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Deformation theory of G-valued pseudocharacters and symplectic determinant laws

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Abstract

We give an introduction to the theory of pseudorepresentations of Taylor, Rouquier, Chenevier and Lafforgue. We refer to Taylor's and Rouquier's pseudorepresentations as *pseudocharacters*. They are very closely related, the main difference being that Taylor's pseudocharacters are defined for a group, where as Rouquier's pseudocharacters are defined for algebras. Chenevier's pseudorepresentations are so-called polynomial laws and will be called *determinant laws*. Lafforgue's pseudorepresentations are a generalization of Taylor's pseudocharacters to other reductive groups G, in that the corresponding notion of representation is that of a G-valued representation of a group. We refer to them as *G-pseudocharacters*.

We survey the known comparison theorems, notably Emerson's bijection between Chenevier's determinant laws and Lafforgue's GL_n-pseudocharacters and the bijection with Taylor's pseudocharacters away from small characteristics.

We show, that duals of determinant laws exist and are compatible with duals of representations. Analogously, we obtain that tensor products of determinant laws exist and are compatible with tensor products of representations. Further the tensor product of Lafforgue's pseudocharacters agrees with the tensor product of Taylor's pseudocharacters.

We generalize some of the results of [Che14] to general reductive groups, in particular we show that the (pseudo)deformation space of a continuous Lafforgue G-pseudocharacter of a topologically finitely generated profinite group Γ with values in a finite field (of characteristic p) is noetherian. We also show, that for specific groups G it is sufficient, that Γ satisfies Mazur's condition Φ_p.

One further goal of this thesis was to generalize parts of [BIP21] to other reductive groups. Let $F/\mathbb{Q}_p$ be a finite extension. In order to carry this out for the symplectic groups Sp_{2d}, we obtain a simple and concrete stratification of the special fiber of the pseudodeformation space of a residual G-pseudocharater of $\mathrm{Gal}(\overline{F}/F)$ into obstructed subloci $\overline{X}_{\overline{\Theta}}^{\mathrm{dec}}$, $\overline{X}_{\overline{\Theta}}^{\mathrm{pair}}$, $\overline{X}_{\overline{\Theta}}^{\mathrm{spcl}}$ of dimension smaller than the expected dimension $n(2n+1)[F:\mathbb{Q}_p]$.

We also prove that Lafforgue's G-pseudocharacters over algebraically closed fields for possibly non-connected reductive groups G come from a semisimple representation. We introduce a formal scheme and a rigid analytic space of all G-pseudocharacters by a functorial description and show, building on our results of noetherianity of pseudodeformation spaces, that both are representable and admit a decomposition as a disjoint sum indexed by continuous pseudocharacters with values in a finite field up to conjugacy and Frobenius automorphisms.

At last, in joint work with Mohamed Moakher, we give a new definition of determinant laws for symplectic groups, which is based on adding a 'Pfaffian polynomial law' to a determinant law which is invariant under an involution. We prove the expected basic properties in that we show that symplectic determinant laws over algebraically closed fields are in bijection with conjugacy classes of semisimple representation and that Cayley-Hamilton lifts of absolutely irreducible symplectic determinant laws to henselian local rings are in bijection with conjugacy classes of representations. We also give a comparison map with Lafforgue's pseudocharacters and show that it is an isomorphism over reduced rings.

Zusammenfassung

Wir geben eine Einführung in die Theorie der Pseudodarstellungen von Taylor, Rouquier, Chenevier und Lafforgue. Wir bezeichnen Taylor's und Rouquier's Pseudodarstellungen als *Pseudocharaktere*. Es gibt einen engen Zusammenhang zwischen diesen Begriffen, der Hauptunterschied besteht darin, dass Taylor's Pseudocharaktere für eine Gruppe definiert werden, während Rouquier's Pseudocharaktere für Algebren definiert werden. Chenevier's Pseudodarstellungen sind sogenannte polynomische Gesetze, die *Determinantengesetze* genannt werden. Lafforgue's Pseudodarstellungen sind eine Verallgemeinerung von Taylor's Pseudodarstellungen auf andere reduktive Gruppen G, d.h. der zugehörige Begriff von Darstellung ist der einer G-wertigen Darstellung einer Gruppe. Wir nenne sie *G-Pseudocharaktere*.

Wir geben einen Überblick über die bekannten Vergleichssätze, wie Emerson's Bijektion zwischen Chenevier's Determinantengesetzen und Lafforgue's GL_n-Pseudocharakteren und die Bijektion zwischen Taylor's Pseudocharakteren und den beiden erstgenannten Begriffen in nicht kleiner Charakteristik.

Wir zeigen, dass Duale von Determinantengesetzen existieren und verträglich mit Dualen von Darstellungen sind. Analog erhalten wir, dass Tensorprodukte von Determinantengesetzen existieren und verträglich mit Tensorprodukten von Darstellungen sind. Weiterhin stimmen Tensorprodukte von Lafforgue's Pseudocharakteren mit Tensorprodukten von Taylor's Pseudocharakteren überein.

Wir verallgemeinern einige der Ergebnisse von [Che14] auf allgemeine reduktive Gruppen. Insbesondere zeigen wir, dass der Pseudodeformationsraum eines stetigen G-Pseudocharakters einer topologisch endlich erzeugten proendlichen Gruppe Γ mit Werten in einem endlichen Körper (von Charakteristik p) noethersch ist. Wir zeigen auch, dass es für spezielle Gruppen G genügt, dass Γ Mazur's Bedingung Φ_p erfüllt.

Ein weiteres Ziel dieser Arbeit war es, Teile von [BIP21] auf andere reduktive Gruppen zu verallgemeinern. Sei $F/\mathbb{Q}_p$ eine endliche Erweiterung. Um das für die symplektischen Gruppen Sp_{2d} durchzuführen, geben wir eine einfache und konkrete Stratifizierung der speziellen Faser des Pseudodeformationsraums eines residuellen Sp_{2d}-Pseudocharakters $\overline{\Theta}$ von $\mathrm{Gal}(\overline{F}/F)$ in obstruierte Unterräume $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ an, deren Dimension kleiner, als die erwartete Dimension $n(2n+1)[F:\mathbb{Q}_p]$ des Gesamtraums ist.

Wir zeigen auch, dass Lafforgue's G-Pseudocharaktere über algebraisch abgeschlossenen Körpern für möglicherweise nicht-zusammenhängende reduktive Gruppen G von einer halbeinfachen Darstellung kommen. Wir führen ein formales Schema und einen rigid-analytischen Raum von allen G-Pseudocharakteren durch eine funktorielle Beschreibung ein, wobei wir auf unsere Ergebnisse zur Noetherschheit der Pseudodeformationsräume zurückgreifen. Wir zeigen dass beide Funktoren darstellbar sind und in eine disjunkte Vereinigung zerfallen, wobei die Indexmenge aus stetigen Pseudodarstellungen mit Werten in einem endlichen Körper bis auf Konjugation und Frobeniusautomorphismen besteht.

Zuletzt geben wir in gemeinsamer Arbeit mit Mohamed Moakher eine neue Definition von Determinantengesetzen für die symplektischen Gruppen, welche darauf basiert einem Determinantengesetz, welches invariant unter einer Involution ist, ein 'Pfaffsches polynomisches Gesetz' hinzuzufügen. Wir zeigen die Eigenschaften die man von Pseudodarstellungen erwartet: Symplektische Determinantengesetze über algebraisch abgeschlossenen Körpern sind in Bijektion mit Äquivalenzklassen von halbeinfachen symplektischen Darstellungen und Cayley-Hamilton Lifts zu henselschen lokalen Ringen eines absolut irreduziblen symplektischen Determinantengesetzes sind in Bijektion mit Äquivalenzklassen von Darstellugnen. Wir geben auch eine Vergleichsabbildung mit Lafforgue's Pseudocharakteren für GL_n an und zeigen, dass diese ein Isomorphismus über reduzierten Ringen ist.

Danksagungen

An erster Stelle möchte ich mich herzlich bei meinem Betreuer Gebhard Böckle bedanken, der mich während der Entstehung dieser Arbeit stets geduldig und hilfsbereit unterstützt hat. Ohne seine Zeit und sein präzises Feedback hätte die Arbeit in dieser Form nicht entstehen können. Auch danke ich ihm für den Vorschlag, mich mit Pseudocharakteren zu beschäftigen, die sich als sehr nützliches Hilfsmittel in der arithmetischen Geometrie erwiesen haben.

Weiterhin bedanke ich mich bei Vytautas Paškūnas für aufschlussreichen Austausch im Rahmen unserer Zusammenarbeit im Laufe des letzten Jahres und bei Mohamed Moakher, der die Entwicklung der Theorie symplektischer Determinantengesetze im letzten Teil dieser Arbeit ermöglicht hat. Nicht zuletzt sei erwähnt, dass die Arbeit mit Moakher in einer frühen Phase von Gesprächen mit Claudio Procesi inspiriert wurde.

Bei technischen Fragen zur Invariantentheorie und guten Filtrierungen hat ein Austausch mit Stephen Donkin und Ariel Weiss zu entscheidenen Fortschritten geführt.

Zuletzt bedanke ich mich bei all meinen Kollegen, in deren fruchtbarem Umfeld immer ein reger Austausch über Mathematik möglich war.

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Contents

Introduction

The first chapter of this thesis is an introduction to the theory of pseudocharacters of Taylor, Rouquier, Chenevier and Lafforgue. We survey the known comparison theorems, notably Emersons bijection between determinant laws and Lafforgue's GL_n-pseudocharacters and the bijection with Taylor's pseudocharacters away from small characteristics.

At this early stage I also got interested in defining natural operations on pseudocharacters: Direct sum, duals and tensor products. Their construction and compatibility with the comparison isomorphisms is the first main result of this thesis.

Theorem A.

1. Duals of determinant laws exist and are compatible with duals of representations (Section 3.10).
2. Tensor products of determinant laws exist (Proposition 3.20) and are compatible with tensor products of representations (Proposition 3.21). Further the tensor product of Lafforgue's pseudocharacters agrees with the tensor product of Taylor's pseudocharacters (Proposition 4.34).

The second main goal of this thesis was to generalize [Che14] to general reductive groups. Firstly, we prove that the deformation space of a continuous Lafforgue G-pseudocharacter of a topologically finitely generated profinite group with values in a finite field is noetherian.

Theorem B (Theorem 6.11, Theorem 6.14)**.** Let L be a p-adic local field with ring of integers $\mathcal{O}_L$ and residue field κ. Let G be a generalized reductive $\mathcal{O}_L$-group scheme, let Γ be a profinite group and let $\overline{\Theta}$ be a continuous G-pseudocharacter of Γ over κ.

1. If Γ is topologically finitely generated, then the G-pseudodeformation ring $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ of $\overline{\Theta}$ is noetherian.
2. Assume that $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$, $p > 2$ in the orthogonal cases and let $\iota : G \to \mathrm{GL}_d$ be the standard representation of G. Then the canonical map $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is surjective. If in addition Γ satisfies Mazur's condition Φ_p, then $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is noetherian.

Secondly, we prove that the formal scheme and the rigid analytic space of pseudocharacters admit a decomposition as a disjoint sum indexed by continuous pseudocharacters with values in a finite field up to conjugacy and Frobenius automorphisms. See Definition 7.5 and Lemma 7.7 for a description of the index set $|\,\mathrm{PC}^{\Gamma}_G\,|$.

Theorem C (Theorem 7.21)**.** Let Γ be a topologically finitely generated profinite group. Let G be a connected reductive group over the ring of integers of a p-adic local field. Define $\tilde{X}_G : \mathrm{An}_K^{\mathrm{op}} \to \mathrm{Set}$ as the functor, that associates to every rigid analytic space $Y \in \mathrm{An}_K$ the set of continuous G-pseudocharacters $\mathrm{cPC}^{\Gamma}_G(\mathcal{O}(Y))$. Then $\tilde{X}_G$ is representable by the quasi-Stein space $\coprod_{z \in |\,\mathrm{PC}^{\Gamma}_G\,|} \mathfrak{X}^{\mathrm{rig}}_{G,z}$.

One further goal of this thesis was to generalize parts of [BIP21] to other reductive groups. In order to carry this out for the symplectic groups Sp_{2d}, we need to analyze the special fiber of the deformation space of Lafforgue's Sp_{2d}-pseudocharacters. We obtain a stratification of the pseudodeformation space into obstructed subloci $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ of lower dimension as follows.

Theorem D (Proposition 6.33, Theorem 6.34, Corollary 6.35)**.** Let $F/\mathbb{Q}_p$ be a finite extension. Let $\overline{\Theta}$ be a continuous Sp_{2n}-pseudocharacter of the absolute Galois group Γ_F of F over κ.

1. $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq n(2n+1)[F:\mathbb{Q}_p] - 4(n-1)[F:\mathbb{Q}_p]$.
2. $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq n^2[F:\mathbb{Q}_p] + 1$.
3. $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F:\mathbb{Q}_p] + 1$.
4. $\dim \overline{X}_{\overline{\Theta}} \leq n(2n+1)[F:\mathbb{Q}_p]$.

If $\overline{\Theta}$ comes from an absolutely irreducible representation, then there are non-special irreducible pseudodeformations of $\overline{\Theta}$ and in (4) equality of dimensions holds.

We expect, that Theorem D is sufficient to prove the main result of [BIP21] for symplectic groups. However in an ongoing project with Vytautas Paškūnas, we will prove the main theorem of [BIP21] for general disconnected reductive groups over the ring of integers of a p-adic local field. So Theorem D can be seen as an alternative approach. It also gives a simple stratification of the pseudodeformation space, whereas the general proof rests upon a less concrete stratification.

In the course of this collaboration we proved the reconstruction theorem for Lafforgue's pseudocharacters for disconnected reductive groups.

Theorem E (Theorem 4.56)**.** Let G be a generalized reductive group scheme over a noetherian commutative ring $\mathcal{O}$. Let Γ be a group. Let k be an algebraically closed field over $\mathcal{O}$ and let $\Theta \in \mathrm{PC}_G^{\Gamma}$. Then there is a G-completely reducible representation $\rho : \Gamma \to G(k)$ with $\Theta_\rho = \Theta$, which is unique up to $G^0(k)$-conjugation.

In early 2020 I turned to working on generalizing determinant laws to symplectic and orthogonal groups. In this time I developped some theory of $*$-determinants. The problem with this definition was, that it is not able to distinguish between symplectic and orthogonal groups, so in the pseudodeformation space might contain both symplectic and orthogonal points.

I started a collaboration with Mohamed Moakher, who has also been working on determinant laws for classical groups. By introducing a Pfaffian polynomial law, he has found a way of asking a $*$-determinant law to be symplectic, which lead to our work on symplectic determinant laws. See the introduction of Section 8 for a detailed list of results.

Einführung

Das erste Kapitel dieser Arbeit ist eine Einführung in die Theorie der Pseudocharaktere von Taylor, Rouquier, Chenevier und Lafforgue. Wir geben einen Überblick über die bekannten Vergleichssätze, insbesondere Emersons Bijektion zwischen Determinantengesetzen und Lafforgues GL_n-Pseudocharakteren und die Bijektion mit Taylors Pseudocharakteren in nicht kleiner Charakteristik.

In dieser frühen Phase begann ich, mich auch für die Definition natürlicher Operationen auf Pseudocharakteren zu interessieren: Direkte Summe, Duale und Tensorprodukte. Ihre Konstruktion und Kompatibilität mit den Vergleichsisomorphismen sind das erste Hauptergebnis dieser Arbeit.

Theorem A.

1. Duale von Determinantengesetzen existieren und sind verträglich mit Dualen von Darstellungen (Section 3.10).
2. Tensorprodukte von Determinantengesetzen existieren (Proposition 3.20) und sind verträglich mit Tensorprodukten von Darstellungen. Weiterhin sind Tensorprodukte von Lafforgue's Pseudocharakteren verträglich mit Tensorprodukten von Taylor's Pseudocharakteren (Proposition 4.34).

Das zweite gesetzte Ziel dieser Arbeit war es, die Hauptergebnisse von [Che14] auf allgemeine reduktive Gruppen zu verallgemeinen. Zuerst beweisen wir, dass der Deformationsraum eines stetigen G-Pseudocharakters nach Lafforgue von einer topologisch endlich erzeugten proendlichen Gruppe mit Werten in einem endlichen Körper noethersch ist.

Theorem B (Theorem 6.11, Theorem 6.14)**.** Sei L ein p-adischer lokaler Körper mit Ganzheitsring $\mathcal{O}_L$ und Restklassenkörper κ. Sei G ein verallgemeinertes reduktives $\mathcal{O}_L$-Gruppenschema, sei Γ eine proendliche Gruppe und sei $\overline{\Theta}$ ein stetiger G-Pseudocharakter von Γ über κ.

1. Ist Γ topologisch endlich erzeugt, so ist der G-Pseudodeformationsring $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ von $\overline{\Theta}$ noethersch.
2. Nehme an $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$, $p > 2$ in den orthogonalen Fällen und sei $\iota : G \to \mathrm{GL}_d$ die Standarddarstellung von G. Dann ist die kanonische Abbildung $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ surjektiv. Falls weiterhin Γ Mazur's Bedingung Φ_p erfüllt, so ist $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ noethersch.

Als zweites beweisen wir, dass das formale Schema und der rigid-analytische Raum von Pseudocharakteren eine Zerlegung als disjunkte Vereinigung indiziert von stetigen Pseudocharakteren mit Werten in einem endlichen Körper bis auf Konjugation und Frobeniusautomorphismus besitzt. Siehe Definition 7.5 und Lemma 7.7 für eine Beschreibung der Indexmenge $|\,\mathrm{PC}^{\Gamma}_{G}\,|$.

Theorem C (Theorem 7.21)**.** Sei Γ eine topologisch endlich erzeugte proendliche Gruppe. Sei G eine zusammenhängende reduktive Gruppe über dem Ganzheitsring eines p-adischen lokalen Körpers. Definiere $\tilde{X}_G : \mathrm{An}_K^{\mathrm{op}} \to \mathrm{Set}$ als den Funktor, der einem rigid-analytischen Raum $Y \in \mathrm{An}_K$ die Menge der stetigen G-Pseudocharaktere $\mathrm{cPC}^{\Gamma}_G(\mathcal{O}(Y))$ zuordnet. Dann ist $\tilde{X}_G$ durch den quasi-Stein rigid-analytischen Raum $\coprod_{z \in |\,\mathrm{PC}^{\Gamma}_G\,|} \mathfrak{X}^{\mathrm{rig}}_{G,z}$ darstellbar.

Ein weiteres Ziel dieser Arbeit war es [BIP21] auf andere reduktive Gruppen zu verallgemeinern. Um das für die symplektischen Gruppen Sp_{2d} durchführen zu können, ist es erforderlich, die spezielle Faser des Deformationsraumes von Lafforgueschen G-Pseudocharakteren zu analysieren. Wir erhalten wie folgt eine Stratifizierung des Pseudodeformationsraums durch obstruierte Teilräume $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ niedrigerer Dimension.

Theorem D (Proposition 6.33, Theorem 6.34, Corollary 6.35)**.** Sei $F/\mathbb{Q}_p$ eine endliche Erweiterung. Sei $\overline{\Theta}$ ein stetiger Sp_{2n}-Pseudocharakter der absoluten Galoisgruppe Γ_F von F über κ.

1. $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq n(2n+1)[F : \mathbb{Q}_p] - 4(n-1)[F : \mathbb{Q}_p]$.
2. $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq n^2[F : \mathbb{Q}_p] + 1$.
3. $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F : \mathbb{Q}_p] + 1$.

4. $\dim \overline{X}_{\overline{\Theta}} \leq n(2n+1)[F : \mathbb{Q}_p]$.

Falls $\overline{\Theta}$ von einer absolut irreduziblen Darstellung kommt, dann gibt es nicht-spezielle irreduzible symplektische Pseudodeformationen von $\overline{\Theta}$ und (4) gilt Gleichheit der Dimensionen.

Wir denken, dass Theorem D genügt, um das Hauptresultat von [BIP21] auf symplektische Gruppen zu übertragen. In einem laufenden Projekt mit Vytautas Paškūnas werden wir das Hauptergebnis von [BIP21] auf allgemeine (auch unzusammenhängende) reduktive Gruppen über dem Ganzheitsring eines p-adischen lokalen Körpers übertragen. Somit kann Theorem D als alternativer Zugang zu einem solchen Ergebnis gesehen werden. Des weiteren liefert Theorem D eine einfache Stratifizierung des Pseudodeformationsraumes, während der allgemeine Beweis auf nicht weiter konkretisierten Unterteilungen basiert.

Im Rahmen dieser Zusammenarbeit benötigen wir auch einen Rekonstruktionssatz für unzusammenhängende reduktive Gruppen, welcher uns in dieser Form neu erscheint.

Theorem E (Theorem 4.56)**.** Sei G ein verallgemeinertes reduktives Gruppenschema über einem noetherschen kommutativen Ring $\mathcal{O}$. Sei Γ eine Gruppe. Sei k ein algebraisch abgeschlossener Körper über $\mathcal{O}$ und sei $\Theta \in \mathrm{PC}_G^{\Gamma}$. Dann gibt es eine G-vollständig reduzible Darstellung $\rho : \Gamma \to G(k)$ mit $\Theta_\rho = \Theta$, welche eindeutig bis auf $G^0(k)$-Konjugation ist.

Im Frühjahr 2020 begann ich damit, Determinantengesetze auf sympmlektische und orthogonale Gruppen zu verallgemeinern. In dieser Zeit habe ich auch eine gewisse Theorie von $*$-Determinanten entwickelt. Diese wird in dieser Arbeit nicht weiter ausgeführt. Das Problem mit einer naiven Definition von war, dass eine $*$-Determinante nicht in der Lage dazu ist zwischen symplektischen und orthogonalen Gruppen zu unterscheiden, insbesondere könnte der Pseudodeformationsraum sowohl symplektische, also auch orthogonale Punkte enthalten.

Ich begann eine Zusammenarbeit mit Mohamed Moakher, der parallel und unabhängig an Determinantengesetzen für die symplektische Gruppe arbeitete. Durch Einführung eines Pfaffschen polynomischen Gesetzes gelang es ihm die Forderung an eine $*$-Determinante symplektisch zu sein zu formulieren. Das führte zu unserer gemeinsamen Arbeit an symplektischen Determinantengesetzen. Siehe die Einführung von Section 8 für eine detailliertere Liste der Ergebnisse.

1 Motivation for pseudocharacters

1.1 p-adic Langlands

The p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is a bijection between unitary Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and continuous representations of the absolute Galois group $\Gamma_{\mathbb{Q}_p}$. A proof of this correspondence for all primes p was given by Colmez, Dospinescu and Paškūnas [CDP13b]. A key ingredient for the surjectivity of this correspondence is the density of crystalline points in the deformation space of a mod p representation of $\Gamma_{\mathbb{Q}_p}$. For $p = 2$ this turned out to be exceptionally difficult and was carried out in [CDP13a].

In view of these results it makes sense to take a step back and ask for density of crystalline points in the framed deformation space of a mod p representation $\overline{\rho} : \Gamma_F \to G(\overline{\mathbb{F}}_p)$ of the absolute Galois group Γ_F of a p-adic local field $F/\mathbb{Q}_p$ valued in a reductive group G. For $G = \mathrm{GL}_d$ this has recently been proved by Böckle, Iyengar and Paškūnas [BIP21; BIP22] along with *Gouvêa's dimension conjecture* [Gou01, Lecture 4]: The universal framed deformation ring $R^{\square}_{\overline{\rho}}$ is a local complete intersection ring of relative dimension $(\dim \mathrm{GL}_d) \cdot (1 + [F : \mathbb{Q}_p])$. The proof of the results of [BIP21] rely on a careful analysis of the special fiber of the universal pseudodeformation space by Böckle and Juschka [BJ19]. For their work, they use Chenevier's notion of determinant laws for pseudocharacters.

The results of Section 6 can be seen as a step in this direction for the symplectic groups Sp_{2d} using the newly constructed deformation spaces of Lafforgue's G-pseudocharacters. Since [Eme18] it is known, that Lafforgue's pseudocharacters are a generalization of Chenevier's determinant laws over any base ring. We give dimension estimates for the symplectic groups Sp_{2d}, $d \geq 1$ analogous to [BJ19]. It is sufficient to give upper bounds for the dimension of certain obstructed subloci to carry out a proof of Gouvea's dimension conjecture following a stategy similar to [BIP21]. In an ongoing project with Vytautas Paškūnas we aim to prove Gouvea's conjecture using certain GIT quotients instead of one of the notions of pseudocharacters discussed in this thesis.

1.2 Foundational questions

There are three fundamental requirements for a reasonable notion of pseudorepresentation over a given commutative base ring A:

1. The functor, that maps a commutative A-algebra B to the set of pseudorepresentations over B should be representable by an affine A-scheme.
2. The pseudorepresentations over an algebraically closed A-field k shall be in bijection with isomorphism classes of semisimple representations over k. We refer to such a statement as a *reconstruction theorem*.
3. Over henselian local A-algebras A' and under mild unobstructedness conditions, e.g. residual irreducibility or multiplicity freeness there should be a bijection between pseudorepresentations over A' and isomorphism classes of representations over A'.

In this text we consider all notions of pseudorepresentation, that are avaible at this point in time:

1. Taylor's pseudocharacters of groups for GL_d. (Section 2)
2. Rouquier's pseudocharacters of algebras (for $d \times d$-matrices M_d). (Section 2.6)
3. Chenevier's d-dimensional determinant laws of algebras (for M_d). (Section 3)
4. Lafforgue's G-pseudocharacters of groups for general reductive groups G. (Section 4)
5. In Section 8 we introduce a new notion of pseudorepresentation of algebras with involution for symplectic groups (or better the symplectic standard matrix algebra (M_{2d}, j)), which is very close to Chenevier's determinant laws and we call them *symplectic determinant laws*.

2 Taylor's pseudocharacters

2.1 Characters of representations

Pseudorepresentations should be seen as an axiomatic generalization of the characteristic polynomial of a representation. We approach these axioms by looking at characteristic polynomials of representations. We first consinder only traces. Let us start with the following classical theorem.

Theorem 2.1. Let Γ be a finite group and let $\rho_1 : \Gamma \to \mathrm{GL}_{d_1}(\mathbb{C})$ and $\rho_2 : \Gamma \to \mathrm{GL}_{d_2}(\mathbb{C})$ for $d_1, d_2 \in \mathbb{N}_0$ be representations of Γ. Assume, that for all $\gamma \in \Gamma$, we have $\operatorname{tr} \rho_1(\gamma) = \operatorname{tr} \rho_2(\gamma)$. Then ρ_1 is isomorphic to ρ_2.

The trace of a representation as a function $\Gamma \to \mathbb{C}$ is also known as its *character*. What are necessary conditions for a map $T : \Gamma \to \mathbb{C}$ to be the character of a representation?

When T is the character of a representation $\rho : \Gamma \to \mathrm{GL}_d(\mathbb{C})$, then the following properties follow from well-known properties of the trace.

(T2) $T(1) = d$.

(T3) $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$.

There is also the *Frobenius trace relation*, which holds for arbitrary $(d+1)$-tuples of $d \times d$-matrices. We will deduce it in Section 2.2.

2.2 The Frobenius trace relation

We start with some elementary insights on matrices and traces. Let A be a commutative ring. Multiplicativity of the trace with respect to tensor products is well-known:

Lemma 2.2. Let V and W be free A-modules of finite rank and let $f \in \operatorname{End}(V)$ and $g \in \operatorname{End}(W)$. Then $f \otimes g$ as an endomorphism of $V \otimes W$ has trace $\operatorname{tr}(f \otimes g) = \operatorname{tr}(f)\operatorname{tr}(g)$.

When V is an A-module, then S_n acts on $V^{\otimes n}$ by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \tag{Symm}$$

for $v_1, \dots, v_n \in V$. This defines a homomorphism of A-algebras $A[S_n] \to \operatorname{End}(V^{\otimes n})$ and we will identify each $\sigma \in S_n$ with its image under this homomorphism. When $f_1, \dots, f_n \in \operatorname{End}(V)$, we obtain an endomorphism $f_1 \otimes \cdots \otimes f_n \in \operatorname{End}(V^{\otimes n})$ defined by

$$(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) := f_1(v_1) \otimes \cdots \otimes f_n(v_n)$$

for $v_1, \dots, v_n \in V$.

Lemma 2.3. Let V be a free A-module of finite rank and $f_1, \dots, f_n \in \operatorname{End}(V)$. Then for a cycle $\sigma = (1\ 2 \dots n) \in S_n$ we have $\operatorname{tr}(\sigma \circ (f_1 \otimes \cdots \otimes f_n)) = \operatorname{tr}(f_1 \circ \cdots \circ f_n)$.

Proof. Fix a basis $(e_1, \dots, e_d)$ of V and denote the dual basis of V^* by $(e_1^*, \dots, e_d^*)$. As $\operatorname{tr}(\sigma \circ (f_1 \otimes \cdots \otimes f_n))$ and $\operatorname{tr}(f_1 \circ \cdots \circ f_n)$ are both multilinear in the arguments $(f_1, \dots, f_n)$ we may assume $f_j = e_{i_j} \cdot e_{\bar{i}_j}^* \in \operatorname{End}(V)$. For simpler notation, we consider the indices of i and $\bar{i}$ modulo n. We obtain

$$f_1 \otimes \cdots \otimes f_n = (e_{i_1} \otimes \cdots \otimes e_{i_n}) \cdot (e_{\bar{i}_1}^* \otimes \cdots \otimes e_{\bar{i}_n}^*)$$

and after identifying S_n with a subset of $\operatorname{End}(V^{\otimes n})$, we have

$$\sigma = (e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(n)}}) \cdot (e_{j_1}^* \otimes \cdots \otimes e_{j_n}^*).$$

The composition is

$$\begin{aligned} \sigma \circ (f_1 \otimes \cdots \otimes f_n) &= (e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(n)}}) \cdot (e_{\bar{i}_1}^* \otimes \cdots \otimes e_{\bar{i}_n}^*) \\ &= (e_{i_{\sigma(1)}} e_{\bar{i}_1}^*) \otimes \cdots \otimes (e_{i_{\sigma(n)}} e_{\bar{i}_n}^*) \end{aligned}$$

and by taking the trace and Lemma 2.2, we have

$$\operatorname{tr}(\sigma \circ (f_1 \otimes \cdots \otimes f_n)) = \begin{cases} 1, & \forall k \in \{1, \dots, n\} : i_{\sigma(k)} = \bar{i}_k, \\ 0, & \text{else.} \end{cases}$$

On the other hand

$$f_1 \circ \cdots \circ f_n = e_{i_1} e^*_{\bar{i}_n} \cdot \begin{cases} 1, & \forall k \in \{1, \dots, n-1\} : \bar{i}_k = i_{k+1}, \\ 0, & \text{else,} \end{cases}$$

and taking the trace gives

$$\operatorname{tr}(f_1 \circ \cdots \circ f_n) = \begin{cases} 1, & \forall k \in \{1, \dots, n\} : i_{\sigma(k)} = \bar{i}_k, \\ 0, & \text{else.} \end{cases}$$

□

Proposition 2.4. Let V be a free A-module of rank d and let $n \geq d+1$. Then

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\sigma = 0$$

seen as an endomorphism $V^{\otimes n} \to V^{\otimes n}$ via Equation (Symm).

Proof. Let $T := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\sigma \in \operatorname{End}(V^{\otimes n})$. Fix a basis $\mathcal{B} = (b_1, \dots, b_d)$ of V. Let $\mathcal{B}^{\otimes n} = \{b_{i_1} \otimes \cdots \otimes b_{i_n} \mid i_1, \dots, i_n \in \{1, \dots, d\}\}$ be the associated basis of $V^{\otimes n}$. In every elementary tensor $b = b_{i_1} \otimes \cdots \otimes b_{i_n} \in \mathcal{B}^{\otimes n}$ at least one basis vector in $\mathcal{B}$ occurs at least twice, say $b_{i_x} = b_{i_y}$. So there is a 2-cycle $\mu = (xy) \in S_n$, such that $b = \mu b$. We have

$$\left(\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\sigma \right) b = \sum_{\sigma \in A_n} (\sigma - \sigma\mu) b = 0$$

and conclude, that $T = 0$. □

We are now ready to prove the Frobenius trace relation. To simplify the formulaion of the statement we introduce the following notation: If $T : R \to A$ is a map from a ring or group R into a commutative ring A and $c = (i_1 \dots i_r) \in S_n$ is a cycle, we define $T_{(c)}(\gamma_1, \dots, \gamma_n) := T(\gamma_{i_1} \cdot \dots \cdot \gamma_{i_r})$. By rotation invariance of the trace, this does not depend on the presentation of the cycle. For a general $\sigma \in S_n$ with cycle decomposition $\sigma = c_1 \dots c_k$, let $T_\sigma(\gamma_1, \dots, \gamma_n) := \prod_{j=1}^k T_{(c_j)}(\gamma_1, \dots, \gamma_n)$.

Proposition 2.5 (Frobenius trace relation)**.** Let V be a free A-module of rank d and $f_1, \dots, f_n \in \operatorname{End}(V)$. Whenever $n \geq d+1$ we have

$$\operatorname{tr}_{V^{\otimes n}} \left(\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\sigma \circ (f_1 \otimes \cdots \otimes f_n) \right) = 0.$$

Here S_n acts on $V^{\otimes n}$ as in Equation (Symm). In particular

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \operatorname{tr}_\sigma (f_1, \dots, f_n) = 0.$$

Proof. The first identity follows from Proposition 2.4. For the second part use Lemma 2.3. □

For a map $T : \Gamma \to A$ from a monoid Γ (later Γ might also be the multiplicative monoid of a ring) into a commutative ring A, such that $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$ we say, that T satisfies the *d-dimensional pseudocharacter identity* if

$$\forall \gamma_1, \dots, \gamma_{d+1} \in \Gamma : \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) T_\sigma(\gamma_1, \dots, \gamma_{d+1}) = 0. \tag{PC}$$

We write $S_{d+1}(T)(\gamma_1, \dots, \gamma_{d+1}) := \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) T_\sigma(\gamma_1, \dots, \gamma_{d+1})$.

When $T : \Gamma \to A$ is the trace of a representation $\rho : \Gamma \to \operatorname{GL}_d(A)$, then the d-dimensional pseudocharacter identity holds:

(T4) $S_{d+1}(T) = 0$.

2.3 Taylor's definition

Let Γ be a monoid and let A be a commutative ring. We first present a definition very close to the original definition of d-dimensional pseudocharacters by Taylor [Tay91, §1.1]. It is a map $T : \Gamma \to A$ into a commutative ring A satisfying the relations of a trace observed in Section 2.1 and Section 2.2. In Section 2.6, we introduce a slightly more general definition, where T is a map from an A-algebra R to A. Pseudocharacters as in Definition 2.6 are recovered by taking for R in Definition 2.24 the monoid ring $A[\Gamma]$.

Definition 2.6 (Taylor)**.** Let A be a commutative ring, Γ a monoid and $d \geq 0$ an integer. A *d-dimensional pseudocharacter* of Γ with values in A is a map $T : \Gamma \to A$, that satisfies the following four axioms.

(T1) $d! \in A^\times$.

(T2) $T(1) = d$.

(T3) $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$.

(T4) T satisfies the d-dimensional pseudocharacter identity, i.e.

$$\sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) T_\sigma(\gamma_1, \dots, \gamma_{d+1}) = 0$$

for all $\gamma_1, \dots, \gamma_{d+1} \in \Gamma$.

We denote by $\mathrm{TPC}_d^\Gamma(A)$ the set of d-dimensional A-valued pseudocharacters of Γ.

The following theorem justifies the term *pseudocharacter*. Pseudocharacters can be thought of as a generalization of traces of representations.

Proposition 2.7. Let A be a commutative ring, $d \geq 0$ an integer with $d! \in A^\times$ and let $\rho : \Gamma \to \mathrm{GL}_d(A)$ be a homomorphism. Then $T := \operatorname{tr} \rho : \Gamma \to A,\ \gamma \mapsto \operatorname{tr} \rho(\gamma)$ is a d-dimensional pseudocharacter.

Proof. (T1) holds by assumption. (T2) and (T3) follow from well-known properties of the trace. (T4) follows from Proposition 2.5. □

So far (T1) is not important, but it will turn out to be relevant for identifying pseudocharacters over algebraically closed fields with equivalence classes of semisimple representations.

Example 2.8. Over $\mathbb{R}$ not every pseudocharacters comes from a representation in the sense of Proposition 2.7: Let $\Gamma = Q_8 = \{\pm 1, \pm i, \pm j \pm k\} \subset \mathbb{H}^\times$ be the quaternion group. Consider the composition

$$T : \Gamma \to \mathbb{H}^\times \xrightarrow{\operatorname{tr}} \mathbb{R},$$

where tr is the reduced trace of the central simple $\mathbb{R}$-algebra $\mathbb{H}$. The reduced trace has all properties (T2)-(T4), since its $\mathbb{C}$-linear extension to $\mathbb{H} \otimes_\mathbb{R} \mathbb{C} \cong M_2(\mathbb{C})$ is a trace. Over $\mathbb{C}$, T comes from the unique irreducible 2-dimensional representation of Q_8, which is of quaternionic type. Hence there is no representation defined over $\mathbb{R}$ with trace T.

2.4 Polarization

If M is a (2×2)-matrix over a commutative ring A, then the characteristic polynomial of M is given by

$$\det(t - M) = t^2 - \operatorname{tr}(M)t + \det(M) \in A[t].$$

A direct calculation shows, that if $2 \in A^\times$, we can recover the determinant from the trace as

$$\det(M) = \frac{\operatorname{tr}(M)^2 - \operatorname{tr}(M^2)}{2},$$

so the coefficients of $\det(t - M)$ are polynomials in $\operatorname{tr}(M)$ and $\operatorname{tr}(M^2)$ with coefficients in $\mathbb{Z}[\frac{1}{2}]$. This procedure is known as *polarization* and will play a central role in the theory of pseudocharacters. In this section, we will make use of elementary symmetric polynomials to prove a more general *polarization formula*.

Definition 2.9 (Elementary symmetric polynomial)**.** Let $0 \le i \le d$ be integers. We define the *i-th elementary symmetric polynomial in d variables* as

$$e_i := e_i^{(d)} := \sum_{\substack{S \subset \{1,\dots,d\} \\ |S|=i}} \prod_{a \in S} x_a$$

in $\mathbb{Z}[x_1, \dots, x_d]$, where S varies over all subsets of $\{1, \dots, d\}$ with i elements. We omit the superscript in contexts with no ambiguity about d.

Equivalently, they are implicitly defined by the equation

$$\prod_{i=1}^{d} (1 + x_i t) = \sum_{i=0}^{d} e_i^{(d)} t^i.$$

For example the elementary symmetric polynomials in $d = 3$ variables are

$$\begin{aligned} e_0 &= 1, \\ e_1 &= x_1 + x_2 + x_3, \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3, \\ e_3 &= x_1x_2x_3. \end{aligned}$$

Elementary symmetric polynomials satisfy the following recursion formula:

Lemma 2.10. Let $1 \le i \le d$ be integers. Then

$$e_i^{(d+1)} = e_i^{(d)} + e_{i-1}^{(d)} x_{d+1}.$$

Proof. Let $S \subset \{1, \dots, d+1\}$ be a subset with i elements. Then either S is a subset of $\{1, \dots, d\}$ or $S \setminus \{d+1\} \subset \{1, \dots, d\}$ is a subset with $i - 1$ elements. The claim follows. ☐

If we think of the variables $x_1, \dots, x_d$ as eigenvalues of a triangular matrix

$$\mathbb{X} = \begin{pmatrix} x_1 & & * \\ & \ddots & \\ & & x_d \end{pmatrix},$$

then the next lemma tells us, that the coefficients of the characteristic polynomial of $\mathbb{X}$ are (up to sign) elementary symmetric polynomials in $x_1, \dots, x_d$.

Lemma 2.11. Let $0 \le i \le d$ be integers. Then

$$\det(t - \mathbb{X}) = \prod_{i=1}^{d} (t - x_i) = \sum_{i=0}^{d} (-1)^i e_i^{(d)} t^{d-i}$$

in $\mathbb{Z}[t, x_1, \dots, x_d]$.

Proof. We proceed by induction. For $d = 0$, there is nothing to prove. Assume, that the claim is true for some $d \ge 0$. Then

$$\begin{aligned} \sum_{i=0}^{d+1} (t - x_i) &= \left(\prod_{i=1}^{d} (t - x_i) \right) \cdot (t - x_{d+1}) \\ &= \left(\sum_{i=0}^{d} (-1)^i e_i^{(d)} t^{d-i} \right) \cdot (t - x_{d+1}) \\ &= \sum_{i=0}^{d} (-1)^i e_i^{(d)} t^{d+1-i} + \sum_{i=0}^{d} (-1)^{i+1} e_i^{(d)} x_{d+1} t^{d-i} \\ &= \sum_{i=0}^{d} (-1)^i e_i^{(d)} t^{d+1-i} + \sum_{i=1}^{d+1} (-1)^i e_{i-1}^{(d)} x_{d+1} t^{d+1-i} \\ &= \sum_{i=0}^{d+1} (-1)^i e_i^{(d+1)} t^{d+1-i}, \end{aligned}$$

where the last step follows from Lemma 2.10, $e_0^{(d)} = 1 = e_0^{(d+1)}$ and $e_d^{(d)} x_{d+1} = e_{d+1}^{(d+1)}$. $\square$

If $\mathbb{X}$ is an upper triangular (3×3)-matrix, then the characteristic polynomial of $\mathbb{X}$ is

$$\mathrm{CP}_{\mathbb{X}}(t) = t^3 - \mathrm{tr}(\mathbb{X})t^2 + q(A)t - \det(\mathbb{X}),$$

where q is a 2-homogeneous polynomial in the (diagonal) entries of $\mathbb{X}$.

As in dimension 2 we see, that $\det(\mathbb{X})$ is a polynomial in $\mathrm{tr}(\mathbb{X})$, $\mathrm{tr}(\mathbb{X}^2)$ and $\mathrm{tr}(\mathbb{X}^3)$ with coefficients in $\mathbb{Z}[\frac{1}{6}]$: Since we have a polarization formula

$$x_1x_2x_3 = \frac{1}{6}((x_1+x_2+x_3)^3 - 3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2) + 2(x_1^3+x_2^3+x_3^3)),$$

we get

$$\det(\mathbb{X}) = \frac{1}{6}(\mathrm{tr}(\mathbb{X})^3 - 3\,\mathrm{tr}(\mathbb{X})\,\mathrm{tr}(\mathbb{X}^2) + 2\,\mathrm{tr}(\mathbb{X}^3)).$$

By Lemma 2.11 have $q(\mathbb{X}) = e_2^{(3)}$. A polarization of $e_2^{(3)}$ is given by

$$e_2^{(3)} = x_1x_2 + x_2x_3 + x_1x_3 = \frac{1}{2}((x_1+x_2+x_3)^2 - (x_1^2+x_2^2+x_3^2)),$$

so

$$q(\mathbb{X}) = \frac{1}{2}(\mathrm{tr}(\mathbb{X})^2 - \mathrm{tr}(\mathbb{X}^2)).$$

On the right hand side, we have the same formula as for the determinant in dimension 2, in fact the general description of the coefficients of the characteristic polynomial in terms of the trace does not depend on the dimension. We introduce a sequence of *polarization polynomials*:

Definition 2.12 (Polarization polynomials)**.** Let $k \geq 0$ be an integer. We define

$$\Delta_k(r_1, \ldots, r_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) \prod_{c \in \mathrm{cycles}(\sigma)} r_{|c|}$$

in $\mathbb{Z}[\frac{1}{k!}][r_1, \ldots, r_k]$, where the product varies over the set of disjoint cycles of σ and $|c|$ denotes the length of a cycle.

Note, that Δ_k is homogeneous of degree k, if one defines the degree of r_i as i.

Example 2.13. For $k = 1, 2, 3, 4$ we obtain

$$\begin{aligned}
\Delta_1(r_1) &= r_1 \\
\Delta_2(r_1, r_2) &= \frac{1}{2}(r_1^2 - r_2) \\
\Delta_3(r_1, r_2, r_3) &= \frac{1}{6}(r_1^3 - 3r_1r_2 + 2r_3) \\
\Delta_4(r_1, r_2, r_3, r_4) &= \frac{1}{24}(r_1^4 - 6r_1^2r_2 + 8r_1r_3 + 3r_2^2 - 6r_4)
\end{aligned}$$

and the polarization formulae read

$$\begin{aligned}
e_2^{(2)} &= \Delta_2(x_1+x_2, x_1^2+x_2^2) \\
e_2^{(3)} &= \Delta_2(x_1+x_2+x_3, x_1^2+x_2^2+x_3^2) \\
e_3^{(3)} &= \Delta_3(x_1+x_2+x_3, x_1^2+x_2^2+x_3^2, x_1^3+x_2^3+x_3^3)
\end{aligned}$$

Definition 2.14 (Power sums)**.** Let $0 \leq i \leq d$ be integers. We define the *i-th power sum in d variables* as

$$\mathcal{S}_i^{(d)} := x_1^i + \cdots + x_d^i$$

in $\mathbb{Z}[x_1, \ldots, x_d]$.

To prove the polarization formula for elementary symmetric polynomials, we need the classical *Newton relations.*

Lemma 2.15 (Newton relations)**.** The symmetric polynomials and the power sums in $\mathbb{Z}[x_1, \ldots, x_n]$ satisfy the recursive relations

$$(m+1)e_{m+1}^{(n)} = \sum_{i+j=m} (-1)^i \mathcal{S}_{i+1}^{(n)} e_j^{(n)} \tag{1}$$

for all $m \geq 0$, where $e_i^{(n)} = 0$ if $i > n$ or $i < 0$. These relations are encoded in the coefficients of the equation

$$-t \frac{\frac{\partial}{\partial t}\det(1+t\mathbb{X})}{\det(1+t\mathbb{X})} = \sum_{k=1}^{\infty} \operatorname{tr}(\mathbb{X}^k)t^k \tag{2}$$

in $\mathbb{Z}[x_1, \ldots, x_n][[t]]$, where $\mathbb{X} := \operatorname{diag}(x_1, \ldots, x_n) \in M_n(\mathbb{Z}[x_1, \ldots, x_n])$.

Proof. We sketch the proof in [Pro07, §2.1.1]. It is clear, that $\det(1+t\mathbb{X}) = \prod_{i=1}^n(1+tx_i) = \sum_{i=0}^n e_i^{(n)}t^i$. Taking the logarithmic derivative $\frac{\partial}{\partial t}\log(\ldots)$, we obtain

$$\frac{\frac{\partial}{\partial t}\det(1+t\mathbb{X})}{\det(1+t\mathbb{X})} = \frac{\partial}{\partial t}\log\left(\det(1+t\mathbb{X})\right) = \sum_{i=1}^n \frac{x_i}{1+tx_i} = \sum_{i=1}^n x_i \sum_{k=0}^{\infty}(-tx_i)^k = \sum_{k=0}^{\infty}(-1)^k \mathcal{S}_{k+1}^n t^k$$

which proves Equation (2), since $\operatorname{tr}(\mathbb{X}^k) = \mathcal{S}_k^n$.

On the other hand, we have

$$\frac{\frac{\partial}{\partial t}\det(1+t\mathbb{X})}{\det(1+t\mathbb{X})} = \frac{\sum_{i=1}^n ie_i^{(n)}t^{i-1}}{\sum_{i=0}^n e_i^{(n)}t^i}$$

Equation (1) follows by multiplying with $\sum_{i=0}^n e_i^{(n)}t^i$ and comparison of coefficients. $\square$

This shows, that the power sums $\mathcal{S}_k^n$ can be expressed as polynomials with integral coefficients in elementary symmetric polynomials. On the other hand the symmetric polynomial $e_i^{(n)}$ can be expressed as a polynomial with coefficients in $\mathbb{Z}[\frac{1}{i!}]$ in power sums. Notice, that this way we get a nontrivial, yet canonical isomorphism between the polynomial rings $\mathbb{Z}[\frac{1}{n!}][e_1^{(n)}, \ldots, e_n^{(n)}]$ and $\mathbb{Z}[\frac{1}{n!}][\mathcal{S}_1^{(n)}, \ldots, \mathcal{S}_n^{(n)}]$.

We will also need the Newton relations for the polarization polynomials in the following form:

Proposition 2.16. For all $i \geq 0$, we have the recursion formula

$$(i+1)\Delta_{i+1} = \sum_{k+l=i} (-1)^k r_{k+1}\Delta_l \tag{3}$$

in $\mathbb{Z}[\frac{1}{i!}][r_1, \ldots, r_{i+1}]$.

Proof. For convenience, we define $\Sigma_n := n!\Delta_n$ for general n, i.e.

$$\Sigma_n = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{c \in \operatorname{cycles}(\sigma)} r_{|c|}$$

Multiplying Equation (3) with $i!$, the claim reduces to

$$\Sigma_{i+1} \stackrel{!}{=} \sum_{k+l=i} (-1)^k \frac{i!}{l!} r_{k+1}\Sigma_l$$

i.e.

$$\sum_{\sigma \in S_{i+1}} \operatorname{sign}(\sigma) \prod_{c \in \operatorname{cycles}(\sigma)} r_{|c|} \stackrel{!}{=} \sum_{k+l=i} (-1)^k \frac{i!}{l!} r_{k+1} \sum_{\sigma \in S_{l+1}} \operatorname{sign}(\sigma) \prod_{c \in \operatorname{cycles}(\sigma)} r_{|c|}$$

Let C_{k+1} be the set of cycles $c \in S_{i+1}$ of order $k+1$ with $c(1) \neq 1$. By elementary combinatorics, $|C_{k+1}| = \frac{i!}{(i-k)!}$. For $c \in C_{k+1}$, let $T_{i+1,c} \subseteq S_{i+1}$ be the subset of permutations $\sigma \in S_{i+1}$, such that the cycle of σ containing 1 is c. We see that S_{i+1} is the disjoint union of the $T_{i+1,c}$ over all $k = 0, \ldots, i$ and all $c \in C_{k+1}$. In particular we obtain for the left hand side:

$$\Sigma_{i+1} = \sum_{k+l=i} \sum_{c \in C_{k+1}} \sum_{\sigma \in T_{i+1,c}} \operatorname{sign}(\sigma) \prod_{c' \in \operatorname{cycles}(\sigma)} r_{|c'|}$$

So our claim reduces to

$$(-1)^k \frac{i!}{l!} r_{k+1} \sum_{\sigma \in S_{l+1}} \operatorname{sign}(\sigma) \prod_{c \in \operatorname{cycles}(\sigma)} r_{|c|} \stackrel{!}{=} \sum_{c \in C_{k+1}} \sum_{\sigma \in T_{i+1,c}} \operatorname{sign}(\sigma) \prod_{c' \in \operatorname{cycles}(\sigma)} r_{|c'|} \tag{4}$$

for fixed $0 \leq k \leq i$ and fixed $l = i - k$. We notice, that on the right hand side of Equation (4) the term

$$\sum_{\sigma \in T_{i+1,c}} \operatorname{sign}(\sigma) \prod_{c' \in \operatorname{cycles}(\sigma)} r_{|c'|}$$

does not depend on $c \in C_{k+1}$: Every cycle containing 1 is conjugate to $(1 \ldots k+1)$ so every set $T_{i+1,c}$ is conjugate to $T_{i+1,(1\ldots k+1)}$. In particular there is a bijection $T_{i+1,c} \to T_{i+1,(1\ldots k+1)}$ preserving the partition of $i+1$ defined by the cycle structure of each $\sigma \in T_{i+1,c}$. So we have

$$\begin{aligned}\sum_{c \in C_{k+1}} \sum_{\sigma \in T_{i+1,c}} \operatorname{sign}(\sigma) \prod_{c' \in \operatorname{cycles}(\sigma)} r_{|c'|} &= \frac{i!}{l!} \sum_{\sigma \in T_{i+1,(1\ldots k+1)}} \operatorname{sign}(\sigma) \prod_{c' \in \operatorname{cycles}(\sigma)} r_{|c'|} \\ &= \frac{i!}{l!} (-1)^k r_{k+1} \sum_{\sigma \in S_l} \operatorname{sign}(\sigma) \prod_{c' \in \operatorname{cycles}(\sigma)} r_{|c'|}\end{aligned}$$

and the last expression is exactly the left hand side of Equation (4). ☐

Now the combinatorial work is done and we can deduce the general polarization formula for elementary symmetric polynomials:

Theorem 2.17 (Polarization formula)**.** Let $d \geq 0$ and $0 \leq i \leq d$. Then

$$e_i^{(d)} = \Delta_i(\mathcal{S}_1^{(d)}, \mathcal{S}_2^{(d)}, \ldots, \mathcal{S}_i^{(d)})$$

as polynomials in $\mathbb{Z}[\frac{1}{i!}][x_1, \ldots, x_d]$.

Proof. We prove the claim by induction on i. For $i = 0$, there is nothing to prove. We assume, that for all $i' < i+1$ the claim is proven. By the classical Newton relations Equation (1) and the inductive hypothesis, the claim reduces to

$$(i+1)\Delta_{i+1}(\mathcal{S}_1^{(d)}, \ldots, \mathcal{S}_{i+1}^{(d)}) = \sum_{k+l=i} (-1)^k \mathcal{S}_{k+1}^{(d)} \Delta_l(\mathcal{S}_1^{(d)}, \ldots, \mathcal{S}_l^{(d)})$$

which reduces formally to Proposition 2.16 by taking $r_a = \mathcal{S}_a^{(d)}$ for all $1 \leq a \leq i+1$. ☐

Now we can describe the coefficients of the characteristic polynomial of a matrix in terms of the trace.

Theorem 2.18. Let $0 \leq i \leq d$ be integers. Let A be a $(d \times d)$-matrix over a commutative ring $\mathcal{O}$, such that $d! \in \mathcal{O}^\times$. If $\mathcal{O}$ is an integral domain, let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A over a fixed algebraic closure of the fraction field of $\mathcal{O}$. In this case we have

$$e_i^{(d)}(\lambda_1, \ldots, \lambda_d) = \Delta_i(\operatorname{tr}(A), \operatorname{tr}(A^2), \ldots, \operatorname{tr}(A^i))$$

If $\mathcal{O}$ is arbitrary, then

$$\det(t - A) = \sum_{i=0}^{d} (-1)^i \Delta_i(\operatorname{tr}(A), \operatorname{tr}(A^2), \ldots, \operatorname{tr}(A^i)) t^{d-i}$$

Proof. We first assume, that $\mathcal{O}$ is a field. Then A is conjugate to an upper triangular matrix $\tilde{A}$ over an algebraic closure of $\mathcal{O}$. We may replace A by $\tilde{A}$ since trace and determinant are invariant under conjugation. So wlog. A is upper triangular and $\mathcal{O}$ is algebraically closed. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of A. Then $\mathrm{tr}(A^i) = \lambda_1^i + \dots + \lambda_d^i$. The first claim follows from Theorem 2.17 and specialization $x_j \mapsto \lambda_j$. For the second part, we have $\det(t - A) = (t - \lambda_1) \cdot \dots \cdot (t - \lambda_d)$ on the left hand side. The claim follows by Lemma 2.11, specialization $x_j \mapsto \lambda_j$ and the first part.

The case when $\mathcal{O}$ is an integral domain is proved by passing to the fraction field. The second formula remains true after passing to a quotient ring. Since every commutative ring is a quotient of an integral domain, this proves the claim for every $\mathbb{Z}[\frac{1}{d!}]$-algebra. $\square$

Corollary 2.19. Let $\mathcal{O}$ be a commutative ring, such that $d! \in \mathcal{O}^\times$ and let $A \in M_d(\mathcal{O})$. Then

$$\det(A) = \frac{1}{d!} \sum_{\sigma \in S_d} \mathrm{sign}(\sigma)\, \mathrm{tr}_\sigma(A, \dots, A)$$

Here $\mathrm{tr}_\sigma(A_1, \dots, A_d) := \prod_{c \in \mathrm{cycles}(\sigma)} \mathrm{tr}_c(A_1, \dots, A_d)$ and $\mathrm{tr}_c(A_1, \dots, A_d) := \mathrm{tr}(A_{i_1} \cdots A_{i_{|c|}})$, where $c = (i_1 \dots i_{|c|})$ is a cycle occurring in the cycle decomposition of σ.

Proof. Setting $t = 0$ in Theorem 2.18, we obtain

$$\det(A) = \Delta_d(\mathrm{tr}(A), \dots, \mathrm{tr}(A^d)) \overset{\text{Def. } 2.12}{=} \frac{1}{d!} \sum_{\sigma \in S_d} \mathrm{sign}(\sigma) \prod_{c \in \mathrm{cycles}(\sigma)} \mathrm{tr}(A^{|c|})$$

and per definition $\mathrm{tr}_\sigma(A, \dots, A) = \prod_{c \in \mathrm{cycles}(\sigma)} \mathrm{tr}(A^{|c|})$. $\square$

Lemma 2.20. Let $A \in M_d(F)$ be a matrix with characteristic polynomial $\prod_{i=1}^d (t - \lambda_i)$, and let $\bigwedge^r A : \bigwedge^r F^d \to \bigwedge^r F^d$ be the action of A on the r-th exterior power of F^d. Then

$$\mathrm{tr}(\bigwedge^r A) = e_r^{(d)}(\lambda_1, \dots, \lambda_d)$$

and moreover

$$\det(t - A) = \sum_{i=0}^d (-1)^i\, \mathrm{tr}(\bigwedge^r A) t^{d-i}$$

Proof. The first part is straightforward calculation and the second part then follows from Lemma 2.11. $\square$

Lemma 2.21. If F is a field of characteristic zero and $A \in M_d(F)$, then for all $i \geq 0$ the trace $\mathrm{tr}(\Lambda^i A)$ is determined by $\mathrm{tr}(A), \dots, \mathrm{tr}(A^i)$.

Proof. Since F has characteristic zero the polarization Theorem 2.18 and Lemma 2.20 give

$$\mathrm{tr}(\bigwedge^i A) = e_i^{(d)}(\lambda_1, \dots, \lambda_d) = \Delta_i(\mathrm{tr}(A), \mathrm{tr}(A^2), \dots, \mathrm{tr}(A^i)).$$

where $\lambda_1, \dots, \lambda_d \in \overline{F}$ are the eigenvalues of A. $\square$

2.5 The characteristic polynomial

Suppose $\rho : \Gamma \to \mathrm{GL}_d(A)$ is a representation of a group and $d! \in A^\times$. By Theorem 2.18, the trace $\mathrm{tr}\,\rho : \Gamma \to A$ remembers the entire characteristic polynomial of $\rho(\gamma)$ for all $\gamma \in \Gamma$. Thanks to axiom (T1), we can define for a general pseudocharacter $T : \Gamma \to A$ all coefficients of the characteristic polynomial using the same polarization formula.

Definition 2.22. Let Γ be a group, let A be a commutative ring with $d! \in A^\times$ and let T be a d-dimensional pseudocharacter of Γ. We define

$$\chi^T(\gamma, t) := \sum_{i=0}^d (-1)^i \Delta_i(T(\gamma), T(\gamma^2), \dots, T(\gamma^i)) t^{d-i}$$

for all $\gamma \in \Gamma$ using the polarization polynomials Δ_i of Definition 2.12. We thereby obtain a map $\chi^T : \Gamma \to A,\ \gamma \mapsto \chi^T(\gamma, t)$.

Proposition 2.23. Let Γ be a group and let A be a commutative ring with $d! \in A^\times$. Then the map

$$\mathrm{TPC}_d^\Gamma(A) \to \mathrm{Map}(\Gamma, A[t]), \quad T \mapsto \chi^T$$

is injective.

Proof. For all $\gamma \in \Gamma$, we have $\Delta_1(T(\gamma)) = T(\gamma)$ and $-\Delta_1(T(\gamma))$ is the t^{d-1}-coefficient of the polynomial $\chi^T(\gamma, t)$. □

From Theorem 2.18 it is clear, that if a pseudocharacter is the trace of a representation ρ, then $\chi^T(\gamma, t) = \det(t - \rho(\gamma))$ for all $\gamma \in \Gamma$. This *compatibility with characteristic polynomials* is a very practical property, that will play a role in several of our arguments.

2.6 Rouquier's definition

In [Rou96, Definition 2.1] Rouquier defines pseudocharacters for algebras in analogy to Taylor's pseudocharacters of groups. Some statements will just be proved in Rouquier's setting, as it is more general.

Definition 2.24. Let $d \geq 1$ and let A be a commutative ring with $d! \in A^\times$ and R be an A-algebra. A *pseudocharacter* of R of dimension d is an A-linear map $T : R \to A$, such that:

(T1) $d! \in A^\times$.

(T2) $T(1) = d$.

(T3) $T(xy) = T(yx)$ for all $x, y \in R$.

(T4) $S_{d+1}(T) = 0$.

If $R = A[\Gamma]$ is a group algebra, then T is determined by its values on Γ and $T|_\Gamma$ is a pseudocharacter in the sense of Definition 2.6. Conversely any pseudocharacter of Γ extends to a pseudocharacter of $A[\Gamma]$. If R is an arbitrary A-algebra, $T : R \to A$ is a d-dimensional pseudocharacter of R and $A \to A'$ is a homomorphism, then $T \otimes_A A' : R \otimes_A A' \to A'$, $r \otimes a \mapsto aT(r)$ is a d-dimensional pseudocharacter of $R \otimes_A A'$. Note, that this notion of base extension is compatible with base extension for pseudocharacters as in Definition 2.6.

Remark 2.25. There are slight variations on the definition of pseudocharacters for algebras in the literature.

1. In [BC09, §1.2.1] a pseudocharacter is required to satisfy condition (T3) plus the existence of some $d \geq 0$, such that (T1) and (T4) hold. The smallest such d is then called the 'dimension' of T. In [BC09, Lem. 1.2.5 (2)] it is shown, that when A is connected and T has dimension d, then (T2) holds.
2. In [Bel12, Definition 3] under the assumption that A is connected, condition (T2) is dropped and condition (T4) is strengthened to the requirement, that d is the smallest integer, such that $S_{d+1}(T) = 0$. It is then shown in [Bel12, Proposition 4], that (T2) follows from this strengthened version of (T4). This is also the definition chosen in [Rou96, Définition 2.1].
3. In [Nys96] condition (T1) is dropped. As we will see in Example 2.29, this leads to undesired behavior.

2.7 Representability

Lemma 2.26. Let $d \geq 0$, let $\mathcal{O}$ be a commutative ring such that $d! \in \mathcal{O}^\times$ and let R be an $\mathcal{O}$-algebra. The functor

$$\mathrm{TPC}_d^R : \mathrm{CAlg}_\mathcal{O} \to \mathrm{Set}, \quad A \mapsto \mathrm{TPC}_d^R(A),$$

which associates to a commutative $\mathcal{O}$-algebra A the set of d-dimensional A-valued pseudocharacters of R is representable by a commutative $\mathcal{O}$-algebra B_d^R.

Compare [Che14, p. 2, footnote 6].

Proof. Define B_d^R to be the quotient of $\mathcal{O}[X_r \mid r \in R]$ by the ideal generated by the following polynomials:

- $X_{r_1+r_2} - X_{r_1} - X_{r_2}$ for all $r_1, r_2 \in R$,
- $X_{ar} - aX_r$ for all $r \in R$ and all $a \in \mathcal{O}$,
- $X_1 - d$,
- $X_{r_1 r_2} - X_{r_2 r_1}$ for all $r_1, r_2 \in R$,
- $\sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) X_\sigma(r_1, \dots, r_{d+1})$ for all $r_1, \dots, r_{d+1} \in R$.

where for any cycle $c = (i_1 \dots i_r) \in S_{d+1}$ we define $X_{(c)}(r_1, \dots, r_{d+1}) := X_{r_{i_1} \cdot \dots \cdot r_{i_r}}$ and for any $\sigma \in S_{d+1}$ with cycle decomposition $\sigma = c_1 \dots c_k$ we set $X_\sigma(r_1, \dots, r_{d+1}) = \prod_{j=1}^k X_{(c_k)}(r_1, \dots, r_{d+1})$. One checks at once, that the *universal pseudocharacter* $R \to B_d^R,\ r \mapsto X_r$ represents TPC_d^R. $\square$

The functor TPC_d^R serves as a substitute for the functor $\mathrm{Rep}_d^R : \mathrm{CAlg}_{\mathcal{O}} \to \mathrm{Set}$, where for any $\mathcal{O}$-algebra A, $\mathrm{Rep}_d^R(A)$ is the set of $\mathrm{GL}_d(A)$-conjugacy classes of $\mathcal{O}$-algebra homomorphisms $R \to M_d(A)$. Note, that Rep_d^R is in general not representable. The trace induces a natural transformation $\mathrm{tr} : \mathrm{Rep}_d^R \to \mathrm{TPC}_d^R$. By the previous lemma an A-valued pseudocharacter of R is the same as an A-point $\mathrm{Spec}(A) \to \mathrm{Spec}(B_d^R)$ over $\mathcal{O}$. We define the *d-dimensional pseudocharacter variety* for R over $\mathcal{O}$ to be the $\mathcal{O}$-scheme $\mathrm{Spec}(B_d^R)$, which represents TPC_d^R.

2.8 Reconstruction theorems

The reconstruction theorem tells us, when a d-dimensional pseudocharacter over an algebraically closed field is the trace of a semisimple representation. This has been proved by Taylor for his pseudocharacters in [Tay91, Theorem 1 (2)] in characteristic 0. It was later proved in positive characteristic $p > d$ by Rouquier [Rou96, Théorème 4.2].

Definition 2.27. Let k be a field and let $T : R \to k$ be a d-dimensional Rouquier pseudocharacter of a k-algebra R. We say, that T is *irreducible*, if T cannot be written as a sum of two pseudocharacters $T_1 + T_2$, of dimensions d_1 and d_2 respectively, with $d_1 + d_2 = d$ and $d_1, d_2 \geq 1$.

Theorem 2.28. Let k be an algebraically closed field of characteristic 0 or $p > d$ and let R be a unital k-algebra. Let $T : R \to k$ be a d-dimensional Rouquier pseudocharacter of R. If T is irreducible, then $R/\ker(T)$ is a d^2-dimensional central simple k-algebra and T is the trace of the absolutely irreducible representation $R \to R/\ker(T) \cong M_d(k)$. In general, T is the trace of a semisimple representation, which is unique up to conjugation.

Proof. See [Rou96, Théorème 4.2] in case T is irreducible. Let $T = T_1 + \cdots + T_k$ be a decomposition of an arbitrary T into irreducible pseudocharacters and let ρ_i be an irreducible representation with trace T_i. The representation $\rho := \rho_1 \oplus \cdots \oplus \rho_k$ has trace T. Uniqueness follows from the Brauer-Nesbitt theorem. $\square$

In Theorem 2.28 the condition on the characteristic of k is necessary, which is illustrated by the following example.

Example 2.29. In characteristic 2 and dimension 2, uniqueness of the representation fails: The representation

$$\rho : C_3 \to \mathrm{GL}_2(\overline{\mathbb{F}}_2)$$

of the cyclic group C_3 with generator γ defined by

$$\rho(\gamma) = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$$

where $\omega^2 + \omega + 1 = 0$ is semisimple and has the same pseudocharacter as the trivial representation, which is 0.

There are also reconstruction theorems over local henselian rings, which are particularly important for comparing the pseudodeformation functor to the deformation functor.

Proposition 2.30. Let $T : R \to A$ be a residually multiplicity-free d-dimensional pseudocharacter over a local henselian factorial domain A. Then T is the trace of a representation $\rho : R \to M_d(A)$.

Proof. See [BC09, Prop. 1.6.1]. □

There is a converse to this result under the assumption, that A is noetherian.

