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Solovay Reducibility and Speedability Outside of left-c.e. Reals

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Abstract

A real number is left-c.e. if it has a left-c.e. approximation, i.e., a computable nondecreasing sequence $a_0, a_1, ...$ of rationals that converges to the real number. Furthermore, a real number α is Solovay reducible to a real number β if there exists a partially computable function g that maps every rational number $q < \beta$ to some rational number $g(q) < \alpha$ such that, for some real constant c and all $q < \beta$, it holds that

$$\alpha - g(q) < c(\beta - q).$$

Solovay reducibility can be used to compare the speed at which left-c.e. numbers can be approximated: if a real number α is Solovay reducible to a left-c.e. real number β, then also α is left-c.e. and, for every left-c.e. approximation of β, there is a left-c.e. approximation of α that converges at least as fast up to a constant factor.

Among the left-c.e. reals, the Martin-Löf random ones have been intensively studied, and it is known that they have several natural equivalent characterizations. For example, by results of Solovay [17] and of Calude, Hertling, Khoussainov and Wang [2], the Martin-Löf random left-c.e. reals are exactly the halting probabilities of universal Turing machines. Furthermore, Kučera and Slaman [8] demonstrated that, within the left-c.e. reals, the Martin-Löf random ones form a highest degree of Solovay reducibility, i.e., a left-c.e. real β is Martin-Löf random if and only if every left-c.e. real α is reducible to β. In fact, they showed that the latter holds via arbitrary left-c.e. approximations of α and β. As a consequence, given any Martin-Löf random left-c.e. reals α and β, they are mutually Solovay reducible to each other via arbitrary left-c.e. approximations $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ of α and β, respectively, hence, there are reals $c > 0$ and d such that, for all n, it holds that

$$c < \frac{\alpha - a_n}{\beta - b_n} < d. \tag{1}$$

Actually more is known: the considered ratios are not only restricted to the interval (c, d) but, by a celebrated theorem of Barmpalias and Lewis-Pye [1], they converge, i.e., the limit

$$\lim_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n},$$

exists and does not depend on the choice of the left-c.e. approximations of α and β.

A left-c.e. real α is ρ-speedable if it has a left-c.e. approximation $a_0, a_1, \ldots$ such that, for some computable function f, it holds that

$$\liminf_{n\to\infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} = \rho,$$

and a left-c.e. real is speedable if it is ρ-speedable for some $\rho < 1$. Merkle and Titov [10] introduced these notions and observed that, by the theorem of Barmpalias and Lewis-Pye, it is immediate that Martin-Löf random left-c.e. reals cannot be speedable, furthermore, they gave a short direct proof of the latter fact.

Solovay reducibility is a standard tool for investigating the class of left-c.e. reals. However, though defined as a binary relation on the set of all reals, Solovay reducibility is only rarely used outside the realm of left-c.e. reals, in fact, is viewed as "badly behaved" there in general [5, Section 9.1]. The main theme of this thesis is that, when investigating all reals, Solovay reducibility should be replaced by monotone Solovay reducibility. The latter reducibility is defined literally the same as Solovay reducibility except that, in addition, it is required that the function g in (1) is nondecreasing, i.e., that $g(q) \leq g(q')$ holds for all q and q' in the domain of g, where $q < q'$. Essentially all results that are shown in what follows suggest that monotone Solovay reducibility should be used when investigating all and not just left-c.e. reals.

First, monotone Solovay reducibility can indeed be considered as an extension of Solovay reducibility since both relations coincide on the set of left-c.e. reals. Monotone Solovay reducibility is a reflexive and transitive relation, hence, induces a degree structure in the usual way. Furthermore, the classes of computable, of left-c.e., of right-c.e., of d.c.e. and of computably approximable, or Δ^0_2, reals are all closed downwards under monotone Solovay reducibility.

Second, when extending the notion of speedability from the left-c.e. to all reals, this is done in terms of monotone Solovay reducibility of a real to itself. The resulting notion of speedability coincides on the set of left-c.e. reals with the notion of speedability for left-c.e. reals that has been previously defined in terms of left-c.e. approximations, whereas a definition in terms of Solovay reducibility would be trivial in so far as it renders all left-c.e. reals speedable.

For the speedability notion defined for left-c.e. reals in terms of left-c.e. approximations, the following is shown. The notion is robust in so far as a real that is ρ-speedable for some $\rho < 1$ is actually ρ-speedable for all $\rho > 0$ via any left-c.e. approximation of the real. Also speedability is a degree property, i.e., in a Solovay degree, either every or no real is speedable. Furthermore, Martin-Löf random left-c.e. reals are never speedable, while all nonhigh left-c.e. reals are speedable. For speedability defined in terms of monotone Solovay reducibility, some of these results extend to all reals, in particular, robustness with respect to the choice of nonzero ρ and the nonspeedability of Martin-Löf random reals. Being Martin-Löf random is not equivalent to being nonspeedable, neither for all reals nor when restricting attention to the left-c.e. reals. The former result is shown below by constructing a right-c.e. counterexample, i.e., a right-c.e. real that is neither Martin-Löf random nor speedable. The latter, more interesting and more difficult result is due to Hölzl and Janicki [7], who constructed a left-c.e. counterexample.

Third, the theorem of Barmpalias and Lewis-Pye allows an equivalent reformulation in terms of monotone Solovay reducibility, which can be extended to all reals. This extension is one of the main results of this thesis. A corresponding reformulation in terms of Solovay reducibility is false in general and is actually false for all left-c.e. reals.

Zusammenfassung

Eine reelle Zahl ist linksberechenbar, wenn sie eine linksberechenbare Approximation besitzt, dass heißt, es gibt eine berechenbare nichtfallende Folge $a_0, a_1, \ldots$ von rationalen Zahlen, die gegen die reelle Zahl konvergiert. Weiter ist eine reelle Zahl α auf eine reelle Zahl β Solovay-reduzierbar, wenn es eine partiell berechenbare Funktion g gibt, die jede rationale Zahl $q < \beta$ auf eine rationale Zahl $g(q) < \alpha$ abbildet, so dass für eine reelle Konstante c und alle $q < \beta$ gilt

$$\alpha - g(q) < c(\beta - q).$$

Mittels der Solovay-Reduzierbarkeit kann die Geschwindigkeit verglichen werden, mit der linksberechenbare Zahlen approximiert werden können. Falls eine reelle Zahl α Solovay-reduzierbar auf eine linksberechenbare reelle Zahl β ist, dann ist α ebenfalls linksberechenbar und gibt es für jede linksberechenbare Approximation von β eine linksberechenbare Approximation von α, die bis auf einen konstanten Faktor mindestens genauso schnell konvergiert.

Unter den linksberechenbaren reellen Zahlen wurden die Martin-Löf-zufälligen intensiv erforscht und es ist bekannt, dass diese verschiedene natürliche äquivalente Charakterisierungen erlauben. Zum Beispiel sind nach Ergebnissen von Solovay [17] und von Calude, Hertling, Khoussainov und Wang [2] die Martin-Löf-zufälligen linksberechenbaren reellen Zahlen genau die Haltewahrscheinlichkeiten universeller Turingmaschinen. Kučera und Slaman [8] konnten zeigen, dass innerhalb der linksberechenbaren reellen Zahlen die Martin-Löf-zufälligen Zahlen einen höchsten Grad der Solovay-Reduzierbarkeit bilden, das heißt, eine linksberechenbare reelle Zahl β ist genau dann Martin-Löf-zufällig, wenn jede linksberechenbare reelle Zahl α auf β reduzierbar ist, und dass letztere Aussage sogar bezüglich beliebiger linksberechenbarer Approximationen von α und β gilt.

Folglich sind beliebige Martin-Löf-zufällige linksberechenbare reelle Zahlen α und β bezüglich beliebiger linksberechenbarer Approximationen $a_0, a_1, \ldots$ und $b_0, b_1, \ldots$ von α beziehungsweise β gegenseitig Solovay-reduzierbar, das heißt, es gibt reelle Zahlen $c > 0$ und d, so dass für alle n gilt

$$c < \frac{\alpha - a_n}{\beta - b_n} < d. \tag{2}$$

Tatsächlich ist weit mehr bekannt, als dass die Werte der hier betrachteten Brüche in einem Intervall (c, d) liegen: nach einem bekannten Satz von Barmpalias und Lewis-Pye [1] konvergieren die Werte sogar: der Grenzwert

$$\lim_{n \to \infty} \frac{\alpha - a_n}{\beta - b_n}$$

existiert und hängt nicht von der Wahl der Linksapproximationen von α und β ab.

Eine linksberechenbare reelle Zahl α ist ρ-beschleunigbar, falls sie eine Linksapproximation $a_0, a_1, \ldots$ besitzt, so dass für eine berechenbare Funktion f gilt

$$\liminf_{n \to \infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} \leq \rho,$$

und sie ist beschleunigbar, falls sie ρ-beschleunigbar für ein $\rho < 1$ ist. Merkle and Titov [10] führten diese Begriffe ein und beobachteten, dass aus dem Satz von Barmpalias and Lewis-Pye sofort folgt, dass Martin-Löf-zufällige linksberechenbare reelle Zahlen nicht beschleunigbar sein können, zusätzlich gaben sie einen kurzen direkten Beweis dieser Tatsache an.

Die Solovay-Reduzierbarkeit ist ein Standardwerkzeug zur Untersuchung der Klasse der linksberechenbaren reellen Zahlen. Obwohl die Solovay-Reduzierbarkeit als binäre Relation auf der Menge aller reellen Zahlen definiert ist, wird sie nur selten außerhalb des Bereichs der reellen Zahlen verwendet und wird dort sogar als im Allgemeinen "badly behaved" angesehen [5, Abschnitt 9.1]. Die zentrale These dieser Arbeit ist, dass bei der Untersuchung aller reellen Zahlen die Solovay-Reduzierbarkeit durch die monotone Solovay-Reduzierbarkeit ersetzt werden sollte. Letztere Reduzierbarkeit ist wörtlich genauso definiert wie die Solovay-Reduzierbarkeit, außer dass zusätzlich verlangt wird, dass die Funktion g in (2) monoton steigend ist, das heißt, es gilt $g(q) \leq g(q')$ für alle q und q' im Definitionsbereich von g mit $q < q'$. Im Wesentlichen alle im Folgenden gezeigten Ergebnisse legen nahe, dass die monotone Solovay-Reduzierbarkeit verwendet werden sollte, wenn alle und nicht nur die linksberechenbaren reelle Zahlen untersucht werden.

Erstens kann die monotone Solovay-Reduzierbarkeit als eine Erweiterung der Solovay-Reduzierbarkeit betrachtet werden, da beide Relationen auf der Menge der linksberechenbaren reellen Zahlen übereinstimmen. Die monotone Solovay-Reduzierbarkeit ist eine reflexive und transitive Relation und induziert daher in der üblichen Weise eine Gradstruktur. Darüber hinaus sind die Klassen der berechenbaren, der linksberechenbaren, der rechtsberechenbaren, der d.c.e. und der berechenbar approximierbaren, oder Δ^0_2, reellen Zahlen alle unter der monotonen Solovay-Reduzierbarkeit nach unten abgeschlossen.

Zweitens wird die Erweiterung des Begriffs der Beschleunigbarkeit von den linksberechenbaren auf alle reellen Zahlen mittels der monotonen Solovay-Reduzierbarkeit als geeignete Reduktion einer reellen Zahl auf sich selbst definiert. Der resultierende Begriff der Beschleunigbarkeit stimmt auf der Menge der linksberechenbaren reellen Zahlen mit dem Begriff der Beschleunigbarkeit für linksberechenbare reelle Zahlen überein, der zuvor mittels linksberechenbarer Approximationen definiert wurde, wohingegen eine Definition mittels Solovay-Reduzierbarkeit insofern trivial wäre, als für letztere alle linksberechenbaren reellen Zahlen beschleunigbar sind.

Für den Begriff der Beschleunigbarkeit, der unter Verwendung von linksberechenbaren Approximationen für linksberechenbare reelle Zahlen definiert ist, wird Folgendes gezeigt. Der Begriff ist robust, insofern als eine reelle Zahl, die ρ-beschleunigbar für ein $\rho < 1$ ist, tatsächlich ρ-beschleunigbar für alle $\rho > 0$ bezüglich beliebiger linksberechenbarer Approximationen der reellen Zahl ist. Außerdem ist die Beschleunigbarkeit eine Gradeigenschaft, das heißt, in einem Solovay-Grad ist entweder jede oder keine reelle Zahl beschleunigbar. Darüber hinaus sind Martin-Löf-zufällige linksberechenbare reelle Zahlen niemals beschleunigbar, während alle linksberechenbaren reellen Zahlen beschleunigbar sind, die nicht hoch im Sinne der Berechenbarkeitstheorie sind. Einige dieser Ergebnisse lassen sich für die monotone Solovay-Reduzierbarkeit auf alle reellen Zahlen erweitern, insbesondere die Robustheit in Bezug auf die Wahl eines Werts $\rho > 0$ und die Unbeschleunigbarkeit von Martin-Löf-zufälligen reellen Zahlen. Martin-Löf-zufällig zu sein ist weder für alle noch nur für die linksberechenbaren reellen Zahlen äquivalent zur Unbeschleunigbarkeit. Das erste Ergebnis wird im Folgenden durch die Konstruktion eines rechtsberechenbaren Gegenbeispiels gezeigt, das heißt, durch eine rechtsberechenbare reelle Zahl, die weder Martin-Löf-zufällig noch beschleunigbar ist. Das zweite, interessantere und schwierigere, Ergebnis ist von Hölzl und Janicki [7], die ein linksberechenbares Gegenbeispiel konstruierten.

Drittens kann der Satz von Barmpalias und Lewis-Pye unter Verwendung der monotonen Solovay-Reduzierbarkeit äquivalent umformuliert werden, so dass sich die Umformulierung auf alle reellen Zahlen erweitern lässt. Diese Erweiterung ist eines der Hauptergebnisse dieser Arbeit. Eine entsprechende Umformulierung mittels Solovay-Reduzierbarkeit ist im Allgemeinen falsch und gilt tatsächlich für keine linksberechenbare reelle Zahl.

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Contents

1 Introduction

Background Research in algorithmic randomness focuses on various notions of randomness for individual sequences, i.e., infinite binary sequences. The three main approaches for formalizing the randomness of a sequence are the following. Compressibility: a sequence is random if its initial segments cannot be effectively compressed. Predictability: a sequence is random if effective betting on the bits of a sequence with fair pay-off yields only a bounded gain. Typicalness: a sequence is random if it is not contained in an effective null cover. The central notion of Martin-Löf randomness can be equivalently characterized via all three approaches, e.g., a sequence is Martin-Löf random if and only if, for almost all n, the initial segment of length n of the segment has prefix-free Kolmogorov complexity of at least n, that is, requires a code of length at least n under any fixed effective and prefix-free coding scheme [5].

Beyond simply classifying sequences as random or nonrandom with respect to a specific notion of randomness, it may be informative to compare the degree of randomness of two sequences, and corresponding notions of relative randomness are a central object of study in algorithmic randomness. According to Downey and Hirschfeldt [5], here the aim is to understand whether a given real number is "more random" than another. For example, the following notion of relative randomness can be derived from the compressibility approach to define randomness. For given n, the prefix-free Kolmogorov complexity of the initial segments of length n of two sequences are compared and, in case the complexity of the segments of the first sequence is for almost all n less than or equal to that of the segments of the second sequence, the first sequence is considered to be at most as random as the second one. The latter notion of relative randomness is rather natural but exhibits some defects: not all sequences are at most as random as every given Martin-Löf random sequences and not even for all pairs of Martin-Löf random both sequences are mutually at most as random as the other one.

A notion of relative randomness that behaves better with respect to Martin-Löf randomness is obtained by an approach, where a first sequence is considered to be at most as random as a second sequence in case, given a point close to the second sequence, one can effectively find some point close to the first sequence. In order to be able to define the corresponding formal notion, which is called Solovay reducibility, we review further notation.

As usual, a sequence $A(0)A(1)\dots$ is viewed as representation of the real with binary expansion $0.A(0)A(1)\dots$. This way, we basically obtain an identification of sequences and reals in the unit interval, except that the representations of dyadic rationals are not unique, e.g., the real 0.1 has two distinct representations $0.1000\dots$ and $0.0111\dots$.

In what follows, we will no longer speak of sequences but of reals, where such reals are meant to be in the unit interval.

A real α is SOLOVAY REDUCIBLE to a real β if there is a constant c and partial computable function g from rationals to rationals such that, for every rational $q < \beta$, the value $g(q)$ is defined and satisfies $g(q) < \alpha$ and

$$\alpha - g(q) < c(\beta - q). \tag{3}$$

In the latter situation, we refer to g as TRANSLATION FUNCTION and to c as SOLOVAY CONSTANT.

Research on Solovay reducibility is usually restricted to the realm of left-c.e. reals. Recall that a real α is LEFT-C.E. in case α has a LEFT-C.E. APPROXIMATION, i.e., a nondecreasing sequence $q_0, q_1, \ldots$ of rationals that is computable and converges to α; as usual, the acronym c.e. stands for computably enumerable. According to Calude et Al. [2], for left-c.e. reals α and β, the fact that α is Solovay reducible to β can be equivalently defined by requiring that there is a constant c such that, for every left-c.e. approximation $b_0, b_1, \ldots$ of β, there is a left-c.e. approximation $a_0, a_1, \ldots$ of α such that we have for all n the inequality

$$\alpha - a_n < c(\beta - b_n). \tag{4}$$

Solovay reducibility is obviously reflexive and transitive, hence, as usual, it partitions the set of all reals into SOLOVAY DEGREES, i.e., maximum subsets of reals that are mutually Solovay reducible to each other, where then a partial order on these degrees is induced canonically.

As already said, we would like to argue that Solovay reducibility is a notion of relative randomness in the sense that, if α is Solovay reducible to β, then α can be considered to be at most as random as β. Accordingly, one would require as a necessary condition that there is a greatest Solovay degree, which is formed by the reals that fall under some notion of randomness. When restricting attention to left-c.e. reals, such an assertion holds for the notion of Martin-Löf randomness.

Left c.e. Martin-Löf random reals, which are also called Ω-numbers, have already been investigated by Solovay [17] when introducing the reducibility that now goes under his name. Such reals have several interesting characterizations, for example, they coincide with the measures of domains of universal prefix-free Turing machines. A celebrated result of Kučera and Slaman [8] dating back to 2001 provides another characterization: the left-c.e. Martin-Löf random reals form a Solovay degree that is above all other Solovay degrees of left-c.e. reals. From the proof of the latter result, it follows that not only all left-c.e. Martin-Löf random reals are mutually Solovay reducible to each other but that, in fact, this works via arbitrary left-c.e. approximations. That is, given any left-c.e. approximations $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ of Martin-Löf random reals α and β, respectively, the inequality (4) holds for some Solovay constant, hence, by symmetry, for some appropriate constant c and all n, it holds that

$$\frac{1}{c} < \frac{\alpha - a_n}{\beta - b_n} < c. \tag{5}$$

So, any two left-c.e. approximations that both converge to some Martin-Löf random reals converge to their respective limits, intuitively speaking, at the same speed up to constant factors. This already surprising result was considerably strengthened by a result of Barmpalias and Lewis-Pye [1, 11], who demonstrated that, for any such left-approximations to Martin-Löf random reals α and β, the limit

$$\lim_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} \tag{6}$$

exists and depends only on α and β but is independent of the considered left-c.e. approximations of α and β.

Monotone Solovay reducibility Solovay reducibility, though defined as a binary relation on the set of all reals, is usually only investigated in a setting of left-c.e. reals and, in fact, is viewed as "badly behaved" in general on arbitrary reals [5, Section 9.1]. There are a few exceptions to the latter statement, though. For example, the Martin-Löf random reals are close upwards under Solovay reducibility, i.e., if a Martin-Löf random real α is Solovay reducible to a real β then also β must be Martin-Löf random. This is shown by arguing that any Martin-Löf test that fails on β could be transformed into one that fails on α by using the translation function witnessing the Solovay reducibility from α to β.

Let monotone Solovay reducibility be introduced by literally the same definition as Solovay reducibility except that, in addition, the translation function g in (3) is required to be nondecreasing, i.e., whenever g is defined on two rationals q_1 and q_2, where $q_1 < q_2$, then it holds that $g(q_1) \leq g(q_2)$. The central theme of this thesis is that, when extending the realm of investigation from the left-c.e. to all reals, instead of Solovay reducibility, its monotone variant should be used.

Arguments in favour of using monotone Solovay reducibility include the following. On the left-c.e. reals, Solovay reducibility and its monotone variant coincide, hence all results obtained for left-c.e. reals remain valid. Monotone Solovay reducibility exhibits several closure properties: the classes of computable, of left-c.e., of right-c.e., of d.c.e., and of computably approximable, or Δ^0_2, reals are all closed downwards under monotone Solovay reducibility. That is, if, for one of these classes, some real is monotone Solovay reducibility to some real in the class, then the former real is in this class as well. Furthermore, several results about speedability of left-c.e. reals can be extended to arbitrary reals when considering a notion of speedability that corresponds to monotone Solovay reducibility, see the next paragraph for details. Finally and most important, as the main result of this thesis, it will be demonstrated that the mentioned result of Barmpalias and Lewis-Pye [1, 11] extends to arbitrary Martin-Löf random reals when considering monotone Solovay reducibility. While some of the mentioned results on monotone Solovay reducibility, e.g., the downwards closure of the left-c.e. reals, are simply inherited from Solovay reducibility, for several others, it can be demonstrated that they only hold for the monotone variant.

Speedability Given a left-c.e. approximation, we may ask whether we can effectively go to a subsequence of the approximation such that the rationals in the subsequence are significantly closer to their limit than the approximation we have started with. Formally, define a speed-up function to be a nondecreasing function f from the set of natural numbers to itself that satisfies $n \leq f(n)$. Then call a left-c.e. approximation $a_0, a_1, \ldots$ ρ-speedable if there is a speed-up function f such that we have

$$\liminf_{n \to \infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} \leq \rho, \tag{7}$$

and call a real α speedable if it has some left-c.e. approximation that is ρ-speedable for some $\rho < 1$. Note that in case we require instead that the fraction in (7) is less than or equal to some $\rho < 1$ for all n, this condition would only be satisfied by computable reals. By the mentioned result of Barmpalias and Lewis-Pye, a left-c.e. Martin-Löf random α cannot be speedable as follows by applying their result to any left-c.e. approximation $a_0, a_1, \ldots$ of α and with β equal to α: if we let b_n be equal to a_n, the limit (6) will be equal to 1, hence it must also be equal to 1 in case we let b_n be equal to $a_{f(n)}$ for any computable speed-up function f.

Results obtained about the speedability of left-c.e. reals include a short direct proof for the nonspeedability of left-c.e. Martin-Löf random reals and that all nonhigh left-c.e. reals are speedable. Furthermore, the notion of speedability exhibits several robustness properties: a left-c.e. real is speedable if and only if it is ρ-speedable for every $\rho > 0$ and with respect to any of its left-c.e. approximations. Also speedability is a degree property in the sense that either all or none of the left-c.e. reals in a Solovay degree are speedable.

By the same pattern, a notion of speedability can be extended on $\mathbb{R}$ by deriving it from monotone Solovay reducibility. In what follows, it is demonstrated that basic results about the speedability of left-c.e. reals extend on $\mathbb{R}$, which provides further indication that Solovay reducibility should be replaced by its monotone variant when considering arbitrary and not just left-c.e. reals. More precisely, Martin-Löf random reals are again nonspeedable, and the notion of speedability on $\mathbb{R}$ is robust in so far as every speedable real is ρ-speedable for every $\rho > 0$.

A defect of monotone Solovay reducibility When considering arbitrary, not necessary left-c.e. reals, Solovay reducibility and monotone Solovay reducibility share the defect that the computable reals are not reducible to all other reals. Indeed, one can construct a right-c.e. real to which no left-c.e., and consequently, no computable real is Solovay reducible [5, Proposition 9.6.1]. Rettinger and Zheng [19] have introduced several, partially somewhat technical variants of Solovay reducibility that are designed in order to overcome this defect. We will conclude the thesis by comparing these variants with monotone Solovay reducibility.

Overview In Chapter 2, basic notation and standard results about Solovay reducibility are reviewed. In Chapter 3, the monotone variant of Solovay reducibility is introduced and some of its basic properties are shown, including the downwards closure of several standard classes of reals. Chapters 4 and 5 deal with the notion of speedability for left-c.e. and for arbitrary reals, respectively. The main result of this thesis, the extension of the theorem of Barmpalias and Lewis-Pye [1, 11] from left-c.e. to arbitrary reals with respect to monotone Solovay reducibility is the content of Chapter 6. Finally, in Chapter 7, several variants of Solovay reducibility introduced by Rettinger and Zheng [19] are reviewed and compared to monotone Solovay reducibility.

Notation As usual, we identify a subset A of the set $\mathbb{N} = \{0, 1, \ldots\}$ of natural number with its characteristic sequence $A(0)A(1)\ldots$, where $A(n)$ is 0 in case n is in A and is 1, otherwise. Infinite binary sequences are identified in turn with the real numbers on the unit interval, where a sequence A is identified with the real that has binary expansion $0.A(0)A(1)\ldots$. This way, for example, the sequences $1000\ldots$ and $0111\ldots$ are formally identified with each other but, in what follows, this will not be relevant.

The classes of natural, rational, dyadic rational and real numbers are denoted by ω, $\mathbb{Q}$, $\mathbb{Q}_2$ and $\mathbb{R}$, respectively. Recall that a rationals number q is dyadic if it can be written in the form $\frac{z}{2^t}$ for an integer z and a natural number t. The denominator t is the length of the binary representation of q and will be denoted as $l(q)$. As usual, for real numbers a and b, where $a \leq b$, we write (a, b) and $[a, b]$ for the open and the closed, respectively, intervals with left endpoint a and right endpoint b and similarly for half-open intervals, e.g., $[0, 1)$ denotes the half-open unit interval of all reals r that fulfill $0 \leq r < 1$. In case $b < a$, all such intervals are equal to the empty set.

We assume that the reader is familiar with standard concepts and basic results of recursion theory such as decidable or recursively enumerable sets, computable sequences of rationals or padding lemma. Note that the term "computable set" refers to a computable subset of natural numbers, which is in the literature often denoted as decidable set. For unexplained concepts and notation, see, for example, the monographs by Downey and Hirschfeldt [5] and by Soare [16].

2 Solovay reducibility on left-c.e. reals

2.1 Computable and left-c.e. reals

We review some characterizations of computable and left-c.e. reals. As usual, we restrict attention to reals in the unit interval in order to facilitate the identification of a set A with the real that has binary expansion $0.A(0)A(1)\ldots$. Accordingly, in what follows, real means real in the unit interval $[0,1]$ unless explicitly specified otherwise.

Definition 1. *A real α is* COMPUTABLE *if $\alpha = 0.A(0)A(1)\ldots$ for some decidable set A. The set of all computable reals will be denoted by* COMP.

The next proposition reviews some well-known equivalent characterizations of computable reals using the notions of left cut, where the third characterization is implicit already in Turing [18], see Section 5.1 in Downey and Hirschfeldt [5]. Recall that the LEFT CUT of a real α is the set $\{q \in \mathbb{Q}| : q < \alpha\}$.

Proposition 2. *For a real α, the following statements are equivalent:*

(a) The real α is computable.

(b) The left cut of α is computable.

(c) There exists a computable sequence $a_0, a_1, \ldots$ of rationals such that, for all naturals numbers n, it holds that

$$\alpha - a_n < 2^{-n}. \tag{8}$$

For given α, a sequence $a_0, a_1, \ldots$ that satisfies Condition (c) in Proposition 2 is said TO WITNESS THAT α IS COMPUTABLE.

Remark 3. *For further use, observe that, for every computable real $\alpha > 0$, there is a sequence $a_0, a_1, \ldots$ of rationals that witnesses that α is computable such that $a_0 < a_1 < \ldots$, hence, in particular, all a_i are strictly less than α.*

In order to obtain such a sequence for a given computable real α, let $a_0, a_1, \ldots$ be a sequence of rationals that witnesses that α is computable. If we let a'_n be equal to the maximum of 0 and $a_{n+2} - 2^{-(n+2)}$, then we have $a'_n < \alpha$ and $\alpha - a'_n < 2^{-(n+1)}$. The two latter properties then also hold for a''_n in place of a'_n, where a''_n is equal to the maximum of $a'_0, \ldots, a'_n$. The latter sequence is nondecreasing but not necessarily strictly increasing; in order to obtain a sequence as required, fix $k > 1$ in ω such that $2^{-k} < a''_0$ and consider the sequence of the values $a''_n - 2^{-(k+n)}$ for $n = 0, 1, \ldots$. Observe that the *sequence $a''_0, a''_1, \ldots$ converges to $\alpha > 0$ and that thus we can assume $a''_0 > 0$ because, otherwise, we can simply shift the start of the sequence to a nonzero element.*

We proceed with reviewing standard notation. As usual, the acronyms c.e. and d.c.e. stand for "computably enumerable" and "difference computably enumerable", respectively.

