_k. □

Lemma 7.27 (Analogue of Lemma 3.3)**.** *The shift map*

$$\begin{aligned}
\varphi : \mathcal{O}\times H_1 &\longrightarrow H_0 \\
(\tau, x) &\longmapsto \Big[t \mapsto x(t - \tau(t))\Big]
\end{aligned}$$

is differentiable with derivative at a point (τ, x) *given by*

$$\begin{aligned}
\mathrm{d}\varphi(\tau,x) : \mathcal{T}\times H_1 &\longrightarrow H_0 \\
(T,\hat{x}) &\longmapsto \varphi(\tau,\hat{x}) - T\cdot\varphi(\tau,\partial_t x).
\end{aligned} \tag{7.22}$$

Proof. We have to show that for every $(\tau, x) \in \mathcal{O}\times H_1$ it is

$$\lim_{\|(T,\hat{x})\|\to 0} \frac{1}{\|(T,\hat{x})\|}\|\varphi(\tau+T, x+\hat{x}) - \varphi(\tau,x) - \varphi(\tau,\hat{x}) + T\cdot\varphi(\tau,\partial_t x)\|_{H_0} = 0,$$

where it is convenient to define the norm of the pair $(T, \hat{x})$ by $\|(T, \hat{x})\|^2 = \|T\|^2_{C^\infty} + \|\hat{x}\|^2_{H_1}$. For convenience, we first use $\varphi(\tau + T, x) = \varphi(\tau, \varphi(T, x))$ and the estimate $\|\varphi(\tau, x)\|^2_{H_0} \leq d(\tau) \cdot c(\tau) \cdot \|x\|^2_{H_0}$ from (7.17) to shift the whole expression to $\tau = 0$. Then we use linearity of φ in the second argument and compute the following:

$$
\begin{aligned}
&\frac{1}{\|(T,\hat{x})\|^2}\|\varphi(\tau + T, x + \hat{x}) - \varphi(\tau, x) - \varphi(\tau, \hat{x}) + T \cdot \varphi(\tau, \partial_t x)\|^2_{H_0} \\
&\qquad \leq \frac{d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\|\varphi(T, x + \hat{x}) - \varphi(0, x) - \varphi(0, \hat{x}) + T \cdot \varphi(0, \partial_t x)\|^2_{H_0} \\
&\qquad = \frac{d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\|\varphi(T, x) + \varphi(T, \hat{x}) - x - \hat{x} + T \cdot \partial_t x\|^2_{H_0} \\
&\qquad \leq \frac{d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\Big(\|\varphi(T, x) - x + T \cdot \partial_t x\|_{H_0} + \|\varphi(T, \hat{x}) - \hat{x}\|_{H_0}\Big)^2 \\
&\qquad = \frac{d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\|\varphi(T, x) - x + T \cdot \partial_t x\|^2_{H_0} \\
&\qquad\qquad + \frac{d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\|\varphi(T, \hat{x}) - \hat{x}\|^2_{H_0} \\
&\qquad\qquad + \frac{2d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\|\varphi(T, x) - x + T \cdot \partial_t x\|_{H_0} \cdot \|\varphi(T, \hat{x}) - \hat{x}\|_{H_0}
\end{aligned}
\tag{7.23}
$$

For the first term in (7.23) we use $\frac{1}{\|(T,\hat{x})\|} \leq \frac{1}{\|T\|_{C^\infty}}$ and Lemma 7.26 (ii) to obtain

$$\frac{d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\|\varphi(T, x) - x + T \cdot \partial_t x\|^2_{H_0} \longrightarrow 0$$

as $T \to 0$. For the second term in (7.23), from $\frac{1}{\|(T,\hat{x})\|} \leq \frac{1}{\|T\|_{C^\infty}}$, Lemma 7.26 (ii), and Hölder's inequality we see that

$$
\begin{aligned}
\frac{d(\tau) \cdot c(\tau)}{\|(T,\hat{x})\|^2}\|\varphi(T, \hat{x}) - \hat{x}\|^2_{H_0} &\leq \frac{d(\tau) \cdot c(\tau)}{\|T\|^2_{C^\infty}}\|T \cdot \partial_t \hat{x}\|^2_{H_0} \\
&\leq d(\tau) \cdot c(\tau) \cdot \|\partial_t \hat{x}\|^2_{H_0} \longrightarrow 0
\end{aligned}
$$

as $\hat{x} \to 0$. For the product in the third term in (7.23) we use the same arguments to treat both factors separately and see that both tend to 0. □

7.4.4 Sc-differentiability and sc-smoothness of the shift map

Let us now show that the time-dependent shift map is sc-smooth, that is sc^k for every $k \geq 1$. We start with $k = 1$.

Proposition 7.28. *The shift map*

$$
\begin{aligned}
\varphi : \mathcal{O} \times \mathrm{H} &\longrightarrow \mathrm{H} \\
(\tau, x) &\longmapsto \Big[t \mapsto x(t - \tau(t))\Big]
\end{aligned}
$$

is sc-differentiable (sc^1).

Proof. We use the characterization of sc-differentiability by Frauenfelder–Weber, see Proposition 3.8, and check conditions (i)-(iii).

(i) The first condition is classical differentiability of $\varphi : \mathcal{O} \times H_1 \to H_0$; this is exactly what we proved in Lemma 7.27.

(ii) The second condition is that the derivative $\mathrm{d}\varphi(\tau, x) : \mathcal{T} \times H_1 \longrightarrow H_0$ extends to a bounded linear operator $\mathrm{d}\varphi(\tau, x) : \mathcal{T} \times H_0 \longrightarrow H_0$. Having a look at the formula

$$\mathrm{d}\varphi(\tau, x)(T, \hat{x}) = \varphi(\tau, \hat{x}) - T \cdot \varphi(\tau, \partial_t x)$$

from (7.22), we see that $\mathrm{d}\varphi(\tau, x)(T, \hat{x}) \in H_0$ is also well-defined for $\hat{x} \in H_0$. Continuity of this extension in $\hat{x}$ follows from continuity of $\varphi : \mathcal{O} \times H_0 \to H_0$ (see Lemma 7.24).

(iii) The third condition is that for every $m \in \mathbb{N}$, if $x \in H_{m+1}$, the derivative $\mathrm{d}\varphi(\tau, x)$ restricts to a map $\mathrm{d}\varphi(\tau, x) : \mathcal{T} \times H_m \longrightarrow H_m$ in such a way that

$$\mathrm{d}\varphi : \big(\mathcal{T} \times H_{m+1}\big) \oplus \big(\mathcal{T} \times H_m\big) \longrightarrow H_m$$

is continuous. This again follows from the formula above, this time in combination with Lemma 7.25.

□

With the same strategy we can show sc-smoothness.

Proposition 7.29. *The shift map*

$$\begin{aligned} \varphi : \mathcal{O} \times \mathrm{H} &\longrightarrow \mathrm{H} \\ (\tau, x) &\longmapsto \Big[t \mapsto x(t - \tau(t))\Big] \end{aligned}$$

is sc-smooth (sc^∞).

Proof. We need to show that φ is k times sc-differentiable (sc^k) for every $k \in \mathbb{N}$. For this, we use the characterization of sc^k in terms of higher sc-differentials by Frauenfelder–Weber, see Proposition 3.9.

(i) The first condition is that $\varphi : \mathcal{O} \times H_k \to H_0$ is k times classically differentiable. We have seen this for $k = 1$ in Lemma 7.27 and Proposition 7.28. For $k = 2$, the chain rule implies that the second derivative of φ in the point $(\tau, x) \in \mathcal{O} \times H_2$ evaluated at $(T_1, \hat{x}_1), (T_2, \hat{x}_2) \in \mathcal{T} \times H_2$ needs to be

$$\mathrm{d}^2\varphi(\tau, x)\big(T_1, \hat{x}_1, T_2, \hat{x}_2\big) = -T_2 \cdot \varphi(\tau, \partial_t \hat{x}_1) - T_1 \cdot \varphi(\tau, \partial_t \hat{x}_2) + T_1 T_2 \cdot \varphi(\tau, \partial_t^2 x).$$

This is the same formula as for the case of time-independent τ in [FW21b, Theorem 6.1]; the only difference is that τ, T_1, T_2 are now functions of time. Inductively, we find that for $(\tau, x) \in \mathcal{O} \times H_k$ there is a k-th derivative given by

$$\begin{aligned} \mathrm{d}^k \varphi(\tau, x)\big(T_1, \hat{x}_1, \dots, T_k, \hat{x}_k\big) \;=\; & \sum_{j=1}^{k} (-1)^{k-1} \cdot T_1 \cdots \widehat{T_j} \cdots T_k \cdot \varphi(\tau, \partial_t^{k-1} \hat{x}_j) \\ & + (-1)^k \cdot T_1 \cdots T_k \cdot \varphi(\tau, \partial_t^k x), \end{aligned} \tag{7.24}$$

where $T_1 \cdots \widehat{T_j} \cdots T_k$ means that T_j is omitted in the product. This classical k-th derivative is evaluated on vectors $(T_1, \hat{x}_1), \dots (T_k, \hat{x}_k) \in \mathcal{T} \times H_k$.

(ii) The second condition is that $\mathrm{d}^k \varphi(\tau, x) : (\mathcal{T} \times H_k) \oplus \cdots \oplus (\mathcal{T} \times H_k) \longrightarrow H_0$ extends continuously to $(\mathcal{T} \times H_{k-1}) \oplus \cdots \oplus (\mathcal{T} \times H_{k-1})$. Indeed, the expression in (7.24) gives a well-defined element of H_0 also for $(T_1, \hat{x}_1), \dots (T_k, \hat{x}_k) \in \mathcal{T} \times H_{k-1}$. The extension is continuous because taking derivatives is continuous, taking products is continuous and $\varphi : \mathcal{O} \times H_0 \to H_0$ is continuous by Lemma 7.24.

(iii) The third condition is concerned with restrictions of $\mathrm{d}^k \varphi(\tau, x)$ to higher levels. For $m \geq k-1$ and $(\tau, x) \in \mathcal{O} \times H_{m+1}$, the extended k-th differential

$$\mathrm{d}^k \varphi(\tau, x) : \underbrace{(\mathcal{T} \times H_{k-1}) \oplus \cdots \oplus (\mathcal{T} \times H_{k-1})}_{k \text{ times}} \longrightarrow H_0$$

from (ii) needs to restrict to a continuous map

$$\mathrm{d}^k \varphi(\tau, x) : (\mathcal{T} \times H_m) \oplus \cdots \oplus (\mathcal{T} \times H_m) \longrightarrow H_{m-k+1},$$

and the evaluation

$$\mathrm{d}^k \varphi : (\mathcal{O} \times H_{m+1}) \oplus \big((\mathcal{T} \times H_m) \oplus \cdots \oplus (\mathcal{T} \times H_m)\big) \longrightarrow H_{m-k+1},$$

needs to be continuous. All this follows from the formula in equation (7.24) together with Lemma 7.25.

□

7.4.5 The sc-Fredholm section

Now we are ready to define the section that cuts out the solutions of the time-dependent delay equation (7.10). The map

$$\begin{aligned} s : \mathcal{O} \times H_1 &\longrightarrow H_0 \\ (\tau, x) &\longmapsto \partial_t x - X(\varphi(\tau, x)) \end{aligned}$$

defines a section

$$\begin{aligned} S : \mathcal{O} \times \mathrm{H}^1 &\longrightarrow \mathcal{O} \times \mathrm{H}^1 \triangleright \mathrm{H} \\ (\tau, x) &\longmapsto \big(\tau, x, s(\tau, x)\big) \end{aligned}$$

with principal part s in the trivial tame strong M-polyfold bundle $\mathcal{O}\times \mathrm{H}^1 \triangleright \mathrm{H} \to \mathcal{O}\times \mathrm{H}^1$. At first sight, this section looks exactly as in Chapter 4, the only difference is that instead of the original shift map $\varphi : \mathbb{R} \times \mathrm{H} \to \mathrm{H}$ it uses the time-dependent shift map $\varphi : \mathcal{O} \times \mathrm{H} \to \mathrm{H}$ established in this chapter. The following is immediate from sc-smoothness of the time-dependent shift map (Proposition 7.29) and the chain rule for sc-smoothness (Theorem 3.5).

Proposition 7.30 (cf. Proposition 4.10)**.** *The section S is sc-smooth. Its vertical sc-differential at the point $(\tau, x) \in (\mathcal{O} \times \mathrm{H}^1)_1 = \mathcal{O} \times H_2$ is*

$$\begin{aligned} \mathrm{d}s(\tau,x) : \mathcal{T} \times H_1 &\longrightarrow H_0 \\ (T,\hat{x}) &\longmapsto \partial_t \hat{x} - \mathrm{d}X(\varphi(\tau,x)) \cdot \varphi(\tau,\hat{x}) + T \cdot \mathrm{d}X(\varphi(\tau,x)) \cdot \varphi(\tau,\partial_t x). \end{aligned} \tag{7.25}$$

In particular, at $(0,x) \in (\mathcal{O} \times \mathrm{H}^1)_1$ this simplifies to

$$\mathrm{d}s(0,x)(T,\hat{x}) = \partial_t \hat{x} - \mathrm{d}X(x) \cdot \hat{x} + T \cdot \mathrm{d}X(x) \cdot \partial_t x. \tag{7.26}$$

To see that S has the sc-Fredholm property, we need to go through the conditions again.

Theorem 7.31 (cf. Theorem 4.12 and Proposition 4.13)**.** *S is a sc-Fredholm section of index* 1.

Proof. We adapt the proof of Theorem 4.12 and check all conditions for time-dependent τ, this time directly centering at a smooth point $(\tau, x) \in \mathcal{O} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n)$. The splitting of the domain is again the obvious one, that is $\mathcal{O} \times \mathrm{H}^1$, so $d = 1$ in Definition 3.21.

(i) s is regularizing: If $(\tau, x) \in (\mathcal{O} \times \mathrm{H}^1)_m = \mathcal{O} \times H_{m+1}$ with $s(\tau,x) \in H_{m+1}$, then

$$\partial_t x = s(\tau,x) + X(\varphi(\tau,x)),$$

is an element of H_{m+1} by Lemma 7.25 and by smoothness of X. Hence $(\tau, x) \in \mathcal{O} \times H_{m+2} = (\mathcal{O} \times \mathrm{H}^1)_{m+1}$ as desired.

(ii) For fixed $\tau \in \mathcal{O}$ and $m \in \mathbb{N}$, the map

$$\begin{aligned} s_{\tau,m} := s(\tau,\cdot) : H_{m+1} &\longrightarrow H_m \\ x &\longmapsto \partial_t x - X(\varphi(\tau,x)) \end{aligned}$$

is classically smooth by Lemma 7.27 and the chain rule, with differential

$$\begin{aligned} \mathrm{d}s_{\tau,m}(x) : H_{m+1} &\longrightarrow H_m \\ \hat{x} &\longmapsto \partial_t \hat{x} - \mathrm{d}X(\varphi(\tau,x)) \cdot \varphi(\tau,\hat{x}). \end{aligned} \tag{7.27}$$

The continuity properties (a) and (b) are proved in a similar way as in the proof of Theorem 4.12; we just need to be careful about referencing the right properties of the time-dependent shift map.

(a) Uniform continuity of $\mathrm{d}s_{\tau,m} : H_{m+1} \longrightarrow \mathcal{L}(H_{m+1}, H_m)$ near x: We need to show that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x, x' \in H_{m+1}$ with $\|x - x'\|_{H_{m+1}} < \delta$ and for all $\hat{x} \in H_{m+1}$ it is

$$\|\mathrm{d}s_{\tau,m}(x)\hat{x} - \mathrm{d}s_{\tau,m}(x')\hat{x}\|_{H_m} \le \varepsilon \cdot \|\hat{x}\|_{H_{m+1}}.$$

From equation (7.27) we get the following:

$$\begin{aligned}
&\|\mathrm{d}s_{\tau,m}(x)\hat{x} - \mathrm{d}s_{\tau,m}(x')\hat{x}\|_{H_m} \\
&\qquad = \|\left(\mathrm{d}X(\varphi(\tau,x')) - \mathrm{d}X(\varphi(\tau,x))\right) \cdot \varphi(\tau,\hat{x})\|_{H_m} \\
&\qquad \le \|\mathrm{d}X(\varphi(\tau,x')) - \mathrm{d}X(\varphi(\tau,x))\|_{\mathcal{C}^m(S^1,\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))} \cdot \underbrace{\|\varphi(\tau,\hat{x})\|_{H_{m+1}}}_{\le\sqrt{d(\tau)\cdot c(\tau)}\cdot\|\hat{x}\|_{H_{m+1}}}
\end{aligned}$$

The last estimate follows from the operator norm inequality for fixed $t \in S^1$, $\tau \in \mathbb{R}$ and linear maps on $\mathbb{R}^n$.

For δ small enough the first factor in this estimate is smaller than ε since $\mathrm{d}X$ is continuous and $\|x - x'\|_{H_{m+1}} < \delta$ implies $\|\varphi(\tau,x) - \varphi(\tau,x')\|_{\mathcal{C}^m} < const \cdot \|\varphi(\tau,x) - \varphi(\tau,x')\|_{H_{m+1}} < const \cdot \sqrt{d(\tau)\cdot c(\tau)} \cdot \delta$.

(b) Given sequences $(\tau_\nu, \hat{x}_\nu)_\nu \subseteq (\mathcal{O} \times \mathrm{H}^1)_m$ with $(\tau_\nu)_\nu \to \tau$ and $\|\hat{x}_\nu\|_{H_{m+1}} \le 1$ such that

$$\|\mathrm{d}s_{\tau_\nu}(x)\hat{x}_\nu\|_{H_m} \to 0,$$

we need to find a subsequence of $(\hat{x}_\nu)_\nu$ (still denoted the same way) such that

$$\|\mathrm{d}s_\tau(x)\hat{x}_\nu\|_{H_m} \to 0.$$

By compactness of the embedding $H_{m+1} \hookrightarrow H_m$ we pick a subsequence $(\hat{x}_\nu)_\nu$ converging in H_m to some $\hat{x}$, and the corresponding subsequence $(\tau_\nu)_\nu$. We add zero and use the triangle inequality as follows:

$$\begin{aligned}
\|\mathrm{d}s_\tau(x)\hat{x}_\nu\|_{H_m} &= \|\partial_t\hat{x}_\nu - \mathrm{d}X(\varphi(\tau,x)) \cdot \varphi(\tau,\hat{x}_\nu)\|_{H_m} \\
&= \|\partial_t\hat{x}_\nu - \mathrm{d}X(\varphi(\tau_\nu,x)) \cdot \varphi(\tau_\nu,\hat{x}_\nu) \\
&\qquad + \mathrm{d}X(\varphi(\tau_\nu,x)) \cdot \varphi(\tau_\nu,\hat{x}_\nu) - \mathrm{d}X(\varphi(\tau,x)) \cdot \varphi(\tau,\hat{x}_\nu)\|_{H_m} \\
&\le \underbrace{\|\partial_t\hat{x}_\nu - \mathrm{d}X(\varphi(\tau_\nu,x)) \cdot \varphi(\tau_\nu,\hat{x}_\nu)\|_{H_m}}_{\to 0 \text{ by assumption}} \\
&\quad + \|\mathrm{d}X(\varphi(\tau_\nu,x)) \cdot \varphi(\tau_\nu,\hat{x}_\nu) - \mathrm{d}X(\varphi(\tau,x)) \cdot \varphi(\tau,\hat{x}_\nu)\|_{H_m}
\end{aligned}$$

By Lemma 7.25 we have $\varphi(\tau_\nu,x) \to \varphi(\tau,x)$ in H_{m+1} (hence also in $\mathcal{C}^m$) as well as $\varphi(\tau_\nu,\hat{x}_\nu) \to \varphi(\tau,\hat{x})$ and $\varphi(\tau,\hat{x}_\nu) \to \varphi(\tau,\hat{x})$ in H_m. By continuity of $\mathrm{d}X$ it follows that

$$\|\mathrm{d}X(\varphi(\tau_\nu,x)) \cdot \varphi(\tau_\nu,\hat{x}_\nu) - \mathrm{d}X(\varphi(\tau,x)) \cdot \varphi(\tau,\hat{x}_\nu)\|_{H_m} \longrightarrow 0.$$

(iii) Since $x \in \mathcal{C}^\infty(S^1, \mathbb{R})$ is a smooth point, by (ii) there are linear maps $\mathrm{d}s_{\tau,m}(x) : H_{m+1} \to H_m$ for all $m \geq 0$. We have to show that these define a linear sc-Fredholm map

$$\mathrm{d}s_\tau(x) : \mathrm{H}^1 \longrightarrow \mathrm{H}$$

with Fredholm index not changing under small changes of τ. This can be done exactly as in the proof of Theorem 4.12, using sc-continuity of the time-dependent shift map (Lemma 7.25) and compactness of the embeddings $H_{m+1} \hookrightarrow H_m$ on and on again.

We have shown that S is a sc-Fredholm section. The statement about the index follows as in the proof of Proposition 4.13, just replacing Lemma 4.14 by the more general Lemma 7.5. □

7.4.6 Transversality and the theorem

In order to prove a statement analogous to the one from Theorem 4.1, the only remaining step is to show transversality of $s : \mathcal{O} \times H_1 \to H_0$ at a pair $(0, x_0)$, where x_0 is a non-degenerate 1-periodic orbit of X. This is immediate, though, from the results of Chapter 4.

Corollary 7.32. *If x_0 is a non-degenerate 1-periodic orbit of X, then $\mathrm{d}s(0, x_0) : \mathcal{T} \times H_1 \to H_0$ is surjective.*

Proof. Let us fix the first input to 0 and consider the map $\mathrm{d}s(0, x_0)(0, \cdot) : H_1 \to H_0, \hat{x} \mapsto \partial_t \hat{x} - \mathrm{d}X(x_0) \cdot \hat{x}$. It does not use any shift, neither time-dependent nor independent, but really equals the corresponding map from Chapter 4. Thus, it is surjective by Proposition 4.18. Hence $\mathrm{d}s(0, x_0) : \mathcal{T} \times H_1 \to H_0$ is also surjective. □

Combining all these results and using the M-polyfold implicit function theorem, we get the following generalization of our main theorem.