Proposition 2.31. Let $d \geq 2$, let A be a noetherian local henselian ring and let R be an A-algebra. If each d-dimensional residually multiplicity free pseudocharacter $T : R \to A$ is the trace of a representation $\rho : R \to M_d(A)$, then A is factorial.

Proof. See [BC09, Thm. 1.6.3]. □

2.9 Kernel of Taylor's pseudocharacters

Definition 2.32. Let $T : \Gamma \to A$ be a d-dimensional pseudocharacter of a group Γ. We define the *kernel* of T as

$$\ker(T) := \{\gamma \in \Gamma \mid \forall \delta \in \Gamma : T(\gamma\delta) = T(\delta)\}$$

Lemma 2.33. In Definition 2.32 $\ker(T)$ is a normal subgroup of Γ.

Proof. Let $\gamma, \gamma' \in \ker(T)$. Clearly $1 \in \ker(T)$. For all $\delta \in \Gamma$, we have $T(\gamma\gamma'\delta) = T(\gamma'\delta) = T(\delta)$, so $\gamma\gamma' \in \ker(T)$. Further $T(\gamma^{-1}\delta) = T(\gamma\gamma^{-1}\delta) = T(\delta)$, so $\gamma^{-1} \in \ker(T)$. We have shown, that $\ker(T)$ is a subgroup of Γ. For all $x \in \Gamma$, we have $T(x\gamma x^{-1}\delta) = T(\gamma x^{-1}\delta x) = T(x^{-1}\delta x) = T(\delta)$, so $x\gamma x^{-1} \in \ker(T)$ and thus $\ker(T)$ is a normal subgroup. □

If $\ker(T) = 0$, we say that T is *faithful*.

2.10 Kernel of Rouquier's pseudocharacters

Definition 2.34. Let $T : R \to A$ be a d-dimensional pseudocharacter of an A-algebra R. We define the *kernel* of T as

$$\ker(T) := \{x \in R \mid \forall y \in R : T(xy) = 0\}$$

The kernel is a two-sided ideal of R. If $\ker(T) = 0$, we say, that T is *faithful*. [BC09, §1.2.4]

Proposition 2.35. Let $T : A[\Gamma] \to A$ be a d-dimensional pseudocharacter of a group algebra $A[\Gamma]$.

1. $\ker(T) \cap (\Gamma - 1) = \ker(T|_\Gamma) - 1$.
2. $A[\ker(T|_\Gamma) - 1] \subseteq \ker(T)$.

Proof. Suppose $\gamma \in \Gamma$ with $\gamma - 1 \in \ker(T)$. Then for all $\delta \in \Gamma$, we have $T((\gamma - 1)\delta) = 0$, in particular $T(\gamma\delta) = T(\delta)$, so $\gamma \in \ker(T|_\Gamma)$. Conversely if $\gamma \in \ker(T|_\Gamma)$, then $T((\gamma - 1)y) = 0$ for all $y \in A[\Gamma]$ by linearity. The second assertion follows from the first. □

Note, that Definition 2.32 is insensitive to base extension, i.e. if $f : A \to A'$ is an injective ring homomorphism, then $\ker(T) = \ker(f \circ T)$. It is important to notice, that the notion of kernel and faithfulness of Taylor's and Rouquier's pseudocharacters need not agree. This is illustrated by the following example.

Example 2.36. Let $\Gamma = \langle\gamma\rangle$ be cyclic with generator γ of order 4 and let Γ act on $\mathbb{Q}^2$ by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let $T : \mathbb{Q}[\Gamma] \to \mathbb{Q}$ be the associated 2-dimensional Rouquier pseudocharacter. The fundamental matrix of the trace pairing $(x, y) \mapsto T(xy)$ on $\mathbb{Q}[\Gamma]$ is

$$\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & -2 & 0 & 2 \\ -2 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

and it follows, that $\ker(T) = (1 + \gamma^2)$. In particular T is not faithful. On the other hand the Taylor pseudocharacter $T|_\Gamma$ is faithful. We conclude, that the inclusion in Proposition 2.35 (2) is strict in this case.

2.11 Direct sum

Every $(d' + 1)$-tuple of $d \times d$-matrices satisfies the d'-dimensional pseudocharacter identity (T4) for all $d' \geq d$. This can be seen by embedding each $d \times d$-matrix into the upper left corner of a $d' \times d'$-matrix and filling the rest with zeros. In the following lemma we prove, that the $(d+1)$-dimensional pseudocharacter identity actually follows formally from the d-dimensional pseudocharacter identity. It is also proved in [Bel09, Lem. 2.2].

Lemma 2.37. Let R be an A-algebra and let $T : R \to A$ be a function, such that $T(xy) = T(yx)$ for all $x, y \in R$. Let $d \geq 0$ be an integer. Then

$$\begin{aligned} S_{d+2}(T)(g_1, \dots, g_{d+2}) + \sum_{i=2}^{d+2} S_{d+1}(T)(g_1 g_i, \dots, \hat{g}_i, \dots, g_{d+2}) \\ = T(g_1) S_{d+1}(T)(g_2, \dots, g_{d+2}), \end{aligned}$$

with $S_d(T)$ defined as in (T4). Explicitly

$$\begin{aligned} \sum_{\sigma \in S_{d+2}} \operatorname{sign}(\sigma) T_\sigma(g_1, \dots, g_{d+2}) + \sum_{i=2}^{d+2} \sum_{\tau \in S_{d+1}} \operatorname{sign}(\tau) T_\tau(g_1 g_i, \dots, \hat{g}_i, \dots, g_{d+2}) \\ = T(g_1) \sum_{\tau \in S_{d+1}} \operatorname{sign}(\tau) T_\tau(g_2, \dots, g_{d+2}) \end{aligned}$$

for all $g_1, \dots, g_{d+2} \in R$. In particular, if in addition T satisfies the d-dimensional pseudocharacter identity (T4), then T satisfies the d'-dimensional pseudocharacter identity for all $d' \geq d$.

Proof. We define for all $i = 2, \dots, d+2$ a map $j_i : S_{d+1} \to S_{d+2}$ on cycles:

$$j_i((a_1 \dots a_k)) := (s_i(a_1) \dots s_i(a_k))$$

where

$$s_i(a) := \begin{cases} 1i, & a = 1 \\ a, & 1 < a < i \\ a + 1, & a \geq i \end{cases}$$

For example for $d = 3$ we have $j_3((12)(34)) = (132)(45)$. The j_i are injective and have disjoint images. By construction for all $i = 2, \dots, d+1$ and all $\tau \in S_{d+1}$ we have $\operatorname{sign}(j_i(\tau)) = -\operatorname{sign}(\tau)$ and $T_\tau(g_1 g_i, \dots, \hat{g}_i, \dots, g_{d+2}) = T_{j_i(\tau)}(g_1, \dots, g_{d+2})$. The second sum cancels entirely with the summands of the first sum, for which σ lies in the image of some j_i.

Every $\sigma \in S_{d+2}$, that does not fix 1 lies in the image of some j_i, hence the complement of the union of the images of the j_i is the stabilizer of 1. We denote this stabilizer by $(S_{d+2})_1$. We are left to show the equality

$$\sum_{\sigma \in (S_{d+2})_1} \operatorname{sign}(\sigma) T_\sigma(g_1, \dots, g_{d+2}) = T(g_1) \sum_{\tau \in S_{d+1}} \operatorname{sign}(\tau) T_\tau(g_2, \dots, g_{d+2})$$

which follows easily by identifying $(S_{d+2})_1$ with S_{d+1}. ☐

For two representations ρ_1, ρ_2 of Γ we can form the direct sum $\rho_1 \oplus \rho_2$ and the tensor product $\rho_1 \otimes \rho_2$. Clearly $\operatorname{tr}(\rho_1 \oplus \rho_2) = \operatorname{tr}(\rho_1) + \operatorname{tr}(\rho_2)$ and $\operatorname{tr}(\rho_1 \otimes \rho_2) = \operatorname{tr}(\rho_1)\operatorname{tr}(\rho_2)$. This leads to the question, whether for a d_1-dimensional pseudocharacter T_1 and a d_2-dimensional pseudocharacter T_2 the function $T_1 + T_2$ is a (d_1+d_2)-dimensional pseudocharacter and whether T_1T_2 is a d_1d_2-dimensional pseudocharacter.

Definition 2.38 (Direct sum of pseudocharacters)**.** Let A be a commutative ring, let $T_1 : R \to A$ be a d_1-dimensional pseudocharacter and let $T_2 : R \to A$ be a d_2-dimensional pseudocharacter in the sense of Rouquier and assume $(d_1+d_2)! \in A^\times$. We define the direct sum $T_1 \oplus T_2$ by

$$(T_1 \oplus T_2)(\gamma) := T_1(\gamma) + T_2(\gamma)$$

Proposition 2.39. Let $d_1, d_2 \geq 0$ and A a commutative ring with $(d_1+d_2)! \in A^\times$. Let $T_1 : R \to A$ be a d_1-dimensional pseudocharacter and $T_2 : R \to A$ a d_2-dimensional pseudocharacter. Then $T_1 + T_2$ is a (d_1+d_2)-dimensional pseudocharacter.

Proof.

(T1) We have $(d_1+d_2)! \in A^\times$ by assumption.

(T2) We have $(T_1 \oplus T_2)(1) = T_1(1) + T_2(1) = d_1 + d_2$. Invariance under cyclic permutations is also clear.

(T3) Let $\gamma_1, \gamma_2 \in \Gamma$. Then

$$(T_1 \oplus T_2)(\gamma_1\gamma_2) = T_1(\gamma_1\gamma_2) + T_2(\gamma_1\gamma_2) = T_1(\gamma_2\gamma_1) + T_2(\gamma_2\gamma_1) = (T_1 \oplus T_2)(\gamma_2\gamma_1)$$

(T4) For some $\sigma \in S_n$ with cycle decomposition $\sigma = c_1 \circ \cdots \circ c_k$, we call $t : \{1, \dots, n\} \to \{1, 2\}$ a *σ-stable coloring*, if t is constant on the supports of all c_i. We denote by C_σ the set of σ-stable colorings.

For any σ-stable coloring $t \in C_\sigma$ we define

$$T^t_\sigma(\gamma_1, \dots, \gamma_n) := \prod_{j=1}^{k} (T_{t(|c_i|)})_{(c_i)}(\gamma_1, \dots, \gamma_n)$$

where $t(|c_i|)$ is the value of t on the support of c_i.

With this notation the relation we want to prove zero reads

$$\begin{aligned} S_{d_1+d_2+1}(T_1+T_2)(\gamma_1, \dots, \gamma_{d_1+d_2+1}) &= \sum_{\sigma \in S_{d_1+d_2+1}} \operatorname{sign}(\sigma)(T_1+T_2)_\sigma(\gamma_1, \dots, \gamma_{d_1+d_2+1}) \\ &= \sum_{\sigma \in S_{d_1+d_2+1}} \operatorname{sign}(\sigma) \sum_{t \in C_\sigma} T^t_\sigma(\gamma_1, \dots, \gamma_{d_1+d_2+1}) \\ &= \sum_{\substack{t:\{1,\dots,d_1+d_2+1\} \\ \to \{1,2\}}} \sum_{\substack{\sigma \in S_{d_1+d_2+1} \\ C_\sigma \ni t}} \operatorname{sign}(\sigma) T^t_\sigma(\gamma_1, \dots, \gamma_{d_1+d_2+1}) \end{aligned}$$

Given a map $t : \{1, \dots, n\} \to \{1, 2\}$ the $\sigma \in S_n$, such that t is σ-stable are exactly those lying in the image of the injective homomorphism

$$\operatorname{Sym}(t^{-1}(\{1\})) \times \operatorname{Sym}(t^{-1}(\{2\})) \to S_n, (\tau_1, \tau_2) \mapsto \left(x \mapsto \begin{cases} \tau_1(x), & t(x) = 1 \\ \tau_2(x), & t(x) = 2 \end{cases} \right)$$

We write $S_{t^{-1}(x)} := \operatorname{Sym}(t^{-1}(\{x\}))$. We obtain

$$\cdots = \sum_{\substack{t:\{1,\dots,d_1+d_2+1\} \\ \to \{1,2\}}} \left(\sum_{\tau_1 \in S_{t^{-1}(1)}} \operatorname{sign}(\tau_1)(T_1)_{\tau_1}((\gamma_\alpha)_{\alpha \in t^{-1}(1)}) \right) \left(\sum_{\tau_2 \in S_{t^{-1}(2)}} \operatorname{sign}(\tau_2)(T_2)_{\tau_2}((\gamma_\beta)_{\beta \in t^{-1}(2)}) \right)$$

and this is zero, because either $\#t^{-1}(1) \geq d_1 + 1$ or $\#t^{-1}(2) \geq d_2 + 1$ and Lemma 2.37 does apply.

$\square$

2.12 Dual

Definition 2.40. Let A be a commutative ring with $d! \in A^\times$ and let $T : \Gamma \to A$ be a d-dimensional pseudocharacter. We define the *dual* $T^\vee$ by

$$T^\vee(\gamma) := T(\gamma^{-1})$$

It is clear, that if T is the trace of a representation ρ, then $T^\vee$ is the trace of ρ^*.

Proposition 2.41. In Definition 2.40 $T^\vee$ is a d-dimensional pseudocharacter.

Proof. We are able to verify properties (T1)-(T4) independently.

(T1) $d! \in A^\times$ holds by assumption.

(T2) $T^\vee(1) = T(1) = d$.

(T3) Let $\gamma_1, \gamma_2 \in \Gamma$. Then

$$T^\vee(\gamma_1\gamma_2) = T(\gamma_2^{-1}\gamma_1^{-1}) = T(\gamma_1^{-1}\gamma_2^{-1}) = T^\vee(\gamma_2\gamma_1)$$

(T4) Let $\gamma_1, \dots, \gamma_{d+1} \in \Gamma$. Let $c = (i_1 \dots i_r)$ be a cycle in S_{d+1}. With the notation of Section 2.2, we have

$$\begin{aligned} T^\vee_{(c)}(\gamma_1, \dots, \gamma_{d+1}) &= T^\vee(\gamma_{i_1} \dots \gamma_{i_r}) \\ &= T(\gamma_{i_r}^{-1} \dots \gamma_{i_1}^{-1}) \\ &= T_{(c^{-1})}(\gamma_1^{-1}, \dots, \gamma_{d+1}^{-1}) \end{aligned}$$

The pseudocharacter relation for $T^\vee$ vanishes:

$$\begin{aligned} \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) T^\vee_\sigma(\gamma_1, \dots, \gamma_{d+1}) &= \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) \prod_{c \in \text{cycles}(\sigma)} T^\vee_{(c)}(\gamma_1, \dots, \gamma_{d+1}) \\ &= \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) \prod_{c \in \text{cycles}(\sigma)} T_{(c^{-1})}(\gamma_1^{-1}, \dots, \gamma_{d+1}^{-1}) \\ &= \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) \prod_{c \in \text{cycles}(\sigma^{-1})} T_{(c)}(\gamma_1^{-1}, \dots, \gamma_{d+1}^{-1}) \\ &= \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) T_{\sigma^{-1}}(\gamma_1^{-1}, \dots, \gamma_{d+1}^{-1}) \\ &= 0 \end{aligned}$$

□

2.13 Tensor product

Definition 2.42 (Tensor product of pseudocharacters)**.** Let A be a commutative ring, let $T_1 : \Gamma \to A$ be a d_1-dimensional pseudocharacter and let $T_2 : \Gamma \to A$ be a d_2-dimensional pseudocharacter and assume, that $(2d_1d_2)! \in A^\times$. We define the tensor product $T_1 \otimes T_2$ by

$$(T_1 \otimes T_2)(\gamma) := T_1(\gamma)T_2(\gamma)$$

Proposition 2.43. In Definition 2.42 $T_1 \otimes T_2$ is a d_1d_2-dimensional pseudocharacter.

Proof.

(T1) $(d_1d_2)! \in A^\times$ holds by assumption.

(T2) $(T_1 \otimes T_2)(1) = T_1(1)T_2(1) = d_1d_2$.

(T3) Let $\gamma_1, \gamma_2 \in \Gamma$. Then

$$\begin{aligned}(T_1 \otimes T_2)(\gamma_1\gamma_2) &= T_1(\gamma_1\gamma_2)T_2(\gamma_1\gamma_2)\\ &= T_1(\gamma_2\gamma_1)T_2(\gamma_2\gamma_1)\\ &= (T_1 \otimes T_2)(\gamma_2\gamma_1)\end{aligned}$$

(T4) By Proposition 4.59, the comparison map $\mathrm{tr} : \mathrm{PC}^{\Gamma}_{\mathrm{GL}_{d_i}}(A) \to \mathrm{TPC}^{\Gamma}_{d_i}(A)$ is a bijection. We will show independently in Proposition 4.34, that tr is compatible with formation of tensor products of Lafforgue's GL_d-pseudocharacters. It thus follows, that $T_1 \otimes T_2$ is a d_1d_2-dimensional pseudocharacter and in particular satisfies (T4).

□

Remark 2.44. We are not able to prove directly, that the product of two pseudocharacters of dimensions d_1 and d_2 satisfies (T4), but we expect that an elementary proof similar to the case of direct sums can be given. However we are not really in need of such an argument, since Taylor's pseudocharacters are not well-behaved in small characteristics anyway (see Example 2.29). See also [BC09, Remark 1.2.9].

Remark 2.45. One can define $\underline{\mathrm{Hom}}(T_1, T_2) := T_1^{\vee} \otimes T_2$.

2.14 The semiring of pseudocharacters

For any commutative monoid $(M, +)$ the Grothendieck group $(G(M), +)$ is an abelian group generated by formal differences of elements of M. Any homomorphism from M into an abelian group factors uniquely over $G(M)$. There is a canonical homomorphism $M \to G(M)$, which is injective if and only if M has the cancellation property.

Assume, that A is a $\mathbb{Q}$-algebra. We know, that the trace gives rise to a homomorphism of commutative semirings

$$\mathrm{tr} : \mathrm{Rep}^{\Gamma}(A) \to \mathrm{Map}(\Gamma, A), \quad [\rho] \mapsto \mathrm{tr}(\rho)$$

for all A. Here $\mathrm{Rep}^{\Gamma}(A) = \bigcup_{d=0}^{\infty} \mathrm{Rep}^{\Gamma}_d(A)$ is the set of isomorphism classes of representations of Γ on free A-modules of rank d endowed with the structure of a semiring given by direct sum $\oplus$ and tensor product $\otimes_A$.

From the perspective just described, we would like to show, that the subset $\mathrm{TPC}^{\Gamma}(A)$ of $\mathrm{Map}(\Gamma, A)$ given by Taylor's pseudocharacters is closed under addition and multiplication and that the dimension of a sum or tensor product pseudocharacter is as expected.

Proposition 2.46. Over a $\mathbb{Q}$-algebra A Taylor's pseudocharacters form a commutative semiring

$$\mathrm{TPC}^{\Gamma}(A) = \bigcup_{d=0}^{\infty} \mathrm{TPC}^{\Gamma}_d(A)$$

with pointwise addition (direct sum Definition 2.38) and multiplication (tensor product Definition 2.42). The dualizing operation $(-)^{\vee}$ is a semiring automorphism of order 2. Further, there is a homomorphism of semirings

$$\mathrm{tr} : \mathrm{Rep}^{\Gamma}(A) \to \mathrm{TPC}^{\Gamma}(A), \quad [\rho] \mapsto \mathrm{tr}(\rho)$$

compatible with the dualizing operations on both sides, where $\mathrm{Rep}^{\Gamma}(A)$ is the semiring of isomorphism classes of representations of Γ on finitely generated free A-modules.

Proof. This is a combination of Proposition 2.39, Proposition 2.43 and Proposition 2.41. □

Proposition 2.47. Let C be an algebraically closed field of characteristic 0. Let $\mathrm{Rep}^{\Gamma,\mathrm{ss}}(C)$ be the subsemiring of $\mathrm{Rep}^{\Gamma}(C)$ generated by semisimple representations. Then the trace map $\mathrm{tr} : \mathrm{Rep}^{\Gamma,\mathrm{ss}}(C) \to \mathrm{TPC}^{\Gamma}(C)$ is an isomorphism of commutative semirings.

Proof. We first note, that by a theorem of Chevalley, the tensor product of any two semisimple $C[\Gamma]$-modules is semisimple. So $\mathrm{Rep}^{\Gamma,\mathrm{ss}}(C)$ is indeed the subsemiring of $\mathrm{Rep}^{\Gamma}(C)$ consisting only of semisimple representations. Surjectivity and injectivity follow from the existence and uniqueness part of Theorem 2.28 applied to $C[\Gamma]$. □

2.15 Induction

It seems possible to define the induction of a Taylor pseudocharacter, just as for characters of finite groups. We first give a definition, without claiming that the result is a pseudocharacter.

Definition 2.48 (Induction)**.** Let Γ be a group and let $\Delta \leq \Gamma$ be a subgroup of finite index $n \geq 1$. Let $T : \Delta \to A$ be a d-dimensional pseudocharacter and assume $(nd)! \in A^\times$. Choose a system of left coset representatives $x_1, \dots, x_n \in \Gamma$ of Δ. We define

$$(\mathrm{Ind}_\Delta^\Gamma T)(\gamma) := \sum_{i=1}^n T_\Delta(x_i^{-1}\gamma x_i)$$

where

$$T_\Delta(\gamma) := \begin{cases} T(\gamma), & \gamma \in \Delta \\ 0, & \text{else.} \end{cases}$$

is the *truncation* of T to Δ.

Proposition 2.49. $\mathrm{Ind}_\Delta^\Gamma T$ in Definition 2.48 does not depend on the choice of coset representatives.

Proof. Let $y_i \in \Gamma$, such that $y_i\Delta = x_i\Delta$. Let $\gamma \in \Gamma$. The elements $x_i^{-1}\gamma x_i$ and $y_i^{-1}\gamma y_i$ only differ by a conjugation by $y_i^{-1}x_i \in \Delta$. So $x_i^{-1}\gamma x_i \in \Delta$ if and only if $y_i^{-1}\gamma y_i \in \Delta$. For the same reason, since T is a central function, $T(x_i^{-1}\gamma x_i) = T(y_i^{-1}\gamma y_i)$. ☐

Proposition 2.50. If $\rho : \Delta \to \mathrm{GL}_d(A)$ is a homomorphism, then $\mathrm{Ind}_\Delta^\Gamma \mathrm{tr}(\rho) = \mathrm{tr}(\mathrm{Ind}_\Delta^\Gamma \rho)$.

Proof. The induced representation can be decomposed as $\mathrm{Ind}_\Delta^\Gamma \rho = \bigoplus_{i=1}^n x_iA^d$ as a free A-module of rank dn. If $\gamma \in \Delta$, then γ acts on x_iA^d as $x_i^{-1}\gamma x_i$ and the trace of this action is $\mathrm{tr}(\rho(x_i^{-1}\gamma x_i))$. If $x_i^{-1}\gamma x_i \notin \Delta$, then $\gamma x_i \notin x_i\Delta$, so γ carries x_iA^d into a different summand x_jA^d and the trace of γ on x_iA^d is 0. It follows, that $\mathrm{Ind}_\Delta^\Gamma \mathrm{tr}(\rho) = \mathrm{tr}(\mathrm{Ind}_\Delta^\Gamma \rho)$. ☐

Proposition 2.51. If in Definition 2.48 A is a reduced ring, then $\mathrm{Ind}_\Delta^\Gamma T$ constitutes an nd-dimensional pseudocharacter of Γ.

Proof. We check the pseudocharacter axioms for $T' := \mathrm{Ind}_\Delta^\Gamma T$.

(T1) $(nd)! \in A^\times$ by assumption.

(T2) $T'(1) = nT(1) = nd$.

(T3) Let $\gamma_1, \gamma_2 \in \Gamma$ with $x_i^{-1}\gamma_1\gamma_2x_i \in \Delta$. Note, that $\gamma_2x_1, \dots, \gamma_2x_n$ is also a system of left coset representatives and by the well-definedness we have just seen, we can use it as well for computation of T':

$$T'(\gamma_1\gamma_2) = \sum_{i=1}^n T_\Delta(x_i^{-1}\gamma_1\gamma_2x_i) = \sum_{i=1}^n T_\Delta((\gamma_2x_i)^{-1}\gamma_2\gamma_1(\gamma_2x_i)) = T'(\gamma_2\gamma_1)$$

(T4) We first embed A into the ring $\prod_\mathfrak{p} \overline{\mathrm{Quot}(A/\mathfrak{p})}$, where $\mathfrak{p}$ varies over all minimal primes ideals of A. By projection to the factors we see, that it is enough to prove the claim for A an algebraically closed field. By the reconstruction theorem Theorem 2.28, Proposition 2.50 and Proposition 2.7 the claim follows.

☐

We expect, that (T4) can be proved without any assumption on A, but the calculations get to complicated to carry this out directly. If A is not reduced, it might be possible to exploit the comparison isomorphism Proposition 4.59 between Taylor's pseudocharacters and Lafforgue's pseudocharacters to prove the claim by constructing induced pseudocharacters on the Lafforgue side. We do not carry this out in this thesis.

3 Determinant laws

Chenevier generalizes Taylor's pseudocharacters in [Che14] to *determinant laws of algebras*. The key idea is to consider *homogeneous multiplicative A-polynomial laws of degree n* from an arbitrary A-algebra R to A instead of a single map $R \to A$ satisfying a certain set of identities. By doing so the formalism is a bit thickened, but the problems one encounters with Taylor's definition in small characteristic (see Example 2.29) are resolved. Necessarily the definitions don't agree in small characteristics. In addition this 'linearization' approach allows us to use the machinery of noncommutative algebra. We give a summary of the theory of determinant laws following the exposition of Carl Wang-Erickson in [Wan13, Chapter 1].

3.1 Motivation

Let Γ be a group, A a commutative ring, $d \geq 0$ and $\rho : \Gamma \to \mathrm{GL}_d(A)$ a homomorphism. The family of characteristic polynomials $(\det(T - \rho(\gamma)))_{\gamma \in \Gamma}$ of the elements of Γ is an invariant for the representation ρ. There is a set of relations between these characteristic polynomials, that hold for any representation. These relations come from invariants of tuples of matrices, which we will discuss later. To express those relations in a convenient way we extend the family of characteristic polynomials to a family of maps indexed by all A-algebras B. A homomorphism ρ is equivalent to an A-algebra homomorphism $\rho : A[\Gamma] \to M_d(A)$ and we can recover the family of characteristic polynomials from the map

$$\det \circ (\rho \otimes A[T]) : A[T][\Gamma] \to A[T]$$

by restricting to elements of the form $T - \gamma \in A[T][\Gamma]$. This map makes sense for any commutative A-algebra B, so we associate to ρ the family of maps

$$D_B : A[\Gamma] \otimes_A B \to B, \quad x \mapsto \det((\rho \otimes B)(\gamma))$$

By definition D satisifies the following properties:

1. D is a natural transformation from the functor
$$-[\Gamma] : \mathrm{CAlg}_A \to \mathrm{Alg}_A, \quad B \mapsto B[\Gamma]$$
that maps any A-algebra B to the group algebra $B[\Gamma]$ over Γ, to the inclusion functor $\mathrm{CAlg}_A \subseteq \mathrm{Alg}_A$.
2. $D_B(1) = 1$ and $D_B(xy) = D_B(x)D_B(y)$ for all commutative A-algebras B and all $x, y \in B[\Gamma]$.
3. $D_B(bx) = b^d D_B(x)$ for all commutative A-algebras B and all $b \in B$.

We will see, that these conditions mean, that D is a d-homogeneous multiplicative A-polynomial law.

3.2 Polynomial laws

Definition 3.1 (A-polynomial law)**.** Let A be a commutative ring, let M and N be arbitrary A-modules and let R and S be not necessarily commutative A-algebras.

1. An *A-polynomial law* $P : M \to N$ is a collection of maps $P_B : M \otimes_A B \to N \otimes_A B$ for each commutative A-algebra B, such that for each homomorphism $f : B \to B'$ of commutative A-algebras, the diagram
$$\begin{tikzcd} M \otimes_A B \arrow[r, "D_B"] \arrow[d, "\mathrm{id} \otimes f"] & N \otimes_A B \arrow[d, "\mathrm{id} \otimes f"] \\ M \otimes_A B' \arrow[r, "D_{B'}"] & N \otimes_A B' \end{tikzcd}$$
commutes.

   In other words, an A-polynomial law is a natural transformation $\underline{M} \to \underline{N}$, where $\underline{M}(B) := M \otimes_A B$ is the 'functor of points' of M. We denote the set of A-polynomial laws from M to N by $\mathcal{P}_A(M, N)$.

2. A polynomial law $P : M \to N$ is called *homogeneous of degree* $d \in \mathbb{N}_0$ or *d-homogeneous*, if for all commutative A-algebras B, all $b \in B$ and all $x \in M \otimes_A B$ we have $P_B(bx) = b^d P_B(x)$. We denote the set of d-homogeneous A-polynomial laws from M to N by $\mathcal{P}_A^d(M, N)$.

3. A polynomial law $P : R \to S$ is called *multiplicative*, if for all commutative A-algebras B, we have $P_B(1_{R\otimes_A B}) = 1_{S\otimes_A B}$ and and all $x, y \in R \otimes_A B$, we have $P_B(xy) = P_B(x)P_B(y)$. We denote the set of d-homogeneous multiplicative A-polynomial laws from R to S by $\mathcal{M}_A^d(R, S)$.

Remark 3.2. There is a geometric interpretation of polynomial laws. If M is a finitely generated free A-module of rank n, then $\underline{M}$ is just the functor of points of the affine space $\mathbb{A}^n$ over A. A polynomial law between finitely generated free A-modules is just a morphism of A-schemes. From this perspective a polynomial law is a 'regular map' between 'spaces' modelled by A-modules. Just as in the case of schemes, these 'regular maps' are solely characterized by a naturality condition on a category of rings. However this point of view doesn't seem relevant to the theory of determinant laws.

Lemma 3.3. Let A be a commutative ring, let R, R' and R'' be A-algebras and let $D : R \to R'$ and $D' : R' \to R''$ be polynomial laws.

1. If D is d-homogeneous and D' is d'-homogeneous, then $D' \circ D$ is dd'-homogeneous.
2. If D and D' are multiplicative, then so is $D' \circ D$.

Proof. These are easy calculations. □

Definition 3.4. Let $P : M \to N$ be an A-polynomial law. The *kernel* of P is the set

$$\ker(P) := \{m \in M \mid \forall B \in \mathrm{CAlg}_A : \forall x \in M \otimes_A B : P(x + m) = P(x)\}$$

(Compare [Wan13, Definition 1.1.5.1])

The kernel of P is a submodule of M. When $\ker(P) = 0$, we say that P is *faithful*. The kernel satisfies the usual universal property, see [Wan13, Lemma 1.1.5.2] and [Che14, Lemma 1.18]. Basic properties of the kernel are shown in [Che14, Lemma 1.19].

3.3 Definition of determinant laws

This focusses on [Che14, §1-§2].

Definition 3.5 (Determinant law)**.** Let A be a commutative ring. A *d-dimensional A-valued determinant law* on R is a multiplicative A-polynomial law $D : R \to A$, that is homogeneous of degree $d \in \mathbb{N}_0$. [Wan13, Definition 1.1.7.1]

If B is a commutative A-algebra, we denote the set of d-dimensional B-valued determinant laws of $R \otimes_A B$ by $\mathrm{Det}_d^R(B)$. If $B \to B'$ is a homomorphism of commutative A-algebras and $D : R \otimes_A B \to B$ is a d-dimensional B-valued determinant law, then restriction of functors defines a d-dimensional B'-valued determinant law $D \otimes_B B' : R \otimes_A B' \to B'$. This is the *base change* of D to B' and defines a map $\mathrm{Det}_d^R(B) \to \mathrm{Det}_d^R(B')$. Base change is functorial, so we obtain a moduli functor

$$\mathrm{Det}_d^R : \mathrm{CAlg}_A \to \mathrm{Set}, \quad B \mapsto \mathrm{Det}_d^R(B)$$

A determinant law can be constructed from a representation using the usual determinant: For any A-algebra R and any A-algebra homomorphism $\rho : R \to M_d(A)$ the collection of maps $D_B := \det \circ (\rho \otimes B) : R \otimes_A B \to B$ is a d-dimensional determinant $D : R \to A$. This defines a map

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Alg}_A}(R, M_d(A)) \to \mathrm{Det}_d^R(A) \\ &\quad (\rho : R \to M_d(A)) \mapsto (B \mapsto \det(\rho \otimes_A B)) \end{aligned}$$

There is a unique 0-dimensional determinant law $D : R \to A$, that we will refer to as the *trivial determinant law.*

If A is an infinite integral domain and $D : R \to A$ is a d-dimensional determinant law, then there is a simpler description of the kernel of D ([Wan13, Lemma 1.1.7.2]):

$$\ker(D) = \{r \in R \mid \forall r' \in R : D(1 + rr') = 1\}$$

3.4 Representability

Definition 3.6 (Divided power algebra)**.** Let A be a commutative ring. Given an A-module M we define the *divided power algebra* over M as the commutative A-algebra $\Gamma_A(M)$ generated by symbols $m^{[i]}$ for all $m \in M$, $i \geq 0$ satisfying the following relations:

1. $m^{[0]} = 1$ for all $m \in M$.
2. $(am)^{[i]} = a^i m^{[i]}$ for all $m \in M$, $a \in A$ and $i \in \mathbb{N}_0$.
3. $m^{[i]}m^{[j]} = \frac{(i+j)!}{i!j!} m^{[i+j]}$ for all $m \in M$ and $i, j \in \mathbb{N}_0$.
4. $(m + m')^{[i]} = \sum_{p+q=i} m^{[p]} m'^{[q]}$ for all $m, m' \in M$ and $i \in \mathbb{N}_0$.

The number $\frac{(i+j)!}{i!j!}$ is an integer, so we don't need an assumption on the characteristic of A. Note, that these relations are compatible with the degree $\deg(m^{[i]}) := i$ for $m \in M$ and $i \in \mathbb{N}_0$. So $\Gamma_A(M)$ is naturally $\mathbb{N}_0$-graded. Denote by $\Gamma^d_A(M)$ the A-submodule generated by monomials of degree d. If R is an A-algebra, then $\Gamma^d_A(R)$ carries the structure of an A-algebra defined by

$$(x_1^{[a_1]} \cdots x_r^{[a_r]}) \cdot (y_1^{[b_1]} \cdots y_s^{[b_s]}) := \sum_{(\gamma_{ij})} \prod_{i=1}^{r} \prod_{j=1}^{s} (x_i \cdot y_j)^{[\gamma_{ij}]}$$

where $x_1, \dots, x_r, y_1, \dots, y_s \in R$, $\sum_{i=1}^r a_i = d$, $\sum_{j=1}^s b_j = d$ and (γ_{ij}) ranges over all families of integers $\gamma_{ij} \geq 0$ with $\sum_{i=1}^r \gamma_{ij} = b_j$ and $\sum_{j=1}^s \gamma_{ij} = a_i$. See [Rob80]. For background on divided power algebras, see [Sta19, 09PD].

Theorem 3.7 (Universal homogeneous polynomial law)**.** Let A be a commutative ring and let R be an A-algebra. The functor $\mathcal{M}^d_A(R, -)$ is representable by $\Gamma^d_A(R)$, i.e. there is a natural bijection

$$\mathcal{M}^d_A(R, S) = \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma^d_A(R), S)$$

with universal object $L^d_R : R \to \Gamma^d_A(R),\ r \mapsto r^{[1]}$.

Proof. See [Rob80, Théorème] or [Wan13, Theorem 1.1.6.5]. □

In particular, if S is commutative, there is a natural bijection

$$\mathcal{M}^d_A(R, S) \cong \mathrm{Hom}_{\mathrm{CAlg}_A}(\Gamma^d_A(R)^{\mathrm{ab}}, S)$$

3.5 Reconstruction theorems

It is natural to ask, under what conditions a determinant law arises from a representation in the sense described in Section 3.3. In this case we say, that D is *split*. Chenevier [Che14, Sec. 2.22] proves some converse results for A an algebraically closed field and A a Henselian local ring.

Theorem 3.8. Let k be an algebraically closed field, R a k-algebra $d \geq 0$. Then the natural map

$$\mathrm{Rep}^R_d(k) \to \mathrm{Det}^R_d(k)$$

induces a bijection between the set of conjugacy classes of d-dimensional semisimple representations $R \to M_d(k)$ and the set of d-dimensional k-valued determinant laws of R. If $D \in \mathrm{Det}^R_d(k)$, then

$$\rho : R \to R/\ker(D) \cong \prod_i M_{d_i}(k)$$

is a semisimple representation with associated determinant law $D = \det \circ \rho$ and $\ker(D) = \ker(\rho)$ and $\sum_i d_i = d$.

Proof. See [Che14, Theorem 2.12] and [Wan13, Theorem 1.3.1.1]. □

Theorem 3.9. Let $D : R \to A$ be a Cayley-Hamilton determinant law over a henselian local ring A with residue field k. If $D \otimes_A k$ comes from an absolutely irreducible representation, then there is an isomorphism $\rho : R \cong M_d(A)$, such that $D = \det \circ \rho$.

Proof. See [Che14, Theorem 2.22 (i)]. □

3.6 The characteristic polynomial

The reference for the following definition is [Che14, §1.10].

Definition 3.10. Let A be a commutative ring, let R be an A-algebra and let $D : R \to A$ be a d-dimensional A-valued determinant law. Then we define for each $r \in R$, the *characteristic polynomial* $\chi^D(r,t) \in A[t]$ by

$$\chi^D(r,t) := D_A(t-r)$$

We will understand the characteristic polynomial as a map $\chi^D : R \to A[t],\ r \mapsto \chi^D(r,t)$. If $R = A[\Gamma]$ is a group ring, we also consider the restriction $\chi^D : \Gamma \to A[t]$. We denote the negative of the coefficient of t^{d-1} by tr_D.

By [Che14, Lemma 1.12 (iii)], tr_D satisfies the d-dimensional pseudocharacter identity.

Proposition 3.11. Let A be a commutative ring and let Γ be group. Then the map

$$\mathrm{Det}_d^\Gamma(A) \to \mathrm{Map}(\Gamma, A[t]), \quad D \mapsto \chi^D$$

is injective.

Proof. By Amitsur's formula [Che14, (1.5)] χ^D determines the values of the maps $D_{A[t_1,\dots,t_n]} : A[\Gamma][t_1,\dots,t_n] \to A$ on elements of the form $\gamma_1 t_1 + \dots + \gamma_n t_n$ with $\gamma_i \in \Gamma$. By naturality we can replace finitely many variables by elements of A, so that the χ^D determines all values of $D_{A[t_1,\dots,t_n]} : A[\Gamma][t_1,\dots,t_n] \to A$ for all $n \geq 1$. Again by naturality this is sufficient to determine D. $\square$

Definition 3.12. Let D be an A-linear d-dimensional determinant law. We define the coefficients $\Lambda_i : R \to A$ of the characteristic polynomial of D by the expansion

$$\chi^D(r,t) = D_{B[t]}(t-r) = \sum_{i=0}^{d} (-1)^i \Lambda_{i,B}(r) t^{d-i} \in B[t]$$

for all $B \in \mathrm{CAlg}_A$.

One can show, that the coefficients Λ_i give rise to i-homogeneous A-polynomial laws.

3.7 Continuous determinant laws

Let Γ be a topological group and let A be a topological ring. We say, that a d-dimensional A-linear determinant law $D \in \mathrm{Det}_d^\Gamma(A)$ is *continuous*, if the coefficients Λ_i of Definition 3.12 of the characteristic polynomial of D give rise to continuous maps $\Lambda_{i,A}|_\Gamma : \Gamma \to A$. This notion of continuity is equivalent to that defined in [Che14, §2.30]. We denote the set of continuous d-dimensional A-linear determinant laws by $\mathrm{cDet}_d^\Gamma(A)$.

If $\rho : \Gamma \to \mathrm{GL}_d(A)$ is a continuous representation, then D_ρ is a continuous determinant law. So we have a map $\mathrm{cRep}_{\mathrm{GL}_d}^{\Gamma,\square}(A) \to \mathrm{cDet}_d^\Gamma(A)$, which is natural in A and Γ.

3.8 Comparison with Taylor's pseudocharacters

Proposition 3.13. Let A be a commutative ring with $d! \in A^\times$ and let R be an A-algebra. Then the map

$$\mathrm{Det}_d^R(A) \to \mathrm{TPC}_d^R(A), \quad D \mapsto \mathrm{tr}_D$$

(see Definition 3.10) from the set $\mathrm{Det}_d^R(A)$ of d-dimensional A-valued determinant laws to the set $\mathrm{TPC}_d^R(A)$ of d-dimensional A-valued Rouquier pseudocharacters of R is a well-defined injection. The map is bijective, if one of the following conditions holds.

1. A is reduced.
2. $2 \in A^\times$ and $d = 2$.
3. $(2d)! \in A^\times$.

Proof. See [Che14, Proposition 1.27, Remark 1.28, Proposition 1.29]. $\square$

3.9 Direct sum

The direct sum of two determinant laws should be defined in such a way, that it corresponds to the direct sum of representations. This has been done by Wang-Erickson in [Wan13, §1.1.11].

Definition 3.14 (Direct sum)**.** Let R be an A-algebra, D_1, D_2 determinant laws of dimension d_1, d_2 of R over A. Then we define the *direct sum* $D := D_1 \oplus D_2$ to be the polynomial law given by

$$D_B(x) := D_{1,B}(x) D_{2,B}(x)$$

for all commutative A-algebas B. [Wan13, Def. 1.1.11.6]

$D_1 \oplus D_2$ is multiplicative and homogeneous of degree $d_1 + d_2$, in particular $D_1 \oplus D_2 \in \mathrm{Det}^R_{d_1+d_2}(A)$.

Lemma 3.15 (Basic properties of the direct sum)**.** Let R be an A-algebra, $d_1, d_2 \geq 0$ and $d := d_1 + d_2$. The direct sum operation

$$\oplus : \mathrm{Det}^R_{d_1} \times_{\operatorname{Spec} A} \mathrm{Det}^R_{d_2} \to \mathrm{Det}^R_d$$

is a morphism of affine A-schemes corresponding to the homomorphism of commuative A-algebras

$$\Gamma^d_A(R)^{\mathrm{ab}} \xrightarrow{\Gamma^d(\Delta)} \Gamma^d_A(R \times R) \longrightarrow \Gamma^{d_1}_A(R)^{\mathrm{ab}} \otimes_A \Gamma^{d_2}_A(R)^{\mathrm{ab}}$$

where $\Delta : R \to R \times R$ is the diagonal and the right map is induced by the isomorphism [Wan13, (1.1.11.1)].

Proof. See [Wan13, Lem. 1.1.11.7]. $\square$

3.10 Dual

Suppose $\rho : R \to M_d(A)$ is an A-linear representation of a unital A-algebra R. Since transposition does not change the determinant of a matrix, we have $\det \circ \rho = \det \circ \top \circ \rho$ as determinant laws. We may see $\det \circ \rho$ as a determinant law on $R^{\mathrm{op}} \to A$. It is clear, that $\det \circ \rho$ is the determinant law attached to the action of R^{op} on the dual module of A^d equipped with the action of R by ρ. In case $R = A[\Gamma]$ is a group ring associated to a group Γ, we can compose $\det \circ \rho$ with the antihomomorphism $\iota : A[\Gamma] \to A[\Gamma],\ \gamma \mapsto \gamma^{-1}$. This leads to the following definition:

Definition 3.16 (Dual)**.** Let $D : R \to A$ be a d-dimensional A-valued determinant law. Then the *dual* of D is defined as $D^\vee := D \circ \iota : R \to A$.

By the above discussion, we have:

Proposition 3.17. Let $\rho : R \to M_d(A)$ be an A-linear representation of a unital A-algebra R. Then $(\det \circ \rho)^\vee = \det \circ \rho^*$.

3.11 Vaccarino's result

For the construction of tensor products in Section 3.12 we will need a theorem of Vaccarino, which we recall in this section.

For a set X, let $\mathbb{Z}\{X\}$ be the free unital ring generated by X. For $d \geq 0$, there is a universal d-dimensional representation of $\mathbb{Z}\{X\}$: Let $A_X(d)$ be the free commutative ring generated by symbols x_{ij} with $1 \leq i, j \leq d$ for each $x \in X$. Then the universal representation $\rho^{\mathrm{univ}}_d : \mathbb{Z}\{X\} \to M_d(A_X(d))$ maps x to the matrix $\mathbb{X}^{(x)} \in M_d(A_X(d))$ with $\mathbb{X}^{(x)}_{ij} = x_{ij}$. Let R be a unital ring and let $\pi : X \to R$ be a map. It extends uniquely to a ring homomorphism $\pi : \mathbb{Z}\{X\} \to R$. To a representation $\rho : R \to M_d(A)$ over a commutative ring A, we can associate a ring homomorphism $\varphi_\rho : A_X(d) \to A$ defined by $x_{ij} \mapsto \rho(\pi(x))_{ij}$. The following diagram commutes

$$\begin{tikzcd}
\mathbb{Z}\{X\} \arrow[r, "\rho^{\mathrm{univ}}_d"] \arrow[d, "\pi"] & M_d(A_X(d)) \arrow[d, "M_d(\varphi_\rho)"] \\
R \arrow[r, "\rho"] & M_d(A)
\end{tikzcd}$$

Let $E_X(d)$ be the subring of $A_X(d)$ generated by the coefficients of the characteristic polynomial of all elements $\rho_d^{\mathrm{univ}}(w)$ for $w \in \mathbb{Z}\{X\}$. The determinant law $\det \circ \rho_d^{\mathrm{univ}} : \mathbb{Z}\{X\} \to A_X(d)$ attached to the universal d-dimensional representation takes values in $E_X(d)$ [Che14, §1.10].

We recall the following deep result of Vaccarino, which ultimately relies on knowledge about relations between the coefficients of the characteristic polynomial of $d \times d$-matrices. From now on, for every commutative ring A, whenever we write $\det : M_d(A) \to A$ we mean the d-homogeneous multiplicative determinant law $M_d(A) \to A$ given by $\det_B : M_d(B) \to B$ for every commutative A-algebra B.

Theorem 3.18 (Vaccarino)**.** Let A be a commutative ring, let X be a set and let $D : \mathbb{Z}\{X\} \to A$ be a d-homogeneous multiplicative polynomial law. Then there is a unique homomorphism $\varphi_D : E_X(d) \to A$, such that

$$\varphi_D(\det(\rho_d^{\mathrm{univ}}(w))) = D(w)$$

for all $w \in \mathbb{Z}\{X\}$.

Proof. See [Che14, §1.10 (1.6)] or [Vac08, Thm. 6.1] and [Vac09, Thm. 28]. $\square$

For us the case when D comes from a representation will also be important. Suppse $\rho : \mathbb{Z}\{X\} \to M_d(A)$ is a ring homomorphism and $D = \det \circ \rho$. Then the diagram

$$\begin{tikzcd} M_d(A_X(d)) \arrow[r, "\det"] \arrow[d, "M_d(\varphi_\rho)"'] & E_X(d) \arrow[d, "\varphi_\rho|_{E_X(d)}"] \\ M_d(A) \arrow[r, "\det"] & A \end{tikzcd}$$

commutes. It follows from the uniqueness part of Theorem 3.18, that $\varphi_D = \varphi_\rho|_{E_X(d)}$.

3.12 Tensor product

As opposed to Section 3.9 and Section 3.10 we rely on Vaccarino's result Theorem 3.18 to construct tensor products of determinant laws. It is difficult to write down an explicit construction of a tensor product determinant law $D_1 \otimes D_2$ from two determinant laws D_1 and D_2, but it is feasible to construct the attached homomorphism $\varphi_{D_1 \otimes D_2}$ from φ_{D_1} and φ_{D_2} and thereby give a definition of $D_1 \otimes D_2$.

As a preparation, we define a tensor product homomorphism $f_\otimes : E_X(d_1 d_2) \to E_X(d_1) \otimes E_X(d_2)$. Recall, that $A_X(d)$ is the coordinate ring of the affine scheme $M_d^X \cong (\mathbb{A}^{d^2})^X$ of X-tuples of $d \times d$-matrices, which carries a rational $\mathrm{GL}_{d,\mathbb{Z}}$-action by simultaneous conjugation. Hence $A_X(d)$ is a rational $\mathrm{GL}_{d,\mathbb{Z}}$-module. It turns out, that $E_X(d)$ is the subring of rational $\mathrm{GL}_{d,\mathbb{Z}}$-invariants of $A_X(d)$: From classical invariant theory (see [DP76]) it is known, that the rational invariants when X is finite are generated by the coefficients of the characteristic polynomial of the matrix coordinate functions of M_d^X and the situation is no different, when X is infinite, since invariants commute with filtered colimits.

From now on, we fix a bijection $\{1, \dots, d_1\} \times \{1, \dots, d_2\} \cong \{1, \dots, d_1 d_2\}$, which determines an isomorphism $\mathbb{Z}^{d_1} \otimes \mathbb{Z}^{d_2} \cong \mathbb{Z}^{d_1 d_2}$. So the tensor product of a $d_1 \times d_1$-matrix with a $d_2 \times d_2$-matrix can be identified with a well-defined $d_1 d_2 \times d_1 d_2$-matrix and we have a homomorphism $\otimes : M_{d_1}(A) \times M_{d_2}(A) \to M_{d_1 d_2}(A)$ for every commutative ring A realizing this tensor product operation. This induces in particular a homomorphism of coordinate rings $g_\otimes : A_X(d_1 d_2) \to A_X(d_1) \otimes A_X(d_2)$ and a homomorphism of group schemes $\otimes : \mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \to \mathrm{GL}_{d_1 d_2}$ again realizing the tensor product. Note, that the map $\otimes : M_{d_1}(A) \times M_{d_2}(A) \to M_{d_1 d_2}(A)$ is $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2}$-equivariant, so it follows, that $g_\otimes$ is equivariant as well. Taking $\mathrm{GL}_{d_1 d_2}$-invariants on the source of $g_\otimes$ and $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2}$-invariants on the target of $g_\otimes$, we obtain a map $f_\otimes : E_X(d_1 d_2) \to E_X(d_1) \otimes E_X(d_2)$ as the restriction of $g_\otimes$.

Definition 3.19 (Tensor product on $\mathbb{Z}\{X\}$)**.** Let A be a commutative ring and let X be a set. Suppose $D_i : \mathbb{Z}\{X\} \to A$ is a d_i-homogeneous multiplicative polynomial law for $i = 1, 2$. Let $\varphi_{D_i} : E_X(d_i) \to A$ be the homomorphisms attached to $D_i \circ \pi$ for $i = 1, 2$ from Theorem 3.18. We define the *tensor product* $D_1 \otimes D_2$ as the $d_1 d_2$-homogeneous multiplicative polynomial law $\mathbb{Z}\{X\} \to A$ attached to the homomorphism $(\varphi_{D_1} \otimes \varphi_{D_2}) \circ f_\otimes$.

It is clear, that by extending scalars from $\mathbb{Z}\{X\}$ to $A\{X\}$, we obtain a notion of tensor product for A-valued determinant laws. In the next proposition we use the homomorphisms theorem to extend this definition to general A-algebras R.

Proposition 3.20 (Tensor product)**.** Let R be a unital A-algebra. Suppose $D_1 : R \to A$ is a d_1-dimensional A-valued determinant law and that $D_2 : R \to A$ is a d_2-dimensional A-valued determinant law. Let $\pi : \mathbb{Z}\{X\} \to R$ be a homomorphism, such that $\pi \otimes A : A\{X\} \to R$ is surjective and let $\varphi_i := \varphi_{D_i \circ \pi} : E_X(d_i) \to A$ be the homomorphisms attached to $D_i \circ \pi$ for $i = 1, 2$ from Theorem 3.18. Then $\ker(\pi \otimes A)$ is contained in $\ker(((\varphi_1 \otimes \varphi_2) \circ f_\otimes \circ \det \circ \rho^{\mathrm{univ}}_{d_1 d_2}) \otimes A)$. In particular the A-valued determinant law $((\varphi_1 \otimes \varphi_2) \circ f_\otimes \circ \det \circ \rho^{\mathrm{univ}}_{d_1 d_2}) \otimes A : A\{X\} \to A$ descends to a well-defined $d_1 d_2$-dimensional A-valued determinant law $D_1 \otimes D_2 : R \to A$ with $\varphi_{D_1 \otimes D_2} = (\varphi_1 \otimes \varphi_2) \circ f_\otimes$.

Proof. By Theorem 3.18, we have $\varphi_i \circ \det \circ \rho^{\mathrm{univ}}_{d_i} = D_i \circ \pi$ for $i = 1, 2$, in particular $\ker(\pi \otimes A) \subseteq \ker((\varphi_i \circ \det \circ \rho^{\mathrm{univ}}_{d_i}) \otimes A)$.

This containment implies: Whenever $w \in A\{X\}$ and $s \in \ker(\pi \otimes A)$, then

$$\varphi_i(\chi^{\det \circ (\rho^{\mathrm{univ}}_{d_i} \otimes A)}(w + s, t)) = \varphi_i(\chi^{\det \circ (\rho^{\mathrm{univ}}_{d_i} \otimes A)}(w, t))$$

in $A[t]$.

The term $f_\otimes(\chi^{\det \circ (\rho^{\mathrm{univ}}_{d_1 d_2} \otimes A)}(w + s, t)) \in (E_X(d_1) \otimes E_X(d_2) \otimes A)[t]$ is the characteristic polynomial of $\rho^{\mathrm{univ}}_{d_1}(w + s) \otimes \rho^{\mathrm{univ}}_{d_2}(w + s) \otimes A$ and the coefficients of the t^k are polynomials in the coefficients of the characteristic polynomials of $\rho^{\mathrm{univ}}_{d_i}(w + s) \otimes A$ for $i = 1, 2$. It thus follows, that

$$(\varphi_1 \otimes \varphi_2)(f_\otimes(\chi^{\det \circ (\rho^{\mathrm{univ}}_{d_1 d_2} \otimes A)}(w + s, t))) = (\varphi_1 \otimes \varphi_2)(f_\otimes(\chi^{\det \circ (\rho^{\mathrm{univ}}_{d_1 d_2} \otimes A)}(w, t)))$$

The existence of $D_1 \otimes D_2$ such that $\varphi_{D_1 \otimes D_2} = (\varphi_1 \otimes \varphi_2) \circ f_\otimes$ follows from the homomorphisms theorem for determinant laws [Che14, Lemma 1.18] and Theorem 3.18. ☐

If $\rho_1 : R \to M_{d_1}(A)$ and $\rho_2 : R \to M_{d_2}(A)$ are representations of R, we write $\rho_1 \otimes \rho_2 : R \to M_{d_1 d_2}(A)$ for $\otimes \circ (\rho_1 \times \rho_2)$. We show, that the construction of Proposition 3.20 is compatible with the tensor product of representations and may thus be called a tensor product of determinant laws.

Proposition 3.21. Let $\rho_i : R \to M_{d_1}(A)$ for $i = 1, 2$ be A-linear representations of a unital A-algebra R with associated determinant laws D_{ρ_i}. Then $D_{\rho_1} \otimes D_{\rho_2} = D_{\rho_1 \otimes \rho_2}$.

Proof. The following argument works after tensoring all algebras in sight with $- \otimes A$, so we assume $A = \mathbb{Z}$ for simplicity of notation. It is sufficient to prove, that $\varphi_{\rho_1 \otimes \rho_2} = (\varphi_{\rho_1} \otimes \varphi_{\rho_2}) \circ f_\otimes$, as the argument below shows, this will hold after any base change.

We have

$$\begin{aligned} (\varphi_{\rho_1} \otimes \varphi_{\rho_2}) \circ f_\otimes \circ \det &= (\varphi_{\rho_1} \otimes \varphi_{\rho_2}) \circ \det \circ M_{d_1 d_2}(f_\otimes) \\ &= \det \circ M_{d_1 d_2}(\varphi_{\rho_1} \otimes \varphi_{\rho_2}) \circ M_{d_1 d_2}(f_\otimes) \\ &= \det \circ M_{d_1 d_2}(\varphi_{\rho_1 \otimes \rho_2}) \end{aligned}$$

Composing with the universal representation $\rho^{\mathrm{univ}}_{d_1 d_2} : \mathbb{Z}\{X\} \to M_{d_1 d_2}(A_X(d_1 d_2))$, we obtain

$$\begin{aligned} (\varphi_{\rho_1} \otimes \varphi_{\rho_2}) \circ f_\otimes \circ \det \circ \rho^{\mathrm{univ}}_{d_1 d_2} &= \det \circ M_{d_1 d_2}(\varphi_{\rho_1 \otimes \rho_2}) \circ \rho^{\mathrm{univ}}_{d_1 d_2} \\ &= \det \circ (\rho_1 \otimes \rho_2) \circ \pi \\ &= \varphi_{\rho_1 \otimes \rho_2} \circ \det \circ \rho^{\mathrm{univ}}_{d_1 d_2} \end{aligned}$$

where the last equality follows from Theorem 3.18. Since by definition of $E_X(d_1 d_2)$ is generated by the coefficients of the characteristic polynomials of elements of $\mathbb{Z}\{X\}$ under the universal representation, we may use the equation

$$((\varphi_{\rho_1} \otimes \varphi_{\rho_2}) \circ f_\otimes \circ \det \circ \rho^{\mathrm{univ}}_{d_1 d_2}) \otimes \mathbb{Z}[t] = (\varphi_{\rho_1 \otimes \rho_2} \circ \det \circ \rho^{\mathrm{univ}}_{d_1 d_2}) \otimes \mathbb{Z}[t]$$

over the single-variable polynomial ring $\mathbb{Z}[t]$ to deduce, that $\varphi_{\rho_1 \otimes \rho_2} = (\varphi_{\rho_1} \otimes \varphi_{\rho_2}) \circ f_\otimes$. ☐

Next, we show that our construction does not depend on the choice of the presentation.

Proposition 3.22. In Proposition 3.20, the tensor product $D_1 \otimes D_2$ does not depend on π.

Proof. We may assume $A = \mathbb{Z}$. It is sufficient to check, that for a surjection $\pi' : \mathbb{Z}\{X'\} \to \mathbb{Z}\{X\}$, the tensor products constructed via π and via $\pi \circ \pi'$ agree. Let $\varphi_i : E_X(d_i) \to A$ be the homomorphisms attached to $D_i \circ \pi$ and let $\varphi_i' : E_{X'}(d_i) \to A$ be the homomorphisms attached to $D_i \circ \pi \circ \pi'$. The homomorphism π' induces transition maps $a(d) : A_{X'}(d) \to A_X(d)$ and $e(d) : E_{X'}(d) \to E_X(d)$ for any integer $d \geq 0$. For clarity we write $f_{\otimes,X} : E_X(d_1d_2) \to E_X(d_1) \otimes E_X(d_2)$ and $f_{\otimes,X'} : E_{X'}(d_1d_2) \to E_{X'}(d_1) \otimes E_{X'}(d_2)$ for $f_\otimes$ in the respective cases. Our goal is to show, that

$$(\varphi_1 \otimes \varphi_2) \circ f_{\otimes,X} \circ e(d_1d_2) \stackrel{!}{=} (\varphi_1' \otimes \varphi_2') \circ f_{\otimes,X'}$$

from which independence of the presentation follows.

From Theorem 3.18, we get

$$\begin{aligned} D_i \circ \pi &= \varphi_i \circ \det \circ \rho^{\text{univ}}_{d_i,X} \\ D_i \circ \pi \circ \pi' &= \varphi_i' \circ \det \circ \rho^{\text{univ}}_{d_i,X'} \end{aligned}$$

where $\rho^{\text{univ}}_{d_i,X} : \mathbb{Z}\{X'\} \to M_{d_i}(A_X(d_i))$ and $\rho^{\text{univ}}_{d_i,X'} : \mathbb{Z}\{X\} \to M_{d_i}(A_{X'}(d_i))$ are the respective universal representations. At the same time, we have by composition with π':

$$\begin{aligned} D_i \circ \pi \circ \pi' &= \varphi_i \circ \det \circ \rho^{\text{univ}}_{d_i,X} \circ \pi' \\ &= \varphi_i \circ \det \circ M_{d_i}(a(d_i)) \circ \rho^{\text{univ}}_{d_i,X'} \\ &= \varphi_i \circ e(d_i) \circ \det \circ \rho^{\text{univ}}_{d_i,X'} \end{aligned}$$

In the above we used, that $e(d_i)$ is the restriction of $a(d_i)$.

By definition, the image generated by $\det \circ \rho^{\text{univ}}_{d_i,X'}$ is $E_{X'}(d_i)$. Hence $\varphi_i' = \varphi_i \circ e(d_i)$. We see, that

$$\begin{aligned} (\varphi_1' \otimes \varphi_2') \circ f_{\otimes,X'} &= (\varphi_1 \otimes \varphi_2) \circ (e(d_1) \otimes e(d_2)) \circ f_{\otimes,X'} \\ &= (\varphi_1 \otimes \varphi_2) \circ f_{\otimes,X} \circ e(d_1d_2) \end{aligned}$$

For the last step, we check that $(a(d_1) \otimes a(d_2)) \circ g_{\otimes,X'} = g_{\otimes,X} \circ a(d_1d_2)$. □

3.13 Examples of tensor products

Assume, that $X = \{x\}$ has one element.

Recall, that $E_X(d)$ is a polynomial ring over $\mathbb{Z}$ generated by the coefficients $s_1, \ldots, s_d$ of the characteristic polynomial of a generic $d \times d$-matrix. By restriction to diagonal matrices, we obtain a homomorphism $E_X(d) \to S(d)$ to the ring $S(d)$ generated by elementary symmetric polynomials in the diagonal entries of a generic $d \times d$-matrix. Since $S(d)$ is known to be a polynomial ring and $E_X(d)$ is generated by $s_1, \ldots, s_d$, this map is an isomorphism. By slight abuse of notation, we write $E_X(d) = \mathbb{Z}[s_1(x), \ldots, s_d(x)]$ and think of x as a single generic matrix coordinate.

We want to give a more explicit description of the map $f_\otimes : E_X(d_1d_2) \to E_X(d_1) \otimes E_X(d_2)$ in case X has one element. It follows from the previous paragraph, that $E_X(d_1) \otimes E_X(d_2)$ is a polynomial ring in elementary symmetric polynomials of the diagonal entries of two different generic matrices x and y, we write $E_X(d_1) \otimes E_X(d_2) = \mathbb{Z}[s_1(x), \ldots, s_{d_1}(x), s_1(y), \ldots, s_{d_2}(y)]$. So $f_\otimes$ is determined by its values $f_\otimes(s_i)$ on the generators $s_1, \ldots, s_{d_1d_2}$ and these values have a unique presentation as polynomials in $s_1(x), \ldots, s_{d_1}(x), s_1(y), \ldots, s_{d_2}(y)$.

In the following examples we will compute the polynomials $f_\otimes(s_i)$ in some special cases. An explicit formula for these polynomials can be given in terms of generating functions for the coefficients of the characteristic polynomial of a tensor product of two matrices, but we don't write it down here.

Example 3.23.

1. If d_1 is arbitrary and $d_2 = 1$, our operation coincides with twisting with a character $\chi = s_1(y)$ as introduced in [BJ19, §4.5]. It is easy to see, that $f_\otimes(s_i) = \chi^i s_i(x)$.

2. If $d_1 = d_2 = 2$ we write $\mathrm{tr}(x) = s_1(x)$, $\det(x) = s_2(x)$ and likewise for y. Using the dyadic product $g_\otimes$ of two diagonal matrices, we see that

$$\begin{aligned} f_\otimes(s_1) &= \mathrm{tr}(x)\,\mathrm{tr}(y) \\ f_\otimes(s_2) &= (\mathrm{tr}(x)^2 - 2\det(x))\det(y) + \det(x)(\mathrm{tr}(y)^2 - 2\det(y)) + 2\det(x)\det(y) \\ f_\otimes(s_3) &= \mathrm{tr}(x)\det(x)\,\mathrm{tr}(y)\det(y) \\ f_\otimes(s_4) &= \det(x)^2\det(y)^2 \end{aligned}$$

We emphasize, that the formulae given in Example 3.23 uniquely characterize the tensor product of pseudocharacters and it is not necessary to look at sets X of cardinality > 1:

Let $D : R \to A$ be a d-dimensional A-valued determinant law. By Theorem 3.18 for every $r \in R$, the homomorphism $\pi : \mathbb{Z}[x] \to R$ with $\pi(x) = r$ induces a unique homomorphism $\varphi : E_X(d) \to R$, such that $\varphi_\pi \circ \det \circ \rho_d^{\mathrm{univ}} = D \circ \pi$. In particular

$$\sum_{i=0}^{d} (-1)^i \varphi_\pi(s_i) t^{d-i} = \chi^D(r, t)$$

Since determinant laws are determined by their characteristic polynomials, D is determined by φ_π for all $\pi : \mathbb{Z}[x] \to R$. Picking a presentation $\Pi : \mathbb{Z}\{X\} \to R$ which contains some element $x_0 \in X$ with $\Pi(x_0) = r$, and the statement of independence Proposition 3.22, we see that $\chi^{D_1 \otimes D_2}(r, t)$ only depends on $\chi^{D_1}(r, t)$ and $\chi^{D_2}(r, t)$ in the way described in Example 3.23.

4 G-pseudocharacters

In this section we develop the basic theory of G-pseudocharacters for generalized reductive groups G. We start with theoretical background on group schemes.

4.1 Group schemes

4.1.1 Reductive groups over fields

In this section, we fix terminology for reductive groups, that we will need lateron. Let G be a linear algebraic group over an algebraically closed field k. Recall, that the *unipotent radical* $\mathrm{R_u}(G)$ of G is defined as the maximal closed unipotent normal k-subgroup scheme of G and the *solvable radical* $\mathrm{R}(G)$ is defined as the maximal closed solvable normal k-subgroup scheme of G.

Definition 4.1.

1. G is *reductive*, if the unipotent radical $\mathrm{R_u}(G)$ is trivial.
2. G is *semisimple*, if the solvable radical $\mathrm{R}(G)$ is trivial.

If G is a finite k-group scheme, then it is automatically constant and reductive. If we require G to be connected, we will explicitly say so.

Definition 4.2. [Ser03, §3.2] Let Γ be an abstract group and G a connected reductive group over an algebraically closed field F. Let $\rho : \Gamma \to G(F)$ be a representation. We say, that

(a) ρ is *G-irreducible*, if there is no proper parabolic subgroup $P \subseteq G$, such that the image of ρ is contained in $P(F)$.

(b) ρ is *G-completely reducible*, if for every parabolic subgroup $P \subseteq G$, such that the image of ρ is contained in $P(F)$, there is a Levi subgroup $L \subseteq P$, such that the image of ρ is contained in $L(F)$.

(c) ρ is *G-indecomposable*, if there is no proper parabolic subgroup $P \subseteq G$ containing a Levi subgroup $L \subseteq P$, such that the image of ρ is contained in $L(F)$.

For $G = \mathrm{GL}_d$ this recovers the usual notions. Serre proves basic properties of these notions. The quantities $h(G)$ and $n(V)$ are defined in [Ser03, §5.1, §5.2].

Theorem 4.3. [Ser03, pp. 5.4+5.5] Let G be a connected reductive group over a field k. Let $\Gamma \subset G(k)$ be a subgroup and V be a rational G-module.

(a) If Γ is G-completely reducible and the characteristic of k is either 0 or $p > n(V)$, then V is a semisimple Γ-representation.

(b) If the characteristic of k is either 0 or $p > n(V)$ and V is a presque fidèle (= kernel is of multiplicative type) semisimple Γ-representation, then Γ is G-completely reducible.

(c) If the characteristic of k is either 0 or $p > 2h(G) - 2$ then the following are equivalent:

  (1) Γ is G-completely reducible.

  (2) $\mathrm{Lie}(G)$ is a semisimple Γ-module.