Definition 4. *A* COMPUTABLE APPROXIMATION *of a real α is a computable sequence of rationals $a_0, a_1, \ldots$ that converges to α. Such a sequence is a* LEFT-C.E. APPROXIMATION *of α if the sequence is nondecreasing, a* RIGHT-C.E. APPROXIMATION *of α if the sequence is nonincreasing and d.c.e. if the sum of $|a_{i+1} - a_i|$ for all i is bounded from above.*

A real α is COMPUTABLY APPROXIMABLE, *sometimes also called Δ^0_2, if there is a computable approximation of α. A real α is* LEFT-C.E *if there is a left-c.e. approximation of α. Similarly, a real α is* RIGHT-C.E. *and* D.C.E. *if there is a right-c.e. approximation and d.c.e. approximation of α, respectively.*

The sets of all left-c.e., all right-c.e. and all d.c.e. reals are denoted by $\mathrm{LEFT{-}CE}$, $\mathrm{RIGHT{-}CE}$ *and* DCE*, respectively.*

We will occasionally use the notation

$$a_n \nearrow \alpha \qquad \text{or} \qquad a_n \searrow \alpha$$

if a sequence $a_0, a_1, \ldots$ tends to α and is nondecreasing or nonincreasing, respectively.

Remark 5. *From time to time, we will tacitly assume that a given left-c.e. approximation $a_0, a_1, \ldots$ of a nonzero real is strictly increasing. Indeed, in case the approximation becomes stable, i.e., if $a_n = a_{n+1} = \ldots$ for some n, we can switch to the sequence $a_n - 2^{-k}, a_{n+1} - 2^{-k+1}, \ldots$ for some k such that $2^{-k} < a_n$, while, otherwise, we can switch to an effectively obtained strictly increasing subsequence.*

Standard results about left- and right-c.e reals include that left-c.e. reals can be characterized via left cuts [15] and that a real is computable if and only if it is both, left- and right-c.e. Motivated by the latter fact, in the literature, left-c.e. and right-c.e. reals are sometimes called lower and upper, respectively, semi-computable reals.

Proposition 6. *A real α is left-c.e. if and only if the left cut of α is computably enumerable.*

Proof. Given a left-c.e. approximation $a_0, a_1, \ldots$, an effective enumeration of the left cut of α is obtained by enumerating all rationals that are strictly smaller than any of the values a_n.

Conversely, given an effective enumeration $q_0, q_1, \ldots$ of the left cut of α, the nondecreasing sequence $a_0, a_1, \ldots$, where a_n is the maximum value among $q_0, \ldots, q_n$, is a left-c.e. approximation of α. □

Proposition 7 (Folklore). *A real α is left-c.e. if and only if $-\alpha$ is right-c.e. Further, α is computable if and only if α is left-c.e and right-c.e.*

Proof. Let α be a real. If there exists a left-c.e. approximation $a_0, a_1, \dots$ of α, then the sequence $-a_0, -a_1, \dots$ is a right-c.e. approximation of $-\alpha$ and vise versa.

In case α is computable, the construction in Remark 3 yields a left-c.e. approximation of α, hence α is left-c.e. A right-c.e. approximation of α can be obtained by a very similar symmetric construction.

Conversely, if $b_0, b_1, \dots$ and $c_0, c_1, \dots$ are left-c.e. and right-c.e. approximations of α, respectively, then we obtain a sequence $a_0, a_1, \dots$ that witnesses that α is computable by letting a_n be equal to b_i for the least i such that $c_i - b_i < 2^{-n}$. □

The following easy lemma is stated for further use.

Lemma 8. *Let α be a real number, and let $a_0, a_1, \dots$ be a computable sequence of rationals such that $\alpha = \liminf_{n\to\infty} a_n$ and $a_n < \alpha$ for all n. Then α is left-c.e.*

Proof. By $a_n < \alpha$, it follows that $\limsup a_n \leq \alpha = \liminf a_n$, hence the sequence $a_0, a_1, \dots$ converges to α. Consequently, the sequence $b_0, b_1, \dots$, where $b_n = \max_{i \leq n} a_i$, is a left-c.e. approximation of α, hence α is left-c.e. □

2.2 Solovay reducibility via translation functions

Solovay reducibility is a standard tool for comparing how fast left-c.e. approximations to two given reals converge.

Definition 9. *A* TRANSLATION FUNCTION *for the pair (α, β), or, less formal, for the reals α and β, is a partially computable function g from the set $\mathbb{Q} \cap [0,1]$ to itself such that g is defined for all rationals $q < \beta$ and, for all such q, it holds that $q(q) < \alpha$.*

A real α is SOLOVAY REDUCIBLE *to a real β, also written as $\alpha \leq_S \beta$, if there is a constant c and a translation function for α and β such that, for all $q < \beta$, it holds that*

$$\alpha - g(q) < c(\beta - q). \tag{9}$$

We refer to the fact that the latter inequality is true for all such q as SOLOVAY PROPERTY *and denote the real c as* SOLOVAY CONSTANT.

Research on Solovay reducibility usually restricts attention to the class of left-c.e. reals. We will argue in Chapter 3 that, in order to extend some results about Solovay reducibility to all reals, it makes sense to require that translation functions are nondecreasing, i.e., whenever such a function g is defined on q_1 and q_2, where $q_1 < q_2$, it holds that $g(q_1) \leq g(q_2)$. By the next Proposition, this additional requirement does not make a difference when Solovay reducibility is considered only for left-c.e. reals.

Proposition 10. *Let α and β be reals, where β is left-c.e. and α is Solovay reducible to β with Solovay constant c. Then α is Solovay reducible to β with Solovay constant c via some nondecreasing translation function.*

Proof. Assume that α is Solovay reducible to β with Solovay constant c via some translation function g_0. Fix some left-c.e. approximation $b_0, b_1, \dots$ of β and define a translation function g as follows. For a given argument q of g, let $k(q)$ be the least index such that $q \leq b_{k(q)}$. If the latter index is undefined, the function g is undefined on q, while, otherwise, let $g(q)$ be equal to the maximum value among $g_0(b_0), \dots, g_0(b_{k(q)})$. Note that g is a nondecreasing translation function that is defined exactly on all $q < \beta$. Furthermore, the function g witnesses that α is Solovay reducible to β with Solovay constant c because, for all $q < \beta$, we have

$$\alpha - g(q) \leq \alpha - g_0(b_{k(q)}) < c(\beta - b_{k(q)}) \leq c(\beta - q). \qquad \square$$

For further use, we state the following observation on translation functions.

Proposition 11. *Let the real α be Solovay reducible to some real β via some translation function g. Then it holds for every sequence $q_0, q_1, \dots$ of rationals that converges to β and fulfills $q_n < \beta$ for all n that*

$$\lim_{n \to \infty} g(q_n) = \alpha. \tag{10}$$

Proof. By assumption on g, for all n, the value $\alpha - g(q_n)$ is defined, nonnegative and bounded from above by $c(\beta - q_n)$, where c is the corresponding Solovay constant. When n goes to infinity, the latter values tend to 0, hence the values $g(q_n)$ tend to α. $\square$

Proposition 12 (Folklore)**.** *The classes of computable reals and of left-c.e. reals are both closed downwards under Solovay reducibility.*

Proof. Let the real α be Solovay reducible to some real β via some translation function g and a Solovay constant c that is a natural number.

First, let β be computable. Let $b_0, b_1, \dots$ be a left-c.e. approximation of β according to Remark 3, i.e., in particular, it holds that $b_k < \beta$ and $\beta - b_k < 2^{-k}$ for all k. Let a_n be equal to $g(b_{n+c})$. Then the sequence $a_0, a_1, \dots$ witnesses that the real α is computable because the sequence is computable and, by assumption on g, we have for all n

$$\alpha - a_n = \alpha - g(b_{n+c}) \leq c(\beta - b_{n+c}) < 2^c \cdot 2^{-(n+c)} = 2^{-n}.$$

Next, assume that β is left-c.e. and fix some left-c.e. approximation $b_0, b_2, \dots$ of β. Then $g(b_n) < \alpha$ for all n and the computable sequence $g(b_0), g(b_1), \dots$ converges to α by Proposition 11. Therefore, the real α is left-c.e. by Lemma 8. $\square$

2.3 Solovay reducibility via index functions

That some real is Solovay reducible to a left-c.e. real can be equivalently expressed in terms of left-c.e. approximations.

Proposition 13. *Let α and β be left-c.e. reals, let $b_0, b_1, \dots$ be a left-c.e. approximation of β, and let c be a constant. Then α is Solovay reducible to β with Solovay constant c if and only if there is a left-c.e. approximation $a_0, a_1, \dots$ of α such that, for all n, we have*

$$\alpha - a_n < c(\beta - b_n). \tag{11}$$

We refer to the fact that the latter inequality is true for all n as INDEX FORM OF THE SOLOVAY PROPERTY.

Proof. First, let α be Solovay reducible to β with Solovay constant c and let g be some translation function that witnesses the latter fact. Now, it suffices to let a_n be the maximum value among $g(b_0), \dots, g(b_n)$ and to observe that (11) follows by the same argument as in proof of Proposition 10.

Conversely, assume that there is a left-c.e. approximations $a_0, a_1, \dots$ of α that satisfies (11). For a given rational q, let $k(q)$ be the least index such that $q \leq b_{k(q)}$ and, if the latter index is defined, let $g(q)$ be equal to $a_{k(q)}$. Then the function g witnesses that α is Solovay reducible to β with Solovay constant c because, by construction, g is a translation function that is defined on all $q < \beta$ and satisfies for all such q

$$\alpha - g(q) = \alpha - a_{k(q)} < c(\beta - b_{k(q)}) \leq c(\beta - q). \qquad \square$$

Next, we list some well-known equivalent variants of the index form of Solovay reducibility.

Definition 14. *An* INDEX FUNCTION *is a nondecreasing function $f\colon \omega \to \omega$.*

Proposition 15 (partly by Calude, Hertling, Khoussainov and Wang, [2])**.** *Let α and β be left-c.e. reals and let c be a positive constant. Then the following statements are equivalent.*

(a) The real α is Solovay reducible to β with Solovay constant c.

(b) For every pair of left-c.e. approximations $a_0, a_1, \dots$ of α and $b_0, b_1, \dots$ of β, there is a computable index function f such that, for all n, it holds that

$$\alpha - a_{f(n)} < c(\beta - b_n). \tag{12}$$

(c) For every pair of left-c.e. approximations $a_0, a_1, \dots$ of α and $b_0, b_1, \dots$ of β, there is a strictly increasing computable index function f such that (12) *holds for all n.*

Proof. We show the chain of implications (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b). Let left-c.e. approximations $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ of α and β, respectively, be given. By (a) and Proposition 13, we can fix a left-c.e. approximation $a'_0, a'_1, \ldots$ of α that satisfies (11) with a_n replaced by a'_n. Let $f(n)$ be equal to the least index ℓ such that $a'_n < a_\ell$. We then have

$$\alpha - a_{f(n)} \leq \alpha - a'_n < c(\beta - b_n).$$

(b) ⇒ (c). It suffices to observe that (12) remains valid when $f(n)$ is replaced by any larger value. So, given an index function f as in (b), the index function $n \mapsto f(0)+\ldots+f(n)$ is as required in (c).

(c) ⇒ (a). Let left-c.e. reals α and β be given. If we fix arbitrary left-c.e. approximations of α and β, we obtain by (c) an index function that witnesses that α is Solovay reducible to β with Solovay constant c. □

In Chapter 4, we will argue that the notion of speedability introduced there should not be defined in terms of partial index functions. However, for left-c.e. reals, Solovay reducibility itself can be equivalently characterized by partial index functions.

Remark 16. *Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be left-c.e. approximations of reals α and β, respectively. Let f be a nondecreasing partial computable function that is defined on infinitely many n such that* (12) *holds for every such n. By dovetailing the computations of $f(0), f(1), \ldots$, we obtain an effective sequence $n_0 < n_1 < \ldots$ of natural numbers in the domain of f. For all n, let $k(n)$ be the least index k such that $n \leq n_k$ and let $f'(n) = \max_{i \leq k(n)} f(n_i)$. By construction, f' is a nondecreasing computable index function that witnesses that α is Solovay reducible to β because, for all n, we have*

$$a - a_{f'(n)} \;=\; a - a_{\max_{i \leq k(n)} f(n_i)} \;\leq\; a - a_{f(n_{k(n)})} \;<\; c(\beta - b_{n_{k(n)}}) \;\leq\; c(\beta - b_n).$$

2.4 Solovay degrees

Solovay reducibility is a reflexive and transitive relation, hence the Solovay degree of a real α can be defined in the usual way as

$$\deg(\alpha) = \{\beta \in [0,1] \colon \alpha \leq_{\mathrm{S}} \beta \text{ and } \beta \leq_{\mathrm{S}} \alpha\},$$

while Solovay reducibility induces canonically a partial order on the Solovay degrees. Propositions 17 and 19 together with Theorem 23 summarize some well-known structural properties of the left-c.e. reals under Solovay reducibility. By the downwards closure of the left-c.e. reals under Solovay reducibility, either all or none of the reals in a Solovay real are left-c.e., so, we call a Solovay degree LEFT-C.E. in case all reals in the degree are left-c.e.

Proposition 17 (Solovay [17], Calude, Hertling, Khoussainov and Wang [2], Downey, Hirschfeldt and Nies [6])**.** *The left-c.e. Solovay degrees under the partial order induced by Solovay reducibility form an upper semilattice, where addition is a join operation.*

Lemma 18. *Every computable real is Solovay reducible to every left-c.e. real.*

Proof. Let α be a computable real, and let β be a left-c.e. real. Let $a_0, a_1, \ldots$ be a left-c.e. approximation of α according to Remark 3, i.e., in particular, it holds that $\alpha - a_k < 2^{-k}$ for all k. Let $b_0, b_1, \ldots$ be a left-c.e. approximation of β that is strictly increasing. For a given index n, let $g(n)$ be equal to the least natural number k such that $2^{-k} < b_{n+1} - b_n$ and let $g'(n)$ be equal to the maximum number in range $g(0), \ldots, g(n)$. Then g is computable and nondecreasing and we have for all n

$$\alpha - a_{g'(n)} \leq \alpha - a_{g(n)} < 2^{-g(n)} < b_{n+1} - b_n < \beta - b_n,$$

thus, the index function g' witnesses that α is Solovay reducible to β. □

Proposition 19 (Folklore)**.** *The computable reals form a Solovay degree that is the least among all left-c.e. Solovay degrees.*

Proof. By Lemma 18, there is a least degree among left-c.e. degrees that contains all computable reals. Since the class of computable reals is closed downwards under Solovay reducibility by Proposition 12, the least degree cannot contain any noncomputable reals. □

2.5 Solovay reducibility and Martin-Löf random left-c.e. reals

The notion of a Martin-Löf random real can be defined via several natural characterizations in terms of Martin-Löf test, martingales and Kolmogorov complexity. In what follows, we will use the following characterization via Solovay tests that contain closed subintervals of the unit interval. This characterization is well-suited for our purposes and is easily seen to yield an equivalent definition of Martin-Löf randomness.

Definition 20. *A* SOLOVAY TEST *is a sequence $U_0, U_1, \ldots$ of closed subintervals $U_i = [p_i, q_i]$ on $[0, 1]$, where $p_0, p_1, \ldots$ and $q_0, q_1, \ldots$ are both computable sequences of rationals that fulfill $p_i \leq q_i$ for every i, such that the sum of the uniform measures of the intervals in this sequence is finite, i.e., such that*

$$\sum_{i=0}^{\infty} |U_i| = \sum_{i=0}^{\infty} (q_i - p_i) < \infty.$$

A real α is MARTIN-LÖF RANDOM*, or, for short,* ML-RANDOM*, if there is no Solovay test $U_0, U_1, \ldots$ such that α is contained in U_i for infinitely many i.*

Solovay [17] demonstrated that the Martin-Löf random left-c.e. reals are closed upwards under Solovay reducibility. In the following theorem, we will show this closure property for all reals.

Theorem 21 (Solovay)**.** *Let the Martin-Löf random real α be Solovay reducible to some real β. Then β is Martin-Löf random.*

Corollary 22. *The class of Martin-Löf random reals is closed upwards under Solovay reducibility.*

Proof of Theorem 21. Let the Martin-Löf random real α be Solovay reducible to some real β via some translation function g and Solovay constant c. For a proof by contradiction, we assume that β is not Martin-Löf random. So we can fix a Solovay test $[p_0, q_0], [p_1, q_1], \ldots$ that fails on β in the sense that β is in $[p_n, q_n]$ for infinitely many indices n. For every such n, the choice of g and c implies that $g(p_n) < \alpha$ and $\alpha - g(p_n) < c(\beta - p_n) < c(q_n - p_n)$, hence we have

$$\alpha \in [g(p_n), g(p_n) + c(q_n - p_n)]. \tag{13}$$

So, the Martin-Löf random real α is contained in infinitely many such intervals, and therefore, in order to obtain the desired contradiction, it suffices to show that the intervals as in (13) for $n = 0, 1, \ldots$ form a Solovay test. The latter holds true because, for given n, the rationals $g(p_n)$ and $g(q_n)$ can be computed and the sum of the Lebesgue measures of the intervals in the test is finite since we have

$$\sum_{n\in\omega} |[g(p_n), g(p_n) + c(q_n - p_n)]| = c\sum_{n\in\omega}(q_n - p_n) = c\sum_{n\in\omega} |[p_n, q_n]|,$$

where $|[a, b]| = b - a$ denotes the Lebesgue measure of an interval $[a, b]$. The last value in this chain is finite since it is just the sum of the Lebesgue measures of the intervals in the given Solovay test. □

Assuming that there is a largest degree among the left-c.e. Solovay degrees, every left-c.e. real is Solovay reducible to every real in this degree. Since there are Martin-Löf random left-c.e. reals, by Corollary 22, all reals in such a largest degree must be Martin-Löf random.

By a celebrated result, which is the culmination of work by several groups of authors, the left-c.e. Solovay degrees possess a largest degree, which contains exactly the Martin-Löf random left-c.e. reals.

Theorem 23 (Solovay [17], Calude, Hertling, Khoussainov and Wang [2], Kučera and Slaman [8])**.** *The left-c.e. Solovay degrees possess a largest degree, which coincides with the class of Martin-Löf random left-c.e. reals.*

3 Monotone Solovay reducibility

In what follows, we want to come up with a notion of Solovay reducibility that is suitable to be applied to arbitrary, not necessarily left-c.e. reals. The definition of Solovay reducibility in terms of translation functions indeed applies to arbitrary reals, albeit we will argue that it is advantageous to use the following monotone variant of Solovay reducibility instead. Recall that a partial function $g\colon \mathbb{Q} \to \mathbb{Q}$ is nondecreasing if we have $g(q_1) \leq g(q_2)$ for all $q_1 < q_2$ in the domain of g.

Definition 24. *A real α is* MONOTONE SOLOVAY REDUCIBLE *to some real β, written $\alpha \leq^{\mathrm{m}}_{\mathrm{S}} \beta$, if α is Solovay reducible to β via a translation function g that is monotonically nondecreasing.*

By the following proposition, monotone Solovay reducibility can indeed be considered as a natural extension of Solovay reducibility on LEFT−CE to arbitrary reals.

Proposition 25. *Solovay reducibility and monotone Solovay reducibility coincide on the set of left-c.e. reals.*

Proof. By definition, monotone Solovay reducibility implies Solovay reducibility. The nontrivial implication in Proposition 25 is immediate by Proposition 10, where it was shown that even the same Solovay constants can be achieved for both reducibilities. □

We can easily see that this extension is proper since, for every (not necessarily left-c.e.) real α and every rational $d > 0$, it holds that $d\alpha \leq^{\mathrm{m}}_{\mathrm{S}} \alpha$ via the nondecreasing translation function $g(q) = dq$ and a constant $\frac{1}{d}$. A nontrivial example of a further real α' such that $\alpha' \leq^{\mathrm{m}}_{\mathrm{S}} \alpha$ can be constructed using the merging operator $\oplus$ for binary representations that we define as follows.

For two infinite binary sequences $A = A(0)A(1)\dots$ and $B = B(0)B(1)\dots$, we write $A \oplus B$ for the infinite binary sequence with elements of A on even places and elements of B on the odd places:

$$\begin{aligned} (A \oplus B)(2n) &= A(n), \\ (A \oplus B)(2n+1) &= B(n). \end{aligned}$$

For two reals $\alpha = 0, A$ and $\beta = 0, B$, the real $0, A \oplus B$ will also be denoted as $\alpha \oplus \beta$:

$$\alpha \oplus \beta = 0, (A \oplus B).$$

Finally, we identify every nondyadic rational with its unique binary representation as an infinite periodic fracture and every dyadic rational $\frac{m}{2^n} = 0, k_1 \dots, k_n$ with its infinite binary representation $0, k_1 \dots, k_n 000 \dots$.

Proposition 26. *Let* $\mathbf{0}$ *be the infinite sequence of zeroes. Then, for every real* α*, it holds that* $\alpha \oplus \mathbf{0} \leq_{\mathrm{S}}^{\mathrm{m}} \alpha$.

Proof. The function $g(q) = q \oplus \mathbf{0}$ is computable, nondecreasing, maps every $q \in [0, \alpha)$ in $q \oplus \mathbf{0} \in [0, \alpha \oplus \mathbf{0})$, and, for every $q < \alpha$, we can easily see that

$$\alpha \oplus \mathbf{0} - q \oplus \mathbf{0} < \alpha - q,$$

hence $\alpha \leq_{\mathrm{S}} \beta$ via d and Solovay constant $c = 1$. □

Monotone Solovay reducibility is reflexive and transitive, hence, it introduces a degree structure in the usual way.

Proposition 27. *Monotone Solovay reducibility is a reflexive and transitive relation.*

Proof. Every real α is monotone Solovay reducible to itself by choosing the identity function as a translation function. Next, let α be monotone Solovay reducible to β via the translation function g_α for α and β, let β be monotone Solovay reducible to γ via the translation function g_β for β and γ and let c_α and c_β be corresponding Solovay constants. Then the composition $g_\beta \circ g_\alpha$ is a computable translation function that is nondecreasing and witnesses that α is monotone Solovay reducible to γ with Solovay constant $c_\alpha \cdot c_\beta$, details are left to the reader. □

Corollary 28. *Monotone Solovay reducibility induces a degree structure on the set of real numbers.*

3.1 Closure properties of monotone Solovay reducibility

Since monotone Solovay reducibility implies Solovay reducibility, every downwards and upwards closure that holds for the latter reducibility also holds for the former one. In particular, by Proposition 12 and Corollary 22, we obtain the following.

Proposition 29. *The classes of computable reals and of left-c.e. reals are both closed downwards under monotone Solovay reducibility.*

The class of Martin-Löf random reals is closed upwards under monotone Solovay reducibility.

We start the investigation of further closer properties and, in general, the degree structure induced by the monotone Solovay reducibility by technical lemmas describing the behavior of a translation function witnessing the monotone Solovay reducibility to some real when applied to arguments below and above this real.

One can also note that the monotonicity of the translation function g from the definition of $\leq_{\mathrm{S}}^{\mathrm{m}}$ on its domain implies that, for every $q > \beta$ lying in $dom(g)$, it holds that $g(q) \geq \alpha$. We will examine the possible behavior of g more precisely in the Proposition 31.

Lemma 30. *Let the real α be monotone Solovay reducible to a real β via translation function g and Solovay constant c. Then, for every $q > \beta$ in the domain of g, it holds that $g(q) \geq \alpha$.*

Proof. Assuming that, for some rational $q > \beta$, it holds that $g(q) < \alpha$, fix some rational $q' < \beta$ that differs from β by strictly less than $\frac{\alpha - g(q)}{c}$. By the Solovay property, we obtain

$$\alpha - g(q') < c(\beta - q') < c\left(\frac{\alpha - g(q)}{c}\right) = \alpha - g(q),$$

which implies $g(q') > g(q)$, and therefore, by $q' < q$, contradicts to the assumption that g is nondecreasing. □

Lemma 31. *Let the translation function g witness that some real α is Solovay reducible to a real β. Let $q_0, q_1, \ldots$ be a sequence of rationals in the domain of g that converges to β.*

Then β is right-c.e. or the sequence $g(q_0), g(q_1), \ldots$ converges to α.

If, in addition, the translation function g is nondecreasing, i.e., witnesses that α is monotone Solovay reducible to β, then β is computable or the sequence $g(q_0), g(q_1), \ldots$ converges to α.

Proof. In case β is a rational, there is nothing to prove, hence we can assume that all q_i differ from β. The Solovay condition for g implies that the subsequence of $g(q_0), g(q_1), \ldots$ of all $g(q_i)$, where $q_i < \beta$, converges to α. Therefore, in case the sequence $g(q_0), g(q_1), \ldots$ does not converge to α, at least one of the two following cases applies.

I There is a rational $p > \alpha$ and infinitely many indices i such that $q_i > \beta$ and $g(q_i) > p$.

II There is a rational $p < \alpha$ and infinitely many indices i such that $q_i > \beta$ and $g(q_i) < p$.

In Case I, let L be the set of indices i, where $g(q_i) > p$. Then the set L can be effectively enumerated, contains only indices i, where $q_i > \beta$, and contains indices i, where q_i is arbitrarily close to β. The latter follows since L is infinite and the sequence $q_0, q_1, \ldots$ converges to β. So, given an effective enumeration $i_0, i_1, \ldots$ of the set L, in case we let for all n

$$b_n = \min\{q_{i_0}, \ldots, q_{i_n}\},$$

the sequence $b_0, b_1, \ldots$ is a right-c.e. approximation of β, hence β is right-c.e.

In Case II, by the Solovay condition for g, we can fix a rational $p_0 < \beta$ such that, for all q fulfilling $p_0 \leq q < \beta$, it holds that $g(q) > p$. Let S be the set of all indices i such that $q_i > p_0$ and $g(q_i) < p$. Then the set L can be effectively enumerated, contains only indices i such that $q_i > \beta$ and contains indices i such that q_i is arbitrarily close to β, hence it follows by essentially the same argument as in Case I that β is right-c.e.

Next, assume that g is in addition nondecreasing. If the sequence $g(q_0), g(q_1), \ldots$ converges to α, we are done, so, we can assume the opposite. Then the argument that β is right-c.e. is still valid. We conclude the proof of the lemma by showing that β is also left-c.e., and thus, it is computable by Proposition 7. By Lemma 30, Case II is ruled out, hence we can fix a rational as in Case I. Again, we can argue that there are $q_i > \beta$ arbitrarily close to β that fulfill $g(q_i) > p$. By monotonicity of g, this implies $g(q) > p$ for all $q > \beta$ in the domain of g, hence β is left-c.e. since its left-cut is equal to the set of all $q < p$ in the domain of g, which can be effectively enumerable. □

Proposition 32. *Let the real α be monotone Solovay reducible to some right-c.e. real β. Then α is right-c.e.*

Proof. Let the translation function g witness that α is monotone Solovay reducible to β, and let $b_0, b_1, \ldots$ be a right-c.e. approximation of β. By Lemma 31, the real β is computable or the computable sequence $g(b_0), g(b_1), \ldots$ converges to α. In the latter case, since the sequence $b_0, b_1, \cdots$ is nonincreasing and by monotonicity of g, also the sequence $g(b_0), g(b_1), \cdots$ is nonincreasing, hence it is a right-c.e. approximation of α, thus, α is right-c.e. In case β is computable, by Proposition 12, also α is computable, hence α is right-c.e. □

Proposition 33. *In case the real α is Solovay reducible to the computably approximable real β, then α is computably approximable or β is right-c.e. In case α is monotone Solovay reducible to β, the real α is computably approximable.*

Proof. Let the translation function g witness that α is Solovay reducible to β, and let $b_0, b_1, \ldots$ be a computable approximation of β. We show the first assertion of the proposition. In case β is right-c.e., we are done. So we can assume the opposite, hence, in particular, all b_i differ from β and there are infinitely many i such that $b_i < \beta$. Observe further that we can assume that g is defined on all b_i because, otherwise, by dovetailing the computations of the values $g(b_0), g(b_1), \ldots$, we obtain a computable infinite subsequence of $b_0, b_1, \ldots$ that contains only numbers in the domain of g, which we could use instead.

By Lemma 31, the computable sequence $g(b_0), g(b_1), \ldots$ converges to α, hence α is computably approximable or the real β is right-c.e. In the latter case, if α is actually monotone Solovay reducible to β, then, by the downwards closure of the right-c.e. reals under monotone Solovay reducibility according to Proposition 12, the real α is right-c.e., and therefore, it is computably approximable. □

Proposition 34. *Let the real α be monotone Solovay reducible to some d.c.e. real β. Then α is d.c.e.*

Proof. Let the translation function g witness that α is Solovay reducible to β, and let $b_0, b_1, \ldots$ be a computable approximation of β. If β is left-c.e. or right-c.e., then so is α by the closure properties of monotone Solovay reducibility and we are done. So we can assume the opposite. Thus, in particular, all b_i differ from β and there are infinitely many i fulfilling $b_i < \beta$ and infinitely many i fulfilling $b_i > \beta$. Observe further that we can assume that g is defined on all b_i because, otherwise, by dovetailing the computations of the values $g(b_0), g(b_1), \ldots$, we obtain a computable infinite subsequence of $b_0, b_1, \ldots$ that contains only numbers in the domain of g, which we could use instead. Let $a_n = g(b_n)$. Then $a_0, a_1, \ldots$ is a computable approximation of α by Lemma 31 and since β is assumed not to be computable.

Next, we define inductively and effectively an infinite strictly increasing sequence $i_0, i_1, \ldots$ of indices. Then the sequence $a_{i_0}, a_{i_1}, \ldots$ is a computable approximation of α and, in order to show that α is d.c.e., it suffices to show that

$$d = \sum_{k=0}^{\infty} |a_{i_{k+1}} - a_{i_k}| = \sum_{k=0}^{\infty} |g(b_{i_{k+1}}) - g(b_{i_k})| < \infty. \tag{14}$$

The sum over the values $a_{i_{k+1}} - a_{i_k}$ for $k = 0, \ldots, t$ is telescopic and evaluates to $a_{t+1} - a_0$, hence the infinite sum over all such values is equal to $\alpha - a_0$. Accordingly, in order to show (14), it suffices to show that the sum over all positive values $a_{i_{k+1}} - a_{i_k}$ is finite.

