Dissertation
submitted to the
Combined Faculty of Mathematics, Engineering and Natural Sciences
of Heidelberg University, Germany
for the degree of
Doctor of Natural Sciences

Put forward by
Fabian Hahner
Born in: Fulda (Germany)
Oral examination: 25.01.2024

The Pure Spinor Superfield Formalism
and
Twisted Supergravity

Referees:
Prof. Johannes Walcher
Prof. Owen Gwilliam

Abstract

This thesis discusses the pure spinor superfield formalism and its applications, specifically in the context of twisted eleven-dimensional supergravity. We start by developing the pure spinor superfield formalism as a framework for the construction of supermultiplets from a graded equivariant module over the ring of functions on the nilpotence variety. This perspective establishes a connection between algebrogeometric properties of the nilpotence variety and the physics of multiplets. Furthermore, it allows for efficient computations by means of homological algebra. After exploring the formalism in various examples, we extend it to the setting of derived geometry, show that this generalization establishes an equivalence of categories, and relate it to Koszul duality. In particular, this result establishes a method to construct superspace descriptions for any multiplet. As an application, we provide an extensive case study of supermultiplets with six-dimensional $\mathcal{N} = (1, 0)$ supersymmetry and classify all multiplets whose derived invariants for the supertranslation algebra define a line bundle on the nilpotence variety. In the second part, we consider eleven-dimensional supergravity and its twists. We compute the maximal twist in the free perturbative limit starting from the L_∞ action of the super Poincaré algebra on the BV complex of component fields. Then, we use the pure spinor superfield formalism to construct a generalization of Poisson–Chern–Simons theory, defined on any supermanifold equipped with an appropriate odd distribution. This theory recovers Cederwall's formulation of eleven-dimensional supergravity, Costello's description of the maximal twist, and gives a pure spinor lift of the interactions in the minimally twisted theory. Compatibility between the pure spinor formalism and twisting implies that all these theories are related by twists. Motivated by holographic duality, we use these methods to explore (twisted) six-dimensional (2,0) supersymmetry. We give a pure spinor construction of the decomposition of the minimally twisted eleven-dimensional supergravity fields into $E(3|6)$-modules and provide an interpretation in terms of supergeometry which hints towards a generalization in the untwisted case.

Zusammenfassung

Diese Arbeit behandelt den Pure-Spinor-Superfeld-Formalismus und seine Anwendungen, insbesondere im Hinblick auf getwistete elfdimensionale Supergravitation. Zunächst entwickeln wir den Formalismus als konzeptuellen Rahmen zur Konstruktion von Supermultipletts aus äquivarianten Moduln über dem Ring der Funktionen auf der Nilpotenzvarietät. Mit dieser Perspektive werden Verbindungen zwischen den algebrogeometrischen Eigenschaften der Nilpotenzvarietät und der Physik von Multipletts offengelegt. Zusätzlich ermöglicht sie es effektive Berechnungen mithilfe von Methoden aus der homologischen Algebra durchzuführen. Nachdem wir den Formalismus auf einige Beispiele angewendet haben, erweitern wir ihn auf den Kontext der derivierten Geometrie, zeigen, dass diese Verallgemeinerung eine Äquivalenz von Kategorien liefert, und verbinden sie mit Koszul-Dualität. Insbesondere zeigen diese Resultate, dass der Formalismus Superfeldbeschreibungen für jedes Multiplett konstruiert. Als Anwendung präsentieren wir eine ausführliche Betrachtung zu Multipletts mit sechsdimensionaler $\mathcal{N} = (1,0)$ Supersymmetrie und klassifizieren hierbei alle Multipletts, deren derivierte Invarianten bezüglich Supertranslationen Geradenbündel über der Nilpotenzvarietät definieren. Im zweiten Teil wenden wir uns elfdimensionaler Supergravitation und deren Twists zu. Wir berechnen den maximalen Twist im perturbativen freien Limes explizit ausgehend von der L_∞-Wirkung der Super-Poincaré-Algebra auf den BV Komplex der Komponentenfelder. Darüber hinaus verwenden wir den Pure-Spinor-Superfeld-Formalismus um eine Verallgemeinerung von Poisson–Chern–Simons-Theorie zu konstruieren, welche auf jeder Supermannigfaltigkeit ausgestattet mit einer passenden ungeraden Distribution definiert ist. Diese Theorie vereinheitlicht Cerderwalls Beschreibung von elfdimensionaler Supergravitation, Costellos Formulierung des maximalen Twists und gibt einen Lift der Wechselwirkungen im minimalen Twist. Motiviert durch die holografische Korrespondenz, nutzen wir diese Methoden um (getwistete) (2,0) Supersymmetrie in sechs Dimensionen zu studieren. Wir beschreiben eine geometrische Konstruktion der Zerlegung der Felder minimal getwisteter elfdimensionaler Supergravitation in $E(3|6)$-Moduln mithilfe des Pure-Spinor-Formalismus. Das entstehende geometrische Bild liefert Hinweise für eine Verallgemeinerung im ungetwisteten Fall.

Published Contents

This thesis is based on results which appeared in the following articles.

[EH23] R. Eager, F. Hahner. *Maximally twisted eleven-dimensional supergravity.* Commun. Math. Phys. 398 (2023). arXiv:2106.15640.

[Eag+22] R. Eager, F. Hahner, I. Saberi, B. R. Williams. *Perspectives on the pure spinor superfield formalism.* J. Geom. Phys. 180 (2022). arXiv:2111.01162.

[EHS23] C. Elliott, F. Hahner, I. Saberi. *The derived pure spinor superfield formalism as an equivalence of categories.* SIGMA 19 (2023) arXiv:2205.14133

[Hah+22] F. Hahner, S. Noja, I. Saberi, J. Walcher. *Six-dimensional supermultiplets from bundles on projective spaces.* (June 2022) arXiv:2206.08388

[HS23] F. Hahner, I. Saberi. *Eleven-dimensional supergravity as a Calabi–Yau twofold* (April 2023) arXiv:2304.12371

[Hah+23] F. Hahner, S. Raghavendran, I. Saberi, B. R. Williams. *in preparation* (2023)

The presentation of the pure spinor superfield formalism in §2 and its derived generalization in §3 is adapted from the material published in [Eag+22] and [EHS23]. Chapter §6 is based on [EH23] and §7 is adapted from [HS23]. Finally, §8 is based on the forthcoming work [Hah+23].

Acknowledgements

First, I would like to thank my supervisor Johannes Walcher for his guidance, advice, and support throughout this project. I am especially grateful for the freedom and trust I have enjoyed during the course of this work.

A huge thank you goes to Ingmar Saberi with whom I have collaborated on (almost) all of the projects that make up this thesis. Your advice has not only been invaluable to this thesis, but has inspired me in many different ways. I am especially grateful for insightful discussions on topics too numerous to mention, often going far beyond math and physics.

Further, I'd like to thank all the other people who collaborated with me on the papers making up this thesis: Richard Eager, Chris Elliott, Simone Noja, Surya Raghavendran, and Brian Williams. Working with you is a pleasure and each of you has taught me amazing things.

Next, I want to thank Owen Gwilliam for agreeing to be the second referee for this thesis, and Matthias Bartelmann and Hans-Christian Schultz-Coulon for serving on the examination committee.

During the three years of my doctoral studies I had the privilege to travel to various places and to discuss with amazing people. This was only possible due to the hospitality and support of many institutions: LMU München, University of Massachusetts Amherst, Boston University, Perimeter Institute for Theoretical Physics, University of Washington, GGI in Florence, MITP in Mainz, and DESY in Hamburg. Further, I want to thank the Cluster of Excellence Structures in general (and especially the YRC), as well as the DFG for the funding which allowed me to write this thesis. While traveling, I learned a tremendous amount from discussions with a number of amazing people. Here's a (probably incomplete) list: Ilka Brunner, Martin Cederwall, Ivan Contreras, Kevin Costello, Niklas Garner, Owen Gwilliam, John Huerta, and Natalie Paquette. Thank you all!

Further, I want to thank all my fellow PhD students in the Physical Mathematics group for their advice, support, and all the interesting discussions. Thanks to Michael Bleher, Lukas Hahn, Sebastian Nill, Steffen Schmidt and Raphael Senghaas. I would also like to thank Simon Heuveline and Jakob Ulmer for interesting discussions during their visits. A special thanks goes to Nora Schrenk for helping with all the paperwork.

Finally, I would like to thank my friends Tristan Daus, Benjamin Haake, Thomas Junkermann, Anna-Sophie Schäfter, and Julia Wietzel for their wonderful companionship during my time in Heidelberg. I want to express my deepest gratitude to Katrin Lehle who supported me at every step of this journey in a million different ways. Zu guter Letzt möchte ich noch meiner Familie, besonders meinen Eltern, für all die Unterstützung danken, die es mir überhaupt erst ermöglicht hat, diesen Weg zu beschreiten.

Contents

Chapter 1

Introduction

Over the last fifty years, the construction and analysis of supersymmetric field theories has been a rewarding endeavor for both theoretical physics and mathematics. Speaking broadly, there are at least two a priori distinct aspects to this effort. The first concerns the construction and classification *of* supersymmetric field theories and asks for systematic procedures to come up with interesting supersymmetric models as well as their interpretation, especially in terms of supergeometry. On the other hand, the second aspect deals with the extraction of interesting mathematical quantities *from* a given supersymmetric field theory, often by restricting to suitable subsectors. This thesis explores both, eventually making the case for them to be addressed simultaneously.

Regarding the former, the essential difficulty lies in the fact that the supersymmetry transformations typically only define a representation on the space of fields after imposing the equations of motion for fermionic fields. Similarly, if gauge fields are present, the algebra is only represented up to gauge transformations. For the purposes of quantization, it is desirable for the symmetry to act on the full space of fields, without regard to the dynamics; to attain this, one must often pass to a more complicated model, which tends to involve additional "auxiliary fields" that do not change the physics of the theory. Such auxiliary fields may or may not exist, depending on the example in which one is interested. Relatedly, it is pleasing to think of supersymmetry as arising from the action of particular geometric symmetries on an appropriate supermanifold, which amounts to giving "superfield" formulations of supersymmetric theories. Since supersymmetry necessarily acts on the space of superfields, giving such a formulation implicitly requires extending the field content of the theory, in one fashion or another, by some set of auxiliary fields as described above. (The literature on supersymmetry is immense, and we cannot hope to give an overview here; for foundational work in the subject, the reader is referred to the collection of reprints [Fer87], or the early review [Soh85] and references therein.)

Finding systematic techniques to construct superfield formulations and extend the range of theories for which they are available has thus been a subject of great interest. Numerous approaches have been developed, including harmonic superspace [Gal+01] and the "rheonomy" approach to supergravity theories [CdF91]. In the first part of this thesis we address this challenge by means of the pure spinor superfield formalism. Techniques based on the "space of pure spinors" for the construction of supersymmetric multiplets date back more than thirty years ago to papers by Nilsson [Nil86] and Howe [How91a; How91b]. Pure spinors were used to great effect in Berkovits' formulation of the superstring [Ber00], and their applications to superfield formulations have been developed in a wide variety of examples, notably in work of Cederwall and collaborators. See [CNT01; CNT02; BN05] for early papers, and [Ced14] for a review with references to further literature.

There have been numerous related studies of ideas connected to the space of pure spinors, for example by Krotov, Losev and collaborators [Ale+07; KL09] and, notably, in the work of Movshev and Schwarz on ten-dimensional supersymmetric Yang–Mills theory [MS04; Mov05a; Mov05b; MS06; Mov15]. See also the related work of Kapranov [Kap21]. This work was further developed in a mathematical context in [GKR07], in [GS09], and in [Gál+16], where connections to the theory of Koszul duality were emphasized. Relatedly, work by Movshev, Schwarz, and Xu on the Lie algebra cohomology of supersymmetry algebras [MSX12; MSX14] made the appearance of the space of square-zero elements in that context clear.

Remark 1.0.1. It is worth making an orienting remark on terminology at this point. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \Pi\mathfrak{g}_-$ be a super Lie algebra. We refer to the space of odd square-zero elements

$$Y = \{Q \in \mathfrak{g}_- \mid [Q, Q] = 0\} \tag{1.1}$$

as the nilpotence variety of $\mathfrak{g}$ [ESW21]. In the physics literature, the term "pure spinor" is often used to refer to points in (or coordinates on) the nilpotence variety. On the other hand, pure spinors, in the sense of Cartan and Chevalley [Che54], are defined to be elements of the spin representation for which the dimension of the annihilator under Clifford multiplication is maximal; the pure spinors form the minimal orbit in the action of the spin group on the (projectivized) spin representation. If $\mathfrak{g}$ is a super Poincaré algebra, these terms are closely related though not identical: pure spinors have vanishing self-brackets in the super Poincaré algebra [ESW21; ES19b], but the converse is not always true. In ten-dimensional minimal supersymmetry, which was the first example studied [Nil86] from this angle, the odd elements of the super Poincaré algebra consist of a single spin representation and the bracket is defined by Clifford multiplication such that the nilpotence variety coincides with the space of pure spinors. This coincidence is responsible for the confusing terminology; the usage of "pure spinor formalism" in this thesis is historically, rather than logically, motivated.

Concerning the extraction of interesting mathematical quantities from a given supersymmetric field theory, the crucial observation is that such theories posses protected subsectors that preserve parts of the supersymmetry and whose observables measure interesting mathematical quantities of the underlying spacetime manifold. Such subsectors can be systematically extracted by twisting. To twist a given supersymmetric field theory, one fixes an odd square-zero symmetry of the theory and then takes invariants by the one-dimensional odd abelian super Lie algebra spanned by the chosen supercharge [Cos13a]. The twisted theory then describes a subsector of the full theory and, for twisting supercharges from the super Poincaré algebra, it is sensitive to underlying topological or holomorphic structures on the spacetime manifold. This gives a way to access invariants of manifolds as observables in twisted supersymmetric field theories, thereby achieving the promised extraction. (The prime example of this procedure is Witten's celebrated expression of the Donaldson invariants of smooth four-manifolds using four-dimensional $\mathcal{N} = 2$ supersymmetry [Wit88]). Even more, as supersymmetric theories are related by a fascinating web of dualities, the subsectors described by their twisted versions are in correspondence too. With the twisted theories being mathematically tractable, twisting on both sides of duality can establish new relations between the mathematical structures on either side. This procedure has led to many fascinating insights originating from a variety of different physical dualities (for example from mirror symmetry, phrased as a duality between two topological phases of string theory extracted by twisting [Hor+03], or the applications of S-duality to the geometric Langlands program [KW07], just to name two); the AdS/CFT correspondence, however, until recently, has mostly been fence-sitting regarding this program. In part this was due to a lacking understanding of twisted supergravity theories as compared to twisted gauge theories. Crucially, the definition of twisted supergravity theories is more subtle than the one of twisted supersymmetric field theories. These subtleties were discussed and resolved in [CL16]. As supergravity theories in ten dimensions arise as low energy effective field theories from various string theories, their twisted cousins are closely related to topological string theory which arises by a twist procedure on the worldsheet. Using this line of reasoning conjectures for twisted supergravity theories were provided and studied through the lens of twisted holography [CL16; Cos17; Cos16; CG18]. Eleven-dimensional supergravity [CJS78] is the low energy limit of M-theory, a conjectural theory believed to unify the different superstring theories [Wit95]. In [Cos16] investigated eleven-dimensional supergravity in the omega background and conjectured a link between the maximal twist and Poisson–Chern–Simons theory.

In this thesis, we develop the pure spinor superfield formalism as a systematic tool for the construction of supersymmetric multiplets. We apply it to a variety of problems, in particular we use it to study twisted supergravity theories and provide powerful target space techniques to directly study the twisted eleven-dimensional supergravity theory in target space, without relying on any relation to the worldsheet models. We will see that this closely links the superspace methods employed in the construction of these theories and the computation of their twists.

1.1 Main results of this thesis

The first part of this thesis (§2-4) address the construction by means of the pure spinor superfield formalism. In the second part (§6-8), we use these tools to study twisted supergravity theories and their holographic duals from a target space perspective.

Let $\mathfrak{n}$ be a supertranslation algebra acted on by a Lie algebra $\mathfrak{g}_0$ and let $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{n}$. We will call such super Lie algebras of super Poincaré type (see Definition 2.3.2 for details) and set the up pure spinor superfield formalism for such algebras.

The (derived) pure spinor superfield formalism. The first chapter of this thesis develops the pure spinor superfield formalism as a systematic framework for the construction of supersymmetric multiplet from modules over the ring of functions on the nilpotence variety. We develop suitable computational techniques by means of homological algebra and homotopy transfer and demonstrate these in various examples. We establish links between the algebraic geometry of the nilpotence variety and the physics of supermultiplet; here the Gorenstein and Cohen–Macaulay properties play an important role.

In §3, we construct a derived generalization of the formalism. We define a category of multiplets and show how the derived formalism establishes an equivalence with the category of $C^\bullet(\mathfrak{n})$-modules. The main result of this chapter says the following.

Theorem (Theorem 3.4.3). *There is an equivalence of dg-categories*

$$A^\bullet : \mathsf{Mult}_{\mathfrak{g}} \leftrightarrows \mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})} : C^\bullet \tag{1.2}$$

between $\mathfrak{g}$-multiplets and $\mathfrak{g}_0$-equivariant modules over the Chevalley–Eilenberg algebra $C^\bullet(\mathfrak{n})$. Here $A^\bullet$ denotes the pure spinor functor, and the inverse functor $C^\bullet = C^\bullet(\mathfrak{n}, -)$ is the functor taking derived invariants with respect to the supertranslation algebra $\mathfrak{n}$.

This equivalence of categories is closely related to Koszul duality. We demonstrate the formalism in several examples and use it to answer some questions on the ordinary (underived) pure spinor superfield formalism.

Supermultiplets in six-dimensions: a case study. The projective nilpotence variety for six-dimensional $\mathcal{N} = (1, 0)$ supersymmetry is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^3$. As the geometry of this space is very well understood, we use this example to provide an extensive case study for the link between algebraic geometry and multiplets provided by the pure spinor superfield formalism. Line bundles on $\mathbb{P}^1 \times \mathbb{P}^3$ are easy to understand. Describing the associated multiplets, we provide the following classification result.

Theorem (Theorem 4.3.1)**.** *The multiplets described in §4.3.2 and §4.3.3 classify, up to quasi-isomorphism, all mutiplets for six-dimensional $\mathcal{N} = (1, 0)$ superymmetry whose derived invariants are graded global section modules of line bundles over the projective nilpotence variety.*

These multiplets also have a natural interpretation in terms of their twists: We will see that one can think of these multiplets as being those whose holomorphic twists have rank one over Dolbeault forms on spacetime.

In addition, we explicitly construct multiplets associated to natural higher-rank equivariant vector bundles, including the tangent and normal bundles as well as their duals. Among the multiplets constructed are the vector multiplet and hypermultiplet, the family of $\mathcal{O}(n)$-multiplets, and the supergravity and gravitino multiplets. Along the way, we tackle various theoretical problems within the pure spinor superfield formalism. In particular, we give some general discussion about the relation of the projective nilpotence variety to multiplets and prove general results on short exact sequences and dualities of sheaves in the context of the pure spinor superfield formalism.

Eleven-dimensional supergravity and its twists. The chapters §6 and §7 deal with (twisted) eleven-dimensional supergravity from a target space perspective.

First, we compute the maximal twist of eleven-dimensional supergravity directly in the component field formulation and thereby verify a conjecture by Costello in the free limit. To this end, we briefly review how the eleven-dimensional supergravity theory arises from the structure sheaf on the nilpotence variety in the pure spinor superfield formalism and describe the L_∞ action of the super Poincaré algebra on the component fields. Starting from these supersymmetry transformations we explicitly compute the maximal twist of the free theory and show that the it matches with Poisson–Chern–Simons theory in the free limit.

In §7, we construct a generalization of Poisson–Chern–Simons theory, defined on any supermanifold equipped with an appropriate filtration of the tangent bundle. This construction recovers interacting eleven-dimensional supergravity in Cederwall's pure spinor formulation, as well as all possible twists of the theory, and does so in a uniform and geometric fashion. Under the assumption of compatibility between the pure spinor superfield formalism and twisting (for which we provide evidence) this proves that Costello's description of the maximal twist is the twist of eleven-dimensional supergravity in its pure spinor description. It also provides a pure spinor lift of the interactions in the minimally twisted theory (for which a conjectural form was given in [RSW23]).

The following picture gives an overview of the different twist calculations of eleven-dimensional supergravity (both at pure spinor cochain and component field level) which are carried out in the literature and in this thesis.

$$\begin{tikzcd}
\boxed{\begin{matrix}\text{Pure spinor}\\ \text{supergravity}\\ \text{(Cederwall; §7.5.4)}\end{matrix}} \arrow[r, "\text{[SW21]}", "\text{(free limit)}"'] \arrow[d, "\begin{matrix}\text{Homotopy}\\ \text{Transfer}\end{matrix}"] & \boxed{\begin{matrix}\text{Minimal}\\ \text{pure spinor twist}\\ \text{(§7.5.3)}\end{matrix}} \arrow[r] \arrow[d, "\begin{matrix}\text{Homotopy}\\ \text{Transfer}\end{matrix}"] & \boxed{\begin{matrix}\text{Maximal}\\ \text{pure spinor twist}\\ \text{(§7.5.2)}\end{matrix}} \arrow[d, "\cong"] \\
\boxed{\begin{matrix}\text{Supergravity}\\ \text{on } \mathbb{R}^{11}\\ \text{(Cremmer–Julia–Scherk)}\end{matrix}} \arrow[r] \arrow[rr, bend right, "\text{§6 (free limit)}"'] & \boxed{\begin{matrix}\text{Minimal twist}\\ \text{on } \mathbb{C}^5 \times \mathbb{R}\\ \text{(Rhagavendran–Saberi–Williams)}\end{matrix}} \arrow[r, "\text{[RSW23]}"] & \boxed{\begin{matrix}\text{Poisson–Chern–Simons}\\ \text{theory on } \mathbb{R}^7 \times \mathbb{C}^2\\ \text{(Costello)}\end{matrix}}
\end{tikzcd}$$

FIGURE 1.1: Twisting eleven-dimensional supergravity using pure spinor superfields. In §7, we construct the three upper nodes of the diagram showing that they are homotopy Poisson–Chern–Simons theories. Conjecture 7.4.3 implies that the upper three nodes are twists of each other. The maximal twist in component fields is computed in the free limit in §6.

.

(Twisted) (2,0) supersymmetry and holography. M-theory allows for M2 and M5 branes. The worldvolume theory on a (stack of) M5 brane is a six-dimensional superconformal field theory with (2,0) supersymmetry and in the framework of AdS_7/CFT_6-correspondence one expects an equivalence to eleven-dimensional supergravity in backreacted geometry. Specifically, Kaluza-Klein modes of the supergravity fields are dual to CFT operators in the worldvolume theory. In §8, we explore a schematic picture of this correspondence by starting from well-established results in the maximal twist together with the structural insights that the pure spinor superfield formalism provides on twisting.

We provide a novel pure spinor construction for the decomposition of minimally twisted supergravity into $E(3|6)$-modules both at component field level and at pure spinor cochain level. The construction hints towards a uniform picture appearing in the maximally, minimally and untwisted cases which we sketch out schematically. In the process, we find a systematic way to construct conformal supergravity theories.

Chapter 2

The pure spinor superfield formalism

2.1 Introduction

Speaking broadly, a classical field theory concerns itself with the study of the sheaf of solutions to particular partial differential equations on the spacetime manifold, or more properly on the site of manifolds equipped with appropriate structure. Over an open set U, one considers solutions to the equations of motion of the theory on U, considered up to gauge equivalence; since the equations of motion that are of physical interest tend to arise from variational principles, we will refer to it with the suggestive notation $\mathrm{Crit}(S)/\mathcal{G}$, where S refers to the action functional and $\mathcal{G}$ to the group of local gauge transformations.

In general, this sheaf has several properties: First and foremost, its sections over U can be thought of as a covariant version of the phase space associated to ∂U [Crn88], and thus have the structure of a symplectic space. (We are passing over numerous technical subtleties in silence; in particular, degeneracies of various kinds can and do occur, notably in the theory of constrained systems. Such examples arise naturally in our context [CNT02; SW23b], though we do not treat degeneracies in any detail here.) As already indicated above, it may not consist just of the space of solutions to the equations of motion, but of its quotient by gauge equivalences. Lastly, since the degrees of freedom of many quantum field theories include fermions, it should most properly be understood as a (possibly singular, stacky, or infinite-dimensional) supermanifold or graded space.

In studying field theories, *symmetries* play a crucial role. Let $\mathfrak{g}$ be a sheaf of Lie algebras. A classical theory has a symmetry by $\mathfrak{g}$ when its sheaf of fields is equipped with a local action of the sheaf $\mathfrak{g}$ by infinitesimal automorphisms. (We will make this more precise in §2.2 below.) Usually, $\mathfrak{g}$ is either a constant or a locally free sheaf (though other examples are possible, notably in holomorphic field theories). In the former case, one refers to a "global" symmetry, and in the latter to a "local" symmetry. By Noether's second theorem, local symmetries correspond to degeneracies in the variational problem of precisely the kind we ruled out above; as such, local symmetries are usually only relevant when gauged, and the terms "local symmetry" and "gauge symmetry" are often used interchangeably.[1] Examples of symmetries abound; for example, any field theory on affine space should admit the Lie algebra of infinitesimal affine transformations (the Poincaré algebra) as a symmetry, reflecting the coordinate invariance (homogeneity and isotropy) of its dynamics.

Since fermions are typically present in the theory, $\mathrm{Vect}(\mathrm{Crit}(S)/\mathcal{G})$ is most naturally not a Lie algebra, but a *graded* or *super* Lie algebra. The most important examples of super Lie algebras extend the Poincaré symmetry by odd spacetime symmetries transforming in the spin representation of the Lorentz group; a field theory that admits an action of such an algebra is called *supersymmetric*. The problem of constructing supersymmetric field theories has a long history in physics, dating back to the first explorations of the subject in the seventies [GL71; GS71; VA72].

It is common wisdom in physics that representations of supersymmetry algebras in typical field theory models can be quite intricate. Often, the supersymmetry algebra closes only on-shell or up to gauge transformations. In other words, while a symmetry of the theory in the above sense can be defined, it does not arise in a straightforward manner from an action on the larger space of fields inside of which the equations of motion are solved. This leads, among other issues, to difficulties in quantizing the theory.

In typical field theory models the structure of supersymmetry transformations roughly falls into four distinct cases:

— There is a set of fields on which the supersymmetry algebra is represented on the nose. This is the case, for example, for the four-dimensional $\mathcal{N} = 1$ chiral multiplet.

— The supersymmetry algebra is only represented after taking the quotient by the action of the gauge group. This happens, for example, for the four-dimensional vector multiplet.

— The supersymmetry algebra is represented only after imposing the equations of motion. Here, the six-dimensional hypermultiplet is an example.

[1] For a field theory whose physical fields are of the form $\mathrm{Crit}(S)/\mathcal{G}$ as above, the Lie algebra of $\mathcal{G}$ is a locally free sheaf of Lie algebras that acts on the degenerate variational problem $\mathrm{Crit}(S)$.

— The supersymmetry algebra is represented only after taking the quotient by gauge transformation and imposing the equations of motion. This most general case appears in ten-dimensional super Yang–Mills theory, among other examples.

The first objective of this note is to formalize these considerations using the language of homotopical algebra; we work in the context of the BRST and BV formalisms, which seek to respectively replace the quotient by gauge symmetries and the imposition of equations of motion by appropriate derived analogues. In §2.2, we set up some necessary preliminaries for this context; in particular, we give a definition of a multiplet that is designed to capture all these different aspects of symmetry in our context.

Once this terminology is established, we turn our attention towards the construction of supermultiplets via the pure spinor superfield formalism; see [Ber01], and especially the review [Ced14] and references therein. Our perspective is somewhat nontraditional. In §2.3 we set up the formalism in a generalized setting (without restricting to supersymmetry algebras of physical interest), clarify its relation to various standard constructions in homological algebra, and give an explicit account of calculational techniques from commutative algebra.

In our interpretation, which builds on that in [ESW21], the pure spinor superfield formalism constructs a supermultiplet out of the datum of an equivariant module over the ring of functions on the nilpotence variety Y of the relevant superalgebra. Speaking roughly, the output of the formalism is a rather large cochain complex that is automatically equipped with a strict action of the supersymmetry algebra—indeed, which is quasi-isomorphic to a standard component-field description of the multiplet in the BRST or BV formalism, but which is free over *superspace* rather than just over the spacetime manifold. We can then recover the usual component-field description by moving from this large resolution to a smaller, quasi-isomorphic cochain complex of vector bundles over spacetime, which is in a certain sense the "minimal" resolution of this kind. A particular filtration on the pure spinor cochain complex produces the component-field formulation in canonical fashion; the set of component fields is identified with the vector bundle associated to the representation of Lorentz and R-symmetry on the Koszul homology of the input module.

One can then transfer the various structures present on the large complex to the component fields, using the homotopy transfer theorem. As we will see, this procedure links the component field description of the multiplet closely to the minimal free resolution of the equivariant module over the ambient polynomial ring. In particular, we find that all supersymmetry transformations without spacetime derivatives can be read off directly from the resolution differential. (This was conjectured by Berkovits in [Ber02].)

Given our presentation of the pure spinor superfield formalism, it is natural to ask questions how algebraic properties of $\mathcal{O}_Y$-modules are related to physical properties of the resulting multiplet. In §2.4, we point out that the Gorenstein property ensures the existence of a pairing on the multiplet; this pairing, however, can admit various different physical interpretations. We furthermore study dualizing modules and explain how the Cohen–Macaulay property is related to antifield multiplets.

Throughout the text we illustrate the procedure with examples in different dimensions and with various amounts of supersymmetry. In particular, we provide a detailed discussion of ten-dimensional super Yang–Mills theory showing how all the different structures present in the component field formulation arise via homotopy transfer.

2.2 Preliminaries

2.2.1 Gradings and basic definitions

Many objects appearing throughout this work (be they vector spaces, vector bundles, associative algebras, or Lie algebras) will carry a grading by $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as well as a differential of bidegree $(1, +)$. We will use the abbreviation "dgs," for "differential graded super," to refer to objects of this sort, at least for emphasis.

Definition 2.2.1. A *dgs vector space* is a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded vector space $E^\bullet$, equipped with a square-zero differential d of bidegree $(1, +)$. Equivalently, $E^\bullet$ is a cochain complex in the category of super vector spaces. We can thus write

$$E = \bigoplus_{k \in \mathbb{Z}} \left(E^k_+ \oplus \Pi E^k_- \right) [-k]. \tag{2.1}$$

The *total parity* $|v| \in \mathbb{Z}/2\mathbb{Z}$ of a homogeneous element $v \in E^n$ is defined by

$$|v| = \begin{cases} n \bmod 2, & v \in E^n_+; \\ n + 1 \bmod 2, & v \in E^n_-. \end{cases} \tag{2.2}$$

We remark that each of these gradings has a clear physical meaning: the integer grading corresponds to the ghost number or cohomological degree, whereas the $\mathbb{Z}/2\mathbb{Z}$ grading corresponds to the intrinsic parity (fermion number modulo two). The total parity denotes the $\mathbb{Z}/2\mathbb{Z}$-grading which arises by forgetting the $\mathbb{Z}$-grading to $\mathbb{Z}/2\mathbb{Z}$ and then totalizing with the intrinsic parity. It is the total parity which governs all signs.

It will often be convenient to introduce an additional piece of data: an auxiliary action of a one-dimensional abelian Lie algebra $\mathbb{R}$ on a dgs vector space. Most results will make sense for arbitrary $\mathbb{R}$-equivariant dgs vector spaces, but our most common examples will have the following behavior.

Definition 2.2.2. A *lifted* dgs vector space is an $\mathbb{R}$-equivariant dgs vector space where the $\mathbb{R}$-action has integral weights, and the $\mathbb{Z}/2\mathbb{Z}$-grading coincides with the $\mathbb{R}$-weight modulo two. We will use round brackets to refer to shifts of the weight grading and reserve square brackets for shifts in cohomological degree.

Let us now introduce some notation that we will use when we discuss lifted dgs vector spaces (and later lifted dgs vector bundles on a smooth manifold X). Let us write

$$E = \bigoplus_{(w,k)\in\mathbb{Z}^2} E^{w,k}(-w)[-k] \tag{2.3}$$

for a lifted dgs vector space, where the first index indicates the decomposition of E by $\mathbb{R}$-weight, and the second index indicates cohomological $\mathbb{Z}$-degree on E. In some of our calculations, we will place the summands in a two-dimensional array, where the two coordinates are given by $w - k$ and k. For example,

$$E = \begin{bmatrix} \cdots & E^{-1,0} & E^{0,0} & E^{1,0} & E^{2,0} & \cdots \\ \cdots & E^{0,1} & E^{1,1} & E^{2,1} & E^{3,1} & \cdots \\ \cdots & E^{1,2} & E^{2,2} & E^{3,2} & E^{4,2} & \cdots \end{bmatrix}. \tag{2.4}$$

such that we recover w as the total degree with respect to this sheared bigrading. The overall parity is then determined by the column. If the differential on our dgs vector bundle decomposes as a sum of homogeneous differential operators of order k, say $D = \sum_{k\geq 0} D_k$, the summand D_k acts—with respect to the sheared grading—with bidegree $(2k-1, 1)$. So the homogeneous summands all increase the vertical degree by 1, but they may modify the horizontal degree by any odd integer ≥ -1.

Definition 2.2.3. A *commutative dgs algebra*, or cdgsa, is a dgs vector space $A^\bullet$ equipped with a bilinear multiplication

$$m_2 : A^\bullet \otimes A^\bullet \to A^\bullet. \tag{2.5}$$

The multiplication is required to be a cochain map of bidegree $(0, +)$; furthermore, it should be commutative with respect to the Koszul sign rule determined by the total parity. That is,

$$ab = (-1)^{|a||b|} ba. \tag{2.6}$$

We remark that a cdgsa is a commutative differential graded algebra in the category of super vector spaces. There is also an obvious notion of a lift of a cdgsa, such that a lifted cdgsa is a commutative differential graded algebra in the category of graded vector spaces. Finally, we can extend our definitions to encompass super A_∞ algebras: a (lifted) super A_∞ algebra $A^\bullet$ is an A_∞ algebra in the category of super (or graded) vector spaces. That is, it is a collection

$$A^\bullet = \bigoplus_{k\in\mathbb{Z}} A^k[-k] \tag{2.7}$$

of super (or graded) vector spaces, equipped with maps m_n of arity n and bidegree $(2-n,+)$ (or $(2-n,0)$) that satisfy the usual A_∞ relations.

Example 2.2.4. Let $V^\bullet$ be a dgs vector space. The *polynomial algebra* $\mathrm{Sym}(V^\bullet)$ is the free dgs-commutative algebra generated by $V^\bullet$. Concretely, it is the quotient

$$\mathrm{Sym}(V^\bullet) = T(V^\bullet)/(xy - (-1)^{|x||y|}yx) \tag{2.8}$$

of the tensor algebra by the ideal generated by all (anti)commutators of homogeneous elements, where (anti)commutativity is determined by the Koszul sign rule for the total parity.

Of course, all of the notions we have introduced for associative algebras have parallels for Lie algebras, which we now quickly introduce. Let $x_1, \dots, x_n$ be homogeneous elements of a dgs vector space $V^\bullet$, and $\sigma \in S_n$ a permutation. Then the *Koszul sign* $\epsilon(x_1, \dots, x_n; \sigma)$ of the permutation is defined by the relation

$$x_1 \cdots x_n = \epsilon(x_1, \dots, x_n; \sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}. \tag{2.9}$$

in the algebra $\mathrm{Sym}(V^\bullet)$. Furthermore define $\chi(\sigma) = (-1)^{\mathrm{sgn}(\sigma)}\epsilon(x_1, \dots, x_n; \sigma)$.

Definition 2.2.5. Let $\mathfrak{g}$ be a (lifted) dgs vector space. A *(lifted) super L_∞ algebra structure* on $\mathfrak{g}$ is a collection of multilinear maps

$$\mu_k \colon \mathfrak{g}^{\times k} \to \mathfrak{g} \tag{2.10}$$

for $k \geq 1$, of bidegree $(2-k,+)$ (or $(2-k,0)$, respectively), such that the following two conditions hold:

(1) Graded skew symmetry. For all $\sigma \in S_k$, $x_i \in \mathfrak{g}$ one has

$$\mu_k\left(x_{\sigma(1)}, \dots, x_{\sigma(k)}\right) = \chi(\sigma)\mu_k\left(x_1, \dots, x_k\right) . \tag{2.11}$$

(2) Higher Jacobi identities. For all $x_i \in \mathfrak{g}$ one has

$$\sum_{i+j=k+1} \sum_{\sigma\in S(i;k)} (-1)^{i(j-1)}\chi(\sigma)\mu_j\left(\mu_i\left(x_{\sigma(1)}, \dots, x_{\sigma(i)}\right), x_{\sigma(i+1)}, \dots, x_{\sigma(k)}\right) = 0\,. \tag{2.12}$$

Here $S(i;k) \subset S_k$ denotes all permutations such that $\sigma(1) \leq \cdots \leq \sigma(i)$ and $\sigma(i+1) \leq \cdots \leq \sigma(k)$. We remark that a (lifted) super L_∞ algebra is just an L_∞ algebra in super (respectively, in graded) vector spaces. We further remark that the datum of a (lifted) super L_∞ algebra structure is equivalent to a square-zero derivation of bidegree $(1,+)$ (or $(1,0)$ in the lifted case) on the free dgs commutative algebra $\mathrm{Sym}(\mathfrak{g}^\vee[-1])$. This derivation $\mathrm{d}_\mathfrak{g}$ defines the complex computing Lie algebra cohomology,

$$C^\bullet(\mathfrak{g}) := \left(\mathrm{Sym}(\mathfrak{g}^\vee[-1])\,,\, \mathrm{d}_\mathfrak{g}\right). \tag{2.13}$$

The shift is with respect to the cohomological degree.

There are some special cases of this definition that we point out. When $\mathfrak{g}$ is supported purely in even parity, we recover the ordinary notion of an L_∞ algebra [HS93; LM95]. On the other hand, when $\mathfrak{g}$ is supported in cohomological degree zero, we recover the notion of a super Lie algebra (or, in the lifted case, a graded Lie algebra). When $\mu_k = 0$ for all $k > 2$, we obtain the notion of a *dg super Lie algebra.*

Example 2.2.6. Let $V^\bullet$ be a (lifted) dgs vector space. Then $\mathrm{End}(V^\bullet)$ is a dg super Lie algebra; the bracket μ_2 is given by the commutator

$$\mu_2(x,y) = [x,y] = xy - (-1)^{|x||y|}yx \tag{2.14}$$

whereas the differential arises via

$$d_{\mathrm{End}(V^\bullet)} = [d, -]\,. \tag{2.15}$$

We remark that $\mathrm{End}(V^\bullet)$ is in fact naturally a dgs associative algebra; the dgs Lie structure is obtained by applying the usual forgetful functor.

Definition 2.2.7. A L_∞ *map* between super L_∞ algebras

$$\Phi\colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$$

is a map of graded super commutative algebras

$$\Phi^* : C^\bullet(\mathfrak{h}) \to C^\bullet(\mathfrak{g}). \tag{2.16}$$

that preserves the augmentation map to constants in degree zero.

Definition 2.2.8. Let $\mathfrak{g}$ be a super L_∞ algebra. An L_∞ *dgs module* is a dgs vector space $V^\bullet$, together with an L_∞ map

$$\mathfrak{g} \rightsquigarrow \mathrm{End}(V^\bullet). \tag{2.17}$$

2.2.2 Homotopy transfer

We will repeatedly make use of the *homotopy transfer theorem* in various contexts. We refrain from giving a general review of homotopy algebraic structures here; the reader is referred to [LV12a; Val14; LM95]. Nonetheless, we will quickly recall the general idea.

It is common knowledge that various mathematical objects—for example sheaves or modules—admit interesting "higher structures." This might include higher sheaf cohomology groups, for example, or more generally other derived functors such as Ext and Tor. These higher structures originate, in some sense, from the "constraints" imposed on these objects: for example, the failure of a module to be free.

To compute higher derived functors, one technique is to replace the object one wants to study by a "resolution." This is a cochain complex of simpler objects (for example, free modules) that is quasi-isomorphic to the complicated object one wants to study. In derived geometry, one views this cochain complex as a replacement of the underlying object.

Just as the equations defining a non-free module lead to higher structures and need to be resolved, many algebraic structures are defined by collections of structure morphisms that satisfy certain strict equations. (For example, one requires associativity in the form $((ab)c) = (a(bc))$, or the Jacobi identity for a Lie bracket.) When such equations are imposed in a cochain complex, they do not play well with homotopy-theoretic operations or notions of equivalence such as quasi-isomorphism. The remedy consists of "resolving" the equations that are imposed on the defining maps of the algebraic structure. In technical language, one resolves the operad defining the algebraic structure one is interested in by a free dg operad. (See [Mar98] for discussion of this perspective.)

There is then a collection of general results, which state that a homotopy algebraic structure may be transferred along *homotopy data* between two quasi-isomorphic cochain complexes. For our purposes, this data is typically provided by a deformation retract (though more general situations are possible [LV12a]), i.e. a diagram

$$\begin{tikzcd} (C^\bullet, \mathrm{d}_C) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (D^\bullet, \mathrm{d}_D) \arrow[l, shift left, "i"] \end{tikzcd} \tag{2.18}$$

with the maps satisfying

$$p \circ i = \mathrm{id}_{C^\bullet} \qquad \text{and} \qquad i \circ p - \mathrm{id}_{D^\bullet} = \mathrm{d}_C \circ h + h \circ \mathrm{d}_C. \tag{2.19}$$

In many examples, the complex on the right hand side will simply be the cohomology of the complex on the left. The transfer of the homotopy algebraic structure can then be obtained in a systematic way by by summing over marked trees in a consistent fashion. Here, vertices are to be labeled with operations of the structure to be transferred, and internal edges with the homotopy. We remark, by [LV12a, Theorem 10.3.15], that the transferred structure is independent of the choice of homotopy data up to isomorphism.

The phenomenon of homotopy transfer is very broad, and encompasses many examples from throughout mathematics, both more and less familiar. We mention some examples:

— A cochain complex is defined by a grading, together with a single endomorphism D of degree $+1$, satisfying the equation $D^2 = 0$. A cochain complex in cochain complexes is a bicomplex: we give a second grading on $(C^\bullet, d)$, together with a square-zero cochain map D. Resolving the equation $D^2 = 0$ gives rise to an operad known as the D_∞ operad: it encodes a sequence of maps D_i of bidegree $(1-i, i)$, which obey the relations

$$dD_n + (-1)^n D_n d = \sum_{i+j=n} (-1)^i D_i D_j. \tag{2.20}$$

Homotopy transfer of D to $H^\bullet(C, d)$ generates a D_∞ module structure whose constituent maps encode the higher differentials of the spectral sequence of the bicomplex. This will play a role for us in describing the relation of pure spinor superfields to their component-field descriptions; see §2.3.

— The operad governing associative algebras is resolved by the A_∞ operad, which has operations $\{m_n\}$ of arity n and degree $2-n$ for all $n \geq 1$. Similarly, the operad governing Lie algebras is resolved by the L_∞ operad, which has bracket operations μ_n of arity n and degree $2-n$ for all $n \geq 1$ as we reviewed explicitly above. For example, transferring the associative algebra structure on de Rham forms to cohomology produces an A_∞ structure with vanishing m_1 and m_2 the ordinary cup product. Higher m_n's correspond to the classical Massey product operations.

— In the BV formalism, a perturbative classical field theory is described by a cyclic local L_∞ algebra whose differential encodes the linearized equations of motion and gauge invariances of the free theory. Homotopy transfer to the cohomology of the differential is related to the interaction picture in quantum field theory; the diagrams that describe the transferred L_∞ structure on on-shell states are precisely tree-level Feynman diagrams, where the homotopy is the Feynman propagator. The operations of the transferred L_∞ structure correspond to tree-level amplitudes [Kaj07; MSW19]. Homotopy transfer of L_∞ structures will be relevant for us when discussing interactions for pure spinor superfields and their relation to the component-field formalism; see §2.5 for an example.

— In the BV or BRST formalism, the symmetries of a field theory are encoded as L_∞ module structures on the complex of fields. Moving to another quasi isomorphic complex of fields (e.g by integrating out an auxiliary field), one can obtain the new module structure via homotopy transfer. We will use this to derive the action of the supersymmetry algebra on the component fields in the pure spinor superfield formalism. An explicit account on the homotopy transfer for module structures is given in Appendix 2.A.

2.2.3 Maurer–Cartan elements and nilpotence varieties

We recall that the Maurer–Cartan equation in an L_∞ algebra $(\mathfrak{g}, \mu_k)$ takes the form

$$\sum_{k\geq 1} \frac{1}{k!}\mu_k\,(x, \ldots, x) = 0. \tag{2.21}$$

Here $x \in \mathfrak{g}$ is an element of degree one; each of the terms in the above equation thus carries degree two. We can clearly generalize this definition to super L_∞ algebras by asking for Maurer–Cartan elements x of bidegree $(1, +)$. Maurer–Cartan elements of this form define deformations of the super L_∞ algebra structure; nontrivial deformations are classified by Maurer–Cartan elements up to gauge equivalence. Nonetheless, we will write $\mathrm{MC}(\mathfrak{g})$ for the naive space of Maurer–Cartan elements; in other words, we do not pass to the space of gauge equivalence classes, preferring to think of $\mathrm{MC}(\mathfrak{g})$ as a space equipped with a $\mathfrak{g}_0$-action by vector fields.

Now, given any super L_∞ algebra, we can forget the $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-grading down to a $\mathbb{Z}/2\mathbb{Z}$-grading by remembering only the total parity. This is enough information to define the appropriate Koszul signs, and μ_k is then simply a multilinear operation with appropriate symmetry properties and parity $(-1)^k$. We will call the resulting object a $\mathbb{Z}/2\mathbb{Z}$-graded L_∞ algebra. We can then ask about the space $Y_\mathfrak{g}$ of *odd* elements satisfying the Maurer–Cartan equation (2.21). Elements of this space will correspond to deformations of $\mathfrak{g}$ as a $\mathbb{Z}/2\mathbb{Z}$-graded L_∞ algebra and there will be an injective map $\mathrm{MC}(\mathfrak{g}) \hookrightarrow Y_\mathfrak{g}$. We call $Y_\mathfrak{g}$ the *nilpotence variety* of $\mathfrak{g}$; when $\mathfrak{g}$ is a super Lie algebra, this agrees with the notion given in [ESW21]. (Whenever there is no danger of confusion we drop the subscript $\mathfrak{g}$.)

For our purposes, Y is an affine scheme; we take

$$Y = \operatorname{Spec} R/I \tag{2.22}$$

where $R = \mathrm{Sym}(\mathfrak{g}_-^\vee)$ is a polynomial ring in commuting variables, and I is the ideal generated by the Maurer–Cartan equations (2.21). In this work we will be mostly concerned with the case where $\mathfrak{g}$ is a super Lie algebra, thus I will be generated by quadratic equations and R/I is a graded ring. Since we view Y as an affine scheme, we will move back and forth freely between discussing the geometry of Y and the graded ring R/I; hopefully, no confusion should arise. Sometimes we may also consider the geometry of the projective scheme $\operatorname{Proj} R/I$ (in particular in §4); in either case, the essential object is the graded ring R/I. The distinction between a variety and a scheme will, in fact, play a role in applications; see §2.3.9.

2.2.4 Multiplets and local modules

In this section, we move towards the setting of field theory by introducing the new ingredient of *locality.*

Local modules. Let X be a manifold thought of as spacetime. By a *dgs vector bundle* on X, we mean a $\mathbb{Z} \times \mathbb{Z}/2$-graded vector bundle

$$E = \bigoplus_{k \in \mathbb{Z}} \left(E_+^k \oplus \Pi E_-^k \right) [-k] \tag{2.23}$$

equipped with a collection of differential operators $D \colon \mathcal{E}_\pm^k \to \mathcal{E}_\pm^{k+1}$ such that $D \circ D = 0$.[2] Here, $\mathcal{E}_\pm^k = \Gamma(X, E_\pm^k)$ denotes the C^∞-sections of $E_\pm^k$.

Suppose that $\mathfrak{g}$ is a super L_∞ algebra. We will define a local $\mathfrak{g}$-module to be a dgs vector bundle on X equipped with a sufficiently local homotopy action of $\mathfrak{g}$.

To give the precise definition we first need a small bit of background. Consider the $\mathbb{Z} \times \mathbb{Z}/2$-graded vector space $\mathcal{E} = \Gamma(X, E)$. As explained in Example 2.2.6, the endomorphisms $\operatorname{End}(\mathcal{E})$ naturally form a dg super Lie algebra: the structure maps consist of the commutator and the differential $[D, -]$. Inside $(\operatorname{End}(\mathcal{E}), [D, -])$ there is a sub dg super Lie algebra consisting of all endomorphisms which are differential operators. We will denote it by $(\mathcal{D}(E), [D, -])$.

Definition 2.2.9. A *local (super L_∞) $\mathfrak{g}$-module* is a dgs vector bundle (E, D) equipped with a super L_∞-map (see Definition 2.2.7):

$$\rho \colon \mathfrak{g} \rightsquigarrow \left(\mathcal{D}(E)\, ,\ [D, -] \right) . \tag{2.24}$$

We will refer to the data of a local $\mathfrak{g}$-module by a triple (E, D, ρ).

[2] We assume that each graded piece E^k is a finite rank vector bundle, but the total rank of E may still be infinite.

The space of sections of any dgs vector bundle is a dgs vector space. The space of sections (over any open set) of a local $\mathfrak{g}$-module $(\mathcal{E}, D)$ is a dgs L_∞ module for the super Lie algebra $\mathfrak{g}$, see Definition 2.2.8.

Concretely, the data of ρ consists of a collection of maps

$$\rho^{(j)} : \mathfrak{g}^{\otimes j} \longrightarrow \mathcal{D}(E), \qquad j \geq 1 \tag{2.25}$$

of cohomological degree $1-j$. These satisfy compatibility relations, the lowest of which reads

$$[\rho^{(1)}(x), \rho^{(1)}(y)] - \rho^{(1)}([x,y]) = [D, \rho^{(2)}(x,y)] \,. \tag{2.26}$$

Note that if the left hand side were zero, then we would have a strict Lie algebra action. Thus, $\rho^{(2)}$ provides a homotopy correcting the failure of $\rho^{(1)}$ to be strict.

One way to unravel this definition is in terms of the cochain complex computing the Lie algebra cohomology of $\mathfrak{g}$. The map ρ is equivalent to an element

$$\rho = \sum_k \rho^{(k)} \in C^\bullet(\mathfrak{g}) \otimes \mathcal{D}(E), \quad \rho^{(k)} \in C^k(\mathfrak{g}) \otimes \mathcal{D}(E) \tag{2.27}$$

of bidegree $(1,+)$ which satisfies the Maurer–Cartan equation

$$\mathrm{d}_{\mathfrak{g}}\rho + \frac{1}{2}[\rho, \rho] = 0 \,. \tag{2.28}$$

Here $\mathrm{d}_{\mathfrak{g}}$ denotes the Chevalley–Eilenberg differential of $\mathfrak{g}$ and $[-,-]$ is the commutator of differential operators.

We observe that ρ determines a super L_∞ structure on $\mathfrak{g} \oplus \mathcal{E}$ in such a way that there is a short exact sequence of L_∞ algebras

$$0 \to \mathcal{E} \to \mathfrak{g} \oplus \mathcal{E} \to \mathfrak{g} \to 0 \tag{2.29}$$

where $\mathcal{E}$ is thought of as an L_∞ algebra with $\mu_k = 0$ for $k > 1$.

Let us take some time to reflect on this definition from the physics point of view. It is well known that the supersymmetry algebra is sometimes only realized on-shell or up to gauge transformations. This is precisely captured in the fact that we used a *super L_∞-map* $\mathfrak{g} \rightsquigarrow \big(\mathcal{D}(E)\,,\, [D,-]\big)$ to define a multiplet instead of a super Lie map. The higher order terms $\rho^{(j)}$ for $j \geq 2$ precisely correspond to closure terms correcting $\rho^{(1)}$ by a gauge transformation or contributions proportional to an equation of motion.

This discussion explains how the supersymmetry algebra acts on the fields of the theory. The operators of the theory consist of functionals of the fields and are denoted by $\mathcal{O}(\mathcal{E})$.

For any point $x \in X$, we can define the local operators via

$$\mathcal{O}_x(\mathcal{E}) = \mathrm{Sym}^\bullet(J^\infty E|_x)^\vee \,, \tag{2.30}$$

where $J^\infty E$ denotes the jet bundle of E. In other words, the local operators at x evaluate polynomials in the fields and derivatives of fields at x. Given a map

$$\rho \colon \mathfrak{g} \rightsquigarrow \big(\mathcal{D}(E)\,,\, [D, -]\big)\,, \tag{2.31}$$

the dual maps $(\rho^{(j)})^\vee$ define an action on the linear local operators, which extends to $\mathcal{O}(\mathcal{E})_x$ via the Leibniz rule. Fixing an element $Q \in \mathfrak{g}$ we can define a map

$$\delta_Q = \sum_j \rho^{(j)}(Q, \ldots, Q)^\vee : \mathcal{O}_x(\mathcal{E}) \longrightarrow \mathcal{O}_x(\mathcal{E})\,, \tag{2.32}$$

which defines the action of $Q \in \mathfrak{g}$ on the operators of the theory.

Local algebras. For completeness, let us briefly remark that there is a natural way to make the symmetry algebra $\mathfrak{g}$ local as well. This is relevant if $\mathfrak{g}$ encodes a gauge symmetry.

Definition 2.2.10. A *local super L_∞ algebra* on a manifold X is a dgs vector bundle $L \to X$, equipped with a collection of polydifferential operators

$$\mu_k : (\mathcal{L})^{\times k} \to \mathcal{L} \tag{2.33}$$

of bidegree $(2-k, +)$ that satisfy the relations of a super L_∞ algebra structure. Here $\mathcal{L} = \Gamma(X, L)$ are the smooth sections of L.

The definition of a local module structure generalizes in obvious fashion. We note that, given a super L_∞ algebra $\mathfrak{g}$, the constant sheaf $\underline{\mathfrak{g}}$ is *not* an example of a local super L_∞ algebra for $d > 0$, since it is not given as the smooth sections of any dgs vector bundle. However, we can remedy this by resolving the constant sheaf by the de Rham complex: $\Omega^\bullet(X) \otimes \mathfrak{g}$ is a local super L_∞ algebra on X. (This example is relevant to Chern–Simons theory.)

Furthermore the above definition is important in the general context of the BV formalism: A perturbative classical field theory in the BV formalism will be equivalent to a local super L_∞ algebra on X, equipped with a trace map of degree -3. We will further review this perspective in what follows.

Multiplets. In the context of supersymmetry, we are interested in local modules that satisfy an additional compatibility condition. For now, let $X = V_{\mathbb{R}} = \mathbb{R}^d$ be a d-dimensional affine space and let $V = \mathbb{C}^d$ be its complexification.[3] The Poincaré group is the group of affine transformations of this space; it is of the form

$$\mathrm{Aff}(V) = \mathrm{Spin}(V) \ltimes V \, . \tag{2.34}$$

The complexified Lie algebra $\mathfrak{aff}(V)$ is

$$\mathfrak{spin}(V) \ltimes V \cong \wedge^2 V \ltimes V. \tag{2.35}$$

A *multiplet* is a local module structure for a dgs L_∞ algebra on an *affine*[4] dgs vector bundle on $\mathbb{R}^d$, where the $\mathfrak{g}$-action is required to be compatible with the action of the affine algebra in a certain sense. We make this precise with the following definition.

Definition 2.2.11. Let E be an affine dgs vector bundle on $X = V_{\mathbb{R}}$, and $\mathfrak{g}$ a super L_∞ algebra equipped with a map

$$\phi : \mathfrak{aff}(V) \to \mathfrak{g} \, . \tag{2.36}$$

A $\mathfrak{g}$-*multiplet* is a local $\mathfrak{g}$-module structure on E, such that the pullback of the module structure along ϕ agrees with the natural action on sections of the affine vector bundle. Concretely, this means that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\rho^{(1)}"] & \mathcal{D}(E) \\ \mathfrak{aff}(V) \arrow[u, "\phi"] \arrow[ur] & \end{tikzcd} \tag{2.37}$$

We think of a multiplet as a derived replacement for the (not necessarily locally free) "supersymmetric sheaf" $H^\bullet(E)$. Even though this sheaf could be regarded as the central object of study in physics, it is more natural from either the BRST/BV perspective or the perspective of derived geometry to just work at the cochain level. There is again a generalization of this definition to local super L_∞ algebras, where the global affine algebra is replaced by a local L_∞ algebra modeling local isometries. We do not pursue this further here.

We briefly note that this definition implies that the image of ϕ is represented strictly on the fields. Furthermore, since the natural action of the affine algebra is effective, the

[3] For the pure spinor superfield formalism it will be useful for us to use complex Lie algebras.

[4] A dgs vector bundle is called affine if the total space carries an action of the affine group such that the projection is equivariant with respect to the action of the affine group on $\mathbb{R}^d$.

above definition requires implicitly that ϕ be injective. So multiplets naturally lead us to study superalgebras that contain the affine algebra as a subalgebra.

We take note of the following examples:

— Let $\mathfrak{h}$ be a Lie algebra, and consider the product $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{aff}(V)$, equipped with the obvious choice of ϕ. Then a $\mathfrak{g}$-multiplet contains a collection of fields transforming in a local representation of $\mathfrak{h}$. Flavor symmetry multiplets are an example of this kind.

— Let $\mathfrak{conf}(V)$ be the Lie algebra of conformal vector fields on V. There is a canonical embedding of $\mathfrak{aff}(V)$ in $\mathfrak{conf}(V)$. Then a $\mathfrak{conf}(V)$-multiplet encodes the notion of a conformally invariant multiplet of fields.

— Let $\mathfrak{g}$ be the super Poincaré algebra. It contains $\mathfrak{aff}(V)$ as a subalgebra, and a $\mathfrak{g}$-multiplet recovers the usual notion of a supermultiplet.

Historically speaking, the construction of interesting multiplets for algebras that were not products was the motivation that led to the origin of supersymmetry; we return to this point (and construct examples of the relevant algebras of physical interest) below.

To conclude this paragraph, we discuss a first example of a multiplet.

Example 2.2.12. Let us give one example of a non-strict multiplet for three dimensional $\mathcal{N} = 1$ supersymmetry. Recall that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$; we denote the two dimensional spinor representation by S and the three-dimensional vector representation by V. We fix $\mathfrak{g}$ to be the super Poincaré algebra, whose underlying $\mathbb{Z}/2\mathbb{Z}$-graded vector space is of the form

$$\mathfrak{g} = (\mathfrak{so}(3) \oplus V) \oplus \Pi S \tag{2.38}$$

where Π indicates the shift into odd parity. The symmetric bracket is induced from the isomorphism $\Gamma\colon \mathrm{Sym}^2(S) \cong V$ of $\mathrm{SU}(2)$-representations. Let us define E to be the trivial vector bundle over $V = \mathbb{R}^3$ with fibers

$$E^{-1}_{+} = \mathbb{C} \qquad E^{0}_{+} = V \qquad E^{0}_{-} = S\,, \tag{2.39}$$

where we have used the subscript $\pm$ to indicate $\mathbb{Z}/2\mathbb{Z}$-degree. The field content consists of a zero-form, a one-form and a fermion field. The differential D operates on sections as the de Rham differential $\mathrm{d} : \Omega^0 \longrightarrow \Omega^1$, and vanishes elsewhere. This dgs vector bundle can be lifted to a $\mathbb{Z} \times \mathbb{Z}$-graded vector bundle where E^0_+ has weight one, and E^0_- has weight two.

We summarize this field content with the below array, using the conventions of §2.2.1:

$$\begin{bmatrix} \Omega^0 & & \\ & \searrow^{\mathrm{d}} & \\ & \Omega^1 & S \end{bmatrix} \tag{2.40}$$

The even part of $\mathfrak{g}$ acts in the standard geometric fashion. For $Q \in \mathfrak{g}_-$ we set

$$\begin{array}{rcccccccc} \rho^{(1)}(Q) & : & S & \longrightarrow & \Omega^1 & , & \psi & \mapsto & \Gamma(Q,\psi) \\ & : & \Omega^1 & \longrightarrow & S & , & A & \mapsto & Q \wedge \partial\!\!\!/ A \\ \rho^{(2)}(Q,Q) & : & \Omega^1 & \longrightarrow & \Omega^0 & , & A & \mapsto & \iota_{[Q,Q]}A\,. \end{array} \tag{2.41}$$

Here ι denotes the contraction of a differential form by a vector field and we view $[Q,Q]$ as a constant vector field on X.

2.2.5 Further structures on multiplets

As we will see in the following sections, the pure spinor superfield formalism naturally produces *multiplets* for the supersymmetry algebra. Some extra data is required to produce a theory out of a multiplet; furthermore, depending on whether or not supersymmetry closes off-shell, the resulting theory may be a BRST or a BV theory, so that the additional data required may differ. There are also conditions on the additional data that ensure that the theory is nondegenerate in an appropriate sense. We set up some formalism for the required additional structure in this section.

BRST data. In the BRST formalism, a perturbative field theory is described by a local super L_∞ algebra $\mathcal{L}$ equipped with a BRST action functional S, which is invariant for the L_∞ structure. The L_∞ structure describes the (higher) infinitesimal gauge transformations and the variation of the BRST action gives rise to the equations of motion.

Definition 2.2.13. A *BRST datum* on the $\mathfrak{g}$-multiplet (F, D, ρ) consists of

— a local super L_∞ structure $\{\mu_k\}$ on $L := F[-1]$ such that $\mu_1 = D$, and whose associated Chevalley–Eilenberg differential we denote by Q_{BRST}; and

— a local functional $S_0 \in \mathcal{O}_{\mathrm{loc}}(F)$ of bidegree $(0,+)$, called the "BRST action functional," which is closed with respect to Q_{BRST}.

This data should be such that all maps in the short exact sequence

$$0 \to \mathcal{L} \to \mathfrak{g} \oplus \mathcal{L} \to \mathfrak{g} \to 0 \tag{2.42}$$

are L_∞ maps, and S_0 is invariant for the L_∞ action ρ.

For physicist readers, the shift by one appearing in $L = F[-1]$ may deserve some comment. Essentially, this arises from the fact that the ghosts for the theory (which are valued in the Lie algebra of the gauge group, and thus carry a Lie algebra structure) sit in cohomological degree -1. The observables of the theory, which are functions on compactly supported sections of F, are Lie algebra cochains of the local L_∞ algebra L; the BRST differential on observables is the Chevalley–Eilenberg differential arising from the gauge algebra structure.

BV data. The (classical) Batalin–Vilkovisky (BV) [BV81; BV84; BV85] formalism is a generalization of the BRST formalism that encodes the equations of motion in a derived way. For a comprehensive review of the classical BV formalism we refer to [CG17; CG21] (see also [Jur+19; Mne17]). We recall the general idea briefly.

Perturbatively, a BV theory is described by a local super L_∞ algebra L_{BV} equipped with an invariant, skew-symmetric, non-degenerate, local pairing of degree -3 (see the definition below). The space of "BV fields" is the space of sections of the bundle $E = L_{\mathrm{BV}}[1]$ given by the shift in $\mathbb{Z}$-degree of the L_∞ algebra encoding the BV theory.

The degree-(-3) local pairing on L_{BV} is equivalent to a local skew pairing of degree -1 on the space of BV fields $\mathcal{E} = \mathcal{L}_{\mathrm{BV}}[1]$, which in turn can be thought of as a (-1)-shifted symplectic structure on the BV fields. As above, the shift is needed so that observables of the classical BV theory can be identified with the Lie algebra cochains of L_{BV}. The (-1)-shifted symplectic structure equips the observables with a degree-$(+1)$ Poisson bracket, often called the antibracket. In turn, the degree-$(+1)$ Chevalley–Eilenberg differential on observables that encodes the super L_∞ algebra structure on L_{BV} is a degree-$(+1)$ Hamiltonian vector field, that can be encoded in the datum of a BV action functional S_{BV}, so that

$$C^\bullet(L_{\mathrm{BV}}) = \big(\mathcal{O}(\mathcal{E})\,,\ \{S_{\mathrm{BV}}, -\}\big)\,. \tag{2.43}$$

The zeroth cohomology of this cochain complex is the space of functions on the critical locus of the BRST action modulo gauge equivalence. The condition that $\{S_{\mathrm{BV}}, -\}$ define a differential is equivalent to the classical master equation

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0\,. \tag{2.44}$$

Of course, proper care must be taken to make rigorous sense of the BV complex above. As we are working perturbatively, the space of BV fields will arise as the space of sections of some graded vector bundle on spacetime. Furthermore, the BV action will be given as the integral of a Lagrangian density of the fields. More details on the BV formalism can be found in [CG17; CG21].

For any multiplet, we will define a notion of "BV datum," which consists of the set of data necessary to construct a BV theory (a (-1)-shifted invariant symplectic pairing, with respect to which the homotopy $\mathfrak{g}$-action is defined by Hamiltonian vector fields, and a BV action functional that is compatible with the action of $\mathfrak{g}$). A BV theory will then consist of a multiplet equipped with a BV datum that satisfies an additional nondegeneracy condition.

Definition 2.2.14. A *BV datum* on a $\mathfrak{g}$-multiplet (E, D, ρ) consists of:

— a graded antisymmetric map

$$\langle -, - \rangle_{\mathrm{loc}} : E \otimes E \longrightarrow \mathrm{Dens}_X \tag{2.45}$$

of bidegree $(-1, +)$, which is fiberwise non-degenerate; and

— a $C^\bullet(\mathfrak{g})$-valued BV action

$$S_{\mathrm{BV},\mathfrak{g}} = \sum_k S^{(k)}_{\mathrm{BV},\mathfrak{g}} \in C^\bullet(\mathfrak{g}) \otimes \mathcal{O}_{\mathrm{loc}}(E), \quad S^{(k)}_{\mathrm{BV},\mathfrak{g}} \in C^k(\mathfrak{g}) \otimes \mathcal{O}_{\mathrm{loc}}(E) \tag{2.46}$$

of bidegree $(0, +)$ of the form

$$S^{(0)}_{\mathrm{BV},\mathfrak{g}}(\Phi) = \int_X \langle \Phi, D\Phi \rangle_{\mathrm{loc}} + I_{\mathrm{BV}}(\Phi) \tag{2.47}$$

where $I_{\mathrm{BV}}(\Phi)$ is a Lagrangian that is at least cubic in the fields and where

$$S^{(k)}_{\mathrm{BV},\mathfrak{g}}(x_1, \ldots, x_k; \Phi) = \int_X \langle \Phi, \rho^{(k)}(x_1, \ldots, x_k)\Phi \rangle_{\mathrm{loc}} \tag{2.48}$$

such that

(i) $\langle -, - \rangle_{\mathrm{loc}}$ is invariant for the L_∞ action ρ;

(ii) the total action $S_{\mathrm{BV},\mathfrak{g}}$ satisfies the $\mathfrak{g}$-equivariant master equation

$$\mathrm{d}_{\mathfrak{g}} S_{\mathrm{BV},\mathfrak{g}} + \frac{1}{2}\{S_{\mathrm{BV},\mathfrak{g}}, S_{\mathrm{BV},\mathfrak{g}}\} = 0. \tag{2.49}$$

If D is elliptic, then $S^{(0)}_{\mathrm{BV},\mathfrak{g}}(\Phi)$ is a $\mathfrak{g}$-equivariant perturbative BV theory in the sense of [CG21]. According to the terminology in loc. cit., this total action $S_{\mathrm{BV},\mathfrak{g}}$ endows

$S^{(0)}_{\mathrm{BV},\mathfrak{g}}(\Phi)$ with the structure of a $\mathfrak{g}$-equivariant theory. We will refer to a multiplet equipped with a BV datum for which D is elliptic as a $\mathfrak{g}$-equivariant *BV theory.*

To go from a multiplet with BRST datum to a multiplet with BV datum, one considers

$$L_{\mathrm{BV}} = L \oplus L^{\vee}[-3] \tag{2.50}$$

which is equipped with a canonical evaluation pairing of degree (-3). The BRST action deforms the obvious L_∞ structure on the direct sum of L with $L^{\vee}[-3]$, thus giving rise to an L_∞ structure on L_{BV} for which the evaluation pairing is invariant (after an application of the homological perturbation lemma, which can be thought of as solving the classical master equation for S_{BV} order by order).

We will say that a multiplet equipped with a BRST datum is a *BRST theory* when the corresponding BV datum itself defines a BV theory, meaning that the kinetic term in the BV action involves an elliptic operator.

Note that, in the way we have set things up, any multiplet can be equipped with a trivial BRST datum, whereas a BV datum may not always exist. In §2.4 we will see that some of the multiplets produced in the pure spinor formalism can be naturally equipped with nondegenerate BV data, while this is not possible for others. Of course, a degenerate BRST datum does not, in itself, define a BRST theory.

For a multiplet with BV datum $(E, D, \rho, \langle ., . \rangle)$, the inner product always allows us to write

$$E = F \oplus F^{\vee}[-1] \,, \tag{2.51}$$

where $F = \oplus_{k \leq 0} E^k_{\mathrm{BV}}$. This induces a splitting on the space of sections

$$\mathcal{E} = \mathcal{F} \oplus \mathcal{F}^{!}[-1] \,. \tag{2.52}$$

Note that this is a splitting on the level of super vector spaces.

Definition 2.2.15. A BV multiplet (E, D, ρ) is *off-shell* if the above splitting exists on the level of $\mathfrak{g}$-modules. Then F is naturally a BRST multiplet, and $(F^{\vee}[-1], D|_{\mathcal{F}^!}, \rho|_{\mathcal{F}^!})$ is called the antifields multiplet for $(F, D|_{\mathcal{F}}, \rho|_{\mathcal{F}})$.

Intuitively, this definition means that it is possible to consider the $\mathfrak{g}$-action separately on the fields and antifields. Then, the equations of motions are not needed to close the algebra and the only corrections for the action come from gauge transformations.

2.3 The pure spinor superfield formalism

2.3.1 A universal construction

Let $\mathfrak{g}$ be a super Lie algebra, and Y its nilpotence variety, viewed as an affine scheme as discussed above. Let M be any (dgs) $\mathfrak{g}$-module, and Γ any graded module for the graded ring R/I. (We can equivalently view Γ as defining a sheaf on $\operatorname{Spec} R/I$.) Then there is a map

$$\rho : \mathfrak{g} \to \operatorname{End}(M) \tag{2.53}$$

defining the $\mathfrak{g}$-module structure, and an obvious map

$$m : \mathfrak{g}_-^\vee \to \operatorname{End}(\Gamma) \tag{2.54}$$

given by left multiplication (after recalling that $\mathfrak{g}_-^\vee$ includes into R/I in weight one). If we consider the tensor product $M \otimes \Gamma$, the above two maps define a map

$$\rho \cdot m : \mathfrak{g}_- \otimes \mathfrak{g}_-^\vee \to \operatorname{End}(M \otimes \Gamma) \tag{2.55}$$

as explained in the following diagram.

$$\begin{tikzcd}
\mathfrak{g}_- \otimes \mathfrak{g}_-^\vee \arrow[r, "\rho\cdot m"] \arrow[d, "\rho\otimes m"] & \operatorname{End}(M\otimes\Gamma) \\
\operatorname{End}(M) \otimes \operatorname{End}(\Gamma) \arrow[d, hook] & \\
\operatorname{End}(M\otimes\Gamma) \otimes \operatorname{End}(M\otimes\Gamma) \arrow[uur] &
\end{tikzcd} \tag{2.56}$$

That is, we apply $\rho \otimes m$, include the resulting element into $\operatorname{End}(M \otimes \Gamma) \otimes \operatorname{End}(M \otimes \Gamma)$ and finally multiply to obtain an endomorphism of $M \otimes \Gamma$.

The map $\rho \cdot m$ equips $M \otimes \Gamma$ with a canonical square-zero differential $\mathcal{D}$, defined to be the image of the canonical element

$$1 \in \mathfrak{g}_- \otimes \mathfrak{g}_-^\vee \cong \operatorname{End}(\mathfrak{g}_-). \tag{2.57}$$

The square of this differential sits in the defining ideal of R/I, and thus is zero for any R/I-module Γ.

Remark 2.3.1. In the case that $\Gamma = R/I$ is the ring of functions, we note that this construction is closely related to the following construction: As in derived geometry, we define the "classifying space" of a super L_∞ algebra $\mathfrak{g}$ to be the derived scheme $B\mathfrak{g}$

whose ring of functions consists of the Lie algebra cochains $C^\bullet(\mathfrak{g})$. Then a version of the associated bundle construction associates a sheaf on $B\mathfrak{g}$ to any $\mathfrak{g}$-module M; the global sections of this sheaf are $C^\bullet(\mathfrak{g}; M)$. In the cases we are interested in, there is a close connection between $\mathcal{O}_Y$ and Lie algebra chochains. This is already a first hint towards the derived formalsim we construct in §3.

2.3.2 The case of interest: from sheaves to multiplets

Let $\mathfrak{n}$ be a two-step nilpotent super Lie algebra, defined by a central extension

$$0 \to \mathfrak{n}_2 \to \mathfrak{n} \to \Pi\mathfrak{n}_1 \to 0 \tag{2.58}$$

of the odd abelian super Lie algebra $\Pi\mathfrak{n}_1$. We imagine such objects as generalizations of supertranslation algebras. The automorphisms of $\mathfrak{n}$, which are all outer, will contain an abelian factor generating scale transformations, with respect to which $\mathfrak{n}_1$ has weight one and $\mathfrak{n}_2$ weight two. There is also a natural map

$$\mathfrak{aut}(\mathfrak{n}) \to \mathfrak{gl}(\mathfrak{n}_2). \tag{2.59}$$

The kernel of this map can be thought of as the R-symmetry algebra; in physical examples, $\mathfrak{aut}(\mathfrak{n})$ will be the product of an orthogonal algebra $\mathfrak{so}(d)$, scale transformations, and the R-symmetry algebra.

All of our constructions will take place in reference to a fixed super Lie algebra $\mathfrak{g}$ of the following type.

Definition 2.3.2. A super Lie algebra $\mathfrak{g}$ is of *super Poincaré type* if it can be written as an extension

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{g}_0 \to 0, \tag{2.60}$$

where $\mathfrak{n}$ is a two-step nilpotent super Lie algebra of the form (2.58) and $\mathfrak{g}_0$ is a Lie algebra equipped with a Lie map $\mathfrak{g}_0 \to \mathfrak{aut}(\mathfrak{n})$.

This means that the $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathfrak{g}$ can be lifted to a $\mathbb{Z}$-grading concentrated in degrees zero, one, and two:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n}_1(-1) \oplus \mathfrak{n}_2(-2). \tag{2.61}$$

In keeping with the above discussion, we regard this as a lifted super L_∞ algebra that is concentrated in cohomological degree zero (and in degrees zero, one and two with repsect to the lifted weight degree); as such, only the binary bracket operation can be nonvanishing for degree reasons.

The most important examples will be super Poincaré algebras; we review how these are constructed below, but just remark here that $\mathfrak{n}_2$ consists of translations and $\mathfrak{n}_1$ of supersymmetries in that case.

As we will see momentarily, $\mathfrak{n}_2$ will play the role of the spacetime on which the multiplet is constructed. We note that much of the construction would go through if $\mathfrak{g}$ were any nonnegatively graded Lie algebra. In such a case, however, the bosonic part of $\mathfrak{g}_{>0}$ may not be abelian, and an interpretation of the construction in terms of multiplets on an affine supermanifold will not be immediate. As such, we do not study any examples of this sort here.

To be very explicit, if we choose a basis d_α of $\mathfrak{n}_1$ and a basis e_μ of $\mathfrak{n}_2$, we can express the bracket in terms of structure constants $f^\mu_{\alpha\beta}$

$$[d_\alpha, d_\beta] = f^\mu_{\alpha\beta} e_\mu \,. \tag{2.62}$$

We denote by $\lambda^1, \dots, \lambda^n$ coordinates on $\mathfrak{n}_1$ dual to the basis d_α. Then the defining ideal I of the nilpotence variety is generated by the equations

$$I = (\lambda^\alpha f^\mu_{\alpha\beta} \lambda^\beta), \tag{2.63}$$

such that the quotient ring

$$R/I = \mathbb{C}[\lambda^1, \dots, \lambda^n]/I, \tag{2.64}$$

is its ring of functions.

We are interested in a particular example of the construction above, where M is taken to be the $\mathfrak{g}$-module consisting of smooth functions on $N = \exp(\mathfrak{n})$ (viewed as a supermanifold). Concretely,

$$M = C^\infty(N) = C^\infty(X) \otimes_{\mathbb{C}} \wedge^\bullet(\mathfrak{n}_1^\vee)\,, \tag{2.65}$$

where we already identified $X = \mathfrak{n}_2$. There are two commuting actions of $\mathfrak{g}$ on M, on the left and the right; we denote these by

$$\mathscr{R}, \mathscr{L} : \mathfrak{g} \longrightarrow \mathrm{End}(M). \tag{2.66}$$

Now, for any graded R/I-module Γ that is equivariant for the $\mathfrak{g}_0$-action, applying the construction above to M (with respect to the *right* action of $\mathfrak{g}$) produces a cochain complex

$$A^\bullet(\Gamma) = (\Gamma \otimes_{\mathbb{C}} M,\, \mathcal{D} = \lambda^\alpha \mathscr{R}(d_\alpha)). \tag{2.67}$$

Explicitly, let x^μ be linear coordinate functions on $\mathfrak{n}_2$ and θ^α be (odd) linear coordinate functions on $\mathfrak{n}_1$, dual to the basis (e_μ, d_α) above. Then the differential is given in coordinates by

$$\mathcal{D} = \lambda^\alpha \mathscr{R}(d_\alpha) = \lambda^\alpha \left(\frac{\partial}{\partial \theta^\alpha} - f^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu} \right), \tag{2.68}$$

where the differential operators act in $M = C^\infty(N)$ and λ^α acts on Γ via the $\mathcal{O}_Y$-module structure. Checking that $\mathcal{D}$ squares to zero explicitly is also straightforward:

$$\begin{aligned} \mathcal{D}^2 &= \lambda^\alpha \lambda^\beta \mathscr{R}(d_\alpha)\mathscr{R}(d_\beta) = \frac{1}{2}\lambda^\alpha\lambda^\beta[\mathscr{R}(d_\alpha), \mathscr{R}(d_\beta)] \\ &= \frac{1}{2}\lambda^\alpha\lambda^\beta\mathscr{R}([d_\alpha, d_\beta]) = \frac{1}{2}\lambda^\alpha\lambda^\beta f^\mu_{\alpha\beta}\mathscr{R}(e_\mu) = 0. \end{aligned} \tag{2.69}$$

$A^\bullet(\Gamma)$ naturally has the structure of a dgs vector space: we assign bidegree $(1, -)$ to λ^α, $(0, -)$ to θ^α, and $(0, +)$ to x^μ. Since $\mathfrak{g}$ admits a natural lift, there is also a natural candidate for a lifted dgs vector space structure, in which λ^α carries bidegree $(1, -1)$, θ^α bidegree $(0, -1)$, and x^μ bidegree $(0, -2)$. However, this lift only defines a sensible bigrading on polynomial functions on $\mathfrak{n}_2$, rather than on all smooth functions. This bigrading is often referenced in the pure spinor superfield literature, often under the names "ghost number" and "dimension." We will not need it in what follows, and will view $A^\bullet(\Gamma)$ just as a dgs vector space. However, a filtration related to the dimension will play an important role for us.

From this discussion, it is clear that $A^\bullet(\Gamma)$ can be viewed as the global sections of an affine dgs vector bundle $E \to X$ over $X = \mathfrak{n}_2$ with typical fiber

$$E^k_x \cong \wedge^\bullet \mathfrak{n}_1^\vee \otimes_{\mathbb{C}} (\Gamma)^k \,. \tag{2.70}$$

This is the underlying vector bundle of the multiplet we construct.

We note some properties of this construction below:

— By construction, the left action of $\mathfrak{n}$ commutes with the differential $\mathcal{D}$. As such, the left action defines a *strict* $\mathfrak{n}$-module structure, which is equivariant with respect to $\mathrm{Aut}(\mathfrak{n})$ and as such can be extended to a strict action of $\mathfrak{g}$.

— There is an obvious sense in which (a subgroup of) the even part of $\mathfrak{g}$ consists of affine transformations acting on M. The $\mathfrak{g}$-action is compatible with this inclusion map, and thus makes $A^\bullet(\Gamma)$ into a $\mathfrak{g}$-multiplet.

— In the definition given above, the notion of a multiplet was designed to capture the notion of a sheaf over spacetime admitting an action of supersymmetry. In physical terms, this sheaf could be thought of as either on-shell or off-shell field configurations up to gauge equivalence. A multiplet, that is a cochain complex of vector bundles with a homotopy action of supersymmetry, can be thought of as a resolution of this sheaf. (This corresponds to studying off-shell supersymmetric theories in the BRST formalism, and on-shell theories in the BV formalism.) The multiplet $A^\bullet(\Gamma)$ goes one step further: it resolves a supersymmetric sheaf not just freely over spacetime, but *freely over superspace*. The action of the supersymmetry algebra is thus just the obvious one on functions on superspace, which is both strict and geometric in nature.

— For any super Lie algebra $\mathfrak{g}$ of super Poincaré type, we can choose as input module the ring of functions, $\Gamma = R/I$. We will sometimes call the the associated multiplet $A^\bullet(R/I)$, the canonical multiplet of $\mathfrak{g}$ (this terminology was introduced in [Ced+23]). It is apparent that $A^\bullet(R/I)$ has the structure of a commutative algebra, and therefore that its cohomology $H^\bullet(A^\bullet(R/I))$ is also an A_∞ algebra in a canonical way. $A^\bullet(R/I)$ is a strict model of this A_∞ structure.

— To sum up, we have constructed a canonical way of associating a multiplet to any equivariant sheaf on Y. Schematically, we depict the construction as an assignment

$$\{\text{Graded equivariant } R/I\text{-modules}\} \xrightarrow{\text{Pure spinor formalism}} \{\mathfrak{g}\text{-Multiplets}\} . \tag{2.71}$$

In §3 we will upgrade this construction to a functor to a suitable category of multiplets and, eventually, into an equivalence of dg-categories.

In many examples, there is further structure available, and $A^\bullet(\Gamma)$ can be equipped with a collection of higher brackets endowing it with the structure of an L_∞ algebra. By homotopy transfer this yields an L_∞ structure on the cohomology. In physically relevant examples, such L_∞ structures precisely correspond to those appearing in the BV or BRST description of the underlying field theory.

To give one example, the ten-dimensional super Yang–Mills multiplet is constructed by considering $A^\bullet(R/I)$ for the ten-dimensional $\mathcal{N} = 1$ supersymmetry algebra. Since $A^\bullet(R/I)$ is a commutative dgs algebra, we can tensor with any finite-dimensional Lie algebra $\mathfrak{h}$. Then $A^\bullet(R/I) \otimes \mathfrak{h}$ is a dgs Lie algebra that freely resolves the L_∞ structure of the BV description of interacting $\mathcal{N} = 1$ super Yang–Mills theory. This description is well-known from work of Berkovits and Cederwall, but we review it in our language below in §2.5 and explicitly derive the standard structures using homotopy transfer.

The general construction we have outlined so far produces a "large" multiplet, which, as outlined above, can be thought of resolving a sheaf over spacetime with an action of supersymmetry. Of course we can just move to the cohomology of our multiplet to recover this sheaf; however, one might wonder whether and how a smaller multiplet resolving the same sheaf can be extracted. For example, one wants to connect the pure spinor multiplet $A^\bullet(\Gamma)$ to typical component field multiplets, that is to a multiplet with finite total rank over spacetime. In fact, there is a general technique for producing "minimal" resolutions of this kind, which was discussed in [MSX12; KL09]. We review it in our language below and give a proof that highlights the relation to standard constructions in algebraic geometry and homological algebra. After that, we will construct our first examples of physically relevant algebras and multiplets.

2.3.3 Filtrations and Koszul homology

The object $A^\bullet(\Gamma)$ that we have constructed admits a natural filtration $F^\bullet A^\bullet(\Gamma)$; understanding the spectral sequence associated to this filtration will allow us to relate the multiplets we are constructing to finite-rank vector bundles over the spacetime X. The filtration is associated to a second integer grading; we will find that, while not all of the structures we are interested in preserve this second grading, they do play nicely with the associated filtration. The filtration degree is defined by the assignments in the following table:

	homological degree	intrinsic parity	filtered weight
x	0	$+$	0
λ	1	$-$	1
θ	0	$-$	1

(2.72)

(These conventions for the filtration follow those used in [SW21].)

Since $C^\infty(X)$ plays no role in the filtration, we are exhibiting $A^\bullet(\Gamma)$ as a filtered dgs vector bundle over X. Moreover, since the filtration plays well with the product structure on the algebra $A^\bullet(R/I)$, it gives rise to the structure of a filtered commutative dgs algebra there. However, we observe that the tautological differential does not respect the integer grading by filtration weight. Recall that, in coordinates,

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} - \lambda^\alpha f^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu}. \tag{2.73}$$

As the notation suggests, the differential is the sum of two terms, which have filtered weight zero and two respectively. The associated graded complex is thus equipped only with the differential $\mathcal{D}_0$, which is independent of smooth functions on X. The associated graded then takes the following form.

$$\mathrm{Gr}\, A^\bullet(\Gamma) = \left(C^\infty(X) \otimes_{\mathbb{C}} (\Gamma \otimes_{\mathbb{C}} \mathbb{C}[\theta^\alpha]) \, , \mathcal{D}_0 = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} \right) \cong C^\infty(X) \otimes_{\mathbb{C}} K^\bullet(\Gamma). \tag{2.74}$$

Here, we have defined the *Koszul homology* of any R-module in standard fashion:

$$K^\bullet(\Gamma) := \left(\Gamma \otimes_{\mathbb{C}} \mathbb{C}[\theta^\alpha]\,,\; \mathcal{D}_0 = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha}\right). \tag{2.75}$$

The fact that Γ is an R/I-module is of course vitally important for our construction, but Koszul homology makes sense for any R-module. In the pure spinor superfield literature, the cohomology of $\mathrm{Gr}\,A^\bullet$ is often referred to as "zero mode cohomology" [Ced14].

If we consider the spectral sequence associated to this filtration, we find that the E_1 page is just given by

$$H^\bullet(\mathrm{Gr}\,A^\bullet(\Gamma)) = C^\infty(X) \otimes_{\mathbb{C}} H^\bullet(K^\bullet(\Gamma)). \tag{2.76}$$

Since Γ is a graded module, the Koszul homology of Γ is a finite-dimensional bigraded representation of the Lorentz group. As such, $H^\bullet(\mathrm{Gr}\,A^\bullet(\Gamma))$ determines a vector bundle over $X = \mathfrak{n}_2 \cong \mathbb{R}^d$ with fibers

$$(E'_x)^k \cong H^\bullet(K^\bullet(\Gamma))^{(k)}\,. \tag{2.77}$$

We emphasize that the homological degree of E' is determined by the internal (weight) grading on Γ, whereas the parity is determined by the homological degree in Koszul homology modulo two. $\mathcal{D}_1$ induces a new differential $\mathcal{D}'$ acting on the sections of this vector bundle via homotopy transfer of D_∞-algebras. In addition, the $\mathfrak{g}$-module structure transfers as well such that $(E', \mathcal{D}', \rho')$ is again a multiplet. This multiplet precisely corresponds to the component field description of multiplets as they are known from the physics literature. The transferred differentials play the role of BRST or BV differentials.

Of course one could go on and consider the full cohomology of $A^\bullet(\Gamma)$. If the transferred differential $\mathcal{D}'$ on the component field level does not already vanish, then the resulting object will no longer be free over spacetime, i.e. it does not consist of vector bundles and thus does not fit our definition of a multiplet. It is, however, still a sheaf on spacetime which carries a $\mathfrak{g}$-action. Physically speaking this sheaf consists of the on-shell, gauge invariant states of the multiplet.

Let us summarize these relations by the following diagram.

$$\begin{array}{cc} (A^\bullet(\Gamma), \mathcal{D}) & \text{Free over superspace} \\ \Big\downarrow{\scriptstyle HT} & \\ (H^\bullet(\mathrm{Gr}A^\bullet), \mathcal{D}') & \text{Free over spacetime} \\ \Big\downarrow{\scriptstyle HT} & \\ (H^\bullet(A^\bullet(\Gamma)), 0) & \text{Not necessarily free} \end{array} \tag{2.78}$$

The compatibility of the differential with the filtration in fact arises from a compatibility of the left and right $\mathfrak{g}$-actions with the filtration, once $\mathfrak{g}$ is filtered in an appropriate way. Using the standard definition of a complete filtered Lie algebra [KN64; Koc77], we can equip $\mathfrak{g}$ with a filtered structure by setting

$$\mathfrak{g} = \mathfrak{g}^{(-1)} \supset \mathfrak{g}^{(0)} = \mathfrak{g}_+. \tag{2.79}$$

We observe that this filtration corresponds to the one we defined above, viewing the pure spinor superfield as constructed from functions on superspace together with the degree-zero Lie algebra cohomology of $\mathfrak{g}_{>0}$ (see §2.6.3).

The associated graded super Lie algebra $\mathrm{Gr}(\mathfrak{g})$ is then the extension of $\mathfrak{g}_0$ by the *abelian* module consisting of $\mathfrak{n}_1 \oplus \mathfrak{n}_2$; said differently, we set the bracket between odd elements to zero. It is immediate that there is a $\mathrm{Gr}(\mathfrak{g})$-module structure on $\mathrm{Gr}\, A^\bullet(\Gamma)$. We will be able to derive this module structure, which consists of "all supersymmetry transformations that are independent of spacetime derivatives," efficiently in examples, using purely algebrogeometric information about Γ.

2.3.4 Examples of interest: Supersymmetry algebras

We are mostly interested in multiplets for supersymmetry algebras on an affine spacetime $X = V_{\mathbb{R}}$. Depending on the dimension, $\mathrm{Spin}(V)$ will have either one or two spinor representations, which we denote by S or $S_\pm$ respectively; furthermore, there will be an equivariant map Γ that witnesses V as a submodule of the tensor square of the spin representation.

We construct the space $\mathfrak{n}_1$ by taking the tensor product of a spin representation with an auxiliary vector space U, which (depending on dimension) may or may not be equipped with either a symmetric or antisymmetric bilinear form. The bracket is constructed from the pairing Γ; if Γ pairs one spin representation with the other (in dimension 0 mod 4), we tensor one spin representation with U and the other with $U^\vee$. If Γ is a symmetric self-pairing (as in dimensions 1, 2, and 3 mod 8), U should have a symmetric bilinear form; similarly, if Γ is an antisymmetric self-pairing on a spin representation (as in dimensions 5, 6, and 7 mod 8), U must be a symplectic vector space. The "degree of extended supersymmetry," denoted $\mathcal{N}$, is the dimension of U as a multiple of the smallest possible dimension (two in the symplectic case and one otherwise). In cases where a self-pairing exists on chiral spin representations (dimension 2 and 6 mod 8), two independent choices of $\mathcal{N}$ are possible, one for each chirality. By abuse of notation, we will also write Γ for the symmetric pairing on $\mathfrak{n}_1$.

