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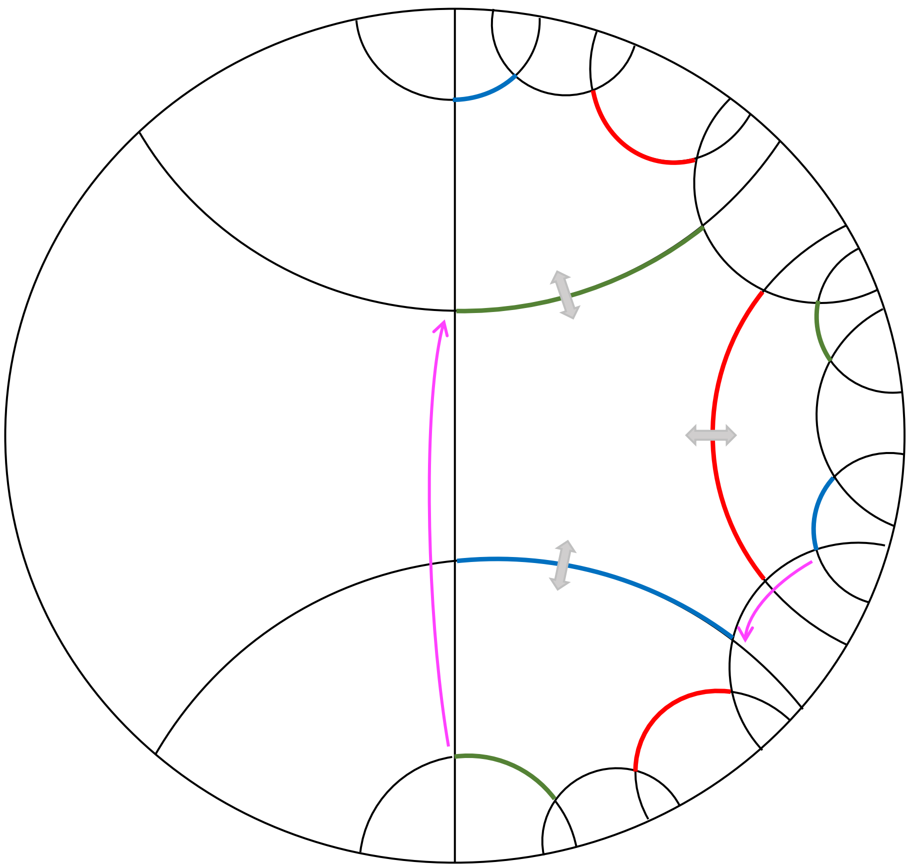
Tag der mündlichen Prüfung:

On arc coordinates for maximal representations



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Abstract

Given a hyperbolic surface with boundary, arc coordinates provide a parametrization of the Teichmüller space. They rely on the choice of a family of arcs which start and end at boundary components and are orthogonal to them. Higher rank Teichmüller theories are a generalization of classical Teichmüller theory and are concerned with the study of representations of the fundamental group of an oriented surface Σ of negative Euler characteristic into simple real Lie groups G of higher rank. It is well known that maximal representations are a higher rank Teichmüller theory for G Hermitian. In this thesis we will discuss how to generalize arc coordinates for maximal representations, focusing on the case where Σ is a pair of pants $\Sigma_{0,3}$ and $G = \mathrm{PSp}(4,\mathbb{R})$. This will be possible by introducing geometric parameters on the space of right-angled hexagons in the Siegel space $\mathcal{X}$, which lead to the visualization of a right-angled hexagon as a polygonal chain inside $\mathbb{H}^2$. We discuss geometric properties of reflections in $\mathcal{X}$ and introduce the notion of maximal representations of a reflection group $W_3 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. We give a parametrization of maximal representations of W_3 into $\mathrm{PSp}^{\pm}(4,\mathbb{R})$, which allows us to parametrize a subset of maximal and Shilov hyperbolic representations into $\mathrm{PSp}(4,\mathbb{R})$.

Zusammenfassung

Bei einer hyperbolischen Fläche mit Rand liefern so genannte Bogenkoordinaten eine Parametrisierung des Teichmüller-Raums. Sie hängen von der Wahl einer Familie von Kurven ab, die an Randkomponenten beginnen und enden und orthogonal zu diesen sind. Höherrangige Teichmüller-Theorien sind eine Verallgemeinerung der klassischen Teichmüller-Theorie und befassen sich mit Darstellungen der Fundamentalgruppe einer orientierten Fläche Σ mit negativer Euler-Charakteristik in einfache reelle Lie-Gruppen G höheren Rangs. Es ist bekannt, dass Maximaldarstellungen eine höherrangige Teichmüller-Theorie für G Hermitesch sind. In dieser Arbeit beschäftigen wir uns mit der Frage, wie Bogenkoordinaten für maximale Darstellungen verallgemeinert werden können, wobei wir uns auf den Fall konzentrieren, in dem Σ die Fläche $\Sigma_{0,3}$ ist und $G = \mathrm{PSp}(4, \mathbb{R})$. Dies wird durch die Einführung geometrischer Parameter auf dem Raum rechtwinkliger Sechsecke im Siegel-Raum $\mathcal{X}$ möglich, die zu einer Visualisierung eines rechtwinkligen Sechsecks als Polygonzug innerhalb von $\mathbb{H}^2$ führen. Wir diskutieren geometrische Eeigenschaften von Spiegelungen in $\mathcal{X}$ und führen den Begriff der maximalen Darstellung einer Spiegelungsgruppe $W_3 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ ein. Wir geben eine Parametrisierung maximaler Darstellungen von W_3 in $\mathrm{PSp}^{\pm}(4, \mathbb{R})$. Das ermöglicht uns, eine Teilmenge maximaler und Shilov-hyperbolischer Darstellungen in $\mathrm{PSp}(4, \mathbb{R})$ zu parametrisieren.

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Contents

1 Introduction

1.1 The space of maximal representations

Given Σ a closed oriented surface of negative Euler characteristic and fundamental group Γ, the Teichmüller space $\mathcal{T}(\Sigma)$ is the parameter space of marked hyperbolic structures on Σ. It is well known that, with the introduction of the holonomy map, one can associate to a point in $\mathcal{T}(\Sigma)$ a discrete and faithful representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ so that the surface Σ is realized by the quotient $\Sigma = \rho(\Gamma)\backslash\mathbb{H}^2$. This representation is well defined up to conjugation by an element in $\mathrm{PSL}(2,\mathbb{R})$ so that the space $\mathcal{T}(\Sigma)$ can be identified with a connected component of the representation variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ which consists entirely of discrete and faithful representations [Gol80].

This phenomenon of the representation variety to admit components consisting only of injective homomorphisms with discrete image is still true if we substitute $\mathrm{PSL}(2,\mathbb{R})$ with a semisimple real Lie group of higher rank G. In this sense higher rank Teichmüller space was developed as a generalization of classical Teichmüller space (see [BIW14], [Wie18], [Poz19] for an introduction to higher Teichmüller theory). More precisely given G a semisimple real Lie group of higher rank, a *higher Teichmüller space* is a subset of $\mathrm{Hom}(\Gamma, G)/G$ which is a union of connected components that consist entirely of discrete and faithful representations. To such a representation ρ we can associate the quotient $\rho(\Gamma)\backslash\mathcal{X}$ where $\mathcal{X}$ is the symmetric space associated to G. The space $\mathcal{X}$ is a non-positively curved Riemannian symmetric manifold of higher rank, where *rank* denotes the maximal dimension of an isometrically embedded flat inside $\mathcal{X}$. The quotient $\rho(\Gamma)\backslash\mathcal{X}$ is a locally symmetric space whose fundamental group is isomorphic to the fundamental group of Σ.

There are two well-known families of higher Teichmüller spaces: Hitchin components and maximal representations. Hitchin components are defined when G is a split real simple Lie group such as $\mathrm{PSL}(n,\mathbb{R})$ or $\mathrm{PSp}(2n,\mathbb{R})$. Maximal representations are defined when G is a Hermitian Lie group such as $\mathrm{PSp}(2n,\mathbb{R})$. Coherently, when $G = \mathrm{PSL}(2,\mathbb{R})$ both Hitchin components and the space of maximal representations coincide with the Teichmüller space $\mathcal{T}(\Sigma)$. Moreover, the only family of split simple Lie groups of Hermitian type is given by $\mathrm{PSp}(2n,\mathbb{R})$ and in this case Hitchin representations are maximal, but not vice-versa [BILW05].

More precisely, Hitchin [Hit92] initiated the study of the connected component in $\mathrm{Hom}(\Gamma, G)/G$ of the composition $\tau \circ \rho$ where $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is the holonomy of a hyperbolization and τ is the irreducible representation $\tau : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$. In general any split real simple Lie group G contains an embedding $\tau : \mathrm{PSL}(2,\mathbb{R}) \to G$ which is unique up to conjugation, and if $G = \mathrm{PSL}(2,\mathbb{R})$ this is the irreducible representation. The Hitchin component is then defined as the connected component of $[\tau \circ \rho] \in \mathrm{Hom}(\Gamma, G)/G$, where $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is the holonomy of a hyperbolization. Hitchin showed, using the theory of Higgs bundles, that the Hitchin component is homeomorphic to the Euclidean space of dimension $\dim(G)(2g-2)$, where g denotes the genus of Σ. Using techniques of dynamical systems Labourie [Lab06] showed that representations in the Hitchin component are discrete and faithful. An indipendent approach to Hitchin components was developed by Fock and Goncharov [FG06], who showed that representations in any Hitchin component are discrete and faithful.

In this thesis we are interested in maximal representations. These are singled out by the Toledo number, which is a generalization of the Euler number and was first introduced by Toledo for representations $\rho : \Gamma \to \mathrm{PU}(1, n)$ [Tol89]. In [Gol88] Goldman showed that the Euler number distinguishes the connected components of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ and that Teichmüller space corresponds to the connected component formed by representations with the maximal value of the Euler number. Burger Iozzi and Wienhard [BILW05], [BIW10] studied the Toledo invariant for general Hermitian Lie groups. A group G is Hermitian if the symmetric space $\mathcal{X}$ associated to G admits a G-invariant complex structure. An equivalent definition is that $\mathcal{X}$ is a Hermitian manifold such that every point $x \in \mathcal{X}$ is the isolated fixed point of an isometric involution s_x. Symmetric spaces which are Hermitian admit a Kähler form $\omega_{\mathcal{X}}$ on $\mathcal{X}$ which allows to associate to every representation $\rho : \Gamma \to G$ a characteristic number, that is the Toledo number T_ρ. The Toledo number is constant on connected components of $\mathrm{Hom}(\Gamma, G)$ and satisfies a Milnor–Wood type inequality

$$|T_\rho| \leqslant \chi(\Sigma)\mathrm{rk}_{\mathcal{X}} \tag{1}$$

where $\mathrm{rk}_{\mathcal{X}}$ is the real rank of $\mathcal{X}$. Maximal representations are the ones for which equality holds in (1).

With these tools Burger Iozzi and Wienhard proved that maximal representations are higher rank Teichmüller spaces. Moreover, they provided an equivalent characterization of maximal representations through the existence of a "well-behaved" boundary map ξ. This is a generalization of a phenomenon in classical Teichmüller space: a representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is the holonomy of a hyperbolization if and only if there exists a continuous monotone equivariant map $\xi : S^1 \to \partial_\infty \mathbb{H}^2$, where $\partial_\infty \mathbb{H}^2$ is a homogeneous $\mathrm{PSL}(2,\mathbb{R})$-space isomorphic to S^1. A crucial difference when considering a general Hermitian group G and the correspondent symmetric space of higher rank $\mathcal{X}$ is that the visual boundary $\partial_\infty \mathcal{X}$ is not a homogeneous G-space and stratifies in orbits isomorphic to partial flag varieties [Ebe96]. These are compact G-homogeneous spaces G/P determined by the choice of a parabolic subgroup P. When considering boundary maps, it is thus natural to consider, instead of maps $\xi : S^1 \to \partial_\infty \mathcal{X}$, maps of the form $\xi_P : S^1 \to G/P$ for a suitable choice of a parabolic subgroup P. In the case of a Hermitian Lie group the parabolic subgroup is the stabiliser of a point in the Shilov boundary $\check{S}$ of the Hermitian symmetric space (this is the set of Lagrangians in the case of $G = \mathrm{PSp}(2n,\mathbb{R})$). The Maslov cocycle induces a partial cyclic order on $\check{S}$ and maximal representations $\mathrm{Hom}^{\max}(\Gamma, G)$ can be characterised as those representations admitting a monotone equivariant boundary map, namely a map $\xi : S^1 \to \check{S}$ such that for every positively oriented triple $(x, y, z) \in (S^1)^3$ the image $(\xi(x), \xi(y), \xi(z))$ is Maslov-positively oriented [BIW10].

A common framework explaining the various higher rank Teichmüller theories was introduced by Guichard and Wienhard [GW18], [GW22], [GLW21] through the notion of Θ-positivity. Hitichin components and maximal representations admit a common characterization in terms of positive structures of flag varieties. For Hitchin components they consider full flag varieties and Lusztig's total positivity [Lus94]. For maximal representations the flag variety is the Shilov boundary of the symmetric space of G and positivity is given by the aforementioned Maslov cocycle. The theory of Θ-positivity generalizes Lusztig's total positivity to a larger class of simple Lie groups (e.g. $\mathrm{SO}(p, q), p \neq q$). Guichard Labourie and Wienhard conjecture that Θ-positive representations also form higher rank Teichmüller spaces. The conjecture has, by now, been proven for the most part in [GLW21] and [BP21].

Another notion that plays an important role in higher rank Teichmüller theory is the notion of Anosov representations. They were introduced by Labourie [Lab06] and further investigated by Guichard and Wienhard [GW12]. Anosov representations are representations of Gromov hyperbolic groups into Lie groups G with strong dynamical properties, defined using continuous equivariant boundary maps. Many key properties of Hitchin and maximal representations, such as being discrete and faithful or admitting ρ-equivariant boundary maps $\xi : S^1 \to G/P$ with respect to a certain parabolic subgroup P, follow from them being Anosov representations. We refer to [Kas18] for a definition and a description of geometric and dynamical properties of Anosov representations.

Till now we have assumed Σ to be a closed surface. There is a related theory for surfaces with punctures or boundary components. The first thing to notice is that when $\partial\Sigma \neq \varnothing$ then Γ is a free group and the whole representation variety is connected. Denote $\Sigma = \Sigma_{g,m}$ a surface of genus g and m boundary components with fundamental group $\Gamma_{g,m}$. Let

$$\Gamma_{g,m} = \langle a_1, b_1, ...a_g, b_g, c_1, ...c_m | \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{m} c_j = 1 \rangle$$

be a presentation where the elements c_i represent loops which are freely homotopic to the corresponding boundary components of $\partial\Sigma$ with positive orientation. Fixing a set $\mathcal{C} = \{C_1, ..., C_m\}$ of conjugacy classes in G one can define the *relative representation variety* as the subspace of $\mathrm{Hom}(\Gamma_{g,m}, G)$ given by

$$\mathrm{Hom}^{\mathcal{C}}(\Gamma_{g,m}, G) = \{\rho \in \mathrm{Hom}(\Gamma_{g,m}, G) | \ \rho(c_i) \in C_i, \ 1 \leqslant i \leqslant m\}$$

The representation variety $\mathrm{Hom}(\Gamma_{g,m}, G)$ is a disjoint union of all the relative representation varieties $\mathrm{Hom}^{\mathcal{C}}(\Gamma_{g,m}, G)$ over all possible choices for $\mathcal{C}$.

The Toledo number is constant on the connected components of $\mathrm{Hom}^{\mathcal{C}}(\Gamma_{g,m}, G)$ but for many choices of conjugacy classes the intersection $\mathrm{Hom}^{\max}(\Gamma_{g,m}, G) \cap \mathrm{Hom}^{\mathcal{C}}(\Gamma_{g,m}, G)$ is empty [BILW05]. The topology and the structure of $\mathrm{Hom}^{\mathcal{C}}(\Gamma_{g,m}, G)/G$ has been studied in [BG99], [DT19], [TT21]. See also [Gol23] for a recent survey.

A more general boundary condition might be imposed by considering

$$\begin{aligned}\mathrm{Hom}^{\check{S}}(\Gamma_{g,m}, G) = \{\rho \in \mathrm{Hom}(\Gamma_{g,m}, G) \mid \rho(c_i) &\text{ has at least} \\ \text{one fixed point in } \check{S}, \quad 1 \leqslant i \leqslant m\}\end{aligned} \tag{2}$$

where $\check{S}$ denotes the Shilov boundary. This is a union of relative character varieties and in this case $\mathrm{Hom}^{\max}(\Gamma_{g,m}, G) \subset \mathrm{Hom}^{\check{S}}(\Gamma_{g,m}, G)$. In particular $\mathrm{Hom}^{\max}(\Gamma_{g,m}, G)$ is a union of connected components of the set $\mathrm{Hom}^{\check{S}}(\Gamma_{g,m}, G)$ [BILW05, Corollary 14].

In this thesis we are interested in maximal representations inside (2) that satisfy a further condition: we will fix a union of conjugacy classes by imposing in (2) that every $\rho(c_i)$ fixes exactly two points in $\check{S}$ on which it acts expandingly (resp. contractingly). This is equivalent for the representation to be Anosov in the sense of [GW12]. We denote this space $\mathrm{Hom}^{\max,\mathrm{Shilov}}(\Gamma_{g,m}, G)$. The definition for the case $G = \mathrm{PSp}(2n, \mathbb{R})$ is given in 7.6. In particular we will study how to generalize arc coordinates for $\mathrm{Hom}^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4, \mathbb{R}))/\mathrm{PSp}(4, \mathbb{R})$. This will be made more clear in the next section.

1.2 The results

Let Σ be an oriented surface of negative Euler characteristic and fundamental group Γ. In this thesis we are interested in the case where $\partial\Sigma \neq \varnothing$.

Parametrizations are a very useful tool to construct examples of representations in higher rank Teichmüller spaces. Several coordinates have been introduced on the space of maximal representations. These often arise as a generalization of well known coordinates on the classical Teichmüller space $\mathcal{T}(\Sigma)$, such as Fenchel–Nielsen coordinates and shear coordinates.

In classical Teichmüller theory Fenchel-Nielsen coordinates are obtained by decomposing the surface in pairs of pants through the choice of a maximal collection of pairwise disjoint simple closed curves. The parametrization of $\mathcal{T}(\Sigma)$ is obtained by recording the length of the curves together with a gluing-parameter which records how much twist is involved in the gluing (see for example [Mar16]). Analogues of Fenchel–Nielsen coordinates on the space of maximal representations were developed by Strubel [Str15].

On the other hand, the construction of shear coordinates on the Teichmüller space of a hyperbolic surface Σ with at least one hole depends on the choice of a triangulation of Σ [Thu22]. Analogues of shear coordinates on the space of maximal representations were developed by Alessandrini Guichard Rogozinnikov and Wienhard [AGRW19].

Other examples of parametrizations of higher Teichmüller spaces are the work of [BD17] generalizing shearing coordinates on the Hitchin component in $\mathrm{PSL}(n, \mathbb{R})$ and its generalizations [MMMZ23], [Pfe22].

In this thesis we are interested in arc coordinates. In classical Teichmüller theory arc coordinates were introduced by Harer [Har86] who defined a complex of arcs on a surface with punctures and boundary components. This arc system allowed him to define a cell complex onto which $\mathcal{T}(\Sigma)$ may be Γ-equivariantly retracted. These coordinates were developed by Penner [Pen87] to decompose decorated Teichmüller space of punctured surface. This decomposition was generalized by [Ush99] [Pen02] for surfaces with boundary. Similar coordinates were used in [Luo07] [Guo09] to show that the Teichmüller space is an open convex polytope and by [Mon09] to express the Weil-Petersson Poisson structure on $\mathcal{T}(\Sigma)$ for a surface with geodesic boundary.