Let i_0 be equal to 0. Assuming that i_k has already been defined, let i_{k+1} be equal to the least index $i > i_k$ such that either $b_i < b_{i_k}$ or

$$b_i > b_{i_k} \qquad \text{and} \qquad g(b_i) - g(b_{i_k}) < 2c(b_i - b_{i_k}). \tag{15}$$

Such an index i always exists. Indeed, in case $b_{i_k} > \beta$, there is some i fulfilling $b_i < b_{i_k}$ because the sequence $b_0, b_1, \ldots$ converges to β. In case $b_{i_k} < \beta$, there is an index i that satisfies (15) because, by choice of g, we have

$$\alpha - g(b_{i_k}) < c(\beta - b_{i_k}),$$

hence, for every i such that $b_i < \beta$ and $\beta - b_i < b_i - b_{i_k}$, it holds that

$$g(b_i) - g(b_{i_k}) < \alpha - g(b_{i_k}) < c(\beta - b_{i_k}) < c(\beta - b_i + b_i - b_{i_k}) < 2c(b_i - b_{i_k}).$$

Finally, observe that, by monotonicity of g, the difference $\delta_k = a_{i_{k+1}} - a_{i_k}$ can only be positive in case $b_{i_{k+1}} > b_{i_k}$, which implies by (15) that

$$\delta_k = g(b_{i_{k+1}}) - g(b_{i_k}) < 2c(b_{i_{k+1}} - b_{i_k}) < 2c \sum_{j=i_k}^{i_k - 1} |g(b_{j+1} - g(b_j))|,$$

hence the infinite sum over all positive values δ_k is finite by (14). □

Rettinger and Zheng [19] introduced a reducibility $\leq_{\mathrm{S}}^{2\mathrm{a}}$ and demonstrated that the set of d.c.e. reals is closed downwards under this reducibility. This yields an alternative proof of Proposition 34 since, for d.c.e. reals, the latter reducibility is implied by monotone Solovay reducibility, see Chapter 7 for details.

We summarize the results on downwards closure under monotone Solovay reducibility obtained so far.

Theorem 35. *The classes of computable, of left-c.e., of right-c.e., of d.c.e. and of computably approximable, or Δ^0_2, reals are all closed downwards under monotone Solovay reducibility.*

Proof. Since Solovay reducibility is implied by its monotone variant, the downwards closure of the classes of computable and of left-c.e. reals under monotone Solovay reducibility follows from the corresponding result for Solovay reducibility in Proposition 12. The closure of the classes of right-c.e., of d.c.e and of computably approximable reals follows by Propositions 32, 33 and 34, respectively. □

3.2 An equivalent characterization of monotone Solovay reducibility on the computably approximable reals

Recall from Proposition 13 the equivalent characterization of Solovay reducibility on the set of left-c.e. reals via its index form, which amounts to requiring a condition of the form $\alpha - a_n < c(\beta - b_n)$. For monotone Solovay reducibility, a similar characterization can be given for the larger set of computably approximable reals.

Proposition 36. *Let α and β be computably approximable reals such that α is monotone Solovay reducible to β with Solovay constant c. Then, for every computable approximation $b_0, b_1, \ldots$ of β, there is a computable approximation $a_0, a_1, \ldots$ of α such that, for every n fulfilling $b_n < \beta$, it holds that*

$$0 < \alpha - a_n < c(\beta - b_n). \tag{16}$$

Proof. First, assume that β is computable. Then α is computable as well by the downwards closure of the set of computable reals under Solovay reducibility. So we are done by defining a computable approximation $a_0, a_1, \ldots$ of α such that every a_n is strictly less than α and differs from α by strictly less than $c(\beta - b_n)$ in case $b_n < \beta$ and by strictly less than 2^{-n} otherwise. In case β is not computable, let g be a translation function that witnesses that α is monotone Solovay reducible to β with Solovay constant c and let a_n be equal to $g(b_n)$. Then the sequence $a_0, a_1, \ldots$ is a computable approximation of α by Lemma 31 that, by choice of g, satisfies (16) for all n such that $b_n < \beta$. □

4 Speedability of left-c.e. reals

4.1 Speedability and speed-up functions

Speed-up functions In this paragraph, we investigate into whether a given left-c.e. real numbers α has left-c.e. approximations $a_0, a_1, \dots$ that can be effectively sped up in the sense that, for some computable function f, there are infinitely many indices n such that $a_{f(n)}$ is significantly closer to α than a_n. The following definition introduces corresponding notions of speedability.

Definition 37. *A function $f\colon \omega \to \omega$ is a* SPEED-UP FUNCTION *if it is nondecreasing and we have $n \leq f(n)$ for all n.*

A left-c.e. approximation $a_0, a_1, \dots$ of a real α is ρ-SPEEDABLE *for some real number ρ if there is a computable speed-up function f such that*

$$\liminf_{n\to\infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} \leq \rho, \tag{17}$$

and the left-c.e. approximation is SPEEDABLE *if it is ρ-speedable for some $\rho < 1$.*

A real number α is ρ-SPEEDABLE WITH RESPECT TO A GIVEN LEFT-C.E. APPROXIMATION *if the approximation converges to α and is ρ-speedable. A real number is* ρ-SPEEDABLE *if it is ρ-speedable with respect to some of its left-c.e. approximations. A real number is* SPEEDABLE *if it is ρ-speedable for some $\rho < 1$, and the notion of* SPEEDABILITY WITH RESPECT TO A GIVEN LEFT-C.E. APPROXIMATION *is defined in the same manner. A left-c.e. approximation is* NONSPEEDABLE *if it is not speedable, and* NONSPEEDABLE *real numbers are defined likewise.*

Note that, by definition of speed-up function f, the value of the fraction in (17) is nonnegative and at most 1 for all n, hence every left-c.e. approximation and thus also every left-c.e. real cannot be ρ-speedable for $\rho < 0$ but is ρ-speedable for all $\rho \geq 1$. Note further that, for a speed-up function f that satisfies (17), we can assume in addition that f is strictly increasing because, otherwise, it suffices to replace f by a computable speed-up function that is strictly increasing and at least as large as f.

The notion of a speedable left-c.e. number has been introduced by Merkle and Titov [10] by a slightly different but obviously equivalent formulation. Introducing the notion of speedability is partially motivated by the fact that Barmpalias and Lewis-Pye [1, Theorem 1.7] have shown implicitly that Martin-Löf left-c.e. random reals are nonspeedable. In what follows, we give a straightforward direct proof of the latter fact and investigate into questions related to the speedability of left-c.e. real numbers.

Speedability and Solovay autoreducibility By Proposition 15, a left-c.e. real α is Solovay reducible to itself with Solovay constant ρ if there is a left-c.e. approximation $a_0, a_1, \dots$ of α and a computable function f such that it holds for all n that

$$\alpha - a_{f(n)} \leq \rho(\alpha - a_n). \tag{18}$$

Here, we can assume that f is an index function because, otherwise, it suffices to replace f by the function $n \mapsto \max\{n, f(0), \dots, f(n)\}$. So, for every $\rho > 0$, any left-c.e. real α that is Solovay autoreducible, i.e., is Solovay reducible to itself, with Solovay constant ρ is ρ-speedable. The reverse implication is not true. Intuitively speaking, the reason is that the upper bound ρ is required for the limit inferior of a sequence of values in the case of ρ-speedability, e.g., it suffices to satisfy the upper bound infinitely often, whereas the upper bound ρ must be satisfied for all values in the case of Solovay autoreducibility. Accordingly, it does not come as a surprise that, in Proposition 49, ρ-speedability will be characterized as infinitely-often Solovay autoreducibility with Solovay constant ρ. For the technical reason that only one of these notions is defined in terms of a limit superior, the equivalence proof does not work directly for identical values of ρ, and therefore, we have to postpone these considerations until we have shown that every speedable left-c.e. real is ρ-speedable for arbitrary $\rho > 0$.

Computable reals are speedable We argue next that, among the left-c.e. reals, exactly the computable reals are Solovay autoreducible with arbitrary Solovay constant $\rho > 0$. By the discussion in the last paragraph, the latter implies that all computable reals are speedable.

Proposition 38. *Every computable real is speedable.*

Proposition 39. *Let α be a left-c.e. real, and let ρ be a real fulfilling $0 < \rho < 1$. Then α is Solovay reducible to itself with Solovay constant ρ if and only if α is computable.*

Proof of Proposition 39. First, assume that α is computable. By Remark 3, we can fix a left-c.e. approximation $a_0, a_1, \dots$ of α that is strictly increasing and fulfills $\alpha - a_k < 2^{-k}$ for all k. Let $f(n)$ be equal to the least $k > n$ such that $2^{-k} < \rho(a_{n+1} - a_n)$. Hence, for every n, it holds that

$$\alpha - a_{f(n)} < 2^{-n} < \rho(a_{n+1} - a_n). \tag{19}$$

Then f is a computable index function that witnesses that α is Solovay reducible to itself with constant ρ because we have for all n

$$\frac{\alpha - a_{f(n)}}{\alpha - a_n} < \frac{\rho(a_{n+1} - a_n)}{\alpha - a_n} < \frac{\rho(a_{n+1} - a_n)}{a_{n+1} - a_n} = \rho,$$

where the first inequality is implied by (19) and the second one by $a_{n+1} < \alpha$.

Conversely, assume that α is Solovay reducible to itself via a computable index function f and Solovay constant $\rho < 1$ with respect to some left-approximation $a_0, a_1, \dots$ of α, i.e., we have (18) for all n. Let $f^{(k)}$ be the k-fold composition of f with itself, e.g., $f^{(2)}(n) = f(f(n))$. Then we have

$$\frac{\alpha - a_{f^{(k)}(n)}}{\alpha - a_n} = \frac{\alpha - a_{f^{(1)}(n)}}{\alpha - a_n} \cdot \frac{\alpha - a_{f^{(2)}(n)}}{\alpha - a_{f^{(1)}(n)}} \cdot \ldots \cdot \frac{\alpha - a_{f^{(k-1]}(n)}}{\alpha - a_{f^{(k-2)}(n)}} \cdot \frac{\alpha - a_{f^{(k]}(n)}}{\alpha - a_{f^{(k-1)}(n)}} \leq \rho^k.$$

Here, $\alpha - a_n$ is at most 1, hence, for given $\varepsilon > 0$, if we choose k so large that $\rho^k < \varepsilon$, then $\alpha - a_{f^{(k)}(n)}$ differs at most by ε from α, hence α is computable. □

4.2 Independence of the constant and of the left-c.e. approximation

For the notion of speedability of left-c.e. reals numbers, we obtain the following dichotomy. Let α be any left-c.e. real number. In case α is nonspeedable, by definition, all left-c.e. approximation s of α are nonspeedable. On the other hand, in case α is speedable, all left-c.e. approximations of α are ρ-speedable for all $\rho > 0$, as is stated in the next theorem.

Theorem 40. *Every speedable left-c.e. real is ρ-speedable for any real number $\rho > 0$ with respect to any of its left-c.e. approximations.*

Theorem 40 is immediate from Lemmas 41 and 42.

Lemma 41. *Let $a_0, a_1, \ldots$ be a left-c.e. approximation that is ρ-speedable for some $\rho > 0$. Then all left-c.e. approximations with the same limit are also ρ-speedable.*

Proof. Let α be the limit of $a_0, a_1, \ldots$, and let $b_0, b_1, \ldots$ another left-c.e. approximation with limit α. Fixing a speed-up function f such that $a_0, a_1, \ldots$ is ρ-speedable via f, we will construct a speed-up function g such that $b_0, b_1, \ldots$ is ρ-speedable via g.

We define g and two other functions j and m by setting for every natural i the values $j(i)$, $m(i)$ and $g(i)$ as follows:

$$\begin{aligned} j(i) &= \max\{0, \max\{j : b_j \leq a_i\}\}, & (20)\\ m(i) &= \max\{0, \max\{j : a_j < b_{i+1}\}\}, & (21)\\ g(i) &= \min\{j : b_j \geq a_{f(m(i))}\}. & (22) \end{aligned}$$

All these functions are obviously total and computable and fulfill for every i the inequalities

$$b_{j(i)} \leq a_i \leq a_{m(j(i))} \quad \text{and} \quad a_{f(i)} \leq a_{f(m(j(i)))} \leq b_{g(j(i))}, \qquad (23)$$

where, on the left side of (23), the first inequality holds by definition of j and the second one by the inequality $m(j(i)) \geq i$ that follows from $a_i < b_{j(i)+1}$. The first inequality on the right side of (23) is implied from the second inequality on the left side by monotonicity of f. Finally, the second inequality on the right side holds by definition of g.

The inequalities in (23) imply together that

$$\frac{\alpha - b_{g(j(i))}}{\alpha - b_{j(i)}} \quad \leq \quad \frac{\alpha - a_{f(n)}}{\alpha - a_n}.$$

The limit inferior of the terms on the right-hand side is at most ρ, hence the same holds for the terms on the left-hand side. Consequently, the left-c.e. approximation $b_0, b_1, \dots$ is ρ-speedable via g.

□

Lemma 42. *Let α be a speedable left-c.e. real number and let $\rho > 0$ be a real number. Then α is ρ-speedable.*

Proof. We can assume $\rho < 1$, since, otherwise, there is nothing to prove. For a proof by contradiction, assume that the conclusion of the lemma is false. We consider the set S of all real numbers ρ' such that α is ρ'-speedable. Let $\rho_{\inf}$ be the infimum of S. By definition, the set S is closed upwards, i.e., contains with some ρ' also all $\rho'' > \rho'$, hence S contains all $\rho' > \rho_{\inf}$ but no $\rho' < \rho_{\inf}$. By assumption on α and ρ, we then have $\rho < \rho_{\inf} < 1$.

So we can fix rational numbers ρ_1, ρ_2 and ρ_2' that satisfy

$$0 < \rho_1 < \rho_{\inf} < \rho_2' < \rho_2 < 1 \qquad \text{and} \qquad \frac{\rho_2}{1-\rho_2} - \frac{\rho_1}{1-\rho_1} < \frac{\rho}{1-\rho} \tag{24}$$

by choosing ρ_1 and ρ_2 distinct from but close enough to $\rho_{\inf}$. Then α is ρ_2'-speedable, so we can fix a left-c.e. approximation $a_0, a_1, \dots$ of α that is ρ_2'-speedable via some computable speed-up function f. Note that, for any real number x in $[0, 1)$, we have

$$\frac{\alpha - a_{f(n)}}{\alpha - a_n} < x \qquad \Longleftrightarrow \qquad \alpha - a_{f(n)} < \frac{x}{1-x}(a_{f(n)} - a_n), \tag{25}$$

which follows by, first, rewriting $\alpha - a_n$ as the sum of $\alpha - a_{f(n)}$ and $a_{f(n)} - a_n$ and, further, rearranging terms. Observe that the equivalence (25) remains valid if we replace both occurrences of " $<$ " by " $>$ " and that both versions of (25) hold for any other speed-up function in place of f. We then have

$$\rho_1 < \frac{\alpha - a_{f(n)}}{\alpha - a_n} < \rho_2 \iff \frac{\rho_1(a_{f(n)} - a_n)}{1-\rho_1} < \alpha - a_{f(n)} < \frac{\rho_2(a_{f(n)} - a_n)}{1-\rho_2}. \tag{26}$$

By choice of ρ_1, ρ_2, ρ_2' and f, the first and thus also the third inequalities in (26), from left to right, hold for almost all n, while the second and the fourth ones hold for infinitely many n. Fix n_0 such that the third strict inequality in (26) holds for all $n \geq n_0$. Let $f'(n) = f(n)$ for all $n < n_0$ and, for all other n, choose $f'(n)$ to be minimum such that we have $n \leq f'(n)$ and

$$a_{f'(n)} > a_{f(n)} + \frac{\rho_1}{1-\rho_1}\left(a_{f(n)} - a_n\right).$$

Then f' is total by choice of n_0 and, for the infinitely many n that satisfy the fourth inequality in (26), we have

$$\begin{aligned}\alpha - a_{f'(n)} &< \alpha - a_{f(n)} - \frac{\rho_1}{1-\rho_1}\left(a_{f(n)} - a_n\right) \\ &< \frac{\rho_2}{1-\rho_2}\left(a_{f(n)} - a_n\right) - \frac{\rho_1}{1-\rho_1}\left(a_{f(n)} - a_n\right) \\ &< \frac{\rho}{1-\rho}\left(a_{f(n)} - a_n\right) < \frac{\rho}{1-\rho}\left(a_{f'(n)} - a_n\right).\end{aligned}$$

The relations hold for these n, from top to bottom and from left to right, by definition of f', by (26), by choice of ρ_1 and ρ_2, and finally by $f(n) \leq f'(n)$.

Thus, by (25), we obtain for all such n that

$$\frac{\alpha - a_{f'(n)}}{\alpha - a_n} < \rho,$$

i.e., the speed-up function f' witnesses that α is ρ-speedable, contradiction. □

Incrementation as a speed-up function We will demonstrate in Theorem 44 of Section 4.3 that Martin-Löf random left-c.e. reals are nonspeedable. The proof of this theorem relies on the following lemma.

Lemma 43. *Let the left-c.e. real number α be ρ-speedable for some $\rho > 0$. Then α is ρ-speedable with respect to some left-c.e. approximation via the function $n \mapsto n+1$.*

Proof. Let α be ρ-speedable with respect to a left-c.e. approximation $a_0, a_1, \dots$ via some computable speed-up function f. Let $g(0) = 0$ and, inductively, let $g(i+1)$ be equal to $f(g(i))$, which can also be written as $g(i) = f^{(i)}(0)$. Let $i(n)$ be the maximum index i such that $g(i) \leq n$. By choice of f and by definition of the functions g and i, we then have for all n and for $i = i(n)$ that

$$g(i) \leq n < g(i+1) = f(g(i)) \leq f(n) < f(g(i+1)) = g(i+2).$$

Thus, for every n, the interval $[n, f(n)]$ is contained in $[g(i(n)), g(i(n)+2)]$, hence

$$\frac{\alpha - a_{g(i(n)+2)}}{\alpha - a_{g(i(n))}} \leq \frac{\alpha - a_{f(n)}}{\alpha - a_n}.$$

This inequality remains valid if we apply the lim inf operator on both sides, hence the limes inferior of the terms on the left-hand side is strictly smaller than ρ. As a consequence, at least one of the left-c.e. approximations $a_{g(0)}, a_{g(2)}, \dots$ and $a_{g(1)}, a_{g(3)}, \dots$ witnesses the lemma assertion. □

4.3 Martin-Löf random and nonhigh left-c.e. real numbers

Martin-Löf random left-c.e. reals are nonspeedable By Theorem 23, the Martin-Löf random reals form the largest degree in the upper semilattice of the Solovay degrees of left-c.e. reals. In what follows, we show the nonspeedability of all such reals. The latter theorem has been implicitly proven by Barmpalias and Lewis-Pye [1] since it is an immediate corollary to their main result.

Theorem 44. *Martin-Löf random left-c.e. real numbers are nonspeedable.*

Proof. For a proof by contradiction, assume that there is a left-c.e. real α that is Martin-Löf random and speedable. By Theorem 42 and Lemma 43, there is a left-c.e approximation $a_0, a_1, \ldots$ of α that is $\frac{1}{3}$-speedable via the speed-up function $n \mapsto n+1$. Consequently, by the definitions of the latter property and of limit inferior, there are then infinitely many n such that

$$\frac{\alpha - a_{n+1}}{\alpha - a_n} < \frac{1}{2}. \tag{27}$$

For every such n, we have

$$2(\alpha - a_{n+1}) < \alpha - a_n = \alpha - a_{n+1} + a_{n+1} - a_n,$$

that is, we have $\alpha - a_{n+1} < a_{n+1} - a_n$. Thus, α is contained in the interval

$$I_n = [a_{n+1}, a_{n+1} + (a_{n+1} - a_n)]$$

for each of the infinitely many n that satisfy (27). So, in order to obtain the desired contradiction, it suffices to show that the intervals $I_0, I_1, \ldots$ form a Solovay test. The latter holds because, for given n, we can compute the rational endpoints of the interval I_n, where the sum of the lengths of these intervals is finite because we have

$$\sum_{n=0}^{\infty} |I_n| = \sum_{n=0}^{\infty} |a_{n+1} - a_n| = \alpha - a_0.$$

□

Nonhigh left-c.e. reals are speedable Recall that, for a real α, the notion of being high is defined in terms of the jump of α but can equivalently be characterized by the following domination property: a real α is NONHIGH if, for every function f that is computable with oracle α, there is a computable function g that is not dominated by f, i.e., where $f(n) < g(n)$ holds for infinitely many n.

Theorem 45. *All nonhigh left-c.e. real number are speedable.*

Proof. Let $\rho > 0$ be a real number. Let α be a nonhigh left-c.e. real number, and let $a_0, a_1, \ldots$ be a left-c.e. approximation of α. Similar to the case of computable α, there is a function f computable in α such that

$$\frac{\alpha - a_{f(n)}}{\alpha - a_n} < \rho \quad \text{for all } n. \tag{28}$$

Since α is nonhigh, there is a computable function g that is not dominated by f, wherein we can assume that g is strictly increasing, and thus, is a speed-up function. Then $a_{f(n)} < a_{g(n)}$ holds for infinitely many n, and therefore, by (28), the left-c.e. approximation $a_0, a_1, \ldots$, and hence also the real α, is ρ-speedable via g. ☐

On the other side, there exists high left-c.e. reals that are speedable. An example of such a real is encoded by the halting problem, as we can see below.

Remind that, according to [5], a real α is called STRONGLY C.E. if $\alpha = 0, A$ for a c.e. set A, see

Proposition 46. *Every strongly c.e. real is speedable.*

Proof. For a c.e. set A, let $a_0, a_1, \ldots$ be a computable enumeration of A. First, we note that this enumeration contains infinitely many FINAL STAGES, i.e., elements a_i that fulfill

$$a_i = \min\{a_k : k \geq i\}. \tag{29}$$

So, fixing by $i_0 < i_1 < \ldots$ the sequence of all indices that fulfills (29), it holds for every i_n that

$$2^{-a_{i_n}} \geq \sum_{k=a_{i_n}+1}^{\infty} 2^{-a_k}. \tag{30}$$

Then, the left-c.e. approximation $(\alpha_n)_{n\in\omega}$ defined as

$$\alpha_n = \sum_{i=0}^{n} 2^{-a_i}$$

is $\frac{1}{2}$-speedable in the sense of Proposition 49(d) because we have for every $n \in \omega$ the inequality

$$\alpha - \alpha_{i_n} = \sum_{k=a_{i_n}+1}^{\infty} 2^{-a_k} \underset{\text{by (30)}}{\leq} \frac{1}{2} \sum_{k=a_{i_n}}^{\infty} 2^{-a_k} = \frac{1}{2}(\alpha - \alpha_{i_n-1}).$$

☐

Since the halting problem obviously encodes a strongly left-c.e. real, the following proposition an immediate consequence from the latter one.

Proposition 47. *The left-c.e. real $0, \emptyset'$, where $\emptyset'$ denotes the halting problem, is speedable.*

4.4 Speedability of left-c.e. reals is a degree property

We have already seen that the Martin-Löf random left-c.e. real numbers form a Solovay degree, where all reals in this degree are nonspeedable by Theorem 44. By definition, the class of nonhigh left-c.e. real numbers is equal to a union of Turing degrees and thus also of Solovay degrees since Solovay reducibility implies Turing reducibility. Every such Solovay degree contains only speedable reals by Theorem 45. By the next theorem, the two latter facts are not a coincidence.

Theorem 48. *Speedability is a degree property with respect to Solovay reducibility in the sense that either all or none of left-c.e. real numbers in a Solovay degree are speedable.*

Proof. Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be left-c.e. approximations of reals α and β, respectively, where α and β are in the same Solovay degree and β is speedable. Then there are computable functions g_1 and g_2 and constants c_1 and c_2 such that α is Solovay reducible to β via g_1 and constant c_1 and β is Solovay reducible to α via g_2 and constant c_2, where we can assume that the functions g_1 and g_2 are strictly increasing. We then have for all n that

$$\alpha - a_{g_1(n)} < c_1(\beta - b_n) \qquad \text{and} \qquad \beta - b_{g_2(n)} < c_2(\alpha - a_n).$$

Next, we argue that, due to the speedability of β, there is a computable speed-up function f such that, for infinitely many n, it holds that

$$\frac{\beta - b_{f(g_2(n))}}{\beta - b_{g_2(n)}} < \frac{1}{2c_1c_2}, \tag{31}$$

that is, the speed-up function f is guaranteed to obtain a certain speed-up on values of the form $b_{g_2(n)}$. By assumption on g_2, the sequence $b_{g_2(0)}, b_{g_2(0)}, \ldots$ is a left-c.e. approximation of β, which is ρ-speedable for every $\rho > 0$ by Theorem 40 and speedability of β. Accordingly, we can fix a computable speed-up function f_0 that witnesses $(3c_1c_2)^{-1}$-speedability of this approximation. Note that f_0 works with indices with respect to the approximation $b_{g_2(0)}, b_{g_2(1)}, \ldots$ in the sense that it maps the index k for $b_{g_2(k)}$ to the index $f_0(k)$ for $b_{g_2(f_0(k))}$. In order to obtain a speed-up function f that works with indices with respect to $b_0, b_1, \ldots$, we define for all n

$$m(n) = \min\{k \in \omega \colon n \leq g_2(k)\} \qquad \text{and} \qquad f(n) = g_2(f_0(m(n))).$$

Observe that f maps any index n of the form $g_2(k)$ to $g_2(f_0(k))$ and that, by choice of f_0 and f, the inequality (31) holds for infinitely many n. For every such n, we then have

$$\frac{\alpha - a_{g_1(f(g_2(n)))}}{\alpha - a_n} \leq \frac{\alpha - a_{g_1(f(g_2(n)))}}{\beta - b_{f(g_2(n))}} \cdot \frac{\beta - b_{f(g_2(n))}}{\beta - b_{g_2(n)}} \cdot \frac{\beta - b_{g_2(n)}}{\alpha - a_n} < c_1 \cdot \frac{1}{2c_1c_2} \cdot c_2 = \frac{1}{2},$$

and consequently, the left-c.e. approximation $a_0, a_1, \ldots$, and thus also the real α, is $\frac{1}{2}$-speedable via the composition $g_1 \circ f \circ g_2$. The latter function is indeed a computable speed-up function by choice of the functions g_1, f and g_2. □

4.5 Alternative characterizations of speedability of left-c.e. reals

For any left-c.e. real α and any of its left-c.e. approximation $a_0, a_1, \ldots$, any speed-up function f witnesses that α is Solovay autoreducible, i.e.., Solovay reducible to itself with Solovay constant 1 with respect to $a_0, a_1, \ldots$ of α because, by $n \leq f(n)$, we have $\alpha - a_{f(n)} \leq \alpha - a_n$ for all n. In case we require in addition that for some speed-up function f and $\rho < 1$ infinitely often actually the Solovay constant ρ is attaine by f, this is equivalent to α being ρ-speedable. The following Proposition states this characterization of speedability of left-c.e. reals in terms of Solovay autoreducibility together with some of its variants that ressemble characterizations of Solovay reducibility and of speedability as stated in Proposition 15 and Lemma 43, respectively.

Proposition 49. *Let α be a left-c.e. real α, and let ρ be a real in the open interval $(0, 1)$. Then the following statements are equivalent.*

(a) The real α is ρ-speedable.

(b) For every left-c.e. approximation $a_0, a_1, \ldots$ of α, there is a computable speed-up function f such that, for infinitely many n, it holds that

$$\alpha - a_{f(n)} \leq \rho(\alpha - a_n). \tag{32}$$

(c) There exists a left-c.e. approximation $a_0, a_1, \ldots$ of α and a computable speed-up function f such that (32) holds for infinitely many n.

(d) There exists a left-c.e. approximation $a_0, a_1, \ldots$ of α such that, for infinitely many n, it holds that

$$\alpha - a_{n+1} \leq \rho(\alpha - a_n). \tag{33}$$

Proof. By definition of speedability, each of (b), (c) and (d) implies (a). Furthermore, the implication from (b) to (c) is immediate.

In order to show the implication from (a) to (b), assume that α is speedable. Fix any left-c.e. approximation $a_0, a_1, \ldots$ of α. By Theorem 40, we can fix a speed-up function f that witnesses that this approximation is $\frac{\rho}{2}$-speedable, i.e., we have

$$\liminf_{n\to\infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} \leq \frac{\rho}{2}. \tag{34}$$

Then, by definition of limes inferior, (32) holds for infinitely many n.

In order to show the implication from (a) to (d), it suffices to observe that α is $\frac{\rho}{2}$-speedable by Theorem 40, hence it satisfies (34) with $n+1$ in place of $f(n)$ by Lemma 43. Then (d) follows again by definition of limes inferior. □

Speed-up functions need to be total The next proposition shows that the notion of speedability of left-c.e. reals is changed when replacing in its definition computable speed-up functions, i.e., functions that are, in particular, total and nondecreasing by partial computable functions that satisfy $n \le f(n)$ but are required neither to be total nor to be nondecreasing. Recall in this connection from Remark 16 that for left-c.e. reals Solovay reducibility has an equivalent characterization via partial computable index functions.

Proposition 50. *Let $a_0, a_1, \ldots$ be a strictly increasing left-c.e. approximation of a real α and let ρ be a real in the open interval $(0, 1)$. Then there is a partial computable function $f\colon \omega \to \omega$ such that $n \le f(n)$ holds for all n in the domain of f and there are infinitely many n in the domain of f that satisfy* (32).