Suppose G is a (now possibly non-connected) reductive group over an algebraically closed field k. In [BMR05, §6] Bate, Martin and Röhrle define a notion of *complete reducibility* of subgroups of $G(k)$. For this the notions of parabolic subgroup and Levi subgroup have to be extended to the non-connected case.

For any cocharacter $\lambda : \mathbb{G}_m \to G$, we call $P_\lambda := \{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$ the *Richardson parabolic* (*R-parabolic*) attached to λ. A subgroup of the form $L_\lambda := Z_G(\lambda(k^\times))$ is called a *Richardson Levi* (*R-Levi*) subgroup of P_λ. These notions agree with the usual notions of parabolic and Levi subgroups in case G is connected [BMR05, Lemma 2.4]. So Definition 4.2 extends to the non-connected case:

Definition 4.4. Let Γ be an abstract group and G a reductive group over an algebraically closed field F. Let $\rho : \Gamma \to G(F)$ be a representation. We say, that

(a) ρ is *G-irreducible*, if there is no proper R-parabolic subgroup $P \subseteq G$, such that the image of ρ is contained in $P(F)$.

(b) ρ is *G-completely reducible*, if for every R-parabolic subgroup $P \subseteq G$, such that the image of ρ is contained in $P(F)$, there is an R-Levi subgroup $L \subseteq P$, such that the image of ρ is contained in $L(F)$.

(c) ρ is *G-indecomposable*, if there is no proper R-parabolic subgroup $P \subseteq G$ containing an R-Levi subgroup $L \subseteq P$, such that the image of ρ is contained in $L(F)$.

We shall also define what a *G-semisimplification* of a G-valued representation in the non-connected case is. For the definition we refer to Appendix A.

4.1.2 Reductive group schemes

Working with deformations of representations valued in other algebraic groups G than GL_n, we have to decide which groups we want to allow for G. Our group G shall be naturally defined over the coefficient ring of some deformation problem, for example the ring of integers of a p-adic local field. In [Con14b, Definition 3.1.1] Brian Conrad introduces reductive and semisimple group schemes over arbitrary base schemes. He requires, that the geometric fibers of G shall be connected, which in particular disallows the orthogonal groups O_n.

Definition 4.5. A *reductive (semisimple) group scheme* over a scheme S is a smooth S-affine S-group scheme G, such that the geometric fibers of G are connected reductive (semisimple) groups.

An S-group scheme D is of *multiplicative type*, if it is fppf-locally diagonalizable, i.e. there is an fppf-covering $\{S_i \to S\}$, such that D_{S_i} is isomorphic to the relative spectrum of the quasi-coherent Hopf algebra $\mathcal{O}_{S_i}[M_i]$ for a finitely generated abelian group M_i, where the comultiplication is given by $\Delta(m) = m \otimes m$ and the antipode is given by $s(m) := m^{-1}$ for $m \in M_i$. An *S-torus* is an S-group scheme of multiplicative type with smooth connected fibers.

If G is a reductive S-group scheme, then a *maximal torus* of G is an S-torus $T \subseteq G$, such that for each geometric point $\overline{s}$ of S, $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$. G admits étale-locally a maximal torus [Con14b, Corollary 3.2.7]. For the slightly technical definition of a *split reductive group* over S, we refer to [Con14b, Definition 5.1.1]. If $S = \mathrm{Spec}(\mathbb{Z})$ and G admits a maximal torus, then G is split [Con14b, Example 5.1.4].

Definition 4.6. A *Chevalley group* is a reductive $\mathbb{Z}$-group scheme, which admits a fiberwise maximal $\mathbb{Z}$-torus.

The following three sets are canonically in bijection [Con14a, Theorem 1.4].

1. Chevalley groups up to $\mathbb{Z}$-isomorphism.
2. Split connected reductive groups over $\mathbb{Q}$ up to $\mathbb{Q}$-isomorphism.
3. Root data up to isomorphism.

Every split connected reductive group G over the fraction field K of a domain $\mathcal{O}$ admits a model over $\mathcal{O}$, which is the base change of a Chevalley group over $\mathbb{Z}$ [Con14a, Theorem 1.2]. If $\mathcal{O}$ is a PID, then every $\mathcal{O}$-model of G is the base change of a Chevalley group [Con14a, Proposition 1.3]. We will use these facts to reduce some of our arguments to Chevalley groups. By a *Chevalley group* over another base than $\mathbb{Z}$ we will always mean the base change of a Chevalley group over $\mathbb{Z}$.

4.1.3 Generalized reductive group schemes

The definition of G-pseudocharacters Definition 4.20 shall be given in a way that also allows for G to be disconnected. Suppose G is a smooth affine group scheme over a commutative ring $\mathcal{O}$, such that the geometric fibers $G_{\overline{s}}$ for $s \in \operatorname{Spec}(\mathcal{O})$ are reductive groups. There is a unique open subgroup scheme $G^0 \subseteq G$, such that $(G^0)_s \cong (G_s)^0$ for all $s \in \operatorname{Spec}(\mathcal{O})$ [Gro66, Corollaire 15.6.5]. We say, that G^0 is the *identity component* of G. Beware, that G^0 is not necessarily a connected scheme. Each G^0_s is geometrically connected [Con14b, Exercise 1.6.5] and it follows, that formation of the identity component $(-)^0$ commutes with any base change. In particular $G^0_{\overline{s}}$ is a connected reductive group, G^0 is an open and closed $\mathcal{O}$-subgroup scheme of G and the quotient G/G^0 exists as a separated étale $\mathcal{O}$-group scheme of finite presentation [Con14b, Proposition 3.1.3]. In general G/G^0 doesn't need to be finite [Con14b, Example 3.1.4].

This leads to the following definition, which includes the orthogonal groups O_n when 2 is invertible in $\mathcal{O}$.

Definition 4.7. Let $\mathcal{O}$ be a commutative ring. A *generalized reductive (generalized semisimple) $\mathcal{O}$-group scheme* G is a smooth affine $\mathcal{O}$-group scheme such that the geometric fibers $G_{\overline{s}}$ for $s \in \operatorname{Spec}(\mathcal{O})$ are reductive (semisimple) groups and the component group G/G^0 is finite over $\mathcal{O}$.

The definition of generalized reductive group scheme is given in [FM88, Definition 2.1] in terms of a short exact sequence.

If G is smooth and affine, G^0 is a reductive group scheme and G/G^0 is finite, then G is generalized reductive. If G is generalized reductive, then G^0 is a reductive group scheme.

If $\mathcal{O}$ is a discrete valuation ring and G is a smooth affine $\mathcal{O}$-group scheme with finite component group, such that the special fiber of G is reductive, then G is already generalized reductive [Con14b, Proposition 3.1.9].

Example 4.8. Here are the main examples we are going to consider.

1. The symplectic group Sp_{2n} over $\mathbb{Z}$ is the scheme-theoretic automorphism group of the standard symplectic bilinear form on $\mathbb{Z}^{2n}$. Sp_{2n} is a semisimple Chevalley group with almost-simple connected geometric fibers.
2. The orthogonal group O_n over $\mathbb{Z}[\frac{1}{2}]$ is the automorphism group of the standard symmetric bilinear form on $\mathbb{Z}^n$. O_n is a non-connected generalized semisimple $\mathbb{Z}[\frac{1}{2}]$-group scheme with almost-simple geometric fibers. The identity component of O_n is the special orthogonal group SO_n and the component group $\mathrm{O}_n / \mathrm{SO}_n$ is the constant $\mathbb{Z}[\frac{1}{2}]$-group scheme $\underline{\mathbb{Z}/2\mathbb{Z}}$. [Con14b, Example 3.1.4]

4.1.4 G-valued representations

Definition 4.9. Let G be an affine group scheme over a scheme S.

1. A *G-valued representation* of a group Γ over an S-scheme T is a homomorphism $\rho : \Gamma \to G(T)$.
2. Denote by $\mathrm{Rep}^{\square,\Gamma}_G : \mathrm{Sch}^{\mathrm{op}}_S \to \mathrm{Set}$ the presheaf on the category of S-schemes, that maps an S-scheme T to the set of homomorphisms $\Gamma \to G(T)$. The group $G(T)$ acts on $\mathrm{Rep}^{\square,\Gamma}_G(T)$ by conjugation.
3. Denote by $\mathrm{Rep}^{\Gamma}_G(T)$ the set of $G(T)$-conjugacy classes of homomorphisms. This also defines a presheaf $\mathrm{Rep}^{\Gamma}_G : \mathrm{Sch}^{\mathrm{op}}_S \to \mathrm{Set}$ on the category of S-schemes.

Note that in the case of affine S and T this coincides with the notion of G-valued representation from Section 4.1.1. The following lemma is standard.

Lemma 4.10. Let S be a scheme and let $(T_i)_{i\in I}$ be a cofiltered system of affine S-schemes. Then the limit $T = \lim_{i\in I} T_i$ exists in the category of S-schemes. Moreover T is S-affine and if $T_i = \underline{\operatorname{Spec}}_S(\mathcal{A}_i)$ for quasi-coherent $\mathcal{O}_S$-algebras $\mathcal{A}_i$, then T is canonically isomorphic to $\underline{\operatorname{Spec}}_S(\mathcal{A})$, where $\mathcal{A} := \operatorname*{colim}_{i\in I} \mathcal{A}_i$ is the colimit in the category of quasi-coherent $\mathcal{O}_S$-algebras.

Proof. Any colimit of quasi-coherent $\mathcal{O}_S$-modules is quasi-coherent [Sta19, 01LA] and $\otimes_{\mathcal{O}_X}$ preserves colimits in both variables [Sta19, 05NB]. From this, we obtain that $(\mathrm{QCoh}(\mathcal{O}_X), \otimes_{\mathcal{O}_X})$ is a cocomplete symmetric monoidal category. Its category of commutative monoids, which in this case is the category of commutative quasi-coherent $\mathcal{O}_S$-algebras $\mathrm{QCohCAlg}(\mathcal{O}_X)$, is cocomplete, see Martin Brandenburg's answer to Mathoverflow question 139968 for a proof. Thus $\mathcal{A}$ exists.

It can be deduced from the fact, that the inclusion functor $\mathrm{QCoh}(\mathcal{O}_S) \to \mathrm{Mod}(\mathcal{O}_S)$ has a monoidal right adjoint (combine [Sta19, 077P] with [Sta19, 01CE] (3)), that the inclusion $\mathrm{QCohCAlg}(\mathcal{O}_S) \to \mathrm{CAlg}(\mathcal{O}_S)$ has a right adjoint, which we will call the *quasi-coherator* $(-)^{\mathrm{qcoh}}$. Thus we can define the relative global sections of an S-scheme $f : T \to S$ by $\underline{\Gamma}_S(T, \mathcal{O}_T) := (f_*\mathcal{O}_T)^{\mathrm{qcoh}}$. We have the usual adjunction

$$\mathrm{Hom}_{\mathrm{QCohCAlg}_{\mathcal{O}_S}}(-, \underline{\Gamma}_S(T, \mathcal{O}_T)) = \mathrm{Hom}_{\mathrm{Sch}_S}(T, \underline{\mathrm{Spec}}_S(-))$$

Let U be any S-scheme. Define $T := \underline{\mathrm{Spec}}_S(\mathrm{colim}_i\, \mathcal{A}_i)$. We have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sch}_S}(U, T) &= \mathrm{Hom}_{\mathrm{QCohCAlg}_{\mathcal{O}_S}}(\mathcal{A}, \underline{\Gamma}_S(U, \mathcal{O}_U)) \\ &= \lim_i \mathrm{Hom}_{\mathrm{QCohCAlg}_{\mathcal{O}_S}}(\mathcal{A}_i, \underline{\Gamma}_S(U, \mathcal{O}_U)) \\ &= \lim_i \mathrm{Hom}_{\mathrm{Sch}_S}(U, T_i)\end{aligned}$$

which proves, that T is indeed the limit of the T_i. $\square$

Theorem 4.11. Let Γ be a group and let S be a scheme. Let G be an affine group scheme over S. The functor $\mathrm{Rep}_G^{\square,\Gamma}$ is representable by an affine S-scheme $X_G^{\square,\Gamma}$.

1. If Γ is finitely generated and G is (locally) of finite type over S, then $X_G^{\square,\Gamma}$ is (locally) of finite type over S.
2. If Γ is finitely presented and G is (locally) of finite presentation over S, then $X_G^{\square,\Gamma}$ is (locally) of finite presentation over S.
3. If Γ is finitely generated, G is of finite type over S and S is noetherian, then $X_G^{\square,\Gamma}$ is noetherian and of finite presentation over S.

In case of a finitely generated group Γ and an affine scheme S, this has been proved by Wang-Erickson in [Wan13, Thm. 1.4.4.5].

Proof. Let $I \subset \Gamma$ be a family of generators of Γ and let $F(I)$ be the free group on I. For any S-scheme T, there is a natural isomorphism between the set of homomorphisms $F(I) \to G(T)$ and $G(T)^I$. The functor $T \mapsto G(T)^I$ is representable by an S-scheme G^I. Here $G^I := \lim_{I' \subset I} G^{I'}$ is the cofiltered limit of affine S-schemes $G^{I'}$ indexed by finite subsets $I' \subset I$. Note, that by Lemma 4.10 this limit exists and is represented by the quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{O}_S(G)^{\otimes I} = \operatorname*{colim}_{I' \subset I} \mathcal{O}_S(G)^{\otimes I'}$. We have

$$\begin{aligned}G^T(I) &= \mathrm{Map}(I, G(T)) \\ &= \mathrm{Hom}(F(I), G(T)) \\ &= \mathrm{Hom}_S(T, G^I) \\ &= \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(G)^{\otimes I}, f_*\mathcal{O}_T)\end{aligned}$$

for any S-scheme $f : T \to S$.

Let $F(J)$ be another free group together with a homomorphism $F(J) \to F(I)$, such that the sequence of groups

$$F(J) \longrightarrow F(I) \longrightarrow \Gamma \longrightarrow 1$$

is exact. We obtain a short exact sequence of groups

$$1 \longrightarrow \mathrm{Hom}(\Gamma, G(T)) \longrightarrow \mathrm{Hom}(F(I), G(T)) \longrightarrow \mathrm{Hom}(F(J), G(T))$$

It follows, that $T \mapsto \mathrm{Hom}(\Gamma, G(T))$ is representable by the quotient $\mathcal{R}_G^{\square,\Gamma}$ of $\mathcal{O}_S(G)^{\otimes I}$ by the image of $\mathcal{O}_S(G)^{\otimes J}$ under the natural map. We put $X_G^{\square,\Gamma} := \underline{\mathrm{Spec}}_S(\mathcal{R}_G^{\square,\Gamma})$.

1. If Γ is finitely generated, then I can be taken to be finite. It follows, that $\mathcal{O}_S(G)^{\otimes I}$ and thus $\mathcal{R}_G^{\square,\Gamma}$ is (locally) of finite type over $\mathcal{O}_S$.

2. If Γ is of finite presentation, then I and J can be taken to be finite. It follows, that $\mathcal{O}_S(G)^{\otimes I}$ and $\mathcal{O}_S(G)^{\otimes J}$ are (locally) of finite presentation over $\mathcal{O}_S$. Thus $\mathcal{R}_G^{\square,\Gamma}$ is (locally) of finite presentation over $\mathcal{O}_S$.
3. By the first step, $\mathcal{R}_G^{\square,\Gamma}$ is of finite type over $\mathcal{O}_S$, in particular noetherian and of finite presentation.

$\square$

4.1.5 Topologizing point sets

Since we work frequently with topologies on point sets of schemes, we want to discuss the general procedure by which all these topologies we are interested in can be obtained. The method is due to Grothendieck and we follow the exposition of [Con12].

Proposition 4.12. Let A be a topological commutative ring. There is a unique way to define a topology on $X(A)$ for all affine A-schemes X of finite type at once, such that the following properties hold.

1. For every morphism $f : X \to Y$ of affine A-schemes of finite type, the map $X(A) \to Y(A)$ is continuous.
2. For every cartesian diagram
$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$
the diagram of topological spaces
$$\begin{tikzcd} (X \times_Z Y)(A) \arrow[r] \arrow[d] & X(A) \arrow[d] \\ Y(A) \arrow[r] & Z(A) \end{tikzcd}$$
is cartesian.
3. For every closed immersion $f : X \to Y$ of affine A-schemes of finite type, the map $X(A) \to Y(A)$ is a topological embedding, i.e. $X(A)$ carries the subspace topology of $Y(A)$.
4. The canonical bijection $A \to \mathbb{A}^1(A)$ is a homeomorphism.

Proof. [Con12, Proposition 2.1]. $\square$

For a finite type affine A-scheme X this topology can be characterized as the coarsest topology on $X(A)$, such that all morphisms of A-schemes $X \to \mathbb{A}^1$ induce a continuous map $X(A) \to A$. It can also be defined by choosing an arbitrary closed immersion $X \to \mathbb{A}^n$ and introducing the subspace topology on $X(A)$ with respect to the injection $X(A) \to A^n$, where A^n carries the product topology.

For every topological commutative A-algebra B, we have $X(B) = X_B(B)$ and we take on $X(B)$ the topology on $X_B(B)$. By choosing an embedding into an affine space, we see that the map $X(B_1) \to X(B_2)$ is continuous for any two topological A-algebras B_1, B_2 and continuous A-homomorphisms $B_1 \to B_2$.

For the proof of Proposition 6.15 we will also need Proposition 4.12 in the following situation: Let κ be a topological field and let A be a finite-dimensional local κ-algebra with residue field κ equipped with the product topology induced by an isomorphism $A \cong \kappa^n$ of κ-vector spaces. If X is an affine A-scheme of finite type, the map $X(A) \to X(\kappa)$ is continuous. If we now take the preimage $Z \subseteq X(A)$ of a Zariski-closed subset $Y(\kappa) \subseteq X(\kappa)$ for some closed A-subscheme $Y \subseteq X$, it is not clear how to identify Z with the A-points of a closed A-subscheme of X, but we still want to describe the topology of Z in a functorial way. This can be done by Weil restriction: The functor $T \mapsto X(A \otimes_\kappa T)$ is representable by an affine κ-scheme $\mathrm{Res}_\kappa^A X$ with $(\mathrm{Res}_{A/\kappa} X)(\kappa) = X(A)$ and the projection $X(A) \to X(\kappa)$ gives rise to a morphism of κ-schemes $\mathrm{Res}_{A/\kappa} X \to X_\kappa$. We now obtain Z as the κ-points of the scheme-theoretic preimage of Y_κ in $\mathrm{Res}_{A/\kappa} X$.

4.1.6 Acyclic G-modules and good filtrations

Let $\mathcal{O}$ be a commutative ring. If V is an $\mathcal{O}$-module with a rational action of an affine $\mathcal{O}$-group scheme G and $\mathcal{O}'$ is an arbitrary commutative $\mathcal{O}$-algebra, then the natural map $V^G \otimes_{\mathcal{O}} \mathcal{O}' \to (V \otimes_{\mathcal{O}} \mathcal{O}')^{G_{\mathcal{O}'}}$ is not always an isomorphism. The entire purpose of this section is to establish conditions under which this map is an isomorphism.

We recall the universal coefficient theorem for rational Ext groups.

Theorem 4.13. Let G be a flat affine group scheme over a Dedekind ring $\mathcal{O}$ and let $\mathcal{O}'$ be a commutative $\mathcal{O}$-algebra. Then for each $\mathcal{O}$-flat G-module N and each finitely generated projective G-module V, we have a short exact sequence

$$0 \to \mathrm{Ext}^n_G(V, N) \otimes_{\mathcal{O}} \mathcal{O}' \to \mathrm{Ext}^n_{G_{\mathcal{O}'}}(V \otimes_{\mathcal{O}} \mathcal{O}', N \otimes_{\mathcal{O}} \mathcal{O}') \to \mathrm{Tor}_1^{\mathcal{O}}(\mathrm{Ext}^{n+1}_G(V, N), \mathcal{O}') \to 0$$

of $\mathcal{O}'$-modules.

Proof. By [Jan03, I.4.4 Lemma] and [Jan03, p. I.4.2], there is a natural identification $\mathrm{Ext}^n_G(V, N) = \mathrm{Ext}^n_G(\mathcal{O}, V^* \otimes_{\mathcal{O}} N) = H^n(G, V^* \otimes_{\mathcal{O}} N)$ and similarly for the middle term. The claim follows from the universal coefficient theorem [Jan03, I.4.18 Proposition (a)]. □

Corollary 4.14. Let G be a flat affine group scheme over a Dedekind ring $\mathcal{O}$, let V be a G-module and let $\mathcal{O}'$ be a commutative $\mathcal{O}$-algebra. Assume, that one of the following holds:

1. $\mathcal{O}'$ is $\mathcal{O}$-flat.
2. $H^1(G, V) = 0$.

Then the natural map $V^G \otimes_{\mathcal{O}} \mathcal{O}' \to (V \otimes_{\mathcal{O}} \mathcal{O}')^{G_{\mathcal{O}'}}$ is an isomorphism.

Proof. By the universal coefficient theorem Theorem 4.13, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}[G^m]^G \otimes_{\mathcal{O}} \mathcal{O}' \longrightarrow \mathcal{O}'[G^m]^G \longrightarrow \mathrm{Tor}_1^{\mathcal{O}}(H^1(G, V), \mathcal{O}') \longrightarrow 0$$

Under both assumptions the claim follows. □

We say, that V is *acyclic*, if the rational cohomology groups $H^i(G, V)$ vanish for all $i > 0$.

If G is a Chevalley group over a principal ideal domain $\mathcal{O}$ with fiberwise maximal $\mathcal{O}$-torus T and Borel subgroup B, we define $H^0(\lambda) := \mathrm{Ind}_B^G \lambda$ and $V(\lambda) := H^0(-w_0\lambda)^*$ for every dominant weight $X(T)_+$ and the longest element w_0 of the Weyl group.

Let V be a G-module. An ascending filtration $V = \bigcup_{i \geq 0} V_i$ of V is *good*, if for all $i \geq 0$, V_{i+1}/V_i is isomorphic to $H^0(\lambda)$ for some $\lambda \in X(T)_+$.

Lemma 4.15. Let G be a Chevalley group over a principal ideal domain $\mathcal{O}$. Let V be a G-module with good filtration. Then V is acyclic.

Proof. If V has finite rank over a PID $\mathcal{O}$, we have $H^i(G, V) = \mathrm{Ext}^i_G(V(0), V) = 0$ for all $i > 0$ by [Jan03, B.9 Lemma (iii)]. If V is not of finite rank, we can choose a good filtration $V = \bigcup_n V_n$ by G-submodules of finite rank and calculate $H^i(G, V) = \varinjlim_n H^i(G, V_n)$ using [Jan03, p. I.4.17]. □

Mathieu's tensor product theorem states, that the tensor product of two modules with good filtration over a connected reductive group over an algebraically closed field admits a good filtration. An integral version of this theorem also holds and we give a proof here in lack of reference.

Theorem 4.16. Let G be a Chevalley group over a principal ideal domain $\mathcal{O}$. Let M and N be G-modules with good filtration. Then $M \otimes_{\mathcal{O}} N$ is a G-module with good filtration.

Proof. We first assume, that M and N are free of finite rank. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}$ with residue field $\kappa := \mathcal{O}/\mathfrak{m}$. By [Jan03, B.9 Lemma (i) $\Rightarrow$ (iv)], $M_\kappa := M \otimes_{\mathcal{O}} \kappa$ and $N_\kappa := N \otimes_{\mathcal{O}} \kappa$ are G_κ-modules with good filtration. Choose a split and fiberwise maximal $\mathcal{O}$-torus $T \subseteq G$ and a Borel subgroup $B \subseteq G$ containing T. By Theorem 4.13, there is an isomorphism $\mathrm{Ext}^1_{G_\kappa}(V(\lambda), M_\kappa) \otimes_\kappa \overline{\kappa} \cong \mathrm{Ext}^1_{G_{\overline{\kappa}}}(V(\lambda), M_{\overline{\kappa}})$ for all dominant weights $\lambda \in X(T)_+$, so the latter group is 0 by [Jan03, B.9 Lemma (iv) $\Rightarrow$ (iii)] applied to M_κ. So by [Jan03, B.9 Lemma (iii) $\Rightarrow$ (i)] $M_{\overline{\kappa}}$ and $N_{\overline{\kappa}}$ are $G_{\overline{\kappa}}$-modules with good filtration. By Mathieu's tensor product theorem [Mat90], which holds for connected reductive groups over algebraically closed fields, see [Jan03, Proposition II.4.21][Kal93, Theorem 4.4.3] $M_{\overline{\kappa}} \otimes_{\overline{\kappa}} N_{\overline{\kappa}}$ has a good filtration.

We now reverse the argument: By [Jan03, B.9 Lemma (i) $\Rightarrow$ (iii)], we have $\mathrm{Ext}^1_G(V(\lambda), M_{\overline{\kappa}} \otimes_{\overline{\kappa}} N_{\overline{\kappa}})$, hence $\mathrm{Ext}^1_G(V(\lambda), M_\kappa \otimes_\kappa N_\kappa)$ for all dominant weights $\lambda \in X(T)_+$. So by [Jan03, B.9 Lemma (iii) $\Rightarrow$ (i)] $M_\kappa \otimes_\kappa N_\kappa$ has a good filtration. Since $\mathfrak{m}$ is arbitrary, we can apply [Jan03, B.9 Lemma (iv) $\Rightarrow$ (i)] to conclude, that $M \otimes_{\mathcal{O}} N$ is a G-module with good filtration.

Now let M and N be arbitrary with good filtrations $M = \bigcup_{i=1}^\infty M_i$ and $N = \bigcup_{j=1}^\infty N_j$. Then $M \otimes_{\mathcal{O}} N = \bigcup_i \bigcup_j M_i \otimes_{\mathcal{O}} N_i$ by [Sta19, 00DD]. By choosing a diagonal sequence, we can define a filtration of $M \otimes_{\mathcal{O}} N$ by good submodules. ☐

Proposition 4.17. Let G be a Chevalley group. Then for all $m \geq 1$, $\mathbb{Z}[G^m]$ equipped with the action of G by conjugation has a good filtration. In particular for every commutative ring $\mathcal{O}$ and every $\mathcal{O}$-algebra $\mathcal{O}'$, the canonical map $\mathcal{O}[G^m]^G \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[G^m]^G$ is an isomorphism.

Proof. In [Jan03, B.8] it is shown, that $\mathbb{Z}[G]$ has a good filtration. Here the action of G is defined by $(g \cdot f)(h) := f(g^{-1}hg)$. By Mathieu's tensor product theorem Theorem 4.16, $\mathbb{Z}[G^m] = \mathbb{Z}[G]^{\otimes m}$ has a good filtration. This proves the first assertion. So $H^1(G, \mathbb{Z}[G^m]) = 0$ by Lemma 4.15. We calculate

$$\mathcal{O}[G^m]^G \otimes_{\mathcal{O}} \mathcal{O}' = (\mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}' = \mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathcal{O}' = \mathcal{O}'[G^m]^G$$

by applying twice Corollary 4.14. ☐

Proposition 4.18. For all $m, n \geq 1$, $\mathbb{Z}[M_n^m]$ equipped with the action of $G = \mathrm{GL}_n$ (resp. $G = \mathrm{SL}_n$) by conjugation has a good filtration. In particular for every commutative ring $\mathcal{O}$ and every $\mathcal{O}$-algebra $\mathcal{O}'$, the canonical map $\mathcal{O}[M_n^m]^G \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[M_n^m]^G$ is an isomorphism.

Proof. Let Std be the standard representation of G. Since the $M_n \cong \mathrm{Std} \otimes \mathrm{Std}^*$ and Std is self-dual, we have $M_n^m \cong \mathrm{Std}^{\otimes 2m}$. By Theorem 4.16 and the formula for symmetric powers of direct sums it is enough to show, that $\mathrm{Sym}^d(\mathrm{Std})$ has a good filtration. But $\mathrm{Sym}^d(\mathrm{Std})$ is a highest weight module, so we are done. ☐

Proposition 4.19. Let $\mathcal{O}$ be a commutative ring with $2 \in \mathcal{O}^\times$ and let $\mathcal{O}'$ be an $\mathcal{O}$-algebra. Then for all $n \geq 0$ the canonical map $\mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{O}_{2n+1}} \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[\mathrm{O}_{2n+1}^m]^{\mathrm{O}_{2n+1}}$ is an isomorphism.

Proof. We have $\mathrm{O}_{2n+1} = \mathrm{SO}_{2n+1} \times \{\pm 1\}$ over $\mathcal{O}$, so we can explicitly compute:

$$\mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{O}_{2n+1}} = \mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{SO}_{2n+1}} = \mathcal{O}[\{\pm 1\}^m] \otimes_{\mathcal{O}} \mathcal{O}[\mathrm{SO}_{2n+1}^m]^{\mathrm{SO}_{2n+1}}$$

We have $\mathcal{O}[\mathrm{SO}_{2n+1}^m]^{\mathrm{SO}_{2n+1}} \otimes_{\mathcal{O}} k = k[\mathrm{SO}_{2n+1}^m]^{\mathrm{SO}_{2n+1}}$ by Proposition 4.17. ☐

4.2 G-valued pseudocharacters

Let $\mathcal{O}$ be a commutative ring and let G be a generalized reductive $\mathcal{O}$-group scheme. By the datum of G, the datum of $\mathcal{O}$ is given and we will drop $\mathcal{O}$ from notations. A G-pseudocharacter will be defined depending on both the coefficient ring $\mathcal{O}$ and a commutative $\mathcal{O}$-algebra A, which corresponds to the base ring A in Section 3.

4.2.1 G-pseudocharacters

The definition of G-pseudocharacter we give is slightly more general than Lafforgue's original definition [Laf18, §11], in that we work over arbitrary base rings $\mathcal{O}$ instead of $\mathcal{O} = \mathbb{Z}$. We will later be interested in the case, that $\mathcal{O}$ is the ring of integers of a p-adic field.

We introduce a special notation for substitutions, which will be particularly important in Definition 4.20 and the proofs of Theorem 4.46 and Proposition 4.47.

Let $\mathrm{FG}(m)$ be the free group on m generators $x_1, \dots, x_m$. Let $\alpha : \mathrm{FG}(m) \to \mathrm{FG}(n)$ be a group homomorphism. Let Γ be an arbitrary group. Then there is a unique map $(-)_\alpha : \Gamma^n \to \Gamma^m,\ \gamma \mapsto \gamma_\alpha$, such that the homomorphism $f_\gamma : \mathrm{FG}(n) \to \Gamma,\ x_i \mapsto \gamma_i$ satisfies $f_\gamma(\alpha(x_j)) = (\gamma_\alpha)_j$ for all $j \in \{1, \dots, m\}$. In other words $(-)_\alpha$ is the induced map $\Gamma^n = \mathrm{Hom}(\mathrm{FG}(n), \Gamma) \to \mathrm{Hom}(\mathrm{FG}(m), \Gamma) = \Gamma^m$. More concretely $w_i = \alpha(x_i)$ is a word in x_j and x_j^{-1} for $j = 1, \dots, n$ and α applied to a tuple $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ is the tuple $(\delta_1, \dots, \delta_m) \in \Gamma^m$ with δ_i the word w_i with x_j substituted by γ_j for $j = 1, \dots, n$.

Similarly we obtain an induced map $(-)_\alpha : G^n \to G^m$. G^0 acts on G^m by $g \cdot (g_1, \dots, g_m) = (gg_1g^{-1}, \dots, gg_mg^{-1})$. This induces a rational action of G^0 on the affine coordinate ring $\mathcal{O}[G^m]$ of G^m. The submodule $\mathcal{O}[G^m]^{G^0} \subseteq \mathcal{O}[G^m]$ is defined as the rational invariant module of the G^0-representation $\mathcal{O}[G^m]$. It is an $\mathcal{O}$-subalgebra, since G^0 acts by $\mathcal{O}$-linear automorphisms. The map $(-)_\alpha : G^n \to G^m$ is G^0-equivariant. So there is an induced homomorphism between the algebras of rational invariants $(-)^\alpha : \mathcal{O}[G^m]^{G^0} \to \mathcal{O}[G^n]^{G^0}$. In the special case, that α is induced by a map of sets $\zeta : \{1, \dots, n\} \to \{1, \dots, m\}$, such that $\alpha(x_i) = x_{\zeta(i)}$, we also write $\gamma_\zeta := \gamma_\alpha$ for $\gamma \in \Gamma^n$ and $f^\zeta := f^\alpha$ for $f \in \mathcal{O}[G^m]^{G^0}$.

Definition 4.20 (G-pseudocharacter)**.** Let Γ be a group and let A be a commutative $\mathcal{O}$-algebra. A *G-pseudocharacter* Θ of Γ over A is a sequence of $\mathcal{O}$-algebra maps

$$\Theta_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$$

for each $m \geq 1$, satisfying the following conditions:

1. For all $n, m \geq 1$, each map $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, every $f \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma_1, \dots, \gamma_n \in \Gamma$, we have
$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)})$$
where $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$.
2. For all $m \geq 1$, for all $\gamma_1, \dots, \gamma_{m+1} \in \Gamma$ and every $f \in \mathcal{O}[G^m]^{G^0}$, we have
$$\Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_{m+1}) = \Theta_m(f)(\gamma_1, \dots, \gamma_m\gamma_{m+1})$$
where $\hat{f}(g_1, \dots, g_{m+1}) = f(g_1, \dots, g_m g_{m+1})$.

We denote the set of G-pseudocharacters of Γ over A by $\mathrm{PC}_G^\Gamma(A)$. If $f : A \to B$ is a homomorphism of $\mathcal{O}$-algebras, then there is an induced map $f_* : \mathrm{PC}_G^\Gamma(A) \to \mathrm{PC}_G^\Gamma(B)$. For $\Theta \in \mathrm{PC}_G^\Gamma(A)$, the image $f_*\Theta$ is called the *scalar extension* of Θ and also denoted with $\Theta \otimes_A B$. This notion of scalar extension shall not be confused with change of the base ring $\mathcal{O}$ of G, which will be discussed in Proposition 4.48 and comes with some subtleties.

If $\iota : G \to H$ is a homomorphism of affine $\mathcal{O}$-group schemes, we define an H-pseudocharacter $\iota(\Theta)$ by letting $\iota(\Theta)_m$ be the composition of Θ_m with the induced map $\mathcal{O}[H^m]^{H^0} \to \mathcal{O}[G^m]^{G^0}$.

In [BHKT, Def. 4.1] a G-pseudocharacter is defined only for Chevalley groups over $\mathbb{Z}$. Some of our proofs do not need this strong assumption.

Every G-valued representation gives rise to a G-pseudocharacter:

Lemma 4.21. Let Γ be a group, let A be a commutative $\mathcal{O}$-algebra and let $\rho : \Gamma \to G(A)$ be a homomorphism. Then the sequence of maps $\Theta_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$ defined by

$$\Theta_m(f)(\gamma_1, \dots, \gamma_m) := f(\rho(\gamma_1), \dots, \rho(\gamma_m))$$

is a G-pseudocharacter $\Theta = (\Theta_m)_{m \geq 1}$, which depends only on ρ up to $G(A)$-conjugation. We write $\Theta_\rho := \Theta$. In particular the map

$$\begin{aligned} \mathrm{Hom}(\Gamma, G(A))/G^0(A) &\to \mathrm{PC}_G^\Gamma(A) \\ \rho &\mapsto \Theta_\rho \end{aligned}$$

is well-defined, where $G^0(A)$ acts by pointwise conjugation.

Proof. Compare [BHKT, p. 4.3]. Let $\gamma_1, \dots, \gamma_n \in \Gamma$ and $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$ and $f \in \mathcal{O}[G^m]^{G^0}$.

$$\begin{aligned}\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) &= f^\zeta(\rho(\gamma_1), \dots, \rho(\gamma_n)) \\ &= f(\rho(\gamma_{\zeta(1)}), \dots, \rho(\gamma_{\zeta(m)})) \\ &= \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}) \\ &= \Theta_m(f)^\zeta(\gamma_1, \dots, \gamma_n)\end{aligned}$$

The second property can be checked by a similar calculation. For $g \in G(A)$ and $f \in \mathcal{O}[G^n]^{G^0}$, we have

$$f(g\rho(\gamma_1)g^{-1}, \dots, g\rho(\gamma_n)g^{-1}) = f(\rho(\gamma_1), \dots, \rho(\gamma_n))$$

since f is invariant under conjugation. □

With Lemma 4.21 in mind, we give an intuitive explanation of this technical definition: A G-pseudocharacter Θ, that comes from a representation, remembers for every $m \geq 0$ for every conjugation invariant regular function on G^m its values on Γ^m, when applied to the representation. Since the coefficients of the characteristic polynomials of a GL_d-representation are conjugation-invariant, Θ remembers all their values and therefore at least the information about the representation, that is carried by the characteristic polynomials.

Lemma 4.22.

(1) For $h : A \to A'$ the map $\mathrm{PC}^\Gamma_{G,A} \to \mathrm{PC}^\Gamma_{G,A'}, \Theta \mapsto h_*\Theta = (h \circ \Theta_n)_{n \geq 1}$ is well-defined.

(2) For $\phi : \Delta \to \Gamma$ the map $\mathrm{PC}^\Gamma_{G,A} \to \mathrm{PC}^\Delta_{G,A}, \Theta \mapsto \phi^*\Theta = (\Theta_n \circ \phi)_{n \geq 1}$ is well-defined.

(3) If $N \leq \Gamma$ is a normal subgroup, then $\pi^* : \mathrm{PC}^{\Gamma/N}_{G,A} \to \mathrm{PC}^\Gamma_{G,A}$ is an injection, that identifies $\mathrm{PC}^{\Gamma/N}_{G,A}$ with the set of pseudocharacters, that take values in $\mathrm{Map}((\Gamma/N)^n, A)$.

Proof. The proof of [BHKT, Lem. 4.4] carries over verbatim. □

Proposition 4.23. Let $\rho : \Gamma \to G(k)$ be a representation over an algebraically closed field k and let ρ^{ss} be some G-semisimplification of ρ. Then $\Theta_\rho = \Theta_{\rho^{\mathrm{ss}}}$ in $\mathrm{PC}^\Gamma_G(k)$.

Proof. Suppose P is minimal and contains $\rho(\Gamma)$, suppose L is an R-Levi of P and let λ be a cocharacter, such that $P = P_\lambda$ and $L = L_\lambda$. Let us write $\rho^{\mathrm{ss}} = c_\lambda \circ \rho = \lim_{t \to 0} \lambda(t)\rho\lambda(t)^{-1}$. The G-pseudocharacter $\Theta_{m,\rho}$ attached to ρ satisfies by definition $\Theta_{\rho,m}(f)(\gamma_1, \dots, \gamma_m) = f(\rho(\gamma_1), \dots, \rho(\gamma_m))$ for all $m \geq 1$ and ρ^{ss} satisfies a similar formula. Since f is G-invariant, the morphism $\mathbb{G}_m \to \mathbb{A}^1, t \mapsto f(\lambda(t)\rho(\gamma_1)\lambda(t)^{-1}, \dots, \lambda(t)\rho(\gamma_m)\lambda(t)^{-1})$ is constant and equal to $f(\rho(\gamma_1), \dots, \rho(\gamma_m))$. Since the limit $\lim_{t \to 0} \lambda(t)\rho\lambda(t)^{-1}$ exists and f is algebraic with separated target $\mathbb{A}^1$, this is equal to $f(\rho^{\mathrm{ss}}(\gamma_1), \dots, \rho^{\mathrm{ss}}(\gamma_m))$ and $\Theta_\rho = \Theta_{\rho^{\mathrm{ss}}}$ follows. □

Theorem 4.24. Let Γ be a group. Assume that one of the following holds:

1. G is a Chevalley group over $\mathbb{Z}$ and k is an algebraically closed field.
2. G is a group scheme over a domain $\mathcal{O}$ of characteristic 0 and k is a field, which contains $\mathcal{O}$, such that G_k is reductive.

Let $\Theta \in \mathrm{PC}^\Gamma_G(k)$. Then there is a finite extension k'/k and a G-completely reducible representation $\rho : \Gamma \to G(k')$ with $\Theta_\rho = \Theta$ and ρ is unique up to $G^0(\overline{k})$-conjugacy.

Proof. The first case is [BHKT, Theorem 4.5]; we can use Proposition 4.17 to identify the k-points of $G^m /\!\!/ G$ with the k-points of $G^m_k /\!\!/ G_k$. Alternatively we can use [Ses77, Theorem 3]. The second case is [Laf18, Proposition 11.7]. □

Remark 4.25. Theorem 4.24 is still true for $G/G^0 \neq 1$ in positive characteristic and can be proved using [Ses77, Theorem 3]. The proof is omitted, as it is not needed for the cases we will consider here.

4.2.2 The kernel of a G-pseudocharacter

We will need kernels of G-pseudocharacters in the proof of Lemma 7.6.

Definition 4.26 (Kernel)**.** Let $\Theta \in \mathrm{PC}_G^\Gamma(A)$ be an arbitrary G-pseudocharacter as in Definition 4.20. We define the *kernel* $\ker(\Theta)$ of Θ as the set of $\delta \in \Gamma$, such that for all $m \geq 1$, all $f \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma_1, \dots, \gamma_m \in \Gamma$, we have

$$\Theta_m(f)(\gamma_1, \dots, \gamma_m\delta) = \Theta_m(f)(\gamma_1, \dots, \gamma_m).$$

Lemma 4.27. $\ker(\Theta)$ in Definition 4.26 is a normal subgroup of Γ.

Proof. It is clear, that $\ker(\Theta)$ is a subgroup of Γ. Let $\delta \in \ker(\Theta)$, $h \in \Gamma$ and $\gamma_1, \dots, \gamma_m \in \Gamma$ for some $m \geq 1$. Then

$$\begin{aligned}\Theta_m(f)(\gamma_1, \dots, \gamma_m h\delta h^{-1}) &= \Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_m h\delta, h^{-1}) \\ &= \Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_m h, h^{-1}) \\ &= \Theta_m(f)(\gamma_1, \dots, \gamma_m)\end{aligned}$$

so $h\delta h^{-1} \in \ker(\Theta)$. □

It is easy to check, that if $\delta \in \ker(\Theta)$, then

$$\Theta_m(f)(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\delta, \gamma_{i+1}, \dots, \gamma_m) = \Theta_m(f)(\gamma_1, \dots, \gamma_m)$$

for every $i = 1, \dots, m$.

We will use this to prove the following homomorphisms theorem. It will also be important in the proof of Lemma 7.6.

Lemma 4.28. Let $\Theta \in \mathrm{PC}_G^\Gamma(A)$ be an arbitrary G-pseudocharacter as in Definition 4.20, let $\Delta \leq \Gamma$ be a normal subgroup and assume, that $\Delta \subseteq \ker(\Theta)$. Then there is a unique G-pseudocharacter $\Theta' \in \mathrm{PC}_G^{\Gamma/\Delta}(A)$, such that Θ is the restriction of Θ' to Γ.

Proof. Uniqueness is clear, since $\Gamma \to \Gamma/\Delta$ is surjective and hence the maps $\mathrm{Map}((\Gamma/\Delta)^m, A) \to \mathrm{Map}(\Gamma^m, A)$ are injective for all $m \geq 1$. We can define Θ' as

$$\Theta'_m(f)(\gamma_1\Delta, \dots, \gamma_m\Delta) := \Theta_m(f)(\gamma_1, \dots, \gamma_m)$$

for all $m \geq 1$, all $f \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma_1, \dots, \gamma_m \in \Gamma$. This is well-defined, since $\Delta \subseteq \ker(\Theta)$. The axioms of a pseudocharacter are easily verified. □

Lemma 4.29. Let $\rho : \Gamma \to G(A)$ be a representation with associated G-pseudocharacter Θ. Then $\ker(\rho) \subseteq \ker(\Theta)$.

Proof. We can define ρ on $\Gamma/\ker(\rho)$. The associated G-pseudocharacter of $\Gamma/\ker(\rho)$ can be inflated to Γ and this turns out to be Θ. □

The converse inclusion is false in general! Here is an example.

Example 4.30. Let $\rho : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ be defined by $\rho(a) := \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$. Then Θ_ρ is the pseudocharacter of the trivial representation. Hence $\ker(\rho) = 1$, but $\ker(\Theta_\rho) = \mathbb{Z}$.

Equality holds, when ρ is G-completely reducible.

Proposition 4.31. Let G be a reductive group over a field k and suppose that one of the assumptions in Theorem 4.24 holds. Let Γ be a group and let $\rho : \Gamma \to G(k)$ be an absolutely G-completely reducible representation with associated G-pseudocharacter Θ. Then $\ker(\rho) = \ker(\Theta)$.

Proof. By Lemma 4.28 Θ factors over a G-pseudocharacter Θ' of $\Gamma/\ker(\Theta)$. By Theorem 4.24 there is a G-completely reducible representation $\rho' : \Gamma/\ker(\Theta) \to G(\overline{k})$ with $\Theta' \otimes_k \overline{k} = \Theta_{\rho'}$. The inflation $\rho'' : \Gamma \to G(\overline{k})$ of ρ' to Γ is still G-completely reducible and conjugate to $\rho \otimes_k \overline{k}$ by an element of $G(\overline{k})$. Hence $\ker(\Theta) \subseteq \ker(\rho'') = \ker(\rho \otimes_k \overline{k}) = \ker(\rho)$. The converse inclusion is Lemma 4.29. □

4.2.3 Direct sum, dual and tensor product

Recall from Section 4.2, that a homomorphism of affine $\mathcal{O}$-group schemes $G \to H$ gives rise to a natural transformation $\mathrm{PC}_G^\Gamma \to \mathrm{PC}_H^\Gamma$. This provides us with an astonishingly easy way to define natural operations on pseudocharacters, such as direct sums, duals and tensor products. Defining such operations for determinant laws is more involved; see e.g. [Wan13, §1.1.11] for a direct sum operation and [BJ19, §4.5] for twisting with a character. It is clear by construction, that these operations will be compatible with the corresponding operations on representations.

Suppose $\Theta \in \mathrm{PC}_{\mathrm{GL}_d}^\Gamma(A)$. Then we can define the *dual* Θ^* by composing with the transpose inverse map $\mathrm{GL}_d \to \mathrm{GL}_d$.

Assume, that $\mathcal{O}$ is a PID. Suppose $\Theta \in \mathrm{PC}_{\mathrm{GL}_a}^\Gamma(A)$ and $\Theta' \in \mathrm{PC}_{\mathrm{GL}_b}^\Gamma(A)$ with $A \in \mathrm{CAlg}_\mathcal{O}$ and $a + b = n$. We will define the *direct sum* $\Theta \oplus \Theta' \in \mathrm{PC}_{\mathrm{GL}_n}^\Gamma(A)$. For $m \geq 1$, we obtain a map $\Theta_m \otimes \Theta'_m : \mathcal{O}[\mathrm{GL}_a^m]^{\mathrm{GL}_a} \otimes_\mathcal{O} \mathcal{O}[\mathrm{GL}_b^m]^{\mathrm{GL}_b} \to \mathrm{Map}(\Gamma^m, A)$. It turns out, that since $\mathcal{O}$ is a PID and by Theorem 4.13, we have $\mathcal{O}[\mathrm{GL}_a^m]^{\mathrm{GL}_a} \otimes_\mathcal{O} \mathcal{O}[\mathrm{GL}_b^m]^{\mathrm{GL}_b} = \mathcal{O}[(\mathrm{GL}_a \times \mathrm{GL}_b)^m]^{\mathrm{GL}_a \times \mathrm{GL}_b}$. The diagonal embedding $\mathrm{GL}_a \times \mathrm{GL}_b \to \mathrm{GL}_n$ induces a map $\mathcal{O}[\mathrm{GL}_n^m]^{\mathrm{GL}_n} \to \mathcal{O}[(\mathrm{GL}_a \times \mathrm{GL}_b)^m]^{\mathrm{GL}_a \times \mathrm{GL}_b}$ and we define $(\Theta \oplus \Theta')_m$ as the composition of this map with $\Theta_m \otimes \Theta'_m$. The compatibility conditions (1) and (2) in Definition 4.20 can be verified directly, but the alternative description of pseudocharacters Corollary 4.45 in the next section provides us with an easier way to see, that $\Theta \oplus \Theta'$ is indeed a pseudocharacter.

As for the direct sum, the *tensor product* $\Theta \otimes \Theta'$ is induced by the dyadic product map $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \to \mathrm{GL}_{d_1 d_2}$ after choice of a bijection $\{1, \dots, d_1\} \times \{1, \dots, d_2\} \cong \{1, \dots, d_1 d_2\}$ as in Section 3.12.

Proposition 4.32. The direct sum of Lafforgue's pseudocharacters is compatible with the direct sum of Taylor's pseudocharacters: Let Γ be a group, A a commutative ring and $d_1, d_2 \geq 0$ with $(d_1 + d_2)! \in A^\times$. Then the diagram

$$\begin{tikzcd}
\mathrm{PC}_{\mathrm{GL}_{d_1}}^\Gamma(A) \times \mathrm{PC}_{\mathrm{GL}_{d_2}}^\Gamma(A) \arrow[r, "\oplus"] \arrow[d, "\mathrm{tr} \times \mathrm{tr}"] & \mathrm{PC}_{\mathrm{GL}_{d_1 d_2}}^\Gamma(A) \arrow[d, "\mathrm{tr}"] \\
\mathrm{TPC}_{d_1}^\Gamma(A) \times \mathrm{TPC}_{d_2}^\Gamma(A) \arrow[r, "\oplus"] & \mathrm{TPC}_{d_1 d_2}^\Gamma(A)
\end{tikzcd}$$

commutes. Here the top arrow is the direct sum constructed in Section 4.2.3, the bottom arrow is the direct sum of Definition 2.38 and the vertical arrows are given by the comparison map Proposition 4.59.

Proof. By definition of the comparison map Proposition 4.59 it is enough to show, that the map

$$\mathbb{Z}[\mathrm{GL}_{d_1+d_2}]^{\mathrm{GL}_{d_1+d_2}} \to \mathbb{Z}[\mathrm{GL}_{d_1}]^{\mathrm{GL}_{d_1}} \otimes \mathbb{Z}[\mathrm{GL}_{d_1}]^{\mathrm{GL}_{d_1}}$$

induced by the direct sum $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \to \mathrm{GL}_{d_1+d_2}$ maps $\mathrm{tr}(\mathbb{X})$ to $\mathrm{tr}(\mathbb{X}_1)+\mathrm{tr}(\mathbb{X}_2)$, where $\mathbb{X} \in \mathrm{GL}_{d_1 d_2}(\mathbb{Z}[\mathrm{GL}_{d_1 d_2}])$ and $\mathbb{X}_i \in \mathrm{GL}_{d_i}(\mathbb{Z}[\mathrm{GL}_{d_1}] \otimes \mathbb{Z}[\mathrm{GL}_{d_2}])$ are the generic matrix coordinates. This is clear by definition. □

Proposition 4.33. The dual of Lafforgue's pseudocharacters is compatible with the dual of Taylor's pseudocharacters: Let Γ be a group, A a commutative ring and $d \geq 0$ with $d! \in A^\times$. Then the diagram

$$\begin{tikzcd}
\mathrm{PC}_{\mathrm{GL}_d}^\Gamma(A) \arrow[r, "\vee"] \arrow[d, "\mathrm{tr} \times \mathrm{tr}"] & \mathrm{PC}_{\mathrm{GL}_d}^\Gamma(A) \arrow[d, "\mathrm{tr}"] \\
\mathrm{TPC}_d^\Gamma(A) \arrow[r, "\vee"] & \mathrm{TPC}_d^\Gamma(A)
\end{tikzcd}$$

commutes. Here the top arrow is the dual constructed in Section 4.2.3, the bottom arrow is the dual constructed in Section 2.12.

Proof. The claim follows, since the map $\mathbb{Z}[\mathrm{GL}_d]^{\mathrm{GL}_d} \to \mathbb{Z}[\mathrm{GL}_d]^{\mathrm{GL}_d}$ induced by the transpose inverse $\mathrm{GL}_d \to \mathrm{GL}_d$ maps $\mathrm{tr}(\mathbb{X})$ to $\mathrm{tr}(\mathbb{X}^{-1})$. □

Proposition 4.34. The tensor product of Lafforgue's pseudocharacters is compatible with the tensor product of Taylor's pseudocharacters: Let Γ be a group, A a commutative ring and $d_1, d_2 \geq 0$ with

$(d_1d_2)! \in A^\times$. Then the diagram

$$\begin{tikzcd}
\mathrm{PC}^\Gamma_{\mathrm{GL}_{d_1}}(A) \times \mathrm{PC}^\Gamma_{\mathrm{GL}_{d_2}}(A) \arrow[r, "\otimes"] \arrow[d, "\mathrm{tr} \times \mathrm{tr}"] & \mathrm{PC}^\Gamma_{\mathrm{GL}_{d_1d_2}}(A) \arrow[d, "\mathrm{tr}"] \\
\mathrm{TPC}^\Gamma_{d_1}(A) \times \mathrm{TPC}^\Gamma_{d_2}(A) \arrow[r, "\otimes"] & \mathrm{TPC}^\Gamma_{d_1d_2}(A)
\end{tikzcd}$$

commutes. Here the top arrow is the tensor product constructed in Section 4.2.3, the bottom arrow is the tensor product constructed in Section 2.13.

Proof. It is enough to show, that the map $\mathbb{Z}[\mathrm{GL}_{d_1d_2}]^{\mathrm{GL}_{d_1d_2}} \to \mathbb{Z}[\mathrm{GL}_{d_1}]^{\mathrm{GL}_{d_1}} \otimes \mathbb{Z}[\mathrm{GL}_{d_1}]^{\mathrm{GL}_{d_1}}$ induced by the tensor product $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \to \mathrm{GL}_{d_1d_2}$ maps $\mathrm{tr}(\mathbb{X})$ to $\mathrm{tr}(\mathbb{X}_1)\,\mathrm{tr}(\mathbb{X}_2)$. This follows from Lemma 2.2, since $\mathrm{tr}(\mathbb{X})$ is mapped to $\mathrm{tr}(\mathbb{X}_1 \otimes \mathbb{X}_2) = \mathrm{tr}(\mathbb{X}_1)\,\mathrm{tr}(\mathbb{X}_2)$. ☐

We shall also need the notion of direct sum of two symplectic pseudocharacters, induced by the natural map $\mathrm{Sp}_{2a} \times \mathrm{Sp}_{2b} \to \mathrm{Sp}_{2n}$ for $a + b = n$, which corresponds to the orthogonal direct sum of symplectic spaces. The procedure for the construction of this direct sum operation is the same as for the general linear group, explained above.

There is also a natural map $\mathrm{GL}_n \to \mathrm{Sp}_{2n}$ induced by mapping a representation V to $V \oplus V^*$ equipped with the symplectic form, which makes V and V^* totally isotropic subspaces, is the canonical pairing on $V \times V^*$ and the negative of the canonical pairing on $V^* \times V$. Even though the map $\mathrm{GL}_n \to \mathrm{Sp}_{2n}$ is not uniquely determined by this description, it is well-defined on conjugacy classes of representations and well-defined on pseudocharacters.

4.2.4 Continuous G-pseudocharacters

We will also need the notion of a continuous G-pseudocharacter. Assume, that G is an affine group scheme over a commutative ring $\mathcal{O}$.

Definition 4.35 (Continuous G-pseudocharacter)**.** Let Γ be a topological group and let A be a commutative topological $\mathcal{O}$-algebra. A G-pseudocharacter $\Theta \in \mathrm{PC}^\Gamma_{\mathcal{O}}(A)$ is *continuous*, if Θ_m takes values in the subset $\mathcal{C}(\Gamma^m, A) \subseteq \mathrm{Map}(\Gamma^m, A)$ of continuous maps for all $m \geq 1$. We write $\mathrm{cPC}^\Gamma_G(A)$ for the set of continuous G-valued pseudocharacters over A.

It is straightforward to verify, that if G is of finite type over $\mathcal{O}$ and $\rho : \Gamma \to G(A)$ is a continuous homomorphism with $G(A)$ topologized as in Proposition 4.12, then Θ_ρ is a continuous G-pseudocharacter.

4.3 $\mathcal{C}$-$\mathcal{O}$-algebras

It turns out to be useful to rephrase the definition of G-pseudocharacters in terms of functors on a category $\mathcal{C}$ with values in $\mathcal{O}$-algebras, which we decided to call '$\mathcal{C}$-$\mathcal{O}$-algebras'. Instances of $\mathcal{C}$-$\mathcal{O}$-algebras appear in [Wei20] under the names FI-, FFM- and FFG-algebra. We develop the basic theory of $\mathcal{C}$-$\mathcal{O}$-algebras and use them to prove existence and basic properties of a fine moduli scheme of G-pseudocharacters.

4.3.1 Generalities

Definition 4.36 ($\mathcal{C}$-$\mathcal{O}$-algebra)**.** Let $\mathcal{O}$ be a commutative ring and let $\mathcal{C}$ be a small category.

1. A *$\mathcal{C}$-$\mathcal{O}$-algebra* is a functor
$$\begin{aligned} A^\bullet : \mathcal{C} &\to \mathrm{CAlg}_{\mathcal{O}} \\ c &\mapsto A^c \end{aligned}$$
into the category of commutative $\mathcal{O}$-algebras $\mathrm{CAlg}_{\mathcal{O}}$.

2. A *$\mathcal{C}$-$\mathcal{O}$-homomorphism* between $\mathcal{C}$-$\mathcal{O}$-algebras is a natural transformation $f^\bullet : A^\bullet \to B^\bullet$.

3. Let $\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{C}}$ be the category of $\mathcal{C}$-$\mathcal{O}$-algebras together with $\mathcal{C}$-$\mathcal{O}$-homomorphisms.
4. A *$\mathcal{C}$-$\mathcal{O}$-subalgebra* of a $\mathcal{C}$-$\mathcal{O}$-algebra $A^\bullet$ is a subfunctor $B^\bullet \subseteq A^\bullet$, such that B^c is an $\mathcal{O}$-subalgebra of A^c for all objects c of $\mathcal{C}$.
5. A *$\mathcal{C}$-$\mathcal{O}$-ideal* is a subfunctor $I^\bullet \subseteq A^\bullet$, such that I^c is an ideal of A^c for all objects c of $\mathcal{C}$.
6. A $\mathcal{C}$-$\mathcal{O}$-homomorphism $f^\bullet : A^\bullet \to B^\bullet$ is *injective* (*surjective*, *bijective*) if f^c is injective (surjective, bijective) for all objects c of $\mathcal{C}$.
7. The *kernel* $\ker(f)^\bullet$ of a $\mathcal{C}$-$\mathcal{O}$-homomorphism $f^\bullet : A^\bullet \to B^\bullet$ is defined by $\ker(f)^c := \ker(f^c)$. It is a $\mathcal{C}$-$\mathcal{O}$-ideal of $A^\bullet$.
8. The *image* $\mathrm{im}(f)^\bullet$ of a $\mathcal{C}$-$\mathcal{O}$-homomorphism $f^\bullet : A^\bullet \to B^\bullet$ is defined by $\mathrm{im}(f)^c := \mathrm{im}(f^c)$. It is a $\mathcal{C}$-$\mathcal{O}$-subalgebra of $B^\bullet$.

$\mathcal{C}$-$\mathcal{O}$-algebras are just commutative $\mathcal{O}$-algebra objects internal to the topos of $\mathcal{C}$-sets, i.e. functors $\mathcal{C} \to \mathrm{Set}$, and all definitions in Definition 4.36 are valid in this generality.

In universal algebra free algebraic structures can be defined on an arbitrary generating set. Analogously a free $\mathcal{C}$-$\mathcal{O}$-algebra is generated by a $\mathcal{C}$-set. This defines a left adjoint to the forgetful functor $\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{C}} \to \mathrm{Set}^{\mathcal{C}}$. The forgetful functor $\mathrm{Set}^{\mathcal{C}} \to \mathrm{Set}^{\mathrm{Ob}(\mathcal{C})}$ also admits a left adjoint. Here the set of objects $\mathrm{Ob}(\mathcal{C})$ of $\mathcal{C}$ is regarded as a discrete category and an $\mathrm{Set}^{\mathrm{Ob}(\mathcal{C})}$ is the same as a family of sets indexed by $\mathrm{Ob}(\mathcal{C})$. Since we are only interested in free $\mathcal{C}$-$\mathcal{O}$-algebras on an $\mathrm{Set}^{\mathrm{Ob}(\mathcal{C})}$, we define the composition of these two left adjoints directly.

Lemma 4.37 (Free commutative $\mathcal{C}$-$\mathcal{O}$-algebra)**.** Let $\mathcal{O}$ be a commutative ring, $\mathcal{C}$ a small category and $T^\bullet$ an $\mathrm{Ob}(\mathcal{C})$-set. Then there is a $\mathcal{C}$-$\mathcal{O}$-algebra $F^\bullet$ together with a map of $\mathrm{Ob}(\mathcal{C})$-sets $\iota : T^\bullet \to F^\bullet$, that satisfies the following universal property:

For every map of $\mathrm{Ob}(\mathcal{C})$-sets $f : T^\bullet \to R^\bullet$ to a $\mathcal{C}$-$\mathcal{O}$-algebra $R^\bullet$, there is a unique homomorphism of $\mathcal{C}$-$\mathcal{O}$-algebras $\bar{f} : F^\bullet \to R^\bullet$, such that $\bar{f} \circ \iota = f$. We call the pair $(F^\bullet, \iota)$ the *free $\mathcal{C}$-$\mathcal{O}$-algebra* on $T^\bullet$. It is unique up to unique isomorphism.

Proof. Let $x \in \mathrm{Ob}(\mathcal{C})$. We define F^x to be the free commutative $\mathcal{O}$-algebra generated by the set

$$\coprod_{y \in \mathcal{C}} \coprod_{\alpha \in \mathrm{Hom}_{\mathcal{C}}(y,x)} T^y$$

For $\alpha : y \to x$ we denote the generator of F^x associated to $t \in T^y$ by ${}^{\alpha}t$. Define $\iota^x : T^x \to F^x, t \mapsto {}^{\mathrm{id}_x}t$ to be the inclusion of T^x into the summand associated to id_x. Define for every morphism $\alpha : x \to y$ of $\mathcal{C}$ an $\mathcal{O}$-homomorphism $\alpha_* : T^x \to T^y$, ${}^{\beta}t \mapsto {}^{\alpha\beta}t$. Now let $f : T^\bullet \to R^\bullet$ be a map of $\mathrm{Ob}(\mathcal{C})$-sets. We define for all $x \in \mathcal{C}$ an $\mathcal{O}$-algebra homomorphism $\bar{f}^x : F^x \to R^x$, ${}^{\beta}t \mapsto \beta_*(f^y(t))$, where $\beta : y \to x$ is a morphism of $\mathcal{C}$ and $t \in T^y \subset F^y$. One easily checks $\bar{f} \circ \iota = f$ and this equation forces uniqueness of $\bar{f}$. By the standard argument $(F^\bullet, \iota)$ is unique up to unique isomorphism. ☐

4.3.2 G-pseudocharacters as $\mathcal{F}$-$\mathcal{O}$-algebra homomorphisms

From now on, we will consider two different small categories for $\mathcal{C}$.

1. Let $\mathcal{M}$ be the category of free monoids $\mathrm{FM}(m)$ on m generators for all $m \geq 1$.
2. Let $\mathcal{F}$ be the category of free groups $\mathrm{FG}(m)$ on m generators for all $m \geq 1$.

A monoid homomorphism between finitely generated free monoids can be understood as a finite sequence of words. Such a sequence also defines a homomorphism between free groups and so we get a canonical functor

$$\mathcal{M} \to \mathcal{F}$$

In particular every $\mathcal{F}$-$\mathcal{O}$-algebra can be restricted to an $\mathcal{M}$-$\mathcal{O}$-algebra.

Example 4.38. Here are the two examples of $\mathcal{F}$-$\mathcal{O}$-algebras we are interested in.

1. If A is an $\mathcal{O}$-algebra, then the functor

$$\mathcal{F} \to \mathrm{CAlg}_{\mathcal{O}},$$
$$\mathrm{FG}(m) \mapsto \mathrm{Map}(\Gamma^m, A)$$

where $\alpha : \mathrm{FG}(n) \to \mathrm{FG}(m)$ is mapped to

$$\alpha_* : \mathrm{Map}(\Gamma^n, A) \to \mathrm{Map}(\Gamma^m, A)$$

where $\alpha_*(f)(\gamma_1, \dots, \gamma_m) := f(\phi(\alpha(x_1)), \dots, \phi(\alpha(x_n)))$, where $\phi : \mathrm{FG}(m) \to \Gamma,\ x_i \mapsto \gamma_i$, defines an $\mathcal{F}$-$\mathcal{O}$-algebra $\mathrm{Map}(\Gamma^\bullet, A)$.

2. Similarly

$$\mathcal{F} \to \mathrm{CAlg}_{\mathcal{O}}$$
$$\mathrm{FG}(m) \mapsto \mathcal{O}[G^m]^{G^0}$$

defines an $\mathcal{F}$-$\mathcal{O}$-algebra: Every homomorphism $\alpha : \mathrm{FG}(n) \to \mathrm{FG}(m)$ induces a morphism of $\mathcal{O}$-schemes $G^m \to G^n$, which in turn induces the desired map $\alpha_* : \mathcal{O}[G^n]^{G^0} \to \mathcal{O}[G^m]^{G^0}$. Note, that since $G^m \to G^n$ is induced by a homomorphism of free groups it is equivariant with respect to diagonal conjugation and hence α_* is well-defined. We will denote this $\mathcal{F}$-$\mathcal{O}$-algebra by $\mathcal{O}[G^\bullet]^{G^0}$.

By definition a G-pseudocharacter Θ is a sequence of maps $\Theta_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$, that behaves natural with respect to two specified types of monoid homomorphisms. Our next goal is to understand, that these types of monoid homomorphisms do already generate all morphisms in $\mathcal{M}$ and make $\Theta_\bullet = (\Theta_m)_{m \geq 0}$ an $\mathcal{M}$-$\mathcal{O}$-homomorphism. We start with generalities on generating sets of morphisms in categories.

Definition 4.39. Let $\mathcal{C}$ be a category and S a system of morphisms $S_{A,B} \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all pairs of objects A, B. Let $\tilde{S}$ be another such system of morphisms.

1. S *generates* $\tilde{S}$, if $\tilde{S}$ is the smallest system of morphisms, that contains S, all identities and for any two composable morphisms $\alpha_1, \alpha_2 \in \tilde{S}$ their composition $\alpha_2 \circ \alpha_1$.

2. S *inv-generates* $\tilde{S}$, if $\tilde{S}$ is the smallest system of morphisms, that contains S, all identities, for any two composable morphisms $\alpha_1, \alpha_2 \in \tilde{S}$ their composition $\alpha_2 \circ \alpha_1$ and for each invertible morphism $\alpha \in \tilde{S}$ its inverse α^{-1}.

Remark 4.40. A system of morphisms S always (inv-)generates a unique system of morphisms, since the conditions in Definition 4.39 are closed under arbitrary intersections. If S generates $\tilde{S}$, then $\tilde{S}$ consists of compositions of morphisms of S and identities. If S inv-generates $\tilde{S}$, then $\tilde{S}$ consists of iterated compositions and inversions of morphisms of S that are invertible in $\mathcal{C}$ and identities.

It is enough to check naturality on (inv-)generating systems of morphisms.

Lemma 4.41. Let $\mathcal{C}$ be a category and S a generating system of the morphisms of $\mathcal{C}$. Let $\mathcal{D}$ be another category, $F, G : \mathcal{C} \to \mathcal{D}$ functors and $\Theta : F \to G$ an *infranatural transformation*, i.e. a collection of morphisms $\Theta_X : FX \to GX$ for each object X of $\mathcal{C}$. Further assume, that Θ is natural for morphisms of $\alpha \in S$, i.e. for all $\alpha \in S$ the diagram

$$\begin{tikzcd} FX \arrow[r, "\Theta_X"] \arrow[d, "F\alpha"] & GX \arrow[d, "G\alpha"] \\ FY \arrow[r, "\Theta_Y"] & GY \end{tikzcd}$$

commutes. Then Θ is a natural transformation. The same is true if S is an inv-generating system.