Theorem 7.33 (cf. Theorem 4.1)**.** *If x_0 is a non-degenerate 1-periodic orbit of X, then there is $\tau_0 > 0$ such that for every delay function $\tau \in \mathcal{O} \subset \mathcal{T}$ with $|\tau|_{\mathcal{C}^\infty} \leq \tau_0$ there exists a (locally unique) smooth 1-periodic solution x_τ of the delay equation* (7.10)*. Moreover, the parametrization $\tau \mapsto x_\tau$ is smooth.*

Remark 7.34. Theorem 7.33 holds for any choice of 1-dimensional subspace $\mathcal{T}$ of $\mathcal{C}^\infty(S^1, \mathbb{R}^n)$. It is an obvious question whether we can find solutions of (7.10) for all τ in a small ball around 0 in $\mathcal{C}^\infty(S^1, \mathbb{R}^n)$. We doubt this. Probably the example from Chapter 5 could be adjusted to find a counterexample. Such a counterexample would need to consist of a vector field X, a non-degenerate 1-periodic orbit x_0 of X, and a sequence $(\tau_\nu)_{\nu \in \mathbb{N}}$ converging to 0 in $\mathcal{C}^\infty(S^1, \mathbb{R}^n)$ such that for each delay function τ_ν there is no 1-periodic solution of (7.10) near x_0. However, in practice it is difficult to show that for a given delay there is no 1-periodic solution of (4.1), and even more difficult that for a given delay function there is no 1-periodic solution of (7.10).

Part II

Compactness for non-local gradient flow lines

Chapter 8

Compactness for non-local gradient flow lines

8.1 Context: Hamiltonian delay equations

Consider a symplectic manifold (M, ω) and and let $H : M \to \mathbb{R}$ be any smooth function, called *Hamiltonian*. By non-degeneracy of the two-form ω, the equation $\omega(X_H, \cdot) = -\mathrm{d}H$ defines a vector field X_H on M, called *Hamiltonian vector field*. Ordinary differential equations of the form $\dot{x} = X_H(x)$ arise naturally in classical mechanics.

The periodic orbits of a Hamiltonian vector field X_H on a symplectic manifold (M, ω) are the critical points of a certain action functional $\mathcal{A}_H : \mathcal{L} \to \mathbb{R}$ on the free loop space $\mathcal{L} = \mathcal{C}^\infty(S^1, M)$. This approach allows to study the set $\mathcal{P}_c$ of contractible periodic orbits by analyzing gradient flow lines on the loop space. In most situations, it is not possible to use actual Morse theory. For instance, the index and co-index of critical points are typically infinite; there are no stable and unstable manifolds; and the equation does not define a flow on the loop space. However, starting with the work of Floer, a lot of technical problems have been solved; the union of these methods is now known as Floer theory. Hamiltonian Floer homology (known as "Morse homology of the action functional") is the homology of a chain complex generated by the set $\mathcal{P}_c$ of contractible periodic orbits of a Hamiltonian vector field, with differential given by "counting" gradient flow lines from one periodic orbit to another. This way, one can, for instance, establish lower bounds on the cardinality of $\mathcal{P}_c$ depending on the topology of the underlying manifold M (cf. Arnold conjecture). A good reference is the beautiful book [AD14] by Michèle Audin and Mihai Damian. Until today, many different Floer type homologies and cohomologies have been defined, resulting in a variety of results on periodic orbits.

One key idea of Floer theory is to translate the ordinary differential equation $\partial_s u(s) = -\nabla \mathcal{A}_H(u(s))$ for maps $u : \mathbb{R} \to W^{1,2}(S^1, M)$, which defines gradient flow lines of $\mathcal{A}_H$, to a partial differential equation like the famous Floer equation

$$\partial_s u(s,t) = J\big(u(s,t)\big)\Big(X_H\big(t, u(s,t)\big) - \partial_t u(s,t)\Big) \tag{8.1}$$

for maps $u : \mathbb{R} \times S^1 \to M$ from the cylinder to our manifold M. Here, $J : TM \to TM$ is an ω-compatible almost complex structure on M. Note that (8.1) is a perturbed

Cauchy–Riemann equation, so there are many tools available to analyse its solution spaces. In particular, under suitable conditions on (M, ω), J and X_H, the moduli space of solutions of (8.1) (modulo reparametrization) with fixed asymptotic periodic orbits is a finite-dimensional manifold, and it has a natural compactification. Loosely speaking, the boundary consists of broken flow lines and J-holomorphic bubble trees.

What does all this have to do with the study of delay differential equations? There have been several approaches to define *Hamiltonian delay equations*. Albers–Frauenfelder–Schlenk [AFS20] give a nice overview of what has been done so far. Their own approach is inspired by the Floer theoretic viewpoint: The periodic solutions of a Hamiltonian delay equation should occur as critical points of an action functional on the loop space. Albers–Frauenfelder–Schlenk present plenty of delay differential equations arising in this way, including the famous Lotka–Volterra equations with delay. They also show how to use polyfold theory to prove the Arnold conjecture for Hamiltonian delay equations of a certain form on symplectically aspherical manifolds.

Of course, the analysis is much more complicated for Hamiltonian delay equations than for classical Hamiltonian equations. In particular, the gradient equation is non-local.[12] Therefore, it does not translate to a PDE on the cylinder, but rather to a DDE. Hence, there is need for a proper theory of non-local gradient flow lines. This should include a suitable analytic setup, Fredholm theory and compactness results. In the end, the goal would be to define Floer homology for Hamiltonian delay equations whose periodic orbits arise as critical points of an action functional, as in [AFS20]. An intermediate goal could be to prove an existence result for at least two periodic orbits of a Hamiltonian delay equation (with fixed period and fixed delay) using a stretching argument (as was done in the master thesis [Sei17] for the case without delay; see Appendix B).

Here we make one step towards compactness of the space of perturbed gradient flow lines of a Hamiltonian delay equation. In the current chapter, we generalize a rather abstract compactness result by Albers–Frauenfelder–Schlenk [AFS19, Theorem 2.4] to the case when the equation depends explicitly on $s \in \mathbb{R}$ and also on an additional parameter $R \in \mathbb{R}_{\geq 0}$. The theorem is formulated in the abstract setting of *unregularized vector fields* in *fractal scale Hilbert spaces*. These spaces are examples of sc-Hilbert spaces defined in polyfold theory (see Definition 3.1), so it indeed makes sense to ask for compactness inside these spaces. However, for understanding the current and the next chapter, it is not necessary to know any definitions from polyfold theory. Our compactness result (Theorem 8.6 below) differs from [AFS19, Theorem 2.4] in that the we allow the unregularized vector field $\mathcal{V}$ to depend both on $s \in \mathbb{R}$ and on $R \in \mathbb{R}_{\geq 0}$ (and that we work in $\mathbb{R}^N$ instead of $\mathbb{R}$).

In Chapter 9, we show that the case of perturbed gradient flow lines of Hamiltonian delay equations is indeed covered by the abstract result.

[12]This sometimes also happens when the original differential equation is local, for instance in Rabinowitz Floer homology (RFH).

8.2 Families of unregularized vector fields

The situation in which we want to define families of unregularized vector fields is nearly the same as in [AFS19, Section 2], namely the setting of fractal scale Hilbert spaces introduced by Frauenfelder [Fra09]. The only difference is that we allow $\mathbb{R}^n$-valued sequences, in order to match the applications.

Definition 8.1. Choose $N \in \mathbb{N}$, and let $f : \mathbb{N} \to (0, \infty)$ be a monotone increasing unbounded function. The *($\mathbb{R}^N$-valued) fractal scale Hilbert space corresponding to f* is defined as follows: Consider the Hilbert space ℓ^2_f as the vector space of all sequences $x = (x_m)_{m\in\mathbb{N}}$ of vectors $x_m \in \mathbb{R}^N$ satisfying

$$\sum_{m=1}^{\infty} f(m) \cdot \|x_m\|^2 < \infty$$

with the inner product given by

$$\langle x, y\rangle_f := \sum_{m=1}^{\infty} f(m)\langle x_m, y_m\rangle.$$

For every $k \in \mathbb{Z}$, denote

$$H_k := \ell^2_{f^k}, \qquad \langle \cdot, \cdot\rangle_{H_k} := \langle \cdot, \cdot\rangle_{f^k}. \tag{8.2}$$

The *($\mathbb{R}^N$-valued) fractal scale Hilbert space corresponding to f* is the sequence

$$\dots H_2 \subset H_1 \subset H_0 \subset H_{-1} \subset \dots$$

Note that every inclusion is dense and each H_k is dual to H_{-k} with respect to the standard inner product on $H_0 = \ell^2$. In fact, $H_0 = \ell^2$ together with the filtration $H_{k+1} \subseteq H_k \subseteq \cdots \subseteq H_0$ is a sc-Hilbert space as in Definition 3.1. See [Fra09], [Kan11] for a detailed study of fractal scale Hilbert spaces. Moreover,

$$\mathcal{F}(x) := \left(\sqrt{f(m)} \cdot x_m\right)_{m\in\mathbb{N}} \tag{8.3}$$

defines an isometric isomorphism $\mathcal{F} : H_{k+1} \to H_k$ for every $k \in \mathbb{Z}$. $\mathcal{F}$ is often called the *fundamental operator* of the fractal scale Hilbert space $(H_k)_{k\in\mathbb{Z}}$.

For the sake of completeness, let us recall the notion of a moving frame from [AFS19]:

Definition 8.2 ([AFS19, Def. 2.1])**.** A *moving frame* is a map $\Phi : H_1 \to \mathcal{L}(H_0, H_0)$ such that for every $x \in H_1$ there is a continuous bilinear map $\mathrm{D}\Phi(x) : H_0 \times H_0 \to H_0$ such that the following axioms are satisfied:

(Φ1) For every $x \in H_1$, the continuous linear map $\Phi(x)$ is an isomorphism.

(Φ2) The map $H_1 \times H_0 \to H_0, (x, v) \mapsto \Phi(x)v$ is continuous.

(Φ3) For $x \in H_1$ and $v \in H_0$, it is

$$\lim_{h \in H_1, \|h\|_{H_1} \to 0} \frac{\|\Phi(x+h)v - \Phi(x)v - \mathrm{D}\Phi(x)(h,v)\|_{H_0}}{\|h\|_{H_1}} = 0.$$

(Φ4) The map $H_1 \times H_0 \times H_0 \to H_0, (x,h,v) \mapsto \mathrm{D}\Phi(x)(h,v)$ is continuous.

(Φ5) For every $x \in H_1$, the isomorphism $\Phi(x) : H_0 \to H_0$ restricts to an isomorphism $\Phi(x) : H_1 \to H_1$ such that the maps

$$\begin{aligned} H_1 \times H_1 &\longrightarrow H_1 \\ (x,v) &\longmapsto \Phi(x)v \\ (x,v) &\longmapsto \Phi(x)^{-1}v \end{aligned}$$

are continuous.

(Φ6) For every $\kappa > 0$ there exists a constant $c_0 = c_0(\kappa)$ such that for every $x \in H_1$ with $\|x\|_{H_1} \leq \kappa$ it is

$$\begin{aligned} \|\Phi(x)\|_{\mathcal{L}(H_0,H_0) \cap \mathcal{L}(H_1,H_1)} &\leq c_0 \\ \|\Phi(x)^{-1}\|_{\mathcal{L}(H_0,H_0) \cap \mathcal{L}(H_1,H_1)} &\leq c_0 \\ \|\mathrm{D}\Phi(x)\|_{\mathrm{Bil}(H_0 \times H_0, H_0)} &\leq c_0, \end{aligned}$$

where $\|\cdot\|_{\mathrm{Bil}(H_0 \times H_0, H_0)}$ denotes the norm as a bilinear map $H_0 \times H_0 \to H_0$.

Remark 8.3. The trivial frame $\Phi \equiv \mathrm{id}_{H_0}$ is a moving frame with $\mathrm{D}\Phi \equiv 0$. We can use it in the application as long as we are only working on $(\mathbb{R}^{2n}, i)$. But if we work with an almost-complex manifold , the complex structure will depend on the point, forcing us to work with a non-trivial moving frame.

Albers–Frauenfelder–Weber defined *unregularized vector* fields $\mathcal{V}$ in [AFS19, Def. 2.2]. Here we want to work with families of such unregularized vector fields.

Definition 8.4. We define a *uniform 2-parameter family of unregularized vector fields* $\big(\mathcal{V}_{R,s}\big)_{R \in \mathbb{R}_{\geq 0}, s \in \mathbb{R}}$ as follows. For each $(R,s) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, $\mathcal{V}_{R,s}$ is a map

$$\mathcal{V}_{R,s} = \mathcal{V}_R(s,\cdot) \in \mathcal{C}^0(H_1, H_0) \cap \mathcal{C}^0(H_2, H_1)$$

such that

($\mathcal{V}$0) The maps

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{C}^0(H_1, H_0) \\ s &\longmapsto \mathcal{V}_{R,s} \end{aligned}$$

(for fixed R) and

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \mathbb{R} \times H_1 &\longrightarrow H_0 \\ (R,s,x) &\longmapsto \mathcal{V}_{R,s}(x) = \mathcal{V}_R(s,x) \end{aligned}$$

are continuous.

Moreover, at every $x \in H_1$ the map $\mathcal{V}_{R,s}$ is differentiable with differential denoted by

$$\mathrm{D}\mathcal{V}_{R,s}(x) : H_1 \longrightarrow H_0$$

and such that the following axioms are satisfied:

($\mathcal{V}1$) Each $\mathrm{D}\mathcal{V}_{R,\cdot}$ is continuous in the compact-open topology, that is, the map

$$\begin{aligned} \mathbb{R} \times H_1 \times H_1 &\longrightarrow H_0 \\ (s, x, \hat{x}) &\longmapsto D\mathcal{V}_{R,s}(x)\hat{x} \end{aligned}$$

is continuous.

($\mathcal{V}2$) There exists a moving frame Φ such that for every $R \in \mathbb{R}_{\geq 0}, s \in \mathbb{R}$ and $x \in H_1$, the map

$$\Phi(x) \circ \mathrm{D}\mathcal{V}_{R,s}(x) \circ \Phi(x)^{-1} - \mathcal{F} : H_1 \longrightarrow H_0$$

extends to a continuous linear operator

$$\mathcal{P}_{R,s}(x) : H_0 \longrightarrow H_0$$

with the property that the maps

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{L}(H_0, H_0) \\ s &\longmapsto \mathcal{P}_{R,s}(x) \end{aligned}$$

(for fixed x) and

$$\begin{aligned} H_1 \times H_0 &\longrightarrow H_0 \\ (x, \hat{x}) &\longmapsto \mathcal{P}_{R,s}(x)\hat{x} \end{aligned}$$

(for fixed s) are continuous.

($\mathcal{V}3$) For every $\kappa > 0$ there exists a constant $c_1(\kappa) > 0$ such that for every (R, s) and $x \in H_1$ with $\|x\|_{H_1} \leq \kappa$ it is

$$\|\mathcal{P}_{R,s}(x)\|_{\mathcal{L}(H_0,H_0)} \leq c_1(\kappa),$$

and if in addition $x \in H_2$, then

$$\|x\|_{H_2} \leq c_1(\kappa) \cdot (\|\mathcal{V}_{R,s}(x)\|_{H_1} + 1).$$

Moreover, concerning the s-dependence, we impose the following condition:

($\mathcal{V}4$) For every pair (R, s) and every $x \in H_1$ the limit

$$\partial_s \mathcal{V}_{R,s}(x) := \lim_{h \to 0} \frac{\mathcal{V}_{R,s+h}(x) - \mathcal{V}_{R,s}(x)}{h} \in H_0$$

exists in H_0 and this defines continuous maps

$$\begin{aligned} \partial_s \mathcal{V}_{R,s} : H_1 &\longrightarrow H_0 \\ x &\longmapsto \partial_s \mathcal{V}_{R,s}(x) \end{aligned}$$

such that $\partial_s \mathcal{V}_{R,s}(x) \in H_1$ if $x \in H_2$. Moreover, for every R we require the map

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{C}^0(H_1, H_0) \\ s &\longmapsto \partial_s \mathcal{V}_{R,s} \end{aligned}$$

to be continuous.

We also require that for every $\kappa > 0$ there is a constant $c_2(\kappa) > 0$ such that for all $x \in H_1$ with $\|x\|_{H_1} \leq \kappa$ and all pairs (R, s) it is

$$\|\partial_s \mathcal{V}_{R,s}(x)\|_{H_0} \leq c_2(\kappa).$$

Remark 8.5.

- Conditions $(\mathcal{V}1)$–$(\mathcal{V}3)$ mean that each $\mathcal{V}_{R,s}$ is an unregularized vector field as defined by Albers, Frauenfelder and Schlenk in [AFS19], with constants $c_1(\kappa)$ that can be chosen uniformly in R and s, and that the familiy satisfies some continuity properties in s.
- Note that we do not require each $\mathcal{V}_{R,s}$ to be elementary (i.e. satisfy assumption $(\mathcal{V}3')$ from [AFS19]). This is because in our main application (see Chapter 9) an estimate of the form

$$\|x\|_{H_1} \leq c_1'(\|\mathcal{V}_{R,s}(x)\|_{H_0} + 1)$$

cannot be satisfied for a pair (R, s) with $\mu_R(s) = 0$, since in this case we can find $x \in H_1$ with $\mathcal{V}_{R,s}(x) = 0$ and $\|x\|_{H_1}$ arbitrarily large.

8.3 The compactness theorem

We want to prove the following theorem:

Theorem 8.6 (s-dependent version of [AFS19, Theorem 2.4]). *Let $\mathcal{V}_{R,s}$ be a family as in Definition 8.4 and for every $\nu \in \mathbb{N}$, let*

$$w_\nu \in \mathcal{C}^0(I_T, H_1) \cap \mathcal{C}^1(I_T, H_0)$$

be a function that satisfies

$$w_\nu(s) \in H_2 \qquad \text{for almost every } s \in I_T \tag{8.4}$$

and

$$\partial_s w_\nu(s) = \mathcal{V}_{R_\nu,s}(w_\nu(s)) \tag{8.5}$$

for all $s \in I_T$ and some $R_\nu \in \mathbb{R}_{\geq 0}$. Suppose that as $\nu \to \infty$, the R_ν converge to some $R_* \in \mathbb{R}_{\geq 0}$, and suppose there is a constant κ such that

$$\|w_\nu\|_{\mathcal{C}^0(I_T,H_1)\cap\mathcal{C}^1(I_T,H_0)} \leq \kappa \tag{8.6}$$

for all $\nu \in \mathbb{N}$. Then there is a subsequence which converges in $\mathcal{C}^0(I_T, H_1) \cap \mathcal{C}^1(I_T, H_0)$ to a limit w_* which satisfies

$$\partial_s w_*(s) = \mathcal{V}_{R_*,s}(w_*(s))$$

for all $s \in I_T$.

Note that this is a only a slight generalization of [AFS19, Theorem 2.4]. Thus, everything in this chapter is along the lines of [AFS19], especially the proofs of Theorem 8.6 and Lemma 8.9 are very similar to the corresponding proofs in [AFS19].

Remark 8.7 (Additional assumption $w_\nu(s) \in H_2$, part 1)**.** The reader might notice that in comparison to [AFS19, Theorem 2.4] we make the additional assumption (8.4). This is because the statement is needed in the proof of Lemma 8.9 (and also in the proof of [AFS19, Lemma 3.1]), and we do not see that it in general follows from the other assumptions. However, in the applications which we have in mind, this assumption will hold true automatically, so it does not cause any intricacies.

Another way of solving the problem – without assuming $w_\nu(s) \in H_2$ rightaway – would be to slightly change the definition of the unregularized vector fields $\mathcal{V}_{R,s}$ in such a way that $w_\nu(s) \in H_2$ does indeed follow from the other assumptions. Again, in all applications which we have in mind, the unregularized vector fields $\mathcal{V}_{R,s}$ meets this modified definition anyway. Hence, both solutions are feasible, and it is purely a matter of taste whether one prefers the additional assumption (8.4) or the modified definition of $\mathcal{V}_{R,s}$. Here we decided for the first one simply because it is more along the lines of [AFS19]. See also Remarks 8.10 and 8.12.

8.4 Two lemmas

We want to prove Theorem 8.6 exactly as it is done for [AFS19, Theorem 2.4] for the case of (R, s)-independent unregularized vector field $\mathcal{V}$. This means establishing an (R, s)-dependent analogue of [AFS19, Lemma 3.1] and combine it with the following lemma by Albers–Frauenfelder–Schlenk.

Lemma 8.8 ([AFS19, Lemma 3.5])**.** *For each $T > 0$, $p \in \mathbb{N}_{\geq 2}$ and $l \in \mathbb{N}$, the inclusion*

$$\iota : \bigcap_{k=0}^{l} W^{k,p}(I_T, H_{l-k}) \longrightarrow \bigcap_{k=0}^{l-1} C^k(I_T, H_{l-1-k})$$

is a compact operator.

(In [AFS19], this lemma is stated and proven in the setting of $\mathbb{R}$-valued fractal scale Hilbert spaces; but the proof works verbatim the same for $\mathbb{R}^N$-valued fractal scale Hilbert spaces.)