Proof. Let $\sigma_0, \sigma_1, \ldots$ be the enumeration of all binary words in length-lexicographical order. For given n, let $f(n)$ be the least index $i \ge n$ such that $a_i > 0.\sigma_n$. Note that f is partial computable and satisfies $n \le f(n)$ for all n in its domain.

For all ℓ, let i_ℓ be the index with respect to $\sigma_0, \sigma_1, \ldots$ of the length ℓ prefix of the fractional part of the binary expansion of α, e.g., in case $\alpha = 0.01\ldots$, the binary word σ_{i_2} is equal to 01. Then we have $i_o < i_1 < \ldots$ and, for all ℓ, we have $i_\ell < 2^{\ell+1}$ and the value $f(i_\ell)$ is defined and satisfies $\alpha - a_{f(i_\ell)} \le 2^{-\ell}$.

In case if there are infinitely many n in the domain of f that satisfy (32), we are done. So we can assume the opposite, i.e., in particular, there is some N such that, for all $\ell \ge N$, it holds that $\rho(\alpha - a_{i_\ell}) \le \alpha - a_{f(i_\ell)}$. Fixing some natural number k such that $2^k < \rho$, we obtain for all $\ell \ge N$ by $i_\ell < 2^{\ell+1}$ that

$$2^{-k}(\alpha - a_{2^{\ell+1}}) \le 2^{-k}(\alpha - a_{i_\ell}) \le \rho(\alpha - a_{i_\ell}) \le \alpha - a_{f(i_\ell)} < 2^{-\ell}. \tag{35}$$

Consequently, we have $\alpha - a_{2^{\ell+1}} < 2^{-\ell+k}$ for all $\ell \ge N$, hence the computable sequence $a_{2^{N+k+1}}, a_{2^{N+k+2}}, \ldots$ witnesses by Proposition 2(2) that the real α is computable, and thus, also ρ-speedable. □

Speedability and slow-down functions By the next proposition, the notion of speedability for left-c.e. reals can also be equivalently characterized via slow-down functions. Here, a SLOW-DOWN FUNCTION is a function $g\colon \omega \to \omega$ that is nondecreasing and unbounded and satisfies $g(n) \le n$ for all n.

Proposition 51. *For every left-c.e. real α and every real $\rho \in (0, 1)$, the following statements are equivalent.*

(a) The real α is ρ-speedable.

(b) For every left-c.e. approximation $a_0, a_1, \ldots$ of α, there is a slow-down function g such that, for infinitely many n, it holds that

$$\alpha - a_n \le \rho(\alpha - a_{g(n)}). \tag{36}$$

(c) *There is a left-c.e. approximation* $a_0, a_1, \dots$ *of* α *and a slow-down function* g *such that* (36) *holds for infinitely many* n.

Proof. The implication from (b) to (c) is immediate.

In order to show that (a) implies (b), assume that the real α is ρ-speedable and fix any left-c.e. approximation $a_0, a_1, \dots$ of α. By the characterization of speedability stated as Item (b) in Proposition 49, there exists a computable speed-up function f such that, for infinitely many n, it holds that

$$\alpha - a_{f(n)} \leq \rho(a - a_n), \tag{37}$$

where, as usual, we can assume that f is strictly increasing. The function g defined by

$$g(n) = \max(\{0\} \cup \{m : f(m) \leq n\}) \tag{38}$$

is a slow-down function by definition and satisfies for all n

$$g(f(n)) = \max(\{0\} \cup \{m : f(m) \leq f(n)\}) \leq n. \tag{39}$$

Now, (b) holds true because f is strictly increasing and, for each of the infinitely many n that satisfy (37), it holds that

$$\alpha - a_{f(n)} \leq \rho(\alpha - a_n) \leq \rho(\alpha - a_{g(f(n))}).$$

It remains to show that (c) implies (a). Fix any left-c.e. approximation $a_0, a_1, \dots$ of α and any slow-down function g such that (36) is satisfied for infinitely many n. The function f defined by

$$f(n) = \min\{m \colon g(m) > n\} \tag{40}$$

is a computable speed-up function since it is by definition nondecreasing and, by $g(n) \leq n$, it holds for all n that

$$f(n) \geq f(g(n)) = \min\{m : g(m) > g(n)\} > n. \tag{41}$$

Furthermore, for the infinitely many n that satisfy (36), it holds that

$$\alpha - a_{f(g(n))} < \alpha - a_n \leq \rho(\alpha - a_{g(n)}),$$

where the set $\{g(n) \colon n \text{ satisfies (36)}\}$ is infinite since the slow-down function g is nondecreasing and unbounded. As a consequence, the real α is ρ-speedable by the characterization of speedability stated as Item (c) in Proposition 49. □

5 Speedability of arbitrary real numbers

5.1 Extending speedability to all reals

Speedability as autoreducibility via nondecreasing translation functions In Chapter 4, we have considered a notion of speedability for left-c.e. reals introduced by Merkle and Titov [10], which is equivalent by Proposition 49 to a special form of Solovay autoreducibility, where some Solovay constant $\rho < 1$ is attained infinitely often.

In this chapter, we extend the concept of speedability to arbitrary reals. Speedability will again be defined in terms of a special form of Solovay autoreducibility. However, we consider now Solovay autoreducibility via translation functions and not, as in the case of speedability for left-c.e. reals, via speed-up functions, i.e., in index form. Furthermore, and more important, we will use monotone Solovay reducibility.

Then speedability of a real α is defined as α being monotone Solovay autoreducible – where the latter again means that α is monotone Solovay reducible to itself – via some translation function g with Solovay constant 1 such that the infimum over all Solovay constants that are achieved on rationals $q < \alpha$ is some $\rho < 1$. For technical reasons that will become clear later, we require in addition that $q < g(q)$ holds for all $q < \alpha$.

Definition 52. *Let g be a partial function from the set of real numbers into itself, and let the real α be a left-sided accumulation point of the domain of α, i.e., for every $\delta > 0$, the domain of g intersects the open interval $(\alpha - \delta, \alpha)$.*[1] *By $\lim_{q \nearrow \alpha} f(q)$, we refer to the* LEFT-SIDED LIMIT *of the function f at α, i.e., the value $\lim_{q \nearrow \alpha} f(q)$ is equal to some real γ if and only if, for every $\epsilon > 0$, there exists some $\delta > 0$ such that the inequality $|f(q) - \gamma| < \epsilon$ holds for every $q \in dom(f) \cap (d - \delta, d)$. The* LEFT-SIDED LIMIT INFERIOR *$\liminf_{q \nearrow \alpha} g(q)$ is defined by*

$$\liminf_{q \nearrow \alpha} = \lim_{\delta \to 0} \inf\{g(p) \colon p \in dom(f) \cap (q - \delta, q)\}.$$

The LEFT-SIDED LIMES SUPERIOR *$\limsup_{q \nearrow \alpha} f(q)$ is introduced in the same way where in the defining formula* inf *is replaced by* sup.

In accordance with the concept of a translation function for two reals in Definition 9, for a given real α, let a TRANSLATION FUNCTION FOR α be a partial function g from the

[1] A right-sided accumulation point can be defined in the obvious way by a symmetric definition, where it is required that all intervals of the form $(\alpha, \alpha+\delta)$ intersect the set under consideration. Apparently, in the literature, the notions of left-sided and right-sided accumulation point are not used uniformly, in particular, there are sources, where the meaning of both terms as defined here is interchanged.

set $\mathbb{Q} \cap [0,1]$ to itself such that g is defined on all rationals $q < \alpha$ and for all such q it holds that $q(q) < \alpha$.

Proposition 53. *Let α be a left-c.e. real, and let ρ be a real in $[0,1)$. The real α is ρ-speedable if and only if there exists a nondecreasing translation function g for α such that $q < g(q)$ for all rationals $q < \alpha$ and it holds that*

$$\liminf_{q \nearrow \alpha} \frac{\alpha - g(q)}{\alpha - q} \leq \rho. \tag{42}$$

Proof. Fix a left-c.e. approximation $a_0, a_1, \dots$ of α that is ρ-speedable via a speed-up function f. We construct a translation function g as required by letting

$$\begin{aligned} m(q) &= \min\{n : a_n > q\} \qquad \text{and} \\ g(q) &= a_{f(m(q))}. \end{aligned}$$

Conversely, given a nondecreasing translation function g that satisfies (42) and the inequality $q < g(q)$ for all $q < \alpha$, we fix a speedable left-c.e. approximation $a_0, a_1, \dots$ of α and define a computable speed-up function f by letting

$$f(n) = \min\{m : a_m \geq g(a_n)\}.$$

Then the left-c.e. approximation $a_0, a_1, \dots$, and thus also α, is ρ-speedable via f. □

According to the latter proposition, the already defined notion of speedability for left-c.e. reals agrees on the set of left-c.e. reals with the notion of speedability for arbitrary reals defined next in terms of nondecreasing translation functions.

Definition 54. *A real α is* ρ-SPEEDABLE *for some nonnegative real ρ if there exist a nondecreasing translation function g for α such that $q < g(q)$ for all rationals $q < \alpha$ and it holds that*

$$\liminf_{q \nearrow \alpha} \frac{\alpha - g(q)}{\alpha - q} \leq \rho. \tag{43}$$

A real is SPEEDABLE *if it is ρ-speedable for some $\rho < 1$ and* NONSPEEDABLE*, otherwise.*

Remark 55. *Suppose that g is a nondecreasing translation function for some real α. By definition of translation function, this implies that, for every rational $q < \alpha$, the value $g(q)$ is defined and satisfies $q \leq g(q) < \alpha$, hence the real α is monotone Solovay autoreducible to itself via g with Solovay constant 1. Note that α is ρ-speedable for some given ρ if and only if the latter condition holds and, in addition,* (43) *is satisfied and $q < g(q)$ holds for all rationals $q < \alpha$.*

The choice of defining speedability in terms of nondecreasing translation functions, which, by Remark 55, corresponds to using monotone Solovay autoreducibility, is crucial. If speedability were defined in terms of arbitrary, not necessarily nondecreasing translation functions, i.e., using Solovay instead of monotone Solovay autoreducibility, by the following proposition the resulting notion would be trivial on the set of left-c.e. reals, hence, in particular, would not coincide with the notion of speedability that we had previously defined for left-c.e. reals in terms of index functions.

Proposition 56. *For every left-c.e. real α and every $\rho \in [0,1)$, there exists a (not necessarily monotonically nondecreasing) translation function g for α such that $q < g(q)$ for all rationals $q < \alpha$ and it holds that*

$$\liminf_{q \nearrow \alpha} \frac{\alpha - g(q)}{\alpha - q} = 0. \tag{44}$$

Proof. Let $a_0, a_1, \ldots$ be a left-c.e. approximation of α. Fix some effective enumeration $q_0, q_1, \ldots$ of the set of rational numbers in the unit interval, and, for every rational q, let $t(q)$ be the minimum index that fulfills $q_{t(q)} = q$. We define the function g as follows. For argument q, let n be minimum such that $q < a_n$. In case no such n can be found, the value $g(q)$ is undefined, otherwise, let $g(q)$ be equal to $a_{n+t(q)}$. Then g is a partial computable function that is defined on all rationals $q < \alpha$ and, for each such q and the corresponding index n, we have $q < a_n \leq a_{n+t(q)} = g(q)$.

We conclude the proof by arguing that (44) holds. For given n and k, let $j(n,k)$ be the least index that fulfills

$$\frac{\alpha - a_{j(n,k)}}{\alpha - a_n} \leq \frac{1}{k}. \tag{45}$$

Fix $n > 0$, and consider the interval $(a_{n-1}, a_n]$. By construction, for every q in this interval, we have $g(q) = a_{n+t(q)}$, while the interval contains reals q with arbitrarily large values of $t(q)$. Thus, for every k, the interval contains some q that fulfills

$$a_{j(n,k)} < a_{n+t(q)} = g(q),$$

and, for such q, inequality (45) holds with $a_{j(n,k)}$ replaced by $a_{g(q)}$. But the a_i converge to α and $n > 0$ was chosen arbitrarily, hence such q can be found arbitrarily close to α, thus, (44) holds true. □

5.2 Independence of the constant

Like in the case of left-c.e. reals, also speedability on arbitrary reals does not depend on the choice of the speeding constant ρ.

Theorem 57. *If a real α is ρ-speedable for some $\rho \in (0,1)$, then it is ρ'-speedable for every other $\rho' < 1$.*

Proof. During the whole proof of the theorem, the symbol q will denote some rational number on the interval $[0,1)$.

First, we mention the equivalence

$$\frac{\alpha - g(q)}{\alpha - q} = x \iff \alpha = g(q) + \frac{x}{1-x}(g(q) - q),$$

wherein the function $x \mapsto \frac{x}{1-x}$ is continuous and monotone increasing on $[0, 1)$.

Assume that this statement does not hold true for some real α, so, there exists a real $\rho_{inf} < 1$ such that α is ρ-speedable for every $\rho > \rho_{inf}$, but not ρ-speedable for every $\rho < \rho_{inf}$.

Let ρ_1, ρ_2 and ρ_3 be three reals fulfilling $0 < \rho_1 < \rho_{inf} < \rho_3 < \rho_2 < 1$, and let $c_1 = \frac{\rho_1}{1-\rho_1}$ and $c_2 = \frac{\rho_2}{1-\rho_2}$ be two rationals fulfilling

$$c_2 > c_1 > c_2 - \frac{1}{4} \min\{c_1, 1\}. \tag{46}$$

Such triple (ρ_1, ρ_2, ρ_3) exists since the function $x \mapsto \frac{1-x}{x}$ is continuous and the set $\mathbb{Q}$ is dense on the interval $(0, 1)$.

The ρ_3-speedability of α implies that there exists a nondecreasing partially computable translation function g such that $q < g(q) \downarrow < \alpha$ for every $q < \alpha$ and a subsequence $q_{i_0}, q_{i_1}, q_{i_2}, \dots \nearrow \alpha$ of the sequence $q_0, q_1, q_2, \dots$ that fulfills for every k the inequality

$$\alpha < g(q_{i_k}) + c_2(g(q_{i_k}) - q_{i_k}). \tag{47}$$

Without loss of generality, assume that $g(q) \le q$ holds for all $q \in dom(g)$; otherwise, for every $q \in dom(g)$ such that $g(q) < q$, we would know that $q \ge \alpha$, and therefore, replacing of all such values $g(q)$ by q will not injure the monotonicity of g and the inequality $q < g(q) < \alpha$ for all $q < \alpha$ nor the inequality (47) for all k.

The ρ_1-nonspeedability of α means that, in particular, α is not ρ_1-speedable via g, hence there exists some rational $q_g < \alpha$ such that, for every $q > q_g$, one holds that

$$\alpha \ge g(q) + c_1(g(q) - q). \tag{48}$$

Without loss of generality, assume $q_g = 0$; otherwise, modify the translation function g without injuring its monotonicity by claiming $g(q) = q_g$ for every rational $q < q_g$.

The ρ_1-nonspeedability of α means further that, for every nondecreasing partially computable function f from $\mathbb{Q}|_{[0,1]}$ to itself that fulfills $q < f(q) \downarrow < \alpha$ for every $q < \alpha$, α is not ρ_1-speedable via f, which implies that there exists some rational $q_f < \alpha$ such that, for every $q > q_f$, one holds that

$$\alpha \ge f(q) + c_1(f(q) - q). \tag{49}$$

The idea of the proof is, to use the information from (47) and (48) for construction of a translation function that will contradict to (49).

Let $q_0, q_1, \dots$ be en effective enumeration without repetition of $dom(g)$ that contains inter alia all rationals on $[0, \alpha)$. First, for every two rationals q_i and q_j in this enumeration that fulfill $0 < q_i < q_j$, we define a computable function $N(q_i, q_j)$ that returns the least index N such that, if we partition $[0, q_i)$ in disjoint set of semi-open intervals of length $d = \frac{q_j - q_i}{2}$ (except the left one which should have length at most d), then, for every such interval I, there exists an index $k \leq N$ such that $q_k \in I$.

Formally, $N(q_i, q_j)$ can be defined as follows:

$$N(q_i, q_j) = \min\big\{K : \forall m \in \{1, \dots, n\} \exists k \leq K \big(q_k \in [q_i - md, q_i - (m-1)d)\big)\big\} \qquad (50)$$

for $d = \frac{q_j - q_i}{2}$ and $n = \lceil \frac{q_i}{q_j - q_i} \rceil$, where $\lceil x \rceil = \max\{k \in \omega : k \geq x\}$ denotes the standard ceiling function of a positive real.

Next, we define for every rational q_i in $[0, \alpha)$ the index sets

$$\begin{aligned} X_i &= \{n : q_n < q_i \wedge n < i\}, \\ Y_i &= \{n : q_i < q_n \wedge n < i\} \quad \text{and} \\ Z_i &= \{n : q_n < q_i \wedge n \leq N(q_i, g(q_i))\} \end{aligned}$$

and introduce three modifications of the translation function g:

$$\begin{aligned} g_0(q_i) &= \min\{g(q_i) + c_1(g(q_i) - q_i), 1\}, \\ g_1(q_i) &= \max\{g_0(q_i), \max_{n \in Z_i}\{g_0(q_n)\}, \max_{n \in X_i}\{g_1(q_n)\}\}, \\ \tilde{g}(q_i) &= \min\{g_1(q_i), \min_{\substack{n \in Y_i \\ q_n \leq g_1(q_i)}} \{\tilde{g}(q_n)\}\}, \end{aligned}$$

where, for any partial function f and any set S, the minimum

$$\min_{s \in S}\{f(s)\}$$

is defined if and only if $f(s)$ is defined for every $s \in S$.

We can straightforwardly obtain from construction of functions g, g_0, g_1 and $\tilde{g}$ that the inequalities

$$q \leq g(q) \leq g_0(q), \qquad g_0(q) \leq g_1(q) \qquad \text{and} \qquad \tilde{g}(q) \leq g_1(q) \qquad (51)$$

hold true for all q lying in domains of corresponding functions.

In the next claim, we will see that $\tilde{g}$ dominates g on its whole domain.

Claim 1. *If $\tilde{g}(q_i)$ is defined for some i, then it fulfills the inequality*

$$g(q_i) \leq \tilde{g}(q_i). \qquad (52)$$

Proof. For a rational $q_i < \alpha$, we fix $k = |Y_i|$ and argue the claim assertion inductively on k.

If $k = 0$, then we have $\tilde{g}(q_i) = g_1(q_i) \downarrow\geq g(q_i)$ by (51).

If $k > 0$, then, supposing $g(q_n) \leq \tilde{g}(q_n)$ holds true for all q_n that such that $\tilde{g}$ is defined and $|Y_n| < k$, we make for q_i the following considerations:

- $g_1(q_i) \downarrow\geq g(q_i)$ by (51);
- for every $n \in Y_i$ such that $q_n \leq g_1(q_i)$, it holds that $\tilde{g}(q_n) \downarrow$ and $Y_n \subseteq Y_i \setminus \{n\}$, hence $|Y_n| < k$. This implies by induction hypothesis for q_n that $g(q_n) \leq \tilde{g}(q_n)$. On the other hand, the inequality $q_i < q_n$ and the monotonicity of g imply that $g(q_i) \leq g(q_n)$.

Thus, the computation of $\tilde{g}(q_i)$ returns minimum of two values not smaller that $g(q_i)$, hence $g(q_i) \leq \tilde{g}(q_i)$. □

Further, all constructed function are defined in all rationals $[0, \alpha)$, and thus, the inequalities (51) and (52) can be unified.

Claim 2. *For every rational $q \in [0, \alpha)$, the functions g_0, g_1 and $\tilde{g}$ are defined in q and fulfill the inequalities*

$$q < g(q) \leq g_0(q) \leq g_1(q) < \alpha \qquad \textit{and} \qquad q < g(q) \leq \tilde{g}(q) \leq g_1(q) < \alpha. \tag{53}$$

Proof. The real α is ρ_3-speedable via g, hence the function g fulfills $q < g(q) \downarrow< \alpha$ for every $q < \alpha$.

The function g_0 fulfills for every $q < \alpha$ the inequality $g_0(q) < \alpha$ by (48).

For a rational $q_i < \alpha$, we fix $k = |X_i|$ and argue inductively on k that

$$g_0(q_i) \leq g_1(q_i) \downarrow< \alpha.$$

If $k = 0$, then we have $g_1(q_i) = g_0(q_i) < \alpha$ already proved.

If $k > 0$, then, supposing $g_1(q_n) \downarrow< \alpha$ holds true for all $q_n < \alpha$ that fulfill $|X_n| < k$, we make for q_i the following considerations:

- $g_0(q_i) < \alpha$ as we have already proved;
- for every $n \in Z_i$, it holds that $q_n < q_i < \alpha$, hence $g_0(q_n) < \alpha$. Therefore, we obtain that $\max_{n \in Z_i}\{g_0(q_n)\} < \alpha$;
- for every $n \in X_i$, it holds that $X_n \subseteq X_i \setminus \{n\}$, hence $|X_n| < k$. This implies by induction hypothesis for $q_n < \alpha$ that $g_1(q_n) \downarrow< \alpha$. Therefore, we obtain that $\max_{n \in X_i}\{g_1(q_n)\} < \alpha$.

Thus, the computation of $g_1(q_i)$ terminates and returns maximum of three values smaller than α, hence $g_0(q_i) \leq g_1(q_i) \downarrow< \alpha$.

Next, for a rational $q_i < \alpha$, we fix $k = |Y_i|$ and argue inductively on k that

$$g(q_i) \leq \tilde{g}(q_i) \downarrow < \alpha.$$

If $k = 0$, then we have $g(q_i) \leq \tilde{g}(q_i) = g_1(q_i) < \alpha$ as we have already proved.

If $k > 0$, then, supposing $\tilde{g}(q_n) \downarrow < \alpha$ holds true for all $q_n < \alpha$ that fulfill $|Y_n| < k$, we make for q_i the following considerations:

- $g(q_i) \leq g_1(q_i) < \alpha$ as we have already proved;
- for every $n \in Y_i$ such that $q_n \leq g_1(q_i)$, it holds that $q_n < \alpha$ and $Y_n \subseteq Y_i \backslash \{n\}$, hence $|Y_n| < k$. This implies by induction hypothesis for $q_n < \alpha$ that $g(q_n) \leq \tilde{g}(q_n) \downarrow < \alpha$. On the other hand, the inequality $q_i < q_n$ and the monotonicity of g imply that $g(q_i) \leq g(q_n)$.

Thus, the computation of $\tilde{g}(q_i)$ terminates and returns minimum of two values smaller than α and not smaller that $g(q_i)$, hence $g(q_i) \leq \tilde{g}(q_i) \downarrow < \alpha$. □

Inter alia, the latter claim implies for g_0 that

$$\forall q < \alpha : g_0(q) = g(q) + c_1(g(q) - q). \tag{54}$$

as well as

$$\forall k \in \omega \quad (g_0(q_{i_k}) \downarrow \quad \wedge \quad g_1(q_{i_k}) \downarrow \quad \wedge \quad \tilde{g}(q_{i_k}) \downarrow) \tag{55}$$

because $q_{i_k} \in [0, \alpha)$ for every k.

Unfortunately, g_0 and g_1 cannot be considered as possible speed-up functions that contradict to (49) because, in general, they are not nondecreasing — in contrast to $\tilde{g}$, as we will see in what follows.

Claim 3. *The function $\tilde{g}$ is monotonically nondecreasing.*

Proof. Given to rationals $q_i, q_j \in dom(\tilde{g})$ such that $0 \leq q_i < q_j \leq 1$, we will prove that $\tilde{g}(q_i) \leq \tilde{g}(q_j)$ for the cases $i > j$ and $i < j$ separately.

Case 1: $i > j$.

In this case, one holds either $g_1(q_i) < q_j$, which immediately implies that

$$\tilde{g}(q_i) \leq g_1(q_i) < q_j \leq \tilde{g}(q_j),$$

where the third inequality holds by Claim 1, or $g_1(q_i) \geq q_j$, which implies that

$$\tilde{g}(q_i) \leq \min_{\substack{q_n \in (q_i, g_1(q_i)] \\ n < i}} \{\tilde{g}(q_n)\} \leq \tilde{g}(q_j),$$

where the left inequality follows from $q_j \in (q_i, g_1(q_i)]$ and $j < i$ and the right one holds by definition of $\tilde{g}(q_1)$.

Case 2: $i < j$.

In this case, we straightforwardly obtain the inequality for the function g_1

$$g_1(q_i) \leq \max_{\substack{q_n < q_j \\ n < j}} \{g_1(q_n)\} \leq g_1(q_j), \tag{56}$$

where the left inequality is implied $q_i < q_j$ and $i < j$ and the left one holds by definition of $g_1(q_j)$.

We prove the property

$$q_i < q_j \wedge i < j \qquad \Longrightarrow \qquad \tilde{g}(q_i) \leq \tilde{g}(q_j) \tag{57}$$

by induction on the amount of such n that we need to compute $\tilde{g}(q_n)$ during the whole computation of $\tilde{g}(q_j)$.

- Induction base: if there is no $n < j$ such that $q_n \in (q_j, g_1(q_j)]$, then one holds that

$$\tilde{g}(q_i) \leq g_1(q_i) \underset{\text{by (56)}}{\leq} g_1(q_j) = \tilde{g}(q_j). \tag{58}$$

- Induction step: for every $n < j$ such that $q_n \in (q_j, g_1(q_j)]$, we obtain either that $q_i < q_n$ and $i > n$, which implies that

$$\tilde{g}(q_i) \leq \tilde{g}(q_n) \tag{59}$$

  by Case 1, or that $q_i < q_n$ and $l(q_i) < l(q_n)$, which implies (59) by induction hypothesis since all computations of $\tilde{g}$ we need to do for computing $\tilde{g}(q_n)$ we still need to do for computing $\tilde{g}(q_j)$.

  Regarding the inequality (59) for all $q_n \in (q_j, g_1(q_j)]$, we obtain that

$$\tilde{g}(q_i) \leq \min_{\substack{q_n \in (q_j, g_1(q_j)] \\ n < j}} \{\tilde{g}(q_n)\},$$

  which implies together with (56) the inequality

$$\tilde{g}(q_i) \leq \min\{g_1(q_j), \min_{\substack{q_n \in (q_j, g_1(q_j)] \\ n < j}} \{\tilde{g}(q_n)\}\} = \tilde{g}(q_j),$$

  that concludes the induction step.

□

Thus, the function $\tilde{g}$ is a well-defined nondecreasing translation function for a real α, therefore, there exists a rational $q_{\tilde{g}} < \alpha$ such that the function $\tilde{g}$ fulfills (49) for every $q > q_{\tilde{g}}$.

To conclude the proof, it remains to argue that α is ρ_1-speedable via $\tilde{g}$ by proving for every element q_{i_k} of the sequence $q_{i_0}, q_{i_1}, q_{i_2}, \cdots \nearrow \alpha$ the inequality

$$\alpha < \tilde{g}(q_{i_k}) + c_1(\tilde{g}(q_{i_k}) - q_{i_k}), \tag{60}$$

that would contradict to the existence of $q_{\tilde{g}}$. Remind that, by (55), $\tilde{g}(q_{i_k})$ if defined for every $k \in \omega$.

For every such q_{i_k}, we can argue that

$$\tilde{g}(q_{i_k}) \geq g(q_{i_k}) + \frac{1}{4}\min\{c_1, 1\}(g(q_{i_k}) - q_{i_k}) \tag{61}$$

by showing the inequality

$$g_1(q_n) \geq g(q_{i_k}) + \frac{1}{4}\min\{c_1, 1\}(g(q_{i_k}) - q_{i_k}) \tag{62}$$

for every $n \leq i_k$ such that $q_n \in [q_{i_k}, \alpha)$ (inter alia, for $n = i_k$).

In order to prove (62), we consider 3 cases of location of q_n and $g(q_n)$ relative to two fixed thresholds

$$m = g(q_{i_k}) - \frac{1}{4}(g(q_{i_k}) - q_{i_k}) \quad \text{and} \quad M = g(q_{i_k}) + \frac{1}{4}\min\{c_1, 1\}(g(q_{i_k}) - q_{i_k}).$$

<u>Case 1</u>: $q_n \in [q_{i_k}, m]$.

Then we obtain (62) by

$$\begin{aligned} g_1(q_n) \geq g_0(q_n) \underset{\text{by (54)}}{=} g(q_n) + c_1(g(q_n) - q_n) \underset{g \text{ is nondecreasing}}{\geq} g(q_{i_k}) + c_1(g(q_{i_k}) - m) \\ = g(q_{i_k}) + c_1(g(q_{i_k}) - (g(q_{i_k}) - \frac{1}{4}(g(q_{i_k}) - q_{i_k}))) = g(q_{i_k}) + \frac{1}{4}c_1(g(q_{i_k}) - q_{i_k}). \end{aligned}$$

<u>Case 2</u>: $g(q_n) \geq M$.

In this case, the inequality $g_1(q_n) \geq g(q_n) \geq M$ holds by Claim 2 and directly implies (62).

<u>Case 3</u>: $m < q_n \leq g(q_n) < M$.