Proof. Follows easily from Remark 4.40 and structural induction. ☐

We now determine generating sets of morphisms for $\mathcal{M}$ and $\mathcal{F}$.

Lemma 4.42. The morphisms of $\mathcal{M}$ are generated by the following two types of homomorphisms:

(1) $\phi : \mathrm{FM}(n) \to \mathrm{FM}(m)$, where $\phi(x_i) := x_{\zeta(i)}$ for each map $\zeta : \{1, \dots, n\} \to \{1, \dots, m\}$.

(2) $\phi : \mathrm{FM}(n) \to \mathrm{FM}(n+1)$ where $\phi(x_i) := x_i$ for all $i < n$ and $\phi(x_n) := x_n x_{n+1}$.

The morphisms of $\mathcal{F}$ are generated by homomorphisms of types (1) and (2) with FM replaced by FG and a third type of homomorphism:

(3) $\phi : \mathrm{FG}(n) \to \mathrm{FG}(n)$ where $\phi(x_i) := x_i$ for all $i < n$ and $\phi(x_n) := x_n^{-1}$.

Proof. This is [Wei20, Lem. 2.1]. Let $\phi : \mathrm{FM}(n) \to \mathrm{FM}(m)$ be a homomorphism with $\phi(x_i) = x_{k_{i1}} \dots x_{k_{il_i}}$. We may write it as a tuple

$$(x_{k_{11}} \dots x_{k_{1l_1}}, \dots, x_{k_{n1}} \dots x_{k_{nl_n}})$$

It is a composition of homomorphisms of type (2) and the homomorphism given by

$$(x_{k_{11}} \dots x_{k_{1l_1}}, \dots, x_{k_{(n-1)1}} \dots x_{k_{(n-1)l_{n-1}}}, x_{k_{n1}}, \dots, x_{k_{nl_n}})$$

Iterated application of permutations of type (1) and homomorphisms of type (2) reduces us to

$$(x_{e_1}, \dots, x_{e_t})$$

where $x_{e_1}, \dots, x_{e_t}$ is the condensed sequence of letters $x_{k_{i1}}, \dots, x_{k_{il_i}}$ for all i. This is already a homomorphism of type (1).

For a homomorphism $\Phi : \mathrm{FG}(n) \to \mathrm{FG}(m)$ one analogously reduces using types (1) and (2) to a sequence

$$(x_{e_1}^{\pm 1}, \dots, x_{e_t}^{\pm 1})$$

Application of permutations and homomorphisms of type (3) reduces us to $(x_{e_1}, \dots, x_{e_t})$. $\square$

Proposition 4.43. We have a canonical bijection between the set of $\mathcal{M}$-$\mathcal{O}$-algebra homomorphisms $\mathcal{O}[G^\bullet]^{G^0} \to \mathrm{Map}(\Gamma^\bullet, A)$ and the set $\mathrm{PC}_G^\Gamma(A)$ of G-pseudocharacters of Γ with values in A.

Proof. We start with a G-pseudocharacter $(\Theta_m)_{m \geq 1}$ and define an association $\tilde{\Theta} : \mathcal{O}[G^\bullet]^{G^0} \to \mathrm{Map}(\Gamma^\bullet, A)$ by setting $\tilde{\Theta}_{\mathrm{FM}(m)} := \Theta_m$. By definition of Θ we know, that $\tilde{\Theta}$ is natural with respect to morphisms $\mathrm{FM}(n) \to \mathrm{FM}(m)$ of type (1) and morphisms $\mathrm{FM}(n) \to \mathrm{FM}(n+1)$ of type (2). By Lemma 4.41 and Lemma 4.42 this implies naturality. Conversely, given a morphism $\tilde{\Theta}$ of $\mathcal{M}$-$\mathcal{O}$-algebras, the associated sequence of algebra maps $\Theta_n := \Theta_{\mathrm{FM}(n)}$ satisfies the required properties by naturality. $\square$

Clearly the morphisms of $\mathcal{F}$ are not generated by homomorphisms of type (1) and (2) with FM replaced by FG: Homomorphisms of type (1) and (2) have the property, that the image of the generators x_i lies in the submonoid spanned by generators. This property is stable under compositions and hence the homomorphism $\mathrm{FG}(1) \to \mathrm{FG}(1)$, $x_1 \mapsto x_1^{-1}$ is not a composition of type (1) or (2) homomorphisms. Fortunately by Lemma 4.41 we only need, that the morphisms of $\mathcal{F}$ are inv-generated by homomorphisms of type (1) and (2) to prove, that any pseudocharacter gives rise to an $\mathcal{F}$-$\mathcal{O}$-algebra homomorphism.

Lemma 4.44. The morphisms of $\mathcal{F}$ are inv-generated by homomorphisms of type (1) and (2) in Lemma 4.42 with FM replaced by FG.

Proof. By Remark 4.40 and Lemma 4.42 it suffices to show, that homomorphisms of type (3) can be written as iterated compositions and inversions of homomorphisms of type (1) and (2). Since by Lemma 4.42 $\mathcal{M}$ is generated by monoid homomorphisms of types (1) and (2), we already know, that all group homomorphisms $\mathrm{FG}(n) \to \mathrm{FG}(m)$, that are induced by monoid homomorphisms $\mathrm{FM}(n) \to \mathrm{FM}(m)$ are generated by group homomorphisms of type (1) and (2).

Let $\phi : \mathrm{FG}(n) \to \mathrm{FG}(n)$ be of type (3). We will use tuple notation for homomorphisms, so $\phi = (\phi(x_1), \dots, \phi(x_n)) = (x_1, \dots, x_{n-1}, x_n^{-1})$. We first give the proof for $n = 2$ and $x_1 = x, x_2 = y$:

$$(x, y^{-1}) = (xy^{-1}, y) \circ (xy, x) \circ (x, x^{-1}y)$$

Since $(xy^{-1}, y) = (xy, y)^{-1}$ and $(x, x^{-1}y) = (x, xy)^{-1}$ we have shown, that ϕ is inv-generated by homomorphisms of type (1) and (2). This argument works analogously for $n \geq 2$. For $n = 1$ we consider the homomorphisms $(y) : \mathrm{FG}(1) \to \mathrm{FG}(2),\ x \mapsto y$ and $(1, x) : \mathrm{FG}(2) \to \mathrm{FG}(1),\ x \mapsto 1,\ y \mapsto x$ and write $(x^{-1}) = (1, x) \circ (x, y^{-1}) \circ (y)$. □

Corollary 4.45. The maps

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}}}(\mathcal{O}[G^\bullet]^{G^0},\ \mathrm{Map}(\Gamma^\bullet, A)) &\to \mathrm{PC}_G^\Gamma(A) \\ \Theta &\mapsto (\Theta_m)_{m \geq 1} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}}}(\mathcal{O}[G^\bullet]^{G^0},\ \mathcal{C}(\Gamma^\bullet, A)) &\to \mathrm{cPC}_G^\Gamma(A) \\ \Theta &\mapsto (\Theta_m)_{m \geq 1} \end{aligned}$$

are bijections.

Proof. The proof of Proposition 4.43 carries over. □

4.3.3 Representability of PC_G^Γ

Theorem 4.46 (Representability of PC_G^Γ)**.** Let Γ be a group and let G be a generalized reductive $\mathcal{O}$-group scheme. The functor $\mathrm{PC}_G^\Gamma : \mathrm{CAlg}_{\mathcal{O}} \to \mathrm{Set}$ is representable by a commutative $\mathcal{O}$-algebra B_G^Γ. There is a universal G-pseudocharacter $\Theta^u \in \mathrm{PC}_G^\Gamma(B_G^\Gamma)$, i.e. for all $m \in \mathbb{N}$, $\mu \in \mathcal{O}[G^m]^G$, $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma^m$, for every $A \in \mathrm{CAlg}_{\mathcal{O}}$ and every $\Theta \in \mathrm{PC}_G^\Gamma(A)$, the associated homomorphism $f_\Theta : B_G^\Gamma \to A$ satisfies $f_\Theta(\Theta_m^u(\mu)(\gamma)) = \Theta_m(\mu)(\gamma)$. As an $\mathcal{O}$-algebra B_G^Γ is generated by $\{\Theta_m^u(\mu)(\gamma) \mid \mu \in \mathcal{O}[G^m]^G,\ \gamma \in \Gamma^m\}$.

In the proof, we only need that G is affine.

Proof. Let $F := \mathcal{O}[\tilde{t}_{\mu,\gamma} \mid m \in \mathbb{N},\ \mu \in \mathcal{O}[G^m]^G,\ \gamma \in \Gamma^m]$ be the free commutative $\mathcal{O}$-algebra generated by the letters $\tilde{t}_{\mu,\gamma}$ for all $m \in \mathbb{N}$, $\mu \in \mathcal{O}[G^m]^G$ and Γ^m. For all $A \in \mathrm{CAlg}_{\mathcal{O}}$ and all $\Theta \in \mathrm{PC}_G^\Gamma(A)$, there is an $\mathcal{O}$-linear map $\tilde{\eta}_\Theta : F \to A,\ \tilde{t}_{\mu,\gamma} \mapsto \Theta_m(\mu)(\gamma)$. Let $\mathfrak{a} \subseteq F$ be the intersection of $\ker(\tilde{\eta}_\Theta)$ for all $A \in \mathrm{CAlg}_{\mathcal{O}}$ and all $\Theta \in \mathrm{PC}_G^\Gamma(A)$. Define $B_G^\Gamma := F/\mathfrak{a}$.

From now on, we let $\eta_\Theta : B_G^\Gamma \to A$ be $\eta_\Theta(x + \mathfrak{a}) := \tilde{\eta}_\Theta(x)$ and $t_{\mu,\gamma} := \tilde{t}_{\mu,\gamma} + \mathfrak{a} \in B_G^\Gamma$. In particular $\eta_\Theta(t_{\mu,\gamma}) = \Theta_m(\mu)(\gamma)$.

For every $A \in \mathrm{CAlg}_{\mathcal{O}}$, we have a map $H_A : \mathrm{PC}_G^\Gamma(A) \to \mathrm{Hom}_{\mathcal{O}}(B_G^\Gamma, A),\ \Theta \mapsto \eta_\Theta$ and these are natural in A. We define the universal pseudocharacter $\Theta^u : \mathcal{O}[G^\bullet]^{G^0} \to \mathcal{C}(\Gamma^\bullet, B_G^\Gamma)$ by $\Theta_m^u(\mu) : \Gamma^m \to B_G^\Gamma,\ \gamma \mapsto t_{\mu,\gamma}$.

We check property (1) in the definition of pseudocharacter for Θ^u, property (2) is similar. Let $\mu \in \mathcal{O}[G^m]^G$ and let $\zeta : \{1, \dots, n\} \to \{1, \dots, m\}$ be some map. Then $\Theta_n^u(\mu^\zeta)(\gamma) = t_{\mu^\zeta,\gamma}$ and $\Theta_m^u(\mu)(\gamma_\zeta) = t_{\mu,\gamma_\zeta}$. Here we write $\gamma_\zeta = (\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(n)})$. We claim, that $t_{\mu^\zeta,\gamma} = t_{\mu,\gamma_\zeta}$. Indeed for every pseudocharacter Θ, we have $\tilde{\eta}_\Theta(\tilde{t}_{\mu^\zeta,\gamma}) = \Theta_n(\mu^\zeta)(\gamma) = \Theta_m(\mu)(\gamma_\zeta) = \tilde{\eta}_\Theta(\tilde{t}_{\mu,\gamma_\zeta})$. So $\tilde{t}_{\mu^\zeta,\gamma} - \tilde{t}_{\mu,\gamma_\zeta} \in \ker(\tilde{\eta}_\Theta)$ and the claim follows by definition of $\mathfrak{a}$.

We see, that for any pseudocharacter Θ, we have $\Theta = \eta_{\Theta *}\Theta^u$ and for every $h \in \mathrm{Hom}_{\mathcal{O}}(B_G^\Gamma, A)$, we have $\eta_{h_*\Theta^u} = h$, so universality of Θ^u and bijectivity of the transformation H follows. □

At this point, we would like to give also a purely categorical proof of Theorem 4.46, which is already inherent in [Zhu20, Remark 2.2.5]. To us the derived perspective is not relevant.

Categorical proof. We use the description of pseudocharacters as $\mathcal{F}$-$\mathcal{O}$-algebra homomorphisms according to Corollary 4.45. We denote by $\mathcal{F}/\Gamma$ the slice category of objects of $\mathcal{F}$ with a fixed homomorphism to Γ. Let

$$B_G^\Gamma := \operatorname*{colim}_{\mathrm{FG}(m) \in \mathcal{F}/\Gamma} \mathcal{O}[G^m]^{G^0}$$

be the colimit in the category of commutative $\mathcal{O}$-algebras indexed over the small category $\mathcal{F}/\Gamma$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}}(B_G^\Gamma, A) &= \lim_{\mathrm{FG}(m)\in\mathcal{F}/\Gamma} \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}}(\mathcal{O}[G^m]^{G^0}, A)\\ &= \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}/\Gamma}}(\mathcal{O}[G^\bullet]^{G^0}, A)\\ &= \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}}}(\mathcal{O}[G^\bullet]^{G^0}, \mathrm{Map}(\Gamma^\bullet, A))\end{aligned}$$

for every $A \in \mathrm{CAlg}_{\mathcal{O}}$, where in the second line A is understood as the constant functor on $\mathcal{F}/\Gamma$. In the last line, we compute the right Kan extension of $A : \mathcal{F}/\Gamma \to \mathrm{CAlg}_{\mathcal{O}}$ along the canonical restriction $p : \mathcal{F}/\Gamma \to \mathcal{F}$ as

$$(\mathrm{Ran}_p\, A)(\mathrm{FG}(m)) = \lim_{\substack{(\mathrm{FG}(n),f)\in\mathcal{F}/\Gamma\\ \varphi\in\mathrm{Hom}(\mathrm{FG}(m),\mathrm{FG}(n))}} A(\mathrm{FG}(n), f) = \lim_{(\mathrm{FG}(n),f)\in\mathcal{F}/\Gamma} \mathrm{Map}(\mathrm{FG}(n)^m, A) = \mathrm{Map}(\Gamma^m, A)$$

using the description of Kan extensions as weighted limits. □

From the categorical proof of Theorem 4.46 it is also clear, that B_G^Γ is finitely generated if Γ and all $\mathcal{O}[G^m]^{G^0}$ are finitely generated. We again give an elementary and a categorical proof.

Proposition 4.47. Let Γ be a finitely generated group, let $\mathcal{O}$ be a commutative ring, which is of finite type over a universally Japanese ring and let G be a generalized reductive $\mathcal{O}$-group scheme. Then B_G^Γ is a finitely generated $\mathcal{O}$-algebra.

Proof. Let r be a number of generators of Γ. We know from [Ses77, Theorem 2 (i)], that $\mathcal{O}[G^r]^{G^0}$ is a finitely generated $\mathcal{O}$-algebra. Let k be a number of $\mathcal{O}$-algebra generators of $\mathcal{O}[G^r]^{G^0}$. Let $s = (s_1, \dots, s_r) \in \Gamma^r$ be generators of Γ and let $f_1, \dots, f_k$ be generators of $\mathcal{O}[G^r]^{G^0}$. With notation as in Theorem 4.46 we claim, that $\{t_{f_i,s} \mid i \in \{1, \dots, k\}\}$ is a system of generators of B_G^Γ. By Theorem 4.46 B_G^Γ is generated by the elements $t_{\mu,\gamma}$ for all $m \geq 1$, all $\mu \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma \in \Gamma^m$. These elements satisfy functoriality properties similar to that of pseudocharacters with respect to the category $\mathcal{F}$, as explained in the proof of Theorem 4.46. Let us fix such an element $t_{\mu,\gamma}$. Every element $\gamma_1, \dots, \gamma_m$ can be written as a product of elements $s_1, \dots, s_r$ and such a presentation determines a homomorphism of free groups $\alpha : \mathrm{FG}(m) \to \mathrm{FG}(r)$, such that the composition with the projection $\mathrm{FG}(r) \twoheadrightarrow \Gamma$, $x_i \mapsto s_i$ maps x_i to γ_i. We have $\gamma = s_\alpha$, so $t_{\mu,\gamma} = t_{\mu,s_\alpha} = t_{\mu^\alpha,s}$. By uniqueness and the defining property, we see, that $t_{\cdot,s} : \mathcal{O}[G^r]^{G^0} \to B_G^\Gamma$ is a homomorphism and it follows, that $t_{\mu^\alpha,s}$ is a product of elements $t_{f_i,s}$. □

Categorical proof. We use the description of B_G^Γ as a colimit as in the categorical proof of Theorem 4.46. If Γ is finitely generated, then $\mathcal{F}/\Gamma$ contains a surjection $\pi : \mathrm{FG}(m) \twoheadrightarrow \Gamma$. For every $\mathrm{FG}(n) \in \mathcal{F}$ every homomorphism $f : \mathrm{FG}(n) \to \Gamma$ factors over π, so the associated map $f_* : \mathcal{O}[G^n]^{G^0} \to B_G^\Gamma$ factors over the map $\pi_* : \mathcal{O}[G^m]^{G^0} \to B_G^\Gamma$ associated to π, which implies, that π_* is surjective. So it suffices, that $\mathcal{O}[G^m]^{G^0}$ is finitely generated. It follows from [Ses77, Theorem 2 (i)], that $\mathcal{O}[G^m]^{G^0}$ is a finitely generated $\mathcal{O}$-algebra. □

Proposition 4.48. Let $\mathcal{O} \to \mathcal{O}'$ be a ring homomorphism, let Γ be a group, let G be a generalized reductive $\mathcal{O}$-group scheme and assume that one of the following holds.

1. $\mathcal{O}'$ is $\mathcal{O}$-flat.
2. G is a Chevalley group.
3. $G = \mathrm{O}_{2n+1}$ and $2 \in \mathcal{O}^\times$.

Then for any $\mathcal{O}'$-algebra A, there is a canonical bijection

$$\mathrm{PC}^\Gamma_{G_{\mathcal{O}'}}(A) \cong \mathrm{PC}^\Gamma_G(A) \tag{5}$$

induced by a canonical isomorphism $\mathcal{O}[G^\bullet]^{G^0} \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[G^\bullet]^{G^0}$ of $\mathcal{F}$-$\mathcal{O}'$-algebras. Moreover, there is a canonical isomorphism $B_G^\Gamma \otimes_{\mathcal{O}} \mathcal{O}' \cong B^\Gamma_{G_{\mathcal{O}'}}$ of $\mathcal{O}'$-algebras.

Proof. By Corollary 4.45 it is enough to show, that $\mathcal{O}[G^m]^{G^0} \otimes_{\mathcal{O}} \mathcal{O}' = \mathcal{O}'[G^m]^{G^0}$ for all $m \geq 1$. In the three cases this follows from Corollary 4.14, Proposition 4.17 and Proposition 4.19 respectively.

We now prove that $B_G^\Gamma \otimes_{\mathcal{O}} \mathcal{O}' \cong B_{G_{\mathcal{O}'}}^\Gamma$. We apply Theorem 4.46 twice and the first assertion once: Let A be an $\mathcal{O}'$-algebra.

$$\mathrm{Hom}_{\mathcal{O}'}(B_{G_{\mathcal{O}'}}^\Gamma, A) \overset{4.46}{=} \mathrm{PC}_{G_{\mathcal{O}'}}^\Gamma(A) \overset{(5)}{\cong} \mathrm{PC}_G^\Gamma(A) \overset{4.46}{=} \mathrm{Hom}_{\mathcal{O}}(B_G^\Gamma, A) = \mathrm{Hom}_{\mathcal{O}'}(B_G^\Gamma \otimes_{\mathcal{O}} \mathcal{O}', A)$$

The claim follows by Yoneda. $\square$

4.4 Characteristic polynomials

Definition 4.49. Let A be a commutative ring and let $\Theta \in \mathrm{PC}_{\mathrm{GL}_d}^\Gamma(A)$. Then we define the *characteristic polynomial* of Θ by

$$\chi^\Theta(\gamma, t) := \sum_{i=0}^{d} (-1)^i \Theta_1(s_i)(\gamma) t^{d-i} \quad \in A[t]$$

where $s_i \in \mathbb{Z}[\mathrm{GL}_d]^{\mathrm{GL}_d}$ are the unique invariant regular functions, which satisfy

$$\det(t - \mathbb{X}) = \sum_{i=0}^{d} (-1)^i s_i(\mathbb{X}) t^{d-i}$$

in $\mathbb{Z}[\mathrm{GL}_d][t]^{\mathrm{GL}_d}$, where $\mathbb{X}$ is the generic matrix coordinate in $\mathrm{GL}_d(\mathbb{Z}[\mathrm{GL}_d])$ which corresponds to the identity under the Yoneda isomorphism.

Proposition 4.50. Let A be a commutative ring. Then the map

$$\mathrm{PC}_{\mathrm{GL}_d}^\Gamma(A) \to \mathrm{Map}(\Gamma, A[t]), \quad \Theta \mapsto \chi^\Theta$$

is injective.

Proof. It suffices to show, that a GL_d-pseudocharacter Θ is determined by the values $\Theta_1(s_i)$ for $1 \leq i \leq d$. By Corollary 5.13, these are generators of the $\mathcal{F}$-$\mathbb{Z}$-algebra $\mathbb{Z}[\mathrm{GL}_d^\bullet]^{\mathrm{GL}_d}$, so the claim follows. $\square$

4.5 Composition with homomorphisms

Lemma 4.51. Let $\rho : G \to H$ be an homomorphism of generalized reductive $\mathcal{O}$-group schemes. Then for $n \geq 0$, the map

$$\rho^* : \mathcal{O}[H^n] \to \mathcal{O}[G^n], \quad f(h_1, \ldots, h_n) \mapsto f(\rho(h_1), \ldots, \rho(h_n))$$

restricts to an is an $\mathcal{O}$-algebra homomorphism

$$\rho^* : \mathcal{O}[H^n]^H \to \mathcal{O}[G^n]^G, \quad f(h_1, \ldots, h_n) \mapsto f(\rho(h_1), \ldots, \rho(h_n))$$

Together these maps define an $\mathcal{F}$-$\mathcal{O}$-algebra homomorphism $\mathcal{O}[H^\bullet]^H \to \mathcal{O}[G^\bullet]^G$.

Proof. For each $n \geq 0$, the map $\rho^* : \mathcal{O}[H^n] \to \mathcal{O}[G^n]$ is just by definition an $\mathcal{O}$-algebra homomorphism. Let $f \in \mathcal{O}[H^n]^H$ and $g \in G$. Then

$$\begin{aligned}(\rho^* f)(g g_1 g^{-1}, \ldots, g g_n g^{-1}) &= f(\rho(g)\rho(g_1)\rho(g)^{-1}, \ldots, \rho(g)\rho(g_n)\rho(g)^{-1}) \\ &= f(\rho(g_1), \ldots, \rho(g_n)) \\ &= (\rho^* f)(g_1, \ldots, g_n)\end{aligned}$$

and thus $\rho^* f \in \mathcal{O}[G^n]^G$. We have to check functoriality on inv-generators of $\mathcal{F}$ according to Lemma 4.42. Let $\zeta : \{1, \ldots, m\} \to \{1, \ldots, n\}$ by any map. Then

$$\begin{aligned}\rho^*(f^\zeta)(g_1, \ldots, g_n) &= f^\zeta(\rho(g_1), \ldots, \rho(g_n)) \\ &= f(\rho(g_{\zeta(1)}), \ldots, \rho(g_{\zeta(n)})) \\ &= f(\rho(g_1), \ldots, \rho(g_n))^\zeta \\ &= (\rho^* f)^\zeta(g_1, \ldots, g_n)\end{aligned}$$

Now assume ρ is a homomorphism. Then

$$\begin{aligned}\rho^*\widehat{f}(g_1,\dots,g_{n+1}) &= \widehat{f}(\rho(g_1),\dots,\rho(g_{n+1}))\\ &= f(\rho(g_1),\dots,\rho(g_n)\rho(g_{n+1}))\\ &= f(\rho(g_1),\dots,\rho(g_n g_{n+1}))\\ &= (\rho^* f)(g_1,\dots,g_n g_{n+1})\\ &= \widehat{\rho^* f}(g_1,\dots,g_{n+1})\end{aligned}$$

□

Lemma 4.52. Let Γ be a group and let $\Theta \in \mathrm{PC}^{\Gamma}_{G}(A)$. If $\rho : G \to H$ is a homomorphism of generalized reductive $\mathcal{O}$-group schemes, then the collection $\rho^*\Theta := (\rho^*\Theta_n)_{n\geq 1}$ is a pseudocharacter $\rho^*\Theta \in \mathrm{PC}^{\Gamma}_{H}(A)$.

Proof. According to Corollary 4.45 Θ is equivalently an $\mathcal{F}$-$\mathcal{O}$-algebra homomorphism $\Theta : \mathcal{O}[G^\bullet]^G \to \mathrm{Map}(\Gamma^\bullet, A)$. By Lemma 4.51 ρ^* defines an $\mathcal{F}$-$\mathcal{O}$-algebra homomorphism $\mathcal{O}[H^\bullet]^H \to \mathcal{O}[G^\bullet]^G$. The claim follows from composability in the category of $\mathcal{F}$-$\mathcal{O}$-algebras, which is just a functor category. □

4.6 The reconstruction theorem

I would like to thank Vytautas Paškūnas for helpful conversation leading to the proof of the following general reconstruction theorem. A variant is used in upcoming joint work with Paškūnas.

Lemma 4.53. Let G be a reductive group over an algebraically closed field k. Let $g = (g_1,\dots,g_n) \in G^n(k)$ and let H be the smallest Zariski closed subgroup of $G(k)$, containing $\{g_1,\dots,g_n\}$. The following are equivalent:

1. The $G^0(k)$-orbit of g is closed.
2. The $G(k)$-orbit of g is closed.
3. H is strongly reductive in G.
4. H is G-completely reducible.

Proof. Let $x_1,\dots,x_r \in G(k)$ be coset representatives of $G(k)/G^0(k)$. So

$$G(k)\cdot g = \bigcup_{i=1}^{r} G^0(k)\cdot x_i g = \bigcup_{i=1}^{r} x_i \cdot (G^0(k)\cdot g)$$

If $G^0(k)\cdot g$ is closed, then all $x_i \cdot (G^0(k)\cdot g)$ are images of $G^0(k)\cdot g$ under multiplication with x_i and therefore also closed. It follows, that (1) implies (2). If $G(k)\cdot g$ is closed, then it contains a closed $G^0(k)$-orbit, which is necessarily of the form $G^0(k)\cdot x_i g$. But then again $G^0(k)\cdot g$ is closed, so (2) implies (1). The equivalence of (2) and (3) is [Ric88, Theorem 16.4]. The equivalence of (3) and (4) is [BMR05, Theorem 3.1]. □

Lemma 4.54. Let G be a generalized reductive group scheme over a noetherian commutative ring $\mathcal{O}$. Let k be an algebraically closed field over $\mathcal{O}$. Then there is a bijection between the following sets induced by $\pi : G^n(k) \to (G^n /\!\!/ G^0)(k)$.

1. $(G^n /\!\!/ G^0)(k)$
2. $G^0(k)$-conjugacy classes of tuples $(g_1,\dots,g_n) \in G^n(k)$, such that the smallest Zariski closed subgroup of $G(k)$ that contains $\{g_1,\dots,g_n\}$ is G-completely reducible.

Proof. Recall from Section 4.1, that G^0 is a reductive group scheme. By [Ses77, Theorem 3], the map $\pi : G^n(k) \to (G^n /\!\!/ G^0)(k)$ is surjective. By [BHKT, Proposition 3.2] for each $x \in (G^n /\!\!/ G^0)(k)$, the fiber $\pi^{-1}(x)$ contains a unique closed $G^0(k)$-orbit. The claim follows from Lemma 4.53. □

Lemma 4.55. Let $\Gamma \subseteq G(k)$ be a subgroup, where G is a reductive group over an algebraically closed field k. Let P and P' be R-parabolic subgroups of G minimal among those which contain Γ. Then $\dim P = \dim P'$ and $|\pi_0(P)| = |\pi_0(P')|$.

Proof. By Lemma A.3, P and P' contain a common R-Levi L. We have $\dim P = \frac{1}{2}(\dim G + \dim L) = \dim P'$ and $|\pi_0(P)| = |\pi_0(L)| = |\pi_0(P')|$. $\square$

The proof of the reconstruction theorem itself is very close to the proof presented in [BHKT, Theorem 4.5] in the case that G is split connected reductive over $\mathbb{Z}$. The main difference is that we prove the result also for groups G with nontrivial component group G/G^0. The validity of Theorem 4.56 has been claimed in the proof of [DHKM, Lemma A.4] without proof.

Theorem 4.56. Let G be a generalized reductive group scheme over a noetherian commutative ring $\mathcal{O}$. Let Γ be a group. Let k be an algebraically closed field over $\mathcal{O}$ and let $\Theta \in \mathrm{PC}_G^\Gamma$. Then there is a G-completely reducible representation $\rho : \Gamma \to G(k)$ with $\Theta_\rho = \Theta$, which is unique up to $G^0(k)$-conjugation.

Proof. For each $n \geq 1$, Θ_n determines for each tuple $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma^n$ an element $\xi_\gamma \in (G^n /\!\!/ G^0)(k)$. The map $G^n(k) \to (G^n /\!\!/ G^0)(k)$ is surjective by Lemma 4.54 and we write $T(\gamma)$ for a representative of ξ_γ contained in the unique closed $G^0(k)$-orbit in $G^n(k)$ over ξ_γ. The representatives $T(\gamma)$ shall be chosen and fixed for each $\gamma \in \Gamma^n$ and each $n \geq 1$ for the rest of the proof.

Let $H(\gamma)$ be the smallest Zariski closed subgroup of $G(k)$, that contains the entries of $T(\gamma)$. By Lemma 4.53 $H(\gamma)$ is G-completely reducible. Let $n(\gamma)$ be the dimension of an R-parabolic subgroup P of G_k minimal among those with $H(\gamma) \subseteq P(k)$ and let $c(\gamma)$ be the cardinality of the component group $\pi_0(P)$. By Lemma 4.55 these numbers are both independent of the choice of P.

Let $N := \sup_{n \geq 1, \gamma \in \Gamma^n} n(\gamma)$ and $C := \sup_{n \geq 1, \gamma \in \Gamma^n; n(\delta)=N} c(\delta)$. We fix an integer $n \geq 1$ and $\delta \in \Gamma^n$, that satisfy the following four conditions:

1. $n(\delta) = N$.
2. $c(\delta) = C$.
3. For any $n' \geq 1$ and $\delta' \in \Gamma^{n'}$ also satisfying (1) and (2), we have $\dim Z_{G_k}(H(\delta)) \leq \dim Z_{G_k}(H(\delta'))$.
4. For any $n' \geq 1$ and $\delta' \in \Gamma^{n'}$ also satisfying (1), (2) and (3), we have $|\pi_0(Z_{G_k}(H(\delta)))| \leq |\pi_0(Z_{G_k}(H(\delta')))|$.

Satisfiability. Condition (1) is satisfiable, since $N \leq \dim G$. Condition (2) is satisfiable, since G has only finitely many conjugacy classes of R-parabolic subgroups ([Mar03, Proposition 5.2 (e)] and [BMR05, Corollary 6.7]) and so C is bounded by the maximal number of components of an R-parabolic subgroup. So the set of pairs (n, δ) satisfying (1) and (2) is not empty. It clear, that (3) and (4) are satisfiable under (1) and (2). $\diamond$

Let $(g_1, \dots, g_n) := T(\delta)$.

Claim A. For all $\gamma \in \Gamma$, there is a unique $g \in G(k)$, such that $(g_1, \dots, g_n, g)$ is $G^0(k)$-conjugate to $T(\delta_1, \dots, \delta_n, \gamma)$.

Proof of existence of g. Let $(h_1, \dots, h_n, h) := T(\delta_1, \dots, \delta_n, \gamma)$. It follows from the substitution properties in the definition of G-pseudocharacter, that $(h_1, \dots, h_n)$ lies over $\xi_\delta \in (G^n /\!\!/ G^0)(k)$.

Let $P \subseteq G_k$ be a minimal R-parabolic among those with $H(\delta_1, \dots, \delta_n, \gamma) \subseteq P(k)$. Since $H(\delta_1, \dots, \delta_n, \gamma)$ is G-completely reducible by find by the very definition of complete reducibility an R-Levi subgroup M_P of P with $H(\delta_1, \dots, \delta_n, \gamma) \subseteq M_P(k)$. Let $N_P := \mathrm{R_u}(P)$ be the unipotent radical of P and let $Q \subseteq M_P$ be an R-parabolic subgroup of M_P minimal among those containing $\{h_1, \dots, h_n\}$. Let M_Q be an R-Levi subgroup of Q and let $h_1', \dots, h_n' \in M_Q(k)$ be the images of $h_1, \dots, h_n$ in $M_Q(k)$ under the map $Q \to M_Q$ determined by the decomposition $Q = M_Q \,\mathrm{R_u}(Q)$. Then the smallest Zariski closed subgroup of M_Q generated by $h_1', \dots, h_n'$ is M_Q-irreducible, as the preimage of an R-parabolic of M_Q in Q is an R-parabolic [BMR05, Lemma 6.2 (ii)]. Therefore G-completely reducible by the non-connected version of [BMR05, Corollary 3.22] as explained in [BMR05, §6.3]. By Lemma 4.54, $h_1', \dots, h_n'$ is $G^0(k)$-conjugate to $T(\delta)$.

The subgroup QN_P of G_k is R-parabolic [BMR05, Lemma 6.2 (ii)] and contains a conjugate t of $T(\delta)$. So we have

$$N = n(\delta) \leq \dim QN_P \leq \dim P \leq N$$

The first inequality follows, since QN_P contains an R-parabolic minimal among those containing t. The second inequality follows, since $QN_P \subseteq P$. The third inequality follows by definition of N. We deduce, that $\dim QN_P = \dim P$. Since $P = M_P \ltimes N_P$ and $Q \subseteq M_P$, we have $\dim Q = \dim M_P$, $Q^0 = M_P^0$ and $|\pi_0(Q)| \leq |\pi_0(M_P)|$.

We also have

$$C = c(\delta) \leq |\pi_0(QN_P)| \leq |\pi_0(P)| \leq C$$

The first inequality follows, since any R-parabolic minimal among those containing t has the same dimension as QN_P. The second inequlity follows from $|\pi_0(Q)| \leq |\pi_0(M_P)|$ and the semidirect product decomposition of P. The third inequality holds, since $N = \dim P = n(\delta_1, \dots, \delta_n, \gamma)$ and $(\delta_1, \dots, \delta_n, \gamma)$ occurs in the supremum in the definition of C.

We conclude, that $QN_P = P$, $Q = M_P$ and $h_i = h_i'$ for all $i = 1, \dots, n$. It follows, that the $G^0(k)$-orbit of $(h_1, \dots, h_n)$ in $G^n(k)$ is closed. By Lemma 4.54, there is some $x \in G^0(k)$, such that $x(h_1, \dots, h_n)x^{-1} = (g_1, \dots, g_n)$. We can take $g := xhx^{-1}$ and the proof of existence is finished. ◇

Proof of uniqueness of g. Fix $\gamma \in \Gamma$ and suppose, that $g, g' \in G(k)$ are such, that $(g_1, \dots, g_n, g)$ and $(g_1, \dots, g_n, g')$ are $G^0(k)$-conjugate to $T(\delta_1, \dots, \delta_n, \gamma)$. In particular, there is some $y \in G^0(k)$, such that $y(g_1, \dots, g_n, g)y^{-1} = (g_1, \dots, g_n, g')$. This means, that $y \in Z_G(g_1, \dots, g_n)$ and our goal is to show, that $y \in Z_G(g_1, \dots, g_n, g)$ for then $g = g'$. There is an inclusion

$$Z_G(g_1, \dots, g_n, g) \subseteq Z_G(g_1, \dots, g_n) \tag{6}$$

Since δ satisfies (1) and (2), $(\delta_1, \dots, \delta_n, \gamma)$ also satisfies (1) and (2). It thus follows from properties (3) and (4) of δ, that in Equation (6) equality holds. ◇

So we have proved claim A and defined a map $\rho : \Gamma \to G(k),\ \gamma \mapsto g$. We have to show, that ρ is a homomorphism.

Claim B. For all $\gamma, \gamma' \in \Gamma$, there are unique $g, g' \in G(k)$, such that $(g_1, \dots, g_n, g, g')$ is $G^0(k)$-conjugate to $T(\delta_1, \dots, \delta_n, \gamma, \gamma')$.

The proof of claim B is similar to the proof of claim A, see [BHKT, Theorem 4.5] for more details.

Claim C. In the situation of claim B, the $G^0(k)$-orbits of $(g_1, \dots, g_n, g)$, $(g_1, \dots, g_n, g')$ and $(g_1, \dots, g_n, gg')$ are closed in $G^{n+1}(k)$.

We only show, that the $G^0(k)$-orbit of $(g_1, \dots, g_n, gg')$ is closed in $G^{n+1}(k)$. The argument for the other two orbits is similar. Let P be an R-parabolic minimal among those containing $\{g_1, \dots, g_n, g, g'\}$. Then P contains $\{g_1, \dots, g_n\}$ and $\dim P = N$ and $|\pi_0(P)| = C$, as before. It follows, that P is minimal among those R-parabolics containing $\{g_1, \dots, g_n\}$. Let M_P be an R-Levi of P containing $\{g_1, \dots, g_n, g, g'\}$, this exists by closedness of the orbit of $(g_1, \dots, g_n, g, g')$. As before, the subgroup generated by $\{g_1, \dots, g_n\}$ is M_P-irreducible, hence G-completely reducible and the same is true for $\{g_1, \dots, g_n, gg'\}$. It follows, that the $G^0(k)$-orbit of $(g_1, \dots, g_n, gg')$ is closed. ◇

By the substitution properties in the definition of G-pseudocharacter, $(g_1, \dots, g_n, g)$ is $G^0(k)$-conjugate to $T(\delta_1, \dots, \delta_n, \gamma)$, $(g_1, \dots, g_n, g')$ is $G^0(k)$-conjugate to $T(\delta_1, \dots, \delta_n, \gamma')$ and $(g_1, \dots, g_n, gg')$ is $G^0(k)$-conjugate to $T(\delta_1, \dots, \delta_n, \gamma\gamma')$. It follows from the uniqueness part of claim A, that $\rho(\gamma) = g$, $\rho(\gamma') = g'$ and $\rho(\gamma\gamma') = gg'$. So ρ is indeed a homomorphism. It can be shown by the same methods, that $\Theta_\rho = \Theta$. By Proposition 4.23, we can replace ρ by its semisimplification ρ^{ss}, which will be G-completely reducible and $\Theta_{\rho^{\mathrm{ss}}} = \Theta$.

We are left to show, that we can recover a G-completely reducible representation $\rho : \Gamma \to G(k)$ from its associated G-pseudocharacter Θ_ρ. For $n \geq 1$ and $\gamma \in \Gamma^n$, let $\xi_\gamma \in (G^n /\!\!/ G^0)(k)$ as before and $T(\gamma) := (\rho(\gamma_1), \dots, \rho(\gamma_n)) \in G^n(k)$. By the non-connected version of [BMR05, Lemma 2.10] as explained in [BMR05, §6.2], we find $\{\delta_1, \dots, \delta_n\} \in \Gamma$, such that for every R-parabolic P and every R-Levi L of P, we have $\rho(\Gamma) \subseteq P$ if and only if $\{\delta_1, \dots, \delta_n\} \subseteq P$ and $\rho(\Gamma) \subseteq L$ if and only if $\{\delta_1, \dots, \delta_n\} \subseteq L$. In particular $(g_1, \dots, g_n) := (\rho(\delta_1), \dots, \rho(\delta_n))$ has closed $G^0(k)$-orbit. After possibly enlarging the tuple $(\delta_1, \dots, \delta_n)$, we may assume that $Z_G(g_1, \dots, g_n)(k) = Z_G(\rho(\Gamma))(k)$.

Let $\gamma \in \Gamma$. We now that $(g_1, \dots, g_n, \rho(\gamma)) = T(\delta_1, \dots, \delta_n, \gamma)$. Suppose $g \in G(k)$ is such that $(g_1, \dots, g_n, g)$

is $G^0(k)$-conjugate to $T(\delta_1, \dots, \delta_n, \gamma)$. So we find $x \in Z_G(g_1, \dots, g_n)(k) = Z_G(\rho(\Gamma))(k)$, such that $x\rho(\gamma)x^{-1} = g$, but this just means $\rho(\gamma) = g$. ☐

4.7 Comparison with determinant laws

In her 2018 dissertation, Kathleen Emerson has shown, that Chenevier's definition of d-dimensional determinant laws and Lafforgue's definition of GL_d-pseudocharacters are equivalent over any base ring. We recall the main result here.

Kathleen Emerson has proven in her 2018 dissertation [Eme18], that there is a bijection between GL_d-valued pseudocharacters and d-dimensional determinant laws over any base ring. In this section we consider GL_d as a group scheme over $\mathbb{Z}$.

Theorem 4.57 (Emerson)**.** *Let A be a commutative ring, Γ a group and $d \geq 1$. Then the map*

$$\mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A) \to \mathrm{Det}^{\Gamma}_d(A), \quad \Theta \mapsto D_\Theta$$

defined in [Eme18, Theorem 4.0.1] is a well-defined bijection.

Emerson's bijection is characterized by the following property: If s_i for $1 \leq i \leq d$ are the coefficients of the characteristic polynomial of a generic matrix in GL_d viewed as elements of $\mathbb{Z}[\mathrm{GL}_d]^{\mathrm{GL}_d}$, then a GL_d-pseudocharacter $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A)$ corresponds to a d-dimensional determinant law $D \in \mathrm{Det}^{\Gamma}_d(A)$ if and only if $\Lambda_{i,A}(\gamma) = \Theta_1(s_i)(\gamma)$ for all $\gamma \in \Gamma$.

In particular $\chi^{\Theta} = \chi^{D_\Theta}$ and if $\rho : \Gamma \to \mathrm{GL}_d(A)$ is a representation, then $D_{\Theta_\rho} = D_\rho$.

Proposition 4.58. *Let A be a commutative topological ring, Γ a topological group and $d \geq 1$. Then $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A)$ is continuous if and only if D_Θ is continuous. In particular the bijection $\mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A) \to \mathrm{Det}^{\Gamma}_d(A)$, $\Theta \mapsto D_\Theta$ in Theorem 4.57 restricts to a bijection $\mathrm{cPC}^{\Gamma}_{\mathrm{GL}_d}(A) \to \mathrm{cDet}^{\Gamma}_d(A)$.*

Proof. First suppose, that Θ is continuous. Then $\Lambda_{i,A}|_\Gamma = \Theta_1(s_i)$ is a continuous map by definition of continuity of Θ, hence D_Θ is continuous.

Conversely, if D_Θ is continuous, then $\Theta_1(s_i)$ is continuous for all $1 \leq i \leq d$. Since the $\mathcal{F}$-$\mathbb{Z}$-algebra $\mathbb{Z}[\mathrm{GL}_d^\bullet]^{\mathrm{GL}_d}$ is generated by $\{s_1, \dots, s_d\}$ and $\det^{-1} = s_d^{-1}$ (see Theorem 5.12), the image of $\mathbb{Z}[\mathrm{GL}_d^\bullet]^{\mathrm{GL}_d}$ is contained $\mathcal{C}(\Gamma^\bullet, A)$, as desired. ☐

4.8 Comparison with Taylor's pseudocharacters

Proposition 4.59. *Let A be a commutative ring with $d! \in A^\times$ and let Γ be a group. Then the map*

$$\mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A) \to \mathrm{TPC}^{\Gamma}_d(A), \quad \Theta \mapsto \Theta_1(\mathrm{tr})$$

from the set $\mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A)$ of A-valued GL_d-pseudocharacters of Γ to the set $\mathrm{TPC}^{R}_d(A)$ of d-dimensional A-valued Taylor pseudocharacters of Γ is a well-defined injection. The map is bijective, if one of the following conditions holds.

1. *A is reduced.*
2. *$2 \in A^\times$ and $d = 2$.*
3. *$(2d)! \in A^\times$.*

Proof. This follows from Proposition 3.13 and Theorem 4.57. ☐

5 Invariant theory of algebraic groups

In order to describe a G-pseudocharacter explicitly by a set of functions satisfying certain relations in the style of Taylor's Definition 2.6 it is vital to understand the algebras of rational invariants $\mathcal{O}[G^m]^G$ in terms of generators and relations. This is achieved by classical invariant theory. If A is an $\mathcal{O}$-algebra on which G acts rationally by algebra automorphisms, we distinguish between two types of theorems:

1. *First fundamental theorem (FFT)*:
   Determine an explicit set of generators of A^G.

2. *Second fundamental theorem (SFT)*:
   Determine an explicit generating set of relations between given generators of A^G.

The first results of this kind in characterstic 0 are due to Frobenius, Sibirskii [Sib67] and Procesi [Pro76]. One can reduce the computation of $\mathbb{Q}[G^m]^G$ to matrix invariants $\mathbb{Q}[M_n^m]^G$ for a faithful representation $G \subseteq M_n$ (see Proposition 5.1). Donkin proved, that if K is an algebraically closed field, the algebras $K[G^m]^G$ are generated by shifted traces of tilting modules [Don92]. This has since been turned into a concrete description of generators of $K[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ by Donkin and $K[\mathrm{Sp}_n^m]^{\mathrm{Sp}_n}$ and $K[\mathrm{O}_n^m]^{\mathrm{O}_n}$ (under the assumption $\mathrm{char}(K) \neq 2$ in the orthogonal case) by Zubkov [Zub94; Zub99]. We can descend generators of invariant algebras to the prime fields $\mathbb{Q}$ and $\mathbb{F}_p$ and lift them to $\mathbb{Z}/p^r$ (see Section 5.1.1, Section 5.1.2). This is sufficient for our applications to deformation theory. Using results on tilting modules it is possible to descend these generators further to $\mathbb{Z}[G^m]^G$ once they are known over fields and defined over $\mathbb{Z}$. We include this argument in Section 8.7. We also obtain a slightly different proof of the first fundamental theorem [DP76, §15.2.1 Theorem 1.10] for $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n}$, $\mathbb{Z}[M_n^m]^{\mathrm{SL}_n}$, $\mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ and $\mathbb{Z}[\mathrm{SL}_n^m]^{\mathrm{SL}_n}$ (see Section 5.1.3).

The second fundamental theorem for $\mathbb{Q}[M_n^m]^{\mathrm{GL}_n}$ has been proven independently by Procesi [Pro76] and Razmyslov [Raz74]. In positive characteristic it is due to Zubkov [Zub99]. In [DP76, Theorem 1.13] de Concini and Procesi prove a second fundamental theorem over $\mathbb{Z}$. The work on semi-invariants of quivers over infinite fields was further developped by Domokos and Zubkov [DZ01].

In [Pro76] Procesi mentions the possibility of *third* and *higher fundamental theorems*. The first fundamental theorem provides us with a surjection $P \to A^G$, where P is a polynomial algebra generated by a formal variable for each explicit generator of A^G. The second fundamental theorem yields a set R of ideal generators of the kernel of $P \to A^G$. So we can define a P-linear surjection from a free P-module $P^{(R)}$ onto the kernel of $P \to A^G$. We can now ask for relations between P-module generators of the kernel of $P^{(R)} \to P$, which would then be a third fundamental theorem. Even though this is a natural question, to my knowledge there is no research towards third fundamental theorems for $\mathbb{Q}[M_n^m]^G$ or $\mathbb{Q}[G^m]^G$.

My original motivation to study classical invariant theory was firstly to give an explicit description of Sp_{2n}-pseudocharacters and secondly to prove, that Sp_{2n}-pseudocharacters are in bijection with symplectic determinant laws as described in Section 8. This requires first and second fundamental theorems for $\mathbb{Z}[\mathrm{Sp}_{2n}^m]^{\mathrm{Sp}_{2n}}$ over $\mathbb{Z}$, which are not yet available. While an explicit description of Lafforgue Sp_{2n}-pseudocharacters seems hard, we might have found a stronger definition of symplectic determinant laws of involutive algebras, which might enable us to prove an Emerson type comparison bijection with Lafforgue's Sp_{2n}-pseudocharacters in case of a group algebra without using a second fundamental theorem, but this is subject to further research and not included in Section 8. What we currently know about a possible comparison map for symplectic determinant laws is discussed in Section 8.6.

As part of this effort I've learned a few methods to compute generators of specific invariant algebras for reductive groups. These are explained in Section 5.2.

Proposition 5.1. Let K be a field of characteristic 0. Let G be a reductive algebraic group over K and let $\iota : G \to \mathrm{GL}_n$ be a faithful rational representation of G. Then the map $K[M_n^m]^G \otimes_K K[\mathbb{A}^m] \to K[G^m]^G$ induced by $G \to M_n \times \mathbb{A}^1,\ g \mapsto (g, \det^{-1}(\iota(g)))$ is surjective.

Proof. The map $G \to M_n \times \mathbb{A}^1,\quad g \mapsto (g, \det^{-1}(\iota(g)))$ is a closed immersion. The claim follows, since G acts trivially on $\mathbb{A}^1$ and the category of rational G-modules is semisimple. □

Note, that Proposition 5.1 is a version of Corollary 5.13 in characteristic 0.

5.1 Invariant theory

The goal of this section is to prove, that the $\mathcal{F}$-$\mathbb{Z}/p^r$-algebras $\mathbb{Z}/p^r[G^\bullet]^{G^0}$ are finitely generated for $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_{2n+1}\}$ and to determine an explicit set of generators. We use a theorem of Zubkov [Zub99] on generators of certain invariant rings over an algebraically closed field and generalize it to $\mathbb{Z}/p^r$-algebras.

Let $\mathbb{X} \in M_d(\mathbb{Z}[x_{ij} \mid i,j \in \{1,\dots,d\}])$ be a generic $d \times d$ matrix, i.e. $\mathbb{X}_{ij} = x_{ij}$ for $1 \le i,j \le d$.

Let $s_i \in \mathbb{Z}[x_{ij}]$ be up to a sign the i-th coefficient of the characteristic polynomial of $\mathbb{X}$:

$$\det(t \cdot I_d - \mathbb{X}) = \sum_{i=0}^{d} (-1)^i s_i(\mathbb{X}) t^{d-i} \quad \in \mathbb{Z}[x_{ij} \mid i,j \in \{1,\dots,d\}][t]$$

If we evaluate $\mathbb{X}$ at a triangular matrix, then s_i is given by the i-th elementary symmetric polynomial in the diagonal entries.

Theorem 5.2 (Zubkov, 1999)**.** Let K be an algebraically closed field. Let G be either the symplectic group $\mathrm{Sp}_{d,K}$ for even $d \ge 2$ or the orthogonal group $\mathrm{O}_{d,K}$ for $d \ge 1$ and assume, that $\mathrm{char}(K) \ne 2$ in the orthogonal case. Let $m \ge 1$. The algebraic group G acts by conjugation on the affine K-scheme $M_d^m \cong \mathbb{A}^{md^2}$. Denote by $K[M_d^m]^G$ the algebra of rational invariants of the coordinate ring $K[M_d^m]$ of M_d^m. Denote by $X_k \in M_d(\mathbb{Z}[x_{ij}^{(k)} \mid i,j \in \{1,\dots,d\}])$ the k-th matrix coordinate of M_d^m. Then:

1. $K[M_d^m]^G$ is generated as a K-algebra by elements of the form

$$s_i(Y_{j_1} \cdots Y_{j_s})$$

   for $i \in \{1,\dots,d\}$ and $s \ge 0$, where $Y_k \in \{X_k, X_k^*\}$ and in the orthogonal case $* = \top$ is transposition and in the symplectic case $* = \mathrm{j}$ is symplectic transposition, i.e. $J(-)^\top J^{-1}$ for $J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$. [Zub99, Theorem 1]

2. The map $K[M_d^m]^G \to K[G^m]^G$ induced by restriction to $G^m \subseteq M_d^m$ is surjective [Zub99, Proposition 3.2]. In particular $K[G^m]^G$ is generated by elements of the form

$$s_i(Y_{j_1} \cdots Y_{j_s})$$

   for $i \in \{1,\dots,d\}$ and $s \ge 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

Zubkov proves Theorem 5.2 for algebraically closed fields and then remarks that the claim holds for all infinite fields by a Zariski density argument [Zub99, Remark 3.2].

5.1.1 Invariants over a field

We now extend Zubkov's Theorem 5.2 to arbitrary fields and to the groups GSp_{2n}, SO_{2n+1} and GO_n when $n \ge 1$.

Proposition 5.3. Let K be a field and let $m \ge 1$.

1. Suppose $G \in \{\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_n\}$ and $d \in \{2n, 2n+1, n\}$ respectively. Assume further $\mathrm{char}(K) \ne 2$ if $G \in \{\mathrm{SO}_{2n+1}, \mathrm{O}_n\}$. Then $K[G^m]^G$ is generated by elements of the form

$$s_i(Y_{j_1} \cdots Y_{j_s})$$

   for $i \in \{1,\dots,d\}$ and $s \ge 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

2. Suppose $G \in \{\mathrm{GSp}_{2n}, \mathrm{GO}_n\}$ and $d \in \{2n, n\}$ respectively. Assume further $\mathrm{char}(K) \ne 2$ if $G = \mathrm{GO}_n$. Then $K[G^m]^G$ is generated by the symplectic (orthogonal) similitude character sim, its inverse sim^{-1} and elements as in (1).

Proof. Let K' be an algebraically closed overfield of K. We first treat the case $G \in \{\mathrm{Sp}_{2n}, \mathrm{O}_n\}$. Let $d = 2n$ in the symplectic case and $d = n$ in the orthogonal case. We have a complex

$$0 \longrightarrow J \longrightarrow K[M_d^m]^G \longrightarrow K[G^m]^G \longrightarrow 0$$

of K-vector spaces, where $J := \ker(K[M_d^m]^G \to K[G^m]^G)$. By faithful flatness it suffices to show, that

$$0 \longrightarrow J \otimes_K K' \longrightarrow K[M_d^m]^G \otimes_K K' \longrightarrow K[G^m]^G \otimes_K K' \longrightarrow 0$$

is exact. By the universal coefficient theorem for rational invariants [Jan03, I.4.18 Proposition], we have isomorphisms $K[M_d^m]^G \otimes_K K' \cong K'[M_d^m]^G$ and $K[G^m]^G \otimes_K K' \cong K'[G^m]^G$, so $J \otimes_K K'$ is the kernel of $K'[M_d^m]^G \to K'[G^m]^G$. The claim follows from Theorem 5.2. For SO_{2n+1}, we note, that the map $K[\mathrm{O}_{2n+1}]^{\mathrm{O}_{2n+1}} \to K[\mathrm{SO}_{2n+1}]^{\mathrm{SO}_{2n+1}}$ is surjective, since $\mathrm{O}_{2n+1} = \mathrm{SO}_{2n+1} \times \{\pm 1\}$.

For the rest of the proof, we argue as in [Wei21, Lemma 3.15]. The natural surjection $\mathrm{Sp}_{2n} \times \mathrm{GL}_1 \to \mathrm{GSp}_{2n}$ induces an inclusion $K[\mathrm{GSp}_{2n}^m]^{\mathrm{GSp}_{2n}} \subseteq K[\mathrm{Sp}_{2n}^m]^{\mathrm{Sp}_{2n}} \otimes_K K[\mathrm{GL}_1^m]$. Here the second factor is generated by the symplectic similitude character sim_i of X_i and its inverse. Since all generators on the right hand side are defined on the left hand side, the map is an isomorphism. For GO_n we argue just the same way. $\square$

5.1.2 Invariants over $\mathbb{Z}/p^r$

In this subsection, fix a prime p and an integer $r \geq 1$. We extend the results over fields to p^r-torsion coefficients by using the theory of good filtrations over $\mathbb{Z}$. We plan to extend the results of this section to general coefficient rings in joint work with Mohamed Moakher. The main purpose here is to demonstrate, that if the coefficients have p^r-torsion the proof is much simpler. We can *lift* invariants using the following variant of Nakayama's lemma.

Lemma 5.4 (Nakayama)**.**

1. Let M be any $\mathbb{Z}/p^r$-module and assume $M/p = 0$. Then $M = 0$.
2. Let $f : M \to N$ be a homomorphism of $\mathbb{Z}/p^r$-modules, such that $\overline{f} : M/p \to N/p$ is surjective. Then f is surjective.

Proof. (1) We have $M = pM$, thus $M = p^r M = 0$. (2) We can apply (1) to $\mathrm{coker}(f)$. $\square$

Lemma 5.5. Let G be a Chevalley group and let $S \subseteq \mathbb{Z}[G^m]^G$ be a subset, that generates $\mathbb{F}_p[G^m]^G$ as a ring. Then S generates $\mathbb{Z}/p^r[G^m]^G$.

Proof. Let $A \subseteq \mathbb{Z}/p^r[G^m]^G$ be the subalgebra generated by S. By Proposition 4.17 $\mathbb{Z}[G^m]$ has a good filtration and in particular is acyclic by Lemma 4.15. We calculate

$$\mathbb{Z}/p^r[G^m]^G \otimes_{\mathbb{Z}/p^r} \mathbb{F}_p = (\mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathbb{Z}/p^r) \otimes_{\mathbb{Z}/p^r} \mathbb{F}_p = \mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathbb{F}_p[G^m]^G$$

applying Corollary 4.14 twice. Hence the inclusion induces a surjection $A/p \twoheadrightarrow (\mathbb{Z}/p^r[G^m]^G)/p$. From Lemma 5.4, we obtain $A = \mathbb{Z}/p^r[G^m]^G$. $\square$

Proposition 5.6. Let $\mathcal{O}$ be a commutative ring, such that $p^r\mathcal{O} = 0$. In the following we denote by X_k a generic group element, which can also be understood as a generic matrix under the standard representation. Let $m \geq 1$ and assume $p > 2$ in the orthogonal cases.

1. Let $n \geq 1$. Then $\mathcal{O}[\mathrm{Sp}_{2n}^m]^{\mathrm{Sp}_{2n}}$ is generated by elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.
2. Let $n \geq 1$. Then $\mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{SO}_{2n+1}}$ is generated by elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

3. Let $n \geq 1$. Then $\mathcal{O}[\mathrm{GSp}_{2n}^m]^{\mathrm{GSp}_{2n}}$ is generated by the symplectic similitude character sim, its inverse sim^{-1} and elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.
4. Let $n \geq 1$. Then $\mathcal{O}[\mathrm{GO}_n^m]^{\mathrm{GO}_n}$ is generated by the orthogonal similitude character sim, its inverse sim^{-1} and elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

Proof. Let $G \in \{\mathrm{Sp}_{2n}, \mathrm{O}_{2n+1}, \mathrm{GSp}_{2n}, \mathrm{GO}_n\}$. Since by Proposition 4.17 all $\mathbb{Z}/p^r[(G^0)^m]$ have a good filtration, we may assume $\mathcal{O} = \mathbb{Z}/p^r$. In all cases, the expected generators are defined as elements of $\mathbb{Z}[G^m]^{G^0}$. The claim now follows from Lemma 5.5 and the generators of $\mathbb{F}_p[G^m]^G$ we have given in Proposition 5.3 (Zubkov). $\square$

5.1.3 Invariants over $\mathbb{Z}$

In [DP76, p. 15.2.1] de Concini and Procesi have determined the generators of $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n}$ and $\mathbb{Z}[M_n^m]^{\mathrm{SL}_n}$, from which the generators of $\mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ and $\mathbb{Z}[\mathrm{SL}_n^m]^{\mathrm{SL}_n}$ can be deduced. We reprove their result using good filtrations and avoiding usage of the formal character of $\mathbb{Z}[M_n^m]$ and the analysis of root subgroups.

Recall the first fundamental theorem on invariants of several matrices.

Theorem 5.7 (De Concini, Procesi)**.** Let K be an algebraically closed field. Then $K[M_n^m]^{\mathrm{GL}_n}$ is generated by elements of the form
$$s_i(X_{j_1} \cdots X_{j_s})$$
for $i \in \{1, \dots, n\}$ and $s \geq 0$.

Proof. See [DP76, Theorem 1.10]. $\square$

The first fundamental theorem for SL_n follows right away:

Proposition 5.8. Let K be an algebraically closed field. Then $K[M_n^m]^{\mathrm{SL}_n}$ is generated by elements of the form
$$s_i(X_{j_1} \cdots X_{j_s})$$
for $i \in \{1, \dots, n\}$ and $s \geq 0$.

Proof. The inclusion of the center $\mathrm{GL}_1 \to \mathrm{GL}_n$ and the inclusion $\mathrm{SL}_n \to \mathrm{GL}_n$ combine to a surjection $\mathrm{SL}_n \times \mathrm{GL}_1 \to \mathrm{GL}_n$. Therefore $K[M_n^m]^{\mathrm{GL}_d} = K[M_n^m]^{\mathrm{SL}_n \times \mathrm{GL}_1} = K[M_n^m]^{\mathrm{SL}_n}$ and we conclude by Theorem 5.7. $\square$

To descend to $\mathbb{Z}$, we need the following lemma.

Lemma 5.9. Let $\mathcal{O}$ be a principal ideal domain and let M and M' be finitely generated free $\mathcal{O}$-modules. Let $f : M \to M'$ be an $\mathcal{O}$-module homomorphism, such that for every $\mathcal{O}$-field K the induced map $M \otimes_{\mathcal{O}} K \to M' \otimes_{\mathcal{O}} K$ is an isomorphism. Then f is an isomorphism.

Proof. Taking K the field of fractions of $\mathcal{O}$ shows, that f is injective and that the cokernel C of f is a finitely generated torsion module. For every prime ideal $0 \neq \mathfrak{p} \subseteq \mathcal{O}$, the sequence
$$M \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} \xrightarrow{\sim} M' \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} \longrightarrow C \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} \longrightarrow 0$$
is exact, which shows, that $C \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} = 0$. It follows, that $C = 0$. $\square$

Theorem 5.10 (De Concini, Procesi)**.** $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n} = \mathbb{Z}[M_n^m]^{\mathrm{SL}_n}$ and is generated by elements of the form
$$s_i(X_{j_1} \cdots X_{j_s})$$
for $i \in \{1, \dots, n\}$ and $s \geq 0$.

Proof. $M_n^m = (\mathrm{Std} \otimes \mathrm{Std}^*)^{\oplus m}$, where Std is the standard representation of GL_n. The standard representation of GL_n is self-dual and has a good filtration. We observe, that $\mathbb{Z}[M_n^m]$ admits a grading by finitely generated free $\mathbb{Z}$-modules $S_d := \mathrm{Sym}^d((M_n^m)^*)$, that is preserved by the action of GL_n. The S_d also have a good filtration, so by Lemma 4.15 and Corollary 4.14 $S_d^{\mathrm{GL}_n} \otimes_{\mathbb{Z}} k = (S_d \otimes_{\mathbb{Z}} k)^{\mathrm{GL}_n}$ for any field k.

Let A be a free commutative $\mathbb{Z}$-algebra generated by variables $t_{(i,(j_1,\dots,j_s))}$ with $i \in \{1,\dots,d\}$ and $s \geq 0$. We let $t_{(i,(j_1,\dots,j_s))}$ have degree si and observe that $A = \bigoplus_{d=0}^{\infty} A_d$ is a graded ring, such that each submodule A_d consisting of homogeneous of degree d elements is a finitely generated free $\mathbb{Z}$-module.

The natural map $A \to \mathbb{Z}[M_n^m]^{\mathrm{GL}_n}$ sending $t_{(i,(j_1,\dots,j_s))}$ to $s_i(X_{j_s} \dots X_{j_s})$ is graded. By Theorem 5.7, the maps $A_d \otimes_{\mathbb{Z}} k \to S_d^{\mathrm{GL}_n} \otimes_{\mathbb{Z}} k$ are surjective for every algebraically closed field k. Hence by Lemma 5.9 the maps $A_d \to S_d$ are surjective and thus $A \to S$ is surjective, proving the first claim. The argument for the SL_n-invariants is the same, using Proposition 5.8. ☐

To pass from invariants of $\mathbb{Z}[M_n^m]$ to invariants of $\mathbb{Z}[\mathrm{GL}_n^m]$ and $\mathbb{Z}[\mathrm{SL}_n^m]$, we use the following general lemma.

Lemma 5.11. Let G be a split reductive group over $\mathbb{Z}$ and let

$$0 \to C \to B \to A \to 0$$

$$0 \to C' \to B' \to A' \to 0$$

be two short exact sequences of G-modules with good filtration. Then the map $(B \otimes B')^G \to (A \otimes A')^G$ is surjective.

Proof. Since good filtration modules are free, the sequences

$$0 \to C \otimes A' \to B \otimes A' \to A \otimes A' \to 0$$

$$0 \to B \otimes C' \to B \otimes B' \to B \otimes A' \to 0$$

are exact. By Mathieu's tensor product theorem Theorem 4.16, the modules $C \otimes A'$ and $B \otimes C'$ have good filtrations, hence by Lemma 4.15 the maps $(B \otimes A')^G \to (A \otimes A')^G$ and $(B \otimes B')^G \to (B \otimes A')^G$ are surjective. ☐

Theorem 5.12. Let $\mathcal{O}$ be a commutative ring, let $m \geq 1$ and let $n \geq 1$.

1. $\mathcal{O}[M_n^m]^{\mathrm{GL}_n}$ and $\mathcal{O}[M_n^m]^{\mathrm{SL}_n}$ are generated by elements of the form

$$s_i(X_{j_1} \cdots X_{j_s})$$

   for $i \in \{1,\dots,n\}$ and $s \geq 0$.

2. $\mathcal{O}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ and $\mathcal{O}[\mathrm{SL}_n^m]^{\mathrm{SL}_n}$ are generated by elements of the form

$$s_i(X_{j_1} \cdots X_{j_s})$$

   for $i \in \{1,\dots,n\}$ and $s \geq 0$ and $\det^{-1}(X_j)$ for $j \in \{1,\dots,m\}$.

Proof. By Proposition 4.17, it is for both GL_n and SL_n sufficient to prove the claim for $\mathcal{O} = \mathbb{Z}$. The closed immersion $\mathrm{GL}_n \to M_n \times \mathbb{A}^1$, $g \mapsto (g, \det(g)^{-1})$ induces a surjection $\mathbb{Z}[(M_n \times \mathbb{A}^1)^m] \to \mathbb{Z}[\mathrm{GL}_n^m]$ of $\mathbb{Z}$-graded $\mathbb{Z}$-modules with GL_n-action, where the graded pieces are finitely generated free $\mathbb{Z}$-modules.

Identifying $\mathbb{Z}[\mathbb{A}^1] = \mathbb{Z}[t]$, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}[M_n] \otimes \mathbb{Z}[t] \xrightarrow{\cdot(t\cdot\det -1)} \mathbb{Z}[M_n] \otimes \mathbb{Z}[t] \longrightarrow \mathbb{Z}[\mathrm{GL}_n] \longrightarrow 0$$

of GL_d-modules, since $t \cdot \det -1$ is an invariant element of the integral domain $\mathbb{Z}[M_d] \otimes \mathbb{Z}[t]$. By Lemma 5.11 and since $\mathbb{Z}[\mathrm{GL}_d]$ and $\mathbb{Z}[M_d]$ have good filtrations (Proposition 4.17, Proposition 4.18), the maps $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n} \otimes \mathbb{Z}[t]^{\otimes m} = \mathbb{Z}[(M_n \times \mathbb{A}^1)^m]^{\mathrm{GL}_n} \to \mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ are surjective. The claim follows from Theorem 5.10.