The supertranlation algebra $\mathfrak{n}$ constructed in this way is extended to the super Poincaré algebra $\mathfrak{g}$ by adding in the automorphisms of $\mathfrak{n}$ in degree zero; these consist of $\mathrm{Lie}(\mathrm{Spin}(V)) = \mathfrak{so}(V)$, together with the automorphisms of U that preserve the pairing if present: either $\mathfrak{gl}(U)$, $\mathfrak{so}(U)$, or $\mathfrak{sp}(U)$, depending on dimension. In physics, this additional automorphism is known as R-symmetry.

The nilpotence varieties of these algebras were studied systematically in [ESW21; ES19b]; most examples were already present in the previous pure spinor literature. It is worth commenting briefly on the connection to the classical notion of a "pure spinor" given by Cartan. Recall that the spin representation of $\mathrm{Spin}(V_{\mathbb{R}})$ is constructed by choosing a maximal isotropic subspace $L \subset V_{\mathbb{C}}$. Then $S = \wedge^{\bullet}(L^{\vee})$, and $V_{\mathbb{C}} = L \oplus L^{\vee}$ acts via Clifford multiplication just by wedging and contracting. (In odd dimensions, $V_{\mathbb{C}} = L \oplus L^{\vee} \oplus (L^{\perp}/L)$, and the single generator in $L^{\perp}/L$ acts diagonally by the parity operator.)

Given the construction of the brackets in $\mathfrak{g}$, it is clear that an element lying in $\wedge^{0}(L^{\vee})$ (tensored with any element of U) is automatically square-zero, and that it will be a "minimal" or holomorphic supercharge. Considered as a projective variety, the space of such elements thus consists of the product of the projective space $P(U)$ and the space $\mathrm{OGr}(n, d)$ of isotropic subspaces $L = \mathbb{C}^n \subset V_{\mathbb{C}} = \mathbb{C}^d$. (Here $n = \lfloor d/2 \rfloor$.) The latter is the space of Cartan pure spinors, the minimal nonzero $\mathrm{Spin}(d)$ orbit in the spin representation. However, we emphasize that the nilpotence variety in general contains many more strata, and may even include nonminimal orbits in the spin representation, quite independently of R-symmetry (as in eleven dimensions).

We will not construct all supersymmetry algebras in detail here (for discussion that uses similar style and notation, see [ESW21]). We will just introduce examples as we need them, beginning with the four-dimensional $\mathcal{N} = 1$ algebra in the next section.

2.3.5 Motivating example: the 4d chiral multiplet

As an explicit example, let us consider the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions. A related discussion of the chiral multiplet already appeared in [ESW21].

Since the dimension is zero modulo four, U carries no pairing and can be taken to be one-dimensional. $\mathfrak{n}_1$ is then $S_+ \oplus S_-$, and the bracket is constructed using the isomorphism

$$S_+ \otimes S_- \cong V \tag{2.80}$$

of $\mathrm{Spin}(4)$ representations. Because this is an isomorphism, the self-bracket of an element $Q \in \mathfrak{n}_1$ is zero precisely when either $Q \in S_+$ or $Q \in S_-$; as such, Y consists of two coordinate planes of the form $\mathbb{C}^2 \subset \mathbb{C}^4$, intersecting at the origin. More precisely,

$$Y = S_+ \cup_{\{0\}} S_- . \tag{2.81}$$

We repeat the same computation in coordinates for emphasis. A general supercharge Q can be written in the form

$$Q = \lambda^\alpha Q_\alpha + \bar{\lambda}^{\dot\beta} \bar{Q}_{\dot\beta} \, . \tag{2.82}$$

Accordingly, the equation $[Q, Q] = 0$ reduces to the four quadratic equations

$$\lambda^\alpha \bar{\lambda}^{\dot\beta} \sigma^\mu_{\alpha\dot\beta} = 0 \, , \tag{2.83}$$

where we used that the isomorphism $S_+ \otimes S_- \cong V$ can, in a basis, be described by the Pauli matrices σ^μ. Multiplying matrices gives the four equations

$$\begin{aligned} \lambda^1\bar{\lambda}^1 + \lambda^2\bar{\lambda}^2 &= 0 \\ \lambda^1\bar{\lambda}^1 - \lambda^2\bar{\lambda}^2 &= 0 \\ \lambda^1\bar{\lambda}^2 + \lambda^2\bar{\lambda}^1 &= 0 \\ \lambda^1\bar{\lambda}^2 - \lambda^2\bar{\lambda}^1 &= 0 \, . \end{aligned} \tag{2.84}$$

Adding and subtracting these equations one finally finds

$$\lambda^1\bar{\lambda}^1 = \lambda^2\bar{\lambda}^2 = \lambda^1\bar{\lambda}^2 = \lambda^2\bar{\lambda}^1 = 0 \, , \tag{2.85}$$

which implies that λ^α or $\bar{\lambda}^{\dot\beta}$ vanish and recovers our result from above.

To construct a multiplet, we have to choose an R/I-module. One possible choice is $\Gamma = \mathbb{C}[\bar{\lambda}_{\dot\alpha}]$, which corresponds to the pushforward of the structure sheaf of S_+ to Y along the inclusion map. We form the pure spinor complex:

$$(A^\bullet(\Gamma) \, , \mathcal{D}) = \left(C^\infty(N) \otimes \mathbb{C}[\bar{\lambda}_{\dot\alpha}] \, , \; \mathcal{D} = \bar{\lambda}_{\dot\alpha} \frac{\partial}{\partial \bar{\theta}_{\dot\alpha}} - \bar{\lambda}^{\dot\alpha} \theta^\alpha \sigma^\mu_{\alpha\dot\alpha} \partial_\mu \right) . \tag{2.86}$$

As emphasized above, we can relate this multiplet to the component field formulation by computing the Koszul homology of Γ. Using $\mathfrak{n}_1 = S_+ \oplus S_-$, we see that the relevant complex can be written as

$$\left(\wedge^\bullet S_+ \otimes \wedge^\bullet S_- \otimes \mathbb{C}[\bar{\lambda}_{\dot\alpha}] \, , \; \mathcal{D}_0 = \bar{\lambda}_{\dot\alpha} \frac{\partial}{\partial \bar{\theta}_{\dot\alpha}} \right) . \tag{2.87}$$

Field	Representative in the $\mathcal{D}_0$-cohomology
ϕ	ϕ
ψ	$\psi\theta$
F	$F\theta_1\theta_2$

TABLE 2.1: Representatives for the $\mathcal{N} = 1$ chiral multiplet in four dimensions organized by θ-degree.

Here we introduced coordinates on S_+ denoted by θ_α and on S_- written as $\bar{\theta}_{\dot{\alpha}}$. Since θ_α does not occur in the differential $\mathcal{D}_0$, we find that the cohomology is a tensor product

$$\wedge^\bullet S_+ \otimes H^\bullet(\wedge^\bullet S_- \otimes \mathbb{C}[\bar{\lambda}_{\dot{\alpha}}]) . \tag{2.88}$$

However, it is easy to see that the second factor is acyclic, i.e. $H^\bullet(\wedge^\bullet S_- \otimes \mathbb{C}[\lambda_\alpha]) = \mathbb{C}$. Thus, reinstalling the spacetime dependence, the $\mathcal{D}_0$-cohomology reads

$$\wedge^\bullet S_+ \otimes C^\infty(V) . \tag{2.89}$$

We immediately see that we are dealing with two scalar fields in degrees 0 and 2 and a Weyl fermion in degree 1. This is precisely the field content of the chiral multiplet. In Table 2.1, we display the corresponding representatives and relate them to the component fields of the chiral multiplet.

It is clear that the differential $\mathcal{D}'_1$ acts trivially on these component fields. Hence, there are also no further terms induced by homotopy transfer. We thus obtain a multiplet described by a of super vector bundle

$$E' = V \times \wedge^\bullet S_+ , \tag{2.90}$$

with differential $\mathcal{D}' = 0$.

As the differential D vanishes, this is one of the rare cases where the supersymmetry algebra acts strictly on the component fields. Expanding $Q = \epsilon^\alpha Q_\alpha$ and $\bar{Q} = \bar{\epsilon}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}$ we have

$$\begin{aligned} \rho(Q) &= \epsilon\mathcal{Q} = \epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} - i(\epsilon\sigma^\mu\bar{\theta})\partial_\mu \\ \rho(\bar{Q}) &= \bar{\epsilon}\bar{\mathcal{Q}} = \bar{\epsilon}^{\dot{\alpha}} \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu\bar{\epsilon})\partial_\mu . \end{aligned} \tag{2.91}$$

The transferrred action only has a $\rho'^{(1)}$ component, which is given explicitly by

$$\begin{aligned}\rho'^{(1)}(Q) &= p \circ \rho(Q) \circ i = \epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} \\ \rho'^{(1)}(\bar{Q}) &= p \circ \rho(\bar{Q}) \circ i = i(\theta\sigma^\mu\bar{\epsilon})\partial_\mu \,.\end{aligned} \tag{2.92}$$

Now we can apply these to the representatives to find

$$\begin{aligned}\rho'^{(1)}(Q)(\phi) &= 0 & \rho'^{(1)}(\bar{Q})(\phi) &= -i\bar{\epsilon}\partial\!\!\!/\phi\theta \\ \rho'^{(1)}(Q)(\psi\theta) &= \epsilon\psi & \rho'^{(1)}(\bar{Q})(\theta^\beta\psi_\beta) &= i\bar{\epsilon}\partial\!\!\!/\psi\theta^1\theta^2 \\ \rho'^{(1)}(Q)(F\theta^1\theta^2) &= \epsilon F\theta & \rho'^{(1)}(\bar{Q})(\theta^1\theta^2 F) &= 0 \,.\end{aligned} \tag{2.93}$$

Writing these relations in terms of operators we obtain the usual supersymmetry transformation rules.

$$\begin{aligned}\delta\phi &= \epsilon\psi \\ \delta\psi &= i\bar{\epsilon}\partial\!\!\!/\phi + \epsilon F \\ \delta F &= -i\bar{\epsilon}\partial\!\!\!/\psi \,.\end{aligned} \tag{2.94}$$

2.3.6 Computational techniques: Koszul homology via free resolutions[5]

In the above example, we were able to compute the cohomology by hand and even could write down explicit representatives easily. In general, such computations are much more convoluted such that we have to rely on more advanced techniques. In this section, we show how the cohomology can be computed from the minimal free resolution of the module Γ and the corresponding Hilbert series. This allows for a fairly direct identification of the ingredients of the multiplet. Further, using tools from the study of spectral sequences, we can write down explicit formulas for the representatives.

Koszul homology and minimal free resolutions. Let us fix a nilpotence variety Y and an R/I-module Γ. To understand the component field description of the multiplet $A^\bullet(\Gamma)$, we are interested in the Koszul homology of Γ. The following proposition shows that we can understand this by considering a free minimal resolution of Γ as an R-module.

Proposition 2.3.3 ([MSX12; KL09]). *Let $L^\bullet \to \Gamma \to 0$ be the minimal free resolution of Γ in free R-modules. Then*

$$H^\bullet(K^\bullet(\Gamma)) \cong L^\bullet \otimes_R \mathbb{C} \,. \tag{2.95}$$

[5]A preliminary version of this section was part of [Hah20].

Proof. We denote the differential on the minimal free resolution $L^\bullet$ by d_L. By definition we have

$$H^k(L^\bullet, d_L) = \begin{cases} \Gamma, & \text{if } k = 0 \\ 0, & \text{else.} \end{cases} \tag{2.96}$$

This implies that there is a quasi-isomorphism

$$\left(\wedge^\bullet \mathfrak{n}_1^\vee \otimes \Gamma \ , \ \mathcal{D}_0\right) \simeq \left(\wedge^\bullet \mathfrak{n}_1^\vee \otimes L^\bullet, \mathcal{D}_0 + d_L\right). \tag{2.97}$$

Note that the right hand side is a bicomplex, such that we can use the associated spectral sequence to compute its cohomology. We start by taking cohomology with respect to $\mathcal{D}_0$ and thus consider

$$\left(\wedge^\bullet \mathfrak{n}_1^\vee \otimes L^\bullet \ , \ \mathcal{D}_0\right). \tag{2.98}$$

It is easy to see that

$$H^k\left(\wedge^\bullet \mathfrak{n}_1^\vee \otimes R[-l] \ , \ \mathcal{D}_0\right) = \begin{cases} \mathbb{C}, & \text{if } k = l \\ 0, & \text{else.} \end{cases} \tag{2.99}$$

This means that we obtain a copy of $\mathbb{C}$ for each generator of $L^\bullet$. In total we get

$$H^\bullet\left(\wedge^\bullet \mathfrak{n}_1^\vee \otimes L^\bullet \ , \ \mathcal{D}_0\right) = L^\bullet \otimes_R \mathbb{C}, \tag{2.100}$$

where the R-module structure on $\mathbb{C}$ is obtained by applying the canonical augmentation (quotienting out the maximal ideal). The differential on the first page is just the morphism induced by d_L. However, since $L^\bullet$ is minimal, d_L contains no constant terms and therefore induces the zero map on the first page. Thus, we find the result as claimed. □

The proposition reduces the task of computing Koszul homology to the task of finding a minimal free resolution of Γ in R-modules. This can easily be done with commutative algebra software such as *Macaulay2* [GS]. In order to identify the field content of the associated component field multiplet, we have to identify the Kosul homology not only as a vector space, but as a representation for the Lorentz and R-symmetry groups. In the next paragraph we show how this can be achieved by means of the Hilbert series; later we will refine the technique (see §4.2.1).

Hilbert series. Let $\Gamma = \bigoplus_{i \geq 0} \Gamma_i$ be a graded R-module generated by finitely many elements in positive degree. The Hilbert series or graded dimension of Γ is the formal power series

$$\mathrm{grdim}(\Gamma) = \sum_{n=0}^{\infty} \dim(\Gamma_n)\ t^n \in \mathbb{Z}[\![t]\!]. \tag{2.101}$$

Let $R = \mathbb{C}[\lambda]$ be the polynomial ring in a single variable λ. Since there is only a single monomial in each degree the Hilbert series takes the form

$$\mathrm{grdim}(R) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \, . \tag{2.102}$$

As the dimension is multiplicative under the tensor product, the Hilbert series of a polynomial ring in n variables $R = \mathbb{C}[\lambda_1, \ldots, \lambda_n] = \mathbb{C}[\lambda_1] \otimes \cdots \otimes \mathbb{C}[\lambda_n]$ is just the product

$$\mathrm{grdim}(R) = \frac{1}{(1-t)^n} \, . \tag{2.103}$$

Now suppose we perform a shift $R(-d)$ with respect to the polynomial degree such that the constants are in degree d. We obtain for the Hilbert series

$$\begin{aligned} \mathrm{grdim}(R(-d)) &= \sum_{n=0}^{\infty} \dim(R(-d)_n) \; t^n \\ &= \sum_{n=0}^{\infty} \dim(R_{n-d}) \; t^n \\ &= t^d \mathrm{grdim}(R) \\ &= \frac{t^d}{(1-t)^n} \, . \end{aligned} \tag{2.104}$$

Thus, considering a free R-module Γ generated by elements in degree $d_1, \ldots, d_k$ we find

$$\mathrm{grdim}(\Gamma) = \frac{t^{d_1} + \cdots + t^{d_k}}{(1-t)^n} \, . \tag{2.105}$$

The Hilbert series is additive with respect to short exact sequences. This means given a sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \, , \tag{2.106}$$

we find

$$\mathrm{grdim}(B) = \mathrm{grdim}(A) + \mathrm{grdim}(C) \, . \tag{2.107}$$

If $L^\bullet$ is a free resolution of Γ, we have a sequence

$$\Gamma \longleftarrow L^0 \longleftarrow L^{-1} \longleftarrow \cdots \longleftarrow L^{-(k-1)} \longleftarrow L^{-k} \longleftarrow 0 \, . \tag{2.108}$$

Then the additivity implies

$$\mathrm{grdim}(\Gamma) = \sum_{j=1}^{k} (-1)^{j-1} \mathrm{grdim}(L^{-j}). \tag{2.109}$$

Using this together with (2.105), we can express the Hilbert series of Γ in terms of the degrees of the basis vectors of the free resolution

$$\mathrm{grdim}(\Gamma) = \sum_{j=1}^{k} (-1)^{j-1} \frac{t^{d_1^j} + \cdots + t^{d_{n_j}^j}}{(1-t)^n} . \tag{2.110}$$

Coming back to our original question, we see that the Hilbert series of Γ as a R-module contains all the information about the degrees of a basis of the minimal free resolution, which in turn coincides with the Koszul cohomology. All we have to do is to store the information about the transformation behavior under Lorentz and R-symmetry in the grading. Therefore, we assign to λ^i the degree

$$\deg(\lambda^i) = (1, w_1^i, \ldots, w_l^i) , \tag{2.111}$$

where $w_1^i, \ldots, w_l^i$ are the weights of the Lorentz and R-symmetry representation. The first entry 1 remembers the weight degree. The Hilbert series then becomes a polynomial in $l+1$ variables $t_0, \ldots, t_l$. Equation (2.110) remains valid, but we have to replace $t^{d_i^j}$ by products of $t_0, \ldots, t_l$ where each factor carries an exponent according to its weight. Initializing such a grading in *Macaulay2* and computing the Hilbert series, we can read off the weights of a basis of the cohomology in each degree, allowing to identify the cohomology as a representation of Lorentz- and R-symmetry.

Identifying representatives. Examining the proof of Proposition 2.3.3 closely, we can deduce a procedure to write down explicit representatives for the cohomology classes that correspond to the component fields of the multiplet. Recall that we used the quasi-isomorphism

$$\big(\wedge^\bullet \mathfrak{n}_1^\vee \otimes \Gamma \ , \ \mathcal{D}_0\big) \simeq \big(\wedge^\bullet \mathfrak{n}_1^\vee \otimes L^\bullet \ , \ \mathcal{D}_0 + d_L\big) . \tag{2.112}$$

On the right side we have a double complex of the form shown in Table 2.2. There are two different spectral sequences computing the total cohomology: the horizontal sequence starting with the differential d_L and the vertical sequence starting with $\mathcal{D}_0$. In the proof of Proposition 2.3.3 we have seen that the latter already gives the exact result for the zero mode cohomology on the first page. The computational procedure amounts to coming to a better understanding of this bicomplex.

It is a fact that any bicomplex can be understood (non-canonically) as the sum of different indecomposable pieces [Ste21]. These pieces are squares

$$\begin{array}{ccc} \bullet & \longleftarrow & \bullet \\ \big\downarrow & & \big\downarrow \\ \bullet & \longleftarrow & \bullet \end{array} \tag{2.114}$$

$$\begin{tikzcd}
\vdots \arrow[d, "\mathcal{D}_0"] & \vdots \arrow[d, "\mathcal{D}_0"] & \vdots \arrow[d, "\mathcal{D}_0"] & \\
L^0 \otimes \wedge^2 \mathfrak{n}_1^\vee \arrow[d, "\mathcal{D}_0"] & L^{-1} \otimes \wedge^2 \mathfrak{n}_1^\vee \arrow[l, "d_L"'] \arrow[d, "\mathcal{D}_0"] & L^{-2} \otimes \wedge^2 \mathfrak{n}_1^\vee \arrow[l, "d_L"'] \arrow[d, "\mathcal{D}_0"] & \dots \arrow[l, "d_L"'] \\
L^0 \otimes \wedge^1 \mathfrak{n}_1^\vee \arrow[d, "\mathcal{D}_0"] & L^{-1} \otimes \wedge^1 \mathfrak{n}_1^\vee \arrow[l, "d_L"'] \arrow[d, "\mathcal{D}_0"] & L^{-2} \otimes \wedge^1 \mathfrak{n}_1^\vee \arrow[l, "d_L"'] \arrow[d, "\mathcal{D}_0"] & \dots \arrow[l, "d_L"'] \\
L^0 \otimes \wedge^0 \mathfrak{n}_1^\vee & L^{-1} \otimes \wedge^0 \mathfrak{n}_1^\vee \arrow[l, "d_L"'] & L^{-2} \otimes \wedge^0 \mathfrak{n}_1^\vee \arrow[l, "d_L"'] & \dots \arrow[l, "d_L"']
\end{tikzcd} \tag{2.113}$$

TABLE 2.2: The bicomplex obtained by using a free resolution to compute Koszul homology

and stairs of different lengths

$$\begin{tikzcd}
 & \bullet \arrow[d] & & & \bullet \arrow[d] & & \\
\bullet & & \bullet & \bullet \arrow[l] & & & \dots \\
 & \bullet & & & \bullet & \bullet \arrow[l] &
\end{tikzcd} \tag{2.115}$$

Here, the bullet denotes the underlying field $\bullet = K$. Crucially, the decomposition can be chosen such that all the arrows are just identity maps. The length of a stair is the number of bullets $\bullet$ occurring.

One can understand the behavior of spectral sequences by thinking about the ways that these indecomposable pieces contribute to cohomology. It is a matter of inspection to see that stairs of even length are acyclic at the E_1 page of one of the two spectral sequences of the bicomplex, but contribute two generators to the E_1 page of the other which cannot be canceled by the differential on that page just for degree reasons. It is thus precisely the (vertically or horizontally oriented) stairs of length $2k$ that contribute to differentials on the E_k page of the corresponding spectral sequence. In contrast, stairs of odd length contribute to the total cohomology of the complex, but do so in a bidegree that depends on which spectral sequence is being considered. If we consider such a stair, we see that the cohomology with respect to the horizontal differential is concentrated at the upper end, while the cohomology with respect to the vertical differential lives at the lower end. They are thus responsible for the breaking of the bigrading to the single homological grading of the total complex.

Now note that the cohomology of our double complex (2.113) is concentrated in the bottom row $L^\bullet \otimes \wedge^0 \mathfrak{n}_1^\vee$ (for the vertical differential $\mathcal{D}_0$) and on the left column $L^0 \otimes \wedge^\bullet \mathfrak{n}_1^\vee$ (for the horizontal differential d_L). This implies that we have odd stairs contributing to the cohomology in the following manner:

$$\begin{tikzcd}
\bullet \arrow[d] & \dots & & & \\
\bullet & \bullet \arrow[l] \arrow[d] & \dots & & \\
\bullet \arrow[u, leftarrow] \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \dots & \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] & \dots
\end{tikzcd} \tag{2.116}$$

Classes in the total cohomology can be represented by elements on either end of the stair. However, if we want to view the representatives as elements in $\wedge^\bullet \mathfrak{n}_1^\vee \otimes \Gamma$, we have to apply the spectral sequence starting with d_L, which amounts to choosing the representatives on the upper end $\wedge^\bullet \mathfrak{n}_1^\vee \otimes L^0$ and then projecting onto the quotient. On the other hand, a basis of the vertical $\mathcal{D}_0$-cohomology is clearly provided by the standard basis $\{e_i\} \subseteq L^{-k} \otimes \wedge^0 \mathfrak{n}_1^\vee = R^{n_k}$. In order to get the desired basis in $\wedge^k \mathfrak{n}_1^\vee \otimes L^0$ we have to walk up the corresponding stair. Since we are now working only with Koszul complexes of maximal ideals in polynomial rings, this can be done explicitly by defining a simple "inverse" or "adjoint" differential to $\mathcal{D}_0$ by the formula[6]

$$\mathcal{D}_0^\dagger = \theta^\alpha \frac{\partial}{\partial \lambda^\alpha}. \tag{2.117}$$

Then our discussion implies the following lemma.

Lemma 2.3.4. *Let $\pi : L_0 \longrightarrow \Gamma$ be the projection. The elements $\pi((\mathcal{D}_0^\dagger d_L)^k e_i)$ form a basis of the cohomology $H^\bullet(\wedge^\bullet \mathfrak{n}_1^\vee \otimes \Gamma)$ in θ-degree k.*

2.3.7 Homotopy transfer to component fields

The new differential acting on the component fields, as well as the action of the supersymmetry algebra and, if present, an L_∞ structure are obtained from the respective structures on $A^\bullet(\Gamma)$ via homotopy transfer. For this we need homotopy data

$$\begin{tikzcd}
(\mathrm{Gr}\, A^\bullet(\Gamma), \mathcal{D}_0) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H^\bullet(\mathrm{Gr}\, A^\bullet(\Gamma)), 0) \arrow[l, shift left, "i"]
\end{tikzcd}\,. \tag{2.118}$$

Using Lemma 2.3.4, we can define an inclusion map

$$i : H^\bullet(\mathrm{Gr}\, A^\bullet(\Gamma)) \hookrightarrow (A^\bullet(\Gamma), \mathcal{D}_0) \tag{2.119}$$

[6]Note that this differential is well defined on $\wedge^\bullet \mathfrak{n}_1^\vee \otimes L^\bullet$, since $L^\bullet$ consists of free R-modules. In particular $\mathcal{D}_0^\dagger$ will in general not descend to a well defined map on $\wedge^\bullet \mathfrak{n}_1^\vee \otimes \Gamma$.

by sending a cohomology class to this representative. This inclusion is a quasi-isomorphism. In addition, choosing a complementary subspace inside $A^\bullet(\Gamma)$ gives the projection p. (We always work equivariantly with respect to Lorentz and R-symmetry.)

The differential. Recall that we decomposed the differential on $A^\bullet(\Gamma)$ as the sum of two pieces, of filtered weight zero and two, respectively:

$$\mathcal{D} = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} - \lambda^\alpha \theta^\beta f^\mu_{\alpha\beta} \frac{\partial}{\partial x^\mu} = \mathcal{D}_0 + \mathcal{D}_1. \tag{2.120}$$

We can thus view $\mathcal{D}_1$ as defining a deformation of the differential on $\operatorname{Gr} A^\bullet(\Gamma)$, which in turn equips $H^\bullet(\operatorname{Gr} A^\bullet(\Gamma))$ with a new differential $\mathcal{D}'$ that is obtained by homotopy transfer of D_∞ structure [DSV15; LV12b; Lap01]. This uses the choice of a homotopy datum to write all of the higher differentials of a spectral sequence as terms in a single differential, acting on the E_1 page, whose cohomology is E_∞. In formulas, we have

$$\mathcal{D}' = \sum_{n=1}^{\infty} \mathcal{D}'_n \tag{2.121}$$

where the pieces are given by

$$\mathcal{D}'_n = p \circ \big((\mathcal{D}_1 h)^{n-1} \mathcal{D}_1 \big) \circ i. \tag{2.122}$$

(Note that, due to our conventions for the filtration, only differentials on *even* pages are non-trivial; the differential on page $2n$ is represented by $\mathcal{D}'_n$ above.) Furthermore, we can fix new homotopy data [Lap01]

$$h' \circlearrowright (A, \mathcal{D}) \underset{i'}{\overset{p'}{\rightleftarrows}} \left(H^\bullet(A, \mathcal{D}_0) \, , \, \mathcal{D}' \right) , \tag{2.123}$$

where

$$\begin{aligned} i' &= \sum_{n=0}^{\infty} i'_n = \sum_{n=0}^{\infty} (h \mathcal{D}_1)^n \circ i \\ p' &= \sum_{n=0}^{\infty} p'_n = p \circ \sum_{n=0}^{\infty} (\mathcal{D}_1 h)^n \\ h' &= \sum_{n=0}^{\infty} h'_n = h \circ \sum_{n=0}^{\infty} (\mathcal{D}_1 h)^n \, . \end{aligned} \tag{2.124}$$

We can use this homtopy data to transfer further structures, such as the action of the supersymmetry algebra or an L_∞ structure, from $A^\bullet(\Gamma)$ to the component field description. Note that, in terms of sum-over-trees formulas, homotopy transfer with respect to the new homotopy data from (2.123) is expressed in terms of (2.118) simply by allowing for unary vertices which are decorated by $\mathcal{D}_1$.

The supersymmetry action. The supersymmetry action is obtained by a homotopy transfer of L_∞ module structures. As a result one obtains an map of super L_∞ algebras

$$\rho' \colon \mathfrak{g} \rightsquigarrow \big(\mathcal{D}(E')\,,\,[\mathcal{D}', -]\big), \tag{2.125}$$

whose component maps can be obtained via sum over trees formulas. For example $\rho'^{(2)}$ is given by

$$\rho'^{(2)}(x_1, x_2) = p' \circ \big(\rho(x_1) \circ h' \circ \rho(x_2) \pm \rho(x_2) \circ h' \circ \rho(x_1)\big) \circ i\,. \tag{2.126}$$

Interestingly, there is a close link between the resolution differential and the action of the supersymmetry algebra. This connection was already conjectured in [Ber02], where it was noticed that the non-derivative supersymmetry transformations and their closure terms appear in the resolution differential of eleven-dimensional supergravity. Using our knowledge on the representatives and the homotopy transfer description of the action of the supersymmetry transformations we can make this observation precise and provide a derivation. Later, we will see that this result is an easy consequence of the equivalence of categories developed in §3 (see Corollary 3.4.8).

We note that the strict part of a non-derivative supersymmetry transformation acts by

$$\mathcal{Q}_0 := \rho_{\partial_x = 0}(Q) = \epsilon^\alpha \frac{\partial}{\partial \theta^\alpha}\,. \tag{2.127}$$

In addition it is easy to see that

$$[\mathcal{Q}_0, \mathcal{D}_0^\dagger] = \epsilon^\alpha \frac{\partial}{\partial \lambda^\alpha} \tag{2.128}$$

and obviously

$$[\mathcal{Q}_0, d_L] = 0\,. \tag{2.129}$$

Now suppose $\mathcal{Q}_0$ acts on a representative in θ-degree k

$$\begin{aligned} \rho'^{(1)}_{\partial_x=0}(f) &= p \circ \mathcal{Q}_0 \circ i(f) \\ &= p \circ \mathcal{Q}_0 \circ \pi (\mathcal{D}_0^\dagger d_L)^k (f^i e_i^{(k)})\,. \end{aligned} \tag{2.130}$$

Here $(e_i^{(k)})$ denotes a basis of $L^{-k} \otimes_R \mathbb{C}$ and $\pi : L^0 \longrightarrow \Gamma$ the projection. Note that $\mathcal{Q}_0 \circ \pi = \pi \circ \mathcal{Q}_0$. In the following, we abbreviate the components of the resolution differential by $d_k := (d_L)_k$. Now we can use the relation (2.128) to bring $\mathcal{Q}_0$ to the right.

We find

$$\rho'^{(1)}_{\partial_x=0}(f) = p \circ \pi \left(\sum_{j=1}^{k} \mathcal{D}_0^\dagger d_1 \dots \mathcal{D}_0^\dagger d_{j-1} \epsilon \frac{\partial}{\partial \lambda} d_j \mathcal{D}_0^\dagger \dots d_k(f) \right) . \tag{2.131}$$

Since we already know the explicit form of the representatives, we can carry out the projection to $\mathcal{D}_0$-cohomology directly. The only remaining term is the following.

$$\rho'^{(1)}_{\partial_x=0}(f) = \pi \left((\mathcal{D}_0^\dagger d_L)^{k-1} \epsilon \frac{\partial}{\partial \lambda} d_k (f^i e_i^{(k)}) \right) . \tag{2.132}$$

Furthermore, only the part of $(d_L)_k$ linear in λ can contribute in $\mathcal{D}_0$-cohomology. Then $\epsilon \frac{\partial}{\partial \lambda}$ simply replaces λ with ϵ in the d_k. Let us denote the resulting map by d_k^ϵ and its components by $(d_k^\epsilon)_i{}^j$. Then we find

$$\begin{aligned} \rho'^{(1)}_{\partial_x=0}(f) &= \pi \left((\mathcal{D}_0^\dagger d_L)^{k-1} (d_k^\epsilon)_i{}^j e_j^{(k-1)} f^i \right) \\ &= \pi \left((\mathcal{D}_0^\dagger d_L)^{k-1} (e_j^{(k-1)}) (d_k^\epsilon)_i{}^j f^i \right) . \end{aligned} \tag{2.133}$$

Identifying the representative in degree $k-1$ and writing the transformation rule dually in terms of operators, we find

$$\delta g^j = (d_k^\epsilon)_i{}^j f^i , \tag{2.134}$$

where g^j denotes the operator corresponding to the respective representative in θ-degree $k-1$. This shows that linear parts in the resolution differential precisely correspond to the strict part of the non-derivative supersymmetry transformations.

This generalizes to the higher components of the supersymmetry action. For $n \geq 2$, the non-derivative part of $\rho^{(n)}$ acts is given by

$$\rho'^{(n)} = p \circ (\mathcal{Q}_0 \circ h \circ \mathcal{Q}_0)^{n-1} \circ i . \tag{2.135}$$

For example one finds for $\rho'^{(2)}$

$$\begin{aligned} \rho'^{(2)}_{\partial_x=0}(Q,Q)(f) &= p \circ \mathcal{Q}_0 \circ h \circ \mathcal{Q}_0 \circ i(f) \\ &= p \circ \mathcal{Q}_0 \circ h \circ \pi \left(\sum_{j=1}^{k} \mathcal{D}_0^\dagger d_1 \dots \mathcal{D}_0^\dagger d_{j-1} \epsilon \frac{\partial}{\partial \lambda} d_j \mathcal{D}_0^\dagger \dots d_k(f) \right) . \end{aligned} \tag{2.136}$$

Now assuming that the homotopy h acts via $h \circ \pi = \pi \circ \mathcal{D}_0^\dagger$ we find using $(\mathcal{D}_0^\dagger)^2 = 0$

$$\begin{aligned} \rho'^{(2)}_{\partial_x=0}(Q,Q)(f) &= p \circ \pi \left(\mathcal{Q}_0 \mathcal{D}_0^\dagger \epsilon \frac{\partial}{\partial \lambda} d_1 \mathcal{D}_0^\dagger \dots d_k(f) \right) \\ &= p \circ \pi \left(\epsilon \frac{\partial}{\partial \lambda} \mathcal{Q}_0 \mathcal{D}_0^\dagger d_1 \mathcal{D}_0^\dagger \dots d_k(f) \right) , \end{aligned} \tag{2.137}$$

where we used that $\epsilon \frac{\partial}{\partial \lambda}$ commutes with both $\mathcal{D}_0^\dagger$ and $\mathcal{Q}_0$.

Now we can again use the relation (2.128) to find

$$\rho'^{(2)}_{\partial_x=0}(Q,Q)(f) = p \circ \pi \left(\epsilon \frac{\partial}{\partial \lambda} \sum_{j=1}^{k} \mathcal{D}_0^\dagger d_1 \dots \mathcal{D}_0^\dagger d_{j-1} \epsilon \frac{\partial}{\partial \lambda} d_j \mathcal{D}_0^\dagger \dots d_k(f) \right) . \tag{2.138}$$

Carrying out the projection p on $\mathcal{D}_0$-cohomology, we see that only one term survives.

$$\rho'^{(2)}_{\partial_x=0}(Q,Q)(f) = \pi \left((\mathcal{D}_0^\dagger d)^{k-1} (e_j^{(k-1)}) (d_k^{\epsilon^2})_i{}^j f^i \right) , \tag{2.139}$$

where $d_k^{\epsilon^2}$ denotes the quadratic part of the resolution differential with λ's replaced by ϵ's. Written in terms of operators this gives a transformation rule

$$\delta g^j = (d_k^{\epsilon^2})_i{}^j f^i . \tag{2.140}$$

Using a similar calculation as above one sees that only the part of order n in the resolution differential contributes to a supersymmetry transformation and we obtain supersymmetry transformation rules of the form

$$\delta g^j = (d_k^{\epsilon^n})_i{}^j f^i . \tag{2.141}$$

Interestingly this provides a direct link between the polynomial degree of the terms in the resolution differential and the homotopy action of the supersymmetry algebra. That is, if the resolution differential is at most quadratic, then the L_∞ module structure will contain at most $\rho'^{(2)}$ corrections.

L_∞ structures. If $(A^\bullet(\Gamma), \mathcal{D})$ carries an L_∞ structure with differential $\mathcal{D}$, this structure can be transferred as well. For this one uses the usual sum over trees formulas. As we will see below, the transferred L_∞ structure on the component fields can encode the structure of gauge transformations and in some cases also interactions. Note that the new L_∞ structure has $\mu_1' = \mathcal{D}'$ the transferred differential. We will see this explicitly in the case of ten-dimensional super Yang–Mills theory.

2.3.8 An example of the technique: the 4d gauge multiplet

To illustrate these techniques, we are going to perform all the necessary calculations for the $d = 4$, $\mathcal{N} = 1$ vector multiplet by hand. Let Y be the nilpotence variety of the $\mathcal{N} = 1$ super Poincaré algebra in four dimensions. We choose the structure sheaf $\mathcal{O}_Y$ as input; in the language of [Ced+23], we construct the canonical multiplet of the $\mathcal{N} = 1$ super Poincaré algebra in four dimensions. Using *Macaulay2* we can compute the minimal free resolution. Its Betti numbers are displayed in the following table.

$$\begin{bmatrix} 1 & - & - & - \\ - & 4 & 4 & 1 \end{bmatrix} \tag{2.142}$$

Here, the horizontal axis denotes degree in θ, while the vertical axis counts powers in λ. To analyze the field content of the multiplet as representations of the Lorentz group, we assign gradings to the generators λ and $\bar{\lambda}$, corresponding to their weights under

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2) \,. \tag{2.143}$$

Concretely this means that we assign the grading

$$\begin{aligned} \deg(\lambda_1) &= (1,1,0) & \deg(\lambda_2) &= (1,-1,0) \\ \deg(\bar{\lambda}_1) &= (1,0,1) & \deg(\bar{\lambda}_2) &= (1,0,-1) \,. \end{aligned} \tag{2.144}$$

Then we examine the numerator of the Hilbert series. We organize the terms by degree in the variable t_0, which indicates the total degree in the complex. In degree 0 we simply obtain 1, which means the field in total degree 0 is a scalar. In degree 2 we find the term

$$-\, t_0^2(t_1 t_2 + t_1 t_2^{-1} + t_1^{-1} t_2 + t_1^{-1} t_2^{-1}) \,. \tag{2.145}$$

Reading off the highest weights we see that the corresponding representation of $SU(2) \times SU(2)$ is

$$[1,1] = [1,0] \otimes [0,1] \,, \tag{2.146}$$

which shows that the field in degree 2 is a vector. In degree 3 we obtain

$$t_0^3(t_1 + t_1^{-1} + t_2 + t_2^{-2}) \,. \tag{2.147}$$

Correspondingly, the representation in degree 3 is a direct sum

$$[1,0] \oplus [0,1] \,. \tag{2.148}$$

Hence, the field in degree 3 is a Dirac fermion. Finally the term of order 4 is just $-t_0^4$ indicating that the field in degree 4 is a scalar. This means that we recover the usual field content of the $d = 4$, $\mathcal{N} = 1$ vector multiplet.

To find representatives with the procedure explained above, we need the differential on the free resolution. The minimal free resolution is of the form

$$R \otimes \left(\mathbb{C} \xleftarrow{(d_L)_1} V \xleftarrow{(d_L)_2} S_+ \oplus S_- \xleftarrow{(d_L)_3} \mathbb{C} \right) \,. \tag{2.149}$$

The differential can be described by the matrices

$$(d_L)_1 = \begin{pmatrix} \lambda_1\bar\lambda_1 & \lambda_1\bar\lambda_2 & \lambda_2\bar\lambda_1 & \lambda_2\bar\lambda_2 \end{pmatrix}$$
$$(d_L)_2 = \begin{pmatrix} 0 & -\bar\lambda_2 & 0 & -\lambda_2 \\ 0 & \bar\lambda_1 & \lambda_2 & 0 \\ -\bar\lambda_2 & 0 & 0 & \lambda_1 \\ \bar\lambda_1 & 0 & \lambda_1 & 0 \end{pmatrix} \tag{2.150}$$
$$(d_L)_3 = \begin{pmatrix} \lambda_1 \\ -\lambda_2 \\ -\bar\lambda_1 \\ \bar\lambda_2 \end{pmatrix} .$$

Choosing a basis $e_{\alpha\dot\alpha}$ of V and $(s_\alpha, \bar s_{\dot\alpha})$ of $S_+ \oplus S_-$, these maps can be conveniently packaged as follows.

$$\begin{array}{ccccccccc} (d_L)_1 & : & V & \longrightarrow & \mathbb{C} & , & A & \mapsto & \lambda^\alpha\bar\lambda^{\dot\alpha}A_{\alpha\dot\alpha} \\ (d_L)_2 & : & S_+\oplus S_- & \longrightarrow & V & , & (\psi,\bar\psi) & \mapsto & (\lambda^\alpha\bar\psi^{\dot\alpha} + \psi^\alpha\bar\lambda^{\dot\alpha})e_{\alpha\dot\alpha} \\ (d_L)_3 & : & \mathbb{C} & \longrightarrow & S_+\oplus S_- & , & D & \mapsto & (\lambda^\alpha s_\alpha - \bar\lambda^{\dot\alpha}\bar s_{\dot\alpha})D \end{array} \tag{2.151}$$

Note that we can apply the isomorphism $S_+ \otimes S_- \cong V$ by a change of basis $e_\mu = (\sigma_\mu)^{\alpha\dot\alpha}e_{\alpha\dot\alpha}$. With this description, it is easy to identify representatives in $\mathcal{D}_0$-cohomology. For example, the vector is represented by

$$A \xmapsto{(d_L)_1} (\lambda\sigma^\mu\bar\lambda)A_\mu \xmapsto{\mathcal{D}_0^\dagger} (\lambda\sigma^\mu\bar\theta + \bar\lambda\sigma^\mu\theta)A_\mu \,. \tag{2.152}$$

For the fermions we find

$$\psi \xmapsto{(d_L)_2} \psi^\alpha\bar\lambda^{\dot\alpha}e_{\alpha\dot\alpha} \xmapsto{\mathcal{D}_0^\dagger} \psi^\alpha\bar\theta^{\dot\alpha}e_{\alpha\dot\alpha} \xmapsto{(d_L)_1} \psi^\alpha\bar\theta^{\dot\theta}\lambda_\alpha\bar\lambda_{\dot\alpha} \xmapsto{\mathcal{D}_0^\dagger} \psi^\alpha\bar\theta^{\dot\alpha}(\lambda_\alpha\bar\theta_{\dot\alpha} + \theta_\alpha\bar\lambda_{\dot\alpha}) \tag{2.153}$$

A similar calculation gives the complex conjugate representative for $\bar\psi$. Finally we can apply the procedure to the auxiliary field.

$$D \xmapsto{\mathcal{D}_0^\dagger\circ(d_L)_3} (\theta s - \bar\theta\bar s)D \xmapsto{(d_L)_2} (\theta^\alpha\bar\lambda^{\dot\alpha} - \lambda^\alpha\bar\theta^{\dot\alpha})e_{\alpha\dot\alpha}D \xmapsto{\mathcal{D}_0^\dagger} 2\theta^\alpha\bar\theta^{\dot\alpha}e_{\alpha\dot\alpha}D \xmapsto{(d_L)_1} 2(\theta\lambda)(\bar\theta\bar\lambda)D \xmapsto{\mathcal{D}_0^\dagger} 2(\theta^2\bar\lambda\bar\theta + \bar\theta^2\lambda\theta)D \tag{2.154}$$

We summarize these representatives in Table 2.3. Note that these representatives are not unique. Other choices are possible; for example one can simplify these representatives by eliminating terms in the image of $\mathcal{D}_0$. For instance the antisymmetric expression

$$\lambda_\alpha\bar\theta_{\dot\alpha} - \bar\lambda_{\dot\alpha}\theta_\alpha \tag{2.155}$$

Field	Representative in the $\mathcal{D}_0$-cohomology
c	c
A	$(\lambda\sigma^\mu\bar{\theta} + \theta\sigma^\mu\bar{\lambda})A_\mu$
ψ	$\psi^\alpha\bar{\theta}^{\dot{\alpha}}(\lambda_\alpha\bar{\theta}_{\dot{\alpha}} + \theta_\alpha\bar{\lambda}_{\dot{\alpha}})$
$\bar{\psi}$	$\bar{\psi}^{\dot{\alpha}}\theta^\alpha(\bar{\lambda}_{\dot{\alpha}}\theta_\alpha + \bar{\theta}_{\dot{\alpha}}\lambda_\alpha)$
D	$(\theta^2\bar{\lambda}\bar{\theta} + \bar{\theta}^2\lambda\theta)D$

TABLE 2.3: Representatives for the $d = 4$, $\mathcal{N} = 1$ vector multiplet organized by θ-degree.

is clearly in the image of $\mathcal{D}_0$. This implies that we could represent the vector equally well by $\lambda_\alpha\bar{\theta}_{\dot{\alpha}}$. Similar observations also hold for the other fields.

Let us now study the different structures arising on the component fields via homotopy transfer.

The differential. By degree reasons, only the first order part $\mathcal{D}_1'$ of the transferred differential $\mathcal{D}'$ can act non-trivially on the component fields. Recall

$$\mathcal{D}_1' = p \circ \mathcal{D}_1 \circ i \,, \tag{2.156}$$

where

$$\mathcal{D}_1 = (\lambda\sigma^\mu\bar{\theta} + \bar{\lambda}\sigma^\mu\theta)\partial_\mu \,. \tag{2.157}$$

The only non-vanishing contribution arises by acting on the ghost. There we find

$$\mathcal{D}_1 c = (\lambda\sigma^\mu\bar{\theta} + \bar{\lambda}\sigma^\mu\theta)\partial_\mu c \,. \tag{2.158}$$

Identifying the representative of the gauge field, we see that the differential is simply the de Rham differential

$$c \mapsto \mathrm{d}c \,. \tag{2.159}$$

Written dually in terms of operators this gives the BRST differential

$$Q_{BRST} A_\mu = \partial_\mu c \,. \tag{2.160}$$

The following picture summarizes the complex on the component field level.

$$
\begin{tikzcd}
\Omega^0(\mathbb{R}^4) \arrow[dr, "\mathrm{d}"] & & & \\
& \Omega^1(\mathbb{R}^4) & \Gamma(\mathbb{R}^4, S_+ \oplus S_-) & \Gamma(\mathbb{R}^4, \mathbb{C})\ .
\end{tikzcd}
\tag{2.161}
$$

The action of the supersymmetry algebra. As explained above, we can read off the non-derivative part of the supersymmetry transformations directly from the resolution differential. This gives transformation rules

$$
\begin{aligned}
\delta c &= (\epsilon\sigma^\mu\bar{\epsilon})A_\mu \\
\delta A_\mu &= \epsilon\sigma_\mu\bar{\psi} + \psi\sigma_\mu\bar{\epsilon} \\
\delta\psi &= \epsilon D \\
\delta\bar{\psi} &= -\bar{\epsilon}D \\
\delta D &= 0\ .
\end{aligned}
\tag{2.162}
$$

Note that there is one higher order component indicating that the action of the supersymmetry algebra is not strict. We will come back to this in a moment.

First, let us investigate the contributions containing derivatives. By degree reasons there cannot appear any higher order contributions containing derivatives, such that we can focus on the strict part. The derivative part of $\rho'^{(1)}$ acts on the representatives by

$$
\mathcal{Q}_1 = \epsilon\sigma^\mu\bar{\theta}\partial_\mu + \theta\sigma^\mu\bar{\epsilon}\partial_\mu\ . \tag{2.163}
$$

For example we can act on the fermions to find

$$
\begin{aligned}
\mathcal{Q}_1(\psi) =& (\bar{\epsilon}\sigma_\mu\theta)\partial_\mu\psi^\alpha\bar{\theta}^{\dot{\alpha}}(\lambda_\alpha\bar{\theta}_{\dot{\alpha}} + \theta_\alpha\bar{\lambda}_{\dot{\alpha}}) \\
=& (\bar{\epsilon}^{\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}\partial_\mu\psi^\alpha)(\lambda_\alpha\theta^\beta\bar{\theta}^2 + \theta_\alpha\theta^\beta\bar{\lambda}\bar{\theta})\ .
\end{aligned}
\tag{2.164}
$$

Projecting to cohomology this equals

$$
\bar{\epsilon}\partial\!\!\!/\psi(\lambda\theta\bar{\theta}^2 + \theta^2\bar{\lambda}\bar{\theta})\ , \tag{2.165}
$$

such that we can identify a transformation rule

$$
\delta D = \bar{\epsilon}\partial\!\!\!/\psi\ . \tag{2.166}
$$

A similar calculation also holds for the complex conjugate $\bar{\psi}$, as well as for the gauge field and yield the usual supersymmetry transformation rules.

This describes the entire L_∞ module structure of the superymmetry algebra on the four-dimensional, $\mathcal{N} = 1$ vector multiplet. The $\rho'^{(1)}$ part resembles the well known supersymmetry transformations from standard physics textbooks. In addition there is one higher correction. Recall that we found a transformation rule

$$\delta c = (\epsilon\sigma^\mu\bar{\epsilon})A_\mu \,. \tag{2.167}$$

This corresponds to a map $\rho'^{(2)}$ given by

$$\rho'^{(2)} : \mathfrak{n}\otimes\mathfrak{n}\otimes\Omega^1 \longrightarrow \Omega^0 \qquad (Q_1\otimes Q_2\otimes A) \mapsto \iota_{[Q_1,Q_2]}A \,. \tag{2.168}$$

We can immediately check that $\rho'^{(2)}$ indeed defines a homotopy correcting for the failure of $\rho'^{(1)}$ to be strict. We clearly have

$$\rho'^{(1)}(Q)(c) = \rho'^{(1)}(\bar{Q}) = 0 \,. \tag{2.169}$$

However, the bracket of Q and $\bar{Q}$ gives a translation which acts via the Lie derivative

$$[Q,\bar{Q}](c) = L_{[Q,\bar{Q}]}(c) \,. \tag{2.170}$$

Thus, according to (2.26) we have to check

$$L_{[Q,\bar{Q}]}(c) = -[D,\rho'^{(2)}(Q,\bar{Q})](c) \,. \tag{2.171}$$

Plugging in $D = \mathrm{d}$ the de Rham differential, we obtain

$$\begin{aligned} L_{[Q,\bar{Q}]}(c) =& -(\mathrm{d}\circ\iota_{[Q_1,Q_2]} - \iota_{[Q_1,Q_2]}\circ\mathrm{d})(c) \\ =& (\iota_{[Q_1,Q_2]}\circ\mathrm{d})(c) \,, \end{aligned} \tag{2.172}$$

where the first term vanishes by degree reasons. We immediately see that this is indeed satisfied due to Cartan's magic formula. This discussion illustrates that the $\rho'^{(2)}$-term indeed provides a homotopy for the failure of $\rho'^{(1)}$ to be strict. In terms of physics terminology, $\rho'^{(2)}$ is a closure term for the supersymmetry action, which closes only up to gauge transformations.

L_∞ **structure.** To treat the non-abelian vector multiplet we can tensor the entire construction with a Lie algebra $\mathfrak{h}$. We notice that R/I is not only an R/I-module, but a ring. Hence, $A^\bullet(R/I)$ carries an algebra structure such that the tensor product $A^\bullet(R/I)\otimes\mathfrak{h}$ comes equipped with an L_∞ structure given by

$$\mu_1 = \mathcal{D}\otimes\mathrm{id}_{\mathfrak{h}} \qquad \mu_2 = m_2\otimes[-,-] \,. \tag{2.173}$$

Here m_2 denotes the multiplication in $A^\bullet(R/I)$. Since the differential does not interfere with the Lie algebra at all, the component fields of the multiplet take values in $H^\bullet(K^\bullet(R/I)) \otimes \mathfrak{h}$. This is just the field content of the abelian version only now taking values in the Lie algbera $\mathfrak{h}$. The transfer of the L_∞ algebra structure to the component fields is very simple. The differential only acts on the ghost fields via the de Rham differential.

$$\mu_1' = \mathrm{d} \otimes \mathrm{id}_{\mathfrak{h}} : \Omega^0 \otimes \mathfrak{h} \longrightarrow \Omega^1 \otimes \mathfrak{h} \,. \tag{2.174}$$

In addition to the differential, only two-ary brackets arise.

$$\begin{array}{llcll} \mu_2' : & \Omega^0 \otimes \mathfrak{h} \times \Omega^0 \otimes \mathfrak{h} & \longrightarrow & \Omega^0 \otimes \mathfrak{h} & \mu_2'(c,c) = [c,c] \\ \mu_2' : & \Omega^0 \otimes \mathfrak{h} \times \Omega^1 \otimes \mathfrak{h} & \longrightarrow & \Omega^1 \otimes \mathfrak{h} & \mu_2'(c,A) = [c,A] \\ \mu_2' : & \Omega^0 \otimes \mathfrak{h} \times \Gamma(X, S_+ \oplus S_-) \otimes \mathfrak{h} & \longrightarrow & \Gamma(X, S_+ \oplus S_-) \otimes \mathfrak{h} & \mu_2'(c,\psi) = [c,\psi] \,. \end{array} \tag{2.175}$$

We can also write these dually as a BRST operator.

$$\begin{aligned} Q_{BRST} c &= -\frac{1}{2}[c,c] \\ Q_{BRST} A &= dc + [A,c] \\ Q_{BRST} \psi &= [\psi, c] \\ Q_{BRST} D &= [D, c] \,. \end{aligned} \tag{2.176}$$

Hence, we recover the usual BRST complex of the $d = 4$, $\mathcal{N} = 1$ gauge multiplet. To equip the multiplet with a BRST datum, we could write the usual component field action for the gauge multiplet. In the terminology of §2.2 this action then makes the multiplet a BRST theory.

2.3.9 Scheme-theoretic properties: three-dimensional $\mathcal{N} = 1$ supersymmetry

In three dimensions we have the isomorphism $\mathrm{Spin}(3) \cong SU(2)$. The vector representation V corresponds to the three-dimensional representation of $SU(2)$, while the spinor representation S is given by the two-dimensional representation. The bracket is provided by the isomorphism

$$\mathrm{Sym}^2(S) \cong V \,. \tag{2.177}$$

Therefore, as a set, the nilpotence variety is simply a point

$$Y = \{0\} \,. \tag{2.178}$$

Even though the nilpotence variety, regarded as a set, is just a point it still may carry an interesting structure as a scheme which allows for the construction of different multiplets. Expanding the equation $[Q, Q] = 0$ in coordinates (λ^1, λ^2) we obtain the equations

$$(\lambda^1)^2 = \lambda^1\lambda^2 = (\lambda^2)^2 = 0\,. \tag{2.179}$$

Clearly, the only solution to these equations is $\lambda^1 = \lambda^2 = 0$. However, the quotient ring $R/I \ncong \mathbb{C}$ differs from just the constants which are the functions on the point considered as an affine variety. This reflects the fact that the affine scheme $\mathrm{Spec}(R/I)$ is not just an ordinary point, but what is called a fat point. As we will see momentarily, this allows us to construct different multiplets from R/I-modules, even though there are no non-trivial square-zero supercharges.

The gauge multiplet. First of all we can consider R/I itself as an equivariant module. This gives rise to the gauge multiplet in three dimensions. The minimal free resolution has the following Betti numbers.

$$\begin{bmatrix} 1 & - & - \\ - & 3 & 2 \end{bmatrix} \tag{2.180}$$

In terms of representations, the free resolution takes the form

$$R \otimes \left(\mathbb{C} \xleftarrow{(d_L)_1} V \xleftarrow{(d_L)_2} S \right) \tag{2.181}$$

with the differentials being described by

$$\begin{array}{ccccccccc} (d_L)_1 & : & V & \longrightarrow & \mathbb{C} & , & A & \mapsto & (\lambda\sigma^\mu\lambda)A_\mu \\ (d_L)_2 & : & S & \longrightarrow & V & , & \psi & \mapsto & (\lambda\sigma^\mu\psi)e_\mu\,. \end{array} \tag{2.182}$$

Thus, we find that the multiplet contains a one-form field together with its ghost as well as a fermion. The only differential acting on the component fields is the de Rham differential

$$c \mapsto \mathrm{d}c \tag{2.183}$$

which encodes the gauge invariance of the one-form. The non-derivative supersymmetry transformations can be read off from the resolution differential and take the usual form.

$$\begin{aligned} \delta c &= (\epsilon\sigma^\mu\epsilon)A_\mu \\ \delta A_\mu &= \epsilon\sigma_\mu\psi \\ \delta\psi &= 0\,. \end{aligned} \tag{2.184}$$

The free superfield. In addition, we can also consider $\mathbb{C} = R/(\lambda^1, \lambda^2)$ as an R/I-module. This yields the free superfield whose Betti numbers we display in the following table.

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \tag{2.185}$$

Indeed, the Koszul homology of this module is just an exterior algebra $\wedge^\bullet S$ and we just recover the usual superspace description of the free superfield.

2.4 From multiplets to theories

In §2.2.5, we introduced the notions of BV and BRST data for multiplets. Under certain conditions on the module Γ, the Koszul homology is naturally equipped with a perfect pairing that equips the corresponding multiplet with a BV datum. This provides an interesting link between the physics of supersymmetric multiplets and the algebraic geometry of $\mathcal{O}_Y$-modules. In fact, the pure spinor formalism provides many such links between algebrogeometric properties of the module Γ and physical properties of the multiplet. We start exploring these in the following section.

2.4.1 Commutative algebra and dualizing complexes

Let us start with a short survey of the relevant notions from commutative algebra. As motivated by the pure spinor superfield formalism, we are mostly interested in quotients of polynomial rings $R = \mathbb{C}[\lambda^1, \ldots, \lambda^n]$ by (quadratic) ideals I. We think of R/I as the ring of functions on a nilpotence variety. The main sources for our discussion are [Eis95; Bas63; Hun99].

Definition 2.4.1. A quotient ring R/I is called a complete intersection, if I can be generated by $r = \mathrm{codim}(R/I) = n - \dim(R/I)$ elements, i.e. $I = (f_1, \ldots, f_r)$.

Intuitively, this definition means that there are no non-trivial relations among the f_i such that each equation cuts the dimension of the corresponding variety by one. Equivalently we can say that $f_1, \ldots, f_r$ forms a regular sequence on R. To be clear we recall the definition.

Definition 2.4.2. Let S be a commutative ring and M a S-module. A sequence $(x_1, \ldots, x_k) \subset S$ is called M-regular if x_i is not a zero divisor in $M/(x_1, \ldots, x_{i-1})$ for all $i = 1, \ldots, k$.

One can define a notion of "size" for modules by asking for the maximal length of a regular sequence in M. The resulting number is called the depth of M.

Definition 2.4.3. The depth of a S-module M with respect to an ideal $J \subseteq S$ is the maximal length of an M-regular sequence in J and will be denoted by $\mathrm{depth}_J(M)$. For local ring $(S, \mathfrak{m})$ with maximal ideal $\mathfrak{m}$, one writes $\mathrm{depth}_{\mathfrak{m}}(M) = \mathrm{depth}(M)$.

We are interested in modules Γ over the polynomial ring R. Maximal ideals then correspond to points $x \in \operatorname{Spec} R$; there exists a unique maximal equivariant ideal in R, corresponding to the skyscraper sheaf at the origin, and consisting of all polynomials with zero constant term. We will always consider the depth with respect to this ideal.

On general grounds, one can show that for any module $\mathrm{depth}(\Gamma) \leq \dim(\Gamma)$. There is an important class of modules for which equality holds. These are called Cohen–Macaulay modules.

Definition 2.4.4. An R-module Γ is called Cohen–Macaulay if $\mathrm{depth}(\Gamma) = \dim(\Gamma)$.[7]

In this context, two equivalent characterizations of the Cohen–Macaulay property will be useful. The first one is in terms of the length of a minimal free resolution.

The Auslander–Buchsbaum formula states

$$\mathrm{depth}(\Gamma) + l_R(\Gamma) = n\,, \tag{2.186}$$

where $l_R(\Gamma)$ is the length[8] of the minimal resolution $L^\bullet$ of Γ in free R-modules.

Proposition 2.4.5. *An R-module Γ is Cohen–Macaulay if the length of its minimal free resolution equals its codimension, i.e.*

$$l_R(\Gamma) = n - \dim_R(\Gamma) = \mathrm{codim}_R(\Gamma)\,. \tag{2.187}$$

Equivalently we can characterize Cohen–Macaulay modules via their Ext-groups. We define the dualizing complex by

$$\omega_\Gamma^\bullet = \mathrm{RHom}_R^\bullet(\Gamma, R) = \mathrm{Hom}_R(L^\bullet, R). \tag{2.188}$$

We note that the cohomology $H^\bullet(\mathrm{RHom}_R^\bullet(\Gamma, R)) \cong \mathrm{Ext}_R^\bullet(\Gamma, R)$. If R/I is Cohen–Macaulay, this cohomology is concentrated in a single degree.

Proposition 2.4.6. *An R-module Γ of dimension q is Cohen–Macaulay if and only if $\mathrm{Ext}_R^k(\Gamma, R) = 0$ for all $k \neq \mathrm{codim}_R(\Gamma) = n - q =: r$.*

[7] Here the correct notion of dimension is the Krull dimension.

[8] The length of a free resolution $L^\bullet = (L^0 \leftarrow L^{-1} \leftarrow \cdots \leftarrow L^{-k} \leftarrow 0)$ is k.

Thus, for Cohen–Macaulay modules, the dualizing complex is in fact quasi-isomorphic to a dualizing module (often also called the canonical module).

Let us now specialize to the ring of functions, i.e. $\Gamma = R/I$. If the canonical module $\omega_{R/I}$ is trivial (free of rank one), the scheme $\mathrm{Spec}(R/I)$ can be thought of as a singular analog to a Calabi–Yau space. This property is called Gorenstein.

Definition 2.4.7. A quotient ring R/I is called Gorenstein if R/I is Cohen–Macaulay and the dualizing module $\mathrm{Ext}_R^{-(n-d)}(R/I, R) = R/I$, where $d = \dim(R/I)$.[9]

Clearly, the Gorenstein property is stronger than the Cohen–Macaulay property. However, to be a complete intersection is an even stronger condition. We thus have the following chain of implications.

$$\text{Complete intersection} \implies \text{Gorenstein} \implies \text{Cohen–Macaulay} \tag{2.189}$$

The key property of Gorenstein rings that is relevant to us is that their minimal free resolutions are self-dual: If R/I is Gorenstein and $(L^\bullet, d_L)$ is a minimal free resolution, then the dual complex $((L^\bullet)^\vee, (d_L)^\vee)$ is, by definition, a resolution of the dualizing module, which, by assumption, is again R/I. Thus, $(L^\bullet, d_L)$ and $((L^\bullet)^\vee, (d_L)^\vee)$ are both minimal free resolutions for R/I and hence, due to the uniqueness of the minimal free resolution, they must be isomorphic.

In fact one can recognize Gorenstein rings conveniently by examining their minimal free resolution:

Proposition 2.4.8. *R/I is a Gorenstein ring $\iff$ The length of the minimal free resolution $L^\bullet$ of R/I is $l_R(R/I) = \mathrm{codim_R(R/I)} =: k$ and $L^{-k} = R$.*

Note that this extends the above statement on the Cohen–Macaulay property. The self-duality of the minimal free resolution induces isomorphisms $L^{-i} \cong (L^{-(k-i)})^\vee$ and thus a non-degenerate pairing

$$L^{-i} \times L^{-(k-i)} \longrightarrow R\,. \tag{2.190}$$

Tensoring both sides with $\mathbb{C}$ we obtain a pairing

$$(L^{-i} \otimes_R \mathbb{C}) \times (L^{-(k-i)} \otimes_R \mathbb{C}) \longrightarrow \mathbb{C}\,. \tag{2.191}$$

As we explained in §2.3.6, $L^\bullet \otimes_R \mathbb{C}$ can be identified with Koszul homology and thus with the component fields of the multiplet. As such, if we feed a Gorenstein ring into

[9] This is not the most general definition, but it suits our setting. In general a ring S is called Gorenstein, if S has finite injective dimension as an S-module. There is also a notion of Gorenstein modules in the literature, but we do not need this level of generality for our discussion.

the pure spinor superfield formalism, we can equip the component fields of the resulting multiplet with a local pairing (a density valued pairing on sections of a vector bundle on spacetime). The parity and homological degree will depend on the properties of the free resolution. These pairings are often of physical interest.

2.4.2 Supplemental structures on multiplets

In some cases these pairings can be used to equip multiplets obtained in the pure spinor superfield formalism with a BV datum. Here, the prime example is ten-dimensional super Yang–Mills theory which we will discuss below. However, this is not the only relevant case. There are other examples of multiplets obtained from Gorenstein rings where the pairing does not give rise to a BV structure; nevertheless, the natural pairings may still be interesting.

As an easy example, let us once again come back to the chiral multiplet for $\mathcal{N} = 1$ supersymmetry in four dimensions. Recall that we obtained the chiral multiplet from the module $\Gamma = \mathbb{C}[\bar{\lambda}_{\dot\alpha}] = \mathbb{C}[\lambda_\alpha, \bar{\lambda}_{\dot\alpha}]/(\lambda_\alpha)$. This is obviously a complete intersection ring, and thus in particular Gorenstein. The minimal free resolution is of the form

$$R \longleftarrow R^2 \longleftarrow R, \tag{2.192}$$

and it is clear what the pairing looks like: $L^0 = R$ pairs with $L^{-2} = R$, while $L^{-1} = R^2$ pairs with itself. Since this is a perfect pairing on Koszul homology, we obtain a local pairing on the component fields. Recall that the scalar field was represented simply by ϕ, the fermion by $\psi = \psi_\alpha \theta^\alpha$ and the auxiliary by $F = F\theta^1\theta^2$. Thus, we get a pairing which is simply induced by the algebra structure on $\wedge^\bullet S_+$ and the projection on the $\theta_1\theta_2$ component

$$\langle a, b\rangle = (ab)|_{\theta_1\theta_2} \,. \tag{2.193}$$

So this pairing gives rise to F-term Lagrangians for the chiral multiplet through the following local pairing on component fields

$$\langle \phi, F\rangle_{\mathrm{loc}} = \phi F, \quad \langle \psi, \psi\rangle_{\mathrm{loc}} = \psi^\alpha \psi_\alpha \,. \tag{2.194}$$

Similar pairings of course exist for other chiral multiplets with more supersymmetry. Furthermore, we could consider the free superfield, arising from the quotient of R by the maximal ideal (or more geometrically from the skyscraper sheaf with value $\mathbb{C}$ on Y supported at the origin). For four-dimensional $\mathcal{N} = 1$ supersymmetry, this module is $\mathbb{C} = \mathbb{C}[\lambda_\alpha, \bar{\lambda}_{\dot\alpha}]/(\lambda_\alpha, \bar{\lambda}_{\dot\alpha})$. Then one gets a pairing which projects on the $\theta^2\bar{\theta}^2$ component. In physics, this pairing gives rise to D-terms.

2.4.3 Constructing cotangent theories: six-dimensional $\mathcal{N} = (1,0)$

If a ring is not Gorenstein, there is no perfect pairing on Koszul homology, and the corresponding multiplet cannot obviously be equipped with a BV structure. (We note that this does *not* mean that such multiplets are never on-shell; in six-dimensional $\mathcal{N} = (2,0)$ supersymmetry [CNT02; SW23b] and ten-dimensional type IIB supersymmetry [ESW21], BV multiplets with degenerate pairings naturally arise. Details on the pairing are given in [SW23b] at the level of the component fields; we do not study the cochain-level origin of such degenerate pairings here, but hope to return to this question in future work.)

For a Cohen–Macaulay module Γ giving rise to a multiplet (E, D, ρ), however, another interesting observation applies: We can consider the dualizing module ω_Γ in the pure spinor superfield formalism. If (L, d_L) is the minimal free resolution of Γ, then $(L^\vee, (d_L)^\vee)$ is the corresponding minimal free resolution of ω_Γ. With the obvious pairing between L and $L^\vee$ we can equip the multiplet corresponding to the direct sum $L \oplus L^\vee[k]$ with a BV datum (for an appropriate shift k). In the terminology of Definition 2.2.15 the resulting BV multiplet is off-shell and ω_Γ gives rise to the antifield multiplet of (E, D, ρ).

On the other hand, if the input module is not Cohen–Macaulay, the cohomology of the dualizing complex will not be concentrated in a single degree, such that we cannot take a single dualizing module to produce an antifield multiplet. Rather, the antifield multiplet will be represented by a dg module. We will see this below in the case of four-dimensional minimal supersymmetry.

Let us now consider the example of six-dimensional $\mathcal{N} = (1,0)$ supersymmetry. There is an accidental isomorphism $\mathrm{Spin}(6) \cong SU(4)$, under which the spinor representation S_+ is identified with the fundamental representation of $SU(4)$ and $S_- = (S_+)^\vee$ with the antifundamental representation. The supertranslation algebra takes the form

$$\mathfrak{n} = V \oplus \Pi(S_+ \otimes U) , \tag{2.195}$$

where $U = (\mathbb{C}^2, \omega)$ is a symplectic vector space. The R-symmetry group is thus $\mathrm{Sp}(1) \cong SU(2)$; corresponding indices will be denoted by i, j. There is an isomorphism

$$\wedge^2 S_+ \cong V , \tag{2.196}$$

which is used to express the bracket as

$$[-,-] = \wedge \otimes \omega . \tag{2.197}$$

Since $\wedge$ is an isomorphism, an element is square-zero precisely when it is of rank one as an element of $S_+ \otimes U$.

In a basis, the supertranslation algebra takes the form

$$[Q^i_\alpha, Q^j_\beta] = \gamma^\mu_{\alpha\beta}\varepsilon^{ij}P_\mu \,. \tag{2.198}$$

Using coordinates λ^α_i, the defining equations of the nilpotence variety Y are given by the 2×2 minors of the matrix

$$\begin{pmatrix} \lambda^1_1 & \lambda^2_1 & \lambda^3_1 & \lambda^4_1 \\ \lambda^1_2 & \lambda^2_2 & \lambda^3_2 & \lambda^4_2 \end{pmatrix}, \tag{2.199}$$

which cut out the space of rank-one matrices. As such Y is a determinantal variety. Taking the ring of functions R/I as the equivariant module in the pure spinor formalism, we recover the $d = 6$, $\mathcal{N} = (1,0)$ gauge multiplet. The Betti numbers of the free resolution are displayed in the following table.

$$\begin{bmatrix} 1 & - & - & - \\ - & 6 & 8 & 3 \end{bmatrix} \tag{2.200}$$

Working equivariantly, one finds that these correspond to a one-form with zero-form gauge invariance, fermions in $S_+ \oplus S_-$, and a triplet of scalars in the adjoint of the R-symmetry group $SU(2)$. We immediately see that the Koszul homology corresponds to the field content of the BRST complex of the gauge multiplet. Since the length of the resolution equals the codimension, R/I is Cohen–Macaulay. This can also be seen as a consequence of the following result on determinantal varieties.

Lemma 2.4.9 ([HE71; Sva74])**.** *Let $R = \mathbb{C}[(x_{ij})]$ for $i = 1 \dots n$ and $j = 1 \dots m$ and I the ideal generated by the $r \times r$ minors of the matrix with entries x_{ij}. Then R/I is a Cohen–Macaulay ring. Further R/I is a Gorenstein ring if and only if $m = n$ or $r = 1$.*

As we are dealing with 4×2 matrices, R/I is not Gorenstein; hence, we cannot expect to equip the multiplet with a BV datum, but only with a BRST datum. However, R/I is Cohen–Macaulay, which means that the dualizing complex is represented by a single sheaf. Thus, we can produce the corresponding antifield multiplet from that sheaf by applying the pure spinor formalism. The dualizing module is

$$\mathrm{Ext}_R^{-\mathrm{codim}(Y)}(R/I, R) = \mathrm{Ext}_R^{-3}(R/I, R) \,. \tag{2.201}$$

Due to the Cohen–Macaulay property, this is the only non-vanishing Ext-module. Its free resolution has the Betti numbers

$$\begin{bmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{bmatrix}. \tag{2.202}$$

Forming the direct sum of the structure sheaf and the dualizing sheaf and shifting appropriately, we obtain a multiplet with the following Betti numbers.

$$\begin{bmatrix} 1 & - & - & - & - & - \\ - & 6 & 8 & 3 & - & - \\ - & - & 3 & 8 & 6 & - \\ - & - & - & - & - & 1 \end{bmatrix} \tag{2.203}$$

This is the expected field content of the BV description for the six-dimensional gauge multiplet. The component multiplet can be equipped with a BV datum by writing the usual action as known from the component formalism. The resulting BV multiplet is off-shell; in fact, it is constructed as the cotangent theory of the corresponding BRST theory. The supersymmetry algebra closes without use of the equations of motion and the antifields can be separated from the fields. Doing this, one recovers the BRST multiplet in components.

One could also consider equipping the pure spinor superfield multiplet with a BRST datum. This was done in [Ced18b], where Cederwall considered a differential operator mapping pure spinor superfields for the structure sheaf to pure spinor superfields for the canonical module. This operator allows one to write a quadratic action functional for the structure sheaf multiplet, which defines a BRST datum for that multiplet.

2.4.4 Failure to be Cohen–Macaulay: the example of four-dimensional $\mathcal{N} = 1$

As we have seen above, the pure spinor superfield formalism applied to the structure sheaf of the $d = 4$, $\mathcal{N} = 1$ nilpotence variety yields the BRST description of the gauge multiplet. The absence of antifields and BV differential is not particularly surprising: The failure of the supersymmetry action to be strict solely comes from gauge transformations; the equations of motions do not need to be imposed. Nevertheless one can ask if and how the corresponding antifield multiplet can be realized independently in the pure spinor superfield formalism. For this purpose, let us compute the dualizing complex of R/I. A model for the dualizing complex is given by

$$\omega^\bullet_{R/I} = \mathrm{RHom}^\bullet_R(R/I, R) = \mathrm{Hom}_R(L^\bullet, R). \tag{2.204}$$

To compute this complex explicitly, we can use the minimal free resolution $L^\bullet \to R/I$ from above. By definition, the differential of the dualizing complex is the dual map $d_L^\vee$ of the resolution differential d_L. In terms of matrices this means that $d_L^\vee$ is represented by the transposed matrices of (2.150). From these matrices, the cohomology can be computed explicitly. We find that

$$H^i(\omega^\bullet_{R/I}) = \begin{cases} \operatorname{coker}\big((\lambda_1, -\lambda_2, -\bar\lambda_1, \bar\lambda_2)\big) \cong \mathbb{C}, & i = 3; \\ \mathbb{C}[\lambda_1, \lambda_2] \oplus \mathbb{C}[\bar\lambda_1, \bar\lambda_2], & i = 2; \\ 0, & \text{otherwise.} \end{cases} \tag{2.205}$$

Note that the codimension of Y is two. If the dualizing complex were to resolve a single module, then $H^\bullet(\omega_{R/I})$ should be concentrated in degree two. Instead, we find a copy of two *disjoint* $\mathbb{C}^2$'s; Y itself, of course, consists of two $\mathbb{C}^2$'s intersecting at the origin. This discrepancy is accounted for homologically by the presence of a single copy of the skyscraper sheaf in degree three. At the end of the day, this means that we cannot find a single (non-dg) dualizing module for R/I to feed into the pure spinor superfield formalism to obtain the antifield multiplet. This phenomenon will occur whenever R/I is not a Cohen–Macaulay ring. We will come back to this example in the context of the derived pure spinor superfield formalism in §3.

2.4.5 A partial dictionary

In this section we summarize some features of the correspondence between algebrogeometric properties of R/I-modules and physical features of the corresponding multiplets. This dictionary is of course by no means complete, but it should serve to provide a quick overview.

— $\Gamma = \mathcal{O}(S')$ *for some hyperplane* $S' \subseteq Y$.

S' is a complete intersection of linear equations. The resulting multiplet is an exterior algebra $\wedge^\bullet S'$, concentrated in homological degree zero. No differentials are transferred to the component field level. The representation of the supersymmetry algebra is strict. Examples include chiral superfields ($S' = S_\pm$), which always exist in dimension 0 (mod 4), and free superfields ($S' = \{0\}$), in any dimension and with any amount of supersymmetry. We emphasize that the free superfield always corresponds to the canonical augmentation of the graded ring R/I.

— $\Gamma = R/I$ *is a complete intersection of quadratic equations.*

The Koszul homology is an exterior algebra generated by the elements $\lambda f^\mu \theta$ in homological degree one. The resulting multiplet can be identified with the de Rham complex $\Omega^\bullet(\mathbb{R}^d)$ on spacetime. The transferred differential acts as the de Rham differential on the component fields; as such, translations act homotopically trivially. Tensoring with a Lie $(d-3)$-algebra, one obtains the BV complex of higher Chern–Simons theory. Odd elements in the supersymmetry algebra act by zero. Examples include the structure sheaves of the three-dimensional $\mathcal{N}=8$ and four-dimensional $\mathcal{N}=4$ supersymmetry algebras; see [Ced08] and [Ced18a], respectively. This sheaf is used, together with another equivariant sheaf, in the construction of the pure spinor resolution of the Bagger–Lambert–Gustavsson model in [Ced08].

— $\Gamma = R/I$ *is a Gorenstein ring, but not a complete intersection.*

The resulting component multiplet is equipped with a local pairing, inherited from the perfect pairing on Koszul homology. For appropriate values of the spacetime dimension and the codimension of Y, this local pairing defines an odd symplectic structure, which may be used to construct a BV datum on the multiplet. The underlying cochain complex always starts with

$$\Omega^0 \xrightarrow{d} \Omega^1 \longrightarrow \dots; \tag{2.206}$$

as such, it always contains at least a one gauge field. By duality, the multiplet ends with the corresponding antifields,

$$\dots \longrightarrow \Omega^{d-1} \xrightarrow{d} \Omega^d. \tag{2.207}$$

Examples include ten-dimensional super Yang–Mills theory and eleven-dimensional supergravity [Ced10c; Ced10a]; see also [SW21; EH23] for treatments using a language close to this work.

— Γ *is Cohen–Macaulay, but not Gorenstein.*

The resulting multiplet will not carry a pairing and thus cannot be equipped with a nondegenerate BV datum. We can interpret the multiplet as a BRST multiplet and obtain the corresponding antifield multiplet from the dualizing module. Here, the structure sheaf of six-dimensional $\mathcal{N}=(1,0)$ supersymmetry is an example. To understand theories of physical interest, though, it may be necessary to consider degenerate pairings (six-dimensional $\mathcal{N}=(2,0)$ supersymmetry and type IIB supergravity are examples).

— Γ *is not Cohen–Macaulay.*

The resulting multiplet usually looks like a BRST multiplet. However, there is really only a dualizing complex instead of a dualizing module. As such, we cannot obtain the antifield multiplet from a single (non-dg) $\mathcal{O}_Y$-module via the pure spinor superfield formalism. An example is the gauge multiplet in four-dimensional $\mathcal{N}=1$ supersymmetry, as discussed above. It would be interesting to consider extending the formalism to dg sheaves on Y.

— Γ *is a Golod ring.*

A ring is Golod if and only if all Massey products on Koszul homology vanish [Fra18]. Recall that, if Γ is assumed to be a ring, the tensor product $A^\bullet(\Gamma)\otimes\mathfrak{h}$ carries an L_∞ structure. Transferring the L_∞ structure to the component fields and then compactifying to a point yields an L_∞ structure which is given by the A_∞ structure on Koszul homology tensored with the Lie algebra $\mathfrak{h}$. The Golod property of Γ implies that this L_∞ structure is strict. For example, the presence of the three-ary product in ten-dimensional super Yang–Mills theory, which after compactification to a point gives rise to a corresponding product in the IKKT matrix model [MS04], witnesses the fact that the ring of functions of the ten-dimensional $\mathcal{N}=1$ nilpotence variety is not Golod.

2.5 Ten-dimensional super Yang–Mills theory

In this section, we present a detailed analysis of ten-dimensional super Yang–Mills theory in the pure spinor superfield formalism. This is the initial example which sparked interest in the formalism [Ber01; CNT02] and was also analyzed in [Ale+07]. As we will see, the canonical multiplet for $\mathcal{N}=1$ supersymmetry in ten dimensions (which we obtain from the structure sheaf $\mathcal{O}_Y$) can be naturally equipped with the full structure of a perturbative interacting BV theory within the pure spinor superfield formalism.

2.5.1 Field content and representatives

We will denote the two 16-dimensional spin representations of Spin(10) by S_+ and S_-. The vector representation is, as always, denoted by V. The defining ideal of the nilpotence variety reads

$$I=(\lambda\gamma^\mu\lambda)\,. \tag{2.208}$$

One finds for the minimal free resolution of R/I the following Betti numbers.

$$\begin{bmatrix} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{bmatrix} \tag{2.209}$$

More concretely, the minimal free resolution of R/I in R-modules takes the form

$$L^\bullet = R \otimes \left(\mathbb{C} \xleftarrow{(d_L)_1} V \xleftarrow{(d_L)_2} S_+ \xleftarrow{(d_L)_3} S_- \xleftarrow{(d_L)_4} V \xleftarrow{(d_L)_5} \mathbb{C} \right) . \tag{2.210}$$

The resolution differential can be described explicitly. Let us choose a basis e_μ of V and s_α of S_+. The corresponding dual basis of $(S_+)^\vee = S_-$ is denoted by s^α.

$$\begin{array}{ccccccccc}
(d_L)_1 & : & V & \longrightarrow & \mathbb{C} & , & A & \mapsto & (\lambda\gamma^\mu\lambda)A_\mu \\
(d_L)_2 & : & S_+ & \longrightarrow & V & , & \chi & \mapsto & (\lambda\gamma^\mu\chi)e_\mu \\
(d_L)_3 & : & S_- & \longrightarrow & S_+ & , & \chi^\vee & \mapsto & (\lambda\gamma^\mu\lambda)(\chi^\vee\gamma_\mu s) - 2(\chi^\vee\lambda)(\lambda s) \\
(d_L)_4 & : & V & \longrightarrow & S_- & , & A^\vee & \mapsto & (\lambda\gamma^\mu s)A^\vee_\mu \\
(d_L)_4 & : & \mathbb{C} & \longrightarrow & V & , & c^\vee & \mapsto & (\lambda\gamma^\mu\lambda)c^\vee e_\mu
\end{array} \tag{2.211}$$

We can perform our procedure to find the representatives. For example, starting with the gauge field,

$$A \xmapsto{(d_L)_1} (\lambda\gamma^\mu\lambda)A_\mu \xmapsto{\mathcal{D}_0^\dagger} (\lambda\gamma^\mu\theta)A_\mu \,, \tag{2.212}$$

so that the elements $(\lambda\gamma^\mu\theta)A_\mu$ represent the one-form in $\mathcal{D}_0$-cohomology. For the gaugino we obtain

$$\chi \xmapsto{(d_L)_2} (\gamma^\mu\lambda)_\alpha\chi^\alpha e_\mu \xmapsto{\mathcal{D}_0^\dagger} (\gamma^\mu\theta)_\alpha\chi^\alpha e_\mu \xmapsto{(d_L)_1} (\lambda\gamma_\mu\lambda)(\gamma^\mu\theta)_\alpha\chi^\alpha \xmapsto{\mathcal{D}_0^\dagger} (\lambda\gamma_\mu\theta)(\gamma^\mu\theta)_\alpha\chi^\alpha \,. \tag{2.213}$$

This means that the gaugino is represented by $(\lambda\gamma_\mu\theta)(\gamma^\mu\theta)_\alpha\chi^\alpha$ in $\mathcal{D}_0$-cohomology. This procedure can also be applied to the antifields

$$\chi^\vee \xmapsto{(d_L)_3\circ\mathcal{D}_0^\dagger} (\lambda\gamma^\mu\theta)(\chi^\vee\gamma_\mu s) \xmapsto{(d_L)_2\circ\mathcal{D}_0^\dagger} (\lambda\gamma^\mu\theta)(\gamma^\nu\theta)_\alpha(\gamma_\mu\chi^\vee)^\alpha e_\nu \xmapsto{(d_L)_1\circ\mathcal{D}_0^\dagger} (\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\gamma_\nu\theta)^\alpha(\gamma_\mu\chi^\vee)_\alpha \tag{2.214}$$

We can simplify the last term to find

$$(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\gamma_\nu\theta)^\alpha(\gamma_\mu\chi^\vee)_\alpha = (\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\gamma_{\mu\nu}\theta)^\alpha\chi^\vee_\alpha \,, \tag{2.215}$$

where $\gamma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}$ denotes the antisymmetrized product of two gamma matrices. Similarly one can track down a representative for the antifield of the one-form field. The result is

$$(\lambda\gamma^\rho\theta)(\lambda\gamma^\nu\theta)(\theta\gamma_{\mu\nu\rho}\theta)A^{+\mu} \,. \tag{2.216}$$

Finally, the antighost can be represented by

$$(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta)c^\vee \,. \tag{2.217}$$

These representatives were already listed in [MSX14]. Let us summarize the results in the following table.

Field	Representative in the $\mathcal{D}_0$-cohomology
c	c
A	$(\lambda\gamma^\mu\theta)A_\mu$
χ	$(\lambda\gamma_\mu\theta)(\chi\gamma^\mu\theta)$
$\chi^\vee$	$(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\gamma_{\mu\nu}\theta\chi^\vee)$
$A^\vee$	$(\lambda\gamma^\rho\theta)(\lambda\gamma^\nu\theta)(\theta\gamma_{\mu\nu\rho}\theta)A^{+\mu}$
$c^\vee$	$(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta)c^\vee$

TABLE 2.4: Representatives for the $d = 10$, $\mathcal{N} = 1$ vector multiplet organized by θ-degree.

2.5.2 The differential

The first order part of the transferred differential is given by

$$\mathcal{D}_1' = p \circ (\lambda\gamma^\mu\theta)\partial_\mu \circ i\,. \tag{2.218}$$

We immediately see that $\mathcal{D}_1'$ acts on the ghost as the de Rham differential.

Furthermore, the differential, $\mathcal{D}_1'$ acts on the gaugino as the Dirac operator,

$$\chi \mapsto \partial\!\!\!/\chi \tag{2.219}$$

encoding the field equation for the gaugino.

Interestingly, this multiplet contains a second order contribution to the differential arising from homotopy transfer. As we will see momentarily, this encodes the equation of motion for the gauge field. Recall that the second order contribution to the transferred differential $\mathcal{D}'$ is given by

$$\mathcal{D}_2' = p \circ (\mathcal{D}_1 \circ h \circ \mathcal{D}_1) \circ i\,. \tag{2.220}$$

To apply $\mathcal{D}_2'$ to the gauge field we need to know how the homotopy h acts on expressions of the form

$$(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)\,. \tag{2.221}$$

Note that the naive guess $h \circ \pi = \pi \circ \mathcal{D}_0^\dagger$ does not work in this case since

$$\mathcal{D}_0^\dagger(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta) = 0 \tag{2.222}$$

by the symmetry of the bracket. However, the result is easily found by a representation theoretic argument. As h acts as a scalar, we are looking for a representative inside

$$\wedge^2 V \subset \wedge^3 S_+ \otimes S_+ \,. \tag{2.223}$$

It is easy to check that there is only one such summand in the decomposition of the right hand-side into irreducibles. This representation is spanned by the elements

$$(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta) \,. \tag{2.224}$$

We set,

$$h\left((\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)\right) = (\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta) \,. \tag{2.225}$$

Equipped with this knowledge we find

$$\begin{aligned} \mathcal{D}_2'\left((\lambda\gamma^\mu\theta)A_\mu\right) =& p\left(\mathcal{D}_1 \circ h\left((\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\mathrm{d}A)_{\mu\nu}\right)\right) \\ =& p\left((\lambda\gamma^\sigma\theta)(\lambda\gamma_\rho\theta)(\theta\gamma^{\mu\nu\rho}\theta)\partial_\sigma(\mathrm{d}A)_{\mu\nu}\right) \,. \end{aligned} \tag{2.226}$$

Projection to the cohomology gives

$$(\lambda\gamma_\nu\theta)(\lambda\gamma_\rho\theta)(\theta\gamma^{\mu\nu\rho}\theta)\partial^\sigma(\mathrm{d}A)_{\sigma\mu}. \tag{2.227}$$

This shows that the transferred differential $\mathcal{D}_2'$ acts via

$$A \mapsto \star\mathrm{d}\star\mathrm{d}A. \tag{2.228}$$

The differentials appearing in the multiplet can be summarized by the following diagram.

$$\begin{tikzcd} \Omega^0(\mathbb{R}^{10}) \arrow[dr, "d"] & & & & & \\ & \Omega^1(\mathbb{R}^{10}) \arrow[drrr, "\star\mathrm{d}\star\mathrm{d}"'] & \Gamma(\mathbb{R}^{10}, S_+) \arrow[dr, "\partial\!\!\!/"] & & & \\ & & & \Gamma(\mathbb{R}^{10}, S_-) & \Omega^1(\mathbb{R}^{10}) \arrow[dr, "\star\mathrm{d}\star"] & \\ & & & & & \Omega^0(\mathbb{R}^{10}) \end{tikzcd} \tag{2.229}$$

2.5.3 The supersymmetry action

We can read off the non-derivative supersymmetry transformations directly from the resolution differential.

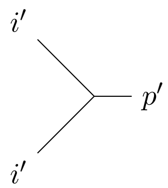
$$\begin{aligned}
\delta c =&(\epsilon\gamma^{\mu}\epsilon)A_{\mu} \\
\delta A_{\mu} =&\epsilon\gamma_{\mu}\chi \\
\delta \chi =&(\epsilon\gamma^{\mu}\epsilon)\chi^{\vee}\gamma_{\mu} - 2\epsilon(\chi^{\vee}\epsilon) \\
\delta\chi^{\vee} =&\epsilon\gamma^{\mu}A^{\vee}_{\mu} \\
\delta A^{\vee}_{\mu} =&(\epsilon\gamma_{\mu}\epsilon)c^{\vee} \\
\delta c^{\vee} =&0
\end{aligned} \tag{2.230}$$

Note that there are two types of closure terms present. For the gauge field, there are again transformation witnessing that the supersymmetry algebra is represented only up to gauge transformations. We already encountered this type of transformation in our discussion of the four-dimensional gauge multiplet. In addition, there are now second order transformations for the gaugino, signaling that the supersymmetry algebra is represented only on-shell.

2.5.4 The L_∞ structure

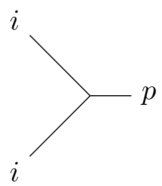
We can define a dgs Lie algebra structure by tensoring $A^\bullet(R/I)$ with a Lie algebra $\mathfrak{h}$. Homotopy transfer gives rise to an L_∞ structure on the component field multiplet. As we will see, this L_∞ structure, together with the pairing, equips the ten-dimensional super Yang–Mills multiplet with the usual structure as an interacting BV theory.