In this case, one holds that

$$\begin{aligned} g(q_n) - q_n &< M - m = \frac{1}{4}\min\{c_1, 1\}(g(q_{i_k}) - q_{i_k}) + \frac{1}{4}(g(q_{i_k}) - q_{i_k}) \leq \frac{1}{2}(g(q_{i_k}) - q_{i_k}), \\ m - q_{i_k} &= g(q_{i_k}) - q_{i_k} - \frac{1}{4}(g(q_{i_k}) - q_{i_k}) = \frac{3}{4}(g(q_{i_k}) - q_{i_k}). \end{aligned}$$

We now from $q < \alpha$ that $g(q_n) > q_n$, thus, for the length $d = \frac{g(q_n) - q_n}{2}$ of intervals used to define of $N(q_n, g(q_n))$, it holds that $m - q_{i_k} \geq 3d$, and therefore, by definition of the function $N(\cdot, \cdot)$, there exists an index $l \leq N(q_n, g(q_n))$ such that $q_{i_k} \leq l \leq m$. The latter inequality implies in particular that $q_l < q_n$, and therefore, that $l \in Z_n$, hence we obtain by definition of g_1 that $g_1(q_n) \geq g_0(q_l)$, wherein it holds by the same argument as in Case 1 that

$$g_0(q_l) \geq g(q_{i_k}) + \frac{1}{4}c_1(g(q_{i_k}) - q_{i_k})$$

in the same way as in Case 1.

So the inequality (61) holds true for every q_{i_k}, hence we can obtain the inequality (60) by

$$\tilde{g}(q_{i_k}) + c_1(\tilde{g}(q_{i_k}) - q_{i_k}) \underset{\text{by (52)}}{\geq} \tilde{g}(q_{i_k}) + c_1(g(q_{i_k}) - q_{i_k})$$

$$\underset{\text{by (61)}}{\geq} g(q_{i_k}) + \frac{1}{4}\min\{c_1, 1\}(g(q_{i_k}) - q_{i_k}) + c_1(g(q_{i_k}) - q_{i_k})$$

$$\underset{\text{by (46)}}{>} g(q_{i_k}) + c_2(g(q_{i_k}) - q_{i_k}) \underset{\text{by (47)}}{\geq} \alpha,$$

that contradicts to the existence of $q_{\tilde{g}}$. □

Similar to the left-c.e. case, Theorem 57 straightforwardly yields an equivalent characterization of ρ-speedability for any given nonzero $\rho < 1$ in terms of a Solovay condition instead of an inequality for $\limsup$.

Proposition 58. *For a real α and a real $\rho \in (0, 1)$, the following statements are equivalent.*

- *(a) The real α is ρ-speedable.*
- *(b) There exists a nondecreasing translation function g such that $q < g(q) < \alpha$ for all $q < \alpha$ and a nondecreasing sequence $q_n \nearrow \alpha$ that fulfills for every n the inequality*

$$\alpha - g(q_n) \leq \rho(\alpha - q_n). \tag{63}$$

5.3 Speedability and Martin-Löf randomness

Recall from Theorem 44 that all left-c.e. Martin-Löf random reals are nonspeedable. In particular, all reals in the largest degree of the upper semilattice of the Solovay degrees of left-c.e. reals are nonspeedable, since this degree coincides with the set of left-c.e. Martin-Löf random reals by Theorem 23.

The nonspeedability of the left-c.e. Martin-Löf random reals is extended to all reals by the next theorem. Then given a monotone Solovay degree that contains a Martin-Löf random real, every monotone Solovay degree above the former degree contains only Martin-Löf random and thus nonspeedable reals, since by Theorem 21 the Martin-Löf random reals are closed upwards under monotone Solovay reducibility, which is a transitive relation.

Theorem 59 (Merkle and Titov [9]). *Martin-Löf random real numbers are nonspeedable.*

Proof. In Chapter 6, the theorem is obtained as a corollary to Theorem 66. A direct proof was given by Merkle and Titov [9]. □

Hölzl and Janicki [7] demonstrated that the implication asserted in Theorem 59 cannot be reversed, even when restricting attention to the class of left-c.e. reals, by constructing a nonspeedable Martin-Löf nonrandom left-c.e. real. The following theorem asserts a similar statement for the easier and somewhat less interesting case of right-c.e. reals, i.e., the implication can neither be reversed on the right-c.e. reals.

Theorem 60. *There exists a right-c.e. real that is Martin-Löf nonrandom and nonspeedable.*

Proof. We construct a right-c.e. approximation $a_0, a_1, \ldots$ that converges to a real α such that α is neither Martin-Löf random nor speedable. Fix computable enumerations $q_0, q_1, \ldots$ and $\phi_0, \phi_1, \ldots$ of all rationals in the unit interval and of all partial computable functions from the set of such rationals to itself, respectively. For latter use, we assume that the enumeration of rationals does not have repetitions. Furthermore, fix a computable enumeration $(e_0, j_0), (e_1, j_1), \ldots$ of all pairs of the form (e, j) such that the partial computable function ϕ_e is defined on argument q_j. Such an enumeration can be obtained by dovetailing the computations of the form $\phi_e(q_j)$ for all e and j.

In case the constructed real α is speedable via some partial computable function g, by definition of speedability, there must be rationals q that are arbitrarily close to α and satisfy $q < g(q) < \alpha$. Accordingly, in case α is speedable, for some index e and all rationals $m < \alpha$,

$$\text{there are infinitely many } q \text{ such that} \qquad m < q < \phi_e(q) < \alpha. \tag{64}$$

On the other hand, in case, for an index e, there is some rational q such that

$$q < \alpha < \phi_e(q), \tag{65}$$

then ϕ_e is not a translation function for α, and thus, in particular, the real α is not speedable via ϕ_e. By Requirement e, we refer to the condition that there is a rational q that satisfies (65). Then, in order to ensure that α is nonspeedable, it suffices to arrange for all indices e that the following implication holds: if (64) holds for all $m < \alpha$, then Requirement e is satisfied. This will be achieved by constructing the real α by means of a standard finite injury priority construction.

At stage 0 of the construction, set $a_0 = 0.5$ and, for all natural numbers e, declare Requirement e to be unsatisfied and set m_e equal to 0. At stage $s > 0$, say that Requirement e requires attention in case $e < s$, Requirement e is currently declared unsatisfied and there is an index $j < s$ such that $e_j = e$ and, for $q = q_j$, we have

$$\max_{i<e} m_i < q < \phi_e(q) < a_{s-1}. \tag{66}$$

In case no index e requires attention at stage s, let $a_s = a_{s-1}$ and end stage s. Otherwise, let e be minimum such that Requirement e requires attention, and say that index e RECEIVES ATTENTION at stage s. Let j be minimum such that $e = e_j$ and (66) is satisfied for $q = q_j$. Assign rationals to m_e and a_s such that we have

$$q < m_e < a_s < \phi_e(q) \qquad \text{and} \qquad a_s - m_e < 2^{-(s+1)}, \tag{67}$$

and DECLARE Requirement e to be SATISFIED. Furthermore, for all $i > e$, DECLARE Requirement i to be UNSATISFIED and let $m_i = 0$, which concludes stage s. In case an index $i > e$ was declared satisfied before, we say that Requirement e INJURES Requirement i at stage s. Occasionally, we will write $m_{j,s}$ for the value of m_j at the end of stage s.

We say a requirement IS SATISFIED at stage s, if it has been declared satisfied at some stage $s' < s$ and has not been declared unsatisfied at any stage s'' where $s' < s'' < s$, and the notion of a requirement BEING UNSATISFIED at stage s is defined similarly with the roles of satisfied and unsatisfied interchanged. In particular, a requirement that is declared satisfied remains satisfied until it is injured at some later stage. We say a requirement is PERMANENTLY SATISFIED after stage s if it is satisfied at all stages $s+1, s+2, \ldots$, which is the case if and only if the requirement is satisfied at stage $s+1$ and not injured later.

So, an index $e < s$ requires attention at stage s in case Requirement e is currently unsatisfied but may now become satisfied because we have found a rational q that satisfies (66), hence can be used in an attempt to ensure that (65) holds via fixing a_s and m_e such that (67) holds. Here, $m_e > q$ is meant as strict lower bound for the values $a_s, a_{s+1}, \ldots$, and, in case this bound is obeyed, we have $q < m_e \leq \alpha$, which implies (65). By definition of stage, the lower bound m_e can only be disobeyed or injured in case later some Requirement $i < e$ receives attention. Accordingly, we say that Requirement i has HIGHER PRIORITY than Requirement e in case $i < e$.

The theorem now follows by the usual verification of a finite injury priority argument. First, we argue by induction over e that every requirement requires attention at most at finitely many stages. The induction base holds true because Requirement 0 has highest priority, and therefore, cannot be injured. Consequently, in case Requirement 0 requires attention at some stage, it receives attention and is then permanently satisfied after this stage, hence cannot require attention again. At the induction step, we consider an index $e > 0$ and, by the induction hypothesis, we can fix a stage s_0 such that no requirement $i < e$ requires attention after stage s_0. Similar to the base case, we can then argue that, in case requirement e receives attention at some stage $s > s_0$, it is permanently satisfied after stage s, hence does not require attention again.

Next, we fix an index e that satisfies (64) for all $m < \alpha$ and show that Requirement e eventually becomes permanently satisfied. Fix a stage s_0 such that none of the Requirements i, where $i < e$, requires attention at any stage $s > s_0$, hence, after stage s_0, Requirement e is never injured and none of the values $m_0, \ldots, m_{e-1}$ changes. Let m be equal to the maximum of the values $m_{0,s}, \ldots, m_{e-1,s}$. By assumption on e, there are infinitely many rationals q that satisfy (64). For all such q and all stages $s > s_0$, the inequality (66) holds by choice of m and $\alpha < a_s$ and, in particular, the value $\phi_e(q)$ is defined. Consequently, we can fix an index $j > s_0$ such that q_j is such a rational and e_j is equal to e. Then Requirement e is satisfied at the end of stage j. The latter is immediate in case Requirement e is already satisfied at the beginning of stage j, while, otherwise, at stage j, it holds that (66), hence Requirement e requires and receives attention and is declared satisfied. Since Requirement e is never injured after stage s_0, it is permanently satisfied after stage j.

It remains to demonstrate that α is Martin-Löf nonrandom. For every s, let I_s be equal to the empty set in case no index requires attention at stage s, while, otherwise, let e be the index that receives attention at stage s and set

$$I_s = [m_{e,s} - 2^{-s}, m_{e,s} + 2^{-s}].$$

The intervals $I_0, I_1, \ldots$ form a Solovay test since they can be effectively enumerated and the sum of their lengths is finite. The value $m_{e,s}$ and, by (67), also the rational a_s are in I_s. In case Requirement e is permanently satisfied after stage s, we have $m_{e,s} \leq \alpha < a_s$, and α is in I_s too. Consequently, in order to show that the real α is Martin-Löf nonrandom, it suffices to show that there are infinitely many indices e such that Requirement e eventually becomes permanently satisfied. Since we have already demonstrated that the latter holds for every index e such that (64) holds for all $m < \alpha$, in turn, it suffices to show that there are infinitely many such e. Recall that $q_0, q_1, \ldots$ is a computable enumeration of the rationals in the unit interval without repetition. Consider the computable function g defined by

$$g(q_n) = \min(\{q_j \colon j < n \text{ and } q_n < q_j\} \cup \{1\}).$$

For given $m < \alpha$, fix an index t, where $m < q_t < \alpha$, and observe that, for the infinitely many $n > t$, where $m < q_n < q_t$, we have

$$m < q_n < g(q_n) < q_t < \alpha.$$

Consequently, for every index e such that g is equal to ϕ_e and for all $m < \alpha$, it holds that (64). By the usual padding argument, we can assume that there are infinitely many such e. Alternatively, we can consider the infinitely partial computable variants of g that are defined only on the arguments in some interval of the form $[0, p]$ and agree there with g. □

6 A generalization of the theorem of Barmpalias and Lewis-Pye

6.1 The rational and the index form of the theorem of Barmpalias and Lewis-Pye

Let $a_0, a_1, \dots$ and $b_0, b_1, \dots$ be left-c.e. approximations of Martin-Löf random reals α and β, respectively. Since the left-c.e. Martin-Löf random reals are Solovay complete for the class of left-c.e. reals, the reals α and β are mutually Solovay reducible to each other, i.e., there are two positive constants c and d such that it holds for all n that

$$c < \frac{\alpha - a_n}{\beta - b_n} < d. \tag{68}$$

Intuitively speaking, the two left-c.e. approximations to α and β proceed at similar speed in the sense that at each step the respective distances to their limit are within a constant multiplicative factors of each other. The latter result was largely improved by Barmpalias and Lewis-Pye, who demonstrated that the ratios of these distances actually converge.

Theorem 61 (Barmpalias and Lewis-Pye [1], index form)**.** *Let α and β be left-c.e. reals, where β is Martin-Löf random. Then there exists a constant $d \geq 0$ such that, for every two left-c.e. approximations $a_n \nearrow \alpha$ and $b_n \nearrow \beta$, it holds that*

$$\lim_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} = d. \tag{69}$$

In what follows, we extend Theorem 61 to arbitrary in place of left-c.e. reals. Such an extension cannot be formulated in terms of left-c.e. approximation, thus we argue in Remark 64 that the following theorem can be viewed as a reformulation of Theorem 61 in terms of translation functions, where we refer to this reformulation as the rational form of the theorem. The extension will then be literally the same as Theorem 62 except that the assumption that α and β are left-c.e. is dropped.

Theorem 62 (Barmpalias and Lewis-Pye, rational form)**.** *Let α and β be left-c.e. reals, where β is Martin-Löf random and α is monotone Solovay reducible to β. Then there exists a constant $d \geq 0$ such that for every function g that witnesses that α is monotone Solovay reducible to β, it holds that*

$$\lim_{q\nearrow\beta} \frac{\alpha - g(q)}{\beta - q} = d. \tag{70}$$

The rational form of the theorem of Barmpalias and Lewis-Pye is formulated in terms of monotone Solovay reducibility, hence the considered translation functions g are nondecreasing. As shown in the following remark, this is necessary in so far, as a similar statement with Solovay reducibility in place of its monotone variant would no longer be equivalent to the index form of the theorem of Barmpalias and Lewis-Pye.

Remark 63. *In the formulation of Theorem 62, the monotonicity of the translation function g is crucial because, with Solovay reducibility in place of its monotone variant, the assertion of the theorem does not hold in general. For a proof, recall that, by Proposition 56, for any left-c.e. real $\alpha \neq 0$ there exists a translation function g such that α is Solovay reducible to itself via g and*

$$\liminf_{q \nearrow \alpha} \frac{\alpha - g(q)}{\alpha - q} = 0.$$

The function g as well as the identity function $id\colon q \mapsto q$ both witness that the real α is Solovay reducible to itself, hence both functions satisfy the assumption of the variant of Theorem 62 with Solovay reducibility in place of its monotone variant. Consequently, the latter variant of the theorem is false because it holds that

$$\liminf_{n\to\infty} \frac{\alpha - g(q_n)}{\beta - q_n} = 0 < 1 = \lim_{n\to\infty} \frac{\alpha - id(q_n)}{\alpha - q_n}.$$

Remark 64. *The index form and the rational form of the theorem of Barmpalias and Lewis-Pye follow easily from each other, hence, can be viewed as reformulations of each other.*

First, we prove that Theorem 61 implies Theorem 62. Let d be as in (69) *and $q_n \nearrow \beta$ be some, not necessary computable, sequence of rationals. Fix some left-c.e. approximation $b_0, b_1, \ldots$ of β, and let f be the nondecreasing function that maps every n to the largest index i such $b_i \leq q_n$. Then f is nondecreasing and, for all n, we have*

$$b_{f(n)} \leq q_n < b_{f(n)+1}, \qquad \textit{hence} \qquad g\left(b_{f(n)}\right) \leq g\left(q_n\right) \leq g\left(b_{f(n)+1}\right)$$

because g is nondecreasing. Consequently, it holds for all n that

$$\frac{\alpha - g(b_{f(n)})}{\beta - b_{f(n)+1}} \leq \frac{\alpha - g(q_n)}{\beta - q_n} \leq \frac{\alpha - g(b_{f(n)+1})}{\beta - b_{f(n)}}. \tag{71}$$

Now, $b_0, b_1, \ldots$ and $b_1, b_2, \ldots$ are left-c.e. approximations of β, while $g(b_0), g(b_1), \ldots$ and $g(b_1), g(b_2), \ldots$ are left-c.e. approximations of α; thus, by applying Theorem 62, we obtain

$$d = \lim_{n\to\infty} \frac{\alpha - g(b_n)}{\beta - b_{n+1}} = \lim_{n\to\infty} \frac{\alpha - g(b_{n+1})}{\beta - b_n} = \lim_{n\to\infty} \frac{\alpha - g(q_n)}{\beta - q_n}.$$

Here, the first two equations hold by choice of d, while the third equation is immediate by the first two equations and (71). *This implies Theorem 61 since the sequence $q_n \nearrow \beta$ has been chosen arbitrarily.*

In order to prove the other direction, we fix for left-c.e. real α and Martin-Löf random left-c.e. β some left-c.e. approximations $b_n \nearrow \beta$ and $a_n \nearrow \alpha$.

Since $\alpha \leq_{\mathrm{S}} \beta$ by Theorem 23, there exists by Proposition 10 some nondecreasing function g such that $\alpha \leq_{\mathrm{S}} \beta$ via g, wherein, from the monotonicity of g, we automatically obtain that $\alpha \leq_{\mathrm{S}}^{\mathrm{m}} \beta$ via g.

We know by rational form of theorem of Barmpalias and Lewis-Pye that the function g and the left-c.e. approximation $b_n \nearrow \beta$ fulfill (70) for some constant d.

For showing that

$$\exists \lim_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} = d.$$

it suffices to show that

$$\liminf_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} \geq d \qquad \textit{and} \qquad \limsup_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} \leq d. \tag{72}$$

Since β is left-c.e. and Martin-Löf random, we know by Theorem 44 that β is nonspeedable, hence, by criterion of speedability given in Proposition 49(c), for every computable nondecreasing index function h, we have

$$\lim_{n\to\infty} \frac{\beta - b_{h(n)}}{\beta - b_n} = 1. \tag{73}$$

To prove the left part of (72), we define

$$h(n) = \max\{n, \min\{k : g(b_k) \geq a_n\}\}. \tag{74}$$

The function h is obviously computable and nondecreasing, hence, fulfills (73).

Therefore, it holds that

$$\liminf_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} = \liminf_{n\to\infty} \left(\frac{\beta - b_{h(n)}}{\beta - b_n} \cdot \frac{\alpha - a_n}{\beta - b_{h(n)}} \right) \underset{\textit{by Lemma } 65(1)}{=}$$

$$\lim_{n\to\infty} \frac{\beta - b_{h(n)}}{\beta - b_n} \cdot \liminf_{n\to\infty} \frac{\alpha - a_n}{\beta - b_{h(n)}} \underset{\textit{by } (73) \textit{ for } h}{=}$$

$$\liminf_{n\to\infty} \frac{\alpha - a_n}{\beta - b_{h(n)}} \underset{\textit{by } (74)}{\geq} \liminf_{n\to\infty} \frac{\alpha - g(b_{h(n)})}{\beta - b_{h(n)}} = d,$$

where the last equality holds true since the limit of a subsequence of some converging sequence coincides with the limit of sequence.

The real β is nonspeedable, hence, by criterion of speedability given in Proposition 51(c), for every computable nondecreasing unbounded index function h fulfilling $h(n) \leq n$ for every $n \in \omega$, we have

$$\lim_{n\to\infty} \frac{\beta - b_{h(n)}}{\beta - b_n} = 1. \tag{75}$$

To prove the right part of (72)*, we define*

$$\tilde{h}(n) = \min\{n, \max: g(b_k) \leq a_n\}. \tag{76}$$

The function $\tilde{h}$ *is obviously computable, nondecreasing, unbounded and fulfills* $\tilde{h}(n) \leq n$ *for every* n*, hence, fulfills* (75)*.*

Therefore, it holds that

$$\limsup_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} = \limsup_{n\to\infty} \left(\frac{\beta - b_{\tilde{h}(n)}}{\beta - b_n} \cdot \frac{\alpha - a_n}{\beta - b_{\tilde{h}(n)}} \right) \underset{\text{by Lemma 65(2)}}{=}$$

$$\lim_{n\to\infty} \frac{\beta - b_{\tilde{h}(n)}}{\beta - b_n} \cdot \limsup_{n\to\infty} \frac{\alpha - a_n}{\beta - b_{\tilde{h}(n)}} \underset{\text{by (75) for } \tilde{h}}{=}$$

$$\limsup_{n\to\infty} \frac{\alpha - a_n}{\beta - b_{\tilde{h}(n)}} \underset{\text{by (76)}}{\leq} \limsup_{n\to\infty} \frac{\alpha - g(b_{\tilde{h}(n)})}{\beta - b_{\tilde{h}(n)}} = d,$$

where the last equality holds true by the same argument as in the previous case.

It remains to proof the following lemma, which we have used in the explanation of Remark 64.

Lemma 65. *For a sequence* $(x_n)_{n\in\omega}$ *of positive reals that converges to some positive limit and a bounded sequence* $(y_n)_{n\in\omega}$ *of positive reals, the following equalities hold true:*

1. $\liminf_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n) \liminf_{n\to\infty}(y_n)$;
2. $\limsup_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n) \limsup_{n\to\infty}(y_n)$.

Proof. We fix $X = \lim_{n\to\infty} x_n > 0$ and $Y = \limsup_{n\to\infty} y_n$.

1. For proving that $XY = \liminf_{n\to\infty}(x_n y_n)$, we note firstly that XY is an accumulation point of $(x_n y_n)_{n\in\omega}$ since, for a index sequence $(n_k)_{k\in\omega}$ such that $Y = \lim_{k\to\infty} y_{n_k}$, we have

$$\lim_{n\to\infty} x_{n_k} y_{n_k} = XY.$$

   If, for some $\epsilon > 0$, the real $XY - \epsilon$ is an accumulation point of $(x_n y_n)_{n\in\omega}$, then, for a index sequence $(n_k)_{k\in\omega}$ such that $XY - \epsilon = \lim_{k\to\infty} x_{n_k} y_{n_k}$, we have

$$\lim_{k\to\infty} y_{n_k} = \frac{\lim_{k\to\infty}(x_{n_k} y_{n_k})}{\lim_{k\to\infty} x_{n_k}} = \frac{XY - \epsilon}{X} < Y,$$

   that contradicts to the choice of Y.

2. similar to the previous assertion with $XY - \epsilon$ replaced by $XY + \epsilon$.

□

6.2 The theorem

The following theorem extends a result of Barmpalias and Lewis [1, 11] from left-c.e. reals to all reals. In what follows, existence of a limit is used in the sense of converging to a real number, i.e., the case that the considered values tend to infinity is excluded.

Theorem 66. *Let α and β be reals, where β is Martin-Löf and α is monotone Solovay reducible to β. Then there exists a constant $d \geq 0$ such that for every function g that witnesses that α is monotone Solovay reducible to β, it holds that*

$$\lim_{q \nearrow \beta} \frac{\alpha - g(q)}{\beta - q} = d.$$

We demonstrate first that the limit considered in the theorem exists. That this limit exists follows rather directly from Claims 1 through 3, which we will state in a minute subsequent to introducing some notation. Claims 1 and 2 follow by arguments that are similar to the ones used in connection with the case of left-c.e. reals [1, 11], whereas the proof of Claim 3 is rather involved and has no counterpart in the left-c.e. case.

We assume that the values of the function g are in the unit interval $[0, 1]$ because, otherwise, we can consider a variant of g that agrees with g except that, on arguments where the value of g is strictly below 0 or strictly above 1, we change the value to 0 and let the function be undefined, respectively. Observe that this variant still satisfies the assumption of the theorem because of $0 < \alpha < 1$.

In the remainder of this proof and unless explicitly stated otherwise, the term interval refers to a closed subinterval of the real numbers that is bounded by rationals. Lebesgue measure is denoted by μ, i.e., the Lebesgue measure, or measure, for short, of an interval U is $\mu(U) = \max U - \min U$. A FINITE TEST is an empty set or a tuple $A = (U_0, \ldots, U_m)$ with $m \geq 0$ where the U_i are not necessarily distinct nonempty intervals. For such a finite test A, its COVERING FUNCTION is

$$\begin{aligned} k_A \colon [0, 1] &\longrightarrow \omega, \\ x &\longmapsto \#\{i \in \{0, \ldots, m\} : x \in U_i\}, \end{aligned}$$

that is, $k_A(x)$ is the number of intervals in A that contain the real number x. Furthermore, the MEASURE of A is $\mu(A) = \sum_{i \in \{0,\ldots,m\}} \mu(U_i)$.

It is easy to see that the measure of a given finite test A can be computed by integrating its covering function on the whole domain $[0, 1]$, i.e., for every finite test A, it holds that

$$\mu(A) = \int_0^1 k_A(x)dx, \tag{77}$$

as follows by induction on the number of intervals contained in the finite test A.

The induction base holds true because, in case $A = \emptyset$, we obviously have

$$\mu(A) = 0 = \int_0^1 0 dx = \int_0^1 k_A(x) dx$$

and, in case $A = (U)$ is a singleton, the function k_A is just the indicator function of U, while the induction step follows from additivity of the integral operator because the function $k_{(U_0,\ldots,U_{n+1})}$ is the sum of $k_{(U_0,\ldots,U_n)}$ and $k_{(U_{n+1})}$.

Observe that by our definition of covering function, the values of the covering functions of the two tests $([0.2, 0.3], [0.3, 0.7])$ and $([0.2, 0.7])$ differ on the argument 0.3. Furthermore, for a given finite test and a rational q, by adding intervals of the form $[q, q]$ the value of the corresponding covering function at q can be made arbitrarily large without changing the measure of the test. However, these observations will not be relevant in what follows since they relate only to the value of covering functions at rationals.

For a given finite subset Q of the domain of g, we will construct a finite test $M(Q)$ by an extension of a construction used by Miller [11] in the left-c.e. case. The construction is effective in the sense that it always terminates and yields the test $M(Q)$ in case it is applied to a finite subset of the domain of g.

For every finite subset Q of the domain of g and every rational p, we let

$$\tilde{k}_Q(p) = k_{M(Q)}(p) \qquad \text{and} \qquad K_Q(p) = \max_{H \subseteq Q} \tilde{k}_H(p).$$

We demonstrate Theorem 66 by a proof by contradiction, i.e., we assume that the limit considered in the theorem does not exist. By this assumption, we obtain a rational e where $0 < e < 1$, which is used in the construction of the test $M(Q)$ for given Q.

The desired contradiction and thus the theorem then follows from the following three claims.

Claim 1. *Let $Q_0 \subseteq Q_1 \subseteq \cdots$ be a sequence of finite sets that converge to the domain of g. Then it holds that*

$$\lim_{n\to\infty} K_{Q_n}(e\beta) = \infty.$$

Claim 2. *For every finite subset Q of the domain of g, it holds that*

$$\int_0^1 \tilde{k}_Q(x) dx = \mu(M(Q)) \leq g(\max Q) - g(\min Q). \tag{78}$$

Claim 3. *For every finite subset Q of the domain of g and for every nonrational real p in $[0, e]$, it holds that*

$$K_Q(p) \leq \tilde{k}_Q(p) + 1. \tag{79}$$

In order to demonstrate Theorem 66, fix some effective enumeration $p_0, p_1, \ldots$ without repetition of the domain of g and, for $n = 0, 1, \ldots$, let $Q_n = \{p_0, \ldots, p_n\}$. We consider a special type of step function with domain $[0, 1]$ that is given by a partition of the unit interval into finitely many intervals with rational endpoints such that the function is constant on the corresponding open intervals but may have arbitrary values at the endpoints. For the scope of this proof, a designated interval of such a step function is an interval that is the closure of a maximum contiguous open interval on which the function attains the same value. I.e., the designated intervals form a partition of the unit interval except that two designated intervals may share an endpoint. Observe that, for every finite subset H of the domain of g, the corresponding cover function $\tilde{k}_H(\cdot)$ is such a step function with values in the natural numbers, and the same holds for the function K_{Q_n} since Q_n has only finitely many subsets. Furthermore, for given n, the designated intervals of the function $K_{Q_n}(\cdot)$ together with the endpoints and function value of every interval are given uniformly effective in n because g is computable and the construction of $M(Q_n)$ is uniformly effective in n.

For all natural numbers i and n, consider the step function K_{Q_n} and its designated intervals. For every such interval, call its intersection with $[0, e]$ its restricted interval. Let X_i^n be the union of all restricted designated intervals where on the corresponding designated interval the function K_{Q_n} attains a value that is strictly larger than 2^{i+2}. Let X_i be the union of the sets $X_i^0, X_i^1, \ldots$.

By our assumption that the values of g are in $[0, 1]$ and by (78), for all n the integral of $\tilde{k}_{Q_n}(p)$ from 0 to 1 is at most 1, hence by (79), the integral of $K_{Q_n}(p)$ from 0 to e is at most 2. Consequently, each set X_i^n has Lebesgue measure of at most $2^{-(i+1)}$. The latter upper bound then also holds for the Lebesgue measure of the set X_i since by the maximization in the definition of K_{Q_n} and

$$Q_0 \subseteq Q_1 \subseteq \cdots, \quad \text{we have} \quad K_{Q_0} < K_{Q_1} < \cdots, \quad \text{hence} \quad X_i^0 \subseteq X_i^1 \subseteq \cdots.$$

By construction, for all i and $n > 0$, the difference $X_i^n \setminus X_i^{n-1}$ is equal to the union of finitely many intervals that are mutually disjoint except possibly for their endpoints, and a list of these intervals is uniformly computable in i and n since the functions K_{Q_n} are uniformly computable in n. Accordingly, the set X_i is equal to the union of a set U_i of intervals with rational endpoints that is effectively enumerable in i and where the sum of the measures of these intervals is at most $2^{-(i+1)}$. By the two latter properties, the sequence $U_0, U_1, \ldots$ is a Martin-Löf test. By Claim 1, the values $K_{Q_n}(e\beta)$ tend to infinity where $e\beta < e$, hence for all n the Martin-Löf random real $e\beta$ is contained in some interval in U_n, a contradiction. This concludes the proof that Claim 1 through 3 together imply that the limit (80) exists.