The same argument using the closed immersion $\mathrm{SL}_n \to M_n$ and the short exact sequence

$$0 \longrightarrow \mathbb{Z}[M_n] \xrightarrow{\cdot(\det -1)} \mathbb{Z}[M_n] \longrightarrow \mathbb{Z}[\mathrm{SL}_n] \longrightarrow 0$$

implies the claim on $\mathbb{Z}[\mathrm{SL}_n^m]^{\mathrm{SL}_n}$. ☐

In upcoming joint work with Mohamed Moakher, we will compute explicit generators of $\mathbb{Z}[G^m]^G$ for $G \in \{\mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$ building on work of Zubkov.

Corollary 5.13. Let $\mathcal{O}$ be a commutative ring.

1. The $\mathcal{F}$-$\mathcal{O}$-algebra $\mathcal{O}[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n}$ is generated by $s_1, \dots, s_n \in \mathcal{O}[\mathrm{GL}_n]^{\mathrm{GL}_n}$.
2. The $\mathcal{F}$-$\mathcal{O}$-algebra $\mathcal{O}[\mathrm{SL}_n^\bullet]^{\mathrm{SL}_n}$ is generated by $s_1, \dots, s_{n-1} \in \mathcal{O}[\mathrm{SL}_n]^{\mathrm{SL}_n}$.

Proof. This follows by inspection of the generators computed in Theorem 5.12 and substitutions. $\square$

Corollary 5.14. Let $r \geq 1$ be an integer and let $\mathcal{O}$ be a commutative ring, such that $p^r\mathcal{O} = 0$. Let $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_{2n+1}\}$ and assume that $p > 2$ in the orthogonal cases. Then the maps $\mathcal{O}[\mathrm{GL}_n^m]^{\mathrm{GL}_n} \to \mathcal{O}[G^m]^{G^0}$ are surjective for all $m \geq 1$. In particular the $\mathcal{F}$-$\mathcal{O}$-algebras $\mathcal{O}[G^\bullet]^{G^0}$ are finitely generated.

Proof. This follows from Corollary 5.13 and Proposition 5.6 and substitutions. $\square$

5.2 Alternative methods for computation of invariants

5.2.1 The successive method

If G is a connected reductive group over an algebraically closed field with Borel subgroup B containing a maximal torus T, there is a Bruhat decomposition $G = \bigcup_{w \in W} BwB$, where W is the Weyl group $W = N_G(T)/T$. The double cosets BwB for $w \in W$ are well-defined. The 'successive method' of computing invariants of a G-module V rests on the observation, that it suffices to take the invariants with respect to subgroups of G, which generate G, separately. The Weyl group does not embed into G in general, but V^T is a W-module. Let $U = R_u(B)$ be the unipotent radical of B. We have the Levi decomposition $B = TU$. It follows, that

$$V^G = (V^T)^W \cap V^U \tag{7}$$

The inclusion '$\subseteq$' is sufficient in most cases.

From a talk of Samit Dasgupta in Essen I learned the following little trick: If B is a Borel subgroup of G, then the orbit Gx of a B-invariant element $x \in V$ is the image of the connected projective variety G/B in V and thus contains only one point. It follows, that $V^G = V^B$. So we can also use

$$V^G = (V^U)^T \tag{8}$$

to compute G-invariants. However in practice it is tedious to first compute U-invariants, so we stick to Equation (7) in the following examples.

5.2.2 Matrix invariants

Example 5.15. As a warmup, we compute the S_2-invariants of $\mathbb{Z}[x_1, x_2]$, where the generator of S_2 interchanges x_1 and x_2. We claim, that $\mathbb{Z}[x_1, x_2]^{S_2}$ is generated by $\mathrm{tr} := x_1 + x_2$ and $\det := x_1x_2$. We can show this by induction over the number of terms of an invariant polynomial. Let $f \in \mathbb{Z}[x_1, x_2]^{S_2}$ be nonconstant with n terms, such that f is not divisible by x_1x_2. Then there is $k \geq 1$ and $\lambda \in \mathbb{Z} \setminus \{0\}$, such that $f = \lambda(x_1^k + x_2^k) + g$ for some $g \in \mathbb{Z}[x_1, x_2]^{S_2}$ with $n - 2$ terms. We are left to show, that the power sums $x_1^k + x_2^k$ are generated by tr and det. This can be shown by induction using

$$x_1^k + x_2^k = (x_1^{k-1} + x_2^{k-1})(x_1 + x_2) - x_1x_2(x_1^{k-2} + x_1^{k-2}).$$

Example 5.16 (Successive method)**.** We first compute the matrix invariants $\mathbb{Z}[M_2]^{\mathrm{GL}_2}$ directly following Equation (7). GL_2 acts on (2×2)-matrices M_2 by conjugation. We can see M_2 as the rational adjoint representation of GL_2 over $\mathbb{Z}$. Let $\mathbb{Z}[M_2]$ be the symmetric algebra on M_2^*. We write $\mathbb{Z}[M_2] = \mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}]$, where x_{ij} is (i,j)-entry of a generic (2×2)-matrix. Let T be the standard diagonal torus of GL_2. The action of T on x_{ij} is given by $t \cdot x_{ij} = t_it_j^{-1}x_{ij}$ with $t = \mathrm{diag}(t_1, t_2) \in T$. We observe, that every monomial in $\mathbb{Z}[M_2]$ spans an invariant T-submodule. Thus the weight space decomposition of $\mathbb{Z}[M_2]$ is given by monomials. We see T as a product $\mathbb{G}_m^2$, where t_1 is the coordinate of the first $\mathbb{G}_m$ and t_2 is the coordinate of the second $\mathbb{G}_m$. A monomial is t_1-invariant if and only if it is a product of x_{11}, x_{22} and $x_{12}x_{21}$. This is equivalent to being t_2-invariant, so we obtain $\mathbb{Z}[M_2]^T = \mathbb{Z}[x_{11}, x_{22}, x_{12}x_{21}]$ as a first intermediate step.

For convenience we substitute $x_1 := x_{11}$, $x_2 := x_{22}$ and $y := x_{12}x_{21}$. We now compute the Weyl-invariants. The Weyl group of GL_2 with respect to the standand torus T is isomorphic to S_2 and acts by $\sigma \cdot x_i = x_{\sigma(i)}$ and $\sigma \cdot y = y$. So it is sufficient to compute the S_2-invariants of $\mathbb{Z}[x_1, x_2]$. We have already done this in Example 5.15 and so we conclude, that $\mathbb{Z}[x_1, x_2, y]^{S_2} = \mathbb{Z}[x_1 + x_2, x_1x_2, y]$.

In the last step, we take invariants under the unipotent subgroup

$$U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

of strictly upper triangular matrices. We compute

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} - ax_{21} & x_{12} + a(x_{11} - x_{22}) - a^2x_{21} \\ x_{21} & x_{22} + ax_{21} \end{pmatrix}$$

over $\mathbb{Z}[M_2][a]$ and observe, that $x_1 + x_2$ and x_{21} are U-invariant. So it is sufficient to compute the U-invariants of $\mathbb{Z}[x_1x_2, y]$. Beware, that the action of U on $\mathbb{Z}[M_2]$ does not preserve the subspace $\mathbb{Z}[x_1 + x_2, x_1x_2, y]$, so $\mathbb{Z}[x_1x_2, y]$ is not an honest U-representation! What we will effectively compute in the last step is $\mathbb{Z}[x_1x_2, y] \cap \mathbb{Z}[M_2]^U$.

As above we let a be the coordinate of $\mathbb{G}_a \cong U$. The action of U is given by

$$a \bullet (x_1x_2) = x_1x_2 + (x_1 - x_2)ax_{21} - (ax_{21})^2$$
$$a \bullet y = y + (x_1 - x_2)ax_{21} - (ax_{21})^2$$

We observe, that $x_1x_2 - y$ is U-invariant. So $\mathbb{Z}[x_1x_2, y] = \mathbb{Z}[x_1x_2 - y, y]$ and we are left to compute the U-invariants of $\mathbb{Z}[y]$. Clearly $\mathbb{Z}[y] \cap \mathbb{Z}[M_2]^U = \mathbb{Z}$. So we end up with $\mathbb{Z}[x_1 + x_2, x_1x_2, y] \cap \mathbb{Z}[M_2]^U = \mathbb{Z}[x_1 + x_2, x_1x_2 - y]$, where $x_1 + x_2$ is the trace and $x_1x_2 - y$ is the determinant of a generic matrix in M_2.

Note, that the entire argument did not use that the coefficient ring is $\mathbb{Z}$. We could now directly conclude $\mathbb{Z}[M_2]^{\mathrm{GL}_2} = \mathbb{Z}[\mathrm{tr}, \det]$ using that the Bruhat decomposition and $B = TU$ are valid over $\mathbb{Z}$, but we don't need this: We have just shown the inclusion $\mathbb{Z}[M_2]^{\mathrm{GL}_2} \subseteq \mathbb{Z}[\mathrm{tr}, \det]$ and conclude by verifying that tr and det are indeed GL_2-invariant.

We can use the argument of Theorem 5.12 to deduce from Example 5.16 directly $\mathbb{Z}[\mathrm{GL}_2]^{\mathrm{GL}_2} = \mathbb{Z}[\mathrm{tr}, \det, \det^{-1}]$. We emphasize, that the benefit of the succesive method is that it is applicable whenever we want to compute invariants for an action of a connected reductive group on a finite-dimensional representation or a symmetric algebra on a finite-dimensional representation. In particular it is theoretically possible to calculate $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n}$ with the successive method, even though we find this not practical to carry out. The goal of the next section is to demonstrate how to obtain $\mathbb{Z}[\mathrm{GL}_2]^{\mathrm{GL}_2}$ from an integral version of Chevalley's restriction theorem.

5.2.3 Chevalley's restriction theorem

We recall the proof of Chevalley's restriction theorem in arbitrary characteristic in lack of an adequate reference.

Theorem 5.17. Let K be an algebraically closed field and let G be a connected semisimple group over K with maximal torus T and Weyl group $W = N_G(T)/T$. Then the restriction map $K[G] \to K[T]$ induces an isomorphism $K[G]^G \cong K[T]^W$ of K-algebras.

Proof. For the proof, we choose a system of positive roots Φ^+. $K[T]$ is an $N_G(T)$-module and hence a W-module. Therefore we get a well-defined map $|_T : K[G]^G \to K[T]^W$. We first show, that $|_T$ is surjective. Let $f \in K[T]^W$. The set of weights $X(T)$ is a K-basis of $K[T]$. So $f = \sum_{\lambda \in X(T)} a_\lambda t^\lambda$, where $a_\lambda \in K$ and t is a generic element of T. Let $\lambda \in X(T)$ be a maximal dominant weight, such that

$a_\lambda \neq 0$. Such a weight exists, since f is W-invariant. Let $\chi_\lambda \in K[G]^G$ be the character of the irreducible G-module with highest weight λ. Recall, that $\chi_\lambda|_T$ only contains weights $\preceq \lambda$. So $f - a_\lambda \chi_\lambda|_T \in K[T]^W$ is a function all of whose maximal dominant weights are $\precneqq$ the maximal dominant weights of f. By induction we conclude, that there is some $\tilde{f} \in K[G]^G$ with $\tilde{f}|_T = f$.

It is now sufficient to show, that $|_T : K[G]^G \to K[T]$ is injective. Let $f \in K[G]^G$, such that $f|_T = 0$. Let $x \in G$ be a regular element, i.e. $Z_G(x_s) = \mathrm{rank}(G)$ for the semisimple part x_s of x. By [Bor91, Theorem 12.3 (1)] x is contained in a Cartan subgroup of G. Since G is connected reductive, such a Cartan subgroup is just a maximal torus. Thus, there is some $h \in G$ with $hxh^{-1} \in T$ and hence $f(x) = f(hxh^{-1}) = 0$. By [Bor91, Theorem 12.3 (1)] the set of regular elements contains a dense open set in G and we conclude $f = 0$. □

Theorem 5.18. Let G be a split connected semisimple group over $\mathbb{Z}$ with fiberwise maximal torus T over $\mathbb{Z}$ and Weyl group W. Then the restriction map $\mathbb{Z}[G] \to \mathbb{Z}[T]$ induces an isomorphism $\mathbb{Z}[G]^G \cong \mathbb{Z}[T]^W$ of rings.

Proof. Surjectivity can be proved by the same argument as in Theorem 5.17 using the Weyl module $\nabla(\lambda)$ in place of the irreducible module of highest weight λ.

We have $\mathbb{Z}[T]^W = \bigoplus_{\lambda \in X^+(T)} \mathbb{Z} \cdot \sum_{w \in W} \lambda^w$, so $\mathbb{Z}[T]^W \otimes \mathbb{Q} = \mathbb{Q}[T]^W$. We also know, that $\mathbb{Z}[G]^G \otimes \mathbb{Q} = \mathbb{Q}[G]^G$ by flatness. We deduce, that the map $\mathbb{Z}[G]^G \to \mathbb{Z}[T]^W$ is injective. □

By Zariski denseness of $\mathrm{GL}_2 \subseteq M_2$, we have $\mathbb{Z}[M_2]^{\mathrm{GL}_2} = \mathbb{Z}[M_2] \cap \mathbb{Z}[\mathrm{GL}_2]^{\mathrm{GL}_2}$ and thus from Theorem 5.18, we directly obtain $\mathbb{Z}[M_2]^{\mathrm{GL}_2} = \mathbb{Z}[\mathrm{tr}, \det]$.

Example 5.19. We will compute $\mathbb{Z}[\mathrm{Sp}_4]^{\mathrm{Sp}_4}$ using Theorem 5.18. The Weyl group of Sp_4 has 8 elements and is generated by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a reflection group on $X(T)$ with

$$T = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_2^{-1} & \\ & & & x_1^{-1} \end{pmatrix}$$

by $\sigma_1(x_1) = x_2$, $\sigma_1(x_2) = x_1$, $\sigma_2(x_1) = x_1$, $\sigma_2(x_2) = x_2^{-1}$. We write $\mathbb{Z}[T] = \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}]$. Taking σ_2-invariants, we obtain $\mathbb{Z}[x_2, x_2^{-1}]^{\sigma_2} = \mathbb{Z}[x_2 + x_2^{-1}]$ by an easy induction over the highest degree term of an invariant. For convenience we next take invariants by

$$\sigma_1\sigma_2\sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and obtain $\mathbb{Z}[x_1, x_1^{-1}]^{\sigma_1\sigma_2\sigma_1} = \mathbb{Z}[x_1 + x_1^{-1}]$. Joining these two results, we obtain

$$\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}]^{\langle \sigma_2, \sigma_1\sigma_2\sigma_1 \rangle} = \mathbb{Z}[x_1 + x_1^{-1}, x_2 + x_2^{-1}]$$

Using Example 5.15, we get

$$\mathbb{Z}[T]^W = \mathbb{Z}[x_1 + x_1^{-1} + x_2 + x_2^{-1}, (x_1 + x_1^{-1})(x_2 + x_2^{-1})]$$

We observe, that $x_1 + x_1^{-1} + x_2 + x_2^{-1}$ is the trace of the standard representation Std. Similarly $(x_1 + x_1^{-1})(x_2 + x_2^{-1}) + 2$ is the trace of $\Lambda^2\,\mathrm{Std}$. It follows, that

$$\mathbb{Z}[\mathrm{Sp}_4]^{\mathrm{Sp}_4} = \mathbb{Z}[\mathrm{tr}, \mathrm{tr}\,\Lambda^2\,\mathrm{Std}]$$

In fact it follows directly from the basis of $\mathbb{Z}[T]^W$ described in the proof of Theorem 5.18, that $\mathbb{Z}[T]^W$ is a polynomial ring generated by sums over W-orbits of fundamental weights. It has already been proved by Chevalley, that $\mathbb{Z}[T]^W$ and therefore $\mathbb{Z}[G]^G$ is a polynomial ring generated by traces of fundamental representations of G.

However all we have seen so far is not sufficient to deduce a first fundamental theorem for $\mathbb{Z}[G^m]^G$. We will survey in the next and last section on invariant theory, what we know about $\mathbb{Z}[G^m]^G$.

5.2.4 Group invariants

We are interested in a first fundamental theorem for $\mathbb{Z}[G^m]^G$ for G a split connected reductive group over $\mathbb{Z}$. The first observation is, that the arguments of Section 8.7 are applicable to general split semisimple groups. Second, the case of split reductive groups reduces to the case of split semisimple groups by taking invariants after restriction to the surjection $Z(G) \times [G, G] \to G$ (see e.g. [Mil12, p. 17.28]). So everything boils down to computation of invariants over algebraically closed fields in which case, we can apply Donkin's theorem [Don92]. However this turns out to be difficult in practice. We give a list of results that compute $K[G^m]^G$ for some reductive group G.

1. Donkin [Don92] computes generators of $K[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$.
2. Zubkov [Zub99] computes generators of $K[\mathrm{Sp}_{2n}^m]^{\mathrm{Sp}_{2n}}$ for all $n \geq 1$.
3. Zubkov [Zub99] computes generators of $K[\mathrm{O}_n^m]^{\mathrm{O}_n}$ for all $n \geq 1$ and $\mathrm{char}(K) \neq 2$.
4. Zubkov [Zub99] computes generators of $K[\mathrm{SO}_n^m]^{\mathrm{SO}_n}$ for odd $n \geq 3$ and $\mathrm{char}(K) \neq 2$.

These results can be adapted to GO_n and GSp_{2n}. They can be interpreted from the perspective of $\mathcal{F}$-K-algebras: The $\mathcal{F}$-K-algebra $K[G^\bullet]^G$ is finitely generated by an explicit set of generators in the cases listed above. We expect, that this is the case for all split reductive groups G. A proof of this (over the ring of integers of a p-adic local field) would be useful in many places: It leads to a simplification of the proofs of Lemma 7.6 and Lemma 7.16 and Lemma 7.6 can probably be generalized to arbitrary profinite groups Γ.

The algebra $K[G^\bullet]^G$ being finitely generated follows from the following stronger statement, of which we are not sure if it holds in general: There is a faithful algebraic representation $G \to \mathrm{GL}_n$, such that the induced homomorphism $K[\mathrm{GL}_n^m]^{\mathrm{GL}_n} \to K[G^\bullet]^G$ is surjective for all $m \geq 1$. A proof of this might be within reach after a more detailed analysis of Donkin's main theorem in [Don92]. It might be sufficient to take a faithful tilting module which generates the category of rational G-modules under tensor products, exterior powers and subquotients, capturing all possible weights of highest weight modules. I have not yet been able to turn this idea into a rigorous proof, as the combinatorics of the Schur algebras that arise remains elusive to me. At last I want to emphasize, that this second conjecture would be a very explicit first fundamental theorem for general reductive groups in arbitrary characteristic and is likely very hard to prove.

It is also remarkable, that in most cases we know of $K[G^\bullet]^G$ is generated by $K[G]^G$. Indeed [Wei20, Theorem 4.3] Weidner shows in characteristic zero, that in this case element-wise conjugacy of G-valued representations implies conjugacy of representations. In this case we say, that G is *acceptable*. Indeed the property of acceptability is also related to multiplicity one theorems, as we have learned from Carl-Wang Erickson. So apart from the theory of pseudocharacters these questions seem to have theoretical relevance.

Let $\mathcal{O}$ be a finitely generated algebra over a Nagata ring and let G be a reductive $\mathcal{O}$-group scheme. Then for all $m \geq 1$, the algebra of rational invariants $\mathcal{O}[G^m]^G$ is finitely generated over $\mathcal{O}$. This follows readily from [Ses77, Thm. 2] applied to G^m. However, we have not found Seshadri's methods to be sufficient to prove a similar non-constructive finite generation result for the $\mathcal{F}$-$\mathcal{O}$-algebras $\mathcal{O}[G^\bullet]^G$.

6 Deformations of G-valued pseudocharacters

We define a deformation space of V. Lafforgue's G-valued pseudocharacters of a profinite group Γ for a (generalized) reductive group G. We show, that our definition generalizes Chenevier's construction [Che14]. We show that the universal pseudodeformation ring is noetherian whenever Γ is topologically finitely generated. For $G = \mathrm{Sp}_{2n}$ we describe three types of obstructed subloci of the special fiber of the universal pseudodeformation space of an arbitrary residual pseudocharacter and give upper bounds for their dimension.

Introduction

Let $F/\mathbb{Q}_p$ be a p-adic local field with absolute Galois group Γ_F. Let L be a p-adic local field with ring of integers $\mathcal{O}_L$ and residue field κ. Let G be a generalized reductive group scheme over $\mathcal{O}_L$ (see Section 4.1.2), which is essentially a model of a possibly disconnected reductive group over $\mathcal{O}_L$. Given a continuous representation $\overline{\rho} : \Gamma_F \to G(\kappa)$, we define the framed deformation functor on the category $\mathfrak{A}_{\mathcal{O}_L}$ of local artinian $\mathcal{O}_L$-algebras with residue field κ by $\mathrm{Def}^{\square}_{\mathcal{O}_L,\overline{\rho}}(A) := \{\rho : \Gamma_F \to G(A) \mid \rho \text{ continuous lift of } \overline{\rho}\}$. The framed deformation functor is pro-representable by a complete local noetherian $\mathcal{O}_L$-algebra $R^{\square}_{G,\overline{\rho}}$ with residue field κ. Inspired by [BIP21, Theorem 1.1], we would like to prove the following conjecture:

Conjecture 6.1. The ring $R^{\square}_{G,\overline{\rho}}$ is a normal, local complete intersection, flat over $\mathcal{O}_L$ and of relative dimension $\dim G_L \cdot ([F : \mathbb{Q}_p] + 1)$ over $\mathcal{O}_L$.

The proof in [BIP21] relies on estimates of certain subloci in the special fiber of the pseudodeformation ring for GL_n. There pseudocharacters in the sense of Chenevier [Che14] are used.

The first main aim of this chapter is to introduce the pseudodeformation ring for generalized reductive group schemes, replacing Chenevier's pseudocharacters by Lafforgue's pseudocharacters as introduced in [Laf18, §11]. We show, that these rings are noetherian for topologically finitely generated profinite groups and in particular for Γ_F.

Theorem B (Theorem 6.11, Theorem 6.14)**.** Let G be a generalized reductive $\mathcal{O}_L$-group scheme, let Γ be a profinite group and let $\overline{\Theta}$ be a continuous G-pseudocharacter of Γ over κ.

1. If Γ is topologically finitely generated, then the G-pseudodeformation ring $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ of $\overline{\Theta}$ is noetherian.
2. Assume that $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$, $p > 2$ in the orthogonal cases and let $\iota : G \to \mathrm{GL}_d$ be the standard representation of G. Then the canonical map $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is surjective. If in addition Γ satisfies Mazur's condition Φ_p, then $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is noetherian.

The second main aim is to give estimates for certain obstructed subloci $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$ and $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ (see Definition 6.25) of the special fiber $\overline{X}_{\overline{\Theta}}$ of $\mathrm{Spec}(R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}})$ analogous to [BIP21, §3.4] and [BJ19], which paves the way for proving Conjecture 6.1 when $G = \mathrm{Sp}_{2n}$.

Theorem D (Proposition 6.33, Theorem 6.34, Corollary 6.35)**.** Let $\overline{\Theta}$ be a continuous Sp_{2n}-pseudocharacter of Γ_F over κ.

1. $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq n(2n+1)[F : \mathbb{Q}_p] - 4(n-1)[F : \mathbb{Q}_p]$.
2. $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq n^2[F : \mathbb{Q}_p] + 1$.
3. $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F : \mathbb{Q}_p] + 1$.
4. $\dim \overline{X}_{\overline{\Theta}} \leq n(2n+1)[F : \mathbb{Q}_p]$.
   If $\overline{\Theta}$ comes from an absolutely irreducible representation, then equality holds and $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \subsetneq \overline{X}_{\overline{\Theta}}$.

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6.1 Deformations of G-pseudocharacters

6.1.1 Coefficient rings

Fix a prime $p > 0$. Let κ be a field of one of the following three types:

1. κ is a finite discrete field.
2. κ is a finite extension of $\mathbb{Q}_p$ equipped with the p-adic topology.
3. κ is a finite extension of $\mathbb{F}_p((t))$ equipped with the t-adic topology.

We introduce a coefficient ring Λ in each of the three cases for κ.

1. In case (1), let Λ be the ring of integers of a p-adic field with residue field κ.
2. In case (2), let $\Lambda = \kappa$.
3. In case (3), let $\Lambda = \kappa$.

By slight abuse of terminology, we will call only such rings Λ *coefficient rings* for κ.

Let $\mathfrak{A}_\Lambda$ be the category of artinian local Λ-algebras with residue field κ. Every A in $\mathfrak{A}_\Lambda$ has a canonical projection $\pi_A : A \to \kappa$ with kernel $\mathfrak{m}_A$ the maximal ideal of A. Note, that $\mathfrak{A}_\Lambda$ admits fiber products [Til96, §2.2]. Every complete local Λ-algebra A with residue field κ is algebraically isomorphic to the inverse limit $\varprojlim A/\mathfrak{m}_A^n$. If A is complete local noetherian Λ-algebra, it can be written as $A = \Lambda[[X_1, \dots, X_r]]/I$, where r is the κ-dimension of the relative cotangent space $t_A^* = \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_\Lambda A)$ of A [Til96, Lem. 5.1].

6.1.2 The universal deformation ring $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$

Definition 6.2. Let κ be a finite or a local field and let Λ be a coefficient ring as in Section 6.1.1 with residue field κ. Let Γ be a profinite group and let G be a generalized reductive Λ-group scheme. Let $\overline{\Theta} \in \mathrm{cPC}^\Gamma_G(\kappa)$ be a continuous G-pseudocharacter of Γ. We define the *deformation functor* of $\overline{\Theta}$

$$\begin{aligned} \mathrm{Def}_{\overline{\Theta}} : \mathfrak{A}_\Lambda &\to \mathrm{Set} \\ A &\mapsto \{\Theta \in \mathrm{cPC}^\Gamma_G(A) \mid \Theta \otimes_A \kappa = \overline{\Theta}\} \end{aligned}$$

that sends an object $A \in \mathfrak{A}_\Lambda$ to the set of continuous G-pseudocharacters Θ of Γ over A with $\Theta \otimes_A \kappa = \overline{\Theta}$.

If A is an arbitrary local topological Λ-algebra with residue field κ, we define $\mathrm{Def}_{\overline{\Theta}}(A)$ analogously. This is notation for a single A and shall not extend the deformation functor $\mathrm{Def}_{\overline{\Theta}}$. To prove pro-representability of the deformation functor we need to show, that it is compatible with taking inverse limits.

Lemma 6.3. Let Λ be a coefficient ring as in Section 6.1.1 and let $A = \varprojlim_i A_i$ be a projective limit of local topological Λ-algebras with $A_i \in \mathfrak{A}_\Lambda$, endowed with the projective limit topology. Let $\overline{\Theta} \in \mathrm{cPC}^\Gamma_G(\kappa)$. Then the natural map $\mathrm{Def}_{\overline{\Theta}}(A) \to \varprojlim_i \mathrm{Def}_{\overline{\Theta}}(A_i)$ is bijective.

Proof. Per definition, we have a pullback diagram

$$\begin{tikzcd} \mathrm{Def}_{\overline{\Theta}}(A) \arrow[r] \arrow[d] & \mathrm{cPC}^\Gamma_G(A) \arrow[d] \\ \{\overline{\Theta}\} \arrow[r] & \mathrm{cPC}^\Gamma_G(\kappa) \end{tikzcd}$$

So it suffices to prove the claim for cPC^Γ_G instead of the deformation functor.

By Corollary 4.45 and since $\mathcal{C}(\Gamma^n, A) = \varprojlim_i \mathcal{C}(\Gamma^n, A_i)$, we have

$$\begin{aligned}\mathrm{cPC}_G^\Gamma(A) &= \mathrm{Hom}_{\mathrm{CAlg}_\Lambda^{\mathcal{T}}}(\Lambda[G^\bullet]^{G^0},\ \mathcal{C}(\Gamma^\bullet, A)) \\ &= \mathrm{Hom}_{\mathrm{CAlg}_\Lambda^{\mathcal{T}}}(\Lambda[G^\bullet]^{G^0},\ \varprojlim_i \mathcal{C}(\Gamma^\bullet, A_i)) \\ &= \varprojlim_i \mathrm{Hom}_{\mathrm{CAlg}_\Lambda^{\mathcal{T}}}(\Lambda[G^\bullet]^{G^0},\ \mathcal{C}(\Gamma^\bullet, A_i)) \\ &= \varprojlim_i \mathrm{cPC}_G^\Gamma(A_i)\end{aligned}$$

□

Theorem 6.4. Let κ be a finite or a local field and let Λ be a coefficient ring for κ. Let Γ be a profinite group and let G be a generalized reductive Λ-group scheme. Let $\overline{\Theta} \in \mathrm{cPC}_G^\Gamma(\kappa)$ be a continuous pseudocharacter. Then the deformation functor

$$\mathrm{Def}_{\overline{\Theta}} : \mathfrak{A}_\Lambda \to \mathrm{Set}$$

is pro-representable by some inverse limit $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ of artinian Λ-algebras with residue field κ, endowed with the inverse limit topology. If κ is finite, then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is pro-p and in particular complete.

If $\overline{\Theta}$ is induced by a continuous representation $\overline{\rho} : \Gamma \to G(\kappa)$, we write $R^{\mathrm{ps}}_{\Lambda,\overline{\rho}} := R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$. If the residue field is a local field, we only have one choice for Λ and we will usually drop it from notations.

Proof. We adapt the proof of [Che14, Proposition 3.3]. Let $B := B_G^\Gamma$ be the Λ-algebra from Theorem 4.46, that represents $\mathrm{PC}_G^\Gamma : \mathrm{CAlg}_\Lambda \to \mathrm{Set}$. Let $\Theta^u \in \mathrm{PC}_G^\Gamma(B)$ be the universal G-pseudocharacter and $\psi : B \to \kappa$ the morphism, that corresponds to $\overline{\Theta}$ under the identification $\mathrm{Hom}_{\mathrm{CAlg}_\Lambda}(B, \kappa) \cong \mathrm{PC}_G^\Gamma(\kappa)$. We define a set $\mathcal{I}$ of ideals of B as follows: An ideal $I \subseteq B$ is in $\mathcal{I}$, if and only if the following three conditions hold:

1. I is contained in the maximal ideal $\mathfrak{m} := \ker(\psi)$ associated with ψ.
2. B/I is artinian and local. If κ is finite, we equip B/I with the discrete topology. If κ is a local field then B/I is a finite-dimensional κ-vector space and we equip B/I with the product topology of κ.
3. The image $\Theta^I := \pi^I_* \Theta^u$ of Θ^u under the map $\mathrm{PC}_G^\Gamma(B) \to \mathrm{PC}_G^\Gamma(B/I)$ induced by the projection $\pi^I : B \to B/I$ is a continuous G-pseudocharacter.

$(\mathcal{I}, \subseteq)$ is a cofiltered poset: If $I, J \in \mathcal{I}$, then we have

1. $I \cap J \subseteq \mathfrak{m}$.
2. The map $\iota : B/(I \cap J) \to B/I \times B/J$ is injective, hence $B/(I \cap J)$ is artinian. Let $\mathfrak{m}'$ be a maximal ideal of B, that contains $I \cap J$. Then $IJ \subset \mathfrak{m}'$, hence either $I \subset \mathfrak{m}'$ or $J \subset \mathfrak{m}'$. In both cases $\mathfrak{m}' = \mathfrak{m}$, since B/I and B/J are local. Hence $B/(I \cap J)$ is local.
3. Note, that ι is a topological embedding. Thus, for the reduction $\Theta^{I \cap J}$ of Θ^u mod $I \cap J$ the homomorphism $\Theta_n^{I \cap J} : B[G^n]^{G^0} \to \mathrm{Map}(\Gamma^n, B/(I \cap J))$ has image in $\mathcal{C}(\Gamma^n, B/(I \cap J))$ for all $n \geq 1$.

Define the topological Λ-algebra

$$R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} := \varprojlim_{I \in \mathcal{I}} B/I$$

The inverse limit is taken in the category of topological Λ-algebras. Let $\pi_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}} : R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to \kappa$ be the map induced by the identification $B/\ker(\psi) \cong \kappa$ and let $\mathfrak{m}_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}} := \ker(\pi_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}})$. Each B/I is a local ring with residue field κ, so an element of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is invertible if and only if its reduction to κ is. This shows, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is local with maximal ideal $\mathfrak{m}_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}}$.

If κ is finite, then each B/I is a finite p-group and $R^{\mathrm{ps}}_{\overline{\Theta}}$ is pro-p and in particular complete.

We show, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ pro-represents $\mathrm{Def}_{\overline{\Theta}}$ and that $\iota_* \Theta^u \in \mathrm{Def}_{\overline{\Theta}}(R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}})$ is the universal deformation of $\overline{\Theta}$, where $\iota : B \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is the canonical map. Assume for the proof, that $\mathrm{Def}_{\overline{\Theta}}$ is defined on the category of local topological Λ-algebras with residue field κ. This way we get uniqueness of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ once we show representability. By Lemma 6.3 we have an isomorphism

$$\mathrm{Def}_{\overline{\Theta}}(R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}) \cong \varprojlim_{I\in\mathcal{I}} \mathrm{Def}_{\overline{\Theta}}(B/I)$$

so it suffices to show representability for artinian rings.

If $A \in \mathfrak{A}_\Lambda$ and $\Theta \in \mathrm{Def}_{\overline{\Theta}}(A)$, then Θ corresponds to a unique homomorphism $\phi : B \to A$, such that $\phi_*\Theta^u = \Theta$ and $\phi \mod \mathfrak{m}_A = \psi$. We will show, that $\ker(\phi) \in \mathcal{I}$. We have $\ker(\phi) \subset \ker(\psi) = \mathfrak{m}$ and $B/\ker(\phi) \subseteq A$ is artinian local. We have to show, that $\pi^{\ker(\phi)}_*\Theta^u$ is continuous. Indeed $\overline{\phi}_*\pi^{\ker(\phi)}_*\Theta^u = \phi_*\Theta^u = \Theta$ is continuous, where $\overline{\phi} : B/\ker(\phi) \to A$ is the map induced by ϕ. Since $\overline{\phi}$ is a topological embedding $\pi^{\ker(\phi)}_*\Theta^u$ is continuous. So there is a unique factorization $B \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to B/\ker(\phi) \to A$ of ϕ over a continuous map $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to A$.

For the converse suppose, that $\varphi : R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to A$ is a continuous local Λ-homomorphism compatible with the projections to κ. We have to show, that the pseudocharacter $\varphi_*\iota_*\Theta^u$ is continuous. It is enough to show, that the universal deformation $\iota_*\Theta^u$ is continuous. Let $\tilde{\pi}^I : R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to B/I$ for $I \in \mathcal{I}$ be the projection map from the definition of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ as an inverse limit. The pseudocharacters $\tilde{\pi}^I_*\iota_*\Theta^u = \pi^I_*\Theta^u$ are continuous by definition of $\mathcal{I}$. For fixed $m \geq 1$ and $f \in \Lambda[G^m]^{G^0}$ the map $(\iota_*\Theta^u)_m(f) : \Gamma^m \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ will be continuous by the universal property of limits. □

Corollary 6.5. Let Γ be a profinite group, let κ be a finite or local field and let Λ be a coefficient ring for κ. Let $\overline{\Theta} \in \mathrm{cPC}^\Gamma_{\mathrm{GL}_d}(\kappa)$ and let $D_{\overline{\Theta}}$ be the determinant law attached to $\overline{\Theta}$ by Theorem 4.57. Then the natural transformation $\mathrm{Def}_{\Lambda,\overline{\Theta}} \to \mathrm{Def}_{\Lambda,D_{\overline{\Theta}}}$ defined as in Proposition 4.58 is a natural bijection. In particular there is a canonical isomorphism $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \cong R^{\mathrm{ps}}_{\Lambda,D_{\overline{\Theta}}}$ of universal pseudodeformation rings.

Proof. This follows from Proposition 4.58 and Theorem 6.4. □

Now that we have proved existence of universal pseudodeformation rings, we observe, that certain completed local rings at dimension 1 points x are pseudodeformation rings for a deformation problem with residue field $\kappa(x)$. It is for this reason, that we also treat cases (2) and (3) from the beginning of this section.

Proposition 6.6. Let Γ be a profinite group. Let κ be a finite field and let Λ be a coefficient ring for κ. Let $\overline{\Theta} \in \mathrm{cPC}^\Gamma_G(\kappa)$ and let $x \in \mathrm{Spec}(R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}})$ be a dimension 1 point and residue field $\kappa(x)$. Assume, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian. By [BIP21, Lemma 3.16] $\kappa(x)$ is a local field and the induced pseudocharacter $\Theta_x \in \mathrm{cPC}^\Gamma_G(\kappa(x))$ is continuous. Let $\mathfrak{p} := \ker(\kappa(x) \otimes_\Lambda R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to \kappa(x))$. Then the following two rings are canonically isomorphic:

1. The universal pseudodeformation ring $R^{\mathrm{ps}}_{\Theta_x}$.
2. The completion of $\kappa(x) \otimes_\Lambda R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ at $\mathfrak{p}$.

The isomorphism is given by the induced map $\kappa(x) \otimes_\Lambda R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to R^{\mathrm{ps}}_{\Theta_x}$.

Proof. The proof of [BJ19, Corollary 4.8.7] goes through in our setting. □

6.1.3 Noetherianity for topologically finitely generated profinite groups

Lemma 6.7. Let Γ be a topological group, $\Delta \subseteq \Gamma$ a dense subgroup, $\mathcal{O}$ a commutative ring and G a generalized reductive $\mathcal{O}$-group scheme. Then for all Hausdorff $\mathcal{O}$-algebras A the restriction

$$\mathrm{cPC}^\Gamma_G(A) \to \mathrm{cPC}^\Delta_G(A)$$

defined by composition with $\mathcal{C}(\Gamma^n, A) \to \mathcal{C}(\Delta^n, A)$ is injective.

This is a generalization of the density argument in [Che14, Ex. 2.31].

Proof. Let $\Theta, \Theta' \in \mathrm{cPC}^{\Gamma}_G(A)$ be such that $\Theta|_{\Delta} = \Theta'|_{\Delta}$. Let $n \geq 0$ and $f \in \mathcal{O}[G^n]^{G^0}$. Then $\Theta_n(f), \Theta'_n(f) : \Gamma^n \to A$ are continuous maps, that agree on the dense subset $\Delta^n \subseteq \Gamma^n$, hence must be equal. $\square$

Lemma 6.8. Let Γ be a group, let G and H be generalized reductive group schemes over a commutative ring $\mathcal{O}$ and let $\iota : G \to H$ be a homomorphism of $\mathcal{O}$-group schemes, such that the induced map of $\mathcal{F}$-$\mathcal{O}$-algebras $\mathcal{O}[H^\bullet]^{H^0} \to \mathcal{O}[G^\bullet]^{G^0}$ is surjective. Let A be a commutative $\mathcal{O}$-algebra and let $\Theta \in \mathrm{PC}^{\Gamma}_G(A)$. Then $\ker(\Theta) = \ker(\iota(\Theta))$.

Proof. By inspection of the Definition 4.26 of kernel. $\square$

Examples, that satisfy the hypotheses of Lemma 6.8 can be obtained from Corollary 5.14.

Proposition 6.9. Let Λ be the ring of integers of a p-adic local field with residue field κ. Let A be a pro-p local Λ-algebra with residue field κ. The following are equivalent:

1. A is noetherian.
2. $\mathfrak{m}_A$ is a finitely generated ideal.
3. $\mathfrak{m}_A/\mathfrak{m}_A^2$ is a finite-dimensional κ-vector space.
4. $\mathfrak{m}_A/(\mathfrak{m}_\Lambda + \mathfrak{m}_A^2)$ is a finite-dimensional κ-vector space.

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ is clear. The proof of $4 \Rightarrow 1$ can be found in Hida's notes [Hid14, Lemma 2.10]. $\square$

Proposition 6.10. Assume, that Λ is the ring of integers of a p-adic local field with residue field κ. Let Γ be a group, let G be a generalized reductive Λ-group scheme and let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$. Then the following are equivalent:

1. $\dim_\kappa(\mathrm{Def}_{\overline{\Theta}}(\kappa[\varepsilon])) < \infty$.
2. $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is a noetherian ring.

Proof. Since $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ represents $\mathrm{Def}_{\overline{\Theta}}$ (Theorem 6.4), the relative tangent space $(\mathfrak{m}_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}}/(\mathfrak{m}_\Lambda + \mathfrak{m}^2_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}}))^*$ of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ over Λ identifies with $\mathrm{Def}_{\overline{\Theta}}(\kappa[\varepsilon])$. Since $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is pro-p, the claim follows from Proposition 6.9. $\square$

Theorem 6.11. Assume, that Λ is the ring of integers of a p-adic local field with residue field κ and that G is a generalized reductive Λ-group scheme. Let Γ be a topologically finitely generated profinite group and let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$. Then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian.

Proof. Let $\Delta \subseteq \Gamma$ be a dense and finitely generated subgroup of Γ. We have a sequence of injections

$$\mathrm{Def}_{\Lambda,\overline{\Theta}}(\kappa[\varepsilon]) \subseteq \mathrm{cPC}^{\Gamma}_G(\kappa[\varepsilon]) \overset{6.7}{\subseteq} \mathrm{cPC}^{\Delta}_G(\kappa[\varepsilon]) \subseteq \mathrm{PC}^{\Delta}_G(\kappa[\varepsilon]) \overset{4.46}{\cong} \mathrm{Hom}_\Lambda(B^{\Delta}_G, \kappa[\varepsilon])$$

By [Sta19, 032W] and [Sta19, 0334] Λ is universally Japanese. By Proposition 4.47, $\mathrm{Hom}_\Lambda(B^{\Delta}_G, \kappa[\varepsilon])$ is a finite-dimensional κ-vector space. By Proposition 6.10 we conclude, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian. $\square$

6.1.4 Noetherianity for profinite groups satisfying Φ_p

The idea of establishing noetherianity of the pseudodeformation rings $R^{\mathrm{ps}}_{\overline{\Theta}}$ for a classical group G in case we only know that our profinite group Γ satisfies Mazur's condition Φ_p is to prove surjectivity of the transition map $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\overline{\Theta}}$ for a suitable rational representation $\iota : G \to \mathrm{GL}_n$, and use noetherianity of $R^{\mathrm{ps}}_{\iota(\overline{\Theta})}$. In this section we give a criterion in terms of invariant theory, which can be applied to other reductive groups once their invariant theory is understood. We found the proofs of this section before the argument in Proposition 4.47 was found, which is of course general and sufficient for applications to absolute Galois groups of local fields.

Lemma 6.12. Let Γ be a group and let $\iota : G \to G'$ be a homomorphism of generalized reductive group schemes over a commutative ring $\mathcal{O}$. Suppose, that the map $\mathcal{O}[G'^{\bullet}]^{G'^0} \to \mathcal{O}[G^{\bullet}]^{G^0}$ of $\mathcal{F}$-$\mathcal{O}$-algebras is surjective. Then the map $\iota^* : B^{\Gamma}_{G'} \to B^{\Gamma}_{G}$ induced by ι is surjective.

Proof. By Theorem 4.46 it is enough to show, that for each $m \geq 1$, each $\mu \in \mathcal{O}[G^m]^{G^0}$ and each $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma^m$, the element $t_{\mu,\gamma} \in B^{\Gamma}_G$ has a preimage in $B^{\Gamma}_{G'}$. By surjectivity of $\mathcal{O}[G'^m]^{G'^0} \to \mathcal{O}[G^m]^{G^0}$, we find some $\mu' \in \mathcal{O}[G'^m]^{G'^0}$ mapping to μ. We claim, that $\iota^*(t_{\mu',\gamma}) = t_{\mu,\gamma}$. Let $A \in \mathrm{CAlg}_{\mathcal{O}}$, $\Theta \in \mathrm{PC}^{\Gamma}_G(A)$ and $f_\Theta : B^{\Gamma}_G \to A$ the homomorphism attached to Θ. Let $f_{\iota(\Theta)} : B^{\Gamma}_{G'} \to A$ be the homomorphism attached to $\iota(\Theta)$. By definition $f_\Theta(\iota^*(t_{\mu',\gamma})) = f_{\iota(\Theta)}(t_{\mu',\gamma}) = \iota(\Theta)_m(\mu')(\gamma) = \Theta_m(\gamma)$. Since this characterizes $\iota^*(t_{\mu',\gamma})$ uniquely, we have $\iota^*(t_{\mu',\gamma}) = t_{\mu,\gamma}$. □

Lemma 6.13. Let Γ be a profinite group. Let G and G' be generalized reductive group schemes over a coefficient ring Λ with finite residue field κ. Let $\iota : G \to G'$ be a homomorphism of Λ-group schemes. Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$ be a continuous pseudocharacter and we denote by $\iota(\overline{\Theta})$ its image in $\mathrm{cPC}^{\Gamma}_{G'}(\kappa)$. Assume, that the homomorphism $B^{\Gamma}_{G'}/p \to B^{\Gamma}_G/p$ is surjective. Then the natural homomorphism $R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is surjective.

Proof. Let $B := B^{\Gamma}_G/p$, $B' := B^{\Gamma}_{G'}/p$, $R := R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}/p$, $R' := R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})}/p$ and let $J := \ker(B' \to B)$. By Nakayama's lemma it is enough to show, that the natural map $j : R' = \varprojlim_{I' \in \mathcal{I}'} B'/I' \to \varprojlim_{I \in \mathcal{I}} B/I = R$ induced by ι is surjective. Here the ideals $\mathcal{I}$ and $\mathcal{I}'$ are defined as in the proof of Theorem 6.4. To an ideal $I \in \mathcal{I}$, we attach the ideal $j^{-1}(I)$ and we claim, that $j^{-1}(I) \in \mathcal{I}'$ and this induces a well-defined map of cofiltered sets $j^{-1} : \mathcal{I} \to \mathcal{I}'$.

As in the proof of Theorem 6.4, let $\psi : B \to \kappa$ be the homomorphism attached to $\overline{\Theta}$. Then $\psi' := \psi \circ j$ is the homomorphism attached to $\iota(\overline{\Theta})$. Let $\mathfrak{m} := \ker(\psi)$ and $\mathfrak{m}' := \ker(\psi')$. We observe, that $\mathfrak{m}' = j^{-1}(\mathfrak{m})$ and thus $j^{-1}(I) \subseteq \mathfrak{m}'$ for all $I \in \mathcal{I}$. For the second property in the definition of $\mathcal{I}'$, we observe, that $B'/j^{-1}(I) \to B/I$ is injective, and surjectivity follows as $pB = 0$. So $B'/j^{-1}(I) \cong B/I$ is finite. Let $\Theta^u \in \mathrm{PC}^{\Gamma}_G(B)$ and $\Theta^{u'} \in \mathrm{PC}^{\Gamma}_{G'}(B')$ be the universal pseudocharacters mod p. The pseudocharacter $\pi^{j^{-1}(I)}_* \Theta^{u'} = \iota(\pi^I_* \Theta^u)$ is continuous as the image of a continuous pseudocharacter.

Next, we claim, that the map $\mathcal{I}' \to \mathcal{I}$, $I' \mapsto j(I' + J)$ is surjective. Indeed, if $I \in \mathcal{I}$, we have just shown, that $j^{-1}(I) \in \mathcal{I}'$ and $j(j^{-1}(I) + J) = j(j^{-1}(I)) = I$. We therefore obtain an isomorphism $R \cong \varprojlim_{I' \in \mathcal{I}'} B'/(I' + J)$ and the map between deformation rings is now a naturally induced map between limits over $\mathcal{I}'$.

The image T of R' in R is compact, since R' is profinite. It is dense, since for all $I' \in \mathcal{I}'$, the map $B'/I' \to B'/(I' + J)$ is surjective. As an inverse limit of Hausdorff spaces R is Hausdorff and hence T is closed in R. It follows, that $T = R$. □

Theorem 6.14. Let $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$ over a coefficient ring Λ with finite residue field κ and assume $p > 2$ in the orthogonal cases. Let $\iota : G \to \mathrm{GL}_d$ be the standard representation of G. Let Γ be a profinite group and let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$ be a continuous pseudocharacter. Then the canonical map $R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is surjective. If in addition Γ satisfies Mazur's condition Φ_p, then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian.

Proof. We have shown in Corollary 5.14, that for $m \geq 1$ the natural maps $\mathcal{O}/p[\mathrm{GL}_d^m]^{\mathrm{GL}_n} \to \mathcal{O}/p[G^m]^{G^0}$ are surjective. It follows from Lemma 6.12, that the maps $B^{\Gamma}_{\mathrm{GL}_{d,\Lambda/p}} \to B^{\Gamma}_{G_{\Lambda/p}}$ are surjective. By Proposition 4.48, we have surjections $B^{\Gamma}_{\mathrm{GL}_d}/p \to B^{\Gamma}_G/p$. Hence we can apply Lemma 6.13 and see, that the map $R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is surjective. □

6.2 Comparing deformations and pseudodeformations

The main purpose of this section is to compare unframed deformation functors to pseudodeformation functors when the residue field of our deformation problem is a finite or a local field. We first prove a version of [BHKT, Theorem 4.10] extended to local residue fields.

Proposition 6.15. Let Γ be a profinite group. Let κ be a finite or local field, let Λ be a coefficient ring for κ and let G be a connected reductive Λ-group scheme. Let $\overline{\rho} : \Gamma \to G(\kappa)$ be a continuous representation and let $\overline{\Theta} \in \mathrm{cPC}_G^{\Gamma_F}(\kappa)$ be the associated pseudocharacter. Assume, that the centralizer of $\overline{\rho}$ is trivial in G^{ad} and that $\overline{\rho}$ is G-completely reducible. Then the natural map of deformation functors $\mathrm{Def}_{\Lambda,\overline{\rho}} \to \mathrm{Def}_{\Lambda,\overline{\Theta}}$ is an isomorphism.

Proof. Let $A \in \mathfrak{A}_\Lambda$ and $\Theta \in \mathrm{Def}_{\Lambda,\overline{\Theta}}(A)$. For any $n \geq 1$, we define affine Λ-schemes of finite type $X_n := G^n$ and $Y_n := G^n /\!\!/ G$ and let $\pi : X_n \to Y_n$ be the projection.

Now fix $n \geq 1$ and $\gamma_1, \dots, \gamma_n \in \Gamma$, such that the scheme-theoretic centralizer $Z_{G_\kappa}(x)$ of $x := (\overline{g}_1, \dots, \overline{g}_n)$ in G_κ coincides with the scheme-theoretic centralizer $Z_{G_\kappa}(\overline{\rho})$ of $\overline{\rho}$ in G_κ. This is possible, as $\kappa[G]$ is a noetherian ring. Thus the image of $Z_{G_\kappa}(x)$ in G_κ^{ad} is trivial by assumption. We may assume by [Mar03, Lemma 9.2], that the subgroup generated by $\overline{\rho}(\gamma_1), \dots, \overline{\rho}(\gamma_n)$ has the same Zariski closure as $\overline{\rho}(\Gamma)$, we denote this topological subgroup of $G(\kappa)$ by H. Since $\overline{\rho}$ is G_κ-completely reducible, by [BMR05, Proposition 2.16] the orbit of x in $X_{n,\kappa}$ is closed.

In [BHKT, Theorem 4.10], the completion of X_n at $x \in X_n(\kappa)$ is defined as the functor $X_n^{\wedge,x} : \mathfrak{A}_\Lambda \to \mathrm{Set}$ defined by $X_n^{\wedge,x}(A) := X_n(A) \times_{X_n(\kappa)} \{x\}$. Similarly, for fixed $h \in H$, we define the completion of X_{n+1} at $y := (x, h) \in X_{n+1}(\kappa)$ and the respective completions of Y_n and Y_{n+1} at $\pi(x)$ and $\pi(y)$. Let $G^{\mathrm{ad},\wedge}$ be the completion of G^{ad} at the neutral element. It is a group functor on $\mathfrak{A}_\Lambda$, representable by a formal Λ-scheme.

In analogy to the completion at a point, we define the completion of X_{n+1} at H as the functor $X_{n+1}^{\wedge,H} : \mathfrak{A}_\Lambda \to \mathrm{Set}$ by $X_{n+1}^{\wedge,H}(A) := X_{n+1}(A) \times_{X_{n+1}(\kappa)} H$, where the map $H \to X_{n+1}(\kappa)$ is given by $h \mapsto (g_1, \dots, g_n, h)$. Similarly we define $Y_{n+1}^{\wedge,H}(A) := Y_{n+1}(A) \times_{Y_{n+1}(\kappa)} H$. We will need these completions to prove continuity of the representation we construct. One can think of completions at H just as putting the completions at single points of H into a continuous family.

Θ_{n+1} determines a natural map $\Lambda[G^{n+1}]^G \to \mathcal{C}(\Gamma, A),\ f \mapsto (\gamma \mapsto \Theta_{n+1}(f)(\gamma_1, \dots, \gamma_n, \gamma))$, which is an element $\alpha \in Y_{n+1}(\mathcal{C}(\Gamma, A)) = \mathcal{C}(\Gamma, Y_{n+1}(A))$. Here $Y_{n+1}(A)$ is endowed with the discrete topology if κ is finite and with the subspace topology of some closed immersion into an affine space over A equipped with the product topology as a κ-vector space.

By the universal property of pullbacks and compatibility of the topologies we have defined on point sets in Section 4.1.5, we obtain a unique continuous map $\beta : \Gamma \to Y_{n+1}^{\wedge,H}(A)$ as indicated in the diagram:

$$\begin{tikzcd}
\Gamma \arrow[r, dotted, "\beta"] \arrow[rr, bend left, "\alpha"] \arrow[dr, "\overline{\rho}"'] & Y_{n+1}^{\wedge,H}(A) \arrow[r] \arrow[d] & Y_{n+1}(A) \arrow[d] \\
 & H \arrow[r] & Y_{n+1}(\kappa)
\end{tikzcd}$$

The proof of [BHKT, Proposition 3.13] goes through verbatim in our setting. Hence $G^{\mathrm{ad},\wedge}$ acts freely on $X_n^{\wedge,x}$ and the projection $X_n^{\wedge,x} \to Y_n^{\wedge,\pi(x)}$ factors through an isomorphism $X_n^{\wedge,x}/G^{\mathrm{ad},\wedge} \to Y_n^{\wedge,\pi(x)}$. In particular $X_n^{\wedge,x}(A) \to Y_n^{\wedge,\pi(x)}(A)$ is surjective and we can choose a preimage $(g_1, \dots, g_n) \in X_n^{\wedge,x}(A)$ of the point in $Y_n^{\wedge,\pi(x)}(A)$ determined by $\Lambda[G^n]^G \to A,\ f \mapsto \Theta_n(f)(\gamma_1, \dots, \gamma_n)$.

For fixed $h \in H$ and $y := (x, h)$, we have two cartesian squares:

$$\begin{tikzcd}
X_{n+1}^{\wedge,y}(A) \arrow[r] \arrow[d] & Y_{n+1}^{\wedge,\pi(y)}(A) \arrow[r] \arrow[d] & \{h\} \arrow[d] \\
X_{n+1}^{\wedge,H}(A) \arrow[r] & Y_{n+1}^{\wedge,H}(A) \arrow[r] & H
\end{tikzcd}$$

As in the proof of [BHKT, Theorem 4.10], the top left arrow is a $G^{\mathrm{ad},\wedge}(A)$-torsor of sets, so $X_{n+1}^{\wedge,y} \to Y_{n+1}^{\wedge,\pi(y)}$ is a $G^{\mathrm{ad},\wedge}$-pseudotorsor. It follows, that $X_{n+1}^{\wedge,H} \to Y_{n+1}^{\wedge,H}$ is a $G^{\mathrm{ad},\wedge}$-pseudotorsor. The square in the following diagram is cartesian, since the horizontal arrows are $G^{\mathrm{ad},\wedge}$-pseudotorsors and the vertical maps are equivariant:

$$\begin{tikzcd}
\Gamma \arrow[r, dotted] \arrow[rr, bend left, "\beta"] \arrow[dr] & X_{n+1}^{\wedge,H}(A) \arrow[r] \arrow[d] & Y_{n+1}^{\wedge,H}(A) \arrow[d] \\
& X_n^{\wedge,x}(A) \arrow[r] & Y_n^{\wedge,\pi(x)}(A)
\end{tikzcd}$$

The map $\Gamma \to X_n^{\wedge,x}(A)$ maps constantly to $(g_1,\dots,g_n)$. By the discussion of the topologies on point sets in Section 4.1.5, the diagram is also cartesian in the category of topological spaces. Again by the universal property, we obtain a continuous map $\Gamma \to X_{n+1}^{\wedge,G(\kappa)}(A)$.

The composition $\Gamma \to X_{n+1}^{\wedge,G(\kappa)}(A) \to X_{n+1}(A) \overset{\mathrm{pr}_{n+1}}{\to} G(A)$ defines the desired ρ with $\Theta_\rho = \Theta$ as in [BHKT, Theorem 4.10]. The second map is continuous by definition of the completion of $X_{n+1}^{\wedge,G(\kappa)}$ as a pullback. The projection pr_{n+1} is continuous by definition of the topologies on point sets Proposition 4.12. So the composition ρ is continuous and this finishes the proof. □

We can now prove a continuous version of Theorem 4.24 for certain residual representations, which will be enough for the proof of Proposition 6.32.

Proposition 6.16. Let Γ be a profinite group, let κ be a finite or a local field, let Λ be a coefficient ring for κ and let G be a Chevalley group over Λ. Suppose $\Theta \in \mathrm{cPC}_G^\Gamma(\mathcal{O}_\kappa)$ is a continuous pseudocharacter, where $\mathcal{O}_\kappa = \kappa$ if κ is finite. If κ is a local field of positive characteristic, assume that Γ is topologically finitely generated and that the reduction $\overline{\Theta}$ of Θ to the residue field k of κ comes from a G-completely reducible representation $\overline{\rho} : \Gamma \to G(k')$ for some finite extension k'/k, which has scheme-theoretically trivial centralizer in $G_{k'}^{\mathrm{ad}}$. Then there exists a continuous (G-completely reducible) representation $\rho : \Gamma \to G(\overline{\kappa})$ with $\Theta_\rho = \Theta$, which is defined over the ring of integers $\mathcal{O}_{\kappa'}$ of a finite extension κ'/κ.

Proof. Suppose κ is finite. Then Theorem 4.24 provides us with a G-completely reducible representation $\rho : \Gamma \to G(\overline{\kappa})$, such that $\Theta_\rho = \Theta$. By [BHKT, Proposition 4.7 (iii)], ρ is continuous. Since Γ is profinite, $\rho(\Gamma)$ is finite. In particular there exists a finite extension κ', such that $\rho(\Gamma) \subseteq G(\kappa')$.

If κ is a local field of characteristic 0, we can argue the same way using [BHKT, Proposition 4.7 (ii)] and [BHKT, Proposition 4.8 (ii)].

Assume κ is a local field of positive characteristic. Let k be the residue field of $\mathcal{O}_\kappa$. By the first step the reduction $\overline{\Theta}$ of Θ to k comes from a continuous G-completely reducible representation $\overline{\rho} : \Gamma \to G(k')$ over a finite extension k'/k, which by our assumption has scheme-theoretically trivial centralizer in $G_{k'}^{\mathrm{ad}}$. Choose a finite extension κ'/κ, such that the residue field of $\mathcal{O}_{\kappa'}$ is k'. So $\Theta \otimes_{\mathcal{O}_\kappa} \mathcal{O}_{\kappa'}$ is a deformation of $\overline{\Theta} \otimes_k k'$. By Proposition 6.15 $\Theta \otimes_{\mathcal{O}_\kappa} \mathcal{O}_{\kappa'}$ thus comes from a continuous deformation $\rho : \Gamma \to G(\mathcal{O}_{\kappa'})$ of $\overline{\rho}$. □

Definition 6.17. We say, that a prime p is *very good* for a simple algebraic group G over an algebraically closed field, if the following conditions hold.

1. $p \nmid n+1$, if G is of type A_n.
2. $p \neq 2$, if G is of type B, C, D, E, F, G.
3. $p \neq 3$, if G is of type E, F, G.
4. $p \neq 5$, if G is of type E_8.

We say, that p is *very good* for a reductive algebraic group G, if it is very good for every simple factor of G^0.

Lemma 6.18. Let Γ be a group. Let $G \subseteq \mathrm{GL}_n$ be a reductive group over an algebraically closed field k of characteristic $p \geq 0$ and let $\rho : \Gamma \to G(k)$ be a G-completely reducible representation, which is in addition irreducible after embedding into $\mathrm{GL}_n(k)$.

Assume, that one of the following holds:

1. p is very good for G^{ad} and G^{ad} is connected.
2. (GL_n, G) is a reductive pair, i.e. $\mathfrak{g}$ is a G-module direct summand of $\mathfrak{gl}_n$.

Then the scheme-theoretic centralizer $Z_{G^{\mathrm{ad}}}(\rho)$ of ρ in G^{ad} is trivial.

Proof. Beware, that $Z_{G^{\mathrm{ad}}}(\rho)$ is defined as follows. For $A \in \mathrm{CAlg}_k$, the group $Z_{G^{\mathrm{ad}}}(\rho)(A)$ is defined as the kernel of the map

$$G^{\mathrm{ad}}(A) \to \mathrm{Hom}(\Gamma, G(A)), \quad g \mapsto g\rho g^{-1}$$

By Schur's lemma $Z_{\mathrm{GL}_n}(\rho)(k) = Z(\mathrm{GL}_n)(k)$. Let $\pi : G \twoheadrightarrow G^{\mathrm{ad}}$ be the canonical projection. By definition, $Z_G(\rho) = \pi^{-1}(Z_{G^{\mathrm{ad}}}(\rho))$ and $Z_G(\rho) = Z_{\mathrm{GL}_n}(\rho) \cap G$. We get $Z(G)(k) \subseteq \pi^{-1}(Z_{G^{\mathrm{ad}}}(\rho)(k)) = Z_G(\rho)(k) = Z_{\mathrm{GL}_n}(\rho)(k) \cap G(k) \subseteq Z(G)(k)$. We conclude, that $Z_{G^{\mathrm{ad}}}(\rho)(k)$ is trivial.

Assuming (1), we see by [Bat+07, Theorem 1.2] since p is very good for G^{ad} and G^{ad} is connected, that $Z_{G^{\mathrm{ad}}}(\rho)$ is smooth and thus trivial as an algebraic group.

Assuming (2), we obtain from [Bat+07, Corollary 2.13], that $Z_G(\rho)$ is smooth. Since GL_n is separable, $Z_{\mathrm{GL}_n}(\rho)$ is also smooth and we have $Z_{\mathrm{GL}_n}(\rho) = Z(\mathrm{GL}_n)$. We can repeat the above calculation without taking points:

$$Z(G) \subseteq \pi^{-1}(Z_{G^{\mathrm{ad}}}) = Z_G(\rho) = Z_{\mathrm{GL}_n}(\rho) \cap G = Z(\mathrm{GL}_n) \cap G \subseteq Z(G)$$

Hence $Z_{G^{\mathrm{ad}}} = 1$. □

Proposition 6.19. Let $\overline{\rho} : \Gamma_F \to G(\kappa)$ be a continuous representation over a finite or local field κ and let Λ be a coefficient ring for κ. Assume, that the unframed deformation functor is representable by $R_{\overline{\rho}}$. We have a presentation $R_{\overline{\rho}} \cong \Lambda[[x_1, \dots, x_r]]/(f_1, \dots, f_s)$, where $r = h^1(\Gamma_F, \mathrm{ad}_{\overline{\rho}})$ and $s = h^2(\Gamma_F, \mathrm{ad}_{\overline{\rho}})$.

Proof. This follows from a standard calculation with cocycles. See e.g. [Til96]. □

Proposition 6.20. Let F be a p-adic local field with absolute Galois group Γ_F. Let κ be a finite or local field of very good characteristic $p \geq 0$ for $G^{\mathrm{ad}}_{\overline{\kappa}}$, let Λ be a coefficient ring for κ and let $G \subseteq \mathrm{GL}_n$ be a Chevalley group over Λ. Let $\overline{\rho} : \Gamma_F \to G(\kappa)$ be an absolutely G-completely reducible continuous representation with associated G-pseudocharacter $\overline{\Theta} \in \mathrm{cPC}^{\Gamma_F}_G(\kappa)$, such that $\overline{\rho}$ is absolutely irreducible after embedding into $\mathrm{GL}_n(\overline{\kappa})$ and such that $H^2(\Gamma_F, \mathfrak{g}_\kappa) = 0$.

Assume, that one of the following holds:

1. p is very good for $G^{\mathrm{ad}}_{\overline{\kappa}}$ and $G^{\mathrm{ad}}_{\overline{\kappa}}$ is connected.
2. $(\mathrm{GL}_{n,\overline{\kappa}}, G_{\overline{\kappa}})$ is a reductive pair, i.e. $\mathfrak{g}_{\overline{\kappa}}$ is a $G_{\overline{\kappa}}$-module direct summand of $\mathfrak{gl}_{n,\overline{\kappa}}$.

Then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is formally smooth over Λ of dimension $\dim \mathfrak{g}_\kappa \cdot [F : \mathbb{Q}_p] + h^0(\Gamma_F, \mathfrak{g}_\kappa) + \dim \Lambda$. In particular $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \cong \Lambda[[x_1, \dots, x_r]]$.

Proof. By Lemma 6.18 the scheme-theoretic centralizer of $\overline{\rho}$ in $G^{\mathrm{ad}}_{\overline{\kappa}}$ is trivial. We can apply Proposition 6.15 to obtain a canonical isomorphism $R_{\overline{\rho}} \cong R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$. By Proposition 6.19, the deformation ring $R_{\overline{\rho}}$ is isomorphic to $\Lambda[[x_1, \dots, x_r]]$, where $r = h^1(\Gamma_F, \mathfrak{g}_\kappa)$. The Euler characteristic formula [BJ19, Theorem 3.4.1] implies, that $\dim R_{\overline{\rho}} = \dim \mathfrak{g}_\kappa \cdot [F : \mathbb{Q}_p] + h^0(\Gamma_F, \mathfrak{g}_\kappa) + \dim \Lambda$. □

6.3 Dimension of $R^{\mathrm{ps}}_{\overline{\Theta}}$

Let $\mathcal{O}_L$ be the ring of integers of a p-adic field L with uniformizer ϖ and residue field κ, let G be a Chevalley group over $\mathcal{O}_L$ and let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$ be a continuous G-pseudocharacter. Let $\overline{X}_{\overline{\Theta}} := \mathrm{Spec}(R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}/\varpi)$, where $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is the universal pseudodeformation ring of $\overline{\Theta}$ with coefficients $\mathcal{O}$ from Theorem 6.4. We define

$$\mathrm{Sp}_{2n}(A) := \{M \in \mathrm{GL}_{2n}(A) \mid M^{-1} = JM^\top J^{-1}\},$$

where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ for every commutative ring A. In this section, we use the methods developed in [BJ19] to estimate the dimension of $\overline{X}_{\overline{\Theta}}$ for $G = \mathrm{Sp}_{2n}$ and $\Gamma = \Gamma_F$ the absolute Galois group of a local field

$F/\mathbb{Q}_p$. We assume throughout, that $p > 2$. Note, that by Theorem 6.11 $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is noetherian, since Γ_F is topologically finitely generated. In lack of reference for this fact, we refer to Chandan Singh Dalawat's answer to Mathoverflow Question # 63094. Let $\iota : \mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$ be the standard representation. By Proposition 4.58, the GL_{2n}-pseudocharacter $\iota(\overline{\Theta})$ corresponds to a unique determinant law $D_{\iota(\overline{\Theta})}$ of dimension $2n$. The pseudodeformation ring $R^{\mathrm{univ}}_{\mathcal{O}_L,D_{\iota(\overline{\Theta})}}$ of $D_{\iota(\overline{\Theta})}$ defined in [BJ19, Proposition 4.7.4] is by Corollary 6.5 canonically isomorphic to $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})}$. We shall use this identification without further mention whenever we cite results from [BJ19].