In this thesis we want to generalize arc coordinates to the space of maximal representations. We will consider the case where $\Sigma = \Sigma_{g,m}$ is a compact orientable smooth surface of genus g and m boundary components. We denote $\Gamma_{g,m}$ the fundamental group $\pi_1(\Sigma_{g,m})$, which is isomorphic to the free group $\mathbb{F}_{2g+m-1}$. An element $\gamma \in \Gamma_{g,m}$ is called *peripheral* if it is represented by a loop that is freely homotopic into a boundary component of $\Sigma_{g,m}$. We can equip $\Sigma_{g,m}$ with a complete hyperbolic structure of finite volume with geodesic boundary. The universal covering $\widetilde{\Sigma}_{g,m}$ of $\Sigma_{g,m}$ is a closed subset of the hyperbolic plane $\mathbb{H}^2$ where boundary curves are geodesics.

Arc coordinates are obtained by decomposing the surface in hexagons through the choice of a maximal collection $\{a_1, ..., a_k\}$ of pairwise disjoint arcs with starting and ending point on a boundary component which are essential and pairwise non-homotopic. For every hexagon in this decomposition there are exactly three alternating edges belonging to $\partial\Sigma_{g,m}$. We denote by E the set of all edges and by E_{bdry} the set of edges lying on a boundary component. For a fixed hyperbolic structure we can always realize the hexagon decomposition of $\Sigma_{g,m}$ in a way such that every edge is a geodesic and every arc is the unique geodesic which is orthogonal to the boundary at both endpoints. For each choice of $\{a_1, ..., a_k\}$ we get a parametrization of the Teichmüller space $\mathcal{T}(\Sigma_{g,m})$: once we fix the lengths $l(a_1), ..., l(a_k)$ there is a unique hyperbolic metric that makes $\Sigma_{g,m} \backslash \bigcup_i a_i$ a union of hyperbolic right-angled hexagons where each hexagon has exactly three alternating edges $a_{i_1}, a_{i_2}, a_{i_3}$ in $E \backslash E_{bdry}$ of length $l(a_{i_1}), l(a_{i_2}), l(a_{i_3})$ respectively, where $i_1, i_2, i_3 \in \{1, ..., k\}$. This is due to the well known fact that given three real numbers $b, c, d > 0$ there exists (up to isometries) a unique right-angled hexagon in $\mathbb{H}^2$ with alternating sides of lengths b, c and d (see for example [Mar16, Lemma 6.2.2]).

A point in the Teichmüller space $\mathcal{T}(\Sigma_{g,m})$ is identified with a maximal representation $\rho : \Gamma_{g,m} \to \mathrm{PSL}(2, \mathbb{R})$. Since we are considering surfaces with geodesic boundary, the image $\rho(\gamma)$ of every element $\gamma \in \Gamma_{g,m}$ is a hyperbolic isometry fixing exactly two points in $\partial\mathbb{H}^2$. The above discussion asserts that once we fix the lengths $l(a_1), ..., l(a_k)$ we can explicitly write (up to conjugation) the maximal representation ρ such that $\Sigma_{g,m} = \rho(\Gamma_{g,m}) \backslash \mathbb{H}^2$. An example for the surface $\Sigma_{0,3}$ (pair of pants) is given in Figure 1, where the fundamental group $\Gamma_{0,3}$ is the free group generated by α and β.

More generally, given a maximal representation $\rho : \Gamma_{g,m} \to \mathrm{PSp}(2n, \mathbb{R})$, the image $\rho(\gamma)$ of every non-peripheral element $\gamma \in \Gamma_{g,m}$ is Shilov hyperbolic (see [Str15]). Equivalently, $\rho(\gamma)$ fixes two transverse Lagrangians l_γ^+ and l_γ^- on which it acts expandingly and contractingly respectively. These Lagrangians are the images $\xi(\gamma^+)$ and $\xi(\gamma^-)$ where $\xi : S^1 \to \mathcal{L}(\mathbb{R}^{2n})$ is the equivariant

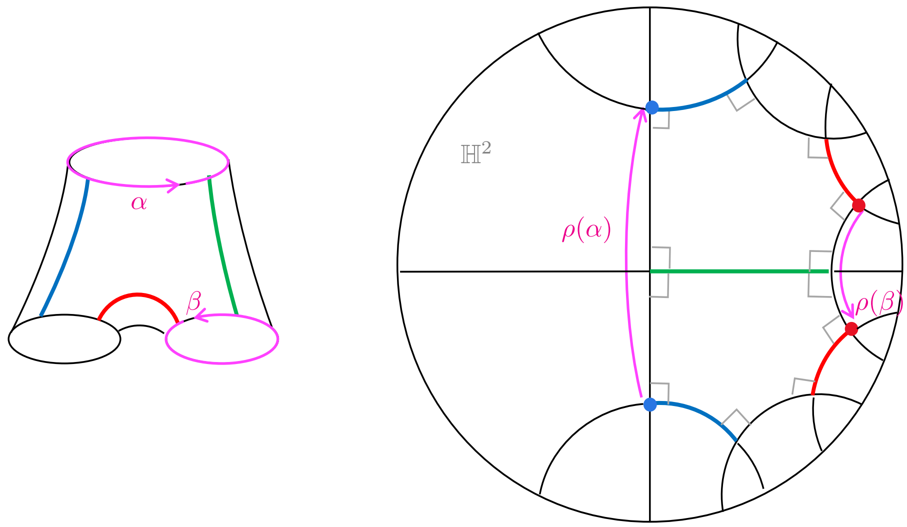


Figure 1: The maximal representation
$\rho : \Gamma_{0,3} \to \mathrm{PSL}(2, \mathbb{R})$

boundary map and $l_\gamma^\pm = \xi(\gamma^\pm)$. We want to parametrize the set of maximal representations where the property of being Shilov hyperbolic is true also for peripheral elements.

Definition. (see 7.6) A maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{PSp}(2n, \mathbb{R})$ will be called *Shilov hyperbolic* if $\rho(\gamma)$ is Shilov hyperbolic for every $\gamma \in \pi_1(\Sigma)$. The set of maximal representations which are Shilov hyperbolic will be denoted by $\mathrm{Hom}^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbb{R}))$. We define $\chi^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbb{R}))$ as the quotient

$$\chi^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbb{R})) := \mathrm{Hom}^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n, \mathbb{R}))/_{\mathrm{PSp}(2n,\mathbb{R})}$$

where $\mathrm{PSp}(2n, \mathbb{R})$ is acting by conjugation: $\rho \sim \rho'$ if there exists $g \in \mathrm{PSp}(2n, \mathbb{R})$ such that $\rho(\gamma) = g\rho'(\gamma)g^{-1}$ for all $\gamma \in \pi_1(\Sigma)$.

The standard example that we will consider is the surface $\Sigma_{0,3}$ (pair of pants), where $\Gamma_{0,3} \cong \mathbb{F}_2 = \langle \alpha, \beta \rangle$. We study in detail how to generalize arc coordinates for the space $\chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4, \mathbb{R}))$. To do this we consider the Siegel space $\mathcal{X}$, the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. We fix the Weyl chamber $\overline{\mathfrak{a}}^+$

$$\overline{\mathfrak{a}}^+ = \{(x_1, x_2) \in \mathbb{R}^2 | \ x_1 \geqslant x_2 \geqslant 0\}$$

and the set of regular vectors inside $\overline{\mathfrak{a}}^+$ will be denoted by $\mathfrak{a}$

$$\mathfrak{a} = \{(x_1, x_2) \in \mathbb{R}^2 | \ x_1 > x_2 > 0\}$$

We further denote by $\mathfrak{d}$ the set

$$\mathfrak{d} = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 = x_2\}$$

The first step is to introduce a parameter space for a right-angled hexagon in $\mathcal{X}$. The subspaces of the Siegel space that play the role of geodesics in $\mathbb{H}^2$ are called $\mathbb{R}$-*tubes* (see Definition 2.12). In Section 4.1 we give the definition of a right-angled hexagon H in $\mathcal{X}$, which is determined by a cyclic sequence of $\mathbb{R}$-tubes

$$H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$$

where any two consecutive tubes are orthogonal. We further define the set of *ordered right-angled hexagons* $\mathcal{H}$ (Definition 4.3). This is the data $(H, \mathcal{Y}_1)$ of a right-angled hexagon together with the choice of a tube $\mathcal{Y}_1$. We distinguish between generic (Definition 4.6) and non-generic hexagons (Section 4.2). A generic hexagon will be parametrized by length parameters $\underline{b}, \underline{c}, \underline{d}$ inside $\mathfrak{a}$ and angle parameters α_1, α_2 lying in $[0, 2\pi)$ (Proposition 4.15). In the non-generic case some length parameters will lie in $\mathfrak{d}$ and some angle parameters will vanish (Propositions 4.20, 4.21 and 4.22). This leads to a geometric visualization of a right-angled hexagon inside $\mathcal{X}$ in terms of a polygonal chain. This is obtained by projecting boundary points to the standard tube $\mathcal{Y}_{0,\infty}$ which is isometric to $\mathbb{R} \times \mathbb{H}^2$ (Lemma 2.28). This is explained in Section 4.4 and is illustrated in Figure 2 below.

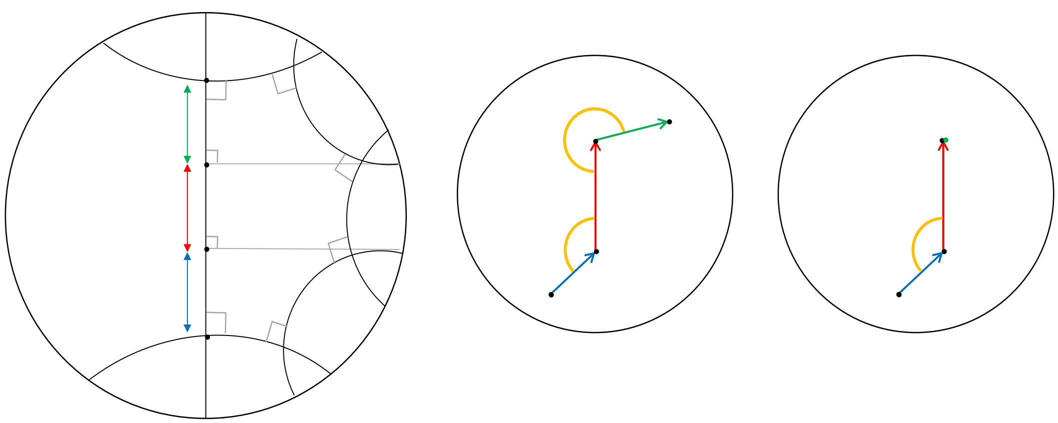


Figure 2: Geometrical interpretation of a hexagon in terms of a polygonal chain in the case of a generic and a non-generic hexagon.

A parameter space which encloses both generic and non-generic hexagons is given by

Theorem. *(see 4.26) The space $\mathcal{H}$ is parametrized up to isometry by*

$$\mathcal{A} = \overline{\mathfrak{a}}^3 \times [0, 2\pi) \times [0, 2\pi)/_\sim$$

The equivalence relation collapses one of the angles to a point in the case where the hexagon degenerates to a non-generic one.

These parameters were firstly introduced with the aim of generalizing ([Mar16, Lemma 6.2.2]). This approach turned out to be very tricky and this is explained in detail in Chapter 5.

Geometric parameters for maximal representations should be thought as the data of lengths and angles which uniquely determine two adjacent hexagons both having three alternating sides of length $\underline{b}, \underline{c}$ and $\underline{d}$ respectively. The maximal representation is then determined by determining the image of the generators of the fundamental group generalizing the geometric construction of Figure 1. The problem is that when extending our hexagon-parameters for two adjacent hexagons we can not guarantee that the constructed hexagons have the same alternating side-lengths. We will therefore construct two adjacent hexagons starting with one hexagon H and obtaining the others by reflecting H across a side (Figure 3). A precise definition of a reflection in the Siegel space $\mathcal{X}$ together with interesting geometric properties is given in Chapter 6.

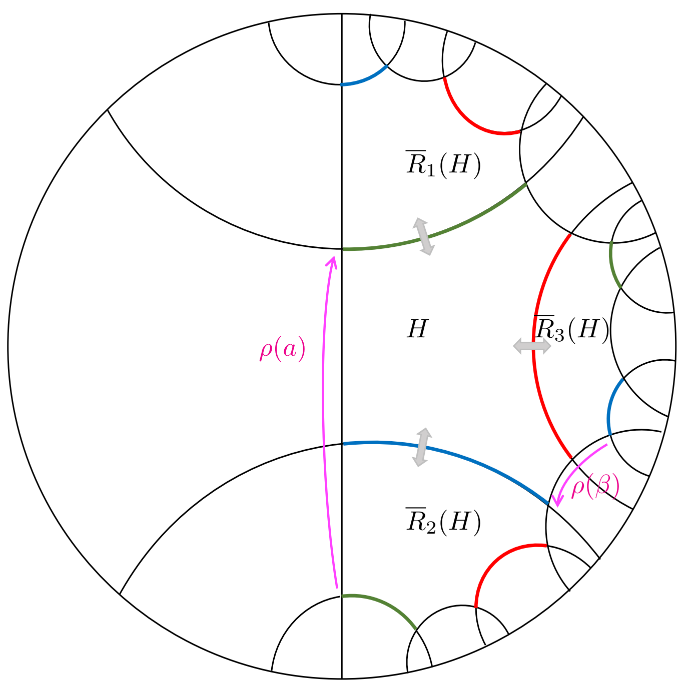


Figure 3: The maximal representation
$\rho : \Gamma_{0,3} \to \mathrm{PSp}(4, \mathbb{R})$

This will lead to the parametrization of a subset

$$\chi^{\mathcal{S}} \subset \chi^{\max,\text{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4,\mathbb{R}))$$

The idea is to see the fundamental group $\Gamma_{0,3}$ as a subgroup of the Coxeter group

$$W_3 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle s_1, s_2, s_3 |\ s_1^2 = s_2^2 = s_3^2 = 1\rangle$$

through the following homomorphism ϕ

$$\begin{aligned} \phi : \Gamma_{0,3} &\to W_3 \\ \alpha &\mapsto s_1 s_2 \\ \beta &\mapsto s_2 s_3 \end{aligned}$$

We will define the notion of maximal representation of the reflection group W_3 into $\mathrm{PSp}^{\pm}(2n,\mathbb{R})$, where $\mathrm{PSp}^{\pm}(2n,\mathbb{R})$ denotes the union of symplectic and antisymplectic matrices (Definition 6.9). Other interesting works introducing maximal representations for orbifold groups are developed in [AC19] [ALS23].

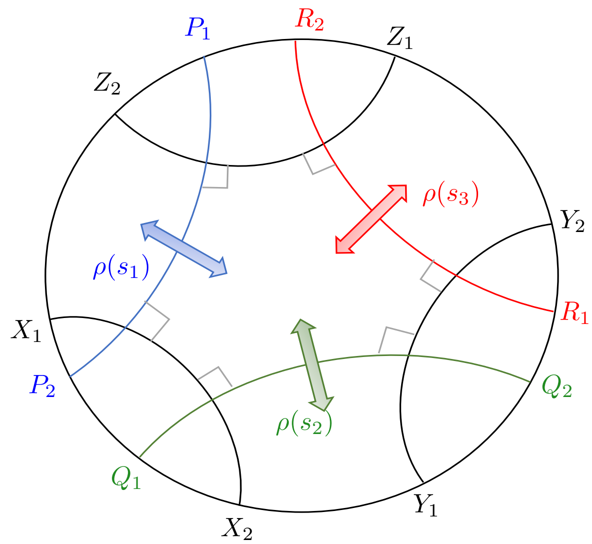


Figure 4: The reflections $\rho(s_1), \rho(s_2), \rho(s_3)$ for $\rho : W_3 \to \mathrm{PSp}^{\pm}(2n,\mathbb{R})$ maximal

Definition. (see 7.8) A representation

$$\rho : W_3 \to \mathrm{PSp}^{\pm}(2n,\mathbb{R})$$

is *maximal* if there exists a maximal 6-tuple of Lagrangians $(P_1, P_2, Q_1, Q_2, R_1, R_2)$ such that $\rho(s_1), \rho(s_2), \rho(s_3)$ are reflections of $\mathcal{X}$ fixing $(P_1, P_2), (Q_1, Q_2), (R_1, R_2)$

respectively and such that

$$\begin{cases} \rho(s_1)(X_1) = X_2 \text{ and } \rho(s_1)(Z_1) = Z_2 \\ \rho(s_2)(X_1) = X_2 \text{ and } \rho(s_2)(Y_1) = Y_2 \\ \rho(s_3)(Y_1) = Y_2 \text{ and } \rho(s_3)(Z_1) = Z_2 \end{cases}$$

where $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are uniquely determined by

$$\mathcal{Y}_{P_1,P_2} \perp \mathcal{Y}_{X_1,X_2} \perp \mathcal{Y}_{Q_1,Q_2} \perp \mathcal{Y}_{Y_1,Y_2} \perp \mathcal{Y}_{R_1,R_2} \perp \mathcal{Y}_{Z_1,Z_2}$$

We provide a parametrization of the set of maximal representations of the reflection group W_3 in the case of $\mathrm{PSp}^{\pm}(4,\mathbb{R})$.

Theorem. *(see 7.21) The set* $\chi^{max}(W_3, \mathrm{PSp}^{\pm}(4,\mathbb{R})) = \mathrm{Hom}^{max}(W_3, \mathrm{PSp}^{\pm}(4,\mathbb{R}))/\mathrm{PSp}(4,\mathbb{R})$ *is parametrized by the parameter space* $\mathcal{S}$*:*

$$\mathcal{S} \subset \mathcal{A} \times \mathcal{K}^3$$

where $\mathcal{A}$ *is the parameter space of a right-angled hexagon and* $\mathcal{K}$ *is the set*

$$\mathcal{K} = \left\{ \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix},\ K \in \mathrm{PO}(2),\ K^2 = \mathrm{Id} \right\}$$

In Section 6.4 we give a geometrical interpretation of the set $\mathcal{K}$ in terms of the polygonal chain associated to a right-angled hexagon. We prove that the restriction to $\Gamma_{0,3}$ of such a maximal representation is maximal and Shilov hyperbolic.

Proposition. *(see 7.18) Fix* $\tilde{\rho} \in \mathrm{Hom}^{max}(W_3, \mathrm{PSp}^{\pm}(4,\mathbb{R}))$*. Then the representation* $\rho := \tilde{\rho}|_{Im(\phi)}$ *is inside* $\mathrm{Hom}^{max, Shilov}(\Gamma_{0,3}, \mathrm{PSp}(4,\mathbb{R}))$.

This allows us to define $\chi^{\mathcal{S}}$ (Definition 7.19) as the image $\chi^{\mathcal{S}} := Im(f)$ where f is the map

$$\begin{aligned} f : \chi^{\max}(W_3, \mathrm{PSp}^{\pm}(4,\mathbb{R})) &\to \chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4,\mathbb{R})) \\ [\tilde{\rho}] &\mapsto [\tilde{\rho}|_{Im(\phi)}] \end{aligned}$$

This leads to a parametrizatin of $\chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}^{\pm}(4,\mathbb{R}))$ by imposing an equivalent relation on $\mathcal{S}$ which identifies the points that have same image under f.

Theorem. *(see 7.23) The set* $\chi^{\mathcal{S}}$ *is parametrized by* $\mathcal{S}/_{\sim}$

In Corollary 7.24 we show that, contrary to the hyperbolic case (Proposition 7.16), the map f is not injective nor surjective.