It remains to show the Claims 1 through 3 and that the limit claimed to exist by Theorem 66 does not depend on g. For a start, we give the construction of the tests of the form $M(Q)$ and gather some facts that will be used subsequently.

The constants c and d and the sets S and T We aim at showing that the limit

$$\lim_{q \nearrow \beta} \frac{\alpha - g(q)}{\beta - q} \tag{80}$$

exists. For a proof by contradiction, we assume the opposite. By assumption of the theorem, the partial computable function g is nondecreasing and defined on all $q < \beta$, and we can pick a positive real c_0 such that for all such q we have $g(q) < \alpha$ and

$$\alpha - g(q) \leq c_0(\beta - q). \tag{81}$$

Thus, in particular, the fractions occurring in (80) are bounded from above by c_0. Consequently, by our assumption that the limit (80) does not exist, we can fix two rational constants c and d where

$$c < d, \quad e = d - c < 1 \qquad \text{and} \qquad \liminf_{q \nearrow \beta} \frac{\alpha - g(q)}{\beta - q} < c < d < \limsup_{q \nearrow \beta} \frac{\alpha - g(q)}{\beta - q}.$$

In particular, β is an accumulation point of both the sets

$$\begin{aligned} S &= \{q < \beta \colon \frac{\alpha - g(q)}{\beta - q} > d\} = \{q < \beta \colon \delta(q) < \alpha - d\beta\}, \\ T &= \{q < \beta \colon \frac{\alpha - g(q)}{\beta - q} < c\} \ = \{q < \beta \colon \gamma(q) > \alpha - c\beta\}. \end{aligned}$$

Here, the partial computional functions γ and δ have the same domain as g and are defined by

$$\gamma(q) = g(q) - cq \qquad \text{and} \qquad \delta(q) = g(q) - dq.$$

By definition of γ and δ and because of $c < d$, the following claim is immediate.

Claim 4. *Whenever $g(q)$ is defined, we have*

$$\gamma(q) - \delta(q) = (d - c)q = ep > 0, \qquad \textit{hence} \qquad \gamma(q) > \delta(q).$$

In particular, the partial function $\gamma - \delta$ is strictly increasing on its domain, hence, for every sequence $q_0 < q_1 < \dots$ of rationals on $[0, \beta)$ that converges to β, the values $g(q_i)$ are defined, and therefore, the values $\gamma(q_i) - \delta(q_i)$ converge strictly increasingly to $(d - c)\beta$.

Claim 5. *The sets S and T are disjoint.*

Proof. The claim holds because for every $q < \beta$, the bounds in the definitions of S and T are strictly farther apart than the values $\gamma(q)$ and $\delta(q)$, i.e., we have

$$\gamma(q) - \delta(q) = (d - c)q < (d - c)\beta = (\alpha - c\beta) - (\alpha - d\beta). \qquad \square$$

The intervals that are used For given rationals p and q, we define the interval

$$R[p,q] = [\gamma(p) - \delta(p), \gamma(q) - \delta(p)].$$

From this definition and the definitions of γ and δ, the following claim is immediate. Note that assertion (iii) in the claim relates to expanding an interval at the right endpoint.

Claim 6. *(i) Any interval of the form $R[p,q]$ has the left endpoint ep.*

(ii) Consider an interval of the form $R[p,q]$. In case $\gamma(p) \leq \gamma(q)$, the interval has length $\gamma(q) - \gamma(p)$, otherwise, the interval is empty. In particular, any interval of the form $R[p,p]$ has length 0.

(iii) Let $R[p,q]$ be a nonempty interval and assume $\gamma(q) \leq \gamma(q')$. Then the interval $R[p,q]$ is a subset of the interval $R[p,q']$, both intervals have the same left endpoint ep and they differ in length by $\gamma(q') - \gamma(q)$.

The following claim, which has already been used in the left-c.e. case [1, 11], will be crucial in the proof of Claim 1.

Claim 7. *Let q be in S and let q' be in T. Then the interval $R[q,q']$ contains $e\beta$.*

Proof. By definition, the interval $R[q,q']$ has the left endpoint eq and the right endpoint $\gamma(q') - \delta(q)$. By definition of the sets S and T, on the one hand, we have $q < \beta$, hence $eq < e\beta$, on the other hand, we have

$$\gamma(q') - \delta(q) > (\alpha - c\beta) - (\alpha - d\beta) = (d-c)\beta = ep. \qquad \square$$

Outline of the construction of the test $M(Q)$ Let $Q = \{q_0 < \cdots < q_n\}$ be a nonempty finite subset of the domain of g, where the notation used to describe Q has its obvious meaning, i.e., Q is the set of $q_0, \ldots, q_n$ and $q_i < q_{i+1}$. We describe the construction of the finite test $M(Q)$, which is an extended version of a construction used by Miller in connection with left-c.e. reals [11]. Using the notation defined in the previous paragraphs, for all i in $\{0, \ldots, n\}$ let

$$\begin{aligned} \delta_i &= \delta(q_i) = g(q_i) - dq_i, \\ \gamma_i &= \gamma(q_i) = g(q_i) - cq_i, \\ I[i,j] &= R[q_i, q_j] = [\gamma(q_i) - \delta(q_i), \gamma(q_j) - \delta(q_i)] = [eq_i, \gamma_j - \delta_i]. \end{aligned}$$

The properties of the intervals of the form $R[p,q]$ extend to the intervals $I[i,j]$, for example, any two nonempty intervals of the form $I[i,j]$ and $I[i,j']$ have the same left endpoint, i.e., $\min I[i,j]$ and $\min I[i,j']$ are the same for all i, j, and j'.

The test $M(Q)$ is constructed in successive steps $j = 0, 1, \ldots, n$, where, at each step j, intervals $U_0^j, \ldots, U_n^j$ are defined. Every such interval U_i^j has the form

$$U_i^j = I[i, \boldsymbol{r}^j(i)] = I[i,k] = R[q_i, q_k] = [\gamma(q_i) - \delta(q_i), \gamma(q_k) - \delta(q_i)]$$

for some index $k \in \{0, \dots, n\}$, where $\boldsymbol{r}^j(\cdot)$ is an index-valued function that maps every index i to such index k that $I[i,k] = U_i^j$.

At step 0, for $i = 0, \dots, n$, we set the values of the function $\boldsymbol{r}^0(i)$ by

$$\boldsymbol{r}^0(i) = i \tag{82}$$

and initialize the intervals U_i^0 as zero-length intervals

$$U_i^0 = I[i, \boldsymbol{r}^0(i)] = I[q_i, q_i] = R[q_i, q_i] = [eq_i, eq_i]. \tag{83}$$

In the subsequent steps, every change to an interval amounts to an expansion at the right end in the sense that for all indices i, the intervals $U_i^0, \dots, U_i^n$ share the same left endpoint, while their right endpoints are nondecreasing. More precisely, as we will see later, for $i = 0, \dots, n$ we have

$$\begin{aligned} eq_i &= \min U_i^0 = \cdots = \min U_i^n, \\ ep &= \max U_i^0 \leq \cdots \leq \max U_i^n, \end{aligned}$$

and thus $U_i^0 \subseteq \cdots \subseteq U_i^n$. After concluding step n, we define the finite test

$$M(Q) = (U_0^n, \dots, U_n^n).$$

In case the right endpoints of two intervals of the form U_i^{j-1} and U_i^j coincide, we say that the interval with index i remains unchanged at step j. Similarly, we will speak informally of the interval with index i, or U_i, for short, in order to refer to the sequence $U_i^0, \dots, U_i^n$ in the sense of one interval that is successively expanded.

Due to technical reasons, for an empty set $\emptyset$, we define $M(\emptyset) = \emptyset$.

A single step of the construction and the index stair During step $j > 0$, we proceed as follows. Let t_0 be the largest index among $\{0, \dots, j-1\}$ such that $\gamma_{t_0} > \gamma_j$, i.e., let

$$t_0 = \arg\max\{q_z : z < j \text{ and } \gamma_z > \gamma_j\} \tag{84}$$

and, in case there is no such index, let $t_0 = -1$.

Next, define indices $s_1, t_1, s_2, t_2, \dots$ inductively as follows. For $h = 1, 2, \dots$, assuming that t_{h-1} is already defined, where $t_{h-1} < j - 1$, let

$$s_h = \max \arg\min\{\delta_x : t_{h-1} < x \leq j-1\}, \tag{85}$$

$$t_h = \max \arg\max\{\gamma_y : s_h \leq y \leq j-1\}. \tag{86}$$

That is, the operator $\arg\min$ yields a set of indices x such that δ_x is minimum among all considered values, and s_h is chosen as the largest index in this set, and similarly for $\arg\max$ and the choice of t_h.

Since we assume that $t_{h-1} < j-1$, the minimization in (85) is over a nonempty set of indices, hence s_h is defined and satisfies $s_h \leq j-1$ by definition. Therefore, the maximization in (86) is over a nonempty index set, hence also t_h is defined.

The inductive definition terminates as soon as we encounter an index $l \geq 0$ such that $t_l = j-1$, which will eventually be the case by the previous discussion and because obviously the values $t_0, t_1, \ldots$ are strictly increasing. For this index l, we refer to the sequence $(t_0, s_1, t_1, \ldots, s_l, t_l)$ as the INDEX STAIR OF STEP j. E.g., in case $l = 1$ the index stair is (t_0, s_1, t_1), and in case $l = 0$, the index stair is (t_0). Note that $l = 0$ holds if and only if even s_1 could not be defined, where the latter in turn holds if and only if t_0 is equal to $j-1$.

Next, for $i = 1, \ldots, n$, we set the values of $\boldsymbol{r}^j(i)$ and define the intervals U_i^j. For a start, in case $l \geq 1$, let

$$\boldsymbol{r}^j(s_1) = j, \tag{87}$$

$$U_{s_1}^j = I[s_1, \boldsymbol{r}^j(s_1)] = I[s_1, j] = [\gamma_{s_1} - \delta_{s_1}, \gamma_j - \delta_{s_1}] \tag{88}$$

and call this a NONTERMINAL EXPANSION of the interval U_{s_1} AT STEP j. In case $l \geq 2$, in addition, let for $h = 2, \ldots, l$

$$\boldsymbol{r}^j(s_h) = t_{h-1}, \tag{89}$$

$$U_{s_h}^j = I[s_h, \boldsymbol{r}^j(s_h)] = I[s_h, t_{h-1}] = [\gamma_{s_h} - \delta_{s_h}, \gamma_{t_{h-1}} - \delta_{s_h}] \tag{90}$$

and call this a TERMINAL EXPANSION of the interval U_{s_h} AT STEP j.

For all remaining indices, the interval with index i REMAINS UNCHANGED at step j, i.e., for all $i \in \{0, \ldots, n\} \setminus \{s_1, \ldots, s_l\}$ let

$$\boldsymbol{r}^j(i) = \boldsymbol{r}^{j-1}(i), \tag{91}$$

$$U_i^j = U_i^{j-1}. \tag{92}$$

The choice of the term "terminal expansion" is motivated by the fact that in case a terminal expansion occurs for the interval with index i at step j, then, at all further steps $j+1, \ldots, n$, the interval remains unchanged, as we will see later.

We conclude step j by defining for $i = 0, \ldots, n$ the half-open interval

$$V_i^j = U_i^j \setminus U_i^{j-1}. \tag{93}$$

That is, during step j the interval with index i is expanded by adding at its right end the half-open interval V_i^j, i.e., we have

$$U_i^j = U_i^{j-1} \,\dot\cup\, V_i^j \qquad \text{where} \qquad |U_i^j| = |U_i^{j-1}| + |V_i^j|. \tag{94}$$

This includes the degenerated case where the interval with index i is not changed, hence V_i^j is empty and has length 0.

In what follows, in connection with the construction of a test of the form $M(Q)$, when appropriate, we will occasionally write t_0^j for the value of t_0 chosen during step j and similarly for other values like s_h in order to distinguish the values chosen during different steps of the construction.

The proof of Claim 1 Now, as the construction of the tests of the form $M(Q)$ has been specified, we can already demonstrate Claim 1. Let $Q_0 \subseteq Q_1 \subseteq \dots$ be a sequence of sets that converges to the domain of g as in the assumption of the claim. Any finite subset H of the domain of g will be a subset of Q_n for all sufficiently large indices n, where then for all such n it holds that $K_H(e\beta) \leq K_{Q_n}(e\beta)$ by definition of K_Q. Consequently, in order to show Claim 1, i.e., that the values $\tilde{k}_{Q_n}(e\beta)$ tend to infinity, it suffices to show that the function $H \mapsto \tilde{k}_H((d-c)\beta)$ is unbounded on the finite subsets H of the domain of g.

Recall that we have defined subsets S and T of the domain of g, which contain only rationals $q < \beta$. Let $r_0 < r_1 < \dots$ be a sequence such that for all indices $i \geq 0$, it holds that

$$r_{2i} \in T, \qquad r_{2i+1} \in S, \qquad \gamma(r_{2i+1}) < \gamma(r_{2i+2}) < \gamma(r_{2i}). \tag{95}$$

Such a sequence can be obtained by the following nonconstructive inductive definition. Let r_0 be an arbitrary number in T. Assuming that r_{2i} has already been defined, let r_{2i+1} be equal to some r in S that is strictly larger than r_{2i}. Note that such r exists since $r_{2i} < \beta$ and β is an accumulation point of S. Furthermore, assuming that r_{2i} and r_{2i+1} have already been defined, let r_{2i+2} be equal to some r in T that is strictly larger than r_{2i+1} and such that the second inequality in (95) holds. Note that such r exists because by definition of T, we have $\gamma(r_{2i}) > \alpha - c\beta$, while β is also an accumulation point of T and $\gamma(r)$ converges to $\alpha - c\beta$ when r tends nondecreasingly to β. Finally, observe that the first inequality in (95) holds automatically for r_{2i+1} in S and r_{2i+2} in T because by Claim 5, the set S is disjoint from T, hence, by definition of T, we have

$$\gamma(r_{2i+1}) \leq \alpha - c\beta < \gamma(r_{2i+2}).$$

Now, let H be equal to $\{r_0, r_1, \dots, r_{2k}\}$ and consider the construction of $M(H)$. For the remainder of this proof, we will use the indices of the r_j in the same way as the indices of the q_j are used in the description of the construction above. For example, for $i = 0, \dots, k-1$, during step $2i+2$ of the construction of $M(H)$, the index t_0 is chosen as the maximum index z in the range $0, \dots, 2i+1$ such that $\gamma(r_{2i+2}) < \gamma(r_z)$. By (95), this means that, in step $2i+2$, the index t_0 is set equal to $2i$ and – since $2i+1$ is the unique index strictly between $2i$ and $2i+2$ – the index stair of this step is $(2i, 2i+1, 2i+1)$. Accordingly, by construction, the interval U^{2i+2}_{2i+1} coincides with the interval $R[r_{2i+1}, r_{2i+2}]$. By Claim 7, this interval, and thus also, its superset U^{2k}_{2i+1} contains $e\beta$. The latter holds for all k different values of i, hence $\tilde{k}_H(e\beta) \geq k$. This concludes the proof of Claim 1 since k can be chosen arbitrarily large.

Some properties of the intervals U^i_j We gather some basic properties of the points and intervals that are used in the construction.

Claim 8. *Let $Q = \{q_0 < \dots < q_n\}$ be a subset of the domain of g. Consider some step j of the construction of $M(Q)$ and let $(t_0, s_1, t_1, \dots, s_l, t_l)$ be the corresponding index stair. Then we have $\gamma_j < \gamma_{t_0}$ in case $t_0 \neq -1$.*

In case the index s_1 could not be defined, i.e., in case $l = 0$, we have $t_0 = j - 1$. Otherwise, i.e., in case $l > 0$, we have

$$t_0 < s_1 < t_1 < \cdots < s_l \le t_l = j - 1 < j, \tag{96}$$

$$\delta_{s_1} < \cdots < \delta_{s_l} < \gamma_{t_l} < \cdots < \gamma_{t_1} \le \gamma_j. \tag{97}$$

Proof. The assertion on the relative size of γ_j and γ_{t_0} is immediate by definition of t_0. In case s_1 cannot be defined, the range between t_0 and j must be empty and $t_0 = j - 1$ follows. Next, we assume $l > 0$ and demonstrate (96) and (97). By definition of the values s_h and t_h, it is immediate that we have $s_h \le t_h < s_{h+1}$ for all $h \in \{1, \ldots, l-1\}$ and have $s_l \le t_l = j - 1$. In order to complete the proof of (96), assume $s_h = t_h$ for some h. Then we have

$$eq_{s_h} = \gamma_{s_h} - \delta_{s_h} = \gamma_{t_h} - \delta_{s_h} \ge \gamma_{j-1} - \delta_{j-1} = eq_{j-1}, \tag{98}$$

where the inequality holds true because $\gamma_{t_h} \ge \gamma_{j-1}$ and $\delta_{s_h} \le \delta_{j-1}$ hold for all h. So we obtain $s_h = t_h = j - 1$ and thus $h = l$ because, otherwise, in case $s_h < j - 1$, we would have $q_{s_h} < q_{j-1}$.

By definition of s_1 and l, it is immediate that, in case $l = 0$, we have $t_0 = j - 1$.

It remains to show (97) in case $l > 0$. The inequality $\gamma_{t_1} \le \gamma_j$ holds because its negation would contradict the choice of t_0 in the range $0, \ldots, j-1$ as largest index with maximum γ-value, as we have $t_0 < t_1 < j$ by (96). In order to show $\delta_{s_l} < \gamma_{t_l}$, it suffices to observe that we have $\delta_{s_l} \le \delta_{t_l}$ by choice of s_l and $s_l \le t_l < j$ and know that $\delta_{t_l} < \gamma_{t_l}$ from Claim 4. In order to show the remaining strict inequalities, fix h in $\{1, \ldots, l-1\}$. By choice of s_h, we have $\delta_{s_h} < \delta_x$ for all x that fulfill $s_h < x \le j-1$ and, since s_{h+1} is among these x, we have $\delta_{s_h} < \delta_{s_{h+1}}$. By a similar argument, it follows that $\gamma_{t_{h+1}} < \gamma_{t_h}$. □

Claim 9. *Let $Q = \{q_0 < \cdots < q_n\}$ be a subset of the domain of g and consider the construction of $M(Q)$. Let i be in $\{0, \ldots, n\}$. Then it holds that*

$$U_i^0 = \cdots = U_i^i. \tag{99}$$

Furthermore, for all steps $j \ge i$ of the construction, it holds that

$$U_i^j = I[i, x], \qquad \text{where} \qquad x \le j. \tag{100}$$

Proof. The equations in (99) hold because the index stair of every step $j \le i$ contains only indices that are strictly smaller than j and thus also than i, hence, by (92), the interval with index i remains unchanged at all such stages.

Next, we demonstrate (100) by induction over all steps $j \ge i$. The base case $j = i$ follows from (99) and because, by definition, we have $U_i^0 = I[i, i]$. At the step $j > i$, we consider its index stair $(t_0, s_1, t_1, \ldots, s_l, t_l)$. Observe that all indices that occur in the index stair are strictly smaller than j. The induction step now is immediate by distinguishing the following three cases. In case $i = s_1$, we have $U_i^j = I[i, j]$. In case $i = s_h$ for some $h > 1$, we have $U_i^j = I[i, t_{h-1}]$. In case i differs from all indices of the form s_h, by (92), the interval with index i remains unchanged at stage j and we are done by the induction hypothesis. □

Claim 10. *Let $Q = \{q_0 < \cdots < q_n\}$ be a subset of the domain of g, and consider the construction of $M(Q)$. Let $j \geq 1$ be a step of the construction, where at least the index s_1 could be defined, and let $(t_0, s_1, t_1, \ldots, s_l, t_l)$ be the index stair of this step. Then, for $h = 1, \ldots, l$, we have*

$$U_{s_h}^{j-1} = I[s_h, t_h] \quad \textit{hence, in particular,} \quad \max U_{s_h}^{j-1} = \gamma_{t_h} - \delta_{s_h}. \tag{101}$$

Consequently, for $i = 0, \ldots, n$, we have

$$U_i^0 \subseteq \cdots \subseteq U_i^n, \qquad \textit{wherein} \qquad \max U_i^0 \leq \max U_i^1 \leq \ldots \leq \max U_i^n. \tag{102}$$

Proof. In order to prove the claim, fix some h in $\{1, \ldots, l\}$. In case $s_h = t_h$, by (96), we have $h = l$ and $s_h = t_h = j - 1$, hence (101) holds true because, by construction and (99), we have

$$U_{s_h}^{j-1} = U_{s_h}^{s_h} = U_{s_h}^0 = I[s_h, s_h] = I[s_h, t_h].$$

So we can assume the opposite, i.e., that s_h and t_h differ. We then obtain

$$t_{h-1} \leq t_0^{t_h} < s_h < t_h < j, \tag{103}$$

where $t_0^{t_h}$, as usual, denotes the first entry in the index stair of step t_h. Here, the last two strict inequalities are immediate by Claim 8 since s_h differs from t_h. In case the first strict inequality was false, again, by Claim 8, we would have $s_h \leq t_0^{t_h} < t_h < j$ as well as $\gamma_{t_h} < \gamma_{t_0^{t_h}}$, which together contradict the choice of t_h. Finally, the first inequality obviously holds in case $h = 1$ and $t_0 = -1$. Otherwise, we have $\gamma_{t_{h-1}} > \gamma_{t_h}$ by (97), as well as $t_{h-1} < t_h$, hence, by definition, the value $t_0^{t_h}$ will not be chosen strictly smaller than t_{h-1}.

By (103), it follows that

$$\{x\colon t_0^{t_h} < x < t_h\} \subseteq \{x\colon t_{h-1} < x < j\}.$$

By definition, the index $s_1^{t_h}$ is chosen as the largest x in the former set that minimizes δ_x, while s_h is chosen from the latter set by the same condition, i.e., as the largest x that minimizes δ_x. Again, by (103), the index s_h is also in the former set, hence must be the largest index minimizing δ_x there. So we have $s_1^{t_h} = s_h$, hence $U_{s_h}^{t_h} = I[s_h, t_h]$ follows from construction.

Next, we argue that $U_{s_h}^{t_h} = U_{s_h}^{t_h}$ by demonstrating that

$$U_{s_h}^{t_h} = U_{s_h}^{t_h+1} = \cdots = U_{s_h}^{j-1},$$

i.e., that at all steps $y = t_h + 1, \ldots, j - 1$, the interval U_{s_h} remains unchanged. For every such step y, by definition of t_h, we have $\gamma_y < \gamma_{t_h}$, hence $s_h < t_h \leq t_0^y$ by choice of t_0^y.

Consequently, the index s_h does not occur in the index stair of step y and we are done by (92).

We conclude the proof of the claim by showing for $i = 1, \ldots, n$ the inequality

$$\max U_i^{j-1} \le \max U_i^j,$$

which then implies $U_i^0 \subseteq \cdots \subseteq U_i^n$ because, by construction, the latter intervals all share the same left endpoint $\min U_i^0 = eq_i$ and j is an arbitrary index in $\{1, \ldots, n\}$.

For indices i that are not equal to some s_h, the interval i remains unchanged at step j and we are done. So we can assume $i = s_h$ for some h in $\{1, \ldots, l\}$; thus, $\max U_i^{j-1} = \gamma_{t_h} - \delta_{s_h}$ follows from (101). The value γ_{t_h} is strictly smaller than both values γ_j and $\gamma_{t_{h-1}}$ by choice of t_0 and t_{h-1}. So we are done because, by construction, in case $h = 1$, we have $\max U_i^j = \gamma_j - \delta_{s_h}$, while, in case $h > 1$, we have $\max U_i^j = \gamma_{t_{h-1}} - \delta_{s_h}$. □

As a corollary of Claim 10, we obtain that, when constructing a test of the form $M(Q)$, any terminal expansion of an interval at some step is, in fact, terminal in the sense that the interval will remain unchanged at all larger steps.

Claim 11. *Let $Q = \{q_0 < \cdots < q_n\}$ be a subset of the domain of g, and consider the construction of $M(Q)$. Let $j \ge 1$ be a step of the construction, where the index s_2 could be defined, and let $(t_0, s_1, t_1, \ldots, s_l, t_l)$ be the index stair of this step. Then, for every $h = 2, \ldots, l$, it holds that $\boldsymbol{r}^j(s_h) = \boldsymbol{r}^n(s_h)$, and therefore, that $U_{s_h}^j = U_{s_h}^n$.*

Proof. For a proof by contradiction, we assume that the claim assertion is false, i.e., we can fix some $h \ge 2$ such that the values $\boldsymbol{r}^j(s_h)$ and $\boldsymbol{r}^n(s_h)$ differ. Let k be the least index in $\{j+1, \ldots, n\}$ such that the values $\boldsymbol{r}^{k-1}(s_h)$ and $\boldsymbol{r}^k(s_h)$ differ, and let $(t_0^k, s_1^k, t_1^k, \ldots)$ be the index stair of step k. Since the interval with index s_h does not remain unchanged at step k, we must have $s_h = s_x^k$ for some $x \ge 1$. In order to obtain the desired contradiction, we distinguish the cases $x = 1$ and $x > 1$. In case $x = 1$, by construction we have

$$t_0^k < s_1^k = s_h < j < k \qquad \text{and} \qquad t_0^k \le t_0 < s_1 < j < k,$$

where all relations are immediate by choice of the involved indices except the nonstrict inequality. The latter inequality holds by choice of t_0^k because, by the chain of relations on the left, we have $t_0^k < j$, and thus, $\gamma_j \le \gamma_k$, while $\gamma_i \le \gamma_j$ holds for $i = t_0 + 1, \ldots, j-1$ by choice of t_0. Now, we obtain as a contradiction that $s_1^k = s_h$ is chosen in the range $t_0^k + 1, \ldots, k-1$ as largest index that has minimum δ-value, where this range includes s_1, hence $\delta_{s_1^k} \le \delta_{s_1}$, while $\delta_{s_1} < \delta_{s_h}$ by $h \ge 2$.

In case $x > 1$, we obtain

$$t_{h-1} = \boldsymbol{r}^j(s_h) = \boldsymbol{r}^{k-1}(s_h) = t_x^k, \tag{104}$$

which contradicts to $t_{h-1} < s_h = s_x^k \le t_x^k$. The equalities in (104) follow, from left to right, from $h \ge 2$, from the minimality condition in the choice of k and, finally, from $s_h = s_x^k$ and Claim 10. □

The explicit description of the intervals of the form $U_{s_h}^{j-1}$ according to Claim 10 now yields an explicit description of the endpoints of the half-open intervals of the form V_i^j, from which in turn we obtain that all such intervals occurring at the same step are mutually disjoint and the sum of their measures is equal to $\gamma_j - \gamma_{j-1}$.

Claim 12. *Let $Q = \{q_0 < \cdots < q_n\}$ be a subset of the domain of g, and consider the construction of $M(Q)$. Let $j > 0$ be a step of the construction.*

If $\gamma_{j-1} \leq \gamma_j$, then it holds for the index stair $(t_0, s_1, t_1, \ldots, s_l, t_l)$ of this step that $l > 0$, i.e., s_1 can be defined and we have

$$\begin{aligned} V_{s_1}^j &= (\gamma_{t_1} - \delta_{s_1}, \gamma_j - \delta_{s_1}], && (105)\\ V_{s_h}^j &= (\gamma_{t_h} - \delta_{s_h}, \gamma_{t_{h-1}} - \delta_{s_h}] \quad \text{for } h \geq 2 \quad (\text{if defined}), && (106)\\ V_i^j &= \emptyset \text{ for } i \text{ in } \{0, \ldots, n\} \setminus \{s_1, \ldots, s_l\}. && (107) \end{aligned}$$

In particular, the half-open intervals $V_0^j, \ldots, V_n^j$ are mutually disjoint and the sum of their Lebesgue measures can be bounded as follows

$$\sum_{i=0}^{n} \mu(V_i^j) = \sum_{h=1}^{l} \mu(V_{s_h}^j) = \gamma_j - \gamma_{j-1}. \tag{108}$$

If $\gamma_{j-1} > \gamma_j$, then the index stair of this step has a form $(j-1)$, i.e., $t_0 = j-1$, $l = 0$, all the intervals $V_0^j, \ldots, V_n^j$ are empty

$$V_i^j = \emptyset \text{ for all } i \tag{109}$$

and the sum of their Lebesgue measures is equal to zero

$$\sum_{i=0}^{n} \mu(V_i^j) = 0. \tag{110}$$

Proof. If $\gamma_{j-1} \leq \gamma_j$, then we have $t_0 \neq j-1$, hence the set $\{x : t_0 < x \leq j-1\}$ used in (85) to define s_1 contains at least one index, namely $j-1$, and therefore, s_1 can be defined.

If $\gamma_{j-1} > \gamma_j$, then we have $t_0 = j-1$, hence the set $\{x : t_0 < x \leq j-1\}$ is empty and s_0 cannot be defined.

Recall that, by construction, the intervals $U_i^0, \ldots, U_i^n$ are all nonempty and have all the same left endpoint $\gamma_i - \delta_i$; thus, we have

$$V_i^j = U_i^j \setminus U_i^{j-1} = (\max U_i^{j-1}, \max U_i^j].$$

This implies (107) in case $\gamma_{j-1} \leq \gamma_j$ and (109) in case $\gamma_{j-1} < \gamma_j$ since, for i not in $\{s_1, \ldots, s_l\}$, the interval with index i remains unchanged at step j, hence V_i^j is empty.