Lemma 8.9 (cf. [AFS19, Lemma 3.1])**.** *Let $\mathcal{V}_{R,s}$ be a family as in Definition 8.4, and fix numbers $\kappa \in \mathbb{R}$ and $0 < T' < T$. Then there is a constant $c = c(\kappa, T)$ such that the following holds: For every $R \in \mathbb{R}_{\geq 0}$ and every function $w \in \mathcal{C}^0(I_T, H_1) \cap \mathcal{C}^1(I_T, H_0)$ satisfying*

$$w(s) \in H_2 \qquad \text{for almost every } s \in I_T \tag{8.7}$$

and

$$\partial_s w(s) = \mathcal{V}_{R,s}(w(s)) \qquad \text{for all } s \in I_T$$

and

$$\|w\|_{\mathcal{C}^0(I_T,H_1)\cap\mathcal{C}^1(I_T,H_0)} \leq \kappa, \tag{8.8}$$

we have $w \in \bigcap_{k=0}^{2} W^{k,2}(I_{T'}, H_{2-k})$ with

$$\|w\|_{\bigcap_{k=0}^{2} W^{k,2}(I_{T'},H_{2-k})} \leq c. \tag{8.9}$$

Remark 8.10 (Additional assumption $w_\nu(s) \in H_2$, part 2)**.** The assumption (8.7) is new compared to [AFS19, Lemma 3.1]. See also Remarks 8.7 and 8.12.

Proof of Lemma 8.9. Denote by Φ the moving frame for the family $(\mathcal{V}_{R,s})_{R,s}$ that exists by assumption ($\mathcal{V}2$) in Definition 8.4. We define

$$\xi := \Phi(w)\partial_s w = \Phi(w)\mathcal{V}_{R,\cdot}(w) \in \mathcal{C}^0(I_T, H_0) \tag{8.10}$$

and observe that, by (8.8) and ($\Phi 6$), it is

$$\|\xi\|_{\mathcal{C}^0(I_T,H_0)} \leq c_0(\kappa) \cdot \kappa. \tag{8.11}$$

Claim A (cf. [AFS19, Claim 3.2]). *If we understand ξ as a map from $I_{T'} \subset I_T$ to $H_{-1} \supset H_0$, then it is $\xi \in \mathcal{C}^1(I_{T'}, H_{-1})$ with derivative given by*

$$\partial_s \xi = \mathrm{D}\Phi(w)\big(\Phi(w)^{-1}\xi, \Phi(w)^{-1}\xi\big) + \Phi(w)\big(\partial_s \mathcal{V}_{R,\cdot}(w)\big) + \mathcal{P}(w)\xi + \mathcal{F}\xi. \tag{8.12}$$

To prove this claim, let us approximate w by smooth functions. As in [AFS19], we write $\varepsilon := \frac{T-T'}{2}$ and choose a smooth cutoff function $\beta \in \mathcal{C}^\infty(I_T, [0,1])$ with

$$\beta(s) = \begin{cases} 1 & \text{if } s \in \overline{I_{T'}} \\ 0 & \text{if } s \in I_T \setminus I_{T-\varepsilon} \end{cases}$$

and a bunp function $\rho \in \mathcal{C}^\infty(\mathbb{R}, [0,\infty))$ with

$$\rho(\sigma) = 0 \text{ for } |\sigma| \geq 1 \qquad \text{and} \qquad \int_{-1}^{1} \rho(\sigma)\, \mathrm{d}\sigma = 1.$$

Still as in [AFS19], for every $\delta > 0$ we set $\rho_\delta(s) := \frac{1}{\delta}\rho\left(\frac{s}{\delta}\right)$ and abbreviate

$$w_\nu := \rho_{\varepsilon/\nu} * (\beta w) \in \mathcal{C}^\infty(I_T, H_1). \tag{8.13}$$

Then each w_ν has compact support in I_T. Moreover, note that the sequence $(w_\nu|_{I_{T'}})_{\nu\in\mathbb{N}}$ converges to w in $\mathcal{C}^0(I_{T'}, H_1) \cap \mathcal{C}^1(I_{T'}, H_0)$. We set

$$\xi_\nu := \Phi(w_\nu)\mathcal{V}_{R,\cdot}(w_\nu) \in \mathcal{C}^0(I_T, H_0). \tag{8.14}$$

Since $\partial_s w_\nu(s) \in H_1$ for all $s \in I_{T'}$, we can use the chain rule and properties $(\Phi 2) - (\Phi 4)$ and $(\mathcal{V}1), (\mathcal{V}4)$ to see that $\xi_\nu \in \mathcal{C}^1(I_T, H_0)$ with derivative

$$\partial_s \xi_\nu = \mathrm{D}\Phi(w_\nu)\Big(\partial_s w_\nu, \mathcal{V}_{R,\cdot}(w_\nu)\Big) + \Phi(w_\nu)\Big(\partial_s \mathcal{V}_{R,\cdot}(w_\nu) + \mathrm{D}\mathcal{V}_{R,\cdot}(w_\nu)\partial_s w_\nu\Big). \tag{8.15}$$

Recall that by $(\mathcal{V}2)$ it is $\Phi(x)\mathrm{D}\mathcal{V}_{R,s}(x) = \big(\mathcal{P}_{R,s}(x) + \mathcal{F}x\big)\Phi(x)$ for all $x \in H_1$. We use this to rewrite (8.15) as

$$\begin{aligned}\partial_s \xi_\nu = {} & \mathrm{D}\Phi(w_\nu)\big(\partial_s w_\nu, \mathcal{V}_{R,\cdot}(w_\nu)\big) + \Phi(w_\nu)\big(\partial_s \mathcal{V}_{R,\cdot}(w_\nu)\big)\\ & + \mathcal{P}_{R,\cdot}(w_\nu)\Phi(w_\nu)\partial_s w_\nu + \mathcal{F}\Phi(w_\nu)\partial_s w_\nu \quad \in \mathcal{C}^0(I_{T'}, H_0).\end{aligned}$$

Claim B. *For $\nu \to \infty$, the restrictions $\partial_s \xi_\nu|_{I_{T'}}$ converge in $\mathcal{C}^0(I_{T'}, H_{-1})$ to the map*

$$\begin{aligned}\eta := {} & \mathrm{D}\Phi(w)\big(\partial_s w, \mathcal{V}_{R,\cdot}(w)\big) + \Phi(w)\big(\partial_s \mathcal{V}_{R,\cdot}(w)\big)\\ & + \mathcal{P}_{R,\cdot}(w)\Phi(w)\partial_s w + \mathcal{F}\Phi(w)\partial_s w \quad \in \mathcal{C}^0(I_{T'}, H_{-1}).\end{aligned}$$

Indeed, since $w_\nu|_{I_{T'}} \to w|_{I_{T'}}$ in $\mathcal{C}^0(I_{T'}, H_1) \cap \mathcal{C}^1(I_{T'}, H_0)$, it is

- $\mathcal{V}_{R,\cdot}(w_\nu) \to \mathcal{V}_{R,\cdot}(w)$ in $\mathcal{C}^0(I_{T'}, H_0)$ by continuity of $\mathcal{V}$,
- $\partial_s \mathcal{V}_{R,\cdot}(w_\nu) \to \partial_s \mathcal{V}_{R,\cdot}(w)$ in $\mathcal{C}^0(I_{T'}, H_0)$ by $(\mathcal{V}4)$, and
- $\Phi(w_\nu)\partial_s w_\nu \to \Phi(w)\partial_s w$ in $\mathcal{C}^0(I_{T'}, H_0)$ by $(\Phi 2)$.

So convergence of the first term follows from $(\Phi 4)$, convergence of the second term follows from $(\Phi 2)$ and convergence of the third term follows from $(\mathcal{V}2)$, all in $\mathcal{C}^0(I_{T'}, H_0)$. The fourth term converges in $\mathcal{C}^0(I_{T'}, H_{-1})$ since $\mathcal{F} : H_0 \to H_{-1}$ is an isometric isomorphism. Together, this proves Claim B.

Let us resume the proof of Claim A. In view of Claim B, the map η is our candidate for the derivative of ξ. Note that H_1 is, with respect to the inner product $\langle \cdot, \cdot \rangle_{H_0}$, the dual space of H_{-1}. For every compactly supported test function $\phi \in C^\infty(I_{T'}, H_1)$ we compute

$$\begin{aligned}\int_{I_{T'}} \langle \eta(s), \phi(s)\rangle_{H_0}\,\mathrm{d}s &= \lim_{\nu\to\infty} \int_{I_{T'}} \langle \partial_s \xi_\nu(s), \phi(s)\rangle_{H_0}\,\mathrm{d}s\\ &= -\lim_{\nu\to\infty} \int_{I_{T'}} \langle \xi_\nu(s), \partial_s \phi(s)\rangle_{H_0}\,\mathrm{d}s\\ &= -\int_{I_{T'}} \langle \xi(s), \partial_s \phi(s)\rangle_{H_0}\,\mathrm{d}s.\end{aligned}$$

Thus, η is a weak derivative of ξ. But we already know that $\eta \in \mathcal{C}^0(I_{T'}, H_{-1})$ is continuous, and so it is the derivative $\partial_s \xi := \eta$. Using the definition of ξ in (8.10) we see that we have proved Claim A.

Next we want to see that ξ and $\partial_s\xi$ are in fact even more regular than we knew.

Claim C (cf. [AFS19, Claim 3.3]). *It is* $\xi \in L^2(I_{T'}, H_1) \cap W^{1,2}(I_{T'}, H_0)$ *with*

$$\|\partial_s\xi\|_{L^2(I_{T'},H_0)} \leq c_3(T,\kappa) \qquad \text{and} \qquad \|\xi\|_{L^2(I_{T'},H_1)} \leq c_3(T,\kappa),$$

where $c_3(T,\kappa) > 0$ *is a constant only depending on* T, κ *and the family* $\mathcal{V}_{R,s}$*, but not on* w *or* R.

To prove Claim C, again we need some approximations. For every $\delta > 0$ set

$$\xi_\delta^\beta := \rho_\delta * (\beta\xi) \in C^\infty(I_T, H_0)$$

and note that these functions are compactly supported inside $I_{T'} \subset I_T$. From (8.12) and since $\mathcal{F}$ is invertible we get that

$$\xi = \mathcal{F}^{-1}\Big(\partial_s\xi - \mathrm{D}\Phi(w)\big(\Phi(w)^{-1}\xi, \Phi(w)^{-1}\xi\big) - \Phi(w)\big(\partial_s\mathcal{V}_{R,\cdot}(w)\big) - \mathcal{P}_{R,\cdot}(w)\xi\Big), \quad (8.16)$$

and so with

$$\rho_\delta * (\mathcal{F}^{-1}\beta\partial_s\xi) = (\partial_s\rho_\delta) * (\mathcal{F}^{-1}\beta\xi) - \rho_\delta * (\mathcal{F}^{-1}(\partial_s\beta)\xi) \qquad (8.17)$$

we can compute

$$\begin{aligned}
\xi_\delta^\beta &= \rho_\delta * (\beta\xi)\\
&= \rho_\delta * \Big(\beta\mathcal{F}^{-1}\Big(\partial_s\xi - \mathrm{D}\Phi(w)\big(\Phi(w)^{-1}\xi, \Phi(w)^{-1}\xi\big) - \Phi(w)\big(\partial_s\mathcal{V}_{R,\cdot}(w)\big) - \mathcal{P}_{R,\cdot}(w)\xi\Big)\Big)\\
&= \rho_\delta * \bigg(\mathcal{F}^{-1}\Big(\beta\partial_s\xi - \mathrm{D}\Phi(w)\big(\Phi(w)^{-1}\xi, \Phi(w)^{-1}\beta\xi\big)\\
&\qquad\qquad - \beta\Phi(w)\big(\partial_s\mathcal{V}_{R,\cdot}(w)\big) - \mathcal{P}_{R,\cdot}(w)\beta\xi\Big)\bigg)\\
&\overset{(8.17)}{=} (\partial_s\rho_\delta) * \big(\mathcal{F}^{-1}\beta\xi\big) - \rho_\delta * \big(\mathcal{F}^{-1}(\partial_s\beta)\xi\big)\\
&\quad - \rho_\delta * \mathcal{F}^{-1}\underbrace{\Big(\mathrm{D}\Phi(w)\big(\Phi(w)^{-1}\xi, \Phi(w)^{-1}\beta\xi\big) + \beta\Phi(w)\big(\partial_s\mathcal{V}_{R,\cdot}(w)\big) + \mathcal{P}_{R,\cdot}(w)\beta\xi\Big)}_{=:\,\text{rest}}\\
&= (\partial_s\rho_\delta) * \Big(\mathcal{F}^{-1}\underbrace{\beta\xi}_{\in\, \mathcal{C}^0(I_T,H_0)}\Big) - \rho_\delta * \Big(\mathcal{F}^{-1}\underbrace{\big((\partial_s\beta)\xi + \text{rest}\big)}_{\in\, \mathcal{C}^0(I_T,H_0) \text{ by } (\Phi 4),\ (\mathcal{V}4) \text{ and } (\mathcal{V}2)}\Big). \qquad (8.18)
\end{aligned}$$

Since $\mathcal{F}^{-1} : H_0 \to H_1$ is an isometric isomorphism, from equation (8.18) we see that $\xi_\delta^\beta \in C^\infty(I_T, H_1)$. Now we compute

$$\begin{aligned}
\partial_s\xi_\delta^\beta - \mathcal{F}\xi_\delta^\beta &= \partial_s\xi_\delta^\beta - \mathcal{F}\Big((\partial_s\rho_\delta) * \big(\mathcal{F}^{-1}\beta\xi\big)\Big) + \mathcal{F}\Big(\rho_\delta * \big(\mathcal{F}^{-1}\big((\partial_s\beta)\xi + \text{rest}\big)\big)\Big)\\
&= \partial_s\xi_\delta^\beta - \partial_s\xi_\delta^\beta + \rho_\delta * \big((\partial_s\beta)\xi + \text{rest}\big)\\
&= \rho_\delta * \big((\partial_s\beta)\xi + \text{rest}\big). \qquad (8.19)
\end{aligned}$$

By Young's inequality, it is

$$\|\rho_\delta * (\dots)\|_{L^2(I_T,H_0)} \leq \|\rho_\delta\|_{L^1(I_T,\mathbb{R})} \cdot \|\dots\|_{L^2(I_T,H_0)},$$

and we have

$$\|\rho_\delta\|_{L^1(I_T,\mathbb{R})} = \int_{-T}^{T} \rho_\delta(s)\ \mathrm{d}s = \int_{-T}^{T} \tfrac{1}{\delta}\rho(\tfrac{s}{\delta})\ \mathrm{d}s = \int_{-T/\delta}^{T/\delta} \rho(\sigma)\ \mathrm{d}\sigma = 1.$$

Hence we can use (8.19) to estimate

$$\begin{aligned}
\|\partial_s\xi_\delta^\beta - \mathcal{F}\xi_\delta^\beta\|_{L^2(I_T,H_0)} &= \big\|\rho_\delta * \big((\partial_s\beta)\xi + \mathrm{rest}\big)\big\|_{L^2(I_T,H_0)} \\
&\le \|(\partial_s\beta)\xi + \mathrm{rest}\|_{L^2(I_T,H_0)} \\
&\overset{(8.11)}{\le} \|\partial_s\beta\|_{L^2(I_T,\mathbb{R})}\cdot c_0(\kappa)\cdot\kappa + \|\ \mathrm{rest}\ \|_{L^2(I_T,H_0)} \\
&\le \|\partial_s\beta\|_{L^2(I_T,\mathbb{R})}\cdot c_0(\kappa)\cdot\kappa \\
&\quad + \|\mathrm{D}\Phi(w)\big(\Phi(w)^{-1}\xi, \Phi(w)^{-1}\beta\xi\big)\|_{L^2(I_T,H_0)} \\
&\quad + \|\beta\Phi(w)\big(\partial_s\mathcal{V}_{R,\cdot}(w)\big)\|_{L^2(I_T,H_0)} \\
&\quad + \|\mathcal{P}_{R,\cdot}(w)\beta\xi\|_{L^2(I_T,H_0)} \\
&\le \|\partial_s\beta\|_{L^2(I_T,\mathbb{R})}\cdot c_0(\kappa)\cdot\kappa + \sqrt{2T}\cdot c_0(\kappa)\cdot\big(c_0(\kappa)\cdot c_0(\kappa)\cdot\kappa\big)^2 \\
&\quad + c_0(\kappa)\cdot c_2(\kappa) + \sqrt{2T}\cdot c_1(\kappa)\cdot c_0(\kappa)\cdot\kappa \\
&=: c_3(T,\kappa).
\end{aligned} \tag{8.20}$$

Here we used (8.11) and the constants κ from (8.8), $c_0(\kappa)$ from (Φ6), $c_1(\kappa)$ from ($\mathcal{V}$3), and $c_2(\kappa)$ from ($\mathcal{V}$4). Note that $c_3(T,\kappa)$ does not depend on δ.

Recall that $\mathcal{F}$ is self-adjoint and that ξ_δ^β has compact support. We use integration by parts to compute

$$\begin{aligned}
\int_{-T}^{T}\langle\partial_s\xi_\delta^\beta, \mathcal{F}\xi_\delta^\beta\rangle_{H_0}\ \mathrm{d}s &= -\int_{-T}^{T}\langle\xi_\delta^\beta, \mathcal{F}\partial_s\xi_\delta^\beta\rangle_{H_0}\ \mathrm{d}s \\
&= -\int_{-T}^{T}\langle\mathcal{F}\xi_\delta^\beta, \partial_s\xi_\delta^\beta\rangle_{H_0}\ \mathrm{d}s \\
&= -\int_{-T}^{T}\langle\partial_s\xi_\delta^\beta, \mathcal{F}\xi_\delta^\beta\rangle_{H_0}\ \mathrm{d}s,
\end{aligned}$$

which implies that

$$\int_{-T}^{T}\langle\partial_s\xi_\delta^\beta, \mathcal{F}\xi_\delta^\beta\rangle\ \mathrm{d}s = 0.$$

Therefore,

$$\begin{aligned}
\|\partial_s\xi_\delta^\beta - \mathcal{F}\xi_\delta^\beta\|_{L^2(I_T,H_0)} &= \int_{-T}^{T}\langle\partial_s\xi_\delta^\beta - \mathcal{F}\xi_\delta^\beta, \partial_s\xi_\delta^\beta - \mathcal{F}\xi_\delta^\beta\rangle\ \mathrm{d}s \\
&= \int_{-T}^{T}\langle\partial_s\xi_\delta^\beta, \partial_s\xi_\delta^\beta\rangle_{H_0}\ \mathrm{d}s + \int_{-T}^{T}\langle\mathcal{F}\xi_\delta^\beta, \mathcal{F}\xi_\delta^\beta\rangle_{H_0}\ \mathrm{d}s \\
&\qquad - 2\int_{-T}^{T}\langle\partial_s\xi_\delta^\beta, \mathcal{F}\xi_\delta^\beta\rangle_{H_0}\ \mathrm{d}s \\
&= \|\partial_s\xi_\delta^\beta\|_{L^2(I_T,H_0)} + \|\mathcal{F}\xi_\delta^\beta\|_{L^2(I_T,H_0)} \\
&= \|\partial_s\xi_\delta^\beta\|_{L^2(I_T,H_0)} + \|\xi_\delta^\beta\|_{L^2(I_T,H_1)}.
\end{aligned} \tag{8.21}$$

Now from (8.21) together with (8.20) we can deduce the two estimates

$$\|\partial_s \xi_\delta^\beta\|_{L^2(I_T,H_0)} \leq c_3(T,\kappa) \qquad \text{and} \qquad \|\xi_\delta^\beta\|_{L^2(I_T,H_1)} \leq c_3(T,\kappa) \tag{8.22}$$

Note that these estimates are independent of δ, and so, as $\delta \to 0$, there is a subsequence of the family $(\xi_\delta^\beta)_{\delta>0}$ that converges weakly in $L^2(I_T,H_1) \cap W^{1,2}(I_T,H_0)$ to some ξ_0^β. But we already know that $\xi_\delta^\beta \to \beta\xi$ strongly in $L^2(I_T,H_0)$, and so

$$\beta\xi = \xi_0^\beta \in L^2(I_T,H_1) \cap W^{1,2}(I_T,H_0).$$

This means in particular that $\beta\xi$ has a weak derivative which we denote by $\partial_s(\beta\xi)$. Taking the limit of (8.22) we see that

$$\|\partial_s(\beta\xi)\|_{L^2(I_T,H_0)} \leq c_3(T,\kappa) \qquad \text{and} \qquad \|\beta\xi\|_{L^2(I_T,H_1)} \leq c_3(T,\kappa). \tag{8.23}$$

Now on the interval $I_{T'}$ the functions ξ and $\beta\xi$ coincide, so we get that

$$\xi \in L^2(I_{T'},H_1) \cap W^{1,2}(I_{T'},H_0).$$

with

$$\|\partial_s\xi\|_{L^2(I_{T'},H_0)} \leq c_3 \qquad \text{and} \qquad \|\xi\|_{L^2(I_{T'},H_1)} \leq c_3.$$

This finishes the proof of Claim C.

Recall that $\xi = \Phi(w)\partial_s w$ and that in the end we want to show that w lies in $\bigcap_{k=0}^2 W^{k,2}(I_{T'},H_{2-k})$ and that it satisfies the estimate (8.9) for some constant c which only depends on κ and T.

Claim D (cf. [AFS19, Claim 3.4]). *For every $0 < T' < T$, the function $\partial_s w = \Phi(w)^{-1}\xi$ has a weak derivative in $L^2(I_{T'},H_0)$ which is given by*

$$\partial_s^2 w = \Phi(w)^{-1}\partial_s\xi - \Phi(w)^{-1}\mathrm{D}\Phi(w)(\partial_s w,\partial_s w). \tag{8.24}$$

As in the proof of Claim A, we will approximate the function by smooth functions to compute a candidate for a weak derivative, and then verify that this candidate really behaves well with test functions.