A motivation for this work is to study compactification of character varieties where similar arguments can be carried out with non-Archimedean Siegel spaces as in [BP17]. We expect applications of this work in the study of the real spectrum compactification of maximal character varieties (see [BIPP21a], [BIPP23]) Of particular interest are rank two groups where [BIPP21b] and [OT23b] [OT23a] suggest a link with flat structures with angle multiple of $\frac{\pi}{2}$. Developing arc coordinates for those would be interesting.

Finally, the parameter space of $\chi^{\mathcal{S}}$ was implemented on a Python program which constructs the generators of a maximal representation into $\mathrm{PSp}(4,\mathbb{R})$. The most important functions appearing in the program are shown in Chapter 8.

The last chapter investigates other methods for the generalization of arc coordindates to maximal representations and is a joint work with Eugen Rogozinnikov.

1.3 Organization of the work

In Chapter 2 we discuss properties of the geometry of the Siegel space $\mathcal{X}$- the symmetric space associated to $\mathrm{Sp}(2n,\mathbb{R})$. We cover the basic definitions and study in detail the geometry of $\mathbb{R}$-tubes in the case of $\mathrm{Sp}(4,\mathbb{R})$.

In Chapter 3 we define the set of generic quintuples and give a parameter space for them (Proposition 3.7). These parameters will be very useful for the parametrization of right-angled hexagons.

Chapter 4 is dedicated to the study of hexagons. We define the set of ordered right-angled hexagons (Definition 4.3) and distinguish between generic (Definition 4.6) and non-generic hexagons (Section 4.2). We introduce a parameter space for both cases (Proposition 4.15 for the generic case and Propositions 4.20, 4.21 and 4.22 for the non-generic case). A parameter space which encloses both generic and non-generic hexagons is given in Theorem 4.26. These parameters will be called *arc coordinates*. Note that this term could be misleading as we have used it also for the parametrization of classical Teichmüller space and for its generalization in the case of maximal representations. Nevertheless, we have decided to keep this name also for the parameters of a hexagon as they are crucial for the construction of parameters for maximal representations and will appear in their parameter space (Theorem 7.23).

In Chapter 5 we show how arc coordinates arise from the idea of generalizing coordinates of a hexagon in $\mathbb{H}^2$ and explain the problems encountered in this approach.

In Chapter 6 we discuss properties of reflections in $\mathbb{H}^2$ and their analogues in the Siegel space $\mathcal{X}$. We define the *reflection set associated to the side of a hexagon* (Definition 6.20) and give a geometric interpretation of it (Section 6.4).

In Chapter 7 we discuss geometric properties of Shilov hyperbolic isometries and we define the set $\chi^{\mathrm{max,Shilov}}(\pi_1(\Sigma),\mathrm{PSp}(2n,\mathbb{R}))$. We further define the notion of a maximal representation from the Coxeter group $W_3 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ into $\mathrm{PSp}^{\pm}(2n,\mathbb{R})$ (Definition 7.8) and provide a parameter space for the $\mathrm{PSp}^{\pm}(4,\mathbb{R})$-case (Theorem 7.21). We further define the set $\chi^{\mathcal{S}} \subset \chi^{\mathrm{max,Shilov}}(\Gamma_{0,3},\mathrm{PSp}(4,\mathbb{R}))$ (Definition 7.19) to which we provide a parameter space (Theorem 7.23).

In Chapter 8 we list and explain the functions of a Python program which implements the proof of Theorem 7.23, that is it constructs the generators of a

$\rho \in \chi^{\mathcal{S}}$ for a given point in its parameter space.

In Chapter 9 we discuss other approaches to parametrize right-angled hexagons in $\mathcal{X}$. This chapter is a joint work with Eugen Rogozinnikov. We discuss the problems that arise when extending these parameters to maximal representations.

2 The Siegel space

2.1 Definition and models

The *Siegel space* $\mathcal{X}$ is the symmetric space associated to the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. Standard references for the theory of symmetric spaces are for example [Hel79], [Ebe96], [Mau04], [Boo86]. Recall that the symplectic group

$$\mathrm{Sp}(2n,\mathbb{R}) = \{M \in \mathrm{SL}(2n,\mathbb{R}) \mid M^T J_n M = J_n\}$$

is the subgroup of $\mathrm{SL}(2n,\mathbb{R})$ preserving the symplectic form $\omega(\cdot,\cdot)$ represented, with respect to the standard basis, by the matrix

$$J_n = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

The group $\mathrm{Sp}(2n,\mathbb{R})$ can also be described as the group of block matrices:

$$\mathrm{Sp}(2n,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A^TC, B^TD \text{ symmetric, and } A^TD - C^TB = \mathrm{Id}_n \right\}$$

If $n = 1$ the group $\mathrm{Sp}(2n,\mathbb{R})$ coincides with $\mathrm{SL}(2,\mathbb{R})$. There are two models commonly used for the Siegel space: the *upper-half space* and the *Borel embedding* model.

1. The *upper-half space model* is a generalization of the upper-half space model of the hyperbolic plane and is given by a specific set of symmetric matrices:

   $$\mathcal{X} = \{X + iY,\ X \in \mathrm{Sym}(n,\mathbb{R}),\ Y \in \mathrm{Sym}^+(n,\mathbb{R})\}$$

   where $\mathrm{Sym}(n,\mathbb{R})$ denotes the set of n-dimensional symmetric matrices with coefficients in $\mathbb{R}$ and $\mathrm{Sym}^+(n,\mathbb{R})$ is the subset of $\mathrm{Sym}(n,\mathbb{R})$ given by positive definite matrices. The group $\mathrm{Sp}(2n,\mathbb{R})$ acts on $\mathcal{X}$ by fractional linear transformations:

   $$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$$

   This action is transitive and the stabilizer of the point $i\mathrm{Id}_n \in \mathcal{X}$ is isomorphic to the group $U(n)$.

2. The *Borel embedding model* is given by

   $$\mathbb{X} = \{l \in \mathcal{L}(\mathbb{C}^{2n})|\ \ i\omega(\sigma(\cdot),\cdot)_{\mathbb{C}}|_{l\times l} \text{ is positive definite}\}$$

   where $\mathcal{L}(\mathbb{C}^{2n})$ is the set of Lagrangians and $\sigma : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ denotes complex conjugation.

An $\mathrm{Sp}(2n,\mathbb{R})$-equivariant identification $\mathcal{X} \mapsto \mathbb{X}$ is induced by the affine chart

$$\iota : \mathrm{Sym}(n,\mathbb{C}) \to \mathcal{L}(\mathbb{C}^{2n})$$

that associates to a symmetric matrix Z the linear subspace of $\mathbb{C}^{2n}$ spanned by the columns of the matrix $\begin{pmatrix} Z \\ \mathrm{Id}_n \end{pmatrix}_{2n\times n}$, where it is easy to show that the symmetry of Z implies $\iota(Z) \in \mathcal{L}(\mathbb{C}^{2n}$.

For the inverse of ι observe that for any $l \in \mathcal{L}(\mathbb{C}^{2n})$ we can always write l as

$$l = \left\langle \begin{pmatrix} z_1^1 \\ \vdots \\ z_1^n \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., \begin{pmatrix} z_n^1 \\ \vdots \\ z_n^n \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\rangle = \langle v_1, ..., v_n \rangle$$

where $z_i^j \in \mathbb{C}$. Then for l to be a Lagrangian it must hold $\omega(v_i, v_j) = 0$, that is

$$\omega(v_i, v_j) = -z_j^i + z_i^j = 0$$

so that l is a subspace of $\mathbb{C}^{2n}$ spanned by the columns of the matrix $\begin{pmatrix} Z \\ \mathrm{Id}_n \end{pmatrix}$ where $Z \in \mathrm{Sym}(n,\mathbb{C})$. The restriction of the affine chart ι to the subspace $\mathrm{Sym}(n,\mathbb{R})$ provides a parametrization of the set of real Lagrangians that are transverse as linear subspaces to $\langle e_1, ..., e_n \rangle$, which will be denoted by l_∞ or just ∞.

Remark 2.1. We have seen that the group $\mathrm{Sp}(2n,\mathbb{R})$ acts on $\mathcal{X}$ by fractional linear transformations. Given $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$ and $Z \in \mathcal{X}$ it holds

$$M \cdot Z = (AZ + B)(CZ + D)^{-1} = (-M) \cdot Z$$

When studying actions on $\mathcal{X}$ it makes therefore sense to consider M and $-M$ to be the same element inside $\mathrm{Sp}(2n,\mathbb{R})$. This means considering the group

$$\mathrm{PSp}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R})/_{\{\pm \mathrm{Id}\}}$$

2.2 Boundary and Lagrangians

The set of real Lagrangians $\mathcal{L}(\mathbb{R}^{2n})$ naturally arises as the unique closed $\mathrm{Sp}(2n,\mathbb{R})$-orbit in the boundary of $\mathcal{X}$ in its Borel embedding and for this reason $\mathcal{L}(\mathbb{R}^{2n})$ is the Shilov boundary of the bounded domain realization of $\mathcal{X}$ (see [Wie04]). Denote by $\mathcal{L}(\mathbb{R}^{2n})^{(k)}$ the set of k-tuples of pairwise transverse Lagrangians.

It is easy to prove that the group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on $\mathcal{L}(\mathbb{R}^{2n})^{(2)}$. Moreover, it has $(n+1)$ orbits in $\mathcal{L}(\mathbb{R}^{2n})^{(3)}$, indexed by the Maslov index: let $(l_1, l_2, l_3) \in \mathcal{L}(\mathbb{R}^{2n})^{(3)}$ and denote by l_3 the unique linear map

$$\begin{aligned} l_3 : l_1 &\to l_2 \\ u &\mapsto v|\ u + v \in l_3 \end{aligned}$$

Using the symplectic form ω we can define the bilinear form β on l_1 as following

$$\beta(u_1, u_2) := \omega(u_1, l_3(u_2))$$

Definition 2.2. The bilinear form β defined as above is called the *Maslov form.*

The Maslov form is symmetric and nondegenerate (see for example [Sou05]). We denote the signature of β by

$$sgn(\beta) = p - q$$

where p is the dimension of a maximal subspace of l_1 on which β is positive definite and q is the dimension of a maximal subspace of l_1 on which β is negative definite.

Definition 2.3. The *Maslov index* of the triple (l_1, l_2, l_3) is the signature of the associated Maslov form β and is denoted by $\mu_n(l_1, l_2, l_3)$.

The Maslov index is cyclically invariant, is invariant under the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathcal{L}(\mathbb{R}^{2n})^{(3)}$ and the group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on the set of triples of pairwise transverse Lagrangians with the same Maslov index [LV80].

The value of the Maslov index is maximal on the orbit of

$$(\langle e_1, ..., e_n\rangle, \langle e_{n+1}, ..., e_{2n}\rangle, \langle e_1 + e_{n+1}, ..., e_n + e_{2n}\rangle) = (l_\infty, 0, \mathrm{Id})$$

It is minimal on the orbit of $(l_\infty, \mathrm{Id}, 0)$ and is zero on the orbit of $(l_\infty, 0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$.

Definition 2.4. (**Maximal triple and maximal m-tuple**) A triple of pairwise transverse Lagrangians is called *maximal* if it is in the $\mathrm{Sp}(2n,\mathbb{R})$-orbit of $(l_\infty, 0, \mathrm{Id})$. An m-tuple $(l_1, ..., l_m)$ is *maximal* if for every $i < j < k$ the triple (l_i, l_j, l_k) is maximal.

Maximal triples are a generalization of positively oriented triples in the circle $S^1 = \partial\mathbb{H}^2$ and they play a central role in the study of maximal representations. It is useful to have a concrete criterion to check when triples of Lagrangian are maximal. The following Lemma can be found for example in [BP17, Lemma 2.10].

Lemma 2.5. *The following hold:*
(1) *Any cyclic permutation of a maximal triple is maximal;*
(2) *The triple (l_∞, X, Y) is maximal if and only if $Y - X$ is positive definite;*
(3) *If $Z - X$ is positive definite, the triple (X, Y, Z) is maximal if and only if $Z - Y$ and $Y - X$ are positive definite.*

From a given maximal m-tuple we can obtain a maximal $(m+k)$-tuple by adding a maximal k-tuple between two consecutive Lagrangians. More precisely:

Lemma 2.6. *Let $(P_1, ..., P_m)$ be a maximal m-tuple. For $i \in \{1, ..., m-1\}$ and $k \geqslant 1$ let $(P_i, Q_1, ..., Q_k, P_{i+1})$ be maximal. Then the $(m+k)$-tuple $(P_1, ..., P_i, Q_1, ..., Q_k, P_{i+1}, ..., P_m)$ is maximal.*

Proof. Up to isometry we reduce to the case where $P_1 = 0, P_m = l_\infty$, that is we consider the $(m+k)$-tuple $(0, P_2, ...P_i, Q_1, ..., Q_k, P_{i+1}, ..., l_\infty)$ where $(0, P_2, ..., P_{m-1}, l_\infty)$ maximal and $(P_i, Q_1, ..., Q_k, P_{i+1})$ maximal. Using Lemma 2.5 result follows immediately. □

2.3 $\mathrm{Sp}(2n, \mathbb{R})$-invariant distances

We introduce a $\mathrm{Sp}(2n, \mathbb{R})$-invariant distance on the symmetric space $\mathcal{X}$. Fix a point p in a maximal flat F and a Weyl chamber $\overline{\mathfrak{a}}^+ \subset T_pF$. This is a fundamental domain for the action of $\mathrm{Sp}(2n, \mathbb{R})$ on the tangent bundle $T\mathcal{X}$. In our case we have

$$\overline{\mathfrak{a}}^+ = \{(x_1, ..., x_n) \in \mathbb{R}^n |\ x_1 \geqslant ... \geqslant x_n \geqslant 0\}$$

A vector in the Weyl chamber is *regular* if all the inequalities are strict, which is equivalent to being contained in a unique flat. The set of regular vectors inside $\overline{\mathfrak{a}}^+$ will be denoted by $\mathfrak{a}$

$$\mathfrak{a} = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 > ... > x_n > 0\}$$

We will further denote by $\mathfrak{d}$ the following set

$$\mathfrak{d} = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 = ... = x_n\}$$

In order to define a vectorial $\mathrm{Sp}(2n, \mathbb{R})$-invariant distance in $\mathcal{X}$ we need to recall from [BP17] the definition of an endomorphism-valued cross-ratio. If $l_1, l_2 \in \mathcal{L}(\mathbb{R}^{2n})$ are transverse (denoted by $l_1 \pitchfork l_2$), we denote by $p_{l_1}^{||l_2} : \mathbb{R}^{2n} \to l_1$ the projection to l_1 parallel to l_2.

Definition 2.7. (**Cross-ratio**) For Lagrangians $l_1, ..., l_4 \in \mathcal{L}(\mathbb{C}^{2n})$ such that $l_1 \pitchfork l_2$ and $l_3 \pitchfork l_4$ the cross-ratio $R(l_1, l_2, l_3, l_4)$ is given in the Borel embedding model by the endomorphism of l_1

$$R(l_1, l_2, l_3, l_4) = p_{l_1}^{||l_2} \circ p_{l_4}^{||l_3}|_{l_1}$$

In the upper half space model the explicit expression for the cross-ratio is given by ([BP17, Lemma 4.2]):

$$R(X_1, X_2, X_3, X_4) = (X_1 - X_2)^{-1}(X_4 - X_2)(X_4 - X_3)^{-1}(X_1 - X_3)$$

where R is expressed with respect to the basis of X_1 given by the columns of the matrix $\begin{pmatrix} X_1 \\ \mathrm{Id}_n \end{pmatrix}$.

The following lemma can be found in [BP17, Lemma 4.3]:

Lemma 2.8. *Assume* $0, Z, X, l_\infty$ *are pairwise transverse. Then*

$$R(0, Z, X, l_\infty) = Z^{-1}X$$

We can now define the vectorial distance $d^{\overline{\mathfrak{a}}^+}$. The fact that the cross-ratio can be used to describe the projection of a pair of points in $\mathcal{X}$ onto the Weyl chamber was proved by Siegel in [Sie43].

Definition 2.9. The *vectorial distance* $d^{\overline{\mathfrak{a}}^+}$ is the projection onto the Weyl chamber $\overline{\mathfrak{a}}^+$:

$$\mathcal{X}^2 \to \overline{\mathfrak{a}}^+$$
$$(X, Z) \mapsto (\log(\lambda_1), ..., \log(\lambda_n))$$

where $\lambda_i = \frac{1+\sqrt{r_i}}{1-\sqrt{r_i}}$ and $1 > r_1 \geqslant ... \geqslant r_n \geqslant 0$ are the eigenvalues of $R(X, \overline{Z}, Z, \overline{X})$

For interesting properties about the distance $d^{\overline{\mathfrak{a}}^+}$ see [Par10],[KLP17]. For other interesting $\mathrm{Sp}(2n, \mathbb{R})$-invariant distances such as the Finsler distance see [FP20]. The following lemma can be found in [FP20, Lemma 2.14]:

Lemma 2.10. *Let* A *and* B *be positive definite symmetric matrices such that the difference* $B - A$ *is positive definite. Let* $\mu_1 \geqslant ... \geqslant \mu_n$ *be the eigenvalues of* $A^{-1}B$*. Then*

$$d^{\overline{\mathfrak{a}}^+}(iA, iB) = (\log \mu_1, ..., \log \mu_n)$$

2.4 Copies of $\mathbb{H}^2$ inside the Siegel space $\mathcal{X}$

Definition 2.11. Let $\mathcal{X}$ be the symmetric space associated to $\mathrm{Sp}(2n, \mathbb{R})$. A *maximal polydisc* in $\mathcal{X}$ is the image of a totally geodesic and holomorphic embedding of the Cartesian product of n copies of $\mathbb{H}^2$ into $\mathcal{X}$.

We will be interested in the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(4, \mathbb{R})$. In this case an example of a maximal polydisc is the image of the following map ψ:

$$\psi : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathcal{X}$$
$$(z_1, z_2) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

We will refer to this polydisc as the *model polydisc* since every other polydisc is translate of our model polydisc by an element in $\mathrm{Sp}(4, \mathbb{R})$ (see [Wol72]). Let (M_1, M_2) be an element of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. Then (M_1, M_2) acts on the model polydisk as following:

$$(M_1, M_2) \cdot \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} = \begin{pmatrix} M_1(z_1) & 0 \\ 0 & M_2(z_2) \end{pmatrix}$$

where

$$M(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az+b)(cz+d)^{-1}$$

is the action on a point $z \in \mathbb{H}^2$ by Möbius transformation. Let Δ be the diagonal embedding given by

$$\Delta : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(4,\mathbb{R})$$

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}$$

then we obtain the following commuting diagram

$$\begin{array}{ccc} \mathbb{H}^2 \times \mathbb{H}^2 & \xrightarrow[(M_1, M_2)\cdot]{} & \mathbb{H}^2 \times \mathbb{H}^2 \\ \psi \big\downarrow & & \big\downarrow \psi \\ \mathcal{X} & \xrightarrow[\Delta((M_1, M_2))]{} & \mathcal{X} \end{array}$$

In particular the set

$$\psi\big((z,z)\big) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \subset \mathcal{X}$$

is a copy of $\mathbb{H}^2$ inside $\mathcal{X}$ and will be called the *diagonal disc*.

2.5 $\mathbb{R}$-tubes

The subspaces of the Siegel space that play the role of geodesics in $\mathbb{H}^2$ are called $\mathbb{R}$-*tubes*. Let $\{a, b\}$ be an unordered pair of transverse Lagrangians.