The binary bracket μ'_2 is given by

$\mu'_2 =$ 

. (2.231)

Expressing this in terms of the unprimed homotopy data, there will be obviously a diagram of the form

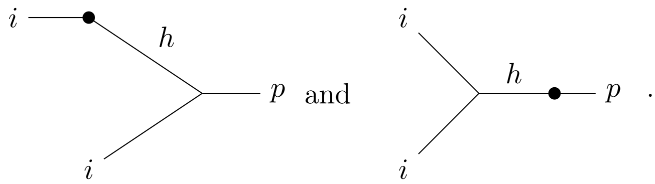


. (2.232)

As we already explored in the case of the four-dimensional $\mathcal{N}=1$ vector multiplet, this diagram encodes the structure of gauge transformations on the component fields. In particular it yields brackets

$$\begin{array}{lcccc} \mu_2' : & \Omega^0 \times \Omega^0 & \longrightarrow & \Omega^0 & \mu_2'(c,c) = [c,c] \\ \mu_2' : & \Omega^0 \times \Omega^1 & \longrightarrow & \Omega^1 & \mu_2'(c,A) = [c,A] \\ \mu_2' : & \Omega^0 \times \Gamma(X,S_+) & \longrightarrow & \Gamma(X,S_+) & \mu_2'(c,\psi) = [c,\psi]\,. \end{array} \tag{2.233}$$

Furthermore, considering degree bounds, we see that only two more diagrams can contribute, namely



(2.234)

Here we marked the unary vertices with a dot, signaling the application of $\mathcal{D}_1$.

From the first type of diagram we obtain

$$p\left((\lambda\gamma^\sigma\theta)h((\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta))\right)[A_\sigma,\partial_\mu A_\nu] = p\left((\lambda\gamma^\sigma\theta)(\lambda\gamma_\rho\theta)(\theta\gamma^{\mu\nu\rho}\theta)\right)[A_\sigma,\partial_\mu A_\nu] \tag{2.235}$$

Using the antisymmetry in μ and ν and projecting onto $\mathcal{D}_0$-cohomology this gives

$$(\lambda\gamma_\nu\theta)(\lambda\gamma_\rho\theta)(\theta\gamma^{\mu\nu\rho}\theta)\,[A^\sigma,(dA)_{\mu\sigma}]\,. \tag{2.236}$$

The second diagram gives a contribution of the form

$$p\left((\lambda\gamma^\sigma\theta)\partial_\sigma h\left((\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)\right)[A_\mu,A_\nu]\right) = p\left((\lambda\gamma^\sigma\theta)(\lambda\gamma_\rho\theta)(\theta\gamma^{\mu\nu\rho}\theta)\right)\partial_\sigma[A_\mu,A_\nu]. \tag{2.237}$$

Projection to the cohomology gives

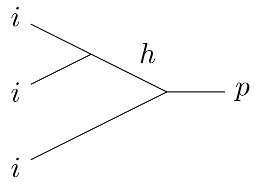
$$(\lambda\gamma_\nu\theta)(\lambda\gamma_\rho\theta)(\theta\gamma^{\mu\nu\rho}\theta)\,\partial^\sigma[A_\mu,A_\sigma]\,. \tag{2.238}$$

Together this gives a transferred binary product

$$\mu_2' : \Omega^1 \times \Omega^1 \longrightarrow \Omega^1 \qquad \mu_2'(A,A)_\mu = [A^\sigma,(dA)_{\mu\sigma}] + \partial^\sigma[A_\mu,A_\sigma]\,. \tag{2.239}$$

By degree reasons, there are no $\mathcal{D}_1$ insertions allowed for μ'_3. Hence, the only contributing diagram is of the following form.

$\mu'_3 =$



(2.240)

This diagram gives a contribution of the form

$$p\left((\lambda\gamma_\rho\theta)(\theta\gamma^{\mu\nu\rho}\theta)(\lambda\gamma^\sigma\theta)[A_\sigma,[A_\mu,A_\nu]]\right) = (\lambda\gamma_\rho\theta)(\lambda\gamma_\nu\theta)(\theta\gamma^{\mu\nu\rho}\theta)\,[A^\sigma,[A_\mu,A_\sigma]]\,. \quad (2.241)$$

This gives a product

$$\mu'_3 : \Omega^1 \times \Omega^1 \times \Omega^1 \longrightarrow \Omega^1 \qquad \mu'_3(A,A,A)_\mu = [A^\sigma,[A_\mu,A_\sigma]]\,. \quad (2.242)$$

Thus, we see that the transferred L_∞ structure equips the multiplet with the usual interactions as expected for ten-dimensional super Yang–Mills theory.

2.5.5 The pairing

The ring R/I is Gorenstein, which implies that the minimal free resolution, and hence the component field formulation of the multiplet, is equipped with a local (in the sense of Definition 2.2.14) pairing. At the level of Koszul homology, the pairing is induced by multiplication and projection to the subspace spanned by the top class

$$(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta)\,. \quad (2.243)$$

This equips the component field multiplet with a BV structure.

We thus obtained the usual description of ten-dimensional super Yang–Mills theory as an interacting BV theory solely by homotopy transfer from the pure spinor superfield description.

2.6 A bestiary of multiplets from modules

In this final section we construct a variety of equivariant R/I-modules and examine the structure of the associated supersymmetric multiplets. We offer some observations connecting certain of these multiplets to constructions in the physics literature, along with some other speculations of various kinds.

2.6.1 Presentations of modules and shift symmetry

Any module Γ over any ring S can be described using a free presentation, that is an exact sequence

$$0 \longleftarrow \Gamma \longleftarrow F_0 \xleftarrow{\varphi} F_1 , \tag{2.244}$$

where F_0 and F_1 are free S-modules. The module can then be identified as the cokernel of the map φ

$$\Gamma \cong \mathrm{coker}\varphi = F_0/\mathrm{Im}(\varphi) . \tag{2.245}$$

As F_0 and F_1 are free, we can think of φ as a matrix with entries in S, these entries give the relations to obtain Γ as a quotient from F_0. In fact, a free presentation is just the start of a free resolution. By resolving kernels we can extend a free presentation to a free resolution

$$0 \longleftarrow \Gamma \longleftarrow F_0 \xleftarrow{\varphi_0} F_1 \xleftarrow{\varphi_1} F_2 \xleftarrow{\varphi_2} \dots . \tag{2.246}$$

For $R = \mathbb{C}[\lambda_1, \dots, \lambda_n]$ it is very easy to study such maps φ; these just correspond to matrices whose entries are polynomials in λ. The cokernels of such maps are then R-modules. For the pure spinor superfield formalism, it is crucial to use R/I-modules as this ensures that the differential $\mathcal{D}$ squares to zero. Suppose we have an R-module defined by a free presentation

$$\varphi : R^n \longrightarrow R^k \qquad \Gamma = \mathrm{coker}(\varphi) . \tag{2.247}$$

The R-module Γ descends to a R/I-module if the image of φ contains $I^{\times k}$. Thus, we can conveniently construct R/I-modules by studying suitable maps between free R-modules. If the map φ is also equivariant with respect to the action of the Lorentz and R-symmetry groups on R, then the resulting module is also equivariant. Hence, such equivariant maps between free R-modules precisely give rise to the desired input for the pure spinor superfield formalism. In the physics literature this procedure was used to construct multiplets in the pure spinor superfield formalism under the name shift symmetry (see [Ced10c; Ced10a; CK11; Ced14]).[10]

[10] The name "shift symmetry" arises from writing out the equivalence relation (2.245) as

$$f_0 \approx f_0 + \varphi(f_1)$$

with explicit representatives $f_0 \in F_0$ and $f_1 \in F_1$.

Example 2.6.1. We can immediately give a free presentation for the quotient rings R/I which we previously considered. The map

$$\varphi : R^d \longrightarrow R, \qquad \varphi = \begin{pmatrix} \lambda\gamma^0\lambda & \dots & \lambda\gamma^{d-1}\lambda \end{pmatrix}. \tag{2.248}$$

realizes the free presentation $\mathrm{coker}(\varphi) = R/I$.

2.6.2 Motivating example of a nontrivial sheaf: the six-dimensional hypermultiplet

As an example to demonstrate this technique, let us construct the six-dimensional hypermultiplet. We already constructed the six-dimensional vector multiplet from the structure sheaf of the nilpotence variety in §2.4.3. Recall that for six-dimensional $\mathcal{N} = (1, 0)$ supersymmetry, the odd part of the supertranslation algebra is

$$S_+ \otimes U \, , \tag{2.249}$$

where S_+ is the fundamental representation of $\mathfrak{su}(4)$ and $U \cong \mathbb{C}^2$ carries the fundamental representation of $\mathfrak{su}(2)$. The polynomial ring R is nothing but the symmetric algebra on $S_+ \otimes U$ and comes with the natural action of $\mathfrak{su}(4) \times \mathfrak{su}(2)$. There is a unique equivariant map

$$S_+ \otimes R \longrightarrow U \otimes R \tag{2.250}$$

which is linear in λ. Choosing a basis for U and S_+, this map is represented by

$$\varphi : S_+ \otimes R \longrightarrow U \otimes R \qquad \varphi = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \end{pmatrix}. \tag{2.251}$$

It is easy to check that the image of φ indeed contains $I^{\times 2}$, thus we can consider $\Gamma = \mathrm{coker}(\varphi)$ as an equivariant R/I-module in the pure spinor superfield formalism.

We display the Betti numbers of the minimal free resolution in the following table.

$$\begin{bmatrix} 2 & 4 & - & - \\ - & - & 4 & 2 \end{bmatrix} \tag{2.252}$$

The representations appearing in the minimal free resolution can be computed using *Macaulay2* via the highest weight package. The minimal free resolution of Γ in R-modules takes the form

$$L^\bullet = R \otimes \left(U \xleftarrow{\varphi} S_+ \xleftarrow{\epsilon} \wedge^3 S_+ \xleftarrow{\varphi^T} U \otimes \wedge^4 S_+ \right) \, , \tag{2.253}$$

and is a special case of the Buchsbaum–Rim complex [BE77] (see [Eis95, Appendix A.2.6] for a textbook presentation and a description of the differential ϵ in terms of the 2×2 minors of φ).

Choosing a basis e^i for U and s_α a basis for S_+, we can write out the differentials in the complex as:

$$\begin{array}{rclcl} d_1 & : & \wedge^1 S_+ \longrightarrow U & \psi & \mapsto & \lambda_i^\alpha \psi_\alpha e^i \\ d_2 & : & \wedge^3 S_+ \longrightarrow \wedge^1 S_+ & \psi^+ & \mapsto & (\lambda_i^\alpha \lambda_j^\beta \epsilon^{ij}) \psi^+_{\alpha\beta\gamma} s_\gamma \\ d_3 & : & U \otimes \wedge^4 S_+ \longrightarrow \wedge^3 S_+ & \phi^+ & \mapsto & \lambda_\alpha^i \phi_i^+ s^\alpha . \end{array} \tag{2.254}$$

In the last differential we identify s^α with $\epsilon^{\alpha\beta\gamma\delta} s_\beta \wedge s_\gamma \wedge s_\delta$ along the isomorphism $\wedge^3 S_+ \cong S_-$. The differential $(\lambda_i^\alpha \lambda_j^\beta \epsilon^{ij})$ is the differential ϵ appearing in the Buchsbaum–Rim complex.

As expected, the hypermultiplet consists of two scalars that form a doublet under $\mathfrak{su}(2)$ as well as fermions in S_+ that are neutral under $\mathfrak{su}(2)$ and their corresponding antifields. The two maps are expected to encode the respective equations of motions. We are thus dealing with an on-shell representation of the supersymmetry algebra. The multiplet can be equipped with a pairing which yields a BV structure.

We can use the zig-zag procedure to find representatives for the fields in the multiplet. These are expressed in terms of the basis e_i of U.

Field	Representative in the $\mathcal{D}_0$-cohomology
ϕ	$\phi_i e^i$
ψ	$\psi_\alpha \theta_i^\alpha e^i$
$\psi^\vee$	$\lambda_i^\alpha \theta_j^\beta \theta_k^\gamma \varepsilon^{ij} \psi^\vee_{\alpha\beta\gamma} e^k$
$\phi^\vee$	$\lambda_i^\alpha \theta_j^\beta \epsilon^{ij} \theta_l^\delta \theta_k^\gamma \varepsilon_{\alpha\beta\gamma\delta} \phi^{+k} e^l$

TABLE 2.5: Representatives for the hypermultiplet in six dimensions organized by θ-degree.

From the resolution differential, we can easily read off the non-derivative supersymmetry transformations.

$$\begin{aligned} \delta\phi_i &= \epsilon_i^\alpha \psi_\alpha \\ \delta\psi_\alpha &= \epsilon_i^\beta \epsilon_j^\gamma \varepsilon^{ij} \psi^\vee_{\alpha\beta\gamma} \\ \delta\psi^\vee_\alpha &= \epsilon^i_\alpha \phi_i^\vee \\ \delta\phi_i^\vee &= 0 \end{aligned} \tag{2.255}$$

Again, we see the quadratic transformation involving the fermion and its antifield showing that the supersymmetry algebra only closes up to the equations of motion.

Consequently, the equations of motions are encoded in the transferred differential $\mathcal{D}'$. There is a first order term $\mathcal{D}'_1$ acting on the fermion. Given the representatives, it is easy to see that $\mathcal{D}'_1$ acts by the Dirac operator

$$\psi \mapsto \partial\!\!\!/ \psi \,. \tag{2.256}$$

Further, there is a second order differential $\mathcal{D}'_2$ induced via homotopy transfer which encodes the field equation of the scalar field and which acts via

$$\mathcal{D}'_2 = p \circ (\mathcal{D}_1 \circ h \circ \mathcal{D}_1) \circ i \,. \tag{2.257}$$

Acting on the scalar, we find

$$\mathcal{D}'_2 \phi = p \left(\mathcal{D}_1 \, h \, (\lambda^{[i}_{[\alpha} \theta^{j]}_{\beta]} \, \partial^{[\alpha\beta]} \, \phi^i e_i) \right) \,. \tag{2.258}$$

By degree reasons, applying the homotopy h to the element in the brackets yields an expression in θ^2. On purely representation theoretic grounds, we can see that there is a unique (up to a non-zero prefactor) expression which comes into question, namely

$$\theta^{(i}_{[\alpha} \theta^{j)}_{\beta]} \, \partial^{[\alpha\beta]} \phi_{(i} e_{j)} \,. \tag{2.259}$$

As a check, we may apply the differential $\mathcal{D}_0$ to that representative. There we obtain

$$\lambda^{(i}_{[\alpha} \theta^{j)}_{\beta]} \, \partial^{[\alpha\beta]} \phi_{(i} e_{j)} \,, \tag{2.260}$$

which, at first sight, does not look like the original element we started with. However, recall that we are working in the module Γ which is the quotient $R^2/\mathrm{Im}(\varphi)$. In particular this means that $\lambda^i e_i = 0$ and hence

$$0 = \lambda^i \theta^j e_i \phi_j = \lambda^{[i} \theta^{j]} e_{[i} \phi_{j]} + \lambda^{(i} \theta^{j)} e_{(i} \phi_{j)} \,, \tag{2.261}$$

such that we indeed get back our original expression (up to a non-zero prefactor). Moving on, we then easily find

$$\mathcal{D}'_2 \phi = (\lambda^\alpha_i \theta^\beta_j \epsilon^{ij} \theta^\delta_l \theta^\gamma_k \varepsilon_{\alpha\beta\gamma\delta}) \, \partial^\mu \partial_\mu \phi^k e^l \,, \tag{2.262}$$

such that the transferred differential indeed encodes the Laplace equation.

Summarizing, the multiplet has the following structure.

$$
\begin{tikzcd}
\Omega^0 \otimes U \arrow[drrr, "\star \mathrm{d} \star \mathrm{d}"'] & \Omega^0 \otimes S_+ \arrow[dr, "\partial\!\!\!/"] & & \\
& & \Omega^0 \otimes S_- & \Omega^0 \otimes U
\end{tikzcd}
\tag{2.263}
$$

This multiplet was defined in Equation (3.2) of [Ced18b] using shift symmetry.

2.6.3 Lie algebra cohomology

Another natural source for equivariant modules are the Lie algebra cohomology groups of the supertranslation algebra $\mathfrak{n}$. This was already noted in [ESW21]. Recall that the Chevalley–Eilenberg complex takes the form

$$
C^\bullet(\mathfrak{n}) = \left(\mathrm{Sym}^\bullet(\mathfrak{n}^\vee[1])\, ,\, \mathrm{d}_{CE}\right)\ . \tag{2.264}
$$

The Chevalley–Eilenberg differential is induced by the dual of the bracket, which is extended to the whole algebra according to the Leibniz rule. For the supertranslation algebra, the $\mathbb{Z} \times \mathbb{Z}/2$ grading of the Chevalley–Eilenberg complex lifts to a $\mathbb{Z} \times \mathbb{Z}$ grading by viewing the supertranslations as a graded Lie algebra as we have done above. If we totalize this bigrading, generators in $\mathfrak{n}_2^\vee$ sit in degree -1 and generators in $\mathfrak{n}_1^\vee$ in degree zero. Identifying, $\mathrm{Sym}^q(\mathfrak{n}_1^\vee) = R = \mathbb{C}[\lambda^\alpha]$ we write

$$
C^{-p}(\mathfrak{n}) = \wedge^p(\mathfrak{n}_2^\vee) \otimes R. \tag{2.265}
$$

Denoting a basis on $\mathfrak{n}_2^\vee$ by v^μ, the Chevalley–Eilenberg differential acts on the generators by

$$
\begin{aligned}
\mathrm{d}_{CE} v^\mu &= \lambda^\alpha f^\mu_{\alpha\beta} \lambda^\beta \\
\mathrm{d}_{CE} \lambda^\alpha &= 0\, .
\end{aligned}
\tag{2.266}
$$

Now two observations turn out to be crucial. First, the zeroth Chevalley–Eilenberg cohomology is nothing else then the ring of functions of the nilpotence variety

$$
H^0(C^\bullet(\mathfrak{n})) = R/I\, . \tag{2.267}
$$

Second, as the Chevalley–Eilenberg complex comes with the structure of a cdgsa, the cohomology is equipped with a multiplication which preserves the grading. Hence, all cohomology groups are $H^0(C^\bullet(\mathfrak{n})) = R/I$-modules and can thus be used as input data for the pure spinor superfield formalism.

The analysis of examples suggests some speculations about dualities between the multiplets associated to Chevalley–Eilenberg cohomology groups in different degrees. For a start, it seems to be the case that the Chevalley–Eilenberg cohomology groups are concentrated in negative degrees up to $n := \dim(V) - \operatorname{codim}(Y)$. In all examples we have checked there is an isomorphism

$$\operatorname{Ext}^{-\operatorname{codim}(Y)}(R/I, R) \cong H^{-n}(C^\bullet(\mathfrak{n})) \,. \tag{2.268}$$

In addition, for the example of ten-dimensional $\mathcal{N} = 1$ supersymmetry, we further observe dualities "up to a copy of the free superfield" for the multiplets associated to $H^i(C^\bullet(\mathfrak{n}))$ and $H^{-n-i}(C^\bullet(\mathfrak{n}))$. Here, we give a first overview on these phenomena; we will formalize some of these findings later (see in particular §7.4.2).

Three-dimensional $\mathcal{N} = 1$. As a motivating example, let us consider again $\mathcal{N} = 1$ supersymmetry in three dimensions. Using *Macaulay2* one can compute the Chevalley–Eilenberg cohomology. Only H^0 and H^{-1} are non-vanishing. The zeroth cohomology is R/I and thus gives rise to the gauge multiplet from §2.3.9. As the length of the minimal free resolution is two—which equals the codimension of Y—we immediately see that R/I is Cohen–Macaulay. The first cohomology group is represented as the cokernel of the map

$$\varphi : R^3 \longrightarrow R^2 \qquad \varphi = \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_2 & \lambda_1 \end{pmatrix} \tag{2.269}$$

The resulting multiplet is the antifield multiplet of the gauge multiplet.

$$\begin{bmatrix} 2 & 3 & - \\ - & - & 1 \end{bmatrix} \tag{2.270}$$

Note that, as discussed in §2.4 we could have also obtained the antifield multiplet from $\operatorname{Ext}^{-2}(R/I, R)$.

Four-dimensional $\mathcal{N} = 1$. The Chevalley–Eilenberg cohomology is concentrated in degrees zero, minus one and, minus two. As the zeroth cohomology is just R/I, the corresponding multiplet is the gauge multiplet. The first cohomology group yields a multiplet with the following Betti numbers.

$$\begin{bmatrix} 4 & 7 & - & & \\ - & - & 6 & 4 & 1 \end{bmatrix} \tag{2.271}$$

Decomposing the minimal free resolution equivariantly, we find

$$L^\bullet = R \otimes \left(S_+ \oplus S_- \xleftarrow{(d_L)_1} \wedge^2 V \oplus \mathbb{C} \xleftarrow{(d_L)_2} \wedge^3 V \oplus \mathbb{C}^2 \xleftarrow{(d_L)_3} S_+ \oplus S_- \xleftarrow{(d_L)_4} \mathbb{C} \right) . \tag{2.272}$$

Thus we see that this multiplet contains a two-form. It would be interesting to interpret this as a field-strength multiplet.

The second Chevalley–Eilenberg cohomology yields two copies of the chiral multiplet.

$$\begin{bmatrix} 2 & 4 & 2 \end{bmatrix} \tag{2.273}$$

Note that this precisely matches with $\mathrm{Ext}^{-2}(R/I, R)$ as described in §2.4.4.

Ten-dimensional $\mathcal{N} = 1$. Let us further study the multiplets associated to the ten-dimensional Chevalley–Eilenberg cohomology of the ten-dimensional $\mathcal{N} = 1$ supertranslation algebra. These cohomology groups were already computed equivariantly in [MSX12] The multiplet associated to the first Chevalley–Eilenberg cohomology has the following Betti numbers.

$$\begin{bmatrix} 16 & 45 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ - & 16 & 250 & 720 & 1874 & 4368 & 8008 & 11440 & 12870 & 11440 & 8008 & 4368 & 1820 & 560 & 120 & 16 & 1 \\ - & - & - & - & 16 & 10 & - & - & - & - & - & - & - & - & - & - & - \end{bmatrix} \tag{2.274}$$

We notice that the graded rank (with respect to the homological degree) of the associated vector bundle over spacetime—which, in physical terms, corresponds to the number of degrees of freedom—is given by

$$\begin{aligned} & -16 - 45 \\ & + 16 + 250 + 720 + 1874 + 4368 + 8008 + 11440 + 12870 \\ & + 11440 + 8008 + 4368 + 1820 + 560 + 120 + 16 + 1 \\ & - 16 - 10 \\ & = 65792 = (2^{15} + 2^{15}) + (128 + 128) . \end{aligned} \tag{2.275}$$

This precisely matches the number of degrees of freedom of the supercurrent multiplet constructed in [HNv82]. Further, recall that the free superfield just corresponds to the exterior algebra $\wedge^\bullet S$ on 16 generators. Hence, its Betti numbers are precisely binomial coefficients $\binom{16}{i}$. We note that (2.274) contains precisely such coefficients, except for a missing 1 in degree $(0, 1)$. However, we can add a trivial pair in degrees $(0, 1)$ and $(1, 0)$. Then we can subtract the respective Betti numbers of the free superfield to obtain the following table.

$$\begin{bmatrix} 16 & 45+1 & - & - & - & - \\ - & - & 130 & 160 & 154 & - \\ - & - & - & - & 16 & 10 \end{bmatrix} \tag{2.276}$$

This is precisely the dual of the Betti table of $H^{-4}(C^\bullet(\mathfrak{n}))$, which is displayed in (2.280). We remark that this "almost-duality" phenomenon is closely analogous to the structure sheaf of 4d $\mathcal{N}=1$; it reflects the failure of the module to be Cohen–Macaulay. We further note that the fields in the first row are a spinor, a two-form, and a scalar; it is tempting to interpret this as a field-strength multiplet, containing the gaugino χ and the field strength F of the gauge field, and subject to certain constraints.

The multiplet associated to $H^{-2}(C^\bullet(\mathfrak{n}))$ is the stress-energy tensor multiplet or supercurrent multiplet. Its Betti table is of the following form.

$$\begin{bmatrix} 120 & 720 & 2130 & 4512 & 8008 & 11440 & 12870 & 11440 & 8008 & 4368 & 1820 & 560 & 120 & 16 & 1 \\ - & - & - & 136 & 160 & 45 & - & - & - & - & - & - & - & - & - \end{bmatrix} \tag{2.277}$$

The supercurrent multiplet can be constructed as

$$J_{\mu\nu\rho} = \mathrm{tr}(\chi\gamma_{\mu\nu\rho}\chi). \tag{2.278}$$

Here $\gamma_{\mu\nu\rho}$ just represents the isomorphism $\wedge^2(S_+) \cong \wedge^3(V)$, and χ is the spinor superfield describing on-shell Yang-Mills theory [HU87] that corresponds to $H^1(C^\bullet(\mathfrak{n}))$. Alternatively it can be described as an abstract superfield satisfying the constraints [HU87]

$$D_\alpha J_{abc} = (\gamma_{[a} J^1_{bc]})_\alpha + (\gamma_{[ab} J^1_{c]})_\alpha + (\gamma_{[abc]} J^1)_\alpha$$

where the superfields $J^1_{bc\alpha}$, $J^1_{c\alpha}$, and J^1_α are three superfields in the representation $[0,1,0,1,0]$, $[1,0,0,0,1]$ and $[0,0,0,1,0]$. The total dimension of the constraints is $560+144+16 = 720$. The leading component of J_{abc} is in the $\wedge^3 V$ representation $[0,0,1,0,0]$ of dimension 120.

Again, introducing trivial pairs and subtracting precisely yields the dual of the Betti table of $H^{-3}(C^\bullet(\mathfrak{n}))$.

$$\begin{bmatrix} 45 & 160 & 136 & - & - & - & - \\ - & - & 144 & 310 & 160 & - & - \\ - & - & - & - & - & 16 & 1 \end{bmatrix} \tag{2.279}$$

The Betti table associated to $H^{-4}(C^\bullet(\mathfrak{n}))$ takes the following form.

$$\begin{bmatrix} 10 & 16 & - & - & - & - \\ - & 54 & 160 & 130 & - & - \\ - & - & - & - & 46 & 16 \end{bmatrix} \tag{2.280}$$

Finally, $H^{-5}(C^\bullet(\mathfrak{n})) \cong R/I$ again yields the vector multiplet. Note that Y is Gorenstein and of codimension five, such that $\mathrm{Ext}^{-5}(R/I, R) \cong R/I$.

2.6.4 Six-dimensional multiplets from line bundles

For six-dimensional $\mathcal{N} = 1$ supersymmetry, the nilpotence variety can be identified with $\mathbb{P}^1 \times \mathbb{P}^3$ using the Segre embedding, hence there is an interesting family of multiplets associated to line bundles. We will study this family in detail in §4 in detail; here we just give a short preview.

Line bundles on $\mathbb{P}^n$ are classified by a single integer $j \in \mathbb{Z}$ and are denoted by $\mathcal{O}(j)$. Using the projections

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^3 & \xrightarrow{\pi_3} & \mathbb{P}^3 \\ \big\downarrow{\scriptstyle \pi_1} & & \\ \mathbb{P}^1 & & \end{array} \tag{2.281}$$

we can define a family of line bundles

$$\mathcal{O}(i, j) := \pi_1^* \mathcal{O}(i) \otimes \pi_3^* \mathcal{O}(j) \tag{2.282}$$

on $\mathbb{P}^1 \times \mathbb{P}^3$. This family has been investigated in the physics literature [KNT18a].

Let us here list the corresponding multiplets for some integers i and j. Clearly $\mathcal{O}(0, 0)$ is just the structure sheaf of the nilpotence variety and hence the corresponding multiplet is the vector multiplet. $\mathcal{O}(1, 0)$ is the hypermultiplet, which we studied above. $\mathcal{O}(2, 0)$ is the antifield multiplet of the vector.

For $\mathcal{O}(3, 0)$ a multiplet with the following Betti numbers arises.

$$\begin{bmatrix} 4 & 12 & 12 & 4 \end{bmatrix} \tag{2.283}$$

The minimal free resolution of the module in R-modules takes the form

$$L^\bullet = R \otimes \left(\mathbb{C}^4 \xleftarrow{(d_L)_1} \mathbb{C}^3 \otimes S_+ \xleftarrow{(d_L)_2} \mathbb{C}^2 \otimes \wedge^2 S_+ \xleftarrow{(d_L)_3} \mathbb{C}^1 \otimes \wedge^3 S_+ \right) , \tag{2.284}$$

The multiplet for $\mathcal{O}(4,0)$ is a building block in the construction of the "relaxed hyper-multiplet" [HST83].

$$\begin{bmatrix} 5 & 16 & 18 & 8 & 1 \end{bmatrix} \tag{2.285}$$

The minimal free resolution of the module in R-modules takes the form

$$L^\bullet = R \otimes \left(\mathbb{C}^5 \xleftarrow{(d_L)_1} \mathbb{C}^4 \otimes S_+ \xleftarrow{(d_L)_2} \mathbb{C}^3 \otimes \wedge^2 S_+ \xleftarrow{(d_L)_3} \mathbb{C}^2 \otimes \wedge^3 S_+ \xleftarrow{(d_L)_4} \mathbb{C}^1 \otimes \wedge^4 S_+ \right), \tag{2.286}$$

The minimal free resolutions are "twisted Lascoux" complexes which are described with their differentials in [DS14].

2.6.5 Conormal modules

Denoting the defining ideal of the nilpotence variety by I, the conormal module is defined as the quotient I/I^2. This gives another interesting module to consider as an input for the pure spinor superfield formalism. The resulting multiplets seem to often correspond to supergravity theories. We demonstrate this in low dimensions.

Three-dimensional $\mathcal{N} = 1$. The resulting multiplet has the following Betti numbers.

$$\begin{bmatrix} 3 & 2 & - \\ - & 5 & 4 \end{bmatrix} \tag{2.287}$$

Investigating the Hilbert series, we find that all occuring representations are irreducible representations of the spin group $\mathrm{Spin}(3) \cong SU(2)$. Thus, the first line contains a vector and a spinor, while the second line can be identified with a symmetric traceless tensor and the four-dimensional part of the decomposition

$$S \otimes V \cong [1] \oplus [3] \,. \tag{2.288}$$

Four-dimensional $\mathcal{N} = 1$. In four dimensions the conormal module yields a multiplet with the following Betti numbers.

$$\begin{bmatrix} 4 & 4 & 1 & - \\ - & 9 & 12 & 4 \end{bmatrix} \tag{2.289}$$

Investigating the Hilbert series we find that the representations in the first line are a vector, a Dirac spinor and a scalar. The nine-dimensional representation in the second line is a symmetric traceless tensor. The twelve-dimensional representation has Dynkin labels $[2,1] \oplus [1,2]$. Thus, the multiplet consists of one spin-2, two spin-$\frac{3}{2}$ and a single spin-1 field. In terms of Dynkin labels, the multiplet takes the following form.

$$\begin{tikzcd} {[1,1]} \arrow[dr] & {[1,0] \oplus [0,1]} \arrow[dr] & {[0,0]} \arrow[dr] & \\ & {[2,2]} & {[2,1] \oplus [1,2]} & {[1,1]} \end{tikzcd} \tag{2.290}$$

Ten-dimensional $\mathcal{N} = 1$. In this case, by a pleasing coincidence, the conormal module coincides with the module $H^{-4}(C^\bullet(\mathfrak{n}))$ constructed above. The resolution was studied in [Kuz18, Corollary 4.4].

2.6.6 Dimensional reduction and restriction to strata: the 4d $\mathcal{N} = 2$ tensor multiplet

There are interesting relations between the nilpotence varieties of supersymmetry algebras in different dimensions, for instance the nilpotence variety of a higher dimensional supersymmetry algebra may sit inside the nilpotence variety of a lower dimensional one. The resulting multiplets will then be related by dimensional reduction. We illustrate this by considering the relation between six-dimensional $\mathcal{N} = (1,0)$ and four-dimensional $\mathcal{N} = 2$ supersymmetry. Recall that we described the nilpotence variety for six-dimensional $\mathcal{N} = (1,0)$ supersymmetry by the 2×2-minors of a 2×4-matrix with entries λ_i^α. As explained in [ESW21] one obtains the nilpotence variety for four-dimensional $\mathcal{N} = 2$ supersymmetry by replacing

$$\lambda_i^\alpha \longrightarrow (\lambda_i^\beta, \bar{\lambda}_i^{\dot{\beta}}) \,, \tag{2.291}$$

and throwing away the two minors which do not mix the different chiralities. Hence, there is an inclusion

$$i : Y(6;1,0) \hookrightarrow Y(4;2) \,, \tag{2.292}$$

whose image we denote by Y_0. In fact the global structure of $Y(4;2)$ is easily described. It consists of three strata; in addition to Y_0 there are two copies of $(S_\pm \otimes U) \cong \mathbb{C}^4$ corresponding to solutions where $\lambda = 0$ or $\bar{\lambda} = 0$ respectively:

$$Y(4;2) = Y_0 \cup Y_1 \cup Y_2 \cong Y(6;1,0) \cup (S_+ \otimes U) \cup (S_- \otimes U) \,. \tag{2.293}$$

Pushing forward the structure sheaf $\mathcal{O}_{Y(6;1,0)}$ along i we thus obtain $\mathcal{O}_{Y_0}$. As we already discussed at multiple occasions, the structure sheaf $\mathcal{O}_{Y(6;1,0)}$ produces the vector multiplet. Clearly, considering $\mathcal{O}_{Y_0}$ in the pure spinor superfield formalism gives a multiplet with the same Betti numbers; only the weights have to be adapted to four dimensions. Resolving $\mathcal{O}_{Y_0}$ equivariantly, we see that the six-dimensional vector splits up into a four-dimensional vector and two scalars. The fermion gives two Dirac fermions in four dimensions and the scalars remain scalars. Hence, the resulting multiplet is precisely the $\mathcal{N} = 2$ vector multiplet in four-dimensions as one can obtain it from dimensional reduction. A similar phenomenon holds in general: given a multiplet in dimension d, we can push the corresponding sheaf forward along the dimensional reduction map to obtain the dimensionally reduced multiplet.

Interestingly, considering $\mathcal{O}_{Y(4;2)}$ as an input in the pure spinor superfield machinery gives a multiplet with the following Betti numbers.

$$\begin{bmatrix} 1 & - & - & - & - \\ - & 4 & - & - & - \\ - & - & 9 & 8 & 2 \end{bmatrix} \tag{2.294}$$

Working equivariantly, the minimal free resolution gives

$$L^\bullet = R\otimes\left(\mathbb{C} \xleftarrow{(d_L)_1} V \xleftarrow{(d_L)_2} \wedge^2 V \oplus \mathbb{C}^3 \xleftarrow{(d_L)_3} (S_+ \otimes U) \oplus (S_- \otimes U) \xleftarrow{(d_L)_4} \mathbb{C}_2 \oplus \mathbb{C}_{-2}\right), \tag{2.295}$$

where $\mathbb{C}^3$ carries the adjoint representation of $SU(2)_R$ and has $U(1)_R$-charge 0 while the two scalars in the top degree have $U(1)_R$-charges $+2$ and -2 as indicated by the subscript. This is the field content of a tensor multiplet as described in [WS06; Jur+19].

Of course we can also restrict to the other strata. The minimal free resolutions are then exterior algebras $\wedge^\bullet(S_\pm \otimes U)$, the resulting multiplets are thus chiral multiplets as described in [dRo+80].

2.A Homotopy transfer for L_∞ modules

Let $(L, \tilde{\mu}_k)$ be a (super) L_∞ algebra and $(V, d_V, \rho^{(j)})$ an L_∞ module for L. As was explained in [Lad04], the L_∞ module structure gives rise to an L_∞ structure on $L \oplus V$. Explicitly we can define (setting $\rho^{(0)} = d_V$)

$$\mu_k((x_1, v_1), \dots, (x_k, v_k)) = \left(\tilde{\mu}_k(x_1, \dots, x_k), \sum_{i=1}^{k} \pm\rho^{(k-1)}(x_1, \dots, \hat{x}_i, \dots, x_k)v_i\right). \tag{2.296}$$

For example, if $(L,[.,.])$ is a (super) Lie algebra and ρ is a strict action, we find

$$\begin{aligned} \mu_1((x,v)) &= (0, d_V v) \\ \mu_2((x_1,v_1),(x_2,v_2)) &= ([x_1,x_2]\ ,\ \rho(x_1)v_2 - \rho(x_2)v_1) \end{aligned} \tag{2.297}$$

All higher order operations vanish. Now suppose we have homotopy data

$$\begin{tikzcd} (V,d_V) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (W,d_W) \arrow[l, shift left, "i"] \end{tikzcd} \tag{2.298}$$

and want to transfer an L_∞ module structure on V to a new L_∞ module structure on W. The fact that these L_∞ module structures can be thought of as L_∞ structures on $L \oplus V$ and $L \oplus W$ suggests to extend the above homotopy data to

$$\begin{tikzcd} (L\oplus V,d_V) \arrow[loop left, "\mathrm{id}\oplus h"] \arrow[r, shift left, "\mathrm{id}\oplus p"] & (L\oplus W,d_W) \arrow[l, shift left, "\mathrm{id}\oplus i"] \end{tikzcd} \tag{2.299}$$

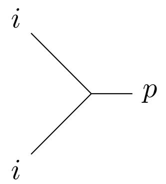
and then to use the usual homotopy transfer for L_∞ structures. Let us denote the transferred L_∞ structure on $L \oplus W$ by μ'_k. We can read off the transferred module action $\rho'^{(k)}$ as follows. Let

$$\pi : L \oplus V \longrightarrow V \tag{2.300}$$

be the obvious projection. Then (2.296) implies

$$\rho'^{(k)}(x_1,\dots,x_k)w = \pi\left(\mu'_{k+1}((x_1,0),\dots,(x_k,0),(x_{k+1},w))\right)\ . \tag{2.301}$$

As usual, the transferred L_∞ structure μ'_k can be calculated by sum over trees formulas. Using this, one can also derive sum over tree formulas for the induced action ρ'. For our purposes we are only interested in the case where $L = \mathfrak{g}$ is a super Lie algebra and ρ is a strict action. As explained above, this means that $(\mathfrak{g} \oplus V, \mu_k)$ is a dgs Lie algebra. In this case the L_∞ structure on $\mathfrak{g} \oplus W$ is computed by the sum over all rooted binary trees by decorating each leaf with the inclusion i, each internal line with the homotopy h, and the root by the projection p. A vertex means the application of the product μ_2. In the case of the binary product one writes:

$\mu'_2 =$ 

.

(2.302)

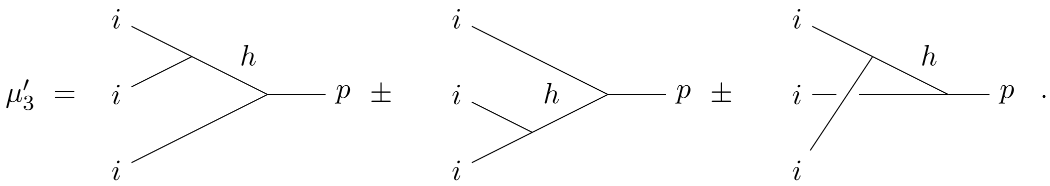
In formulas this means

$$\mu_2' \left((x_1, w_1), (x_2, w_2)\right) = \left([x_1, x_2]\ ,\ p(\rho(x_1)i(w_2) \pm \rho(x_2)i(w_1))\right)\ . \tag{2.303}$$

Accordingly we find for the L_∞ module action ρ'

$$\rho'^{(1)} = p \circ \rho \circ i\ . \tag{2.304}$$

In the case of μ_3' we can write

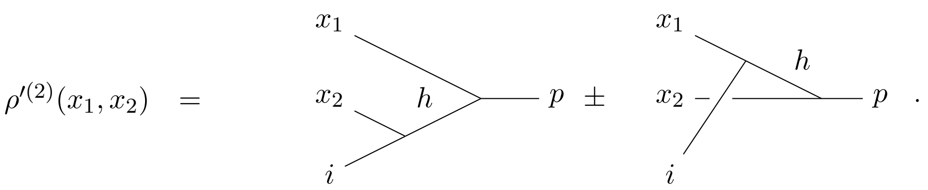


(2.305)

This gives for $\rho'^{(2)}$

$$\rho'^{(2)}(x_1, x_2) = p \circ (\rho(x_1)h\rho(x_2) \pm \rho(x_2)h\rho(x_1)) \circ i\ . \tag{2.306}$$

In this manner we can also obtain a general sum over trees representation for $\rho'^{(k)}$ in terms of ρ. Using equations (2.301) and (2.296) we see that $\rho'^{(k)}$ can be obtained from binary rooted trees with $k+1$ leaves by the following rules. Label the first k leaves by elements $x_1, \ldots, x_k$ and the last one by the inclusion i. Keep only those trees where there are no vertices connecting two elements of $\mathfrak{g}$. As usual, each internal line carries the homotopy h and the root is decorated by p. A vertex now means "apply $\rho(x_i)$". For example we can write for $\rho'^{(2)}$:



(2.307)

Clearly this recovers (2.306).

Chapter 3

Derived pure spinor superfields

3.1 Introduction

As established in the previous chapter, the pure spinor superfield formalism constructs a supersymmetric multiplet out of the datum of a graded module over the ring of functions on the nilpotence variety of a supertranslation algebra. It is natural to ask for a characterization of all multiplets which arise in this manner. In §3.5.2, we already described an example of a multiplet which cannot be constructed using the technique.[1] There, we linked the failure to the geometry of the underlying nilpotence variety: it is not Cohen–Macaulay, so that the dualizing complex of its ring of functions has cohomology in multiple degrees and is not quasi-isomorphic to a single homogeneous module. At the level of multiplets, the structure sheaf gives rise to the four-dimensional $\mathcal{N} = 1$ vector multiplet. On general grounds, one expects that the dualizing module gives rise to the dual (or antifield) multiplet. However, due to the failure of the Cohen–Macaulay property, this does not work on the nose. Instead, the appearance of the dualizing complex suggests that a generalization of the pure spinor superfield formalism to the world of derived algebraic geometry is necessary.

In this chapter, we tackle these questions systematically, and show that a derived generalization of the pure spinor formalism can be used to produce *every* supermultiplet in a very general setting.

As before, let $\mathfrak{g}$ be a super Lie algebra of super Poincaré type with supertranslation subalgebra $\mathfrak{n}$. Building on the terminology of the previous chapter, we will introduce the dg-category of $\mathfrak{g}$-multiplets which we denote by $\mathsf{Mult}_{\mathfrak{g}}$. The main result of this chapter (Theorem 3.4.3), stated somewhat informally, establishes an equivalence of categories

[1] The obstruction to constructing this multiplet using the pure spinor formalism was previously observed by Martin Cederwall.

between between $\mathfrak{g}$-multiplets and $\mathfrak{g}_0$-equivariant modules over the Chevalley–Eilenberg algebra $C^\bullet(\mathfrak{n})$.

$$\mathsf{Mult}_{\mathfrak{g}} \leftrightarrows \mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})} \tag{3.1}$$

We will view the functor from left to right as a natural derived enhancement of the pure spinor construction. Indeed, taking Lie algebra cochains of (2.58), we obtain an exact sequence

$$\mathbb{C} \to C^\bullet(\Pi\mathfrak{n}_1) \to C^\bullet(\mathfrak{n}) \to C^\bullet(\mathfrak{n}_2) \to \mathbb{C} \tag{3.2}$$

of bigraded cdgas, witnessing $C^\bullet(\mathfrak{n})$ as an R-algebra, where $R := C^\bullet(\Pi\mathfrak{n}_1) \cong \mathrm{Sym}^\bullet(\mathfrak{n}_1)$ is the free commutative algebra on $\mathfrak{n}_1^\vee$. If we totalize the bigrading, $C^\bullet(\mathfrak{n})$ is concentrated in non-positive degrees and its degree-zero cohomology is then R/I, the ring of functions on the nilpotence variety of $\mathfrak{g}$. Furthermore, there is a natural equivariant map

$$C^\bullet(\mathfrak{n}) \to R/I, \tag{3.3}$$

so that any R/I-module is a $C^\bullet(\mathfrak{n})$-module in a natural way. When applied to modules of this sort, our derived enhancement agrees on the nose with the standard pure spinor formalism.

From this point of view, we can view $C^\bullet(\mathfrak{n})$ as a derived enhancement of the ring of functions on the nilpotence variety: its spectrum is an affine derived scheme whose underlying classical affine scheme is exactly the affine nilpotence variety, but whose derived data remembers the higher cohomology of $\mathfrak{n}$ with respect to the totalized grading.

Remark 3.1.1. While we will not need any technology from the theory of derived algebraic geometry in this work, we refer the interested reader to the article [Toë14] of Toën for a survey.

The functor in (3.1) from $\mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})}$ to multiplets is defined by extending the construction in §2. It is closely connected to more standard versions of Koszul duality in at least two ways. First of all, it can be thought of as arising from the dual pair $(C^\bullet(\mathfrak{n}), U(\mathfrak{n}))$. The standard Koszul duality functor is generated by the kernel

$$(\mathcal{K}', \mathcal{D}') = (C^\bullet(\mathfrak{n}) \otimes U(\mathfrak{n})\ ,\ X^i \otimes n_i), \tag{3.4}$$

where $\{X^i\}$ is a basis for $\mathfrak{n}^\vee$, therefore a set of generators for $C^\bullet(\mathfrak{n})$, and n_i denotes the corresponding basis of $\mathfrak{n}$ (thus set of generators for $U(\mathfrak{n})$).

If we now view $C^\infty(N)$ as a right module for $U(\mathfrak{n})$, where $\mathfrak{n}$ acts by right-invariant vector fields—in particular, n_i by the vector field $D_i := \rho(n_i)$—we can modify this kernel to obtain a $(C^\bullet(\mathfrak{n}), C^\infty(N))$-bimodule of the form

$$(\mathcal{K}, \mathcal{D}) = (C^\bullet(\mathfrak{n}) \otimes C^\infty(N)\,,\; X^i \otimes D_i). \tag{3.5}$$

Our functor is defined as the integral transform associated to this kernel. From this perspective, we can interpret the functor heuristically in two steps:

1. First form the Koszul dual $U(\mathfrak{n})$-module of a module over the Chevalley–Eilenberg complex.
2. Then apply the associated bundle construction to obtain a dg-vector bundle over the supergroup N, with a residual right N-action. In particular, we can forget down to a dg-vector bundle on the even part V of N, with a geometric action of the supersymmetry algebra $\mathfrak{n}$.

It is another version of Koszul duality which is perhaps most closely connected. Kapranov [Kap91] established a version of Koszul duality that provides a Quillen equivalence between the model categories of D-modules on a space and $\Omega^\bullet$-modules over the same space. Our construction can be understood as a version of this equivalence in the case where that space is the nilpotent super Lie group N; it relates translation-invariant natural vector bundles to modules over the translation-invariant differential forms, which are precisely the Lie algebra cochains. The connection to the notion of "multiplet" in the physics literature has, as far as we know, not been appreciated before.

While Koszul duality provides a natural language in which to understand the pure spinor construction, our proof of Theorem 3.4.3, and all of our discussion of examples, will proceed by direct computation using the kernel $(\mathcal{K}, \mathcal{D})$. This is meant to emphasize that the technique provides not just an abstract equivalence, but a set of efficient and practical computational techniques. We also emphasize that the version of Koszul duality encapsulated in the pure spinor formalism naturally produces dg vector bundles whose sections are sheaves on the site of manifolds equipped with appropriate structure: in the case of standard super Poincaré algebras, this means that one knows how to place the resulting multiplet on any Riemannian manifold. We see this as being connected to ideas in Cartan geometry for the model space $N \cong G/G_0$, and hope to return to versions of the formalism on non-flat spacetimes (for nontrivial or more general Cartan geometries) in future work.

Finally, it is worth remarking that the functor from right to left in (3.1) is also easy to understand: it is just taking the derived $\mathfrak{n}$-invariants of a multiplet. This is intuitively satisfying in various respects. For example, the twist of a multiplet consists of the derived invariants of some abelian odd subalgebra of $\mathfrak{n}$, whereas its dimensional reduction consists of the invariants of an abelian even subalgebra of $\mathfrak{n}$. The fact that a multiplet can be recovered from the datum of its derived $\mathfrak{n}$-invariants thus says, in a sense, that it is equivalent to all of its possible twists, considered as a natural family over the derived classifying space $B\mathfrak{n}$.

The upshot of this result is that, after an appropriate derived upgrade, *every* supermultiplet has a pure spinor superfield description. We can use the equivalence of categories to shed some light on the motivating questions concerning the underived pure spinor formalism discussed above. For example, the following is an immediate consequence:

Theorem (Corollary 3.4.6)**.** *A given multiplet* (E, D, ρ) *lies in the image of the* underived *pure spinor formalism if and only if the Chevalley–Eilenberg cohomology* $H^\bullet(\mathfrak{n}, \mathcal{E})$ *is concentrated in a single degree.*

We illustrate the derived formalism with some applications and examples; we also show that the derived formalism gives rise to a construction of certain maps relating the multiplets associated to the different Chevalley–Eilenberg cohomology groups of the supertranslation algebra, associated to a filtration on $C^\bullet(\mathfrak{n})$. We give explicit calculations for examples with minimal supersymmetry in dimensions three and four.

Further directions

The realization of the pure spinor formalism as an equivalence of dg-categories offers potential insight to numerous more involved applications. We give a few indications of possible directions here:

1. Given any multiplet $M = (E, D, \rho)$, associated to any supersymmetry algebra $\mathfrak{g}$ with associated supertranslation algebra $\mathfrak{n}$, it is possible to realize a $C^\bullet(\mathfrak{n})$-module Γ generating M via the pure spinor formalism: one can simply set $\Gamma = C^\bullet(\mathfrak{n}, \mathcal{E})$. These modules have cohomology consisting of a graded sum of finitely generated modules over the ring of functions on the nilpotence variety, together with additional information encoding the action of the generators of $C^\bullet(\mathfrak{n})$ in nonzero total degree. By the results of [SW21], they efficiently and fully encode the information of the various *twists* of the original multiplet M: one can compute the stalk of the module at a classical point $Q\colon C^\bullet(\mathfrak{n}, \mathcal{E}) \to \mathbb{C}$, obtaining descriptions of different twists for different orbits under the action of $\mathrm{Spin}(n)$ and the group of R-symmetries. We refer to the discussion in [ES19b; ESW20; ESW21] for details of this classification.

   It would, for example, be interesting to explicitly understand the modules associated to multiplets such as the $\mathcal{N} = (1, 0)$ tensor multiplet in six dimensions.

2. One can use the Batalin–Vilkovisky (BV) formalism to construct *interacting* supersymmetric classical field theories using the pure spinor formalism. In the BV formalism, an interacting supersymmetric field theory is encoded by a cyclic L_∞ structure on a multiplet M, that is, an L_∞ algebra structure together with a pairing of degree -3. This additional data can be carried along the pure spinor functor; it is enough to define a Lie algebra structure and shifted symplectic pairing internal to equivariant $C^\bullet(\mathfrak{n})$-modules. This will require a bit more input; it is not straightforward to define a natural tensor product on the category of multiplets, so one would need to work instead with the D-modules obtained by taking the sheaf of sections of the multiplet with its natural tensor product. One would then aim use the equivalence to establish a "convolution" monoidal structure $*$ making the functor monoidal.

   For example, as computed in [CNT02] and discussed in [SW21, §7.3] and [EH23], if $\mathfrak{n}$ is the eleven-dimensional $\mathcal{N} = 1$ supertranslation algebra, the eleven-dimensional supergravity multiplet arises from the zeroth cohomology of $C^\bullet(\mathfrak{n})$, placed in degree -3 with respect to the natural weight grading, as a $C^\bullet(\mathfrak{n})$-module. In order to obtain an *interaction*, it would suffice to define, up to homotopy, a Lie structure on this module. It would be interesting to understand the action functionals of [Ced10c; Ced10a] in this language, and to use them to connect to component actions for perturbative supergravity, either twisted or untwisted. An interacting BV theory conjecturally describing the minimal twist of eleven-dimensional supergravity was studied in [RSW23]. We come back to this question in §7.

3. There are several sheaves over the (derived) nilpotence variety that automatically carry Lie structures. For example, if $\mathfrak{h}$ is a finite-dimensional semisimple Lie algebra, one can form the tensor product $H^0(\mathfrak{n}) \otimes \mathfrak{h}$.[2] It has been known for some time that this Lie algebra describes interacting ten-dimensional super Yang–Mills theory in the BV formalism; the correct gauge algebras for lower-dimensional super Yang–Mills theories are also obtained in this fashion.

   In dimensions higher than eleven we expect $C^\bullet(\mathfrak{n})$ to have no higher cohomology, so $\mathrm{Spec}(C^\bullet(\mathfrak{n}))$ will be purely classical. Equivalently, the nilpotence variety is expected to be a complete intersection according to Hartshorne's conjecture: roughly speaking, the nilpotence variety is determined by a system of n equations, while the number of variables is of order $2^{n/2}$. Hartshorne's conjecture states that any smooth projective variety in $\mathbb{P}^n$ with codimension $< n/3$ is a complete intersection.

[2] When we discuss Lie algebra cohomology $H^\bullet(\mathfrak{n})$, we will view $\mathfrak{n}$ as being a $\mathbb{Z}$-graded object with respect to the weights of the natural rescaling action. As such, the complex $C^\bullet(\mathfrak{n})$ is naturally bigraded; we refer here to the zeroth cohomology with respect to the total grading. We will discuss these degree conditions further in §3.3.1.

The codimension condition applies to nilpotence varieties with minimal supersymmetry in dimension ≥ 12, although these varieties are not smooth (but see for instance the recent article [ESS21] for version of this result that would be applicable to the example of nilpotence varieties in high dimensions). However, it is often possible to obtain non-trivial cohomology by taking—rather than the entire Chevalley–Eilenberg complex—a non-trivial quotient associated to an orbit closure within the nilpotence variety.

To give a further example, the tangent sheaf $\mathrm{Der}(C^\bullet(\mathfrak{n}))$ carries a Lie bracket for all supertranslation algebras $\mathfrak{n}$. While the associated multiplet is not generally associated to a BV theory—it does not typically carry a -1-shifted symplectic pairing—one can always build a BV theory by applying the cotangent theory construction and considering the multiplet $M \otimes M^![-1]$, where $M^! = M^\vee \otimes \Omega^{\mathrm{top}}(V)$ is the density-valued dual. This procedure allows for the construction of a very general family of interacting classical supersymmetric field theories.

4. The equivalence of categories allows for a program to classify families of multiplets starting from the algebraic geometry of the (derived) nilpotence variety. We explore this direction in §4, where (among other things) a description of all six-dimensional $\mathcal{N} = (1,0)$ multiplets whose derived invariants form a single line bundle on the projective nilpotence variety is given.

3.2 The category of multiplets

In §2, we introduced the notion of a $\mathfrak{g}$-multiplet for a super Lie algebra $\mathfrak{g}$, now we move on to define a category of multiplets $\mathsf{Mult}_{\mathfrak{g}}$. To do so, we proceed in two steps: First, we discuss strict multiplets with strict morphisms, then we move on to the homotopy-theoretic generalization.

3.2.1 Strict multiplets

As before, for a dgs vector bundle (E, D) over a base manifold X, we denote the space of global smooth sections by $\mathcal{E} = \Gamma(X, E)$. The endomorphisms $\mathrm{End}(\mathcal{E})$ form a dgs Lie algebra, where the bracket is given by the commutator and the differential is $[D, -]$. Inside $(\mathrm{End}(\mathcal{E}), [D, -])$ there is a sub dgs Lie algebra consisting of all endomorphisms which act on sections via differential operators. We denote this subalgebra by $(\mathcal{D}(E), [D, -])$.

Definition 3.2.1. A *strict local dgs* $\mathfrak{g}$*-module* is a triple (E, D, ρ) where (E, D) is a dgs vector bundle and

$$\rho\colon \mathfrak{g} \longrightarrow (\mathcal{D}(E), [D, -]) \tag{3.6}$$

is a map of dgs Lie algebras. Here the super Lie algebra $\mathfrak{g}$ is viewed as a dgs Lie algebra in cohomological degree zero with trivial differential.

Remark 3.2.2. This definition (as well as many of the following) has a natural generalization for $\mathfrak{g}$ a dgs Lie algebra. Since we are ultimately interested in the pure spinor superfield formalism, we restrict our attention to super Lie algebras with no cohomological grading.

Note that, since a super Lie algebra $\mathfrak{g}$ has no differential, ρ commutes with the differential on the dgs vector bundle,

$$[D, \rho(x)] = 0 \quad \forall x \in \mathfrak{g} \, . \tag{3.7}$$

It is standard to encode a $\mathfrak{g}$-module structure ρ on (E, D) as a dgs Lie algebra structure on the direct sum $\mathfrak{g} \oplus \mathcal{E}$. Concretely we set for the unary and binary operations

$$\begin{aligned} \mu_1(x_1, \sigma_1) &= (0, D\sigma_1) \\ \mu_2((x_1, \sigma_1), (x_2, \sigma_2)) &= ([x_1, x_2] \, , \, \rho(x_1)\sigma_2 - (-1)^{|\sigma_1||x_2|}\rho(x_2)\sigma_1) \, , \end{aligned} \tag{3.8}$$

where $x_1, x_2 \in \mathfrak{g}$ and $\sigma_1, \sigma_2 \in \mathcal{E}$.

There is an obvious notion of morphisms between strict local $\mathfrak{g}$-modules.

Definition 3.2.3. A *strict morphism* of strict local $\mathfrak{g}$-modules (E, D, ρ) and (E', D', ρ') is a map of cochain complexes

$$\psi \colon \mathcal{E} \longrightarrow \mathcal{E}' \tag{3.9}$$

realized by differential operators such that

$$\psi \circ \rho(x) = \rho'(x) \circ \psi \tag{3.10}$$

for all $x \in \mathfrak{g}$.

A strict morphism $\psi \colon (E, D, \rho) \longrightarrow (E', D', \rho')$ gives rise to a strict morphism of the associated dgs Lie algebras by setting $\tilde{\psi} = \mathrm{id}_{\mathfrak{g}} \times \psi \colon \mathfrak{g} \oplus \mathcal{E} \longrightarrow \mathfrak{g} \oplus \mathcal{E}'$. Conversely it is easy to check that every strict morphism of dgs Lie algebras of that form gives rise to a strict morphism of $\mathfrak{g}$-modules. We call ψ a quasi-isomorphism if it is a quasi-isomorphism of dgs vector bundles; equivalently $\tilde{\psi}$ is a quasi-isomorphism of dgs Lie algebras.

Now, suppose that (E, D) is a dgs vector bundle over an affine space V (so, in the notation used above, $V = X$). Let us additionally assume that $\mathfrak{g}$ is equipped with a map of super Lie algebras

$$\phi \colon \mathfrak{aff}(V) \longrightarrow \mathfrak{g}. \tag{3.11}$$

In essence, when we refer to strict $\mathfrak{g}$-*multiplets*, we are referring to strict local $\mathfrak{g}$-modules for which the affine transformations acts in a geometric way.

Definition 3.2.4. A *strict* $\mathfrak{g}$-*multiplet* (E, D, ρ) on V is an affine dgs vector bundle over V equipped with a strict local $\mathfrak{g}$-module structure

$$\rho\colon \mathfrak{g} \longrightarrow \mathcal{D}(E) \tag{3.12}$$

such that the pullback of the module structure along ϕ agrees with the natural action on sections of the affine vector bundle. Concretely, this means that the following diagram commutes.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\rho"] & \mathcal{D}(E) \\ \mathfrak{aff}(V) \arrow[u, "\phi"] \arrow[ur, "\mathrm{aff}"'] & \end{tikzcd} \tag{3.13}$$

Here aff denotes the natural action of the affine algebra on E. We define the category of strict multiplets with strict morphisms to be the full subcategory of the category of strict local $\mathfrak{g}$-modules with objects strict $\mathfrak{g}$-multiplets. We denote this category by $\mathsf{Mult}_{\mathfrak{g}}^{\text{strict}}$.

Remark 3.2.5. Note that a morphism of strict local $\mathfrak{g}$-modules is automatically compatible with the action of $\mathfrak{aff}(V)$. For example let $\psi\colon (E, D, \rho) \longrightarrow (E', D', \rho')$ be a strict morphism. Then we have

$$\psi \circ \mathrm{aff}(x) = \psi \circ \rho(\phi(x)) = \rho'(\phi'(x)) \circ \psi = \mathrm{aff}'(x) \circ \psi\,. \tag{3.14}$$

Therefore it is sensible to define $\mathsf{Mult}_{\mathfrak{g}}^{\text{strict}}$ as a full subcategory of strict local $\mathfrak{g}$-modules.

3.2.2 Homotopy theory of multiplets

The appropriate homotopy theoretic generalization of strict $\mathfrak{g}$-module structures are L_∞ $\mathfrak{g}$-modules. In the context of this work, we will refer to such homotopy module structures simply as module structures and choose to emphasize whenever a module is strict instead. In this spirit we can easily define (not necessarily strict) local $\mathfrak{g}$-modules and $\mathfrak{g}$-multiplets by replacing Lie maps in the above definitions with L_∞ maps.

Recall that a local $\mathfrak{g}$-module is just a dgs vector bundle (E, D) together with an L_∞ map of L_∞ algebras

$$\rho\colon \mathfrak{g} \rightsquigarrow \mathcal{D}(E)\,. \tag{3.15}$$

We can expand ρ in component maps

$$\rho^{(k)}\colon \mathfrak{g}^{\otimes k} \longrightarrow \mathcal{D}(E)[1-k] \qquad k \geq 1 \tag{3.16}$$

satisfying a series of compatibility relations. As usual, we recover strict modules if and only if $\rho^{(k)} = 0$ for all $k \geq 2$.

Similarly to the strict case, we can conveniently encode a $\mathfrak{g}$-module structure ρ as an L_∞ structure on $\mathfrak{g} \oplus \mathcal{E}$. To this end we supplement the operations (3.8) by the following brackets for $k \geq 3$. For details we refer to [Lad04; All10].

$$\mu_k((x_1, v_1), \ldots, (x_k, v_k)) = \left(0\,,\, \sum_{i=1}^{k} \pm \rho^{(k-1)}(x_1, \ldots, \hat{x}_i, \ldots, x_k) v_i\right) \tag{3.17}$$

One can define a morphism $\psi\colon (E, D, \rho) \longrightarrow (E', D', \rho')$ of local $\mathfrak{g}$-modules by component maps

$$\psi_n \colon \mathfrak{g}^{\otimes n-1} \otimes \mathcal{E} \longrightarrow \mathcal{E}', \tag{3.18}$$

which are given by differential operators and satisfy a series of compatibility relations (see [All10] for details). We recover strict morphisms by restricting to those where the only component map is ψ_1. Again, we can describe such morphisms by morphisms of the associated L_∞ algebras

$$\tilde{\psi}\colon \mathfrak{g} \oplus \mathcal{E} \rightsquigarrow \mathfrak{g} \oplus \mathcal{E}' \tag{3.19}$$

by supplementing $\tilde{\psi}_1 = \mathrm{id}_{\mathfrak{g}} \times \psi_1$ with

$$\tilde{\psi}_k((x_1, \sigma_1), \ldots, (x_k, \sigma_k)) = (0\,,\, \sum_{i=1}^{k} \pm \psi_k(x_1, \ldots, \hat{x}_i, \ldots, x_k, \sigma_i)) \tag{3.20}$$

for $k \geq 2$. ψ is a quasi-isomorphism of local $\mathfrak{g}$-modules if ψ_1 is a quasi-isomorphism of cochain complexes, or equivalently if $\tilde{\psi}$ is a quasi-isomorphism of L_∞ algebras. In general, encoding module structures as L_∞ structures is very convenient because it allows the use of many known tools from the theory of L_∞ algebras, like homotopy transfer for instance.

Remark 3.2.6. An equivalent realization of the notion of a non-strict morphism between a pair of local L_∞ algebras $\mathfrak{g}, \mathfrak{g}'$ is provided by considering the Chevalley–Eilenberg chain complex $C_\bullet(\mathfrak{g})$ of an L_∞ algebra $\mathfrak{g}$. This is a cocommutative dg coalgebra whose underlying graded coalgebra is $\mathrm{Sym}^\bullet(\mathfrak{g}[-1])$, with differential induced from the L_∞ structure on $\mathfrak{g}$ as a sum of terms given by the L_∞ brackets on $\mathfrak{g}$. The data of a morphism $\psi\colon \mathfrak{g} \to \mathfrak{g}'$ of local L_∞ algebras is equivalent to that of a morphism of cocommutative dg coalgebras

$$\psi' \colon C_\bullet(\mathfrak{g}) \to C_\bullet(\mathfrak{g}')$$

given by differential operators. This interpretation allows us to view local L_∞ algebras as objects of a dg-category, so that we can discuss (for example) homotopies between L_∞ algebra morphisms. When the source $\mathfrak{g}$ is finite-dimensional, we can dually consider morphisms of commutative dg algebras $C^\bullet(\mathfrak{g}') \to C^\bullet(\mathfrak{g})$. This is for example the case for the local $\mathfrak{g}$-module structures $\rho\colon \mathfrak{g} \rightsquigarrow \mathcal{D}(E)$ appearing in the definition of multiplets.

Replacing the strict morphisms ρ in the definition of strict multiplets by an L_∞-map, we recover the definition of a multiplet as presented in Definition 2.2.11.

Definition 3.2.7. We define the dg-category of $\mathfrak{g}$-multiplets to be the full subcategory of local $\mathfrak{g}$-modules with objects being $\mathfrak{g}$-multiplets and denote it by $\mathsf{Mult}_\mathfrak{g}$.

We will sometimes wish to refer to the full subcategory of strict multiplets, but allowing arbitrary morphisms.

Definition 3.2.8. Denote by $\mathsf{Mult}_\mathfrak{g}^{\text{strict-ob}}$ the full sub dg-category of $\mathsf{Mult}_\mathfrak{g}$ generated by strict multiplets.

We can obtain homotopy categories from the dg-categories $\mathsf{Mult}_\mathfrak{g}$ as well as $\mathsf{Mult}_\mathfrak{g}^{\text{strict}}$ by replacing the hom space $\mathrm{Hom}_{\mathsf{Mult}_\mathfrak{g}}(E,E')$ by its zeroth cohomology $H^0(\mathrm{Hom}_{\mathsf{Mult}_\mathfrak{g}}(E,E'))$. We denoted the resulting categories by $\mathrm{Ho}(\mathsf{Mult}_\mathfrak{g})$ and $\mathrm{Ho}(\mathsf{Mult}_\mathfrak{g}^{\text{strict}})$. Speaking physically, quasi-isomorphisms of $\mathfrak{g}$-multiplets correspond to *perturbative* equivalences of multiplets, where by "perturbative" here we mean equivalences of the derived formal neighborhoods of a point in the classical moduli field space. Therefore isomorphism classes in the homotopy category correspond to perturbatively distinct multiplets.

Remark 3.2.9. Because every L_∞ algebra can be strictified [KM95, Part II Corollary 1.6], the natural inclusion

$$\mathrm{Ho}(\mathsf{Mult}_\mathfrak{g}^{\text{strict-ob}}) \to \mathrm{Ho}(\mathsf{Mult}_\mathfrak{g}) \tag{3.21}$$

is an equivalence of categories, where $\mathsf{Mult}_\mathfrak{g}^{\text{strict-ob}}$ denotes the full subcategory of multiplets spanned by strict objects.

3.2.3 Linear structure

We can define the direct sum of two $\mathfrak{g}$-multiplets as follows.

Definition 3.2.10. The *direct sum* of two $\mathfrak{g}$-multiplets (E,D,ρ) and (E',D',ρ') is defined to be the multiplet

$$(E,D,\rho) \oplus (E',D',\rho') = (E \oplus E'\ ,\ D \oplus D'\ ,\ \rho \oplus \rho')\,. \tag{3.22}$$

Remark 3.2.11. In contrast, defining a tensor product on the category of $\mathfrak{g}$-multiplets is not straightforward. Considering the D-modules of global sections, one can take the tensor product in the category of D-modules, but this does not take the additional structures on a multiplet into account. As alluded to in the introduction, one can might also try using the equivalence established in §3.4 to define a product on the category which makes the functor monoidal. We will not discuss this issue further in this work.

For further reference, we also define the dual of a $\mathfrak{g}$-multiplet. In physics context, these are usually called antifield multiplets

Definition 3.2.12. The *dual* of a $\mathfrak{g}$-multiplet (E, D, ρ), also referred to as the associated *antifield multiplet* is the $\mathfrak{g}$-multiplet $(E^!, D^\vee, \rho^\vee)$. Here $E^!$ is the linear dual vector bundle to E twisted by the canonical bundle. The action $\rho^\vee$ denotes the map

$$\rho^\vee : \mathfrak{g} \rightsquigarrow \mathcal{D}(E^!) \tag{3.23}$$

given by $\rho^{\vee(k)}(Q_1, \ldots, Q_k) = \rho^{(k)}(Q_1, \ldots, Q_k)^\vee$.

Note that we will typically be working over the flat space $\mathbb{R}^n$, so we may choose a trivialization of the canonical bundle if we wish to identify $E^!$ with $E^\vee$.

3.3 Derived pure spinor superfields

In this section, we define the derived generalization of the pure spinor superfield construction, which will provide one of the two functors that witness the equivalence of categories. We begin by setting up the general context in which we want to work.

Let us choose a basis d_α for $\mathfrak{n}_1$ and e_μ for $\mathfrak{n}_2$ such that we can expand the symmetric bracket in structure constants[3]

$$[d_\alpha, d_\beta] = f^\mu_{\alpha\beta} e_\mu \,. \tag{3.24}$$

We denote by $R = \mathrm{Sym}^\bullet(\mathfrak{n}_1^\vee) = \mathbb{C}[\lambda^\alpha]$ the ring of polynomial functions on $\mathfrak{n}_1$. For Q in $\mathfrak{n}_1$, the equation $[Q, Q] = 0$ defines an ideal I in R, explicitly given by

$$I = (\lambda^\alpha f^\mu_{\alpha\beta} \lambda^\beta) \,. \tag{3.25}$$

The pure spinor superfield formalism gives a systematic tool to construct $\mathfrak{g}$-multiplets from the input datum of a graded $\mathfrak{g}_0$-equivariant R/I-module. Here we generalize the pure spinor superfield construction to $\mathfrak{g}_0$-equivariant $C^\bullet(\mathfrak{n})$-modules and show that it defines a functor.

[3]In the context of super Poincaré algebras, these structure constants are typically expressed in terms of the matrix elements of the gamma matrices.

3.3.1 The category of $C^\bullet(\mathfrak{n})$-modules

Recall that the Chevalley–Eilenberg complex of $\mathfrak{n}$ takes the form

$$C^\bullet(\mathfrak{n}) = \big(\mathrm{Sym}^\bullet(\mathfrak{n}^\vee[1])\,,\,\mathrm{d}_{\mathrm{CE}}\big) \tag{3.26}$$

where the Chevalley–Eilenberg differential d_{CE} is induced by the dual of the bracket. Here the notation $\mathfrak{n}^\vee[1]$ means that we shift $\mathfrak{n}^\vee$ down in cohomological degree by one. The Chevalley–Eilenberg complex has a $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-grading endowing it with the structure of a dgs algebra. In our present case, since the the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathfrak{n}$ lifts to a $\mathbb{Z}$-grading where $\mathfrak{n}_i$ has weight i, $C^\bullet(\mathfrak{n})$ can be given a $\mathbb{Z}\times\mathbb{Z}$-grading. Totalizing this grading, the generators of $\mathfrak{n}_1^\vee$ sit in degree 0 while the generators of $\mathfrak{n}_2^\vee$ live in degree -1. We denote these generators by λ^α and v^μ respectively. We can thus identify

$$C^{-p}(\mathfrak{n}) = \wedge^p\mathfrak{n}_2^\vee\otimes R \tag{3.27}$$

with respect to the totalized grading.

The Chevalley–Eilenberg differential acts on these generators according to

$$\begin{aligned} \mathrm{d}_{\mathrm{CE}}\lambda^\alpha &= 0 \\ \mathrm{d}_{\mathrm{CE}}v^\mu &= \lambda^\alpha f^\mu_{\alpha\beta}\lambda^\beta\,. \end{aligned} \tag{3.28}$$

In coordinates we will often write the Chevalley–Eilenberg algebra in the form

$$(C^\bullet(\mathfrak{n})\,,\,\mathrm{d}_{\mathrm{CE}}) = \left(\mathbb{C}[\lambda^\alpha, v^\mu]\,,\,\mathrm{d}_{\mathrm{CE}} = \lambda^\alpha f^\mu_{\alpha\beta}\lambda^\beta\frac{\partial}{\partial v^\mu}\right)\,. \tag{3.29}$$

Using this description, it is immediate to see $H^0(\mathfrak{n}) = R/I$.

$C^\bullet(\mathfrak{n})$ is a dgs algebra, therefore we can consider dgs modules over it.

Definition 3.3.1. A $C^\bullet(\mathfrak{n})$-module is a dgs vector space $(\Gamma, \mathrm{d}_\Gamma)$ together with a morphism

$$(C^\bullet(\mathfrak{n})\,,\,\mathrm{d}_{\mathrm{CE}}) \longrightarrow (\mathrm{End}(\Gamma)\,,\,[\mathrm{d}_\Gamma, -]), \tag{3.30}$$

of dgs algebras.

Definition 3.3.2. A morphism of $C^\bullet(\mathfrak{n})$-modules $(\Gamma, \mathrm{d}_\Gamma)$ and $(\Gamma', \mathrm{d}_{\Gamma'})$ is a cochain map

$$f\colon (\Gamma, \mathrm{d}_\Gamma) \longrightarrow (\Gamma', \mathrm{d}_{\Gamma'}) \tag{3.31}$$

such that

$$f(x\cdot_\Gamma\gamma) = x\cdot_{\Gamma'} f(\gamma)\,. \tag{3.32}$$

In other words, f is just a morphism of dgs modules.

Note that $\mathfrak{g}_0$ acts on both $\mathfrak{n}_1$ and $\mathfrak{n}_2$ and thus also on $C^\bullet(\mathfrak{n})$.

Definition 3.3.3. A $C^\bullet(\mathfrak{n})$ module Γ is called $\mathfrak{g}_0$-*equivariant* if Γ is also a representation of $\mathfrak{g}_0$ and the module structure map is $\mathfrak{g}_0$-equivariant. We further assume that each degree Γ^k is finite dimensional as a complex vector space. We will denote the dg-category of $\mathfrak{g}_0$-equivariant $C^\bullet(\mathfrak{n})$-modules by $\mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})}$.

Our main results will take place in this category of $\mathfrak{g}_0$-equivariant modules.

3.3.2 The derived pure spinor superfield formalism

Recall that $\mathfrak{n}$ is a two-step nilpotent super Lie algebra and let $N = \exp(\mathfrak{n})$ be the associated nilpotent super Lie group. Left and right translations induce two commuting, $\mathfrak{aut}(\mathfrak{n})$-equivariant actions of $\mathfrak{n}$ on N by vector fields

$$\mathscr{L}, \mathscr{R} \colon \mathfrak{n} \longrightarrow \mathrm{Vect}(N). \tag{3.33}$$

Let us choose coordinates x^μ on the abelian group N_2 and θ^α on N_1. In terms of these coordinates, the vector fields given by the action of the odd elements can be described as follows.

$$\begin{aligned} \mathscr{R}(d_\alpha) &= \frac{\partial}{\partial\theta^\alpha} - f^\mu_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^\mu} \\ \mathscr{L}(d_\alpha) &= \frac{\partial}{\partial\theta^\alpha} + f^\mu_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^\mu} \end{aligned} \tag{3.34}$$

In addition, the even elements simply act by derivatives

$$\mathscr{R}(e_\mu) = \mathscr{L}(e_\mu) = \frac{\partial}{\partial x^\mu} \tag{3.35}$$

As before, the free superfield is the strict $\mathfrak{g}$-multiplet with $\mathcal{E} = C^\infty(N)$, vanishing differential, and module structure given by the left translations $\mathscr{L}$.

Let us now generalize the pure spinor superfield formalism to a derived setting.

Definition 3.3.4. We define the *pure spinor functor* $A^\bullet \colon \mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})} \longrightarrow \mathsf{Mult}^{\mathrm{strict}}_{\mathfrak{g}}$ by setting

$$A^\bullet(\Gamma) = (C^\infty(N) \otimes_{\mathbb{C}} \Gamma \, , \, \mathcal{D}) \, , \tag{3.36}$$

for an object $(\Gamma, \mathrm{d}_\Gamma)$. The differential $\mathcal{D}$ is constructed using the right action $\mathscr{R}$ and the $C^\bullet(\mathfrak{n})$-module structure on Γ. Explicitly, it takes the form

$$\mathcal{D} = \lambda^\alpha \mathscr{R}(d_\alpha) + v^\mu \mathscr{R}(e_\mu) + \mathrm{d}_\Gamma \, . \tag{3.37}$$

For a morphism $f\colon \Gamma \longrightarrow \Gamma'$ we define

$$A^\bullet(f) = \mathrm{id}_{C^\infty(N)} \otimes f\colon A^\bullet(\Gamma) \longrightarrow A^\bullet(\Gamma')\,. \tag{3.38}$$

A few comments are in order.

— The differential $\mathcal{D}$ squares to zero precisely since Γ is a $C^\bullet(\mathfrak{n})$-module.

— We can equip $A^\bullet(\Gamma)$ with the structure of a dgs vector bundle over the spacetime $V := N_2$ by placing $C^\infty(N)$ in cohomological degree zero. Note that in particular the differential is of bidegree $(1,+)$.

— Further, $\mathfrak{g}$ acts on $C^\infty(N)$ via left translations and on Γ by the trivial extension of the $\mathfrak{g}_0$-module structure which was part of the input datum. The tensor product of these two action makes $A^\bullet(\Gamma)$ into a strict multiplet.

— It is immediate to check that $A^\bullet(f)$ is a strict morphism of strict multiplets. Since f is a morphism of $C^\bullet(\mathfrak{n})$-modules we have

$$A^\bullet(f) \circ \mathcal{D} = \mathcal{D}' \circ A^\bullet(f)\,. \tag{3.39}$$

In addition,

$$A^\bullet(f) \circ \mathscr{L} = \mathscr{L} \circ A^\bullet(f) \tag{3.40}$$

obviously follows from the definition.

— $A^\bullet$ is additive. The direct sum of two $C^\bullet(\mathfrak{n})$-modules is mapped to the direct sum of the respective multiplets, $A^\bullet(\Gamma \oplus \Gamma') = A^\bullet(\Gamma) \oplus A^\bullet(\Gamma')$.

This construction is a direct generalization of the pure spinor superfield formalism as described in [Eag+22]. To see this, we notice that the category of graded equivariant of R/I-modules sits as a subcategory inside $\mathsf{Mod}^{\mathrm{strict}}_{C^\bullet(\mathfrak{n})}$, namely precisely as those modules concentrated in cohomological degree zero. Indeed, every R/I-module is a $C^\bullet(\mathfrak{n})$-module by the map

$$C^\bullet(\mathfrak{n}) = \mathbb{C}[\lambda^\alpha, v^\mu] \longrightarrow R/I \qquad (\lambda^\alpha, v^\mu) \mapsto \lambda^\alpha\,. \tag{3.41}$$

The other way round, let $(\Gamma, \mathrm{d}_\Gamma)$ be a $C^\bullet(\mathfrak{n})$-module concentrated in cohomological degree zero. Then the differential d_Γ vanishes and v^μ acts trivially for degree reasons. Therefore one has, for γ any element of Γ,

$$0 = \mathrm{d}_\Gamma(v^\mu \cdot \gamma) = (d_{\mathrm{CE}} v^\mu)\gamma = (\lambda f^\mu \lambda)\gamma\,, \tag{3.42}$$

which endows Γ with the structure of an R/I-module.

Restricting the functor $A^\bullet$ to graded equivariant R/I-modules, we obtain a functor

$$A^\bullet_{R/I} : \mathsf{Mod}^{\mathfrak{g}_0}_{R/I} \longrightarrow \mathsf{Mult}^{\mathrm{strict}}_{\mathfrak{g}} \tag{3.43}$$

where the output simplifies to

$$A^\bullet_{R/I}(\Gamma) = (C^\infty(N) \otimes_{\mathbb{C}} \Gamma \, , \, \lambda^\alpha \mathscr{R}(d_\alpha)) \tag{3.44}$$

such that we precisely recover the pure spinor superfield formalism as presented in §2.

Further, this is a derived generalization of the pure spinor superfield construction in the sense that $C^\bullet(\mathfrak{n})$ can be viewed as a derived replacement of the ring R/I. We will therefore refer to this construction as the derived pure spinor superfield formalism.

Remark 3.3.5. We can alternatively view this geometrically as a derived enhancement of the affine nilpotence variety $\mathrm{Spec}(R/I)$. We can view a multiplet as arising from a quasi-coherent sheaf over the nilpotence variety, but this point of view requires forgetting some of the data given by the $C^\bullet(\mathfrak{n})$ action. The philosophy of derived algebraic geometry suggests instead retaining this information by viewing a multiplet as arising from a coherent sheaf over the affine derived scheme $\mathrm{Spec}(C^\bullet(\mathfrak{n}))$—the derived analogue of the nilpotence variety.

3.3.3 The multiplet associated to $C^\bullet(\mathfrak{n})$

As a first example we can plug $C^\bullet(\mathfrak{n})$ itself into the pure spinor functor $A^\bullet$ and study the associated multiplet.

Lemma 3.3.6. *There is a natural equivalence*

$$A^\bullet(C^\bullet(\mathfrak{n})) \simeq \Omega^\bullet(N). \tag{3.45}$$

Proof. First, we can describe

$$A^\bullet(C^\bullet(\mathfrak{n})) = (C^\infty(N) \otimes C^\bullet(\mathfrak{n}) \, , \, \mathcal{D} = \lambda^\alpha \mathscr{R}(d_\alpha) + v^\mu \mathscr{R}(e_\mu) + \mathrm{d}_{\mathrm{CE}}) \ . \tag{3.46}$$

Let us write V for the even part N_2 of N, viewed as an affine space. We can identify $C^\infty(N) = C^\infty(V) \otimes \mathbb{C}[\theta^\alpha]$ and $C^\bullet(\mathfrak{n}) = \mathbb{C}[\lambda^\alpha, v^\mu]$. The differential takes the explicit form

$$\mathcal{D} = \lambda^\alpha \frac{\partial}{\partial\theta} - \lambda^\alpha f^\mu_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^\mu} + v^\mu \frac{\partial}{\partial x^\mu} + (\lambda^\alpha f^\mu_{\alpha\beta}\lambda^\beta)\frac{\partial}{\partial v^\mu} \ . \tag{3.47}$$

On the other hand, N is parallelizable, therefore its de Rham complex takes the form

$$\Omega^\bullet(N) \cong C^\infty(N) \otimes \mathrm{Sym}^\bullet(\mathfrak{n}_1^\vee) \otimes \wedge^\bullet \mathfrak{n}_2^\vee \ . \tag{3.48}$$

Identifying $d\theta^\alpha = \lambda^\alpha$ and $dx^\mu = v^\mu$, the de Rham differential takes the form

$$\mathrm{d_{dR}} = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + v^\mu \frac{\partial}{\partial x^\mu} \,. \tag{3.49}$$

Further, $\mathfrak{g}$ acts on the de Rham complex via left translation making it a strict multiplet. The map defined in coordinates via

$$(A^\bullet(C^\bullet(\mathfrak{n}))\,,\,\mathcal{D}\,,\mathscr{L}) \longrightarrow (\Omega^\bullet(N)\,,\,\mathrm{d_{dR}}\,,\mathscr{L}) \qquad (x^\mu,\lambda^\alpha,\theta^\alpha,v^\mu) \mapsto (x^\mu,\lambda^\alpha,\theta^\alpha,v^\mu{+}\lambda f^\mu\theta) \tag{3.50}$$

is a quasi-isomorphism of $\mathfrak{g}$-multiplets. Therefore, we can identify the multiplet associated to $C^\bullet(\mathfrak{n})$ itself as the differential forms on the super Lie group N. □

Remark 3.3.7. We can further compute cohomology with respect to $\mathcal{D}_0 = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha}$ and obtain a deformation retract to the de Rham complex on the spacetime manifold V:

$$\begin{tikzcd} (\Omega^\bullet(N)\,,\,\mathcal{D}_0) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (\Omega^\bullet(V)\,,\,0) \arrow[l, shift left, "i"] \end{tikzcd} \tag{3.51}$$

Here, i is the embedding of the factor of polynomial degree 0 in the λ and θ variables, and p is the obvious projection. The homotopy h is given by $h = \theta^\alpha \frac{\partial}{\partial\lambda^\alpha}$. The induced differential via homotopy transfer is the de Rham differential on V. The module structure, however, is no longer strict. In fact the strict part now vanishes, but there are now quadratic pieces appearing. These take the form

$$\rho^{(2)}(Q,Q) = \iota_{[Q,Q]} \colon \Omega^k(V) \longrightarrow \Omega^{k-1}(X)\,. \tag{3.52}$$

Here we view $[Q,Q]$ as a constant vector field on V and ι denotes the contraction of a differential form with a vector field.

There is, of course, a further quasi-isomorphism of $\mathfrak{g}$-modules to the trivial $\mathfrak{g}$-module $\mathbb{C}$. In this spirit, each of the multiplets $A^\bullet(C^\bullet(\mathfrak{n}))$, $\Omega^\bullet(N)$, and $\Omega^\bullet(V)$ can be viewed as free resolutions of the trivial module—either free over spacetime (i.e. as $C^\infty(V)$-modules), or even free over superspace (i.e. as $C^\infty(N)$-modules). Note, however, that the trivial module does not form a $\mathfrak{g}$-multiplet since the translations do not act geometrically. Therefore this last quasi-isomorphism is just a quasi-isomorphism of $\mathfrak{g}$-modules.

3.3.4 Component fields and homotopy transfer

The multiplet $A^\bullet(\Gamma)$ given by applying the pure spinor functor does not at first glance resemble the more standard component field formulations known from physics. These component field multiplets are distinguished by the fact that they are given by dgs vector bundles whose total rank (as vector bundles over spacetime) is finite. In other words, component field multiplets are usually defined with the assumption that they only contain a finite number of component fields. With some care, it is always possible to choose a homotopy representative for $A^\bullet(\Gamma)$ that satisfies this condition, as long as Γ satisfies a finiteness condition.

Lemma 3.3.8. *Let Γ be a $C^\bullet(\mathfrak{n})$-module such that only finitely many cohomology groups $H^\bullet(\Gamma)$ are non-vanishing and each of these is finitely generated as an R-module. Then $A^\bullet(\Gamma)$ is quasi-isomorphic to a multiplet of finite rank.*

In the rest of this section, we will explain an algorithm that provides explicit finitely generated component field representatives for $A^\bullet(\Gamma)$, thus establishing the lemma. We already discussed a procedure to extract such a component field multiplet out of the underived model $A^\bullet_{R/I}(\Gamma)$ in §2.3.6. In essence, this is done by homotopy transfer: one identifies a retraction to another quasi-isomorphic complex and then applies homotopy transfer to the structures present for the multiplet. By construction, this yields a quasi-isomorphic and thus physically equivalent multiplet. Crucially, the component field multiplets obtained in that way are not necessarily strict anymore. Higher-order terms in the module structure can arise during the transfer.

Let us begin by summarizing the procedure for the underived pure spinor formalism $A^\bullet_{R/I}$ and then describe the generalization to $A^\bullet$.

Minimal models for $A^\bullet_{R/I}$. Recall that the differential on $(A^\bullet_{R/I}(\Gamma), \mathcal{D})$ admits an obvious splitting

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} - \lambda^\alpha f^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu} \,, \tag{3.53}$$

which can be viewed by equipping $A^\bullet_{R/I}(\Gamma)$ with the filtration discussed in §2.3.3 and splitting the differential into the terms that preserve and lower filtered degree. We can take cohomology with respect to $\mathcal{D}_0$ and then perform homotopy transfer along the diagram

$$\begin{tikzcd} (A^\bullet(\Gamma)\,,\,\mathcal{D}_0) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H^\bullet(A^\bullet(\Gamma),\mathcal{D}_0),\,0) \arrow[l, shift left, "i"] \end{tikzcd} . \tag{3.54}$$

We observe, by [LV12a, Theorem 10.3.15], that although the transfer data depends on a choice of section for the projection onto the cohomology, the multiplet obtained by homotopy transfer is independent of this choice up to isomorphism. In this example, it is always "minimal" in the sense that its differential does not contain terms of order zero in differential operators. Intuitively, this means that we cannot take any further cohomology without leaving the category of multiplets. (In more general examples coming from $C^\bullet(\mathfrak{n})$-modules, “minimal” is not related to having no terms of order zero in the differential; the massive Klein–Gordon field is a simple example.)

Starting from a multiplet of the form $A^\bullet_{R/I}(\Gamma)$, we will refer to the corresponding minimal multiplet as $\mu A^\bullet_{R/I}(\Gamma)$. One can identify the $\mathcal{D}_0$-cohomology with the Koszul cohomology of Γ, tensored with functions on spacetime,

$$H^\bullet(A^\bullet_{R/I}(\Gamma)) = C^\infty(V) \otimes H^\bullet(K^\bullet(\Gamma)) \, . \tag{3.55}$$

The Koszul cohomology is conveniently computed by a minimal free resolution L of Γ in R-modules. The minimal multiplet takes the form

$$\mu A^\bullet_{R/I}(\Gamma) = \big(C^\infty(V) \otimes (L \otimes_R \mathbb{C}) \, , \, \mathcal{D}' \, , \, \rho' \big) \, , \tag{3.56}$$

where $\mathcal{D}'$ is the differential induced from $\mathcal{D}_1$ and ρ' the module structure induced from $\mathscr{L}$ via homotopy transfer. For Γ a finitely generated module, Hilbert’s syzygy theorem states that the minimal free resolution exists, consists of finitely generated modules and its length is less or equal than $\dim(\mathfrak{n}_1)$ (see for example [Eis95, Theorem 1.13]; for a discussion in the equivariant case we refer to [Gal16, Proposition 2.4.9, Remark 2.4.10]). Therefore, $\mu A^\bullet_{R/I}(\Gamma)$ is indeed of finite rank over spacetime, i.e. it provides a reasonable component field multiplet.

Minimal models for $A^\bullet$. Let us now discuss a generalization of this procedure to the functor $A^\bullet$. We will assume in this section that Γ carries an action of the abelian Lie algebra $\mathbb{R}$ compatible with the action of $\mathbb{R}$ on $\mathfrak{n}$ where $\mathfrak{n}_i$ has $\mathbb{R}$-weight i. This will be used to construct a filtration at the very end of this section (however, this assumption is not required for Lemma 3.3.8).