As before, we use the functions $w_\nu \in \mathcal{C}^\infty(I_T,H_1)$ defined in (8.13), but this time we do not work with ξ_ν as before, but with

$$\zeta_\nu := \Phi(w_\nu)\partial_s w_\nu \in \mathcal{C}^1(I_T,H_0).$$

Recall (from the proof of Claim B) that $\zeta_\nu = \Phi(w_\nu)\partial_s w_\nu$ converges to $\Phi(w)\partial_s w = \xi$ in $\mathcal{C}^0(I_{T'},H_0)$, in particular pointwise. By (Φ3) and (Φ4), the map

$$s \longmapsto \Phi(w_\nu(s)) \in \mathcal{L}(H_0,H_0)$$

is differentiable, and so $\partial_s w_\nu = \Phi(w_\nu)^{-1}\zeta_\nu$ is differentiable with derivative

$$\begin{aligned}
\partial_s^2 w_\nu &= \partial_s\big(\Phi(w_\nu)^{-1}\zeta_\nu\big)\\
&= \Phi(w_\nu)^{-1}\partial_s\zeta_\nu + \partial_s\big(\Phi(w_\nu)^{-1}\big)\zeta_\nu\\
&= \Phi(w_\nu)^{-1}\partial_s\zeta_\nu - \Big(\Phi(w_\nu)^{-1}\circ\partial_s\big(\Phi(w_\nu)\big)\circ\Phi(w_\nu)^{-1}\Big)\zeta_\nu\\
&= \Phi(w_\nu)^{-1}\partial_s\zeta_\nu - \Phi(w_\nu)^{-1}\circ\Big(\partial_s\big(\Phi(w_\nu)\big)\Big)\partial_s w_\nu\\
&= \Phi(w_\nu)^{-1}\partial_s\zeta_\nu - \Phi(w_\nu)^{-1}\circ\mathrm{D}\Phi(w_\nu)\big(\partial_s w_\nu,\partial_s w_\nu\big) \quad \in \mathcal{C}^0(I_T,H_0).
\end{aligned} \tag{8.25}$$

From this we see that if $\partial_s w$ has a weak derivative, then it needs to be given by (8.24). To verify that (after restriction to $I_{T'} \subset I_T$) this is indeed the case, pick any compactly supported test function $\phi \in \mathcal{C}^\infty(I_{T'}, H_0)$. We claim that

$$\begin{aligned}
\int_{I_{T'}} &\langle \partial_s w, \partial_s \phi\rangle_{H_0} \,\mathrm{d}s \\
&= \int_{I_{T'}} \left\langle \Phi(w)^{-1}\xi, \partial_s\phi\right\rangle_{H_0} \mathrm{d}s \\
&\overset{\text{(i)}}{=} \lim_{\nu\to\infty} \int_{I_{T'}} \left\langle \Phi(w_\nu)^{-1}\zeta_\nu, \partial_s\phi\right\rangle_{H_0} \mathrm{d}s \\
&\overset{\text{(ii)}}{=} -\lim_{\nu\to\infty} \int_{I_{T'}} \Big\langle \Phi(w_\nu)^{-1}\partial_s\zeta_\nu - \Phi(w_\nu)^{-1} \circ \mathrm{D}\Phi(w_\nu)\big(\partial_s w_\nu, \partial_s w_\nu\big), \phi\Big\rangle_{H_0} \mathrm{d}s \\
&\overset{\text{(iii)}}{=} -\int_{I_{T'}} \Big\langle \Phi(w)^{-1}\partial_s\xi - \Phi(w)^{-1} \circ \mathrm{D}\Phi(w)\big(\partial_s w, \partial_s w\big), \phi\Big\rangle_{H_0} \mathrm{d}s.
\end{aligned}$$

Indeed:

(i) We know that $\zeta_\nu \to \xi$ pointwise and that $w_\nu \to w$ in $\mathcal{C}^0(I_{T',H_1})$. With ($\Phi$2) it follows that $\Phi(w_\nu)^{-1}\zeta_\nu \to \Phi(w)^{-1}\xi$ pointwise. Moreover, since $\|w_{nu}(s)\|_{H_1} \leq \kappa$ for all $s \in I_{T'}$, and so by (Φ6) the functions $\left\langle \Phi(w_\nu)^{-1}\zeta_\nu, \partial_s\phi\right\rangle_{H_0}$ are (for large enough $\nu \in \mathbb{N}$) bounded by some constant depending on ξ, ϕ and $c_0(\kappa)$. So this equality follows from the dominated convergence theorem.

(ii) This equality follows by partial integration using the formula for $\partial_s(\Phi(w_\nu)^{-1}\zeta_\nu)$ in (8.25), since all functions are compactly supported inside $I_{T'}$.

(iii) For this equality, let us consider the two terms on the left side of the inner product separately. The first identity that needs to be shown,

$$\lim_{\nu\to\infty} \int_{I_{T'}} \Big\langle \Phi(w_\nu)^{-1}\partial_s\zeta_\nu, \phi\Big\rangle_{H_0} \mathrm{d}s = \int_{I_{T'}} \Big\langle \Phi(w)^{-1}\partial_s\xi, \phi\Big\rangle_{H_0} \mathrm{d}s,$$

is equivalent to

$$\lim_{\nu\to\infty} \int_{I_{T'}} \Big\langle \zeta_\nu, \partial_s\Big(\big(\Phi(w_\nu)^{-1}\big)^T\phi\Big)\Big\rangle_{H_0} \mathrm{d}s = \int_{I_{T'}} \Big\langle \xi, \partial_s\Big(\big(\Phi(w)^{-1}\big)^T\phi\Big)\Big\rangle_{H_0} \mathrm{d}s$$

and this follows – as in (i) – from the dominated convergence theorem, since we already know pointwise convergence and get uniform bounds from $\|w_{nu}(s)\|_{H_1} \leq \kappa$ together with (Φ6). The second one is

$$\begin{aligned}
\lim_{\nu\to\infty} \int_{I_{T'}} &\Big\langle \Phi(w_\nu)^{-1} \circ \mathrm{D}\Phi(w_\nu)\big(\partial_s w_\nu, \partial_s w_\nu\big), \phi\Big\rangle_{H_0} \mathrm{d}s \\
&= \int_{I_{T'}} \Big\langle \Phi(w)^{-1} \circ \mathrm{D}\Phi(w)\big(\partial_s w, \partial_s w\big), \phi\Big\rangle_{H_0} \mathrm{d}s
\end{aligned}$$

and again follows from pointwise convergence, the bounds from (Φ6), and the dominated convergence theorem.

This finishes the proof of Claim D.
It remains to show the estimate (8.9). We start by estimating

$$\begin{aligned}
\|\partial_s^2 w\|_{L^2(I_{T'},H_0)} &\overset{\text{Claim D}}{\leq} \|\Phi(w)^{-1}\partial_s\xi\|_{L^2(I_{T'},H_0)} + \|\Phi(w)^{-1}\mathrm{D}\Phi(w)(\partial_s w,\partial_s w)\|_{L^2(I_{T'},H_0)} \\
&\overset{(8.8),(\Phi6)}{\leq} c_0(\kappa)\cdot\|\partial_s\xi\|_{L^2(I_{T'},H_0)} + \sqrt{2T}\cdot c_0(\kappa)^2\cdot\kappa^2 \\
&\overset{\text{Claim C}}{\leq} c_0(\kappa)\cdot c_3(T,\kappa) + \sqrt{2T}\cdot c_0(\kappa)^2\cdot\kappa^2. \qquad (8.26)
\end{aligned}$$

From this we get

$$\begin{aligned}
\|w\|^2_{W^{2,2}(I_{T'},H_0)} &= \|\partial_s^2 w\|^2_{L^2(I_{T'},H_0)} + \|w\|^2_{W^{1,2}(I_{T'},H_0)} \\
&\leq \|\partial_s^2 w\|^2_{L^2(I_{T'},H_0)} + 2T\|w\|^2_{\mathcal{C}^1(I_{T'},H_0)} \\
&\overset{(8.8),(8.26)}{\leq} \left(c_0(\kappa)\cdot c_3(T,\kappa) + \sqrt{2T}\cdot c_0(\kappa)^2\cdot\kappa^2\right)^2 + 2T\kappa^2 \\
&=: c_4(T,\kappa).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|\partial_s w\|_{L^2(I_{T'},H_1)} &\overset{(8.10)}{=} \|\Phi(w)^{-1}\xi\|_{L^2(I_{T'},H_1)} \\
&\overset{(\Phi6),\ \text{Claim C}}{\leq} c_0(\kappa)\cdot c_3(T,\kappa) \qquad (8.27)
\end{aligned}$$

and so

$$\begin{aligned}
\|w\|^2_{W^{1,2}(I_{T'},H_1)} &= \|\partial_s w\|^2_{L^2(I_{T'},H_1)} + \|w\|^2_{L^2(I_{T'},H_1)} \\
&\overset{(8.27),(8.8)}{\leq} \left(c_0(\kappa)\cdot c_3(T,\kappa)\right)^2 + 2T\kappa^2 \\
&=: c_5(T,\kappa).
\end{aligned}$$

Finally, using assumption (8.7) from this lemma and assumption ($\mathcal{V}3$) from Definition 8.4,

$$\begin{aligned}
\|w\|_{L^2(I_{T'},H_2)} &= \left(\int_{I_{T'}} \|w(s)\|^2_{H_2}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\overset{(8.8),(\mathcal{V}3)}{\leq} \sqrt{2T'}c_1(\kappa)\left(\|\mathcal{V}_{R,\cdot}(w)\|_{L^2(I_{T'},H_1)} + 1\right) \\
&= \sqrt{2T'}c_1(\kappa)\left(\|\partial_s w\|_{L^2(I_{T'},H_1)} + 1\right) \\
&\leq \sqrt{2T}c_1(\kappa)\left(\|w\|_{W^{1,2}(I_{T'},H_1)} + 1\right) \\
&\leq \sqrt{2T}c_1(\kappa)\left(c_5(T,\kappa) + 1\right) \\
&=: c_6(T,\kappa).
\end{aligned}$$

So with

$$c := \max\{c_4(T,\kappa),\ c_5(T,\kappa),\ c_6(T,\kappa)\}$$

we have verified (8.9). □

Remark 8.11. If the moving frame Φ for the family $\mathcal{V}_{R,s}$ is trivial (that is, $\Phi \equiv \mathrm{id}_{H_0}$), then the bound (8.8) in the assumption of Lemma 8.9 can be weakened to

$$\|w\|_{L^2(I_T,H_1)\cap W^{1,2}(I_T,H_0)} \leq \kappa.$$

Remark 8.12 (Additional assumption $w_\nu(s) \in H_2$, part 3)**.** We comment again on assumption (8.7), see also Remarks 8.7 and 8.10.

Note that (8.7) was only used in the very end of the proof of Lemma 8.9, where we needed $w(s) \in H_2$ for almost every $s \in I_{T'}$ in order to use ($\mathcal{V}$) and to estimate $\|w\|_{L^2(I_{T'},H_2)}$. It is tempting to say that $w \in W^{2,2}(I_{T'}, H_0) \cap W^{1,2}(I_{T'}, H_1)$ implies $w(s) \in H_2$ for almost every $s \in I_{T'}$. However, we do not see how this should follow with the given definition of the unregularized vector fields $\mathcal{V}_{R,s}$, so we decided to make the additional assumption.

Another possibility to solve the problem would be to change the definition of the unregularized vector fields $\mathcal{V}_{R,s}$ in the following way.

Recall that condition ($\mathcal{V}2$) in Definition 8.4 requires that after conjugation with a moving frame Φ, the linearization $\mathrm{d}\mathcal{V}_{R,s}(x) : H_1 \to H_0$ at $x \in H_1$ splits as the sum of the fundamental operator $\mathcal{F} : H_1 \to H_0$ and something which extends to a continuous linear operator $\mathcal{P}_{R,s}(x) : H_0 \to H_0$ (with certain continuity properties),

$$\Phi(x) \circ \mathrm{D}\mathcal{V}_{R,s}(x) \circ \Phi(x)^{-1} = \mathcal{P}_{R,s}(x) + \mathcal{F}. \tag{8.28}$$

This made it possible to write $\xi = \Phi(w)\partial_s w = \Phi(w)\mathcal{V}_{R,\cdot}(w)$ as the preimage of an element of H_0 under $\mathcal{F}$, see (8.16). That way we could show that $\partial_s w$, which was originally an element of $\mathcal{C}^0(I_T, H_0)$, lies in $L^2(I_{T'}, H_1)$. Let us now for a moment assume that the moving frame Φ is trivial, and modify condition ($\mathcal{V}2$). Instead of the splitting above, we assume that $\mathcal{V}_{R,s} : H_1 \to H_0$ itself splits as the sum of $\mathcal{F} : H_1 \to H_0$ and something which extends to a (non-linear) map $\mathcal{Q}_{R,s} : H_0 \to H_0$ (with certain continuity properties),

$$\mathcal{V}_{R,s} = \mathcal{Q}_{R,s} + \mathcal{F}. \tag{8.29}$$

Then we can see that

$$w = \mathcal{F}^{-1}\big(\partial_s w - \mathcal{Q}(w)\big)$$

takes values in H_2. Moreover, when we linearize (8.29) at $x \in H_1$, we find

$$\mathrm{d}\mathcal{V}_{R,s}(x) = \mathrm{d}\mathcal{Q}_{R,s}(x) + \mathcal{F}.$$

Hence, with $\mathcal{P}_{R,s}(s) = \mathrm{d}\mathcal{Q}_{R,s}(x)$ we get back the splitting (8.28) from the original version of ($\mathcal{V}2$), and the conditions on $\mathcal{P}_{R,s}$ can be written in terms of $\mathcal{Q}_{R,s}$. If we pass to a non-trivial moving frame, things get a little more complicated. Conjugation with a moving frame does not make sense for the non-linear map $\mathcal{V}_{R,s}$. Instead, we may assume that

$$\mathcal{V}_{R,s}(x) = I(x)\big(\mathcal{Q}_{R,s}(x) + \mathcal{F}x\big), \tag{8.30}$$

where $I(x) : H_0 \to H_0$ is an isomorphism which respects the scales and depends on $x \in H_1$ in a "nice" way[13]. Then we find that

$$w = \mathcal{F}^{-1}\big(I(x)^{-1}(\partial_s w) - \mathcal{Q}(w)\big)$$

indeed takes values in H_2. Unfortunately, since we used postcomposition instead of conjugation, linearizing (8.30) does not exactly give back a splitting of the form (8.28), but something slightly different. Thus, if we really chose to modify the definition of unregularized vector fields $\mathcal{V}$ in this way, we would have to adjust the proof of Lemma 8.9 at every point where the exact form of the splitting was used.

8.5 Proof of the compactness theorem

Proof of Theorem 8.6. For any fixed $0 < T' < T$ we can apply Lemma 8.9 to all functions w_ν and get that

$$\Big(w_\nu\big|_{I_{T'}}\Big)_{\nu\in\mathbb{N}} \subseteq \bigcap_{k=0}^{2} W^{k,p}(I_T, H_{2-k})$$

is uniformly bounded, that is,

$$\Big\| w_\nu\big|_{I_{T'}} \Big\|_{\bigcap_{k=0}^{2} W^{k,2}(I_{T'},H_{2-k})} \leq c(\kappa, T').$$

On the other hand, from Lemma 8.8 we know that the inclusion

$$\bigcap_{k=0}^{2} W^{k,p}(I_{T'}, H_{2-k}) \longrightarrow \bigcap_{k=0}^{1} C^k(I_{T'}, H_{1-k})$$

is compact, and so $\big(w_\nu|_{I_{T'}}\big)_{\nu\in\mathbb{N}}$ has a convergent subsequence in $\mathcal{C}^0(I_{T'}, H_0)\cap\mathcal{C}^1(I_{T'}, H_0)$. Denote the limit by w_*. Since every function w_ν satisfies

$$\partial_s w_\nu(s) = \mathcal{V}_{R_\nu,s}(w_\nu(s))$$

for all $s \in I_T$, this limit satisfies the corresponding equation with R_* for all $s \in I_{T'}$. Letting $T' < T$ grow bigger and bigger and choosing a diagonal sequence, we find a subsequence of $(w_\nu)_{\nu\in\mathbb{N}}$ converging to a limit $w_* \in \mathcal{C}^0(I_T, H_0)\cap\mathcal{C}^1(I_T, H_0)$ that satisfies that equation for all $s \in I_T$. □

[13]In the application described in Chapter 9, when $\mathcal{V}_{R,s}(x) = -J(x)\big(\partial_t x - \mu_R(s)\mathcal{X}(x)\big)$, we have (8.30) with $I(x) = -i \cdot J(x)$ and $\mathcal{Q}_{R,s}(x) = i\mu_R(s)\mathcal{X}(x) - \frac{1}{2}x$.

Chapter 9

The main application of the compactness theorem

Important applications of the compactness result from Chapter 8 are in the context of (perturbed) gradient flow lines of a Hamiltonian delay equation (cf. Section 8.1). In Appendix 9), we outline a stretching argument which should prove the existence of at least two periodic orbits of a Hamiltonian delay equation on a suitable symplectic manifold. Here, we want to show that the compactness part of that argument indeed fits into the setting of Chapter 8, so that Theorem 8.6 can be applied.

For fixed $R \in \mathbb{R}_{\geq 0}$, the equation considered in Appendix B reads

$$\partial_s w(s) = -J\big(w(s)\big)\Big(\partial_t w(s) - \mu_R(s) \cdot \mathcal{X}(w(s))\Big). \tag{9.1}$$

Here, w is a sufficiently regular function $w : \mathbb{R} \to L^2(S^1, \mathbb{R}^{2n})$; μ_R denotes a family of cutoff functions parametrized by $R \in \mathbb{R}_{\geq 0}$, see Remark 9.2 below; and $\mathcal{X} : L^2(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is a map coming from a vector field X on $\mathbb{R}^{2n}$. It is defined by $\mathcal{X}(x)(t) := X_t(x(t - \tau))$, where $\tau \in \mathbb{R}$ is the delay parameter.

For the argument outlined in Appendix B, one needs the space of perturbed gradient flow lines to be compact in a suitable topology. That is, consider a sequence of pairs (R_ν, w_ν) where each w_ν satisfies equation (9.1) for the corresponding R_ν. Further assume that $R_\nu \to R_*$ for $\nu \to \infty$, and that the w_ν satisfy certain gradient bounds. Theorem 8.6 should imply that in this case there is a sufficiently regular w_* such that $w_\nu \to w_*$ and w_* satisfies the equation for R_*.

To make this rigorous, we need to describe the fractal scale Hilbert space H_k, $k \geq 0$ (including the corresponding monotone unbounded function $f : \mathbb{N} \to \mathbb{R}$) and show that the right hand side of equation (9.1) defines a *uniform 2-parameter family of unregularized vector fields* as in Definition 8.4.

Remark 9.1. In the setting for Floer theory without delay (as in [AFS19, Section 5]), the map $\mathcal{X}$ would be defined by $\mathcal{X}(x)(t) := X_t(x(t))$. Note that in that case, one can consider $-J(p) \cdot X_t(p), p \in \mathbb{R}^{2n}$ as a new vector field, and then apply $-J(w(s))$ only to the first summand in (9.1), as in [AFS19, Section 5]. In the same way, we could work with $\mathcal{X}(x)(t) := J(x(t) \cdot X_t(x(t - \tau)$ and apply $-J(w(s))$ only to the first summand on the right hand side in equation (9.1).

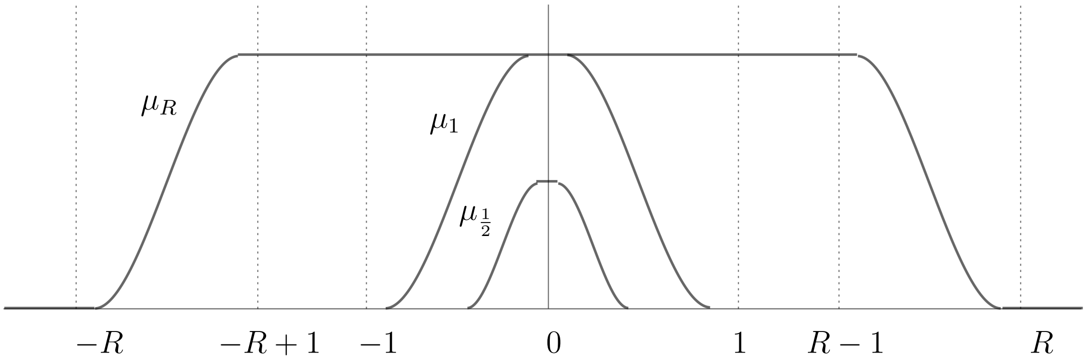


Figure 9.1: Sketch of $\mu_R : \mathbb{R} \to \mathbb{R}$ for several values of R.

Remark 9.2 (The cutoff functions μ_R)**.** Here and in Appendix B, we use a smooth family of smooth cutoff functions $\mu_R : \mathbb{R} \to [0, 1]$, where $R \in \mathbb{R}_{\geq 0}$. For Section 9.3 and Section 9.4 it is enough to know that this family is as nice as can be: Everything is smooth, and we can take the derivatives to be uniformly bounded.

For reasons of completeness, we here include everything that one should assume about the family for making the stretching argument work. It is a family of smooth functions $\mu_R : \mathbb{R} \to [0, 1]$ smoothly varying in $R \in [0, \infty)$ with the following properties:

- $\mu_R(s) = 0$ for $s \leq -R + \delta$ and for $s \geq R - \delta$ for some $\delta > 0$,
- $\mu_R(s) = 1$ for $-R + 1 \leq s \leq R - 1$ (this can only happen for $R \geq 1$),
- $\mu_R(s) \leq R$ for all $s \in \mathbb{R}$
- $\mu_0 \equiv 0$,
- $\frac{\mathrm{d}}{\mathrm{d}s}\mu_R(s)$ is bounded uniformly in R and s and
- $\frac{\mathrm{d}}{\mathrm{d}R}\mu_R(s)$ is bounded uniformly in R and s.