6.3.1 Symplectic representations

Definition 6.21. Let Γ be a group and let V be a representation of Γ on a finite-dimensional vector space over an algebraically closed field. We say, that V is *symplectic*, if there exists a non-degenerate antisymmetric Γ-invariant k-bilinear form $\beta : V \times V \to k$.

With Definition 6.21, being symplectic is a property of usual representations. We also know this under the name of *quaternionic representations*. Fixing a non-degenerate antisymmetric Γ-invariant $\beta : V \times V \to k$, a symplectic representation is a homomorphism $\Gamma \to \mathrm{Sp}(V,\beta)$, where $\mathrm{Sp}(V,\beta)$ is the subgroup of $\mathrm{GL}(V)$ consisting of endomorphisms $\Phi \in \mathrm{GL}(V)$ with $\beta(\Phi(x),\Phi(y)) = \beta(x,y)$ for all $x, y \in V$. The structure theory of bilinear forms [Jac85, Theorem 6.3] tells us that $\mathrm{Sp}(V,\beta)$ is isomorphic to the standard symplectic group $\mathrm{Sp}_{2n}(k)$, where $2n = \dim V$.

For semisimple symplectic representations we have the following structure theorem.

Proposition 6.22. Every semisimple symplectic representation of a group Γ over an algebraically closed field k is a direct sum of representations of one of the following two types.

1. An irreducible symplectic representation.
2. A direct sum $V \oplus V^*$, where V is an arbitrary irreducible representation.

Proof. Let V be a symplectic representation.

We proceed by induction over $\dim V$. If $\dim V = 0$ there is nothing to show. We assume $\dim V > 0$. Let W be an irreducible subrepresentation of V and assume, that $\beta : W \times W \to k$ is non-degenerate. In particular W is an irreducible symplectic representation. Then $W^\perp$ is non-degenerate and Γ-invariant and we may assume $W^\perp$ has the desired form. This implies the claim.

We now assume, that V has no irreducible subrepresentation on which β is non-degenerate. Let W be any irreducible subrepresentation of V. Since β is non-degenerate, there is an irreducible subrepresentation $W' \neq W$, such that $\beta : W \times W' \to k$ is non-degenerate. β is non-degenerate on $W \oplus W'$, so $(W \oplus W')^\perp$ is non-degenerate and Γ-invariant. As in the previous case, this implies the claim. □

This motivates the following terminology. We say that a symplectic representation V is *symplectically decomposable*, if it can be written as the direct sum of two nonzero symplectic representations, and *symplectically indecomposable* otherwise. There are exactly two types of symplectically indecomposable representations: Those which are irreducible under the standard embedding into GL_{2n} and those which are a direct sum $W \oplus W^*$ for some irreducible representation W.

When $p > 2$, two semisimple symplectic representations over an algebraically closed field are conjugate over Sp_{2n} if and only if they are conjugate over GL_{2n}. This is a consequence of the fact, that when $p > 2$ the notions of Sp_{2n}-semisimplicity and GL_{2n}-semisimplicity coincide [Ric88, Corollary 16.10] and the uniqueness part of Theorem 4.24. So being symplectic can be seen as a property of GL_{2n}-conjugacy classes of semisimple representations. It is easy to check, that a representation of the form $W \oplus W^*$ for some arbitrary representation W is always symplectic. We call these *representations of pair type*. In general a semisimple symplectic representation is a direct sum of irreducible symplectic representations and representations of pair type.

It can actually deduced from the theory of Lafforgue's pseudocharacters and the first fundamental theorems of invariant theory for the general linear and symplectic groups, that semisimple symplectic representations over an algebraically closed field k are conjugate over $\mathrm{GL}_{2n}(k)$ if and only if they are conjugate by a symplectic matrix. The proof we give requires technique developed in the next section, but it is a surprisingly basic application of G-pseudocharacters.

Proposition 6.23. Let $\rho_1, \rho_2 : \Gamma \to \mathrm{Sp}_{2n}(k)$ be Sp_{2n}-completely reducible representations over an algebraically closed field k. Assume, that there is a matrix $M \in \mathrm{GL}_{2n}(k)$, such that $\rho_1 = M\rho_2 M^{-1}$. Then there is a matrix $N \in \mathrm{Sp}_{2n}(k)$, such that $\rho_1 = N\rho_2 N^{-1}$.

Proof. Let us for simplicity of the argument assume that the algebraic groups GL_{2n} and Sp_{2n} are defined over k. The equation $\rho_1 = M\rho_2 M^{-1}$ implies, that the associated GL_{2n}-pseudocharacters $\Theta_{\rho_1}, \Theta_{\rho_2} : k[\mathrm{GL}_{2n}^{\bullet}]^{\mathrm{GL}_{2n}} \to \mathrm{Map}(\Gamma^{\bullet}, k)$ are equal via $\mathcal{F}$-k-algebras (Lemma 4.21). We are using the characterization of G-pseudocharacters of Corollary 4.45. At the same time, the associated Sp_{2n}-pseudocharacters $\Theta'_{\rho_1}, \Theta'_{\rho_2} : k[\mathrm{Sp}_{2n}^{\bullet}]^{\mathrm{Sp}_{2n}} \to \mathrm{Map}(\Gamma^{\bullet}, k)$ are mapped to $\Theta_{\rho_1}, \Theta_{\rho_2}$ under the standard representation $\mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$. By the first fundamental theorems for GL_{2n} [Don92] and Sp_{2n} [Zub99], the homomorphism of $\mathcal{F}$-k-algebras $k[\mathrm{GL}_{2n}^{\bullet}]^{\mathrm{GL}_{2n}} \to k[\mathrm{Sp}_{2n}^{\bullet}]^{\mathrm{Sp}_{2n}}$ induced by the standard representation is (objectwise) surjective, hence $\Theta'_{\rho_1} = \Theta'_{\rho_2}$. From the reconstruction theorem Theorem 4.56 and Sp_{2n}-complete reducibility it follows, that ρ_1 and ρ_2 are conjugate by a matrix $N \in \mathrm{Sp}_{2n}(k)$. $\square$

6.3.2 Subdivision of $\overline{X}_{\overline{\Theta}}$

For a point $x \in \overline{X}_{\overline{\Theta}}$, there is a natural G-pseudocharacter $\Theta_x \in \mathrm{PC}_G^{\Gamma}(\overline{\kappa(x)})$ defined after choice of an algebraic closure $\overline{\kappa(x)}$ of the residue field $\kappa(x)$ of x. Let P be a property of G-completely reducible representations over an algebraically closed field, which is stable under G-conjugation and passage to algebraically closed sub- and overfields. We say x has property P, if the G-completely reducible representation attached to Θ_x by Theorem 4.24 has property P. If Q is a property of representations into GL_{2n}, we say that a representation ρ into Sp_{2n} has property Q, if ρ followed by the standard representation $\iota : \mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$ has property Q.

In their analysis [BJ19] of the special fiber of the pseudodeformation space for GL_n, Böckle and Juschka have noticed that irreducible points need not be unobstructed. They have found a convenient characterization of obstructed irreducible points [BJ19, Lemma 5.1.1], which allows them to find good dimension bounds for the obstructed locus. We recall their definition [BJ19, Definition 5.1.2]. It turns out, that for $G = \mathrm{Sp}_{2n}$ the dimension of the locus of special points for GL_{2n} is still small enough to get the desired estimates.

Definition 6.24. Let k be an algebraically closed $\mathbb{Z}_p$-field and let $\rho : \Gamma_F \to \mathrm{GL}_{2n}(k)$ be an irreducible representation. We say, that ρ is *special*, if one of the following holds.

1. $\zeta_p \notin F$ and $\rho \cong \rho(1)$.
2. $\zeta_p \in F$ and there is some degree p Galois extension F'/F, such that $\rho|_{\Gamma_{F'}}$ is reducible.

Definition 6.25. We define the following subsets of $\overline{X}_{\overline{\Theta}}$.

1. $\overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}}$ is the subset of non-special points.
2. $\overline{X}_{\overline{\Theta}}^{\mathrm{spcl}}$ is the subset of special points.
3. $\overline{X}_{\overline{\Theta}}^{\mathrm{pair}}$ is the subset of points of pair type.
4. $\overline{X}_{\overline{\Theta}}^{\mathrm{dec}}$ is the subset of symplectically decomposable points.
5. For any of the above subsets $\dot{\overline{X}}_{\overline{\Theta}}^{?} := \overline{X}_{\overline{\Theta}}^{?} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}_{\mathcal{O},\overline{\Theta}}}\}$.

Proposition 6.26. $\overline{X}_{\overline{\Theta}} = \overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}} \mathbin{\dot\cup} \overline{X}_{\overline{\Theta}}^{\mathrm{spcl}} \mathbin{\dot\cup} (\overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{pair}})$.

Proof. This follows directly from Proposition 6.22. $\square$

Lemma 6.27. Suppose $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_2 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2n}}(\kappa)$ with $\overline{\Theta}_1 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2a}}(\kappa)$, $\overline{\Theta}_2 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2b}}(\kappa)$ and $a+b=n$, where the direct sum is a direct sum of symplectic pseudocharacters as explained in Section 4.2.3. Then the map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\overline{\Theta}_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\overline{\Theta}_2}$ induced by $(\Theta_1, \Theta_2) \mapsto \Theta_1 \oplus \Theta_2$ is finite.

Proof. Let $\iota : \mathrm{Sp}_{2d} \to \mathrm{GL}_{2d}$ be the canonical embedding and let $\iota(\overline{\Theta})$ be the associated GL_{2d}-pseudocharacter, similarly for $\iota_i(\overline{\Theta}_i)$. By Lemma 6.13 $R^{\mathrm{ps}}_{\overline{\Theta}}$ is a quotient of $R^{\mathrm{ps}}_{\iota(\overline{\Theta})}$ and similarly for $R^{\mathrm{ps}}_{\overline{\Theta}_i}$. We know from [BIP21, Lemma 3.23], that the map $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)}$ is finite. It follows, that the induced map $R^{\mathrm{ps}}_{\overline{\Theta}} \to (R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)}) \widehat{\otimes}_{R^{\mathrm{ps}}_{\iota(\overline{\Theta})}} R^{\mathrm{ps}}_{\overline{\Theta}}$ is finite. Since there is a natural surjection $R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)} \twoheadrightarrow R^{\mathrm{ps}}_{\overline{\Theta}_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\overline{\Theta}_2}$, the natural map $(R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)}) \widehat{\otimes}_{R^{\mathrm{ps}}_{\iota(\overline{\Theta})}} R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\overline{\Theta}_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\overline{\Theta}_2}$ is surjective. ☐

Lemma 6.28. Let $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_1^* \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2n}}(\kappa)$ be a symplectic representation as explained at the end of Section 4.2.3 with $\overline{\Theta}_1 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{GL}_n}(\kappa)$. Then the map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\overline{\Theta}_1}$ induced by $\Theta_1 \mapsto \Theta_1 \oplus \Theta_1^*$ is finite.

Proof. As in the proof of Lemma 6.27, the map $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)}$ is finite. By affineness, the map $R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)}$ induced by $\Theta_1 \mapsto (\Theta_1, \Theta_1^*)$ is surjective. So the composition $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)}$ is finite and induced by $\Theta_1 \mapsto \Theta_1 \oplus \Theta_1^*$. Tensoring with $R^{\mathrm{ps}}_{\overline{\Theta}}$, we obtain a finite map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{R^{\mathrm{ps}}_{\iota(\overline{\Theta})}} R^{\mathrm{ps}}_{\overline{\Theta}} \cong R^{\mathrm{ps}}_{\overline{\Theta}_1}$. The last isomorphism can be seen to hold by considering the corresponding deformation functors. ☐

Proposition 6.29.

1. The natural map $\overline{X}_{\overline{\Theta}} \to \overline{X}_{\iota(\overline{\Theta})}$ is a closed immersion.
2. $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ is closed in $\overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$.
3. $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$ is closed in $\overline{X}_{\overline{\Theta}}$.
4. $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$ is closed in $\overline{X}_{\overline{\Theta}}$.
5. $\overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}}$ is open in $\overline{X}_{\overline{\Theta}}$.

Proof.

1. By Theorem 6.14, the map $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is surjective.
2. $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ is the preimage of $\overline{X}^{\mathrm{spcl}}_{\iota(\overline{\Theta})}$ under the closed immersion $\overline{X}_{\overline{\Theta}} \to \overline{X}_{\iota(\overline{\Theta})}$. Since $\overline{X}^{\mathrm{nspcl}}_{\iota(\overline{\Theta})}$ is open in $\overline{X}_{\iota(\overline{\Theta})}$ by [BJ19, Theorem 4.5.1 (ii)], $\overline{X}^{\mathrm{spcl}}_{\iota(\overline{\Theta})}$ is closed in $\overline{X}^{\mathrm{irr}}_{\iota(\overline{\Theta})}$ and the claim follows.
3. $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$ is the union of the images of finitely many maps as in Lemma 6.28.
4. $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$ is the union of the images of finitely many maps as in Lemma 6.27.
5. $\overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$ is open in $\overline{X}_{\overline{\Theta}}$, as the complement of $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \cup \overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$ (see Proposition 6.26). The subset $\overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}} \subseteq \overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$ is the complement of $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$, which is closed in $\overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$. Hence $\overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}}$ is open in an open subset of $\overline{X}_{\overline{\Theta}}$.

☐

Lemma 6.30. Let $\bar{f} : \kappa \to \kappa'$ be a homomorphism between either two finite or two local fields. Let $f : \Lambda \to \Lambda'$ be a local homomorphism of complete noetherian local rings with residue fields κ and κ' respectively and assume, that f reduces to $\bar{f}$ on residue fields. Let Γ be a profinite group and let G be an affine Λ-group scheme. Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$ and define $\overline{\Theta}' := \overline{\Theta} \otimes_\kappa \kappa'$. Then the natural map

$$R^{\mathrm{ps}}_{\Lambda',\overline{\Theta}'} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \widehat{\otimes}_\Lambda \Lambda'$$

induced by

$$\mathrm{Def}_{\Lambda,\overline{\Theta}}(A) \to \mathrm{Def}_{\Lambda',\overline{\Theta}'}(A \otimes_\Lambda \Lambda'), \quad \Theta \mapsto \Theta \otimes_\Lambda \Lambda'; \qquad A \in \mathfrak{A}_\Lambda$$

is an isomorphism.

Proof. The proof of [BJ19, Proposition 4.7.6] carries over in our setting. ☐

6.3.3 Dimension bounds for $G = \mathrm{Sp}_{2n}$

The following proposition is the analog of [BJ19, Lemma 5.1.6] for $G = \mathrm{Sp}_{2n}$.

Lemma 6.31. Let k be a field with $2 \in k^\times$. Then the symplectic Lie algebra $\mathfrak{sp}_{2n,k}$ is a direct summand of $\mathfrak{gl}_{2n,k}$ and of $\mathfrak{sl}_{2n,k}$ and the corresponding projection maps $\mathfrak{gl}_{2n,k} \twoheadrightarrow \mathfrak{sp}_{2n,k}$ and $\mathfrak{sl}_{2n,k} \twoheadrightarrow \mathfrak{sp}_{2n,k}$ are equivariant for the adjoint action of the symplectic group Sp_{2n}.

Proof. Recall, that $\mathfrak{sp}_{2n,k} = \{M \in \mathfrak{gl}_{2n,k} \mid JM^\top + MJ = 0\}$, where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Right multiplication with J is an isomorphism of k-vector spaces $-\cdot J : \mathfrak{gl}_{2n,k} \to M_{2n}(k)$ and identifies $\mathfrak{sp}_{2n,k}$ with the subspace of symmetric $2n \times 2n$ matrices. The symmetrization map $a : M_{2n}(k) \to M_{2n}(k),\ M \mapsto \frac{1}{2}(M + M^\top)$ shows, that symmetric matrices are a direct summand of $M_{2n}(k)$. The map $\mathfrak{gl}_{2n}(k) \to \mathfrak{gl}_{2n}(k),\ M \mapsto a(MJ)J^{-1}$ is equivariant for the adjoint action of Sp_{2n} on $\mathfrak{gl}_{2n}(k)$: Suppose $M \in M_{2n}(k)$ and $A \in \mathrm{Sp}_{2n}(k)$: Then

$$a(AMA^{-1}J)J^{-1} = \frac{1}{2}(AMA^{-1} + J^{-1}(A^{-1})^\top M^\top A^\top J^{-1})$$

and

$$Aa(MJ)J^{-1}A^{-1} = \frac{1}{2}(AMA^{-1} + AJ^{-1}M^\top J^{-1}A^{-1}) = \frac{1}{2}(AMA^{-1} + J^{-1}(A^{-1})^\top M^\top A^\top J^{-1})$$

using $A \in \mathrm{Sp}_{2n}(k)$, so that $A^{-1} = JA^\top J^{-1}$ and $J^\top = J^{-1}$. We also obtain, that the projection map $\mathfrak{gl}_{2n,k} \twoheadrightarrow \mathfrak{sp}_{2n,k}$ is split by the inclusion and equivariant for the adjoint action of Sp_{2n}. Since $\mathfrak{sp}_{2n,k} \subseteq \mathfrak{sl}_{2n,k}$, the restriction $\mathfrak{sl}_{2n,k} \to \mathfrak{sp}_{2n,k}$ is still split by the inclusion and Sp_{2n}-equivariant. ☐

Proposition 6.32. Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2n}}(\kappa)$ with κ a finite field of characteristic $p > 2$ and let Λ be a coefficient ring for κ. Let $x \in U := \dot{X}^{\mathrm{irr}}_{\Lambda,\overline{\Theta}}$ be a closed point. By [BIP21, Lemma 3.16] the residue field $\kappa(x)$ of x is a local field. Let $R^{\mathrm{ps}}_{\Theta_x}$ be the universal pseudodeformation ring of the Sp_{2n}-pseudocharacter Θ_x attached to x. By Proposition 6.16, there is a finite extension κ' of $\kappa(x)$, such that $\Theta'_x := \Theta_x \otimes_{\kappa(x)} \kappa'$ is induced by a continuous absolutely irreducible representation $\overline{\rho} : \Gamma_F \to G(\kappa')$.

1. (a) Suppose, that x is non-special. Then $R^{\mathrm{ps}}_{\Theta'_x}$ is regular of dimension $n(2n+1) \cdot [F : \mathbb{Q}_p]$.
   (b) If in addition $U^{\mathrm{nspcl}} \neq \emptyset$, then U^{nspcl} is regular and equidimensional of dimension $n(2n+1) \cdot [F : \mathbb{Q}_p] - 1$.
2. Suppose, that $\zeta_p \notin F$ and that x is special. Then $\dim R^{\mathrm{ps}}_{\Theta'_x} \in \{n(2n+1) \cdot [F : \mathbb{Q}_p], n(2n+1) \cdot [F : \mathbb{Q}_p] + 1\}$.
3. If $\zeta_p \notin F$, then $\dim U \leq n(2n+1) \cdot [F : \mathbb{Q}_p]$.

Proof. Ad (1) (a). If $\zeta_p \notin F$, then by [BJ19, Lemma 5.1.1 Case I], we have $H^2(\Gamma_F, \mathfrak{gl}_{2n,\kappa'}) = 0$. Since 2 is invertible in κ', by Lemma 6.31 $\mathfrak{sp}_{2n,\kappa'}$ is a direct summand of $\mathfrak{gl}_{2n,\kappa'}$ and so $H^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$. If $\zeta_p \in F$, then we have $H^2(\Gamma_F, \mathfrak{sl}_{2n}) = 0$ by [BJ19, Lemma 5.1.1 case II]. By Lemma 6.31, $\mathfrak{sp}_{2n}$ is also a direct summand of $\mathfrak{sl}_{2n}$. It follows, that $H^2(\Gamma_F, \mathfrak{sp}_{2n}) = 0$. Let $R^{\mathrm{ps}}_{\Theta'_x}$ be the universal pseudodeformation ring of Θ'_x over a coefficient ring $\Lambda' \supseteq \Lambda$ with residue field κ'. By Proposition 6.20 $R^{\mathrm{ps}}_{\Theta'_x}$ is regular of dimension $\dim \mathfrak{sp}_{2n,\kappa'} \cdot [F : \mathbb{Q}_p] + h^0(\Gamma_F, \mathfrak{sp}_{2n,\kappa'})$. By Schur's lemma $h^0(\Gamma_F, \mathfrak{gl}_{2n,\kappa'}) = 1$. Clearly $H^0(\Gamma_F, \mathfrak{gl}_{2n,\kappa'})$ is spanned by the diagonal matrices in $\mathfrak{gl}_{2n,\kappa'}$. These are not contained in $\mathfrak{sp}_{2n,\kappa'}$, hence $h^0(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$.

Ad (1) (b). Assume, that x is non-special. By Proposition 6.6, the universal pseudodeformation ring $R^{\mathrm{ps}}_{\Theta_x}$ can be identified with the completion of $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_\Lambda \kappa(x)$ at the kernel of the natural map $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_\Lambda \kappa(x) \to \kappa(x)$ attached to x. Since x is a 1-dimensional point of $R^{\mathrm{ps}}_{\overline{\Theta}}$ with residue characteristic p, it follows from [BJ19, Lemma 3.3.3], that x is a regular point of dimension $n(2n+1) \cdot [F : \mathbb{Q}_p] - 1$ of U^{nspcl}. Let $U^{\mathrm{sing}} \subseteq U^{\mathrm{nspcl}}$ be the closed subscheme of singular points. By [Sta19, 02J4] and [Sta19, 01TB], the closed points are dense in U^{sing}. But since all closed points of U^{nspcl} are regular, U^{sing} must be empty. Since closed points are dense in U^{nspcl}, it follows that U^{nspcl} is equidimensional of dimension $n(2n+1) \cdot [F : \mathbb{Q}_p] - 1$.

Ad (2). As in (1)(a) $h^0(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$. Since x is special, we have $\overline{\rho} \cong \overline{\rho}(1)$ by [BJ19, Lemma 5.1.1 Case (I)]. We have $H^2(\Gamma_F, \mathfrak{gl}_{2n,\kappa'}) \cong \mathrm{Hom}_{\Gamma_F}(\overline{\rho}, \overline{\rho}(1)) \cong \kappa'$ since $\overline{\rho}$ is irreducible, hence $h^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) \leq 1$. The case when $h^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$ is already covered in Proposition 6.20, so we assume $h^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 1$. By the Euler characteristic formula [BJ19, Theorem 3.4.1]

$$h^1(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = n(2n+1)[F:\mathbb{Q}_p] + 1$$

and by Proposition 6.19, $R^{\mathrm{ps}}_{\Theta'_x}$ is a quotient of $\kappa'[[x_1, \dots, x_{n(2n+1)[F:\mathbb{Q}_p]+1}]]$ by an ideal generated by at most one element, so the assertion follows.

Ad (4). Let $x \in U$ be a closed point. Cases (1) and (2) imply, that $\dim R^{\mathrm{ps}}_{\Theta_x} \leq n(2n+1)[F:\mathbb{Q}_p]+1$. As in (1)(b), identifying $R^{\mathrm{ps}}_{\Theta_x}$ with a completion of $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_\Lambda \kappa(x)$ and applying [BJ19, Lemma 3.3.3], we see that U has dimension $\leq n(2n+1)[F:\mathbb{Q}_p]$. □

Proposition 6.33. Assume $G = \mathrm{Sp}_{2n}$. Then $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F:\mathbb{Q}_p]+1$. In particular if $n[F:\mathbb{Q}_p] \geq 3$ and if $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq \dim \overline{X}_{\overline{\Theta}} - 2$.

Proof. Since $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ is a closed subspace of $\overline{X}^{\mathrm{spcl}}_{\iota(\overline{\Theta})}$ by Proposition 6.29 and the latter can be identified with the special locus of the pseudodeformation space of the determinant law $\overline{D}$ attached to $\iota(\overline{\Theta})$ by Theorem 4.57, we can take the estimate [BJ19, Theorem 5.3.1 (i)] to obtain $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F:\mathbb{Q}_p]+1$. If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then $\dim \overline{X}_{\overline{\Theta}} \geq \dim \overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}} = n(2n+1)[F:\mathbb{Q}_p]$ by Proposition 6.32 (1)(b). We get $\dim \overline{X}_{\overline{\Theta}} - \dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \geq n[F:\mathbb{Q}_p] - 1 \geq 2$. □

Theorem 6.34. Assume $G = \mathrm{Sp}_{2n}$.

1. $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq n(2n+1)[F:\mathbb{Q}_p] - 4(n-1)[F:\mathbb{Q}_p]$.
   In particular, if $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq \dim \overline{X}_{\overline{\Theta}} - 4$.
2. $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq n^2[F:\mathbb{Q}_p]+1$.
   In particular, if $\overline{X}_{\overline{\Theta}}$ contains a non-special point and $n[F:\mathbb{Q}_p] \geq 2$, then $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq \dim \overline{X}_{\overline{\Theta}} - 3$.
3. $\dim \overline{X}_{\overline{\Theta}} \leq n(2n+1)[F:\mathbb{Q}_p]$.
   In particular, if $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then equality holds.

Proof. We make an induction over n, so we assume the entire theorem has been proved for all $n' < n$. Since our assertions are only about dimensions, by Lemma 6.30 we may assume that $\iota(\overline{\Theta})$ comes from a representation $\Gamma_F \to \mathrm{GL}_{2n}(\kappa)$ and that the irreducible constituents are absolutely irreducible.

1. If $n = 1$, then the decomposable locus $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$ is empty, so we may assume $n \geq 2$. There are up to isomorphism only finitely many ways to write $\overline{\Theta}$ as a direct sum of two symplectic pseudocharacters $\overline{\Theta}_1 \oplus \overline{\Theta}_2$. Here the notion of direct sum is that for symplectic pseudocharacters, introduced in Section 4.2.3. By Lemma 6.27, the map

$$\iota^{\mathrm{dec}}_{\overline{\Theta}_1,\overline{\Theta}_2} : \overline{X}_{\overline{\Theta}_1} \widehat{\times}_{\mathcal{O}} \overline{X}_{\overline{\Theta}_2} \to \overline{X}_{\overline{\Theta}}$$

is finite. We have an inclusion

$$\overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \subseteq \bigcup_{\overline{\Theta}_1 \oplus \overline{\Theta}_2 = \overline{\Theta}} \iota^{\mathrm{dec}}_{\overline{\Theta}_1,\overline{\Theta}_2}(\overline{X}_{\overline{\Theta}_1} \widehat{\times}_{\mathcal{O}} \overline{X}_{\overline{\Theta}_2})$$

where the right hand side is a closed subset of $\overline{X}_{\overline{\Theta}}$. Suppose $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_2$ is a decomposition into an Sp_{2a}-pseudocharacter $\overline{\Theta}_1$ and an Sp_{2b}-pseudocharacter $\overline{\Theta}_2$ for $a + b = n$ with $a, b \geq 1$. Then since $\iota^{\mathrm{dec}}_{\overline{\Theta}_1,\overline{\Theta}_2}$ is finite and by part (3) of the inductive hypothesis, we have

$$\begin{aligned} \dim \iota^{\mathrm{dec}}_{\overline{\Theta}_1,\overline{\Theta}_2}(\overline{X}_{\overline{\Theta}_1} \widehat{\times} \overline{X}_{\overline{\Theta}_2}) &\leq \dim(\overline{X}_{\overline{\Theta}_1} \widehat{\times} \overline{X}_{\overline{\Theta}_2}) \\ &\leq a(2a+1)[F:\mathbb{Q}_p] + b(2b+1)[F:\mathbb{Q}_p] \end{aligned}$$

Calculating

$$\begin{aligned}
&n(2n+1)[F:\mathbb{Q}_p] - a(2a+1)[F:\mathbb{Q}_p] - b(2b+1)[F:\mathbb{Q}_p] \\
&= 4ab[F:\mathbb{Q}_p] \geq (\min_{\substack{a+b=n \\ a,b\geq 1}} 4ab)\cdot[F:\mathbb{Q}_p] = 4(n-1)[F:\mathbb{Q}_p]
\end{aligned}$$

we obtain the desired bound

$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \leq n(2n+1)[F:\mathbb{Q}_p] - 4(n-1)[F:\mathbb{Q}_p]$$

If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then by Proposition 6.32 (1)(b), we have a lower bound $\dim \overline{X}_{\overline{\Theta}} \geq n(2n+1)[F:\mathbb{Q}_p]$. Since $4(n-1)[F:\mathbb{Q}_p] \geq 4$, this implies the assertion.

2. There are finitely many ways to write $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_1^*$ for some GL_n-pseudocharacter $\overline{\Theta}_1$ and we may assume, that there is at least one way. The sum yields an Sp_{2n}-pseudocharacter, as explained in Section 4.2.3. By Lemma 6.28, the map

$$\iota_{\overline{\Theta}_1}^{\mathrm{pair}} : \overline{X}_{\overline{\Theta}_1} \to \overline{X}_{\overline{\Theta}}$$

induced by $\Theta_1 \mapsto \Theta_1 \oplus \Theta_1^*$ is finite. We have an inclusion

$$\overline{X}_{\overline{\Theta}}^{\mathrm{pair}} \subseteq \bigcup_{\overline{\Theta}_1 \oplus \overline{\Theta}_1^* = \overline{\Theta}} \iota_{\overline{\Theta}_1}^{\mathrm{pair}}(\overline{X}_{\overline{\Theta}_1})$$

and the estimate

$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{pair}} \leq \dim \overline{X}_{\overline{\Theta}_1} = n^2[F:\mathbb{Q}_p] + 1$$

where the last equality follows from [BJ19, p. 5.4.1] after applying the bijection Corollary 6.5. If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, we obtain a lower bound as in step (1) and the estimate $n(n+1)[F:\mathbb{Q}_p] - 1 \geq 3$ implies the assertion.

3. Let us recollect all upper bounds, we have established.

$$\begin{aligned}
\dim \overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}} &\overset{6.32\ (1)(b)}{\leq} n(2n+1)\cdot[F:\mathbb{Q}_p] \\
\dim \overline{X}_{\overline{\Theta}}^{\mathrm{spcl}} &\overset{6.33}{\leq} 2n^2\cdot[F:\mathbb{Q}_p] + 1 \\
\dim \overline{X}_{\overline{\Theta}}^{\mathrm{dec}} &\overset{(1)}{\leq} n(2n+1)[F:\mathbb{Q}_p] - 4(n-1)[F:\mathbb{Q}_p] \\
\dim \overline{X}_{\overline{\Theta}}^{\mathrm{pair}} &\overset{(2)}{\leq} n^2[F:\mathbb{Q}_p] + 1
\end{aligned}$$

Using the stratification $\overline{X}_{\overline{\Theta}} = \overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{spcl}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{pair}}$ from Proposition 6.26, we obtain the desired dimension bound for $\overline{X}_{\overline{\Theta}}$. If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, we obtain equality from Proposition 6.32 (1)(b).

□

Corollary 6.35. Assume $G = \mathrm{Sp}_{2n}$ and that $\overline{\Theta}$ comes from a residual representation $\overline{\rho} : \Gamma_F \to \mathrm{Sp}_{2n}(\kappa)$, which is absolutely irreducible under the standard embedding into $\mathrm{GL}_{2n}(\kappa)$. Then $\dim \overline{X}_{\overline{\Theta}} = n(2n+1)[F:\mathbb{Q}_p]$ and in particular $\overline{X}_{\overline{\Theta}}$ contains a non-special point.

Proof. By Proposition 6.15 and Lemma 6.18 $\overline{X}_{\overline{\Theta}}$ identifies with the deformation functor of $\overline{\rho}$. From [Til96, Proposition 5.7] and the Euler characteristic formula [BJ19, Theorem 3.4.1], we know, that $\overline{X}_{\overline{\Theta}} \geq h^1(\Gamma_F, \mathfrak{sp}_{2n}) - h^2(\Gamma_F, \mathfrak{sp}_{2n}) = h^0(\Gamma_F, \mathfrak{sp}_{2n}) + n(2n+1)[F:\mathbb{Q}_p]$. By absolute irreducibility and Schur's lemma $h^0(\Gamma_F, \mathfrak{sp}_{2n}) = 0$. So from Proposition 6.33, we see, that the special locus $\overline{X}_{\overline{\Theta}}^{\mathrm{spcl}}$ is strictly contained in $\overline{X}_{\overline{\Theta}}$ and there must be a non-special point in $\overline{X}_{\overline{\Theta}}$. □

Remark 6.36. It is likely that the arguments of Section 6.3.3 carry over to $G = \mathrm{GSp}_{2n}$ with minor modifications. It is also likely that in future work we will be able to deduce the existence of non-special points for arbitrary residual Sp_{2n}- and GSp_{2n}-pseudocharacters, so that in Theorem 6.34 (3) equality holds.

7 The rigid analytic space of G-pseudocharacters

Let Γ be a profinite group, that satisfies Mazur's condition Φ_p. In [Che14, Thm. D] Chenevier shows, that the functor $X_d : \mathrm{An}_{\mathbb{Q}_p}^{\mathrm{op}} \to \mathrm{Set}$ on the category $\mathrm{An}_{\mathbb{Q}_p}$ of rigid analytic spaces over $\mathbb{Q}_p$, that associates to every $Y \in \mathrm{An}_{\mathbb{Q}_p}$ the set $\mathrm{cDet}_d^{\Gamma}(\mathcal{O}(Y))$ of continuous d-dimensional determinant laws with values in the global sections $\mathcal{O}(Y)$, is representable by a quasi-Stein rigid analytic space. Here $\mathcal{O}(Y)$ carries the topology of uniform convergence on open affinoid subsets. By Proposition 4.58 the set $\mathrm{cDet}_d^{\Gamma}(\mathcal{O}(Y))$ identifies with $\mathrm{cPC}_{\mathrm{GL}_d}^{\Gamma}(\mathcal{O}(Y))$. The goal of this section is to generalize Chenevier's construction to generalized reductive group schemes.

We fix notations:

- Let Γ be a topologically finitely generated profinite group.
- Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, uniformizer ϖ and residue field $\mathbb{F}$.
- Let Aff_K be the category of affinoid K-algebras.
- Let An_K be the category of rigid analytic spaces over K.
- Let G be a generalized reductive group scheme over $\mathcal{O}_K$.

Definition 7.1. Define $X_G : \mathrm{Aff}_K \to \mathrm{Set}$ as the functor, that associates to every affinoid K-algebra A the set of continuous G-pseudocharacters $\mathrm{cPC}_G^{\Gamma}(A)$.

All of Chenevier's results carry over in case $G = \mathrm{GL}_d$ by base change from $\mathbb{Q}_p$ to K. Using invariant theory, it is certainly possible to give a direct construction of X_G from X_{GL_d} for the classical groups SL_n, Sp_n, GSp_n, O_n or GO_n. They will be closed subspaces of X_{GL_d}. We will not do this, but instead give directly a functorial construction for general G, which does not depend on the choice of a faithful representation of G.

7.1 The formal scheme of G-pseudocharacters

Before we construct the p-adic analytic space of G-pseudocharacters, we define an auxiliary functor on the level of admissible $\mathcal{O}_K$-algebras, which will turn out to be representable by a disjoint union of formal spectra of deformation rings of residual representations, recovering [Che14, Cor. 3.14] in case $G = \mathrm{GL}_d$.

Definition 7.2. Let A be a complete Hausdorff commutative topological ring. We say, that A is *admissible*, if 0 has a neighborhood basis of ideals, there is an ideal $I \subseteq A$, called *ideal of definition*, such that an ideal $J \subseteq A$ is open if and only if there is some $n \geq 1$, such that $I^n \subseteq J$.

Lemma 7.3. Let A be a commutative topological ring. The following are equivalent:

1. A is complete linearly topologized and has an ideal of definition. This is the notion of admissiblity defined in [Sta19, 07E8].
2. A is, in the category of commutative topological rings, isomorphic to a cofiltered limit of discrete rings $\varprojlim_{\lambda} A_\lambda$, where the index category posesses a final object 0 and the transition maps $A_\lambda \to A_0$ are surjective with nilpotent kernel. This is the notion of admissibility defined in [Che14, §3.9].

This is [Gro60, Lemme 0.7.2.2], we recall the proof for convenience of the reader.

Proof.

$\underline{(1) \implies (2)}$ Let $I \subset A$ be an ideal of definition. Since A is complete, we have $A \cong \varprojlim_{n \geq 1} A/I^n$. The final object of our index category is A/I. The kernel of the projection map $A/I^n \to A/I$ is nilpotent for all n.

$\underline{(2) \implies (1)}$ As an inverse limit of discrete rings, A is complete. We claim, that $I := \ker(A \to A_0)$ is an ideal of definition. Let $U \subset A$ be an open neighborhood of 0. We have to show, that U contains a power of I. By definition of the topology on the projective limit, there is a finite number of indices $\lambda_1, \dots, \lambda_n$

and open neighborhoods $U_i \subset A_{\lambda_i}$ of $0 \in A_{\lambda_i}$, such that $\bigcap_{i=1}^n \pi_{\lambda_i}^{-1}(U_i) \subset U$, where $\pi_\lambda : A \to A_\lambda$ are the projection maps. Since the A_λ are discrete, we can take $U_i = \{0\}$. Since the index category is cofiltered, there is an index object μ, that admits maps $\mu \to \lambda_i$ for all $i = 1, \dots, n$. It follows, that $\ker(\pi_\mu) \subset U$. Let m be a natural number, such that $\ker(A_\mu \to A_0)^m = 0$. We conclude $I^m \subset \ker(\pi_\mu) \subset U$. □

Definition 7.4. Let $\mathfrak{X}_G : \mathrm{Adm}_{\mathcal{O}_K} \to \mathrm{Set}$ be the functor, that attaches to an admissible $\mathcal{O}_K$-algebra A the set of continuous pseudocharacters $\mathrm{cPC}_G^\Gamma(A)$.

Next, we will define a set which will lateron be the index set of a disjoint decomposition of X_G into open subspaces.

Definition 7.5. We denote by $|\mathrm{PC}_G^\Gamma| \subset \mathrm{PC}_G^\Gamma$ the subset of closed points z with finite residue field k_z, such that the canonical G-pseudocharacter $\Theta_z \in \mathrm{PC}_G^\Gamma(k_z)$ attached to z is continuous for the discrete topology on k_z.

Although the above definition is possible in general, we will always assume, that the $\mathcal{O}_K$-algebra B_G^Γ representing PC_G^Γ is finitely generated.

Lemma 7.6. Let A be a discrete $\mathcal{O}_K$-algebra and $\Theta \in \mathrm{cPC}_G^\Gamma(A)$. Then Θ factors over an open normal subgroup $\Delta \leq \Gamma$.

Proof. The idea is the same as in the proof of Proposition 4.47. Let $\sigma = (\sigma_1, \dots, \sigma_r) \in \Gamma^r$ be a tuple of topological generators of Γ and let Σ be the subgroup generated by $\sigma_1, \dots, \sigma_r$. By [Ses77, Theorem 2 (i)], $\mathcal{O}_K[G^{r+1}]^{G^0}$ is a finitely generated $\mathcal{O}_K$-algebra. Let $f_1, \dots, f_s \in \mathcal{O}_K[G^{r+1}]^{G^0}$ be a set of $\mathcal{O}_K$-algebra generators. Since A is discrete and Γ^{r+1} is a profinite set, a map $\Theta_{r+1}(f_i) : \Gamma^{r+1} \to A$ is constant on a finite partition of open subsets of Γ^{r+1}. Such a partition can be refined to consist of open sets in a topological basis of Γ^{r+1}. So we can assume, that the partition of Γ^{r+1} consists of products of sets in a topological basis of Γ. Refining further, we can assume, that the basis of Γ consists of cosets of an open normal subgroup Δ_i of Γ. We take $\Delta := \bigcap_{i=1}^s \Delta_i$ and observe, that for all $\gamma \in \Gamma^{r+1}$, all $\delta \in \Delta$ and all $f \in \mathcal{O}_K[G^{r+1}]^{G^0}$, we have $\Theta_{r+1}(f)(\gamma_1, \dots, \gamma_r, 1) = \Theta_{r+1}(f)(\gamma_1, \dots, \gamma_r, \delta)$.

Let $m \geq 0$, $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma^m$, $f \in \mathcal{O}_K[G^m]^{G^0}$ and $\delta \in \Delta$. Our goal is to show, that $\Theta_m(f)(\gamma_1, \dots, \gamma_m) = \Theta_m(f)(\gamma_1, \dots, \gamma_m\delta)$ and therefore $\Delta \subseteq \ker(\Theta)$ (see Definition 4.26). Since $\Theta_m(f)$ is continuous and A is discrete, we can choose $\gamma' = (\gamma_1', \dots, \gamma_m') \in \Sigma^m$ close enough to γ, such that both $\Theta_m(f)(\gamma_1, \dots, \gamma_m) = \Theta_m(f)(\gamma_1', \dots, \gamma_m')$ and $\Theta_m(f)(\gamma_1, \dots, \gamma_m\delta) = \Theta_m(f)(\gamma_1', \dots, \gamma_m'\delta)$ hold. There is a homomorphism of free groups $\alpha : \mathrm{FG}(m) \to \mathrm{FG}(r)$, such that the composition with the projection $\mathrm{FG}(r) \twoheadrightarrow \Gamma,\ x_i \mapsto s_i$ maps x_i to γ_i'. We extend α to a homomorphism $\tilde{\alpha} : \mathrm{FG}(m+1) \to \mathrm{FG}(r+1)$, such that $\tilde{\alpha}(x_{m+1}) = x_{r+1}$. Let $\eta : \mathrm{FG}(m) \to \mathrm{FG}(m+1)$ be defined by $\eta(x_i) := x_i$ for $i \leq m-1$ and $\eta(x_m) = x_m x_{m+1}$.

Using, what we have just proved, we conclude:

$$\begin{aligned}
\Theta_m(f)(\gamma_1, \dots, \gamma_m\delta) &= \Theta_m(f)(\gamma_1', \dots, \gamma_m'\delta) \\
&= \Theta_{m+1}(f^\eta)(\gamma_1', \dots, \gamma_m', \delta) \\
&= \Theta_{r+1}((f^\eta)^{\tilde{\alpha}})(\sigma_1, \dots, \sigma_r, \delta) \\
&= \Theta_{r+1}((f^\eta)^{\tilde{\alpha}})(\sigma_1, \dots, \sigma_r, 1) \\
&= \Theta_{m+1}(f^\eta)(\gamma_1', \dots, \gamma_m', 1) \\
&= \Theta_m(f)(\gamma_1', \dots, \gamma_m') \\
&= \Theta_m(f)(\gamma_1, \dots, \gamma_m)
\end{aligned}$$

By the homomorphisms theorem Lemma 4.28, Θ factors over a unique pseudocharacter of Γ/Δ. □

We have a more explicit description of $|\mathrm{PC}_G^\Gamma|$:

Lemma 7.7. There is a canonical bijection between $|\mathrm{PC}_G^\Gamma|$ and the set of continuous G-completely reducible representations $\Gamma \to G(\overline{\mathbb{F}})$ up to $G(\overline{\mathbb{F}})$-conjugation and the $\mathbb{F}$-linear Frobenius action on $G(\overline{\mathbb{F}})$ on the coefficients.

Proof. Let $\mathcal{S}$ be the set of continuous G-completely reducible representations $\Gamma \to G(\overline{\mathbb{F}})$ modulo the action of $G(\overline{\mathbb{F}})$ by conjugation and modulo the action of the $\mathbb{F}$-Frobenius of $\overline{\mathbb{F}}$ on the entries of $G(\overline{\mathbb{F}})$. Let $\mathcal{S} \to$

$|\operatorname{PC}_G^\Gamma|$ be the map, that maps an equivalence class $[\rho]$ to the well-defined and unique point in the image of $\operatorname{Spec}(\overline{\mathbb{F}}) \to \operatorname{PC}_G^\Gamma$ attached to Θ_ρ. Surjectivity follows from the reconstruction theorem Theorem 4.56 together with the fact, that a continuous pseudocharacter over $\overline{\mathbb{F}}$ factors over an open normal subgroup Lemma 7.6. For injectivity suppose $\rho, \rho' : \Gamma \to G(\overline{\mathbb{F}})$ are such, that the attached pseudocharacters Θ_ρ and $\Theta_{\rho'}$ are supported on the same point $z \in |\operatorname{PC}_G^\Gamma|$. Then there are $\mathbb{F}$-homomorphisms $f, f' : k_z \to \overline{\mathbb{F}}$, such that $\Theta_z \otimes_{k_z,f} \overline{\mathbb{F}} = \Theta_\rho$ and $\Theta_z \otimes_{k_z,f'} \overline{\mathbb{F}} = \Theta_{\rho'}$. We can take a power of the $\mathbb{F}$-Frobenius $\varphi : \overline{\mathbb{F}} \to \overline{\mathbb{F}}$, such that $\varphi \circ f = f'$, in particular $\Theta_\rho \otimes_{\overline{\mathbb{F}},\varphi} \overline{\mathbb{F}} = \Theta_{\rho'}$. The uniqueness part Theorem 4.56 tells us, that $\rho \otimes_{\overline{\mathbb{F}},\varphi} \overline{\mathbb{F}}$ and ρ' are conjugate. □

Lemma 7.8. Let Γ be a finite group and let G be a generalized reductive group scheme over $\mathcal{O}_K/\varpi^r\mathcal{O}_K$. Then B_G^Γ is finite as a set.

Proof. We first show that $B_G^\Gamma \otimes \overline{\mathbb{F}}$ is a finite-dimensional $\overline{\mathbb{F}}$-vector space. By Proposition 4.47, we already know that $B_G^\Gamma \otimes \overline{\mathbb{F}}$ is a finitely generated $\overline{\mathbb{F}}$-algebra. By Theorem 4.56 the canonical map $\operatorname{Rep}_G^{\Gamma,\square}(\overline{\mathbb{F}}) \twoheadrightarrow \operatorname{PC}_G^\Gamma(\overline{\mathbb{F}}) = \operatorname{Hom}_{\mathrm{CAlg}_{\overline{\mathbb{F}}}}(B_G^\Gamma \otimes \overline{\mathbb{F}}, \overline{\mathbb{F}})$ is surjective. But $\operatorname{Rep}_G^{\Gamma,\square}(\overline{\mathbb{F}})$ is finite, so $B_G^\Gamma \otimes \overline{\mathbb{F}}$ has finitely many $\overline{\mathbb{F}}$-points and thus its nilreduction $(B_G^\Gamma \otimes \overline{\mathbb{F}})_{\mathrm{red}}$ must be a finite product of $\overline{\mathbb{F}}$ with itself. The nilradical $N := \operatorname{Nil}(B_G^\Gamma \otimes \overline{\mathbb{F}})$ is finitely generated and hence nilpotent. So by induction each N^i is a finitely generated $(B_G^\Gamma \otimes \overline{\mathbb{F}})_{\mathrm{red}}$-module. It follows, that $B_G^\Gamma \otimes \overline{\mathbb{F}}$ is a finite-dimensional $\overline{\mathbb{F}}$-vector space. Hence B_G^Γ/ϖ is finite. Since B_G^Γ is ϖ^r-torsion, there is a finite descending sequence

$$B_G^\Gamma \supseteq \varpi B_G^\Gamma \supseteq \varpi^2 B_G^\Gamma \supseteq \cdots \supseteq 0$$

With quotients $\varpi^i B_G^\Gamma / \varpi^{i+1} B_G^\Gamma$. These are finitely generated B_G^Γ/ϖ-modules, hence finite and thus B_G^Γ is finite. □

Lemma 7.9. Let A be an admissible $\mathcal{O}_K$-algebra. Let $\Theta \in \operatorname{cPC}_G^\Gamma(A)$ be a continuous pseudocharacter. Let $A' \subseteq A$ be the closure of the $\mathcal{O}_K$-subalgebra of A generated by $\Theta_n(f)(\gamma_1, \dots, \gamma_n)$ for all $n \geq 1$, all $f \in \mathcal{O}_K[G^n]^{G^0}$ and all $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$. Then A' is an admissible profinite $\mathcal{O}_K$-subalgebra of A.

Proof. Assume, that A is discrete. Then there is some $r \geq 1$, such that $\varpi^r A = 0$. By Proposition 4.48, Θ factors over the $G_{\mathcal{O}_K/\varpi^r}$-pseudocharacter $\Theta/\varpi^r := \Theta \otimes_{\mathcal{O}_K} \mathcal{O}_K/\varpi^r$. By Lemma 7.6 Θ/ϖ^r factors through an open subgroup $\Delta \leq \Gamma$. The representing ring $B_{G_{\mathcal{O}_K/\varpi^r}}^{\Gamma/\Delta}$ of $\operatorname{PC}_{G_{\mathcal{O}_K/\varpi^r}}^{\Gamma/\Delta}$ is finite by Lemma 7.8. By Theorem 4.46, A' is the image of the map $B_{G_{\mathcal{O}_K/\varpi^r}}^{\Gamma/\Delta} \to A$ attached to Θ/ϖ^r, in particular A' is finite, hence admissible.

Now let $A = \varprojlim_\lambda A_\lambda$ be a presentation of A as an inverse limit of discrete rings as in Lemma 7.3. Let $\pi_\lambda : A \to A_\lambda$ be the canonical projection and let $\Theta_\lambda := \pi_{\lambda *}\Theta$. Since A_λ is discrete, the image A'_λ of A' in A_λ is finite by the previous step. Since $\ker(A'_\lambda \to A'_0) \subseteq \ker(A_\lambda \to A_0)$, the former kernel is nilpotent for all λ. It follows from Lemma 7.3, that $A' = \varprojlim_\lambda A'_\lambda$ is admissible. □

We have just shown, that Θ can be uniquely descended to a continuous A'-valued pseudocharacter.

Definition 7.10. If A' in Lemma 7.9 is local, we say that Θ is *residually constant*.

In Lemma 7.13, we will see that A' is a finite product of local profinite admissible $\mathcal{O}_K$-algebras. So if Θ is not residually constant it is essentially a finite product of residually constant pseudocharacters, defined over different connected components of A'. This picture will be crucial for the description of the functor of points of the generic fiber in Theorem 7.21.

Suppose Θ is residually constant. In Lemma 7.9 the natural map $B_G^\Gamma \to A'_0$ (with A'_0 as in Lemma 7.3) is surjective by definition. The radical of the kernel of this map does not depend on the choice of the presentation of A' as an inverse limit as in Lemma 7.3. It is a maximal ideal of B_G^Γ with finite residue field and therefore determines a closed point $z \in |\operatorname{PC}_G^\Gamma|$. The residue field of A' is canonically isomorphic to the residue field $k(z)$ of z. Therefore Θ can be reduced to a continuous $k(z)$-valued pseudocharacter along the map $A' \to k(z)$. This reduction is the pseudocharacter Θ_z attached to z.

Proposition 7.11. Let A be a local profinite admissible $\mathbb{Z}_p$-algebra with residue field k. Then A admits a unique Teichmüller lift $\omega : k^\times \to A^\times$.

Proof. Let $\mathfrak{m}$ be the maximal ideal of A. The residue field k is finite of order $q := |k|$ a power of p. We have $A^\times = A \setminus \mathfrak{m}$. First, suppose A is finite. It follows, that $|A^\times| = |A| - |\mathfrak{m}|$ and $A^\times$ is a product of finite groups E and T with $|E| = q - 1$ and $T = |\mathfrak{m}|$ of p-power order. Since p does not divide $q-1$, T must be in the kernel of the canonical projection $A^\times \to k^\times$. It follows, that the inverse of the restriction of the projection to E is a unique Teichmüller lift. If $A = \varprojlim_\lambda A_\lambda$, all Teichmüller lifts $\omega_\lambda : k^\times \to A_\lambda^\times$ constructed in the previous step are compatible and define a Teichmüller lift for A. □

Theorem 7.12. Let $z \in |\operatorname{PC}_G^\Gamma|$ and let $\mathfrak{X}_{G,z} : \operatorname{Adm}_{\mathcal{O}_K} \to \operatorname{Set}$ be the functor, that attaches to an admissible $\mathcal{O}_K$-algebra A the set $\mathfrak{X}_{G,z}(A)$ of continuous pseudocharacters $\Theta \in \operatorname{cPC}_G^\Gamma(A)$, such that Θ is residually constant and equal to Θ_z. Then $\mathfrak{X}_{G,z}$ is representable by $R_{\rho_z}^{\mathrm{ps}}$, which is a complete noetherian local $\mathcal{O}_K$-algebra with residue field $k(z)$.

Proof. Let $\Theta \in \mathfrak{X}_{G,z}(A)$. By Lemma 7.9, Θ descends to an A'-valued pseudocharacter for some admissible profinite $\mathcal{O}_K$-subalgebra $A' \subseteq A$, which we will also denote by Θ. Using the Teichmüller lift of A' (Proposition 7.11), we see that there is a finite unramified extension L/K, such that $\mathcal{O}_L$ has residue field $k(z)$ and A' is an $\mathcal{O}_L$-algebra. By Proposition 4.48 Θ can be regarded as a $G_{\mathcal{O}_L}$-pseudocharacter. As such it is a lift of Θ_z in the pseudodeformation functor $\operatorname{Def}_{\Theta_z} : \mathfrak{A}_{\mathcal{O}_L} \to \operatorname{Set}$. It follows, that $\operatorname{Def}_{\Theta_z}$ and $\mathfrak{X}_{G,z}$ are naturally isomorphic as functors on $\mathfrak{A}_{\mathcal{O}_L}$. By Theorem 6.11, the pseudodeformation functor $\operatorname{Def}_{\Theta_z}$ is representable by a complete noetherian local $\mathcal{O}_L$-algebra with residue field $k(z)$. □

From now on, we denote by $\mathfrak{X}_{G,z}$ the formal scheme $\operatorname{Spf}(R_{\rho_z}^{\mathrm{ps}})$.

Lemma 7.13. Let A be a profinite admissible $\mathcal{O}_K$-algebra. Then A is a finite product of local profinite admissible $\mathcal{O}_K$-algebras.

We emphasize, that Lemma 7.13 holds independently of any noetherianity hypothesis.

Proof. We only show, that A is a finite product of local rings, the rest of the claim then follows easily. Let $\mathfrak{m}$ be a maximal ideal of A and let I be an ideal of definition of A. Then $\{(I^n + \mathfrak{m})/\mathfrak{m}\}_{n \geq 1}$ is a system of open subgroups of $A/\mathfrak{m}$, that induces the quotient topology of $A/\mathfrak{m}$. But $I^n + \mathfrak{m}$ is either $\mathfrak{m}$ or A, so $A/\mathfrak{m}$ is either discrete or indiscrete. Since $\mathbb{F}$ is discrete and there is a continuous injection $\mathbb{F} \to A/\mathfrak{m}$ induced by the natural map $\mathcal{O}_K \to A$, we have that $A/\mathfrak{m}$ is discrete, hence finite. So there is some $n \geq 1$, such that $I^n + \mathfrak{m} = \mathfrak{m}$, hence $I \subseteq \mathfrak{m}$.

We know that $\mathfrak{m}A/I$ is a maximal ideal of A/I and by [Mat70, (24.C)], we know that A/I has only finitely many maximal ideals. It follows, that A has only finitely many maximal ideals. Since A is commutative, it follows that A is semilocal and thus the claim follows from [Mat70, (24.C)]. □

Lemma 7.14. Let A be an admissible local $\mathcal{O}_K$-algebra and let $\Theta \in \operatorname{cPC}_G^\Gamma(A)$. Then Θ is residually constant.

Proof. According to Lemma 7.9, there is an admissible profinite subring $A' \subseteq A$, over which Θ is defined. From Lemma 7.13, we obtain a system of primitive orthogonal idempotents for A', which also leads to a product decomposition of A. It follows, that the only nonzero idempotent of A' is 1 and that A' is local. □

Corollary 7.15. The functor $\mathfrak{X}_G : \operatorname{Adm}_{\mathcal{O}_K} \to \operatorname{Set}$ is representable by the coproduct $\coprod_{z \in |\operatorname{PC}_G^\Gamma|} \mathfrak{X}_{G,z}$ in the category of formal schemes over $\mathcal{O}_K$.

Proof. It is clear, that on the level of Zariski sheaves on $\operatorname{Adm}_{\mathcal{O}_K}$, there is an injective natural transformation $\coprod_{z \in |\operatorname{PC}_G^\Gamma|} \mathfrak{X}_{G,z} \to \mathfrak{X}_G$. We want to show surjectivity. Let A be an admissible $\mathcal{O}_K$-algebra. If $\Theta \in \mathfrak{X}_G(A)$, then by Lemma 7.9 Θ is defined over a profinite admissible $\mathcal{O}_K$-algebra, so we may assume A is profinite. Then by Lemma 7.13 A is a finite product $A = \prod_i A_i$ of local profinite $\mathcal{O}_K$-algebras A_i.

Since every continuous G-pseudocharacter over an admissible local $\mathcal{O}_K$-algebra is automatically residually constant (Lemma 7.14), the map of sets $(\coprod_z \mathfrak{X}_{G,z})(A_i) = \coprod_z \mathfrak{X}_{G,z}(A_i) \to \mathfrak{X}_G(A_i)$ is bijective for all i. This will be used in the third equality below. Recall also, since the decomposition of A is finite, we have $\operatorname{Spf}(A) = \coprod_i \operatorname{Spf}(A_i)$ in the category $\operatorname{FSch}_{\mathcal{O}_K}$ of formal $\mathcal{O}_K$-schemes.

We calculate

$$\begin{aligned}\mathrm{Hom}_{\mathrm{FSch}_{\mathcal{O}_K}}(\mathrm{Spf}(A),\ \coprod\nolimits_z \mathfrak{X}_{G,z}) &= \prod\nolimits_i \mathrm{Hom}_{\mathrm{FSch}_{\mathcal{O}_K}}(\mathrm{Spf}(A_i),\ \coprod\nolimits_z \mathfrak{X}_{G,z}) \\ &= \prod\nolimits_i \coprod\nolimits_z \mathrm{Hom}_{\mathrm{FSch}_{\mathcal{O}_K}}(\mathrm{Spf}(A_i),\ \mathfrak{X}_{G,z}) \\ &= \prod\nolimits_i \mathrm{Hom}_{\mathrm{FSch}_{\mathcal{O}_K}}(\mathrm{Spf}(A_i),\ \mathfrak{X}_G) \\ &= \mathrm{Hom}_{\mathrm{FSch}_{\mathcal{O}_K}}(\mathrm{Spf}(A),\ \mathfrak{X}_G)\end{aligned}$$

□

7.2 The rigid analytic space of G-pseudocharacters

The goal of this subsection is to construct the p-adic analytic space of G-pseudocharacters, which will be obtained by taking Berthelot's generic fiber (see [Ber96, (0.2.6)] or [De 95, §7]) of $\mathfrak{X}_G$. Let $\mathrm{FSch}^{\mathrm{lnad}}_{\mathcal{O}_K}$ be the category of locally noetherian adic formal schemes $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$ such that the mod ϖ reduction $\mathfrak{X}_{\mathrm{red}}$ of $\mathfrak{X}$ is a scheme locally of finite type over $\mathrm{Spec}(\mathbb{F})$.

We briefly recall the features of Berthelot's functor. It is a functor

$$\begin{aligned}(\quad)^{\mathrm{rig}} : \mathrm{FSch}^{\mathrm{lnad}}_{\mathcal{O}_K} &\to \mathrm{An}_K \\ \mathfrak{X} &\mapsto \mathfrak{X}^{\mathrm{rig}}\end{aligned}$$

from $\mathrm{FSch}^{\mathrm{lnad}}_{\mathcal{O}_K}$ to the category of rigid analytic spaces over K.

If $\mathfrak{X}$ is of the form $\mathrm{Spf}(A)$ for some quotient $A = \mathcal{O}_K[[x_1, \dots, x_n]]/(f_1, \dots, f_s)$ of a formal power series ring $\mathcal{O}_K[[x_1, \dots, x_n]]$, the space $\mathfrak{X}^{\mathrm{rig}}$ will be a closed analytic subvariety of the rigid analytic open unit disk $\mathbb{D}^n$ of dimension n, defined by vanishing of the functions $f_1, \dots, f_s$ interpreted as analytic functions on $\mathbb{D}^n$.

If A is an affinoid K-algebra, a *model* of A is a continuous open $\mathcal{O}_K$-algebra homomorphism $\mathcal{A} \to A$ for some admissible $\mathcal{O}_K$-algebra $\mathcal{A}$, such that the induced map $\mathcal{A}[1/\varpi] \to A$ is an isomorphism. For a fixed model $\mathcal{A} \to A$, there is a canonical map

$$\iota_{\mathcal{A}} : \mathfrak{X}_G(\mathcal{A}) \to X_G(A)$$

that maps a continuous pseudocharacter with values in $\mathcal{A}$ to its base change to A.

We also have a natural map

$$\iota : \varinjlim_{\mathcal{A}} \mathfrak{X}_G(\mathcal{A}) \to X_G(A) \tag{9}$$

where the colimit on the left hand side is taken over the category of all models of A with continuous ring homomorphisms over A. The next goal is to show, that ι is bijective. For d-dimensional determinant laws (i.e. $G = \mathrm{GL}_d$ here by Emerson's isomorphism) and $K = \mathbb{Q}_p$ and this has been shown by Chenevier in [Che14, Lemma 3.15].

Lemma 7.16. Assume, that G is connected. Let A be an affinoid K-algebra and let $\Theta \in X_G(A)$.

1. For all $m \geq 1$, all $f \in \mathcal{O}_K[G^m]^{G^0}$ and all $\gamma \in \Gamma^m$, we have that $\Theta_m(f)(\gamma)$ is contained in the subring A° of power-bounded elements of A.
2. Assume, that Γ is topologically finitely generated. Then ι in Equation (9) is bijective.
3. Assume, that Γ is topologically finitely generated. If A is reduced, then $\mathfrak{X}_G(A^\circ) = X_G(A)$.

Proof.

1. An element of an affinoid K-algebra is power-bounded if and only if for every maximal ideal $\mathfrak{m} \subseteq A$, its image in $A/\mathfrak{m}$ is power-bounded. This follows from [BGR84, Proposition 6.2.3/1] and the boundedness of the supremum norm [BGR84, §6.2.1 and Corollary 3.8.2/2]. We may thus assume, that A is a finite field extension of K and that A° the ring of integers of A. The claim follows directly from [BHKT, Theorem 4.8 (i)].

2. ι is injective, as every model $\mathcal{A}$ of A maps to a ϖ-torsionfree model (take the image of $\mathcal{A}$ in A) and for a torsionfree model $\mathcal{A}$, the map $\iota_{\mathcal{A}}$ is injective. We are left to show surjectivity of ι, so let $\Theta \in X_G(A)$ and let $\mathcal{A} \subseteq A$ be some (torsionfree) model of A. Since we assume, that Γ is topologically finitely generated, we can choose a finitely generated dense subgroup $\Sigma \subseteq \Gamma$. Let $\sigma_1, \dots, \sigma_r \in \Sigma$ be group generators of Σ. Let $f_1, \dots, f_s \in \mathcal{O}_K[G^r]^{G^0}$ be $\mathcal{O}_K$-algebra generators, which we find by [Ses77, Theorem 2 (i)].

   We define a compact subset $C := \bigcup_{i=1}^s \Theta_r(f_i)(\Gamma^r) \subseteq A$. As $\mathcal{A}$ is an open subset of A, C meets only finitely many additive translates of $\mathcal{A}$ in A. So there are $k_i \in C$ with $i = 1, \dots, t$, such that

$$C \subseteq \sum_{i=1}^{t} (k_i + \mathcal{A})$$

   We claim that the algebra $\mathcal{A}' := \mathcal{A}\langle k_1, \dots, k_s\rangle$ (the closure of $\mathcal{A}[k_1, \dots, k_s]$ in A) is a model of A containing C. First, since $\mathcal{A}$ is open in A, $\mathcal{A}'$ is also open. It is also clear, that $\mathcal{A}'[1/\varpi] = A$. For admissibility of $\mathcal{A}'$, we note, that by (1) each of the k_i is power-bounded, so there is a continuous surjection by a Tate algebra $\mathcal{A}\langle T_1, \dots, T_s\rangle \to \mathcal{A}'$ mapping $T_i \mapsto k_i$, and this map is also open, since after inverting ϖ, we obtain a surjection $A\langle T_1, \dots, T_s\rangle \twoheadrightarrow A$, which is open and a quotient map by the open mapping theorem for p-adic Banach spaces [BGR84, §2.8.1]. It follows, that $\mathcal{A}'$ is a complete Hausdorff ring, which carries the I-adic topology for some ideal of definition of $\mathcal{A}$ and is therefore admissible.

   We claim, that Θ actually takes values in $\mathcal{A}'$, so that Θ is the image of a pseudocharacter in $\mathfrak{X}_G(\mathcal{A}')$, as desired. Let $m \geq 1$, $f \in \mathcal{O}_K[G^m]^{G^0}$ and $\delta \in \Sigma^m$. As in the proof of Lemma 7.6, we find a homomorphism $\alpha : \mathrm{FG}(m) \to \mathrm{FG}(r)$, such that $\Theta_m(f)(\delta) = \Theta_r(f^\alpha)(\sigma)$. Since f^α is in the $\mathcal{O}_K$-algebra span of the f_i and $\Theta_r(f_i)(\sigma) \in \mathcal{A}'$ by construction, we find that $\Theta_r(f^\alpha)(\sigma) \in \mathcal{A}'$. Overall, we have shown that $\Theta_m(f)(\Sigma^m) \subseteq \mathcal{A}'$. Since $\Theta_m(f) : \Gamma^m \to A$ is continuous, Γ^m is compact, A is Hausdorff and $\mathcal{A}'$ is closed in A, we conclude that $\Theta_m(f)(\Gamma^m) \subseteq \mathcal{A}'$ and therefore Θ takes values in $\mathcal{A}'$.

3. This is a direct consequence of (2), since if A is reduced, then it is known that A^0 is the terminal model of A [Che14, §3.14.1].

□

Definition 7.17. Let $z \in |\mathrm{PC}_G^\Gamma|$ and define for every affinoid K-algebra A the set $X_{G,z}(A)$ as the set of $\Theta \in X_G(A)$, such that there exists a model $\mathcal{A} \to A$, such that Θ is the image of a pseudocharacter $\tilde{\Theta} \in \mathfrak{X}_{G,z}(\mathcal{A})$.

Suppose A is an affinoid K-algebra and x is a point in the maximal spectrum of A with residue field L. We know, that L is a finite extension of K.

Definition 7.18. The *reduction map* at x is defined as $\mathrm{red}_x : X_G(A) \to |\mathrm{PC}_G^\Gamma|$, where for $\Theta \in X_G(A)$, $\mathrm{red}_x(\Theta)$ shall be the reduction of the unique pseudocharacter $\tilde{\Theta} \in X_G(\mathcal{O}_L)$ (see Lemma 7.16 (3)) mapping to $\Theta \otimes_A L$.

Definition 7.19.

1. Define $\tilde{X}_G : \mathrm{An}_K^{\mathrm{op}} \to \mathrm{Set}$ as the functor, that associates to every rigid analytic space $Y \in \mathrm{An}_K$ the set of continuous G-pseudocharacters $\mathrm{cPC}_G^\Gamma(\mathcal{O}(Y))$.
2. For $z \in |\mathrm{PC}_G^\Gamma|$, let $\tilde{X}_{G,z}$ be the subset of $\tilde{X}_G$ of G-pseudocharacters Θ, such that for all $x \in \mathrm{Specmax}(A)$, the specialization Θ_z of Θ at z defined as the image of Θ under $\tilde{X}_G(A) \to \tilde{X}_G(k_x) \to \tilde{X}_G(\mathcal{O}_{k_x})$ is residually equal to z.