Recall that the differential on $A^\bullet(\Gamma)$ takes the form

$$\mathcal{D} = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} - \lambda^\alpha f^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu} + \mathrm{d}_\Gamma + v^\mu \frac{\partial}{\partial x^\mu} \, . \tag{3.57}$$

One can construct a finite-dimensional component field model in two steps. First, one takes cohomology with respect to the internal differential of the module d_Γ and performs homotopy transfer along the diagram

$$\begin{tikzcd} (A^\bullet(\Gamma) \, , \, \mathrm{d}_\Gamma) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (C^\infty(N) \otimes H^\bullet(\Gamma)), 0) \arrow[l, shift left, "i"] \end{tikzcd} \, . \tag{3.58}$$

The differential and the $C^\bullet(\mathfrak{n})$-module structure of the resulting multiplet may contain additional pieces induced from homotopy transfer. Since $H^\bullet(\Gamma)$ is a $C^\bullet(\mathfrak{n})$-module with vanishing differential, each homogeneous summand $H^k(\Gamma)$ carries the structure of an R/I-module. Thus, we can proceed by taking cohomology with respect to the Koszul differential $\mathcal{D}_0 = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha}$, and applying the homotopy transfer along a retraction on to this cohomology. As before, the $\mathcal{D}_0$-cohomology is computed by minimal free resolutions of the individual $H^k(\Gamma)$ in R-modules, and as before, it is independent of the choice of transfer datum. The resulting multiplet is thus of the form

$$C^\infty(V) \otimes (\bigoplus_k L_k^\bullet \otimes_R \mathbb{C}) \,, \tag{3.59}$$

where $L_k^\bullet$ is the minimal free resolution of $H^k(\Gamma)$. As long as the cohomology $H^\bullet(\Gamma)$ is bounded, this is already a finite rank vector bundle over spacetime. The field content of this multiplet is just the field content of the direct sum of all the minimal multiplets associated to the cohomology groups, i.e.

$$\bigoplus_k \mu A^\bullet(H^k(\Gamma)) \,. \tag{3.60}$$

At this stage we have already obtained a finite-dimensional model, as required by Lemma 3.3.8. However, additional acyclic differentials induced by homotopy transfer can still be present. At this final stage, we will use the filtration on each multiplet $\mu A^\bullet(H^k(\Gamma))$ by weight with respect to the $\mathbb{R}$-action on Γ. We define $\mu A^\bullet(\Gamma)$ by taking the cohomology with respect to the summand of the total differential of filtered degree zero (in other words, we pass to the E_1 page of the associated spectral sequence), and apply homotopy transfer. We will illustrate this procedures using examples in §3.5.

3.4 An equivalence of categories

We now show that the derived pure spinor superfield formalism provides an equivalence of categories between the dg categories of $\mathfrak{g}_0$-equivariant $C^\bullet(\mathfrak{n})$-modules and $\mathfrak{g}$-multiplets. This implies in particular that, up to quasi-isomorphism, every $\mathfrak{g}$-multiplet can be constructed using the derived pure spinor superfield formalism.

Remark 3.4.1. Recall that for a general Lie algebra there is a Koszul duality equivalence between the dg-categories of $C_\bullet(\mathfrak{g})$-comodules and $U(\mathfrak{g})$-modules. If $\mathfrak{g}$ is, for instance, finite dimensional, we can instead consider $C^\bullet(\mathfrak{g})$-modules. For a relevant discussion in a similar context, see [Cos13b, §7 and §8]. Relatedly, Kapranov [Kap91] gives a formulation of Koszul duality that establishes a Quillen equivalence between the categories of D-modules and $\Omega^\bullet$-modules on the same space. Our results show that the pure spinor formalism admits a natural derived generalization that is closely related to Kapranov's construction; however, in our setting, one is working on a supermanifold, and asks for appropriate equivariance conditions. Alternatively, our procedure can be viewed in two steps, as an explicit form of commutative/Lie Koszul duality tailored to the examples in question, combined with an associated bundle construction that realizes a $U(\mathfrak{n})$-module as an $\mathfrak{n}$-equivariant dg vector bundle over $V = N_2$.

3.4.1 The inverse functor: derived $\mathfrak{n}$-invariants

Any $\mathfrak{g}$-module is in particular a $\mathfrak{n}$-module, and we can thus take derived invariants with respect to $\mathfrak{n}$. For multiplets, this defines a functor in the opposite direction to the functor $A^\bullet$, assigning a strict $C^\bullet(\mathfrak{n})$-module to a multiplet. The resulting $C^\bullet(\mathfrak{n})$-module is also $\mathfrak{g}_0$-equivariant. The functor takes the form

$$C^\bullet(\mathfrak{n}, -)\colon \ \mathsf{Mult}_{\mathfrak{g}}^{\text{strict}} \longrightarrow \mathsf{Mod}_{C^\bullet(\mathfrak{n})}^{\mathfrak{g}_0}, \tag{3.61}$$

It maps the multiplet (E, D, ρ) to $C^\bullet(\mathfrak{n}, \mathcal{E})$; this Chevalley–Eilenberg complex can be written more explicitly as

$$C^\bullet(\mathfrak{n}, \mathcal{E}) = (C^\bullet(\mathfrak{n}) \otimes \mathcal{E}\ ,\ \mathrm{d}_{\mathrm{CE}} + D + \lambda^\alpha \rho(d_\alpha) + v^\mu \rho(e_\mu))\ . \tag{3.62}$$

It is a strict $C^\bullet(\mathfrak{n})$-module; the action is on $C^\bullet(\mathfrak{n})$ via multiplication and on $\mathcal{E}$ via the identity. Morphisms are mapped according to the rule

$$\psi \mapsto \mathrm{id}_{C^\bullet(\mathfrak{n})} \otimes \psi. \tag{3.63}$$

Remark 3.4.2. The assignment on the level of objects is also well defined for not necessarily strict modules. Then the higher-order terms of the $\mathfrak{g}$-module structure enter the differential such that the term $\lambda^\alpha \rho(d_\alpha)$ is replaced by

$$\lambda \cdot \rho := \sum_{k=1}^{\infty} \lambda^{\alpha_1} \dots \lambda^{\alpha_k} \rho^{(k)}(d_{\alpha_1}, \dots, d_{\alpha_k})\ . \tag{3.64}$$

Notice that the output is still a strict $C^\bullet(\mathfrak{n})$-module.

There are several ways to intuitively understand the fact that the inverse functor is given by the derived invariants of supertranslations. One can of course appeal to the general structure of Koszul duality. On a more down-to-earth level, one can recall that the component fields of a supermultiplet in the usual pure spinor superfield formalism correspond to the generators of the minimal free resolution of that module over $R = C^\bullet(\Pi\mathfrak{n}_1)$. The resolution differential agrees with those supersymmetry transformations that are order zero in spacetime derivatives. Since spacetime derivatives act by zero on translation-invariant sections of E, it is clear that one can think of the minimal free resolution of the module as arising from the derived $\Pi\mathfrak{n}_1$-invariants of translation-invariant sections: R consists of the Lie algebra cochains of $\Pi\mathfrak{n}_1$, the generators of the free graded R-module arise from the component fields of the multiplet, and the differential encodes the action of supersymmetry on translation-invariant sections. It is clear that this story is just a two-step procedure to compute derived $\mathfrak{n}$-invariants by first taking $\mathfrak{n}_2$-invariants, and then accounting for the action of supersymmetry.

To flesh this story out, we now describe the module $C^\bullet(\mathfrak{n}, \mathcal{E})$ in some more detail and sketch how its cohomology can be computed. Recall that, since $\mathcal{E}$ is a multiplet, the action of $\mathfrak{n}_2$ is just given by derivatives along the coordinate directions

$$\rho(e_\mu) = \frac{\partial}{\partial x^\mu} \, . \tag{3.65}$$

Thus, taking cohomology with respect to the term $v^\mu \rho(e_\mu)$ in the differential means restricting to translation invariant sections of $\mathcal{E}$ and eliminating v^μ. Denoting the fiber of E over 0 by E_0 we find a quasi-isomorphism

$$C^\bullet(\mathfrak{n}, \mathcal{E}) \simeq \left(R \otimes E_0^\bullet \, , \; \lambda \cdot \rho_{\text{constants}}\right) \, . \tag{3.66}$$

Note that $E_0^\bullet$ carries a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading since it comes from a dgs vector bundle. In addition, there is an integer grading by polynomial degree in λ present. In the following, we will label cohomology groups by the former degree. Then each cohomology group is an $\mathfrak{g}_0$-equivariant R/I-module graded by polynomial degree in λ. If the cohomology is concentrated in a single degree, the complex on the right hand-side can be viewed as the minimal free resolution of the R/I-module forming the cohomology. Note that this is indeed a minimal free resolution, since all terms in the differential are of nonzero polynomial degree in λ. It will follow from the theorem below that this R/I-module is precisely the algebraic input datum the multiplet can be constructed from in the pure spinor superfield formalism, i.e. by applying the functor $A^\bullet_{R/I}$.

3.4.2 Main theorem and proof

Let us now show that the functors $A^\bullet$ and $C^\bullet(\mathfrak{n}, -)$ induce an equivalence of dg-categories between the homotopy categories of multiplets and equivariant $C^\bullet(\mathfrak{n})$-modules.

Theorem 3.4.3. *$A^\bullet$ and $C^\bullet(\mathfrak{n}, -)$ provide an equivalence of dg-categories between $\mathsf{Mult}_{\mathfrak{g}}^{\textit{strict-ob}}$ and $\mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})}$.*

Proof. We first show that there is an equivalence of dg-functors

$$\mathrm{id}_{\mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})}} \to C^\bullet \circ A^\bullet\,. \tag{3.67}$$

We will naturally construct quasi-isomorphisms

$$\Gamma \simeq C^\bullet(\mathfrak{n}, A^\bullet(\Gamma)) \tag{3.68}$$

for each equivariant $C^\bullet(\mathfrak{n})$-module $(\Gamma, \mathrm{d}_\Gamma)$.

We can explicitly describe $C^\bullet(\mathfrak{n}, A^\bullet(\Gamma))$ by

$$\left(\mathbb{C}[\lambda', v'] \otimes C^\infty(N) \otimes \Gamma\,,\; \mathrm{d}_\Gamma + (\lambda' f^\mu \lambda')\frac{\partial}{\partial v'^\mu} + v'^\mu \frac{\partial}{\partial x^\mu} + \lambda'^\alpha \mathscr{L}(d_\alpha) + \lambda^\alpha \mathscr{R}(d_\alpha) + v^\mu \mathscr{R}(e_\mu)\right). \tag{3.69}$$

Here, we made a notational distinction between the generators of the $C^\bullet(\mathfrak{n})$ in the construction (denoted by λ' and v') and the action of $C^\bullet(\mathfrak{n})$ on Γ (denoted by λ and v).

The differential contains a piece of the form

$$\mathrm{d}_{\mathrm{dR}} = v'^\mu \frac{\partial}{\partial x^\mu} + \lambda'^\alpha \frac{\partial}{\partial \theta^\alpha}\,. \tag{3.70}$$

Thus, we can identify

$$\left(C^\bullet(\mathfrak{n}, A^\bullet(\Gamma))\,,\; v'^\mu \frac{\partial}{\partial x^\mu} + \lambda'^\alpha \frac{\partial}{\partial \theta^\alpha}\right) = (\Omega^\bullet(N)\,,\; \mathrm{d}_{\mathrm{dR}}) \otimes \Gamma\,. \tag{3.71}$$

Since the de Rham complex on N is acyclic, this complex is quasi-isomorphic to Γ. We can fix homotopy data

$$\begin{tikzcd} (\Omega^\bullet(N) \otimes \Gamma, \mathrm{d}_{\mathrm{dR}}) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (\Gamma,\, 0) \arrow[l, shift left, "i"] \end{tikzcd}\,. \tag{3.72}$$

It is easy to see that the only induced differential on the right hand side is d_Γ, so that we obtain a quasi-isomorphism uniformly for all choices of Γ:

$$C^\bullet(\mathfrak{n}, A^\bullet(\Gamma)) \simeq (\Gamma, \mathrm{d}_\Gamma). \tag{3.73}$$

Now, for any morphism $f\colon \Gamma \to \Gamma'$ of $C^\bullet(\mathfrak{n})$-modules, we can identify

$$C^\bullet(A^\bullet(f)) = \mathrm{id}_{C^\bullet(\mathfrak{n})} \otimes \mathrm{id}_{C^\infty(N)} \otimes f. \tag{3.74}$$

So there is a commutative square

$$
\begin{tikzcd}
\Gamma \arrow[r] \arrow[d, "f"] & C^\bullet(\mathfrak{n}, A^\bullet(\Gamma)) \arrow[d, "A^\bullet(f)"] \\
\Gamma' \arrow[r] & C^\bullet(\mathfrak{n}, A^\bullet(\Gamma'))
\end{tikzcd}
\tag{3.75}
$$

inducing an equivalence of hom complexes $\mathrm{Hom}(\Gamma, \Gamma') \to \mathrm{Hom}(C^\bullet(\mathfrak{n}, A^\bullet(\Gamma)), C^\bullet(\mathfrak{n}, A^\bullet(\Gamma')))$.

Conversely, we will construct an equivalence of dg-functors

$$A^\bullet \circ C^\bullet \to \mathrm{id}_{\mathsf{Mult}_{\mathfrak{g}}^{\text{strict-ob}}} . \tag{3.76}$$

Let (E, D, ρ) be a $\mathfrak{g}$-multiplet. We can describe $A^\bullet(C^\bullet(\mathfrak{n}, \mathcal{E}))$ explicitly by

$$\left(\mathbb{C}[\lambda, v] \otimes \mathcal{E} \otimes C^\infty(N) \, , \; D + \lambda f^\mu \lambda \frac{\partial}{\partial v^\mu} + \lambda \cdot \rho + v^\mu \rho(e_\mu) + \lambda^\alpha \mathscr{R}(d_\alpha) + v^\mu \mathscr{R}(e_\mu)\right) . \tag{3.77}$$

We denote coordinates on the base of the vector bundle E by x^μ and on N_2 by y^μ. In this notation, we find

$$v^\mu \left(\rho(e_\mu) + \mathscr{R}(e_\mu)\right) = v^\mu \left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu}\right) . \tag{3.78}$$

Taking cohomology with respect to this piece, we obtain a quasi-isomorphic complex of the form

$$\left(\mathbb{C}[\lambda, \theta] \otimes \mathcal{E} \, , \; D + \lambda \cdot \rho + \lambda^\alpha \mathscr{R}(d_\alpha)\right) . \tag{3.79}$$

The differential contains a piece corresponding to the Koszul differential

$$\mathrm{d}_K = \mathcal{D}_0 = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} . \tag{3.80}$$

Clearly, taking cohomology with respect to the Koszul differential we arrive at $\mathcal{E}$. Concretely let us consider homotopy data

$$
\begin{tikzcd}
(\mathbb{C}[\lambda, \theta] \otimes \mathcal{E}, \mathcal{D}_0) \arrow[loop left, "\mathcal{D}_0^\dagger"] \arrow[r, shift left, "p"] & (\mathcal{E}, 0) \arrow[l, shift left, "i"]
\end{tikzcd} ,
\tag{3.81}
$$

where the homotopy is given by $\mathcal{D}_0^\dagger = \theta \frac{\partial}{\partial \lambda}$, while i is the obvious inclusion and p evaluates at $\lambda = \theta = 0$. It is easy to see that the induced differential is just D. Thus, we find homotopy data

$$\begin{tikzcd} (\mathbb{C}[\lambda,\theta]\otimes\mathcal{E}, d) \arrow[loop left, "\mathcal{D}_0'^\dagger"] \arrow[r, shift left, "p'"] & (\mathcal{E},\, D) \arrow[l, shift left, "i'"] \end{tikzcd}\,, \tag{3.82}$$

providing a quasi-isomorphism of cochain complexes (here d denotes the full differential in (3.79)). The induced maps are given by

$$\begin{aligned} i' &= \sum_{n=0}^{\infty}(\mathcal{D}_0^\dagger(D+\mathcal{D}_1+\lambda\cdot\rho))^n\circ i \\ p' &= p\circ\sum_{n=0}^{\infty}(\mathcal{D}_0^\dagger(D+\mathcal{D}_1+\lambda\cdot\rho))^n = p \\ \mathcal{D}_0'^\dagger &= \mathcal{D}_0^\dagger\circ\sum_{n=0}^{\infty}((D+\mathcal{D}_1+\lambda\cdot\rho)\mathcal{D}_0^\dagger)^n\,. \end{aligned} \tag{3.83}$$

By construction, $p\circ\mathcal{D}_0^\dagger = 0$ and thus $p' = p$. We note that these sums are all finite by degree reasons, since the left hand side in (3.82) is concentrated in finitely many degrees with respect to the grading by polynomial degree in the θ-variables, and in each expression in equation (3.83), the operator being raised to the power n within the sum raises this θ-degree. Therefore for n sufficiently large, all terms in the sum defining our maps vanish.

We have to check that this not only provides a quasi-isomorphism of cochain complexes, but of multiplets. Therefore we transfer the module structure induced by $\mathscr{L}$ to the right hand side and check that it agrees with the original module structure ρ of the multiplet.

Explicitly, the transferred module structure can be described by

$$\begin{aligned} \rho^{(k)}_{\mathscr{L}}(Q,\dots,Q) &= p\mathscr{L}(Q)(\mathcal{D}_0'^\dagger\mathscr{L}(Q))^{k-1}i' \\ &= p\mathscr{L}(Q)(\mathcal{D}_0^\dagger\sum_{n=0}^{\infty}((D+\mathcal{D}_1+\lambda\cdot\rho)\mathcal{D}_0^\dagger)^n\mathscr{L}(Q))^{k-1}\sum_{j=0}^{\infty}(\mathcal{D}_0^\dagger(D+\mathcal{D}_1+\lambda\cdot\rho))^j\circ i. \end{aligned} \tag{3.84}$$

Since p projects onto $\lambda=\theta=0$, only terms of order zero in λ and θ can contribute. It is easy to see that the only such term is

$$\rho^{(k)}_{\mathscr{L}}(Q,\dots,Q) = p\epsilon\frac{\partial}{\partial\theta}\left(\mathcal{D}_0^\dagger\epsilon\frac{\partial}{\partial\theta}\right)^{k-1}\mathcal{D}_0^\dagger\lambda\cdot\rho^{(k)}i\,. \tag{3.85}$$

Here we expressed Q in a basis, $Q=\epsilon^\alpha d_\alpha$. Noting that

$$[\mathcal{D}_0^\dagger,\epsilon\frac{\partial}{\partial\theta}] = \epsilon\frac{\partial}{\partial\lambda}, \tag{3.86}$$

we deduce

$$\rho^{(k)}_{\mathscr{L}}(Q,\dots,Q) = \rho^{(k)}(Q,\dots,Q)\,. \tag{3.87}$$

This shows that $A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E})) \simeq (E,D,\rho)$ as multiplets.

As before, for any morphism $\psi\colon \mathcal{E} \to \mathcal{E}'$ we can realize

$$A^\bullet(C^\bullet(\psi)) = C^\bullet(\psi) \otimes \mathrm{id}_{C^\infty(N)}\ . \tag{3.88}$$

So there is a commutative diagram

$$\begin{tikzcd} A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E})) \arrow[r] \arrow[d, "C^\bullet(\psi)"] & \mathcal{E} \arrow[d, "\psi"] \\ A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E}')) \arrow[r] & \mathcal{E}' \end{tikzcd} \tag{3.89}$$

inducing an equivalence of hom spaces, and hence we have an equivalence of dg-categories. □

Remark 3.4.4. The equivalence of dg-categories automatically induces an equivalence of the underlying homotopy categories. Hence, each multiplet is perturbatively equivalent to a multiplet constructed via the derived pure spinor superfield formalism.

3.4.3 Some consequences of the theorem

Let us now discuss some consequences of the equivalences of categories.

Corollary 3.4.5. *Let (E,D,ρ) be any multiplet. $A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E}))$ is a strictification.*

Note that this does not imply that there exist a strict finitely generated component field multiplet that is equivalent to $\mathcal{E}$. In particular, $\mathcal{E}$ need not admit an auxiliary field formulation in the usual sense. The strictification $A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E}))$ typically contains infinitely many component fields.

We can further use the equivalence of categories to derive some statements on the (underived) pure spinor superfield formalism, i.e. the functor $A^\bullet_{R/I}$.

Corollary 3.4.6. *The essential image of the functor $A^\bullet_{R/I}$ consists of those multiplets for which $H^\bullet(\mathfrak{n},\mathcal{E})$ is concentrated in a single $\mathbb{Z}$-degree after totalization of the natural $\mathbb{Z}\times\mathbb{Z}$-grading.*

This gives a description of all multiplets which can be constructed via the pure spinor superfield formalism. We already argued in §2.4.4 that the antifield multiplet of the four-dimensional vector multiplet cannot be constructed via $A^\bullet_{R/I}$. We come back to this example in §3.5.2 where we compute the relevant cohomologies and explain how the multiplet is built in the derived formalism.

Corollary 3.4.7. *The functors $A^\bullet_{R/I}$ and $H^\bullet(\mathfrak{n}, \mathcal{E})$ provide an equivalence of categories between the essential image of $A^\bullet_{R/I}$ and the category of graded $\mathfrak{g}_0$-equivariant R/I-modules.*

Suppose we have an R/I-module Γ with associated minimal multiplet $\mu A^\bullet_{R/I}(\Gamma)$. As discussed earlier, the fields of $\mu A^\bullet_{R/I}(\Gamma)$ take values in the minimal free resolution of Γ. In §2.3.7, we already observed a close link between the supersymmetry module structure and the resolution differential. We can now see this result as a consequence of the equivalence of categories. Specifically, we can pull back $\mu A^\bullet_{R/I}(\Gamma)$ along the inclusion

$$\{0\} \hookrightarrow V\,. \tag{3.90}$$

This restricts the multiplet to the fiber $\mu A^\bullet(\Gamma)_0$. The multiplet $\mu A^\bullet_{R/I}(\Gamma)_0$ carries a module structure for the odd abelian super Lie algebra $\Pi\mathfrak{n}_1$. This module structure coincides with the resolution differential in the following sense.

Corollary 3.4.8. *Let Γ be an R/I-module and let (L, d_L) be its minimal free resolution in R-modules. Let us identify the fiber*

$$\mu A^\bullet_{R/I}(\Gamma)_0 = L \otimes_R \mathbb{C}. \tag{3.91}$$

The map generated over R by the $\Pi\mathfrak{n}_1$-module structure on $\mu A^\bullet_{R/I}(\Gamma)_0$

$$\rho_{\text{constants}} \colon \mu A^\bullet_{R/I}(\Gamma)_0 \otimes \Big(\bigoplus_k \mathfrak{n}_1^{\otimes k}\Big) \longrightarrow \mu A^\bullet_{R/I}(\Gamma)_0 \tag{3.92}$$

coincides with the resolution differential. In coordinates, we can express this as

$$\lambda \cdot \rho_{\text{constants}} = \sum_k \rho^{(k)}_{\text{constants}}(\lambda^{\alpha_1} d_{\alpha_1}, \dots, \lambda^{\alpha_k} d_{\alpha_k}) = \mathrm{d}_L, \tag{3.93}$$

where d_α is a basis for $\mathfrak{n}_1$.

Proof. By construction we have $H^\bullet(\mathfrak{n}, \mu A^\bullet(\Gamma)) = \Gamma$. But by (3.66) we know that there is a quasi-isomorphism

$$C^\bullet(\mathfrak{n}, \mu A^\bullet(\Gamma)) \simeq (\mu A^\bullet(\Gamma)_0 \otimes R, \lambda \cdot \rho_{\text{constants}})\ . \tag{3.94}$$

Thus, the cochain complex on the right is the minimal free resolution of Γ in R-modules and we obtain the desired result. ☐

In practice this means that the resolution differential contains all the information on the supersymmetry transformations which are of order zero in the derivatives. This result was conjectured by Berkovits in [Ber02].

Let us now state some results on the duality operations in the category of multiplets.

Corollary 3.4.9. *Let (E, D, ρ) be a $\mathfrak{g}$-multiplet and let $(E^\vee, D^\vee, \rho^\vee)$ be the respective dual (or antifield) multiplet. If these are both quasi-isomorphic to a multiplet in the image of $A^\bullet_{R/I}$ and we have*

$$(E, D, \rho) \simeq A^\bullet_{R/I}(\Gamma) \,, \tag{3.95}$$

then there is also a quasi-isomorphism

$$(E^\vee, D^\vee, \rho^\vee) \simeq A^\bullet_{R/I}(\mathrm{Ext}^{n-q}_R(\Gamma, R)) \,. \tag{3.96}$$

Here, $n = \dim(\mathfrak{n}_1)$ and $q = \dim_R(\Gamma)$.

Proof. Move to the minimal multiplet $\mu A^\bullet_{R/I}(\Gamma) \simeq (E, D, \rho)$. Corollary 3.4.8 implies that

$$C^\bullet(\mathfrak{n}, \mathcal{E}) \simeq C^\bullet(\mathfrak{n}, \mu A^\bullet_{R/I}(\Gamma)) \simeq (L, d_L) \,, \tag{3.97}$$

where (L, d_L) is the minimal free resolution of Γ in R-modules. There is a quasi-isomorphism

$$(E^\vee, D^\vee, \rho^\vee) \simeq \left(\mu A^\bullet_{R/I}(\Gamma)\right)^\vee \,. \tag{3.98}$$

This in turn implies that

$$C^\bullet(\mathfrak{n}, \mathcal{E}^\vee) \simeq C^\bullet\left(\mathfrak{n}, \mu A^\bullet_{R/I}(\Gamma)^\vee\right) \simeq (L^\vee, d_L^\vee) \,, \tag{3.99}$$

where $(L^\vee, d_L^\vee)$ is the dual of the minimal free resolution (L, d_L). By assumption the derived invariants of $(E^\vee, D^\vee, \rho^\vee)$ are concentrated in a single degree. Hence, $(L^\vee, d_L^\vee)$ again resolves a single R/I-module, namely $\mathrm{Ext}^{n-q}_R(\Gamma, R)$. □

In general, however, taking dual multiplets can lead outside of the image of $A^\bullet_{R/I}$. In fact, the above proof implies that $(A^\bullet_{R/I}(\Gamma))^\vee$ is in the essential image of $A^\bullet_{R/I}$ precisely when Γ is a Cohen–Macaulay R-module. In general this may not be the case—the dual of the vector multiplet in four-dimensional $\mathcal{N} = 1$ supersymmetry is an example of this type, that does not occur in the image of the underived pure spinor functor.

If Γ is not Cohen–Macaulay, we can still compute the derived invariants of the associated multiplet to find

$$C^\bullet(\mathfrak{n}, A^\bullet_{R/I}(\Gamma)^\vee) \simeq \mathrm{RHom}_R(\Gamma, R) \simeq \mathrm{Ext}^\bullet_R(\Gamma, R) \,. \tag{3.100}$$

We can therefore deduce the following natural description for the *dual* of a multiplet obtained using the (underived) pure spinor formalism.

Corollary 3.4.10. *Let Γ be an R/I-module and $\mu A^\bullet_{R/I}(\Gamma)$ the associated component field multiplet. Then there is a $C^\bullet(\mathfrak{n})$-module structure on the Ext-algebra $\mathrm{Ext}^\bullet_R(\Gamma, R)$ such that*

$$\mu A^\bullet_{R/I}(\Gamma)^\vee \simeq A^\bullet(\mathrm{Ext}^\bullet_R(\Gamma, R)) \,. \tag{3.101}$$

3.4.4 First examples

Let us discuss the pure spinor formalism as an equivalence of categories in a few simple examples.

Example 3.4.11. First, let us consider the situation where there is no supersymmetry at all. Let V be an d-dimensional vector space. Let $\mathfrak{g}$ denote its Poincaré Lie algebra of infinitesimal isometries, and let $\mathfrak{n}$ denote its Lie algebra of translations, viewed as a dg Lie algebra concentrated in degree 2. So $C^\bullet(\mathfrak{n}) \cong \mathrm{Sym}^\bullet(\mathfrak{n}^\vee[1])$ is an exterior algebra in d generators $v_1, \dots, v_d$ of degree -1. We are, therefore, interested in $\mathfrak{g}_0 \cong \mathfrak{so}(d)$-equivariant dg-modules over the exterior algebra. We restrict attention to those modules with finite-dimensional cohomology in each degree.

On the other hand, the dg-category $\mathsf{Mult}_{\mathfrak{g}}$ of multiplets is nothing but the dg-category of Poincaré equivariant dg-vector bundles on the affine space V: the $\mathfrak{g}$ action is completely determined (and all multiplets are automatically strict). Again, let us restrict attention to bundles with finite-dimensional cohomology in each degree. Such multiplets are determined by their restriction to a formal neighborhood of the origin, which is a Poincaré equivariant dg-module over the completed Weyl algebra

$$\hat{D}_d = \mathbb{C}[\partial_1, \dots, \partial_d][\![z_1, \dots, z_d]\!]/([\partial_i, z_i] - 1). \tag{3.102}$$

Translation equivariance guarantees that all such modules are induced from $\mathfrak{so}(d)$-equivariant modules over $\mathbb{C}[\partial_1, \dots, \partial_d]$. Our statement then reduces to ordinary Koszul duality as a relationship between modules over an exterior and a commutative algebra.

Example 3.4.12. Let Σ be a finite-dimensional vector space. Let us now briefly discuss the example where $\mathfrak{g} = \mathfrak{n} = \mathbb{R} \oplus \Pi\Sigma$ as a graded vector space, with Lie bracket given by a non-degenerate inner product $\Sigma^{\otimes 2} \to \mathbb{R}$. This is the background for *supersymmetric classical mechanics*. Now

$$C^\bullet(\mathfrak{n}) \cong (\mathbb{C}[v, \lambda^1, \dots, \lambda^{\mathcal{N}}] \,,\, \mathrm{d}_{\mathrm{CE}}), \tag{3.103}$$

where v is an odd generator, λ^i are even generators, $\mathrm{d}_{\mathrm{CE}}\lambda^i = 0$ for all i, and $\mathrm{d}_{\mathrm{CE}}v = (\lambda^1)^2 + \cdots + (\lambda^{\mathcal{N}})^2$ (so the λ^i are linearly dual to a choice of orthonormal basis of Σ).

Let $(\Gamma, \mathrm{d}_\Gamma)$ be a $C^\bullet(\mathfrak{n})$-module. The derived version of the pure spinor formalism associates to Γ the multiplet

$$A^\bullet(\Gamma) = \left(C^\infty(\mathbb{R}_t) \otimes \mathbb{C}[\theta_1, \dots, \theta_{\mathcal{N}}] \otimes \Gamma \ , \ \mathrm{d}_\Gamma + (v - \lambda^a \theta_a) \frac{\partial}{\partial t} + \lambda^a \frac{\partial}{\partial \theta^a} \right), \tag{3.104}$$

where $\theta^1, \dots, \theta^{\mathcal{N}}$ are, again, odd generators, and the expression $\lambda^a \theta_a$ implicitly uses the choice of inner product on Σ.

On the other hand, we can study multiplets on $\mathbb{R}$ for the super Lie algebra $\mathfrak{n}$ directly. These are affine dgs-vector bundles on $\mathbb{R}$ equipped with $\mathcal{N}$ commuting odd symmetries, each of which squares to the action of translation. One can see this structure more concretely from the expression of equation (3.104) by applying homotopy transfer to take the cohomology with respect to the final summand $\lambda^a \frac{\partial}{\partial \theta^a}$ of the differential.

There is a rich theory underlying the classification of multiplets in supersymmetric mechanics; see for example [FG05]. It would be interesting to investigate the connections between this existing work and the point of view described here.

3.5 Applications

In this final section, we discuss some examples of multiplets in the light of the derived formalism. This also provides some insight to some of the curiosities of the underived pure spinor superfield formalism. In particular, we connect the multiplets associated to different Lie algebra cohomology groups by curvature maps and construct antifield multiplets for four-dimensional $\mathcal{N} = 1$ supersymmetry. In addition, we show how both the on- and off-shell version of the chiral multiplet can be constructed.

3.5.1 Multiplets from Lie algebra cohomology

We noticed in §3.3.3 that the multiplet associated to $C^\bullet(\mathfrak{n})$ can be identified with the de Rham complex of the super Lie group N. On the other hand, it was already appreciated in the previous literature that the individual Lie algebra cohomology groups of $\mathfrak{n}$ give an interesting class of R/I-modules which yield multiplets via the underived pure spinor construction $A^\bullet_{R/I}$. We studied examples for various super Poincaré algebras in §2.6.3; see also [MSX12]. The derived formalism gives a new tool to study these multiplets and to highlight the relations among the multiplets associated to the different Lie algebra cohomology groups.

To start with, recall that when the nilpotence variety $\mathrm{Spec}(R/I)$ associated to $\mathfrak{n}$ is a complete intersection, the Chevalley–Eilenberg cohomology of $\mathfrak{n}$ is concentrated in degree zero. The associated multiplet to $H^0(\mathfrak{n})$ is then simply quasi-isomorphic to the de Rham complex on the supertranslation group $\Omega^\bullet(N)$.

In contrast, for cases with higher Lie algebra cohomology, there is no direct quasi-isomorphism between the multiplets associated to $C^\bullet(\mathfrak{n})$ and $H^\bullet(\mathfrak{n})$. However—as advertised in §3.3.4—we can use the derived formalism to study the multiplets associated to the individual Lie algebra cohomology groups and their relationships. As discussed, there is a quasi-isomorphism identifying the multiplet associated to $C^\bullet(\mathfrak{n})$ with the de Rham complex on spacetime,

$$A^\bullet(C^\bullet(\mathfrak{n})) \simeq \Omega^\bullet(N)\,. \tag{3.105}$$

On the other hand, we can first take cohomology with respect to the Chevalley–Eilenberg differential. Let us filter the complex $A^\bullet(C^\bullet(\mathfrak{n}))$ by the total degree on the ring $C^\bullet(\mathfrak{n})$: the internal $\mathbb{Z}$-degree plus the weight: the Chevalley–Eilenberg cohomology has degree one for this filtration, and the remaining piece of the differential has degree zero. The homotopy transfer theorem leads to an equivalence

$$A^\bullet(C^\bullet(\mathfrak{n})) \simeq \left(\bigoplus_k A^\bullet \left(H^k(\mathfrak{n}) \right) \,,\, \mathcal{D}' \right)\,, \tag{3.106}$$

where the differential $\mathcal{D}'$ on the right hand side is induced by homotopy transfer, and consists of a sum of terms

$$\mathcal{D}'_{j,k} \colon A^\bullet \left(H^k(\mathfrak{n}) \right) \to A^\bullet \left(H^{k-j}(\mathfrak{n}) \right) \tag{3.107}$$

for $j \geq 1$. An alternative way of understanding these differentials is to consider higher differentials in the spectral sequence associated to the filtration discussed above, by total degree on $C^\bullet(\mathfrak{n})$.

For example, let us consider the case $j = 1$. The process we have just discussed gives rise to differential operators between the associated component field multiplets:

$$\nabla \colon \mu A^\bullet(H^k(\mathfrak{n})) \longrightarrow \mu A^\bullet(H^{k-1}(\mathfrak{n}))\,. \tag{3.108}$$

As we will see in an example momentarily, these operators can intuitively be thought of in the case $k = 0$ by viewing $\mu A^\bullet(H^{-1})$ as the field strength multiplet of $\mu A^\bullet(H^0)$, with

∇ acting as the field strength or curvature map. In particular, the multiplet associated to H^0 always contains a summand corresponding to a p-form abelian gauge field for some p. Since the deformation that we describe deforms the sum of the minimal multiplets $\mu A^\bullet(H^k(\mathfrak{n}))$ to (an object quasi-isomorphic to) the de Rham complex on spacetime, one expects that the operator mapping $\mu A^\bullet(H^0)$ to $\mu A^\bullet(H^{-1})$ will include a de Rham differential that carries the p-form to a $(p+1)$-form "field strength" component field in the multiplet $\mu A^\bullet(H^{-1})$. In examples, we will see that this is the case.

We will see in §7 that these operators play a big role in the construction of interactions. Broadly speaking, they provide a general coordinate-free description of operators of the type "R^a", considered by Cederwall in constructing pure spinor superfield actions; see [Ced10b]. These operators appear, for example, in the discussion of interacting eleven-dimensional supergravity in [Ced10c; Ced10a], and for the Dirac–Born–Infeld action in [CK11].

Example 3.5.1. Consider $\mathcal{N} = 1$ supersymmetry in three dimensions. Recall that the supertranslation algebra is of the form

$$\mathfrak{n} = S(-1) \oplus V(-2) \tag{3.109}$$

where S is the two-dimensional spin representation of $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, and V is the three-dimensional vector representation of $\mathrm{Spin}(3)$, with the bracket being induced from the equivariant isomorphism $\mathrm{Sym}^2(S) \cong V$. We can choose bases for these representations to obtain generators and relations for the Chevalley–Eilenberg complex as follows:

$$C^\bullet(\mathfrak{n}) = \mathbb{C}[\lambda^\alpha, v^\mu], \qquad \mathrm{d}_{\mathrm{CE}} v^1 = (\lambda^1)^2\ ,\ \mathrm{d}_{\mathrm{CE}} v^2 = \lambda^1\lambda^2\ ,\ \mathrm{d}_{\mathrm{CE}} v^3 = (\lambda^2)^2\ . \tag{3.110}$$

It will sometimes be convenient to write the differential more compactly as

$$\mathrm{d}_{\mathrm{CE}} v^{(\alpha\beta)} = \lambda^{(\alpha}\lambda^{\beta)}. \tag{3.111}$$

The Lie algebra cohomology is easily computed.

$$H^k = \begin{cases} R/I & \text{if } k = 0 \\ \left((\lambda^2 v^1 - \lambda^1 v^2)\mathbb{C}[\lambda^1] \oplus (\lambda^2 v^2 - \lambda^1 v^3)\mathbb{C}[\lambda^2]\right)/I_{-1} & \text{if } k = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.112}$$

The ideal I_{-1} appearing in the case where $k = -1$ is spanned by $\lambda^2(\lambda^2 v^1 - \lambda^1 v^2) - \lambda^1(\lambda^2 v^2 - \lambda^1 v^3)$. As explained in §2.3.9, $\mu A^\bullet(H^0)$ is identified with the 3d $\mathcal{N} = 1$ gauge multiplet and $\mu A^\bullet(H^{-1})$ with the corresponding antifield multiplet. The direct sum of these multiplets takes the following form.

$$\mu A^\bullet(H^0) \oplus \mu A^\bullet(H^{-1}) = \begin{bmatrix} \Omega^0 & & & \\ & \searrow^{\mathrm{d}} & & \\ & \Omega^1 & S & \\ & S & \Omega^2 & \\ & & \searrow^{\mathrm{d}} & \\ & & & \Omega^3 \end{bmatrix} \tag{3.113}$$

This direct sum is not yet isomorphic to the de Rham complex, but we can clearly see how the differential can be deformed such that this is the case: one has to add an acyclic piece which cancels the two spin representations and an additional differential of order one forming the de Rham differential between Ω^1 and Ω^2.

Let us now see how this deformation arises from the differential on $A^\bullet(C^\bullet(\mathfrak{n}))$ via homotopy transfer. We start by taking cohomology with respect to the Chevalley–Eilenberg differential d_{CE} and fix homotopy data

$$\begin{tikzcd} (A^\bullet(C^\bullet(\mathfrak{n}))\,,\, \mathrm{d}_{\mathrm{CE}}) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (C^\infty(N) \otimes H^\bullet(\mathfrak{n}),\, 0) \arrow[l, shift left, "i"] \end{tikzcd}\ , \tag{3.114}$$

such that we find induced differentials of the form $\mathcal{D}' = p\mathcal{D}i + p\mathcal{D}h\mathcal{D}i + \dots$. By degree reasons, only the first and the second summand can contribute non-zero terms. Taking cohomology with respect to $\mathcal{D}_0$ and transferring to the minimal multiplets we finally obtain the map

$$\nabla \colon \mu A^\bullet(H^0) \longrightarrow \mu A^\bullet(H^{-1})\,, \tag{3.115}$$

which we view as a field strength map. Let us now study this map explicitly using our basis. Running the techniques described in §2.3.7 (see also [KL09] for an earlier account), one finds the following representatives for the component fields in $\mu A^\bullet(H^0)$.

$$\begin{bmatrix} 1 & - & - \\ - & \lambda^{(\alpha}\theta^{\beta)} & \lambda^\alpha\theta^2 \end{bmatrix}. \tag{3.116}$$

For $\mu A^\bullet(H^{-1})$ one finds

$$\begin{bmatrix} \lambda_\alpha v^{(\alpha\beta)} & \lambda^{(\alpha}\theta^{\beta)} v^{\gamma\delta} & - \\ - & - & \theta^2\lambda_\alpha\lambda_\beta v^{(\alpha\beta)} \end{bmatrix}, \tag{3.117}$$

where now the rows indicate degree 2 and 3. We can fix a homotopy h for d_{CE} by

$$h(\lambda^\alpha\lambda^\beta) = v^{(\alpha\beta)}\,. \tag{3.118}$$

There will be terms in ∇ of the form

$$p\lambda\frac{\partial}{\partial\theta}h\lambda\frac{\partial}{\partial\theta}i\,. \tag{3.119}$$

Acting on representatives of the fermions in the multiplet $\mu A^\bullet(H^0)$, we find

$$\lambda^\alpha\theta^2 \mapsto \lambda^\alpha\lambda^\beta\theta_\beta \mapsto v^{(\alpha\beta)}\theta_\beta \mapsto \lambda_\beta v^{(\alpha\beta)}\,. \tag{3.120}$$

This is a map between $\mu A^\bullet(H^0)$ and $\mu A^\bullet(H^{-1})$, which induces an acyclic deformation on their direct sum.

In addition, ∇ contains terms of order one given by

$$pv\frac{\partial}{\partial x}i\,. \tag{3.121}$$

Investigating the representatives, it is clear that this acts as a de Rham differential between the one-form in $\mu A^\bullet(H^0)$ and the two-form in $\mu A^\bullet(H^{-1})$. This maps the one-form gauge field present in $\mu A^\bullet(H^0)$ to its field strength, which is part of $\mu A^\bullet(H^{-1})$.

3.5.2 Antifield multiplets in four dimensions

Let us now come back to the multiplet which provided the original motivation for the derived formalism. For four-dimensional $\mathcal{N}=1$ supersymmetry, $\mu A^\bullet(R/I)$ can be identified with the vector multiplet. In §2.4.4, we argued that the corresponding antifield multiplet, cannot be constructed in the underived pure spinor superfield formalism. Here, we describe the antifield multiplet using component fields first and then calulate the derived $\mathfrak{n}$-invariants. As expected we find that the cohomology is not concentrated in a single degree such that the multiplet is not in the essential image of the functor $A^\bullet_{R/I}$.

Let (E, D, ρ) denote the antifield multiplet to the vector multiplet on $\mathbb{R}^4$. It is concentrated in weight 0 to 4 and cohomological degree 0 and 1, and can be concretely described in array notation as

$$\mu A^\bullet(R/I)^\vee = \left[\begin{tikzcd} \Omega^0(\mathbb{R}^4) & \Gamma(\mathbb{R}^4, S_+\oplus S_-) & \Omega^1(\mathbb{R}^4) \arrow[dr, "\star d\star"] & \\ & & & \Omega^0(\mathbb{R}^4) \end{tikzcd}\right]\,. \tag{3.122}$$

We can also describe the L_∞ action of the supertranslation algebra $\mathfrak{n}$ concretely. We will only need the action on constant sections. There, odd elements act by the following formula:

$$\begin{array}{llcccclcl}
\rho^{(1)}_{\text{constants}}(Q) & : & S_+ \oplus S_- & \longrightarrow & \Omega^0 & , & (\psi^\vee, \bar{\psi}^\vee) & \mapsto & Q_+ \wedge \psi^\vee + Q_- \wedge \bar{\psi}^\vee \\
 & : & \Omega^1 & \longrightarrow & S_+ \oplus S_- & , & A^\vee & \mapsto & Q_+ \wedge A^\vee + Q_- \wedge A^\vee \\
\rho^{(2)}_{\text{constants}}(Q_1, Q_2) & : & \Omega^0 & \longrightarrow & \Omega^1 & , & c^\vee & \mapsto & \Gamma(Q_1, Q_2) \otimes c^\vee,
\end{array} \tag{3.123}$$

where we have denoted the fields by $(\psi^\vee, \bar{\psi}^\vee) \in \Gamma(\mathbb{R}^4, S_+ \oplus S_-)$, $A^\vee \in \Omega^1(\mathbb{R}^4)$, and $c^\vee \in \Omega^0(\mathbb{R}^4)$ in degree one. We have similarly denoted the positive and negative helicity summands of $Q \in S_+ \oplus S_-$ by Q_+ and Q_- respectively.

Let us write E_0 for the fiber of E over $0 \in \mathbb{R}^4$ (not to be confused with the summand of degree zero), so concretely

$$E_0 = \left[\begin{array}{cccc} \mathbb{R} & S_+ \oplus S_- & \mathbb{R}^4 & \\ & & & \mathbb{R} \end{array} \right]. \tag{3.124}$$

We can use this to describe the Chevalley–Eilenberg complex of $\mathfrak{n}$ with coefficients in our multiplet, using the L_∞ action of $\mathfrak{n}$ on the sheaf $\mathcal{E}$ of sections of E. We find the following.

Proposition 3.5.2.

$$C^\bullet(\mathfrak{n}, \mathcal{E}) \cong (E_0 \otimes \mathrm{Sym}(S_+ \oplus S_-), \mathrm{d}_\rho), \tag{3.125}$$

where d_ρ *is generated over* $\mathrm{Sym}(S_+ \oplus S_-)$ *by the sum of the terms*

$$\rho^{(1)}|_{\text{constants}} \colon E_0 \otimes (S_+ \oplus S_-) \to E_0 \tag{3.126}$$

$$\rho^{(2)}|_{\text{constants}} \colon E_0 \otimes \mathrm{Sym}^2(S_+ \oplus S_-) \to E_0 \tag{3.127}$$

obtained by restricting the L_∞ *action* ρ *to constant sections of* $\mathcal{E}$*.*

Proof. Note that

$$C^\bullet(\mathfrak{n}) \cong (\mathrm{Sym}((\mathbb{R}^4)^\vee[1])) \otimes \mathrm{Sym}(S_+ \oplus S_-), \mathrm{d}_{\mathrm{CE}}), \tag{3.128}$$

where we have identified $S_\pm$ with its dual using its canonical inner product. We obtain the given description by taking the cohomology by the operator dual to the action of the algebra of translations on $\mathcal{E}$; the result is quasi-isomorphic to $C^\bullet(\mathfrak{n}, \mathcal{E})$, no additional homotopical correction terms appear. □

Let us now discuss the cohomology of the Chevalley–Eilenberg complex.

Proposition 3.5.3. *We have an isomorphism*

$$H^\bullet(\mathfrak{n}, \mathcal{E}) \cong \mathbb{C} \oplus (\mathrm{Sym}(S_+) \oplus \mathrm{Sym}(S_-))[-1]. \tag{3.129}$$

Proof. This is a straightforward calculation using the description that we have just given. In the weight zero term of E_0, all elements are d_ρ-closed, but all such elements other than constants are also d_ρ-exact. In weight 1 in E_0, the d_ρ-closed elements are generated over $1 \otimes \mathrm{Sym}(S_+ \oplus S_-)$ by $\wedge^2 S_+ \oplus \wedge^2 S_- \subseteq (S_+ \oplus S_-) \otimes (S_+ \oplus S_-)$. When we quotient by d_ρ-exact elements we are left with $\big(\wedge^2 S_+ \otimes \mathrm{Sym}(S_-)\big) \oplus \big(\wedge^2 S_- \otimes \mathrm{Sym}(S_+)\big)$. In weight 2 in E_0 the closed and exact elements coincide, and in weight > 2 in E_0 there are no d_ρ-closed elements. $\square$

From our general results, we know abstractly that $A^\bullet(C^\bullet(\mathfrak{n}, \mathcal{E}))$ is dual to the vector multiplet. It is instructive to calculate the component field formulation for this multiplet explicitly using the recipe presented in §3.3.4 in order to see how the is related to the corresponding calculation in the underived pure spinor superfield formalism.

Our calculation will follow the same outline as the calculation we performed in the previous section. We will first study the multiplet associated to each of the two cohomology groups of $C^\bullet(\mathfrak{n}, \mathcal{E})$ in isolation, then we will compute the additional differential in $A^\bullet(H^\bullet(\mathfrak{n}, \mathcal{E}))$ relating these two individual terms, obtained by homotopy transfer.

First, recall that

$$\begin{aligned} A^\bullet(\mathbb{C}) &\simeq C^\infty(N) \\ &\simeq \Gamma(\mathbb{R}^4, \mathrm{Sym}(S_+[1] \oplus S_-[1])). \end{aligned} \tag{3.130}$$

For the non-trivial summand of the cohomology, we can compute that

$$\begin{aligned} A^\bullet(\mathrm{Sym}(S_+)) &\simeq (\Gamma(\mathbb{R}^4, \mathrm{Sym}(S_+) \oplus \mathrm{Sym}(S_+[1] \oplus S_-[1])), \mathcal{D}) \\ &\simeq \Gamma(\mathbb{R}^4, \mathrm{Sym}(S_-[1]), \end{aligned} \tag{3.131}$$

where the differential $\mathcal{D}$ is generated by the degree one isomorphism $S_+[1] \to S_+$. Similarly

$$A^\bullet(\mathrm{Sym}(S_+)) \simeq \Gamma(\mathbb{R}^4, \mathrm{Sym}(S_+[1])). \tag{3.132}$$

So altogether, when we compute $A^\bullet(H^\bullet(\mathfrak{n}, \mathcal{E}))$, we obtain a multiplet with the following Betti numbers:

$$\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ - & 2 & 4 & 2 & - \end{bmatrix}. \tag{3.133}$$

Now, let us compute the correction terms that allow us to obtain $A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E}))$ in full. As discussed in the previous section, §3.5.1, there is an additional differential coming from homotopy transfer. We will compute this differential in coordinates.

Running the procedure described in §2.3.7, we find the following local coordinates for our multiplet:

$$\begin{bmatrix} 1 & (\theta_\alpha,\bar\theta_{\dot\alpha}) & (\theta^2,\theta_\alpha\bar\theta_{\dot\alpha},\bar\theta^2) & (\theta^2\theta_\alpha,\bar\theta^2\bar\theta_{\dot\alpha}) & \theta^2\bar\theta^2 \\ - & (\lambda s,\bar\lambda\bar s) & (\lambda s\theta_\alpha,\bar\lambda\bar s\bar\theta_{\dot\alpha}) & (\lambda s\theta^2,\bar\lambda\bar s\bar\theta^2) & - \end{bmatrix}. \tag{3.134}$$

Let us unpack the notation. Recall that we identified $C^\bullet(\mathfrak{n},\mathcal{E})$ with $(E_0\otimes \mathrm{Sym}(S_+\oplus S_-),\mathrm{d}_\rho)$. We use $\{s^\alpha,\bar s^{\dot\alpha}\}$ for a basis of $S_+\oplus S_-$ in the first factor, and $\{\lambda^\alpha,\bar\lambda^{\dot\alpha}\}$ for a basis of $S_+\oplus S_-$ in the second factor, and e.g. λs the obvious contraction. We then use $\{\theta_\alpha,\bar\theta_{\dot\alpha}\}$ for the linear odd coordinate functions $S_+^\vee\oplus S_-^\vee\subseteq C^\infty(N)$.

With this concrete basis in hand, let us now investigate the additional pieces in the differential coming from homotopy transfer. For this purpose, let us fix a homotopy h for the differential d_ρ. We have

$$\mathrm{d}_\rho(s_\alpha)=\lambda_\alpha\,, \tag{3.135}$$

and therefore we can set

$$h(\lambda_\alpha)=s_\alpha \tag{3.136}$$

and similarly for $\bar\lambda$ and $\bar s$. There are two nontrivial terms contributing to the transferred differential

$$\mathcal{D}_0 h\mathcal{D}_0(\theta^2)=\lambda s \tag{3.137}$$

and again similarly for the complex conjugates. We see that this induces a differential that cancels some of the representative basis elements in pairs. The remaining representatives are

$$\begin{bmatrix} 1 & (\theta,\bar\theta) & \theta\bar\theta & - \\ - & - & - & \lambda s\theta^2+\bar\lambda\bar s\bar\theta^2 \end{bmatrix} \tag{3.138}$$

It is immediate to see that these representatives span the Spin(4) representations occurring in (3.122). In addition there is a differential of order one described by

$$\mathcal{D}_0 h\mathcal{D}_1(\theta^\alpha\bar\theta^{\dot\beta}A_{\alpha\dot\beta})=(\lambda s\theta^2+\bar\lambda\bar s\bar\theta^2)\partial^\mu A_\mu\,. \tag{3.139}$$

We thus recover the anticipated description of the vector antifield multiplet.

3.5.3 The chiral multiplet revisited

Let us discuss one further example in the derived formalism, which will illustrate the relation between on- and off-shell multiplets within the formalism, i.e. the appearance of non-trivial L_∞ actions of the supersymmetry algebra. We will again work specifically with four-dimensional $\mathcal{N}=1$ supersymmetry.

The minimal multiplet $\mu A^\bullet_{R/I}(\Gamma)$ associated to the module $\Gamma = \mathrm{Sym}(S_+)$ is equivalent to the BRST version of the chiral multiplet. Constructing the associated cotangent theory yields the standard off-shell BV theory of the chiral multiplet whose component fields include a scalar field ϕ, a chiral spinor field ψ, and an auxiliary scalar field F, as well as their associated complex conjugates and antifields. Of course, one can integrate out the auxiliary field F and obtain an equivalent BV theory, but the supersymmetry algebra action is now no longer strict. This is referred to as the on-shell formulation of the chiral multiplet. This is discussed in the L_∞-module language in [SW20], but the idea is much older, for example, the related example of an $\mathcal{N}=2$ hypermultiplet is discussed in the on-shell language in [Bau+90].

Let us again start from the component field formulation for these multiplets and compute their derived $\mathfrak{n}$-invariants. Plugging in this module in the derived pure spinor formalism we find that the minimal multiplet $\mu A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E}))$ can be explicitly identified with the on-shell formulation for the chiral multiplet described above. The off-shell formulation including the auxiliary field is given by a quasi-isomorphic non-minimal multiplet.

The component fields of the chiral multiplet in the on-shell formulation take the following form.

$$E = \left[\begin{tikzcd} \Omega^0\otimes\mathbb{C}^2 \arrow[drrr, "\star\mathrm{d}\star\mathrm{d}"'] & \Omega^0\otimes(S_+\oplus S_-) \arrow[dr, "\partial\!\!\!/"] & & \\ & & \Omega^0\otimes(S_+\oplus S_-) & \Omega^0\otimes\mathbb{C}^2 \ . \end{tikzcd}\right] \tag{3.140}$$

In order to describe the $\mathfrak{n}$-action, we will denote the component fields by $(\phi,\bar\phi)$ in degree zero, weight zero and $(\psi,\bar\psi)$ in degree zero, weight one and their respective antifields in degree one by $(\phi^\vee,\bar\phi^\vee)$ and $(\psi^\vee,\bar\psi^\vee)$. The odd elements of $\mathfrak{n}$ act in the following way.

$$\begin{array}{lclcll} \rho^{(1)}(Q) & : & S_+\oplus S_- \longrightarrow \Omega^0\otimes\mathbb{C}^2 & , & (\psi,\bar\psi) \mapsto & p_+(Q)\wedge\psi + p_-(Q)\wedge\bar\psi) \\ & : & \Omega^0\otimes\mathbb{C}^2 \longrightarrow S_+\oplus S_- & , & (\phi,\bar\phi) \mapsto & Q\wedge\partial\!\!\!/(\phi+\bar\phi) \\ \rho^{(2)}(Q,Q) & : & S_+\oplus S_- \longrightarrow S_+\oplus S_- & , & (\psi^+,\bar\psi^+) \mapsto & p_-(Q)\otimes p_+(Q)\wedge\psi^\vee \\ & & & & & +p_+(Q)\otimes p_-(Q)\wedge\bar\psi^\vee \end{array} \tag{3.141}$$

With this description we can compute the cohomology of the Chevalley–Eilenberg complex. We will use the following notation. Let $(e,\bar e)$ and $(s,\bar s)$ denote bases of $\mathbb{C}^2$ and $S_+\oplus S_-$ respectively. As in the previous section, let us use $\lambda_\alpha,\bar\lambda_{\dot\alpha}$ to denote even basis elements for $C^\bullet(\mathfrak{n})$. One finds the following description for the Chevalley–Eilenberg cohomology.

$$H^\bullet(\mathfrak{n},\mathcal{E}) \cong (\mathrm{Sym}(S_-)e \oplus \mathrm{Sym}(S_+)\bar{e}) + \big(\mathrm{Sym}(S_+)\lambda s \oplus \mathrm{Sym}(S_-)\bar{\lambda}\bar{s}\big)\,[-1]. \tag{3.142}$$

Again, we see that the cohomology is not concentrated in a single degree.

We can calculate the multiplet associated to this module following a method similar to the previous section, beginning with the multiplets associated to the summands of the cohomology described above, and then computing the correction terms associated to homotopy transfer. We obtain a quasi-isomorphic multiplet

$$\left(\bigoplus_k A^\bullet(H^k(\mathfrak{n},\mathcal{E}))\ ,\ \mathcal{D}',\rho'\right)\ , \tag{3.143}$$

where the new differential $\mathcal{D}'$ contains terms induced via homotopy transfer. Individually, both $\mu A^\bullet(H^0(\mathfrak{n},\mathcal{E}))$ and $\mu A^\bullet(H^1(\mathfrak{n},\mathcal{E}))$ contain the field content of a chiral and an antichiral BRST multiplet. We have the following Betti numbers for the sum of the multiplets induced from the individual cohomology groups:

$$\begin{bmatrix} 2 & 4 & 2 & - \\ - & 2 & 4 & 2 \end{bmatrix} \tag{3.144}$$

There are explicit elements representing the cohomology which take the forrm

$$\begin{bmatrix} (e,\bar{e}) & (\theta_\alpha e, \bar{\theta}_{\dot{\alpha}}\bar{e}) & (\theta^2 e, \bar{\theta}^2, \bar{e}) & - \\ - & (\lambda s, \bar{\lambda}\bar{s}) & (\lambda s\bar{\theta}_{\dot{\alpha}}, \bar{\lambda}\bar{s}\theta)_\alpha & (\lambda s\bar{\theta}^2, \bar{\lambda}\bar{s}\theta^2) \end{bmatrix}. \tag{3.145}$$

Now let's take a look at the induced differentials under homotopy transfer. For an explicit homotopy h we can choose

$$h(\lambda_\alpha e) = s_\alpha \qquad h(\bar{\lambda}_{\dot{\alpha}}\bar{e}) = \bar{s}_{\dot{\alpha}}\ . \tag{3.146}$$

Applying this to the representatives, we find

$$\begin{aligned} p\mathcal{D}_0 h\mathcal{D}_0 i(F) &= p\mathcal{D}_0 h\mathcal{D}_0(F\theta^2 e) = p((\lambda s)F) = F \\ p\mathcal{D}_1 h\mathcal{D}_0 i(\psi) &= p\mathcal{D}_1 h\mathcal{D}_0(\psi\theta e) = p((\lambda s)\bar{\theta}\partial\!\!\!/\psi) = \partial\!\!\!/\psi \\ p\mathcal{D}_1 h\mathcal{D}_1 i(\phi) &= p\mathcal{D}_1 h\mathcal{D}_1(\phi e) = p((\lambda s)\bar{\theta}^2\partial^2\phi) = \partial^2\phi\ . \end{aligned} \tag{3.147}$$

Similar results hold for the complex conjugate fields. Thus, we find that there are induced differentials of order zero, one and two. With respect to the weight grading, these operators alter the weight grading by minus one, one and three respectively. We can summarize the multiplet in the following diagram.

$$
\left[\begin{tikzcd}
\Omega^0\otimes\mathbb{C}^2 \arrow[drrr, "\star\mathrm{d}\star\mathrm{d}"'] & \Omega^0\otimes(S_+\oplus S_-) \arrow[dr, "\not{\partial}"] & \Omega^0\otimes\mathbb{C}^2 \arrow[dl, "\mathrm{id}"'] & \\
 & \Omega^0\otimes\mathbb{C}^2 & \Omega^0\otimes(S_+\oplus S_-) & \Omega^0\otimes\mathbb{C}^2\ .
\end{tikzcd}\right] \tag{3.148}
$$

This is precisely the off-shell BV model for the chiral multiplet. Taking cohomology with respect to the acyclic differential we obtain the minimal multiplet $\mu A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E}))$ which is precisely the original on-shell formulation that we started with.

Chapter 4

Six-dimensional supermultiplets from bundles on projective spaces

4.1 Introduction

With the pure spinor superfield formalism and its derived generalization firmly established, we turn our attention towards using the technique systematically in applications. One natural possibility is to use the equivalence of categories to look for patterns as well as possible classification results in the category of multiplets starting from the (derived) algebraic geometry of $C^\bullet(\mathfrak{n})$-modules. To get the best mileage out of the formalism, it is natural to start in a setting where the category of $C^\bullet(\mathfrak{n})$-modules, or at least the category of equivariant sheaves on the nilpotence variety, is relatively easy to understand.

A promising candidate is $\mathcal{N} = (1,0)$ supersymmetry in six dimensions, where the nilpotence variety Y is the space of two-by-four matrices of rank one; as such, the corresponding projective variety $\mathcal{Y} = \operatorname{Proj} R/I$ is just $\mathbb{P}^1 \times \mathbb{P}^3$, sitting inside $\mathbb{P}^7$ via the Segre embedding. There is a great abundance of geometrically interesting equivariant vector bundles on $\mathcal{Y}$. Taking the direct sum of the global sections of all twists of these bundles, we obtain graded equivariant modules over the ring of functions on the nilpotence variety, so that we can study the associated multiplets. In particular, we classify all multiplets originating from line bundles over $\mathcal{Y} = \mathbb{P}^1 \times \mathbb{P}^3$; among others, this recovers the family of so-called "$\mathcal{O}(n)$-multiplets" studied in the literature [KNT17; KNT18b; GR+98; LTM12; GIO87], and encompasses the vector multiplet and its antifield muliplet, as well as the hypermultiplet. (These three examples have already been studied via the pure spinor superfield formalism in [CN08; Ced18b].)

Roughly speaking, we provide a link between vector bundles on $\mathcal{Y}$ and $\mathfrak{g}$-multiplets in two steps, by combining the connection between quasicoherent sheaves on $\operatorname{Proj} R/I$ and graded R/I-modules (which is standard algebraic geometry) with the pure spinor superfield construction. Concretely, we convert a sheaf on $\mathcal{Y}$ into a module by forming its graded module of global sections (i.e. by taking the sum of the global sections of all its twists). Equivariant vector bundles form a subcategory of equivariant quasi-coherent sheaves; conversely, one can assign a sheaf on $\mathcal{Y}$ to each module, though it is important to note that the two operations are not inverses in general. Following the results on twisting pure spinor superfields in [SW21], we argue that modules whose associated sheaf is trivial correspond to multiplets that are perturbatively trivial in every twist.

In turn, the category of graded equivariant R/I-modules sits as a subcategory inside equivariant $C^\bullet(\mathfrak{n})$-modules, which is equivalent to the category of multiplets. We can summarize the situation with the following diagram:

$$\mathsf{LineBundles}^{\mathfrak{g}_0}_{\mathcal{Y}} \hookrightarrow \mathsf{QCoh}^{\mathfrak{g}_0}_{\mathcal{Y}} \underset{\sim}{\overset{\Gamma_*}{\rightleftarrows}} \mathsf{Mod}^{\mathfrak{g}_0}_{R/I} \hookrightarrow \mathsf{Mod}^{\mathfrak{g}_0}_{C^\bullet(\mathfrak{n})} \underset{C^\bullet}{\overset{A^\bullet}{\rightleftarrows}} \mathsf{Mult}_{\mathfrak{g}} \tag{4.1}$$

Since the inverse functor to the pure spinor superfield construction is given by taking derived $\mathfrak{n}$-invariants, classifying all multiplets associated to line bundles thus amounts to classifying all multiplets whose derived $\mathfrak{n}$-invariants are the graded global section module of a line bundle on the projective nilpotence variety. An alternative characterization of such multiplets is via their twists, which are necessarily holomorphic for minimal supersymmetry in six dimensions; in keeping with the results of [SW21], one expects that the holomorphic twist of such a multiplet is of rank one over Dolbeault forms on the spacetime, and we verify this below.

In addition, in §4.4, we extend the results of §2 by relating the duality theory of multiplets to sheaves on the nilpotence variety. To this end, we study the Cohen–Macaulay property and prove that, in good situations, the antifield multiplet can be constructed using the dualizing sheaf on the projective nilpotence variety.

Moving on, in §4.5 we develop some general methods regarding short exact sequences of sheaves in the pure spinor superfield formalism. These can be used to tackle higher-rank bundles; generalizations would allow for the construction of the multiplet associated to any higher-rank bundle via a resolution into a chain complex of sums of line bundles, though we do not pursue this in detail here. Our results show that the multiplet associated to a nontrivial extension of two sheaves is a deformation of the direct sum of the multiplets associated to each sheaf by a further differential, and we study such deformations explicitly at the level of component-field presentations of various multiplets.

In §4.6 and 4.7, we then use the Euler exact sequence, as well as the normal and conormal bundle sequences, to explicitly construct the multiplets associated to the tangent bundle, the normal bundle, and their duals. Several of these multiplets are of obvious physical interest; in particular, we identify the supergravity multiplet with the conormal bundle, and the gravitino multiplet with the pullback of the tangent bundle to the ambient space.

4.2 Preliminaries

4.2.1 Computational techniques for pure spinor superfields

We work in the setting established in the previous chapters, i.e. $\mathfrak{g}$ is a Lie algebra of super Poincaré type with super translation subalgebra $\mathfrak{n}$, corresponding nilpotence variety Y and ring of functions R/I.

Recall that, given an R/I module Γ, a minimal component field formulation of the multiplet $A^\bullet(\Gamma)$ can be canonically constructed. As before, we denote the minimal component field multiplet by $\mu A^\bullet(\Gamma)$. The fields of this component field multiplet take values in the Koszul homology of Γ, which can be computed by means of a minimal free resolution (L, d_L) of Γ in free R-modules. When we want to refer to the underlying vector bundle of $\mu A^\bullet(\Gamma)$ — which is the associated bundle of the bigraded Lorentz representation on $H^\bullet(K^\bullet(\Gamma))$ — we will write $\mu A^\bullet(\Gamma)^\#$. This notation will in general apply to any multiplet and denote (a natural bigraded lift of) its underlying $\mathbb{Z} \times \mathbb{Z}/2$-graded vector bundle, considered without the data of the differential and the module structure.

As usual, the minimal free resolution appearing in this context are bigraded, by cohomological degree and by the weight grading on R such that $L^\bullet$ is nonpositively graded and the differential d_L has cohomological degree one and weight zero. We will often write resolutions as

$$L^\bullet = \bigoplus_{k \geq 0} W_k \otimes R[k], \tag{4.2}$$

where W_k is the finite-dimensional weighted $\mathfrak{g}_0$-representation in which the generators of L^{-k} transform. Recall that Proposition 2.3.3 establishes an isomorphism of bigraded $\mathfrak{g}_0$-modules between the generators of $L^\bullet$ and the Koszul homology such that the fields of the minimal multiplet take values in the representations W_k.

In §2.3.6, we described a technique to extract the representations appearing in the minimal multiplet from the Hilbert series by assigning suitable weights to the generators. Here, we use a variant of the procedure by considering the *equivariant* Hilbert series as a formal power series in the representation ring of $\mathfrak{g}_0$.[1] Therefore, we define

$$\mathrm{Hilb}(\Gamma) = \sum_{k=0}^{\infty} \Gamma_k \, t^k \in \mathrm{Rep}(\mathfrak{g}_0)[\![t]\!]. \tag{4.3}$$

We can then rewrite the Hilbert series in the form

$$\begin{aligned} \mathrm{Hilb}(\Gamma) &= \left[\sum_{d=0}^{\infty} \mathrm{Sym}^d(\mathfrak{n}_1^\vee) \, t^d \right] \otimes \left[\sum_{k,\ell} (-1)^\ell W_k^\ell t^k \right] \\ &= \mathrm{Hilb}(R) \cdot \mathrm{Hilb}(\chi(W^\bullet)). \end{aligned} \tag{4.4}$$

Comparing coefficients order by order, one obtains a system of equations which allows to identify $\chi(W^\bullet)$, and thus (at least in favorable cases) $W^\bullet$ itself:

$$\begin{aligned} \chi(W^\bullet)_0 &= \Gamma_0 \\ \chi(W^\bullet)_1 &= \Gamma_1 - \mathfrak{n}_1^\vee \otimes \chi(W^\bullet)_0 \\ &\vdots \\ \chi(W^\bullet)_k &= \Gamma_k - \sum_{d=1}^{k} \mathrm{Sym}^d(\mathfrak{n}_1^\vee) \otimes \chi(W^\bullet)_{k-d}. \end{aligned} \tag{4.5}$$

This technique is used frequently in the work of Cederwall and collaborators, and we will apply it in examples in what follows.

4.2.2 The projective nilpotence variety for six-dimensional $\mathcal{N} = 1$

We already introduced the six-dimensional $\mathcal{N} = (1,0)$ super Poincaré algebra and the corresponding nilpotence variety in §2.4.3. Recall that there are exceptional isomorphisms $\mathfrak{so}(6) \cong \mathfrak{sl}_4$ for the Lorentz symmetry and $\mathfrak{sp}(1) \cong \mathfrak{sl}_2$ for the R-symmetry such that the nilpotence variety and all the bundles we consider on it carry an action by $\mathfrak{sl}_2 \times \mathfrak{sl}_4$. Since the bracket of odd elements in the supertranslation algebra is given by wedging on the spin representations and the symplectic form on the R-symmetry space $U = (\mathbb{C}^2, \omega)$, a supercharge $Q \in \mathfrak{n}_1$ is square zero if and only if the rank of the associated linear map $(S_+)^\vee \longrightarrow U$ is less or equal then one. In terms of coordinates this means that the defining ideal I of the nilpotence variety is spanned by the 2×2 minors of the matrix with entries λ_i^α,

$$\begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \end{pmatrix}. \tag{4.6}$$

[1] The representation ring of $\mathfrak{g}_0$ is the free abelian group on the set of finite-dimensional irreducible $\mathfrak{g}_0$-representations, with the multiplication induced by the tensor product of representations.

Accordingly, the nilpotence variety $Y = \operatorname{Spec} R/I$ can be thought of as the space of rank one matrices inside $M^{2\times 4}(\mathbb{C})$. Its projective version $\mathcal{Y} = \operatorname{Proj} R/I$ can be identified with the product of two projective spaces via the Segre embedding. In more detail, the square-zero supercharges are precisely those which can be written as

$$Q = \xi \otimes r \quad \text{with} \quad \xi \in S_+ \, , \; r \in U. \tag{4.7}$$

Interpreting $[r_0 : r_1]$ and $[\xi_0 : \cdots : \xi_3]$ as homogeneous coordinates on $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively identifies $\mathcal{Y}$ with the image of the Segre embedding

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^3 \longrightarrow \mathbb{P}^7 \qquad ([r_0 : r_1], [\xi_0 : \ldots \xi_3]) \mapsto [r_0\xi_0 : \cdots : r_1\xi_3] \, . \tag{4.8}$$

We can thus explore supermultiplets in six dimensions using the algebraic geometry of projective spaces.

4.2.3 From sheaves on projective schemes to modules

Clearly, the R/I-modules serving as inputs for the pure spinor functor $A^\bullet$ are closely related to sheaves of $\mathcal{O}_Y$-modules on Y: For any affine scheme $X = \operatorname{Spec} S$ there is an equivalence of categories between quasi-coherent sheaves of $\mathcal{O}_X$-modules and S-modules. Explicitly, this equivalence is given by taking global sections

$$\mathsf{QCoh}_{\mathcal{O}_X} \longrightarrow \mathsf{Mod}_S \qquad \mathcal{F} \mapsto \Gamma(\operatorname{Spec}(S), \mathcal{F}) \, , \tag{4.9}$$

and conversely assigning

$$\mathsf{Mod}_S \longrightarrow \mathsf{QCoh}_{\mathcal{O}_X} \qquad M \mapsto \tilde{M} \, , \tag{4.10}$$

where $\tilde{M}$ is defined by the requirement $\tilde{M}(D_f) = M_f$ for all $f \in S$.[2] If S is graded, one can think of the grading as defining a $\mathfrak{gl}_1$-action on $\operatorname{Spec} S$; it is then possible to define an equivalence between graded S-modules and quasicoherent sheaves of $\mathcal{O}_X$-modules on X that are equivariant for rescalings.

One can thus always think of the input to the (underived) pure spinor superfield formalism geometrically as a $(\mathfrak{g}_0 \oplus \mathfrak{gl}_1)$-equivariant sheaf on the affine nilpotence variety. It is tempting to ask if one can picture the situation using the geometry of sheaves on $\mathcal{Y} = \operatorname{Proj} R/I$. Here, the situation is geometrically compelling, but a bit less unequivocal. From a graded S-module M, we can construct a quasi-coherent sheaf on $\operatorname{Proj} S$ by setting $\tilde{M}(D_f) = (M_f)_0$. By the definition of the Proj-construction we have $\tilde{S} = \mathcal{O}_{\operatorname{Proj} S}$.

[2]Here $D_f \subseteq \operatorname{Spec} S$ denotes all prime ideals of S not containing f and M_f the localization of M at f.

The twisting sheaves are defined by

$$\mathcal{O}_{\mathrm{Proj}\, S}(n) = \widetilde{S(n)}\,. \tag{4.11}$$

For a sheaf $\mathcal{F}$ on $\mathrm{Proj}\, S$, we define the associated S-module to be

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathrm{Proj}\, S, \mathcal{F}(n))\,. \tag{4.12}$$

We will call $\Gamma_*(\mathcal{F})$ the graded global section module of $\mathcal{F}$. In general, these assignments no longer give an equivalence of categories, but we can still use $\Gamma_*(-)$ to construct large families of input data for the pure spinor superfield formalism from sheaves on the projective version of the nilpotence variety. This is in particular useful in the case of $\mathcal{N} = (1,0)$ supersymmetry in six dimensions, since—as we explained above—the projective version of the nilpotence variety can be identified with $\mathbb{P}^1 \times \mathbb{P}^3$ and equivariant sheaves on this space are very well understood geometrically.

What the projective perspective misses. Contrary to the affine case, the functors $\sim$ and Γ_* do not yield an equivalence of categories. While it is true that

$$\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F} \tag{4.13}$$

for any quasicoherent sheaf $\mathcal{F}$, it can happen that $\Gamma_*(\tilde{M})$ is not isomorphic to the original module M. Let us restrict to the case where $S = R$ is a polynomial ring and M is a finitely generated graded module. Consider the class $\mathcal{C}$ of modules M such that $M_n = 0$ for n large enough. One finds that these are precisely the modules which are in the kernel of $\sim$. One has the following result:

Proposition 4.2.1 ([Ser55]). *Let M be a graded S-module. Then*

$$\tilde{M} = 0 \iff M \in \mathcal{C}\,. \tag{4.14}$$

For the pure spinor superfield formalism, this means that multiplets corresponding to modules which are concentrated in finitely many degrees cannot be obtained from sheaves on the projective nilpotence variety. One such example is the free superfield $A^\bullet(\mathbb{C})$ which is constructed from the trivial module $\mathbb{C}$ (thought of as the quotient of R by the maximal ideal corresponding to the origin). The corresponding sheaf on the affine nilpotence variety is the skyscraper sheaf with value $\mathbb{C}$ at the origin; the associated sheaf on the projective nilpotence variety is trivial.

In general, such sheaves must have zero-dimensional support. The support of an equivariant sheaf must consist of a union of orbits of the P_0-action; since we only consider sheaves that are equivariant for rescaling, the origin is the unique zero-dimensional orbit, so that any module in the kernel of $\sim$ defines a sheaf supported entirely at the origin.

Remark 4.2.2. It is natural to wonder how conditions on the support of a sheaf translate into properties of the corresponding multiplet. An intuitive answer is suggested by the results of [SW21] on twisting in the pure spinor formalism. There, it was noted that deforming a super Poincaré-type algebra by a square-zero supercharge commutes with forming the pure spinor multiplet of the structure sheaf. When $\mathcal{Y}$ is smooth (as is the case here), only holomorphic twists are available, and the computations in [SW21] imply that the holomorphic twist of a given multiplet is freely generated over the Dolbeault complex on spacetime by the stalk of the corresponding sheaf at the holomorphic supercharge. We do not explain this in detail here, but will remark from time to time on the physical interpretations of our results that it suggests.

In keeping with Remark 4.2.2, we expect that multiplets corresponding to sheaves in the kernel of $\sim$ are precisely those that are perturbatively trivial in every possible twist. We note that the free superfield falls into this class.

4.2.4 Some natural equivariant vector bundles

In the bulk of this chapter we are going to consider various vector bundles over the nilpotence variety $\mathcal{Y} \cong \mathbb{P}^1 \times \mathbb{P}^3$ and construct the associated multiplets using the pure spinor superfield formalism. For later reference and completeness, we now introduce the bundles that will appear later on.

Line bundles. The product space geometry of the nilpotence variety $\mathcal{Y} \cong \mathbb{P}^1 \times \mathbb{P}^3$ makes it easy to describe all of its line bundles. Indeed, holomorphic line bundles are classified up to isomorphism by the Picard group $\mathrm{Pic}(\mathcal{Y}) \cong H^1(\mathcal{Y}, \mathcal{O}^*_{\mathcal{Y}})$, which can be easily computed using the exponential short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\mathbb{P}^1\times\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3} \longrightarrow \mathcal{O}^*_{\mathbb{P}^1\times\mathbb{P}^3} \longrightarrow 0 \tag{4.15}$$

and its related long exact sequence in cohomology. In particular, one finds the isomorphism $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^3) \cong \mathbb{Z} \oplus \mathbb{Z}$, which tells that every line bundle on the product variety $\mathbb{P}^1 \times \mathbb{P}^3$ arises from line bundles defined on its factors $\mathbb{P}^1$ and $\mathbb{P}^3$. (Recall that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$

for any $n \geq 1$.) In fact, given the structural projections

$$
\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^3 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
\mathbb{P}^1 & & \mathbb{P}^3
\end{tikzcd}
\tag{4.16}
$$

from $\mathbb{P}^1 \times \mathbb{P}^3$ to its cartesian components, the line bundles on $\mathbb{P}^1 \times \mathbb{P}^3$ are all given by the exterior tensor product of a pair line bundles defined over $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively. In other words,

$$
\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(n,m) = \mathcal{O}_{\mathbb{P}^1}(n) \boxtimes \mathcal{O}_{\mathbb{P}^3}(m) = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}} \pi_3^*\mathcal{O}_{\mathbb{P}^3}(m)\,, \quad (n,m) \in \mathbb{Z}^{\oplus 2}. \tag{4.17}
$$

Note that the generators of the Picard group $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^3)$ are given by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,0)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(0,1)$; the connecting (iso)morphism $\delta_2 : \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^3) \to \mathbb{Z} \oplus \mathbb{Z}$ thus carries $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(n,m)$ to (n,m), and tensor product yields an isomorphism (of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}$-modules)

$$
\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(n,m) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(k,l) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(n+k,m+l) \tag{4.18}
$$

for any $(n,m),(k,l) \in \mathbb{Z}^{\oplus 2}$. We will often use the shorthand $\mathcal{O}(n,m) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(n,m)$. Finally, we will denote a k-twisting sheaf for the nilpotence variety $\mathcal{Y}$ by

$$
\mathcal{O}_{\mathcal{Y}}(k) := \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(k,k). \tag{4.19}
$$

Tangent and Cotangent Bundles. Similarly, tangent and cotangent bundles on a product variety can be reconstructed by the tangent and cotangent bundles of its Cartesian components. In fact, the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^3$ is given by the exterior direct sum

$$
\mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^3} \cong \pi_1^*\mathcal{T}_{\mathbb{P}^1} \oplus \pi_3^*\mathcal{T}_{\mathbb{P}^3} =: \mathcal{T}_{\mathbb{P}^1} \boxplus \mathcal{T}_{\mathbb{P}^3}. \tag{4.20}
$$

Note that $\mathcal{T}_{\mathbb{P}^1}$ is a line bundle and one has $\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(+2)$, while $\mathcal{T}_{\mathbb{P}^3}$ is an ample non-decomposable vector bundle of rank three. The tangent bundle on any projective space $\mathbb{P}^n$ sits in the Euler exact sequence

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(+1) \otimes V_{n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n} \longrightarrow 0, \tag{4.21}
$$

where V_{n+1} is a $(n+1)$-dimensional complex vector space that carries the fundamental representation of $\mathfrak{sl}_{n+1}$. The Euler exact sequence (4.21) is a short exact sequence of $\mathfrak{sl}_{n+1}$-equivariant sheaves; this will play a role in §4.6, when we will study the multiplet associated to the tangent bundle $\mathcal{T}_{\mathcal{Y}}$ of the nilpotence variety.

In a similar fashion, the cotangent bundle $\Omega^1_{\mathcal{Y}} := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}}(\mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3})$ of the nilpotence variety $\mathcal{Y}$ is given by the exterior direct sum

$$\Omega^1_{\mathbb{P}^1\times\mathbb{P}^3} \cong \pi_1^*\Omega^1_{\mathbb{P}^1} \oplus \pi_3^*\Omega^1_{\mathbb{P}^3} = \Omega^1_{\mathbb{P}^1} \boxplus \Omega^1_{\mathbb{P}^3}, \tag{4.22}$$

where now $\Omega^1_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. Note that, taking the dual of the Euler sequence (4.21), one finds

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \otimes V^\vee_{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0, \tag{4.23}$$

which in turn describes the cotangent bundle on any projective space $\mathbb{P}^n$.

Normal and Conormal Bundles. Let us now consider $\mathcal{Y}$ via its Segre embedding $\sigma : \mathcal{Y} \hookrightarrow \mathbb{P}^7$. (Recall that this embedding is canonically associated to the datum of the supertranslation algebra $\mathfrak{n}$.) Having introduced the tangent bundle $\mathcal{T}_{\mathcal{Y}}$, one defines the normal bundle $\mathcal{N}_{\mathcal{Y}/\mathbb{P}^7}$ of $\mathcal{Y}$ in $\mathbb{P}^7$ to be quotient bundle $\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}/\mathcal{T}_{\mathcal{Y}}$, where $\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}} := \sigma^*\mathcal{T}_{\mathbb{P}^7}$. As such, the normal bundle sits in the exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{Y}} \xrightarrow{d\sigma} \mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}} \longrightarrow \mathcal{N}_{\mathcal{Y}/\mathbb{P}^7} \longrightarrow 0, \tag{4.24}$$

of vector bundles on $\mathcal{Y}$, which will be referred to as the normal bundle exact sequence. Dualizing (4.24), one obtains the exact sequence defining the conormal bundle:

$$0 \longrightarrow \mathcal{N}^\vee_{\mathcal{Y}/\mathbb{P}^7} \longrightarrow \Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}} \xrightarrow{d\sigma^\vee} \Omega^1_{\mathcal{Y}} \longrightarrow 0. \tag{4.25}$$

The conormal bundle $\mathcal{N}^\vee_{\mathcal{Y}/\mathbb{P}^7}$ is thus the kernel of the morphism of vector bundles $d\sigma^\vee : \Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}} \to \Omega^1_{\mathcal{Y}}$. Another characterization of the conormal bundle $\mathcal{N}^\vee_{\mathcal{Y}/\mathbb{P}^7}$ is possible using the sheaf of ideals $\mathcal{J}_{\mathcal{Y}}$, which is defined as the kernel of the morphism of sheaves $\sigma^\sharp : \mathcal{O}_{\mathbb{P}^7} \to \sigma_*\mathcal{O}_{\mathcal{Y}}$. In fact, there is a natural isomorphism of vector bundles on $\mathcal{Y}$ given by $\sigma^*(\mathcal{J}_{\mathcal{Y}}/\mathcal{J}^2_{\mathcal{Y}}) \cong \mathcal{N}^\vee_{\mathcal{Y}/\mathbb{P}^7}$. In the following, since no confusion regarding the ambient space can arise, we will denote the normal and conormal bundles with respect to the Segre embedding by $\mathcal{N}_{\mathcal{Y}}$ and $\mathcal{N}^\vee_{\mathcal{Y}}$.

4.3 A family of multiplets from line bundles

4.3.1 General procedure

Let us now classify all multiplets associated to the infinite family of line bundles $\mathcal{O}(n,m)$. We will denote the multiplets by

$$\mu A^\bullet(n,m) := \mu A^\bullet(\Gamma_*(\mathcal{O}(n,m))). \tag{4.26}$$

As a first observation, we note that the construction exhibits the following symmetry under twists of line bundles:

$$\begin{aligned}\Gamma_*(\mathcal{O}(n+k,m+k)) &= \bigoplus_{d\in\mathbb{Z}} H^0(\mathcal{O}(n+k+d,m+k+d)) \\ &= \bigoplus_{d\in\mathbb{Z}} H^0(\mathcal{O}(n+d,m+d))(k) \\ &= \Gamma_*(\mathcal{O}(n,m))(k).\end{aligned} \tag{4.27}$$

This implies that the multiplets $\mu A^\bullet(n,m)$ and $\mu A^\bullet(n+k,m+k)$ agree up to a total degree shift. Since the weight grading of a graded equivariant R/I-module becomes the cohomological grading of the corresponding multiplet, we have that

$$\mu A^\bullet(n+k,m+k) = \mu A^\bullet\left(\Gamma_*(\mathcal{O}(n,m))(k)\right) = \mu A^\bullet\left(\Gamma_*(\mathcal{O}(n,m))\right)[k] = \mu A^\bullet(n,m)[k]. \tag{4.28}$$

It is thus sufficient to consider the line bundles $\mathcal{O}(n,0)$ and $\mathcal{O}(0,m)$ for $n,m \geq 0$. (Equivalently, one could also consider the family $\mathcal{O}(n,0)$ for $n \in \mathbb{Z}$.)

We will identify the field content of the multiplets using the technique sketched above in §4.2.1. We resum the equivariant Hilbert series, working in the ring of formal power series with coefficients in the representation ring of $\mathfrak{sl}_2 \times \mathfrak{sl}_4$, and read off the equivariant structure of the minimal free resolution from its numerator.

Recall that $\Gamma_*(\mathcal{O}(n,m))_d = \mathbb{C}[x_0,x_1]_{n+d} \otimes \mathbb{C}[y_0,\dots,y_3]_{m+d}$. The monomials of degree d are the d-th symmetric power of the defining representation of the corresponding group of linear transformations, so that we have

$$\Gamma_*(\mathcal{O}(n,m))_d = [n+d|m+d,0,0]. \tag{4.29}$$

in terms of Dynkin labels for $\mathfrak{sl}_2 \times \mathfrak{sl}_4$. Thus, the equivariant Hilbert series takes the form

$$\mathrm{Hilb}(n,m) := \mathrm{Hilb}(\Gamma_* \mathcal{O}(n,m)) = \sum_{d=-\min(n,m)}^{\infty} [n+d|m+d,0,0]\, t^d. \tag{4.30}$$

Following §4.2.1, we rewrite the Hilbert series using the identity

$$\mathrm{Hilb}(n,m) = \mathrm{Hilb}(R) \cdot \mathrm{Hilb}\left(\chi(W^\bullet(n,m))\right), \tag{4.31}$$

and solve for $\chi(W^\bullet(n,m))$. The equations (4.5) become

$$\begin{aligned}
\chi(W^\bullet(n,m))_0 &= [n|m,0,0] \\
\chi(W^\bullet(n,m))_1 &= [n+1|m+1,0,0] - [1|1,0,0] \otimes \chi(W^\bullet(n,m))_0 \\
&\;\;\vdots \\
\chi(W^\bullet(n,m))_k &= [n+k|m+k,0,0] - \sum_{d=1}^{k} \mathrm{Sym}^d([1|1,0,0]) \otimes \chi(W^\bullet(n,m))_{k-d}\,.
\end{aligned} \tag{4.32}$$

In what follows, we solve these equations case by case.

4.3.2 The bundles $\mathcal{O}(n,0)$ for $n \geq 0$

We begin with the case of the bundles $\mathcal{O}(n,0)$ for nonnegative n. As we will see, these bundles include the vector multiplet, its antifield multiplet, and the hypermultiplet, as well as an infinite family of strict component-field multiplets associated to $\mathcal{O}(n,0)$ with $n \geq 3$.

Computation of the Betti numbers. We specialize the Hilbert series (4.30) to the case at hand. At the level of the graded dimension,

$$\mathrm{grdim}(n,0) = \sum_{d=0}^{\infty} (n+d+1) \frac{(d+3)(d+2)(d+1)}{6}\, t^d, \tag{4.33}$$

which can be rewritten as a derivative of a geometric series

$$\mathrm{grdim}(n,0) = \frac{1}{6} \frac{\partial^3}{\partial t^3} t^{3-n} \frac{\partial}{\partial t} \sum_{d=0}^{\infty} t^{d+n+1} = \frac{1}{6} \frac{\partial^3}{\partial t^3} t^{3-n} \frac{\partial}{\partial t} \frac{t^{n+1}}{1-t}. \tag{4.34}$$

Performing the derivatives, the general result can be expressed in the following form.

$$\mathrm{grdim}(n,0) = \frac{(n+1) - 4nt + 6(n-1)t^2 - 4(n-2)t^3 + (n-3)t^4}{(1-t)^8} \tag{4.35}$$

The coefficients of the numerator now correspond to the Betti numbers of the associated multiplet.