A possible family is sketched in figure 9.1.

9.1 The spaces H_k

In our application we will use the same setting as in [AFS19, Section 5]: We want to have

$$H_k \cong W^{k,2}(S^1, \mathbb{R}^{2n}) \tag{9.2}$$

as long as we are working with the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$ (see Section 9.3) and

$$H_k \cong W^{k+1,2}(S^1, \mathbb{R}^{2n}) \tag{9.3}$$

for the case of a general almost complex structure J on $\mathbb{R}^{2n}$ (see Section 9.4). Moreover, we want the fundamental operator $\mathcal{F} : H_{k+1} \to H_k$ to be given by

$$\begin{aligned} \mathcal{F} : W^{k+1,2}(S^1, \mathbb{R}^n) &\hookrightarrow W^{k,2}(S^1, \mathbb{R}^n) \\ z &\mapsto -i\partial_t z + \tfrac{1}{2} z. \end{aligned} \tag{9.4}$$

In (9.2) and (9.3) above, by "$\cong$" we linear bijections on each level which commute with the inclusions and with the fundamental operators on both sides and respect the norms on both sides. Note that we do not need isometries (which would respect also the inner product); this is because in the end, for the compactness result in Theorem 8.6 we only care about the norm, not about the inner product.

It is implicitly used in [AFS19] that such isomorphisms exist, so that the compactness result formulated in the setting of fractal scale Hilbert spaces can be used in the setting of Sobolev spaces $W^{k,2}(S^1, \mathbb{R}^{2n})$. In this section, we give a detailed proof of that fact in the case of (9.2). The case of (9.3) follows then by a simple shift of the monontone unbounded function $f : \mathbb{N} \to (0, \infty)$.

Readers without interest in these details can directly move on to Section 9.2.

Lemma 9.3. *There is a monotone increasing unbounded function $f : \mathbb{N} \to (0, \infty)$ such that for the $\mathbb{R}^{2n}$-valued fractal scale Hilbert space $(H_k = \ell^2_{f^k})_{k \in \mathbb{Z}}$ there are linear bijections $H_k \cong W^{k,2}(S^1, \mathbb{R}^{2n})$ for every $k \geq 0$ which commute with the inclusions $H_{k+1} \hookrightarrow H_k$ and $W^{k+1,2}(S^1, \mathbb{R}^{2n}) \hookrightarrow W^{k,2}(S^1, \mathbb{R}^{2n})$.*

Proof. Let us start with $k = 0$. We identify $\mathbb{R}^{2n} \cong \mathbb{C}$. Then every $z \in L^2(S^1, \mathbb{R}^{2n}) = W^{0,2}(S^1, \mathbb{R}^{2n})$ can be written as a Fourier series

$$z(t) = \sum_{j \in \mathbb{Z}} e^{2\pi i j t} z_m, \tag{9.5}$$

where by a slight abuse of notation i denotes the matrix

$$\begin{pmatrix} i & 0 & \dots & 0 \\ 0 & i & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & i \end{pmatrix} \in \mathbb{C}^{n \times n} \tag{9.6}$$

and each z_j is an element of $\mathbb{C}^n = \mathbb{R}^{2n}$. Since z is square integrable, the series in (9.5) converges at almost every $t \in S^1$, and so we can understand (9.5) as an equation of L^2-functions. This identifies $z \in L^2(S^1, \mathbb{R}^{2n})$ with the sequence $(z_j)_{j \in \mathbb{Z}}$. However, we need a sequence of vectors indexed by $\mathbb{N}$. Therefore, consider the following bijection:

$$\begin{aligned} \chi : \mathbb{Z} &\to \mathbb{N} \\ j &\mapsto \begin{cases} 2j + 1 & j \geq 0 \\ -2j & j < 0 \end{cases} \\ \left. \begin{matrix} m \text{ odd} & \dfrac{m-1}{2} \\ m \text{ even} & \dfrac{-m}{2} \end{matrix} \right\} &\mapsfrom m \end{aligned} \tag{9.7}$$

Then by

$$x_m := z_{\chi^{-1}(m)}, \qquad m \in \mathbb{N} \tag{9.8}$$

we get a sequence indexed by $\mathbb{N}$. The condition that z is square integrable translates to

$$\sum_{m\in\mathbb{N}} \|x_m\|^2 = \sum_{j\in Z} \|z_j\|^2 < \infty.$$

This means we have established a linear bijection $L^2(S^1, \mathbb{R}^{2n}) \cong H_0$.

Let us now move on to $k > 0$. From equation (9.5) we see that a formula for the k^{th} derivative of $z \in L^2(S^1, \mathbb{R}^{2n})$ it is given by

$$z^{(k)}(t) = \sum_{j\in\mathbb{Z}} (2\pi i j)^k e^{2\pi i j t} z_j,$$

and so the first k derivatives of z are square integrable if and only if

$$\sum_{j\in\mathbb{Z}} (2\pi j)^{2k} \|z_j\|^2 < \infty,$$

which is equivalent to

$$\sum_{j\in\mathbb{Z}} (2\pi j + \tfrac{1}{2})^{2k} \|z_j\|^2 < \infty. \tag{9.9}$$

We define

$$\begin{aligned} &\bar{f} : \mathbb{Z} \to \mathbb{R}_{>0} \\ &\bar{f}(j) := (2\pi j + \tfrac{1}{2})^2. \end{aligned}$$

Then $f := \bar{f} \circ \chi^{-1} : \mathbb{N} \to \mathbb{R}_{>0}$ is a monotone increasing unbounded function, and

$$\sum_{m\in\mathbb{N}} f^k(m) \|x_m\|^2 = \sum_{m\in\mathbb{N}} \bar{f}^k(\chi^{-1}(m)) \|x_m\|^2 = \sum_{j\in\mathbb{Z}} \bar{f}^k(j) \|z_j\|^2 = \sum_{j\in\mathbb{Z}} (2\pi j + \tfrac{1}{2})^{2k} \|z_j\|^2.$$

Thus, z is an element of $W^{k,2}(S^1, \mathbb{R}^{2n})$ if and only if the corresponding sequence $(x_m)_{m\in\mathbb{N}}$ is an element of $H_k = \ell^2_{f^k}$. Hence we have constructed linear bijections $H_k \cong W^{k,2}(S^1, \mathbb{R}^{2n})$ for all $k \geq 0$ and seen that they commute with the inclusions. □

On every H_k, the inner product (8.2) induces the norm

$$\|x\|^2_{H_k} = \sum_{m\in\mathbb{N}} f(m)^k \|x_m\|^2.$$

By the linear bijections above, this norm can be pushed forward to a norm $\|\cdot\|_{H_k}$ on $W^{k,2}(S^1, \mathbb{R}^{2n})$, which then reads

$$\|z\|^2_{H_k} = \sum_{j\in\mathbb{Z}} \bar{f}(j)^k \|z_j\|^2. \tag{9.10}$$

Of course, we need this norm to be equivalent to the usual Sobolev norm $\|\cdot\|_{W^{k,2}}$. Note that we do not need equivalence of the inner products arising in the same way.

Lemma 9.4. *For every k, the norm $\|\cdot\|_{H_k}$ from* (9.10) *is equivalent to the usual Sobolev norm $\|z\|^2_{W^{k,2}} = \sum_{j=0}^k \|\partial_t^j z\|^2_{L^2}$.*

Proof. Let us start with the case $k = 0$. Take any $z \in W^{0,2}(S^1, \mathbb{R}^{2n})$. Recall that i denotes the matrix (9.6) and that each Fourier coefficient z_j is an element of $\mathbb{C}^n$. We have

$$\begin{aligned}
\|z\|^2_{W^{0,2}} = \|z\|^2_{L^2} &= \left\| \sum_{j\in\mathbb{Z}} e^{2\pi i j\cdot} z_j \right\|^2_{L^2} \\
&= \int_{t=0}^1 \left\| \sum_{j\in\mathbb{Z}} e^{2\pi i j t} z_j \right\|^2_{\mathbb{C}^n} \mathrm{d}t \\
&= \sum_{j\in\mathbb{Z}} \|e^{2\pi i j\cdot}\|^2_{L^2(S^1,\mathbb{C}^{n\times n})} \|z_j\|^2 \\
&= \sum_{j\in\mathbb{Z}} \|z_j\|^2 \\
&= \|z\|^2_{H_0},
\end{aligned}$$

since $(t \mapsto e^{2\pi i j t})_{j\in\mathbb{Z}}$ is an orthonormal linearly independent subset of $L^2(S^1, \mathbb{C}^{n\times n})$. So for $k = 0$ the norms are not only equivalent, but equal.

For general $k \geq 0$, we use the formula for the derivatives of x and see that

$$\begin{aligned}
\|z\|^2_{W^{k,2}} = \sum_{l=0}^k \|\partial_t^l z\|^2_{L^2} &= \sum_{l=0}^k \left\| \sum_{j\in\mathbb{Z}} (2\pi i j)^l e^{2\pi i j\cdot} z_j \right\|^2_{L^2} \\
&= \sum_{l=0}^k \left\| \sum_{j\in\mathbb{Z}} (2\pi i j)^l e^{2\pi i j\cdot} \right\|^2_{L^2(S^1,\mathbb{C}^{n\times n})} \|z_j\|^2_{\mathbb{C}^n} \\
&= \sum_{j\in\mathbb{Z}} \sum_{l=0}^k (2\pi j)^{2l} \|z_j\|^2_{\mathbb{C}^n}.
\end{aligned}$$

This used again the orthonormal set from above. For every fixed k one can find constants $c_k, C_k > 0$ such that for every $j \in \mathbb{Z}$ it is

$$c_k \leq \tfrac{1}{\bar{f}^k(j)} \sum_{l=0}^k (2\pi j)^{2l} = \sum_{l=0}^k \frac{(2\pi j)^{2l}}{(2\pi j + \frac{1}{2})^{2k}} \leq C_k$$

and so

$$c_k \cdot \sum_{j\in\mathbb{Z}} \bar{f}^k(j)\|z_j\|^2 \leq \sum_{j\in\mathbb{Z}} \sum_{l=0}^k (2\pi j)^{2l} \|z_j\|^2 \leq C_k \cdot \sum_{j\in\mathbb{Z}} \bar{f}^k(j)\|z_j\|^2$$

for all sequences $(z_j)_j \subset \mathbb{C}^n$. Thus, the norms $\|\cdot\|_{W^{k,2}}$ and $\|\cdot\|_{H_k}$ are equivalent. □

Lemma 9.5. *Under the identifications $H_k \cong W^{k,2}(S^1, \mathbb{R}^{2n})$ from above, the fundamental operator $\mathcal{F}(z) := -i\partial_t z + \frac{1}{2}z$ defined in* (9.4) *corresponds to the one defined in* (8.3)*, where $f = \bar{f} \circ \chi^{-1} : \mathbb{N} \to \mathbb{R}_{>0}$.*

Proof. With the definition of $\mathcal{F}$ in (9.4) and $z(t) = \sum_{j\in\mathbb{Z}} e^{2\pi ijt} z_j$ we get that

$$\begin{aligned}\mathcal{F}(z)(t) &= -i\partial_t z(t) + \tfrac{1}{2} z(t) \\ &= -i\sum_{j\in\mathbb{Z}} (2\pi ij)\cdot e^{2\pi ijt} z_j + \tfrac{1}{2}\sum_{j\in\mathbb{Z}} e^{2\pi ijt} z_j \\ &= \sum_{j\in\mathbb{Z}} \big(2\pi j + \tfrac{1}{2}\big)\cdot e^{2\pi ijt} z_j \\ &= \sum_{j\in\mathbb{Z}} \sqrt{\bar{f}(j)}\cdot e^{2\pi ijt} z_j,\end{aligned}$$

so applying $\mathcal{F}$ means multiplying every z_j with $\sqrt{\bar{f}(j)}$. Hence, if $(x_m)_{m\in\mathbb{N}}$ denotes the $\mathbb{N}$-indexed sequence for z as defined in (9.8), then the $\mathbb{N}$-indexed sequence for $\mathcal{F}(z)$ is given by

$$\bar{x}_m := \sqrt{\bar{f}(\chi^{-1}(m))}\cdot z_{\chi^{-1}(m)} = \sqrt{f(m)}\cdot z_{\chi^{-1}(m)} = \sqrt{f(m)}\cdot x_m, \qquad m\in\mathbb{N}.$$

This exactly matches the definition of the fundamental operator $\mathcal{F}$ in (8.3). □

9.2 The map $\mathcal{X}$ and the cutoff function μ_R

Let $X : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a smooth time-dependent vector field, and choose a parameter $\tau \in \mathbb{R}$. We define a map $\mathcal{X} : L^2(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ by

$$\mathcal{X}(x)(t) = X_t(x(t-\tau)) \tag{9.11}$$

for all $t \in S^1$. We now establish some properties of $\mathcal{X}$ that will be used below in Sections 9.3 and 9.4.

Lemma 9.6. *If $\mathcal{X}$ is defined by (9.11), then it has the following properties:*

(i) $\mathcal{X} : W^{1,2}(S^1, \mathbb{R}^{2n}) \longrightarrow L^2(S^1, \mathbb{R}^{2n})$ *is continuous.*

(ii) $\mathcal{X} : W^{1,2}(S^1, \mathbb{R}^{2n}) \longrightarrow L^2(S^1, \mathbb{R}^{2n})$ *is continuously differentiable with differential*

$$\mathrm{D}\mathcal{X} : W^{1,2}(S^1, \mathbb{R}^{2n}) \times W^{1,2}(S^1, \mathbb{R}^{2n}) \longrightarrow L^2(S^1, \mathbb{R}^{2n})$$

defined by $\big(\mathrm{D}\mathcal{X}(x)\hat{x}\big)(t) = \mathrm{D}X_t(x(t-\tau))\hat{x}(t-\tau)$.

(iii) $\mathcal{X}$ *restricts to continuous maps*

$$\begin{aligned}\mathcal{X} &: W^{2,2}(S^1, \mathbb{R}^{2n}) \longrightarrow W^{1,2}(S^1, \mathbb{R}^{2n}) \\ \mathcal{X} &: W^{3,2}(S^1, \mathbb{R}^{2n}) \longrightarrow W^{2,2}(S^1, \mathbb{R}^{2n}).\end{aligned}$$

(iv) $\mathrm{D}\mathcal{X}$ *is also continuous when interpreted as*

$$\mathrm{D}\mathcal{X} : W^{2,2}(S^1, \mathbb{R}^{2n}) \times W^{1,2}(S^1, \mathbb{R}^{2n}) \longrightarrow W^{1,2}(S^1, \mathbb{R}^{2n}).$$

Proof. (i) We use the embedding $W^{1,2} \hookrightarrow \mathcal{C}^0$: If a sequence $(x_n)_{n\in\mathbb{N}}$ converges to some x in $W^{1,2}(S^1,\mathbb{R}^{2n})$, then $x_n(t) \to x(t)$ for all $t \in S^1$. Thus $X_t(x_n(t-\tau)) \to X_t(x(t-\tau))$ for all t, in particular $\mathcal{X}(x_n) \to \mathcal{X}(x)$ in L^2.

(ii) For fixed $x \in H_1 \hookrightarrow \mathcal{C}^0$, the matrix norm $\|\mathrm{D}X_t(x(t-\tau))\|_{\mathcal{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})}$ is bounded, and so $\big(\mathrm{D}\mathcal{X}(x)\hat{x}\big)(t) := \mathrm{D}X_t(x(t-\tau))\hat{x}(t-\tau)$ defines a bounded linear operator $\mathrm{D}\mathcal{X}(x) \in \mathcal{L}(W^{1,2},L^2)$. To see that this operator really qualifies as the derivative of $\mathcal{X}$, we note that $\|\hat{x}(t)\| \leq \|\hat{x}\|_{\mathcal{C}^0} \leq \mathrm{const}\cdot\|\hat{x}\|_{W^{1,2}}$ for all $t \in S^1$, which implies $\frac{1}{\|\hat{x}\|_{W^{1,2}}} \leq \frac{\mathrm{const}}{\|\hat{x}(t)\|}$, and compute

$$\begin{aligned}
&\lim_{\|\hat{x}\|_{W^{1,2}}\to 0} \frac{\|\mathcal{X}(x+\hat{x}) - \mathcal{X}(x) - \big(\mathrm{D}\mathcal{X}(x)\hat{x}\big)\|_{L^2}^2}{\|\hat{x}\|_{W^{1,2}}^2} \\
&\quad\leq \mathrm{const}\cdot \lim_{\|\hat{x}\|_{W^{1,2}}\to 0} \\
&\quad\int_0^1 \frac{\|X_t(x(t\text{-}\tau)+\hat{x}(t\text{-}\tau)) - X_t(x(t\text{-}\tau)) - \mathrm{D}X_t(x(t\text{-}\tau))\hat{x}(t\text{-}\tau)\|^2}{\|\hat{x}(t)\|^2}\,\mathrm{d}t \\
&\quad= 0.
\end{aligned}$$

In the last line we used that $\|\hat{x}\|_{W^{1,2}} \to 0$ implies $\|\hat{x}(t)\| \to 0$ for every $t \in S^1$ and so the expression inside the integral converges to 0 for every t by definition of the differential $\mathrm{D}X_t(x(t-\tau))$. This shows that $\mathrm{D}\mathcal{X}(x)$ is the differential of $\mathcal{X}$ in x.

We need to show that this is a continuous derivative, in the sense that the map

$$\begin{aligned}
\mathrm{D}\mathcal{X} : W^{1,2}(S^1,\mathbb{R}^{2n}) &\longrightarrow \mathcal{L}\big(W^{1,2}(S^1,\mathbb{R}^{2n}), L^2(S^1,\mathbb{R}^{2n})\big) \\
x &\longmapsto \mathrm{D}\mathcal{X}(x)
\end{aligned}$$

is continuous. Again consider a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x in $W^{1,2}(S^1,\mathbb{R}^{2n})$ and thus pointwise. We compute that

$$\begin{aligned}
&\|\mathrm{D}\mathcal{X}(x) - \mathrm{D}\mathcal{X}(x_n)\|_{\mathcal{L}(W^{1,2},L^2)}^2 \\
&\quad= \sup_{\|\hat{x}\|_{W^{1,2}}\leq 1} \|\mathrm{D}\mathcal{X}(x)\hat{x} - \mathrm{D}\mathcal{X}(x_n)\hat{x}\|_{L^2}^2 \\
&\quad\leq \sup_{\|\hat{x}\|_{W^{1,2}}\leq 1} \int_0^1 \underbrace{\|\mathrm{D}X_t(x(t-\tau)) - \mathrm{D}X_t(x_n(t-\tau))\|_{\mathcal{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})}^2}_{\to 0 \text{ by continuity of } \mathrm{D}X_t} \cdot \|\hat{x}(t-\tau)\|^2\,\mathrm{d}t \\
&\quad\longrightarrow 0.
\end{aligned}$$

This shows continuity of $D\mathcal{X}$.

(iii) For continuity of $\mathcal{X} : W^{2,2}(S^1,\mathbb{R}^{2n}) \to W^{1,2}(S^1,\mathbb{R}^{2n})$, we use the embedding $W^{2,2} \hookrightarrow \mathcal{C}^1$ to see that if $x_n \to x$ in $W^{2,2}(S^1,\mathbb{R}^{2n})$, then

$$\partial_t\big(\mathcal{X}(x_n)\big)(t) = \big(\partial_t X\big)(t)(x_n(t-\tau)) + \mathrm{D}X_t(x_n(t-\tau))\partial_t x_n(t-\tau)$$

converges to

$$\partial_t\big(\mathcal{X}(x)\big)(t) = \big(\partial_t X\big)(t)(x(t-\tau)) + \mathrm{D}X_t(x(t-\tau))\partial_t x(t-\tau)$$

for every $t \in S^1$, in particular $\partial_t\big(\mathcal{X}(x_n)\big) \to \partial_t\big(\mathcal{X}(x)\big)$ in L^2.

For continuity of $\mathcal{X} : W^{3,2}(S^1, \mathbb{R}^{2n}) \to W^{2,2}(S^1, \mathbb{R}^{2n})$, in the same way we use the embedding $W^{3,2} \hookrightarrow \mathcal{C}^2$ to see that if $x_n \to x$ in $W^{2,2}(S^1, \mathbb{R}^{2n})$, then $\partial_t^2\big(\mathcal{X}(x_n)\big) \to \partial_t^2\big(\mathcal{X}(x)\big)$ pointwise and thus in L^2.

(iv) Given sequences $x_n \to x$ in $W^{2,2}$ and $\hat{x}_n \to \hat{x}$ in $W^{1,2}$, we have to show that $\mathrm{D}\mathcal{X}(x_n)\hat{x}_n \to \mathrm{D}\mathcal{X}(x)\hat{x}$ in $W^{1,2}$. Convergence in L^2 was shown in (ii) already, so what remains to show is that $\partial_t\big(\mathrm{D}\mathcal{X}(x_n)\hat{x}_n\big) \to \partial_t\big(\mathrm{D}\mathcal{X}(x)\hat{x}\big)$ in L^2. We compute

$$\begin{aligned}\partial_t\big(\mathrm{D}\mathcal{X}(x_n)\hat{x}_n\big)(t) = \Big(\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}X_t\big) \big(\partial_t x_n(t-\tau)\big)\Big) \cdot \hat{x}_n(t-\tau) \\ + \mathrm{D}X_t\big(x_n(t-\tau)\big) \cdot \partial_t \hat{x}_n(t-\tau).\end{aligned}$$

Because of the embeddings $W^{1,2} \hookrightarrow \mathcal{C}^0$ and $W^{2,2} \hookrightarrow \mathcal{C}^1$, we know that

$$\partial_t x_n(t-\tau) \to \partial_t x(t-\tau), \qquad \hat{x}_n(t-\tau) \to \hat{x}(t-\tau), \qquad x_n(t-\tau) \to x(t-\tau)$$

converge pointwise. Together with smoothness of the family X_t this implies that the whole expression converges in L^2.