The proofs of Lemma 7.20 and Theorem 7.21 are the same as the proofs of [Che14, Lemma 3.16] and [Che14, Theorem 3.17].

Lemma 7.20. Assume, that G is connected. Suppose A is an affinoid K-algebra and $z \in |\mathrm{PC}_G^\Gamma|$. Then

$$X_{G,z}(A) = \{\Theta \in X_G(A) \mid \forall x \in \mathrm{Specmax}(A) : \mathrm{red}_x(\Theta) = z\}$$

Proof. Let $\Theta \in X_G(A)$, so that for all $x \in \operatorname{Specmax}(A)$, we have $\operatorname{red}_x(\Theta) = z$. By Lemma 7.16 (2), there is some model $\mathcal{A} \to A$ and some $\Theta' \in \mathfrak{X}_G(\mathcal{A})$ that maps to Θ. Let $A' \subseteq \mathcal{A}$ be the ring attached to Θ' as in Lemma 7.9. We know, that A' is a product of local $\mathcal{O}_K$-algebras $\prod_{i=1}^n A'_i$. The idempotents of this decomposition induce a decomposition of A into a product $\prod_{i=1}^n A_i$. Let $x_i \in \operatorname{Specmax}(A_i)$ be a closed point with residue field L_i. By assumption, the kernel of the composition $B_G^\Gamma \to A'_i \to \mathcal{O}_{L_i}/\mathfrak{m}_{\mathcal{O}_{L_i}}$ is the maximal ideal of B_G^Γ, that corresponds to z. By definition of A', the map $B_G^\Gamma \to A' \to A'/\operatorname{Jac}(A')$ is surjective and thus A' itself must be local. This shows, that Θ' is residually constant and residually equal to Θ_z, so $\Theta' \in \mathfrak{X}_{G,z}(A')$. It follows, that $\Theta \in X_{G,z}(A)$. □

Lemma 7.20 in particular implies, that $\tilde{X}_{G,z}$ is representable by $\mathfrak{X}_{G,z}^{\mathrm{rig}}$.

Theorem 7.21. Assume, that G is connected. Then $\tilde{X}_G$ is representable by the quasi-Stein space $\coprod_{z \in |\operatorname{PC}_G^\Gamma|} \mathfrak{X}_{G,z}^{\mathrm{rig}}$.

Proof. To verify, that $\mathfrak{X}_G^{\mathrm{rig}} = \coprod_{z \in |\operatorname{PC}_G^\Gamma|} \mathfrak{X}_{G,z}^{\mathrm{rig}}$ represents $\tilde{X}_G$ it is enough to check that the functor of points agree on affinoid analytic spaces $Y \in \operatorname{An}_K$, since $\tilde{X}_G$ and the functor of points of $\mathfrak{X}_G^{\mathrm{rig}}$ are sheaves for the Zariski topology on An_K. We have

$$\begin{aligned} \operatorname{Hom}_{\operatorname{An}_K}(Y, \mathfrak{X}_G^{\mathrm{rig}}) &= \varinjlim_{\mathcal{Y} \to Y} \operatorname{Hom}_{\operatorname{FSch}/\mathcal{O}_K}(\mathcal{Y}, \mathfrak{X}_G) \\ &= \varinjlim_{\mathcal{Y} \to Y} \mathfrak{X}_G(\mathcal{O}(\mathcal{Y})) \\ &= X_G(\mathcal{O}(Y)) = \tilde{X}_G(Y) \end{aligned}$$

Here the first equality is the universal property of Berthelot's generic fiber functor [De 95, §7.1.7.1], the third equality is using Lemma 7.16 (2). □

Remark 7.22. In [PQ23] we will show, that a continuous representation $\rho : \Gamma \to G(A)$ is $G^0(A)$-conjugate to a representation with values in $G(A^\circ)$. In particular the same arguments show, that Lemma 7.16, Lemma 7.20 and Theorem 7.21 hold for generalized reductive group schemes.

Remark 7.23. It would also have been possible to take the adic generic fiber $\mathfrak{X}_G^{\mathrm{ad}} \times_{\operatorname{Spa}(\mathcal{O}_K)} \operatorname{Spa}(K)$ of the adic space $\mathfrak{X}_G^{\mathrm{ad}}$ attached to $\mathfrak{X}_G$. It is canonically isomorphic to X_G^{ad}. Although we found no advantage in the usage of adic spaces so far, this point of view might be more natural for further applications.

8 Symplectic determinant laws (joint with M. Moakher)

In joint work with Mohamed Moakher, we have developed a notion of *symplectic determinant law* for Sp_{2d} ($d \geq 1$) over $\mathbb{Z}[\frac{1}{2}]$ in analogy to Chenevier's definition [Che14] for GL_d. We give a classification of symplectic determinant laws over fields and show that every symplectic determinant law over an algebraically closed field comes from a unique semisimple symplectic representation. We prove, that the natural map from the GIT quotient of framed symplectic representations into the moduli scheme of symplectic determinant laws is a finite universal homeomorphism. We also establish a comparison bijection with Lafforgue's Sp_{2n}-pseudocharacters provided the coefficient ring is reduced. At last we compute generators of the invariant algebras $A[M_d^m]^G$ and $A[G^m]^G$, where $G \in \{\mathrm{Sp}_d, \mathrm{O}_d, \mathrm{GSp}_d, \mathrm{GO}_d\}$ over a commutative ring A generalizing results of Zubkov [Zub99].

Introduction

In [Che14] Chenevier has given a definition of pseudocharacters of algebras over arbitrary base rings using the notion of multiplicative d-homogeneous polynomial laws. He calls them *determinant laws* and we follow this terminology. The goal of this paper is to give a definition of determinant laws of involutive algebras for the symplectic groups Sp_{2n} over arbitrary $\mathbb{Z}[\frac{1}{2}]$-algebras and study their general properties in analogy to Chenevier's determinant laws.

The first and most important result we obtain is that geometric points of our symplectic pseudocharacter variety (see Proposition 8.15) are in bijection with conjugacy classes semisimple symplectic representations. This is the symplectic analog of [Che14, Theorem 2.12].

Theorem E (Theorem 8.28)**.** Let $\overline{k}$ be an algebraically closed field ($2 \in \overline{k}^{\times}$) and let (R, σ) be an involutive $\overline{k}$-algebra. There is a bijection between isomorphism classes of semisimple $2d$-dimensional symplectic representations of (R, σ) over $\overline{k}$ and $2d$-dimensional symplectic determinant laws of (R, σ) over $\overline{k}$ given by sending $\rho : (R, \sigma) \to (M_{2d}(\overline{k}), \mathrm{j})$ to $(\det \circ \rho, \mathrm{Pf} \circ \rho)$.

Secondly, we obtain a description of Cayley-Hamilton $*$-determinant laws lifting absolutely irreducible symplectic (or orthogonal) representations. This is the symplectic and orthogonal analog of [Che14, Theorem 2.22].

Theorem F (Proposition 8.30)**.** Let R be an A-algebra with involution equipped with a d-dimensional Cayley-Hamilton $*$-determinant $D : R \to A$ such that $\overline{D} = \det \circ \overline{\rho}$ for some absolutely irreducible orthogonal (resp. symplectic) representation $\overline{\rho} : (\overline{R}, \sigma) \to (M_d(k), ^{\top})$ (resp. $(M_d(k), \mathrm{j})$). Then there exists an isomorphism of involutive algebras $\rho : (R, \sigma) \to (M_d(A), ^{\top})$ (resp. $(M_d(A), \mathrm{j})$) lifting $\overline{\rho}$ such that $D = \det \circ \rho$.

We also study the connection between the moduli stack of symplectic representations and the quotient stack of framed symplectic representations and obtain the following expected equivalences. This is the symplectic analog of [Wan13, Theorem 1.4.1.4].

Theorem G (Theorem 8.33)**.** The canonical functors

$$[\mathrm{SpRep}^{\square,2d}_{(R,*)} / \mathrm{Sp}_{2d}] \xrightarrow{\sim} \mathrm{SpRep}^{2d}_{(R,*)} \qquad \text{and} \qquad [\mathrm{SpRep}^{\square,2d}_{(R,*)} / \mathrm{PGSp}_{2d}] \xrightarrow{\sim} \overline{\mathrm{SpRep}}^{2d}_{(R,*)}$$

are equivalences of étale stacks on Sch/S. On the left hand sides we take the étale stack quotient.

There is a natural comparison map between the GIT quotient of framed symplectic representations and the pseudocharacter variety. We prove that it is a finite universal homeomorphism. This is almost the symplectic analog of [Wan18, Theorem 2.20].

Theorem H (Theorem 8.34)**.** $\nu : \mathrm{SpRep}^{\square,2d}_{(R,*)} /\!\!/ \mathrm{Sp}_{2d} \to \mathrm{SpDet}^{2d}_{(R,*)}$ is a finite universal homeomorphism.

We would be happy to show, that ν is an isomorphism in characteristic 0, but we have run into difficulties that come from lack of knowledge about the relations between the natural generators (see Proposition 8.43 and Proposition 8.45) of the invariant algebras $\mathbb{Z}[M_{2d}^m]^{\mathrm{Sp}_{2d}}$ and $\mathbb{Z}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}}$. We expect, that an analog of Vaccarino's theorem [Vac09] for involutive $\mathbb{Q}$-algebras would be sufficient.

We also obtain a bijection between Lafforgue's pseudocharacters and symplectic determinant laws for reduced rings. This is a weakened symplectic analog of [Eme18, Theorem 4.0.1].

Theorem I (Proposition 8.41)**.** Let A be a reduced commutative $\mathbb{Z}[\frac{1}{2}]$-algebra. Then the map $\mathrm{PC}^{\Gamma}_{\mathrm{Sp}_{2d}}(A) \to \mathrm{SpDet}^{\Gamma}_{2d}(A)$ defined in Proposition 8.38 is bijective.

Again, the proof of a full analog of [Eme18, Theorem 4.0.1] cannot be carried out without a symplectic analog of Vaccarino's theorem, this time over $\mathbb{Z}$. We expect to resolve these issues in future work by strenghening the definition of symplectic determinant laws, circumventing the problem of determining relations between invariants.

At last, we adapt Zubkov's results [Zub99] on generators of invariant algebras for the symplectic groups over algebraically closed fields to $\mathbb{Z}$. The method can be used to compute generators of invariant algebras of various different kinds over $\mathbb{Z}$, once the results over algebraically closed fields are available. So we see this as an interesting technical result in its own right.

Theorem J (Proposition 8.43, Proposition 8.45)**.**

1. Let Sp_{2d} act rationally by simultaneous conjugation on the scheme of m-tuples of $2d \times 2d$-matrices M_{2d}^m and thereby on the coordinate ring $\mathbb{Z}[M_{2d}^m]$. Then the invariant algebra $\mathbb{Z}[M_{2d}^m]^{\mathrm{Sp}_{2d}}$ is generated by the elements
$$(X_1, \dots, X_m) \mapsto \sigma_i(Y_{j_1} \cdot \dots \cdot Y_{j_s})$$
where every matrix Y_i is either X_i or the symplectic transpose X_i^{j} and $\sigma_i(X)$ is the i-th coefficient of the characteristic polynomial of X.

2. The invariant algebra $\mathbb{Z}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}}$ is generated by the restriction of the elements defined in (1)
$$(X_1, \dots, X_m) \mapsto \sigma_i(Y_{j_1} \cdot \dots \cdot Y_{j_s})$$
along the closed embedding $\mathrm{Sp}_{2d} \subseteq M_{2d}$. Note, that symplectic transpose becomes inversion in Sp_{2d}.

8.1 Notations

Let A be a commutative ring.

1. $J := \begin{pmatrix} 0 & \mathrm{id}_d \\ -\mathrm{id}_d & 0 \end{pmatrix} \in M_{2d}(A)$
2. Transposition of matrices in $M_n(A)$ is $(-)^\top$. It is also called the *orthogonal standard involution* of $M_n(A)$.
3. The *symplectic standard involution* $(-)^{\mathrm{j}} : M_{2d}(A) \to M_{2d}(A)$ is defined by $M^{\mathrm{j}} := JM^\top J^{-1}$.
4. We define the *symplectic group* $\mathrm{Sp}_{2n}(A) := \{M \in \mathrm{GL}_{2n}(A) \mid M^{-1} = JM^\top J^{-1}\}$.
5. If $(R, *)$ is an involutive ring, let $R^+ := \{x \in R \mid x^* = x\}$ and $R^- := \{x \in R \mid x^* = -x\}$. We say, that the elements of R^+ are *symmetric* and the elements of R^- are *antisymmetric*.
6. The *swap involution* is defined as
$$\mathrm{swap} : M_d(A) \times M_d(A) \to M_d(A) \times M_d(A),\ (a, b) \mapsto (b^\top, a^\top).$$
7. CAlg_A is the category of commutative A-algebras.

8.2 Polynomial laws

Chenevier's original definition [Che14] of determinant laws is based on the notion of polynomial laws. The basic references are [Rob80; BC09; Che14; Wan13]. We give the basic definitions and explain how to introduce the structure of an algebra with involution on the graded pieces of a divided power algebra. We consider a commutative ring A.

Definition 8.1. Let M and N be any A-modules and let R and S be not necessarily commutative A-algebras.

1. An *A-polynomial law* $P : M \to N$ is a collection of maps $P_B : M \otimes_A B \to N \otimes_A B$ for each commutative A-algebra B, such that for each homomorphism $f : B \to B'$ of commutative A-algebras, the diagram

$$\begin{tikzcd} M \otimes_A B \arrow[r, "D_B"] \arrow[d, "\mathrm{id} \otimes f"] & N \otimes_A B \arrow[d, "\mathrm{id} \otimes f"] \\ M \otimes_A B' \arrow[r, "D_{B'}"] & N \otimes_A B' \end{tikzcd}$$

   commutes. In other words, an A-polynomial law is a natural transformation $\underline{M} \to \underline{N}$, where $\underline{M}(B) := M \otimes_A B$ is the *functor of points* of M. We denote the set of A-polynomial laws from M to N by $\mathcal{P}_A(M, N)$.

2. A polynomial law $P : M \to N$ is called *homogeneous of degree* $d \in \mathbb{N}_0$ or *d-homogeneous*, if for all commutative A-algebras B, all $b \in B$ and all $x \in M \otimes_A B$ we have $P_B(bx) = b^d P_B(x)$. We denote the set of d-homogeneous A-polynomial laws from M to N by $\mathcal{P}^d_A(M, N)$.

3. A polynomial law $P : R \to S$ is called *multiplicative*, if for all commutative A-algebras B, we have $P_B(1_{R \otimes_A B}) = 1_{S \otimes_A B}$ and for all $x, y \in R \otimes_A B$, we have $P_B(xy) = P_B(x)P_B(y)$. We denote the set of d-homogeneous multiplicative A-polynomial laws from R to S by $\mathcal{M}^d_A(R, S)$.

4. If R and S are equipped with A-linear involutions, both denoted by $*$, we say that a polynomial law $P : R \to S$ *preserves the involution* if $P_B(x^*) = P_B(x)^*$ for every commutative A-algebra B, and all $x \in R \otimes B$.

5. A d-dimensional *determinant law* on R is a d-homogeneous multiplicative polynomial law $D : R \to A$.

6. If $* : R \to R$ is an A-linear involution, a d-dimensional *$*$-determinant law* on $(R, *)$ is a d-homogeneous multiplicative polynomial law $D : R \to A$, which preserves the involution $*$.

Definition 8.2. Let $P : M \to N$ be an A-polynomial law. We define $\ker(P) \subseteq M$ as a sub A-module whose elements are the $m \in M$ such that for every commutative A-algebra B, $b \in B$ and $m' \in M \otimes_A B$, we have:

$$P(m \otimes b + m') = P(m')$$

Definition 8.3. Let R be an A-algebra, and $P : R \to A$ be a d-homogeneous A-polynomial law. For a commutative A-algebra B an element $r \in R \otimes_A B$, we define its characteristic polynomial by:

$$\chi^P(r, t) := P_{B[t]}(t - r) \in B[t]$$

For an integer $n \geq 1$, $r_1, \dots, r_n \in R$, and ordered tuple of integers $\alpha = (\alpha_1, \dots, \alpha_n)$, we consider the function $\chi^P_\alpha : R^n \to R$ defined by:

$$\chi^P(t_1 r_1 + \dots + t_n r_n, t_1 r_1 + \cdots t_n r_n) = \sum_\alpha \chi^P_\alpha(r_1, \dots, r_n) t^\alpha \in R[t]$$

where $t^\alpha = \prod_{i=1}^n t^{\alpha_i}$. Note that $\chi^P_\alpha \equiv 0$ if $\sum_i \alpha_i \neq d$.

It is debatable, whether *characteric polynomial* is an appropriate name in Definition 8.3. This definition will only be applied in case P is a determinant law or P is the Pfaffian of a symplectic determinant law.

We will now describe a few representability results for polynomial laws, that are already explained in [Che14].

Recall that for any commutative ring A and any A-module M, the divided power algebra $\Gamma_A(M)$ is the commutative graded A-algebra generated by the symbols $m^{[i]}$ in degree i for $m \in M$, $i \in \mathbb{N}_0$, which is subject to the following relations:

- $m^{[0]} = 1$ for all $m \in M$.
- $(am)^{[i]} = a^i m^{[i]}$ for all $a \in A$, $m \in M$.
- $m^{[i]}m^{[j]} = \frac{(i+j)!}{i!j!} m^{[i+j]}$ for all $i, j \in \mathbb{N}_0$, $m \in M$.
- $(m + m')^{[i]} = \sum_{p+q=i} m^{[p]}m'^{[q]}$ for all $i \in \mathbb{N}_0$, $m, m' \in M$.

We denote by $\Gamma^d_A(M)$ the d-th graded piece of $\Gamma_A(M)$. It represents the functor $\mathcal{P}^d_A(M, -) : \mathrm{Mod}_A \to \mathrm{Set}$ with the universal d-homogeneous polynomial law given by $P^{\mathrm{univ}} : M \to \Gamma^d_A(M),\ m \mapsto m^{[d]}$. We have $\mathcal{P}^d_A(M, N) \cong \mathrm{Hom}_A(\Gamma^d_A(M), N)$ for any A-module N.

For an A-algebra R, we can equip $\Gamma^d_A(R)$ with the structure of an A-algebra as follows:

The map $R \oplus R \to R \otimes_A R,\ (r, r') \mapsto r \otimes r'$ is homogeneous of degree 2 and is compatible with $- \otimes_A B$ for any $B \in \mathrm{CAlg}_A$. Thus it gives rise to a 2-homogeneous A-polynomial law. Composing this map with the universal d-homogeneous polynomial law $R \otimes_A R \to \Gamma^d_A(R \otimes_A R)$, we obtain a $2d$-homogeneous polynomial law $R \oplus R \to \Gamma^d_A(R \otimes_A R)$. By the universal property of $\Gamma^{2d}_A(R \oplus R)$, we get a morphism of A-modules:

$$\eta : \Gamma^{2d}_A(R \oplus R) \to \Gamma^d_A(R \otimes_A R)$$

There is canonical isomorphism $\Gamma^{2d}_A(R \oplus R) \cong \bigoplus_{p+q=2d} \Gamma^p_A(R) \otimes_A \Gamma^q_A(R)$ (see [Wan13, §1.1.11]) and η kills $\Gamma^p_A(R) \otimes_A \Gamma^q_A(R)$ for $p \neq q$. From the multiplication map $\theta : R \otimes_A R \to R$, we obtain an A-linear map

$$\Gamma^d_A(R) \otimes_A \Gamma^d_A(R) \xrightarrow{\eta} \Gamma^d_A(R \otimes_A R) \xrightarrow{\Gamma^d_A(\theta)} \Gamma^d_A(R)$$

defining the structure of an A-algebra on $\Gamma^d_A(R)$. In fact, we have a natural isomorphism $\mathcal{M}^d_A(R, S) \cong \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma^d_A(R), S)$ for any commutative A-algebra S.

If R is equipped with an A-linear involution $*$, we want to equip $\Gamma^d_A(R)$ with an induced involution. For this, let R^{op} be the opposite algebra of R. Then $*$ induces an isomorphism $R \cong R^{\mathrm{op}}$. We define the A-linear maps $s : R \oplus R \to R \oplus R,\ (a, b) \mapsto (b, a)$ and $s' : R \otimes_A R \to R \otimes_A R,\ a \otimes b \to b \otimes a$, and we have a commutative diagram

$$\begin{tikzcd}
\Gamma^d_A(R) \otimes \Gamma^d_A(R) \arrow[r, "\eta"] \arrow[d, "\Gamma^{2d}_A(s)"] & \Gamma^d_A(R \otimes_A R) \arrow[r, "\Gamma^d(\theta^{\mathrm{op}})"] \arrow[d, "\Gamma^d_A(s')"] & \Gamma^d_A(R) \arrow[d, "\mathrm{id}"] \\
\Gamma^d_A(R) \otimes \Gamma^d_A(R) \arrow[r, "\eta"] & \Gamma^d_A(R \otimes_A R) \arrow[r, "\Gamma^d(\theta)"] & \Gamma^d_A(R)
\end{tikzcd}$$

which shows that we have a canonical isomorphism $\Gamma^d_A(R^{\mathrm{op}}) \cong \Gamma^d_A(R)^{\mathrm{op}}$. Here $\theta^{\mathrm{op}} : R \otimes_A R \to R, a \otimes b \mapsto ba$ is the multiplication of R^{op} and $\Gamma^d_A(R) \otimes \Gamma^d_A(R)$ is identified with a subset of $\Gamma^{2d}_A(R \oplus R)$.

Definition 8.4. Let $(R, *)$ be an A-algebra with involution. We define the involution $*$ on $\Gamma^d_A(R)$ by the isomorphism

$$\Gamma^d_A(R) \xrightarrow{\Gamma^d_A(*)} \Gamma^d_A(R^{\mathrm{op}}) \cong \Gamma^d_A(R)^{\mathrm{op}}$$

Since the above diagram is compatible with tensoring with any $B \in \mathrm{CAlg}_A$, the isomorphism $\Gamma^d_A(R) \otimes_A B \cong \Gamma^d_B(R \otimes_A B)$ is compatible with the involution.

8.3 Symplectic representations

Definition 8.5. Let $(R, *)$ be an involutive A-algebra and B a commutative A-algebra. A *symplectic representation* of $(R, *)$ is a homomorphism of involutive A-algebras $(R, *) \to (M_{2d}(B), \mathrm{j})$. We let

$\mathrm{SpRep}^{\square,2d}_{(R,*)}$ be the functor of symplectic representations of $(R,*)$:

$$\begin{aligned}\mathrm{SpRep}^{\square,2d}_{(R,*)} : \mathrm{CAlg}_A &\to \mathrm{Set}\\ B &\mapsto \{\text{symplectic representations } (R,*) \to (M_{2d}(B), \mathrm{j})\}\end{aligned}$$

Lemma 8.6. The functor $\mathrm{SpRep}^{\square,2d}_{(R,*)}$ is representable by a commutative A-algebra $A[\mathrm{SpRep}^{\square,2d}_{(R,*)}]$. We let $u^{\mathrm{univ}} : (R,*) \to (M_{2d}(A[\mathrm{SpRep}^{\square,2d}_{(R,*)}]), \mathrm{j})$ be the universal representation. If R is a finitely generated A-algebra, then $A[\mathrm{SpRep}^{\square,2d}_{(R,*)}]$ is a finitely generated A-algebra.

Proof. If $R = A\langle x_i, x_i^*\rangle_{i\in S}$ is a free (non-commutative) A-algebra with involution on a set S, then clearly $A[\mathrm{SpRep}^{\square,2d}_{(R,*)}] = A[\xi^{(i)}_{h,k}]$ is the polynomial algebra over A in the variables $\xi^{(i)}_{h,k}$, $i \in S$, $1 \le h, k \le 2d$ and $u^{\mathrm{univ}}(x_i) = \xi_i = (\xi^{(i)}_{h,k})_{h,k}$.

For a general A-algebra with involution R, there is a presentation $R = A\langle x_i, x_i^*\rangle / I$ for some involution-stable two-sided ideal I of $A\langle x_i, x_i^*\rangle$, respecting the involution. Then $u^{\mathrm{univ}}(I)$ generates a two-sided ideal in $M_{2d}(A[\xi^{(i)}_{h,k}])$, which is as any two-sided ideal in a matrix algebra, of the form $M_{2d}(J)$, with J an ideal of $A[\xi^{(i)}_{h,k}]$. Then the universal map for R is given by:

$$\begin{tikzcd} A\langle x_i, x_i^*\rangle \arrow[r] \arrow[d] & M_{2d}(A[\xi^{(i)}_{h,k}]) \arrow[d] \\ R \arrow[r, "u^{\mathrm{univ}}"] & M_{2d}(A[\xi^{(i)}_{h,k}]/J) \end{tikzcd}$$

By the universal property $M_{2d}(A[\xi^{(i)}_{h,k}]/J)$ is independent of the presentation of R. $\square$

8.4 Symplectic determinant laws

8.4.1 Definition and basic properties

The definition of symplectic determinant laws is based on the following observation. Let A be a commutative ring with $2 \in A^\times$ and let $M \in M_{2d}(A)$ be a matrix with $M^{\mathrm{j}} = M$. We will call such matrices *symplectically symmetric*. Then

$$M = \begin{pmatrix} D & B \\ C & D^\top \end{pmatrix}$$

where $D \in M_d(A)$ is arbitrary and $B, C \in M_d(A)$ are antisymmetric. The matrix

$$MJ = \begin{pmatrix} -B & D \\ -D^\top & C \end{pmatrix} = JM^\top = -J^\top M^\top = -(MJ)^\top$$

is alternating and therefore the Pfaffian $\mathrm{Pf}(MJ)$ is defined. We have $\det(M) = \det(MJ) = \mathrm{Pf}(MJ)^2$.

Definition 8.7. A $2d$-dimensional symplectic determinant law on an involutive A-algebra $(R,*)$ with coefficients in a commutative A-algebra B is the data of an A-linear $2d$-dimensional $*$-determinant law $D : R \to B$ together with a homogeneous polynomial law $P : R^+ \to B$ of degree d, such that $P^2 = D_{|R^+}$ and $P(1) = 1$.

Example 8.8. Let A be a commutative ring and let $\rho : (R,*) \to (M_{2n}(A), \mathrm{j})$ be symplectic representation. Define for any commutative A-algebra B:

1. $D_B : R \otimes_A B \to M_{2n}(B)$ by $D_B(r \otimes b) := b^{2n} \det(\rho(r))$.
2. $P_B : R^+ \otimes_A B \to M_{2n}(B)$ by $P_B(r \otimes b) := b^n \,\mathrm{Pf}(\rho(r)J)$.

Then (D, P) is a symplectic determinant of $(R,*)$ over A.

Lemma 8.9. Let $(R,*)$ be an A-algebra with involution equipped with a symplectic determinant (D,P). Then for every commutative A-algebra B, any $x \in R \otimes_A B$, and any $y \in R^+ \otimes_A B$ such that $P_B(y)$ is a non zero divisor, we have that:

$$P_B(xyx^*) = D_B(x)P_B(y)$$

Proof. For a fixed y as in the statement, consider the polynomial laws $Q_1 : R \otimes_A B \to B,\ x \mapsto P(xyx^*)$ and $Q_2 : R \otimes_A B \to B,\ x \mapsto D(x)P(y)$. Then it is clear that $Q_1^2 = Q_2^2$, and so evaluating at the formal power series ring $B[[t]]$, we have:

$$(Q_1(tx - t + 1) - Q_2(tx - t + 1))(Q_1(tx - t + 1) + Q_2(tx - t + 1)) = 0$$

The evaluation of the second summand at $t = 0$ gives $2P_B(y)$, thus $Q_1(tx-t+1)+Q_2(tx-t+1)$ is a non zero divisor. And so, $Q_1(tx - t + 1) = Q_2(tx - t + 1)$ whose evaluation at $t = 1$ gives us the result. $\square$

We record the following property discovered in [CC21, Proposition 3.1]:

Lemma 8.10. Let $(R,*)$ be an A-algebra with involution equipped with a symplectic determinant (D,P) of dimension $2d$. Then for any commutative A-algebra B any commuting elements $x,y \in R^+ \otimes_A B$, we have that $xy \in R^+ \otimes_A B$ and :

$$P_B(xy) = P_B(x)P_B(y)$$

Proof. The fact that $xy \in R^+ \otimes_A B$ is immediate. Now we introduce the commuting elements $1+t_1x, 1+t_2y \in R^+ \otimes_A B[t_1,t_2]$, and the polynomials:

$$Q_x = P_B(1 + t_1x), \quad Q_y = P_B(1 + t_2y), \quad Q_{xy} = P_B((1 + t_1x)(1 + t_2y))$$

in $B[t_1,t_2]$. The Q_x is a polynomial in t_1 of degree at most d whose coefficient of t^d is $P_B(x)$. Similarly Q_y is a polynomial in t_2 of degree at most d whose coefficient of t^d is $P_B(y)$, and Q_{xy} is a polynomial in t_1,t_2 whose coefficient of $t_1^d t_2^d$ is $P_B(xy)$. Thus to prove the statement, it suffices to show the equality $Q_xQ_y = Q_{xy}$, which can be checked inside the power series ring $B[[t_1,t_2]]$.
Note that for every power series $g \in B[[t_1,t_2]]^\times$ with $g(0,0) \in B^\times$ and every square root $f_0 \in B^\times$ of $g(0,0)$, there exists a unique power series $f \in B[[t_1,t_2]]^\times$ with $f(0,0) = f_0$ such that $f^2 = g$. This can be seen by considering the power series expansion of the square root function at 1. Using this fact, the equality $Q_{xy}^2 = Q_x^2Q_y^2$ (coming from multiplicativity of D), and $Q_{xy}(0,0) = Q_y(0,0) = Q_x(0,0)$, we ge that $Q_xQ_y = Q_{xy}$ as desired. $\square$

For an A-algebra with involution $(R,*)$, and a symplectic determinant $(D,P) : (R,*) \to A$, we introduce the polynomial laws

$$\Lambda_i : R \to A \quad \text{for } 1 \le i \le 2d$$
$$\mathcal{T}_j : R^+ \to A \quad \text{for } 1 \le j \le d$$

defined for any A-algebra B by the formulas:

$$\chi^D(r,t) := D_B(t - r) = \sum_{i=0}^{2d} (-1)^i \Lambda_{i,B}(r) t^{2d-i}, \quad r \in R \otimes_A B$$
$$\chi^P(r,t) := P_B(t - r) = \sum_{i=0}^{d} (-1)^i \mathcal{T}_{i,B}(r) t^{d-i}, \quad r \in R^+ \otimes_A B$$

The following result explains how the characteristic polynomial of P is related to the characteristic polynomial of D when restricted to symmetric elements. In particular, we see that a symplectic determinant law (D,P) is determined by D.

Proposition 8.11. If $D : R \to A$ and $P,P' : R^+ \to A$ are polynomial laws, such that (D,P) and (D,P') are symplectic determinant laws, then $P = P'$. Further, we have the recursion formula

$$\Lambda_i|_{R^+} = \sum_{j=0}^{i} \mathcal{T}_j \mathcal{T}_{i-j}$$

for $1 \le i \le 2d$ with $\mathcal{T}_i = 0$ for $i > d$.

Proof. Since $1 = P'(1) = P(1)$, we have $\mathcal{T}_0 = \mathcal{T}'_0$. The coefficients Λ_i for $0 \le i \le 2d$ of D are defined by the equation

$$\chi^D(r,t) = D_{B[t]}(t-r) = \sum_{i=0}^{2d} (-1)^i \Lambda_{i,B}(r) t^{2d-i} \in B[t]$$

and similarly the coefficients $\mathcal{T}_i$ for $0 \le i \le d$ of P (and $\mathcal{T}'_i$ of P') are defined by

$$\chi^P(r,t) = P_{B[t]}(t-r) = \sum_{i=0}^{d} (-1)^i \mathcal{T}_{i,B}(r) t^{d-i} \in B[t]$$

for all A-algebras B. By comparing the coefficients $\mathcal{T}_i$ and $\mathcal{T}'_i$ and the coefficients Λ_i using $\chi^D(\cdot,t)|_{R^+} = \chi^P(\cdot,t)^2 = \chi^{P'}(\cdot,t)^2$ we obtain

$$\Lambda_i|_{R^+} = \sum_{j=0}^{i} \mathcal{T}_j \mathcal{T}_{i-j} = \sum_{j=0}^{i} \mathcal{T}'_j \mathcal{T}'_{i-j}$$

for $1 \le i \le 2d$ and $\mathcal{T}_d = P$ and $\mathcal{T}'_d = P'$. For $i = 0$, we know, that $1 = \Lambda_0 = \mathcal{T}_0^2 = \mathcal{T}_0'^2$.

By induction over the above equations and using $2 \in A^\times$, we obtain $\mathcal{T}'_i = \mathcal{T}_i$ for all $0 \le i \le d$, in particular $P' = P$. ☐

Taking $r = 1$, we see that $\mathcal{T}_i(1) = \pm\binom{d}{i}$ for $0 \le i \le d$ and the assumption $P(1) = \mathcal{T}_d(1) = 1$ implies $\mathcal{T}_i(1) = \binom{d}{i}$ by downward induction.

Example 8.12. Let Γ be a group. By [Che14, Lemma 1.9], the datum of a 2-dimensional determinant law $D : A[\Gamma] \to A$ is equivalent to the datum of a pair of functions $(d,t) : \Gamma \to A$ such that $d : \Gamma \to A^\times$ is a group homomorphism, and t is a function satisfying $t(1) = 2$ and for all $\gamma, \gamma' \in \Gamma$ the following two equations:

(a) $t(\gamma\gamma') = t(\gamma'\gamma)$,

(b) $d(\gamma)t(\gamma^{-1}\gamma') - t(\gamma)t(\gamma') + t(\gamma\gamma') = 0$.

Here the functions t and d are obtained from the determinant law D by considering the characteristic polynomial $\chi^D(x,\gamma) = x^2 - t(\gamma)x + d(\gamma) \in A[x]$ for all $\gamma \in \Gamma$. In particular they are defined as functions $t, d : A[\Gamma] \to A$ and we have the usual polarization formula

$$d(r) = \frac{t(r)^2 - t(r^2)}{2} \tag{10}$$

for all $r \in A[\Gamma]$.

We are interested in the case, that D is a symplectic determinant law in the sense of Definition 8.7. Note, that this means that D is a determinant law for $\mathrm{Sp}_2 = \mathrm{SL}_2$. So we require that there exists a 1-homogeneous A-polynomial law $P : A[\Gamma]^+ \to A$ with $P^2 = D|_{A[\Gamma]^+}$ and $P(1) = 1$. So let us assume such a P exists. By [Che14, Example 1.2 (i)] P is determined by the A-linear map $P_A : A[\Gamma]^+ \to A$. By Proposition 8.11, we have $P_A(r) = \frac{1}{2}t(r)$ for all $r \in A[\Gamma]^+$. Evaluating the equation $P_A^2 = d|_{A[\Gamma]^+}$ at $\gamma + \gamma^{-1}$ for some $\gamma \in \Gamma$, we thus obtain

$$\frac{1}{4}t(\gamma + \gamma^{-1})^2 = d(\gamma + \gamma^{-1}) \tag{11}$$

Equation (10) gives

$$d(\gamma + \gamma^{-1}) = d(\gamma) + d(\gamma^{-1}) + t(\gamma)t(\gamma^{-1}) - 2 \tag{12}$$

Combining Equation (12) with Equation (11) we get:

$$\frac{1}{4}t(\gamma + \gamma^{-1})^2 = d(\gamma) + d(\gamma^{-1}) + t(\gamma)t(\gamma^{-1}) - 2$$

and thus

$$t(\gamma)^2 + 2t(\gamma)t(\gamma^{-1}) + t(\gamma^{-1})^2 - 2t(\gamma^2) - 2t(\gamma^{-2}) - 8 = 0$$

In Definition 8.7, we also require that the determinant law D is invariant for the A-linear involution on $A[\Gamma]$ extending inversion $\Gamma \to \Gamma,\ \gamma \mapsto \gamma^{-1}$. This implies that the functions t, d are invariant under the inversion map. So we have

$$4t(\gamma)^2 - 4t(\gamma^2) - 8 = 0$$

and hence $d(\gamma) = 1$ by Equation (10).

Example 8.13. For $d = 4$, one finds that:

$$P = \frac{1}{2}\Lambda_4 - \frac{1}{4}\Lambda_1\Lambda_3 + \frac{1}{16}\Lambda_1^2\Lambda_2 + \frac{1}{8}\Lambda_2^2 - \frac{3}{128}\Lambda_1^4$$

In particular we see that the recursion formulas of Proposition 8.11 provide us with a way to define P as a d-homogeneous A-polynomial law on the entire algebra R for every $2d$-dimensional determinant law D when $2 \in A^\times$. Requiring, that D is symplectic is requiring that P satfies the conditions in Definition 8.7.

Lemma 8.14. Let $(R, *)$ be an A-algebra with involution equipped with a symplectic determinant law (D, P). Then $\ker(D)$ is stable under $*$ and $\ker(D) \cap R^+ \subseteq \ker(P)$. In particular for every $*$-ideal $I \subseteq \ker(D)$, (D, P) factors uniquely through a symplectic determinant law $(\overline{D}, \overline{P}) : (R/I, *) \to A$.

Proof. Since D is $*$-invariant, it follows that $\ker(D)$ is a $*$-ideal. Using [Che14, Lemma 1.19] we have that:

$$\ker(D) = \{r \in R \mid \forall B \in \mathrm{CAlg}_A,\ \forall m \in R \otimes_A B,\ \forall i \geq 1,\ \Lambda_i(rm) = 0\}$$

By Proposition 8.11, we know that P can be expressed as a polynomial in the Λ_i, thus to show that $r \in \ker(D) \cap R^+$ is in $\ker(P)$, it suffices to show that $\Lambda_i(r \otimes b + m) = \Lambda_i(m)$ for all commutative A-algebras B, $b \in B$ and $m \in R^+ \otimes_A B$. But this follows from the definition of the Λ_i and the definition of $\ker(D)$.

Since $2 \in R^\times$, we have a surjection $R^+ \twoheadrightarrow (R/I)^+$ and $(R/I)^+$ is identified with $R^+/(I \cap R^+)$. Since $I \cap R^+ \subseteq \ker(D) \cap R^+ \subseteq \ker(P)$, P descends to a well-defined A-polynomial law $\overline{P} : (R/I)^+ \to A$ satisfying the desired properties. $\square$

Proposition 8.15. Let $(R, *)$ be an A-algebra with involution. Then the functor

$$\begin{aligned}\mathrm{SpDet}^{2d}_{(R,*)} : \mathrm{CAlg}_A &\to \mathrm{Set}\\ B &\mapsto \{\text{symplectic determinant laws } (D, P) : R \to B\}\end{aligned}$$

is represented by a commutative A-algebra denoted by $A[\mathrm{SpDet}^{2d}_{(R,*)}]$. If R is a finitely generated A-algebra, then $A[\mathrm{SpDet}^{2d}_{(R,*)}]$ is a finitely generated A-algebra.

Proof. We let I be the ideal of $\mathrm{Sym}_A(\Gamma^d_A(R^+))$ generated by the element $[1]^d - 1$. Then the ring $\mathrm{Sym}_A(\Gamma^d_A(R^+))/I$ represents the functor which associates to a commutative A-algebra B the set of homogeneous polynomial laws P of degree d such that $P(1) = 1$. Using the isomorphism

$$\begin{aligned}\Gamma_A(R^+ \times R^+) &\xrightarrow{\sim} \Gamma_A(R^+) \otimes_A \Gamma_A(R^+)\\ [(r_1, r_2)]^i &\mapsto \sum_{p+q=i} [r_1]^p \otimes [r_2]^q\end{aligned}$$

we get a morphism of A-modules:

$$\widetilde{\varphi} : \Gamma^{2d}_A(R^+) \xrightarrow{\Gamma_A(\Delta)} \Gamma_A(R^+ \times R^+) \twoheadrightarrow \Gamma^d_A(R^+) \otimes_A \Gamma^d_A(R^+) \to \mathrm{Sym}_A(\Gamma^d_A(R^+))/I$$

which for $[r_1]^{i_1} \cdots [r_m]^{i_m} \in \Gamma^{2d}_A(R^+)$ with $i_1 + \cdots + i_m = d$, is given by:

$$\widetilde{\varphi}([r_1]^{i_1} \cdots [r_m]^{i_m}) = \sum_{p_1, q_1, \ldots, p_m, q_m} ([r_1]^{p_1} \cdots [r_m]^{p_m}) \odot ([r_1]^{q_1} \cdots [r_m]^{q_m})$$

where the sum runs over the integers p_j, q_j satisfying $p_j + q_j = i_j$ and $p_1 + \cdots + p_m = q_1 + \cdots + q_m = d$. Here $\odot$ denotes the product in the symmetric algebra.
Therefore we get a morphism of A-algebras $\varphi : \mathrm{Sym}_A(\Gamma^{2d}_A(R^+)) \to \mathrm{Sym}_A(\Gamma^d_A(R^+))/I$.

On the other hand, the canonical map $\Gamma^{2d}_A(R^+) \to \Gamma^{2d}_A(R)$ induces a morphism of commutative A-algebras $\mathrm{Sym}_A(\Gamma^{2d}_A(R^+)) \to \Gamma^{2d}_A(R)^{\mathrm{ab}}$. Then we take:

$$A[\mathrm{SpDet}^{2d}_{(R,*)}] = (\Gamma^{2d}_A(R)^{\mathrm{ab}}/*) \otimes_{\mathrm{Sym}_A(\Gamma^{2d}_A(R^+)),\varphi} \mathrm{Sym}_A(\Gamma^{d}_A(R^+))/I$$

Here $\Gamma^{2d}_A(R)^{\mathrm{ab}}/*$ is the quotient of $\Gamma^{2d}_A(R)^{\mathrm{ab}}$ by the ideal generated by $\gamma - \gamma^*$ for $\gamma \in \Gamma^{2d}_A(R)^{\mathrm{ab}}$. $\square$

We can define direct sums for symplectic determinant laws. On the level of representations it corresponds to the orthogonal direct sum of symplectic spaces carrying an equivariant group action. We will use the direct sum to state the structure theorem Proposition 8.27 for symplectic determinants over fields.

Lemma 8.16. Let A be a commutative ring, let $(R,*)$ be an involutive A-algebra and let (D_1, P_1) and (D_2, P_2) be symplectic determinant laws of $(R,*)$ over A of dimension $2d_1$ and $2d_2$ respectively. Then (D_1D_2, P_1P_2) is a symplectic determinant law of dimension $2(d_1+d_2)$.

We also write $(D_1, P_1) \oplus (D_2, P_2)$ for (D_1D_2, P_1P_2) in analogy to the direct sum of representations.

Proof. As in [Che14, §2.1], D_1D_2 is a determinant law of dimension $2(d_1+d_2)$ and one checks, that it is a $*$-determinant. Similarly $P_1P_2 : R^+ \to A$ is homogeneous of degree d_1+d_2. Further $(P_1P_2)^2 = D_1|_{R^+}D_2|_{R^+}$ and $(P_1P_2)(1) = 1$. $\square$

Remark 8.17. Let A be a commutative ring and $(R,*)$ be an involutive A-algebra. If (D_1, P_1) and (D_2, P_2) are the symplectic determinants attached respectively to the symplectic representations $\rho_1 : (R,*) \to (M_{2d_1}(A), \mathrm{j})$ and $\rho_2 : (R,*) \to (M_{2d_2}(A), \mathrm{j})$ then $(D_1, P_1) \oplus (D_2, P_2)$ is the symplectic determinant attached to $\rho_1 \oplus \rho_2$.

Proposition 8.18. Let (D_1, P_1) and (D_2, P_2) be symplectic determinant laws of dimensions $2d_1$ and $2d_2$ satisfying $\mathrm{CH}(P_i) \subseteq \ker(D_i)$ for $i = 1, 2$. Then $\mathrm{CH}(P_1P_2) \subseteq \ker(D_1D_2)$.

Proof. Suppose (D_1, P_1) and (D_2, P_2) are symplectic determinant laws of dimensions $2d_1$ and $2d_2$. Suppose $\mathrm{CH}(P_i) \subseteq \ker(D_i)$. We will show, that $\mathrm{CH}(P_1P_2) \subseteq \ker(D_1D_2)$. Let $P := P_1P_2$ and $D := D_1D_2$. Recall Definition 8.3 of the functions $\chi^{P_i}_\alpha$, where $\alpha \in \mathbb{N}_0^n$ with $\sum_{j=1}^n \alpha_j = d_i$. For $r_1, \dots, r_n \in R$, the equation

$$\chi^P(r_1t_1 + \dots + r_nt_n) = \chi^{P_1}(r_1t_1 + \dots + r_nt_n)\chi^{P_2}(r_1t_1 + \dots + r_nt_n)$$

in $R[t_1, \dots, t_n]$ implies

$$\chi^P_\alpha(r_1, \dots, r_n) = \sum_{\alpha'+\alpha''=\alpha} \chi^{P_1}_{\alpha'}(r_1, \dots, r_n)\chi^{P_2}_{\alpha''}(r_1, \dots, r_n)$$

by comparing the coefficients of t^α. To check that $D(1 + \chi^P_\alpha(r_1, \dots, r_n)r) = 1$ for all $r \in R$, it suffices to check that $D_i(1 + \chi^P_\alpha(r_1, \dots, r_n)r) = 1$ for all $r \in R$. This is clear, since

$$\sum_{\alpha'+\alpha''=\alpha} \chi^{P_1}_{\alpha'}(r_1, \dots, r_n)\chi^{P_2}_{\alpha''}(r_1, \dots, r_n)r \in \ker(D_1) \cap \ker(D_2)$$

and by [Che14, Lemma 1.19]. $\square$

8.4.2 Symplectic determinant laws over Azumaya algebras

Recall, that an Azumaya algebra over a commutative ring A (where we still assume that $2 \in A^\times$) is a unital A-algebra R, such that there is an étale covering $\{A \to B_i\}_{i\in I}$, such that for all $i \in I$ the B_i-algebra $R \otimes_A B_i$ is isomorphic to a matrix algebra of positive rank over B_i. The rank of this matrix algebra may vary over $\mathrm{Spec}(A)$ and we will assume, that the rank is constant on $\mathrm{Spec}(A)$. In this section, we explain what we mean by a symplectic, orthogonal or unitary involution of an Azumaya algebra of constant rank.

Definition 8.19. Let A be a commutative ring and let R be an Azumaya algebra of constant rank d^2 over A. Let σ be an A-linear involution of R.

1. We say, that σ is an *involution of the first kind of symplectic (orthogonal) type*, if there is an étale covering $\{A \to B_i\}_{i \in I}$, such that the involution $\sigma \otimes 1$ on $R \otimes_A B_i$ is induced by an antisymmetric (symmetric) non-degenerate bilinear form $\beta : V \times_{B_i} V \to L$ on a locally free B_i-module V of constant rank d with values in a locally free B_i-module L of constant rank 1.

2. Let $A^\circ \to A$ be an étale covering of degree 2, where A° is connected, and let σ be an A°-linear involution on R. We say, that σ is an *involution of the second kind (wrt. A°)* or *unitary involution*, if it restricts to the unique nontrivial A°-linear automorphism of A.

We put $A^\circ := A$ by convention if σ is of the first kind.

Proposition 8.20. Let (R, σ) be an Azumaya algebra of constant rank d^2 over A with involution of the first or second kind over A°. Then étale locally over A°, (R, σ) has one of the following three forms.

1. $(M_d(A^\circ), \mathrm{j})$, if σ is symplectic.
2. $(M_d(A^\circ), \top)$, if σ is orthogonal.
3. $(M_d(A^\circ) \times M_d(A^\circ), \mathrm{swap})$, if σ is unitary.

Following the book [Knu91], we define a symplectic determinant law for an Azumaya algebra R over a commutative ring A equipped with a symplectic involution.

Let R be an Azumaya algebra over A of rank d^2 with an involution σ. Let S be a faithfully flat A-algebra, such that we have a splitting $\alpha : S \otimes_A R \xrightarrow{\sim} M_d(S)$ of R over S and let $\widetilde{\sigma} = \alpha(1 \otimes \sigma)\alpha^{-1}$ be the induced involution on $M_d(S)$. The map $x \mapsto \widetilde{\sigma}(x^\top)$ is an automorphism of $M_d(S)$. We can choose S so that $\widetilde{\sigma}(x) = u(x^\top)u^{-1}$ for some suitably chosen $u \in \mathrm{GL}_d(S)$ and all $x \in M_d(S)$. The fact that $\widetilde{\sigma}^2 = \mathrm{id}$ implies that $u^\top = \epsilon u$ for some $\epsilon \in \mu_2(S)$. By [Knu91, p. 8.1.1], one can choose S so that $\epsilon \in \mu_2(A)$ and this element is independent of the choice of S. We call it the *type* of the involution σ on R. An involution of type 1 is called an *orthogonal involution*, and an involution of type -1 is called a *symplectic involution*.

To define the Pfaffian on an Azumaya algebra, we first consider the case of the endomorphism ring $\mathrm{End}_A(V)$ for V a finitely generated projective A-module of rank $2d$. In this case, one can show by glueing local Pfaffians (c.f. [Knu91, p. 9.2.1]), the existence of a (unique) map

$$\mathrm{Pf} : \wedge^2 V \to \wedge^{2d} V$$

that commutes with base change and that is given by the usual Pfaffian if V is free.

In order to generalize this construction, let R be an Azumaya algebra of rank $4d^2$ over A such that its class in the Brauer group $\mathrm{Br}(A)$ is of order 2. Let $\varphi : R \otimes R \xrightarrow{\sim} \mathrm{End}_A(P)$ be an isomorphism of A-algebras for P a faithfully flat finitely generated projective A-module. We call the triple (R, P, φ) a *2-torsion datum*. The triple $(\mathrm{End}_A(V), V \otimes_A V, \mathrm{can})$, where V is a finitely generated projective A-module and can is the canonical isomorphism $\mathrm{can} : \mathrm{End}_A(V) \otimes_A \mathrm{End}_A(V) \xrightarrow{\sim} \mathrm{End}_A(V \otimes_A V)$ is called a *split datum*. By [Knu91, p. 9.3.1], any 2-torsion datum (R, P, φ) admits a splitting by a faithfully flat étale A-algebra S, i.e. an isomorphism

$$(\alpha, g) : (R, P, \varphi) \otimes S \xrightarrow{\sim} (\mathrm{End}_A(V), V \otimes_A V, \mathrm{can})$$

of 2-torsion data for a finitely generated projective S-module V.

By [Knu91, p. 8.4.1], there exists an element $u \in (R \otimes_A R)^\times$ such that for any splitting (α, g), the conjugation $i_u : R \otimes_A R \to R \otimes_A R$ is the switch map, i.e. $i_u(r \otimes r') = r' \otimes r$ for all $r, r' \in R$, and $(\alpha \otimes \alpha) \circ (1 \otimes u) : V \otimes_S V \to V \otimes_S V$ is also the switch map ω_V. The element $\psi := \varphi(u) \in \mathrm{End}_S(P)$ is called the *module involution* of P with respect to φ and we call the set

$$\mathcal{S}_-(P) = \{x - \psi(x) \mid x \in P\}$$

the set of *alternating elements* of P. We shall identify $\mathcal{S}_-(V) = \{x - \omega_V(x) \mid x \in V \otimes_S V\}$ with $\wedge^2 V$ through the map $x \otimes y - y \otimes x \mapsto x \wedge y$.

Theorem 8.21. [Knu91, p. 9.3.2] For any 2-torsion datum (R, P, φ) with R of rank $4d^2$, there exists up to isomorphism a unique invertible A-module $\mathrm{Pf}(P)$ and a map $\mathrm{Pf} : \mathcal{S}_-(P) \to \mathrm{Pf}(P)$, which is unique once $\mathrm{Pf}(P)$ is fixed, with the following properties:

1. If $(R, P, \varphi) = (\mathrm{End}_R(V), V \otimes V, \mathrm{can})$, then $\mathrm{Pf}(P) = \wedge^{2d} V$, and Pf is the Pfaffian $\wedge^2 V \to \wedge^{2d} V$.
2. For any commutative A-algebra B, there exist canonical isomorphisms $\gamma_B : \mathrm{Pf}(P) \otimes_A B \to \mathrm{Pf}(P \otimes_A B)$ such that $\gamma_B(1 \otimes \mathrm{Pf}(x)) = \mathrm{Pf}(1 \otimes x)$, $x \in \mathcal{S}_-(P)$.
3. Viewing P as an (R, R^{op})-bimodule through φ, we have $\mathrm{Pf}(rxr) = N_R(r)\,\mathrm{Pf}(x)$ where $N_R : R \to A$ is the reduced norm, and $\mathrm{Pf}(\lambda x) = \lambda^d\,\mathrm{Pf}(x)$ for all $r \in R$, $x \in P$ and $\lambda \in A$.

Now suppose that R is equipped with an involution σ of type $\epsilon \in \mu_2(A)$. We have an isomorphism $\varphi_\sigma : R \otimes_A R \xrightarrow{\sim} \mathrm{End}_R(A)$ given by $\varphi_\sigma(r \otimes r')(x) = rx\sigma(r')$. Thus (R, R, φ_σ) is a 2-torsion datum and using [Knu91, p. 9.5.1], we can show that:

$$\mathcal{S}_-(R) = \{x - \epsilon\sigma(x) \mid x \in R\}$$

Moreover, in this case there is a canonical nonsingular pairing:

$$\widetilde{\sigma} : \mathrm{Pf}(R) \times \mathrm{Pf}(R) \to A$$

such that $\widetilde{\sigma}(\mathrm{Pf}(r), \mathrm{Pf}(r)) = \mathrm{Nrd}_R(r)$ for $r \in \mathcal{S}_-(R)$ (c.f. [Knu91, p. 9.5.2]).

Proposition 8.22. [Knu91, p. 9.5.4] Let R be an Azumaya algebra over A with an involution σ of symplectic type, then $\mathrm{Pf}(R) \cong A$ and the pairing $\widetilde{\sigma}$ is the trivial one.

Therefore if R is an Azumaya algebra over A equipped with a symplectic involution σ, we use the explicit generator of $\mathrm{Pf}(R)$ given in the proof of [Knu91, p. 9.5.4], to get an A-valued function that we denote by $\mathrm{Prd}_R : R \to A$. Explicitly, let $\alpha : R \otimes_A S \xrightarrow{\sim} M_{2d}(S)$ be a splitting such that $(\alpha \circ (1 \otimes \sigma) \circ \alpha^{-1})(x) = ux^\top u^{-1}$ with u an alternating matrix in $M_{2d}(S)$. The matrix $u^{-1}(\alpha(1 \otimes x))$ for $x \in \mathcal{S}_-(R)$ is skew-symmetric and

$$\mathrm{Prd}_R(x) = \mathrm{Pf}(u)\,\mathrm{Pf}(u^{-1}(\alpha(1 \otimes x))), \quad x \in \mathcal{S}_-(R) = R^+$$

and we have that $\mathrm{Prd}_R(x)^2 = \mathrm{Nrd}_R(x)$. Since the construction is stable under base change, we can make the following definition:

Definition 8.23. The pair $(\mathrm{Nrd}_R, \mathrm{Prd}_R)$ defines a symplectic determinant law on (R, σ).

8.4.3 Symplectic determinant laws over fields

The goal of this subsection is to give a precise structure theorem for symplectic determinants over general fields of characteristic $\neq 2$. It is the symplectic analog of [Che14, Thm. 2.16]. A crucial ingredient in the GL_n-case is the Artin-Wedderburn theorem. Here we need a version of the Artin-Wedderburn theorem for semisimple rings with involution.

If D is a division algebra over a field K and $\dagger : D \to D$ is an involution, we extend $\dagger$ to a map $\dagger : M_n(D) \to M_n(D)$ by defining $A^\dagger$ by $(A^\dagger)_{kl} := (A_{kl})^\dagger$ for $A \in M_n(D)$. Note, that $\dagger \circ \top = \top \circ \dagger$ is an involution on $M_n(D)$, but in general neither $\dagger$ nor $\top$ is an involution. In the following, we identify $Z(M_n(D))$ with $Z(D)$.

Proposition 8.24. Let k be a field. Let R be a semisimple k-algebra, such that every simple factor of R is finite-dimensional over its center. Let $* : R \to R$ be a k-linear involution. Then $(R, *)$ is isomorphic as an involutive k-algebra to a product

$$(R, *) \cong \prod_{i=1}^{t} (T_i, \iota_i)$$

for some $t \in \mathbb{N}_{\geq 1}$, where the involutive rings $(T, \iota) := (T_i, \iota_i)$ have one of the following three forms:

(I-II) $T = M_n(D)$ is an $n \times n$ matrix algebra over a finite-dimensional division k-algebra D with center K and $\iota(A) = SA^{\dagger\top}S^{-1}$ for some $S \in M_n(D)^\times$, where $\dagger : D \to D$ is a K-linear involution and $S^{\dagger\top} = \pm S$. If $\mathrm{char}(k) \neq 2$, then the involution ι is of the same type as $\dagger$ if and only if $S^{\dagger\top} = S$.

(IIIa) $T = M_n(D)$ is an $n \times n$ matrix algebra over a finite-dimensional division k-algebra D with center L and $\iota(A) = SA^{\dagger\top}S^{-1}$ for some $S \in M_n(D)^\times$, where $\dagger : D \to D$ is a K-linear involution for some index 2 subfield K of L with L/K separable, $\dagger$ is not the identity on L, and $S^{\dagger\top} = S$.

(IIIb) $T = M_n(D) \times M_n(D^{\mathrm{op}})$ for some finite-dimensional divison algebra D over k and $\iota(a, b^{\mathrm{op}}) = (b^\top, (a^{\mathrm{op}})^\top)$.

Proof. Applying Artin-Wedderburn to R, we see, that R is isomorphic to a finite product $\prod_{j=1}^{s} M_{n_j}(D_j)$ of matrix algebras $M_{n_j}(D_j)$ over finite-dimensional division algebras D_j over k. This product decomposition corresponds to a unique set of orthogonal central primitive idempotents $e_1, \dots, e_s \in R$ with $e_1 + \cdots + e_s = 1$. The involution $*$ defines a bijection $* : \{e_1, \dots, e_s\} \to \{e_1, \dots, e_s\}$. This defines a $*$-stable partition of $\{e_1, \dots, e_s\}$ into singletons $\{e_j\}$, when $e_j^* = e_j$ (case (I-II-IIIa)) and pairs $\{e_j, e_j^*\}$ otherwise (case (IIIb)). Let t be the number of classes of this partition and choose some numbering of the partition by $i \in \{1, \dots, t\}$.

Since e_jR is $*$-stable for all j with $e_j^* = e_j$ and $(e_j + e_j^*)R$ is $*$-stable for all j, we obtain $*$-stable k-algebras T_i with $T_i = M_{n_j}(D_j)$ in case (I-II-IIIa) and $T_i = M_{n_j}(D_j) \times M_{n_{j'}}(D_{j'})$ in case (IIIb) with j', such that $e_{j'} = e_j^*$. The involution $*$ induces an isomorphism $M_{n_j}(D_j) \cong M_{n_{j'}}(D_{j'})^{\mathrm{op}}$. Thus, we may assume, that $M_{n_{j'}}(D_{j'}) = M_{n_j}(D_j)^{\mathrm{op}}$.

We obtain a product decomposition of R into $*$-stable algebras T_i, with either $T_i = M_{n_j}(D_j)$ or $T_i = M_{n_j}(D_j) \times M_{n_j}(D_j^{\mathrm{op}})$ for $i \in \{1, \dots, t\}$. Let $\iota_i : T_i \to T_i$ be the restriction of $*$ to T_i.

Fix i and let $(T, \iota) := (T_i, \iota_i)$.

(I-II-IIIa) These follow from [Knu+98, Chapter I, Proposition 2.20].

(IIIb) Suppose $T = M_n(D) \times M_n(D)^{\mathrm{op}}$. We know, that $\iota(\{0\} \times M_n(D)^{\mathrm{op}}) \subseteq M_n(D) \times \{0\}$. It defines an anti-isomorphism $M_n(D)^{\mathrm{op}} \to M_n(D)$. Composed with the anti-isomorphism $\mathrm{op} : M_n(D) \to M_n(D)^{\mathrm{op}}$, we get an automorphism $\beta : M_n(D)^{\mathrm{op}} \to M_n(D)^{\mathrm{op}}$ This gives an isomorphism of involutive k-algebras $(\mathrm{id}, \beta) : (T, \iota) \to (T, \mathrm{swap})$, where $(a, b^{\mathrm{op}})^{\mathrm{swap}} = (b, a^{\mathrm{op}})$. After identifying $M_n(D)^{\mathrm{op}}$ with $M_n(D^{\mathrm{op}})$ using transposition, the claim follows.

□

Example 8.25. Let K/k be a field extension, and let $k^s \subseteq K$ be the maximal separable extension of k inside K. We assume that $f = [k^s : k]$ is finite. If $\mathrm{char}(k) = p > 0$, assume there is an integer $q \in p^{\mathbb{N}}$ such that $K^q \subseteq k^s$. We take q minimal with this property. If $p = 0$ we take $q = 1$.

Let (R, σ) be a K-algebra with involution and let (D, P) be a symplectic determinant law of (R, σ) over k. We consider the following cases:

(I) (R, σ) is a central simple algebra over K with a symplectic involution. Then (D, P) is power of:

$$\mathrm{Norm}_{k^s/k} \circ F^q \circ \mathrm{Nrd}_R : R \to k$$
$$\mathrm{Norm}_{k^s/k} \circ F^q \circ \mathrm{Prd}_R : R^+ \to k$$

This follows from [Che14, Lemma 2.17] and the existence and uniqueness Proposition 8.11 of the Pfaffian.

(II) (R, σ) is a central simple algebra over K with an orthogonal involution. Suppose that after base change $R \otimes K' = M_r(K')$, then there exists some $m \in \mathbb{N}_{>0}$ such that $D(\mathrm{diag}(t, 1, \dots, 1)) = t^{qfm} \in K'[t]$. The existence of P forces qfm to be even. Then if m is even, (D, P) is a power of:

$$\mathrm{Norm}_{k^s/k} \circ F^q \circ \mathrm{Nrd}_R^2 : R \to k$$
$$\mathrm{Norm}_{k^s/k} \circ F^q \circ \mathrm{Nrd}_R : R^+ \to k$$

If qf is even, This follows from [Che14, Lemma 2.17], existence and uniqueness of the Pfaffian.

(III) (R, σ) is a central simple algebra over an étale K-algebra L of degree 2 equipped with a unitary involution over L/K. In other words L is either $K \times K$ and $R = E \times E^{\mathrm{op}}$ with E a central simple algebra over K, or L is a separable field extension of K and R is a central simple algebra over L. Also σ is K-linear and restricts to the nontrivial element of $\mathrm{Aut}_K(L)$. Then (D, P) is a power of:

$$\mathrm{Norm}_{k^s/k} \circ F^q \circ \mathrm{Norm}_{L/K} \circ \mathrm{Nrd}_R : R \to k$$
$$\mathrm{Norm}_{k^s/k} \circ F^q \circ \mathrm{Nrd}_R : R^+ \to k$$

This is because Nrd_R on R^+ takes values in K. Indeed in the first case, we have that σ is given by $\sigma(a,b) = (\iota(b), \iota(b))$ with $\iota : E \to E^{\mathrm{op}}$ an isomorphism of central simple algebras over K. So for $(a, \iota(a)) \in R^+$ with $a \in E$, we have

$$\mathrm{Nrd}_R(a, \iota(a)) = (\mathrm{Nrd}_E(a), \mathrm{Nrd}_{E^{\mathrm{op}}}(\iota(a))) = (\mathrm{Nrd}_E(a), \mathrm{Nrd}_E(a))$$

The second case follows from the first case by base change [Knu+98, §2, Proposition 2.15].

Example 8.26. There is an infinite field extension K/k, such that there is a determinant $D : K \to k$. Indeed, we can take $k := \mathbb{F}_p(t_i^p \mid i \in \mathbb{N})$ and $K = \mathbb{F}_p(t_i \mid i \in \mathbb{N})$. The extension K/k is infinite with $K^p = k$, so the Frobenius $F^p : K \to k$ is a determinant.

Proposition 8.27. Let $(D, P) : (R, \sigma) \to k$ be a $2d$-dimensional symplectic determinant. Then there is an isomorphism

$$(R/\ker(D), \overline{\sigma}) \cong \prod_{i=1}^{s} (R_i, \sigma_i)$$

of involutive k-algebras, where each (R_i, σ_i) is equipped with a symplectic determinant (D_i, P_i) which takes one of the forms (I)-(III) described in the example 8.25, and where:

$$(D, P) = \left(\prod_{i=1}^{s} D_i \circ \pi_i, \prod_{i=1}^{s} P_i \circ \pi_i \right)$$

with $\pi_i : R \to R_i$ are the projections given by the isomorphism.

Proof. This proposition follows from [Che14, Theorem 2.16] and 8.24. □

Theorem 8.28. Let $\overline{k}$ be an algebraically closed field and let (R, σ) be an involutive $\overline{k}$-algebra. There is a bijection between isomorphism classes of semisimple $2d$-dimensional symplectic representations of (R, σ) over $\overline{k}$ and $2d$-dimensional symplectic determinant laws of (R, σ) over $\overline{k}$ given by sending $\rho : (R, \sigma) \to (M_{2d}(\overline{k}), \mathrm{j})$ to $(\det \circ \rho, \mathrm{Pf} \circ \rho)$.

Proof. Let (D, P) be a symplectic determinant of (R, σ) over $\overline{k}$. By proposition 8.27 there is a decomposition

$$(R/\ker(D), \overline{\sigma}) \cong \prod_{i=1}^{s} (R_i, \sigma_i)$$

where the R_i are K_i-algebras of the form described in example 8.25 for some extension field K_i/k.

Arguing as in the proof of [Wan13, Theorem 1.3.1.3], we have $K_i = \overline{k}$ for all i. Thus we have the following three cases:

(I) $(R_i, \sigma_i) \cong (M_{2n_i}(\overline{k}), \mathrm{j})$. We let $\rho_i : (R, \sigma) \to (M_{2n_i}(\overline{k}), \mathrm{j})$ be the corresponding symplectic representation.

(II) $(R_i, \sigma_i) \cong (M_{n_i}(\overline{k}), \top)$. We let

$$\begin{aligned} \rho_i : (R, \sigma) &\to (M_{2n_i}(\overline{k}), \mathrm{j}) \\ r &\mapsto \begin{pmatrix} \pi_i(r) & 0 \\ 0 & \pi_i(r) \end{pmatrix} \end{aligned}$$

(III) $(R_i, \sigma_i) \cong (M_{n_i}(\overline{k}) \times M_{n_i}(\overline{k}), \mathrm{swap})$. We let

$$\begin{aligned} \rho_i : (R, \sigma) &\to (M_{2n_i}(\overline{k}), \mathrm{j}) \\ r &\mapsto \begin{pmatrix} \mathrm{pr}_1(\pi_i(r)) & 0 \\ 0 & \mathrm{pr}_2(\pi_i(r))^{\mathrm{op}} \end{pmatrix} \end{aligned}$$

In these three cases (D_i, P_i) is of the form $(\det \circ \rho_i, \mathrm{Pf} \circ \rho_i)$. In particular (D, P) is of the form $(\det \circ \rho, \mathrm{Pf} \circ \rho)$, where $\rho = \bigoplus_{i=1}^{s} \rho_i$. Since R surjects onto the R_i, the ρ_i are semisimple and thus ρ is semisimple.