Definition 2.12. ($\mathbb{R}$-***tube***) The $\mathbb{R}$-*tube* associated to $\{a, b\}$ is the set

$$\mathcal{Y}_{a,b} = \{l \in \mathcal{X} \mid R(a, l, \sigma(l), b) = -\mathrm{Id}\}$$

It can be proven (see [BP17]) that $\mathcal{Y}_{a,b}$ is a totally geodesic subspace of $\mathcal{X}$ of the same real rank as $\mathcal{X}$ and that it is the parallel set of the Riemannian singular geodesics whose endpoints in the visual boundary of $\mathcal{X}$ are the Lagrangians a and b. The group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on $\mathcal{L}(\mathbb{R}^{2n})^{(2)}$ and for every $g \in \mathrm{Sp}(2n,\mathbb{R})$ it holds $g \cdot \mathcal{Y}_{a,b} = \mathcal{Y}_{ga,gb}$. Up to the symplectic group action we can therefore reduce to a model $\mathbb{R}$-tube, the one with endpoints 0 and l_∞. In the upper-half space model this will be called the *standard tube* and consists of matrices of the form

$$\mathcal{Y}_{0,\infty} = \{iY | Y \in \mathrm{Sym}^+(n, \mathbb{R})\}$$

Intersection patterns of $\mathbb{R}$-tubes in the Siegel space reflect the intersection patterns of geodesics in the hyperbolic plane. This is shown in the following result, which can be found in [FP20, Proposition 2.16].

Proposition 2.13. *If (l_1, l_2, l_3, l_4) is maximal, the intersection $\mathcal{Y}_{l_1,l_3} \cap \mathcal{Y}_{l_2,l_4}$ consists of a single point and $\mathcal{Y}_{l_1,l_2} \cap \mathcal{Y}_{l_3,l_4}$ is empty.*

Definition 2.14. (**Orthogonal $\mathbb{R}$-tubes**) Two $\mathbb{R}$-tubes $\mathcal{Y}_{a,b}$ and $\mathcal{Y}_{c,d}$ are orthogonal if they are orthogonal as submanifolds of the symmetric space (where there is a well defined $\mathrm{Sp}(2n, \mathbb{R})$-invariant scalar product).

Remark 2.15. The orthogonality relation can be expressed as a property of the cross-ratio of the boundary points: if (a, c, b, d) is maximal, the $\mathbb{R}$-tubes $\mathcal{Y}_{a,b}$ and $\mathcal{Y}_{c,d}$ are orthogonal if and only if $R(a, c, b, d) = 2\mathrm{Id}$ (see [BP17, Definition 4.14]).

Denote by $((a, b)) := \{l \in \mathcal{L} |\ (a, l, b) \text{ is maximal}\}$ and by

$$p_{a,b} : \mathcal{X} \cup ((a, b)) \to \mathcal{Y}_{a,b}$$

the orthogonal projection. It will be useful to have concrete expressions for the orthogonal projection to $((a, b))$ when $(a, b) = (0, l_\infty)$ and for the Weyl chamber distance between two orthogonally projected points. Recall that we identify $\mathrm{Sym}(n, \mathbb{R})$ with the Lagrangians in $\mathcal{L}(\mathbb{R}^{2n})$ that are transverse to l_∞ via the restriction of the affine chart $\iota : \mathrm{Sym}(n, \mathbb{C}) \to \mathcal{L}(\mathbb{C}^{2n})$. Both of the following lemmas can be found in [FP20, Lemma 2.24 and 2.25].

Lemma 2.16. *For any $A \in \mathrm{Sym}^+(n, \mathbb{R})$ the $\mathbb{R}$-tubes $\mathcal{Y}_{A,-A}$ and $\mathcal{Y}_{0,\infty}$ are orthogonal and their unique intersection point is iA. In particular $p_{0,\infty}(A) = iA$.*

Lemma 2.17. *If $(a, x, y, b) \in \mathcal{L}(\mathbb{R}^{2n})^4$ is a maximal 4-tuple and $p_{a,b}$ is the orthogonal projection onto $\mathcal{Y}_{a,b}$, the distance*

$$d^{\overline{\mathfrak{a}}^+}(p_{a,b}(x), p_{a,b}(y)) = (\log \mu_1, ..., \log \mu_n)$$

where μ_i are the eigenvalues of the cross-ratio $R(a, x, y, b)$.

2.6 Computing orthogonal tubes

In Chapter 4 we will define right-angled hexagons in the Siegel space $\mathcal{X}$ and introduce a suitable parameter space for them. A crucial tool to construct right-angled hexagons is computing orthogonal $\mathbb{R}$-tubes. For this reason, this section lists concrete criteria to determine them.

Lemma 2.18. *Let (P_1, P_2, P_3, P_4) be a maximal 4-tuple. Then there exists a unique tube $\mathcal{Y}_{P_5,P_6}$ orthogonal to both $\mathcal{Y}_{P_1,P_4}$ and $\mathcal{Y}_{P_2,P_3}$.*

Proof. Up to $\mathrm{Sp}(2n,\mathbb{R})$-action we can consider

$$(P_1, P_2, P_3, P_4) = (0, \mathrm{Id}, P, \infty)$$

By Lemma 2.16 the tubes orthogonal to $\mathcal{Y}_{0,\infty}$ are of the form $Y_{-Q,Q}$ where $Q \in \mathrm{Sym}^+(n,\mathbb{R})$. We want to find Q such that the triple (Id, Q, P) is maximal and such that $\mathcal{Y}_{-Q,Q} \perp \mathcal{Y}_{\mathrm{Id},P}$. By the orthogonality condition (see Remark 2.15) this happens if and only if $R(P, -Q, \mathrm{Id}, Q) = 2\mathrm{Id}$. Developing the left-hand side we obtain:

$$2(P+Q)^{-1}Q(Q-\mathrm{Id})^{-1}(P-\mathrm{Id}) = 2(P+Q)^{-1}\big((Q-\mathrm{Id})Q^{-1}\big)^{-1}(P-\mathrm{Id}) =$$

$$= 2\big((\mathrm{Id}-Q^{-1})(P+Q)\big)^{-1}(P-\mathrm{Id}) = 2\mathrm{Id}$$

This simplifies to

$$P + Q - Q^{-1}P - \mathrm{Id} = P - \mathrm{Id}$$

We obtain

$$Q^2 = P \text{ which has unique solution } Q = \sqrt{P}$$

In particular $P \in \mathrm{Sym}^+(n,\mathbb{R})$ as $(0, \mathrm{Id}, P, \infty)$ is maximal (see Lemma 2.5). ☐

Lemma 2.19. *Let $(0, \mathrm{Id}, P, \infty)$ be a maximal quadruple. Then $\mathcal{Y}_{-\sqrt{P},\sqrt{P}}$ is the unique $\mathbb{R}$-tube orthogonal to both $\mathcal{Y}_{0,\infty}$ and $\mathcal{Y}_{\mathrm{Id},P}$.*

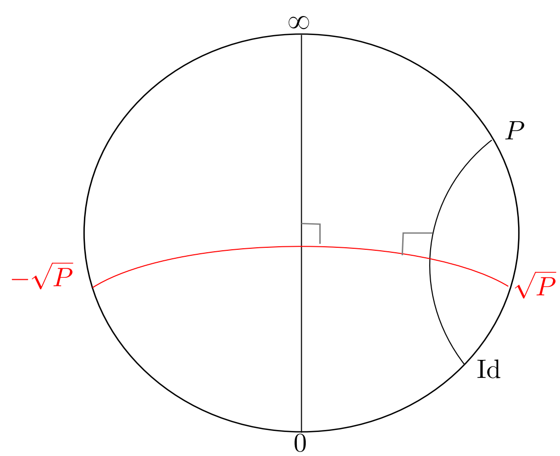


Figure 5: The tube $\mathcal{Y}_{-\sqrt{P},\sqrt{P}}$ orthogonal to both $\mathcal{Y}_{\mathrm{Id},P}$ and $\mathcal{Y}_{0,\infty}$

Proof. Follows directly from proof the of Lemma 2.18 The configuration of the tubes $\mathcal{Y}_{0,\infty} \perp \mathcal{Y}_{-\sqrt{P},\sqrt{P}} \perp \mathcal{Y}_{\mathrm{Id},P}$ is illustrated in Figure 5. ☐

In the more general case we have:

Lemma 2.20. *Let (P_1, P_2, P_3, P_4) be a maximal quadruple. Then the unique tube orthogonal to both $\mathcal{Y}_{P_1,P_4}$ and $\mathcal{Y}_{P_2,P_3}$ is $\mathcal{Y}_{Z_1,Z_2}$ where*

$$Z_1 = g^{-1}\big(-\sqrt{gP_3}\big),\ Z_2 = g^{-1}\big(\sqrt{gP_3}\big)$$

$$g = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & (P_1 - P_4)^{-1} \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -P_4 \\ 0 & \mathrm{Id} \end{pmatrix}$$

and $A = \sqrt{(P_1 - P_4)(P_2 - P_1)^{-1}(P_2 - P_4)}$

Proof. The matrix $g \in \mathrm{Sp}(2n, \mathbb{R})$ is an isometry such that

$$g \cdot (P_1, P_2, P_4) = (0, \mathrm{Id}, \infty)$$

Result follows from Lemma 2.19. □

Lemma 2.21. *Let $(P_1, P_2, P_3, P_4, P_5, P_6)$ be a maximal 6-tuple and let Q_1, Q_2, Q_3, Q_4 be such that*

$$\mathcal{Y}_{P_1,P_2} \perp \mathcal{Y}_{Q_1,Q_2} \perp \mathcal{Y}_{P_3,P_4} \perp \mathcal{Y}_{Q_3,Q_4} \perp \mathcal{Y}_{P_5,P_6}$$

Then the quadruple (P_3, Q_2, Q_3, P_4) is maximal.

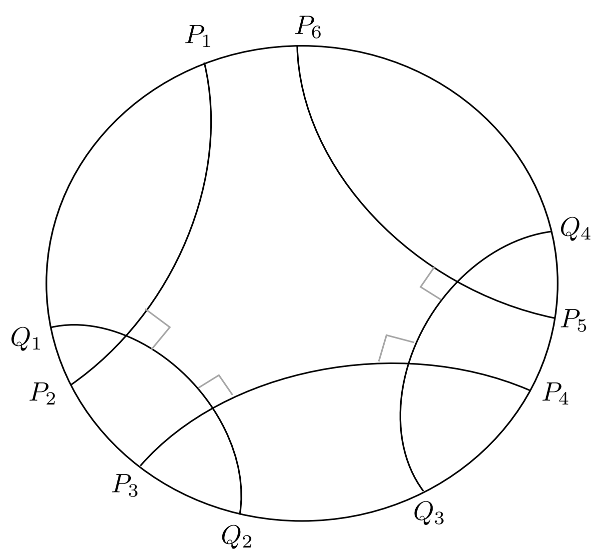


Figure 6: The quadruple (P_3, Q_2, Q_3, P_4) is maximal

Proof. Let $g \in \mathrm{Sp}(2n, \mathbb{R})$ be such that

$$g \cdot (Q_1, Q_2) = (\infty, 0) \text{ and } g \cdot (P_3, P_4) = (-\mathrm{Id}, \mathrm{Id})$$

We obtain the tubes

$$g \cdot \mathcal{Y}_{P_1,P_2} = \mathcal{Y}_{-M,M},\ g \cdot \mathcal{Y}_{P_5,P_6} = \mathcal{Y}_{P,Q}$$

for some M, P, Q positive definite matrices. The tube $g \cdot \mathcal{Y}_{Q_3,Q_4} = \mathcal{Y}_{X,Y}$ is such that

$$\mathcal{Y}_{-\mathrm{Id},\mathrm{Id}} \perp \mathcal{Y}_{X,Y} \perp \mathcal{Y}_{P,Q} \tag{3}$$

where P and Q are positive definite matrices (Figure 7).

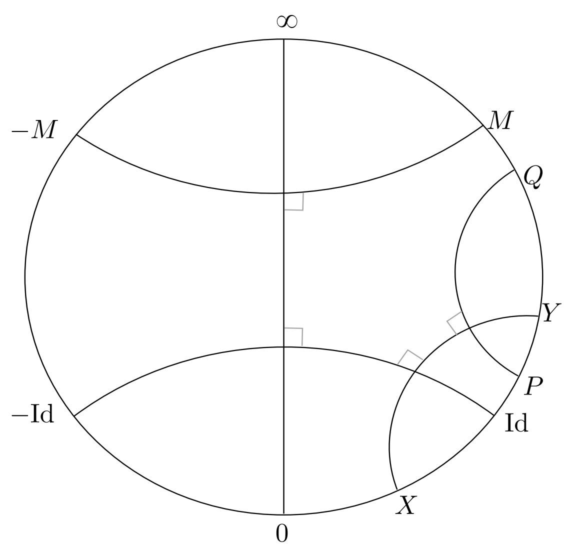


Figure 7: The quadruple $(0, X, \mathrm{Id}, Y)$ is maximal

By construction of orthogonal tubes we know $(-\mathrm{Id}, 0, \mathrm{Id})$ and (X, Id, Y) maximal (Lemma 2.19). It is not hard to show that the matrix X needs to be positive definite for the condition (3) to be satisfied. It follows $(-\mathrm{Id}, 0, X, \mathrm{Id})$ maximal and so is its preimage (P_3, Q_2, Q_3, P_4). ☐

We end this section by giving some concrete expressions to find two orthogonal tubes when one of them is of the form $\mathcal{Y}_{-P,P}$ for a positive definite matrix P. This configuration will turn out to be very useful when defining the parameter space of right-angled hexagons.

Lemma 2.22. *Let $(0, P_1, P_2, \infty)$ be a maximal quadruple. Then*

$$\mathcal{Y}_{-P_1,P_1} \perp \mathcal{Y}_{P_1P_2^{-1}P_1,P_2} \text{ and } \mathcal{Y}_{-P_2,P_2} \perp \mathcal{Y}_{P_1,P_2P_1^{-1}P_2}$$

Proof. For the first case it is sufficient to find $X \in \mathrm{Sym}(n, \mathbb{R})$ such that $R(X, P_1, P_2, -P_1) = 2\mathrm{Id}$ (see Remark 2.15). Developing the left-hand side we obtain

$$(X - P_1)^{-1}(-2P_1)(-P_1 - P_2)^{-1}(X - P_2) = 2\mathrm{Id}$$

which can be rewritten

$$P_1(P_1 + P_2)^{-1}(X - P_2) = (X - P_1)$$

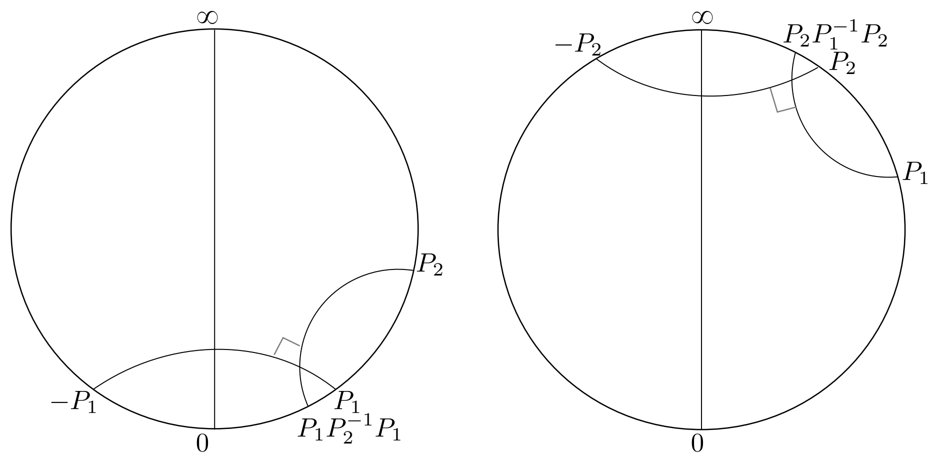


Figure 8: Expressions to find the unique orthogonal tubes

and this simplifies to

$(X-P_2) = (P_1+P_2)P_1^{-1}(X-P_1) = (\mathrm{Id}+P_2P_1^{-1})(X-P_1) = X-P_1+P_2P_1^{-1}X-P_2$

We obtain

$$P_2P_1^{-1}X = P_1$$

Result follows. The proof for the second case is the same. The configuration of the orthogonal tubes $\mathcal{Y}_{-P_1,P_1} \perp \mathcal{Y}_{P_1P_2^{-1}P_1,P_2}$ and $\mathcal{Y}_{-P_2,P_2} \perp \mathcal{Y}_{P_1,P_2P_1^{-1}P_2}$ is illustrated in Figure 8.

$\square$

2.7 Orientation of boundary points: the $\mathrm{Sp}(4,\mathbb{R})$-case

In this section we investigate orientation of boundary points. From now on we will consider the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(4,\mathbb{R})$. This leads us to consider real Lagrangians $\mathcal{L}(\mathbb{R}^4)$ in the boundary of $\mathcal{X}$. Recall that for $l \in \mathcal{L}(\mathbb{R}^4)$ the dual space is given by

$$l^* = \{\lambda : l \to \mathbb{R} \text{ linear}\}$$

If v_1, v_2 is a basis of l then v_1^*, v_2^* is a basis of l^* where $v_i^*(v_j) = \delta_{ij}$ and δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

For $l_1 \pitchfork l_2$ then $l_2 \cong l_1^*$ through the map

$$\begin{aligned} l_2 &\to l_1^* \\ u &\mapsto \omega(u, \cdot) \end{aligned} \tag{4}$$

where ω is the symplectic form represented by the matrix $\begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix}$.

The inverse map is given by

$$\begin{aligned} &l_1^* \to l_2 \\ &\lambda \mapsto u|\ \lambda(v) = \omega(u,v)\ \forall v \in l_1 \end{aligned} \tag{5}$$

In particular take a basis vector v_i of l_1. Then v_i^* can be seen as the vector $u \in l_2$ such that

$$v_i^*(v) = \omega(u,v) \quad \forall v \in l_1$$

where $v_i^*(v_j) = \delta_{ij}$. We obtain $v_i^* \cong u \in l_2$ where

$$\omega(u, v_i) = 1 \text{ and } \omega(u, v_j) = 0$$

Definition 2.23. Let $l_1 \in \mathcal{L}(\mathbb{R}^4), l_1 = \langle v_1, v_2 \rangle$ and fix the orientation $\mathfrak{o}^{l_1}$ on l_1 given by the volume form $v_1 \wedge v_2$. For $l_2 \pitchfork l_1$ the *orientation on l_2 induced by l_1* is given by $v_1^* \wedge v_2^*$ and will be denoted by $\mathfrak{o}^{l_2} \leftarrow \mathfrak{o}^{l_1}$.

Recall that we denote by $\mathcal{L}(\mathbb{R}^n)^{(k)}$ the set of k-tuples of pairwise transverse Lagrangians.