Let us write out these Betti numbers concretely for all n. There are three special cases when $n \in \{0, 1, 2\}$. For $n = 0$ one finds

$$\mathrm{grdim}\mu A^\bullet(0,0)^\# = \begin{bmatrix} 1 & - & - & - \\ - & 6 & 8 & 3 \end{bmatrix}, \tag{4.36}$$

which corresponds to the vector multiplet. For $n = 1$, we obtain

$$\mathrm{grdim}\mu A^\bullet(1,0)^\# = \begin{bmatrix} 2 & 4 & - & - \\ - & - & 4 & 2 \end{bmatrix}, \tag{4.37}$$

which corresponds to the hypermultiplet. For $n = 2$, the result reads

$$\mathrm{grdim}\mu A^\bullet(2,0)^\# = \begin{bmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{bmatrix}, \tag{4.38}$$

which corresponds to the antifield multiplet of the vector multiplet. Finally, for $n \geq 3$, the resulting Betti numbers take the general form

$$\mathrm{grdim}\mu A^\bullet(n,0)^\# = \begin{bmatrix} n+1 & 4n & 6(n-1) & 4(n-2) & n-3 \end{bmatrix}. \tag{4.39}$$

Equivariant decomposition. The above recursive relations are easily solved, either by hand or with the help of a computer program such as LiE [LCL]. Let us again first consider the three special cases where $n \in \{0, 1, 2\}$. For $n = 0$ we obtain

$$\begin{aligned} W_0 &= [0|0,0,0] \\ W_1 &= 0 \\ W_2 &= -[0|0,1,0] \\ W_3 &= [1|0,0,1] \\ W_4 &= -[2|0,0,0]. \end{aligned} \tag{4.40}$$

Thus the resulting multiplet takes the form

$$\mu A^\bullet(0,0)^\# = \begin{bmatrix} \Omega^0 & & & \\ & \Omega^1 & \mathbb{C}^2 \otimes S_- & \Omega^0 \otimes \mathbb{C}^3 \end{bmatrix} \tag{4.41}$$

where the three scalar fields live in the adjoint representation of the R-symmetry group. (Here and in the following tables showing the field content of multiplets $\mathbb{C}^n$ will always denote the unique irreducible n-dimensional representation of $\mathfrak{sl}_2$.) This corresponds to the vector multiplet of six-dimensional $\mathcal{N} = (1,0)$ supersymmetry. For $n = 1$, we find

$$\begin{aligned} W_0 &= [1|0,0,0] \\ W_1 &= -[0|1,0,0] \\ W_2 &= 0 \\ W_3 &= -[0|0,0,1] \\ W_4 &= [1|0,0,0]. \end{aligned} \tag{4.42}$$

We can thus identify $\mu A^\bullet(1,0)$ as the hypermultiplet

$$\mu A^\bullet(1,0)^{\#} = \begin{bmatrix} \Omega^0 \otimes \mathbb{C}^2 & S_+ & & \\ & & S_- & \Omega^0 \otimes \mathbb{C}^2 \end{bmatrix} \tag{4.43}$$

For $n = 2$

$$\begin{aligned} W_0 &= [2|0,0,0] \\ W_1 &= -[1|1,0,0] \\ W_2 &= -[0|0,1,0] \\ W_3 &= 0 \\ W_4 &= -[0|0,0,0]. \end{aligned} \tag{4.44}$$

The resulting multiplet $\mu A^\bullet(2,0)$ is the antifield multiplet of the vector multiplet.

$$\mu A^\bullet(2,0)^{\#} = \begin{bmatrix} \Omega^0 \otimes \mathbb{C}^3 & S_+ \otimes \mathbb{C}^2 & \Omega^1 & \\ & & & \Omega^0 \end{bmatrix} \tag{4.45}$$

Finally for $n \geq 3$, the general form is

$$\begin{aligned} W_0 &= [n|0,0,0] \\ W_1 &= -[n-1|1,0,0] \\ W_2 &= [n-2|0,1,0] \\ W_3 &= -[n-3|0,0,1] \\ W_4 &= [n-4|0,0,0]. \end{aligned} \tag{4.46}$$

Thus, $\mu A^\bullet(n,0)$ for $n \geq 3$ are of the form

$$\mu A^\bullet(n,0)^{\#} = \tag{4.47}$$
$$\Big[\ \mathbb{C}^{n+1} \quad \mathbb{C}^n \otimes S_+ \quad \mathbb{C}^{n-1} \otimes \wedge^2 S_+ \quad \mathbb{C}^{n-2} \otimes \wedge^3 S_+ \quad \mathbb{C}^{n-3} \otimes \wedge^4 S_+\ \Big]$$

This family of multiplets was described in the physics literature under the name $\mathcal{O}(n)$-multiplets [KNT17; KNT18b; GR+98; LTM12; GIO87].

Supersymmetry module structure and interpretation. Given our results so far, it is easy to give an explicit description of the module $\Gamma_*(\mathcal{O}(n,0))$ as a cokernel of a map between free R-modules as well as to describe their minimal free resolutions in R-modules. We already discussed the cases $n = 0$ and $n = 1$ discussed in §2. For $n \geq 1$, we are looking for a map

$$\varphi_n : \mathbb{C}^n \otimes S_+ \otimes R \longrightarrow \mathbb{C}^{n+1} \otimes R \tag{4.48}$$

which is linear in λ^α_i and equivariant under $\mathfrak{sl}_2 \times \mathfrak{sl}_4$. Up to a non-zero constant prefactor, there is a unique such map which can be described in components by

$$F \mapsto \lambda^\alpha_{(i_n} F_{i_1 \dots i_{n-1})\,\alpha}\ . \tag{4.49}$$

Resolving $\Gamma_*(\mathcal{O}(n,0)) = \mathrm{coker}(\varphi_n)$ one recovers the field content of the multiplets described above. In addition, the resolution differential encodes the part of the $\mathfrak{g}$-module structure acting by differential operators of degree zero.

Let us describe the minimal free resolution and the module structure for the cases $n \geq 3$. This will provide an intuitive interpretation of $\mu A^\bullet(n,0)$: it is a multiplet whose observables are generated by the degree-n monomials in the observables of the $\mathcal{O}(1,0)$-multiplet, i.e. the hypermultiplet. One can thus imagine that the fields of the hypermultiplet map to the fields of the $\mathcal{O}(n,0)$ multiplet via a n-fold covering, dual to the inclusion map on observables.

In components, the resolution differential

$$(d_L)_i : \mathbb{C}^{n-i} \otimes \wedge^i S_+ \otimes R \longrightarrow \mathbb{C}^{n-i+1} \otimes \wedge^{i-1} S_+ \otimes R \qquad i = 1 \dots 4 \tag{4.50}$$

is described by contracting along S_+ and symmetrizing along the $\mathfrak{sl}_2$-representation, for example

$$[(d_L)_2 F]_{i_1 \dots i_{n-1} \alpha} = \lambda^\beta_{(i_{n-1}} F_{i_1 \dots i_{n-2})\,\alpha\beta}\ . \tag{4.51}$$

This translates into supersymmetry transformation rules of the form

$$\delta F_{i_1 \dots i_m} = \epsilon^{\alpha}_{(i_n} F_{i_1 \dots i_{n-1})\alpha} \, . \tag{4.52}$$

Recall that we identified the $\mu A^\bullet(1,0)$ as the hypermultiplet. Let us denote the linear observables in physical fields of the hypermultiplet by ϕ_i and ψ_α. This suggests to identify the linear observables of the $\mathcal{O}(n,0)$-multiplet as polynomials of degree n in the linear observables of the hypermultiplet, as follows:

$$\begin{aligned} F_{i_1 \dots i_n} &= \phi_{i_1} \dots \phi_{i_n} \\ F_{i_1 \dots i_{n-1}\,\alpha} &= \phi_{i_1} \dots \phi_{i_{n-1}} \psi_\alpha \\ &\vdots \\ F_{i_1 \dots i_{n-4}\,\alpha\beta\gamma\delta} &= \phi_{i_1} \dots \phi_{i_{n-4}} \psi_\alpha \psi_\beta \psi_\gamma \psi_\delta . \end{aligned} \tag{4.53}$$

Further, recall that for the hypermultiplet the module structure of the supersymmetry algebra contains terms of the form

$$\delta \phi_i = \epsilon_i^\alpha \psi_\alpha \, . \tag{4.54}$$

By the Leibniz rule, this precisely induces the supersymmetry transformations of the $\mathcal{O}(n,0)$-multiplet we recorded above. Thus, we can view, for $n \geq 3$, $\mu A^\bullet(n,0)$ as consisting of polynomials of degree n in the linear observables of $\mu A^\bullet(1,0)$. Intuitively, this can be viewed as a remnant of the statement $\mathcal{O}(n,0) = \mathcal{O}(1,0)^{\otimes n}$ after applying the pure spinor superfield formalism. We remark that a special case of this is already visible in the action for supersymmetric Yang–Mills theory coupled to hypermultiplets studied in [Ced18b]. There, an action is written that reproduces the minimal coupling of the gauge sector to matter; the relevant term is cubic, containing two hypermultiplets and one gauge field. From our perspective, this makes use of the identification of the $\mathcal{O}(2,0)$ multiplet both as the dual to the vector multiplet and as governing quadratic functionals on the hypermultiplet.

It is straightforward to compute the holomorphic twist of these multiplets uniformly for $n \geq 3$, and we sketch this briefly here. Following Remark 4.2.2, we expect to find that the twist is of rank one over Dolbeault forms on $\mathbb{C}^3$. Choosing a holomorphic supercharge fixes a complex structure on $\mathbb{R}^6$ and a polarization of the R-symmetry space (or equivalently a choice of Cartan subalgebra of $\mathfrak{sl}_2$.) We decompose

$$S_+ \cong \mathbb{C} \oplus V, \quad S_- \cong V^\vee \oplus \mathbb{C} \tag{4.55}$$

as $\mathfrak{sl}_3$-representations. Here $V = V_3$ is the three-dimensional fundamental representation of $\mathfrak{sl}_3$. Using the non-derivative supersymmetry transformations indicated above, we see that the complex of fields takes the following form.

$$
\begin{tikzcd}
\boxed{\mathbb{C}} & & & & & & & \\
 & \mathbb{C} & \boxed{V} & & & & & \\
\mathbb{C} \arrow[ur] & & & V & \boxed{V^\vee} & & & \\
 & \mathbb{C} & V \arrow[ur] & & & V^\vee & \boxed{\mathbb{C}} & \\
\mathbb{C} \arrow[ur] & & & V & V^\vee \arrow[ur] & & & \mathbb{C} \\
 & \mathbb{C} & V \arrow[ur] & & & V^\vee & \mathbb{C} \arrow[ur] & \\
\mathbb{C} \arrow[ur] & & & V & V^\vee \arrow[ur] & & & \\
 & \mathbb{C} & V \arrow[ur] & & & & & \\
\mathbb{C} \arrow[ur] & & & & & & &
\end{tikzcd}
\tag{4.56}
$$

Here, the vertical axis represents the $\mathfrak{sl}_2$-weight with respect to the fixed Cartan; we have drawn the diagram for $n = 4$, but the pattern is clear. For each n, the surviving fields are precisely isomorphic to Dolbeault forms on $\mathbb{C}^3$, and the remaining (derivative-dependent) components of the holomorphic supercharge generate the $\overline{\partial}$ operator. Due to the twisting homomorphism, the twist naturally resolves holomorphic sections of $K^{n/2}$:

$$\mathcal{O}(n,0)^Q \simeq \Omega^{0,\bullet}(\mathbb{C}^3) \otimes K^{n/2}. \tag{4.57}$$

(This clearly generalizes the results for $n = 0$, 1, and 2, which are well-known.)

4.3.3 The bundles $\mathcal{O}(0, m)$ for $m \geq 0$

Computation of the Betti numbers. The Hilbert series specializes to

$$\operatorname{grdim}(0,m) = \sum_{d=0}^{\infty} (d+1) \frac{(m+d+3)(m+d+2)(m+d+1)}{6}\, t^d\,, \tag{4.58}$$

which can be rewritten as

$$\operatorname{grdim}(0,m) = \frac{1}{6} \frac{\partial}{\partial t} t^{1-m} \frac{\partial^3}{\partial t^3} \frac{t^{m+3}}{1-t}\,. \tag{4.59}$$

Again, we can bring the Hilbert series into a form such that we can read off the Betti numbers of the associated multiplet.

$$\frac{1}{(1-t)^8}\Bigg[(1+\frac{11}{6}m+m^2+\frac{n^3}{6})-(m^3+5m^2+6m)t-(\frac{10}{3}m^3+10m^2-\frac{4}{3}m-8)t^3 \\ +(\frac{5}{2}m^3+5m^2-\frac{9}{2}m-3)t^4-(m^3+m^2-2m)t^5+(\frac{m^3}{6}-\frac{m}{6})t^6\Bigg] \tag{4.60}$$

It is immediate to see that for $m=0$ we recover the result from above. Let us in addition give the Betti tables for some small values of m. For $m=1$, we find

$$\mathrm{grdim}\mu A^{\bullet}(0,1)=\begin{bmatrix}4 & 12 & 12 & 4\end{bmatrix}\,. \tag{4.61}$$

For $n=2$ one obtains

$$\mathrm{grdim}\mu A^{\bullet}(0,2)=\begin{bmatrix}10 & 40 & 65 & 56 & 28 & 8 & 1\end{bmatrix}\,. \tag{4.62}$$

Equivariant decomposition. Solving the equations (4.32) one finds the following representations appearing in $\mu A^{\bullet}(0,m)$.

$$\begin{aligned} W_0 &= [0|m,0,0] \\ W_1 &= -[1|m-1,1,0] \\ W_2 &= [0|m-2,2,0]+[2|m-1,0,1] \\ W_3 &= -[1|m-2,1,1]-[3|m-1,0,0] \\ W_4 &= [0|m-2,0,2]+[2|m-2,1,0] \\ W_5 &= -[1|m-2,0,1] \\ W_6 &= [0|m-2,0,0] \end{aligned} \tag{4.63}$$

Presentation and equivariant resolution. We can describe the module $\Gamma_*(\mathcal{O}(0,1))$ explicitly as the cokernel of a map of free R-modules

$$\psi_1 : (\wedge^2 S_+ \otimes \mathbb{C}^2)\otimes R \longrightarrow S_+ \otimes R\,. \tag{4.64}$$

For degree reasons, the map should be linear in λ. It is easy to check that there is, up to non-zero constant prefactors, a unique such map explicitly given by

$$G \mapsto \lambda_i^{\alpha} G^i_{[\alpha\beta]} s^{\beta}\,. \tag{4.65}$$

Here s^β denotes a basis of S_+. The modules $\Gamma_*(\mathcal{O}(0,m))$ are obtained by taking symmetric products. It can be checked explicitly (for example using a computer program such as *Macaulay2* [GS]) that the minimal free resolutions of these modules reproduce the multiplets described above.

4.3.4 A classification result

The results above describe all multiplets for six-dimensional $\mathcal{N}=(1,0)$ supersymmetry which can be obtained from line bundles on the nilpotence variety $\mathbb{P}^1\times\mathbb{P}^3$. Based on the equivalence of categories between multiplets and $C^\bullet(\mathfrak{n})$-modules developed in §3 this can be viewed as a classification result as follows. Given an R/I-module Γ, the derived $\mathfrak{n}$-invariants of the associated multiplet are concentrated in degree zero and we have

$$H^\bullet(\mathfrak{n},\mu A^\bullet(\Gamma))=\Gamma\,. \tag{4.66}$$

Conversely, given a multiplet (E,D,ρ) such that its derived $\mathfrak{n}$-invariants are concentrated in degree zero one can identify

$$\mu A^\bullet(C^\bullet(\mathfrak{n},\mathcal{E}))\simeq(E,D,\rho)\,. \tag{4.67}$$

Therefore we obtain the following theorem.

Theorem 4.3.1. *The above multiplets classify, up to quasi-isomorphism, all multiplets for six-dimensional $\mathcal{N}=(1,0)$ supersymmetry such that $H^\bullet(\mathfrak{n},\mathcal{E})$ is the graded global section module of a single line bundle on the projective nilpotence variety.*

As remarked above (Remark 4.2.2), the interpretation of the input module as the Chevalley–Eilenberg cohomology with coefficients in the multiplet provides an interesting conceptual link to the twists of the multiplet involved. Twisting by a supercharge Q takes invariants of the multiplet with respect to the abelian subalgebra spanned by that supercharge. The cohomology groups $H^\bullet(\mathfrak{n},\mathcal{E})$ define a sheaf on the nilpotence variety which, by the result of [SW21], encodes all the information on the twists of the original multiplet. In fact, one expects that the twist by a square-zero supercharge $Q\in Y$ is determined by the stalk of that sheaf at Q.

In our example, we see that—as the derived invariants of all the multiplets above are line bundles—the stalk at any point is isomorphic to $\mathcal{O}_{\mathcal{Y},x}$. Our nilpotence variety $\mathcal{Y}=\mathbb{P}^1\times\mathbb{P}^3$ only has one stratum corresponding to the holomorphic twist. Correspondingly, as we have seen above, the holomorphic twists of the above multiplets always have rank one over Dolbeault forms on $\mathbb{C}^3$.

This intuition makes many aspects of the physical behavior of the multiplets and their twists manifest. For example, we can take any of the above multiplets and dimensionally reduce to a four-dimensional $\mathcal{N} = 2$ multiplet. In this case, the nilpotence variety is reducible and has three different components, one of which is the image of the six-dimensional $\mathcal{N} = (1, 0)$ nilpotence variety under the dimensional reduction map [ESW21]. The other two components correspond to the Donaldson–Witten twist, which does not descend from a square-zero supercharge in six dimensions. The Chevalley–Eilenberg cohomology with coefficients in the dimensionally reduced multiplets is obtained by pushing forward along the inclusion $Y_{6D} \hookrightarrow Y_{4D}$. Clearly, any supercharge corresponding to a Donaldson–Witten twist is outside of the support of the resulting sheaf, so that the respective stalks are trivial. Following Remark 4.2.2, one thus expects that the Donaldson–Witten twists of all multiplets arising by dimensional reduction are perturbatively trivial. We hope to give a more complete account of extensions of the methods developed in [SW21] to general multiplets in future work.

4.4 Antifield multiplets and duality

4.4.1 General observations

Given any multiplet $\mu A^\bullet(\Gamma)$, one may form the dual (or antifield) multiplet $\mu A^\bullet(\Gamma)^\vee$ by dualizing the underlying vector bundle, the differential and the supersymmetry module structure. Via the pure spinor superfield formalism, the operation of taking the antifield multiplet corresponds, in good cases, to taking the dualizing module of the input module Γ. We already recognized this in §2.4; here we explore this direction further and link it to statements in terms of sheaves on the nilpotence variety.

Given a Cohen–Macaulay module Γ, the multiplet $\mu A^\bullet(\Gamma)$ is described by the minimal free resolution of Γ in R-modules. The antifield multiplet $\mu A^\bullet(\Gamma)$ is described by the dual of that minimal free resolution, which is, by definition, a minimal free resolution of the dualizing module $\mathrm{Ext}^r_R(\Gamma, R)$. Therefore we can identify for Cohen–Macaulay modules Γ,

$$\mu A^\bullet(\Gamma)^\vee = \mu A^\bullet(\mathrm{Ext}^r_R(\Gamma, R)) \,. \tag{4.68}$$

If Γ is not Cohen–Macaulay, this is no longer true. Then the dual of the minimal free resolution of Γ is no longer a resolution of a single module, but in fact a model for the dualizing complex of Γ. Its cohomology is the Ext-algebra $\mathrm{Ext}^\bullet_R(\Gamma, R)$.

The relation to sheaves. Here, we are interested in modules which arise from sheaves on the nilpotence variety $\mathcal{Y}$ via Γ_*. In this case, we can link the above statements on duality to more geometric notions for sheaves on projective schemes.

Therefore, let us consider a Cohen–Macaulay projective scheme $\iota : X \hookrightarrow \mathbb{P}^n$ of codimension r. In this setting the dualizing sheaf of X is a vector bundle, denoted by ω_X°. Explicitly, it can be defined in terms of the ambient projective space as

$$\omega_X^\circ = \mathcal{E}\mathrm{xt}^r_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n}) \,. \tag{4.69}$$

Let us further assume that $\Gamma_*(\mathcal{O}_X)$ is Cohen–Macaulay as an $R = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$-module. Then the following holds.

Proposition 4.4.1. *Let $\mathcal{F} \in \mathsf{Coh}(X)$ be a coherent sheaf on X and $\Gamma_*(\mathcal{F})$ is its associated $R = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$-module. Then there exists a natural isomorphism*

$$\mathrm{Ext}^r_R(\Gamma_*(\mathcal{F}), R) \cong \Gamma_* \, \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_X^\circ)(n+1), \tag{4.70}$$

so that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\Gamma_*"] \arrow[d, "\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* -, \omega_X^\circ)"'] & \Gamma_*(\mathcal{F}) \arrow[d, "\mathrm{Ext}^r_R(-, R)"] \\
\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_X^\circ) \arrow[r, "\Gamma_*"] & \mathrm{Ext}^r_R(\Gamma_*(\mathcal{F}), R).
\end{tikzcd} \tag{4.71}$$

Proof. One has

$$\begin{aligned}
\mathrm{Ext}^r_R(\Gamma_*(\mathcal{F}), R) &\cong \mathrm{Ext}^r_R(\Gamma_*(\mathcal{F})(-n-1) \,,\, R(-n-1)) \\
&\cong \mathrm{Ext}^r_R(\Gamma_*(\mathcal{F})(-n-1) \otimes_R \Gamma_*(\mathcal{O}_X) \,,\, R(-n-1)) \,,
\end{aligned} \tag{4.72}$$

where the second isomorphism follows from the fact that the sheaf $\mathcal{F}$ is supported on X and $\widetilde{\Gamma_*(\mathcal{O}_X)} \cong \mathcal{O}_X$. By derived hom-tensor adjunction [Huy06] one has

$$\begin{aligned}
\mathrm{Ext}^r_R(\Gamma_*(\mathcal{F}), R) &\cong \mathrm{Hom}_R(\Gamma_*(\mathcal{F})(-n-1) \,,\, \mathrm{Ext}^r_R(\Gamma_*(\mathcal{O}_X), R(-n-1))) \\
&= \mathrm{Hom}_R(\Gamma_*(\mathcal{F})(-n-1) \,,\, \mathrm{Ext}^r_R(\Gamma_*(\mathcal{O}_X), \Gamma_*(\omega_{\mathbb{P}^n}))) \,,
\end{aligned} \tag{4.73}$$

where we used that $\Gamma_*(\omega_{\mathbb{P}^n}) = R(-n-1)$ in the second step. Notice that, by assumption $\Gamma_*(\mathcal{O}_X)$ is Cohen–Macaulay as an R-module, hence the only non-zero Ext-module in the derived hom-tensor adjunction is indeed the dualizing module $\mathrm{Ext}^r_R(\Gamma_*(\mathcal{O}_X), R(-n-1))$.

Further, note that Hom_R in (4.73) denotes graded morphisms of all degrees. For morphisms of degree zero, we have the adjunction [Vak; Har77]

$$\mathrm{Hom}_R^{\deg=0}(M, \Gamma_*(\mathcal{H})) = \mathrm{Hom}_{\mathbb{P}^n}(\tilde{M}, \mathcal{H}) \tag{4.74}$$

between the functors $\Gamma_* : \mathsf{QCoh}_{\mathcal{O}_X} \to \mathsf{Mod}_R$ and $\widetilde{(-)} : \mathsf{Mod}_R \to \mathsf{QCoh}_{\mathcal{O}_X}$. Upon using $\widetilde{\Gamma_*(\mathcal{G})} = \mathcal{G}$, this implies that

$$\mathrm{Hom}_R^{\deg=0}(\Gamma_*(\mathcal{G}), \Gamma_*(\mathcal{H})) = \mathrm{Hom}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H})\ . \tag{4.75}$$

Shifting and summing on both sides, one reconstructs the graded morphisms:

$$\begin{aligned}
\mathrm{Hom}_R(\Gamma_*(\mathcal{G}), \Gamma_*(\mathcal{H})) &= \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H}(k)) = \bigoplus_{k\in\mathbb{Z}} \Gamma \circ (\mathcal{H}\mathrm{om}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H}(k))) \\
&= \bigoplus_{k\in\mathbb{Z}} \Gamma \circ (\mathcal{H}\mathrm{om}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H}) \otimes \mathcal{O}_{\mathbb{P}^n}(k)) \\
&= \Gamma_*(\mathcal{H}\mathrm{om}_{\mathbb{P}^n}(\mathcal{G}, \mathcal{H})),
\end{aligned} \tag{4.76}$$

where we have used that $\Gamma \circ \mathcal{H}\mathrm{om}_{\mathbb{P}^n} = \mathrm{Hom}_{\mathbb{P}^n}$, where Γ is the global section functor. Deriving the above functors, one gets a local-to-global spectral sequence which, in our case, yields the isomorphism

$$\mathrm{Ext}^r_R(\Gamma_*(\mathcal{O}_X), \Gamma_*(\omega_{\mathbb{P}^n})) \cong \Gamma_*(\mathcal{E}\mathrm{xt}^r_{\mathbb{P}^n}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^n}))\ . \tag{4.77}$$

Plugging this into (4.73), we finally obtain

$$\begin{aligned}
\mathrm{Ext}^r_R(\Gamma_*(\mathcal{F}), R) &\cong \Gamma_*\left(\mathcal{H}\mathrm{om}_{\mathbb{P}^n}(\mathcal{F}(-n-1), \mathcal{E}\mathrm{xt}^r_{\mathbb{P}^n}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^n}))\right) \\
&\cong \Gamma_*\left(\mathcal{H}\mathrm{om}_{\mathbb{P}^n}(\mathcal{F}, \mathcal{E}\mathrm{xt}^r_{\mathbb{P}^n}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^n}))(n+1)\right) \\
&\cong \Gamma_*\left(\mathcal{H}\mathrm{om}_{\mathbb{P}^n}(\mathcal{F}, \omega_X^\circ)(n+1)\right)\ ,
\end{aligned} \tag{4.78}$$

which concludes the proof. □

This result establishes that the dualizing module of $\Gamma_*(\mathcal{F})$ arises geometrically from the sheaf $\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_*\mathcal{F}, \omega_X^\circ)(n+1)$. The above proposition, together with (4.68), implies the following corollary.

Corollary 4.4.2. *If $\Gamma_*(\mathcal{F})$ is a Cohen–Macaulay R-module, then we have*

$$\mu A^\bullet(\Gamma_*(\mathcal{F}))^\vee \cong \mu A^\bullet(\Gamma_*(\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_*\mathcal{F}, \omega_X^\circ)))\ . \tag{4.79}$$

It is possible to prove whether or not a sheaf $\mathcal{F}$ gives rise to a Cohen–Macaulay module via Γ_* by studying its sheaf cohomology. In particular the following result holds [Kol13].

Proposition 4.4.3. *Let X be a Cohen-Macaulay projective scheme and let $\mathcal{L}$ be an ample line bundle on it. Given a coherent sheaf $\mathcal{F}$ on X, then $\Gamma_*(\mathcal{F})$ is a Cohen-Macaulay R-module if and only if $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes k}) = 0$ for any $0 < i < \dim(X)$ for any $k \in \mathbb{Z}$.*

4.4.2 Duality and line bundles

As a case study for the general results above, let us consider the multiplets $\mu A^\bullet(n, 0)$ for $n \in \mathbb{Z}$. For a start, recall that the field content of the multiplet $\mu A^\bullet(n, 0)$ takes values in the minimal free resolution of the R-module $\Gamma_*(\mathcal{O}_(n, 0))$. Therefore, given the results in the previous section, we can easily read off $l_R(\Gamma_*(\mathcal{O}_(n, 0)))$:

$$l_R(\Gamma_*(\mathcal{O}(n,0))) = \begin{cases} 3 & \text{for } n \in \{-1, 0, 1, 2, 3\} \\ 4 & \text{for } n > 3 \\ 6 & \text{for } n < -2. \end{cases} \tag{4.80}$$

Notice that all the modules $\Gamma_*(\mathcal{O}(n, 0))$ come from line bundles supported on the nilpotence variety $\mathcal{Y} \cong \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, which is of codimension 3 in $\mathbb{P}^7$. From this, we can infer the following lemma.

Lemma 4.4.4. *The R-module $\Gamma_*(\mathcal{O}(n, 0))$ is Cohen–Macaulay if and only if $n \in \{-1, 0, 1, 2, 3\}$.*

Therefore, for n in this range, we have

$$\mu A^\bullet(n, 0)^\vee \cong \mu A^\bullet(\mathrm{Ext}^3_R(\Gamma_*(\mathcal{O}(n, 0)), R)) \, . \tag{4.81}$$

Lemma 4.4.4 can also be proved directly by studying the sheaf cohomology of the line bundles $\mathcal{O}(n, m)$ and using Proposition 4.4.3. Indeed, we can choose $\mathcal{L} = \mathcal{O}(1, 1)$ as an ample line bundle and use the Künneth theorem to verify that the middle cohomologies $H^i(\mathcal{Y}, \mathcal{O}(n + k, k))$ vanish for $i = 1, 2, 3$ and for all k precisely when $n \in \{-1, 0, 1, 2, 3\}$.

Furthermore, since $\mathcal{Y} = \mathbb{P}^1 \times \mathbb{P}^3$, the dualizing sheaf can be described explicitly as the exterior tensor product of the respective dualizing sheaves on the factors. Explicitly,

$$\omega^\circ_{\mathcal{Y}} = \pi_1^* \omega_{\mathbb{P}^1} \otimes \pi_3^* \omega_{\mathbb{P}^3} = \mathcal{O}(-2, -4) \, . \tag{4.82}$$

Using this together with Theorem 4.4.1 gives

$$\begin{aligned} \mathrm{Ext}^3_R(\Gamma_*(\mathcal{O}(n, 0)), R) &= \Gamma_*(\mathcal{O}(n, 0)^\vee \otimes \mathcal{O}(-2, -4)) \\ &= \Gamma_*(\mathcal{O}(-n - 2, -4)) \\ &= \Gamma_*(\mathcal{O}(2 - n, 0))(4) \, . \end{aligned} \tag{4.83}$$

Thus, we obtain

$$\mu A^\bullet(2-n,0)[4] = \mu A^\bullet \left(\mathrm{Ext}^3_R(\Gamma_*(\mathcal{O}(n,0)),R)\right) . \tag{4.84}$$

In the range where $\Gamma_*(\mathcal{O}(n,0))$ is Cohen–Macaulay, this implies

$$\mu A^\bullet(n,0)^\vee \cong \mu A^\bullet(2-n,0)[4] . \tag{4.85}$$

This can be viewed as a remnant of Serre duality for line bundles on the multiplet side.

4.5 Short exact sequences

In this section, we discuss some general conclusions that can be drawn about short exact sequences of vector bundles in the context of the pure spinor superfield formalism, and then move on to study some concrete examples in our six-dimensional setting. The sections that follow will study the multiplets associated to the tangent and normal bundles and their duals, and apply these results in the context of natural short exact sequences in which those vector bundles appear. As such, we are motivated both by abstract considerations—having understood that the pure spinor construction is a functor, it is natural to ask about it not just on single objects, but on diagrams of objects—and, as throughout this thesis, by concrete computational examples.

4.5.1 General observations

Let

$$0 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma'' \longrightarrow 0 \tag{4.86}$$

be a short exact sequence of graded equivariant R/I-modules. Applying $A^\bullet$ is an exact functor; we thus obtain a short exact sequence of strict multiplets

$$0 \longrightarrow A^\bullet(\Gamma') \longrightarrow A^\bullet(\Gamma) \longrightarrow A^\bullet(\Gamma'') \longrightarrow 0. \tag{4.87}$$

Up to perturbative equivalence, this is the end of the story. However, we are often interested in component-field descriptions, and therefore specifically in the minimal multiplets $\mu A^\bullet(\Gamma)$, $\mu A^\bullet(\Gamma')$, and $\mu A^\bullet(\Gamma'')$. To investigate the relationship at this level, we note the following: Each homogeneous degree Γ_k is a finite-dimensional representation of $\mathfrak{sl}_2 \times \mathfrak{sl}_4$; restricting the above short exact sequence to degree k gives a short exact sequence of $\mathfrak{sl}_2 \times \mathfrak{sl}_4$-representations.

$$0 \longrightarrow \Gamma'_k \longrightarrow \Gamma_k \longrightarrow \Gamma''_k \longrightarrow 0 \tag{4.88}$$

Since $\mathfrak{sl}_2 \times \mathfrak{sl}_4$ is semisimple, all finite-dimensional representations are completely decomposable, and the sequence splits for all $k \in \mathbb{Z}$. We thus have

$$\Gamma_k \cong \Gamma'_k \oplus \Gamma''_k \tag{4.89}$$

as $\mathfrak{sl}_2 \times \mathfrak{sl}_4$-representations. This implies for the equivariant Hilbert series,

$$\mathrm{Hilb}(\Gamma) = \mathrm{Hilb}(\Gamma') + \mathrm{Hilb}(\Gamma'') \tag{4.90}$$

and thus for the field content of the respective multiplets

$$\chi(W^\bullet_\Gamma) = \chi(W^\bullet_{\Gamma'}) + \chi(W^\bullet_{\Gamma''}) \, . \tag{4.91}$$

In practical terms this means that the direct sum of $\mu A^\bullet(\Gamma')$ and $\mu A^\bullet(\Gamma'')$ admits a deformation to $\mu A^\bullet(\Gamma)$,

$$\mu A^\bullet(\Gamma) = \big[\mu A^\bullet(\Gamma') \oplus \mu A^\bullet(\Gamma'')\big]^{\mathrm{Deform}} = \big[\mu A^\bullet(\Gamma' \oplus \Gamma'')\big]^{\mathrm{Deform}} . \tag{4.92}$$

Recall that the differential is given by the right action together with the module structure on Γ,

$$\mathcal{D} = \lambda^\alpha R(Q_\alpha) \, . \tag{4.93}$$

Thus, the deformation on the direct sum of the multiplets precisely corresponds to a deformation of the module structure on $\Gamma' \oplus \Gamma''$ such that

$$\big[\Gamma' \oplus \Gamma''\big]^{\mathrm{Deform}} \cong \Gamma \, . \tag{4.94}$$

This deformation of the module structure arises from the class of Γ inside $\mathrm{Ext}^1(\Gamma', \Gamma'')$. Even more explicitly, we notice that Γ' sits inside Γ as a submodule; therefore the deformation of the module structure is characterized by a map

$$R/I \times \Gamma'' \longrightarrow \Gamma' \, . \tag{4.95}$$

We can summarize these findings by the following lemma.

Lemma 4.5.1. *Let $0 \to \Gamma' \to \Gamma \to \Gamma'' \longrightarrow 0$ be a short exact sequence of graded equivariant R/I-modules. Then the deformation of the module structure on $\Gamma' \oplus \Gamma''$ determined by this sequence induces a deformation on the respective multiplets*

$$\mu A^\bullet(\Gamma) \cong \big[\mu A^\bullet(\Gamma') \oplus \mu A^\bullet(\Gamma'')\big]^{\mathrm{Deform}} \, . \tag{4.96}$$

In this chapter we often deal with short exact sequences of equivariant sheaves on $\mathcal{Y}$. Let

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \tag{4.97}$$

be such a sequence. First, we observe that taking the tensor product with a line bundle keeps the sequence exact, thus we obtain short exact sequences

$$0 \longrightarrow \mathcal{F}'(k) \longrightarrow \mathcal{F}(k) \longrightarrow \mathcal{F}''(k) \longrightarrow 0 \tag{4.98}$$

for all $k \in \mathbb{Z}$. Second, a short exact sequence induces a long exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{F}'(k)) \longrightarrow H^0(\mathcal{F}(k)) \longrightarrow H^0(\mathcal{F}''(k)) \xrightarrow{\delta} H^1(\mathcal{F}'(k)) \longrightarrow \dots \tag{4.99}$$

Thus, if the map δ vanishes (for example due to $H^1(\mathcal{F}'(k))$ being zero) for all k, we obtain a short exact sequence on the global sections

$$0 \longrightarrow H^0(\mathcal{F}'(k)) \longrightarrow H^0(\mathcal{F}(k)) \longrightarrow H^0(\mathcal{F}''(k)) \longrightarrow 0\,, \tag{4.100}$$

and therefore a short exact sequence of graded equivariant R/I-modules

$$0 \longrightarrow \Gamma_*(\mathcal{F}') \longrightarrow \Gamma_*(\mathcal{F}) \longrightarrow \Gamma_*(\mathcal{F}'') \longrightarrow 0\,. \tag{4.101}$$

and we find ourselves in the situation described above.

Remark 4.5.2. It is worth recalling that extensions of sheaves are often interpreted as related to interactions or bound states in mathematical physics. Just for example, in topological string theory, B-branes are identified with coherent sheaves on the target space, which is typically a Calabi–Yau threefold. As emphasized in early work on the subject [Sha99; Dou01; Asp04, for example], a nontrivial extension sequence of the form

$$0 \to A \to B \to C \to 0 \tag{4.102}$$

indicates that B should be thought of as a bound state of the branes A and C. (Making this interpretation precise led to the identification of the category of B-branes with the *derived* category of coherent sheaves.)

In our setting, as explained, the extension defines a deformation of the module structure, which in turn deforms the differential on the multiplet. Thinking in the context of the Batalin–Vilkovisky formalism, a deformation of the differential can in turn be thought of as a deformation of the quadratic part of the BV action. As such, the new differentials we consider on component fields can be interpreted, at least schematically, as (quadratic) supersymmetric interactions between the multiplets $\mu A^\bullet(\Gamma')$ and $\mu A^\bullet(\Gamma'')$, such that the deformed multiplet has derived supertranslation invariants Γ.

4.5.2 Excursion: three-dimensional $\mathcal{N} = 1$

Let us illustrate the findings from above in the case of three-dimensional $\mathcal{N} = 1$ supersymmetry. Thus, let $R = \mathbb{C}[\lambda^1, \lambda^2]$ be the polynomial ring in two variables and

$$I = ((\lambda^1)^2, \lambda^1\lambda^2, (\lambda^2)^2) = R_{\geq 2} \,. \tag{4.103}$$

the defining ideal of the nilpotence variety

We are interested in the following short exact sequence

$$0 \longrightarrow S(-1) \longrightarrow R/I \longrightarrow R/R_{\geq 1} \longrightarrow 0 \,, \tag{4.104}$$

where the first map is given by sending a basis s^α to the generators λ^α and the second map is the obvious projection. Note that the module structure on the direct sum $R/R_{\geq 1} \oplus S(-1)$ is trivial; the deformation which makes it isomorphic to R/I is simply given by

$$R/I \times R/R_{\geq 1} \longrightarrow S(-1) \qquad (\lambda^\alpha, 1) \mapsto s^\alpha \,. \tag{4.105}$$

Let us now study the associated multiplets. $\mu A^\bullet(R/I)$ is a free superfield and $\mu A^\bullet(S(-1))$ is a free superfield with values in the spinor representation shifted to cohomological degree 1. Their direct sum is described as follows.

$$\mu A^\bullet(S(-1))^\# \oplus \mu A^\bullet(R/R_{\geq 1})^\# = \begin{bmatrix} \mathbb{C} & S & \mathbb{C} \\ & & \\ S & \Omega^1 \oplus \mathbb{C} & S \end{bmatrix} \tag{4.106}$$

Using the procedure from §3.3.4, we find the following representatives for the component fields

$$\begin{bmatrix} 1 & \theta^\alpha & \theta^1\theta^2 \\ s^\alpha & \theta^\alpha s^\beta & \theta^1\theta^2 s^\alpha \end{bmatrix} \,. \tag{4.107}$$

Deforming the module structure by (4.105), induces a non-trivial differential. From

$$\mathcal{D} = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha} + \lambda^{(\alpha}\theta^{\beta)} \frac{\partial}{\partial x^{(ab)}} \,, \tag{4.108}$$

and using the representatives, it is easy to see that the direct sum of multiplets is deformed to

$$\left[\begin{tikzcd}[column sep=small] \mathbb{C} \arrow[dr, "\mathrm{d}"] & S \arrow[dl] & \mathbb{C} \arrow[dl] \\ S & \Omega^1 \oplus \mathbb{C} & S \end{tikzcd}\right], \tag{4.109}$$

where every arrow directed down and left is an identity morphism. On the other hand, the multiplet associated to R/I is the gauge multiplet.

$$\mu A^\bullet(R/I) = \left[\begin{tikzcd}[column sep=small] \Omega^0 \arrow[dr, "\mathrm{d}"] & & \\ & \Omega^1 & S \end{tikzcd}\right] \tag{4.110}$$

It is immediate to see that the above deformation is quasi-isomorphic to this multiplet.

4.5.3 The Euler sequence for $\mathbb{P}^1$

Let us now discuss a family of short exact sequences in six dimensions. Identifying $\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, the Euler exact sequence for $\mathbb{P}^1$ reads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0\,. \tag{4.111}$$

Note that this is a sequence of equivariant sheaves and that $\mathbb{C}^2$ carries the fundamental representation of $\mathfrak{sl}_2$. Twisting by $\mathcal{O}_{\mathbb{P}^1}(n)$ and pulling back along π_1 we obtain a family of short exact sequences of equivariant sheaves on $\mathcal{Y}$.

$$0 \longrightarrow \mathcal{O}(n,0) \longrightarrow \mathcal{O}(n+1,0) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(n+2,0) \longrightarrow 0\,. \tag{4.112}$$

Let us restrict to the case $n \geq 0$ for the moment. Twisting by $\mathcal{O}_Y(k) = \mathcal{O}(k,k)$ we obtain the sequences

$$0 \longrightarrow \mathcal{O}(n+k,k) \longrightarrow \mathcal{O}(n+k+1,k) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(n+k+2,k) \longrightarrow 0\,. \tag{4.113}$$

The relevant first cohomology group is $H^1(\mathcal{O}(n+k,k))$ which is easily seen to vanish for all $k \in \mathbb{Z}$ by the Künneth theorem. Thus, we obtain for all $n \geq 0$ a short exact sequence of graded equivariant R/I-modules,

$$0 \longrightarrow \Gamma_*(\mathcal{O}(n,0)) \longrightarrow \Gamma_*(\mathcal{O}(n+1,0)) \otimes \mathbb{C}^2 \longrightarrow \Gamma_*(\mathcal{O}(n+2,0)) \longrightarrow 0\,. \tag{4.114}$$

Let us study the associated multiplets.

$n = 0$. Recall that $\mu A^\bullet(0,0)$ is the vector multiplet, $\mu A^\bullet(1,0)$ the hypermultiplet and $\mu A^\bullet(2,0) = \mu A^\bullet(0,0)^\vee$ the antifield multiplet of the vector. Therefore, $\mu A^\bullet(\mathcal{O}(2,0) \otimes \mathbb{C}^2) = \mu A^\bullet(\mathcal{O}(2,0)) \otimes \mathbb{C}^2$ is a doublet of hypermultiplets with values in the fundamental representation of the R-symmetry $\mathfrak{sl}_2$. Let us arrange the direct sum as follows.

$$\mu A^\bullet(0,0)^\# \oplus \mu A^\bullet(2,0)^\# = \begin{bmatrix} \mathbb{C} \oplus \mathbb{C}^3 & S_+ \otimes \mathbb{C}^2 & \Omega^1 & & \\ & & \Omega^1 & S_- \otimes \mathbb{C}^2 & \mathbb{C} \oplus \mathbb{C}^3 \end{bmatrix} \tag{4.115}$$

We can deform it by adding an acyclic differential relating the the two one-forms, the Dirac operator for the fermions, and the Laplacian for the scalar fields.

$$[\mu A^\bullet(0,0) \oplus \mu A^\bullet(2,0)]^{\mathrm{Deform}} = \left[\begin{tikzcd} \mathbb{C} \oplus \mathbb{C}^3 \arrow[drrr, "\star d \star d"'] & S_+ \otimes \mathbb{C}^2 \arrow[dr, "\not{\partial}"] & \Omega^1 \arrow[dl, "\mathrm{id}"] & \\ & \Omega^1 & S_- \otimes \mathbb{C}^2 & \mathbb{C} \oplus \mathbb{C}^3 \end{tikzcd}\right] \tag{4.116}$$

Taking cohomology with respect to the acyclic part of the differential (i.e. integrating out the auxiliary field) and recalling that for $\mathfrak{sl}_2$-representations $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}^3$, we immediately see that we recover the hypermultiplet with values in $\mathbb{C}^2$.

Interestingly there is another BV theory which can be formed out of $\mu A^\bullet(0,0)$ and $\mu A^\bullet(0,2)$. Adding both multiplets with an appropriate shift and deforming the resulting complex one obtains the BV theory of the vector multiplet; we discussed this in §2.4.3. Denoting the vector multiplet by E, these findings can be summarized by stating that the cotangent theory $T^\vee[-1]E$ corresponds to the BV theory describing the vector multiplet, while the construction we presented above corresponds to $(T^\vee[1]E)[-1]$ which is seen to be equivalent to the hypermultiplet. Let us remark that all these considerations are purely perturbative.

$n = 1$. Proceeding analogously, we can define a deformation on the direct sum of $\mu A^\bullet(1,0)$ and $\mu A^\bullet(3,0)$ that renders it quasi-isomorphic to $\mu A^\bullet(2,0) \otimes \mathbb{C}^2$:

$$\begin{gathered} [\mu A^\bullet(1,0) \oplus \mu A^\bullet(3,0)]^{\mathrm{Deform}} = \\ \left[\begin{tikzcd} \mathbb{C}^2 \oplus \mathbb{C}^4 & S_+ \otimes (\mathbb{C} \oplus \mathbb{C}^3) & \wedge^2 S_+ \otimes \mathbb{C}^2 \arrow[dr, "\star d \star"'] & S_- \arrow[dl, "\mathrm{id}"'] \\ & & S_- & \mathbb{C}^2 \end{tikzcd}\right] \\ \simeq \mu A^\bullet(2,0) \otimes \mathbb{C}^2. \end{gathered} \tag{4.117}$$

Here, we used the isomorphisms $V \cong \wedge^2 S_+$ and $S_- \cong \wedge^3 S_+$.

$n = 2$. Similarly, there is a deformation of $\mu A^\bullet(2,0) \oplus \mu A^\bullet(4,0)$ giving $\mu A^\bullet(3,0) \otimes \mathbb{C}^2$.

$$[\mu A^\bullet(2,0) \oplus \mu A^\bullet(4,0)]^{\text{Deform}} = \left[\begin{tikzcd} \mathbb{C}^3 \oplus \mathbb{C}^5 & S_+ \otimes (\mathbb{C}^2 \oplus \mathbb{C}^4) & \wedge^2 S_+ \otimes (\mathbb{C} \oplus \mathbb{C}^3) & \wedge^3 S_+ \otimes \mathbb{C}^2 & \mathbb{C} \arrow[dl, "\mathrm{id}"] \\ & & & \mathbb{C} & \end{tikzcd}\right] \simeq \mu A^\bullet(3,0) \otimes \mathbb{C}^2. \tag{4.118}$$

$n \geq 3$. For $n \geq 3$, we interpreted $\mu A^\bullet(n,0)$ as receiving an n-fold covering map from the hypermultiplet, witnessed by the isomorphism between its observables and the subalgebra of hypermultiplet observables with polynomial degree divisible by n. The short exact sequence gives a relation between $\mu A^\bullet(n+1,0) \otimes \mathbb{C}^2$ and $\mu A^\bullet(n,0) \oplus \mu A^\bullet(n+2,0)$:

$$\mu A^\bullet(n,0) \oplus \mu A^\bullet(n+2,0) = \tag{4.119}$$
$$\left[\; \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+3} \quad (\mathbb{C}^n \oplus \mathbb{C}^{n+2}) \otimes S_+ \quad (\mathbb{C}^{n-1} \oplus \mathbb{C}^{n+1}) \otimes \wedge^2 S_+ \quad (\mathbb{C}^{n-2} \oplus \mathbb{C}^n) \otimes \wedge^3 S_+ \quad \mathbb{C}^{n-3} \oplus \mathbb{C}^{n-1} \;\right]$$
$$\simeq \mu A^\bullet(n+1,0) \otimes \mathbb{C}^2.$$

Here, identifying $\mu A^\bullet(n+1) \otimes \mathbb{C}^2$ just amounts to the decomposition rule $\mathbb{C}^n \otimes \mathbb{C}^2 \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^{n-1}$. In other words, considering pairs of hypermultiplet observables of polynomial degree $(n+1)$ and regarding such pairs as transforming in the fundamental representation of the R-symmetry $\mathfrak{sl}_2$, we can either symmetrize with respect to the R-symmetry index (yielding $\mu A^\bullet(n+2,0)$) or antisymmetrize to land in $\mu A^\bullet(n,0)$. Note, however, that the polynomial degree of the observables involved does *not* change; the multiplet $\mu A^\bullet(n,0)$, rather than its realization via a map from the hypermultiplet, is the fundamental object.

4.6 The normal bundle exact sequence

In the next two sections, we use short exact sequences of geometric bundles on $\mathbb{P}^1 \times \mathbb{P}^3$ to extend our survey of multiplets to higher-rank bundles. We start with the tangent and normal bundles, using the defining short exact sequence that relates them; in the following section, we will consider the dual of this sequence and work out the multiplets involved explicitly.

As recalled above, the tangent bundle of the projectivized nilpotence variety $\mathcal{T}_\mathcal{Y}$, the restriction $\mathcal{T}_{\mathbb{P}^7}|_\mathcal{Y}$ of the tangent bundle of the ambient $\mathbb{P}^7$ to $\mathcal{Y}$, and the normal bundle

$\mathcal{N}_{\mathcal{Y}}$ sit in the normal bundle exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{Y}} \longrightarrow \mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}} \longrightarrow \mathcal{N}_{\mathcal{Y}} \longrightarrow 0\,. \tag{4.120}$$

Since $H^1(\mathcal{T}_{\mathcal{Y}}(k)) = 0$ for all $k \in \mathbb{Z}$, this short exact sequence induces a short exact sequence on global sections. Thus, applying Γ_*, we obtain a short exact sequence of R/I-modules.

$$0 \longrightarrow \Gamma_*(\mathcal{T}_{\mathcal{Y}}) \longrightarrow \Gamma_*(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}) \longrightarrow \Gamma_*(\mathcal{N}_{\mathcal{Y}}) \longrightarrow 0 \tag{4.121}$$

We apply this short exact sequence to study the associated multiplets and their relations to one another. Again, we will find that there is a deformation of $\mu A^\bullet(\mathcal{T}_{\mathcal{Y}}) \oplus \mu A^\bullet(\mathcal{N}_{\mathcal{Y}})$ which is quasi-isomorphic to $\mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}})$. (Here and in the following section, we often will suppress the graded global section functor when we are talking about the associated multiplets, i.e. for a sheaf $\mathcal{F}$, we set $A^\bullet(\mathcal{F}) := A^\bullet(\Gamma_*(\mathcal{F}))$.)

4.6.1 Tangent bundle

Cohomology and Hilbert Series. Recall that, as seen above, the tangent bundle to the nilpotence variety is given by the exterior sum

$$\mathcal{T}_{\mathcal{Y}} = \pi_1^*\mathcal{T}_{\mathbb{P}^1} \oplus \pi_3^*\mathcal{T}_{\mathbb{P}^3} = \mathcal{T}_{\mathbb{P}^1} \boxplus \mathcal{T}_{\mathbb{P}^3}, \tag{4.122}$$

where $\pi^*\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}(2,0)$. Accordingly, the resulting multiplet will be given by a direct sum,

$$\mu A^\bullet(\mathcal{T}_{\mathcal{Y}}) = \mu A^\bullet(2,0) \oplus \mu A^\bullet(\pi_3^*\mathcal{T}_{\mathbb{P}^3}). \tag{4.123}$$

We have already identified $\mu A^\bullet(2,0)$ as the antifield multiplet of the vector multiplet in §4.3.2, so we are left with studying $\mu A^\bullet(\pi_3^*\mathcal{T}_{\mathbb{P}^3})$, which amounts to computing the zeroth cohomology of

$$\pi_3^*\mathcal{T}_{\mathbb{P}^3}(k) = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \pi_3^*\mathcal{T}_{\mathbb{P}^3}(k) = \mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{T}_{\mathbb{P}^3}(k). \tag{4.124}$$

By the Künneth theorem, we have

$$H^0(\pi_3^*\mathcal{T}_{\mathbb{P}^3}(k)) = H^0(\mathcal{O}_{\mathbb{P}^1}(k)) \otimes H^0(\mathcal{T}_{\mathbb{P}^3}(k)). \tag{4.125}$$

which in turn reduces the problem to compute $H^0(\mathcal{T}_{\mathbb{P}^3}(k))$. Twisting the Euler exact sequence (4.21) by $\mathcal{O}_{\mathbb{P}^3}(k)$, we find

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(k+1) \otimes S_- \longrightarrow \mathcal{T}_{\mathbb{P}^3}(k) \longrightarrow 0\,, \tag{4.126}$$

where $S_- \cong \mathbb{C}^4$. The long cohomology sequence, for the relevant cases $k \geq 0$, reduces to the following short exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(k)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(k+1)) \otimes \mathbb{C}^4 \longrightarrow H^0(\mathcal{T}_{\mathbb{P}^3}(k)) \longrightarrow 0, \tag{4.127}$$

since $H^1(\mathcal{O}_{\mathbb{P}^3}(k)) = 0$ for any $k \geq 0$. As a consequence, one has

$$h^0(\mathcal{T}_{\mathbb{P}^3}(k)) = 4h^0(\mathcal{O}_{\mathbb{P}^3}(k+1)) - h^0(\mathcal{O}_{\mathbb{P}^3}(k)) = 4\binom{k+4}{3} - \binom{k+3}{3} = \frac{1}{2}(k+2)(k+3)(k+5). \tag{4.128}$$

The resulting Hilbert series is easily resummed, giving

$$\begin{aligned} \operatorname{grdim}(\pi_3^* \mathcal{T}_{\mathbb{P}^3}) &= \sum_{k=0}^{\infty} \frac{1}{2}(k+1)(k+2)(k+3)(k+5)t^k \\ &= \frac{15 - 48t + 54t^2 - 24t^3 + 3t^4}{(1-t)^8}, \end{aligned} \tag{4.129}$$

such that the Betti numbers of the associated multiplet are

$$\operatorname{grdim} \mu A^\bullet(\mathcal{T}_{\mathbb{P}^3}) = \begin{bmatrix} 15 & 48 & 54 & 24 & 3 \end{bmatrix} . \tag{4.130}$$

Equivariant Decomposition. Since the Euler exact sequence splits, we find in terms of representations of $\mathfrak{sl}_2 \times \mathfrak{sl}_4$

$$H^0(\mathcal{T}_{\mathbb{P}^3}(k)) = [0|k+1,0,0] \otimes S_- - [0|k,0,0] = [0|k+1,0,1] , \tag{4.131}$$

and hence by (4.125)

$$H^0((\pi_3^* \mathcal{T}_{\mathbb{P}^3})(k)) = [k|k+1,0,1] . \tag{4.132}$$

Running our machinery, we obtain the representations

$$\begin{aligned} W_0 &= \quad [0|0,1,0] \\ W_1 &= -[1|0,1,1] - [1|1,0,0] \\ W_2 &= \quad [0|0,1,0] + [2|0,0,2] + [2|0,1,0] \\ W_3 &= -[1|0,0,1] - [3|0,0,1] \\ W_4 &= \quad [2|0,0,0]. \end{aligned} \tag{4.133}$$

Let us summarize the field content.

$$\mu A^\bullet(\pi_3^*\mathcal{T}_{\mathbb{P}^3})^\# = \tag{4.134}$$
$$\Big[\ \Omega^2 \quad S_-\otimes V\otimes\mathbb{C}^2 \quad V\oplus\Omega^3_-\otimes\mathbb{C}^3\oplus V\otimes\mathbb{C}^3 \quad (\mathbb{C}^2\oplus\mathbb{C}^4)\otimes S_- \quad \mathbb{C}^3\ \Big]$$

4.6.2 Restriction of $\mathcal{T}_{\mathbb{P}^7}$ to the nilpotence variety

Cohomology and Hilbert series. Since restriction to a smooth subvariety is an exact functor, it is easy to describe the vector bundle $\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)$ as the quotient bundle sitting in the restriction of the ordinary k-twisted Euler exact sequence of the embedding space $\mathbb{P}^7$ of $\mathcal{Y}$, i.e.

$$0\longrightarrow \mathcal{O}_{\mathcal{Y}}(k,k)\longrightarrow \mathcal{O}_{\mathcal{Y}}(k+1,k+1)\otimes[1|0,0,1]\longrightarrow \mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)\longrightarrow 0\,, \tag{4.135}$$

where we have used that $\mathcal{O}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)\cong\mathcal{O}_{\mathcal{Y}}(k,k)$. Observing that $H^1(\mathcal{O}_{\mathcal{Y}}(k,k))$ vanishes for any $k\geq 0$, we find the short exact sequence in cohomology

$$0\longrightarrow H^0(\mathcal{O}_{\mathcal{Y}}(k,k))\longrightarrow H^0(\mathcal{O}_{\mathcal{Y}}(k+1,k+1))\otimes[1|0,0,1]\longrightarrow H^0(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k))\longrightarrow 0. \tag{4.136}$$

This yields the formula

$$\begin{aligned} h^0(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) &= 8(k+2)\binom{k+4}{3}-(k+1)\binom{k+3}{3}\\ &=\frac{4}{3}(k+2)(k+4)(k+3)(k+2)-\frac{1}{6}(k+1)(k+3)(k+2)(k+1)\end{aligned} \tag{4.137}$$

for the dimensions of the spaces of global sections of $\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)$. Notice that this also accounts for the special case $k=-1$, when $H^0(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(-1))\cong H^0(\mathcal{O}_{\mathcal{Y}})\otimes[1|0,0,1]\cong[1|0,0,1]$. The Hilbert series of $\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}$ is found to be

$$\mathrm{grdim}(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}})=\frac{8-t-48t^2+70t^3-32t^4+3t^5}{t(1-t)^8}. \tag{4.138}$$

Equivariant decomposition. Since (4.136) splits, we find on the level of representations

$$H^0(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k))=[k+1|k+1,1,0,0]\otimes[1|0,0,1]-[k|k,0,0]\,. \tag{4.139}$$

The associated representations appearing in $\mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}})$ are

$$\begin{gathered} W_0=[1|0,0,1],\qquad W_1=-[0|0,0,0],\qquad W_2=-[1|0,1,1]-[1|1,0,0],\\ W_3=[0|0,0,2]+[2|0,0,2]+2[0|0,1,0]+[2|0,1,0],\\ W_4=-2[1|0,0,1]-[3|0,0,1],\qquad W_5=[2|0,0,0].\end{gathered} \tag{4.140}$$

Explicitly, the field content of $\mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}})$ is summarized in the array

$$\mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}})^\# = \left[\begin{array}{cccccc} S_- \otimes \mathbb{C}^2 & \Omega^0 & & & & \\ & S_- \otimes V \otimes \mathbb{C}^2 & \begin{array}{c}\Omega^1 \otimes (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^3) \\ \Omega^3_- \otimes (\mathbb{C} \oplus \mathbb{C}^3)\end{array} & & S_- \otimes (\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^4) & \mathbb{C}^3 \end{array}\right]. \tag{4.141}$$

This multiplet looks like a gravitino multiplet, containing a spin-3/2 (Rarita–Schwinger) field, but no metric or other degree of freedom corresponding to a particle of spin two.

4.6.3 Normal bundle

Cohomology and Hilbert series. Using our results on the cohomology of the bundles $\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)$ and $\mathcal{T}_{\mathcal{Y}}(k)$, it is easy to compute the cohomology of $\mathcal{N}_{\mathcal{Y}}(k)$ by means of the twisted normal bundle exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathcal{Y}}(k) \longrightarrow \mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k) \longrightarrow \mathcal{N}_{\mathcal{Y}}(k) \longrightarrow 0\,. \tag{4.142}$$

Since $H^1(\mathcal{T}_{\mathcal{Y}}(k)) = 0$ for any k, as can be seen upon using and Künneth theorem in combination with the twisted Euler exact sequence to evaluate the first cohomology group of $\mathcal{T}_{\mathbb{P}^3}(k)$, then one finds a short exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{T}_{\mathcal{Y}}(k)) \longrightarrow H^0(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) \longrightarrow H^0(\mathcal{N}_{\mathcal{Y}}(k)) \longrightarrow 0. \tag{4.143}$$

This implies that for $k \geq -1$ one finds the Betti numbers

$$h^0(\mathcal{N}_{\mathcal{Y}}(k)) = h^0(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) - h^0(\mathcal{T}_{\mathcal{Y}}(k))\,. \tag{4.144}$$

Using our previous results for $h^0(\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}(k))$ and $h^0(\mathcal{T}_{\mathcal{Y}}(k))$ we can deduce

$$h^0(\mathcal{N}_{\mathcal{Y}}(k)) = \frac{1}{2}(k+3)^2(k+2)(k+5)\,. \tag{4.145}$$

Finally, resumming the Hilbert series yields

$$\mathrm{grdim}(\mathcal{N}_{\mathcal{Y}}) = \frac{8 - 19t + 8t^2 + 10t^3 - 8t^4 + t^5}{t(1-t)^8}\,. \tag{4.146}$$

Equivariant decomposition. For the equivariant decomposition, we identify

$$H^0(\mathcal{N}_{\mathcal{Y}}(k)) = [k+2|k+1,0,1]\,. \tag{4.147}$$

Using this, we can find the representations appearing in $\mu A^\bullet(\mathcal{N}_\mathcal{Y})$:

$$\begin{gathered} W_0 = [1|0,0,1], \qquad W_1 = -[0|0,0,0] - [0|1,0,1] - [2|0,0,0], \\ W_2 = [1|1,0,0], \qquad W_3 = [0|0,0,2], \qquad W_4 = -[1|0,0,1], \qquad W_5 = [0|0,0,0]. \end{gathered} \tag{4.148}$$

Explicitly, the multiplet takes the following form:

$$\mu A^\bullet(\mathcal{N}_\mathcal{Y})^\# = \left[\begin{array}{cccccc} S_-\otimes\mathbb{C}^2 & \mathbb{C}\oplus\Omega^2\oplus\mathbb{C}^3 & S_+\otimes\mathbb{C}^2 & & & \\ & & & \Omega^3_- & S_-\otimes\mathbb{C}^2 & \mathbb{C} \end{array}\right]. \tag{4.149}$$

4.6.4 Deformation

As in the example of the Euler sequence for $\mathbb{P}^1$, we find that the field contents of the direct sums of the multiplets associated to tangent and normal bundle does not match the field content of $\mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_\mathcal{Y})$, i.e.

$$\mu A^\bullet(\mathcal{T}_\mathcal{Y})^\# \oplus \mu A^\bullet(\mathcal{N}_\mathcal{Y})^\# \neq \mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_\mathcal{Y})^\#. \tag{4.150}$$

This is again related to the fact that the normal exact sequence does not split as a sequence of R/I-modules. However, there is a deformation of the direct sum such that the resulting multiplet recovers the field content of $\mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_\mathcal{Y})$ up to quasi-isomorphism:

$$\mu A^\bullet(\mathcal{T}_{\mathbb{P}^7}|_\mathcal{Y}) \simeq [\mu A^\bullet(\mathcal{T}_\mathcal{Y})^\# \oplus \mu A^\bullet(\mathcal{N}_\mathcal{Y})^\#]^{\mathrm{Deform}} =$$
$$\left[\begin{tikzcd} S_-\otimes\mathbb{C}^2 & \mathbb{C}\oplus\boxed{\Omega^2\oplus\mathbb{C}^3} \arrow[dl, "\mathrm{id}"'] & \boxed{S_+\otimes\mathbb{C}^2} \arrow[dl, "\mathrm{id}"'] & & \\ \boxed{\Omega^2\oplus\mathbb{C}^3} & \begin{matrix}\boxed{S_+\otimes\mathbb{C}^2} \\ S_-\otimes V\otimes\mathbb{C}^2\end{matrix} & \begin{matrix}\Omega^3_-\otimes(\mathbb{C}\oplus\mathbb{C}^3) \\ \Omega^1\otimes(\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}^3)\end{matrix} & S_-\otimes(\mathbb{C}^2\oplus\mathbb{C}^2\oplus\mathbb{C}^4) & \boxed{\mathbb{C}}\oplus\mathbb{C}^3 \arrow[dl, "\mathrm{id}"'] \\ & & & \boxed{\mathbb{C}} & \end{tikzcd}\right]. \tag{4.151}$$

4.7 The conormal bundle exact sequence

The cotangent bundle, the conormal bundle, and the restriction of the cotangent bundle of the ambient $\mathbb{P}^7$ to the nilpotence variety sit in the conormal exact sequence, which is the dual of (4.120):

$$0 \longrightarrow \mathcal{N}_{\mathcal{Y}}^{\vee} \longrightarrow \Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}} \longrightarrow \Omega^1_{\mathcal{Y}} \longrightarrow 0. \tag{4.152}$$

In the same fashion as above, we now study the cohomology of these sheaves and their associated multiplets.

4.7.1 Cotangent bundle

Cohomology and Hilbert series. As explained above, the cotangent bundle of the nilpotence variety $\mathcal{Y}$ is given by the exterior sum

$$\Omega^1_{\mathcal{Y}} = \pi^*\Omega^1_{\mathbb{P}^1} \oplus \pi_3^*\Omega^1_{\mathbb{P}^3} = \Omega^1_{\mathbb{P}^1} \boxplus \Omega^1_{\mathbb{P}^3}, \tag{4.153}$$

where $\Omega^1_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. As a consequence, the associated multiplet is again a direct sum

$$\mu A^\bullet(\Omega^1_{\mathcal{Y}}) = \mu A^\bullet(-2,0) \oplus \mu A^\bullet(\pi_3^*\Omega^1_{\mathbb{P}^3})\,. \tag{4.154}$$

The multiplet $\mu A^\bullet(\mathcal{O}_{\mathcal{Y}}(-2,0))$, arising from the cotangent bundle of $\mathbb{P}^1$, was already described in §4.3.3, therefore we are left with describing $\mu A^\bullet(\pi_3^*\Omega^1_{\mathbb{P}^3})$. To this end, we need to study the zeroth cohomology of

$$\pi_3^*\Omega^1_{\mathbb{P}^3}(k) = \pi^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \pi_3^*\Omega^1_{\mathbb{P}^3}(k) = \mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \Omega^1_{\mathbb{P}^3}(k). \tag{4.155}$$

The Künneth theorem implies that

$$H^0(\pi_3^*\Omega^1_{\mathbb{P}^3}(k)) = H^0(\mathcal{O}_{\mathbb{P}^1}(k)) \otimes H^0(\Omega^1_{\mathbb{P}^3}(k)), \tag{4.156}$$

reducing the problem to compute the dimension of the zeroth cohomology of $\Omega^1_{\mathbb{P}^3}(k)$. This can be obtained by Bott formulas [OSS80] or by explicitly studying the twist of the dual of the Euler exact sequence for $\Omega^1_{\mathbb{P}^3}$,

$$0 \longrightarrow \Omega^1_{\mathbb{P}^3}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(k-1) \otimes \mathbb{C}^4 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(k) \longrightarrow 0. \tag{4.157}$$

In order to obtain a short exact sequence of modules, we have to check that the connection morphism in the corresponding long exact sequence in cohomology vanishes. Clearly, if $k < 0$ then $H^0(\Omega^1_{\mathbb{P}^3}(k)) = 0$. If $k = 0$, then this corresponds to the Hodge number of $\mathbb{P}^3$ and in particular one finds $h^0(\Omega^1_{\mathbb{P}^3}) = 0 = h^{1,0}(\mathbb{P}^3)$. For $k = 1$ it is easy seen that the map

$$\varphi_{k=1} : H^0(\mathcal{O}_{\mathbb{P}^3}) \otimes \mathbb{C}^4 \to H^0(\mathcal{O}_{\mathbb{P}^1}(1)), \tag{4.158}$$

given by $\mathbb{C}^4 \ni (c_0, \ldots, c_3) \mapsto \sum_{i=0}^3 c_i X_i$, for $\{X_0, \ldots, X_1\}$ global sections of $\mathcal{O}_{\mathbb{P}^3}(1)$ is an isomorphism and hence $H^0(\Omega^1_{\mathbb{P}^3}(1)) = 0$. On the other hand in the case $k > 1$ the map $\varphi_{k>1} : H^0(\mathcal{O}_{\mathbb{P}^3}) \otimes \mathbb{C}^4 \to H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ is only surjective so that $H^1(\Omega^1_{\mathbb{P}^3}(k)) = 0$ and $\ker(\varphi_{k>1}) = H^0(\Omega^1_{\mathbb{P}^3})$.

It follows that for $k > 1$ one has

$$h^0(\Omega^1_{\mathbb{P}^3}(k)) = 4\binom{k+2}{k-1} - \binom{k+3}{k} = \frac{1}{2}(k+2)(k+1)(k-1). \tag{4.159}$$

In turn, this implies that

$$h^0(\pi_3^*\Omega^1_{\mathbb{P}^3}(k)) = \frac{1}{2}(k+1)^2(k+2)(k-1), \tag{4.160}$$

and the related Hilbert series gives

$$\mathrm{grdim}(\pi_3^*\Omega^1_{\mathbb{P}^3}) = t^2 \frac{18 - 64t + 89t^2 - 64t^3 + 28t^4 - 8t^5 + t^6}{(1-t)^8}. \tag{4.161}$$

Equivariant decomposition. For the equivariant decomposition, we identify

$$H^0(\Omega^1_{\mathbb{P}^3}(k)) = [0|k-2,1,0]\,, \tag{4.162}$$

and find the following representations in $\mu A^\bullet(\pi_3^*\Omega^1_{\mathbb{P}^3})$.

$$\begin{aligned}
W_0 &= [2|0,1,0] \\
W_1 &= -[1|0,0,1] - [1|1,1,0] - [3|0,0,1] \\
W_2 &= [0|0,0,0] + [0|0,2,0] + [0|1,0,1] + [2|0,0,0] + [2|1,0,1] + [4|0,0,0] \\
W_3 &= -[1|0,1,1] - [1|1,0,0] - [3|1,0,0] \\
W_4 &= [0|0,0,2] + [2|0,1,0] \\
W_5 &= -[1|0,0,1] \\
W_6 &= [0|0,0,0]
\end{aligned} \tag{4.163}$$

The resulting multiplet takes the following form.

$$\mu A^\bullet(\pi_3^*\Omega^1_{\mathbb{P}^3})^\# = \tag{4.164}$$

$$\left[\begin{array}{ccccccc}
\Omega^1 \otimes \mathbb{C}^3 & \begin{matrix} \mathbb{C}^2 \otimes V \otimes S_+ \\ \mathbb{C}^4 \otimes S_- \end{matrix} & \begin{matrix} \mathrm{Sym}^2(V) \oplus \Omega^2 \\ \mathbb{C}^3 \oplus \mathbb{C}^3 \otimes \Omega^2 \oplus \mathbb{C}^5 \end{matrix} & \begin{matrix} \mathbb{C}^2 \otimes S_- \otimes V \\ S_+ \otimes \mathbb{C}^4 \end{matrix} & \begin{matrix} \Omega^3_- \\ \Omega^1 \otimes \mathbb{C}^3 \end{matrix} & \mathbb{C}^2 \otimes S_- & \mathbb{C}
\end{array}\right]$$

4.7.2 Restriction of $\Omega^1_{\mathbb{P}^7}$ to the nilpotence variety

Cohomology and Hilbert series. Dually to the case of $\mathcal{T}_{\mathbb{P}^7}|_{\mathcal{Y}}$, the cohomology of $\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}$ and its twists is studied by restricting the dual of the Euler exact sequence for the ambient space $\mathbb{P}^7$ to Y. This gives

$$0 \longrightarrow \Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k) \longrightarrow \mathcal{O}(k-1,k-1)\otimes\mathbb{C}^8 \longrightarrow \mathcal{O}_{\mathcal{Y}}(k,k) \longrightarrow 0. \tag{4.165}$$

Studying the related long exact cohomology sequence, it is easy to see that if $k \leq 1$ then $H^0(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) = 0$. In the remaining case, when $k > 1$, the space of global sections $H^0(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k))$ is actually non-zero and the long cohomology sequence splits since the polynomial map

$$H^0(\mathcal{O}_{\mathcal{Y}}(k-1,k-1))\otimes\mathbb{C}^8 \xrightarrow{(X_i,Y_j)} H^0(\mathcal{O}_{\mathcal{Y}}(k,k)) \tag{4.166}$$

is surjective, so one gets the short exact sequence

$$0 \to H^0(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) \to H^0(\mathcal{O}_{\mathcal{Y}}(k-1,k-1))\otimes\mathbb{C}^8 \to H^0(\mathcal{O}_{\mathcal{Y}}(k,k)) \to 0. \tag{4.167}$$

This says that

$$\begin{aligned} h^0(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) &= 8k\binom{k+2}{k-1} - (k+1)\binom{k+3}{k} \\ &= \frac{4}{3}k(k+2)(k+1)k - \frac{1}{6}(k+1)(k+3)(k+2)(k+1). \end{aligned} \tag{4.168}$$

The related Hilbert series of $\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}$ can be resummed easily to give

$$\operatorname{grdim}(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}) = t^2\frac{34-112t+137t^2-80t^3+28t^4-8t^5+t^6}{(1-t)^8} \tag{4.169}$$

Equivariant decomposition. In terms of representations, the sequence gives

$$H^0(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) = [k-2|k,0,0] + [k|k-2,1,0] + [k-2|k-2,1,0]\,. \tag{4.170}$$

Using these results, we can deduce the field content of $\mu A^\bullet(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}})$.

$$\begin{aligned}
W_0 &= [0|0,1,0] + [0|2,0,0] + [2|0,1,0] \\
W_1 &= -2[1|0,0,1] - 2[1|1,1,0] - [3|0,0,1] \\
W_2 &= [0|0,0,0] + [0|0,2,0] + [0|1,0,1] + 2[2|0,0,0] + 2[2|1,0,1] + [4|0,0,0] \\
W_3 &= -\ [1|0,1,1] - [1|1,0,0] - 2[3|1,0,0] \\
W_4 &= [0|0,0,2] + [2|0,1,0] \\
W_5 &= -\ [1|0,0,1] \\
W_6 &= [0|0,0,0]
\end{aligned} \tag{4.171}$$

The resulting multiplet takes the following form.

$$\mu A^\bullet(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}})^{\#} = \tag{4.172}$$

$$\left[\begin{matrix}
\begin{matrix}\Omega^1\otimes(\mathbb{C}\oplus\mathbb{C}^3) \\ \mathrm{Sym}^2 S_+\end{matrix} &
\begin{matrix}(\mathbb{C}^2\otimes V\otimes S_-)^{\oplus 2} \\ \mathbb{C}^4\otimes S_-\end{matrix} &
\begin{matrix}\mathrm{Sym}^2(V)\oplus\Omega^2 \\ \Omega^2\otimes\mathbb{C}^3\oplus\mathbb{C}^5 \\ \Omega^2\otimes\mathbb{C}^3\oplus\mathbb{C}^3\end{matrix} &
\begin{matrix}\mathbb{C}^2\otimes V\otimes S_- \\ (S_+\otimes\mathbb{C}^4)^{\oplus 2}\end{matrix} &
\begin{matrix}\mathrm{Sym}^2(S_-) \\ S_-\otimes\mathbb{C}^2\end{matrix} &
S_-\otimes\mathbb{C}^2 & \mathbb{C}
\end{matrix}\right]$$

4.7.3 The conormal bundle and its supergravity multiplet

Cohomology and Hilbert series. Having available the cohomology of the cotangent bundle and the restriction of $\Omega^1_{\mathbb{P}^7}$ to the nilpotence variety $\mathcal{Y}$, one can study the conormal bundle and its related multiplet in a similar fashion as for the normal bundle above, i.e. by considering k-twists of the conormal exact sequence (4.25):

$$0 \longrightarrow \mathcal{N}^\vee_{\mathcal{Y}}(k) \longrightarrow \Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k) \longrightarrow \Omega^1_{\mathcal{Y}}(k) \longrightarrow 0. \tag{4.173}$$

The issue one faces following this approach is that the related long exact cohomology sequence starts with a four-term sequence in the relevant case $k > 1$

$$0 \to H^0(\mathcal{N}^\vee_{\mathcal{Y}}(k)) \to H^0(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}}(k)) \to H^0(\Omega^1_{\mathcal{Y}}(k)) \to H^1(\mathcal{N}^\vee_{\mathcal{Y}}(k)) \to 0, \tag{4.174}$$

and it is not completely trivial to establish the vanishing of the group $H^1(\mathcal{N}^\vee_{\mathcal{Y}}(k))$ for any $k \geq 1$. We refer to the appendix of [Hah+22] for this verification.

Using the above results, one computes

$$h^0(\mathcal{N}^\vee_{\mathcal{Y}}(k)) = 8k\binom{k+2}{3} - 2k\binom{k+3}{3} - (k^2-1)\binom{k+2}{2} = \frac{1}{2}(k+1)(k-1)^2(k+2). \tag{4.175}$$

The related Hilbert series is resummed to give

$$\mathrm{grdim}(\mathcal{N}^\vee_{\mathcal{Y}}) = \sum_{k=2}^{\infty} \frac{1}{2}(k+1)(k-1)^2(k+2)t^k = t^2 \frac{6 - 8t - 17t^2 + 40t^3 - 28t^4 + 8t^5 - t^6}{(1-t)^8} \,. \tag{4.176}$$

Equivariant decomposition. In terms of representations the conormal exact sequence implies

$$H^0(\mathcal{N}^\vee_{\mathcal{Y}}(k)) = [k-2|k-2,1,0] \,. \tag{4.177}$$

We find the following field content for $\mu A^\bullet(\mathcal{N}^\vee_{\mathcal{Y}})$.

$$\begin{gathered} W_0 = [0|0,1,0], \qquad W_1 = -[1|0,0,1], \qquad W_2 = -[0|0,2,0] + [2|0,0,0], \\ W_3 = [1|0,1,1], \qquad W_4 = -[0|0,0,2] - [2|0,1,0], \qquad W_5 = -[1|0,0,1], \qquad W_6 = [0|0,0,0]. \end{gathered} \tag{4.178}$$

In summary, the multiplet takes the form

$$\mu A^\bullet(\mathcal{N}^\vee_{\mathcal{Y}})^\# = \begin{bmatrix} V & S_- \otimes \mathbb{C}^2 & \mathbb{C}^3 & & & \\ & \mathrm{Sym}^2_0(V) & (V \otimes S_-)_{\frac{3}{2}} \otimes \mathbb{C}^2 & V \otimes \mathbb{C}^3 \oplus \Omega^3_- & S_- \otimes \mathbb{C}^2 & \mathbb{C} \,. \end{bmatrix} \tag{4.179}$$

This is precisely the field content of six-dimensional $\mathcal{N} = (1,0)$ supergravity, presented as the "type-II Weyl multiplet" [LTM12].

4.7.4 Deformation

There is again a deformation of $\mu A^\bullet(\Omega^1_{\mathcal{Y}})^\# \oplus \mu A^\bullet(\mathcal{N}^\vee_{\mathcal{Y}})^\#$ such that the result is quasi-isomorphic to $\mu A^\bullet(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}})^\#$:

$$\mu A^\bullet(\Omega^1_{\mathbb{P}^7}|_{\mathcal{Y}})^\# \simeq [\mu A^\bullet(\Omega^1_{\mathcal{Y}})^\# \oplus \mu A^\bullet(\mathcal{N}^\vee_{\mathcal{Y}})^\#]^{\mathrm{Deform}} = \tag{4.180}$$

$$\left[\begin{array}{ccccccc} \begin{matrix} \Omega^1 \otimes \mathbb{C}^3 \\ \mathrm{Sym}^2 S_+ \\ V \end{matrix} & \begin{matrix} \mathbb{C}^2 \otimes V \otimes S_- \\ \mathbb{C}^4 \otimes S_- \\ \mathbb{C}^2 \otimes (V \otimes S_+)_{\frac{3}{2}} \\ S_- \otimes \mathbb{C}^2 \end{matrix} & \begin{matrix} \mathrm{Sym}^2(V) \oplus \Omega^2 \\ \Omega^2 \otimes \mathbb{C}^3 \oplus \mathbb{C}^5 \\ \boxed{\mathrm{Sym}^2_0(V)} \\ \Omega^2 \otimes \mathbb{C}^2 \\ \mathbb{C}^3 \oplus \mathbb{C}^3 \end{matrix} & \begin{matrix} \mathbb{C}^2 \otimes V \otimes S_- \\ (S_+ \otimes \mathbb{C}^4)^{\oplus 2} \\ \boxed{\mathbb{C}^2 \otimes (V \otimes S_-)_{\frac{3}{2}}} \end{matrix} & \begin{matrix} \boxed{\Omega^3_-} \\ \boxed{\Omega^1 \otimes \mathbb{C}^3} \\ \mathrm{Sym}^2(S_-) \\ S_- \otimes \mathbb{C}^2 \end{matrix} & \begin{matrix} \boxed{S_- \otimes \mathbb{C}^2} \\ S_- \otimes \mathbb{C}^2 \end{matrix} & \begin{matrix} \boxed{\mathbb{C}} \\ \mathbb{C} \end{matrix} \\ & & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow \\ & \boxed{\mathrm{Sym}^2_0(V)} & \boxed{\mathbb{C}^2 \otimes (V \otimes S_-)_{\frac{3}{2}}} & \boxed{\Omega^1 \otimes \mathbb{C}^3 \oplus \mathbb{C}^2} & \boxed{S_- \otimes \mathbb{C}^2} & \boxed{\mathbb{C}} & \end{array}\right]$$

Chapter 5

Zwischenspiel: twisting and holography

We have already encountered twists of supersymmetric field theories at various places in this thesis at an ad-hoc level; a proper introduction seems overdue. This chapter is meant to set the stage for the second part of this thesis which treats the eleven-dimensional supergravity theory and its twists. For this purpose, we first give an brief introduction to twisted supersymmetric field theories and then provide a short review of some relevant aspects from the twisted holography program. Among many other things, these works established conjectures on the twists of supergravity theories in ten and eleven-dimensions by using tools from topological string theory. In §6 and §7, we study these conjectures then directly from a target space perspective.

5.1 Twisting supersymmetric field theories

5.1.1 The nilpotence variety as the moduli space of twists

Let G be a super Lie group with super Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \Pi\mathfrak{g}_-$ and consider a field theory $\mathcal{T}$ with symmetry G. Broadly speaking, twisting means taking invariants with respect to the odd abelian subalgebra spanned by a square-zero element $Q \in \mathfrak{g}_-$. Concretely, this is achieved by adding the action of Q to the BRST or BV differential of the theory (we'll discuss what this means in more detail below). Let us denote the twisted theory by $\mathcal{T}^Q$.[1]

[1]In addition to deforming the differential, often a twisting morphism which modifies the action of G_+ on the theory is applied. For the moment this is not essential for our discussion.

Thus, given a G-equivariant field theory $\mathcal{T}$, the nilpotence variety of $\mathfrak{g}$ is the natural moduli space of twists for $\mathcal{T}$. The nilpotence variety decomposes into orbits under the action of the Lie group G_+. As G_+ acts on $\mathcal{T}$ by symmetries, physically inequivalent twists are labeled by the G_+-orbits of $Y_{\mathfrak{g}}$.

Fixing a square-zero element $Q \in Y$, we can deform the super Lie algebra of symmetries itself $\mathfrak{g}$ to a differential super Lie algebra

$$(\mathfrak{g}\,,\ [Q,-])\,. \tag{5.1}$$

Its cohomology $\mathfrak{g}_Q := H^\bullet((\mathfrak{g},[Q,-]))$ is again a super Lie algebra and consists of residual symmetries in the twisted theory. In particular, all elements in $\mathfrak{g}$ which are in the image of $[Q,-]$ act trivially on the twisted theory. The residual symmetry algebra $\mathfrak{g}_Q$ again has a nilpotence variety $Y_{\mathfrak{g}_Q}$ encoding the further twists of the theory.

5.1.2 Twists for the super Poincaré algebra

For the purpose of this thesis, we are interested in the case where $\mathfrak{g}$ is a super Poincaré algebra. Then, the relevant orbit stratification is induced by the Poincaré group and R-symmetry. Since the translations act trivially, this reduces to the Lorentz group and R-symmetry. The nilpotence varieties for super Poincaré algebras and their orbit structures were studied and classified in [ES19b; ESW21]. In a first rough overview the properties of the different twists by elements from the super Poincaré algebra can be understood in terms of the number of translations which act non-trivial on the twisted theory. This is precisely measured by the cohomology group

$$H^2(\mathfrak{g},[Q,-]) = V/\mathrm{Im}([Q,-])\,, \tag{5.2}$$

consisting of those translations which are not in the image of $[Q,-]$. On general grounds (see for example [ES19b]), for non-vanishing Q, the dimension of $H^2((\mathfrak{g},[Q,-]))$ is at most half the dimension of the vector representation

$$\dim H^2(\mathfrak{g},[Q,-]) \leq \frac{1}{2}\dim V. \tag{5.3}$$

Depending on the dimension of this cohomology group, one distinguishes different cases.

— If $H^2(\mathfrak{g},[Q,-]) = 0$, all translations act trivially on the twisted theory. The supercharge Q is called topological.

— If the dimension of V is even and the inequality is saturated, precisely half of the translations act trivially on the twisted theory. Choosing such a supercharge induces a complex structure on V; the supercharge Q is called holomorphic.

— In the general case, when more than half but not all translations act trivially, the supercharge Q is called mixed.

In general, twisting a field theory on $\mathbb{R}^d$ by a supercharge with

$$\dim(H^2(\mathfrak{g}, [Q, -])) = k \tag{5.4}$$

yields a topologic-holomorphic theory formulated on $\mathbb{R}^{d-2k} \times \mathbb{C}^k$. We say that Q has $d - 2k$ topological directions (and $2k$ holomorphic directions).

Further terminology arises from the orbit stratification induced from the action of the Lorentz and R-symmetry groups on the nilpotence varieties. Twists lying in the minimal orbits are called minimal. In even dimensions, these are precisely the holomorphic twists; in odd dimensions, they have precisely one topological direction. Twists in the top strata are called maximal; they correspond to the smooth points of the nilpotence variety. More generally, the stratification of the nilpotence variety tells us which twists can be obtained as further deformations from other twists. The maximal twists are precisely those having no further deformations. (See [ESW21] for more information on this terminology.)

5.1.3 Twisting in the BV formalism

In order to compute twists in practice, one has to specify a model for the theory $\mathcal{T}$ at hand. One natural choice is to work within the BV formalism (where all relevant information about the theory is already encoded in a differential such that the deformation by a square-zero element is straightforward). Recall that this means to model $\mathcal{T}$ by a local cyclic L_∞ algebra $(L, \mu_k, \langle -, - \rangle)$ with pairing of degree -3 thought of as the space of fields. The classical observables are described by the factorization algebra of Chevalley–Eilenberg cochains

$$\mathrm{Obs}(\mathcal{T}) = (\mathrm{C}^\bullet(L)\,,\, Q_{BV} = \mathrm{d}_{CE})\,, \tag{5.5}$$

where the Chevalley–Eilenberg differential is called the BV differential. By assumption, the L_∞ algebra L carries an L_∞ $\mathfrak{g}$-module structure. For each square-zero element $Q \in Y_{\mathfrak{g}}$, this module structure induces a differential

$$\delta_Q : \mathrm{Obs}(\mathcal{T}) \longrightarrow \mathrm{Obs}(\mathcal{T}), \tag{5.6}$$

such that we can consider the deformation[2]

$$(\mathrm{C}^\bullet(L)\,,\, Q_{BV}+\delta_Q) = \mathrm{Obs}(\mathcal{T})^Q, \tag{5.7}$$

constituting the factorization algebra of observables of the twisted theory (for more details see [Cos13a]). In order to describe the twisted theory, one typically moves to a smaller quasi-isomorphic version of $\mathrm{Obs}(\mathcal{T})^Q$ by eliminating acyclic pairs. Further, there is a new local L_∞ algebra L^Q whose Chevalley–Eilenberg cochains are isomorphic to $\mathrm{Obs}(\mathcal{T})^Q$. This L_∞ algebra describes the fields of the twisted theory.

Equivalently, twisting in the BV formalism can be described by means of the BV action. This amounts to deforming,

$$S^Q_{BV}[\Phi] = S_{BV}[\Phi] + \sum_i \int_M \langle \Phi, \rho^{(i)}(Q,\dots,Q)(\Phi)\rangle. \tag{5.8}$$

Physically, this can be interpreted as putting the field theory into a background where the constant ghosts associated with the symmetry transformations of $\mathfrak{g}$ take the value Q (here we take the background values for all other fields as trivial, though more general situations are possible). This point of view is in particular relevant to supergravity theories, where the supersymmetry transformations are gauged. In this case, this point of view becomes the defining feature [CL16] such that a twisted supergravity theory is one placed in such a background. Some fields then decouple from the rest of the theory such that they cease influencing the dynamics. Integrating out these fields corresponds to the elimination of acyclic pairs in $\mathrm{Obs}(\mathcal{T})^Q$.

Remark 5.1.1 (Twists of multiplets). While twisting supersymmetric field theories is ultimately what we are interested in, it is clear that the above description of twists is compatible with the definition of multiplets as introduced in §2. Thus, one can easily consider twists of multiplets which are not equipped with a BV datum (and in fact we briefly did this in §4.3.2).

5.1.4 Aside: Twisting in representation theory

Independent of the field theoretic background, the concept of twisting plays an important role in the representation theory of simple super Lie algebras (see [DS05; Gor+22]). Twisting there comes under the name of the Duflo–Serganova functor

$$DS_Q : \mathsf{Mod}_{\mathfrak{g}} \longrightarrow \mathsf{Mod}_{\mathfrak{g}_Q} \qquad (M,\rho)\mapsto H^\bullet(M,\rho(Q)) \tag{5.9}$$

[2] In addition, one typically performs a regrading in order to guarantee that the deformed differential is of uniform degree. If this is not possible, the twisted theory is only $\mathbb{Z}/2\mathbb{Z}$-graded.

which assigns to every $\mathfrak{g}$-module a corresponding $\mathfrak{g}_Q = H^\bullet(\mathfrak{g}, [Q, -])$-module. The modules $DS_Q(M)$ form a family over the nilpotence variety; the support of this family is a subvariety of $Y_\mathfrak{g}$ and is called the associated variety of M.

Associated to any supersymmetric field theory comes a super Hilbert space of states where the application of such results is natural.

5.1.5 Twisting and curved backgrounds

While the field theories we are interested in exist in any appropriately structured background geometry M, twists—as defined in the previous section—typically do not. In essence, this is due to most backgrounds breaking supersymmetry (see for example [FS11]). For starters, M has to admit covariantly constant spinors for the twisting procedure to make sense globally.

Starting from the opposite direction, one can consider the twist of the theory on a model geometry preserving supersymmetry (in our case $\mathbb{R}^d$ equipped with the standard euclidean metric) and globalize the twisted theory from there. However, since the presence of the twisting supercharge breaks the Lorentz (and R-symmetry) of the untwisted theory to $H^0(\mathfrak{g}, [Q, -])$, this typically come with requirements on the holonomy group of the underlying manifold. This is, in particular unfortunate for topological twists, where the twisted theory is supposed to compute smooth or topological invariants. In some cases this can be addressed by a *twisting morphism.*

Definition 5.1.2. Let $Q \in Y$ be a square-zero supercharge. A twisting morphism for Q is a morphism of Lie groups $f : \mathrm{SO}(d) \longrightarrow G_R$ such that Q is scalar under the new $\mathrm{SO}(d)$-action defined by composing with $\mathrm{id} \times f : \mathrm{SO}(d) \longrightarrow \mathrm{SO}(d) \times G_R$.

In essence, a twisting morphism thus mixes the action of the Lorentz on the fields with the one of the R-symmetry group such that we recover an action of a full copy of the Lorentz group on the fields of the theory which allows globalization to a general manifold M. Whenever twisting morphisms exist, the supercharge is topological; the converse however is not true [ES19b].

In the general case, the twist on $\mathbb{R}^d$ defines a theory on $\mathbb{R}^{d-2k} \times \mathbb{C}^k$ and this theory does not globalize to an arbitrary manifold M, but only to a manifold equipped with an appropriate transverse holomorphic fibration. Somewhat pictorially, we can summarize this approach to the twisting procedure in the following picture.

$$
\begin{tikzcd}
\boxed{\text{Theory on } \mathbb{R}^d} \arrow[r, dashed] \arrow[d, "\text{twist}"] & \boxed{\text{Theory on } M} \arrow[d, dashed] \\
\boxed{\text{Theory on } \mathbb{R}^{d-2k} \times \mathbb{C}^k} \arrow[r, "\text{globalize}"] & \boxed{\text{Theory on } M_{THF}}
\end{tikzcd}
$$

FIGURE 5.1: The twisted theory on a flat background globalizes to a theory on a manifold with a THF structure. It would be interesting to see how this square could be completed by twisting the physical theory in a more general background M.