□

9.3 $\mathcal{V}_{R,s}$ with $J \equiv i$

With the standard (constant) complex structure i on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, the uniform elementary family of unregularized vector fields will be defined by

$$\mathcal{V}_{R,s}(x) := -i\Big(\partial_t x - \mu_R(s)\mathcal{X}(x)\Big) \tag{9.12}$$

for $x \in H_1$. The map $\mathcal{X}$ and the family of cutoff functions $\mu_R : \mathbb{R} \to [0,1]$ were defined in Section 9.2.

Recall that $\mathcal{X}$ involves a shift by a "delay" parameter $\tau \in \mathbb{R}$. With $\tau = 0$ we recover the easier case of no delay, where equation (8.5) is local and the maps $w : \mathbb{R} \to H_0$ can be treated as maps $u : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$.

Proposition 9.7. *The family* $(\mathcal{V}_{R,s})_{R \in \mathbb{R}_{\geq 0}, s \in \mathbb{R}}$ *defined in equation* (9.12) *is a uniform 2-parameter family of unregularized vector fields as in Definition 8.4.*

Proof. This proof is mostly taken from [AFS19, Section 5.1, pp. 23–24]. The only differences are that here the definition of $\mathcal{X}$ in equation (9.11) involves a delay τ; that we apply $-i$ also to the second summand in (9.12), see Remark 9.1; and that we have the s-dependent factor $\mu_R(s)$. Moreover, we include more detailed computations than in [AFS19].

Recall from Lemma 9.4 that the norms on $H_0 = L^2$ and $H_1 = W^{1,2}$ coming from the setting of sequences used in Definition 8.4 are equivalent to the usual L^2 and $W^{1,2}$ norms, so we can work with the latter ones.

We have to check that if $x \in H_1$, then $\mathcal{V}_{R,s}(x) \in H_0$, and if $x \in H_2$, then $\mathcal{V}_{R,s}(x) \in H_1$. This is clear from (9.12) and (9.11). Moreover, the maps $\mathcal{V}_{R,s} : H_1 \to H_0$ and $\mathcal{V}_{R,s} : H_2 \to H_1$ are continuous: $\partial_t : H_{k+1} \to H_k$ is continuous for every $k \in \mathbb{N}$, and $\mathcal{X} : H_1 \to H_0$ and $\mathcal{X} : H_2 \to H_1$ are continuous by Lemma 9.6 (i),(iii). Strong continuity in s follows from the fact that each μ_R is smooth in s. Furthermore we see that evaluation of $\mathcal{V}_{\cdot,s}$ in R is continuous since $\mu_R(s) \in \mathbb{R}$ depends continuously on R. For fixed (R, s) and $x \in H_1$ we have a differential

$$\begin{aligned}
&\mathrm{D}\mathcal{V}_{R,s}(x) : H_1 = W^{1,2}(S^1, \mathbb{R}^{2n}) \longrightarrow L^2(S^1, \mathbb{R}^{2n}) = H_0 \\
&\mathrm{D}\mathcal{V}_{R,s}(x)\hat{x} = -i\Big(\partial_t \hat{x} - \mu_R(s) \cdot \mathrm{D}\mathcal{X}(x)\hat{x}\Big),
\end{aligned}$$

where $\mathrm{D}\mathcal{X}$ is as in Lemma 9.6 (ii).

We now have to verify $(\mathcal{V}1)$, $(\mathcal{V}2)$, $(\mathcal{V}3)$ and $(\mathcal{V}4)$.

$(\mathcal{V}1)$ This property is continuity of $\mathrm{D}\mathcal{V}_{R,\cdot}$ after evaluation in s, x and $\hat{x}$. It is immediate from the formula for $\mathrm{D}\mathcal{V}_{R,s}(x)\hat{x}$ and continuity of $\mathrm{D}\mathcal{X} : W^{1,2} \to \mathcal{L}(W^{1,2}, L^2)$.

$(\mathcal{V}2)$ Let us use the constant moving frame $\Phi \equiv \mathrm{id}_{H_0}$. We get

$$\begin{aligned}
\mathcal{P}_{R,s}(x)\hat{x} &= \mathrm{D}\mathcal{V}_{R,s}(x)\hat{x} - \mathcal{F}(\hat{x}) \\
&= -\mu_R(s) \cdot i \cdot \mathrm{D}\mathcal{X}(x)\hat{x} - \tfrac{1}{2}\hat{x},
\end{aligned} \tag{9.13}$$

which indeed extends to $\hat{x} \in H_0 = L^2(S^1, \mathbb{R}^{2n})$. Again, this is certainly strongly continuous in $s \in \mathbb{R}$, and also in $x \in H_1$ since $\mathrm{D}\mathcal{X} : H_1 \to \mathcal{L}(H_1, H_0)$ is continuous by Lemma 9.6(ii).

$(\mathcal{V}3)$ By continuity of the embedding $H_1 = W^{1,2} \hookrightarrow \mathcal{C}^0$, all images of maps $x \in H_1$ with $\|x\|_{H_1} \leq \kappa$ are contained in a big ball B in $\mathbb{R}^{2n}$ with radius depending only on κ. Smoothness of the vector field X_t now implies that there is a constant $C(\kappa)$ depending only on κ and X such that

$$\|X_t(x(t-\tau))\|_{\mathbb{R}^{2n}} \leq C(\kappa) \tag{9.14}$$

$$\|(\partial_t X_t)(x(t-\tau))\|_{\mathbb{R}^{2n}} \leq C(\kappa) \tag{9.15}$$

$$\|\mathrm{D}X_t(x(t-\tau))\|_{\mathcal{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})} \leq C(\kappa) \tag{9.16}$$

for all such x with $\|x\|_{H_1} \leq \kappa$ and all $t \in S^1$. So from equation (9.13) we see that

$$\begin{aligned}
\|\mathcal{P}_{R,s}(x)\|_{\mathcal{L}(H_0,H_0)} &\leq \|\mathrm{D}\mathcal{X}(x)\|_{\mathcal{L}(H_0,H_0)} + \frac{1}{2} \\
&= \sup_{\|\hat{x}\|_{H_0} \leq 1} \|\mathrm{D}\mathcal{X}(x)\hat{x}\|_{H_0} + \frac{1}{2} \\
&= \sup_{\|\hat{x}\|_{H_0} \leq 1} \left(\int_0^1 \|\mathrm{D}X_t(x(t-\tau))(\hat{x}(t-\tau))\|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} + \frac{1}{2} \\
&\leq C(\kappa) + \frac{1}{2},
\end{aligned} \tag{9.17}$$

where we used (9.16) in the last line.

Moreover we can use (9.14), (9.15) and (9.16) to estimate

$$\begin{aligned}\|\mathcal{X}(x)\|_{H_1} &\leq \|\mathcal{X}(x)\|_{L^2} + \|\partial_t(\mathcal{X}(x))\|_{L^2} \\ &\leq \|\mathcal{X}(x)\|_{L^2} + \|(\partial_t X)(x(\cdot - \tau))\|_{L^2} + \|\mathrm{D}X_t(x(\cdot - \tau))\partial_t x(\cdot - \tau)\|_{L^2} \\ &\leq C(\kappa) + C(\kappa) + C(\kappa)\cdot\kappa \\ &= (2+\kappa)C(\kappa) \qquad (9.18)\end{aligned}$$

Now if $x \in H_2$ and $\|x\|_{H_1} \leq \kappa$, then

$$\begin{aligned}\|\mathcal{V}_{R,s}(x)\|_{H_1} &= \| - i\partial_t x - \mu_R(s)\mathcal{X}(x)\|_{H_1} \\ &\geq \|i\partial_t x\|_{H_1} - \|\mu_R(s)\mathcal{X}(x)\|_{H_1} \\ &\geq \|\partial_t x\|_{H_1} - \|\mathcal{X}(x)\|_{H_1} \\ &\geq \|\partial_t x\|_{H_1} - (2+\kappa)C(\kappa). \qquad (9.19)\end{aligned}$$

We further note that

$$\begin{aligned}\|x\|_{H_2} = \|x\|_{W^{2,2}} &= \left(\|x\|^2_{L^2} + \|\partial_t x\|^2_{L^2} + \|\partial_t^2 x\|^2_{L^2}\right)^{\frac{1}{2}} \\ &\leq \|x\|_{L^2} + \left(\|\partial_t x\|^2_{L^2} + \|\partial_t^2 x\|^2_{L^2}\right)^{\frac{1}{2}} \\ &\leq \kappa + \|\partial_t x\|_{H_1}\end{aligned}$$

and continue the estimate (9.19) by

$$\begin{aligned}\|\mathcal{V}_{R,s}(x)\|_{H_1} &\geq \|\partial_t x\|_{H_1} - (2+\kappa)C(\kappa) \\ &\geq \|x\|_{H_2} - (\kappa + (2+\kappa)C(\kappa)).\end{aligned}$$

This implies (using that without loss of generality $\kappa + (2+\kappa)C(\kappa) \geq 1$) that

$$\begin{aligned}\|x\|_{H_2} &\leq \|\mathcal{V}_{R,s}(x)\|_{H_1} + \kappa + (2+\kappa)C(\kappa) \\ &\leq (\kappa + (2+\kappa)C(\kappa)\Big(\|\mathcal{V}_{R,s}(x)\|_{H_1} + 1\Big). \qquad (9.20)\end{aligned}$$

Combining the estimates (9.17) and (9.20), with

$$c_1(\kappa) := \max\left\{C(\kappa) + \frac{1}{2}, \kappa + (2+\kappa)C(\kappa)\right\}$$

we have verified ($\mathcal{V}3$).

($\mathcal{V}4$) For the derivative in s-direction in some $x \in H_1$ we find

$$\begin{aligned}&\partial_s\mathcal{V}_{R,s} : H_1 \longrightarrow H_0 \\ &\partial_s\mathcal{V}_{R,s}(x) = -\mu_R'(s)\cdot i\cdot\mathcal{X}(x).\end{aligned}$$

This is continuous by Lemma 9.6 (i). Continuity of $\partial_s\mathcal{V}_{R,s}$ in $s \in \mathbb{R}$ follows from smoothness of μ_R.

For the estimate recall the constant $C(\kappa)$ which we introduced above for ($\mathcal{V}3$). For every $x \in H_1$ with $\|x\|_{H_1} \leq \kappa$ and every $(R, s) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ we have

$$\begin{aligned}\|\partial_s \mathcal{V}_{R,s}(x)\|_{H_0} &= |\mu_R'(s)| \cdot \|\mathcal{X}(x)\|_{L^2} \\ &\leq C \cdot C(\kappa) \\ &=: c_2(\kappa)\end{aligned}$$

with some constant $C > 0$ such that $|\mu_R'(s)| \leq C$ for all $s \in \mathbb{R}$.

So our family $(\mathcal{V}_{R,s})_{R \in \mathbb{R}_{\geq 0}, s \in \mathbb{R}}$ matches Definition 8.4. □

9.4 $\mathcal{V}_{R,s}$ with general J

Now we start to work with a general almost complex structure J on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. This means that we define the family of unregularized vector fields by

$$\mathcal{V}_{R,s}(x) := -J(x)\Big(\partial_t x - \mu_R(s)\mathcal{X}(x)\Big) \tag{9.21}$$

for $x \in H_1$, where $\mathcal{X}$ is defined in (9.11) and involves a delay by $\tau \in \mathbb{R}$. Here, $J(x)$ varies with the function x, so from now on we will have to work with a non-trivial moving frame Φ.

Since $J(p)$ varies smoothly with $p \in \mathbb{R}^{2n}$, there is a smooth function $\Psi : \mathbb{R}^{2n} \to \mathrm{GL}(\mathbb{R}^{2n})$ into the invertible $2n$-matrices with

$$\Psi(p) \circ J(p) \circ \Psi(p)^{-1} = i \qquad \text{for all } p \in \mathbb{R}^{2n}. \tag{9.22}$$

For $x \in H_1$ and $v \in H_0$ we want to define $\Phi(x)v$ by

$$\big(\Phi(x)v\big)(t) := \Psi\big(x(t)\big)v(t). \tag{9.23}$$

Unfortunately though, with $H_0 = L^2$ and $H_1 = W^{1,2}$ as above, this does not qualify as a moving frame. Loops $x \in L^2$ do not have to be continuous, and so smoothness of Ψ does not suffice to deduce properties ($\Phi2$), ($\Phi4$) and ($\Phi6$) of a moving frame. It seems that ($\Phi2$) and ($\Phi4$) could be satisfied if we assume additional properties of Ψ, but the estimate $\|\mathrm{D}\Phi(x)\|_{\mathrm{Bil}(H_0)} \leq c_0$ will not be satisfied.

This means that from now on we have to work with

$$H_k := W^{k+1,2}(S^1, \mathbb{C}^n),$$

as mentioned already in Section 9.1.

Proposition 9.8 ([AFS19, Prop. 5.2])**.** *With $H_k = W^{k+1,2}(S^1, \mathbb{C}^n)$, equation* (9.23) *defines a moving frame $\Phi : H_1 \to \mathcal{L}(H_0, H_0)$ as in Definition 8.2, with derivative $\mathrm{D}\Phi(x) \in \mathcal{L}(H_0 \times H_0, H_0)$ defined by $\mathrm{D}\Phi(x)(h, v)(t) := \mathrm{D}\Psi(x(t))(h(t), v(t))$.*

Proposition 9.9. *The family $(\mathcal{V}_{R,s})_{R \in \mathbb{R}_{\geq 0}, s \in \mathbb{R}}$ defined by equation* (9.21) *is a uniform 2-parameter family of unregularized vector fields as in Definition 8.4.*

Proof. This proof is mostly taken from [AFS19, Section 5.1, pp. 22-23]. The only differences are that here the definition of $\mathcal{X}$ in equation (9.11) involves a delay τ; that we apply $-J(x)$ also to the second summand in (9.12), see Remark 9.1; and that we have the s-dependent factor $\mu_R(s)$. Moreover, we include more detailed computations than in [AFS19].

Recall that for reasons of the moving frame Φ defined in (9.23) we needed to change the scales from $H_k = W^{k,2}$ to $H_k = W^{k+1,2}$. Thus also in the parts where no moving frame is involved we cannot just copy the proof of Proposition 9.7 but have to adjust the norms everywhere.

Since $\partial_t : H_{k+1} \to H_k$ is continuous for all $k \in \mathbb{N}$ and $\mathcal{X} : H_{k+1} \to H_k$ is continuous for $k = 0, 1$ by Lemma 9.6 (iii), from (9.21) we see that every $\mathcal{V}_{R,s}$ is in $\mathcal{C}^0(H_1, H_0) \cap \mathcal{C}^0(H_2, H_1)$.

Strong continuity in s and weak continuity in R (needed for axiom ($\mathcal{V}0$)) can be shown as in the proof of Prop. 9.7. For $x \in H_1$, we compute that the differential is

$$\begin{aligned}
&\mathrm{D}\mathcal{V}_{R,s}(x) : H_1 = W^{2,2}(S^1, \mathbb{R}^{2n}) \longrightarrow W^{1,2}(S^1, \mathbb{R}^{2n}) = H_0 \\
&\mathrm{D}\mathcal{V}_{R,s}(x)\hat{x} = -\mathrm{D}J(x)\big(\hat{x}, \partial_t x\big) - J(x)\partial_t\hat{x} + \mathrm{D}J(x)\big(\hat{x}, \mu_R(s)\cdot\mathcal{X}(x)\big) + \mu_R(s)\cdot\mathrm{D}\mathcal{X}(x)\hat{x}
\end{aligned}$$

with $\mathrm{D}\mathcal{X}$ as in Lemma 9.6 (ii). It is a bounded linear operator by Lemma 9.6 (iv) and since $\|\hat{x}\|_{\mathcal{C}^0}$ is controlled by $\|\hat{x}\|_{H_1}$ and J is smooth.

We now need to verify ($\mathcal{V}1$) – ($\mathcal{V}4$).

($\mathcal{V}1$) Continuity of

$$\begin{aligned}
\mathbb{R} \times H_1 \times H_1 &\longrightarrow H_0 \\
(s, x, \hat{x}) &\longmapsto D\mathcal{V}_{R,s}(x)\hat{x}
\end{aligned}$$

follows from continuity of $\mathrm{D}\mathcal{X} : W^{2,2} \times W^{2,2} \to W^{1,2}$ (see Lemma 9.6 (iv)) together with continuity of $\mathrm{D}J : \mathbb{R}^{2n} \to \mathcal{L}(\mathbb{R}^{2n}, \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})) \cong \mathcal{L}(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \mathbb{R}^{2n})$ and continuity of $\mu_R : \mathbb{R} \to \mathbb{R}$.

($\mathcal{V}2$) Consider the moving frame Φ from (9.23). For every $(R, s) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $x, \hat{x} \in H_1$, we see that

$$\begin{aligned}
\Big(\Phi(x) \circ D\mathcal{V}_{R,s}(x) \circ \Phi(x)^{-1}\Big)\hat{x} = &-\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big) \\
&- \Phi(x)\Big(J(x)\partial_t\big(\Phi(x)^{-1}\hat{x}\big)\Big) \\
&+ \Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \mu_R(s)\cdot\mathcal{X}(x)\big)\Big) \\
&+ \Phi(x)\Big(\mu_R(s)\cdot\mathrm{D}\mathcal{X}(x)\big(\Phi(x)^{-1}\hat{x}\big)\Big).
\end{aligned}$$

From

$$\begin{aligned}
\partial_t\big(\Phi(x)^{-1}\hat{x}\big) &= \partial_t\big(\Phi(x(t))^{-1}\big)\hat{x} + \Phi(x)^{-1}\partial_t\hat{x} \\
&= -\Phi(x)\Big(\mathrm{D}\Phi(x)\big(\partial_t x, \Phi(x)^{-1}\hat{x}\big)\Big) + \Psi(x)^{-1}\partial_t\hat{x}
\end{aligned}$$

together with (9.23) and (9.22) we get that

$$\Phi(x)\Big(J(x)\partial_t\big(\Phi(x)^{-1}\hat{x}\big)\Big) = -i\mathrm{D}\Phi(x)\big(\partial_t x, \Phi(x)^{-1}\hat{x}\big) + i\partial_t\hat{x}$$

and so

$$\begin{aligned}\Big(\Phi(x)\circ D\mathcal{V}_{R,s}(x)\circ\Phi(x)^{-1}\Big)\hat{x} = &-\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\\ &+ i\mathrm{D}\Phi(x)\big(\partial_t x, \Psi(x)^{-1}\hat{x}\big)\\ &- i\partial_t\hat{x}\\ &+ \Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \mu_R(s)\cdot\mathcal{X}(x)\big)\Big)\\ &+ \Phi(x)\Big(\mu_R(s)\cdot\mathrm{D}\mathcal{X}(x)\big(\Phi(x)^{-1}\hat{x}\big)\Big).\end{aligned}$$

Recall the fundamental operator $\mathcal{F}$ from (9.4). Now

$$\begin{aligned}\mathcal{P}_{R,s}(x)\hat{x} := &\Big(\Phi(x)\circ D\mathcal{V}_{R,s}(x)\circ\Phi(x)^{-1}\Big)\hat{x} - \mathcal{F}\hat{x}\\ = &-\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\\ &+ i\mathrm{D}\Phi(x)\big(\partial_t x, \Phi(x)^{-1}\hat{x}\big)\\ &- i\tfrac{1}{2}\hat{x}\\ &+ \Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \mu_R(s)\cdot\mathcal{X}(x)\big)\Big)\\ &- \Phi(x)\Big(\mu_R(s)\cdot\mathrm{D}\mathcal{X}(x)\big(\Phi(x)^{-1}\hat{x}\big)\Big)\end{aligned} \tag{9.24}$$

extends to a continuous linear operator

$$\mathcal{P}_{R,s}(x) : H_0 \longrightarrow H_0.$$

Indeed, there is no derivative of $\hat{x}$ involved, thus we have $\mathcal{P}_{R,s}(x)\hat{x} \in H_0$ for $\hat{x} \in H_0$, and we already know that $\Phi(x) \in \mathcal{L}(H_0, H_0)$ by definition of a moving frame, $\mathrm{D}\Phi(x) \in \mathcal{L}(H_0 \times H_0, H_0)$ by (Φ4), and $\mathrm{D}\mathcal{X}(x) \in \mathcal{L}(H_0, H_0)$ by Lemma 9.6 (iv).

Continuity of $\mathbb{R} \to \mathcal{L}(H_0, H_0)$, $s \mapsto \mathcal{P}_{R,s}(x)$ (for fixed $x \in H_1$) follows from continuity of μ_R. For continuity of

$$\begin{aligned} H_1 \times H_0 &\longrightarrow H_0\\ (x, \hat{x}) &\longmapsto \mathcal{P}_{R,s}(x)\hat{x}\end{aligned}$$

we use that

- $\Phi : H_1 \times H_0 \to H_0$ is continuous by (Φ2),
- $\mathrm{D}\Phi : H_1 \times H_0 \times H_0 \to H_0$ is continuous by (Φ4),
- $\mathrm{D}\mathcal{V}_{R,s} : H_1 \times H_1 \to H_0$ is continuous by ($\mathcal{V}$1),
- $\mathrm{D}\mathcal{X} : H_1 \times H_0 \to H_0$ is continuous by Lemma 9.6 (iv), and

– $x \in W^{2,2} \hookrightarrow \mathcal{C}^1$, $\Phi(x)^{-1}\hat{x} \in W^{1,2} \hookrightarrow \mathcal{C}^0$ and smoothness of J imply that also the first term of (9.24) is continuous with respect to the $H_0 = W^{1,2}$ topology in the target.