Proposition 2.24. *Let $(l_1, l_2) \in \mathcal{L}(\mathbb{R}^4)^{(2)}$ and fix an orientation $\mathfrak{o}^{l_1}$ of l_1. Then $\mathfrak{o}^{l_1}$ coincides with the induced orientation $\mathfrak{o}^{l_1} \leftarrow \mathfrak{o}^{l_2} \leftarrow \mathfrak{o}^{l_1}$.*

Proof. Let $g \in \mathrm{Sp}(4,\mathbb{R})$ such that $g(l_1, l_2) = (l_\infty, 0)$. Put

$$l_\infty = \langle e_1, e_2 \rangle,\ 0 = \langle e_3, e_4 \rangle$$

Let $\mathfrak{o}^{l_\infty}$ be the orientation on l_∞ given by the volume form $e_1 \wedge e_2$. We want to show that this orientation coincides with $\mathfrak{o}^{l_\infty} \leftarrow \mathfrak{o}^0 \leftarrow \mathfrak{o}^{l_\infty}$. The orientation on $0 \cong l_\infty^*$ induced by $l_\infty = \langle e_1, e_2 \rangle$ is given by $e_1^* \wedge e_2^*$. In the identification (5) the vector $e_1^* \in l_\infty^*$ is the vector $u \in 0 = \langle e_3, e_4 \rangle$ such that

$$\omega(u, e_1) = 1 \text{ and } \omega(u, e_2) = 0$$

where we have the following equalities for ω:

$$\begin{aligned} \omega(e_1, e_2) &= 0 \\ \omega(e_1, e_3) &= 1 \\ \omega(e_1, e_4) &= 0 \\ \omega(e_2, e_3) &= 0 \\ \omega(e_2, e_4) &= 1 \\ \omega(e_3, e_4) &= 0 \end{aligned}$$

We obtain $u = e_1^* = -e_3$ and similarly $e_2^* = -e_4$, so that $\mathfrak{o}^0 \leftarrow \mathfrak{o}^{l_\infty}$ is given by the volume form $-e_3 \wedge -e_4 = e_3 \wedge e_4$. The orientation $\mathfrak{o}^{l_\infty} \leftarrow \mathfrak{o}^0 \leftarrow \mathfrak{o}^{l_\infty}$ is therefore given by $e_3^* \wedge e_4^*$, where this time $e_3^*, e_4^* \in 0^* \cong l_\infty$. In the same way as before one can show that this orientation corresponds exactly to $e_1 \wedge e_2$.

□

Remark 2.25. Observe that the proof of Proposition 2.24 works more generally for the $\mathrm{Sp}(4n, \mathbb{R})$ case, that is $(l_1, l_2) \in \mathcal{L}(\mathbb{R}^{2n})^{(2)}$. It is false for the $\mathrm{Sp}(4n+2, \mathbb{R})$-case, where $(l_1, l_2) \in \mathcal{L}(\mathbb{R}^{2n+1})^{(2)}$ and where an odd number of minus signs is appearing in the volum form.

Recall that we denote by $\mu_n(l_1, l_2, l_3)$ the Maslov index of a triple in $\mathcal{L}(\mathbb{R}^n)^{(3)}$ (see Section 2.2). Given $(l_1, l_2, l_3) \in \mathcal{L}(\mathbb{R}^4)^{(3)}$ we obtain $\mu_2(l_1, l_2, l_3) \in \{-2, 0, 2\}$.

Proposition 2.26. *Let $(l_1, l_2, l_3) \in \mathcal{L}(\mathbb{R}^4)^{(3)}$ and fix an orientation $\mathfrak{o}^{l_1}$ of l_1. Then the induced orientation $\mathfrak{o}^{l_3} \leftarrow \mathfrak{o}^{l_1}$ coincides with $\mathfrak{o}^{l_3} \leftarrow \mathfrak{o}^{l_2} \leftarrow \mathfrak{o}^{l_1}$ if $\mu_2(l_1, l_2, l_3) \neq 0$ and it does not coincide if $\mu_2(l_1, l_2, l_3) = 0$.*

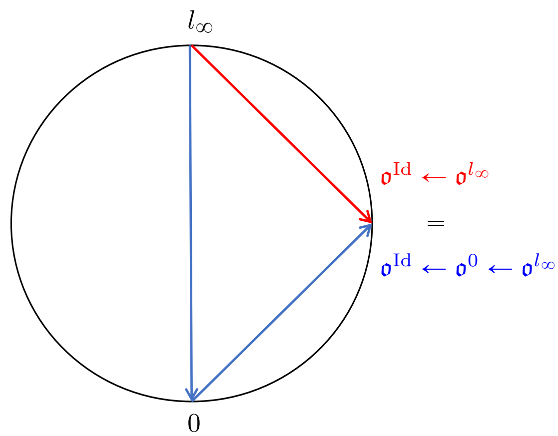


Figure 9: Induced orientation on a positive triple

Proof. 1) $\mu_2(l_1, l_2, l_3) > 0$.

Let $g \in \mathrm{Sp}(4, \mathbb{R})$ such that $g(l_1, l_2, l_3) = (l_\infty, 0, \mathrm{Id})$. Put

$$l_\infty = \langle e_1, e_2 \rangle, \ 0 = \langle e_3, e_4 \rangle, \ \mathrm{Id} = \langle e_1 + e_3, e_2 + e_4 \rangle = \langle \epsilon_1, \epsilon_2 \rangle$$

Let $\mathfrak{o}^{l_\infty}$ be the orientation on l_∞ given by the volume form $e_1 \wedge e_2$. We want to show that $\mathfrak{o}^{\mathrm{Id}} \leftarrow \mathfrak{o}^{l_\infty}$ coincides with $\mathfrak{o}^{\mathrm{Id}} \leftarrow \mathfrak{o}^{0} \leftarrow \mathfrak{o}^{l_\infty}$ (see Figure 9). The orientation on $\mathrm{Id} \cong l_\infty^*$ induced by $l_\infty = \langle e_1, e_2 \rangle$ is given by $e_1^* \wedge e_2^*$. In the identification (5) the vector $e_1^* \in l_\infty^*$ is the vector $u \in \mathrm{Id} = \langle \epsilon_1, \epsilon_2 \rangle$ such that

$$\omega(u, e_1) = 1 \text{ and } \omega(u, e_2) = 0$$

Recall again the following equalities for ω:

$$\begin{aligned}
\omega(e_1, e_2) &= 0 \\
\omega(e_1, e_3) &= 1 \\
\omega(e_1, e_4) &= 0 \\
\omega(e_2, e_3) &= 0 \\
\omega(e_2, e_4) &= 1 \\
\omega(e_3, e_4) &= 0
\end{aligned}$$

We obtain $u = e_1^* = -\epsilon_1$ and similarly $e_2^* = -\epsilon_2$, so that $\mathfrak{o}^{\mathrm{Id}} \leftarrow \mathfrak{o}^{l_\infty}$ is given by the volume form $-\epsilon_1 \wedge -\epsilon_2 = \epsilon_1 \wedge \epsilon_2$.

In the same way one can show that the orientation on $0 = \langle e_3, e_4 \rangle$ induced by $l_\infty = \langle e_1, e_2 \rangle$ is given by $e_3 \wedge e_4$ and that $\mathfrak{o}^{\mathrm{Id}} \leftarrow \mathfrak{o}^0 \leftarrow \mathfrak{o}^{l_\infty}$ is given by $\epsilon_1 \wedge \epsilon_2$.

2) $\mu_2(l_1, l_2, l_3) < 0$.
Let $g \in \mathrm{Sp}(4, \mathbb{R})$ such that $g(l_1, l_2, l_3) = (l_\infty, \mathrm{Id}, 0)$. The proof is very similar to 1), where this time $\mathfrak{o}^0 \leftarrow \mathfrak{o}^{l_\infty}$ is given by $e_3 \wedge e_4$ and coincides with $\mathfrak{o}^0 \leftarrow \mathfrak{o}^{\mathrm{Id}} \leftarrow \mathfrak{o}^{l_\infty}$.

3) $\mu_2(l_1, l_2, l_3) = 0$.
Let $g \in \mathrm{Sp}(4, \mathbb{R})$ such that $g(l_1, l_2, l_3) = (l_\infty, 0, m)$ where m is the Lagrangian
$$m = \langle e_1 + e_3, -e_2 + e_4 \rangle = \langle \epsilon_1, f_2 \rangle$$
and coincides with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the $\mathrm{Sp}(4, \mathbb{R})$-equivariant identification $\mathcal{X} \mapsto \mathbb{X}$ (see Section 2.1). The proof is again very similar to the previous cases, where this time $\mathfrak{o}^m \leftarrow \mathfrak{o}^{l_\infty}$ is given by $-\epsilon_1 \wedge -f_2$ whereas $\mathfrak{o}^m \leftarrow \mathfrak{o}^0 \leftarrow \mathfrak{o}^{l_\infty}$ is given by $\epsilon_1 \wedge -f_2$.

$\square$

2.8 The symmetric spaces $\mathcal{X}_{\mathrm{GL}(n,\mathbb{R})}$ and $\mathcal{X}_{\mathrm{SL}(n,\mathbb{R})}$

Recall that the standard model for the symmetric space associated to $\mathrm{GL}(n, \mathbb{R})$ is
$$\mathcal{X}_{\mathrm{GL}(n,\mathbb{R})} = \mathrm{Sym}^+(n, \mathbb{R})$$
We endow $\mathcal{X}_{\mathrm{GL}(n,\mathbb{R})}$ with the distance given by

$$d_{\mathrm{GL}}(X, Y) = \sqrt{\sum_{i=1}^n (\log \lambda_i)^2}$$

where λ_i are the eigenvalues of XY^{-1}. With this choice of d_{GL} the natural identification $\mathcal{X}_{\mathrm{GL}(n,\mathbb{R})} = \mathcal{Y}_{0,\infty}$ is an isometry (where $\mathcal{Y}_{0,\infty}$ is equipped with the induced Riemannian metric).

Recall also that the symmetric space associated to $\mathrm{SL}(n,\mathbb{R})$ is

$$\mathcal{X}_{\mathrm{SL}(n,\mathbb{R})} = \{X \in \mathrm{Sym}^+(n,\mathbb{R}) |\ \det(X) = 1\}$$

Similarly, we endow $\mathcal{X}_{\mathrm{SL}(n,\mathbb{R})}$ with the distance given by

$$d_{\mathrm{SL}}(X,Y) = \sqrt{\sum_{i=1}^{n} (\log \lambda_i)^2}$$

where λ_i are the eigenvalues of XY^{-1}. In particular, the symmetric space $\mathcal{X}_{\mathrm{SL}(2,\mathbb{R})}$ can be identified with the hyperbolic upper-half plane
$\mathbb{H}^2 = \{z = x + iy |\ x, y \in \mathbb{R},\ y > 0\}$ via the following map:

$$\begin{aligned} \mathbf{h} : \mathcal{X}_{\mathrm{SL}(2,\mathbb{R})} &\to \mathbb{H}^2 \\ B &\mapsto \left[B\right] \cdot i \end{aligned}$$

where B is an element of $\mathrm{PSL}(2,\mathbb{R})$ and acts on $\mathbb{H}^2$ via Möbius transformations. The inverse of $\mathbf{h}$ is given by

$$\begin{aligned} \mathbf{h}^{-1} : \mathbb{H}^2 &\to \mathcal{X}_{\mathrm{SL}(2,\mathbb{R})} \\ z &\mapsto \sqrt{AA^T} \end{aligned}$$

where $A \in \mathrm{PSL}(2,\mathbb{R})$ and $A \cdot i = z$.

Remark 2.27. If we endow $\mathbb{H}^2$ with the distance $d_{\mathbb{H}^2}$ relative to the standard metric $\frac{dx^2+dy^2}{y^2}$ on the upper-half plane then $\mathbf{h}$ is not an isometry and in general $(\mathcal{X}_{\mathrm{SL}(2,\mathbb{R})}, d_{\mathrm{SL}})$ and $(\mathbb{H}^2, d_{\mathbb{H}^2})$ are not isometric. It holds

$$d_{\mathrm{SL}}(B_1, B_2) = \frac{1}{\sqrt{2}} d_{\mathbb{H}^2}(\mathbf{h}(B_1), \mathbf{h}(B_2))$$

2.9 The geometry of the standard tube $\mathcal{Y}_{0,\infty}$

Let us consider the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(4,\mathbb{R})$. This case provides nice geometric interpretations in the construction of right-angled hexagons inside $\mathcal{X}$. A precise definition of hexagon will be given in Section 4.1. Hexagons are the building blocks that will be glued together to compute maximal representations. The geometric features arising in the case of $\mathrm{Sp}(4,\mathbb{R})$ can already be seen in the description of the standard tube $\mathcal{Y}_{0,\infty}$, as stated in the following lemma:

Lemma 2.28. *The tube $\mathcal{Y}_{0,\infty}$ is isometrically identified with $\mathbb{R} \times \mathbb{H}^2$.*

Proof. As seen in Section 2.8 there is a natural identification $\mathcal{Y}_{0,\infty} = \mathcal{X}_{\mathrm{GL}(2,\mathbb{R})}$, where $\mathcal{X}_{\mathrm{GL}(2,\mathbb{R})}$ is the set of positive definite symmetric matrices. The map

$$\begin{aligned} f = \pi^{\mathbb{R}} \times \pi^{\mathbb{H}^2} : \mathcal{X}_{\mathrm{GL}(2,\mathbb{R})} &\to \mathbb{R} \times \mathcal{X}_{\mathrm{SL}(2,\mathbb{R})} \\ Q &\mapsto \Big(\frac{\log \det Q}{\sqrt{2}}, \frac{Q}{\sqrt{\det Q}} \Big) \end{aligned}$$

is a bijection, with inverse

$$\begin{aligned} f^{-1} : \mathbb{R} \times \mathcal{X}_{\mathrm{SL}(2,\mathbb{R})} &\to \mathcal{X}_{\mathrm{GL}(2,\mathbb{R})} \\ (r, B) &\mapsto \sqrt{e^{\sqrt{2}r}} B \end{aligned}$$

When $\mathbb{R} \times \mathcal{X}_{\mathrm{SL}(2,\mathbb{R})}$ and $\mathcal{X}_{\mathrm{GL}(2,\mathbb{R})}$ are considered as metric spaces (endowed with $d_{\mathbb{R}} \times d_{\mathrm{SL}}$ and d_{GL} respectively), the map f is an isometry (see [FP20] Lemma 2.17). The identification $\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$ follows from $\mathcal{X}_{\mathrm{SL}(2,\mathbb{R})} = \mathbb{H}^2$ (see Section 2.8). In particular all copies of $\mathbb{H}^2$ in $\mathcal{Y}_{0,\infty}$ are canonically identified. Observe that to turn $\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$ into an isometric identification we have to scale the metric $d_{\mathbb{H}^2}$ by a factor $k = \frac{1}{\sqrt{2}}$ (Remark 2.27). □

For any $A \in \mathrm{Sym}^+(2,\mathbb{R})$ the point obtained by projecting A orthogonally on $\mathcal{Y}_{0,\infty}$ is $p_{0,\infty}(A) = iA$ (Lemma 2.16). We give the following:

Definition 2.29. Given $A \in \mathrm{Sym}^+(2,\mathbb{R})$ the *hyperbolic component* of A is the $\mathbb{H}^2$-component of iA in the isometric identification $\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$, that is the point $\pi^{\mathbb{H}^2}(p_{0,\infty}(A))$. Similarly the $\mathbb{R}$-component of A is the point $\pi^{\mathbb{R}}(p_{0,\infty}(A))$ and will be called *level* of A.

Consider the upper-half space model $\mathbb{H}^2 = \{x + iy | \ x, y \in \mathbb{R}, \ y > 0\}$. Then for any fixed level in $\mathbb{R} \times \mathbb{H}^2$ the set of diagonal matrices coincides with the y-axis of $\mathbb{H}^2$, where the set $\Big\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1 > \lambda_2, \ \lambda_1 \cdot \lambda_2 = 1 \Big\}$ consists of points "above" $i \in \mathbb{H}^2$ in the vertical y-axis of the hyperbolic plane. The isometries stabilizing the standard tube $\mathcal{Y}_{0,\infty}$ are of the form:

$$\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0,\infty) = \Big\{ \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}, A \in \mathrm{GL}(2,\mathbb{R}) \Big\} \simeq \mathrm{GL}(2,\mathbb{R})$$

Remark 2.30. Let $iX, iY \in \mathcal{Y}_{0,\infty}$ and $(d_1, d_2) \in \overline{\mathfrak{a}}^+$ such that

$$d^{\overline{\mathfrak{a}}^+}(iX, iY) = (d_1, d_2)$$

We can associate to (d_1, d_2) another vector

$$(r, h) = \big(r(d_1, d_2), h(d_1, d_2)\big)$$

that has a geometric interpretation in the cylinder $\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$. The vector (r, h) based at iX has first coordinate r equal to the difference between the levels of iX and iY

$$r = d^{\mathbb{R}}(\pi^{\mathbb{R}}(iX), \pi^{\mathbb{R}}(iY))$$

and second coordinate h equal to the distance between the two points in $\mathbb{H}^2$

$$h = d^{\mathbb{H}^2}(\pi^{\mathbb{H}^2}(iX), \pi^{\mathbb{H}^2}(iY))$$

The vector (r, h) is illustrated in Figure 10.

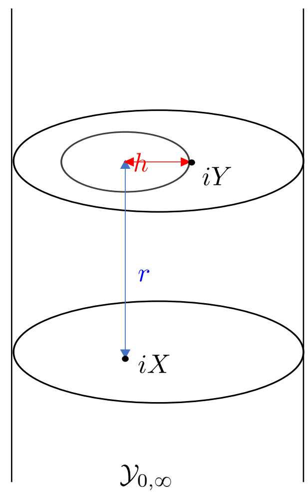


Figure 10: The geometric interpretation of (r, h) for two points iX, iY in $\mathcal{Y}_{0,\infty}$

It is not hard to show that given $d^{\overline{\mathfrak{a}}^+}(iX, iY) = (d_1, d_2)$ then

$$r = \frac{d_1 + d_2}{\sqrt{2}} \text{ and } h = (d_1 - d_2)$$

The vector (r, h) also gives a geometric condition for the maximality of the triple (l_∞, X, Y) i.e. for the matrix $Y - X$ to be positive definite (see Lemma 2.5). It holds ([FP20, Corollary 2.21])

$$Y - X \text{ positive definite} \iff r > \frac{1}{\sqrt{2}} h$$

The following Lemma of linear algebra will play a crucial role in the definition of meaningful parameters for hexagons inside $\mathcal{X}$.

Lemma 2.31. *For any $M \in \mathrm{Sym}^+(2, \mathbb{R})$ with distinct eigenvalues there exist unique S, Q in $\mathrm{PSO}(2)$ and $\mathrm{PO}(2) \backslash \mathrm{PSO}(2)$ respectively such that*

$$SMS^T = QMQ^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ where } \lambda_1 > \lambda_2$$

Proof. Let v_1, v_2 be orthonormal eigenvectors relative to the eigenvalues $\lambda_1 > \lambda_2 > 0$ respectively and let L denote the orthogonal matrix

$$L = \begin{pmatrix} [v_1] \, [v_2] \end{pmatrix}$$

If $\det L = 1$, it is a standard fact of linear algebra that

$$S = \begin{pmatrix} [\ v_1^T\ ] \\ [\ v_2^T\ ] \end{pmatrix} = L^T$$

is the unique element of PSO(2) such that $SMS^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 > \lambda_2$. Put

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S = \begin{pmatrix} -[\ v_1^T\ ] \\ [\ v_2^T\ ] \end{pmatrix}$$

Then $\det Q = -1$ and Q is the desired matrix in PO(2)\PSO(2).
If $\det L = -1$ then the two diagonalizing matrices are $Q = L^T$ and $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Q$.