5.2 A panoramic view on twisted holography

Twists are especially of interest in combination with dualities: twisting on either side of one can both establish fascinating relations between different areas of mathematics as well as provide mathematically rigorous instances of a physical duality principle. To set the stage for the following chapters, we now give a schematic overview on the twisted holography program as initiated by Costello, Li, Gaiotto and others (see for example [Cos07; CL16; CG18; Cos17; Cos16; Gin+22; CP21]).

The top-down approach to holography typically knows three ingredients: the underlying worldsheet string theory, the induced worldvolume gauge theory on a configuration of branes, and a closed string field theory in the backreacted geometry. Holographic dualities establish equivalences between the latter two of those, by viewing them as instances of the first system.

For illustrational purpose, let us briefly sketch the original argument provided by Maldacena [Mal98]. Consider type IIB superstring theory in $\mathbb{R}^{10}$ with a stack of N D3 branes situated along a subspace $\mathbb{R}^4 \subset \mathbb{R}^{10}$. There are two fundamentally different perspectives on these branes

1. Open strings end on D branes. There is a low energy effective theory on the worldvolume of the brane, here given by $\mathcal{N} = 4$ super Yang–Mills theory on $\mathbb{R}^4$.
2. D branes are sources for the fields in the closed string sector and as such deform the background geometry. The low energy effective description is a supergravity theory in the backreacted geometry X^{back}, here type IIB supergravity.

Taking the first perspective, one obtains an effective theory at low energies consisting of three sectors: the worldvolume theory describing the dynamics of open strings, type

IIB supergravity on $\mathbb{R}^{10}$ corresponding to the dynamics of the closed strings, and a third piece describing interactions between the two. In an appropriate limit (the decoupling limit [Mal98]), open and closed string modes decouple such that the third piece vanishes.

On the other hand, in the appropriate limit, the theory describing closed strings in the backreacted geometry, also decomposes into two decoupled sectors: closed strings near the brane and closed strings far away from the brane. Examining X^{back}, one finds that the former is described by type IIB supergravity in $AdS_5 \times S^5$, while the latter is given by type IIB supergravity in $\mathbb{R}^{10}$.

Identifying both perspectives and matching the sectors, one arrives at an equivalence between $\mathcal{N} = 4$ super Yang–Mills theory in $\mathbb{R}^4$ and type IIB supergravity in $AdS_5 \times S^5$, historically the first instance of the AdS/CFT correspondence.

The twisted holography program aims to study such correspondences in the twisted setting. Thus, in order to arrive at a comprehensive understanding of twisted holography, three different kinds of twists are relevant: twists of the supersymmetric worldvolume gauge theories, twists of supergravity theories in backreacted backgrounds, and, finally, the twists of the underlying worldsheet string theory from which the former two are expected to arrive as effective field theories.

From the worldsheet perspective, a twisted version of the theory is most naturally described as a topological string theory. Results by Costello [Cos07] and Lurie [Lur09] show that specifying such a theory is equivalent to fixing a Calabi–Yau A_∞ category, which can be interpreted as a category of branes. (We are working at a very impressionistic level here and ignore many technical details.)

At a qualitative level, it is useful to picture the category of branes as follows. Its objects are the branes itself, while the morphisms are open string configurations connecting branes,

$$\mathrm{Hom}(A, B) = \{\text{Open strings connecting from } A \text{ to } B\}, \tag{5.10}$$

with composition, A_∞ structure, and the Calabi–Yau pairing $\mathrm{tr}_A : \mathrm{Hom}(A, A) \longrightarrow \mathbb{C}$ given by the respective diagrams joining open strings.

Fixing a category of twisted branes $\mathcal{C}$, one can reconstruct both the worldvolume gauge theory on the brane (describing open strings), as well as the closed string sector in the following way [Cos07].

1. Fix an object $C \in \mathcal{C}$. The worldvolume theory is modeled by $\mathrm{RHom}_{\mathcal{C}}(C, C)$
2. The closed string field theory is described by the cyclic cochains of $\mathcal{C}$, $CC^\bullet(\mathcal{C})$.

From the perspective of topological string theory, two distinct twists of the worldsheet model are central, the A and the B twist. The corresponding A and B model can be thought of as topological phases of underlying string theory. Correspondingly, there are categories of A and B branes.

A. $\mathcal{C} = \mathrm{Fuk}(X)$. The category of branes in the A twist is the Fukaya category of target space, where branes are Lagrangian submanifolds.

B. $\mathcal{C} = \mathrm{Coh}(X)$. The category of branes in the B twist is the derived category of coherent sheaves on target space.

Let us review the following classic example.

Example 5.2.1 (Holomorphically twisted type IIB with D3 branes; see [CL16]). Let $\mathcal{C} = \mathrm{Coh}(\mathbb{C}^5)$ and consider a single branes along $\mathbb{C}^2 \subset \mathbb{C}^5$. This brane is represented by the object $\mathcal{O}_{\mathbb{C}^2} \in \mathrm{Ob}(\mathcal{C})$.

We compute the worldvolume theory in two steps. First we resolve $\mathcal{O}_{\mathbb{C}^2}$ in free objects inside $\mathcal{C}$. For this, we use a Koszul resolution giving,

$$K^\bullet = \big(\mathcal{O}_{\mathbb{C}^5} \otimes \mathbb{C}[\varepsilon^1, \varepsilon^2, \varepsilon^3]\,,\; d_K z_i = \varepsilon^i\big)\,, \tag{5.11}$$

Thus, we find

$$\mathrm{RHom}_{\mathcal{C}}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_{\mathbb{C}^2}) = \mathrm{Hom}_{\mathcal{C}}(K^\bullet, \mathcal{O}_{\mathbb{C}^2}) = \mathcal{O}_{\mathbb{C}^2} \otimes \mathbb{C}[\varepsilon^1, \varepsilon^2, \varepsilon^3]. \tag{5.12}$$

In a second step, we resolve in smooth functions on $\mathbb{C}^2$ to get a field theory on the brane. This results in

$$\big(\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \mathbb{C}[\varepsilon^1, \varepsilon^2, \varepsilon^3]\,,\; \bar\partial\big)\,. \tag{5.13}$$

Hence, the worldvolume theory is holomorphic Chern–Simons theory. Slightly more general, considering a stack of N branes represented by $\mathcal{O}_{\mathbb{C}^2}^{\oplus N}$, we obtain holomorphic Chern–Simons theory with gauge algebra $\mathfrak{gl}_N$. This indeed matches with the holomorphic twist of $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions as first computed by Baulieu [Bau11].

On the other hand, using the Hochschild–Kostant–Rosenberg theorem one finds that the closed string sector is described by BCOV theory, modeled by the cochain complex

$$\big(\mathrm{PV}^{\bullet,\bullet}(\mathbb{C}^5)[\![t]\!]\,,\; \bar\partial + t\partial\big)\,, \tag{5.14}$$

where t is a parameter placed in degree two. The backreaction was carried out in [CL16] by including appropriate branes as sources in BCOV theory. They find that there is a non-vanishing five-form flux, such that BCOV theory on $\mathbb{C}^5 \setminus \mathbb{C}^2$ with this background five form flux gives the candidate for holomorphically twisted type IIB supergravity on $AdS_5 \times S^5$.

Crucially, for the purpose of this thesis, this procedure produces conjectural description for twisted supergravity theories starting from a category of twisted branes. In the case of the above example, this yielded the conjectural description of the minimal twist of type IIB supergravity in terms of BCOV theory [CL16].

Pictorially, the different field theories and their twists are related as summarized by the following diagram.

$$\begin{tikzcd}
 & & \boxed{\Omega \text{ background}} \\
\boxed{\mathrm{RHom}_{\mathcal{C}}(C,C)} \arrow[rr, bend left, leftrightarrow, "\text{twisted holography}"] & \boxed{\text{Category of branes } \mathcal{C}} \arrow[l] \arrow[r] \arrow[d, shift left] & \boxed{\begin{matrix} CC^\bullet(\mathcal{C}) \\ + \text{ backreaction} \end{matrix}} \arrow[u, dashed] \\
 & \boxed{2d \text{ TCFT}} \arrow[u, shift left] & \\
\boxed{\text{Worldvolume theory}} \arrow[uu, "\text{twist}"] \arrow[rr, bend right, leftrightarrow, "\text{holographic duality}"'] & \boxed{\text{WS string theory}} \arrow[l] \arrow[r] \arrow[u, "\text{twist}"] & \boxed{\text{Supergravity in } X^{\text{back}}} \arrow[uu, "\text{twist}"']
\end{tikzcd}$$

FIGURE 5.2: A schematic overview of the twisted holography program

The bottom row represents the physical (i.e. untwistd) holographic duality arising from worldsheet string theory. The top row sketches the procedure employed by Costello and Li to arrive at conjectural descriptions for the twisted worldvolume gauge theories and supergravity theories. However, by definition, these twisted theories arise as twists from the full physical worldvolume gauge and supergravity theories respectively. In order to provide proofs for these conjectures and to give a more complete understanding of Figure 5.2 providing direct calculations of the twists of supergravity theories in target space is indispensable (in the diagram this corresponds to starting in the bottom right corner and following the arrow upwards).

In addition, there are cases of interest where a worldsheet perspective based on topological string theory is not readily available. This is in particular the case for eleven-dimensional supergravity which is the low energy limit of M-theory. (Still, using dualities and results from ten-dimensions the maximal twist of eleven-dimensional supergravity was conjectured to be Poisson–Chern–Simons theory by Costello [Cos; Cos16] and tested [RY19].) In the following two chapters, we aim to understand twisted supergravity from a target space perspective.

Chapter 6

Maximally twisted eleven-dimensional supergravity

6.1 Introduction

Eleven-dimensional supergravity [CJS78] is the low energy limit of M-theory, a conjectural theory that is believed to unify type I, II, and heterotic superstring theories [Wit95]. It realizes the maximal dimension that has a supersymmetric representation with particles of spin at most two [Nah78], and the action of eleven-dimensional supergravity is expected to be unique [CJS78]. M-theory compactifications on manifolds with G_2 holonomy result in four-dimensional field theories with minimal supersymmetry and have been intensely studied in relation to non-perturbative string dualities and phenomenology.

In this chapter, we consider the maximal twist of eleven-dimensional supergravity starting from the component field BV complex of the theory. This twist is of mixed type (topological in seven, and holomorphic in the remaining four directions) such that the twist of supergravity in a flat background defines a theory on $\mathbb{R}^7 \times \mathbb{C}^2$. More generally, we can put the twisted theory on manifolds $M^7 \times M^4$ of $G_2 \times \mathrm{SU}(2)$ holonomy. Costello conjectured the maximal twist to be given by Poisson–Chern–Simons theory [Cos; Cos16; RY19]. As a free BV theory, Poisson–Chern–Simons theory on $\mathbb{R}^7 \times \mathbb{C}^2$ is given by the cochain complex

$$\left(\Omega^\bullet(\mathbb{R}^7) \otimes \Omega^{0,\bullet}(\mathbb{C}^2)\,,\, D^{\mathrm{tw}}\right), \tag{6.1}$$

where the differential D^{tw} decomposes into

$$D^{\mathrm{tw}} = \mathrm{d}_{\mathbb{R}^7} \otimes 1 + 1 \otimes \overline{\partial}_{\mathbb{C}^2}\,. \tag{6.2}$$

Here $\mathrm{d}_{\mathbb{R}^7}$ is the de Rham differential on $\mathbb{R}^7$ and $\overline{\partial}_{\mathbb{C}^2}$ is the Dolbeault differential on $\mathbb{C}^2$. In addition, there are interactions given by the Poisson bracket, for this chapter we restrict our attention to the free theory. We will come back to the interactions in §7. The generalization to $M^7 \times M^4$ is straightforward.

The aim of this chapter is to compute the maximal twist explicitly in component fields and thereby make contact with the formulation of the full supergravity theory as originally envisioned by Cremmer–Julia–Scherk [CJS78]. To this end, we will show how to obtain the fields and BV differential by directly twisting the component fields of eleven-dimensional supergravity in the BV formalism [BV81]. After the twist, the three-form $C^{(3)}$ with its ghost system $C^{(2)}, C^{(1)}, C^{(0)}$, the spin-3/2 Rarita–Schwinger field ψ, and all of their corresponding antifields organize into a differential form $\mathcal{A} \in \Omega^\bullet(\mathbb{R}^7) \otimes \Omega^{0,\bullet}(\mathbb{C}^2)$, as conjectured by Costello. Its components are displayed in Table 6.1.

	$\Omega^0(\mathbb{R}^7)$	$\Omega^1(\mathbb{R}^7)$	$\Omega^2(\mathbb{R}^7)$	$\Omega^3(\mathbb{R}^7)$	$\Omega^4(\mathbb{R}^7)$	$\Omega^5(\mathbb{R}^7)$	$\Omega^6(\mathbb{R}^7)$	$\Omega^7(\mathbb{R}^7)$
$\Omega^{0,0}(\mathbb{C}^2)$	$C^{(0)}$	$C^{(1)}$	$C^{(2)}$	$C^{(3)}$	ψ	$\psi^\vee$	$C^{(3)\vee}$	$C^{(2)\vee}$
$\Omega^{0,1}(\mathbb{C}^2)$	$C^{(1)}$	$C^{(2)}$	$C^{(3)}$	ψ	$\psi^\vee$	$C^{(3)\vee}$	$C^{(2)\vee}$	$C^{(1)\vee}$
$\Omega^{0,2}(\mathbb{C}^2)$	$C^{(2)}$	$C^{(3)}$	ψ	$\psi^\vee$	$C^{(3)\vee}$	$C^{(2)\vee}$	$C^{(1)\vee}$	$C^{(0)\vee}$

TABLE 6.1: Fields in maximally twisted supergravity

We will derive the conjectured form of the twisted fields and differential starting from the manifestly covariant formulation of eleven-dimensional supergravity [Ber02; Ced10c; Ced10a; BG18] in the pure spinor superfield formalism [Ber00; Ber05; Ced14]. We use this formalism to obtain the BV complex of the three-form multiplet in eleven dimensional supergravity, including the full action of the supersymmetry algebra on the component fields. These results are then used to carry out the actual twist on the level of component fields. This gives an explicit understanding of the fields in the twisted theory as well as the formation of trivial pairs in terms of the fields of the untwisted supergravity multiplet.

The traditional approach to eleven-dimensional supergravity in superspace [BH80; CF80; Ced+00a; Ced+00b; Ced+05] starts with the supervielbein and imposes conventional constraints [GSW80; GS80] on torsions and curvatures. We will make some speculative remarks about the twist of the supervielbein at the end. A partially off-shell formulation of eleven-dimensional supergravity adapted to manifolds of $G_2 \times \mathrm{SU}(2)$ holonomy is given in [Bec+17; Bec+18; Bec+21] and is closely related to the twisted theory.

We will work in Euclidean signature. We hope to return to the twist of the higher order terms and the formulation in Lorentzian signature in subsequent work.

Organization. The rest of this chapter is structured as follows. In §6.2, we describe the types of available twists in eleven-dimensional supergravity in general and the $G_2 \times \mathrm{SU}(2)$ invariant maximal twist in particular. In §6.3 we briefly review how the eleven-dimensional supergravity arises in the pure spinor superfield formalism. We introduce the BV complex and describe the action of supersymmetry on its component fields. Finally, in §6.4 we describe the decomposition of the fields and supersymmetry transformations with respect to $G_2 \times \mathrm{SU}(2)$. We then use the decomposition to determine the fields surviving the partial topological twist and the resulting action of the modified BV differential. We conclude with some thoughts on further directions in §6.5.

6.2 The two twists of eleven-dimensional supergravity

Eleven-dimensional supergravity can be twisted in two distinct ways that correspond to the two orbits of the nilpotence variety [ESW21]. Let us quickly review the relevant structure of the nilpotence variety.

Recall that, in any dimension, the Dirac spinor representation S is obtained from a maximal isotropic subspace $L \subset V$ of the vector representation V by setting

$$S = \wedge^{\bullet} L^{\vee} \, . \tag{6.3}$$

S forms a Clifford module for $Cl(V)$ and thus in particular a representation of $\mathfrak{so}(V)$. In the case where $d = \dim(V)$ is odd, this representation is irreducible. As we are interested in eleven-dimensional supergravity, we restrict to this case for the moment.

For $Q \in S$, the annihilator with respect to Clifford multiplication

$$\mathrm{Ann}(Q) = \{v \in V | v \cdot Q = 0\} \tag{6.4}$$

gives an isotropic subspace $\mathrm{Ann}(Q) \subset V$. Q is called a *Cartan pure spinor* if $\mathrm{Ann}(Q)$ is maximal isotropic. Every Cartan pure spinor is square zero, the converse, however, is in general not true. More generally, one can define the varieties

$$\mathrm{PS}_k = \{Q \in S \mid \dim(L) - \dim(\mathrm{Ann}(Q)) \leq k\} \, , \tag{6.5}$$

which define a filtration

$$\mathrm{PS}_0 \subseteq \mathrm{PS}_1 \subseteq \dots \mathrm{PS}_n = S \, . \tag{6.6}$$

Let V_{11} denote the vector representation of Spin(11) and S_{11} the Dirac spinor representation. The dimension of S is 32 and its symmetric square decomposes as

$$\mathrm{Sym}^2(S) \cong V \oplus \wedge^2 V \oplus \wedge^5 V. \tag{6.7}$$

The $\mathcal{N} = 1$ super Poincare algebra in eleven dimensions takes the form,

$$\mathfrak{g} = \mathfrak{so}(V) \oplus S(-1) \oplus V(-2), \tag{6.8}$$

where the bracket of two degree one elements is given by the projection onto the vector representation in the decomposition (6.7).

In coordinates, the nilpotence variety can be explicitly described by the eleven equations

$$\lambda^\alpha \Gamma^\mu_{\alpha\beta} \lambda^\beta = 0\,. \tag{6.9}$$

This variety is closely related but not identical with the variety of Cartan pure spinors; in fact, one finds $Y = \mathrm{PS}_3$ [ES19a]. The variety of Cartan pure spinors sits inside Y as a subvariety $\mathrm{PS}_0 \subset \mathrm{PS}_3 = Y$. It is the singular locus of Y and can be explicitly described by imposing the additional equations

$$\lambda \Gamma^{\mu\nu} \lambda = 0\,. \tag{6.10}$$

In addition, PS_0 is also the minimal orbit of the even part of the super Poincaré algebra and thereby its points correspond to the minimal twists of the eleven-dimensional supergravity theory. These are topological in one direction and holomorphic in the remaining ten; the stabilizer of such a supercharge is SU(5).

The second orbit consists of points away from the singular locus. There, the stabilizer of a supercharge is $G_2 \times \mathrm{SU}(2)$. This corresponds to the maximal twist of eleven-dimensional supergravity that we will study in this chapter. The twist exists on manifolds with $G_2 \times \mathrm{SU}(2)$ holonomy [Mov11; Cos16; ESW21].

Let us elaborate a little further on the maximal twist. The spinor representation in eleven dimensions decomposes as

$$S_{11} = S_7 \otimes S_4\,. \tag{6.11}$$

The Dirac spinor representation in four dimensions, S_4, decomposes into Weyl spinor representations S_+ and S_-:

$$S_4 = \wedge^\bullet L_4^\vee = \wedge^{\mathrm{even}} L_4^\vee \oplus \wedge^{\mathrm{odd}} L_4^\vee =: S_+ \oplus S_-\,. \tag{6.12}$$

Identifying the group $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$, S_+ and S_- are the fundamental representations of $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$, respectively. Restricting to G_2, the spinor representation S_7 further decomposes as

$$S_7 = 1_{G_2} \oplus V_{G_2} \,, \tag{6.13}$$

where V_{G_2} is the seven-dimensional representation of G_2. Thus, we have the decomposition

$$S_{11} = (1_{G_2} \oplus V_{G_2}) \otimes (\wedge^0 L_4^\vee \oplus \wedge^2 L_4^\vee \oplus S_-) \,. \tag{6.14}$$

As a representation of $G_2 \times \mathrm{SU}(2)_- \times \mathrm{U}(1)_L$, where $\mathrm{U}(1)_L$ is the Cartan subgroup of $\mathrm{SU}(2)_+$ this gives

$$S_{11} = [(00) \oplus (10)] \otimes \left(\mathbf{1}^{-1} \oplus \mathbf{1}^{+1} \oplus \mathbf{2}^{0}\right) \,. \tag{6.15}$$

Here, we introduced Dynkin labels for the G_2-representation. $\mathrm{SU}(2)\times\mathrm{U}(1)$-representations are labeled by the dimension of the $\mathrm{SU}(2)$-representation, with the $\mathrm{U}(1)$-charge as a superscript. To study the maximal twist, we choose a square zero supercharge

$$Q \in 1_{G_2} \otimes \wedge^0 L_4^\vee = (00)\mathbf{1}^{-1} \,. \tag{6.16}$$

Thus, we immediately see that Q is invariant under the action of $G_2 \times \mathrm{SU}(2)_-$ and has $\mathrm{U}(1)_L$-charge -1,.

The normal space to the nilpotence variety is spanned by the supercharges

$$Q_m \in (V_{G_2} \otimes \wedge^2 L_4^\vee), \tag{6.17}$$

$$Q_{\dot\alpha} \in (1_{G_2} \otimes S_-). \tag{6.18}$$

They satisfy the following relations

$$[Q, Q_m] = P_m \tag{6.19}$$

$$[Q, Q_{\dot\alpha}] = P_{-\dot\alpha} \,. \tag{6.20}$$

Here we already used that the vector representation decomposes under $G_2\times\mathrm{SU}(2)\times\mathrm{U}(1)$ as

$$V_{11} = (10) \oplus \mathbf{2}^{-1} \oplus \mathbf{2}^{1} \,. \tag{6.21}$$

Our conventions are that indices $m, n, \ldots$ are indices for the seven-dimensional vector representation, while $\dot\alpha, \dot\beta, \ldots$ correspond to $\mathrm{SU}(2)_-$.

The above relations explicitly show that, as announced earlier, the twisted theory is indeed topological in seven and holomorphic in the remaining four directions.

6.3 Eleven-dimensional supergravity in the pure spinor superfield formalism

The canonical multiplet $A^\bullet(R/I)$ associated to the nilpotence variety of the super Poincaré algebra in eleven dimensions is the eleven-dimensional supergravity multiplet [How91a; Ber02; CNT02]. In this chapter, we are interested in the computation of the maximal twist in component fields. For this purpose, we need the full L_∞ action of the super Poincaré algebra on the component fields which we obtain using pure spinor superfield techniques (in particular Corollary 3.4.8).

6.3.1 Representatives for component fields

In the following, let $V_{11} = V$ and $S_{11} = S$ be the vector and spinor representations of Spin(11) respectively. As before, the component fields take values in the minimal free resolution of R/I in free R-modules

$$\mu A^\bullet(R/I) \cong (L^\bullet \otimes_R \mathbb{C}) \otimes C^\infty(V)\,, \tag{6.22}$$

In our case, the minimal free resolution $L^\bullet$ takes the form

$$\begin{aligned} R\otimes\Big(\mathbb{C} &\xleftarrow{d_1} V \xleftarrow{d_2} \wedge^2V\oplus V \xleftarrow{d_3} \wedge^3V\oplus \mathrm{Sym}^2(V)\oplus S \xleftarrow{d_4} S\otimes V \\ &\xleftarrow{d_5} S\otimes V \xleftarrow{d_6} \wedge^3V\oplus \mathrm{Sym}^2(V)\oplus S \xleftarrow{d_7} \wedge^2V\oplus V \xleftarrow{d_8} \mathbb{C}\Big)\,. \end{aligned} \tag{6.23}$$

The resolution differential was already described in [Ber02]. Let us choose a basis (e_μ) of V and (s_α) of S. We will need the maps $d_1, \dots d_5$. In this basis they take the following form.

$$\begin{array}{llllll}
d_1 & : & V \longrightarrow \mathbb{C} & C^{(1)} & \mapsto & (\lambda\Gamma^\mu\lambda)C^{(1)}_\mu \\[1em]
d_2 & : & \wedge^2V\oplus V \longrightarrow V & v & \mapsto & (\lambda\Gamma^{\mu\nu}\lambda)v_\mu e_\nu \\
 & & & C^{(2)} & \mapsto & (\lambda\Gamma^\mu\lambda)C^{(2)}_{\mu\nu}e^\nu \\[1em]
d_3 & : & \wedge^3V\oplus \mathrm{Sym}^2(V)\oplus S \longrightarrow \wedge^2V\oplus V & C^{(3)} & \mapsto & (\lambda\Gamma^\mu\lambda)C^{(3)}_{\mu\nu\rho}(e^\nu\wedge e^\rho) \\
 & & & g & \mapsto & \left((\lambda\Gamma^\mu\lambda)e_\nu + \eta_{\rho\sigma}(\lambda\Gamma^{\sigma\nu}\lambda)(e^\mu\wedge e^\rho)\right)g_{\mu\nu} \\
 & & & \omega & \mapsto & \left((\lambda\Gamma^\mu)_\alpha e_\mu + \frac{1}{2}(\lambda\Gamma_{\mu\nu})_\alpha(e^\mu\wedge e^\nu)\right)\omega^\alpha
\end{array}$$

$$d_4 \quad : \quad S\otimes V \longrightarrow \wedge^3 V \oplus \mathrm{Sym}^2(V)\oplus S \qquad \psi \quad \mapsto \quad -(\lambda\Gamma^\mu\lambda)\psi^\alpha_\mu s_\alpha + \tfrac{1}{2}(\lambda\Gamma^{\mu\nu})_\alpha(\lambda\Gamma_\mu)^\beta\psi_{\nu\beta}s^\alpha + \tfrac{1}{2}(\lambda\Gamma_\mu)^\alpha\psi_{\nu\alpha}(e^{(\mu}\otimes e^{\nu)}) + \tfrac{1}{4}(\lambda\Gamma_{\nu\rho})^\alpha\psi_{\mu\alpha}e^\mu\wedge e^\nu\wedge e^\rho$$

$$d_5 \quad : \quad S\otimes V\longrightarrow S\otimes V \qquad \psi^\vee \quad \mapsto \quad (\lambda M^{\alpha\beta}_{\mu\nu}\lambda)\psi^{\vee\nu}_\beta v^\mu\otimes s_\alpha. \tag{6.24}$$

We do not specify the tensor $M^{\alpha\beta}_{\mu\nu}$ here, but just remark that it is a rather complicated expression in terms of Γ-matrices. The $\mathcal{D}_0$-cohomology is bigraded by λ and θ. The component fields organize according to degree in λ and θ according to Table 6.2. As usual, the λ-degree is linked to the cohomological grading (or ghost degree). Here, they are related by a shift of three, i.e. the physical fields in ghost degree zero sit in λ-degree three.

λ \ θ	0	1	2	3	4	5	6	7	8	9
0	$C^{(0)}$									
1		$C^{(1)}$								
2			$C^{(2)}, v_\mu$	ω						
3				$C^{(3)}, g_{\mu\nu}$	ψ					
4						$\psi^\vee$	$C^{(3)\vee}, g^\vee_{\mu\nu}$			
5							$\omega^\vee$	$C^{(2)\vee}, v^\vee_\mu$		
6									$C^{(1)\vee}$	
7										$C^{(0)\vee}$

TABLE 6.2: θ and λ degrees for the supergravity three-form BV multiplet

We can run the machinery developed in §2 to find explicit representatives for the component fields. For example we find for the one-form

$$C^{(1)} \xmapsto{d_1} (\lambda\Gamma^\mu\lambda)C^{(1)}_\mu \xmapsto{\mathcal{D}_0^\dagger} (\lambda\Gamma^\mu\theta)C^{(1)}_\mu\,, \tag{6.25}$$

such that the one-form field is represented by $(\lambda\Gamma^\mu\theta)C^{(1)}_\mu$.

Similarly one finds for the two-form

$$C^{(2)} \xmapsto{d_2} (\lambda\Gamma^\mu\lambda)C^{(2)}_{\mu\nu}e^\nu \xmapsto{\mathcal{D}_0^\dagger} (\lambda\Gamma^\mu\theta)C^{(2)}_{\mu\nu}e^\nu \xmapsto{d_1} (\lambda\Gamma^\nu\lambda)(\lambda\Gamma^\mu\theta)C^{(2)}_{\mu\nu} \xmapsto{\mathcal{D}_0^\dagger} (\lambda\Gamma^\nu\theta)(\lambda\Gamma^\mu\theta)C^{(2)}_{\mu\nu}\,, \tag{6.26}$$

such that the two-form is represented by $(\lambda\Gamma^\nu\theta)(\lambda\Gamma^\mu\theta)C^{(2)}_{\mu\nu}$. Likewise, the three-form is represented by $(\lambda\Gamma^\nu\theta)(\lambda\Gamma^\mu\theta)(\lambda\Gamma^\rho\theta)C^{(3)}_{\mu\nu\rho}$.

Let us continue with the vector ghost v

$$v \overset{d_2}{\longmapsto} (\lambda\Gamma^{\mu\nu}\lambda)v_\nu e_\mu \overset{\mathcal{D}_0^\dagger}{\longmapsto} (\lambda\Gamma^{\mu\nu}\theta)v_\nu e_\mu \overset{d_1}{\longmapsto} (\lambda\Gamma_\mu\lambda)(\lambda\Gamma^{\mu\nu}\theta)v_\nu \overset{\mathcal{D}_0^\dagger}{\longmapsto} (\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu\nu}\theta)v_\nu \,. \tag{6.27}$$

Thus, the representative is $(\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu\nu}\theta)v_\nu$. For the graviton we find with a similar calculation $(\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu(\nu}\theta)(\lambda\Gamma^{\rho)}\theta)g_{\rho\nu}$.

Performing this procedure one can find representatives for the gravitino and its ghost. The results are summarized in Table 6.3.

Field	Representative in $\mathcal{D}_0$-cohomology
$C^{(0)}$	$C^{(0)}$
$C^{(1)}$	$(\lambda\Gamma^\mu\theta)C^{(1)}_\mu$
$C^{(2)}$	$(\lambda\Gamma^\mu\theta)(\lambda\Gamma^\nu\theta)C^{(2)}_{\mu\nu}$
v	$(\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu\nu}\theta)v_\nu$
ω	$\left[(\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu\nu}\theta)(\theta\Gamma_\nu)_\alpha + \frac{1}{2}(\lambda\Gamma^\mu\theta)(\lambda\Gamma^\nu\theta)(\theta\Gamma_{\mu\nu})\right]\omega^\alpha$
$C^{(3)}$	$(\lambda\Gamma^\mu\theta)(\lambda\Gamma^\nu\theta)(\lambda\Gamma^\rho\theta)C^{(3)}_{\mu\nu\rho}$
g	$(\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu(\nu}\theta)(\lambda\Gamma^{\rho)}\theta)g_{\rho\nu}$
ψ	$[(\lambda\Gamma^\mu\theta)(\lambda\Gamma^\nu\theta)(\lambda\Gamma^\rho\theta)(\theta\Gamma_{\nu\rho})_\alpha - (\lambda\Gamma^\mu\theta)(\lambda\Gamma^{\nu\rho}\theta)(\lambda\Gamma_\nu\theta)(\theta\Gamma_\rho)_\alpha]\,\psi^\alpha_\mu$

TABLE 6.3: Representatives for the fields in 11D supergravity organized by θ-degree.

6.3.2 The BV differential

The differential D acting on the component fields is obtained by transferring $\mathcal{D}_1$ to the $\mathcal{D}_0$-cohomology. In general, this is done by a homotopy transfer of D_∞-algebras but here we are only interested in the lowest order term that acts on the representatives simply by the usual formula of $\mathcal{D}_1$,

$$\mathcal{D}_1 = (\lambda\Gamma^\mu\theta)\partial_\mu \,. \tag{6.28}$$

This gives part of the differential, that is first order in derivatives. For example, we can act on the $C^{(0)}$ ghost

$$\mathcal{D}_1(C^{(0)}) = (\lambda\Gamma^\mu\theta)\partial_\mu C^{(0)} \,. \tag{6.29}$$

Thus, we see that the differential corresponds to the de Rham differential. This obviously generalizes to $C^{(1)}$ and $C^{(2)}$ such that we see that the ghost system of the three-form indeed corresponds to the usual ghost system of a higher form field. Moving on to the diffeomorphism ghost v_μ for the graviton, we find

$$\mathcal{D}_1((\lambda\Gamma_\mu\theta)(\theta\Gamma^{\mu\nu}\theta)v_\nu) = (\lambda\Gamma_\mu\theta)(\theta\Gamma^{\mu\nu}\theta)(\lambda\Gamma^\rho\theta)\partial_\rho v_\nu \,. \tag{6.30}$$

From our calculations of the representatives, we know that only the part where ρ and ν are symmetrized corresponds to a non-trivial cohomology class. Thus, we find

$$\mathcal{D}_1(v) = (\lambda\Gamma_\mu\theta)(\theta\Gamma^{\mu(\nu}\theta)(\lambda\Gamma^{\rho)}\theta)(\partial_\rho v_\nu + \partial_\nu v_\rho) \,. \tag{6.31}$$

Written dually in terms of operators, we find that the BV operator acts by

$$Q_{BV} g_{\mu\nu} = \partial_\mu v_\nu + \partial_\nu v_\mu = (\mathcal{L}_v \eta)_{\mu\nu} \,, \tag{6.32}$$

which is indeed the expected gauge transformation for the graviton.

A similar story also holds for the gravitino and its ghost. There we find

$$\mathcal{D}_1(\omega) = (\lambda\Gamma^\rho\theta)\left[(\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu\nu}\theta)(\theta\Gamma_\nu)_\alpha + (\lambda\Gamma^\mu\theta)(\lambda\Gamma^\nu\theta)(\theta\Gamma_{\mu\nu})_\alpha\right]\partial_\rho\omega^\alpha \,. \tag{6.33}$$

This gives a gauge transformation

$$Q_{BV}\psi_\mu^\alpha = \partial_\mu\omega^\alpha \,. \tag{6.34}$$

Thus, we see that $\mathcal{D}_1$ encodes the usual gauge transformations, expected for the field content. Furthermore, one expects $\mathcal{D}_1$ to encode the Rarita–Schwinger equation between the gravitino and its antifield. In addition, homotopy transfer is expected to induce a second order differential giving the linearized equations of motions of the graviton and the three-form field.

6.3.3 The action of supersymmetry

Using Corollary 3.4.8 as well as the explicit forms of the representatives, we are able to deduce the L_∞ action of the supersymmetry algebra on the component fields.

The three-form ghost system. We begin with the ghost system of the three-form. From degree reasons, it is obvious that $\rho^{(1)}$ acts trivially on the ghost system for the three-form. Thus, we have

$$\rho^{(1)}(C^{(0)}) = \rho^{(1)}(C^{(1)}) = \rho^{(1)}(C^{(2)}) = 0 \,. \tag{6.35}$$

However, this is corrected by higher order contributions. Examining the resolution differential, we find maps

$$\rho^{(2)}(Q,Q) = \iota_{[Q,Q]} : \Omega^i(M) \longrightarrow \Omega^{i-1}(M)\,, \tag{6.36}$$

for $i = 1,2,3$. Written dually for operators, this gives a supersymmetry transformation rule

$$\delta C^{(i)}_\mu = (\epsilon\Gamma_\mu\epsilon)C^{(i)}\,. \tag{6.37}$$

However, these transformations will not cancel any components in the twist since there the relevant supercharge satisfies $[Q,Q]=0$ and thus the above maps all vanish.

The diffeomorphism ghost. The only non-derivative transformation for the diffeomorphism ghost appears in $\rho^{(2)}$. It takes the form

$$\begin{aligned}\rho^{(2)}(Q,Q)(v) &= \rho^{(2)}(Q,Q)((\lambda\Gamma_\mu\theta)(\theta\Gamma^{\mu\nu}\lambda)v_\nu) \\ &= (\lambda\Gamma_\mu\theta)(\epsilon\Gamma^{\mu\nu}\epsilon)v_\nu\end{aligned} \tag{6.38}$$

and thus gives a transformation rule

$$\delta C^{(1)}_\mu = (\epsilon\Gamma_{\mu\nu}\epsilon)v^\nu. \tag{6.39}$$

In addition, there is a $\rho^{(1)}$-piece involving a derivative that can be seen to give rise to the usual supersymmetry transformation between the diffeomorphism and supertranslation ghost [Ber02]

$$\delta\omega_\alpha = -\frac{1}{2}(\epsilon\Gamma^{\mu\nu})_\alpha\partial_\mu v_\nu\,. \tag{6.40}$$

The gravitino ghost. For the gravitino ghost, we obtain

$$\rho^{(1)}(Q)(\omega) = (\lambda\Gamma_\mu\theta)(\lambda\Gamma^{\mu\nu}\theta)(\epsilon\Gamma_\nu\omega) + \frac{1}{2}(\lambda\Gamma_\mu\theta)(\lambda\Gamma_\nu\theta)(\epsilon\Gamma^{\mu\nu}\omega)\,. \tag{6.41}$$

This gives two supersymmetry transformations

$$\begin{aligned}\delta v_\mu &= \epsilon\Gamma_\mu\omega \\ \delta C^{(2)}_{\mu\nu} &= \frac{1}{2}\epsilon\Gamma_{\mu\nu}\omega\,.\end{aligned} \tag{6.42}$$

In this way, one obtains the full higher order corrections to the supersymmetry transformations and encode them in the differential δ. We summarize the full non-derivative supersymmetry transformations in Table 6.4. These results first appeared in [Ber02]. In

Operator ϕ	Transformation rule $\delta\phi$
$C^{(0)}$	$\delta C^{(0)} = (\epsilon\Gamma^{\mu}\epsilon)C^{(1)}_{\mu}$
$C^{(1)}$	$\delta C^{(1)}_{\mu} = (\epsilon\Gamma^{\nu}\epsilon)C^{(2)}_{\mu\nu} + (\epsilon\Gamma_{\mu\nu}\epsilon)v^{\nu}$
$C^{(2)}$	$\delta C^{(2)}_{\mu\nu} = \frac{1}{2}\epsilon\Gamma_{\mu\nu}\omega + (\epsilon\Gamma^{\rho}\epsilon)C^{(3)}_{\mu\nu\rho} + (\epsilon\Gamma_{[\mu\rho}\epsilon)g^{\rho}{}_{\nu]}$
v	$\delta v_{\mu} = \epsilon\Gamma_{\mu}\omega + (\epsilon\Gamma^{\nu}\epsilon)g_{\mu\nu}$
ω	$\delta\omega_{\alpha} = (\epsilon\Gamma^{\mu}\epsilon)\psi_{\alpha\mu} + \frac{1}{2}(\epsilon\Gamma^{\mu\nu})_{\alpha}(\epsilon\Gamma_{\mu})^{\beta}\psi_{\beta\nu}$
$C^{(3)}$	$\delta C^{(3)}_{\mu\nu\rho} = \frac{1}{4}\epsilon\Gamma_{[\mu\nu}\psi_{\rho]}$
g	$\delta g_{\mu\nu} = \frac{1}{2}\epsilon\Gamma_{(\mu}\psi_{\nu)}$
ψ	$\delta\psi^{\alpha}_{\mu} = (\epsilon M^{\alpha\beta}_{\mu\nu}\epsilon)\psi^{\vee\nu}_{\beta}$

TABLE 6.4: Non-derivative supersymmetry transformations

addition, we list the transformations including derivatives for the gravitino and its ghost in Table 6.5.

Operator ϕ	Transformation rule $\delta\phi$
ω	$\delta\omega_{\alpha} = (\epsilon\Gamma^{\mu\nu})_{\alpha}\partial_{\mu}v_{\nu}$
ψ	$\delta\psi_{\mu} = (\Gamma_{\mu}^{\nu\rho\sigma\tau} - 8\Gamma^{\rho\sigma\tau}\delta^{\nu}_{\mu})G^{(4)}_{\nu\rho\sigma\tau}\epsilon$

TABLE 6.5: Supersymmetry transformations with derivatives

6.4 Twisting the free theory

In this section, we will show that the fields of the twisted theory arrange into a differential form

$$\mathcal{A} \in \Omega^{\bullet}(\mathbb{R}^7) \otimes \Omega^{0,\bullet}(\mathbb{C}^2)\,. \tag{6.43}$$

The strategy to establish this result is clear: we restrict the supersymmetry transformations from Table 6.4 to our $G_2 \times \mathrm{SU}(2)$ invariant supercharge and look for fields that form trivial pairs under δ. In the twisted theory these fields decouple and can be neglected. To find such cancellations we have to decompose the field content as well as the supersymmetry transformations equivariantly under $G_2 \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

As a result, we will see that only certain components of the three-form, the three-form ghost system, the gravitino, and the corresponding antifields play a role in the twisted theory. These fields then arrange into the differential form described above. We will further see that the twisted differential takes the form

$$D^{\mathrm{tw}} = \mathrm{d}_{\mathbb{R}^7} \otimes 1 + 1 \otimes \overline{\partial}_{\mathbb{C}^2} . \tag{6.44}$$

Before we continue, let us briefly remark on the different gradings present in the untwisted and twisted theories. As a BV theory, eleven-dimensional supergravity comes, by definition, with a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading by cohomological (ghost) degree and internal parity. As described earlier, the ghost degree corresponds to the λ-degree up to a shift by three. The maximal twist, viewed as an interacting BV theory, will only be graded by $\mathbb{Z}/2\mathbb{Z}$. Nevertheless, it can be useful to consider $\mathbb{Z}$-gradings on the fields of the twisted theory for the purpose of the calculation (these are then broken by the interaction to $\mathbb{Z}/2\mathbb{Z}$). This is mostly, because the fields of the twisted theory are naturally organized by their form degrees (even though these are not compatible with interactions) and it is instructive to see how these form degrees arise from the fields in the untwisted theory.

To this end, we can consider a $\mathbb{Z}\times\mathbb{Z}$-grading on the fields of the untwisted theory (again, ignoring the interactions) given by the λ-degree d_λ and the $\mathrm{U}(1)_L$-charge $d_{\mathrm{U}(1)_L}$. After twisting, the new BV operator $Q_{BV} + \delta_Q$ breaks the $\mathbb{Z}\times\mathbb{Z}$-grading on the space of fields E to the $\mathbb{Z}$-grading

$$d^Q = d_\lambda - d_{\mathrm{U}(1)_L} , \tag{6.45}$$

in the twisted theory. Note that D^{tw} is not homogenous with respect to this grading since $\overline{\partial}_{\mathbb{C}^2}$ operator carries $\mathrm{U}(1)_L$-charge -1. The new degree of a component of $\mathcal{A}$ is simply its de Rham form degree on $\mathbb{R}^7$. Alternatively, note that the twisted BV differential preserves the total form degree and we can assign a total form degree to the components of $\mathcal{A}$. We observe that for component fields in $\mathcal{A}$ the total form degree agrees with their original θ-degree.

6.4.1 Decomposition of the field content

We now decompose the field content into representations of $G_2 \times \mathrm{SU}(2)_- \times \mathrm{U}(1)_L$. To do this, recall the following sequence of inclusions

$$\mathrm{Spin}(11) \supset \mathrm{Spin}(7) \times \mathrm{SU}(2)_- \times \mathrm{U}(1)_L \supset G_2 \times \mathrm{SU}(2)_- \times \mathrm{U}(1)_L . \tag{6.46}$$

The branching of the relevant representations from $\mathrm{Spin}(11)$ to $\mathrm{Spin}(7)\times\mathrm{SU}(2)_-\times\mathrm{U}(1)_L$ is described by Table 6.6. Here we are using Dynkin labels to identify the $\mathrm{Spin}(11)$ and $\mathrm{Spin}(7)$ representations. We identify $\mathrm{SU}(2) \times \mathrm{U}(1)$-representations by the dimension of the $\mathrm{SU}(2)$-representation and denote the $\mathrm{U}(1)$-charge as a superscript. Recall that the vector representation V has Dynkin label (10000) and its second and third exterior

Spin(11)	$\mathrm{Spin}(7)\times\mathrm{SU}(2)_-\times\mathrm{U}(1)_L$
(00000)	$(000)\mathbf{1}^0$
(10000)	$(000)(\mathbf{2}^{-1}+\mathbf{2}^1)\oplus(100)\mathbf{1}^0$
(00001)	$(001)(\mathbf{1}^{-1}+\mathbf{1}^1+\mathbf{2}^0)$
(01000)	$(000)(\mathbf{1}^{-2}+\mathbf{1}^0+\mathbf{3}^0+\mathbf{1}^2)\oplus(010)\mathbf{1}^0\oplus(100)(\mathbf{2}^{-1}+\mathbf{2}^1)$
(00100)	$(000)(\mathbf{2}^{-1}+\mathbf{2}^1)\oplus(002)\mathbf{1}^0\oplus(010)(\mathbf{2}^{-1}+\mathbf{2}^1)\oplus(100)(\mathbf{1}^{-2}+\mathbf{1}^0+\mathbf{3}^0+\mathbf{1}^2)$
(20000)	$(000)(\mathbf{3}^{-2}+\mathbf{1}^0+\mathbf{3}^0+\mathbf{3}^2)\oplus(100)(\mathbf{2}^{-1}+\mathbf{2}^1)+(200)\mathbf{1}^0$
(10001)	$(001)(\mathbf{2}^{-2}+\mathbf{3}^{-1}+\mathbf{1}^{-1}+(\mathbf{2}^0)^{\oplus 2}+\mathbf{1}^1+\mathbf{3}^1+\mathbf{2}^2)\oplus(101)(\mathbf{1}^{-1}+\mathbf{2}^0+\mathbf{1}^1)$

TABLE 6.6: Branching of $\mathrm{Spin}(11)\to\mathrm{Spin}(7)\times\mathrm{SU}(2)_-\times\mathrm{U}(1)_L$-representations.

powers are labeled by (01000) and (00100). The spinor representation S has Dynkin label (00001). Furthermore, the gravitino representation already decomposes as a Spin(11) representation according to

$$S\otimes V\cong(00001)\oplus(10001)\,. \tag{6.47}$$

Finally, the graviton transforms in the representation

$$\mathrm{Sym}^2 V\cong(20000)\oplus(00000)\,. \tag{6.48}$$

We also need the branching rules for $\mathrm{Spin}(7)\to G_2$, which we collect in Table 6.7.

Spin(7)	G_2
(000)	(00)
(100)	(10)
(001)	$(10)\oplus(00)$
(010)	$(01)\oplus(10)$
(002)	$(00)\oplus(10)\oplus(20)$
(101)	$(01)\oplus(10)\oplus(20)$
(200)	(20)

TABLE 6.7: Branching of $\mathrm{Spin}(7)\to G_2$-representations.

From these branching rules, we can already develop some expectation how the computation of the maximal twist could play out. This works on any product manifold $M^7\times M^4$ of $G_2\times\mathrm{SU}(2)$-holonomy. Clearly, the three-form and its ghosts $C^{(p)}$ split into forms in $\Omega^i(M^7)\otimes\Omega^{j_1,j_2}(M^4)$, where $i+j_1+j_2=p$ is the total form degree. Thus, in the light of the conjecture, we expect all components with non-zero holomorphic form degree ($j_1 \neq 0$) to cancel in the twisted theory.

We now consider the decomposition of the gravitino field ψ_μ^α. It transforms in the product of the Spin(11) vector and spinor representations. We first consider its decomposition under $\mathrm{Spin}(11) \to \mathrm{Spin}(7) \times \mathrm{SU}(2)_-$. We will later see that the only components that survive in the twisted multiplet have index μ transforming in a Spin(7)-vector representation whose components we denote by m.

On a manifold of G_2 holonomy exterior powers of the cotangent bundle decompose into irreducible G_2 representations [Joy07]. This induces the following decomposition on differential forms.

$$\begin{array}{cccccccc}
\Omega^0_1 & \Omega^1_7 & \Omega^2_7 & \Omega^3_1 & \Omega^4_1 & \Omega^5_7 & \Omega^6_7 & \Omega^7_1 \\
 & & \oplus & \oplus & \oplus & \oplus & & \\
 & & \Omega^2_{14} & \Omega^3_7 & \Omega^4_7 & \Omega^5_{14}. & & \\
 & & & \oplus & \oplus & & & \\
 & & & \Omega^3_{27} & \Omega^4_{27} & & &
\end{array} \tag{6.49}$$

Here, we denote the sections of each irreducible piece by Ω^k_l, where the subscript denotes the respective dimension of the G_2-representation.

The spin 1/2 and spin 3/2 fields on M^7 decompose as [CG+18; HS19]

$$\Sigma_{1/2} \cong \Omega^0_1 \oplus \Omega^1_7 \tag{6.50}$$

$$\Sigma_{3/2} \cong \Omega^1_7 \oplus \Omega^2_{14} \oplus \Omega^3_{27}. \tag{6.51}$$

Using the above decomposition and the $\mathrm{Spin}(11) \to \mathrm{Spin}(7) \times \mathrm{SU}(2) \times \mathrm{U}(1)_L$ branchings in Table 6.6, and the isomorphisms

$$\Sigma_{3/2} \oplus \Sigma_{1/2} \cong \left(\Omega^1_7 \oplus \Omega^2_{14} \oplus \Omega^3_{27}\right) \oplus \left(\Omega^0_1 \oplus \Omega^1_7\right) \tag{6.52}$$

$$\cong \Omega^2 \oplus \Omega^3, \tag{6.53}$$

we see that the gravitino, given by a pair of spin 3/2 and spin 1/2 fields on a G_2 holonomy manifold, can be identified with a pair of two- and three-forms on the manifold. We will find that the components of the gravitino that survive the twist are contained in the representation

$$(S_+ \oplus S_-) \otimes (\Sigma_{3/2} \oplus \Sigma_{1/2}) \cong (S_+ \oplus S_-) \otimes (\Omega^2 \oplus \Omega^3)\,. \tag{6.54}$$

However, not all of these components survive. We will find that the surviving components are $\Omega^3 \otimes \wedge^0 L_4^\vee$, $\Omega^3 \otimes S_-$, and $\Omega^2 \otimes \wedge^2 L_4^\vee$. The gravitino has λ-degree 3 in the untwisted theory and the representations $\wedge^0 L_4^\vee, S_-, \wedge^2 L_4^\vee$ have U(1)-charge $-1, 0$, and 1, respectively. Thus, their new twisted degree defined by (6.45) are 4, 3, and 2. The components surviving the twist are therefore in $\Omega^4(M^7) \otimes \Omega^{0,0}(M^4)$, $\Omega^3(M^7) \otimes \Omega^{0,1}(M^4)$, and $\Omega^2(M^7) \otimes \Omega^{0,2}(M^4)$, where we have used the isomorphism $\Omega^3 \cong \Omega^4$ to ensure that the gravitino has its correct twisted degree.

The components of the three-form and its ghosts $C^{(p)}$, $p = 0 \dots 3$ and the gravitino along with their antifields that survive the twist therefore give exactly the right field content to be described by a form

$$\mathcal{A} \in \Omega^\bullet(M^7) \otimes \Omega^{0,\bullet}(M^4). \tag{6.55}$$

6.4.2 Decomposition of the supersymmetry transformations

We now determine the supersymmetry transformations for the twisting supercharge Q. For the moment, we are only interested in the supersymmetry transformations without derivatives since these are the ones responsible for the formation of trivial pairs. The transformations with derivatives will later be used to determine the twisted BV differential. Recall that the spin representation S decomposes as

$$[(00) \oplus (10)] \left(\mathbf{1}^{-1} + \mathbf{1}^{1} + \mathbf{2}^{0}\right). \tag{6.56}$$

This means that we can decompose the parameter ϵ from Table 6.4 into

$$\epsilon \to \left(\epsilon_-, \epsilon_+, \epsilon_{\dot\alpha}, \epsilon_{-m}, \epsilon_{+m}, \epsilon_{m\dot\alpha}\right). \tag{6.57}$$

Here m is an index for the seven-dimensional representation of G_2. To act by Q, we specify $\epsilon_- = 1$ and set all other components to zero.

On general grounds, these transformation take a very simple form. As explained above, the supercharge Q is invariant under $G_2 \times \mathrm{SU}(2)$ and has U(1) charge -1. As a consequence, δ_Q is an $G_2 \times \mathrm{SU}(2)$-equivariant map. By decomposing the field content into irreducible $G_2 \times \mathrm{SU}(2)$-representations, δ_Q splits up as a map between these irreducibles. However, since δ_Q is equivariant, we can apply Schur's lemma and find, first, that there can not be any non-trivial maps between non-isomorphic components and, second, transformations between isomorphic $G_2 \times \mathrm{SU}(2)$-representations are always of the form $\alpha \cdot \mathrm{id}$ for some $\alpha \in \mathbb{C}$. Thus, to check whether there are any trivial pairs, we only have to see if there is a non-vanishing map between isomorphic representations. In addition, δ_Q carries a U(1)-charge that simply equals minus the number of ϵ's appearing in the transformation, which can be used as a further criterion to establish that certain maps vanish.

To check whether or not supersymmetry transformation yields a trivial pair we need to decompose Γ-matrices.

Gamma matrix decomposition. In eleven dimensions the symmetric square of the spin representation decomposes as

$$\mathrm{Sym}^2 S \cong V \oplus \wedge^2 V \oplus \wedge^5 V\,. \tag{6.58}$$

Accordingly, there are maps denoted by Γ^μ, $\Gamma^{\mu\nu}$ and $\Gamma^{\mu_1\cdots\mu_5}$ given by projecting onto the summands in this decomposition. So for example, Γ^μ is given by the composition

$$\begin{tikzcd} \mathrm{Sym}^2(S) \arrow[r, "\cong"] \arrow[dr, "\Gamma^\mu"'] & V \oplus \wedge^2 V \oplus \wedge^5 V \arrow[d] \\ & V \end{tikzcd}\,. \tag{6.59}$$

Recall the spin representation S decomposes under $G_2 \times \mathrm{SU}(2) \times \mathrm{U}(1)$ as

$$S \to \mathbf{1}^{-1} + \mathbf{1}^{1} + \mathbf{2}^{0} + (10)(\mathbf{1}^{-1} + \mathbf{1}^{1} + \mathbf{2}^{0})\,. \tag{6.60}$$

We are interested in $\epsilon_-\Gamma^\mu\epsilon$ and $\epsilon_-\Gamma^{\mu\nu}\epsilon$, where $\epsilon_- \in \mathbf{1}^{-1}$ in the above decomposition and ϵ is arbitrary. This means we are looking at a map $\mathbf{1}^{-1} \otimes S \to V$ or $\mathbf{1}^{-1} \otimes S \to \wedge^2 V$, respectively. The representations V and $\wedge^2 V$ decompose as

$$\begin{aligned} V &\to \mathbf{2}^{1} \oplus \mathbf{2}^{-1} \oplus (10) \\ \wedge^2 V &\to (\mathbf{1}^{-2} \oplus \mathbf{1}^{0} \oplus \mathbf{3}^{0} \oplus \mathbf{1}^{2}) \oplus (10)(\mathbf{2}^{-1} \oplus \mathbf{2}^{1}) \oplus (10) \oplus (01). \end{aligned} \tag{6.61}$$

We can now compare this with the decomposition of $\mathbf{1}^{-1} \otimes S$ and read off the following results for Γ^μ:

$$\begin{aligned} \epsilon_-\Gamma^\mu\epsilon_- &= 0 \\ \epsilon_-\Gamma^\mu\epsilon_+ &= 0 \\ \epsilon_-\Gamma^\mu\epsilon_{\dot\alpha} &\in \mathbf{2}^{-1} \\ \epsilon_-\Gamma^\mu\epsilon_{+m} &\in (10) \\ \epsilon_-\Gamma^\mu\epsilon_{-m} &= 0 \\ \epsilon_-\Gamma^\mu\epsilon_{m\dot\alpha} &= 0. \end{aligned} \tag{6.62}$$

For $\Gamma^{\mu\nu}$ we find:

$$\begin{array}{rcl}
\epsilon_-\Gamma^{\mu\nu}\epsilon_- & \in & \mathbf{1}^{-2} \\
\epsilon_-\Gamma^{\mu\nu}\epsilon_+ & \in & \mathbf{1}^{0} \\
\epsilon_-\Gamma^{\mu\nu}\epsilon_{\dot\alpha} & = & 0 \\
\epsilon_-\Gamma^{\mu\nu}\epsilon_{+m} & = & 0 \\
\epsilon_-\Gamma^{\mu\nu}\epsilon_{-m} & = & 0 \\
\epsilon_-\Gamma^{\mu\nu}\epsilon_{m\dot\alpha} & \in & (10)\,\mathbf{2}^{-1} .
\end{array} \tag{6.63}$$

For example, we immediately see that all terms of the form $\epsilon_-\Gamma^\mu\epsilon_-$ vanish and hence do not affect the twist; this is of course nothing else but the condition for Q to be square zero.

Let us start examining the supersymmetry transformations. Note that we are ignoring any potential non-zero scalar coefficients α as we are only interested in the formation of trivial pairs.

Furthermore, we are only considering cancellations between the fields of the multiplet as well as between the gravitino and its antifield. Since the action of supersymmetry respects the pairing on the BV complex, the same cancellations also occur for the respective antifields.

The zero-form $C^{(0)}$. For the zero-form ghost, we obviously have $\delta_Q C^{(0)} = 0$. Since there is no supersymmetry transformation generating $C^{(0)}$, it descends to a field in the twisted theory.

The diffeomorphism ghost v. Next we consider the diffeomorphism ghost v_μ. It decomposes into components

$$v_\mu \to (v_m, v_{+\dot\alpha}, v_{-\dot\alpha}) . \tag{6.64}$$

We have a supersymmetry transformation of the form

$$\delta_Q v_\mu = \epsilon\Gamma_\mu\omega . \tag{6.65}$$

The gravitino ghost ω lives in the spinor representation and hence decomposes according to (6.57). From the Γ-matrix decomposition in (6.62), we know that $\epsilon_-\Gamma_\mu\omega$ is only non-vanishing for the components $\omega_{\dot\alpha}$ and ω_{+m} of ω. Thus, we get up to potential non-zero prefactors

$$\delta_Q v_m = \omega_{+m} \tag{6.66}$$

and

$$\delta_Q v_{-\dot\alpha} = \omega_{\dot\alpha} \,. \tag{6.67}$$

Finally we have,

$$\delta_Q v_{+\dot\alpha} = 0 \,. \tag{6.68}$$

Thus, we already find that some components of the diffeomorphism ghost v form trivial pairs with parts of the gravitino ghost. In addition, it is interesting to note that $\delta_Q v_{+\dot\alpha} = 0$. In light of the conjecture, we expect that $v_{+\dot\alpha}$ will not be part of the twisted multiplet. Hence, it should be in the image of δ_Q, forming a trivial pair with another field. Indeed, we will momentarily find that $v_{+\dot\alpha}$ cancels the holomorphic part of the one-form $C^{(1)}$.

The one-form $C^{(1)}$. For the field $C^{(1)}$, we have a supersymmetry transformation rule

$$\delta_Q C^{(1)}_\mu = (\epsilon_- \Gamma_{\mu\nu} \epsilon_-) v^\nu \,. \tag{6.69}$$

From the Γ-matrix decomposition, we know $\epsilon_- \Gamma_{\mu\nu} \epsilon_- \in \mathbf{1}^{-2}$. Thus, we immediately find

$$\delta_Q C^{(1)}_m = 0 \tag{6.70}$$

and

$$\delta_Q C^{(1)}_{+\dot\alpha} = 0 \,. \tag{6.71}$$

In addition, we have

$$\delta_Q C^{(1)}_{-\dot\alpha} = v_{+\dot\alpha} \,. \tag{6.72}$$

This shows that $C^{(1)}_{-\dot\alpha}$ and $v_{+\dot\alpha}$ form a trivial pair and thus do not appear in the twisted theory. Recall that the choice $(\epsilon_-, \epsilon_+, \epsilon_{\dot\alpha}) = (1, 0, 0)$ defines a complex structure on $\mathbb{R}^4 \cong \mathbb{C}^2$. The four-dimensional vector representation decomposes as

$$V_4 = S_+ \otimes S_- = \mathbf{2}^1 \oplus \mathbf{2}^{-1} \,. \tag{6.73}$$

The representation $\mathbf{2}^{-1}$ corresponds to holomorphic and $\mathbf{2}^1$ to the antiholomorphic components. Thus, we see that, for this complex structure, the components $C^{(1)}_{-\dot\alpha}$ form the holomorphic parts of the one-form ghost $C^{(1)}$. As expected, only the anti-holomorphic part of the one-form plays a role in the twisted theory.

We can alternatively describe the cancellation using complex geometry. With respect to the complex structure on $\mathbb{C}^2$,

$$\Omega = (\epsilon_- \Gamma_{\mu\nu} \epsilon_-) dx^\mu \wedge dx^\nu \tag{6.74}$$

defines a holomorphic $(2,0)$-form. Introducing coordinates $(z^{\dot\alpha}, \bar z^{\dot\alpha})$ on $V = \mathbf{2}^{-1} \oplus \mathbf{2}^{1}$, the holomorphic $(2,0)$-form simplifies to

$$\Omega = dz^1 \wedge dz^2 \,. \tag{6.75}$$

This allows us to rewrite the supersymmetry transformation of the one-form ghost as

$$\delta_Q C^{(1)} = \iota_v \Omega = v_{+\dot\alpha} dz^{\dot\alpha} \,. \tag{6.76}$$

Thus, we again see that the holomorphic components of $C^{(1)}$ cancel with the diffeomorphism ghost.

The two-form field $C^{(2)}$. Let us continue with the supersymmetry transformation of the two-form

$$\delta_Q C^{(2)}_{\mu\nu} = \frac{1}{2}\epsilon_-\Gamma_{\mu\nu}\omega + \epsilon_-\Gamma_{[\mu\rho}\epsilon_- g^{\rho}_{\nu]} \,. \tag{6.77}$$

The two-form and the graviton decompose into components

$$\begin{aligned} C^{(2)}_{\mu\nu} &\to (C^{(2)}_{mn}, C^{(2)}_{m+\dot\alpha}, C^{(2)}_{m-\dot\alpha}, C^{(2)}_{2}, C^{(2)}_{0}, C^{(2)}_{(\dot\alpha\dot\beta)}, C^{(2)}_{-2}) \\ g_{\mu\nu} &\to (g_{mn}, g_{m+\dot\alpha}, g_{m-\dot\alpha}, g_{2(\dot\alpha\dot\beta)}, g_{(\dot\alpha\dot\beta)}, g_0, g_{-2(\dot\alpha\dot\beta)}, h) \,. \end{aligned} \tag{6.78}$$

Consulting the Γ-matrix decomposition in (6.63), we get

$$\begin{aligned} \delta_Q C^{(2)}_{mn} &= 0 \\ \delta_Q C^{(2)}_{+m\dot\alpha} &= 0 \\ \delta_Q C^{(2)}_{-m\dot\alpha} &= \omega_{m\dot\alpha} + g_{+m\dot\alpha} \\ \delta_Q C^{(2)}_{2} &= 0 \\ \delta_Q C^{(2)}_{0} &= \omega_+ \\ \delta_Q C^{(2)}_{(\dot\alpha\dot\beta)} &= g_{2(\dot\alpha\dot\beta)} \\ \delta_Q C^{(2)}_{-2} &= \omega_- + g_0 \,. \end{aligned} \tag{6.79}$$

Thus, we find that the components

$$C^{(2)}_{-m\dot\alpha} \quad C^{(2)}_{0} \quad C^{(2)}_{(\dot\alpha\dot\beta)} \quad C^{(2)}_{-2} \tag{6.80}$$

do not appear in the twisted multiplet, while

$$C^{(2)}_{mn} \quad C^{(2)}_{+m\dot\alpha} \quad C^{(2)}_{2} \tag{6.81}$$

are in the kernel of δ_Q and thus, since there are no supersymmetry transformations that could make these exact, part of the twisted multiplet. Note again that this matches with the expectation that only $(0,\bullet)$-forms on $\mathbb{C}^2$ play a role in the twisted multiplet.

Note that we can rewrite the piece of the supersymmetry transformation (6.77) involving the graviton using the holomorphic (2,0)-form Ω as

$$\delta_Q C^{(2)} = \iota_{g^\rho_\nu \partial_\rho} \Omega \wedge dx^\nu \; . \tag{6.82}$$

However, due to the symmetry properties of the graviton, this transformation alone does not cancel all holomorphic component of the two-form. So one really needs the supersymmetry ghost to cancel the singlet $C^{(2)}_0$.

The three-form field $C^{(3)}$. For the three-form field, we have a supersymmetry transformation of the form

$$\delta_Q C^{(3)}_{\mu\nu\rho} = \frac{1}{4}\epsilon_- \Gamma_{[\mu\nu}\psi_{\rho]} \; . \tag{6.83}$$

The three-form decomposes into components

$$C^{(3)}_{\mu\nu\rho} \to (C^{(3)}_{mnp}, C^{(3)}_{mn+\dot\alpha}, C^{(3)}_{mn-\dot\alpha}, C^{(3)}_{m-2}, C^{(3)}_{m0}, C^{(3)}_{m(\dot\alpha\dot\beta)}). \tag{6.84}$$

To decompose this transformation, we write for the gravitino

$$\psi^\alpha_\mu = \xi^\alpha \otimes \chi_\mu \tag{6.85}$$

where ξ^α takes values in S and χ_μ in V. From (6.63), we see that ξ^α has to live in

$$\mathbf{1}^{-1} \oplus \mathbf{1}^1 \oplus (10)\mathbf{2}^0 \tag{6.86}$$

to get a non-zero result. Decomposing $(\mathbf{1}^{-1} \oplus \mathbf{1}^1 \oplus (10)\mathbf{2}^0) \otimes V$ into irreducibles, we can identify the decomposed transformations. The results are listed in Table 6.8.

The supersymmetry ghost ω. The non-derivative part of the supersymmetry transformation of ω_α reads

$$\delta_Q \omega_\alpha = \frac{1}{2}(\epsilon_- \Gamma^{\mu\nu})_\alpha (\epsilon_- \Gamma_\mu \psi_\nu) \; . \tag{6.87}$$

Again decomposing the gravitino as we did for the three-form field and using the decomposition (6.62), we find that ξ^α has to take values in

$$\mathbf{2}^0 \oplus (10)\mathbf{1}^1 \; . \tag{6.88}$$

Tensoring with the vector representation V and identifying matching representations gives the result listed below.

The graviton $g_{\mu\nu}$. The supersymmetry transformation

$$\delta_Q g_{\mu\nu} = \frac{1}{2}\epsilon_- \Gamma_{(\mu}\psi_{\nu)} \tag{6.89}$$

again only allows for ξ to come from $\mathbf{2}^0 \oplus (10)\mathbf{1}^1$. As before, we just list the results in Table 6.8.

In Table 6.8, we collect all decomposed non-derivative supersymmetry transformations. There M is an index for the 14-dimensional representation (01) of G_2. It appears in the variation

$$\delta_Q C^{(3)}_{mn-\dot\alpha} = \psi_{M\dot\alpha} + \psi_{m\dot\alpha} \tag{6.90}$$

where the notation describes the decomposition $\wedge^2(10) \to (10)\oplus(01)$ of G_2-representations.

Operator ϕ	Transformation rule $\delta_Q\phi$
$C^{(0)}$	0
$C^{(1)}_m$, $C^{(1)}_{+\dot\alpha}$, $C^{(0)}_{-\dot\alpha}$	0, 0, $v_{+\dot\alpha}$
$C^{(2)}_{mn}$, $C^{(2)}_{+m\dot\alpha}$, $C^{(2)}_{-m\dot\alpha}$, $C^{(2)}_{2}$, $C^{(2)}_{0}$, $C^{(2)}_{(\dot\alpha\dot\beta)}$, $C^{(2)}_{-2}$	0,0, $\omega_{m\dot\alpha} + g_{+m\dot\alpha}$, 0, ω_+, $g_{2(\dot\alpha\dot\beta)}$, $\omega_- + g_0$
v_m, $v_{+\dot\alpha}$, $v_{-\dot\alpha}$	ω_{+m}, 0, $\omega_{\dot\alpha}$
ω_+, ω_-, $\omega_{\dot\alpha}$, ω_{-m}, ω_{+m}, $\omega_{m\dot\alpha}$	0, ψ_+, 0, ψ_{+m}, 0, $\psi_{2m\dot\alpha}$
$C^{(3)}_{mnp}$, $C^{(3)}_{mn+\dot\alpha}$, $C^{(3)}_{mn-\dot\alpha}$, $C^{(3)}_{m-2}$, $C^{(3)}_{m0}$, $C^{(3)}_{m(\dot\alpha\dot\beta)}$	0, 0, $\psi_{M\dot\alpha} + \psi_{m\dot\alpha}$, ψ_{m-}, ψ_{m+}, $\psi_{m+(\dot\alpha\dot\beta)}$
$C^{(3)}_{m2}$, $C^{(3)}_{-\dot\alpha}$, $C^{(3)}_{+\dot\alpha}$	0, $\psi_{\dot\alpha}$, $\psi_{2\dot\alpha}$
g_{mn}, $g_{m+\dot\alpha}$, $g_{m-\dot\alpha}$, $g_{2(\dot\alpha\dot\beta)}$, $g_{(\dot\alpha\dot\beta)}$, g_0, $g_{-2(\dot\alpha\dot\beta)}$, h	ψ_{mn+}, $\psi_{2m\dot\alpha}$, $\psi_{m\dot\alpha}$, 0, $\psi_{+(\dot\alpha\dot\beta)}$, ψ_+, $\psi_{-(\dot\alpha\dot\beta)}$, ψ_+
ψ	$\delta_Q\psi^\alpha_\mu = (\epsilon_- M^{\alpha\beta}_{\mu\nu}\epsilon_-)\psi^{\vee\nu}_\beta$

TABLE 6.8: Decomposed supersymmetry transformations

6.4.3 Supersymmetry variation of the gravitino

The non-derivative supersymmetry transformation of the gravitino reads

$$\delta\psi^\alpha_\mu = (\epsilon M^{\alpha\beta}_{\mu\nu}\epsilon)\psi^{\vee\nu}_\beta \,. \tag{6.91}$$

This transformation reflects the fact that the supersymmetry algebra acts only up to the equations of motions of the gravitino. Correspondingly, there is a quadratic term in antifields appearing in the BV action [Bau+90; Ber02]

$$S^{(2)} \propto (\epsilon M \epsilon)\psi^\vee \psi^\vee \,. \tag{6.92}$$

The transformation (6.91) is responsible for the remaining cancellations between of the gravitino. To argue that indeed the correct components of ψ cancel, we change our strategy. As the structure of $M^{\alpha\beta}_{\mu\nu}$ is very complicated, we will not decompose it directly under $G_2 \times \mathrm{SU}(2)$. Instead we give an indirect argument.

For this, recall that (6.91) precisely represents the homotopy correcting for the failure of the linear supersymmetry transformations to define a strict representation. Denoting the linearized part of the supersymmetry transformation by δ^{lin}_Q and the quadratic transformation of the gravitino by δ^{quad}_Q, we have

$$\begin{aligned}
[\delta^{\mathrm{lin}}_Q, \delta^{\mathrm{lin}}_Q]\psi &= \delta^{\mathrm{lin}}_{[Q,Q]}\psi + \delta^{\mathrm{quad}}_Q Q_{BV}\psi^\vee \\
&= \delta^{\mathrm{quad}}_Q Q_{BV}\psi^\vee \\
&= (\epsilon_- M \epsilon_-) Q_{BV}\psi^\vee \,,
\end{aligned} \tag{6.93}$$

where we have used the fact that Q is square zero in the second equality.

Thus, we can try to learn something about the quadratic transformation by studying two consecutive linear transformations applied to the gravitino. Recall that a linear transformation applied to the gravitino transforms it to the field strength of three-form,

$$\delta^{\mathrm{lin}}_Q \psi_\mu = (\Gamma_\mu^{\ \nu\rho\sigma\tau} - 8\Gamma^{\rho\sigma\tau}\delta^\nu_\mu) G^{(4)}_{\nu\rho\sigma\tau}\epsilon_- \,, \tag{6.94}$$

while the three-form transforms to the gravitino

$$\delta^{\mathrm{lin}}_Q C^{(3)}_{\mu\nu\rho} = \frac{1}{4}\epsilon_- \Gamma_{[\mu\nu}\psi_{\rho]} \tag{6.95}$$

Decomposing the gravitino and applying these tranformations, there are two distinct cases: Whenever the result is non-zero, the linearized piece fails to be a Lie map and a homotopy is present for such components. In other words, evaluating the quadratic transformation (6.91) on such a component gives a non-zero result and a trivial pair forms. On the other hand, when a component is in the kernel of two consecutive linear transformations, then a homotopy is not strictly necessary and it is possible that the component is also in the kernel of the quadratic transformation such that the corresponding field descends to the twisted theory.

This reasoning suggests to view the cancellations between components of the gravitino and its antifield as a two-step procedure. First, the linearized transformation identifies a piece of ψ with a component of $G^{(4)} = \mathrm{d}C^{(3)}$. Then we can act with another linearized transformation to obtain a component of $\psi^\vee$. Clearly the U(1)-charges of components connected in this way satisfy

$$d_{\mathrm{U}(1)}(\psi^\vee) = d_{\mathrm{U}(1)}(G^{(4)}) + 1 = d_{\mathrm{U}(1)}(\psi) + 2\,. \tag{6.96}$$

To investigate the kernel of two consecutive linearized transformations, recall from Table 6.8 that the components components

$$C^{(3)}_{mnp}\,, C^{(3)}_{mn+\dot\alpha}\,, C^{(3)}_{m2} \tag{6.97}$$

are in the kernel of δ_Q. They correspond to the differential forms

$$\Omega^3(\mathbb{R}^7)\otimes\Omega^{0,0}(\mathbb{C}^2)\oplus\Omega^2(\mathbb{R}^7)\otimes\Omega^{0,1}(\mathbb{C}^2)\oplus\Omega^1(\mathbb{R}^7)\otimes\Omega^{0,2}(\mathbb{C}^2). \tag{6.98}$$

Investigating the complex of differential forms, it is easy to see that the components of the field strengths living in

$$\Omega^4(\mathbb{R}^7)\otimes\Omega^{0,0}(\mathbb{C}^2)\oplus\Omega^3(\mathbb{R}^7)\otimes\Omega^{0,1}(\mathbb{C}^2)\oplus\Omega^2(\mathbb{R}^7)\otimes\Omega^{0,2}(\mathbb{C}^2) \tag{6.99}$$

can only arise from the three-form components above. In particular, these components of the field strength are then also in the kernel δ_Q. Components of the gravitino who are transformed to such a component of the field strength by the first linear supersymmetry transformation are annihilated by the second one. These components thus are expected to descend to the twisted theory.

With this information, we can analyze the remaining compoments of the gravitino. In Table 6.9, we display the $G_2\times \mathrm{SU}(2)$-equivariant decomposition of the gravitino, its antifield, and the field strength organized by U(1)-charges. All components of ψ and $\psi^\vee$ that form trivial pairs with other fields according to Table 6.8 are indicated with an arrow.

We circle the components of the field strength which are in the kernel with red dashed lines and the corresponding components of the gravitino in blue. We then expect these to descend to the twisted theory. In addition, we also circle the surviving dual components of the gravitino antifield in blue. For example, with U(1)-charge 1, there appears a representation

$$(10)\mathbf{1}\oplus(01)\mathbf{1} \tag{6.100}$$

in the decomposition of the gravitino which can be identified with $\Omega^2\otimes\Omega^{0,2}$ after applying δ_Q^{lin}. Indeed, these components descend to the twist as they cannot form any trivial pairs with any components from the gravitino antifield due to their U(1)-charges alone.

Similarly, we circle pieces in blue with U(1)-charge 0 and -1 which can be identified with the differential forms $\Omega^3 \otimes \Omega^{0,1}$ and $\Omega^2 \otimes \Omega^{0,2}$ respectively.

On the other hand, we see that different pieces of the gravitino are mapped to components of the field strength which are not part of the kernel of δ_Q^{lin}. These then can have $[\delta_Q^{\mathrm{lin}}, \delta_Q^{\mathrm{lin}}]\psi \neq 0$, such that a cancellation is possible. In Table 6.9 we indicate such components, the corresponding intermediate components of the field strength and the respective partners from $\psi^\vee$ with green rectangles.

In this way, one can understand all cancellation except one subtlety. For U(1)-charge zero, there is a leftover representation $(00)\mathbf{2}$. We expect that this component of the gravitino cancels with the respective component of the antifield with U(1)-charge 2. However, since the only field strength component which could serve as intermediary is in the kernel of δ_Q^{lin}, we cannot understand this cancellation in the above manner. For a more complete understanding, a direct investigation of the homotopy seems necessary.

6.4.4 Summary of cancellations

We summarize the cancellations obtained in the previous sections in Table 6.10. The fields that do not form trivial pairs are circled in blue. They form the multiplet $\mathcal{A} \in \Omega^\bullet(\mathbb{R}^7) \otimes \Omega^{0,\bullet}(\mathbb{C}^2)$ and appear in Table 6.1. The bi-directional strike-through arrows indicate cancellations that occur between ψ and its anti-field $\psi^\vee$ found in §6.4.3.

Special care should be taken for the variations of the components of $C^{(2)}$ that cancel with a linear combination of components of the graviton and supersymmetry ghost

$$\delta_Q C^{(2)}_{-m\dot{\alpha}} = \omega_{m\dot{\alpha}} + g_{+m\dot{\alpha}} \tag{6.101}$$

$$\delta_Q C^{(2)}_{-2} = \omega_- + g_0 \tag{6.102}$$

that occur in (6.79). A subsequent variation yields

$$\delta_Q \omega_{m\dot{\alpha}} = -\delta_Q g_{+m\dot{\alpha}} = \psi_{2m\dot{\alpha}} \tag{6.103}$$

$$\delta_Q \omega_- = -\delta_Q g_0 = \psi_+ \tag{6.104}$$

which is consistent with $\delta_Q^2 C^{(2)} = 0$. These extra cancellations are indicated by the strike-through arrows with labels x and y.

Field	2	1	0	-1	-2
ψ	$\cancel{(00)\mathbf{2}} \oplus \cancel{(10)\mathbf{2}}$	$\boxed{(10)\mathbf{1} \oplus (01)\mathbf{1}}$ $\cancel{(00)\mathbf{1}} \oplus \cancel{(10)\mathbf{1}} \oplus \cancel{(20)\mathbf{1}}$ $\cancel{(00)\mathbf{3}} \oplus \cancel{(10)\mathbf{3}}$ $\cancel{(00)\mathbf{1}} \oplus \cancel{(10)\mathbf{1}}$	$\cancel{(00)\mathbf{2}} \oplus \cancel{(10)\mathbf{2}} \oplus \cancel{(01)\mathbf{2}}$ $\boxed{(00)\mathbf{2} \oplus (10)\mathbf{2} \oplus (20)\mathbf{2}}$ $\cancel{(10)\mathbf{2}}$ $(00)\mathbf{2} \oplus \boxed{(10)\mathbf{2}}$	$\cancel{(10)\mathbf{1}} \oplus \boxed{(01)\mathbf{1}}$ $\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1} \oplus (20)\mathbf{1}}$ $\cancel{(00)\mathbf{3}} \oplus \boxed{(10)\mathbf{3}}$ $\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1}}$	$\boxed{(00)\mathbf{2} \oplus (10)\mathbf{2}}$
$G^{(4)}$	$\boxed{(10)\mathbf{1} \oplus (01)\mathbf{1}}$	$\boxed{(00)\mathbf{2} \oplus (10)\mathbf{2} \oplus (20)\mathbf{2}}$ $\boxed{(10)\mathbf{2}}$	$\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1} \oplus (20)\mathbf{1}}$ $\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1} \oplus (10)\mathbf{3}}$ $\boxed{(01)\mathbf{1}} \oplus (01)\mathbf{3}$	$(20)\mathbf{2} \oplus (10)\mathbf{2}$ $\boxed{(00)\mathbf{2} \oplus (10)\mathbf{2}}$	$(10)\mathbf{1} \oplus (01)\mathbf{1}$
$\psi^{\vee}$	$(00)\mathbf{2} \oplus \boxed{(10)\mathbf{2}}$	$\cancel{(10)\mathbf{1}} \oplus \boxed{(01)\mathbf{1}}$ $\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1} \oplus (20)\mathbf{1}}$ $\cancel{(00)\mathbf{3}} \oplus \boxed{(10)\mathbf{3}}$ $\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1}}$	$\cancel{(00)\mathbf{2}} \oplus \cancel{(10)\mathbf{2}} \oplus \cancel{(01)\mathbf{2}}$ $\boxed{(00)\mathbf{2} \oplus (10)\mathbf{2} \oplus (20)\mathbf{2}}$ $\cancel{(10)\mathbf{2}}$ $\boxed{(00)\mathbf{2} \oplus (10)\mathbf{2}}$	$\boxed{(10)\mathbf{1} \oplus (01)\mathbf{1}}$ $\cancel{(00)\mathbf{1}} \oplus \cancel{(10)\mathbf{1}} \oplus \cancel{(20)\mathbf{1}}$ $\cancel{(00)\mathbf{3}} \oplus \cancel{(10)\mathbf{3}}$ $\cancel{(00)\mathbf{1}} \oplus \cancel{(10)\mathbf{1}}$	$\cancel{(00)\mathbf{2}} \oplus \cancel{(10)\mathbf{2}}$

TABLE 6.9: Decomposition of the non-linear gravitino supersymmetry variation

Field	Spin(11)	2	1	0	-1	-2
$C^{(0)}$	(00000)			$\boxed{(00)\mathbf{1}}$		
$C^{(1)}$	(10000)		$\boxed{(00)\mathbf{2}}$	$\boxed{(10)\mathbf{1}}$	$\cancel{(00)\mathbf{2}}^{a}$	
$C^{(2)}$	(01000)	$\boxed{(00)\mathbf{1}}$	$\boxed{(10)\mathbf{2}}$	$\cancel{(00)\mathbf{1}}^{b} \oplus \cancel{(00)\mathbf{3}}^{c} \oplus \boxed{(10)\mathbf{1} \oplus (01)\mathbf{1}}$	$\cancel{(10)\mathbf{2}}^{d,x}$	$\cancel{(00)\mathbf{1}}^{d,y}$
v	(10000)		$\cancel{(00)\mathbf{2}}^{a}$	$\cancel{(10)\mathbf{1}}^{e}$	$\cancel{(00)\mathbf{2}}^{f}$	
ω	(00001)		$\cancel{(00)\mathbf{1}}^{b} \oplus \cancel{(10)\mathbf{1}}^{e}$	$\cancel{(00)\mathbf{2}}^{f} \oplus \cancel{(10)\mathbf{2}}^{g,x}$	$\cancel{(00)\mathbf{1}}^{h,y} \oplus \cancel{(10)\mathbf{1}}^{h}$	
$C^{(3)}$	(00100)	$\boxed{(10)\mathbf{1}}$	$\cancel{(00)\mathbf{2}}^{i} \oplus \boxed{(10)\mathbf{2} \oplus (01)\mathbf{2}}$	$\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1} \oplus (20)\mathbf{1}}$	$\cancel{(00)\mathbf{2}}^{k} \oplus \cancel{(10)\mathbf{2}}^{k} \oplus \cancel{(01)\mathbf{2}}^{k}$	$\cancel{(10)\mathbf{1}}^{l}$
				$\cancel{(10)\mathbf{1}}^{j} \oplus \cancel{(10)\mathbf{3}}^{j}$		
g	(20000)	$\cancel{(00)\mathbf{3}}^{c}$	$\cancel{(10)\mathbf{2}}^{d}$	$\cancel{(00)\mathbf{1}}^{d} \oplus \cancel{(00)\mathbf{3}}^{m} \oplus \cancel{(20)\mathbf{1}}^{m}$	$\cancel{(10)\mathbf{2}}^{o}$	$\cancel{(00)\mathbf{3}}^{p}$
	(00000)			$\cancel{(00)\mathbf{1}}^{n}$		
ψ	(10001)	$\cancel{(00)\mathbf{2}}^{i} \oplus \cancel{(10)\mathbf{2}}^{g}$	$\boxed{(10)\mathbf{1} \oplus (01)\mathbf{1}}$	$\cancel{(00)\mathbf{2}}^{k} \oplus \cancel{(10)\mathbf{2}}^{k} \oplus \cancel{(01)\mathbf{2}}^{k}$	$\cancel{(10)\mathbf{1}}^{l} \oplus \cancel{(01)\mathbf{1}}^{q}$	$\cancel{(00)\mathbf{2}}^{q} \oplus \cancel{(10)\mathbf{2}}^{q}$
			$\cancel{(00)\mathbf{1}}^{h} \oplus \cancel{(10)\mathbf{1}}^{h} \oplus \cancel{(20)\mathbf{1}}^{m}$	$\boxed{(00)\mathbf{2} \oplus (10)\mathbf{2} \oplus (20)\mathbf{2}}$	$\boxed{(00)\mathbf{1} \oplus (10)\mathbf{1} \oplus (20)\mathbf{1}}$	
			$\cancel{(00)\mathbf{3}}^{m} \oplus \cancel{(10)\mathbf{3}}^{j}$	$\cancel{(10)\mathbf{2}}^{o}$	$\cancel{(00)\mathbf{3}}^{p} \oplus \cancel{(10)\mathbf{3}}^{q}$	
	(00001)		$\cancel{(00)\mathbf{1}}^{n} \oplus \cancel{(10)\mathbf{1}}^{j}$	$\cancel{(00)\mathbf{2}}^{q} \oplus \cancel{(10)\mathbf{2}}^{q}$	$\cancel{(00)\mathbf{1}}^{q} \oplus \cancel{(10)\mathbf{1}}^{q}$	

TABLE 6.10: Cancellations of fields under Q. Fields are decomposed into $G_2 \times \mathrm{SU}(2)_- \times \mathrm{U}(1)_L$-representations.

6.4.5 The twisted differential

Recall that the BV differential of the twisted theory is the sum of two terms

$$Q_{BV}^{\mathrm{tw}} = Q_{BV} + \delta_Q \,. \tag{6.105}$$

We already examined how the non-derivative part of δ_Q leads to the formation of various trivial pairs; now we turn towards the parts containing derivatives in order to see how they act on the twisted multiplet.

The BV operator Q_{BV}^{tw} is dual to a differential D^{tw} acting on the fields of the twisted multiplet. We already know that D acts as the de Rham differential on the three-form ghost system. Under $G_2 \times \mathrm{SU}(2)$ the de Rham differential decomposes

$$\mathrm{d} = \mathrm{d}_{\mathbb{R}^7} + \bar{\partial}_{\mathbb{C}^2} + \partial_{\mathbb{C}^2} \,. \tag{6.106}$$

As only $(0,\bullet)$-forms are part of the twisted multiplet, this restricts to

$$\mathrm{d}_{\mathbb{R}^7} + \bar{\partial}_{\mathbb{C}^2} \,. \tag{6.107}$$

In addition, D acts on the gravitino by the Rarita–Schwinger equation. Identifying the gravitino as a spinor valued one-form, $\psi \in \Omega^1(M) \otimes S$, the Rarita–Schwinger operator can be understood as a composition of the exterior differential and Clifford multiplication [HS19]. From this, one can see that it also acts by $\mathrm{d}_{\mathbb{R}^7} + \bar{\partial}_{\mathbb{C}^2}$ on the relevant pieces of the gravitino.

Finally, there is a contribution to D^{tw} coming from the supersymmetry transformation (6.94). This transformation also acts by $\mathrm{d}_{\mathbb{R}^7} + \bar{\partial}_{\mathbb{C}^2}$ and provides the missing differential between $C^{(3)}$ and ψ.

In summary, the twisted multiplet can thus be described by the cochain complex

$$\left(\Omega^\bullet(\mathbb{R}^7) \otimes \Omega^{0,\bullet}(\mathbb{C}^2) \,,\, D^{\mathrm{tw}} = \mathrm{d}_{\mathbb{R}^7} + \bar{\partial}_{\mathbb{C}^2}\right) \,, \tag{6.108}$$

as conjectured by Costello.

Interestingly, the form of the differential can also be deduced directly from the explicit formulas in the pure spinor formalism. Recall that $\mathcal{D}_1$ acts on the representatives by

$$\mathcal{D}_1 = (\lambda \Gamma^\mu \theta)\partial_\mu \,, \tag{6.109}$$

and that the one-form was represented by the cohomology classes $C_\mu^{(1)}(\lambda\Gamma^\mu\theta)$. As we already know that the twisted multiplet forms the exterior algebra $\Omega^\bullet(\mathbb{R}^7) \otimes \Omega^{0,\bullet}(\mathbb{C}^2)$, we see that $\mathcal{D}_1$ simply acts by taking derivatives and wedging with the corresponding component of the one-form, i.e. precisely by $\mathrm{d}_{\mathbb{R}^7} + \bar{\partial}_{\mathbb{C}^2}$.

In addition the derivative part of the supersymmetry transformation acts by

$$Q_{\partial_x} = (\epsilon_- \Gamma^\mu \theta)\partial_\mu \, . \tag{6.110}$$

From the Gamma matrix decomposition (6.57), we see

$$(\epsilon_- \Gamma^\mu \theta) \in \mathbf{2}^{-1} \oplus (10) \, . \tag{6.111}$$

Identifying the corresponding components with $d\bar{z}^{\dot{\alpha}}$ and dx^m, we once again see that Q_{∂_x} acts as desired.

Another more roundabout way of understanding the appearance of the de Rham differential is as follows. Recall that the gravitino field on M^7 can be organized into $\Omega^2 \oplus \Omega^3$ when M^7 has G_2 holonomy. Since there are $b^2(M^7)+b^3(M^7)$ zero modes of the gravitino on M^7 [Fon10; CG+18; Wan91; HS19], this suggests that the BV differential acts by the de Rham differential

$$\mathrm{d}_{\mathrm{d}R} : \Omega^2 \oplus \Omega^3 \to \Omega^3 \oplus \Omega^4. \tag{6.112}$$

We note that the appearance of the de Rham and Dolbeault differential is similar to the holomorphic twist of ten-dimensional abelian super Yang–Mills theory on $\mathbb{C}^5$ (see [ES19a]). In that case, the analogous BV differential between the gaugino χ and its antifield expresses the Dirac equation. The relevant part of the differential in the twisted theory is

$$Q_{BV}(\chi^{mn})^\vee = i\epsilon^{mnpqr}\overline{\partial}_p \chi_{qr}, \tag{6.113}$$

and only involves the Dolbeault operator on $\Omega^{0,\bullet}(\mathbb{C}^5)$.

6.5 Conclusions and future directions

While the above calculation establishes the maximal twist on the level of the free theory, the interactions remain opaque. The main advantage of the component field approach is its immediacy. We could explicitly see how the fields of the physical theory arrange to the twisted theory and thereby obtains direct insights on the twisted degrees of freedom. However, as the interactions of supergravity theories expressed in component fields can be quite complicated (in particular they are non-polynomial in the metric), approaching them in the same fashion seems out of reach at the moment. In the next chapters, we fully leverage the pure spinor superfield formalism to address the twist of the interacting theory at pure spinor cochain level rather than at component field level.