($\mathcal{V}3$) From (9.24) we see that

$$
\begin{aligned}
\|\mathcal{P}_{R,s}(x)\|_{\mathcal{L}(H_0,H_0)} &= \sup_{\|\hat{x}\|_{H_0}\le 1} \|\mathcal{P}_{R,s}(x)\hat{x}\|_{H_0} \\
&\le \sup_{\|\hat{x}\|_{H_0}\le 1} \Bigg(\Big\|\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\Big\|_{H_0} \\
&\qquad + \big\|\mathrm{D}\Phi(x)\big(\partial_t x, \Phi(x)^{-1}\hat{x}\big)\big\|_{H_0} \\
&\qquad + \big\|\tfrac{1}{2}\hat{x}\big\|_{H_0} \\
&\qquad + \|\Phi(x)\big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \mu_R(s)\cdot \mathcal{X}(x)\big)\big)\|_{H_0} \\
&\qquad + \Big\|\Phi(x)\Big(\mu_R(s)\cdot \mathrm{D}\mathcal{X}(x)\big(\Phi(x)^{-1}\hat{x}\big)\Big)\Big\|_{H_0}\Bigg),
\end{aligned}
\tag{9.25}
$$

where $H_0 = W^{1,2}$. This means we have to show that for $\|x\|_{H_1} \le \kappa$ and $\|\hat{x}\|_{H_0} \le 1$, the L^2 norms of all these five expressions and of their derivatives are bounded independently of x.

We use the embedding $H_1 = W^{2,2} \hookrightarrow \mathcal{C}^1$ to see that the union of images

$$
\bigcup_{x\in H_1, \|x\|_{H_1}\le\kappa} x(S^1)\cup \partial_t x(S^1) \quad \subset \quad \mathbb{R}^{2n}
$$

is contained in a big ball of radius depending only on κ. So by smoothness of J, Ψ and X there is a constant $D(\kappa)$ depending only on κ, J, Ψ and X such that for all $x \in H_1$ with $\|x\|_{H_1}$ and for all $t \in S^1$ it is

$$
\begin{aligned}
\|x(t)\|_{R^{2n}} &\le D(\kappa) \\
\|\partial_t x(t)\|_{R^{2n}} &\le D(\kappa) \\
\|\Psi(x(t))\|_{\mathcal{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\Psi(x(t))^{-1}\|_{\mathcal{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\mathrm{D}\Psi(x(t))\|_{\mathrm{Bil}(\mathbb{R}^{2n}\times\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\mathrm{D}\Psi(x(t))\|^{-1}_{\mathrm{Bil}(\mathbb{R}^{2n}\times\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\mathrm{D}\Psi(\partial_t x(t))\|_{\mathrm{Bil}(\mathbb{R}^{2n}\times\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\mathrm{D}J(x(t))\|_{\mathrm{Bil}(\mathbb{R}^{2n}\times\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\mathrm{DD}J(x(t))\|_{\mathrm{Tril}(\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\mathrm{D}X_t\big(x(t-\tau)\big)\|_{\mathcal{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa) \\
\|\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}X_t\right)\left(\partial_t x(t-\tau)\right)\|_{\mathcal{L}(\mathbb{R}^{2n},\mathbb{R}^{2n})} &\le D(\kappa)
\end{aligned}
\tag{9.26}
$$

et cetera for all the other corresponding expressions that show up in the derivatives of the maps in (9.28). Moreover, because of $H_0 \hookrightarrow \mathcal{C}^0$ there is a constant E

such that

$$\|\hat{x}(t)\|_{R^{2n}} \leq E$$

for all $t \in S^1$ and all $\hat{x} \in H_0$ with $\|\hat{x}\|_{H_0} \leq 1$. These pointwise estimates together with the L^2-estimates $\|x\|_{H_1} \leq \kappa$ and $\|\hat{x}\|_{H_0} \leq 1$ can be used to estimate (9.25) further by a constant $F(\kappa)$ that does not depend on x.

As an example, we estimate the first term in (9.25) as follows:

$$\begin{aligned}
&\Big\|\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\Big\|_{H_0} \\
&\qquad \leq \Big\|\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\Big\|_{L^2} \\
&\qquad\qquad + \Big\|\partial_t\Big(\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\Big)\Big\|_{L^2} \\
&\qquad \leq \Big\|\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\Big\|_{L^2} \\
&\qquad\qquad + \Big\|\mathrm{D}\Phi(\partial_t x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\Big\|_{L^2} \\
&\qquad\qquad + \Big\|\Phi(x)\mathrm{DD}J(x)\big(\partial_t x, \Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big\|_{L^2} \\
&\qquad\qquad + \Big\|\Phi(x)\mathrm{D}J(x)\big(\mathrm{D}\Phi(x)^{-1}(\partial_t x, \hat{x}), \partial_t x\big)\Big\|_{L^2} \\
&\qquad\qquad + \Big\|\Phi(x)\mathrm{D}J(x)\big(\Phi(x)^{-1}\partial_t\hat{x}, \partial_t x\big)\Big\|_{L^2} \\
&\qquad\qquad + \Big\|\Phi(x)\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t^2 x\big)\Big\|_{L^2} \\
&\qquad = \left(\int_0^1 \Big\|\Psi(x(t))\Big(\mathrm{D}J(x(t))\big(\Psi(x(t))^{-1}\hat{x}(t), \partial_t x(t)\big)\Big)\Big\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\qquad + \left(\int_0^1 \Big\|\mathrm{D}\Psi(\partial_t x(t))\Big(\mathrm{D}J(x(t))\big(\Psi(x(t))^{-1}\hat{x}(t), \partial_t x(t)\big)\Big)\Big\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\qquad + \left(\int_0^1 \Big\|\Psi(x(t))\mathrm{DD}J(x(t))\big(\partial_t x(t), \Psi(x(t))^{-1}\hat{x}(t), \partial_t x(t)\big)\Big\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\qquad + \left(\int_0^1 \Big\|\Psi(x(t))\mathrm{D}J(x(t))\big(\mathrm{D}\Psi(x(t))^{-1}(\partial_t x(t), \hat{x}(t)), \partial_t x(t)\big)\Big\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\qquad + \left(\int_0^1 \Big\|\Psi(x(t))\mathrm{D}J(x(t))\big(\Psi(x(t))^{-1}\partial_t\hat{x}(t), \partial_t x(t)\big)\Big\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\qquad + \left(\int_0^1 \Big\|\Psi(x(t))\mathrm{D}J(x(t))\big(\Psi(x(t))^{-1}\hat{x}(t), \partial_t^2 x(t)\big)\Big\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad \leq D(\kappa)^4\|\hat{x}\|_{L^2} + D(\kappa)^4\|\hat{x}\|_{L^2} + D(\kappa)^5\|\hat{x}\|_{L^2} + D(\kappa)^5\|\hat{x}\|_{L^2} \\
&\qquad\qquad + D(\kappa)^4\|\partial_t\hat{x}\|_{L^2} + D(\kappa)^3 E\big\|\partial_t^2 x\big\|_{L^2} \\
&\qquad \leq 2D(\kappa)^5 + 3D(\kappa)^4 + D(\kappa)^3 E\kappa
\end{aligned}$$

Similarly estimating the other terms in (9.25), we see that there is a constant $F(\kappa)$

(depending on the constants $D(\kappa)$, E and κ) such that

$$\begin{aligned}\|\mathcal{P}_{R,s}(x)\|_{\mathcal{L}(H_0,H_0)} \leq \sup_{\|\hat{x}\|_{H_0}\leq 1} \Bigg(& \Big\|\Phi(x)\Big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \partial_t x\big)\Big)\Big\|_{H_0} \\ & + \big\|\mathrm{D}\Phi(x)\big(\partial_t x, \Phi(x)^{-1}\hat{x}\big)\big\|_{H_0} \\ & + \big\|\tfrac{1}{2}\hat{x}\big\|_{H_0} \\ & + \|\Phi(x)\big(\mathrm{D}J(x)\big(\Phi(x)^{-1}\hat{x}, \mu_R(s)\cdot\mathcal{X}(x)\big)\big)\|_{H_0} \qquad (9.27) \\ & + \Big\|\Phi(x)\Big(\mu_R(s)\cdot\mathrm{D}\mathcal{X}(x)\big(\Phi(x)^{-1}\hat{x}\big)\Big)\Big\|_{H_0}\Bigg) \\ \leq F(\kappa) & \qquad (9.28)\end{aligned}$$

For ($\mathcal{V}3$), it remains to prove the additional estimate for $x \in H_2$. In (9.18), we estimated $\|\mathcal{X}(x)\|_{H_1}$ for $H_1 = W^{1,2}$ and $\|x\|_{H_1} \leq 1$ using the constant $C(\kappa)$ and equations (9.14), (9.15) and (9.16). In the same way, for $H_1 = W^{2,2}$ and $\|x\|_{H_1} \leq 1$ we can use the constant $D(\kappa)$ to estimate

$$\|\mathcal{X}(x)\|_{H_1} \leq G(\kappa), \qquad (9.29)$$

where $G(\kappa)$ is a constant that only depends on κ and $D(\kappa)$ and not on x. The computation is similar to the one in (9.18), just involving more derivatives.

For $x \in H_2$ and $\|x\|_{H_1} \leq \kappa$, we use (9.29) to see that

$$\begin{aligned}\|\mathcal{V}_{R,s}(x)\|_{H_1} &= \|-J(x)(\partial_t x - \mu_R(s)\mathcal{X}(x))\|_{H_1} \\ &\geq \|\partial_t x\|_{H_1} - \|\mu_R(s)\mathcal{X}(x)\|_{H_1} \\ &\geq \|\partial_t x\|_{H_1} - \|\mathcal{X}(x)\|_{H_1} \\ &\geq \|\partial_t x\|_{H_1} - G(\kappa). \qquad (9.30)\end{aligned}$$

We compute

$$\begin{aligned}\|x\|_{H_2} = \|x\|_{W^{3,2}} &= \big(\|x\|_{L^2}^2 + \|\partial_t x\|_{L^2}^2 + \|\partial_t^2 x\|_{L^2}^2 + \|\partial_t^3 x\|_{L^2}^2\big)^{\frac{1}{2}} \\ &\leq \|x\|_{L^2} + \big(\|\partial_t x\|_{L^2}^2 + \|\partial_t^2 x\|_{L^2}^2 + \|\partial_t^3 x\|_{L^2}^2\big)^{\frac{1}{2}} \\ &\leq \kappa + \|\partial_t x\|_{H_1}\end{aligned}$$

and continue the estimate (9.30) by

$$\begin{aligned}\|\mathcal{V}_{R,s}(x)\|_{H_1} &\geq \|\partial_t x\|_{H_1} - G(\kappa) \\ &\geq \|x\|_{H_2} - (\kappa + G(\kappa)).\end{aligned}$$

This implies (using that without loss of generality $\kappa + G(\kappa) \geq 1$) that

$$\begin{aligned}\|x\|_{H_2} &\leq \|\mathcal{V}_{R,s}(x)\|_{H_1} + \kappa + G(\kappa) \\ &\leq (\kappa + G(\kappa)\Big(\|\mathcal{V}_{R,s}(x)\|_{H_1} + 1\Big). \qquad (9.31)\end{aligned}$$

Combining the estimates (9.28) and (9.31), with

$$c_1(\kappa) := \max\{F(\kappa), \kappa + G(\kappa)\}$$

we have verified ($\mathcal{V}3$).

($\mathcal{V}4$) The parameter $s \in \mathbb{R}$ only appears in the second term of $\mathcal{V}_{R,s}$ (defined in (9.21)). Thus the derivative in s-direction in some $x \in H_1$ is

$$\begin{aligned}&\partial_s \mathcal{V}_{R,s} : H_1 \longrightarrow H_0\\ &\partial_s \mathcal{V}_{R,s}(x) = -\mu_R'(s) J(x) \cdot \mathcal{X}(x),\end{aligned}$$

and this is continuous by Lemma 9.6 (iii).

For the estimate, recall the constant $D(\kappa)$ that was introduced in (9.26). We use the estimate (9.29) to see that for every $x \in H_1$ with $\|x\|_{H_1} \leq \kappa$ it is

$$\|\mathcal{X}(x)\|_{H_0} \leq \|\mathcal{X}(x)\|_{H_1} \leq G(\kappa).$$

and thus for every $(R, s) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ we have

$$\begin{aligned}\|\partial_s \mathcal{V}_{R,s}(x)\|_{H_0} &= |\mu_R'(s)| \cdot \|\mathcal{X}(x)\|_{H_0}\\ &\leq C \cdot G(\kappa)\\ &=: c_2(\kappa)\end{aligned}$$

with some constant $C > 0$ such that $|\mu_R'(s)| \leq C$ for all $s \in \mathbb{R}$. □

We have shown that equation (9.1) fits into the setting of Chapter 8. Before one can apply Theorem 8.6, one still needs to find an estimate of the form (8.6). See Appendix B for a brief discussion and further references.

Chapter 10

Conclusion

In this thesis, we have described two instances of cases where methods from symplectic geometry can be used to study periodic solutions of delay differential equations.

In Part I, we used polyfold theory to show the existence of a smooth family of delay orbits parametrized by delay near a given non-degenerate orbit with zero delay. First we proved this for equations of the type $\dot{x}(t) = X_t(x(t-\tau))$ on $\mathbb{R}^n$, then we generalized it to several other types of equations. We expect that it can be further generalized, for instance to equations distributed delay (that is, involving an integral like $\int_{-T}^{0} x(t-\tau)\mathrm{d}\tau$, also known as delay integro-differential equations). It would be great to tackle also the question of neutral DDEs (that is, equations of the form $\dot{x}(t) = F_t(\dot{x}(t-\tau))$ or more complicated) or state-dependent delay, but we are in doubt that this can be done with polyfold theory alone. It would be very interesting to bring together the current research on neutral DDEs and state-dependent delay with polyfold methods.

In Part II, we proved a compactness theorem which can be applied to what should be (perturbed) gradient flow lines connecting periodic orbits of Hamiltonian delay equations. As mentioned in Section 8.1, the compactness result is one step towards proving the existence of such periodic Hamiltonian delay orbits with a given delay. A next task would be to complete this proof by carefully adjusting all the steps of the stretching argument described in Appendix B. The long-term objective is to define a Floer homology for Hamiltonian delay equations and (hopefully) prove some kind of Arnold conjecture.

Appendix A

Proofs for classical differentiability

In this appendix we give proofs for the facts that were mentioned in §4.2. The following basic observation is repeatedly used throughout this appendix.

Remark A.1. From $\|\varphi(\tau, x)\|_{H_m} = \|x\|_{H_m}$ and linearity of φ in the second variable we conclude that $\varphi(\tau, \cdot)$ is an H_m-isometry, that is, $\|\varphi(\tau, x) - \varphi(\tau, y)\|_{H_m} = \|x - y\|_{H_m}$. Therefore, for every $x \in H_m$, every sequence $(x_i)_i \subset H_m$ and every $\tau \in \mathbb{R}$ it is

$$\varphi(\tau, x_i) \to \varphi(\tau, x) \qquad \Longleftrightarrow \qquad x_i \to x.$$

Proof of Lemma 4.2. Let us recall that continuity of $\varphi : \mathbb{R} \to \mathcal{L}(H_m, H_m)$ with respect to the compact-open topology means the following: For sequences $(\tau_i)_{i \in \mathbb{N}}$ in $\mathbb{R}$ and $(x_i)_{i \in \mathbb{N}}$ in H_m, if $\tau_i \to \tau$ in $\mathbb{R}$ and $x_i \to x$ in H_m as $i \to \infty$, it follows that

$$\varphi(\tau_i, x_i) \longrightarrow \varphi(\tau, x)$$

in H_m as $i \to \infty$. We first note that it is enough to prove continuity at $\tau = 0$ since

$$\varphi(\tau_i, x_i) \to \varphi(\tau, x) \qquad \Longleftrightarrow \qquad \varphi(\tau_i - \tau, x_i) \to x$$

by the previous remark. This means that for any $\varepsilon > 0$ we need to show that

$$\|x - \varphi(\tau_i, x_i)\|_{H_m} \leq \varepsilon$$

for i sufficiently large. As a first step, we show the claim in the case of a constant sequence $x_i \equiv x \in H_m$. For $m = 0$, this is Lemma 2.1 from [FW21b], and we extend their proof to the case $m \neq 0$.

The map x may not be smooth, but it can be approximated in H_m by smooth elements. Fix $\varepsilon > 0$ and choose $\bar{x} \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$ with

$$\|\bar{x} - x\|_{H_m} \leq \frac{\varepsilon}{6}.$$

Now $\bar{x}$ and its derivatives $\partial_t^k \bar{x}, k = 0, \ldots, m$, are uniformly continuous, thus

$$\|\partial_t^k \bar{x}(t) - \partial_t^k \bar{x}(t - \tau_i)\|_{\mathbb{R}^n} \leq \frac{\varepsilon}{6(m+1)} \qquad \text{for all } t \in S^1$$

for all $k = 0, \dots, m$ and i sufficiently large. In particular, the H_0-distance satisfies

$$\|\partial_t^k \bar{x} - \varphi(\tau_i, \partial_t^k \bar{x})\|_{H_0} \leq \frac{\varepsilon}{6(m+1)}$$

and we can estimate

$$\begin{aligned}
\|\bar{x} - \varphi(\tau_i, \bar{x})\|_{H_m} &\leq \sum_{k=0}^{m} \|\partial_t^k \bar{x} - \partial_t^k \varphi(\tau_i, \bar{x})\|_{H_0} \\
&= \sum_{k=0}^{m} \|\partial_t^k \bar{x} - \varphi(\tau_i, \partial_t^k \bar{x})\|_{H_0} \\
&\leq (m+1) \cdot \frac{\varepsilon}{6(m+1)} = \frac{\varepsilon}{6}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x - \varphi(\tau_i, x)\|_{H_m} &\leq \|x - \bar{x}\|_{H_m} + \|\bar{x} - \varphi(\tau_i, \bar{x})\|_{H_m} + \|\varphi(\tau_i, \bar{x}) - \varphi(\tau_i, x)\|_{H_m} \\
&= \|x - \bar{x}\|_{H_m} + \|\bar{x} - \varphi(\tau_i, \bar{x})\|_{H_m} + \|\bar{x} - x\|_{H_m} \\
&\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.
\end{aligned}$$

In particular, we have proved the statement for any constant sequence $x_i \equiv x \in H_m$. Now for the general case, let $(x_i)_i \subseteq H_m$ be a sequence converging to x. For $\varepsilon > 0$ and i sufficiently large we established

$$\|x - \varphi(\tau_i, x)\|_{H_m} \leq \frac{\varepsilon}{2}.$$

After increasing i even further, we may assume that $\|x - x_i\|_{H_m} \leq \frac{\varepsilon}{2}$ since $x_i \to x$ converges in H_m. All in all we get

$$\begin{aligned}
\|x - \varphi(\tau_i, x_i)\|_{H_m} &\leq \|x - \varphi(\tau_i, x)\|_{H_m} + \|\varphi(\tau_i, x) - \varphi(\tau_i, x_i)\|_{H_m} \\
&= \|x - \varphi(\tau_i, x)\|_{H_m} + \|x - x_i\|_{H_m} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

as desired. □

To prove Lemma 4.4, we need the following elementary lemma about difference quotients of H_1-functions.

Lemma A.2. *Let $x \in H_1$. Then the following holds:*

(i) $\left\| \frac{\varphi(T,x) - x}{T} \right\|_{H_0} \leq \|\partial_t x\|_{H_0}$ *for $T \in \mathbb{R} \setminus \{0\}$.*

(ii) $\lim_{T \to 0} \left\| \frac{\varphi(T,x) - x}{T} - \partial_t x \right\|_{H_0} = 0.$

Proof. Since the map $x \in H_1$ is, in particular, weakly differentiable, we get

$$\|x(t+T) - x(t)\| \leq \int_0^1 \|\partial_t x(t+sT)\| |T| \, \mathrm{d}s$$

for every $t \in S^1$, $T \in \mathbb{R}$. Squaring this and using the Cauchy-Schwarz inequality leads to

$$\|x(t+T) - x(t)\|^2 \leq |T|^2 \int_0^1 \|\partial_t x(t+sT)\|^2 \, \mathrm{d}s$$

which is of course equivalent to

$$\frac{\|x(t+T) - x(t)\|^2}{T^2} \leq \int_0^1 \|\partial_t x(t+sT)\|^2 \, \mathrm{d}s.$$

Now we integrate over $t \in S^1$ and get

$$\begin{aligned}
\left\| \frac{\varphi(-T,x) - x}{T} \right\|_{H_0}^2 &= \int_0^1 \frac{\|x(t+T) - x(t)\|^2}{T^2} \, \mathrm{d}t \\
&\leq \int_0^1 \int_0^1 \|\partial_t x(t+sT)\|^2 \, \mathrm{d}s \, \mathrm{d}t \\
&= \|\partial_t x\|_{H_0}^2.
\end{aligned}$$

Using $\|\varphi(T,x) - x\|_{H_0} = \|\varphi(-T,x) - x\|_{H_0}$ the first assertion follows.

To show (ii), we approximate x in H_1 by smooth functions $x_k \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$. Using the triangle inequality we compute

$$\begin{aligned}
\left\| \frac{\varphi(T,x) - x}{T} - \partial_t x \right\|_{H_0} \leq & \left\| \frac{\varphi(T,x) - x}{T} - \frac{\varphi(T,x_k) - x_k}{T} \right\|_{H_0} \\
& + \left\| \frac{\varphi(T,x_k) - x_k}{T} - \partial_t x_k \right\|_{H_0} + \|\partial_t x_k - \partial_t x\|_{H_0}
\end{aligned}$$

for every k. The second term on the right-hand side goes to 0 for $T \to 0$ because x_k is smooth. The first term is bounded by $\|\partial_t x - \partial_t x_k\|_{H_0}$ according to (i) using linearity of φ in its second argument. Therefore, the second assertion follows. □

Proof of Lemma 4.4. We have to show that

$$\lim_{\|(T,\hat{x})\| \to 0} \tfrac{1}{\|(T,\hat{x})\|} \|\varphi(\tau + T, x + \hat{x}) - \varphi(\tau, x) - \varphi(\tau, \hat{x}) + T \cdot \varphi(\tau, \partial_t x)\|_{H_0} = 0,$$

where it is convenient to define the norm of the pair $(T, \hat{x})$ by $\|(T,\hat{x})\|^2 = |T|^2 + \|\hat{x}\|_{H_1}^2$.