□

2.10 Geometric interpretation of diagonalization matrix

The group $\mathrm{PSp}(4, \mathbb{R})$ acts on a point $iM \in \mathcal{Y}_{0,\infty}$ via fractional linear transformations (Section 2.1). Recall the following identifications (see Sections 2.8 and 2.9):

$$\mathrm{Sym}^+(2, \mathbb{R}) = \mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$$

In this identification the identity matrix is identified with the point $i \in \mathbb{H}^2$ in the 0-level of $\mathcal{Y}_{0,\infty}$. Moreover, all copies of $\mathbb{H}^2$ in $\mathcal{Y}_{0,\infty}$ are canonically identified. For a matrix $S \in \mathrm{PSO}(2)$ we want to interpret the action

$$\mathrm{PSp}(4, \mathbb{R}) \ni \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \cdot (iM) = iSMS^T \tag{6}$$

as a transformation which fixes the level of M and rotates its hyperbolic component around $i \in \mathbb{H}^2$. This will be very useful in the description of parameters for generic quintuples (Chapter 3). If we consider the action on $\mathbb{H}^2$ through Möbius transformations we see that

$$\mathrm{Stab}_{\mathrm{PSL}(2,\mathbb{R})}(i) = \mathrm{PSO}(2)$$

For $\theta \in [0, \pi)$ the action (6) of a matrix $S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{PSO}(2)$ can be interpreted as a clockwise rotation of angle 2θ around $i \in \mathbb{H}^2$. For every $S \in \mathrm{PSO}(2)$ and every $\theta \in [0, \pi)$ there is a unique way to write S as a rotation matrix of the form

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \sim \begin{pmatrix} \cos(\pi+\theta) & -\sin(\pi+\theta) \\ \sin(\pi+\theta) & \cos(\pi+\theta) \end{pmatrix} = -S$$

Given M positive definite with distinct eigenvalues we interpret the unique $S \in \mathrm{PSO}(2)$ for which

$$SMS^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \lambda_1 > \lambda_2$$

as the angle formed by the semi-axis $\{(0, y) | \ y > 1\}$ inside $\mathbb{H}^2$ and the geodesic segment connecting the hyperbolic components of Id and M (Figure 11). This will be very practical for a concrete visualization of hexagon-parameters.

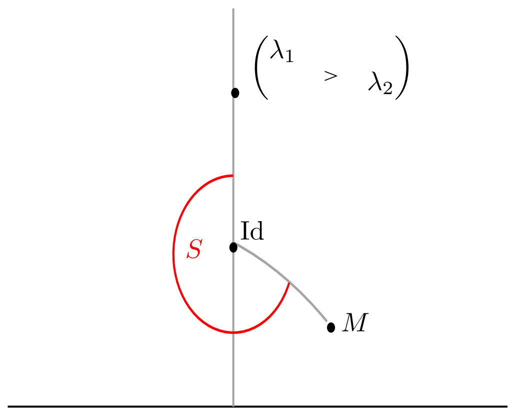


Figure 11: Geometric interpretation of diagonalization matrix S as an angle 2θ

In this thesis we will use both the matrix and the angle notation: angle parameters will be denoted by S or α depending on the context, where

$$S = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \alpha \in [0, 2\pi)$$

Remark 2.32. (*Drawing angles "on the left"*)
The matrix $S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ acts on M by clockwise rotation of center Id and angle 2θ on the $\mathbb{H}^2$-component of the standard tube $\mathcal{Y}_{0,\infty}$. For this reason to draw the angle parameters we will consider the oriented geodesic going from M to Id and draw the angle on the left of it.

Remark 2.33. For $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1 \neq \lambda_2$, the stabilizer $\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0, \mathrm{Id}, \Lambda, \infty)$ is given by

$$\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0, \mathrm{Id}, \Lambda, \infty) = \left\{ \mathrm{Id}, \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \right\} \cong \mathbb{Z}/2\mathbb{Z}$$

where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Observe that in the identification $\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$ the action of the matrix

$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in \mathrm{PSp}(4, \mathbb{R}), \ r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

on $\mathcal{Y}_{0,\infty}$ is a reflection across the y-axis of $\mathbb{H}^2$: for any

$$\mathbb{H}^2 \ni a + ib \overset{(\S 2.8)}{=} \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \in \mathrm{Sym}^+(2, \mathbb{R})$$

it holds

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_1 & -m_2 \\ -m_2 & m_3 \end{pmatrix} \overset{(\S 2.8)}{=} -a + ib \in \mathbb{H}^2$$

Put

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \text{ and } M^r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

If M is a point of angle $\alpha \in [0, 2\pi)$ from $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 > \lambda_2$, then M^r is a point of angle $(2\pi - \alpha)$ from $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 > \lambda_2$ (Figure 12).

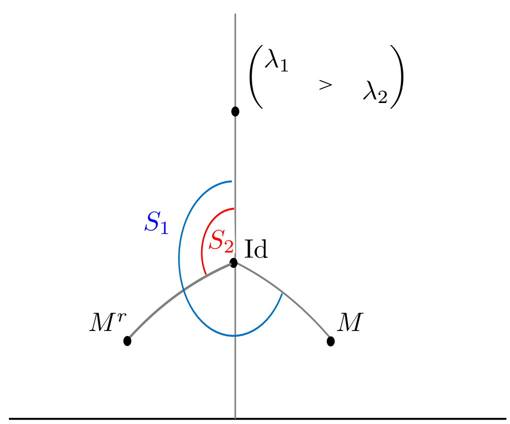


Figure 12: The point M^r is obtained by reflecting M across the y-axis

To see this using the angle interpretation of diagonalization matrices take $S_1 \in \mathrm{PSO}(2)$ diagonalizing M as in Lemma 2.31. It holds:

$$S_1 M S_1^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = S_1 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M^r \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_1^T$$

so that

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_1 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{S_2 \in SO(2)} M^r \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_1^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{S_2^T}$$

If

$$S_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

then

$$S_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(2\pi-\theta) & -\sin(2\pi-\theta) \\ \sin(2\pi-\theta) & \cos(2\pi-\theta) \end{pmatrix}$$

Remark 2.34. More generally let A, B be symmetric positive definite matrices such that $A^{-1}B$ has distinct eigenvalues $\lambda_1 > \lambda_2$. Then the stabilizer $\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0, A, B, \infty)$ is given by

$$\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0, A, B, \infty) = \left\{ \mathrm{Id}, \begin{pmatrix} \sqrt{A} P^T r P \sqrt{A^{-1}} & 0 \\ 0 & \sqrt{A^{-1}} P^T r P \sqrt{A} \end{pmatrix} \right\}$$

where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and P is the unique matrix in PSO(2) such that

$$P(\sqrt{A^{-1}} B \sqrt{A^{-1}}) P^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Geometrically we should interpret the non-trivial element of this stabilizer as a reflection across the hyperbolic component of the standard tube across the geodesic going through A and B (Figure 13).

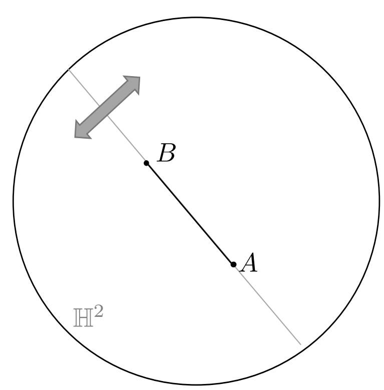


Figure 13: The non-trivial element of the stabilizer $\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0, A, B, \infty)$ in the Poincaré disk model of $\mathbb{H}^2$

2.11 Orientation of the hyperbolic component of $\mathcal{Y}_{0,\infty}$

Let us consider the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(4,\mathbb{R})$. In this section we want to investigate the relation between fixing an orientation of $\mathbb{H}^2 \subset \mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$ and fixing the orientation of the Lagrangian $l_\infty \in \mathcal{L}(\mathbb{R}^4)$ in the boundary of $\mathcal{X}$. It will be useful to use the interplay between symmetric matrices and Lagrangian subspaces. Recall the two models of the Siegel space, the *upper-half space model*

$$\mathcal{X} = \{X + iY,\ X \in \mathrm{Sym}(2,\mathbb{R}), Y \in \mathrm{Sym}^+(2,\mathbb{R})\}$$

and the *Borel embedding model*

$$\mathbb{X} = \{l \in \mathcal{L}(\mathbb{C}^4)|\ \ i\omega(\sigma(\cdot),\cdot)_{\mathbb{C}}|_{l\times l} \text{ is positive definite}\}$$

where $\sigma : \mathbb{C}^4 \to \mathbb{C}^4$ denotes complex conjugation and ω is the symplectic form represented by the matrix $\begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix}$.
The $\mathrm{Sp}(4,\mathbb{R})$-equivariant identification $\mathcal{X} \mapsto \mathbb{X}$ is induced by the affine chart

$$\iota : \mathrm{Sym}(2,\mathbb{C}) \to \mathcal{L}(\mathbb{C}^4)$$

that associates to a symmetric matrix Z the linear subspace of $\mathbb{C}^4$ spanned by the columns of the matrix $\begin{pmatrix} Z \\ \mathrm{Id}_2 \end{pmatrix}$.

Proposition 2.35. *Consider the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(4,\mathbb{R})$. Choosing an orientation of $\mathbb{H}^2$ inside $\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$ is equivalent to choosing an orientation of $\mathbb{P}(\infty) \simeq \mathbb{P}(0)$ where $\infty, 0 \in \mathcal{L}(\mathbb{R}^4)$.*

Proof. Fix a basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$. Consider the two transverse Lagrangians $0 = \langle e_3, e_4\rangle$ and $\infty = \langle e_1, e_2\rangle$. The standard tube

$$\mathcal{Y}_{0,\infty} = \{iY|\ Y \in \mathrm{Sym}^+(2,\mathbb{R})\}$$

is isometrically identified with $\mathbb{R} \times \mathbb{H}^2$ (see Lemma 2.28). The hyperbolic plane inside $\mathcal{Y}_{0,\infty}$ is identified with the symmetric space associated to $\mathrm{SL}(2,\mathbb{R})$, that is

$$\mathcal{X}_{\mathrm{SL}(2,\mathbb{R})} = \{X \in \mathrm{Sym}^+(2,\mathbb{R})|\ \det(X) = 1\}$$

All copies of $\mathbb{H}^2$ inside $\mathcal{Y}_{0,\infty}$ are canonically identified and stabilized by the set of matrices

$$\left\{ \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, R \in \mathrm{O}(2) \right\} \simeq \mathrm{O}(2) \tag{7}$$

acting $\mathrm{Sp}(4,\mathbb{R})$-equivariantly in the identification $\mathcal{X} \overset{\iota}{\mapsto} \mathbb{X}$.

Consider the geodesic ray

$$\gamma(t) = \begin{pmatrix} ie^t & 0 \\ 0 & ie^{-t} \end{pmatrix}$$

which lies inside the hyperbolic component of $\mathcal{Y}_{0,\infty}$ and converges to the Lagrangians l_+, l_- where

$$\gamma(t) \xrightarrow[t\to\infty]{} l_+ = \langle e_1, e_4\rangle$$

$$\gamma(t) \xrightarrow[t\to-\infty]{} l_- = \langle e_2, e_3\rangle$$

To fix an orientation of $\mathbb{H}^2$ inside $\mathcal{Y}_{0,\infty}$ it is sufficient to orient its visual boundary $\partial_\infty\mathbb{H}^2$. Recall that $0 \pitchfork \infty$ where

$$0 = \langle e_3, e_4\rangle \text{ and } \infty = \langle e_1, e_2\rangle$$

In Section 2.10 we have investigated the action of orthogonal matrices on $\mathbb{H}^2 \subset \mathcal{Y}_{0,\infty} = \mathbb{R}\times\mathbb{H}^2$ and we have interpret matrices as angles in the hyperbolic component. The set in (7) acts preserving the Lagrangians 0 and ∞ respectively and the visual boundary of the hyperbolic component is realized by the O(2)-orbit of the Lagrangian $l_+ = \langle e_1, e_4\rangle$. It is identified to $\mathbb{P}(\infty)$ through the following map

$$\begin{aligned} \partial_\infty\mathbb{H}^2 &\to \mathbb{P}(\infty) \\ l &\mapsto l\cap\infty \end{aligned}$$

To fix an orientation of $\mathbb{H}^2$ inside $\mathcal{Y}_{0,\infty}$ it is therefore sufficient to orient $\mathbb{P}(\infty)$. This set is canonically identified with $\mathbb{P}(0)$. To see this let us note by $v^{\perp\omega}$ the set

$$v^{\perp\omega} = \{u\in\mathbb{R}^4 \mid \omega(v,u) = 0\}$$

In particular $v \in v^{\perp\omega}$ and $\dim(v^{\perp\omega})$=3. Then $\mathbb{P}(0)$ and $\mathbb{P}(\infty)$ are identified through the map

$$\begin{aligned} \mathbb{P}(0) &\to \mathbb{P}(\infty) \\ [v] &\mapsto [v^{\perp\omega}\cap\infty] \end{aligned}$$

□

2.12 Isometries reflecting the hyperbolic component

In Section 2.11 we have studied how to orient the hyperbolic component of the standard tube $\mathcal{Y}_{0,\infty}$. The group of isometries stabilizing the standard tube is the group:

$$\left\{\begin{pmatrix} A & 0 \\ 0 & A^{-T}\end{pmatrix}, A\in \mathrm{GL}(2,\mathbb{R})\right\} \in \mathrm{PSp}(4,\mathbb{R})$$

Proposition 2.36. *For $A \in \mathrm{GL}(2,\mathbb{R})$ let f_A be an isometry stabilizing the standard tube $\mathcal{Y}_{0,\infty}$:*

$$\begin{aligned} f_A : \mathcal{Y}_{0,\infty} &\to \mathcal{Y}_{0,\infty} \\ iY &\mapsto iAYA^T \end{aligned}$$

Then f_A is reversing the orientation of the hyperbolic component of $\mathcal{Y}_{0,\infty}$ if and only if $\det A < 0$

Proof. Recall that

$$\mathcal{Y}_{0,\infty} = \chi_{\mathrm{GL}_{2,\mathbb{R}}} = \mathrm{Sym}^+(2,\mathbb{R}) = \mathbb{R} \times \chi_{\mathrm{SL}_{2,\mathbb{R}}} = \mathbb{R} \times \mathbb{H}^2$$

The isometry f_A is linear in Y and its differential is the map

$$X \mapsto AXA^T$$

for any tangent vector $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \cong \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. For $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ the tangent vector $AXA^T = df_A(X)$ can be rewritten as

$$\begin{pmatrix} a_1^2 x_1 + 2a_1a_2x_2 + a_2^2x_3 \\ a_1a_3x_1 + (a_1a_4 + a_2a_3)x_2 + a_2a_4x_3 \\ a_3^2x_1 + 2a_3a_4x_2 + a_4^2x_3 \end{pmatrix} = \begin{pmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ a_1a_3 & a_1a_4 + a_2a_3 & a_2a_4 \\ a_3^2 & 2a_3a_4 & a_4^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$\det \begin{pmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ a_1a_3 & a_1a_4 + a_2a_3 & a_2a_4 \\ a_3^2 & 2a_3a_4 & a_4^2 \end{pmatrix} = (\det A)^3$$

The map f_A is therefore reversing the orientation of the tube $\mathcal{Y}_{0,\infty}$ if and only if $\det A < 0$. To finish the proof we need to show that only the orientation of the hyperbolic component can be reversed, not the orientation of the $\mathbb{R}$-component.

The action of a $A \in \mathrm{GL}(2,\mathbb{R})$ on the $\mathbb{R}$-component of $iY \in \mathbb{R} \times \mathbb{H}^2$ is a translation: if we note by r the $\mathbb{R}$-component of iY then the $\mathbb{R}$-component of $iAYA^T$ is given by $r + \frac{2\log|\det A|}{\sqrt{2}}$. The map f_A is therefore preserving the orientation on the $\mathbb{R}$-component of the tube. □

Remark 2.37. For an isometry $g = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \in \mathrm{PSp}(4,\mathbb{R})$ whether or not g is reversing the orientation of the hyperbolic component of $\mathcal{Y}_{0,\infty}$ is intrinsic and only depends on the sign of $\det A$. This is true in general for any element in $\mathrm{PSp}(4,\mathbb{R})$ which is conjugate to $\begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}$ for an $A \in \mathrm{GL}(2,\mathbb{R})$.

Definition 2.38. An isometry $g \in \mathrm{PSp}(4,\mathbb{R})$ conjugate to $\begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}$ with $A \in \mathrm{GL}(2,\mathbb{R})$ is called *reflecting* (resp. *non-reflecting*) if $\det A < 0$ (resp. > 0).

3 Parameters for quintuples

3.1 The sets $\mathcal{Q}^{gen}$ and $\mathcal{Q}^{st}$

As already mentioned, our goal is to define parameters for right-angled hexagons. A precise definition of hexagon will be given in Section 4.1. To study meaningful parameters we first parametrize a set $\mathcal{Q}^{gen}$ consisting of specific ordered quintuples of Lagrangians at the boundary of $\mathcal{X}$ which we call generic (Definition 3.3). Recall that we can identify Lagrangians and symmetric matrices through the map ι introduced in Section 2.1. Up to isometry we can consider quintuples of Lagrangians where the first and the last Lagrangian are 0 and ∞ respectively and we can further diagonalize one symmetric matrix choosing an order on the diagonal. This choice will lead us to define the set of standard quintuples $\mathcal{Q}^{st}$ and we will focus on the case where $\mathcal{X}$ is the symmetric space associated to $\mathrm{Sp}(4,\mathbb{R})$. To define generic and standard quintuples we first need to define generic quadruples.

Definition 3.1. (**Generic quadruple**) Let (P,X,Y,Q) be a maximal quadruple and let $\mu_1,...,\mu_n$ be the eigenvalues of the cross-ratio $R(P,X,Y,Q)$. The quadruple (P,X,Y,Q) is said to be *generic* if for any $i \neq j$ it holds $\mu_i \neq \mu_j$.

Remark 3.2. Recall that we denote by $p_{P,Q}$ the orthogonal projection on the tube $\mathcal{Y}_{P,Q}$. Let $\underline{b}$ be the vector obtained by the orthogonal projection of X and Y on the tube $\mathcal{Y}_{P,Q}$ (Figure 14)

$$\underline{b} = d^{\overline{\mathfrak{a}}^+}(p_{P,Q}(X), p_{P,Q}(Y))$$

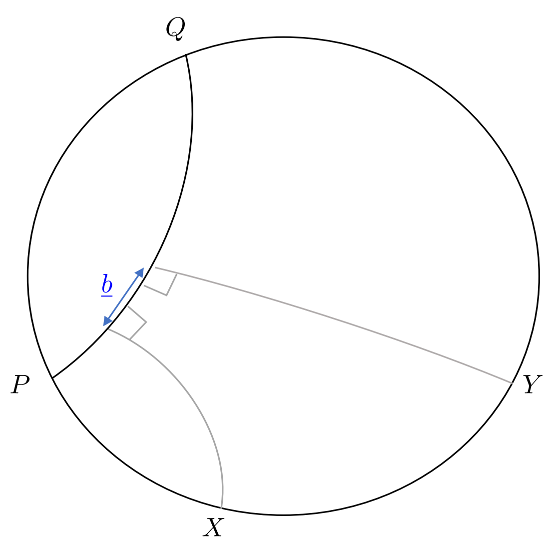


Figure 14: The quadruple (P,X,Y,Q) is generic if and only if vector b is regular

From Lemma 2.17 it is easy to see that the quadruple (P, X, Y, Q) is generic if and only if the vector $\underline{b}$ is a regular vector of the Weyl chamber (see Section 2.3 for the definition of a regular vector).

Definition 3.3. The set of *generic quintuples* $\mathcal{Q}^{gen}$ is given by:

$$\mathcal{Q}^{gen} := \{(P, X, Y, Z, Q) \text{ maximal}|\ (P, X, Y, Q) \text{ and } (P, Y, Z, Q) \text{ generic } \}$$

Remark 3.4. Observe that the definition of generic quintuple strongly depends on the order of the quintuple: given (P, X, Y, Z, Q) generic it is not necessarily true that a cyclic permutation of the quintuple is generic.