First hints can already be obtained by looking at "field strength" formulations of supergravity; in eleven-dimensional supergravity there is a "super-vielbein" multiplet (as opposed to the "three-form multiplet" we studied in this chapter). The super-vielbein multiplet contains the graviton, gravitino, and 4-form field strength $G^{(4)}$ as its physical fields. It is used in the traditional superspace formulation of supergravity. It is natural to expect that the twisted fields of the super-vielbein multiplet organize into a differential form

$$\partial \mathcal{A} \in \Omega^\bullet(M^7) \otimes \Omega^{1,\bullet}(M^4), \tag{6.114}$$

with leading component $v_{+\dot{\alpha}}$ from the diffeomorphism ghost. In fact, we will see that precisely this happens and that the super-vielbein multiplet is best thought of as being associated to the first Chevalley–Eilenberg cohomology of the residual supertranslation algebra and is indeed a field strength in the sense discussed in §3 (see also [CNT02]).

In general, addressing twists directly in component field could prove to be useful in linking mathematical and physical approaches to the holographic duality. For instance, cojectural twists of type IIB supergravity were described in [CL15; CL16] and for the minimal twist of eleven-dimensional supergravity in [RSW23]. In a particular limit, holographic duality relates weakly coupled type IIB supergravity on products of five-dimensional AdS space AdS_5 with arbitrary Sasaki-Einstein manifolds SE^5 to four-dimensional supersymmetric gauge theories. A different form of the conjecture relates the weak coupling limit of M-theory on the products $AdS_4 \times SE^7$ to three-dimensional supersymmetric gauge theories. The cone over the Sasaki–Einstein manifold is a local Calabi–Yau manifold. One corollary of the conjecture is the equivalence of the superconformal index [Rö06; Kin+07] under gauge-gravity duality. The gravity superconformal index was computed in terms of holomorphic invariants of the Calabi–Yau manifold in [EST14; ES15]. The corresponding field theory index was later shown to be most directly computed in the holomorphic twist [ES19a; SW23c]. In [RW22] the minimally twisted eleven-dimensional supergavity theory described in [RSW23] was used to compute superconformal indices and matched with corresponding results from the physics literature. Twist computations in component fields naturally bridge the gap between these different versions of index calculations and therey between physical and mathematical approaches to holography.

In addition, one expects that a further twist of the one considered in this chapter can be used to derive twisted M-theory in the Ω-background [Cos16] following [OY19]. This could provide a physical origin for the applications in [GO19; OZ21] by coupling a twisted M5-brane [SW23b] to twisted M-theory. Finally, we hope that twisted M-theory can shed new light on topological M-theory [Hit00; GS04; Dij+05; GV05; Bec+16], which is believed to unify the Kähler [BS96] and Kodaira–Spencer theories of topological gravity.

Chapter 7

Eleven-dimensional supergravity as a Calabi–Yau twofold

7.1 Introduction

Since the first supersymmetric field theories were constructed, it has been a goal to understand their properties and simplify their construction using superspace techniques. This motivation has perhaps been largest in the case of supergravity theories. The geometric nature of the theory of Einstein gravity, which is constructed using a covariant least-action principle on the space of metrics of Lorentzian signature, has motivated much research that tries to give an equally pithy formulation of supergravity theories as governing moduli problems of (deformations of) particular natural geometric structures on superspace.

Among all supergravity theories of physical interest, perhaps the most exceptional is eleven-dimensional supergravity, which was first constructed by Cremmer, Julia, and Scherk in 1978 [CJS78], and which is expected to be the low energy limit of M-theory [Wit95]. M-theory has yet to be constructed, although expectations exist that a worldsheet construction as a theory of fundamental membranes might be possible. While the component-field formulation of this theory is relatively streamlined—in addition to the metric, the theory contains only a gravitino and an abelian three-form gauge field with Chern–Simons term—it proved difficult to even formulate the theory in superspace, and a superspace least action principle remained out of reach. Part of the difficulty can be attributed to attempts to find sets of auxiliary fields that could be used to represent supersymmetry off shell, which was seen as a necessary prerequisite.

A major leap forward was taken in work of Cederwall, who applied the pure spinor superfield formalism to construct a superspace description of perturbative eleven-dimensional supergravity using the BV formalism. The relation of eleven-dimensional pure spinors to supergravity dates back at least to [How91b]. The connection had been sharpened in [CNT02], which observed that a particular eleven-dimensional pure spinor superfield reproduced the BV supergravity multiplet. In [Ced10c], a candidate cubic interaction term for this multiplet was constructed; in [Ced10a], Cederwall went on to extend this by a somewhat subtle quartic term in the BV action functional, and to prove that the result satisfies the BV master equation, thus giving a consistent, manifestly supersymmetric interacting theory that—since the theory is expected to be unique—must be eleven-dimensional supergravity itself. (Pure spinor techniques were also used from a first-quantized perspective to give new models of the supermembrane; see Berkovits' work in [Ber02], generalizing his formulation of the superstring.) The pure spinor description thus not only formulates the theory on superspace, but also dramatically simplifies the structure of its interactions: a non-polynomial action for the component fields is replaced by a quartic polynomial. Nonetheless, it does not provide a geometric origin for the quartic polynomial in question. Neither does it give an interpretation of the moduli problem it describes in terms of deformations of the superspace geometry itself.

Later, and in disjoint fashion, further progress was made on twisted versions of eleven-dimensional supergravity. Twists of supergravity theories were defined by Costello and Li in [CL16], generalizing the standard notion of a twist of a supersymmetric field theory. Using worldsheet techniques from topological string theory, they gave a proposed description of the holomorphic twist of type IIB supergravity. Costello and Li's theory is a version of BCOV theory [Ber+94], for which the moduli-theoretic interpretation is clear; it is related to the Kodaira–Spencer theory of deformations of Calabi–Yau structure. In [Cos16], Costello went on to investigate eleven-dimensional supergravity in the omega background; his proposed description links the maximal twist of eleven-dimensional supergravity to Poisson–Chern–Simons theory.

Poisson–Chern–Simons theory is simple to describe in the BV formalism. Its fields are given by the Dolbeault complex of $(0,\bullet)$-forms on a Calabi–Yau twofold, tensored with the de Rham complex on $\mathbb{R}^7$ (or, more generally, a G_2-manifold; for nonperturbative issues related to G_2-manifolds, see [Dij+05; DZOZ22] and references therein). The interactions are determined by an L_∞ structure on the fields, which is in fact strict: the Lie bracket is the Poisson bracket of holomorphic functions induced by the Calabi–Yau form, whose inverse is a holomorphic Poisson bivector. This theory has two essential features. Firstly, it also has a moduli-theoretic interpretation. The Lie algebra of holomorphic functions with the Poisson bracket is a one-dimensional central extension of holomorphic Hamiltonian vector fields. Since the symplectic structure is the holomorphic volume form, these are also divergence-free vector fields, and can thus be also thought of as related to the moduli space of deformations of Calabi–Yau structures. Secondly, the central extension equips the fields of Poisson–Chern–Simons theory with a *commutative* structure; the interactions define not just a dg Lie structure, but a dg Poisson algebra structure. Recalling that the observables of a three-dimensional TQFT are equipped with an E_3-algebra structure, which is equivalent to an even-shifted Poisson structure, we see that this formulation is at least suggestive of a first-quantized origin. (Note, though, that there are subtleties in defining E_3 algebra structures on theories of this type; see [EW21].)

Recent work has pushed our understanding of twisted eleven-dimensional supergravity further; all approaches have either used dualities or target-space techniques, since no worldsheet description is available. Using the component field formulation on target space, we computed the maximal twist in the free limit in §6. Pure spinor techniques were applied in [SW21] to give concise and computationally straightforward descriptions of the twists of supergravity multiplets. This led to the first direct computations of the minimally twisted eleven-dimensional and type IIB supergravity multiplets, the latter confirming Costello and Li's proposal at the free level. In [RSW23], a consistent interacting $\mathbb{Z}/2\mathbb{Z}$-graded BV theory was defined on the minimally twisted eleven-dimensional supergravity multiplet. Surprisingly, the cohomology of this theory on flat space is a one-dimensional L_∞ central extension of the exceptional infinite-dimensional simple super Lie algebra $E(5|10)$ [Kac77]. Other exceptional simple super Lie algebras also play fundamental roles in holomorphic M-theory [RW22; SW23a].

In this chapter, we take a step towards bringing some of these lines of work together by exploiting a powerful and seemingly underappreciated analogy between the geometric structures in play on each case. Thinking of Poisson–Chern–Simons theory (after localizing six directions with omega backgrounds) as a theory in five dimensions, we note that the theory must be equipped with a transversely holomorphic foliation that lets us think of the geometry as locally isomorphic to $\mathbb{C}^2 \times \mathbb{R}$. The THF structure is an (involutive) three-dimensional subbundle of the complexified tangent bundle. Similarly, the minimally twisted theory is most generally defined on eleven-dimensional manifolds equipped with a six-dimensional complex distribution.

Flat superspace itself is also canonically equipped with a distribution, spanned by the left–invariant odd vector fields. However, since all bosonic translations are in the image of brackets of supersymmetry transformations, this distribution is as far from being integrable as possible. It is thus not possible to naively draw a connection between these two structures. A clue to the resolution is provided by the theory of Dolbeault cohomology for almost complex manifolds, recently developed in [CW21]. This theory uses the distribution $T^{(0,1)}$ to define a filtration of the de Rham complex. The differential on the associated graded measures the nonintegrability of the distribution; passing to its cohomology and transferring the D_∞ structure defined by the remaining terms in the de Rham differential provides a new filtered complex, which they use as a replacement for the Hodge filtration. Passing to the associated graded of this new filtration defines their analogue of the Dolbeault complex.

If we apply the same construction to the de Rham complex on superspace, we can identify the term in the differential encoding the nonintegrability of the odd distribution with the Chevalley–Eilenberg differential of the supertranslation algebra. The "generalized Dolbeault complex" that appears is nothing other than the sum of the pure spinor multiplets associated to the cohomology groups of the supertranslation algebra; the cohomology in degree $-k$ plays the role of the Dolbeault complex resolving holomorphic $(k, 0)$-forms. In particular, the canonical supermultiplet of [Ced+23] appears playing the role of the holomorphic functions, and we think of it—equipped with its commutative structure—as the appropriate structure sheaf with which to equip the spacetime. In eleven dimensions, this is eleven-dimensional supergravity.

The analogy with complex geometry allows one to find ready generalizations of many interesting notions: the complex dimension is the degree of the highest Lie algebra cohomology of the supertranslations; a Calabi–Yau structure is a trivialization of (the multiplet of) top cohomology as a module over the structure sheaf. In this analogy, eleven-dimensional supergravity, and all of its twists, are Calabi–Yau twofolds. We use this to construct a family of theories we call *homotopy Poisson–Chern–Simons theories*. The construction uses the derived bracket technique of [KS96], as generalized by [Vor05], and is entirely analogous to the standard construction of the Poisson bracket. However, because we work in a derived setting, the corresponding L_∞ structure is in general not strict. Applying our construction recovers Cederwall's quartic interaction functional in geometric fashion, as well as Costello's maximal twist. Furthermore, it gives a pure spinor lift of the interactions of the minimal twist. It then follows from the results of [SW21], which state that the twist of a canonical multiplet is the canonical multiplet of the twisted supersymmetry algebra, that these theories are all related by twisting, proving Costello's conjecture on the maximal twist at the full interacting level.

7.2 Flag structures and generalized Dolbeault complexes

Throughout, we work in the category of graded super vector spaces, often equipped with a G-action. The grading and the parity are independent; thus, an object is graded by $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the $\mathbb{Z}/2\mathbb{Z}$ factor determines the monoidal structure. We will also consider cochain complexes; these are then equipped with three integer gradings, called cohomological degree, weight grading, and intrinsic parity, and the Koszul sign is determined by the totalization of cohomological degree and intrinsic parity. Our conventions are always cohomological.

7.2.1 Weighted flag structures

We begin with some very general considerations, related to the type of geometric intuition we will draw on in the sequel. The essential point is to notice that certain (super or graded generalizations of) filtered structures, as studied by Tanaka, are present both on (almost) complex manifolds and on the superspaces of interest in physics. (Other examples abound, but these are the two that will interest us here.) The resulting analogy between superspaces and almost complex manifolds will let us construct a sheaf of commutative differential graded algebras on such a manifold, which reproduces Dolbeault cohomology for complex manifolds, as well as its generalization to almost complex manifolds as defined in [CW21]. When we apply our techniques to superspaces, the construction naturally reproduces a particular supermultiplet in the pure spinor formalism. This is the multiplet assigned to the structure sheaf of the nilpotence variety, termed the canonical multiplet in [Ced+23].

Geometrically, we will be interested in manifolds (including supermanifolds or graded manifolds) that are equipped with distributions. Recall that a *distribution* on a manifold M is a subbundle $D \subset TM$ of the tangent bundle. A distribution is said to be *involutive* if the space of vector fields lying in D is a subalgebra of vector fields on M with respect to the Lie bracket. By Frobenius' theorem, involutive distributions are integrable, i.e. there exists a submanifold of M whose tangent bundle is D.

More generally, we can consider a *flag of distributions*, which is defined to be a finite sequence

$$0 \subset D_1 \subset \cdots \subset D_k = TM \tag{7.1}$$

of subbundles of the tangent bundle, each contained in the next. We require that this flag is chosen to be compatible with the Lie bracket of vector fields, in the sense that

$$[\Gamma(D_i), \Gamma(D_j)] \subset \Gamma(D_{i+j}). \tag{7.2}$$

When $k = 2$, this condition is vacuous; all of our examples will be of this type, so that only a single distribution D_1 is relevant. In any case, the flag of distributions gives $\mathrm{Vect}(M)$ the structure of a filtered Lie algebra.

Given a flag of distributions in the tangent bundle, we can ask what corresponding structure appears on the de Rham forms of M. To do this, we can filter $\Omega^\bullet(M)$ in the following way. We observe that the cotangent bundle is equipped with a dual series of quotients of the form

$$T^*M = D_k^\vee \to D_{k-1}^\vee \to \cdots \to D_1^\vee \to 0. \tag{7.3}$$

Since we want to filter the de Rham forms by subalgebras, rather than by successive quotients, we define a negatively graded filtration on the cotangent bundle by taking

$$F^{-i}T^*M = \ker(T^*M \to D_i^\vee), \tag{7.4}$$

with respect to the map defined by (7.3). We can extend this multiplicatively to a non-positive filtration $F^\bullet\Omega^\bullet(M)$ of the de Rham forms, which is then automatically preserved by the differential.

As an example, consider the flag of distributions on an almost complex manifold defined by taking

$$0 \subset D_1 = T^{(0,1)}X \subset D_2 = T_{\mathbb{C}}X. \tag{7.5}$$

The filtration $F^\bullet$ can be thought of as assigning weight -1 to $\mathrm{d}\bar{z}$ and weight -2 to $\mathrm{d}z$. As is clear from (7.19) below, this filtration is compatible with the de Rham differential for any almost complex structure.

Note that this filtration, although it is compatible with the de Rham differential, is *not* the standard Hodge filtration. Nor is it particularly convenient in applications. To recover the Hodge filtration, one needs to construct a new filtration $F_+^\bullet\Omega^\bullet(M)$, defined by taking

$$F_+^i\Omega^\bullet(M) = \bigoplus_{k+j=i} F^j\Omega^k(M). \tag{7.6}$$

In the example of an almost-complex manifold, we then have that

$$F_+^{-i}\Omega^\bullet(M) = \Omega^{\geq i,\bullet}(M). \tag{7.7}$$

When the complex structure is not integrable, the de Rham differential does not preserve $F_+^\bullet$; see §7.2.2 below.

Compatible weight gradings. Matters are simplified when we have a decomposition of the tangent bundle via a positive integer grading that induces the flag of distributions we are interested in. We will refer to such a grading as a *weight grading*. It consists of a direct sum decomposition of the tangent space of the form

$$TM = \bigoplus_{1\leq j\leq k} T_j M, \tag{7.8}$$

such that the flag of distributions we are interested in is recovered by taking

$$D_k = \bigoplus_{1\leq j\leq k} T_j M. \tag{7.9}$$

In fact, both for almost complex manifolds and superspaces, there is a canonical choice of such a splitting: in the first case, we take the eigenspaces of J, and in the second, we take the eigenspaces of the parity operator $(-)^F$.

Motivated by the previous considerations we now give definitions which are meant to abstractly model the structures that are present on the de Rham complex of a manifold equipped with a flag of distributions (and perhaps with a compatible weight grading).

Definition 7.2.1. Let $(\Omega^\bullet, \mathrm{d})$ be a cdga. A *flag structure* on $\Omega^\bullet$ is a decreasing filtration $F_+^\bullet\Omega^\bullet$ of finite length that is compatible with the differential. A *weighted flag structure* on $\Omega^\bullet$ consists of a weight grading for $\Omega^\bullet$ in non-positive degrees, with respect to which the differential decomposes into pieces of non-positive weight. In other words, the differential preserves the decreasing filtration associated to the weight degree.

From our perspective, there are (at least) three important and natural examples of flag structures. The first of these is an essential motivating example: the Hodge filtration on the de Rham complex of an (almost) complex manifold. The second is more obviously related to the examples related to supersymmetric field theory that we have in mind as applications: any flat superspace is equipped with a canonical distribution, defined by considering the span of all translation-invariant odd vector fields. More generally, the supermanifolds that are valid backgrounds for supersymmetric field theories or supergravity theories are equipped with a maximally non-involutive odd distribution, modelling the local supersymmetry transformations. (This is well-known; consider, for example, the definition of a super Riemann surface [Fri+86; RSV88; Wit19]. The idea goes back at least to Manin in [Man85; Man84].) The third centers around the observation that our definition is closely connected to a set of structures that appear in the theory of *Tanaka prolongation* for filtered structures.[1] We will not delve deeply into connections to that theory, or to parabolic geometry more broadly, here, though these are certainly of great interest. We will return to them in future work; for now, the interested reader is referred to [Tan70; Zel09; CS09].

[1]We owe deep thanks to John Huerta for calling our attention to the relevance of Tanaka's work.

In some sense, the usefulness of the definition lies in the fact that it brings the three classes of examples under one roof. In particular, our main application—to eleven-dimensional supergravity—will rely on exploiting the analogy between instances of the first two types. To get to these examples, we need to construct the generalization of Dolbeault cohomology to this more general setting. This will be done in the next section. We then move on to discuss examples in §7.2.3.

7.2.2 D_∞ algebras from weighted flag structures

Given a weighted flag structure, we can regrade $\Omega^\bullet$ with respect to the sum of the weight grading and the cohomological grading. (The filtration associated to this totalized grading recovers $F^\bullet_+$.) Having done this, the differential d decomposes as a sum of terms

$$\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_0 + \mathrm{d}_{-1} + \cdots \tag{7.10}$$

with respect to the totalized grading. (All terms have cohomological degree one.)

We observe that d_1 itself defines a differential of square zero. (This is the differential on $\operatorname{Gr} F^\bullet(\Omega^\bullet)$.) We will now choose to regard this differential as "internal," and the additional terms $\mathrm{d}_0 + \mathrm{d}_{-1} + \cdots$ as defining a further structure on $\operatorname{Gr} F^\bullet(\Omega^\bullet)$.

Recall that a square-zero endomorphism can be thought of as the defining data of an algebra structure over the operad D governing square-zero differentials. (See, for example, [Val14].) This operad has a single operation d_0 of arity one, subject to the relation that its concatenation with itself vanishes. We view it as a dg operad in totalized degree zero. (From the perspective of the D_∞ structure, the cohomological degree is the *totalized* degree.)

A D-algebra in cochain complexes is thus almost the same thing as a bicomplex, except for the fact that the second grading has been forgotten. We could restore it by giving an action of $U(1)$ on the operad D with respect to which the nontrivial operation has weight one, and asking for an equivariant D-algebra structure on a weighted cochain complex.

Due to the relation $\mathrm{d}_0^2 = 0$, the operad D is not free, and does not play well with quasi-isomorphisms. As is standard in homotopical algebra, we must replace D by a freely generated dg operad that resolves it. This operad D_∞ is generated by one operation d_i for each nonpositive i, all of which have arity one. The conditions defining a D_∞ algebra structure in cochain complexes amount to the condition that the total differential

$$\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_0 + \mathrm{d}_{-1} + \cdots \tag{7.11}$$

is of square zero, where d_1 is the internal differential of the cochain complex and d_i for $i \leq 0$ encode the D_∞ algebra structure. A weighted flag structure thus defines a D_∞ algebra structure on $\mathrm{Gr}\, F^\bullet(\Omega^\bullet)$ with respect to the totalized degree.

Homotopy transfer. Since D_∞ is a good homotopy replacement for D, one can use homotopy transfer of D_∞ algebra structures to pass between different quasi-isomorphic models. This encodes, in particular, the higher differentials of the spectral sequence of a bicomplex. We thus consider the cohomology

$$W^\bullet := H^\bullet\left(\mathrm{Gr}\, F^\bullet(\Omega^\bullet)\right) = H^\bullet\left(\Omega^\bullet, \mathrm{d}_1\right). \tag{7.12}$$

Since d_1 is homogeneous for the weight grading, $W^\bullet$ is again bigraded, by weight and cohomological degree—or equivalently, by cohomological degree and totalized degree. We will find it more convenient to work with the totalized degree in the sequel.

We can apply the homotopy transfer theorem for D_∞ algebras [LV12b] in order to obtain a new D_∞ algebra structure on $W^\bullet$. In concrete terms, this is done by fixing a retraction

$$\begin{tikzcd} (\Omega^\bullet\,,\ \mathrm{d}_1) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (W^\bullet,\, 0) \arrow[l, shift left, "i"] \end{tikzcd}. \tag{7.13}$$

The structure sheaf $A^\bullet$; geometric interpretation. When applied to a weighted flag structure, the output of the above construction is a cdga $W^\bullet$ with zero internal differential, equipped with a bigrading and a D_∞ structure. We will denote the terms of the D_∞ structure by d'_i for $i \leq 0$; the term d'_i has cohomological degree one and totalized degree i. As such, $\mathrm{d}' = \sum \mathrm{d}'_i$ is a square-zero differential of cohomological degree one, which now *does* respect the filtration $F^\bullet_+ W^\bullet$ associated to the totalized degree.

If we like, we can therefore repeat the procedure from above. $\mathrm{Gr}\, F^\bullet_+ W^\bullet$ will be a bigraded cdga with a differential of totalized degree zero. If we were to shift the totalized grading up by the cohomological degree *again*, we would get a D_∞ structure on $\mathrm{Gr}\, F^\bullet_+ W^\bullet$ with respect to that new grading. However, we will not have cause to do this. Instead, we will regard $\mathrm{Gr}\, F^\bullet_+ W^\bullet = (W^\bullet, \mathrm{d}'_0)$ as the fundamental object. We will allow ourselves to refer to this object as the *generalized Dolbeault complex.*

We have seen above that, for an integrable complex structure, $F^\bullet_+$ is nothing other than the Hodge filtration. As was worked out in [CW21], $F^\bullet_+ W^\bullet$ is the correct object to replace the standard Hodge filtration (and thus the standard Dolbeault cohomology) for non-integrable complex structures. We will discuss this in detail in examples in the next section.

The complex geometry of a complex manifold is governed by its sheaf of holomorphic functions; a good derived replacement for this sheaf is the sheaf $\Omega^{0,\bullet}$ of Dolbeault forms that smoothly resolves it. There is an obvious generalization of this structure sheaf in our setting as well. $W^\bullet$ is negatively graded with respect to the totalized grading, so that we can decompose it as a sum

$$W^\bullet = \bigoplus_{i \leq 0} W^{i,\bullet} \tag{7.14}$$

of homogeneous subspaces. This splitting is compatible with the differential on $\mathrm{Gr}\, F_+^\bullet W^\bullet$. As such, we can consider the cdga $A^\bullet := (W^{0,\bullet}, \mathrm{d}_0')$ sitting in totalized degree zero; this should be viewed as the structure sheaf of the geometry we are considering. As we will see in the next section, applying this construction to examples arising from superspaces produces the canonical supermultiplet—and therefore, among other physically important examples, the eleven-dimensional supergravity multiplet. Pursuing this analogy with complex geometry further will allow us to produce the interactions of eleven-dimensional supergravity from a holomorphic Poisson structure on this ringed space, reproducing and generalizing work of Cederwall [Ced10c; Ced10a].

7.2.3 Examples of weighted flag structures

Complex manifolds. Let X be a complex manifold; locally, we can equip X with corresponding coordinates $(z^i, \bar{z}^i)$. We consider the de Rham complex on X,

$$\left(\Omega^\bullet(X)\,,\ \mathrm{d} = \partial + \bar\partial\right). \tag{7.15}$$

The de Rham differential d splits into holomorphic and antiholomorphic pieces, the operators ∂ and $\bar\partial$. The cohomological grading is by form degree; to define the weight grading, we assign $\mathrm{d}\bar{z}$ weight zero and $\mathrm{d}z$ weight -1. This coresponds to the filtration

$$0 \subset D_1 = T^{(0,1)}X \subset D_2 = T_{\mathbb{C}}X \tag{7.16}$$

of the complexified tangent bundle, which we have refined to give a weighted flag structure by choosing

$$T_1X = T^{(0,1)}X, \quad T_2X = T^{(1,0)}X. \tag{7.17}$$

(Note that, for integrable complex structures, D_1 is as far as possible from being bracket-generating.) In this example, it is clear that the terms of the decomposition of the differential are

$$\mathrm{d}_1 = 0, \quad \mathrm{d}_0 = \bar\partial, \quad \mathrm{d}_{-1} = \partial, \tag{7.18}$$

with all higher terms vanishing. As a result, $W^\bullet$ can be identified with $\Omega^\bullet$, and $A^\bullet$ is the Dolbeault complex $\Omega^{0,\bullet}(X)$.

Almost complex manifolds. Nothing in the construction of the weighted flag structure above depended on the integrability of the complex structure. In fact, the construction generalizes immediately to almost complex manifolds, with the difference that D_1 is no longer involutive. Correspondingly, the internal differential d_1 no longer vanishes. We recover the theory of Dolbeault cohomology for almost complex manifolds, as worked out in [CW21].

On an almost complex manifold, the de Rham differential decomposes as

$$\mathrm{d} = \overline{\mu} + \bar{\partial} + \partial + \mu, \tag{7.19}$$

where μ and its complex conjugate $\overline{\mu}$ are related to the Nijenhuis tensor. No other terms are present. Defining the weighted flag structure considered above, we see that

$$\mathrm{d}_1 = \overline{\mu}, \quad \mathrm{d}_0 = \bar{\partial}, \quad \mathrm{d}_{-1} = \partial, \quad \mathrm{d}_{-2} = \mu. \tag{7.20}$$

Crucially, the Dolbeault differential $\bar{\partial}$ no longer squares to zero, such that standard Dolbeault cohomology is no longer well defined. But we can nevertheless construct $W^\bullet$ by first passing to the cohomology of $\bar{\mu}$:

$$W^\bullet = H^\bullet(\Omega^\bullet(X), \bar{\mu}). \tag{7.21}$$

This reproduces the construction of the Dolbeault cohomology of an almost complex manifold, as defined in [CW21]. Homotopy transfer as D_∞ algebras then produces a D_∞ structure on $W^\bullet$, which plays the role of the Hodge-to-de-Rham spectral sequence in this case.

We note that the first term in the differential, $\mathrm{d}_1 = \overline{\mu}$, can be thought of as encoding the failure of the corresponding flag of distributions to be integrable. (In the theory of filtered structures, one would say that the symbol of the flag of distributions fails to be abelian.) This is further illustrated by the next examples.

Superspaces and the canonical supermultiplet. Let $\mathfrak{n}$ be a supertranslation algebra in the sense of §2.2: a consistently $\mathbb{Z}$-graded super Lie algebra supported in degrees one and two. In our conventions here, which differ slightly, $\mathfrak{n}_1$ has weight one and odd internal parity, whereas $\mathfrak{n}_2$ has weight two and even internal parity. Let $N = \exp(\mathfrak{n})$ be the corresponding flat superspace. The de Rham complex

$$(\Omega^\bullet(N), \mathrm{d}_{\mathrm{dR}}) = \left(C^\infty(T_+)[\theta, \mathrm{d}\theta, \mathrm{d}x] \, , \, \mathrm{d}x \frac{\partial}{\partial x} + \mathrm{d}\theta \frac{\partial}{\partial \theta} \right) \tag{7.22}$$

is then a cdga equipped with a weight grading.

We can define a flag of distributions in TN by choosing D_1 to be spanned by the odd left-invariant vector fields $\left(\mathrm{Vect}(N)^N\right)_-$, and D_2 to be just TN. In physical examples in three or more dimensions, D_1 is always bracket-generating, since every translation is the square of some supercharge. Thus, the distribution we consider is maximally noninvolutive.

This flag of distributions defines a weighted flag structure on $\Omega^\bullet(N)$. Concretely, we can express the de Rham complex in a left-invariant basis

$$\lambda = \mathrm{d}\theta, \qquad v = \mathrm{d}x + \lambda\theta. \tag{7.23}$$

Then, the de Rham differential takes the form

$$\mathrm{d}_{\mathrm{dR}} = \lambda^2 \frac{\partial}{\partial v} + \lambda \left(\frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial x} \right) + v \frac{\partial}{\partial x}. \tag{7.24}$$

Note that we suppress the contractions in the notation when there is no ambiguity. The weight grading on the de Rham complex is just given by the polynomial degree in v. The differential splits according to

$$\begin{aligned} \mathrm{d}_1 &= \lambda^2 \frac{\partial}{\partial v}, \\ \mathrm{d}_0 &= \lambda \frac{\partial}{\partial \theta} - \lambda\theta \frac{\partial}{\partial x}, \\ \mathrm{d}_{-1} &= v \frac{\partial}{\partial x} \, . \end{aligned} \tag{7.25}$$

As we wil see later, the generalized Dolbeault complex,

$$W^\bullet = \left(H^\bullet(\Omega^\bullet(N), \mathrm{d}_1) \, , \, \mathrm{d}_0' \right), \tag{7.26}$$

has a natural interpretation within the pure spinor superfield formalism. In particular the degree zero piece $W^{0,\bullet}$ coincides with the canonical multiplet of $\mathfrak{n}$ [Ced+23]; the analogue of the Dolbeault resolution of holomorphic p-forms is given by the multiplet associated to the $(-p)$-th Lie algebra cohomology of the supertranslation algebra $\mathfrak{n}$, with respect to the totalized degree. We already investigated examples of such multiplets in §2.6.3; the acyclic deformation of the differential arising from the strictly negative terms in d was defined, and worked out concretely in examples, in §3.5.1.

These multiplets were discussed in detail in physical examples in [Eag+22]; the acyclic deformation of the differential arising from the strictly negative terms in d was defined, and worked out concretely in examples, in [EHS23].

Further examples; (flat) distributions of constant symbol. In the previous sections, we have already gone through the examples that will interest us in detail in the remainder of this work. Our main aim here is to set up the analogy between almost complex geometry and superspace by viewing them both as weighted flag structures, and to exploit this to give a geometric construction of interacting eleven-dimensional supergravity and its twists. However, numerous other structures could be viewed through this lens, and we feel it would be profitable to do so. We give a partial list of such examples, to which we hope to return in future work.

— Any manifold equipped with a Tanaka structure [AD17, Definition 1] has a flag structure on its de Rham complex.

— Let $\mathfrak{n}$ be a super Lie algebra equipped with a positive weight grading. Following [Zel09], we can consider the flat Tanaka structure with constant symbol $\mathfrak{n}$. By definition, this is the simply connected super Lie group $N = \exp(\mathfrak{n})$, equipped with the flag of distributions spanned by the left-invariant vector fields in $\mathfrak{n}_{\leq j}$. We observe that flat superspace is a particular example of such a flat Tanaka structure, with symbol the supertranslation algebra. It should be possible to consider non-strict examples (super L_∞ algebras with positive weight gradings), using results of Getzler [Get09].

— Any Lie algebra equipped with a finite-length positive filtration gives rise to a flag structure on its Chevalley–Eilenberg cochains.

— Any filtered Lie algebroid gives rise to a flag structure on its Lie algebroid cochains. This is a clear generalization, both of the previous example and of a flag of distributions in the tangent bundle of a manifold. It should be possible to extend this definition to Courant algebroids, following [Roy99], and then to understand potential connections to exceptional generalized geometry. In particular, connections of Tanaka prolongation to tensor hierarchy algebras [Pal14] should be interesting to explore.

7.3 Poisson–Chern–Simons theories via derived brackets

7.3.1 Holomorphic Poisson–Chern–Simons theory

In this section, we briefly review the construction of the standard Poisson–Chern–Simons theory, defined on a product of a Calabi–Yau twofold and an odd-dimensional smooth manifold. The theory is $\mathbb{Z}$-graded only when the smooth manifold is one-dimensional. Poisson–Chern–Simons theory was related to the maximal twist of eleven-dimensional supergravity in a particular omega background by Costello in [Cos16].

Let X be a Calabi–Yau twofold with holomorphic volume form Ω. In complex dimension two, Ω is also a holomorphic symplectic structure. We denote the corresponding holomorphic Poisson bivector by $\pi = \Omega^{-1}$.

Recall from §7.2.3 above that the totalized grading places $\mathrm{d}z$ in degree -1 and $\mathrm{d}\bar{z}$ in degree zero. Our construction above recovers the standard Dolbeault complex (equipped with a nonstandard grading):

$$W^\bullet = \Omega^\bullet(X), \quad \mathrm{d}_0 = \bar{\partial}, \quad \mathrm{d}_{-1} = \partial. \tag{7.27}$$

Contracting with π defines an isomorphism of $\Omega^{0,\bullet}(X)$-modules

$$\pi : \left(\Omega^{2,\bullet}(X)\,,\,\bar{\partial}\right) \longrightarrow \left(\Omega^{0,\bullet}(X)\,,\,\bar{\partial}\right), \qquad \alpha \mapsto \pi \vee \alpha. \tag{7.28}$$

One can now use this data to equip the Dolbeault complex $\Omega^{0,\bullet}(X)$ with the structure of a cyclic L_∞ algebra. This can be done in two steps:

1. Turn $\Omega^\bullet(X)$ into a BV algebra.
2. Define the Poisson bracket on $\Omega^{0,\bullet}(X)$ as a derived bracket of the BV bracket.

For the first step, note that the commutator $\Delta = [\pi, \partial]$ defines a second-order differential operator acting on $\Omega^\bullet(X)$, satisfying $\Delta^2 = 0$ and $\Delta(1) = 0$. Hence, we can define the Koszul bracket on $\Omega^\bullet(X)$ by

$$\{\alpha, \beta\} = (-1)^{|\alpha|}(\Delta(\alpha\beta) - \Delta(\alpha)\beta) - \alpha\Delta(\beta)\,, \tag{7.29}$$

making $(\Omega^\bullet(X), 1, \Delta, \{-,-\})$ into a BV algebra. This construction is due to Koszul [Kos85].

For the second step, we employ the derived bracket construction with respect to the differential ∂, as described by [KS96]. The derived bracket is defined by

$$[-,-]_\partial := \{\partial(-),-\}. \tag{7.30}$$

Crucially, this bracket does not turn all of $\Omega^\bullet(X)$ into a Lie algebra; only after restricting to an abelian subalgebra (with respect to the underived bracket $\{-,-\}$) does $[-,-]_\partial$ have the right symmetry properties. It is easy to check that the Dolbeault complex $\Omega^{0,\bullet}(X)$ is indeed such a subalgebra; from this, it follows that

$$\left(\Omega^{0,\bullet}(X)\,,\ \bar\partial\,,\ [-,-]_\partial\right) \tag{7.31}$$

is a dg Lie algebra.

Evaluating $[-,-]_\partial$ on $\alpha,\beta \in \Omega^{0,\bullet}(X)$, we find

$$[\alpha,\beta]_\partial = \{\partial\alpha,\beta\} = \pi(\partial\alpha \wedge \partial\beta)\,, \tag{7.32}$$

recovering the well known formula for the Poisson bracket. Together with the pairing induced by wedging with the holomorphic volume form Ω and integration, this makes $(\Omega^{0,\bullet}(X),\bar\partial,[-,-]_\partial)$ into a cyclic L_∞ algebra—indeed, a local L_∞ algebra with a cyclic structure of degree -2. Tensoring with an odd-dimensional smooth manifold gives a quasi-isomorphic cdga that is a local L_∞ algebra with an odd-shifted cyclic structure on the product manifold. The corresponding $\mathbb{Z}/2\mathbb{Z}$-graded BV theory is called holomorphic Poisson–Chern–Simons theory.

7.3.2 Homotopy Poisson–Chern–Simons theory

We now generalize the above setting to the context of §7.2 in order to construct a "homotopy" version of Poisson–Chern–Simons theory.

Let $(\Omega^\bullet,\mathrm{d})$ be a cdga equipped with a weighted flag structure, and let $(W^\bullet,\mathrm{d}')$ be the corresponding generalized Dolbeault complex. Let us assume that, with respect to the totalized grading, $W^\bullet$ is concentrated in degrees 0, -1, and -2. For degree reasons, the differential then splits into three pieces

$$\mathrm{d}' = \mathrm{d}'_0 + \mathrm{d}'_{-1} + \mathrm{d}'_{-2}\,. \tag{7.33}$$

Explicitly, these terms arise via homotopy transfer along the diagram (7.13).

$$\begin{aligned} \mathrm{d}'_0 &= i \circ \mathrm{d}_0 \circ p \\ \mathrm{d}'_{-1} &= i \circ (\mathrm{d}_0 h \mathrm{d}_0 + \mathrm{d}_{-1}) \circ p \\ \mathrm{d}'_{-2} &= i \circ \big((\mathrm{d}_0 h)^2 \mathrm{d}_0 + \mathrm{d}_0 h \mathrm{d}_{-1} + \mathrm{d}_{-1} h \mathrm{d}_0 \big) \circ p \end{aligned} \tag{7.34}$$

Note that the square zero condition for d' implies the following identities:

$$\begin{aligned} (\mathrm{d}'_0)^2 &= 0 \\ [\mathrm{d}'_{-1}, \mathrm{d}'_0] &= 0 \\ (\mathrm{d}'_{-1})^2 + [\mathrm{d}'_0, \mathrm{d}'_{-2}] &= 0 \\ [\mathrm{d}'_{-1}, \mathrm{d}'_{-2}] &= 0 \\ (\mathrm{d}'_{-2})^2 &= 0. \end{aligned} \tag{7.35}$$

Here, the bracket $[-,-]$ denotes the commutator of endomorphisms. As all terms are of cohomological degree one, these are all symmetric. We further assume that there is an isomorphism

$$\pi : (W^{-2,\bullet}\,,\ \mathrm{d}'_0) \longrightarrow (W^{0,\bullet}\,,\ \mathrm{d}'_0) \tag{7.36}$$

of $W^{0,\bullet}$-modules.

In summary, the D_∞ structure and the pairing π act on $W^{\bullet,\bullet}$ as indicated by the following diagram.

$$\begin{tikzcd} W^{0,\bullet} \arrow[loop above, "\mathrm{d}'_0"] \arrow[r, "\mathrm{d}'_{-1}"] \arrow[rr, bend right, "\mathrm{d}'_{-2}"'] & W^{-1,\bullet} \arrow[loop above, "\mathrm{d}'_0"] \arrow[r, "\mathrm{d}'_{-1}"] & W^{-2,\bullet} \arrow[loop above, "\mathrm{d}'_0"] \arrow[ll, bend right=60, "\pi"'] \end{tikzcd} \tag{7.37}$$

From this data, we now construct an L_∞ structure on $W^{0,\bullet}$. For this purpose we perform the appropriate generalizations of the steps described in §7.3.1.

1. Turn $W^\bullet$ into a BV_∞ algebra.

2. Define an L_∞ structure on $A^\bullet = W^{0,\bullet}$ using a derived bracket construction.

We will see that both steps can be viewed as instances of the derived bracket construction described by [Vor05; BV16].

We begin by recalling the definition of a BV_∞ algebra.

Definition 7.3.1. A BV_∞ algebra $(A, \Delta, 1)$ is a unital graded commutative algebra over $\mathbb{C}$ together with a degree one linear map $\Delta : A \longrightarrow A[\![t]\!]$ which can be expanded as

$$\Delta = \frac{1}{t}\sum_{k=1}^{\infty} t^k \Delta_k, \tag{7.38}$$

such that Δ_k is a differential operator of order at most k and

$$\Delta^2 = 0 \quad \text{and} \quad \Delta(1) = 0. \tag{7.39}$$

One can equip both $A[\![t]\!]$ and A with L_∞ structures in the following way. By identifying an element $a \in A$ by the endomorphism given by left multiplication with a, we can embed A as an abelian subalgebra into its graded Lie algebra of endomorphisms, $(\mathrm{End}(A), [-,-])$. The other way round, evaluating an endomorphism at the unit gives a right inverse to this embedding. One can define a series a series of brackets on $A[\![t]\!]$ by the following formulas [Vor05].

$$\{a_1, \ldots, a_n\}_t = [\ldots[\Delta, a_1], \ldots, a_n](1) \tag{7.40}$$

This makes $A[\![t]\!]$ into an L_∞ algebra. Note that the unary bracket is just given by Δ, while the binary bracket is then given by the well known formula for BV algebras

$$\{a_1, a_2\} = \Delta(a_1 a_2) - \Delta(a_1)a_2 - (-1)^{|a_1|} a_1 \Delta(a_2). \tag{7.41}$$

In general, the n-ary bracket can be thought of as measuring the failure of the $(n-1)$-ary bracket to be a multiderivation with respect to the algebra structure.

Further, we can extract an L_∞ algebra structure on A by taking an appropriate limit for the parameter t. We define

$$\{a_1, \ldots, a_n\} = \lim_{t\to 0} \frac{1}{t^{n-1}} \{a_1, \ldots, a_n\}_t. \tag{7.42}$$

The limit makes sense because Δ_k is a differential operator of order at most k. Note that, for this L_∞ structure, the n-ary operation is generated by Δ_n, i.e.

$$\{a_1, \ldots, a_n\} = [\ldots[\Delta_n, a_1], \ldots, a_n](1)\ . \tag{7.43}$$

Coming back to our setting, we define the operator

$$\Delta = \Delta_1 + t\Delta_2 + t^2\Delta_3 = \mathrm{d}'_0 + t[\pi, \mathrm{d}'_{-1}] + t^2[\pi, [\pi, \mathrm{d}'_{-2}]] \tag{7.44}$$

on $W^\bullet[\![t]\!]$. A direct calculation shows the following proposition.

Proposition 7.3.2. *$(W^\bullet, \Delta, 1)$ is a BV_∞ algebra. Furthermore, $W^{0,\bullet}$ is an abelian subalgebra, and $W^{<0,\bullet}$ is a subalgebra with respect to the bracket $\{-,-\}$.*

Proof. These statements can be shown by direct calculations. For example, we can examine $\Delta^2 = 0$ order by order in t. Recall the identities (7.35) for the D_∞-algebra structure on $W^\bullet$. At order t^0, $\Delta^2 = 0$ is just the square-zero condition for d'_0, while the t^1-term vanishes since d'_{-1} and d'_0 anti-commute. For the t^2-piece we find

$$[\pi, \mathrm{d}'_{-1}]^2 + [\mathrm{d}'_0, \pi \mathrm{d}'_{-2} \pi]. \tag{7.45}$$

Recall that $(\mathrm{d}'_{-1})^2 = -[\mathrm{d}'_0, \mathrm{d}'_{-2}]$. For degree reasons, the only term contributing to the first summand is $\pi(\mathrm{d}'_{-1})^2\pi$, for which we find

$$\pi(\mathrm{d}'_{-1})^2\pi = -\pi[\mathrm{d}'_0, \mathrm{d}'_{-2}]\pi = -[\mathrm{d}'_0, \pi\mathrm{d}'_{-2}\pi], \tag{7.46}$$

using compatibility between the the pairing and d'_0. All higher order pieces vanish for degree reasons. The other claims are verified by similar calculations and degree arguments. □

Proposition 7.3.2 sets the stage for the second step. We now apply the derived bracket construction to the differential

$$\mathrm{d}^t = \mathrm{d}'_0 + t\mathrm{d}'_{-1} + t^2\mathrm{d}'_{-2}. \tag{7.47}$$

Again, this first endows $W^{0,\bullet}[\![t]\!]$ with an L_∞ structure

$$\mu^t_n(a_1, \dots, a_n) = \{\dots\{\mathrm{d}^t, a_1\}, \dots a_n\} \tag{7.48}$$

and then finally $W^{0,\bullet}$ by taking the limit

$$\mu_n = \lim_{t\to 0} \frac{1}{t^{n-1}} \mu^t_n \tag{7.49}$$

The L_∞ structure then takes the following form

$$\begin{aligned} \mu_1(\alpha) &= \mathrm{d}'_0\alpha \\ \mu_2(\alpha,\beta) &= \{\mathrm{d}'_{-1}\alpha, \beta\} \\ \mu_3(\alpha,\beta,\gamma) &= \{\{\mathrm{d}'_{-2}\alpha, \beta\}, \gamma\}. \end{aligned} \tag{7.50}$$

It is useful to express this L_∞ structure in terms of the pairing π.

Proposition 7.3.3. *For $\alpha, \beta, \gamma \in W^{0,\bullet}$ we have*

$$\begin{aligned} \mu_2(\alpha,\beta) &= \pi(\mathrm{d}'_{-1}\alpha \cdot \mathrm{d}'_{-1}\beta) \\ \mu_3(\alpha,\beta,\gamma) &= \pi(\mathrm{d}'_{-2}\alpha \cdot \pi(\mathrm{d}'_{-1}\beta \cdot \mathrm{d}'_{-1}\gamma)). \end{aligned} \tag{7.51}$$

Proof. For μ_2 we have

$$\{\mathrm{d}'_{-1}\alpha, \beta\} = (-1)^{|\alpha|} \left(\pi \mathrm{d}'_{-1}(\mathrm{d}'_{-1}\alpha \cdot \beta) - (\pi(\mathrm{d}'_{-1})^2\alpha) \cdot \beta\right) , \tag{7.52}$$

where we already used that $[\pi, \mathrm{d}'_{-1}]\beta = 0$ by degree reasons. Using that d'_{-1} is a derivation for the multiplication, we find the desired result.

For μ_3, note that

$$\begin{aligned} \{\mathrm{d}'_{-2}\alpha, \beta\} &= (-1)^{|\alpha|} \left(\mathrm{d}'_{-1}\pi(\mathrm{d}'_{-2}\alpha \cdot \beta) - (\mathrm{d}'_{-1}\pi\mathrm{d}'_{-2}\alpha) \cdot \beta\right) \\ &= (\pi\mathrm{d}'_{-2}\alpha) \cdot \mathrm{d}'_{-1}\beta \in W^{-1,\bullet}, \end{aligned} \tag{7.53}$$

where we used that π is an isomorphism of $W^{0,\bullet}$-modules in the second step. Thus, we find

$$\begin{aligned} \{\{\mathrm{d}'_{-2}\alpha, \beta\}, \gamma\} &= \{(\pi\mathrm{d}'_{-2}\alpha)\mathrm{d}'_{-1}\beta, \gamma\} \\ &= (-1)^{|\alpha|+|\beta|} \left[\pi\mathrm{d}'_{-1}\left((\pi\mathrm{d}'_{-2}\alpha)(\mathrm{d}'_{-1}\beta)\gamma\right) - \pi\mathrm{d}'_{-1}\left((\pi\mathrm{d}'_{-2}\alpha)(\mathrm{d}'_{-1}\beta)\right) \cdot \gamma\right] \\ &= \pi\left((\pi\mathrm{d}'_{-2}\alpha)\, \mathrm{d}'_{-1}\beta \cdot \mathrm{d}'_{-1}\gamma\right) \end{aligned} \tag{7.54}$$

Again, using that π is a map of $W^{0,\bullet}$-modules, we find the desired result. $\square$

In the examples we are interested in and which we will discuss in the following sections, $W^{0,\bullet}$ is local, i.e. arising as a sheaf of L_∞ algebras on some manifold, and equipped with a pairing making it a cyclic L_∞ algebra. In these instances, $(W^{0,\bullet}, \mathrm{d}'_0, \mu_2, \mu_3)$ defines a perturbative interacting BV theory, perhaps after tensoring with the de Rham complex of a smooth manifold to correct for the parity of the cyclic structure. Since the L_∞ structure describing the interactions is no longer strict, we refer to such a theory as a homotopy Poisson–Chern–Simons theory.

7.4 Calabi–Yau twofolds from certain Gorenstein rings

The construction of interactions in homotopy Poisson–Chern–Simons theory can be applied to supersymmetric field theories and their twists just by working with the example of §7.2.3—that is, with the standard odd distribution on superspace. As we will show more explicitly below, this automatically places us in the context of the pure spinor superfield formalism. It remains only to check which superspaces give rise to weighted flag structures satisfying the conditions of §7.3. Of the standard superspaces that appear in physics, there are precisely three examples, corresponding to eleven-dimensional minimal supersymmetry and its two distinct twists.

We begin by discussing the compatibility between the pure spinor superfield formalism and twisting; the observations here extend [SW21]. Then we remark on the algebraic conditions required for the generalized Dolbeault complex $(W^\bullet, \mathrm{d}_0)$ of a superspace to have the properties of the Dolbeault complex of a Calabi–Yau twofold, and thus to give rise to a homotopy Poisson–Chern–Simons theory using the techniques of §7.3. In §7.5 below, we will show that the resulting theories are eleven-dimensional supergravity and its maximal and minimal twists.

7.4.1 Pure spinor superfields for twisted field theories

Let $\mathfrak{g}$ be a super Lie algebra of super Poincaré type and $\mathfrak{n}$ the corresponding supertranslation subalgebra. As witnessed in various places throughout this thesis, there is a correspondence between supertranslation algebras and generating sets of quadratic ideals in polynomial rings. Let $R = \mathrm{Sym}^\bullet(\mathfrak{n}_1^\vee)$ denote the ring of polynomial functions of $\mathfrak{n}_1$ and I the quadratic ideal generated by the equations $[Q, Q] = 0$ for $Q \in \mathfrak{n}_1$. As usual, the quotient ring R/I is the ring of functions of the nilpotence variety. Conversely, we can produce a super Lie algebra of supertranslation type from any finite sequence of quadratic equations. Let $R = \mathbb{C}[\lambda_1, \dots, \lambda_n]$ be the polynomial ring in n variables and I an ideal generated by the equations,

$$I = (\lambda^\alpha f^\mu_{\alpha\beta} \lambda^\beta), \qquad \mu = 1 \dots d, \quad \alpha, \beta = 1 \dots n. \tag{7.55}$$

We define $\mathfrak{n}$ to be the two-step nilpotent super Lie algebra

$$\mathfrak{n} = \Pi S(-1) \oplus V(-2), \tag{7.56}$$

equipped with the indicated weight grading. Here $S \cong \mathbb{C}^n$, $V \cong \mathbb{C}^d$, and the only non-trivial bracket is the map

$$[-,-] : \mathrm{Sym}^2(S) \longrightarrow V, \tag{7.57}$$

generated by the equations (7.55)—in other words, with structure constants $f^\mu_{\alpha\beta}$.

Recall that applying the pure spinor superfield functor to $C^\bullet(\mathfrak{n})$ itself recovers $\Omega^\bullet(N)$, expressed in the left-invariant frame discussed in §7.2.3

$$A^\bullet(C^\bullet(\mathfrak{n})) = (C^\infty(N) \otimes C^\bullet(\mathfrak{n})\,, \mathcal{D}) \cong (\Omega^\bullet(N), \mathrm{d}_{\mathrm{dR}}), \tag{7.58}$$

where the differential splits according to (7.25); the internal differential d_1 now coincides with the Chevalley–Eilenberg differential d_{CE} on $C^\bullet(\mathfrak{n})$. Taking cohomology with respect to d_1, we thus recover that

$$W^\bullet = A^\bullet(H^\bullet(\mathfrak{n})); \tag{7.59}$$

the differential d_0 is the standard pure spinor superfield differential, so that the generalized Dolbeault complex in totalized degree k—the analogue of the holomorphic $(-k)$-forms—consists of the supermultiplet associated by $A^\bullet_{R/I}$ to the Lie algebra cohomology group $H^k(\mathfrak{n})$, again in the totalized grading. In particular, the role of the would-be structure sheaf is played by the canonical multiplet associated to the ring R/I itself. This justifies our notation $A^\bullet = A^\bullet_{R/I}(R/I) = (W^{0,\bullet}, \mathrm{d}_0')$ from above.

Pure spinor superfields and twisting. Fixing an element $Q \in Y$, we can twist the algebra itself by defining a dg Lie algebra $(\mathfrak{g}\,,\ [Q,-])$. Its cohomology $\mathfrak{g}_Q = H^\bullet(\mathfrak{g},[Q,-])$ is again a graded Lie algebra in degrees zero to two and should be viewed as the residual symmetry algebra of any theory twisted by Q; we denote its nilpotence variety (which encodes the possible further twists of the Q-twisted theory) by Y_Q.

We call the positively graded piece of the cohomology

$$\mathfrak{n}_Q = H^{>0}(\mathfrak{g},[Q,-]) \tag{7.60}$$

the twisted supertranslation algebra. Sometimes it is convenient to work with a quasi-isomorphic dg model for $\mathfrak{n}_Q$ which keeps all the even translations in degree two. To this end we define a dg Lie algebra $\tilde{\mathfrak{n}}_Q$ by throwing away the degree zero piece of $(\mathfrak{g},[Q,-])$ while simultaneously replacing its degree one piece by the cokernel of the adjoint action of Q,

$$\tilde{\mathfrak{n}}_Q = (\mathfrak{n}_1/\mathrm{Im}([Q,-])(-1) \oplus \mathfrak{n}_2(-2)\,,\ [Q,-])\,. \tag{7.61}$$

Given any $C^\bullet(\mathfrak{n})$-module Γ, the twist by Q of the multiplet associated to Γ,

$$A^\bullet(\Gamma)^Q = (A^\bullet(\Gamma)\,,\ \mathcal{D} + \mathscr{L}(Q))\,, \tag{7.62}$$

is a multiplet for the dg Lie algebra $(\mathfrak{g}, \mathrm{ad}_Q)$. On the other hand, we can also apply the pure spinor superfield formalism directly to the residual supersymmetry algebra $(\mathfrak{g}, \mathrm{ad}_Q)$ (or equivalently its cohomology $\mathfrak{g}_Q$). As the formalism provides an equivalence of categories, there is a $C^\bullet(\mathfrak{n}_Q)$-module (or equivalenty $C^\bullet(\tilde{\mathfrak{n}}_Q)$) Γ_Q such that

$$A^\bullet(\Gamma_Q) \simeq A^\bullet(\Gamma)^Q. \tag{7.63}$$

Explicitly, we can take derived $\tilde{\mathfrak{n}}_Q$-invariants on both sides of this equation to find

$$\Gamma_Q \simeq C^\bullet(\tilde{\mathfrak{n}}_Q\,,\, A^\bullet(\Gamma)^Q). \tag{7.64}$$

Before the derived formalism was available, Saberi and Williams already recognized in examples that the twist of the canonical multiplet is equivalent to the canonical multiplet of the twisted supersymmetry algebra [SW21],

$$A^\bullet(\mathcal{O}_Y)^Q \simeq A^\bullet(\mathcal{O}_{Y_Q}). \tag{7.65}$$

With the above observations at hand, we can now show this in general.

Theorem 7.4.1. *For the structure sheaf one has $(\mathcal{O}_Y)_Q \simeq \mathcal{O}_{Y_Q}$, i.e. the twist of the canonical multiplet is equivalent the canonical multiplet of the twisted algebra.*

Proof. The proof is a short cohomology calculation similar to those in §3. Recall that the degree one piece $(\tilde{\mathfrak{n}}_Q)_1 = \mathfrak{n}_1/\mathrm{Im}([Q,-])$ Let us choose a splitting (as vector spaces)

$$\mathfrak{n}_1 = (\tilde{\mathfrak{n}}_Q)_1 \oplus \mathrm{Im}([Q,-]) \tag{7.66}$$

and let us correspondingly organize the generators of $C^\bullet(\mathfrak{n})$ as $(\lambda^1,\dots,\lambda^k,\lambda^{k+1},\dots,\lambda^n)$ such that the first k correspond to the generators of $\tilde{\mathfrak{n}}_Q$.

We then have

$$\begin{aligned}(\mathcal{O}_Y)_Q &= C^\bullet(\tilde{\mathfrak{n}}_Q\,,\, A^\bullet(\mathcal{O}_Y)^Q)\\ &= \left(C^\infty(N)\otimes\mathbb{C}[\tilde{\lambda},\tilde{v}]\otimes R/I\,,\, \mathscr{L}(Q)+\lambda^\alpha\mathscr{R}(d_\alpha)+\mathrm{d}_{CE}+\tilde{\lambda}^i\mathscr{L}(d_i)+\tilde{v^\mu}\frac{\partial}{\partial x^\mu}\right).\end{aligned} \tag{7.67}$$

Here, we denote the generators of $C^\bullet(\tilde{\mathfrak{n}}_Q)$ by $\tilde{\lambda}^i$ and $\tilde{v}^\mu$. Note that, while the index i for $\tilde{\lambda}$ only runs from 1 to k, the index μ runs over all spacetime coordinates. The generators of the ring R are (as usual) denoted by λ^α. We now use the filtration by polynomial degree in $\tilde{v}$ and take cohomology with respect to the degree one piece of the differential, given by $\tilde{v}\frac{\partial}{\partial x}$. This yields

$$(\mathcal{O}_Y)_Q \simeq \left(\mathbb{C}[\theta,\tilde{\lambda}^i,\lambda^\alpha]/(\lambda^2)\,,\,(\lambda^\alpha+\epsilon^\alpha)\frac{\partial}{\partial\theta^\alpha}+\tilde{\lambda}^i\frac{\partial}{\partial\theta^i}\right), \tag{7.68}$$

where we expanded the twisting supercharge Q into the basis $Q = \epsilon^\alpha d_\alpha$. Finally, we see that the cohomology is given by

$$(\mathcal{O}_Y)_Q \simeq \mathbb{C}[\tilde{\lambda}^i]/((\tilde{\lambda}^i + \epsilon^i)^2) = \mathcal{O}_{Y_Q}. \tag{7.69}$$

□

Whenever Q is a maximal twist, the twisted nilpotence variety Y_Q is just a point such that the associated canonical multiplet is given by a tensor product of the de Rham and Dolbeault complexes such that the above implies the following corollary.

Corollary 7.4.2. *Let Q be a maximal twist with k surviving translations´, then*

$$A^\bullet(\mathcal{O}_Y)^Q \simeq \left(\Omega^\bullet(\mathbb{R}^{d-2k}), \mathrm{d}\right) \otimes \left(\Omega^{0,\bullet}(\mathbb{C}^k), \bar{\partial}\right). \tag{7.70}$$

These considerations mean that the operation of twisting is, in a sense, fully internal to the superspace: any construction which relies only on the "(almost) complex geometry" of the weighted flag structure of a superspace, as encoded in its generalized Dolbeault complex, should behave in the same way in any twist. (Recall, for example, that the full Dolbeault complex can be reconstructed algebraically from $\Omega^{0,\bullet}$ by considering the module of Kähler differentials. We can thus think of the acyclic D_∞ structure we construct on $W^\bullet$ as related to the algebraic de Rham cohomology of the affine dg scheme $\operatorname{Spec} A^\bullet$.)

In light of the above considerations, we can bootstrap information about this maximal twist: if we have a description of an interacting theory that uses only information about the complex geometry of $\mathbb{C}^n$ (or, more precisely, the THF structure on $\mathbb{C}^n \times \mathbb{R}^{d-2n}$), then the same construction (appropriately generalized to take into account the non-involutiveness of the underlying distribution) should give a pure spinor model for the untwisted interacting theory—or for any other twist—when applied to the corresponding generalized Dolbeault complex.

More specifically, compatibility between the pure spinor superfield construction and twisting at the interacting level can be formulated in the following way. Consider a multiplet $A^\bullet(\Gamma)$ and assume that it is further equipped with an L_∞ structure making it an interacting BV theory. As discussed above, there is a quasi-isomorphism in the category of multiplets

$$A^\bullet(\Gamma)^Q \simeq A^\bullet(\Gamma^Q). \tag{7.71}$$

Explicitly, such a quasi-isomorphism can be obtained by a cohomology computation using the techniques presented in [SW21].

We can use this to formulate homotopy data

$$\begin{tikzcd} (A^\bullet(\Gamma))^Q \arrow[loop left, "h"] \arrow[r, shift left, "p"] & A^\bullet(\Gamma^Q) \arrow[l, shift left, "i"] \end{tikzcd} \tag{7.72}$$

and perform the homotopy transfer of the L_∞ structure along this diagram to $A^\bullet(\Gamma^Q)$.

Conjecture 7.4.3 (Saberi)**.** *When the L_∞ structure on $A^\bullet(\Gamma)$ is local on* $\operatorname{Spec} A^\bullet$ *(i.e. there is a BV action in terms of the pure spinor superfield), the interactions are compatible with twisting in the sense that the homotopy transfer* (7.72) *is formal.*

7.4.2 The defect, the effective dimension, and the maximal twist

To apply the construction of §7.3.2 in the pure spinor superfield formalism, we thus need to specify conditions that guarantee the existence and appropriate properties of the pairing π. In particular, we would like the generalized Dolbeault complex $W^\bullet$ to exhibit the properties of the Dolbeault complex of a Calabi–Yau twofold.

Let us fix a supertranslation algebra $\mathfrak{n}$ with corresponding polynomial ring $R = \mathrm{Sym}^\bullet(\mathfrak{n}_1^\vee)$ together with $\dim(\mathfrak{n}_2)$ generators for the quadratic ideal I and nilpotence variety Y. We call the number

$$\operatorname{def}(\mathfrak{n}) = \dim(Y) - (\dim(\mathfrak{n}_1) - \dim(\mathfrak{n}_2)) = \dim(\mathfrak{n}_2) - \operatorname{codim}(Y) \tag{7.73}$$

the *defect* of $\mathfrak{n}$.[2] Roughly, it measures how far the generators of the ideal I are from forming a regular sequence. The following proposition shows that the defect governs the support of the Chevalley–Eilenberg cohomology of $\mathfrak{n}$, and thus the "complex dimension" of $\operatorname{Spec} A^\bullet$.

Proposition 7.4.4. *Let R/I be a Cohen–Macaulay ring. Then,* $\operatorname{def}(\mathfrak{n})$ *is the smallest non-negative number such that $H^{-i}(\mathfrak{n}) \neq 0$ for all $i \geq \operatorname{def}(\mathfrak{n})$.*

Proof. Recall that the Chevalley–Eilenberg complex of $\mathfrak{n}$ is the Koszul complex on our set of generators for the ideal I. Let $H^{-n}(\mathfrak{n})$ be the top cohomology group. By depth sensitivity (see for example [Eis95, Theorem 17.4]) of the Koszul complex one has

$$\operatorname{depth}(I, R) = \dim(V) - n. \tag{7.74}$$

The Cohen–Macaulay condition implies $\operatorname{depth}(I, R) = \operatorname{codim}(Y)$ implies the claim. □

[2] If Y is not equidimensional, we take $\dim(Y)$ to denote the maximum of the dimensions of its irreducible pieces.

We can further define a local version of the defect for any orbit in the nilpotence variety. For $Q \in Y$ we set

$$\mathrm{def}(Q) = \dim(V) - \mathrm{codim}(P_0 \cdot Q). \tag{7.75}$$

The following lemma shows that the defect of Q is equal to the number of surviving translations in a twist by Q.

Lemma 7.4.5. $\mathrm{def}(Q) = \dim(H^2(\mathfrak{n}, [Q, -]))$.

Proof. Recall that

$$H^2(\mathfrak{n}, [Q, -]) \cong V/\mathrm{Im}([Q, -]) . \tag{7.76}$$

The map $[Q, -]$ induces an isomorphism

$$\mathfrak{n}_1/\ker([Q, -]) \longrightarrow \mathrm{Im}([Q, -]) \subseteq V. \tag{7.77}$$

Let $P_0 \cdot Q$ denote the orbit of Q inside Y. Recall that $P_0 \cdot Q$ sits inside $\mathfrak{n}_1$ by the inclusion $i : (P_0 \cdot Q) \hookrightarrow \mathfrak{n}_1$. The ambient space splits into directions tangent and normal to the orbit:

$$\mathfrak{n}_1 \cong T_Q(P_0 \cdot Q) \oplus N_Q(P_0 \cdot Q). \tag{7.78}$$

We can identify the tangent space with $\ker([Q, -])$ and the normal space with the quotient $\mathfrak{n}_1/\ker([Q, -])$. Thus, we find in particular

$$\mathrm{codim}(P_0 \cdot Q) = \dim(N_Q(P_0 \cdot Q)) = \dim(\mathfrak{n}_1/\ker([Q, -])). \tag{7.79}$$

and therefore

$$\begin{aligned} \mathrm{def}(Q) &= \dim(V) - \dim(\mathfrak{n}_1/\ker([Q, -])) \\ &= \dim(V/\mathrm{Im}([Q, -])) = \dim(H^2(\mathfrak{n}, [Q, -])), \end{aligned} \tag{7.80}$$

proving the claim. □

It follows from the proposition that the defect of $\mathfrak{n}$ is the local defect evaluated at a maximal twist lying in an orbit of maximal dimension. (Note that this is neither the maximum, nor the minimum, value of the local defect; Y need not be—and often is not—equidimensional.)

Gorenstein rings of defect two. Let us fix a supertranslation algebra $\mathfrak{n}$ of defect two such that the quotient ring R/I is both Gorenstein and strongly Cohen–Macaualay.[3]

[3] A quotient ring R/I is called strongly Cohen–Macaulay, when all Koszul homology groups (for R/I viewed as an R-module) are Cohen–Macaulay [Gol05].

By construction, the zeroth Chevalley–Eilenberg cohomology of $\mathfrak{n}$ yields,

$$H^0(\mathfrak{n}) = R/I. \tag{7.81}$$

Further, $H^\bullet(\mathfrak{n})$ is concentrated in degrees $0, -1$ and -2. Since R/I is strongly Cohen–Macaulay, $H^\bullet(\mathfrak{n})$ is a Poincaré duality algebra [AG71; Gol05]. In particular, we have

$$H^{-2}(\mathfrak{n}) \cong \mathrm{Ext}_R^{-\operatorname{codim}(Y)}(R/I, R) \cong R/I, \tag{7.82}$$

where we used the Gorenstein property for the last identification. Thus, there is an isomorphism of $A^\bullet(H^0(\mathfrak{n}))$-modules

$$\pi : \left(A^\bullet(H^{-2}(\mathfrak{n}))\ ,\ \mathrm{d}_0'\right) \longrightarrow \left(A^\bullet(H^0(\mathfrak{n}))\ ,\ \mathrm{d}_0'\right). \tag{7.83}$$

As we assumed that the defect of the supertranslation algebra equals two, transfer of the D_∞ along (7.13) yields an induced D_∞ structure given by (7.34).

Hence, we are in the situation described in §7.3.2 and can construct an L_∞ structure on $A^\bullet(H^0(\mathfrak{n}))$. Furthermore, the Gorenstein property implies that there is another pairing on $A^\bullet(H^0(\mathfrak{n}))$, (see §2.4), making it a cyclic L_∞ algebra and hence an interacting BV theory (after taking the product with an odd-dimensional smooth manifold to adjust the parity of the cyclic structure, if necessary).

7.5 Eleven-dimensional supergravity, both twisted and not

As mentioned above, there are three significant examples of "Calabi–Yau twofolds" that arise from superspaces relevant to physics. They are all connected to eleven-dimensional supergravity: either the full theory, or one of its two twists. In this section, we review the construction of these Gorenstein rings of defect two, and then construct the corresponding homotopy Poisson–Chern–Simons theories. These recover Cederwall's pure spinor formulation of eleven-dimensional supergravity, Costello's description of the maximal twist in terms of holomorphic Poisson–Chern–Simons theory, and a (conjectural) pure spinor lift of the interactions of minimally twisted eleven-dimensional supergravity described in [RSW23]. We also recall how the rings are related to one another by twists of the corresponding super Poincaré algebras, which, under the assumption of compatibility between the pure spinor construction and twisting, shows that the three interacting theories are also obtained from one another—in particular, from eleven-dimensional supergravity—by taking the corresponding twist

7.5.1 Eleven-dimensional supersymmetry and its twists

Let V denote the vector representation for Spin(11) and S the unique spinor representation of dimension 32. The super Poincaré algebra in eleven dimensions is of the form

$$\mathfrak{g} = \mathfrak{so}(V) \oplus S(-1) \oplus V(-2). \tag{7.84}$$

The nilpotence variety $Y \subset S$ is of dimension 23, so that $\mathrm{def}(Y) = 2$. Furthermore, its coordinate ring, which is the quotient of polynomial functions on S by the quadratic ideal generated by the eleven gamma matrices, is a Gorenstein ring. In this sense, the generalized Dolbeault cohomology of eleven-dimensional superspace describes a Calabi–Yau twofold. Furthermore, the structure sheaf of this space is nothing other than the eleven-dimensional supergravity multiplet, described with a pure spinor superfield in the BV formalism [How91b; CNT02]. As was emphasized in [SW21; Ced+23], eleven-dimensional supergravity is a canonical supermultiplet, and is thus equipped with a *commutative* structure on the space of fields.

Twists. As we have discussed in §6.2, the nilpotence variety decomposes into two orbits for $\mathrm{Spin}(V)$, as such, there are two distinct twists available. Recall that a maximal twist is a smooth point of Y, whereas the minimal twist corresponds to a singular point. As is well-known [BN05], the singularities take the form of the cone over the projective variety $\mathrm{Gr}(2, 5)$. The stabilizer of a minimal supercharge is $\mathrm{SU}(5)$, whereas the stabilizer of a maximal supercharge is $G_2 \times \mathrm{SU}(2)$.

Applying the pure spinor functor to the coordinate rings of the twisted nilpotence varieties Y_Q, one obtains the BV complexes of the free twisted theories. We can now apply our results to construct interactions for these theories in all these cases in a uniform way, realizing them as homotopy Poisson–Chern–Simons theories.

We will begin by describing the maximal twist, and work up to the full theory.

7.5.2 The maximal twist

In [Cos16] a description of the maximal twist in terms of Poisson–Chern–Simons theory was proposed. We computed the twist in the free limit using component fields in §6; it was also realized as a further twist of the minimal twist in [RSW23]. Now, we finally address the interacting theory with pure spinor methods.

Twisting the supersymmetry algebra. The maximal twist on flat spacetime is defined on $\mathbb{R}^7 \times \mathbb{C}^2$. As in §6, we begin by decomposing all relevant Spin(11)-representations to $G_2 \times \mathrm{SU}(2) \times \mathrm{U}(1)$. Recall that, under this subgroup, the vector representation of Spin(11) decomposes as

$$V = V_7 \oplus L \oplus L^\vee, \tag{7.85}$$

where V_7 is the seven-dimensional irreducible representation of G_2 and $L \cong \mathbf{2}^1$ (as well as $L^\vee \cong \mathbf{2}^{-1}$) as $\mathrm{SU}(2) \times \mathrm{U}(1)$-representations. The spin representation gives

$$S = (\mathbf{1}_{G_2} \oplus V_7) \otimes (\mathbf{2}^0 \oplus \mathbf{1}^1 \oplus \mathbf{1}^{-1}). \tag{7.86}$$

We immediately see that S contains two copies of the trivial representation of $G_2 \times \mathrm{SU}(2)$, coming with U(1) weights ± 1. These correspond to the maximal square-zero supercharges. For definiteness, we choose

$$Q \in \mathbf{1}_{G_2} \otimes \mathbf{1}^{-1}. \tag{7.87}$$

Remembering that $\mathfrak{so}(V) \cong \wedge^2 V$ and that, as G_2-representations,

$$\wedge^2 V_7 \cong V_7 \oplus \mathfrak{g}_2, \tag{7.88}$$

we can decompose the dg Lie algebra $(\mathfrak{g}, [Q, -])$ as shown in Table 7.1. Here, the arrows

$$\begin{tikzcd}
V_7 \arrow[r] & V_7 \otimes \mathbf{1}^{-1} & \\
\mathfrak{g}_2 & V_7 \otimes \mathbf{1}^{1} \arrow[r] & V_7 \\
V_7 \otimes \mathbf{2}^{1} \arrow[r] & V_7 \otimes \mathbf{2}^{0} & \mathbf{2}^{1} \\
V_7 \otimes \mathbf{2}^{-1} & \mathbf{2}^{0} \arrow[r] & \mathbf{2}^{-1} \\
\mathbf{1}^{0} \arrow[r] & \mathbf{1}^{-1} & \\
\mathbf{1}^{2} \arrow[r] & \mathbf{1}^{1} & \\
\mathbf{1}^{-2} & & \\
\mathbf{3}^{0} & &
\end{tikzcd} \tag{7.89}$$

TABLE 7.1: Decomposition under the stabilizer

represent the map $[Q, -]$. By Schur's lemma, all non-vanishing arrows are multiples of the identity; thus it is immediate to compute the cohomology. Identifying the holomorphic translations as $\mathbf{2}^1 = L$, we find a purely even Lie algebra of the form

$$\mathfrak{g}_Q = H^\bullet(\mathfrak{g}, [Q, -]) = \big(\mathfrak{g}_2 \oplus \mathfrak{sl}(L) \oplus V_7 \otimes \mathbf{2}^{-1} \oplus \mathbf{1}^{-2}\big) \oplus L(-2). \tag{7.90}$$

We note that the positively graded piece $\mathfrak{n}_Q$ is just the abelian even algebra L. The dg model $\tilde{\mathfrak{n}}_Q$ is of the form

$$\begin{array}{ccc} V_7 \otimes \mathbf{1}^1 & \longrightarrow & V_7 \\ \mathbf{2}^0 & \longrightarrow & \mathbf{2}^{-1} \\ & & \mathbf{2}^1 \end{array} . \tag{7.91}$$

Correspondingly, $\mathcal{O}_{Y_Q} = \mathbb{C}$, and the nilpotence variety is just a point. Note that both the dimension as well as the codimension are zero. As there are two surviving translations, the defect is thus $\mathrm{def}(\mathfrak{n}_Q) = 2$.

We can now apply the formalism of §7.2.3 to the twisted supertranslation algebra $\mathfrak{n}_Q$. The weighted flag structure takes D_1 to be the zero section and D_2 to be the full (holomorphic) tangent bundle. Doing so, we recover the negatively graded algebraic de Rham complex of $\mathbb{C}^2$:

$$\Omega^\bullet = \mathbb{C}[z_1, z_2][\mathrm{d}z_1, \mathrm{d}z_2], \tag{7.92}$$

with $\mathrm{d}z_i$ in totalized degree -1. The differential d_1 is trivial, and $W^\bullet = \Omega^\bullet$; the "structure sheaf," which is the canonical multiplet of $\mathfrak{n}_Q$, just consists of holomorphic functions on $\mathbb{C}^2$.

In order to give a representation as a multiplet living on $V = \mathbb{R}^7 \times \mathbb{C}^2$, we can resolve in smooth functions over V; this recovers the Dolbeault complex of $(0, \bullet)$ forms on $\mathbb{C}^2$. We note that this can be obtained directly by considering the canonical multiplet of the dg model $\tilde{\mathfrak{n}}_Q$:

$$A^\bullet(\mathcal{O}_{Y_Q}) \simeq \left(\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^\bullet(\mathbb{R}^7) \, , \, \bar{\partial}_{\mathbb{C}^2} + \mathrm{d}_{\mathbb{R}^7}\right). \tag{7.93}$$

In either case, this corresponds to the field content of the maximal twist of eleven-dimensional supergravity.

We thus find ourselves in the setting of $\mathbb{Z}/2\mathbb{Z}$-graded holomorphic Poisson–Chern–Simons theory. Constructing the L_∞ structure recovers the interactions of Poisson–Chern–Simons described in §7.3.1.

We note that the vanishing of the Chevalley–Eilenberg differential on the twisted supertranslation algebra (which directly follows from maximality of the twist) ensures that we end up with Poisson–Chern–Simons theory instead of its homotopy version. This is a general feature of maximal twists. Nonetheless, applying our construction to a non-integrable complex structure would have given rise to a non-strict Poisson–Chern–Simons theory with nonvanishing 3-ary bracket.

7.5.3 The minimal twist

The minimal twist was computed in the free limit at the pure spinor cochain level in [SW21]. Interactions for the component fields were proposed (and numerous consistency checks perfomed) in [RSW23].

Twisting the supersymmetry algebra. The stabilizer of a minimal square-zero supercharge $Q \in Y$ is isomorphic to SU(5). Choosing such a Q is equivalent to the choice of a maximal isotropic subspace $L \subset V$. The vector representation then decomposes as

$$V = L \oplus L^\vee \oplus \mathbb{C}. \tag{7.94}$$

The twisted super Poincaré algebra $(\mathfrak{g}, [Q, -])$ and its cohomology $\mathfrak{g}_Q$ were analyzed in [SW21]. The positively graded piece of the cohomology is found to be

$$\mathfrak{n}_Q \cong \Pi \wedge^2 L(-1) \oplus \wedge^4 L(-2), \tag{7.95}$$

where the bracket of two odd elements is given by the wedge product. (The parentheses refer to shifts in the weight grading.) The nilpotence variety Y_Q is isomorphic to the affine cone over the the Grassmannian $\mathrm{Gr}(2,5)$ of two-planes inside a five-dimensional vector space. One can equivalently think of this as the space of bilinear skew forms of rank two on $L^\vee$. As an affine variety, we have $\dim(Y_Q) = 7$, and therefore

$$\mathrm{def}(\mathfrak{n}_Q) = 7 - (10 - 5) = 2. \tag{7.96}$$

$\mathcal{O}_{Y_Q}$ is also Gorenstein, so that we can apply our procedure to construct interactions for $A^\bullet(\mathcal{O}_{Y_Q})$. By [SW21], the pure spinor multiplet $A^\bullet(\mathcal{O}_{Y_Q})$ is equivalent to the minimal twist of the supergravity multiplet. Our procedure thus constructs interactions for minimally twisted supergravity on the pure spinor cochain level, corresponding to a suggestion in [Ced21]. We expect that the interacting theory with this field content constructed in [RSW23] can be obtained from this cochain-level description via homotopy transfer, thus rigorously proving that the twisted eleven-dimensional supergravity theory of [RSW23]—which is intimately related to the exceptional simple linearly compact super Lie algebra $E(5|10)$—is in fact the twist of eleven-dimensional supergravity.

From above, we know that $W^\bullet$ can be constructed by considering the pure spinor multiplets associated to the Lie algebra cohomology groups of $\mathfrak{n}_Q$. The cochains are given by

$$C^\bullet(\mathfrak{n}_Q) \cong \wedge^\bullet L^\vee \otimes R, \tag{7.97}$$

where we identified

$$\mathrm{Sym}^\bullet(\mathfrak{n}_1^\vee) = R = \mathbb{C}[\lambda^{ab}]. \tag{7.98}$$

We think of λ^{ab} as a basis on $(\mathfrak{n}_Q)_1^\vee = (\wedge^2 L)^\vee$ for $a, b = 1, \ldots, 5$, and make use of the isomorphism $\wedge^4 L \cong L^\vee$. Further, we can think of $L^\vee$ as constant holomorphic one-forms on $L = \mathbb{C}^5$ with basis $\{\mathrm{d}z^a\}$. The Chevalley–Eilenberg differential is of the form

$$\mathrm{d}_{CE} = \lambda^{ab}\lambda^{cd}\varepsilon_{abcde}\frac{\partial}{\partial(\mathrm{d}z_e)}. \tag{7.99}$$

As expected for a Gorenstein ring of defect two, the cohomology is concentrated in degrees $0, -1$ and -2:

$$H^k(\mathfrak{n}_Q) \cong \begin{cases} R/I & k \in \{0, -2\} \\ M & k = -1 \\ 0 & \text{else,} \end{cases} \tag{7.100}$$

where M is the cokernel of the map

$$\phi : R \otimes \wedge^2 L \longrightarrow R \otimes L \qquad e_a \wedge e_b \mapsto \varepsilon^{abcde}\lambda_{cd}e_e. \tag{7.101}$$

Here $\{e_a\}$ is a basis of L. As R/I-modules, $H^0(\mathfrak{n}_Q)$ is freely generated by the unit 1, while $H^{-2}(\mathfrak{n}_Q)$ is freely generated by $\lambda_{ab}\mathrm{d}z^a\mathrm{d}z^b$.

After tensoring with de Rham forms on $\mathbb{R}$ in order to resolve freely over $\mathbb{C}^5 \times \mathbb{R}$, we can describe $\Omega^\bullet$ with the quasi-isomorphic complex

$$\left(\Omega^\bullet_{\mathrm{dR}}(\mathbb{C}^5) \otimes \mathbb{C}[\lambda^{ab}, \theta^{ab}]\,,\ \partial_{\mathbb{C}^5} + \bar\partial_{\mathbb{C}^5} + \mathscr{R} + \mathrm{d}_{CE}\right) \otimes (\Omega^\bullet(\mathbb{R})\,,\ \mathrm{d}_{\mathbb{R}})\,, \tag{7.102}$$

where

$$\mathscr{R} = \lambda\left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x}\right) \tag{7.103}$$

is the standard pure spinor differential. Here, the spatial coordinate x is one of $(z, \overline{z})$. Note that, with respect to the description in §7.2.3, θ is an odd function on the superspace N, whereas the one-forms are λ, $\mathrm{d}z$, and $\mathrm{d}\overline{z}$. The weighted flag structure places $\mathrm{d}z$ in totalized degree -1 and everything else in degree zero. We can identify

$$\mathrm{d}_1 = \mathrm{d}_{CE}, \quad \mathrm{d}_0 = \mathscr{R} + \bar\partial + \mathrm{d}_{\mathbb{R}}, \quad \mathrm{d}_{-1} = \partial. \tag{7.104}$$

We construct the generalized Dolbeault complex $W^\bullet$ according to the standard procedure, using the formulas for the transferred D_∞ structure above (7.34). The weighted pieces of the generalized Dolbeault complex are the pure spinor multiplets associated to the modules of (7.100). In contrast to the maximal twist, a piece of degree -2 arises, such that there is a non-vanishing map

$$\mathrm{d}'_{-2} : A^\bullet(H^0(\mathfrak{n}_Q)) \longrightarrow A^\bullet(H^{-2}(\mathfrak{n}_Q)), \tag{7.105}$$

signaling that the induced L_∞ structure will not be strict.

The Gorenstein property guarantees that there is an isomorphism

$$\pi : \left(W^{-2,\bullet}, \mathrm{d}'_0\right) \longrightarrow \left(W^{0,\bullet}, \mathrm{d}'_0\right). \tag{7.106}$$

Explicitly, π is induced from the isomorphism between $H^{-2}(\mathfrak{n}_Q)$ and $H^0(\mathfrak{n}_Q)$; thus, in terms of representatives, we have

$$\pi(\lambda_{ab}\mathrm{d}z^a\mathrm{d}z^b) = 1. \tag{7.107}$$

Hence, we obtain an L_∞ algebra structure on $A^\bullet(H^0(\mathfrak{n}_Q))$ by the formulas in Proposition 7.3.3.

7.5.4 Eleven-dimensional supergravity

Recall that the canonical multiplet associated to the eleven-dimensional supertranslation algebra is the supergravity multiplet. In [Ced10c] and [Ced10a], Cederwall constructed a consistent quartic BV action functional, recovering interacting eleven-dimensional supergravity in the pure spinor superfield formalism. We now recover these interactions as an instance of homotopy Poisson–Chern–Simons theory.

Lie algebra cohomology and $W^\bullet$. The Chevalley–Eilenberg cochains of the untwisted supertranslation algebra take the form

$$C^\bullet(\mathfrak{n}) = \left(\wedge^\bullet V^\vee \otimes R\,,\, \mathrm{d}_{CE}\right), \tag{7.108}$$

where $R = \mathrm{Sym}^\bullet(S^\vee) = \mathbb{C}[\lambda^\alpha]$ is the polynomial ring in $\{\lambda^\alpha\}$ with $\alpha = 1, \ldots, 32$. Fixing a basis $\{v_\mu\}$ of $V^\vee$, the Chevalley–Eilenberg differential takes the form

$$\mathrm{d}_{CE} = \lambda^\alpha \Gamma^\mu_{\alpha\beta}\lambda^\beta \frac{\partial}{\partial v^\mu}. \tag{7.109}$$

Again, Chevalley–Eilenberg cohomology is concentrated in degrees $0, -1$ and -2, with $H^0(\mathfrak{n})$ and $H^{-2}(\mathfrak{n})$ both being isomorphic to the ring of functions on the nilpotence variety $\mathcal{O}_Y = R/I$. The cohomology in degree -2 is spanned by the class

$$(\lambda^\alpha \Gamma^{\mu\nu}_{\alpha\beta}\lambda^\beta)v_\mu v_\nu. \tag{7.110}$$

Eleven-dimensional interactions. Applying the pure spinor superfield construction, we construct the generalized Dolbeault complex as the sum of the multiplets associated to the modules from the previous section. (We note that the multiplet $W^{-1,\bullet}$ physically corresponds to a field-strength multiplet for $W^{0,\bullet}$; this fact was already appreciated in [CNT02].)

As always, the weighted flag structure on the de Rham complex of superspace induces a D_∞-module structure on $W^\bullet$, where d'_0 is the standard pure spinor differential and d'_{-1} and d'_{-2} are both nontrivial. Restricting these differentials to $W^{0,\bullet}$ recovers Cederwall's differential operators constructed in [Ced10c; Ced10a], where

$$\mathrm{d}'_{-1} : W^{0,\bullet} \longrightarrow W^{-1,\bullet}, \tag{7.111}$$

corresponds to "R" and

$$\mathrm{d}'_{-2} : W^{0,\bullet} \longrightarrow W^{-2,\bullet} \tag{7.112}$$

corresponds to "T". Together with π induced from

$$\pi(\lambda^\alpha \Gamma^{\mu\nu}_{\alpha\beta} \lambda^\beta v_\mu v_\nu) = 1, \tag{7.113}$$

this yields an L_∞-structure on $W^{0,\bullet}$.

Chapter 8

Differential operators and twisted (2,0) supersymmetry

8.1 Introduction

In the previous chapter, we established an analogy between superspace geometry as employed by the pure spinor superfield formalism and almost complex geometry. In this perspective, the canonical multiplet $A^\bullet(R/I)$ plays the role of the structure sheaf on superspace. It is natural to ask for the analogs of standard geometric constructions in this language. For example, we can consider derivations of the cdgsa $A^\bullet(R/I)$; these then model "holomorphic vector" fields on superspace. More generally, it makes sense to consider differential operators on the canonical multiplet.

As explored above, one advantage of this perspective is that it allows for uniform descriptions of the full theory and all of its twist. This line of thought allowed us to construct the eleven-dimensional supergravity theory and its twists in a unified way as homotopy Poisson–Chern–Simons theories. M-theory allows for M5 branes and M2 branes. The effective theory on the worldvolume of a stack of N indistinguishable M5 branes is a six-dimensional superconformal field theory with $(2,0)$ supersymmetry. This theory, often simply called the $(2,0)$ theory (of type A_{N-1}), is of somewhat elusive nature; famously it contains a two-form gauge field with self-dual curvature and it is expected to not admit a formulation in terms of a variational principle. Holographic duality states that the eleven-dimensional supergravity theory in backreacted geometry (in this case $AdS_7 \times S^4$) is dual to such a worldvolume theory in the limit where N is large. This duality is best understood in the maximal twist [Cos16]. Further, the maximal twist of the A_{N-1} theory has been investigated and linked to W_N algebras [Yag12; BRR15]. Given compatibility between twisting and the pure spinor superfield formalism, it makes sense to ask for a lift of these findings to the minimal and eventually untwisted cases. In this chapter, we take first steps towards such a program.

In recent work [RSW23], the component fields of minimally twisted eleven-dimensional supergravity were linked to the infinite-dimensional super Lie algebra $E(5|10)$. In [RW22], a decomposition of $E(5|10)$ into $E(3|6)$-modules was investigated and used to compute superconformal indices in the six-dimensional (2,0) theory. We construct a pure spinor lift of this decomposition, relate it to differential operators on the canonical multiplet, and further to line bundles over the twisted nilpotence variety. In addition, we lift the comparison of these pieces to minimally twisted supergravity to pure spinor cochain level. Finally, we offer some speculations on the untwisted case.

8.2 Differential operators on the canonical multiplet

As usual, let $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{n}$ be super Lie algebra of super Poincaré type with supertranslation algebra $\mathfrak{n}$. We start by studying the derivations of the cdga $A^\bullet(R/I)$ and view the resulting multiplet $\mathrm{Der}(A^\bullet(R/I))$ as an analog of holomorphic vector fields on $\operatorname{Spec} A^\bullet(R/I)$. Moving on, we consider differential operators of any order k to obtain a family of multiplets $\mathrm{Diff}^k(A^\bullet(R/I))$ which for $k = 0$ recovers the canonical multiplet itself and for $k = 1$ coincides with the derivations. We now describe the construction of these multiplets.

Recall that, for a cdgs A a linear map $D : A \longrightarrow A$ is called a differential operator of order k if for all $a_0, \dots, a_k \in A$ we have $[a_0, [a_1, \dots [a_k, D] \dots]] = 0$. Differential operators are filtered by order

$$A^\bullet(R/I) \subseteq \mathrm{Diff}^{\leq 1}(A^\bullet(R/I)) \subseteq \mathrm{Diff}^{\leq 2}(A^\bullet(R/I)) \subseteq \dots \subseteq \mathrm{Diff}(A^\bullet(R/I)), \tag{8.1}$$

such that differential operators of degree precisely k can be defined as the quotient

$$\mathrm{Diff}^k(A^\bullet(R/I)) = \mathrm{Diff}^{\leq k}/\mathrm{Diff}^{\leq k-1}(A^\bullet(R/I)). \tag{8.2}$$

Note that the derivations $\mathrm{Der}(A^\bullet(R/I))$ differential operators of arbitrary degree $\mathrm{Diff}(A^\bullet(R/I))$ are naturally equipped with a dg Lie structure induced by the commutator.

It turns out that the multiplet $\mathrm{Der}(A^\bullet(R/I))$ is arises via the pure spinor functor from a simple sheaf on the nilpotence variety Y. Thinking of the points of Y as possible twists of a theory with $\mathfrak{n}$-supersymmetry, the stalk of this sheaf at $Q \in Y$ consists of those bosonic spacetime translations that survive in the Q-twist. We sum this up in the following theorem.

Theorem 8.2.1. *Let Γ be the R/I-module, defined as the cokernel of the map*

$$(\lambda\gamma)^{\mu}_{\beta} : \mathfrak{n}_1 \otimes R/I \longrightarrow \mathfrak{n}_2 \otimes R/I. \tag{8.3}$$

Then there is an equivalence of multiplets

$$\mathrm{Der}(A^{\bullet}(R/I)) \simeq A^{\bullet}(\Gamma). \tag{8.4}$$

Note that the support of the sheaf $\mathrm{coker}(\lambda\gamma)$, if viewed as an R-module, does not necessarily lie within the support of the sheaf R/I. This means that the use of coefficients R/I in (8.3) is essential.