To prove that the map is injective, let us consider two semisimple representations ρ_1 and ρ_2 of R over $\overline{k}$ of dimension $2d$ that have the same symplectic determinant. By [Che14, Theorem 2.12], ρ_1 and ρ_2 are conjugated by an element $g \in \mathrm{GL}_{2d}(\overline{k})$. We need to show that we can take $g \in \mathrm{Sp}_{2d}(\overline{k})$.

Since the product of copies of the symplectic group embeds diagonally in a symplectic group up to conjugation, it suffices to check this for direct summands of ρ_1 and ρ_2. We can match the irreducible symplectic subrepresentations of ρ_1 and ρ_2. An irreducible subrepresentation of ρ_1, which is contained in an indecomposable symplectic subrepresentation of ρ_1 that is not irreducible, is mapped into an indecomposable symplectic subrepresentation of ρ_2 that is also not irreducible. Thus, we can assume that ρ_1 and ρ_2 are indecomposable as symplectic representations.

We distinguish two cases:

(a) ρ_1 and ρ_2 are irreducible as representations. In this case, they are both surjective onto $M_{2d}(\overline{k})$, so that $\mathrm{Inner}(g) \in \mathrm{Aut}((M_{2d}(\overline{k}), \mathrm{j})) = \mathrm{PSp}_{2d}(\overline{k})$.

(b) The representations are of the form $\rho_i = \rho_{i,1} \oplus \rho_{i,2}$ with $\rho_i(r^\sigma) = (\rho_{i,2}(r)^\top, \rho_{i,1}(r)^\top)$. There exist $g_1, g_2 \in \mathrm{GL}_d(\overline{k})$ such tht $\rho_{1,1} = g_1 \rho_{2,1} g_1^{-1}$ and $\rho_{1,2} = g_2 \rho_{2,2} g_2^{-1}$. The compatibility of the representations with the involution implies that $g_2 = g_1^{-1,\top}$, and so $\mathrm{diag}(g_1, g_2) = \mathrm{diag}(g_1, g_1^{-1,\top}) \in \mathrm{Sp}_{2d}(\overline{k})$.

$\square$

Corollary 8.29. Let (R, σ) be an involutive k-algebra equipped with a symplectic determinant (D, P) over k of dimension $2d$. Assume, that $R/\ker(D)$ is a finitely generated k-algebra. Then there exists a finite field extension k'/k and a symplectic representation $\rho : (R \otimes_k k', \sigma) \to (M_{2d}(k'), \mathrm{j})$ such that $(D \otimes_k k', P \otimes_k k') = (\det \circ \rho, \mathrm{Pf} \circ \rho)$.

Proof. By Lemma 8.14, we may assume, that $\ker(D) = 0$ and that R is a finitely generated k-algebra. By Theorem 8.28, let $\rho_{\overline{k}} : (R \otimes_k \overline{k}, \sigma \otimes \mathrm{id}_{\overline{k}}) \to (M_d(\overline{k}), \mathrm{j})$ be a symplectic representation with $D \otimes_k \overline{k} = \det \circ \rho_{\overline{k}}$ and $P \otimes_k \overline{k} = \mathrm{Pf} \circ \rho_{\overline{k}}|_{(R \otimes_k \overline{k})^+}$. Then the image $\rho_{\overline{k}}(R) \subseteq M_d(\overline{k})$ is as a k-subalgebra generated by finitely many matrices in $M_d(\overline{k})$, hence there is a finite field extension k'/k, such that $\rho_{\overline{k}}(R) \subseteq M_d(k')$. We have $\rho_{\overline{k}}(R^+) \subseteq M_d(k') \cap M_d(\overline{k})^+ = M_d(k')^+$. Thus the restriction of $\rho_{\overline{k}}$ to k' defines a symplectic representation $\rho : (R \otimes_k k', \sigma \otimes \mathrm{id}_{k'}) \to (M_d(k'), \mathrm{j})$.

For every commutative k'-algebra B we obtain a diagram

$$\begin{tikzcd}
R \otimes_{k'} B \arrow[rr, "D_B"] \arrow[dr, "\rho\otimes\mathrm{id}"'] \arrow[dd] & & B \arrow[dd] \\
& M_d(B) \arrow[ur, "\det"'] \arrow[dd] & \\
R \otimes_{k'} (B \otimes_{k'} \overline{k}) \arrow[rr, "D_{B\otimes_{k'}\overline{k}}"] \arrow[dr, "\rho\otimes\mathrm{id}"'] & & B \otimes_{k'} \overline{k} \\
& M_d(B \otimes_{k'} \overline{k}) \arrow[ur, "\det"'] &
\end{tikzcd}$$

By the functorialities of D, det and the base changes of ρ, we know that every square commutes. The bottom triangle commutes by Theorem 8.28. The vertical maps are all injective and so it follows, that the top triangle commutes, hence $\det \circ \rho = D \otimes_k k'$. We proceed similarly for the Pfaffian. $\square$

8.4.4 Symplectic determinant laws over Henselian local rings

We fix a Henselian local ring A with maximal ideal $\mathfrak{m}_A$ and residue field k, and we suppose that $2 \in A^\times$.

Proposition 8.30. Let R be an A-algebra with involution equipped with a d-dimensional Cayley-Hamilton $*$-determinant $D : R \to A$ such that $\overline{D} = \det \circ \overline{\rho}$ for some absolutely irreducible orthogonal (resp. symplectic) representation $\overline{\rho} : (\overline{R}, \sigma) \to (M_d(k), {}^\top)$ (resp. $(M_d(k), \mathrm{j})$). Then there exists an isomorphism of involutive algebras $\rho : (R, \sigma) \to (M_d(A), {}^\top)$ (resp. $(M_d(A), \mathrm{j})$) lifting $\overline{\rho}$ such that $D = \det \circ \rho$.

Proof. First we treat the orthogonal case. By [Che14, Theorem 2.22], we know that $R \cong M_d(A)$ and that $(\overline{R}/\ker(\overline{D}), \overline{\sigma}) \cong (M_d(k), {}^\top)$. We let $\epsilon_{ij} \in M_d(k)$ be the matrix with 1 at the (i,j) entry and 0 elsewhere. By [Che14, Lemma 2.10 (i)], we know that $\mathrm{rad}(R) = \ker(R \to \overline{R}/\ker(\overline{D}))$. This allows us to use the proof of [BC09, Lemma 1.8.2] to show the existence of σ-fixed orthogonal idempotents e_{ii} lifting ϵ_{ii} for $1 \leq i \leq d$ with $\sum_i e_{ii} = 1$. Since A is local, we have a decomposition $A^d = e_{11}A^d \oplus \cdots \oplus e_{dd}A^d$ into free of rank 1 summands, and we may choose generators α_i of $e_{ii}A^d$ such that the base change matrix $g \in \mathrm{GL}_d(A)$ from the canonical basis of A^d to the basis $(\alpha_1, \ldots, \alpha_d)$ reduces to the identity modulo $\mathfrak{m}_A$. Thus after conjugaction by g, we can suppose that e_{ii} is the matrix with 1 at the (i,i) entry and 0 elsewhere. By [Alj+21, Remark 3.4.19], every automorphism of $M_d(A)$ is inner, so there exists an invertible matrix $P \in \mathrm{GL}_d(A)$ such that $\sigma(M)^\top = P^{-1}M^\top P$ for every $M \in M_d(A) \cong R$. It follows from the fact that $\sigma(e_{ii}) = e_{ii}$ that we have $P = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_i \equiv 1 \mod \mathfrak{m}_A$. Since A is Henselian and $2 \in A^\times$, there exists elements $\lambda_i' \in A$ such that $\lambda_i'^2 = \lambda_i$. Letting $Q = \mathrm{diag}(\lambda_1', \ldots, \lambda_d')$, we get an isomorphism of involutive algebras $(M_d(A), \sigma) \to (M_d(A), {}^\top) : M \mapsto QMQ^{-1}$ which is what we want.

The symplectic case reduces to the orthogonal case after conjugating the involution by J. ☐

8.5 Moduli of symplectic representations

8.5.1 Setting

Let A be a noetherian commutative ring with $2 \in A^\times$ and let $(R, *)$ be a finitely generated A-algebra with involution. Let $d \geq 1$ be an integer. The goal of this section is to compare the moduli of $2d$-dimensional symplectic representations of $(R, *)$ to the space of symplectic determinants of dimension $2d$.

We put $S = \mathrm{Spec}(A)$ and recall the following functors on S-schemes. They are defined in analogy to [Wan18, Definition 2.1].

Definition 8.31.

1. Define the functor on S-schemes to $\mathrm{SpRep}^{\square,2d}_{(R,*)} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ by setting

   $$\mathrm{SpRep}^{\square,2d}_{(R,*)}(X) := \{A\text{-algebra morphisms } (R,*) \to (M_{2d}(\Gamma(X, \mathcal{O}_X)), \mathrm{j}) \text{ respecting the involution}\}$$

2. We also define a functor $\mathrm{SpRep}^{2d}_{(R,*)} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Gpd}$ by setting

   $$\begin{aligned} \mathrm{ob}\ \mathrm{SpRep}^{2d}_{(R,*)}(X) := \{&V/X \text{ a rank } 2d \text{ vector bundle,} \\ &b : V \times V \to \mathcal{O}_X \text{ a non-singular skew-symmetric } \mathcal{O}_X\text{-bilinear form,} \\ &\text{and an } A\text{-algebra morphism } \rho : (R,*) \to (\Gamma(X, \mathrm{End}_{\mathcal{O}_X}(V)), \sigma_b) \text{ respecting the involution}\} \end{aligned}$$

   An isomorphism of two objects (V, b, ρ) and (V', b', ρ') is an isomorphism $\alpha : V \to V'$, such that $b' \circ (\alpha \times \alpha) = b$ and $\Gamma(X, \mathrm{End}_{\mathcal{O}_X}(\alpha)) \circ \rho = \rho'$.

3. We also define a functor $\overline{\mathrm{SpRep}}^{2d}_{(R,*)} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Gpd}$ by setting

   $$\begin{aligned} \mathrm{ob}\ \overline{\mathrm{SpRep}}^{2d}_{(R,*)}(X) := \{&(\mathcal{E}, \sigma) \text{ a rank } 4d^2 \text{ Azumaya algebra over } X \text{ equipped with a symplectic involution,} \\ &\text{and an } A\text{-algebra morphism } \rho : (R,*) \to (\Gamma(X, \mathcal{E}), \sigma) \text{ respecting the involution}\} \end{aligned}$$

   An isomorphism of two objects $(\mathcal{E}, \sigma, \rho)$ and $(\mathcal{E}', \sigma', \rho')$ is an isomorphism $\alpha : \mathcal{E} \to \mathcal{E}'$ of Azumaya algebras over $\mathcal{O}_X$, such that $\alpha \circ \rho = \rho'$.

$\mathrm{SpRep}^{\square,2d}_{(R,*)}$ is representable by an affine scheme, which is of finite type over S, if R is finitely generated over A. The functors $\mathrm{SpRep}^{2d}_{(R,*)}$ and $\overline{\mathrm{SpRep}}^{2d}_{(R,*)}$ are (2-)representable by categories fibered in groupoids over Sch/S.

Lemma 8.32. Let X be a scheme and $d \geq 1$.

1. There is a natural bijection of pointed sets between the set of symplectic vector bundles of rank $2d$ on Sch/X up to isomorphism and the set of étale Sp_{2d}-torsors on Sch/X up to isomorphism.

2. There is a natural bijection of pointed sets between the set of Azumaya algebras of rank $4d^2$ equipped with a symplectic involution on Sch/X up to isomorphism and the set of étale PGSp_{2d}-torsors on Sch/X up to isomorphism.

Proof. We first observe, that symplectic vector bundles are the same in the Zariski and in the étale topology. This follows from the equivalence of categories [Sta19, 03DX], which is also used in the proof of Hilbert's Theorem 90 [Sta19, 03P7] in the case of line bundles.

The bijection between étale symplectic vector bundles and étale Sp_{2d}-torsors is now the standard one: Take an étale symplectic vector bundle $(\mathcal{V}, \sigma)$ to the étale $\mathcal{I}som$-sheaf

$$\mathcal{I}som((\mathcal{V}, \sigma), (\mathcal{O}_X^{2d}, \mathrm{std}))(U) := \mathrm{Isom}((\mathcal{V}, \sigma)|_U, (\mathcal{O}_X^{2d}, \mathrm{std})|_U)$$

with Sp_{2d}-action induced by the standard action on $\mathcal{O}_X^{2d}$. It follows directly from local triviality of $(\mathcal{V}, \sigma)$, that $\mathcal{I}som((\mathcal{V}, \sigma), (\mathcal{O}_X^{2d}, \mathrm{std}))$ is an Sp_{2d}-torsor.

Take an étale Sp_{2d}-torsor $\mathcal{T}$ to the étale sheaf quotient $\mathcal{T} \times^{\mathrm{Sp}_{2d}} \mathcal{O}_X^{2d} := (\mathcal{T} \times^{\mathrm{Sp}_{2d}} \mathcal{O}_X^{2d}) / \mathrm{Sp}_{2d}$, which by local triviality of $\mathcal{T}$ is again easily seen to be an étale symplectic vector bundle.

By the same argument using that the automorphism group of the standard Azumaya algebra with symplectic involution is PGSp_{2d} and that Azumaya algebras are étale locally trivial, we see that the groupoid of Azumaya algebras with symplectic involution is equivalent to the groupoid of étale PGSp_{2d}-torsors. $\square$

Theorem 8.33. The canonical functors

$$[\mathrm{SpRep}_{(R,*)}^{\square,2d} / \mathrm{Sp}_{2d}] \xrightarrow{\sim} \mathrm{SpRep}_{(R,*)}^{2d} \qquad \text{and} \qquad [\mathrm{SpRep}_{(R,*)}^{\square,2d} / \mathrm{PGSp}_{2d}] \xrightarrow{\sim} \overline{\mathrm{SpRep}}_{(R,*)}^{2d}$$

are equivalences of étale stacks on Sch/S. On the left hand sides we take the étale stack quotient.

The proof follows closely the proof of [Wan13, Theorem 1.4.1.4]. We remark, that the result is a version of [Wan13, Theorem 1.4.4.6] for representations of algebras instead of groups.

Proof. By [Sta19, 003Z] it is enough to show, that the functors induce equivalences of fiber categories (which are groupoids). For the purpose of this proof the stacks will be described as pseudofunctors from $(\mathrm{Sch}/S)^{\mathrm{op}}$ to the $(2,1)$-category of groupoids in the sense of [Sta19, 003V].

$[\mathrm{SpRep}_{(R,*)}^{\square,2d} / \mathrm{Sp}_{2d}]$ parametrizes for each S-scheme $t : T \to S$ pairs $(f : \mathcal{G} \to T,\ \mathcal{G} \to \mathrm{SpRep}_{(R,*)}^{\square,2d}) \in [\mathrm{SpRep}_{(R,*)}^{\square,2d} / \mathrm{Sp}_{2d}](T)$, where $\mathcal{G}$ is an étale Sp_{2d}-torsor over T and $\mathcal{G} \to \mathrm{SpRep}_{(R,*)}^{\square,2d}$ is an Sp_{2d}-equivariant map of S-schemes.

Using Lemma 8.32, we attach to $\mathcal{G}$ a symplectic vector bundle (V, b) on T. Since $\mathcal{G}(\mathcal{G})$ contains $\mathrm{id}_{\mathcal{G}}$, (V, b) is canonically trivialized over $\mathcal{G}$. The composition

$$f^* t^* R \to (M_{2d}(\mathcal{O}_{\mathcal{G}}), \mathrm{j}) \to \mathrm{End}_{\mathcal{O}_{\mathcal{G}}}(f^* V, \sigma_b)$$

can be descended to a map $t^* R \to \mathrm{End}_{\mathcal{O}_{\mathcal{G}}}(V, \sigma_b)$ using Sp_{2d}-equivariance of $\mathcal{G} \to \mathrm{SpRep}_{(R,*)}^{\square,2d}$. The functor $\mathcal{G} \mapsto (V, b)$ realizes the identification Lemma 8.32 between symplectic vector bundles and Sp_{2d}-torsors. In particular it induces an equivalence between the groupoid of symplectic vector bundles and the groupoid of Sp_{2d}-torsors.

To show, that the functor $[\mathrm{SpRep}_{(R,*)}^{\square,2d} / \mathrm{Sp}_{2d}](T) \to \mathrm{SpRep}_{(R,*)}^{2d}(T)$ is an equivalence, we give a functor in the other direction. It is then formal to verify that this realizes an equivalence of groupoids.

An object of $\mathrm{SpRep}_{(R,*)}^{2d}(T)$ is a triple (V, b, ρ) as in Definition 8.31. We define an Sp_{2d}-torsor $\mathcal{G}$ over T by setting

$$\mathcal{G}(X) := \mathrm{Isom}_{\mathcal{O}_X}((x^* V, b),\ (\mathcal{O}_X^{\oplus 2d}, b_{\mathrm{std}}))$$

for all T-schemes $x : X \to T$. Here b_{std} is the standard symplectic form and isomorphisms shall preserve the bilinear forms. $\mathcal{G}$ is represetable by a flat scheme $f : \mathcal{G} \to T$ of finite presentation over T [You, Theorem 3.24]. The identity map in $\mathcal{G}(\mathcal{G})$ corresponds to an isomorphism $f^* V \xrightarrow{\sim} \mathcal{O}_{\mathcal{G}}^{\oplus 2d}$ compatible with b and b_{std}. The composition

$$(f^* t^* R, *) \xrightarrow{\rho} (\mathrm{End}_{\mathcal{O}_{\mathcal{G}}}(f^* V), \sigma_b) \xrightarrow{\sim} (\mathrm{End}_{\mathcal{O}_{\mathcal{G}}}(\mathcal{O}_{\mathcal{G}}^{\oplus 2d}), \mathrm{j})$$

defines a representation in $\mathrm{SpRep}^{\square,2d}_{(R,*)}(\mathcal{G})$, so we obtain a map $\mathcal{G} \to \mathrm{SpRep}^{\square,2d}_{(R,*)}$. The latter is Sp_{2d}-equivariant, for the action of Sp_{2d} realizes a change of basis. We have constructed an object of $[\mathrm{SpRep}^{\square,2d}_{(R,*)} / \mathrm{Sp}_{2d}](T)$.

The equivalence $[\mathrm{SpRep}^{\square,2d}_{(R,*)} / \mathrm{PGSp}_{2d}] \xrightarrow{\sim} \overline{\mathrm{SpRep}}^{2d}_{(R,*)}$ follows by an analogous argument. We only mention, that PGSp_{2d} is the automorphism group scheme of (M_{2d}, j). We are using Lemma 8.32 to identify étale PGSp_{2d}-torsors and Azumaya algebras with symplectic involution. $\square$

8.5.2 Comparison with the GIT quotient

Assume, that A is noetherian and that R is finitely generated over A.

By [Alp14, Theorem 9.1.4] the canonical map $[\mathrm{SpRep}^{\square,2d}_{(R,*)} / \mathrm{Sp}_{2d}] \to \mathrm{SpRep}^{\square,2d}_{(R,*)} /\!\!/ \mathrm{Sp}_{2d}$ is an adequate moduli space. Since the canonical map $[\mathrm{SpRep}^{\square,2d}_{(R,*)} / \mathrm{Sp}_{2d}] \to \mathrm{SpRep}^{2d}_{(R,*)}$ is an equivalence of stacks (8.33), the map $\phi : \mathrm{SpRep}^{2d}_{(R,*)} \to \mathrm{SpRep}^{\square,2d}_{(R,*)} /\!\!/ \mathrm{Sp}_{2d}$ is an adequate moduli space as well.

The map $\psi^{\square} : \mathrm{SpRep}^{\square,2d}_{(R,*)} \to \mathrm{SpDet}^{2d}_{(R,*)}$ given by mapping a representation to its determinant factors over the stack quotient and thus through a map $\psi : \mathrm{SpRep}^{2d}_{(R,*)} \to \mathrm{SpDet}^{2d}_{(R,*)}$, which in turn factors through the adequate moduli space ϕ. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{SpRep}^{2d}_{(R,*)} \arrow[d, "\psi"] \arrow[dr, "\phi"] & \\
\mathrm{SpDet}^{2d}_{(R,*)} & \mathrm{SpRep}^{\square,2d}_{(R,*)} /\!\!/ \mathrm{Sp}_{2d} \arrow[l, "\nu"]
\end{tikzcd}$$

Recall, that an *adequate homeomorphism* is an integral universal homeomorphism, which is a local isomorphism at all points with residue field of characteristic 0 (see [Alp14, Definition 3.3.1]).

Theorem 8.34. ν is a finite universal homeomorphism.

We follow closely the structure of the proof of [Wan18, Theorem 2.20].

Proof. We know, that ν is surjective and radicial, since ν is a bijection on geometric points [Pro76, Theorem 15.4]. Hence by [Gro67, Corollaire 18.12.11] it suffices to show, that ν is integral and by [Sta19, 01WM] it suffices to show, that ν is universally closed.

We will apply the valuative criterion for universally closed morphisms in the version of [Gro61, Remarques 7.3.9 (i)].

Let B be a complete discrete valuation ring with an algebraically closed residue field and fraction field K.

We will show, that given a diagram of A-schemes:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\alpha"] \arrow[d] & \mathrm{SpRep}^{\square,2d}_{(R,*)} /\!\!/ \mathrm{Sp}_{2d} \arrow[d, "\nu"] \\
\operatorname{Spec} B \arrow[r, "{(D,P)}"] & \mathrm{SpDet}^{2d}_{(R,*)}
\end{tikzcd}$$

there exists a finite field extension K''/K and letting B'' be the integral closure of B in K'' there is a morphism $f : \operatorname{Spec} B'' \to \mathrm{SpRep}^{2d}_{(R,*)}$, such that $\phi \circ f$ fits in the diagram

$$\begin{tikzcd}
\operatorname{Spec} K'' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r, "\alpha"] \arrow[d] & \mathrm{SpRep}^{\square,2d}_{(R,*)} /\!\!/ \mathrm{Sp}_{2d} \arrow[d, "\nu"] \\
\operatorname{Spec} B'' \arrow[r] \arrow[urr, dashed, "\phi \circ f"] & \operatorname{Spec} B \arrow[r] & \mathrm{SpDet}^{2d}_{(R,*)}
\end{tikzcd}$$

and thereby verifies the valuative criterion.

Let (D,P) be the symplectic determinant of $(R,*)$ associated to the point $\operatorname{Spec} B \to \mathrm{SpDet}^{2d}_{(R,*)}$. Our Theorem 8.28 together with [Wan18, 2.19 (1)] implies, that there is a $\overline{K}$-linear semisimple symplectic representation $\rho : (R \otimes_A \overline{K}, *) \to (M_{2d}(\overline{K}), \mathrm{j})$ such that the corresponding point $\operatorname{Spec} \overline{K} \to \mathrm{SpRep}^{2d}_{(R,*)}$ lies above α:

$$\begin{tikzcd} \operatorname{Spec} \overline{K} \arrow[r, "\rho"] \arrow[d] & \mathrm{SpRep}^{2d}_{(R,*)} \arrow[d, "\phi"] \\ \operatorname{Spec} K \arrow[r, "\alpha"] & \mathrm{SpRep}^{\square,2d}_{(R,*)} /\!\!/ \mathrm{Sp}_{2d} \end{tikzcd}$$

By [Che14, Theorem 2.12] and [Che14, Lemma 2.8] we have $\ker(\rho) \cap (R \otimes_A K) = \ker(D \otimes_A K)$, so the action of $R \otimes_A K$ on $\overline{K}^n$ factors through $(R \otimes_A K)/\ker(D \otimes_A K)$, which is finite-dimensional over K by [Wan18, Corollary 2.14]. By Corollary 8.29 there is a finite extension K'/K and a symplectic representation $\rho : R \otimes_A K' \to (M_{2d}(K'), \mathrm{j})$, which induces $(D \otimes_A K', P \otimes_A K')$.

Let B' be the integral closure of B in K'. Let $V' := (K')^{2d}$ be the K'-vector space realizing ρ. Let $L \subseteq V'$ be a B'-lattice and as in the proof of [Wan18, Theorem 2.20], we may assume that L is R-stable. The symplectic bilinear form on V' restricts to a B'-bilinear form $\beta : L \times L \to K'$; beware that we don't know a priori whether β has values in B'. Choose a basis $x_1, \dots, x_{2d}$ of L and let F be the fundamental matrix of β, i.e. $F_{ij} = \beta(x_i, x_j)$. Let ϖ be a uniformizer of B'. Then $\det(F) = a\varpi^r$, where $a \in (B')^\times$ and $r \in \mathbb{Z}$.

We find a finite extension K''/K', such that there is an element $z \in K''$ with $z^{4d} = \varpi^{-r}$. Let B'' be the integral closure of B' (and B) in K''. Let $L'' := L \otimes_{B'} B''$, which is a lattice in $V'' := V' \otimes_{K'} K''$ with basis $x_1, \dots, x_{2d}$. β extends to a B''-bilinear form $\beta : L'' \times L'' \to K''$ with fundamental matrix F. The rescaled lattice zL'' has basis $zx_1, \dots, zx_{2d}$ and fundamental matrix z^2F. It follows, that $\det(z^2F) = a$ and thus β is non-degenerate on zL''. So there is a representation on the B''-lattice zL'' compatible with (the involution induced by) β, which gives $\rho \otimes K''$ after extension of scalars. To obtain an actual symplectic representation $R \otimes_A B'' \to (M_{2d}(B''), \mathrm{j})$, we use [MH74, Corollary 3.5]: Every non-degenerate bilinear form over B'' is congruent to the standard symplectic form. $\square$

8.6 Comparison With Lafforgue's pseudocharacters

We recall the definition of Lafforgue's pseudocharacters for reductive groups. See [Laf18, §11] for the original definition and [BHKT, Definition 4.1] for a definition in the context of deformation theory. For GL_n Lafforgue's definition has been proven to be equivalent to Chenevier's notion of determinant laws [Eme18, Theorem 4.0.1]. We expect, that the bijection constructed in [Eme18, Theorem 4.0.1] restricts to a bijection between Lafforgue's pseudocharacters for the symplectic groups and symplectic determinant laws over commutative $\mathbb{Z}[\frac{1}{2}]$-algebras. The main goal of this section is prove this conjecture in some special cases.

Recall for the next definition, that a reductive group scheme over $\mathbb{Z}$ is connected by definition [Con14b]. Only $G = \mathrm{GL}_d$ and $G = \mathrm{Sp}_{2d}$ will be relevant here.

Definition 8.35 (G-pseudocharacter)**.** Let G be a reductive $\mathbb{Z}$-group scheme, let Γ be an abstract group and let A be a commutative ring. A *G-pseudocharacter* Θ of Γ over A is a sequence of ring homomorphisms

$$\Theta_m : \mathbb{Z}[G^m]^G \to \mathrm{Map}(\Gamma^m, A)$$

for each $m \geq 1$, satisfying the following conditions[1]:

1. For all $n, m \geq 1$, each map $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, every $f \in \mathbb{Z}[G^m]^G$ and all $\gamma_1, \dots, \gamma_n \in \Gamma$, we have
$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)})$$
where $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$.

[1]Here G acts on G^m by $g \cdot (g_1, \dots, g_m) = (gg_1g^{-1}, \dots, gg_mg^{-1})$. This induces a rational action of G on the affine coordinate ring $\mathbb{Z}[G^m]$ of G^n. The submodule $\mathcal{O}[G^m]^G \subseteq \mathcal{O}[G^m]$ is defined as the rational invariant module of the G-representation $\mathcal{O}[G^m]$. It is an $\mathcal{O}$-subalgebra, since G acts by $\mathcal{O}$-linear automorphisms.

2. For all $m \geq 1$, for all $\gamma_1, \dots, \gamma_{m+1} \in \Gamma$ and every $f \in \mathbb{Z}[G^m]^G$, we have

$$\Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_{m+1}) = \Theta_m(f)(\gamma_1, \dots, \gamma_m \gamma_{m+1})$$

where $\hat{f}(g_1, \dots, g_{m+1}) = f(g_1, \dots, g_m g_{m+1})$.

We denote the set of G-pseudocharacters of Γ over A by $\mathrm{PC}_G^\Gamma(A)$. If $f : A \to B$ is a ring homomorphism, then there is an induced map $f_* : \mathrm{PC}_G^\Gamma(A) \to \mathrm{PC}_G^\Gamma(B)$. This defines a functor $\mathrm{PC}_G^\Gamma : \underline{\mathrm{CRing}} \to \mathrm{Set}$, which is representable by a commutative ring B_G^Γ Theorem 4.46.

The definition of G-pseudocharacter can be brought into a more convenient and practical form. Let $\mathcal{F} := \{\mathrm{FG}(m) \mid m \geq 1\}$ be the category of finitely generated free groups $\mathrm{FG}(m)$ on m letters. Then the associations $\mathbb{Z}[G^\bullet]^G : \mathrm{FG}(m) \mapsto \mathbb{Z}[G^m]^G$ and $\mathrm{Map}(\Gamma^\bullet, A) : \mathrm{FG}(m) \mapsto \mathrm{Map}(\Gamma^m, A)$ give rise to functors $\mathcal{F} \to \underline{\mathrm{CRing}}$. It can be proved, that there is a natural bijection

$$\mathrm{PC}_G^\Gamma(A) \cong \mathrm{Nat}(\mathbb{Z}[G^\bullet]^G, \mathrm{Map}(\Gamma^\bullet, A))$$

for any commutative ring A, where Nat stands for natural transformations. See Corollary 4.45 for more details.

Now assume, that $G = \mathrm{Sp}_{2d}$. For $m \geq 1$, the Sp_{2d}-module $\mathbb{Z}[\mathrm{Sp}_{2d}^m]$ has a good filtration by [Jan03, §B.8] and Mathieu's tensor product theorem [Mat90] (which holds also over the integers, see e.g. Theorem 4.16) and $H^i(\mathrm{Sp}_{2d}, \mathbb{Z}[\mathrm{Sp}_{2d}^m]) = 0$ for all $i > 0$ [Jan03, §B.9]. In particular for any homomorphism of commutative rings $A \to B$, we have

$$B[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \cong \mathbb{Z}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \otimes_{\mathbb{Z}} B \cong (\mathbb{Z}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \otimes_{\mathbb{Z}} A) \otimes_A B \cong A[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \otimes_A B$$

We recall the definition of characteristic polynomials of Lafforgue pseudocharacters here.

Definition 8.36. Let A be a commutative ring and let $\Theta \in \mathrm{PC}_{\mathrm{GL}_d}^\Gamma(A)$. Then we define the *characteristic polynomial* of Θ by

$$\chi^\Theta(\gamma, t) := \sum_{i=0}^{d} (-1)^i \Theta_1(s_i)(\gamma) t^{d-i} \quad \in A[t]$$

where $s_i \in \mathbb{Z}[\mathrm{GL}_d]^{\mathrm{GL}_d}$ are the unique invariant regular functions, which satisfy

$$\det(t - \mathbb{X}) = \sum_{i=0}^{d} (-1)^i s_i(\mathbb{X}) t^{d-i}$$

in $\mathbb{Z}[\mathrm{GL}_d][t]^{\mathrm{GL}_d}$, where $\mathbb{X}$ is the generic matrix coordinate in $\mathrm{GL}_d(\mathbb{Z}[\mathrm{GL}_d])$ which corresponds to the identity under the Yoneda isomorphism.

Proposition 8.37. Let A be a commutative ring. Then the map

$$\mathrm{PC}_{\mathrm{GL}_d}^\Gamma(A) \to \mathrm{Map}(\Gamma, A[t]), \quad \Theta \mapsto \chi^\Theta$$

is injective.

Proof. It suffices to show, that a GL_d-pseudocharacter Θ is determined by the values $\Theta_1(s_i)$ for $1 \leq i \leq d$. By Corollary 5.13, these are generators of the $\mathcal{F}$-$\mathbb{Z}$-algebra $\mathbb{Z}[\mathrm{GL}_d^\bullet]^{\mathrm{GL}_d}$, so the claim follows. $\square$

Now we are in shape to define a comparison map in one direction:

Proposition 8.38. Let $\Theta^u \in \mathrm{PC}_{\mathrm{Sp}_{2d}}^\Gamma(B_{\mathrm{Sp}_{2d}}^\Gamma)$ be the universal Sp_{2d}-pseudocharacter and let C be a commutative $B_{\mathrm{Sp}_{2d}}$-algebra. We have seen, that $C[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \cong B_{\mathrm{Sp}_{2d}}^\Gamma[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \otimes_{B_{\mathrm{Sp}_{2d}}^\Gamma} C$ and Θ_m^u induces a homomorphism $\Theta_{m,C}^u : C[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \to \mathrm{Map}(\Gamma^m, C)$ for all $m \geq 1$. We define maps

$$D_C : C[\Gamma] \to C, \quad \sum_{i=1}^m c_i \gamma_i \quad \mapsto \quad \Theta_{m,C}^u\left(\det\left(\sum_{i=1}^m c_i \mathbb{X}_i\right)\right)(\gamma_1, \dots, \gamma_m)$$

$$P_C : C[\Gamma]^+ \to C, \quad \sum_{i=1}^m c_i(\gamma_i + \gamma_i^{-1}) \quad \mapsto \quad \Theta_{m,C}^u\left(\mathrm{Pf}\left(\sum_{i=1}^m c_i(\mathbb{X}_i + \mathbb{X}_i^{-1})\right)\right)(\gamma_1, \dots, \gamma_m)$$

Then D is a $2d$-dimensional determinant law over $B^\Gamma_{\mathrm{Sp}_{2d}}$ and P is a d-homogeneous polynomial law with $P^2 = D|_{B^\Gamma_{\mathrm{Sp}_{2d}}[\Gamma]^+}$ and $P(1) = 1$. In particular we have a natural map $\mathrm{PC}^\Gamma_{\mathrm{Sp}_{2d}}(A) \to \mathrm{SpDet}^\Gamma_{2d}(A)$ for every commutative $\mathbb{Z}[\frac{1}{2}]$-algebra A.

Here $\mathbb{X}_i \in \mathrm{Sp}_{2d}(\mathbb{Z}[\mathrm{Sp}^m_{2d}])$ is the generic i-th coordinate, i.e. the unique element of $\mathrm{Sp}_{2d}(\mathbb{Z}[\mathrm{Sp}^m_{2d}])$, such that for all commutative rings A and all tuples $T = (T_1, \dots, T_m) \in \mathrm{Sp}_{2d}(A)^m$, we have $\mathbb{X}_i(T_1, \dots, T_m) = \mathrm{Sp}_{2d}(T)(\mathbb{X}_i) = T_i$.

(D, P) is formally a symplectic determinant law attached to the universal Sp_{2d}-pseudocharacter over $\mathbb{Z}$, but in Definition 8.7 we require that 2 shall be invertible on the base, so we refrain from calling it a symplectic determinant law.

Proof. The way the maps are defined is functorial, so clearly D and P are polynomial laws. We check $2d$-homogeneity of D:

$$\Theta^u_{m,C}\left(\det\left(\sum_{i=1}^m cc_i\mathbb{X}_i\right)\right) = c^{2d}\Theta^u_{m,C}\left(\det\left(\sum_{i=1}^m c_i\mathbb{X}_i\right)\right)$$

for all $m \geq 1$, $C \in \underline{\mathrm{CRing}}$, $c, c_i \in C$ and $\mathbb{X}_i \in \mathrm{Sp}_{2d}(\mathbb{Z}[\mathrm{Sp}^m_{2d}])$ the generic matrix corresponding to the i-th projection $\mathrm{Sp}^m_{2d} \to \mathrm{Sp}_{2d}$ by Yoneda. d-homogeneity of P follows similarly. For multiplicativity, we notice, that

$$\Theta^u_{m+m',C}\left(\det\left(\sum_{i=1}^m c_i\mathbb{X}_i\right)\right)\Theta^u_{m+m',C}\left(\det\left(\sum_{j=1}^{m'} c'_j\mathbb{X}_{m+j}\right)\right) = \Theta^u_{m+m',C}\left(\det\left(\sum_{i=1}^m\sum_{j=1}^{m'} c_ic'_j\mathbb{X}_i\mathbb{X}_{m+j}\right)\right)$$

Define

$$\mu := \det\left(\sum_{i=1}^m\sum_{j=1}^{m'} c_ic'_j\mathbb{X}_i\mathbb{X}_{m+j}\right) \qquad \mu' := \det\left(\sum_{i=1}^m\sum_{j=1}^{m'} c_ic'_j\mathbb{X}_{i+(j-1)m}\right)$$

Now

$$\begin{aligned}&\Theta^u_{m+m',C}(\mu)(\gamma_1, \dots, \gamma_m, \gamma'_1, \dots, \gamma'_{m'})\\ =&\Theta^u_{mm',C}(\mu')(\gamma_1\gamma'_1, \gamma_1\gamma'_2, \dots, \gamma_m\gamma'_{m'})\end{aligned}$$

holds by a suitable substitution in an $\mathcal{F}$-$\mathbb{Z}$-algebra. $*$-invariance of D, $P^2 = D|_{C[\Gamma]^+}$ and $P(1) = 1$ follow by a similar substitution. $\square$

Let $\Theta \in \mathrm{PC}^\Gamma_{\mathrm{Sp}_{2d}}(A)$ and write (D_Θ, P_Θ) for the image of Θ under the natural map $\mathrm{PC}^\Gamma_{\mathrm{Sp}_{2d}}(A) \to \mathrm{SpDet}^\Gamma_{2d}(A)$. We observe, that the comparison map is compatible with characteristic polynomials: We have $\Theta = \varphi_\Theta(\Theta^u)$ for the arrow $\varphi_\Theta : \mathbb{Z}[\frac{1}{2}][\mathrm{PC}^\Gamma_{\mathrm{Sp}_{2d}}] \to A$ associated to Θ. Then $D_\Theta = \varphi_\Theta \circ \tilde{D}$ with $\tilde{D}$ associated to Θ^u as in Proposition 8.38. We see that the comparison map is compatible with characteristic polynomials:

$$\begin{aligned}\chi^{D_\Theta}(\gamma, t) &= \varphi_\Theta(\chi^{\tilde{D}}(\gamma, t))\\ &= \varphi_\Theta(\tilde{D}_{\mathbb{Z}[t]}(t - \gamma))\\ &= \varphi_\Theta(\Theta^u_1(\det(t - \mathbb{X}_1))(\gamma))\\ &= \varphi_\Theta(\chi^{\Theta^u}(\gamma, t))\\ &= \chi^\Theta(\gamma, t)\end{aligned}$$

for all $\gamma \in \Gamma$.

Lemma 8.39. The map $\mathrm{PC}^\Gamma_{\mathrm{Sp}_{2d}}(A) \to \mathrm{SpDet}^\Gamma_{2d}(A)$ defined in Proposition 8.38 is injective.

Proof. Indeed, the map $\mathrm{PC}^\Gamma_{\mathrm{Sp}_{2d}}(A) \to \mathrm{PC}^\Gamma_{\mathrm{GL}_{2d}}(A)$ induced by the standard representation $\iota : \mathrm{Sp}_{2d} \to \mathrm{GL}_{2d}$ is injective, since the maps $\mathbb{Z}[\mathrm{GL}^m_{2d}]^{\mathrm{GL}_{2d}} \twoheadrightarrow \mathbb{Z}[\mathrm{Sp}^m_{2d}]^{\mathrm{Sp}_{2d}}$ are surjective by Proposition 8.45. The map $\mathrm{SpDet}^\Gamma_{2d}(A) \to \mathrm{Det}^\Gamma_{2d}(A)$ forgetting the Pfaffian is injective by Proposition 8.11. The claim follows, since we have a bijection $\mathrm{PC}^\Gamma_{\mathrm{GL}_{2d}}(A) \to \mathrm{Det}^\Gamma_{2d}(A)$ by [Eme18, Theorem 4.0.1]. $\square$

Lemma 8.40. If $A \to B$ is an injective homomorphism of $\mathbb{Z}[\frac{1}{2}]$-algebras and $\mathrm{PC}^{\Gamma}_{\mathrm{Sp}_{2d}}(B) \to \mathrm{SpDet}^{\Gamma}_{2d}(B)$ defined in Proposition 8.38 is a bijection, then $\mathrm{PC}^{\Gamma}_{\mathrm{Sp}_{2d}}(A) \to \mathrm{SpDet}^{\Gamma}_{2d}(A)$ is a bijection.

Proof. By Lemma 8.39 it is enough to show surjectivity. Let $(D, P) \in \mathrm{SpDet}^{\Gamma}_{2d}(A)$. By Proposition 8.11, the natural map $\mathrm{SpDet}^{\Gamma}_{2d}(A) \to \mathrm{Det}^{\Gamma}_{2d}(A)$ is injective. By assumption $(D \otimes_A B, P \otimes_A B) \in \mathrm{SpDet}^{\Gamma}_{2d}(B)$ corresponds to a unique $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{Sp}_{2d}}(B)$. We know, that $\chi^D(\gamma, t) = \chi^{D\otimes_A B}(\gamma, t)$ for all $\gamma \in \Gamma$ and thus (for the standard representation $\iota : \mathrm{Sp}_{2d} \to \mathrm{GL}_{2d}$) the coefficients of $\chi^{\iota(\Theta)}(\gamma, t)$ lie in A. By Proposition 8.45, these coefficients determine $\Theta_1 : \mathbb{Z}[\mathrm{Sp}_{2d}]^{\mathrm{Sp}_{2d}} \to \mathrm{Map}(\Gamma, B)$. But also by Corollary 8.46 these elements already generate the entire $\mathcal{F}$-$\mathbb{Z}$-algebra $\mathbb{Z}[\mathrm{Sp}^{\bullet}_{2d}]^{\mathrm{Sp}_{2d}}$ and thereby all homomorphisms $\Theta_m : \mathbb{Z}[\mathrm{Sp}^m_{2d}]^{\mathrm{Sp}_{2d}} \to \mathrm{Map}(\Gamma^m, B)$. By considering substitutions with morphisms of the category $\mathcal{F}$, it follows, that the image of Θ_m is contained in $\mathrm{Map}(\Gamma^m, A)$. Thus Θ descends to an Sp_{2d}-pseudocharacter in $\mathrm{PC}^{\Gamma}_{\mathrm{Sp}_{2d}}(A)$, which we will also call Θ. It remains to show, that Θ is indeed mapped to (D, P), but this follows from the compatibility of the comparison map with characteristic polynomials: We have $\chi^{\iota(\Theta)}(\gamma, t) = \chi^{D\otimes_A B}(\gamma, t) = \chi^D(\gamma, t)$. Since D is determined by χ^D and P is determined by P, Θ is necessarily mapped to (D, P). ☐

Proposition 8.41. Let A be a reduced commutative $\mathbb{Z}[\frac{1}{2}]$-algebra. Then the map $\mathrm{PC}^{\Gamma}_{\mathrm{Sp}_{2d}}(A) \to \mathrm{SpDet}^{\Gamma}_{2d}(A)$ defined in Proposition 8.38 is bijective.

Proof. By Lemma 8.39 it is enough to show surjectivity. If $(D, P) \in \mathrm{SpDet}^{\Gamma}_{2d}(A)$, we know by [Eme18, Theorem 4.0.1], that there is some $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_{2d}}(A)$, that maps to D. So it is enough to show, that Θ_m factors over $\mathbb{Z}[\mathrm{Sp}^m_{2d}]^{\mathrm{Sp}_{2d}}$ for all $m \geq 1$.

By Lemma 8.40, the proof of the proposition reduces to the case of an algebraically closed field in three steps: First embed $A \to \prod_{\mathfrak{p}} A/\mathfrak{p}$, where $\mathfrak{p}$ varies over all prime ideals of A. Second, representable functors preserve products, so the claim for $\prod_{\mathfrak{p}} A/\mathfrak{p}$ reduces to the claim for $A/\mathfrak{p}$. Third, embed an integral domain $A/\mathfrak{p}$ into an algebraic closure of its fraction field. If A is an algebraically closed field, then by Theorem 8.28 there is a semisimple representation $\rho : \Gamma \to \mathrm{Sp}_{2d}(A)$, that induces (D, P). The Sp_{2d}-pseudocharacter induced by ρ is necessarily mapped to (D, P). ☐

8.7 Symplectic and orthogonal matrix invariants

The main theorem of [Zub99] is stated as follows:

Theorem 8.42. Let $G = \mathrm{Sp}_d$ or O_d and let K be an algebraically closed field (of characteristic $\neq 2$ in the orthogonal case). Then the invariant algebra $K[M^m_d]^G$ is generated by the elements

$$(X_1, \ldots, X_m) \mapsto \sigma_i(Y_{j_1} \cdots Y_{j_s})$$

where every matrix Y_i is either X_i or the symplectic (or orthogonal) transpose $X_i^{\mathfrak{j}}$ and $\sigma_i(X)$ is the i-th coefficient of the characteristic polynomial of X.

Using this theorem, we use the idea of Donkin (cf. [Don92]) to find generators of the symplectic invariants of several matrices with integral coefficients.

Proposition 8.43. Let $G = \mathrm{Sp}_d$, then the invariant algebra $\mathbb{Z}[M^m_d]^G$ is generated by the elements

$$(X_1, \ldots, X_m) \mapsto \sigma_i(Y_{j_1} \cdots Y_{j_s})$$

defined above.

Proof. Let us write $\widetilde{R} = \mathbb{Z}[M^m_d]^G$ and let $R \subseteq \widetilde{R}$ be the subalgebra generated by the functions defined in the statement of the proposition. We need to show that this inclusion is an equality.

Note that the algebra of regular functions on m matrices has a natural grading $K[M^m_d] = \bigoplus_{\alpha\in\mathbb{N}^m} K[M^m_d]_\alpha$ defined by giving to the (i, j)-entry $x^{(l)}_{i,j}$ of the l-th matrix X_l $(1 \leq l \leq m)$ the degree $(0, .., 1, .., 0)$ (the 1 is in the l-th position). In particular, the grading on $\mathbb{C}[M^m_d]$ induces a grading on R and $\widetilde{R}$.

By [Don92, §3], $K[M^m_d]_\alpha$ has a good filtration as a $\mathrm{GL}_d(K)$-module. But as mentioned in the proof of [Don94, Theorem 3.9], the restriction to Sp_d of a GL_d-module with a good filtration has a good filtration.

From [Don90, Proposition 1.2a(iii)], we get that $\dim K[M_d^m]_\alpha^G$ is the coefficient of the character of the Weyl module $\nabla(0)$ in the expension of the character of the G-module $K[M_d^m]$ as a $\mathbb{Z}$-linear combination of the characters of $\nabla(\lambda)$ for $\lambda \in X^+$ (loc.cit.). In particular $d_\alpha = \dim K[M_d^m]_\alpha^G$ is the same for all algebraically closed fields K.

Let $R_\alpha := R \cap \mathbb{Z}[M_d^m]_\alpha$. Since $\mathbb{C} \otimes_\mathbb{Z} R_\alpha = \mathbb{C} \otimes_\mathbb{Z} \widetilde{R}_\alpha = \mathbb{C}[M_d^m]_\alpha$, we get that $\operatorname{rank}_\mathbb{Z} R_\alpha = \operatorname{rank}_\mathbb{Z} \widetilde{R}_\alpha = d_\alpha$. Also by Theorem 8.42 we have a sequence of morphisms

$$\begin{tikzcd} K \otimes_\mathbb{Z} R_\alpha \arrow[r] \arrow[rr, bend left, two heads] & K \otimes_\mathbb{Z} \widetilde{R}_\alpha \arrow[r] & K[M_d^m]_\alpha^G \end{tikzcd}$$

where all of the vector spaces have the same dimension d_α, so all of the arrows are isomorphisms. In particular, we have $K \otimes_\mathbb{Z} R_\alpha \cong K \otimes_\mathbb{Z} \widetilde{R}_\alpha$ for every algebraically closed field K, and so $R_\alpha = \widetilde{R}_\alpha$ which is enough to conclude. □

Consider the general symplectic group GSp_{2n} over $\mathbb{Z}$ whose funtor of points is as follows:

$$\mathrm{GSp}_{2n}(A) = \{g \in \mathrm{M}_{2n}(A) \mid gg^* = \lambda(g) \cdot \mathrm{id} \text{ for some } \lambda(g) \in A^*\}$$

There is a natural embedding of GSp_{2n} inside $\mathrm{M}_{2n} \times \mathbb{A}^1$ given by $g \mapsto (g, \lambda(g)^{-1})$. This way, we see that $\mathbb{Z}[\mathrm{GSp}_{2n}] = \mathbb{Z}[c_{i,j}, \delta]/I$ where I is the ideal generated by the relation $\det(c_{i,j})_{i,j} \cdot \delta^n = 1$ and the relations coming from the identity $(c_{i,j})_{i,j} \cdot (c_{i,j})^*_{i,j} = \delta^{-1} \cdot \mathrm{id}$. Thus we see that for the grading on $\mathbb{Z}[c_{i,j}, \delta]$ such that $\deg(c_{i,j}) = 1$ and $\deg(\delta) = -2$, I is homogeneous and so the grading can be transferred to $\mathbb{Z}[\mathrm{GSp}_{2n}]$. More generally we will consider the graded ring

$$\mathbb{Z}[\mathrm{GSp}_{2n}^m] = \bigoplus_{\alpha \in \mathbb{Z}^m} \mathbb{Z}[\mathrm{GSp}_{2n}^m]_\alpha \tag{13}$$

Lemma 8.44. Let K be an algebraically closed field, then the invariant algebra $K[\mathrm{GSp}_d^k]^{\mathrm{Sp}_d}$ is generated by the functions

$$(X_1, \ldots, X_k) \mapsto \sigma_i(X_{j_1} \cdots X_{j_s}) \quad \text{and} \quad (X_1, \ldots, X_k) \mapsto \lambda^{-1}(X_i)$$

where $\sigma_i(X)$ is the i-th coefficient of the characteristic polynomial of X.

Proof. The proof is based on a remark in [Zub99, §3]. Consider the canonical morphism of algebraic groups $\pi : \mathrm{Sp}_d \times \mathbb{G}_m \to \mathrm{GSp}_d$ which is surjective since it is surjective on the K-points (see [Knu+98, Proposition 22.3]). The same is true for $\pi^{\otimes k}$ so we get an embedding $(\pi^{\otimes k})^* : K[\mathrm{GSp}_d^k] \hookrightarrow K[(\mathrm{Sp}_d \times \mathbb{G}_m)^k]$. Note that for $h \in \mathrm{Sp}_d(K)$ and $f \in K[\mathrm{GSp}_d^k]$, we have $h \cdot ((\pi^{\otimes k})^* f) = (\pi^{\otimes k})^*(\pi^{\otimes k}(h) \cdot f)$ (the action is by conjugation). Therefore we get that $(\pi^{\otimes k})^*(K[\mathrm{GSp}_d^k]^{\mathrm{Sp}_d}) = (\pi^{\otimes k})^*(K[\mathrm{GSp}_d^k]) \cap K[(\mathrm{Sp}_d \times \mathbb{G}_m)^k]^{\mathrm{Sp}_d} = (\pi^{\otimes k})^*(K[\mathrm{GSp}_d^k]) \cap K[\mathrm{Sp}_d^k]^{\mathrm{Sp}_d} \otimes K[\mathbb{G}_m^k]$ hence the result. □

Proposition 8.45. The invariant algebra $\mathbb{Z}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}}$ is generated by the elements

$$(X_1, \ldots, X_m) \mapsto \sigma_i(Y_{j_1} \cdots Y_{j_s})$$

defined in Theorem 8.42.

Proof. Let T be a maximal torus of Sp_{2d} and let $(\pi_n)_{n\geq 1}$ be an ascending sequence of finite saturated subsets of $X^+(T)$ such that $\bigcup_{n\geq 1} \pi_n = \pi = X^+(T)$, which is possible since Sp_{2d} is semisimple. For a field K, let O_τ be the truncation functor associated to a finite saturated subset $\tau \subseteq X^+(T^m)$ whose definition and properties we are going to use are given in [Jan03, §A]. This definition makes sense over $\mathbb{Z}$ for a finite saturated τ by setting $O_\tau(\mathbb{Z}[\mathrm{Sp}_{2d}^m]) := O_\tau(\mathbb{Q}[\mathrm{Sp}_{2d}^m]) \cap \mathbb{Z}[\mathrm{Sp}_{2d}^m]$, which is a finitely generated free $\mathbb{Z}$-module. We have for any field K ([Jan03, §A.24]):

$$O_\tau(\mathbb{Z}[\mathrm{Sp}_{2d}^m]) \otimes_\mathbb{Z} K = O_\tau(K[\mathrm{Sp}_{2d}^m]) \tag{14}$$

For the cartesian power $\pi^m = X^+(T)^m$, we have $\pi^m = \bigcup_{n\geq 1} \pi_n^m$ and π_n^m are finite saturated subsets for the group Sp_{2d}^m. By definition, we have $O_{\pi^m}(\mathbb{Q}[\mathrm{Sp}_{2d}^m]) = \mathbb{Q}[\mathrm{Sp}_{2d}^m]$ and since $O_{\pi^m}(\mathbb{Q}[\mathrm{Sp}_{2d}^m]) = \bigcup_{n\geq 1} O_{\pi_n^m}(\mathbb{Q}[\mathrm{Sp}_{2d}^m])$ ([Jan03, §A.1]), we get that $(O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m]))_{n\geq 1}$ is an ascending filtration of $\mathbb{Z}[\mathrm{Sp}_{2d}^m]$.

Now let R be the subalgebra of $\mathbb{Z}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}}$ generated by the elements in the statement of the proposition and let $R_n := R \cap O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m])$. By [Jan03, Lemma A.15], for any field K $O_{\pi_n^m}(K[\mathrm{Sp}_{2d}^m])$ is finite-dimensional and admits a good filtration as an $\mathrm{Sp}_{2d}^m \times \mathrm{Sp}_{2d}^m$-module (for the left action induced by left multiplication by the first factor and inverse right multiplication by the second factor on Sp_{2d}^m) with factors $\nabla(\lambda) \otimes \nabla(-w_0\lambda)$ for $\lambda \in \pi_n^m$. By [Don94, Theorem 3.3], the tensor product of two induced modules $\nabla(\lambda) \otimes \nabla(\lambda')$ admits a good filtration, hence $O_{\pi_n}(K[\mathrm{Sp}_{2d}^m])$ admits a good filtration as an Sp_{2d}^m-module under conjugation. But by [Jan03, Lemma I.3.8], $\nabla(\lambda) = \otimes_i \nabla(\lambda_i)$ for $\lambda = (\lambda_i)_{1 \leq i \leq m} \in X^+(T^m)$, so by the same argument as before, we get that $O_{\pi_n^m}(K[\mathrm{Sp}_{2d}^m])$ admits a good filtration as an Sp_{2d}-module. It follows from [Jan03, Lemma B.9] that $O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m])$ admits a good filtration as an Sp_{2d}-module, hence by [Don90, Proposition 1.2a (iii)]

$$\mathrm{rank}_{\mathbb{Z}}\, O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}} = \dim_K O_{\pi_n^m}(K[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}} =: d_n$$

for any field K.

We have an exact sequence

$$0 \to R_n \to \mathbb{Z}[\mathrm{Sp}_{2d}^m] \to (\mathbb{Z}[\mathrm{Sp}_{2d}^m]/R) \times (\mathbb{Z}[\mathrm{Sp}_{2d}^m]/O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m]))$$

so tensoring with $\mathbb{Q}$ gives an exact sequence

$$0 \to R_n \otimes \mathbb{Q} \to \mathbb{Q}[\mathrm{Sp}_{2d}^m] \to (\mathbb{Q}[\mathrm{Sp}_{2d}^m]/(R \otimes \mathbb{Q})) \times (\mathbb{Q}[\mathrm{Sp}_{2d}^m]/O_{\pi_n^m}(\mathbb{Q}[\mathrm{Sp}_{2d}^m]))$$

By [Zub99, Proposition 3.2], we have $R \otimes \mathbb{Q} = \mathbb{Q}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}}$, so the kernel of the rightmost arrow is $\mathbb{Q}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \cap O_{\pi_n^m}(\mathbb{Q}[\mathrm{Sp}_{2d}^m]) = O_{\pi_n^m}(\mathbb{Q}[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}}$.

Hence $R_n \otimes_{\mathbb{Z}} \mathbb{Q} = O_{\pi_n}(\mathbb{Q}[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}}$, and in particular we get that $\mathrm{rank}_{\mathbb{Z}} R_n = d_n$.

We claim, that R_n is cotorsionfree in R: We know, that by definition R/R_n embeds into $\mathbb{Z}[\mathrm{Sp}_{2d}^m]/O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m])$, which is free.

Let K be an algebraically closed field. The top map in the following diagram is an isomorphism by [Zub99, Proposition 3.2].

$$\begin{tikzcd}
R \otimes K \arrow[r, "\cong"] & K[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}} \\
R_n \otimes K \arrow[u, hook] \arrow[r, hook] & O_{\pi_n^m}(K[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}} \arrow[u, hook]
\end{tikzcd}$$

So the bottom map injective. Since $\mathrm{rank}_{\mathbb{Z}} R_n = d_n$, it must be an isomorphism. We deduce, that in the following diagram all maps are isomorphisms.

$$\begin{tikzcd}
R_n \otimes_{\mathbb{Z}} K \arrow[r] \arrow[rr, bend left] & O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}} \otimes_{\mathbb{Z}} K \arrow[r] & O_{\pi_n^m}(K[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}}
\end{tikzcd}$$

Since this is true for every algebraically closed field K, the map $R_n \to O_{\pi_n^m}(\mathbb{Z}[\mathrm{Sp}_{2d}^m])^{\mathrm{Sp}_{2d}}$ of finitely generated free $\mathbb{Z}$-modules is an isomorphism. So $R = \mathbb{Z}[\mathrm{Sp}_{2d}^m]^{\mathrm{Sp}_{2d}}$, as desired. ☐

Corollary 8.46. The $\mathcal{F}$-$\mathbb{Z}$-algebra $\mathbb{Z}[\mathrm{Sp}_{2d}^{\bullet}]^{\mathrm{Sp}_{2d}}$ is generated by the elements $\sigma_1, \dots, \sigma_{2d}$ defined in Theorem 8.42.

Proof. This follows directly from Proposition 8.45 and subsitutions with morphisms from $\mathcal{F}$. ☐

Remark 8.47. The statements of Proposition 8.43 and Proposition 8.45 hold after replacing $\mathbb{Z}$ by an arbitrary commutative ring A. Since the Sp_d-modules $\mathbb{Z}[M_d^m]$ and $\mathbb{Z}[\mathrm{Sp}_d^m]$ have good filtrations, taking invariants commutes with tensoring with A, so $A[M_d^m]^{\mathrm{Sp}_d} = \mathbb{Z}[M_d^m]^{\mathrm{Sp}_d} \otimes_{\mathbb{Z}} A$ and $A[\mathrm{Sp}_d^m]^{\mathrm{Sp}_d} = \mathbb{Z}[\mathrm{Sp}_d^m]^{\mathrm{Sp}_d} \otimes_{\mathbb{Z}} A$. The same arguments go through for the odd orthogonal groups O_d ($d \geq 3$) using Zubkov's computation [Zub99] of generators of these invariant rings over an algebraically closed field. The arguments in Lemma 8.44 can be adapted to the general orthogonal groups GO_d ($d \geq 2$) and we obtain the same generators with the inverse of the orthogonal similitude character in place of the symplectic similitude character.

A On common R-Levis and G-semisimplification

The content of this section of the appendix also occurs in [PQ23]. As this work is not yet available, the relevant preparations for the proof of reconstruction theorem for disconnected groups Theorem 4.56 are taken from there.

Let G be a (possibly disconnected) reductive group over an algebraically closed field κ.

Lemma A.1. Let P and Q be R-parabolic subgroups of G. Then $P \cap Q$ contains a maximal torus of G.

Proof. By [Mar03, Rmk. 5.3], P^0 and Q^0 are parabolic subgroups of G^0. By [BT65, §2.4] $P^0 \cap Q^0$ contains a maximal torus T of G^0. T is also a maximal torus of G. ☐

Lemma A.2. Let P and Q be R-parabolic subgroups of G. Assume, that Q contains an R-Levi of P and P contains an R-Levi of Q. Then P and Q have a common R-Levi.

Proof. Let T be a maximal torus of G contained in $P \cap Q$ (Lemma A.1). Let L be an R-Levi of P, which contains T and let M be an R-Levi of Q, which contains T. Existence and uniqueness of L and M follow from [BMR05, Cor. 6.5]. Since by assumption $P \cap Q$ contains an R-Levi of P as well as an R-Levi of Q, the maps

$$P \cap Q \to P/R_u(P)$$

$$P \cap Q \to Q/R_u(Q)$$

are surjective. Since by [BMR05, Lem. 6.2 (iii)], $P \cap Q = (L \cap M)R_u(P \cap Q)$, we obtain surjections

$$L \cap M \to P/R_u(P)$$

$$L \cap M \to Q/R_u(Q)$$

Hence $P = (L \cap M)R_u(P)$. Since $P = L \cdot R_u(P)$ and $L \cap R_u(P) = 1$, we have $L \cap M = L$. Similarly, we have $L \cap M = M$. ☐

Lemma A.3. Let H be a closed subgroup of G. Let P and Q be R-parabolic subgroups of G, both minimal among R-parabolics containing H. Then P and Q have a common R-Levi.

Proof. The group $(P \cap Q)R_u(Q)$ contains H and is contained in Q and by [BMR05, Cor. 6.9], $(P \cap Q)R_u(Q)$ is R-parabolic. By minimality of Q, we have $Q = (P \cap Q)R_u(P)$. Again by [BMR05, Cor. 6.9], P contains an R-Levi subgroup of Q. Similarly Q contains an R-Levi of P. We can apply Lemma A.2. ☐

Our definition of G-semisimplification of representations is close to the definition of G-semisimplification of subgroups of $G(\kappa)$ given in [BMR20, Definition 4.1]. The main difference is, that we have to keep track of the map from Γ to $G(\kappa)$.

Definition A.4. Let $\rho : \Gamma \to G(\kappa)$ be a homomorphism. Let P be an R-parabolic of G, such that $\rho(\Gamma) \subseteq P(\kappa)$ and such that P is minimal among all R-parabolics with this property. Let L be an R-Levi of P. We have a canonical surjective homomorphism $c_{P,L} : P \to L$. We define the *G-semisimplification* ρ^{ss} of ρ with respect to P and L as the composition $\Gamma \xrightarrow{\rho} P(\kappa) \xrightarrow{c_{P,L}} L(\kappa) \to G(\kappa)$.

When $G = \mathrm{GL}_n$ we recover the usual notion of semisimplification of n-dimensional Γ-representations, which is defined as the direct sum of the Jordan-Hölder factors of ρ. By definition $\rho^{\mathrm{ss}}(\Gamma)$ is a G-semisimplification of the subgroup $\rho(\Gamma)$ in the sense of [BMR20, Definition 4.1]. It is thus immediate, that ρ^{ss} is G-completely reducible. If Γ is a topological group, κ is a topological field and ρ is continuous, then ρ^{ss} is continuous, but the converse is false in general.

Proposition A.5. In Definition A.4, the $G^0(k)$-conjugacy class of ρ^{ss} does not depend on the choice of P and L.

Proof. Let $\rho^{\mathrm{ss},i} = c_{P_i,L_i} \circ \rho$ for $i = 1, 2$ be two semisimplifications of ρ with respect to an R-parabolic P_i and an R-Levi L_i respectively. We first assume $P := P_1 = P_2$. Then there exists some $u \in R_u(P)$, such that $uL_1u^{-1} = L_2$. Since the square in the following diagram commutes, we obtain $uc_{P,L_1}u^{-1} = c_{P,L_2}$ and thus in particular $u\rho^{\mathrm{ss},1}u^{-1} = \rho^{\mathrm{ss},2}$.

$$\begin{tikzcd}
P \arrow[r] \arrow[rr, bend left, "c_{L_1}"] \arrow[dr] \arrow[drr, bend right, "c_{L_2}"'] & P/R_u(P) \arrow[d, equal] & L_1 \arrow[l, "\sim"'] \arrow[d, "u(-)u^{-1}"] \\
 & P/R_u(P) & L_2 \arrow[l, "\sim"']
\end{tikzcd}$$

If $P_1 \neq P_2$, we can apply Lemma A.3 to find a common R-Levi L of P_1 and P_2. By [BMR05, Lemma 6.2 (iii)], we have $P_1 \cap P_2 = L \cdot (R_u(P_1) \cap R_u(P_2))$. It follows, that the following diagram commutes.

$$\begin{tikzcd}
P_1 \cap P_2 \arrow[r] \arrow[d] & P_1 \arrow[d, "c_{P_1,L}"] \\
P_2 \arrow[r, "c_{P_2,L}"] & L
\end{tikzcd}$$

This implies $c_{P_1,L} \circ \rho = c_{P_2,L} \circ \rho$. We obtain from the first step, that there are $u_i \in R_u(P_i)$ with $u_i(c_{P_i,L_i} \circ \rho)u_i^{-1} = c_{P_i,L} \circ \rho$. ☐

Corollary A.6. Let $\rho : \Gamma \to G(\kappa)$ be a homomorphism. Then ρ is G-completely reducible if and only if ρ and ρ^{ss} are $G^0(\kappa)$-conjugate.

Proof. The reverse direction is clear. Suppose, that ρ is G-completely reducible and that ρ^{ss} is some G-semisimplification of ρ. Let P be a minimal R-parabolic, such that $\rho(\Gamma) \subseteq P(\kappa)$. Since ρ is G-completely reducible, there exists some R-Levi L of P, such that $\rho(\Gamma) \subseteq L(\kappa)$. In particular $\rho = c_{P,L} \circ \rho$. We can apply Proposition A.5 to conclude, that $c_{P,L} \circ \rho$ and ρ^{ss} are conjugate. ☐

Proposition A.7. Let $\rho : \Gamma \to G(\kappa)$ be a homomorphism. Then the determinant laws attached to $\tau \circ \rho$ and $\tau \circ \rho^{\mathrm{ss}}$ agree.

Proof. Let λ be a cocharacter, such that $\rho^{\mathrm{ss}} = \lim_{t\to 0} \lambda(t)\rho\lambda(t)^{-1}$. Let $D_{\tau\circ\rho} : \kappa[\Gamma] \to \kappa$ be the determinant law attached to $\tau \circ \rho$ and let $D_{\rho^{\mathrm{ss}}}$ be the determinant law attached to $\tau \circ \rho^{\mathrm{ss}}$. We also have a family of determinant laws $D : \kappa[\Gamma] \to \kappa[t, t^{-1}]$ over $\mathbb{G}_m$ given by $D_A : A[\Gamma] \to A[t, t^{-1}], r \mapsto \det(((\tau(\lambda(t))(\tau \circ \rho)\tau(\lambda(t))^{-1}) \otimes \mathrm{id}_A)(r))$, which is actually constant in t and equal to $D_{\tau\circ\rho}$. So this family extends uniquely to a family over $\mathbb{A}^1$. Since the limit of $\lambda(t)\rho\lambda(t)^{-1}$ as $t \to 0$ exists and formation of the determinant is algebraic, we obtain $D^{t=0} = D_{\tau\circ\rho^{\mathrm{ss}}}$ and hence $D_{\tau\circ\rho} = D_{\tau\circ\rho^{\mathrm{ss}}}$. ☐

Remark A.8. In general $(\tau \circ \rho)^{\mathrm{ss}}$ is not isomorphic to $\tau \circ \rho^{\mathrm{ss}}$. But it follows from Proposition A.7, that $D_{\tau\circ\rho^{\mathrm{ss}}} = D_{\tau\circ\rho}$ and in particular that $(\tau \circ \rho^{\mathrm{ss}})^{\mathrm{ss}}$ is isomorphic to $(\tau \circ \rho)^{\mathrm{ss}}$.

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