In case $\gamma_{j-1} < \gamma_j$, we obtain (110) directly from (109) by

$$\sum_{i=0}^{n} \mu(V_i^j) = \sum_{i=0}^{n} \mu(\emptyset) = 0,$$

so, from now on, we assume that $\gamma_{j-1} \leq \gamma_j$ and, as we have seen before, $l > 0$.

In order to obtain (105) and (106) in this case, it suffices to observe that $\max U_{s_h}^j$ is equal to $\gamma_j - \delta_{s_1}$ in case $h = 1$ and is equal to $\gamma_{t_{h-1}} - \delta_{s_h}$ in case $h \geq 2$, respectively, while $\max U_{s_h}^{j-1} = \gamma_{t_h} - \delta_{s_h}$ for $h = 1, \ldots, l$ by Claim 10.

Next, we show that the half-open intervals $V_0^j, \ldots, V_n^j$ are mutually disjoint. These intervals are all empty except for $V_{s_1}^j, \ldots, V_{s_l}^j$. In case the latter list contains at most one interval, we are done. So we can assume $l \geq 2$. Disjointedness of $V_0^j, \ldots, V_n^j$ then follows from

$$\min V_{s_l}^j < \max V_{s_l}^j < \cdots < \min V_{s_1}^j < \max V_{s_1}^j.$$

These inequalities hold because, for $h = 2, \ldots, l$, by Claim 8, we have $\gamma_{t_{h-1}} > \gamma_{t_h}$ and $\delta_{s_{h-1}} < \delta_{s_h}$, which together with (105) and (106) yield

$$\gamma_{t_h} - \delta_{s_h} = \min V_{s_h}^j < \max V_{s_h}^j = \gamma_{t_{h-1}} - \delta_{s_h} < \gamma_{t_{h-1}} - \delta_{s_{h-1}} = \min V_{s_{h-1}}^j.$$

Since the intervals $V_0^j, \ldots, V_n^j$ are mutually disjoint, the Lebesgue measure of their union is equal to

$$\begin{aligned}\sum_{i=0}^{n} \mu(V_i^j) = \sum_{h=1}^{l} \mu(V_{s_h}^j) &= \mu(V_{s_1}^j) + \sum_{h=2}^{l} \mu(V_{s_h}^j) \\ &= \gamma_j - \gamma_{t_1} + \sum_{h=2}^{l} (\gamma_{t_{h-1}} - \gamma_{t_h}) = \gamma_j - \gamma_{t_l} = \gamma_j - \gamma_{j-1},\end{aligned}$$

where the last two equations are implied by evaluating the telescoping sum and because t_l is equal to $j - 1$ by Claim 8, respectively. □

The proof of Claim 2 Using the results on the intervals V_i^j in Claim 12, we can now easily demonstrate Claim 2. We have to show for every subset $Q = \{q_0 < \cdots < q_n\}$ of the domain of g that

$$\mu(M(Q)) \leq g(q_n) - g(q_0). \tag{111}$$

This inequality holds true because we have

$$\begin{aligned}\mu(M(Q)) &= \sum_{U \in M(Q)} \mu(U) = \sum_{i=0}^{n} \mu(U_i^n) = \sum_{i=0}^{n} \sum_{j=1}^{n} \mu(V_i^j) = \sum_{j=1}^{n} \sum_{i=0}^{n} \mu(V_i^j) \\ &\leq \sum_{j=1}^{n} (\max\{\gamma_j - \gamma_{j-1}, 0\}) \leq \sum_{j=1}^{n} (\max\{g(q_j) - g(q_{j-1}), 0\}) \leq g(q_n) - g(q_0).\end{aligned}$$

In the first line, the first equality holds by definition of $\mu(M(Q))$ and the second and the third equalities hold by construction of $M(Q)$ and by (93), respectively.

In the second line, the equality holds because, for every j, we have

$$\sum_{i=0}^{n} \mu(V_i^j) = \max\{\gamma_j - \gamma_{j-1}, 0\}$$

due to the following argumentation: in case $\gamma_{j-1} \leq \gamma_j$, we obtain from Claim 12, (108), that $\sum_{i=0}^{n} \mu(V_i^j) = \gamma_j - \gamma_{j-1} \geq 0$ and, in case $\gamma_{j-1} > \gamma_j$, we obtain from Claim 12, (110), that $\sum_{i=0}^{n} \mu(V_i^j) = 0$.

Finally, the inequality in the second line holds because the difference $g(q_n) - g(q_0)$ can be rewritten as a telescoping sum

$$g(q_n) - g(q_0) = (g(q_n) - g(q_{n-1})) + (g(q_{n-1}) - g(q_{n-2})) + \ldots + (g(q_1) - g(q_0))$$

and, for every j from 1 to n, we have

$$\max\{\gamma_j - \gamma_{j-1}, 0\} \leq g(q_j) - q(q_{j-1})$$

due to the following argumentation: in case $\gamma_{j-1} \leq \gamma_{q_j}$, we have

$$\gamma_j - \gamma_{j-1} = \gamma(q_j) - \gamma(q_{j-1}) = g(q_j) - cq_j - (g(q_{j-1}) - cq_{j-1}) \leq g(q_j) - g(q_{j-1})$$

and, in case $\gamma_{j-1} > \gamma_{q_j}$, we direclty have

$$0 \leq g(q_j) - g(q_{j-1}),$$

where, in both cases, the inequalities are implied by motonony of g for arguments $q_{j-1} < q_j$.

Preliminaries for the proof of Claim 3 The following claim asserts that, when adding to a finite subset Q of the domain of g one more rational that is strictly larger than all members of Q, the cover function of the test corresponding to Q increases at most by one on all nonrational arguments.

Claim 13. *Let Q be a finite subset of the domain of g. Then, for every real $p \in [0, 1]$, it holds that*

$$\tilde{k}_{Q \setminus \{\max Q\}}(p) \leq \tilde{k}_Q(p) \leq \tilde{k}_{Q \setminus \{\max Q\}}(p) + 1. \tag{112}$$

Proof. Let $Q = \{q_0 < \cdots < q_n\}$ be a finite subset of the domain of g. We consider the constructions of the tests $M(Q \setminus \{q_n\})$ and $M(Q)$ and denote the intervals constructed in the latter test by U_i^j, as usual. The steps 0 through n of both constructions are essentially identical up to the fact that, in the latter construction, in addition, the interval U_n^0 is initialized as $[eq_n, eq_n]$ in step 0 and then remains unchanged. Accordingly, the test $M(Q \setminus \{q_n\})$ consists of the intervals $U_0^{n-1}, \ldots, U_{n-1}^{n-1}$, hence, the first inequality in (112) holds true because the test $M(Q)$ is then obtained by expanding these intervals. More precisely, in the one additional step of the construction of $M(Q)$, these intervals and the interval $U_n^{n-1} = U_n^0$ are expanded by letting

$$U_i^n = U_i^{n-1} \cup V_i^n \quad \text{for } i = 0, \ldots, n.$$

The intervals $V_0^n, \ldots, V_n^n$ are mutually disjoint by Claim 12. Consequently, the cover functions of both tests can differ at most by one, hence also the second inequality in (112) holds true. □

The following somewhat technical claim will be used in the proof of Claim 15.

Claim 14. *Let $Q = \{q_0 < \cdots < q_n\}$ be a subset of the domain of g and let $p \in (0, 1]$ be a real number. Let i, j, k be indices such that $q_0 \le q_i < q_j < q_k < p$,*

$$ep < \gamma_i - \delta_j \quad \text{and} \quad ep < \gamma_k - \delta_j. \tag{113}$$

Let $Q_i = \{q_0, \ldots, q_i\}$ and $Q_k = \{q_0, \ldots, q_k\}$. Then the following inequality holds true:

$$\tilde{k}_{Q_i}(ep) < \tilde{k}_{Q_k}(ep). \tag{114}$$

Proof. Let $s = \max \arg\min\{\delta_x : i < x < k\}$, and let

$$i' = \max \arg\max\{\gamma_y : i \le y < s\} \quad \text{and} \quad k' = \max \arg\max\{\gamma_y : s < y \le k\}.$$

The following inequalities are immediate by definition

$$\delta_s \le \delta_j, \qquad \gamma_s < \gamma_i \le \gamma_{i'}, \qquad \gamma_s < \gamma_k \le \gamma_{k'}, \tag{115}$$

except the two strict upper bounds for γ_s. The first of these bounds, i.e., $\gamma_s < \gamma_i$, follows from

$$\gamma_s - \delta_s = eq_s < ep < \gamma_i - \delta_j \le \gamma_i - \delta_s,$$

where the inequalities hold, from left to right, by $q_s < q_k < p$, by (113) and by (115). By an essentially identical argument, this chain of relations remains valid when γ_i is replaced by γ_k, which shows the second bound, i.e., $\gamma_s < \gamma_k$.

We denote the intervals that occur in the construction of $M(Q)$ by U_i^j, as usual. As in the proof of Claim 13, we can argue that the construction of the test $M(Q_i)$ is essentially identical to initial parts of the construction of $M(Q_k)$ and of $M(Q_n)$ and that a similar remark holds for the tests $M(Q_k)$ and $M(Q)$. Accordingly, we have

$$M(Q_i) = (U_0^i, \ldots, U_i^i) \qquad \text{and} \qquad M(Q_k) = (U_0^k, \ldots, U_i^k, U_{i+1}^k, \ldots, U_k^k),$$

For $x = 1, \ldots, i$, the interval U_x^i is a subset of U_x^k by $i < k$ and Claim 10. Hence it suffices to show

$$ep \in U_s^k, \tag{116}$$

because the latter statement implies by $i < s < k$ that

$$\tilde{k}_{Q_i}(ep) + 1 \leq \tilde{k}_{Q_k}(ep).$$

We will show (116) by proving that ep is strictly larger than the left endpoint and is strictly smaller than the right endpoint of the interval U_s^k. The assertion about the left endpoint, which is equal to $\gamma_s - \delta_s = eq_s$, holds true because the inequalities $s < k$ and $q_k < p$ imply together that $q_s < p$.

In order to demonstrate the assertion about the right endpoint, we distinguish two cases.

Case 1: $\gamma_{i'} > \gamma_{k'}$. In this case, let $(t_0, s_1, t_1, \ldots)$ be the index stair of the step k'. Then we have

$$i \leq i' \leq t_0 < s < k' \leq k, \tag{117}$$

where all inequalities are immediate by choice of i' and k' except the second and the third one. Both inequalities follow from the definition of t_0: the second one together with the case assumption, the third one because, by $\gamma_s < \gamma_{k'}$ and by choice of k', no value among $\gamma_s, \ldots, \gamma_{k'-1}$ is strictly larger than $\gamma_{k'}$.

By (117), it is immediate that the set $\{t_0+1, \ldots, k'-1\}$ contains s and is a subset of the set $\{i+1, \ldots, k-1\}$. By definition, the indices s_1 and s minimize the value of δ_j among the indices j in the former and in the latter set, respectively, hence we have $s = s_1$. By construction, in step k', the right endpoint of the interval $U_s^{k'}$ is then set to $\gamma_{k'} - \delta_s$. So we are done with Case 1 because we have

$$ep < \gamma_k - \delta_j \leq \gamma_{k'} - \delta_s = \max U_s^{k'} \leq \max U_s^k,$$

where the first inequality holds by assumption of the claim, the second holds by (115), and the last one holds by $k' \leq k$ and Claim 10.

Case 2; $\gamma_{i'} \leq \gamma_{k'}$. In this case, let

$$r = \min\{y : s < y \leq k \wedge \gamma_{i'} \leq \gamma_y\} \tag{118}$$

and let $(t_0, s_1, t_1, \ldots\}$ be the index stair of the step r. By choice of s and by $r \leq k$, all values among $\delta_{s+1}, \ldots, \delta_{r-1}$ are strictly larger than δ_s, hence we have $s_1 \leq s$ by choice of s_1. Accordingly, the index

$$m = \max\{h > 0 : s_m \leq s\} \tag{119}$$

is well-defined. Next, we argue that, actually, it holds that $s_m = s$. Otherwise, i.e., in case $s_m < s$, by choice of s_m and since s is chosen as largest index in the range $i+1, \ldots, k-1$ that has minimum δ-value, we must have $s_m \leq i$, and thus,

$$t_0 < s_m \leq i \leq i' < s < r.$$

Therefore, the index i' belongs to the index set used to define t_m according to (86), while the values $\gamma_{i'+1}, \ldots, \gamma_{r-1}$ are all strictly smaller than $\gamma_{i'}$. The latter assertion follows for the indices in the considered range that are strictly smaller, equal and strictly larger than s from choice of i', from (115) and from choice of r, respectively. It follows that $t_m \leq i'$, hence s_{m+1} exists and is equal to s by minimality of δ_s and by choice of s_{m+1} in the range $t_m + 1, \ldots, r - 1$ which contains s by $i' < s < r$. But, by definition of m, we have $s < s_{m+1}$, a contradiction. Consequently, we have $s_m = s$.

Observe that we have

$$ep < \gamma_i - \delta_j \leq \gamma_{i'} - \delta_s \leq \gamma_r - \delta_s, \tag{120}$$

where the inequalities hold, from left to right, by assumption of the claim, by (115) and by choice of r.

In case $m = 1$, we are done because then we have by construction

$$\gamma_r - \delta_s = \max U_s^r \leq \max U_s^k, \tag{121}$$

hence ep is indeed strictly smaller than the right endpoint of the interval U_s^k.

So, from now one, we can assume $m > 1$. Then s_{m-1} and t_{m-1} are defined, and the upper bound of the interval U_s^r is set equal to $\gamma_{t_{m-1}} - \delta_s$ by (90). Consequently, in case $\gamma_{i'} \leq \gamma_{t_{m-1}}$, both of (120) and (121) hold true with γ_r replaced by $\gamma_{t_{m-1}}$, and we are done by essentially the same argument as in case $m = 1$.

We conclude the proof of the claim assertion by demonstrating the inequality $\gamma_{i'} \leq \gamma_{t_{m-1}}$. The index s is chosen in the range $i+1, \ldots, k-1$ as largest index with minimum δ-value. The latter range contains the range $i+1, \ldots, r-1$ because we have $i < s < r \leq k$. The index s_{m-1} differs from $s = s_m$ and is chosen as the largest index with minimum δ-value among indices that are less than or equal to $r - 1$, hence $s_{m-1} \leq i$. By $i \leq i' < s < r$, the index i' belongs to the range $s_{m-1}, \ldots, r-1$, from which t_{m-1} is chosen as largest index with maximum δ-value according to (86), hence we obtain that $\gamma_{i'} \leq \gamma_{t_{m-1}}$. □

Claim 15. *Let Q be a nonempty finite set of rationals and let p be a nonrational real in $[0, 1]$. In case $p > \max Q$, it holds that*

$$\tilde{k}_Q(ep) = K_Q(ep). \tag{122}$$

Proof. The inequality $\tilde{k}_Q(ep) \leq K_Q(ep)$ is immediate by definition of $K_Q(ep)$. We show the reverse inequality $\tilde{k}_Q(ep) \geq K_Q(ep)$ by induction on the size of Q.

In the base case, let Q be empty or a singleton set. The induction claim holds in case Q is empty, because then Q is its only subset, as well as in case Q is a singleton, because then K_Q is equal to the maximum of $\tilde{k}_Q$ and $\tilde{k}_\emptyset$, where the latter function is identically 0.

In the induction step, let Q be of size at least 2. For a proof by contradiction, assume that the induction claim does not hold true for Q, i.e., that there exist a subset H of Q such that

$$\tilde{k}_Q(ep) < \tilde{k}_H(ep). \tag{123}$$

Then we have the following chain of inequalities

$$\tilde{k}_Q(ep) \geq \tilde{k}_{Q\setminus\{\max Q\}}(ep) \geq \tilde{k}_{H\setminus\{\max Q\}}(ep) \geq \tilde{k}_H(ep) - 1 \geq \tilde{k}_Q(ep), \tag{124}$$

where the first and the third inequalities hold true by Claim 13, the second one holds by the induction hypothesis for the set $Q \setminus \{\max Q\}$ and the fourth one by (123). The first and the last values in the chain (124) are identical, thus, the chain remains true when we replace all inequality symbols by equation symbols, i.e., we obtain

$$\tilde{k}_Q(ep) = \tilde{k}_{Q\setminus\{\max Q\}}(ep) = \tilde{k}_{H\setminus\{\max Q\}}(ep) = \tilde{k}_H(ep) - 1 = \tilde{k}_Q(ep). \tag{125}$$

Since $\tilde{k}_H(ep)$ is strictly larger than $\tilde{k}_{H\setminus\{\max Q\}}(ep)$, the set H must contain $\max Q$, hence $\max Q$ and $\max H$ coincide and H has size at least two.

Now, let $Q = \{q_0, \ldots, q_n\}$, where $q_0 < \cdots < q_n$, and let $H = \{q_{z(0)}, \ldots, q_{z(n_H)}\}$, where $z(0) < \ldots < z(n_H)$. Furthermore, let $Q_i = \{q_0, \ldots, q_i\}$ for $i = 0, \ldots, n$ and let $H_i = \{q_{z(0)}, \ldots, q_{z(i)}\}$ for $i = 0, \ldots, n_H$. So, the set Q has size $n+1$, its subset H has size $n_H + 1$ and the function z transforms indices with respect to H into indices with respect to Q. For example, since the maxima of Q and H coincide, we have $z(n_H) = n$.

In what follows, we consider the construction of $M(H)$. The index stairs that occur in this construction contain indices with respect to H, i.e., for example, the index t_0 refers to $q_{z(t_0)}$. A similar remark holds for the intervals that occur in the construction of $M(H)$, i.e., for such an interval U_s^t, we have

$$\max U_s^t = \gamma_{z(t)} - \delta_{z(s)} = \gamma(q_{z(t)}) - \delta(q_{z(s)}).$$

However, as usual, for a given index i, we write γ_i for $\gamma(q_i)$ and δ_i for $\delta(q_i)$.

For every interval of the form U_s^t that occurs in some step of the construction of $M(H)$, the left endpoint $eq_{z(s)}$ of this interval is strictly smaller than ep by assumption of the claim, hence, for every such interval, it holds that

$$ep \in U_s^t \qquad \text{if and only if} \qquad ep \leq \max U_s^t. \tag{126}$$

Let $(t_0, s_1, t_1, \ldots, s_l)$ be the index stair of step n_H of the construction of $M(H)$, i.e., of the last step, and recall that these indices are chosen with respect to H, e.g., the index t_0 stands for $q_{z(t_0)}$. By the third equality in (125), for some index $h \in \{1, \ldots, l\}$, the interval $V_{s_h}^{n_H}$ added during this step contains ep, that is,

$$ep \in V_{s_h}^{n_H} = U_{s_h}^{n_H} \setminus U_{s_h}^{n_H - 1}. \tag{127}$$

By the explicit descriptions for the left and right endpoint of $V_{s_h}^{n_H}$ according to Claim 12, we obtain

$$\gamma_{z(t_1)} - \delta_{z(s_1)} < ep \leq \gamma_{z(n_H)} - \delta_{z(s_1)} \qquad \text{if } h = 1, \tag{128}$$

$$\gamma_{z(t_h)} - \delta_{z(s_h)} < ep \leq \gamma_{z(t_{h-1})} - \delta_{z(s_h)} \leq \gamma_{z(n_H)} - \delta_{z(s_h)} \qquad \text{if } h > 1. \tag{129}$$

So, in the last step of the construction of $M(H)$, the real ep is covered via the expansion of the interval with index s_h. We argue next that, in the construction of $M(H)$, the last stage before stage n_H, in which ep is covered by the expansion of some interval, must be strictly smaller than s_h, i.e., we show

$$\tilde{k}_{H_{s_h}}(ep) = \tilde{k}_{H\setminus\{\max H\}}(ep). \tag{130}$$

For a proof by contradiction, assume that this equation is false. Then there is a stage x of the construction of $M(H)$ with index stair $(t'_0, s'_1, t'_1, \ldots, s'_{l'}, t'_{l'})$ and some index i in $\{1, \ldots, l'\}$ such that

$$s_h < x < n_H \qquad \text{and} \qquad ep \in V^x_{s'_i} = U^x_{s'_i} \setminus U^x_{s'_i}. \tag{131}$$

Observe that the indices in this index stair are indices with respect to the set H_x but coincide with indices with respect to the set H because H_x is an initial segment of H in the sense that H_x contains the least $x+1$ members of H. In particular, the index transformation via the function z works also for the indices in this index stair, for example, the index t'_0 refers to $z(t'_0)$.

We have $s'_i \leq x$ because, otherwise, the interval $V^x_{s'_i}$ would be empty by Claim 9. Furthermore, the indices s_h and s'_i must be distinct because ep is contained in both of the intervals $V^{n_H}_{s_h}$ and $V^x_{s'_i}$, while the former interval is disjoint from the interval $V^x_{s_h}$ by $V^x_{s_h} \subseteq U^x_{s_h} \subseteq U^{n_H-1}_{s_h}$ and $U^{n_H-1}_{s_h} \cap V^{n_H}_{s_h} = \emptyset$.

Next, we argue that

$$\gamma_{z(t'_i)} \leq \gamma_{z(x)} \leq \gamma_{z(t_h)} \leq \gamma_{z(n_H)} < \gamma_{z(t_0)} \qquad \text{and} \qquad \gamma_{z(x)} < \gamma_{z(t_{h-1})}. \tag{132}$$

In the chain on the left, the last two inequalities hold by $t_0 < t_h < n_H$ and by definition of t_0. The first inequality holds because, otherwise, in stage x, the index $t'_i > t'_0$ would have been chosen in place of t'_0. The second inequality holds by choice of t_h as largest index in the range $s_h, \ldots, n_H - 1$ that has maximum γ-value and because this range contains x. From the latter inequality then follows the single inequality on the right since $\gamma_{z(t_h)} < \gamma_{z(t_{h-1})}$ holds by definition of index stair. From (132), we now obtain

$$t_{h-1} \leq t'_0 \qquad \text{and} \qquad \delta_{z(s_h)} \leq \delta_{z(s'_1)} \leq \delta_{z(s'_i)}. \tag{133}$$

Here, the first inequality is implied by the right part of (132) and choice of t'_0. The last inequality holds by definition of index stair. The remaining inequality holds because s_h and s'_1 are chosen as largest indices with minimum δ-value in the ranges $t_{h-1}+1, \ldots, n_H-1$ and $t'_0 + 1, \ldots, x-1$, respectively, where the latter range is a subset of the former one by the just demonstrated first inequality and since x is in H.

Now, we obtain as a contradiction to (127) that ep is in $U^{n_H-1}_{s_h}$ since we have

$$ep \leq \max U^x_{s'_i} \leq \gamma_{z(x)} - \delta_{z(s'_i)} \leq \gamma_{z(t_h)} - \delta_{z(s_h)} = \max U^{n_H-1}_{s_h}.$$

Here, the first inequality holds because ep is in $U^x_{s'_i}$ by choice of i and x. The second inequality holds because, by construction, $\max U^x_{s'_i}$ is equal to $\gamma_{z(x)} - \delta_{z(s'_i)}$ in case $i = 1$

and is equal to $\gamma_{z(t_{h-1})} - \delta_{z(s'_i)}$ in case $h > 1$, where $\gamma_{z(t_{h-1})} \leq \gamma_{z(x)}$. The third inequality holds by (132) and (133) and the final equality holds by Claim 10. This concludes the proof of (130).

By (130), during the stages $s_h + 1, \ldots, n_H - 1$, none of the expansions of any interval covers ep. Now, let y be the minimum index in the range $t_{h-1}, \ldots, s_h$ such that, during the stages $y + 1, \ldots, s_h$, none of the expansions of any interval covers ep, i.e.,

$$y = \min\{k\colon t_{h-1} \leq k \leq s_h \text{ and } \tilde{k}_{H_k}(ep) = \tilde{k}_{H_{s_h}}(ep)\}. \tag{134}$$

Note that y is an index with respect to the set H. We demonstrate that the index y satisfies

$$ep \leq \gamma_{z(y)} - \delta_{z(s_h)}. \tag{135}$$

For further use, note that inequality (135) implies that y and s_h are distinct because, otherwise, since we have $\max Q < p$, we would obtain the contradiction:

$$ep \leq \gamma_{z(y)} - \delta_{z(s_h)} = \gamma_{z(s_h)} - \delta_{z(s_h)} = eq_{z(s_h)}.$$

Now, we show (135). Assuming $y = t_{h-1}$, the inequality is immediate by (128) and choice of t_0 in case $h = 1$ and by (129) in case $h > 1$. So, in the remainder of the proof of (135), we can assume $t_{h-1} < y$.

By choice of y, we have $\tilde{k}_{H_{y-1}}(p) \neq \tilde{k}_{H_y}(p)$, that implies $\tilde{k}_{H_{y-1}}(p) < \tilde{k}_{H_y}(p)$ by Claim 10. Consequently, for the index stair $(t''_0, s''_1, t''_1, \ldots, s''_{l''}, t''_{l''})$ of step y of the construction of the test $M(H)$, there exists an index $j \in \{1, \ldots, l''\}$ such that ep is in $V^y_{s''_j}$. Thus, in particular, it holds that

$$ep \leq \max U^y_{s''_j} \leq \gamma_{z(y)} - \delta_{z(s''_j)} \tag{136}$$

because, by construction, the value $\max U^y_{s''_j}$ is equal to $\gamma_{z(y)} - \delta_{z(s''_j)}$ in case $j = 1$ and is equal to $\gamma_{z(t''_{j-1})} - \delta_{z(s''_j)}$ in case $h > 1$, where $\gamma_{z(t''_{j-1})} \leq \gamma_{z(y)}$. By (136), it is then immediate that, in order to demonstrate (135), it suffices to show that

$$\delta_{z(s_h)} \leq \delta_{z(s''_j)}. \tag{137}$$

The latter inequality follows in turn if we can show that

$$t_{h-1} \leq t''_0 \leq t''_{j-1} < y \leq s_h < n_H, \tag{138}$$

because the indices s_h and s''_j are chosen as largest indices with minimum δ-value in the ranges $t_{h-1} + 1, \ldots, n_H - 1$ and $t''_{j-1} + 1, \ldots, y - 1$, respectively, where the latter range is a subset of the former.

We conclude the proof of (137), and thus also of (135), by showing (138). The second to last inequality holds by choice of y, and all other inequalities hold by definition of index stair, except for the first one. Concerning the latter, by our assumption $t_{h-1} < y$, by $y < n_H$ and by choice of t_{h-1}, we obtain that $\gamma_{z(y)} < \gamma_{z(t_{h-1})}$, which implies that $t_{h-1} \leq t''_0$ by choice of t''_0.

Now, we can conclude the proof of the claim. For the indices $z(y) < z(s_h) < z(n_H) = n$ and the set Q, by (128), (129) and (135), all assumptions of Claim 14 are satisfied, hence the claim yields that

$$\tilde{k}_{Q_{z(s_h)}}(ep) < \tilde{k}_Q(ep). \tag{139}$$

Now, we obtain the contradiction

$$\tilde{k}_Q(ep) = \tilde{k}_{H\setminus\{\max Q\}}(ep) = \tilde{k}_{H_{s_h}}(ep) \le \tilde{k}_{Q_{z(s_h)}}(ep) < \tilde{k}_Q(ep),$$

where the relations follow, from left to right, by (125), by (130), by the induction hypothesis for the set $Q_{z(s_h)}$ and by (139). □

Claim 16. *Let $Q = \{q_0 < \cdots < q_n\}$ be a subset of the domain of g and, for $z = 0, \ldots, n$, let $Q_z = \{q_0, \ldots, q_z\}$. Let p be a nonrational real such that, for some index x in $\{1, \ldots, n\}$, it holds that $p \in [0, q_x]$ and*

$$\tilde{k}_{Q_{x-1}}(ep) \ne \tilde{k}_{Q_x}(ep). \tag{140}$$

Then it holds that

$$\tilde{k}_{Q_x}(ep) = \tilde{k}_{Q_{x+1}}(ep) = \cdots = \tilde{k}_{Q_n}(ep).$$

Proof. We denote the intervals considered in the construction of the test $M(Q)$ by U_i^j, as usual. Again, we can argue that the construction of a test of the form $M(Q_z)$, where $z \le n$, is essentially identical to an initial part of the construction of $M(Q)$, and that accordingly such a test $M(Q_z)$ coincides with $(U_0^z, \ldots, U_z^z)$.

Let $(t_0, s_1, t_1, \ldots, s_l, t_l)$ be the index stair of step x in the construction of $M(Q_n)$. By (140), there is an index h in $\{1, \ldots, l\}$ such that ep is in $V_{s_h}^x$, hence

$$eq_{s_h} = \min U_{s_h}^{x-1} \le \max U_{s_h}^{x-1} = \gamma_{t_h} - \delta_{s_h} < ep \le \gamma_x - \delta_{s_h}. \tag{141}$$

Here, the two equalities hold by definition of the interval and by Claim 10, respectively. The strict inequality holds because ep is assumed not to be in $U_{s_h}^{x-1}$. The last inequality holds because ep is assumed to be in $U_{s_h}^{x-1}$, while, by construction, the right endpoint of the latter interval is equal to $\gamma_x - \delta_{s_h}$ in case $h = 1$ and is equal to $\gamma_{t_h} - \delta_{s_h}$ with $\gamma_{t_h} \le \gamma_x$, otherwise.