Using $\|\varphi(\tau,x)\|_{H_0} = \|x\|_{H_0}$, we compute the following:

$$
\begin{aligned}
&\tfrac{1}{\|(T,\hat{x})\|^2}\|\varphi(\tau+T,x+\hat{x})-\varphi(\tau,x)-\varphi(\tau,\hat{x})+T\cdot\varphi(\tau,\partial_t x)\|_{H_0}^2\\
&\quad= \tfrac{1}{\|(T,\hat{x})\|^2}\|\varphi(T,x+\hat{x})-\varphi(0,x)-\varphi(0,\hat{x})+T\cdot\varphi(0,\partial_t x)\|_{H_0}^2\\
&\quad= \tfrac{1}{\|(T,\hat{x})\|^2}\|\varphi(T,x)+\varphi(T,\hat{x})-x-\hat{x}+T\cdot\partial_t x\|_{H_0}^2\\
&\quad\le \tfrac{1}{\|(T,\hat{x})\|^2}\Big(\|\varphi(T,x)-x+T\cdot\partial_t x\|_{H_0}+\|\varphi(T,\hat{x})-\hat{x}\|_{H_0}\Big)^2\\
&\quad= \tfrac{1}{\|(T,\hat{x})\|^2}\|\varphi(T,x)-x+T\cdot\partial_t x\|_{H_0}^2\\
&\qquad+\tfrac{1}{\|(T,\hat{x})\|^2}\|\varphi(T,\hat{x})-\hat{x}\|_{H_0}^2\\
&\qquad+\tfrac{2}{\|(T,\hat{x})\|^2}\|\varphi(T,x)-x+T\cdot\partial_t x\|_{H_0}\cdot\|\varphi(T,\hat{x})-\hat{x}\|_{H_0}
\end{aligned}
\tag{A.1}
$$

For the first term in (A.1) we use $\frac{1}{\|(T,\hat{x})\|} \le \frac{1}{|T|}$ and obtain

$$\tfrac{1}{\|(T,\hat{x})\|^2}\|\varphi(T,x)-x+T\cdot\partial_t x\|_{H_0}^2 \le \left\|\frac{\varphi(T,x)-x}{T}+\partial_t x\right\|_{H_0}^2 \longrightarrow 0$$

as $T\to 0$, where we used Lemma A.2 (ii). For the second term in (A.1) we similarly use $\frac{1}{\|(T,\hat{x})\|} \le \frac{1}{\|\hat{x}\|_{H_1}}$ and see

$$\tfrac{1}{\|(T,\hat{x})\|^2}\|\varphi(T,\hat{x})-\hat{x}\|_{H_0}^2 \le \tfrac{1}{\|\hat{x}\|_{H_1}^2}\|\partial_t\hat{x}\|_{H_0}^2\cdot T^2 \longrightarrow 0$$

as $T\to 0$, where we used Lemma A.2 (i). For the product in the third term in (A.1) we use the same arguments to treat both factors separately and see that both tend to 0. □

Proof of Lemma 4.6 (sketch). Using the formula (4.5) for the first derivative of φ, we get that if a second derivative of φ exists at the point $(\tau,x)\in\mathbb{R}\times H_m$, then it has to be

$$\mathrm{d}^2\varphi(\tau,x)\Big((T_1,\hat{x}_1),(T_2,\hat{x}_2)\Big) = -T_2\cdot\varphi(\tau,\partial_t\hat{x}_1)-T_1\cdot\varphi(\tau,\partial_t\hat{x}_2)+T_1\cdot T_2\cdot\varphi(\tau,\partial_t^2 x)$$

which is well-defined if $x\in H_2$. Iteratively computing what an m-th derivative of φ at (τ,x) should look like, we see that as a multilinear map

$$
\begin{aligned}
\mathrm{d}^m\varphi(\tau,x): \underbrace{(\mathbb{R}\times H_m)\times\cdots\times(\mathbb{R}\times H_m)}_{m\text{ times}} &\longrightarrow H_0\\
\Big((T_1,\hat{x}_1),\ldots,(T_m,\hat{x}_m)\Big) &\longmapsto \cdots+(-1)^m\prod_{i=1}^m T_i\cdot\varphi(\tau,\partial_t^m x)
\end{aligned}
\tag{A.2}
$$

it has a lot of summands that involve shifts by τ of the maps $\hat{x}_m$, $\partial_t\hat{x}_{m-1}$, $\partial_t^2\hat{x}_{m-2},\ldots$, $\partial_t^{m-1}\hat{x}_1$ and $\partial_t^m x$. Thus it needs an m-th derivative of x. To show that (A.2) really meets the definition of a derivative, one can estimate each summand exactly as in the proof of Lemma 4.4. Again, one can show that all these derivatives are continuous, so the map $\varphi:\mathbb{R}\times H_m\to H_0$ is $\mathcal{C}^m$. □

Appendix B

A stretching argument

In this appendix, we give a rough sketch of the stretching argument from [Sei17] which was mentioned in Chapters 8 and 9. We also comment on how to generalize it to the setting of Hamiltonian delay equations.

In the master thesis [Sei17], the following statement (which can be deduced from Hamiltonian Floer theory) is proven by a stretching argument.

Theorem B.1. *Let (M, ω) be a closed, symplectically aspherical manifold. Assume that $H : S^1 \times M \to \mathbb{R}$ is a Hamiltonian function (degenerate or not) such that the Hamiltonian flow $\phi_H^t : M \to M$ has only finitely many 1-periodic orbits. Then there are two contractible 1-periodic orbits x and y with $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$, where $\mathcal{A}_H : \mathcal{C}^\infty_{contr}(S^1, M) \to \mathbb{R}$ denotes the Hamiltonian action functional.*

The argument in [Sei17] is roughly[14] as follows:

(I) Contractible 1-periodic orbits of ϕ_H^1 are exactly the critical points of $\mathcal{A}_H$.

(II) The L^2-gradient of $\mathcal{A}$ is given by $\nabla \mathcal{A}_H(x) = J(x)\big(\partial_t x - X_H(x)\big)$, where X_H denotes the Hamiltonian vector field, J is an ω-compatible almost complex structure, and the metric on M is deduced from ω and J.

(III) Choose a family of cutoff functions $\mu_R : \mathbb{R} \to [0, 1]$, $R \in \mathbb{R}_{\geq 0}$ with

- $\mu_R(s) = 0$ for $s \leq -R + \delta$ and for $s \geq R - \delta$ for some $\delta > 0$,
- $\mu_R(s) = 1$ for $-R + 1 \leq s \leq R - 1$ (this can only happen for $R \geq 1$),

see Figure 9.1.

(IV) For each $R \in \mathbb{R}_{\geq 0}$, consider the following perturbed Floer equation for maps $u : \mathbb{R} \times S^1 \to M$:

$$\partial_s u(s,t) + J\big(u(s,t)\big)\Big(\partial_t u(s,t) - \mu_R(s) X_{H_t}\big(u(s,t)\big)\Big) = 0 \tag{B.1}$$

[14] Not all the steps are mentioned here. For instance, for the analysis part it makes sense to compactify the cylinder $\mathbb{R} \times S^1$ to the sphere S^2 and to write (B.1) in a coordinate-free way. However, these details would rather distract from the main argument, so they are omitted here.

For $R = 0$ this is the Cauchy–Riemann equation; for $R \to \infty$ it converges to the unperturbed Floer equation

$$\partial_s u(s,t) + J\big(u(s,t)\big)\Big(\partial_t u(s,t) - X_{H_t}\big(u(s,t)\big)\Big) = 0. \tag{B.2}$$

(V) Choose a point $P \in M$ which is not part of a periodic orbit of ϕ_H^t, and define a moduli space by

$$\mathcal{M} := \big\{(R,u) \mid R \in \mathbb{R}_{\geq 0}, u \in \mathcal{C}^\infty(\mathbb{R} \times S^1, M), u(0,0) = P, (R,u) \text{ satisfies (B.1)}\big\}$$

plus some topological condition on u.

(VI) For $R = 0$, equation (B.1) has exactly one solution (namely the constant one).

(VII) For suitable J, the moduli space $\mathcal{M}$ is cut out of $\mathbb{R} \times W^{1,2}(\mathbb{R} \times S^1, M)$ by a transverse Fredholm section of index 1, therefore it is a 1-dimensional manifold with boundary contained in $\{0\} \times W^{1,2}(\mathbb{R} \times S^1, M)$.

(VIII) If $\mathcal{M}$ was compact, we would have found a compact 1-dimensional manifold with boundary consisting of one point – a contradiction. Therefore, $\mathcal{M}$ is not compact.

(IX) So-called bubbling cannot happen because of symplectic asphericity of (M, ω), and so-called breaking is excluded by the condition $u(0,0) = P$. Therefore, the only source of non-compactness is that there are solutions $(R_\nu, u_\nu)_{\nu \in \mathbb{N}}$ with $R_\nu \to \infty$.

(X) These solutions converge to a solution of the unperturbed Floer equation.

(XI) Such an unperturbed Floer cylinder $u : \mathbb{R} \times S^1 \to M$ is a gradient flow line of $\mathcal{A}_H$ connecting two critical points of $\mathcal{A}_H$. Hence, there are two 1-periodic orbits x and y of ϕ_H^t with $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$.

We expect that the stretching argument can be generalized to periodic orbits of Hamiltonian delay equations in the sense of [AFS20]. Recall from Section 8.1 that [AFS20] define a *Hamiltonian delay equation* on a manifold M by the property that its periodic solutions arise as critical points of an action functional $\mathcal{A} : \mathcal{C}^\infty_{\text{contr}}(S^1, M) \to \mathbb{R}$ on the loop space. This functional will typically use several functions $H, K, \dots : J \to \mathbb{R}$ which we may call Hamiltonians.

Most of the steps above do not directly generalize to the case of Hamiltonian delay equations, but have to be adjusted. For instance, in order to generalize step (II), we will need further assumptions on $\mathcal{A}$ to ensure that it has a gradient of the form

$$\nabla \mathcal{A}(x) = J(x)\Big(\partial_t x - \mathcal{X}(x)\Big),$$

where the delay equation is encoded in a map $\mathcal{X}$ from the loop space to itself, which should satisfy certain continuity and differentiability properties. In the case of a delay equation given by a vector field X in $\mathbb{R}^{2n}$, we would have $\mathcal{X}(x)(t) = X_t\big(x(t - \tau)\big)$ for a fixed delay parameter $\tau \in \mathbb{R}$ (cf. Section 9.2). Another step which should be treated carefully is step (XI); this is closely related to [AF13].

The compactness theorem from Chapters 8 and 9 of this thesis are part of generalizing step (X): Given a sequence of pairs $(R_\nu, u_\nu) \in \mathcal{M}$, $\nu \in \mathcal{M}$, with $R_\nu \to R_* \in \mathbb{R}_{\geq 0}$, one needs to show that a subsequence of $(u_\nu)_\nu$ converges to some u_* such that $(R_*, u_*) \in \mathcal{M}$. So one first needs to translate this to the setting of fractal scale Hilbert spaces from Chapter 8, as shown in Chapter 9. Next, one should argue why there is a bound of the form (8.6). This can probably be done by considering only (perturbed) gradient flow lines with action in a fixed compact interval (yielding a bound on $\|\partial_s w_\nu\|_{L^2(I_T,H_0)}$) and combining this with the bubbling analysis from step (IX) and elliptic regularity, see also [AFS19, Remark 2.6]. Then one can apply our compactness result (Theorem 8.6) to see that $(u_\nu)_\nu$ converges to some u_* in $\mathcal{C}^0(I_T, H_1) \cap \mathcal{C}^1(I_T, H_0)$, that is on compact subsets $I_T \subset \mathbb{R}$ and in a somewhat weak topology. Finally, using the Arzelà–Ascoli theorem and elliptic regularity, one should be able to show that convergence is actually in $\mathcal{C}^\infty_{\text{loc}}\big(\mathbb{R}, \mathcal{C}^\infty(S^1, \mathbb{R}^{2n})\big)$.

Bibliography

[AB21] Pablo Amster and Melanie Bondorevsky, *Persistence and periodic solutions in systems of delay differential equations*, Applied Mathematics and Computation **403** (2021), 126193.

[AD14] Michèle Audin and Mihai Damian, *Morse Theory and Floer Homology*, Universitext (Berlin. Print), Springer, 2014.

[AF] Peter Albers and Urs Frauenfelder, *A Variational Approach to Givental's Nonlinear Maslov Index.*

[AF13] ———, *Exponential decay for sc-gradient flow lines*, Journal of Fixed Point Theory and Applications **13** (2013), 571–586.

[AFS19] Peter Albers, Urs Frauenfelder, and Felix Schlenk, *A compactness result for non-local unregularized gradient flow lines*, Journal of Fixed Point Theory and Applications **21** (2019), no. 1, 34.

[AFS20] ———, *Hamiltonian delay equations – examples and a lower bound for the number of periodic solutions*, Advances in Mathematics **373** (2020), 107319.

[AH90] Ovide Arino and Moulay Lhassan Hbid, *Periodic solutions for retarded differential systems close to ordinary ones*, Nonlinear Analysis: Theory, Methods and Applications **14** (1990), no. 1, 23 – 34.

[AO08] Ravi P. Agarwal and Donal O'Regan, *An introduction to Ordinary Differential Equations*, Universitext, Springer New York, 2008.

[AS20] Peter Albers and Irene Seifert, *Periodic delay orbits and the polyfold implicit function theorem*, 2020, arXiv:2011.14828, to be published in CMH.

[BDS+21] Franziska Beckschulte, Ipsita Datta, Irene Seifert, Anna-Maria Vocke, and Katrin Wehrheim, *A Polyfold Proof of Gromov's Non-squeezing Theorem*, Research Directions in Symplectic and Contact Geometry and Dynamics (Bahar Acu, Catherine Cannizzo, Dusa McDuff, Ziva Myer, Yu Pan, and Lisa Traynor, eds.), Association for Women in Mathematics Series, Springer, 2021, pp. 1–45.

[CF09] Kai Cieliebak and Urs Frauenfelder, *A Floer homology for exact contact embeddings*, Pacific Journal of Mathematics **239** (2009), no. 2, 251–316.

[CO18] Kai Cieliebak and Alexandru Oancea, *Symplectic homology and the Eilenberg–Steenrod axioms*, Algebraic & Geometric Topology **18** (2018), no. 4, 1953–2130.

[DvGVLW95] Odo Diekmann, Stephan A. van Gils, Sjoerd M. Verduyn Lunel, and Hans-Otto Walther, *Delay equations*, Springer New York, 1995.

[EGH00] Yakov M. Eliashberg, Alexander Givental, and Helmut Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. **Special Volume – Part II** (2000), 560–673.

[FFGW12] Oliver Fabert, Joel Fish, Roman Golovko, and Katrin Wehrheim, *Polyfolds: A first and second look*, EMS Surveys in Mathematical Sciences **3** (2012), 131–208.

[FH] Joel Fish and Helmut Hofer, *Polyfold Constructions: Tools, Techniques, and Functors*, book in preparation. Preprints available as [FH18a] and [FH18b].

[FH18a] ———, *Lectures on Polyfolds and Symplectic Field Theory*, 2018, arXiv:1808.07147, to appear as part of [FH].

[FH18b] ———, *Polyfold and SFT Notes I: A Primer on Polyfolds and Construction Tools*, 2018, arxiv:1806.07025, to appear as part of [FH].

[FO99] Kenji Fukaya and Kaoru Ono, *Arnold conjecture and Gromov–Witten invariant*, Topology **38** (1999), no. 5, 933–1048.

[Fra09] Urs Frauenfelder, *Fractal scale Hilbert spaces and scale Hessian operators*, 2009, arXiv:0912.1154.

[Fra20] ———, *Nullity bounds for certain Hamiltonian delay equations*, 2020, arXiv:2005.07535.

[FW21a] Benjamin Filippenko and Katrin Wehrheim, *A Polyfold Proof of the Arnold Conjecture*, Selecta Mathematica **28** (2021), no. 11.

[FW21b] Urs Frauenfelder and Joa Weber, *The shift map on Floer trajectory spaces*, J. Symplectic Geom. **19** (2021), no. 2, 351 – 397.

[FZW19] Benjamin Filippenko, Zhengyi Zhou, and Katrin Wehrheim, *Counterexamples in scale calculus*, Proceedings of the National Academy of Sciences **116** (2019), no. 18, 8787–8797.

[Hal77] Jack K. Hale, *Theory of Functional Differential Equations*, Springer New York, 1977.

[Har16] Ferenc Hartung, *Differentiability of solutions with respect to the delay function in functional differential equations*, Electronic Journal of Qualitative Theory of Differential Equations **2016** (2016), no. 73, 1–16.

[HKWW06] Ferenc Hartung, Tibor Krisztin, Hans-Otto Walther, and Jianhong Wu, *Chapter 5: Functional Differential Equations with State-Dependent Delays: Theory and Applications*, Handbook of Differential Equations: Ordinary Differential Equations (A. Cañada, P. Drábek, and A. Fonda, eds.), vol. 3, North-Holland, 2006, pp. 435 – 545.

[HL91] J. K. Hale and Luiz A.C Ladeira, *Differentiability with respect to delays*, Journal of Differential Equations **92** (1991), 14–26.

[HQ06] My Lhassan Hbid and Redouane Qesmi, *Periodic solutions for functional differential equations with periodic delay close to zero*, Electronic Journal of Differential Equations **2006** (2006), no. 141, 1–12.

[HWZ09] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, *A general Fredholm theory. II. Implicit function theorems*, Geom. Funct. Anal. **19** (2009), no. 1, 206–293.

[HWZ10] ———, *Sc-smoothness, retractions and new models for smooth spaces*, Discrete and Continuous Dynamical Systems **28** (2010), no. 2, 665–788.

[HWZ17] ———, *Applications of Polyfold Theory I: The Polyfolds of Gromov–Witten Theory*, American Mathematical Society, 2017.

[HWZ21] ———, *Polyfold and Fredholm Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 2021.

[Kan11] Jungsoo Kang, *Local invariant for scale structures on mapping spaces*, 2011, arXiv:1108.0327.

[Kua93] Yang Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Inc, 1993.

[KY74] James L. Kaplan and James A. Yorke, *Ordinary differential equations which yield periodic solutions of differential delay equations*, Journal of Mathematical Analysis and Applications **48** (1974), no. 2, 317 – 324.

[LN20] Alejandro López Nieto, *Periodic orbits of delay equations with monotone feedback and even-odd symmetry*, 2020, arXiv:2002.01313.

[Moe02] Ieke Moerdijk, *Orbifolds as Groupoids: an Introduction*, Orbifolds in Mathematics and Physics. Proceedings of a conference on mathematical aspects of orbifold string theory, Madison, WI, USA (Alejandro Adem, Jack Morava, and Yongbin Ruan, eds.), American Mathematical Society, 2002, pp. 205–222.

[MP88] John Mallet-Paret, *Morse decompositions for delay-differential equations*, Journal of Differential Equations **72** (1988), no. 2, 270 – 315.

[MS12] Dusa McDuff and Dietmar Salamon, *J-holomorphic Curves and Symplectic Topology, second edition*, American Mathematical Society, Providence, Rhode Island, 2012.

[Nis19] Junya Nishiguchi, *C^1-smooth dependence on initial conditions and delay: spaces of initial histories of Sobolev type, and differentiability of translation in L^p*, Electronic Journal of Qualitative Theory of Differential Equations **91** (2019), 1–32.

[Nus73] Roger D. Nussbaum, *Periodic solutions of some nonlinear, autonomous functional differential equations. ii*, Journal of Differential Equations **14** (1973), no. 2, 360 – 394.

[Rih21] Fathalla A. Rihan, *Delay Differential Equations and Applications to Biology*, Forum for Interdisciplinary Mathematics, Springer Singapore, 2021.

[Sal99] Dietmar Salamon, *Lectures on Floer homology*, Symplectic geometry and topology **7** (1999), 143–229.

[Sei17] Irene Seifert, *Fixed points of Hamiltonian diffeomorphisms on closed symplectically aspherical manifolds*, Master's thesis, WWU Münster, Germany, 2017, master thesis, to be found on `https://www.mathi.uni-heidelberg.de/~geodyn/theses.html`.

[Sie12] Jan Sieber, *Finding periodic orbits in state-dependent delay differential equations as roots of algebraic equations*, Discrete and Continuous Dynamical Systems - A **32** (2012), 2607.

[Smi11] Hal Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer New York, 2011.

[Wal72] Wolfgang Walter, *Gewöhnliche Differentialgleichungen*, Springer, 1972.

[Wal19] Hans-Otto Walther, *Differentiability in Fréchet spaces and delay differential equations*, Electronic Journal of Qualitative Theory of Differential Equations **2019** (2019), 1–44.

[Weh12] Katrin Wehrheim, *Fredholm notions in scale calculus and Hamiltonian Floer theory*, 2012, arXiv:1209.4040.

[Wen15] Chris Wendl, *Lectures on Holomorphic Curves in Symplectic and Contact Geometry*, 2015, lecture notes. Preprint arXiv:1011.1690, updates and a newer version on the author's webpage.

[Wen16] ———, *Lectures on Symplectic Field Theory*, 2016, book in preparation. Preprint arXiv:1612.01009, updates and a newer version on the author's webpage.

[Wen18] Chris Wendl, *Holomorphic Curves in Low Dimensions*, Lecture Notes in Mathematics, Springer, 2018.