We will see in the next section how the parametrization of $\mathcal{Q}^{gen}$ is connected with the parametrization of right-angled hexagons of $\mathcal{X}$. Let us now consider the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. We give the following

Definition 3.5. The set of *standard quintuples* $\mathcal{Q}^{st} \subset \mathcal{Q}^{gen}$ is given by

$$\mathcal{Q}^{st} := \{(0, X, \mathrm{Id}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \infty) \in \mathcal{Q}^{gen}|\ \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 > \lambda_2\ \}$$

Remark 3.6. Recall that the set of isometries stabilizing the standard tube $\mathcal{Y}_{0,\infty}$ is the group:

$$\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(\mathcal{Y}_{0,\infty}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}, A \in \mathrm{GL}(2, \mathbb{R}) \right\}$$

Recall also that for a diagonal matrix Λ with different eigenvalues we have

$$\mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0, \mathrm{Id}, \Lambda, \infty) = \left\{ \mathrm{Id}, \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \right\} \cong \mathbb{Z}/2\mathbb{Z}$$

where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. For any $(P, X, Y, Z, Q) \in \mathcal{Q}^{gen}$ we can always find a $g \in \mathrm{PSp}(4, \mathbb{R})$ (more details in proof of Proposition 3.7) such that

$$g \cdot P = 0,\ g \cdot Y = \mathrm{Id},\ g \cdot Z = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\ g \cdot Q = \infty$$

where $\lambda_1 > \lambda_2$. It is therefore clear that

$$\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathcal{Q}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$$

Recall that we denote by $\mathfrak{a}$ the set of regular vectors of the Weyl chamber, that is the set

$$\mathfrak{a} = \{(x_1, x_2) \in \mathbb{R}^2|\ x_1 > x_2 > 0\}$$

We can now state the following:

Proposition 3.7. *The set* $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$ *is parametrized by*

$$\mathfrak{a} \times \mathfrak{a} \times \mathrm{PSO}(2)/_\sim$$

where for $S, S' \in \mathrm{PSO}(2)$ *we have the following equivalence relation:*

$$S \sim S' \iff S' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

The parametrization is given by

$$\Big((c_1, c_2), (d_1, d_2), [S]\Big) \mapsto \left[\left(0, S^T \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix} S,\ \mathrm{Id},\ \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}, \infty\right)\right] \in \mathcal{Q}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$$

with inverse map

$$[(P, X, Y, Z, Q)] \mapsto \Big(\ (c_1, c_2) = d^{\mathfrak{a}^+}\big(p_{P,Q}(X), p_{P,Q}(Y)\big), (d_1, d_2) = d^{\mathfrak{a}^+}\big(p_{P,Q}(Y), p_{P,Q}(Z)\big), [S]\ \Big)$$

where

$$S(gX)S^T = \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix},\ \frac{1}{e^{c_2}} > \frac{1}{e^{c_1}}$$

and g *is a map in* $\mathrm{PSp}(4,\mathbb{R})$ *such that*

$$g(P, X, Y, Z, Q) \in \mathcal{Q}^{st}$$

The parameter space of $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$ *can be rewritten as*

$$\mathfrak{a}^3 \times [0, 2\pi)/_\sim$$

where for $\alpha \in [0, 2\pi)$ *it holds*

$$S = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

and the equivalence relation is given by

$$\alpha \sim \alpha' \iff \alpha' = 2\pi - \alpha$$

Proof. We first show how to find parameters $(\underline{c}, \underline{d}, [S])$ for a given quintuple $[(P, X, Y, Z, Q)]$ in $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$. We want to use the fact that (Remark 3.6) :

$$\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathcal{Q}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$$

Let $(P, X, Y, Z, Q) \in \mathcal{Q}^{gen}$. Up to isometry we can consider $P = 0$ and $Q = \infty$. Put

$$\underline{c} = (c_1, c_2) = d^{\mathfrak{a}^+}\big(p_{0,\infty}(X), p_{0,\infty}(Y)\big) \in \mathfrak{a}$$

$$\underline{d} = (d_1, d_2) = d^{\mathfrak{a}^+}\big(p_{0,\infty}(Y), p_{0,\infty}(Z)\big) \in \mathfrak{a}$$

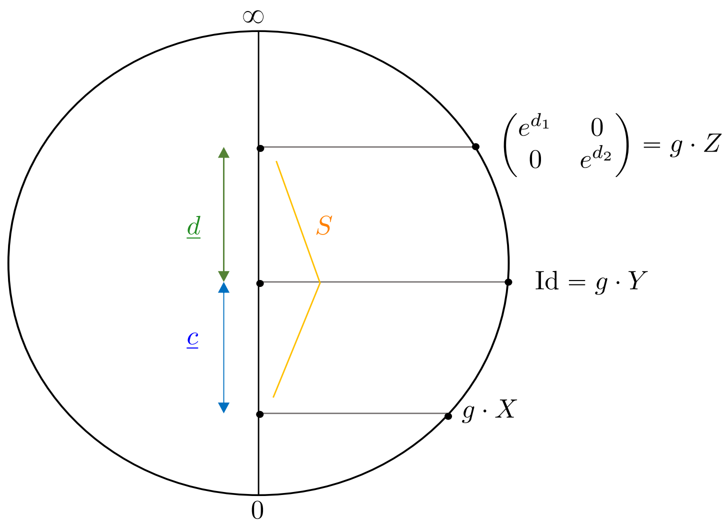


Figure 15: The isometry g sends the quintuple (P, X, Y, Z, Q) to a standard one

Let $g \in \mathrm{Stab}(\mathcal{Y}_{0,\infty})$ be such that (Figure 15)

$$g \cdot Y = \mathrm{Id}$$

$$g \cdot Z = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}$$

Recall that

$$\mathrm{Stab}(\mathcal{Y}_{0,\infty}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}, A \in \mathrm{GL}(2, \mathbb{R}) \right\} \in \mathrm{PSp}(4, \mathbb{R})$$

The first equality forces $\mathcal{A} = \mathcal{O}\sqrt{Y^{-1}}$ where $\mathcal{O} \in \mathrm{PO}(2)$. The second equality forces $\mathcal{O} = P, Q$ where P and Q are the unique matrices in $\mathrm{PSO}(2)$ and $\mathrm{PO}(2) \backslash \mathrm{PSO}(2)$ respectively (see Lemma 2.31) such that

$$P(\sqrt{Y^{-1}} Z \sqrt{Y^{-1}})P^T = Q(\sqrt{Y^{-1}} Z \sqrt{Y^{-1}})Q^T = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}$$

Accordingly, the two only possibilities for g are:

$$g_1 = \begin{pmatrix} P\sqrt{Y^{-1}} & 0 \\ 0 & P\sqrt{Y} \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} Q\sqrt{Y^{-1}} & 0 \\ 0 & Q\sqrt{Y} \end{pmatrix}$$

It holds:

$$g_1 X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g_2 X \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let S be the unique matrix in $\mathrm{PSO}(2)$ such that $S g_1 X S^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for $\lambda_1 > \lambda_2$ (see Lemma 2.31). The triple $(0, g_1 X, \mathrm{Id})$ is maximal and the map g is preserving the Weyl chamber distance, by Lemma 2.17 we deduce $\lambda_1 = \frac{1}{e^{c_2}}$, $\lambda_2 = \frac{1}{e^{c_1}}$. Furthermore $S' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unique matrix in PSO(2) such that $S' g_2 X S'^T = \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix}$. The point $g_2 X$ is the image of $g_1 X$ through a reflection on the hyperbolic component $g_2 X = (g_1 X)^r$ (see Remark 2.33) and the two quintuples

$$\mathcal{Q}^{st} \ni (0, g_1 X, \mathrm{Id}, \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}, \infty) \quad \text{and} \quad (0, g_2 X, \mathrm{Id}, \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}, \infty) \in \mathcal{Q}^{st}$$

are equivalent in $\mathcal{Q}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$. The third parameter $[S] \in \mathrm{PSO}(2)/_\sim$ is given by the diagonalization matrix and has the geometric interpretation of an angle: for $S \in \mathrm{PSO}(2)$ we write S as the matrix (see Section 2.10)

$$S = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \alpha \in [0, 2\pi)$$

and the equivalence relation is the identification of angle α with angle $(2\pi - \alpha)$ (see Figure 12). We obtain parameters

$$(\underline{c}, \underline{d}, [S]) \in \mathfrak{a} \times \mathfrak{a} \times \mathrm{PSO}(2)/_\sim = \mathfrak{a} \times \mathfrak{a} \times [0, 2\pi)_\sim$$

The parameter S or α will be called the *angle parameter* of the generic quintuple and provides information about the angle between the hyperbolic components of X and Z (Figure 16). We will draw the angle on the left as explained in Remark 2.32.

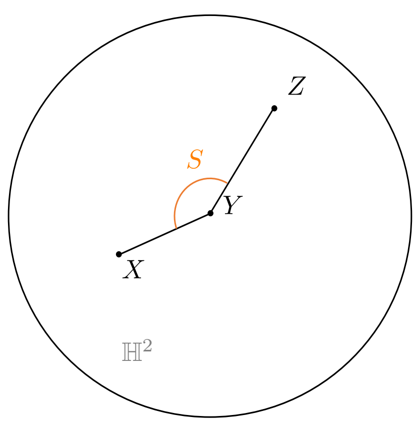


Figure 16: Geometric interpretation of the angle parameter S in the Poincaré disk model of $\mathbb{H}^2$

For the inverse map, to any element of the parameter space

$$\mathfrak{a} \times \mathfrak{a} \times \mathrm{PSO}(2)/_{\sim}$$

we can associate a unique quintuple inside $\mathcal{Q}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$. To $\big((c_1, c_2), (d_1, d_2), [S]\big)$ in $\mathfrak{a} \times \mathfrak{a} \times \mathrm{PSO}(2)/_{\sim}$ we associate the standard quintuple

$$(P, X, Y, Z, Q) = \left(0, S^T \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix} S,\ \mathrm{Id},\ \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}, \infty\right)$$

Then X is a matrix such that $d^{\mathfrak{a}^+}(iX, i\mathrm{Id}) = (c_1, c_2)$ and $SXS^T = \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix}$ where $\frac{1}{e^{c_2}} > \frac{1}{e^{c_1}}$. For $S' \sim S$ we obtain an equivalent quintuple (P, X', Y, Z, Q) inside $\mathcal{Q}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$ where

$$(P, X', Y, Z, Q) = (P, X^r, Y, Z, Q) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \cdot (P, X, Y, Z, Q), \quad \text{where } r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

□

Remark 3.8. **(**$\mathrm{PSO}(2)$ **and not** $\mathrm{SO}(2)$**)** By definition of the parameter $S \in \mathrm{PSO}(2)/_{\sim}$, the matrix S is such that

$$SMS^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \lambda_1 > \lambda_2$$

for M a positive definite symmetric matrix. It makes therefore sense to consider S inside $\mathrm{PSO}(2)$ to guarantee unicity of the diagonalization matrix i.e. S is equivalent to $(-S)$ as $SMS^T = (-S)M(-S^T)$.

Remark 3.9. (*Genericity condition*) The hypothesis of a quintuple (P, X, Y, Z, Q) to be generic is essential for the parameter space $\mathfrak{a} \times \mathfrak{a} \times \mathrm{PSO}(2)/_{\sim}$ to be well defined: the uniqueness of the angle parameter is strictly related to Lemma 2.31, which holds only for matrices with different eigenvalues.

Corollary 3.10. *The set* $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$ *is parametrized by*

$$\mathfrak{a} \times \mathfrak{a} \times [0, \pi]$$

Proof. The equivalence relation of Proposition 3.7 is given by $\alpha \sim 2\pi - \alpha$. We can always choose $\alpha \in [0, \pi]$ as representative of the equivalence class. □

To conclude this section we state two technical lemmas that will be very useful in the description of parameters of right-angled hexagons.

Lemma 3.11. *Let* $p = \big((c_1, c_2), (d_1, d_2), [S]\big) \in \mathfrak{a} \times \mathfrak{a} \times \mathrm{PSO}(2)/_{\sim}$ *and let* X, Y *be positive definite such that* $d^{\mathfrak{a}^+}(iX, iY) = (c_1, c_2)$. *Then the unique* Z *such that* $(0, X, Y, Z, \infty)$ *corresponds to* p *in the parametrization of Proposition 3.7 is given by*

$$Z = \sqrt{Y} R^T S \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S^T R \sqrt{Y}$$

where R is the unique matrix in PSO(2) *such that*

$$R(\sqrt{Y}^{-1}X\sqrt{Y}^{-1})R^T = \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix}$$

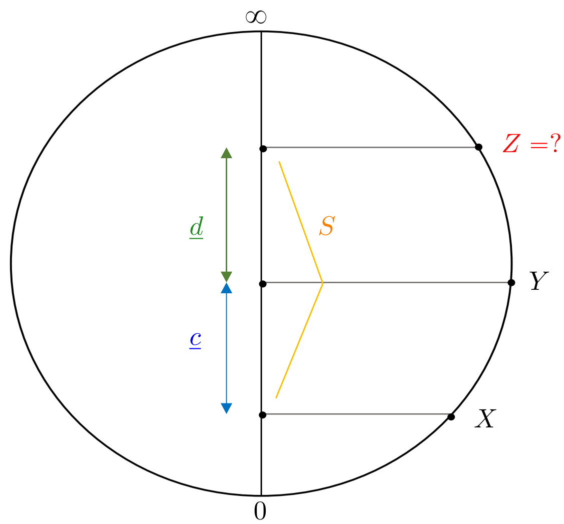


Figure 17: Given $X, Y, \underline{c}, \underline{d}, S$ we want to find Z

Proof. It is easy to check that for such a Z

$$d^{\mathfrak{a}^+}(iY, iZ) = (d_1, d_2)$$

By Proposition 3.7 we know that $SgXS^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for $\lambda_1 > \lambda_2$ and where g is such that $g \cdot (0, X, Y, Z, \infty) \in \mathcal{Q}^{st}$, that is g such that

$$g \cdot Y = \mathrm{Id}, \quad g \cdot Z = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}$$

Then $g = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}$ where $A = S^T R\sqrt{Y}^{-1}$ and

$$g \cdot X = S^T R\sqrt{Y}^{-1} X \sqrt{Y}^{-1} S R^T$$

It holds

$$S(g \cdot X)S^T = R(\sqrt{Y}^{-1}X\sqrt{Y}^{-1})R^T = \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix}, \text{ where } \frac{1}{e^{c_2}} > \frac{1}{e^{c_1}}$$

To finish the proof we need to check that for $S' \sim S$ we obtain the same point in $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$. Take S' such that

$$S' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$Z' = \sqrt{Y} R^T S' \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S'^T R \sqrt{Y}$$

Take $h \in \mathrm{Stab}(\mathcal{Y}_{0,\infty}) \cong \mathrm{GL}(2, \mathbb{R})$

$$h = \begin{pmatrix} \sqrt{Y} R^T S' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S^T R \sqrt{Y}^{-1} & 0 \\ 0 & \sqrt{Y}^{-1} R^T S' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S^T R \sqrt{Y} \end{pmatrix}$$

Then it holds

$$h \cdot (0, X, Y, Z, \infty) = (0, X, Y, Z', \infty)$$

so that $[(0, X, Y, Z, \infty)] = [(0, X, Y, Z', \infty)]$ in $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$. Geometrically the map h can be seen as a reflection in the $\mathbb{H}^2$-component across the geodesic passing through X and Y, as shown in Figure 18 below. □

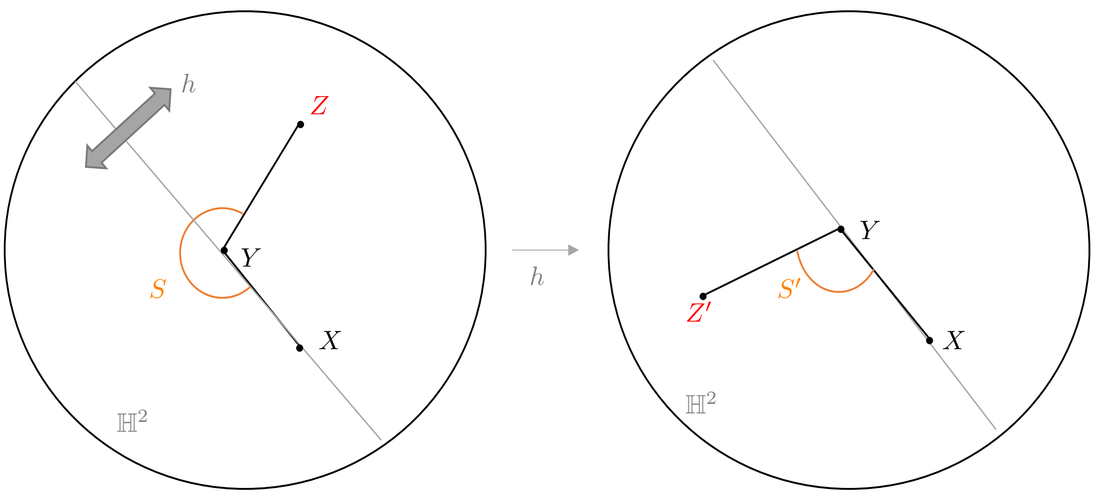


Figure 18: The map h in the Poincaré disk model of $\mathbb{H}^2$

Lemma 3.12. *Let* $p = \big((c_1, c_2), (d_1, d_2), [S]\big)$ *be inside* $\mathfrak{a} \times \mathfrak{a} \times \mathrm{PSO}(2)/_\sim$ *and let* Y, Z *be positive definite such that* $d^{\mathfrak{a}^+}(iY, iZ) = (d_1, d_2)$*. Then the unique* X *such that* $(0, X, Y, Z, \infty)$ *corresponds to* p *in the parametrization of Proposition 3.7 is given by*

$$X = \sqrt{Y} P^T S^T \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix} S P \sqrt{Y}$$

where P *is the unique matrix in* $\mathrm{PSO}(2)$ *such that*

$$P(\sqrt{Y}^{-1} Z \sqrt{Y}^{-1}) P^T = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}$$

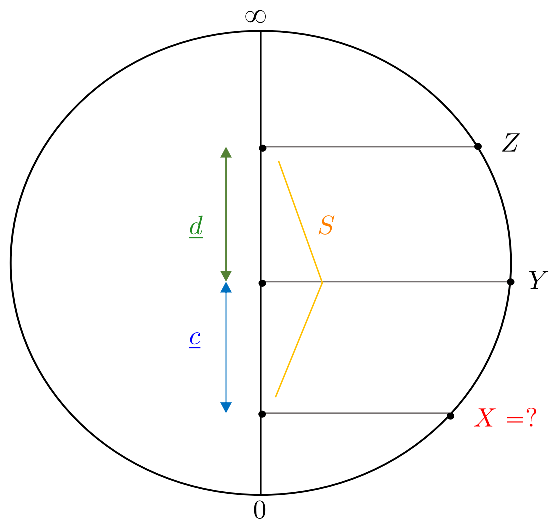


Figure 19: Given $Y, Z, \underline{c}, \underline{d}, S$ we want to find X

Proof. It is easy to check that for such X

$$d^{\mathfrak{a}^+}(iX, iY) = (c_1, c_2)$$

Furthermore take

$$g = \begin{pmatrix} P\sqrt{Y}^{-1} & 0 \\ 0 & P\sqrt{Y} \end{pmatrix}$$

then $g \cdot Y = \mathrm{Id},\ g \cdot Z = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix}$ and

$$g \cdot X = P\sqrt{Y}^{-1} X \sqrt{Y}^{-1} P^T = S^T \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix} S$$

so that

$$S(g \cdot X)S^T = \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix}, \text{ where } \frac{1}{e^{c_2}} > \frac{1}{e^{c_1}}$$

To finish the proof we need to check that for $S' \sim S$ we obtain the same point in $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$. Take S' such that

$$S' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$X' = \sqrt{Y} P^T S'^T \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix} S' P \sqrt{Y} = \sqrt{Y} P^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S^T \begin{pmatrix} \frac{1}{e^{c_2}} & 0 \\ 0 & \frac{1}{e^{c_1}} \end{pmatrix} S \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P \sqrt{Y}$$

Take $h \in \mathrm{Stab}(\mathcal{Y}_{0,\infty}) \cong \mathrm{GL}(2,\mathbb{R})$

$$h = \begin{pmatrix} \sqrt{Y} P^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P \sqrt{Y}^{-1} & 0 \\ 0 & \sqrt{Y}^{-1} P^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P \sqrt{Y} \end{pmatrix}$$

Then it holds

$$h \cdot (0, X, Y, Z, \infty) = (0, X', Y, Z, \infty)$$

so that $[(0, X, Y, Z, \infty)] = [(0, X', Y, Z, \infty)]$ in $\mathcal{Q}^{gen}/_{\mathrm{PSp}(4,R)}$. Geometrically the map h can be seen as a reflection in the $\mathbb{H}^2$-component across the geodesic passing through Y and Z (similar to Figure 18). $\square$

4 Parameters for right-angled hexagons

In this chapter we define right-angled hexagons in the Siegel space $\mathcal{X}$ and give meaningful parameters for them. A right-angled hexagon will be formed by six $\mathbb{R}$-tubes and we will define the space of ordered hexagons $\mathcal{H}$ consisting of hexagons together with the choice of one $\mathbb{R}$-tube (Definition 4.3). We distinguish between generic (Definition 4.6) and non-generic hexagons (Section 4.2). We introduce a parameter space for both cases (Proposition 4.15 for the generic case and Propositions 4.20, 4.21 and 4.22 for the non-generic case). A parameter space which encloses both generic and non-generic hexagons is given in Theorem 4.26. These parameters will be called *arc coordinates* and aim to generalize the parameters for a right-angled hexagon inside $\mathbb{H}^2$. It is well known that in $\mathbb{H}^2$ we can parametrize a right-angled hexagon by giving the length of three alternating side (see for example [Mar16, Lemma 6.2.2]). The hope is to prove a similar result for hexagons inside $\mathcal{X}$, where beyond length parameters we will introduce angle parameters. This approach will turn out to be tricky as explained in Chapter 5. The consequence is that we will not be able to extend our parameters to the space of adjacent hexagons having same alternating-side lengths, which is very useful to compute maximal representations. This will be solved by introducing reflections and symmetric hexagons and will be explained in Chapter 6.