For a proof by contradiction, we assume that the conclusion of the claim is false. So we can fix an index $y \in \{x+1, \ldots, n\}$ such that

$$\tilde{k}_{Q_x}(ep) = \tilde{k}_{Q_{y-1}}(ep) < \tilde{k}_{Q_y}(ep).$$

Let $(t'_0, s'_1, t'_1, \ldots, s'_{l'}, t'_{l'})$ be the index stair of step y of the construction of $M(Q_n)$. By essentially the same argument as in the case of (141), we can fix an index i in $\{1, \ldots, l'\}$ such that

$$eq_{s'_i} = \min U_{s'_i}^{y-1} \le \max U_{s'_i}^{y-1} = \gamma_{t'_i} - \delta_{s'_i} < ep \le \gamma_y - \delta_{s'_i}. \tag{142}$$

By assumption, the real p is in $[0, q_x]$ and, together with (141) and (142), we obtain $q_{s_h} < p \leq q_x$ and $q_{s'_i} < p \leq q_x$. Consequently, we have

$$s_h < x \qquad \text{and} \qquad s'_i < x \tag{143}$$

(where the left inequality also follows from definition of index stair). In particular, we have $t'_0 < x$, which implies by $x < y$ and choice of t'_0 that

$$\gamma_y < \gamma_x. \tag{144}$$

In order to derive the desired contradiction, we distinguish the two cases that are left open by (143) for the relative sizes of the indices s_h, s'_i and x.

Case 1: $s_h < s'_i < x$. Since s_h is chosen in the range $t_{h-1}+1, \ldots, x-1$ as largest index with minimum δ-value, we obtain by case assumption that

$$\delta_{s_h} < \delta_{s'_i}. \tag{145}$$

Furthermore, it holds that

$$t_0 < s_h \leq t'_{i-1} < s'_i < x < y. \tag{146}$$

Here, the first and the third inequalities hold by definition of index stair. The two last inequalities hold by (143) and by choice of y, respectively. The remaining second inequality holds because, otherwise, in case $t'_{i-1} < s_h$, the range $t'_{i-1}+1, \ldots, y-1$, from which s'_i is chosen as largest index with minimum δ-value, would contain s_h, which contradicts (145).

Now, we obtain a contradiction, which concludes Case 1. Due to $t_0 < t'_{i-1} < x$ and definition of t_0, we have $\gamma_{t'_{i-1}} \leq \gamma_x$. The latter inequality contradicts the fact that t'_{i-1} is chosen in the range $s'_{i-1}+1, \ldots, y-1$ as largest index with maximum γ-value, where the latter range contains x by (146) and $s'_{i-1} < s'_i$.

Case 2: $s'_i < s_h < x$. In this case, we have

$$\delta_{s'_i} < \delta_{s_h} \qquad \text{and} \qquad \gamma_x \leq \gamma_{t'_i}. \tag{147}$$

Here, the first inequality holds since s'_i is chosen as largest index with minimum δ-value from a range that, by case assumption, contains s_h. The second inequality holds since t'_i is chosen in the range $s'_i+1, \ldots, y-1$ as largest index with maximum γ-value, where this range contains x by case assumption and $x < y$.

Now, we obtain a contradiction, which concludes Case 2, since we have

$$ep < \gamma_x - \delta_{s_h} < \gamma_{t'_i} - \delta_{s'_i} < ep, \tag{148}$$

where the inequalities hold, from left to right, by (141), by (147) and by (142).

So we obtain in both cases a contradiction, which concludes the proof of the claim. □

The proof of Claim 3 Let $Q = \{q_0 < \cdots < q_n\}$ be a subset of the domain of g, where $q_n < 1$. For $z = 0, \ldots, n$, let $Q_z = \{q_0, \ldots, q_z\}$, and let p be an arbitrary nonrational real in $[0, 1]$. In order to demonstrate Claim 3, it suffices to show

$$K_Q(ep) \leq \tilde{k}_Q(ep) + 1. \tag{149}$$

Since p was chosen as an arbitrary nonrational real in $[0, 1]$, this easily implies the assertion of Claim 3, i.e., that $K_Q(p') \leq \tilde{k}_Q(p') + 1$ for all nonrational p' in $[0, e]$.

By construction, for all subsets H of Q, all intervals in the test $M(H)$ have left endpoints of the form $\gamma(q_i) - \delta(q_i) = eq_i$. Consequently, in case $p < q_0$, none of such intervals contains ep, hence $K_{Q_n}(ep) = 0$ and we are done.

So, from now on, we can assume $q_0 < p$. Then, among $q_0, \ldots, q_n$, there is a maximum value that is smaller than p and we let

$$j = \max\{i \in \{0, \ldots, n\} : q_i < p\} \tag{150}$$

be the corresponding index. It then holds that

$$K_{Q_j}(ep) = \tilde{k}_{Q_j}(ep) \leq \tilde{k}_{Q_{j+1}}(ep) \leq \cdots \leq \tilde{k}_{Q_{n-1}}(ep) \leq \tilde{k}_Q(ep), \tag{151}$$

where the equality is implied by choice of j and Claim 15 and the inequalities hold by Claim 13.

Fix some subset H of Q that realizes the value $K_Q(ep)$ in the sense that

$$K_Q(ep) = \tilde{k}_H(ep). \tag{152}$$

Next, we show that, for the set H, we have

$$\tilde{k}_H(ep) \leq \tilde{k}_{H \cap Q_j}(ep) + 1. \tag{153}$$

In case $\tilde{k}_H(ep) \leq \tilde{k}_{H \cap Q_j}(ep)$, we are done. Otherwise, let x be the least index in the range $j + 1, \ldots, n$ such that $\tilde{k}_{H \cap Q_x}(ep)$ differs from $\tilde{k}_{H \cap Q_{x-1}}(ep)$. Then (152) follows from

$$\tilde{k}_{H \cap Q_j}(ep) + 1 = \tilde{k}_{H \cap Q_{x-1}}(ep) + 1 = \tilde{k}_{H \cap Q_x}(ep) = \tilde{k}_{H \cap Q}(ep) = \tilde{k}_H(ep),$$

where the equalities hold, from left to right, by choice of x, by Claim 13, by Claim 16 and since H is a subset of Q. Now, we have

$$K_Q(ep) = \tilde{k}_H(ep) \leq \tilde{k}_{H \cap Q_j}(ep) + 1 \leq K_{Q_j}(ep) + 1 \leq \tilde{k}_Q(ep) + 1,$$

where the relations hold, from left to right, by choice of H, by (153), because $H \cap Q_j$ is a subset of Q_j and, finally, by (151).

This conclude the proof of (149) and thus also of Claim 3.

Proof that the limit is unique So, we have demonstrated that the limit asserted in Theorem 66 exists and is finite. It remains to show that this limit does not depend on the function that witnesses the monotone Solovay reducibility from α to β. For a proof by contradiction, assume that the reducibility is not only witnessed by g but also by some function f, where the limits for g and f differ. By symmetry, without loss of generality, we can then pick rationals c and d such that

$$\lim_{q \nearrow \beta} \frac{\alpha - g(q)}{\beta - q} < c < d < \lim_{q \nearrow \beta} \frac{\alpha - f(q)}{\beta - q}, \tag{154}$$

Recall that, by assumption on f and g, both functions are defined on all rationals $q < \beta$ and both limits in (154) exist and are finite.

By (154), for every rational $q < \beta$ that is close enough to β, it holds that

$$\frac{\alpha - g(q)}{\beta - q} < c \qquad \text{and} \qquad d < \frac{\alpha - f(q)}{\beta - q}.$$

Fix some rational $p < \beta$ such that the two latter inequalities are both true for all rationals q in the interval $[p, \beta]$. We then have for all such q

$$d - c < \frac{(\alpha - f(q)) - (\alpha - g(q))}{\beta - q} = \frac{g(q) - f(q)}{\beta - q}, \tag{155}$$

and consequently, letting $e = d - c$,

$$eq < e\beta < eq + g(q) - f(q) \quad \text{for all rationals } q \text{ in } [p, \beta], \tag{156}$$

where the lower bound is immediate by $q < \beta$ and the upper bound follows by multiplying the first and the last terms in (155) by $\beta - q$ and rearranging. Let

$$D = \{q \in [0, 1]\colon f \text{ and } g \text{ are both defined on } q \text{ and } f(q) < g(q)\}.$$

For every q in D, define the intervals

$$I_q = [f(q), g(q)] \qquad \text{and} \qquad U_q = [eq, eq + g(q) - f(q)].$$

Fix some effective enumeration $q_0, q_1, \ldots$ of D. We define inductively a subset S of the natural numbers and let S_n be the intersection of S with $\{0, , \ldots, n\}$. Let 0 be in S and, for $n > 0$, assuming that S_n has already been defined, let

$$n + 1 \in S \text{ if and only if } \text{ for all } i \text{ in } S_n, \text{ the intervals } I_{q_i} \text{ and } I_{q_{n+1}} \text{ are disjoint.}$$

The intervals U_{q_n}, where n is in S, form a Solovay test. First, these intervals can be effectively enumerated since S is computable by construction. Second, the sum of the lengths of these intervals is at most 1 because, for every q in D, the intervals U_q and I_q have the same length by definition, while the intervals I_{q_n}, where n is in S, are mutually disjoint by definition of S.

So in order to obtain the desired contradiction, it suffices to show that the Martin-Löf random real $e\beta$ is covered by the Solovay test just defined, i.e., that there are infinitely many i in S such that the interval U_{q_i} contains $e\beta$. By definition of these intervals and (156), here it suffices in turn to show that there are infinitely many i such that i is in S and q_i is in $[p, \beta]$. To this end, we fix some arbitrary natural number n and show that there is such $i > n$.

Since the values $f(q)$ converge from below to α when q tends from below to β, we can fix an index $i_0 > n$ such that $q_{i_0} \in [p, \beta]$, and in addition, we have

$$\text{(i) } g(p) < f(q_{i_0}) \quad \text{and} \quad \text{(ii) } g(q_i) < f(q_{i_0}) \text{ for all } i \text{ in } S_n \text{ where } q_i < \beta. \tag{157}$$

In case i_0 is selected, we are done. Otherwise, there must be some $i_1 < i_0$ in S such that $I_{q_{i_1}}$ has a nonempty intersection with $I_{q_{i_0}}$. We fix such an index i_1 and conclude the proof that the proof by showing that we must have $n < i_1$ and $q_{i_1} \in [p, \beta]$. In order to prove the latter, we show for q in D that, in case $q < p$ and in case $q > \beta$, the intervals I_q and $I_{q_{i_0}}$ are disjoint. In the former case, by monotonicity of g, the right endpoint $g(q)$ of the interval I_q is strictly smaller than the left endpoint $f(q_{i_0}) > g(p)$ of $I_{q_{i_0}}$. In the latter case, the left endpoint $f(q)$ of I_q is at least as large as α, and thus, is strictly larger than the right endpoint of $I_{q_{i_0}}$ since f maps $[0, \beta)$ is $[0, \alpha)$ as a translation function. Otherwise, i.e., in case $f(q) < \alpha$, since the values $f(q')$ converge from below to α when q' tends from below to β, there would be $q' < \beta$ where $f(q) < f(q')$, contradicting the monotonicity of f.

It remains to show that $n < i_1$, i.e., that i_1 is not in S_n. But, for any index i in S_n, the intervals U_{q_i} and $U_{q_{i_0}}$ are disjoint as follows in case $\beta < q_i$ from the discussion in the preceding paragraph and follows in case $q_i < \beta$ from (ii) in (157).

This concludes the proof of uniqueness of the limit point as well as the proof of Theorem 66.

Corollaries of Theorem 66 The proof of Theorem 66 can be adjusted to yield the following corollary: it suffices to replace the enumeration of the domain of g by the sequence $b_0, b_1, \ldots$.

Corollary 67. *Let α be a real and let β be a Martin-Löf random computably approximable real with computable approximation $b_n \to \beta$ where $b_n < \beta$ for infinitely many n. In case α is monotone Solovay reducible to β via some partial function g, the limit*

$$\lim_{\substack{n\to\infty \\ b_n<\beta}} \frac{\alpha - g(b_n)}{\beta - b_n}$$

exists and does neither depend on the considered computable approximation of β nor on the choice of the partial function g witnessing that α is monotone Solovay reducible to β.

Finally, we also obtain a proof of Theorem 59. Recall that this theorem asserts that Martin-Löf random reals are never speedable, and that Merkle and Titov [9] observed that the latter fact is a straightforward consequence of Theorem 66 but also gave a short direct proof.

In order to see that Theorem 66 implies that Martin-Löf random reals are never speedable, fix some Martin-Löf random real α. By Theorem 66, there is a real d such that, for every partial function g that witnesses that the real α is monotone Solovay reducible to itself, it holds that

$$\lim_{q \nearrow \alpha} \frac{\alpha - g(q)}{\alpha - q} = d. \tag{158}$$

The identity function $q \mapsto q$ is such a function, hence d must be equal to 1. On the other hand, by Remark 55, also every translation function g that witnesses that α is speedable, i.e., is ρ-speedable for some $\rho < 1$, is such a function, where then, by definition of ρ-speedable, it holds that

$$\liminf_{q \nearrow \alpha} \frac{\alpha - g(q)}{\alpha - q} \leq \rho, \tag{159}$$

By $d = 1$, the latter contradicts to (158), hence α cannot be speedable.

7 Variants of Solovay reducibility outside of the left-c.e. reals

7.1 Reducibilities that use absolute distance

Zheng and Rettinger proposed in [19] another way to extend the notion of Solovay reducibility outside of left-c.e. reals in order to preserve the property of "not slower convergence" of a given approximation by considering the absolute distance from the actual element of it to its limit point.

The following reducibility is defined for computably approximable reals and, according [19], Theorem 3.2.(2), coincides on LEFT−CE with the standard Solovay reducibility.

Definition 68 (Zheng and Rettinger, [19])**.** *Let α and β be two Δ^0_2 reals.*

*α is $S2a$-reducible to β, written $\alpha \leq^{2a}_{S} \beta$, if there exist two computable approximations $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ of α and β, respectively, and a constant $c \in \mathbb{R}$ that satisfy for every $n \in \omega$ the $\leq^{2a}_{S}$-*Solovay property*:*

$$|\alpha - a_n| \leq c(|\beta - b_n| + 2^{-n}). \tag{160}$$

Note that the mapping $b_n \mapsto a_n$ in the latter definition is not required to be monotone.

Rettinger and Zheng have also proved in [13], Theorem 3.7, that d.c.e. reals form a field on the set of Δ^0_2 reals which is closed downwards relative to $\leq^{2a}_{S}$ with Martin-Löf random left-c.e. reals as a highest degree.

Theorem 69 (Rettinger and Zheng, [13])**.** *For a Martin-Löf random left-c.e. real Ω and a Δ^0_2 real β, $\alpha \leq^{2a}_{S} \Omega$ if and only if $\alpha \in$ DCE.*

Moreover, the reducibility $\leq^{2a}_{S}$ can be extended from the set of Δ^0_2 reals to all reals using the idea of translation function that maps, unlike in the standard notion of Solovay reducibility, dyadic rationals into dyadic rationals.

Definition 70. *Let α and β be two reals.*

*α is $S1a$-reducible to β, written $\alpha \leq^{1a}_{S} \beta$, if there exist a partial computable function $g : \mathbb{Q}_2 \to \mathbb{Q}_2$ and a constant $c \in \mathbb{R}$ such that β is an accumulation point of $dom(g)$ that satisfies the $\leq^{1a}_{S}$-*Solovay property*:*

$$\forall q \in dom(g) : |\alpha - f(q)| \leq c(|\beta - q| + 2^{-l(q)}), \tag{161}$$

where $l(q)$ denotes the length of binary representation of q.

According to [19], Theorem 4.2., the reducibilities $\leq_S^{1a}$ and $\leq_S^{2a}$ coincide of the set of Δ_2^0 reals.

The closure downwards of d.c.e. reals relative to $\leq_S^{2a}$-reducibility is one of the main results of [13].

Proposition 71 (Rettinger and Zheng, [13])**.** *The set* DCE *is closed downwards relative to $\leq_S^{2a}$.*

7.2 Relations between the variants of Solovay reducibilities on real numbers

We already know that both reducibilities $\leq_S^{1a}$ and $\leq_S^{2a}$ introduced in the previous section coincide on the set of Δ_2^0 reals and, moreover, coincide with $\leq_S$ of the set of left-c.e. reals.

In this section, we prove that the ordinary Solovay reducibility $\leq_S$ implies $\leq_S^{1a}$ on $\mathbb{R}$, and therefore, $\leq_S^{2a}$ on the set of Δ_2^0 reals.

Proposition 72. *Let $\alpha, \beta \in [0,1]$ be reals fulfilling $\alpha \leq_S \beta$ via some Solovay constant c. Then, for every $\tilde{c} > c$, it holds that $\alpha \leq_S^{1a} \beta$ via $\tilde{c}$ as a Solovay constant.*

Proof. Let $\alpha, \beta \in [0,1]$ be two reals such that $\alpha \leq_S \beta$ via some translation function g (defined on all rationals in $[0, \beta)$) and a Solovay constant c, and let $q_0, q_1, \ldots$ with $q_0 = 0$ be an effective enumeration of $\mathbb{Q}_2 \cap dom(g)$ that contains inter alia all dyadic rationals on $[0, \beta)$.

In means in particular that, for every index n such that $q_n < \beta$, the Solovay condition (11) holds true for q_n:

$$0 < \alpha - g(q_n) < c(\beta - q_n). \tag{162}$$

First, given a real $\tilde{c} > c$, we fix some dyadic rationals d and $\tilde{d}$ such that $c < d < \tilde{d} < \tilde{c}$) and a natural N such that

$$(1 - 2^{-N})d < \tilde{d} \quad \text{and} \quad \tilde{c} - \tilde{d} < 2^{-N}. \tag{163}$$

To prove the proposition for the constant $\tilde{c}$, we construct a partially computable function $\tilde{g}$ that maps dyadic rationals into dyadic rationals and an index sequence $n_0, n_1, \ldots$ such that $q_{n_0}, q_{n_1}, \ldots$ is an enumeration of $dom(\tilde{g})$, the real β an accumulation point of $q_{n_0}, q_{n_1}, \ldots$ and, for every $k \in \omega$, it holds that

$$|\alpha - \tilde{g}(q_{n_k})| < \tilde{c}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}). \tag{164}$$

Then it will mean that $\alpha \leq_{\mathrm{S}}^{1\mathrm{a}} \beta$ via the translation function $\tilde{g}$ and the constant $\tilde{c}$.

We set $n_0 = 0$ and, due to the technical reasons, $n_{-1} = -1$.

Step k: Assuming $n_0 < \cdots < n_{k-1}$ to be defined, we continue the enumeration of $dom(g)$ from the index $n_{k-1} + 1$ on until we meet an index n such that q_n is a dyadic rational and, for every q' such that

$$q' < q_n \quad \text{and} \quad l(q') \leq l(q_n) + N, \tag{165}$$

the following conditions are fulfilled:

$$g(q') \downarrow \quad , \tag{166}$$
$$-d \cdot 2^{-l(q_n)} < \quad g(q_n) - q(q') \quad , \tag{167}$$
$$g(q_n) - q(q') \quad < d((q_n - q') + 2^{-l(q_n)}). \tag{168}$$

After we find such n, we define

$$\tilde{g}(q_n) = 0, (g(q_n) \upharpoonright (N + l(q_n))), \tag{169}$$

set $n_k = n$, and go to the step $k + 1$.

We start the proof that the construction step k terminates from noting for every $i \in \omega$ that $0, (\beta \upharpoonright i)$ is a dyadic rational on the interval $[0, \beta)$, hence $0, (\beta \upharpoonright i) \in dom(g)$. It allows to define an (in general noncomputable) index sequence $m_0, m_1, \ldots$ such that, for every $i > 0$, it holds that

$$q_{m_i} = 0, (\beta \upharpoonright i). \tag{170}$$

We fix one (of all but finitely many) number $i \in \omega$ that fulfills $m_i \geq n_{k-1} + 1$. From $(\beta \upharpoonright i) < \beta$, we obtain the Solovay property (162) for q_{m_i}.

Next, fixing an index $q' < q_{m_i}$ that fulfills $l(q') \leq l(q_{m_i}) + N$, we can obtain a lower bound for $g(q_{m_i}) - q(q')$:

$$g(q_{m_i}) - g(q') \underset{\text{by } q'<\beta}{>} g(q_{m_i}) - \alpha \underset{\substack{\text{by (162)} \\ \text{and } d>c}}{>} d(q_{m_i} - \beta) = -d(\beta - 0, (\beta \upharpoonright i)) > -d \cdot 2^{-i}.$$

Therefore, the index m_i suffices for q' fulfilling (165) the conditions (166) and (167) since $l(q_{m_i}) = l(\beta \upharpoonright i) = i$.

The condition (168) for m_i and q' is implied from the following inequality:

$$g(q_{m_i}) - g(q') \underset{\text{by (162)}}{\leq} \alpha - g(q') \underset{\substack{\text{by (162)} \\ \text{and } d>c}}{<} d(\beta - q') = d((q_{m_i} - q') + (\beta - q_{m_i}))$$

$$\underset{\text{by (170)}}{=} d((q_{m_i} - q') + (\beta - (\beta \upharpoonright i))) \leq d((q_{m_i} - q') + 2^{-i}) = d((q_{m_i} - q') + 2^{-l(q_{m_i})}).$$

Therefore, the stage k should be terminated by enumerating the index $n = m_i$ or earlier.

Note that the termination proof implies as well that, for every i, the dyadic rational q_{m_i} lies in $dom(\tilde{g})$ since k can be chosen such that $n_{k-1} < m_i \leq n_k$; in the latter case, we obtain that $n_k = m_i$ since the index m_i fulfills all requirements for n_k, as we already know from the termination proof. Hence, the sequence of dyadic rationals $(0, (\beta \upharpoonright i))_{i\in\omega}$ lies in $dom(g)$ and obviously tends to β.

To conclude the proof of proposition, it remains to argue that every element q_{n_k} of the constructed sequence fulfills the property (164).

We consider 4 cases of positions of q_{n_k} and $g(q_{n_k})$ relative to β and α, respectively.

- $q_{n_k} < \beta$, $g(q_{n_k}) < \alpha$.

  In this case, the inequality (164) follows directly from Solovay condition (162) for q_{n_k} due to

$$|\alpha - g(q_{n_k})| = \alpha - g(q_{n_k}) < c(\beta - q_{n_k}) \underset{\text{by } c<\tilde{d}}{\leq} \tilde{d}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}).$$

- $q_{n_k} < \beta$, $g(q_{n_k}) \geq \alpha$.

  This case is impossible since it contradicts to the Solovay condition (162) for q_{n_k}.

- $q_{n_k} \geq \beta$, $g(q_{n_k}) < \alpha$.

  In this case, we obtain for the greatest dyadic rational $\tilde{q} < \beta (< q_{n_k})$ such that $l(\tilde{q}) \leq l(q_{n_k}) + N$ the inequality

$$\tilde{q} < \beta \leq \tilde{q} + 2^{-l(q_{n_k})-N} \leq q_{n_k}, \tag{171}$$

  and therefore, that

$$|\alpha - g(q_{n_k})| = \alpha - g(q_{n_k}) \leq \underbrace{(\alpha - g(\tilde{q}))}_{\substack{<d(\beta-\tilde{q}) \text{ by } (162)\\ \text{for } q_n=\tilde{q} \text{ and } d>c}} + \underbrace{(g(\tilde{q}) - q_{n_k})}_{<d\cdot 2^{-l(q_{n_k})} \text{by } (167)}$$

$$< d((\beta - \tilde{q}) + 2^{-l(q_{n_k})}) \underset{\text{by } (171)}{\leq} d(2^{-l(q_{n_k})-N} + 2^{-l(q_{n_k})}) =$$

$$= d(1 - 2^{-N})(2^{-l(q_{n_k})}) \underset{\text{by } (163)}{<} \tilde{d} \cdot 2^{-l(q_{n_k})} \leq \tilde{d}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}).$$

- $q_{n_k} \geq \beta$, $g(q_{n_k}) \geq \alpha$.

  In this case, we obtain for $\tilde{q}$ defined as in the previous case that $\tilde{q}$ fulfills (171), and therefore, that

$$|\alpha - g(q_{n_k})| = g(q_{n_k}) - \alpha < g(q_{n_k}) - g(\tilde{q}) \underset{\text{by } (168)}{<} d((q_{n_k} - \tilde{q}) + 2^{-l(q_{n_k})})$$

$$= d((q_{n_k} - \beta) + (\beta - \tilde{q}) + 2^{-l(q_{n_k})}) \underset{\text{by } (171)}{\leq} d(|\beta - q_{n_k}| + 2^{-l(q_{n_k})} + 2^{-l(q_{n_k}-N)})$$

$$= d|\beta - q_{n_k}| + d(1-2^{-N})2^{-l(q_{n_k})} \underset{\text{by } (163)}{\leq} d|\beta - q_{n_k}| + \tilde{d}\cdot 2^{-l(q_{n_k})} \underset{\text{by } d<\tilde{d}}{\leq} \tilde{d}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}).$$

Consequently, for every $q_{n_k} \in dom(\tilde{g})$, it holds that

$$|\alpha - g(q_{n_k})| < \tilde{d}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}), \tag{172}$$

that implies the inequality (164) in the following way:

$$\begin{aligned}
|\alpha - \tilde{g}(q_{n_k})| \quad &\leq \quad \underbrace{|\alpha - g(q_{n_k})|}_{<\tilde{d}(|\beta - q_{n_k}|+2^{-l(q_{n_k})}) \text{ by } (172)} + \underbrace{|g(q_{n_k}) - \tilde{g}(q_{n_k})|}_{<2^{-N-l(q_{n_k})} \text{ by } (169)} \\
&< \quad \tilde{d}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}) + 2^{-N-l(q_{n_k})} \\
&\underset{by\ (163)}{<} \quad \tilde{d}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}) + (\tilde{c} - \tilde{d})2^{-l(q_{n_k})} < \tilde{c}(|\beta - q_{n_k}| + 2^{-l(q_{n_k})}).
\end{aligned}$$

□

The next corollary is straightforwardly implied from the latter proposition, the coincidence of $\leq_S^{1a}$ and $\leq_S^{2a}$ for computably approximable reals and the closure downwards of Δ_2^0 reals relative to Solovay reducibility.

Corollary 73. *For a real α and a computably approximable real β, it holds that*

$$\alpha \leq_S^{m} \beta \quad \Longrightarrow \quad \alpha \leq_S \beta \quad \Longrightarrow \quad \alpha \leq_S^{2a} \beta.$$

Bibliography

[1] George Barmpalias and Andrew Lewis-Pye. *Differences of halting probabilities.* Journal of Computer and System Sciences 89:349–360 (2017).

[2] Christian Calude, Peter Hertling, Bakhadyr Khoussainov, and Yongge Wang. Recursively enumerable reals and Chaitin Ω numbers. In M. Morvan, C. Meinel, and D. Krob, editors, STACS 98. 15th Annual Symposium on Theoretical Aspects of Computer Science 1998. Proceedings, volume 1373 of Lecture Notes in Computer Science, pp 596–606. Springer, Berlin (1998).

[3] Gregory Chaitin. *A theory of program size formally identical to information theory.* Journal of the Association for Computing Machinery, 22:329–340 (1975).

[4] Rodney Downey, Evan Griffiths. *Schnorr randomness.* The Journal of Symbolic Logic, 69:533–554 (2004).

[5] Rodney Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity.* Springer, Berlin (2010).

[6] Rodney Downey, Denis Hirschfeldt, and André Nies. Randomness, computability and density. SIAM Journal on Computing, 31:1169–1183 (2002).

[7] Rupert Hölzl and Philip Janicki. *Benign approximations and non-speedability.* Technical report, `arXiv:2303.11986v2`, submitted for further publication (2023).

[8] Antonín Kučera and Theodore Slaman. *Randomness and recursive enumerability.* SIAM Journal on Computing, 31:199–211 (2001).

[9] Wolfgang Merkle and Ivan Titov. *Speedability and randomness.* Talk at CCA 2021. Eighteenth International Conference on Computability and Complexity in Analysis (2021).

[10] Wolfgang Merkle and Ivan Titov. *Speedable left-c.e. numbers.* Henning Fernau, editor, CSR 2020. Computer Science - Theory and Applications - 15th International Computer Science Symposium in Russia, 2020. Proceedings, volume 12159 of Lecture Notes in Computer Science, pp 303–313. Springer (2020).

[11] Joseph Miller. On work of Barmpalias and Lewis-Pye: a derivation on the d.c.e. reals. In Adam Day et al., editors. Computability and Complexity: Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday, pp 644–659. Springer, Lecture Notes in Computer Science (2017).

[12] Kenshi Miyabe, Andre Nies and Frank Stephan. *Randomness and Solovay degrees.* Journal of Logic and Analysis 10:3:1–13 (2018).

[13] Robert Rettinger and Xizhong Zheng. *Solovay reducibility on d-c.e real numbers.* Lecture Notes in Computer Science 3595:359–368 (2005).

[14] Claus Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie.* Volume 218 of Lecture Notes in Mathematics. Springer-Verlag, Berlin–New York (1971).

[15] Robert Soare. *Cohesive sets and recursively enumerable Dedekind cuts.* Pacific Journal of Mathematics, 31:215–231 (1969).

[16] Robert Soare. *Recursively Enumerable Sets and Degrees.* Springer-Verlag, Berlin Heidelberg (1987).

[17] Robert Solovay. *Draft of paper (or series of papers) on Chaitin's work.* Unpublished notes, 215 pages, (1975).

[18] Alan Turing. *On computable numbers with an application to the Entscheidungsproblem.* Proceedings of the London Mathematical Society, 42:230–265, 1936. Correction in Proceedings of the London Mathematical Society 43:544–546 (1937)

[19] Xizhong Zheng and Robert Rettinger. *On the extensions of reducibility.* COCOON 2004: Computing and Combinatorics, pp 360–369 (2004).