Proof. We can expand a general derivation in coordinates as

$$\delta = X^{\mu}_{(x)}\frac{\partial}{\partial x^{\mu}} + X^{\alpha}_{(\theta)}\frac{\partial}{\partial \theta^{\alpha}} + X^{\alpha}_{(\lambda)}\frac{\partial}{\partial \lambda^{\alpha}} \tag{8.5}$$

with coefficient functions $X^{i}_{(A)} \in C^{\infty}(N) \otimes R/I$ for $A \in \{x, \theta, \lambda\}$ and $X_{(\lambda)}$ subject to the additional constraint

$$X^{\alpha}_{(\lambda)}\gamma^{\mu}_{\alpha\beta}\lambda^{\beta} = 0 \quad \forall \mu. \tag{8.6}$$

The differential acts as follows

$$[\mathcal{D}, \delta] = \mathcal{D}(X^{i}_{(A)})\frac{\partial}{\partial A^{i}} \pm X^{\alpha}_{(\theta)}\lambda^{\beta}\gamma^{\mu}_{\alpha\beta}\frac{\partial}{\partial x^{\mu}} \pm X^{\alpha}_{(\lambda)}\left(\frac{\partial}{\partial \theta^{\alpha}} - \gamma^{\mu}_{\alpha\beta}\theta^{\beta}\frac{\partial}{\partial x^{\mu}}\right). \tag{8.7}$$

Note that the first term in the differential is internal in the sense that it only acts on the coefficient functions of the derivation. We first take cohomology with respect to the second and the third term in the differential. Denoting with $K \subseteq R/I \otimes \mathfrak{n}_1$ the subset satisfying the constraint (8.6), these terms act via

$$\begin{tikzcd} C^{\infty}(N) \otimes R/I \otimes \mathfrak{n}_2 & C^{\infty}(N) \otimes R/I \otimes \mathfrak{n}_1 \arrow[l, "(\lambda\gamma^{\mu})_{\beta}"'] & C^{\infty}(N) \otimes K \arrow[l, hook'] \arrow[ll, bend left, "(\gamma^{\mu}\theta)_{\beta}"] \end{tikzcd} \tag{8.8}$$

such that its cohomology indeed coincides with $A^{\bullet}(\Gamma)$ as graded vector spaces. Further, the induced differential from the first "internal" piece in (8.7) precisely coincides with the pure spinor differential on $A^{\bullet}(\Gamma)$ making it an equivalence of multiplets. $\square$

From the module $\Gamma = \mathrm{coker}(\lambda\gamma)$, we can construct the family of modules $S^{k}\Gamma$ such that

$$A^{\bullet}(S^{k}\Gamma) \simeq \mathrm{Diff}^{k}(A^{\bullet}(R/I)) \tag{8.9}$$

by taking symmetric powers. Explicitly, with (8.3) defining Γ as a cokernel, the cokernel of the map

$$(S \otimes S^{k-1}V) \otimes R/I \longrightarrow S^k V \otimes R/I, \tag{8.10}$$

is $S^k\Gamma$.

Derivations and conformal supergravity. The multiplet $\mathrm{Der}(A^\bullet(R/I))$ has a meaningful interpretation in terms of conformal supergravity. Conformal supergravity theories have been constructed in various dimensions and with various amounts of supersymmetry (see for example [BSVP86; BRW83]); typically by the following recipe. One starts with a supersymmetric gauge theory and computes the conserved currents associated to the supersymmetry transformations. This gives three currents: The energy-momentum tensor associated to the even translations, the supercurrent associated to supertranslation, and a current associated to the R-symmetry. These current operators can be represented by quadratic polynomials in the fundamental field operators of the underlying gauge multiplet. Applying the supersymmetry transformation rules to these expressions generates a subrepresentation inside all local operators of the gauge theory, the supercurrent multiplet. The conformal supergravity theory is defined as the theory which couples to this supercurrent multiplet. It contains spin-2 degrees of freedom coupling to the energy momentum tensor, a spin-3/2 field coupling to the supercurrent and so on. Since the currents obey conservation equations, the fields in the conformal supergravity theory are gauged by the supertranslation algebra and R-symmetry.

Whenever the gauge theory is realized as a canonical multiplet, we find that $\mathrm{Der}(A^\bullet(R/I))$ precisely recovers the field content of these conformal supergravity theories. We list these results in §8.6.

8.3 Preliminaries

8.3.1 (Twisted) six-dimensional $(2,0)$ supersymmetry

The super Poincaré algebra and its nilpotence variety. In six-dimensions there is an exceptional isomorphism identifying $\mathrm{Spin}(6) \cong SU(4)$; under this identification, the two spinor representations S_+ and S_- correspond to the fundamental and antifundamental representations. There are $\mathrm{Spin}(6)$-equivariant isomorphisms $\wedge^2 S_\pm \cong V$, where V denotes the six-dimensional vector representation.

The six-dimensional $(2,0)$ supertranslation algbebra is

$$\mathfrak{n} = (S_+ \otimes U_2)(-1) \oplus V(-2)\,, \tag{8.11}$$

where $U_2 = (\mathbb{C}^4, \omega)$ is a four dimensional symplectic vector space and the bracket is provided by the isomorphism $\wedge^2 S_+ \cong V$ and the symplectic form. The R-symmetry group is hence $\mathrm{Sp}(2)$ such that the super Poincaré algebra is of the form

$$\mathfrak{g} = (\mathfrak{so}(6) \oplus \mathfrak{sp}(2)) \ltimes \mathfrak{n}. \tag{8.12}$$

The nilpotence variety decomposes into two orbits corresponding to the two distinct twists [ESW21; ES19b; SW23b]. The orbits are distinguished by the rank of the elements under the tensor product decomposition in $\mathfrak{n}_1$. The maximal twists correspond to rank two elements which are of the form

$$Q = \xi_1 \otimes r_1 + \xi_2 \otimes r_2 \quad \text{with} \quad \xi_1, \xi_2 \in S_+ \quad r_1, r_2 \in U_2. \tag{8.13}$$

They are topological in four directions and holomorphic in the remaining two. Correspondingly, the defect of the supertranslation algebra is $\mathrm{def}(\mathfrak{n}) = 1$. The minimal twists correspond to rank one elements

$$Q = \xi_1 \otimes r_1. \tag{8.14}$$

As a supercharge of rank one is automatically square-zero, the orbit of minimal supercharges corresponds to the space of rank one matrices four-by-four matrices. As such the minimal orbit is a Segre variety whose projectivization is $\mathbb{P}^3 \times \mathbb{P}^3$. Further, the minimal supercharges are holomorphic; their stabilizer is $SL(3)$.

The canonical multiplet and its twists. The canonical multiplet of the $(2,0)$ super Poincaré algebra is the abelian (2,0) tensor multiplet [CNT02; ESW21] whose component fields consist of a self-dual two-form gauge field, a scalar field with values in the five-dimensional vector representation of the R-symmetry group $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$, and chiral fermions taking values in the fundamental representation U_2 of the R-symmetry group. The theory is a presymplectic BV theory in the sense of [SW23b].

Let $Q \in Y$ be a maximal square-zero supercharge. Again, the twisted nilpotence variety Y_Q is a point and no further twists are possible. The maximal twist was computed using component fields techniques in [SW23b] and is described by the following dg Lie algebra

$$\left(\Omega^{0,\bullet}(\mathbb{C}) \otimes \Omega^{\bullet}(\mathbb{R}^4)\ ,\ \bar{\partial}_{\mathbb{C}} + \mathrm{d}_{\mathbb{R}^4}\right). \tag{8.15}$$

Let $Q \in Y$ now be a minimal square-zero supercharge. The choice of such a supercharge is equivalent to the choice of a maximal isotropic subspace $L \subset V$ and a line $\rho \subset U_2$. The twisted super Poincaré algebra was described in [SW21]; for the twisted supertranslation algebra we have

$$\mathfrak{n}_Q = (L \otimes U^\circ)(-1) \oplus \wedge^2 L(-2)\,, \tag{8.16}$$

where U° denotes the symplectic reduction $U_2//\rho$. The bracket is given by the wedge product and the symplectic form on U°.

Odd square-zero elements in $\mathfrak{n}_Q$ are of rank one with respect to the tensor product decomposition of $(\mathfrak{n}_Q)_1$; thus, the twisted nilpotence variety Y_Q can be identified as the space of two-by-three matrices with rank less or equal the one (the ideal I is spanned by the two-by-two minors of a two-by-three matrix). As such, we can identify the projective version of the nilpotence variety as

$$\mathcal{Y}_Q \cong \mathbb{P}^1 \times \mathbb{P}^2 \tag{8.17}$$

via the Segre embedding. The orbit structure on $\mathcal{Y}_Q$ corresponds to the natural $\mathfrak{sl}(2) \times \mathfrak{sl}(3)$-action on $\mathbb{P}^1 \times \mathbb{P}^2$. Clearly, the twisted nilpotence variety only has a single orbit which corresponds to the maximal twist realized as a further twist of the minimal one.

The canonical multiplet of $\mathfrak{g}_Q$, and thus the minimal twist of the abelian (2,0) theory, was described in [SW21]. The twist was previously computed using component field methods in [SW23b]. Explicitly its component fields take the form,

$$\mu A^\bullet(\mathcal{O}_{Y_Q}) = \left[\begin{tikzcd} \Omega^{0,\bullet}(\mathbb{C}^3) \arrow[dr, "\partial"] & & \\ & \Omega^{1,\bullet}(\mathbb{C}^3) & U^\circ \end{tikzcd}\right]. \tag{8.18}$$

8.3.2 The maximally twisted example

Let us briefly review the maximally twisted case from our perspective. These results have been worked out in great detail in the literature [Cos16; BRR15; Yag12].

Let Q be a maximal square-zero supercharge in the (2,0) supertranslation algebra. The canonical multiplet consists just of the Dolbeault-de Rham complex on $\mathbb{C} \times \mathbb{R}^4$; its minimal model is just holomorphic functions on $\mathbb{C}$ such that the superspace underlying our discussion is just $\mathbb{C}$. Therefore, the constructions of derivations and differential operators are immediate: the derivations of the canonical multiplet can be modeled by holomorphic vector fields $\mathrm{Vect}(\mathbb{C})$ and the differential operators as $\mathrm{Diff}(\mathbb{C})$ accordingly.

W_N-algebras and the boundary theory In [Yag12] it was argued that the algebra of operators on N M5 branes in the omega background is the W_N algebra. Further, in [BRR15] a subsector of the six-dimensional (2,0) theory associated to a Lie algebra $\mathfrak{g}$ was identified which can be described by the W-algebra $\mathcal{W}_{\mathfrak{g}}$; this subsector is constructed by a twist with a square-zero supercharge of the superconformal algebra of "mixed type" (i.e. of the form $Q + S$, where Q is an ordinary supercharge in the super Poincaré algebra and S is a conformal supercharge). By the arguments described in [OY19], this construction corresponds to the omega deformation of the theory and thereby contains the information on the maximal twist. In the large N limit, the W_N algebra becomes the $W_{1+\infty}$ algebra.

Maximally twisted holography Recall that the maximal twist of the bulk theory is described by Poisson–Chern–Simons theory on $\mathbb{C}^2 \times \mathbb{R}^7$,

$$\left(\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^{\bullet}(\mathbb{R}^7)\ ,\ \bar{\partial}_{\mathbb{C}^2} + \mathrm{d}_{\mathbb{R}^7}\ ,\ \{-,-\}_{PB}\right). \tag{8.19}$$

The minimal model is described by holomorphic functions on $\mathbb{C}^2$ equipped with the Poisson bracket. Introducing coordinates (z, w) on $\mathbb{C}^2$ the Poisson bracket acts as

$$\{z, w\}_{PB} = 1. \tag{8.20}$$

This theory, placed in the omega background, and its relation to a stack of M5 branes was discussed in [Cos16] (see also [Cos17] for the corresponding story with M2 branes). For our purposes, we can introduce M5 branes along

$$\mathbb{C}_z \times \mathbb{R}^4 \times \{0\} \subset \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{R}^7, \tag{8.21}$$

such that w is the coordinate for the holomorphic direction transverse to the brane.

On the other hand, considering differential operators on $\mathbb{C}$ (now thought of as the part of the brane's worldvolume with holomorphic dependence), the relevant commutation relation is

$$[z, \partial_z] = 1. \tag{8.22}$$

Already at this level, we can see how identifying the scaling in the transverse direction with the order of the differential operator makes the comparison between $(\mathrm{Diff}(\mathbb{C}), [-,-])$ and $(\mathcal{O}(\mathbb{C}^2), \{-,-\}_{PB})$ apparent. This can be made much more precise, crucially one has to include backreaction and work out the Kaluza-Klein compactification (see [Cos16] where this is worked out in the omega background).

8.4 The minimal twist

Let us now fix a minimal square-zero supercharge Q in the six-dimensional (2,0) supertranslation algebra; as usual $\mathfrak{n}_Q$ denotes the twisted algebra, and $A^\bullet(\mathcal{O}_{Y_Q})$ is the twist of the canonical multiplet.

8.4.1 Differential operators, $E(3|6)$, and line bundles

Derivations and $\mathcal{O}(0,1)$. Theorem 8.2.1 shows that $\mathrm{Der}(A^\bullet(\mathcal{O}_{Y_Q})) \simeq A^\bullet(\Gamma_{(0,1)})$, where $\Gamma_{(0,1)}$ (the notation will become clear in a moment) is the cokernel of the map induced by the bracket,

$$(L \otimes U^\circ) \otimes R/I \longrightarrow L^\vee \otimes R/I. \tag{8.23}$$

For the component fields of the multiplet, one recovers the content of $E(3|6)$:

$$\mu\,\mathrm{Der}(A^\bullet(\mathcal{O}_{Y_Q}))^\# = \left[\Omega^{2,\bullet}(\mathbb{C}^3) \quad U^\circ \otimes \Omega^{1,\bullet}(\mathbb{C}^3) \quad S^2U^\circ \otimes \Omega^{0,\bullet}(\mathbb{C}^3)\right]. \tag{8.24}$$

We note that $\mathrm{Der}(A^\bullet(\mathcal{O}_{Y_Q}))$ is naturally equipped with a dgs Lie structure, which—by homotopy transfer—gives rise to an L_∞ structure on the component fields. We expect this L_∞ structure to coincide with the Lie bracket on $E(3|6)$.

Further, the projective nilpotence variety is equipped with two natural projections,

$$\begin{tikzcd} & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^2 \end{tikzcd} . \tag{8.25}$$

such that all line bundles arise via pullbacks

$$\mathcal{O}(n,m) := \pi_1^*\mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}} \pi_2^*\mathcal{O}_{\mathbb{P}^2}(m). \tag{8.26}$$

It is easy to see that the module $\Gamma_{(0,1)}$ can be viewed as the graded global section module of the line bundle $\mathcal{O}(0,1)$,

$$\Gamma_{(0,1)} = \Gamma_*(\mathcal{O}(0,1)) = \bigoplus_{k\in\mathbb{Z}} H^0(\mathcal{O}(k,k+1)). \tag{8.27}$$

Differential operators and $\mathcal{O}(0,m)$. We can take symmetric powers of the module $\Gamma_{(0,1)}$ to obtain the modules which give rise to differential operators of higher degrees via the pure spinor functor. Geometrically, these then are graded global section modules of the line bundles $\mathcal{O}(0,m) \cong \mathcal{O}(0,1)^{\otimes m}$,

$$\Gamma_{(0,m)} = S^m \Gamma_{(0,1)} = \Gamma_*(\mathcal{O}(0,m)). \tag{8.28}$$

The corresponding multiplets

$$A^\bullet(0,m) := A^\bullet(\Gamma_{(0,m)}) \cong \mathrm{Diff}^m(A^\bullet(\mathcal{O}_{Y_Q})). \tag{8.29}$$

describe differential operators of degree m on the canonical multiplet. We find the following field content for their component fields.

$$\mu A^\bullet(0,m)^\# = \tag{8.30}$$

$$\Big[[0|m,0] \quad [1|m-1,1] \quad [0|m-2,2] \oplus [2|m-1,0] \quad [1|m-2,1] \quad [0|m-2,0] \Big]. \tag{8.31}$$

More multiplets from line bundles. Taking a step back, we can use the pure spinor superfield formalism to construct the family of multiplets for the twisted $(2,0)$ algebra corresponding to the family of line bundles $\mathcal{O}(n,m)$. This is done by the same techniques as in §4 (where the nilpotence variety was $\mathbb{P}^1 \times \mathbb{P}^3$). As before, we denote the component field multiplets associated to these modules bundles by

$$\mu A^\bullet(\Gamma_*(\mathcal{O}(n,m))) = \mu A^\bullet(n,m). \tag{8.32}$$

Recall that every homogeneous piece of the graded global section module is a finite dimensional representation of $\mathfrak{sl}_2 \times \mathfrak{sl}_3$. In terms of Dynkin labels, we can write for our line bundles:

$$\Gamma_*(\mathcal{O}(n,m))_d = H^0(\mathcal{O}(n+d, m+d)) = [n+d|m+d,0]. \tag{8.33}$$

Again, the multiplets $\mu A^\bullet(n,m)$ and $\mu A^\bullet(n+k, m+k)$ are identical up to a degree shift. Thus, it is sufficient to consider the line bundles $\mathcal{O}(n,0)$ and $\mathcal{O}(0,m)$ for $m,n \geq 0$ and it remains to describe the multiplets $A^\bullet(n,0)$ for $n > 0$.

Investigating the component fields, we find that the multiplet $\mu A^\bullet(1,0)$ is the antifield multiplet of the canonical multiplet.

$$\mu A^\bullet(1,0)^\# = \begin{bmatrix} U^\circ & \Omega^{1,\bullet}(\mathbb{C}^3) & \\ & & \Omega^{0,\bullet}(\mathbb{C}^3) \end{bmatrix} \tag{8.34}$$

For $\mu A^\bullet(2,0)$ one finds

$$\mu A^\bullet(2,0)^\# = \Big[\ S^2U^\circ \otimes \Omega^{0,\bullet}(\mathbb{C}^3) \quad U^\circ \otimes \Omega^{1,\bullet}(\mathbb{C}^3) \quad \Omega^{2,\bullet}(\mathbb{C}^3),\ \Big], \tag{8.35}$$

which is the dual of the derivations (8.24) which we identified with $E(3|6)$.

For $n \geq 3$ one finds the following field content.

$$\mu A^\bullet(n,0)^\# = \Big[\ S^nU^\circ \otimes \Omega^{0,\bullet} \quad S^{n-1}U^\circ \otimes \Omega^{1,\bullet} \quad S^{n-2}U^\circ \otimes \Omega^{2,\bullet} \quad S^{n-3}U^\circ \otimes \Omega^{3,\bullet}\ \Big] \tag{8.36}$$

We can also describe the graded global section modules of these line bundles explicitly as cokernels. Let us denote the ring of polynomial functions on $\mathfrak{n}_Q^{6d}$ by $R = \mathrm{Sym}^\bullet(\mathfrak{n}_Q^{6d}) = \mathbb{C}[\lambda_i^\mu]$ with $\mu = 1, \ldots, 3$ and $i = 1, 2$. Then, for example $\Gamma_*(\mathcal{O}(1,0))$ is the cokernel of the map

$$\varphi : \mathbb{C}^3 \otimes R \longrightarrow U^\circ \otimes R \tag{8.37}$$

that is given in components by the matrix

$$\varphi = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 \end{pmatrix}. \tag{8.38}$$

In general, for $n \geq 1$, $\Gamma_*(\mathcal{O}(n,0))$ is the cokernel of the map

$$S^{n-1}U^\circ \otimes \mathbb{C}^3 \otimes R \longrightarrow S^nU^\circ \otimes R \quad , \quad F \mapsto \lambda^\mu_{(i_n} F_{i_1 \ldots i_{n-1})\mu} \tag{8.39}$$

and the multiplets described above can be obtained by considering their minimal free resolutions.

8.4.2 Duality and the Cohen–Macaulay condition

Based on the component field multiplet constructed above (which take values in the minimal free resolutions of the input modules), we can immediately deduce the following.

Lemma 8.4.1. *The R-modules $\Gamma_*(\mathcal{O}(n,0))$ are Cohen–Macaulay if and only if $n \in \{-1,0,1,2\}$.*

In fact, we can compute the Ext-groups explicitly for all line bundles. Recall that the dualizing sheaf on $\mathbb{P}^1 \times \mathbb{P}^2$ is $\mathcal{O}(-2,-3)$ and that the codimension of the nilpotence variety is two. On general grounds, one therefore finds

$$\mathrm{Ext}_R^{-2}(\Gamma_{(n,0)}, R) \cong \Gamma_{(-n-2,-3)} = \Gamma_{(1-n,0)}(3) = \Gamma_{(0,n-1)}(n+2). \tag{8.40}$$

For $n \in \{-1, 0, 1, 2\}$, this is the only Ext-module; outside of this range, there is an additional contribution. For $n \geq 3$ the only other non-vanishing Ext-group is $\mathrm{Ext}^3_R(\Gamma_*(\mathcal{O}(n,0)), R)$, which has the following equivariant decomposition

$$\mathrm{Ext}_R^{-3}(\Gamma_{(n,0)}, R) = \bigoplus_{k=0}^{n-3} [n-3-k|k,0](-k+3). \tag{8.41}$$

In addition for the $\Gamma_{(0,m)}$ with $m \geq 2$, there is an additional contribution

$$\mathrm{Ext}_R^{-4}(\Gamma_{(0,m)}, R) = \bigoplus_{k=0}^{m-2} [k|m-k-2,0](-k+2). \tag{8.42}$$

For n in the Cohen–Macaulay range, we have

$$\mu A^\bullet(n,0)^\vee = \mu A^\bullet(1-n,0)[3] = \mu A^\bullet(0,n-1)[n+2]. \tag{8.43}$$

Outside this range, the dualizing complex is not quasi-isomorphic to a single module, but only to the above Ext-algebras.

In summary, we constructed four series of multiplets each parametrized by a natural number: the multiplets $A^\bullet(n,0)$, $A^\bullet(0,m)$, and their respective antifield multiplets. Only within the Cohen–Macaulay range these overlap. We will now see how these appear in the comparison to minimally twisted eleven-dimensional supergravity.

8.4.3 The comparison to minimally twisted supergravity

Minimally twisted eleven-dimensional supergravity and $E(5|10)$. The component fields of minimally twisted eleven-dimensional supergravity at free level were described in [SW21; Ced21], interactions were conjectured (and numerous consistency checks performed) in [RSW23]. Explicitly, the free component fields on $\mathbb{C}^5 \times \mathbb{R}$ are the $\mathbb{Z}/2\mathbb{Z}$ graded BV theory described by the following cochain complex.

$$\left[\begin{tikzcd}[column sep=small] (\Omega^{0,\bullet}(\mathbb{C}^5), \bar\partial) \arrow[dr, "\partial"] & & & \\ & (\Omega^{1,\bullet}(\mathbb{C}^5), \bar\partial) & (\mathrm{PV}^{1,\bullet}(\mathbb{C}^5), \bar\partial) \arrow[dr, "\partial_\Omega"] & \\ & & & (\mathrm{PV}^{0,\bullet}(\mathbb{C}^5), \bar\partial) \end{tikzcd}\right] \otimes (\Omega^\bullet(\mathbb{R}), \mathrm{d}) \tag{8.44}$$

Further, in [RSW23] it is also shown that the minimal model of the eleven-dimensional supergravity multiplet is an L_∞ central extension of the infinite-dimensional simple super Lie algebra $E(5|10)$.

Recall that $E(5|10)$ can be described as follows. Its even piece consists of divergence free vector fields on $\mathbb{C}^5$,

$$E(5|10)_+ = \mathrm{Vect}_0(\mathbb{C}^5), \tag{8.45}$$

while its odd piece is closed two forms

$$E(5|10)_- = \Omega^2_{cl}(\mathbb{C}^5). \tag{8.46}$$

The bracket of two even elements is just the bracket of vector fields, even elements act on the odd piece via the Lie derivative, and the bracket of two odd elements is defined by

$$[\alpha, \beta] = \iota_{\Omega^{-1}}(\alpha \wedge \beta), \tag{8.47}$$

where Ω^{-1} is the holomorphic volume form on $\mathbb{C}^5$. It is immediate to see that he cohomology of the complex (8.44) coincides with $E(5|10)$ up tp a copy of $\mathbb{C}$; this copy is responsible for the central extension [RSW23].

Relating twisted supersymmetry in six and eleven dimensions. Recall that choosing a minimal square-zero supercharge $Q \in \mathfrak{g}^{11d}$ fixes a decomposition $V_{11} = L \oplus L^\vee \oplus \mathbb{C}$ and that the twisted supertranslation algebra $\mathfrak{n}_Q^{11d}$ takes the form

$$\mathfrak{n}_Q \cong \wedge^2 L(-1) \oplus L^\vee(-2). \tag{8.48}$$

It is clear that this algebra embeds into $E(5|10)$ as constant two-forms and vector fields.

We now decompose the five-dimensional isotropic subspace of holomorphic translations $L^\vee$ as

$$L^\vee = Z^\vee \oplus W^\vee \,, \tag{8.49}$$

where Z is of dimension three and W is of dimension two. It is intuitive to think of this decomposition in terms of the worldvolume of an M5 brane with Z corresponding to the holomorphic directions along the brane and W to the holomorphic direction transverse to the brane.

Then the exterior square decomposes as

$$\wedge^2 L = \wedge^2 Z \oplus (Z \otimes W) \oplus \wedge^2 W. \tag{8.50}$$

Using this decomposition, we can equip $\mathfrak{n}_Q^{11d}$ with the following $\mathbb{Z} \times \mathbb{Z}$-grading.

$$\begin{array}{c|cc} & 1 & 2 \\ \hline -1 & \wedge^2 W & \\ 0 & Z \otimes W & Z^\vee \\ 1 & \wedge^2 Z & W^\vee \end{array} \tag{8.51}$$

By identifying the $Z^\vee$ with the space of holomorphic translations determined by the minimal twist in six dimensions and W with the the residual R-symmetry representation U°, we see that the subalgebra sitting in degree zero corresponds to the minimally twisted six-dimensional $(2,0)$ algebra $\mathfrak{n}_Q^{6d}$.

Similar to the symmetry enhancement of $\mathfrak{n}_Q^{11d}$ to $E(5|10)$, the residual supersymmetry algebra $\mathfrak{n}_Q^{6d}$ also enhances to the infinite dimensional super Lie algebra $E(3|6)$ [RW22]. The even piece of $E(3|6)$ consists of holomorphic vector fields and $\mathfrak{sl}_2$-valued holomorphic functions on $\mathbb{C}^3$

$$E(3|6)_+ = \mathrm{Vect}(\mathbb{C}^3) \oplus \mathcal{O}(\mathbb{C}^3) \otimes \mathfrak{sl}_2, \tag{8.52}$$

while its odd piece is holomorphic one forms with values in the fundamental representation of $\mathfrak{sl}_2$,

$$E(3|6)_- = \Omega^1(\mathbb{C}^3) \otimes \mathbb{C}^2. \tag{8.53}$$

Further, the relation between $\mathfrak{n}_Q^{11d}$ and $\mathfrak{n}_Q^{6d}$ is compatible with these symmetry enhancements, i.e. there is an additional grading on $E(5|10)$ such that $E(3|6)$ is a subalgebra sitting in degree zero and such that every graded piece of $E(5|10)$ is an $E(3|6)$-module [KR01].

The comparison at component field level. In [RW22] the aforementioned decomposition of $E(5|10)$ into $E(3|6)$-modules was investigated in the context of minimally twisted eleven-dimensional supergravity and used to propose a holographic approach to the six-dimensional superconformal index. This decomposition, called the fivebrane decomposition, is concentrated in degrees above -1, and has $E(3|6)$ sitting in degree zero. The individual graded components were explicitly described in [RW22].

Observation 8.4.2. The degree n piece of the fivebrane decomposition constructed in [RW22] is identical to $\mu A^\bullet(n+2,0)^\vee$.

In terms of holography, it is instructive to think of the different pieces appearing in the decomposition as analogous to the Kaluza-Klein modes of the bulk theory (as we haven't included the backreaction in any meaningful way, this comparison is more at a schematic level). These modes are dual to CFT operators living on the boundary via a coupling prescription. Mathematically, this coupling prescription is realized by Koszul duality [CL16; CP21]. One can speculate that the appearance of the duals in the comparison 8.4.2 is related to this picture.

In the maximal twist, we were able to directly identify the different orders of differential operators with powers in the coordinate transverse to the brane. Here, however, the situation is slightly more complicated. Recall that the differential operators on the canonical multiplet correspond to the line bundles $\mathcal{O}(0,m)$ with $m \geq 0$. However, the multiplets appearing the comparison 8.4.2 are the antifield multiplets of those associated to $\mathcal{O}(n,0)$. We can think of these as being generated by the dualizing complexes of $\mathcal{O}(n,0)$. In the Cohen–Macaulay range, the dualizing complex is quasi-isomorphic to a dualizing sheaf and can be identifed (up to a global shift) with a line bundle of the form $\mathcal{O}(0,m)$. In general, however, an additional Ext-group is present, such that the multiplets in the decomposition 8.4.2 differ from differential operators on superspace. It is tempting to speculate that forming the procedure of computing the dualizing complex on the nilpotence variety acts as a proxy for Koszul duality in this holographic comparison.

We note that this complication is not visible in the maximal twist as the maximally twisted nilpotence variety is just a point and all modules are automatically Cohen–Macaulay.

The comparison at pure spinor cochain level. Recall that the twisted nilpotence variety of eleven-dimensional minimal supersymmetry is the cone over the affine Grassmannian $Gr(2,5)$. In terms of the decomposition (8.49), we can describe its ring of functions as follows. We identify polynomial functions on $(\mathfrak{n}_Q^{11d})_1$ as

$$R = \mathbb{C}[\lambda_{ww}, \lambda_{zw}, \lambda_{zz}], \tag{8.54}$$

with

$$\lambda_{ww} \in \wedge^2 W \quad \lambda_{zw} \in Z \otimes W \quad \lambda_{zz} \in \wedge^2 Z. \tag{8.55}$$

The defining ideal of the nilpotence variety takes the form

$$I = (\lambda_{zw}^2 + \lambda_{ww}\lambda_{zz}, \lambda_{zz}\lambda_{zw}). \tag{8.56}$$

In this way, the canonical multiplet can be described as

$$A^{\bullet}_{11d}(\mathcal{O}_{Gr(2,5)}) = (C^\infty(Z \times W) \otimes \mathbb{C}[\theta_{ww}, \theta_{zw}, \theta_{zz}] \otimes R/I\ ,\ \mathcal{D})\,, \tag{8.57}$$

where the differential is of the form

$$\begin{aligned}\mathcal{D} =& \lambda_{zz}\frac{\partial}{\partial\theta_{zz}} + \lambda_{zw}\frac{\partial}{\partial\theta_{zw}} + \lambda_{ww}\frac{\partial}{\partial\theta_{ww}} + (\lambda_{zz}\theta_{zw} + \lambda_{zw}\theta_{zz})\frac{\partial}{\partial w} \\ &+ (\lambda_{zz}\theta_{ww} + \lambda_{ww}\theta_{zz} + 2\lambda_{zw}\theta_{zw})\frac{\partial}{\partial z}.\end{aligned} \tag{8.58}$$

Recall that the ring of functions on the minimally twisted six-dimensional nilpotence variety is obtained from the above variety by intersecting with the plane where $\lambda_{ww} = 0 = \lambda_{zz}$ such that its ring of functions is given by $\mathbb{C}[\lambda_{zw}]/(\lambda_{zw}^2)$.

We now start from the canonical multiplet in eleven dimensions and apply a spectral sequence to find a quasi-isomorphism to the multiplets obtained from line bundles over the minimally twisted nilpotence variety in six dimensions. This can be seen as a lift of the identification 8.4.2 to pure spinor cochain level.

Theorem 8.4.3. *There is a quasi-isomorphism*

$$A^\bullet_{11d}(\mathcal{O}_{Gr(2,5)}) \simeq \bigoplus_{n\geq 1} A^\bullet_{6d}(n,0)^\vee. \tag{8.59}$$

Proof. We observe that we can filter the complex $A^\bullet(\mathcal{O}_{Gr(2,5)})$ by assigning weight zero to any combination of λ variables and weight one to the remainder. Considering this filtration together with the canonical filtration allows us to construct a spectral sequence beginning with the cohomology of any individual term $\lambda\partial/\partial\theta$.

We want to compute the cohomology with respect to the differential

$$\tilde{\mathcal{D}} = \lambda_{ww}\frac{\partial}{\partial\theta_{ww}} + \lambda_{zz}\frac{\partial}{\partial\theta_{zz}}. \tag{8.60}$$

This can be done in the following way. First, one replaces R/I with a minimal free resolution (L, d_L) in free R-modules, such that one obtains a quasi-isomorphic complex

$$\left(A^\bullet(\mathcal{O}_{Gr(2,5)}), \tilde{\mathcal{D}}\right) \simeq \left(C^\infty(Z\times W)\otimes\mathbb{C}[\theta_{ww},\theta_{zw},\theta_{zz}]\otimes L^\bullet\ ,\ \tilde{\mathcal{D}} + d_L\right). \tag{8.61}$$

The precise form of the minimal free resolution can easily be computed using Macaulay2. Working with the complex on the right hand side, we now consider the spectral sequence that first takes cohomology with respect to $\tilde{\mathcal{D}}$. As we have resolved in free R-modules, the λ_{ww} and θ_{ww} as well as λ_{zz} and θ_{zz} form trivial pairs. Thus, one obtains for the complex on the first page

$$\left(C^\infty(Z\times W)\otimes\mathbb{C}[\theta_{zw}]\otimes(L^\bullet\otimes_R\mathbb{C}[\lambda_{zw}])\ ,\ d_L|_{\lambda_{ww}=0=\lambda_{zz}}\right), \tag{8.62}$$

where the differential is just obtained by restricting the resolution differential. In a second step, we take cohomology with respect to the remaining pieces of the resolution differential. One finds that the cohomology is concentrated in degrees zero and one. In degree one we find,

$$C^\infty(Z \times W) \otimes \mathbb{C}[\theta_{zw}, \lambda_{zw}]/(\lambda_{zw}^2) = A^\bullet_{6d}(0,0). \tag{8.63}$$

The degree one piece is identified as

$$C^\infty(Z \times W) \otimes \mathbb{C}[\theta_{zw}] \otimes \Gamma_{(1,0)}. \tag{8.64}$$

In terms of representatives in the total complex, we can think of the degree one piece as being generated by the elements $\lambda_{zw}\theta_{zz}$.

It now suffices to remember that the total complex carries an additional differential of the form $\lambda_{zw}\theta_{zz}\frac{\partial}{\partial w}$. This induces a map between the piece in resolution degrees zero and one. Expanding in polynomial degree in the variable w, this map is of the form

$$C^\infty(Z)[\theta_{zw}, \lambda_{zw}]/(\lambda_{zw}^2) \otimes \mathrm{Sym}^k(W) \longrightarrow C^\infty(Z)[\theta_{zw}] \otimes \Gamma_{(1,0)} \mathrm{Sym}^{k-1}(W). \tag{8.65}$$

This map can be explicitly analyzed and the cohomology computed using computer algebra software. As a result, one finds

$$\bigoplus_{n \geq 1} C^\infty(Z)[\theta_{zw}] \otimes \mathrm{Ext}^\bullet_R(\Gamma_{(n,0)}, R). \tag{8.66}$$

In the Cohen–Macaulay range, i.e. for $n = 1$ and $n = 2$, this already gives the desired result (see (8.43)). For $n \geq 3$ there are two Ext-groups present and there is an additional differential arising via homotopy transfer which provides the equivalence to the antifield multiplet $\mu A^\bullet(n,0)^\vee$. Explicitly, let h be a homotopy for the differential $\lambda_{zw}\theta_{zz}\frac{\partial}{\partial w}$; then the differential is given by terms of the form

$$\left(\lambda_{zw}\frac{\partial}{\partial\theta_{zw}}\right) \circ h \circ \left(\lambda_{zw}\frac{\partial}{\partial\theta_{zw}}\right) \tag{8.67}$$

which provide maps

$$C^\infty(Z)[\theta_{zw}] \otimes \mathrm{Ext}^{-3}_R(\Gamma_{(n,0)}, R) \longrightarrow C^\infty(Z)[\theta_{zw}] \otimes \mathrm{Ext}^{-2}_R(\Gamma_{(n,0)}, R). \tag{8.68}$$

□

8.5 Untwisted physics

Working with the full nilpotence variety of (2,0) supersymmetry, one can hope to spell out the untwisted version of this story. Here, we take first steps in this direction by computing the derivations of the abelian tensor multiplet (which is the canonical multiplet of six-dimensional (2,0) supersymmetry), investigating the some symmetric powers of the corresponding modules and their Ext-groups.

It is straightforward to compute the multiplet corresponding to the tangent sheaf of $\operatorname{Spec} A^\bullet$. Its component fields in degree one are those of the $\mathcal{N} = (2,0)$ conformal supergravity multiplet of Bergshoeff, Sezgin, and van Proeyen [BSVP86]. In our conventions, the underlying dg vector bundle takes the following form:

$$\begin{tikzcd}[column sep=small]
\mathrm{Vect} \arrow[dr] & S_+ \otimes [01] \arrow[dr] & \Omega^0 \otimes \mathfrak{sp}(2) \arrow[dr] & & & \\
& \mathrm{Sym}^2(T\mathbb{R}^6)_0 & \Sigma^+_{3/2} \otimes [01] & \begin{matrix} \Omega^1 \otimes \mathfrak{sp}(2) \\ \Omega^3_+ \otimes [10] \end{matrix} & S_- \otimes [11] & \Omega^0 \otimes [20]
\end{tikzcd} \tag{8.69}$$

Here, we use B_2 Dynkin labels, so that [10] is the five-dimensional vector representation of Spin(5) and [01] the four-dimensional spin representation of Spin(5)—equivalently, the defining representation of $\mathfrak{sp}(2)$. $\Sigma^+_{3/2}$ denotes the spin-3/2 piece in the tensor product $\Omega^1 \otimes S_+$.

In principle, one can now move on and mimic our procedure from the minimal twist. Let Γ denote the module such that $A^\bullet(\Gamma) \simeq \mathrm{Der}(A^\bullet(R/I))$ as specified in Theorem 8.2.1. Forming symmetric powers, one can compute the multiplets of higher order differential operators. Based on the appearance of the dualizing complexes in the minimal twist, one should then move on to investigate the Ext-groups $\mathrm{Ext}^\bullet_R(S^k\Gamma, R)$.

For $k = 1$, we immediately that the module Γ is Cohen-Macaulay such that the only non-vanishing Ext-group is $\mathrm{Ext}^5_R(\Gamma, R)$. The associated multiplet is then the dual of (8.69) and represents the currents to which the conformal supergravity multiplet couples.

For $k \geq 2$, the modules $S^k\Gamma$ are no longer Cohen–Macaulay. In analogy to the minimally twisted case, one expects the multiplets $A^\bullet(\mathrm{Ext}^5_R(S^k\Gamma, R))^\vee$ to appear in an untwisted version of the comparison theorem 8.4.3, or respectively $\mu A^\bullet(\mathrm{Ext}^5_R(S^k\Gamma, R))^\vee$ at component field level. Unfortunately, the computation of these Ext-groups and the associated multiplets can be quite difficult in general. For $k = 2$, we find a multiplet with the following Betti numbers

$$\mathrm{grdim}(\mu A^\bullet(\mathrm{Ext}^5_R(S^2\Gamma, R))^\vee) = \begin{bmatrix} 10 & 80 & 250 & 400 & 350 & 160 & 30 \end{bmatrix}. \tag{8.70}$$

For $k = 3$, the Betti numbers are the following

$$\mathrm{grdim}(\mu A^\bullet(\mathrm{Ext}^5_R(S^3\Gamma, R))^\vee) = \begin{bmatrix} 1 & 16 & 110 & 400 & 840 & 1056 & 786 & 320 & 55 \end{bmatrix}. \quad (8.71)$$

We plan to investigate these multiplets and their relation to the Kaluza-Klein modes of eleven-dimensional supergravity in future work. Here, we simply notice the following: Recall that the abelian tensor multiplet contains a scalar field ϕ with values in the five-dimensional vector representation of the R-symmetry group; the Dynkin label of this representation is $[1, 0]$. The top component of the conformal supergravity multiplet (8.69) is a scalar field with values in the representation $[2, 0]$. As explained in [BSVP86], this is the field coupling to a quadratic current in the scalar field ϕ. The top components of the multiplets in (8.70) and (8.71) are of dimensions $30 = \dim([3, 0])$ and $55 = \dim([4, 0])$ such that one is lead to suspect that these couple to cubic and quartic expressions in ϕ.

8.6 Conformal supergravity from derivations

In the following, let us collect the field contents of the multiplets $\mathrm{Der}(A^\bullet(R/I))$ for various dimensions and amounts of supersymmetry. For this purpose, we use the description by Theorem 8.2.1 together with the techniques for the extraction of component fields described described in §2. The calculations were performed using Macaulay2.

8.6.1 Dimension one

In one dimension and $\mathcal{N}$ supercharges, the nilpotence variety is defined by the single quadratic equation,

$$\lambda_1^2 + \cdots + \lambda_{\mathcal{N}}^2 = 0. \quad (8.72)$$

The map $\lambda\gamma$, is simply given by the matrix

$$(\lambda_1, \ldots, \lambda_{\mathcal{N}}), \quad (8.73)$$

whose cokernel is just $\mathbb{C}$. Hence, $\mathrm{Der}(A^\bullet(R/I))$ can be identified with the free superfield,

$$\mathrm{Der}(A^\bullet(R/I)) \simeq C^\infty(\mathbb{R}) \otimes \wedge^\bullet(\mathbb{C}^{\mathcal{N}}). \quad (8.74)$$

8.6.2 Dimension three

Recall that $\mathrm{Spin}(3) = \mathrm{SU}(2)$; we denote the two-dimensional spin representation by S and the three dimensional vector representation by V. The supertranslation algebra is of the form

$$S(-1) \otimes U \oplus V(-2), \tag{8.75}$$

where $U \cong (\mathbb{C}^{\mathcal{N}}, (-,-))$ is equipped with a non-degenerate symmetric bilinear form and the bracket is induced by the isomorphism $\mathrm{Sym}^2(S) \cong V$ and $(-,-)$.

$\mathcal{N} = 1$. For the field content of $\mathrm{Der}(A^\bullet(R/I))$ we find,

$$\begin{bmatrix} \mathrm{Vect}(\mathbb{R}^3) & \Omega^0 \otimes S & \\ & & \\ & \mathrm{Sym}^2_0(T\mathbb{R}^3) & \Sigma_{3/2} \\ & & \end{bmatrix}. \tag{8.76}$$

Here $\Sigma_{3/2} = [3]$ denotes the spin $3/2$ representation of $\mathrm{SU}(2)$. We note that this multiplet coincides with the multiplet associated to the conormal module I/I^2 which we discussed in §2.6.

$\mathcal{N} = 2$. For $\mathcal{N} = 2$ one finds the following field content for $\mathrm{Der}(A^\bullet(R/I))$.

$$\begin{bmatrix} \mathrm{Vect}(\mathbb{R}^3) & \Omega^0 \otimes S \otimes \mathbb{C}^2 & \Omega^0 & \\ & & & \\ & \mathrm{Sym}^2_0(T\mathbb{R}^3) & \Sigma_{3/2} \otimes \mathbb{C}^2 & \Omega^1 \\ & & & \end{bmatrix} \tag{8.77}$$

8.6.3 Dimension four

We identify $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$ and denote the two two-dimensional spin representations by $S_\pm$. The supertranslation algebra is of the form

$$\left[(S_+ \otimes U) \oplus (S_- \otimes U^\vee)\right](-1) \oplus V(-2), \tag{8.78}$$

where the bracket is induced by the isomorphism $V \cong S_+ \otimes S_-$ and the natural pairing between U and its dual. We denote the Dirac spin representation by $S_+ \oplus S_- = S$ and the spin 3/2-pieces in the tensor product $V \otimes S$ as $\Sigma_{3/2} = \Sigma^+_{3/2} \oplus \Sigma^-_{3/2}$

$\mathcal{N} = 1$. One finds the following field content, which matches the conformal supergravity multiplet discussed in [FVP12].

$$\begin{bmatrix} \mathrm{Vect}(\mathbb{R}^4) & \Omega^0 \otimes S_+ \oplus S_- & \Omega^0 & & \\ & & & & \\ & \mathrm{Sym}^2_0(T\mathbb{R}^4) & \Sigma^+_{3/2} \oplus \Sigma^-_{3/2} & \Omega^1 & \\ & & & & \end{bmatrix} \tag{8.79}$$

$\mathcal{N} = 2$. Again, we find the following field content, matching the 'Weyl multiplet' of conformal supergravity as discussed in [DVV80].

$$\begin{bmatrix} \mathrm{Vect}(\mathbb{R}^4) & \Omega^0 \otimes S \otimes U & \Omega^0 \otimes \mathfrak{u}(2)_R & & & \\ & & & & & \\ & \mathrm{Sym}^2_0(T\mathbb{R}^4) & \Sigma_{3/2} \otimes U & \Omega^2 \oplus \Omega^1 \otimes \mathfrak{u}(2)_R & S \otimes U & \Omega^0 \\ & & & & & \end{bmatrix} \tag{8.80}$$

8.6.4 Dimension six

$\mathcal{N} = (1, 0)$. One finds the field content of the Weyl multiplet as presented in [BSVP86; CVP11]. Note that this is the same field content we obtained from the conormal module in §4.

$$\begin{bmatrix} \mathrm{Vect}(\mathbb{R}^6) & \Omega^0 \otimes S_+ \otimes U & \mathfrak{sp}(1) & & & \\ & & & & & \\ & \mathrm{Sym}^2_0(T\mathbb{R}^6) & \Sigma^+_{3/2} & \Omega^1 \otimes \mathfrak{sp}(1) \oplus \Omega^3_- & \Omega^0 \otimes S_+ \otimes U & \Omega^0 \end{bmatrix} \tag{8.81}$$

$\mathcal{N} = (2, 0)$. See (8.69).

8.6.5 Dimension ten

$\mathcal{N} = 1$. One finds a multiplet with the following field content.

$$\begin{bmatrix} \mathrm{Vect}(\mathbb{R}^{10}) & \Omega^0 \otimes S_+ & & & & \\ & \mathrm{Sym}^2_0(T\mathbb{R}^{10}) & \Omega^1 \otimes S_+ & \Omega^3 \oplus \Omega^1 & & \\ & & & & \Omega^2 \oplus \Omega^0 & \Omega^0 \otimes S_- \end{bmatrix} \tag{8.82}$$

This multiplet is closely related to the gravity multiplets constructed in [BRW83] and [BR82].

Chapter 9

Outlook

In the following, we outline some directions for future research that fit naturally with the work presented in this thesis.

Pure spinor superfields on coset spaces. As presented in this thesis, the pure spinor superfield formalism constructs multiplets on flat spacetime $\mathbb{R}^d$. Of course, once constructed, we can then put these multiplets on any appropriately structured manifold, however, this procedure will break supersymmetry in most cases. The fact that multiplets constructed via pure spinors primarily live on $\mathbb{R}^d$ stems from the requirement on the super Lie algebra to be of super Poincaré type. The construction then associates a multiplet on the supertranslation group given by $N = \exp(\mathfrak{n}) = \exp(\mathfrak{g}/\mathfrak{g}_0)$ (whose even piece is $\mathbb{R}^d$). However, supersymmetric theories also exist in other backgrounds, and there the natural super Lie algebras in consideration are not of super Poincaré type. Here, the most interesting examples are Anti de Sitter spaces. For instance, in ten dimensions $AdS_5 \times S^5$ arises as the even subspace of the quotient

$$AdS_5 \times S^5 \cong (\mathrm{PSU}(2,2|4)/(\mathrm{SO}(1,4) \times \mathrm{SO}(5)))_{\mathrm{even}} \,. \tag{9.1}$$

Generalizing the pure spinor superfield formalism to such pairs of super Lie algebras seems to be a promising way to construct supersymmetric theories in curved backgrounds. Given the relation between pure spinors and twisting this could also pave the way to direct twist calculations in backreacted geometries.

Superconformal nilpotence varieties and twists. In some supersymmetric field theories, the action of the super Poincaré algebra enhances to an action of the superconformal algebra. The natural moduli space of twists for such superconformal field theories is the nilpotence variety of the superconformal algebra. Let $\mathfrak{g}$ denote a super Poincaré algebra and $\mathfrak{U}$ the corresponding superconformal algebra. Clearly, there is an injective map of the nilpotence varieties,

$$Y_{\mathfrak{g}} \hookrightarrow Y_{\mathfrak{U}}, \tag{9.2}$$

signaling that every twist of the underlying supersymmetric theory also defines a twist of the superconformal theory. In general, however, this map is not surjective, i.e. the superconformal theory admits additional twists. Typically, one denotes supercharges in the super Poincaré algebra by Q and the additional superconformal supercharges by S. Mixed twists of the form $Q + S$ played a big role in their relation to chiral algebras [Bee+15] as well as to the omega background [OY19]. A thorough investigation of these nilpotence varieties and the corresponding twist with an eye towards the pure spinor superfield formalism (in particular in negatively curved backgrounds as suggested in the previous paragraph) seems worthwhile. This is in particular true in the context of twisted holography, where the additional twists in the super Poincaré algebra are dual to twists of the supergravity theory in the backreacted geometry.

The pure spinor superstring: Relating worldsheet and target space. In this thesis, we developed the pure spinor superfield formalism as a method for the construction and analysis of supersymmetric theories in target space. Of course pure spinor methods also play a big role in worldsheet superstring theory [Ber00]. Understanding worldsheet twists in this formalism, their relation to techniques from topological string theory as well as to the target space perspective developed in this work seems to be a crucial for developing a more holistic understanding of twisted holography.

Bibliography

[AD17] D. Alekseevsky and L. David. “Prolongation of Tanaka structures: an alternative approach”. *Annali di Matematica Pura ed Applicata (1923-)* 196 (2017), pp. 1137–1164. URL: https://doi.org/10.1007/s10231-016-0610-7.

[Ale+07] V. Alexandrov, D. Krotov, A. Losev, and V. Lysov. “On Pure Spinor Superfield Formalism”. *JHEP* 10 (2007), p. 074. arXiv: 0705.2191 [hep-th].

[All10] M. P. Allocca. *L_∞ Algebra Representation Theory.* Thesis (Ph.D.)–North Carolina State University. 2010. URL: https://repository.lib.ncsu.edu/bitstream/handle/1840.16/7035/etd.pdf.

[Asp04] P. S. Aspinwall. “D-branes on Calabi-Yau manifolds”. *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2003): Recent Trends in String Theory.* Mar. 2004, pp. 1–152. arXiv: hep-th/0403166.

[AG71] L. Avramov and E. Golod. “Homology algebra of the Koszul complex of a local Gorenstein ring”. *Mathematical notes of the Academy of Sciences of the USSR* 9.1-3 (1971), pp. 30–32. URL: https://doi.org/10.1007/BF01405047.

[BV16] D. Bashkirov and A. A. Voronov. “The BV formalism for L_∞-algebras”. *Journal of Homotopy and Related Structures* 12.2 (2016), pp. 305–327. arXiv: 1410.6432 [math.QA]. URL: https://doi.org/10.1007%2Fs40062-016-0129-z.

[Bas63] H. Bass. “On the ubiquity of Gorenstein rings”. *Mathematische Zeitschrift* 82.1 (1963), pp. 8–28. URL: https://doi.org/10.1007/BF01112819.

[BV81] I. A. Batalin and G. A. Vilkovisky. “Gauge Algebra and Quantization”. *Phys. Lett. B* 102 (1981), pp. 27–31. URL: https://doi.org/10.1016/0370-2693(81)90205-7.

[BV84] I. A. Batalin and G. A. Vilkovisky. “Closure of the gauge algebra, generalized Lie equations, and Feynman rules”. *Nucl. Phys. B* 234 (1984), pp. 106–124. URL: https://doi.org/10.1016/0550-3213(84)90227-X.

[BV85] I. A. Batalin and G. A. Vilkovisky. “Existence theorem for gauge algebra”. *J. Math. Phys.* 26 (1985), pp. 172–184. URL: https://doi.org/10.1063/1.526780.

[Bau11] L. Baulieu. “SU(5)-invariant decomposition of ten-dimensional Yang-Mills supersymmetry”. *Phys. Lett. B* 698 (2011), pp. 63–67. arXiv: 1009.3893 [hep-th].

[Bau+90] L. Baulieu, M. P. Bellon, S. Ouvry, and J.-C. Wallet. “Balatin-Vilkovisky analysis of supersymmetric systems”. *Phys. Lett. B* 252 (1990), pp. 387–394. URL: https://doi.org/10.1016/0370-2693(90)90557-M.

[Bec+18] K. Becker, M. Becker, D. Butter, and W. D. Linch. “$N = 1$ supercurrents of eleven-dimensional supergravity”. *JHEP* 05 (2018), p. 128. arXiv: 1803.00050 [hep-th].

[Bec+16] K. Becker, M. Becker, S. Guha, W. D. Linch, and D. Robbins. “M-theory potential from the G_2 Hitchin functional in superspace”. *JHEP* 12 (2016), p. 085. arXiv: 1611.03098 [hep-th].

[Bec+21] K. Becker, D. Butter, W. D. Linch, and A. Sengupta. “Components of Eleven-dimensional Supergravity with Four Off-shell Supersymmetries” (Jan. 2021). arXiv: 2101.11671 [hep-th].

[Bec+17] K. Becker et al. “Eleven-dimensional supergravity in 4D, $N = 1$ superspace”. *JHEP* 11 (2017), p. 199. arXiv: 1709.07024 [hep-th].

[BRR15] C. Beem, L. Rastelli, and B. C. van Rees. “$\mathcal{W}$ symmetry in six dimensions”. *JHEP* 05 (2015), p. 017. arXiv: 1404.1079 [hep-th].

[Bee+15] C. Beem et al. “Infinite chiral symmetry in four dimensions”. *Comm. Math. Phys.* 336.3 (2015), pp. 1359–1433.

[BR82] E. Bergshoeff and M. de Roo. “The Supercurrent in Ten-dimensions”. *Phys. Lett. B* 112 (1982), pp. 53–58.

[BRW83] E. Bergshoeff, M. de Roo, and B. de Wit. “Conformal Supergravity in Ten-dimensions”. *Nucl. Phys. B* 217 (1983). Ed. by A. Salam and E. Sezgin, p. 489.

[BSVP86] E. Bergshoeff, E. Sezgin, and A. Van Proeyen. “Superconformal Tensor Calculus and Matter Couplings in Six-dimensions”. *Nucl. Phys. B* 264 (1986). Ed. by A. Salam and E. Sezgin. [Erratum: Nucl.Phys.B 598, 667 (2001)], p. 653.

[Ber00] N. Berkovits. “Super Poincare covariant quantization of the superstring”. *JHEP* 04 (2000), p. 018. arXiv: hep-th/0001035.

[Ber01] N. Berkovits. "Covariant quantization of the superparticle using pure spinors". *JHEP* 09 (2001), p. 016. arXiv: hep-th/0105050.

[Ber02] N. Berkovits. "Towards covariant quantization of the supermembrane". *JHEP* 09 (2002), p. 051. arXiv: hep-th/0201151.

[Ber05] N. Berkovits. "Pure spinor formalism as an N=2 topological string". *JHEP* 10 (2005), p. 089. arXiv: hep-th/0509120.

[BG18] N. Berkovits and M. Guillen. "Equations of motion from Cederwall's pure spinor superspace actions". *JHEP* 08 (2018), p. 033. arXiv: 1804.06979 [hep-th].

[BN05] N. Berkovits and N. Nekrasov. "The character of pure spinors". *Lett. Math. Phys.* 74 (2005), pp. 75–109. arXiv: hep-th/0503075.

[Ber+94] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes". *Comm. Math. Phys.* 165.2 (1994), pp. 311–427. URL: http://projecteuclid.org/euclid.cmp/1104271134.

[BS96] M. Bershadsky and V. Sadov. "Theory of Kähler gravity". *Int. J. Mod. Phys. A* 11 (1996), pp. 4689–4730. arXiv: hep-th/9410011.

[BH80] L. Brink and P. S. Howe. "Eleven-Dimensional Supergravity on the Mass-Shell in Superspace". *Phys. Lett. B* 91 (1980), pp. 384–386. URL: https://doi.org/10.1016/0370-2693(80)91002-3.

[BE77] D. A. Buchsbaum and D. Eisenbud. "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3". *American Journal of Mathematics* 99.3 (1977), pp. 447–485.

[CS09] A. Cap and J. Slovák. *Parabolic geometries I.* 154. American Mathematical Soc., 2009.

[CdF91] L. Castellani, R. d'Auria, and P. Fré. *Supergravity and superstrings: a geometric perspective (in three volumes)*. World Scientific Publishing Company, 1991.

[Ced08] M. Cederwall. "N=8 superfield formulation of the Bagger-Lambert-Gustavsson model". *JHEP* 09 (2008), p. 116. arXiv: 0808.3242 [hep-th].

[Ced10a] M. Cederwall. "$D = 11$ supergravity with manifest supersymmetry". *Mod. Phys. Lett. A* 25 (2010), pp. 3201–3212. arXiv: 1001.0112 [hep-th].

[Ced10b] M. Cederwall. "Operators on pure spinor spaces". *AIP Conference Proceedings*. Vol. 1243. 1. American Institute of Physics. 2010, pp. 51–59.

[Ced10c] M. Cederwall. "Towards a manifestly supersymmetric action for eleven-dimensional supergravity". *JHEP* 01 (2010), p. 117. arXiv: 0912.1814 [hep-th].

[Ced14] M. Cederwall. "Pure spinor superfields – an overview". *Springer Proc. Phys.* 153 (2014). Ed. by S. Bellucci, pp. 61–93. arXiv: 1307.1762 [hep-th].

[Ced18a] M. Cederwall. "An off-shell superspace reformulation of $D = 4$, $\mathcal{N} = 4$ super Yang–Mills theory". *Fortschritte der Physik* 66.1 (2018), p. 1700082. URL: http://dx.doi.org/10.1002/prop.201700082.

[Ced18b] M. Cederwall. "Pure spinor superspace action for $D = 6$, $\mathcal{N} = 1$ super Yang–Mills theory". *JHEP* 05 (2018), p. 115. arXiv: 1712.02284 [hep-th].

[Ced21] M. Cederwall. "$SL(5)$ supersymmetry". *Fortsch. Phys.* 69.11-12 (2021), p. 2100116. arXiv: 2107.09037 [math.RT].

[Ced+00a] M. Cederwall, U. Gran, M. Nielsen, and B. E. W. Nilsson. "Generalized 11-dimensional supergravity". *International Conference on Quantization, Gauge Theory, and Strings: Conference Dedicated to the Memory of Professor Efim Fradkin*. Oct. 2000. arXiv: hep-th/0010042.

[Ced+00b] M. Cederwall, U. Gran, M. Nielsen, and B. E. W. Nilsson. "Manifestly supersymmetric M theory". *JHEP* 10 (2000), p. 041. arXiv: hep-th/0007035.

[Ced+05] M. Cederwall, U. Gran, B. E. W. Nilsson, and D. Tsimpis. "Supersymmetric corrections to eleven-dimensional supergravity". *JHEP* 05 (2005), p. 052. arXiv: hep-th/0409107.

[Ced+23] M. Cederwall, S. Jonsson, J. Palmkvist, and I. Saberi. "Canonical supermultiplets and their Koszul duals" (2023). arXiv: 2304.01258 [hep-th].

[CK11] M. Cederwall and A. Karlsson. "Pure spinor superfields and Born–Infeld theory". *JHEP* 11 (2011), p. 134. arXiv: 1109.0809 [hep-th].

[CNT01] M. Cederwall, B. E. W. Nilsson, and D. Tsimpis. "The Structure of maximally supersymmetric Yang-Mills theory: Constraining higher order corrections". *JHEP* 06 (2001), p. 034. arXiv: hep-th/0102009.

[CNT02] M. Cederwall, B. E. W. Nilsson, and D. Tsimpis. "Spinorial cohomology and maximally supersymmetric theories". *JHEP* 02 (2002), p. 009. arXiv: hep-th/0110069.

[CN08] M. Cederwall and B. E. Nilsson. "Pure spinors and $d = 6$ super Yang–Mills" (2008). arXiv: 0801.1428 [hep-th].

[Che54] C. C. Chevalley. *The algebraic theory of spinors*. Columbia University Press, New York, 1954, pp. viii+131.

[CW21] J. Cirici and S. O. Wilson. "Dolbeault cohomology for almost complex manifolds". *Advances in Mathematics* 391 (2021), p. 107970. arXiv: 1809.01416 [math.DG].

[CVP11] F. Coomans and A. Van Proeyen. "Off-shell N=(1,0), D=6 supergravity from superconformal methods". *JHEP* 02 (2011). [Erratum: JHEP 01, 119 (2012)], p. 049. arXiv: 1101.2403 [hep-th].

[Cos] K. Costello. "Twisted M theory, the Maulik-Okounkov Yangian, and the AdS dual of the Beem-Rastelli twist of the (2,0) theory". URL: http://www.birs.ca/events/2015/5-day-workshops/15w5154/videos/watch/201505251537-Costello.html.

[Cos07] K. Costello. "Topological conformal field theories and Calabi-Yau categories". *Adv. Math.* 210 (2007), pp. 165–214. arXiv: math/0412149.

[Cos13a] K. Costello. "Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4". *Pure Appl. Math. Q.* 9.1 (2013), pp. 73–165. arXiv: 1111.4234 [math.QA].

[Cos13b] K. Costello. "Supersymmetric gauge theory and the Yangian" (Mar. 2013). arXiv: 1303.2632 [hep-th].

[Cos16] K. Costello. "M-theory in the Omega-background and 5-dimensional non-commutative gauge theory" (Oct. 2016). arXiv: 1610.04144 [hep-th].

[Cos17] K. Costello. "Holography and Koszul duality: the example of the $M2$ brane" (May 2017). arXiv: 1705.02500 [hep-th].

[CG18] K. Costello and D. Gaiotto. "Twisted Holography" (Dec. 2018). arXiv: 1812.09257 [hep-th].

[CG17] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*. Vol. 31. New Mathematical Monographs. Cambridge University Press, Cambridge, 2017, pp. ix+387.

[CG21] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory. Vol. 2*. Vol. 41. New Mathematical Monographs. available at http://people.mpim-bonn.mpg.de/gwilliam. Cambridge University Press, Cambridge, 2021.

[CL15] K. Costello and S. Li. "Quantization of open-closed BCOV theory, I" (May 2015). arXiv: 1505.06703 [hep-th].

[CL16] K. Costello and S. Li. "Twisted supergravity and its quantization" (June 2016). arXiv: 1606.00365 [hep-th].

[CP21] K. Costello and N. M. Paquette. "Twisted Supergravity and Koszul Duality: A case study in AdS_3". *Commun. Math. Phys.* 384.1 (2021), pp. 279–339. arXiv: 2001.02177 [hep-th].

[CF80] E. Cremmer and S. Ferrara. "Formulation of Eleven-Dimensional Supergravity in Superspace". *Phys. Lett. B* 91 (1980), pp. 61–66.

[CJS78] E. Cremmer, B. Julia, and J. Scherk. "Supergravity Theory in Eleven-Dimensions". *Phys. Lett. B* 76 (1978), pp. 409–412.

[Crn88] Č. Crnković. "Symplectic geometry of the covariant phase space". *Classical and Quantum Gravity* 5.12 (1988), p. 1557.

[CG+18] T. C. da C. Guio, H. Jockers, A. Klemm, and H.-Y. Yeh. "Effective Action from M-Theory on Twisted Connected Sum G_2-Manifolds". *Commun. Math. Phys.* 359.2 (2018), pp. 535–601. arXiv: 1702.05435 [hep-th].

[dRo+80] M. de Roo, J. W. van Holten, B. de Wit, and A. Van Proeyen. "Chiral Superfields in $N = 2$ Supergravity". *Nucl. Phys. B* 173 (1980), pp. 175–188. URL: https://doi.org/10.1016/0550-3213(80)90449-6.

[WS06] B. de Wit and F. Saueressig. "Off-shell $\mathcal{N} = 2$ tensor supermultiplets". *JHEP* 09 (2006), p. 062. arXiv: hep-th/0606148.

[DVV80] B. De Wit, J. Van Holten, and A. Van Proeyen. "Transformation rules of $N = 2$ supergravity multiplets". *Nuclear Physics B* 167.1 (1980), pp. 186–204. URL: https://www.sciencedirect.com/science/article/pii/055032138090125X.

[DZOZ22] M. Del Zotto, J. Oh, and Y. Zhou. "Evidence for an algebra of G_2 instantons". *JHEP* 08 (2022), p. 214. arXiv: 2109.01110 [hep-th].

[Dij+05] R. Dijkgraaf, S. Gukov, A. Neitzke, and C. Vafa. "Topological M-theory as unification of form theories of gravity". *Adv. Theor. Math. Phys.* 9.4 (2005), pp. 603–665. arXiv: hep-th/0411073.

[DS14] W. Donovan and E. Segal. "Window shifts, flop equivalences and Grassmannian twists". *Compos. Math.* 150.6 (2014), pp. 942–978. URL: https://doi.org/10.1112/S0010437X13007641.

[DSV15] V. Dotsenko, S. Shadrin, and B. Vallette. "De Rham cohomology and homotopy Frobenius manifolds". *Journal of the European Mathematical Society* 17.3 (2015), pp. 535–547. URL: http://dx.doi.org/10.4171/JEMS/510.

[Dou01] M. R. Douglas. "D-branes, categories and N=1 supersymmetry". *J. Math. Phys.* 42 (2001), pp. 2818–2843. arXiv: hep-th/0011017.

[DS05] M. Duflo and V. Serganova. "On associated variety for Lie superalgebras" (2005). arXiv: math/0507198 [math.RT].

[EH23] R. Eager and F. Hahner. “Maximally Twisted Eleven-Dimensional Supergravity”. *Commun. Math. Phys.* 398.1 (2023), pp. 59–88. arXiv: 2106.15640 [hep-th].

[Eag+22] R. Eager, F. Hahner, I. Saberi, and B. R. Williams. “Perspectives on the pure spinor superfield formalism”. *J. Geom. Phys.* 180 (2022), p. 104626. arXiv: 2111.01162 [hep-th].

[ES19a] R. Eager and I. Saberi. “Holomorphic field theories and Calabi–Yau algebras”. *International Journal of Modern Physics A* (2019). arXiv: 1805.02084 [hep-th].

[ESW21] R. Eager, I. Saberi, and J. Walcher. “Nilpotence varieties”. *Annales Henri Poincaré* 22.4 (2021), pp. 1319–1376. arXiv: 1807.03766 [hep-th].

[ES15] R. Eager and J. Schmude. “Superconformal Indices and M2-Branes”. *JHEP* 12 (2015), p. 062. arXiv: 1305.3547 [hep-th].

[EST14] R. Eager, J. Schmude, and Y. Tachikawa. “Superconformal Indices, Sasaki-Einstein Manifolds, and Cyclic Homologies”. *Adv. Theor. Math. Phys.* 18.1 (2014), pp. 129–175. arXiv: 1207.0573 [hep-th].

[Eis95] D. Eisenbud. *Commutative algebra*. Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995, pp. xvi+785. URL: https://doi.org/10.1007/978-1-4612-5350-1.

[EHS23] C. Elliott, F. Hahner, and I. Saberi. “The Derived Pure Spinor Formalism as an Equivalence of Categories”. *SIGMA* 19 (2023), p. 022. arXiv: 2205.14133 [math-ph].

[ES19b] C. Elliott and P. Safronov. “Topological twists of supersymmetric algebras of observables”. *Commun. Math. Phys.* 371.2 (2019), pp. 727–786. arXiv: 1805.10806 [math-ph].

[ESW20] C. Elliott, P. Safronov, and B. Williams. “A taxonomy of twists of supersymmetric Yang–Mills theory” (2020). arXiv: 2002.10517 [math-ph].

[EW21] C. Elliott and B. R. Williams. “Holomorphic Poisson Field Theories”. *Higher Structures* 5 (Dec. 2021). arXiv: 2008.02302 [math-ph].

[ESS21] D. Erman, S. V. Sam, and A. Snowden. “Strength and Hartshorne’s conjecture in high degree”. *Math. Z.* 297.3-4 (2021), pp. 1467–1471. URL: https://doi.org/10.1007/s00209-020-02564-y.

[FG05] M. Faux and S. J. Gates. “Adinkras: A graphical technology for supersymmetric representation theory”. *Phys. Rev. D* 71 (6 2005), p. 065002. URL: https://link.aps.org/doi/10.1103/PhysRevD.71.065002.

[Fer87] S. Ferrara, ed. *Supersymmetry: Lectures And Reprints (in two volumes)*. North Holland/World Scientific, 1987.

[FS11] G. Festuccia and N. Seiberg. "Rigid Supersymmetric Theories in Curved Superspace". *JHEP* 06 (2011), p. 114. arXiv: 1105.0689 [hep-th].

[Fon10] A. Font. "Heterotic strings on G_2 orbifolds". *JHEP* 11 (2010), p. 115. arXiv: 1009.4422 [hep-th].

[Fra18] R. Frankhuizen. "Massey products and the Golod property for simplicially resolvable rings" (2018). arXiv: 1806.07887 [math.AT].

[FVP12] D. Z. Freedman and A. Van Proeyen. *Supergravity*. Cambridge, UK: Cambridge Univ. Press, May 2012.

[Fri+86] D. Friedan et al. "Notes on string theory and two-dimensional conformal field theory". *Unified string theories* 162 (1986).

[GO19] D. Gaiotto and J. Oh. "Aspects of Ω-deformed M-theory" (July 2019). arXiv: 1907.06495 [hep-th].

[Gal16] F. Galetto. "Propagating weights of tori along free resolutions". *Journal of Symbolic Computation* 74 (2016), pp. 1–45. arXiv: 1406.1900 [math.AC].

[GIO87] A. S. Galperin, E. A. Ivanov, and V. I. Ogievetsky. "Duality Transformations and Most General Matter Selfcoupling in $N = 2$ Supersymmetry". *Nucl. Phys. B* 282 (1987), p. 74.

[Gal+01] A. S. Galperin, E. Ivanov, V. I. Ogievetsky, and E. S. Sokatchev. *Harmonic superspace*. Cambridge University Press, 2001.

[Gál+16] I. Gálvez, V. Gorbounov, Z. Shaikh, and A. Tonks. "The Berkovits complex and semi-free extensions of Koszul algebras". *Ann. Fac. Sci. Toulouse Math. (6)* 25.2-3 (2016), pp. 363–384. URL: https://doi.org/10.5802/afst.1497.

[GS80] S. J. Gates Jr. and W. Siegel. "Understanding Constraints in Superspace Formulations of Supergravity". *Nucl. Phys. B* 163 (1980), pp. 519–545. URL: https://doi.org/10.1016/0550-3213(80)90414-9.

[GSW80] S. J. Gates Jr., K. S. Stelle, and P. C. West. "Algebraic Origins of Superspace Constraints in Supergravity". *Nucl. Phys. B* 169 (1980), pp. 347–364. URL: https://doi.org/10.1016/0550-3213(80)90037-1.

[GS04] A. A. Gerasimov and S. L. Shatashvili. "Towards integrability of topological strings. I. Three-forms on Calabi-Yau manifolds". *JHEP* 11 (2004), p. 074. arXiv: hep-th/0409238.

[GS71] J.-L. Gervais and B. Sakita. "Field theory interpretation of supergauges in dual models". *Nuclear Physics B* 34.2 (1971), pp. 632–639.

[Get09] E. Getzler. "Lie theory for nilpotent L_∞ algebras". *Annals of Mathematics* (2009), pp. 271–301. arXiv: math/0404003 [math.AT].

[Gin+22] G. Ginot, O. Gwilliam, A. Hamilton, and M. Zeinalian. "Large N phenomena and quantization of the Loday-Quillen-Tsygan theorem". *Advances in Mathematics* 409 (2022), p. 108631. arXiv: 2108.12109 [math.QA].

[GL71] Y. A. Gol'fand and E. P. Likhtman. "Extension of the algebra of Poincaré group generators and violation of P invariance". *JETP Lett.* 13 (1971), p. 323.

[Gol05] E. Golod. "On duality in the homology algebra of a Koszul complex". *J Math Sci* 128 (2005), pp. 3381–3383. URL: https://doi.org/10.1007/s10958-005-0276-y.

[GR+98] F. Gonzalez-Rey, M. Rocek, S. Wiles, U. Lindstrom, and R. von Unge. "Feynman rules in N=2 projective superspace: 1. Massless hypermultiplets". *Nucl. Phys. B* 516 (1998), pp. 426–448. arXiv: hep-th/9710250.

[GS09] V. Gorbounov and V. Schechtman. "Homological algebra and divergent series". *SIGMA* 5 (2009), p. 034. arXiv: 0712.3670 [math.AG].

[Gor+22] M. Gorelik, C. Hoyt, V. Serganova, and A. Sherman. "The Duflo-Serganova functor, vingt ans après" (2022). arXiv: 2203.00529 [math.RT].

[GKR07] A. L. Gorodentsev, A. S. Khoroshkin, and A. N. Rudakov. "On syzygies of highest weight orbits". *Moscow Seminar on Mathematical Physics. II.* Vol. 221. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 2007, pp. 79–120. URL: https://doi.org/10.1090/trans2/221/05.

[GV05] P. A. Grassi and P. Vanhove. "Topological M theory from pure spinor formalism". *Adv. Theor. Math. Phys.* 9.2 (2005), pp. 285–313. arXiv: hep-th/0411167.

[GS] D. R. Grayson and M. E. Stillman. *Macaulay2, a software system for research in algebraic geometry.* Available at http://www.math.uiuc.edu/Macaulay2/.

[Hah20] F. Hahner. "The BV Formalism, L_∞-algebras, and Their Applications to Supersymmetry and Chern–Simons theory" (2020). Master's thesis, Universität Heidelberg.

[Hah+22] F. Hahner, S. Noja, I. Saberi, and J. Walcher. "Six-dimensional supermultiplets from bundles on projective spaces" (June 2022). arXiv: 2206.08388 [math-ph].

[Hah+23] F. Hahner, S. Raghavendran, I. Saberi, and B. R. Williams. "" (2023). in preparation.

[HS23] F. Hahner and I. Saberi. "Eleven-dimensional supergravity as a Calabi-Yau twofold" (Apr. 2023). arXiv: 2304.12371 [math-ph].

[Har77] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977, pp. xvi+496.

[HS93] V. Hinich and V. Schechtman. "Homotopy Lie algebras". *Adv. Soviet Math* 16.2 (1993), pp. 1–28.

[Hit00] N. J. Hitchin. "The Geometry of Three-Forms in Six Dimensions". *J. Diff. Geom.* 55.3 (2000), pp. 547–576. arXiv: math/0010054.

[HE71] M. Hochster and J. A. Eagon. "Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci". *American Journal of Mathematics* 93.4 (1971), pp. 1020–1058. URL: https://doi.org/10.2307/2373744.

[HS19] Y. Homma and U. Semmelmann. "The Kernel of the Rarita–Schwinger Operator on Riemannian Spin Manifolds". *Commun. Math. Phys.* 370.3 (2019), pp. 853–871. arXiv: 1804.10602 [math.DG].

[Hor+03] K. Hori et al. *Mirror symmetry*. Vol. 1. Clay mathematics monographs. Providence, USA: AMS, 2003.

[HNv82] P. Howe, H. Nicolai, and A. van Proeyen. "Auxiliary fields and a superspace Lagrangian for linearized ten-dimensional supergravity". *Physics Letters B* 112.6 (1982), pp. 446–450. URL: https://doi.org/10.1016/0370-2693(82)90845-0.

[How91a] P. S. Howe. "Pure spinor lines in superspace and ten-dimensional supersymmetric theories". *Phys. Lett. B* 258 (1991), pp. 141–144. URL: https://doi.org/10.1016/0370-2693(91)91221-G.

[How91b] P. S. Howe. "Pure spinors, function superspaces, and supergravity theories in ten and eleven dimensions". *Physics Letters B* 273.1-2 (1991), pp. 90–94. URL: https://doi.org/10.1016/0370-2693(91)90558-8.

[HST83] P. S. Howe, K. S. Stelle, and P. K. Townsend. "The Relaxed Hypermultiplet: An Unconstrained $\mathcal{N} = 2$ Superfield Theory". *Nucl. Phys. B* 214 (1983), pp. 519–531. URL: https://doi.org/10.1016/0550-3213(83)90249-3.

[HU87] P. S. Howe and A. Umerski. "Anomaly multiplets in six dimensions and ten dimensions". *Phys. Lett. B* 198 (1987), pp. 57–60. URL: https://doi.org/10.1016/0370-2693(87)90158-4.

[Hun99] C. Huneke. "Hyman Bass and ubiquity: Gorenstein rings". *Contemporary Math. vol 243* (1999), pp. 55–78. arXiv: math/0209199.

[Huy06] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford mathematical monographs. Oxford University Press, 2006.

[Joy07] D. D. Joyce. *Riemannian holonomy groups and calibrated geometry*. Vol. 12. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2007, pp. x+303.

[Jur+19] B. Jurčo, L. Raspollini, C. Sämann, and M. Wolf. "L_∞ algebras of classical field theories and the Batalin–Vilkovisky formalism". *Fortsch. Phys.* 67.7 (2019), p. 1900025. arXiv: `1809.09899 [hep-th]`.

[Kac77] V. G. Kac. "Lie superalgebras". *Advances in Math.* 26.1 (1977), pp. 8–96.

[KR01] V. G. Kac and A. Rudakov. "Representations of the Exceptional Lie Superalgebra E(3,6) II: Four Series of Degenerate Modules". *Comm. Math. Phys.* 222.3 (2001), pp. 611–661. arXiv: `math-ph/0012050`.

[Kaj07] H. Kajiura. "Noncommutative homotopy algebras associated with open strings". *Reviews in Mathematical Physics* 19.01 (2007), pp. 1–99. URL: `http://dx.doi.org/10.1142/S0129055X07002912`.

[Kap91] M. Kapranov. "On dg-modules over the de Rham complex and the vanishing cycles functor" (1991), pp. 57–86.

[Kap21] M. Kapranov. "Supergeometry in mathematics and physics". *New spaces in physics—formal and conceptual reflections*. Cambridge Univ. Press, Cambridge, 2021, pp. 114–152. arXiv: `1512.07042 [math.AG]`.

[KW07] A. Kapustin and E. Witten. "Electric-Magnetic Duality And The Geometric Langlands Program". *Commun. Num. Theor. Phys.* 1 (2007), pp. 1–236. arXiv: `hep-th/0604151`.

[Kin+07] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju. "An Index for 4 dimensional super conformal theories". *Commun. Math. Phys.* 275 (2007), pp. 209–254. arXiv: `hep-th/0510251`.

[KN64] S. Kobayashi and T. Nagano. "On filtered Lie algebras and geometric structures I". *Journal of Mathematics and Mechanics* 13.5 (1964), pp. 875–907.

[Koc77] R. M. Koch. "On filtered Lie algebras". *Indiana University Mathematics Journal* 26.1 (1977), pp. 115–124. URL: `http://www.jstor.org/stable/24891326`.

[Kol13] J. Kollár. *Singularities of the Minimal Model Program*. Cambridge Tracts in Mathematics. Cambridge University Press, 2013.

[KS96] Y. Kosmann-Schwazbach. "From Poisson algebras to Gerstenhaber algebras". *Annales de l'Institut Fourier* 46.5 (1996), pp. 1243–1274. URL: `http://eudml.org/doc/75211`.

[Kos85] J.-L. Koszul. "Crochet de Schouten-Nijenhuis et cohomologie". fr. Astérisque S131 (1985). URL: http://www.numdam.org/item/AST_1985__S131__257_0/.

[KL09] D. Krotov and A. Losev. "Quantum field theory as effective BV theory from Chern–Simons". *Nuclear Physics B* 806.3 (2009), pp. 529–566. URL: http://dx.doi.org/10.1016/j.nuclphysb.2008.07.021.

[KNT17] S. M. Kuzenko, J. Novak, and S. Theisen. "New superconformal multiplets and higher derivative invariants in six dimensions". *Nucl. Phys. B* 925 (2017), pp. 348–361. arXiv: 1707.04445 [hep-th].

[KNT18a] S. M. Kuzenko, J. Novak, and S. Theisen. "Non-conformal supercurrents in six dimensions". *JHEP* 02 (2018), p. 030. arXiv: 1709.09892 [hep-th].

[KNT18b] S. M. Kuzenko, J. Novak, and S. Theisen. "Non-conformal supercurrents in six dimensions". *JHEP* 02 (2018), p. 030. arXiv: 1709.09892 [hep-th].

[Kuz18] A. G. Kuznetsov. "On linear sections of the spinor tenfold. I". *Izv. Ross. Akad. Nauk Ser. Mat.* 82.4 (2018), pp. 53–114. URL: https://doi.org/10.4213/im8756.

[KM95] I. Kříž and J. P. May. "Operads, algebras, modules and motives". *Astérisque* 233 (1995), iv+145pp.

[Lad04] T. Lada. "L_∞ algebra representations". *Applied Categorical Structures* 12.1 (2004), pp. 29–34.

[LM95] T. Lada and M. Markl. "Strongly homotopy Lie algebras". *Communications in algebra* 23.6 (1995), pp. 2147–2161.

[Lap01] S. V. Lapin. "Differential perturbations and D_∞-differential modules". *Sbornik Mathematics* 192 (2001).

[LTM12] W. D. Linch III and G. Tartaglino-Mazzucchelli. "Six-dimensional Supergravity and Projective Superfields". *JHEP* 08 (2012), p. 075. arXiv: 1204.4195 [hep-th].

[LV12a] J.-L. Loday and B. Vallette. *Algebraic operads*. Vol. 346. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2012, pp. xxiv+634. URL: https://doi.org/10.1007/978-3-642-30362-3.

[LV12b] J.-L. Loday and B. Vallette. *Algebraic operads*. Vol. 346. Springer Science & Business Media, 2012.

[Lur09] J. Lurie. "On the Classification of Topological Field Theories" (May 2009). arXiv: 0905.0465 [math.CT].

[MSW19] T. Macrelli, C. Sämann, and M. Wolf. "Scattering amplitude recursion relations in Batalin-Vilkovisky–quantizable theories". *Phys. Rev. D* 100.4 (2019), p. 045017. arXiv: 1903.05713 [hep-th].

[Mal98] J. M. Maldacena. "The Large N limit of superconformal field theories and supergravity". *Adv. Theor. Math. Phys.* 2 (1998), pp. 231–252. arXiv: hep-th/9711200.

[Man84] Y. I. Manin. "New directions in geometry". *Russian Mathematical Surveys* 39.6 (1984), p. 51.

[Man85] Y. I. Manin. "Geometry of supergravity and Schubert supercells". *Journal of Soviet Mathematics* 31 (1985), pp. 3362–3373.

[Mar98] M. Markl. "Homotopy algebras via resolutions of operads". *Proceedings of the 19th Winter School Geometry and Physics* (1998). arXiv: 9808101 [math].

[Mne17] P. Mnev. "Lectures on Batalin-Vilkovisky formalism and its applications in topological quantum field theory" (July 2017). arXiv: 1707.08096 [math-ph].

[MS04] M. Movshev and A. S. Schwarz. "On maximally supersymmetric Yang-Mills theories". *Nucl. Phys. B* 681 (2004), pp. 324–350. arXiv: hep-th/0311132.

[MS06] M. Movshev and A. S. Schwarz. "Algebraic structure of Yang-Mills theory". *Prog. Math.* 244 (2006), pp. 473–523. arXiv: hep-th/0404183.

[Mov11] M. V. Movshev. "Geometry of a desingularization of eleven-dimensional gravitational spinors" (May 2011). arXiv: 1105.0127 [hep-th].

[Mov05a] M. Movshev. "On deformations of Yang–Mills algebras" (2005). arXiv: 0509119 [hep-th].

[Mov05b] M. Movshev. "Yang–Mills theories in dimensions 3, 4, 6, and 10 and bar duality" (2005). arXiv: 0503165 [hep-th].

[MSX12] M. V. Movshev, A. Schwarz, and R. Xu. "Homology of Lie algebra of supersymmetries and of super-Poincaré Lie algebra". *Nucl. Phys. B* 854 (2012), pp. 483–503. arXiv: 1106.0335 [hep-th].

[MSX14] M. V. Movshev, A. Schwarz, and R. Xu. "Integral invariants in flat superspace". *Nucl. Phys. B* 884 (2014), pp. 28–43. arXiv: 1403.1997 [hep-th].

[Mov15] M. V. Movshev. "Geometry of pure spinor formalism in superstring theory". *Adv. Math.* 268 (2015), pp. 201–240. URL: https://doi.org/10.1016/j.aim.2014.09.014.

[Nah78] W. Nahm. "Supersymmetries and their Representations". *Nucl. Phys.* B135 (1978), p. 149.

[Nil86] B. E. W. Nilsson. "Pure spinors as auxiliary fields in the ten-dimensional supersymmetric Yang–Mills theory". *Classical and Quantum Gravity* 3.2 (1986), p. L41.

[OY19] J. Oh and J. Yagi. "Chiral algebras from Ω-deformation". *JHEP* 08 (2019), p. 143. arXiv: 1903.11123 [hep-th].

[OZ21] J. Oh and Y. Zhou. "A domain wall in twisted M-theory" (May 2021). arXiv: 2105.09537 [hep-th].

[OSS80] C. Okonek, M. Schneider, and H. Spindler. *Vector Bundles on Complex Projective Spaces*. Springer New York, NY, 1980. URL: https://doi.org/10.1007/978-1-4757-1460-9.

[Pal14] J. Palmkvist. "The tensor hierarchy algebra". *J. Math. Phys.* 55 (2014), p. 011701. arXiv: 1305.0018 [hep-th].

[RSW23] S. Raghavendran, I. Saberi, and B. R. Williams. "Twisted Eleven-Dimensional Supergravity". *Commun. Math. Phys.* 402.2 (2023), pp. 1103–1166. arXiv: 2111.03049 [math-ph].

[RW22] S. Raghavendran and B. R. Williams. "A holographic approach to the six-dimensional superconformal index" (Oct. 2022). arXiv: 2210.07910 [math-ph].

[RY19] S. Raghavendran and P. Yoo. "Twisted S-duality" (2019). arXiv: 1910.13653.

[RSV88] A. A. Rosly, A. S. Schwarz, and A. A. Voronov. "Geometry of superconformal manifolds". *Communications in Mathematical Physics* 119 (1988), pp. 129–152.

[Roy99] D. Roytenberg. *Courant algebroids, derived brackets, and even symplectic supermanifolds*. University of California, Berkeley, 1999.

[Rö06] C. Römelsberger. "Counting chiral primaries in N = 1, d=4 superconformal field theories". *Nucl. Phys. B* 747 (2006), pp. 329–353. arXiv: hep-th/0510060.

[SW20] I. Saberi and B. R. Williams. "Twisted characters and holomorphic symmetries". *Lett. Math. Phys.* 110.10 (2020), pp. 2779–2853. URL: https://doi.org/10.1007/s11005-020-01319-4.

[SW21] I. Saberi and B. R. Williams. "Twisting pure spinor superfields, with applications to supergravity" (2021). arXiv: 2106.15639 [math-ph].

[SW23a] I. Saberi and B. R. Williams. to appear. 2023.

[SW23b] I. Saberi and B. R. Williams. "Constraints in the BV formalism: Six-dimensional supersymmetry and its twists". *Adv. Math.* 412 (2023), p. 108806. arXiv: 2009.07116 [math-ph].

[SW23c] I. Saberi and B. R. Williams. "Superconformal Algebras and Holomorphic Field Theories". *Annales Henri Poincare* 24.2 (2023), pp. 541–604. arXiv: 1910.04120 [math-ph].

[Ser55] J.-P. Serre. "Faisceaux algébriques cohérents". *Annals of Mathematics* 61 (1955), p. 197.

[Sha99] E. Sharpe. "D-branes, derived categories, and Grothendieck groups". *Nuclear Physics B* 561.3 (1999), pp. 433–450.

[Soh85] M. F. Sohnius. "Introducing supersymmetry". *Physics Reports* 128.2-3 (1985), pp. 39–204.

[Ste21] J. Stelzig. "On the structure of double complexes". *Journal of the London Mathematical Society* 104.2 (2021), pp. 956–988. arXiv: 1812.00865 [math.RT].

[Sva74] T. Svanes. "Coherent cohomology on schubert subschemes of flag schemes and applications". *Advances in Mathematics* 14.4 (1974), pp. 369–453. URL: https://www.sciencedirect.com/science/article/pii/0001870874900395.

[Tan70] N. Tanaka. "On differential systems, graded Lie algebras, and pseudo-groups". *Journal of Mathematics of Kyoto University* 10.1 (1970), pp. 1–82.

[Toë14] B. Toën. "Derived algebraic geometry". *EMS Surv. Math. Sci.* 1.2 (2014), pp. 153–240. URL: https://doi.org/10.4171/EMSS/4.

[Vak] R. Vakil. *Foundations of Algebraic Geometry*. URL: http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf.

[Val14] B. Vallette. "Algebra + homotopy = operad". *Symplectic, Poisson, and noncommutative geometry* 62 (2014), pp. 229–290.

[LCL] M. A. A. van Leeuwen, A. M. Cohen, and B. Lisse. *LiE, A package for Lie group computations.* Available at http://www-math.univ-poitiers.fr/~maavl/LiE/.

[VA72] D. V. Volkov and V. P. Akulov. "Possible universal neutrino interaction". *JETP Lett.* 16 (1972), p. 367.

[Vor05] T. Voronov. "Higher derived brackets and homotopy algebras". *Journal of Pure and Applied Algebra* 202.1-3 (2005), pp. 133–153. arXiv: math/0304038 [math.QA].

[Wan91] M. Y. Wang. "Preserving parallel spinors under metric deformations". *Indiana Univ. Math. J.* 40.3 (1991), pp. 815–844. URL: https://doi.org/10.1512/iumj.1991.40.40037.

[Wit88] E. Witten. "Topological Quantum Field Theory". *Commun. Math. Phys.* 117 (1988), p. 353. URL: https://doi.org/10.1007/BF01223371.

[Wit95] E. Witten. "String theory dynamics in various dimensions". *Nucl. Phys. B* 443 (1995), pp. 85–126. arXiv: hep-th/9503124.

[Wit19] E. Witten. "Notes On Super Riemann Surfaces And Their Moduli". *Pure Appl. Math. Quart.* 15.1 (2019), pp. 57–211. arXiv: 1209.2459 [hep-th].

[Yag12] J. Yagi. "Compactification on the \Omega-background and the AGT correspondence". *JHEP* 09 (2012), p. 101. arXiv: 1205.6820 [hep-th].

[Zel09] I. Zelenko. "On Tanaka's prolongation procedure for filtered structures of constant type". *SIGMA* 5 (2009), p. 094. URL: https://doi.org/10.3842%2Fsigma.2009.094.