4.1 Definition of hexagon, the sets $\mathcal{H}, \mathcal{H}^{gen}$ and $\mathcal{H}^{st}$

Definition 4.1. A *right-angled hexagon* in $\mathcal{X}$ is a cyclic sequence of six $\mathbb{R}$-tubes $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ where any two consecutive tubes are orthogonal and such that

$$\mathcal{Y}_1 = \mathcal{Y}_{P_1,P_2},\ \mathcal{Y}_2 = \mathcal{Y}_{Q_1,Q_2},\ \mathcal{Y}_3 = \mathcal{Y}_{P_3,P_4},\ \mathcal{Y}_4 = \mathcal{Y}_{Q_3,Q_4},\ \mathcal{Y}_5 = \mathcal{Y}_{P_5,P_6},\ \mathcal{Y}_6 = \mathcal{Y}_{Q_5,Q_6}$$

for a maximal 12-tuple $(P_1, Q_6, Q_1, P_2, P_3, Q_2, Q_3, P_4, P_5, Q_4, Q_5, P_6)$.

The maximal 12-tuple determining a right hexagon H in $\mathcal{X}$ is illustrated in Figure 20.

Definition 4.2. Let $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ be a right-angled hexagon in $\mathcal{X}$. We define the *stabilizer of* H and denote it by $\mathrm{Stab}(H)$ the stabilizer

$$\mathrm{Stab}(H) = \{g \in \mathrm{PSp}(2n, \mathbb{R}) |\ g \cdot \mathcal{Y}_i = \mathcal{Y}_i,\ i \in \{1, ..., 6\}\}$$

Definition 4.3. The set $\mathcal{H}$ of *ordered right-angled hexagons* in $\mathcal{X}$ is defined by

$$\mathcal{H} := \{(H, \mathcal{Y}_1) |\ H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6] \text{ right-angled hexagon } \}$$

We want to be able to determine a point $(H, \mathcal{Y}_1)$ inside $\mathcal{H}$ by giving the data of an ordered maximal 6-tuple. There are many ways to do this, as explained in the following lemma.

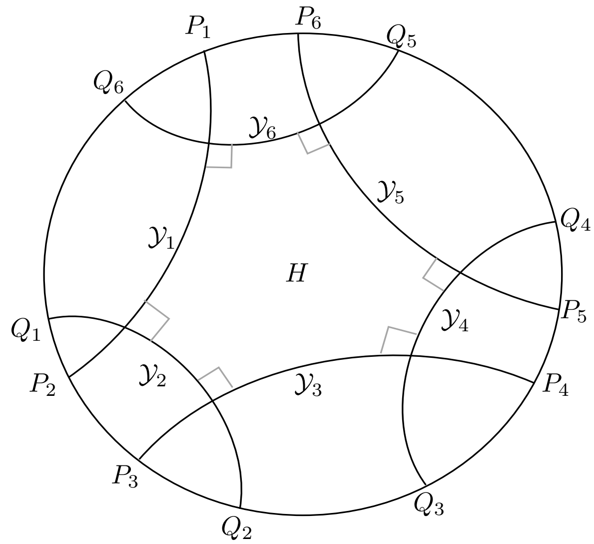


Figure 20: The maximal 12-tuple determining the right-angled hexagon $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$

Lemma 4.4. *Let* $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ *be a right-angled hexagon with associated maximal 12-tuple* $(P_1, Q_6, Q_1, P_2, P_3, Q_2, Q_3, P_4, P_5, Q_4, Q_5, P_6)$*. Then* $(H, \mathcal{Y}_1) \in \mathcal{H}$ *is uniquely determined by the following ordered maximal 6-tuples:*

$$(P_1, P_2, P_3, P_4, P_5, P_6) \tag{9}$$

$$(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) \tag{10}$$

$$(P_2, Q_2, P_4, P_5, Q_5, P_1) \tag{11}$$

Proof. Given the maximal 6-tuple $(P_1, P_2, P_3, P_4, P_5, P_6)$ we use Lemma 2.20 to uniquely determine $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ such that

$$\mathcal{Y}_{P_1,P_2} \perp \mathcal{Y}_{Q_1,Q_2} \perp \mathcal{Y}_{P_3,P_4} \perp \mathcal{Y}_{Q_3,Q_4} \perp \mathcal{Y}_{P_5,P_6} \perp \mathcal{Y}_{Q_6,Q_5} \perp \mathcal{Y}_{P_1,P_2} \tag{12}$$

The hexagon H is determined by the $\mathbb{R}$-tubes in (12) and we put $\mathcal{Y}_1 = \mathcal{Y}_{P_1,P_2}$. The quadruples $(P_1, Q_6, Q_1, P_2), (P_3, Q_2, Q_3, P_4), (P_5, Q_4, Q_5, P_6)$ are maximal by Lemma 2.21 and we obtain a maximal 12-tuple
$(P_1, Q_6, Q_1, P_2, P_3, Q_2, Q_3, P_4, P_5, Q_4, Q_5, P_6)$ by Lemma 2.6. The proof for the 6-tuple in (10) is similar and we put again $\mathcal{Y}_1 = \mathcal{Y}_{P_1,P_2}$ where the 6-tuple $(P_1, P_2, P_3, P_4, P_5, P_6)$ is uniquely determined by the orthogonality conditions in (12).
Given $(P_2, Q_2, P_4, P_5, Q_5, P_1)$ maximal we construct the hexagon H as following: let $g \in \mathrm{Sp}(2n, \mathbb{R})$ such that $g \cdot (P_1, P_2) = (\infty, 0)$. Let us denote

$$g(Q_2) = A,\ g(P_4) = B,\ g(P_5) = C,\ g(Q_5) = D$$

We use Lemma 2.20 and Lemma 2.22 to uniquely determine the right-angled hexagon $H_{0,A,B,C,D,\infty}$ as shown in in Figure 21 below.

The maximality of the 12-tuple at the boundary is again guaranteed by Lemma 2.21. We now put $H = g^{-1}(H_{0,A,B,C,D,\infty})$ and $\mathcal{Y}_1 = Y_{P_1,P_2}$. ∎

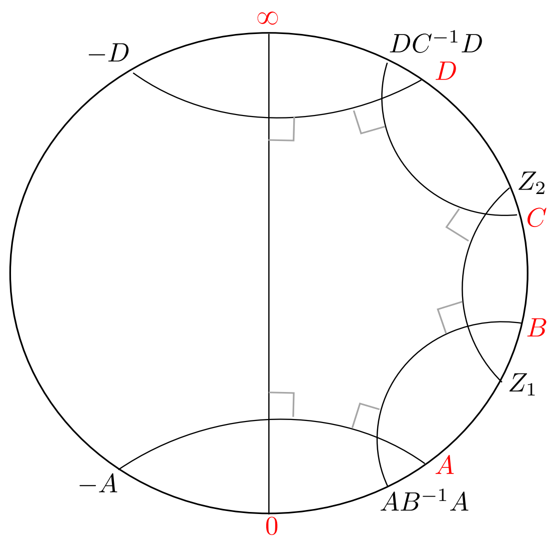


Figure 21: The right-angled hexagon $H_{0,A,B,C,D,\infty}$ is uniquely determined by the maximal 6-tuple in red $(0, A, B, C, D, \infty)$

Notation 4.5. In this thesis we will use a maximal 6-tuple as in (11) to uniquely determine a right-angled hexagon $(H, \mathcal{Y}_1)$ inside $\mathcal{H}$. In order to simplify the notation we will write the 6-tuple as (P, A, B, C, D, Q). By writing

$$H = (P, A, B, C, D, Q)$$

we will refer to the hexagon $(H, \mathcal{Y}_{P,Q})$ where H is uniquely determined by (P, A, B, C, D, Q) as shown in Lemma 4.4 (11). The choice of the tube $\mathcal{Y}_{P,Q}$ is therefore encoded in the order of the maximal 6-tuple. When $P = 0$ and $Q = \infty$ then A, B, C, D are positive definite matrices and we obtain the hexagon $(H, \mathcal{Y}_{0,\infty})$ where H is shown in Figure 21. In particular the maximal 12-tuple associated to H is given by

$$H = (\infty, -D, -A, 0, AB^{-1}A, A, Z_1, B, C, Z_2, D, DC^{-1}D)$$

where Z_1, Z_2 are uniquely defined by requiring

$$\mathcal{Y}_{AB^{-1}A,B} \perp \mathcal{Y}_{Z_1,Z_2} \perp \mathcal{Y}_{C,DC^{-1}D}$$

Recall Definition 3.1 for the notion of a generic quadruple. Let us now give the following

Definition 4.6. The set of *generic hexagons* $\mathcal{H}^{gen} \subset \mathcal{H}$ is given by ordered 6-tuples of the form

$$\mathcal{H}^{gen} := \{(P, A, B, C, D, Q) \text{ maximal} \mid (P, A, B, Q), (P, B, C, Q), (P, C, D, Q) \text{ generic}\}$$

Remark 4.7. Let $(H, \mathcal{Y}_1)$ be a generic hexagon determined by the maximal 6-tuple (P, A, B, C, D, Q) where $\mathcal{Y}_1 = \mathcal{Y}_{P,Q}$. Then both (P, A, B, C, Q) and (P, B, C, D, Q) are inside $\mathcal{Q}^{gen}$. In particular let $\underline{b}, \underline{c}, \underline{d}$ be the three vectors (Figure 22)

$$\underline{b} = d^{\overline{\mathfrak{a}}^+}(p_{P,Q}(A), p_{P,Q}(B))$$

$$\underline{c} = d^{\overline{\mathfrak{a}}^+}(p_{P,Q}(B), p_{P,Q}(C))$$

$$\underline{d} = d^{\overline{\mathfrak{a}}^+}(p_{P,Q}(C), p_{P,Q}(D))$$

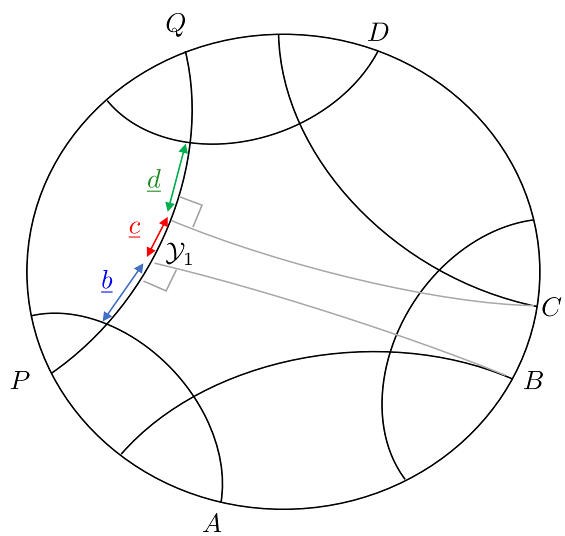


Figure 22: The hexagon $(H, \mathcal{Y}_1)$ is generic if and only if $\underline{b}, \underline{c}, \underline{d}$ are regular

By Lemma 2.17 it is easy to see that the hexagon $(H, \mathcal{Y}_1)$ is generic if and only if the vectors $\underline{b}, \underline{c}, \underline{d}$ are regular.

Let us now focus on the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$.

Definition 4.8. The set of *standard hexagons* $\mathcal{H}^{st} \subset \mathcal{H}^{gen}$ is given by

$$\mathcal{H}^{st} := \{(0, A, \mathrm{Id}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, D, \infty) \in \mathcal{H}^{gen} |\ \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 > \lambda_2\}$$

Remark 4.9. Similarly to what we have seen for quintuples (Remark 3.6) for any $(H, \mathcal{Y}_1) \in \mathcal{H}^{gen}$ we can always find an isometry $g \in \mathrm{PSp}(4, \mathbb{R})$ such that $g \cdot (H, \mathcal{Y}_1) = (gH, \mathcal{Y}_{0,\infty}) \in \mathcal{H}^{st}$. Given a standard hexagon $(H^{st}, \mathcal{Y}_{0,\infty}) \in \mathcal{H}^{st}$ its stabilizer is the group

$$\mathrm{Stab}(H^{st}) = \left\{\mathrm{Id}, \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\right\} \cong \mathbb{Z}/2\mathbb{Z}$$

where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It holds

$$\mathcal{H}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathcal{H}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$$

4.2 Non-generic hexagons: the set $\mathcal{H}^{nongen}$

In this section we define non-generic right-angled hexagons. Recall that by definition a generic hexagon $(H, \mathcal{Y}_1) \in \mathcal{H}^{gen}$ is given by an ordered 6-tuple $H = (P, A, B, C, D, Q)$ where three quadruples $(P, A, B, Q), (P, B, C, Q)$ and (P, C, D, Q) are generic and $\mathcal{Y}_1 = \mathcal{Y}_{P,Q}$ (Definition 4.6). When the hexagon is non-generic some of these quadruples fail to be generic. We define three different types of non-generic hexagons depending on how many quadruples in $H = (P, A, B, C, D, Q)$ fail to be generic: a non-generic hexagon of type k is a hexagon where k quadruples are non-generic. Let us start with the following

Definition 4.10. (**Non-generic quadruple**) Let (P, X, Y, Q) be a maximal quadruple and let (μ_1, μ_2) be the eigenvalues of the cross-ratio $R(P, X, Y, Q)$. The quadruple (P, X, Y, Q) is said to be *non-generic* if $\mu_1 = \mu_2$.

Definition 4.11. The set $\mathcal{H}^{nongen}_{type1}$ is given by

$$\mathcal{H}^{nongen}_{type1} := \mathcal{H}^{nongen}_{type1.1} \cup \mathcal{H}^{nongen}_{type1.2} \cup \mathcal{H}^{nongen}_{type1.3}$$

where

$$\mathcal{H}^{nongen}_{type1.1} := \{(P, A, B, C, D, Q) \text{ maximal} |\, (P, A, B, Q) \text{ non-generic}, (P, B, C, Q), (P, C, D, Q) \text{ generic}\}$$

$$\mathcal{H}^{nongen}_{type1.2} := \{(P, A, B, C, D, Q) \text{ maximal} |\, (P, B, C, Q) \text{ non-generic}, (P, A, B, Q), (P, C, D, Q) \text{ generic}\}$$

$$\mathcal{H}^{nongen}_{type1.3} := \{(P, A, B, C, D, Q) \text{ maximal} |\, (P, C, D, Q) \text{ non-generic}, (P, A, B, Q), (P, B, C, Q) \text{ generic}\}$$

Definition 4.12. The set $\mathcal{H}^{nongen}_{type2}$ is given by

$$\mathcal{H}^{nongen}_{type2} := \mathcal{H}^{nongen}_{type2.1} \cup \mathcal{H}^{nongen}_{type2.2} \cup \mathcal{H}^{nongen}_{type2.3}$$

where

$$\mathcal{H}^{nongen}_{type2.1} := \{(P, A, B, C, D, Q) \text{ maximal} |\, (P, A, B, Q), (P, B, C, Q) \text{ non-generic}, (P, C, D, Q) \text{ generic}\}$$

$$\mathcal{H}^{nongen}_{type2.2} := \{(P, A, B, C, D, Q) \text{ maximal} |\, (P, A, B, Q), (P, C, D, Q) \text{ non-generic}, (P, B, C, Q) \text{ generic}\}$$

$$\mathcal{H}^{nongen}_{type2.3} := \{(P, A, B, C, D, Q) \text{ maximal} |\, (P, B, C, Q), (P, C, D, Q) \text{ non-generic}, (P, A, B, Q) \text{ generic}\}$$

Definition 4.13. The set $\mathcal{H}^{nongen}_{type3}$ is given by

$$\mathcal{H}^{nongen}_{type3} := \{(P, A, B, C, D, Q) \text{ maximal} |\, (P, A, B, Q), (P, B, C, Q), (P, C, D, Q) \text{ non-generic}\}$$

Proposition 4.14.

$$\mathcal{H} = \mathcal{H}^{gen} \cup \mathcal{H}^{nongen}_{type1} \cup \mathcal{H}^{nongen}_{type2} \cup \mathcal{H}^{nongen}_{type3}$$

Proof. The inclusion $\mathcal{H}^{gen} \cup \mathcal{H}^{nongen}_{type1} \cup \mathcal{H}^{nongen}_{type2} \cup \mathcal{H}^{nongen}_{type3} \subset \mathcal{H}$ is trivial.
Let $(H, \mathcal{Y}_1) \in \mathcal{H}$. By Lemma 4.4 we can uniquely determine $(H, \mathcal{Y}_1)$ from a maximal 6-tuple $(H, \mathcal{Y}_1) = (P, A, B, C, D, Q)$ where $\mathcal{Y}_1 = \mathcal{Y}_{P,Q}$. Let $\underline{b}, \underline{c}, \underline{d}$ be the three vectors in Figure 22

$$\underline{b} = d^{\overline{\mathfrak{a}}^+}(p_{P,Q}(A), p_{P,Q}(B))$$

$$\underline{c} = d^{\overline{\mathfrak{a}}^+}(p_{P,Q}(B), p_{P,Q}(C))$$

$$\underline{d} = d^{\overline{\mathfrak{a}}^+}(p_{P,Q}(C), p_{P,Q}(D))$$

The hexagon $(H, \mathcal{Y}_1)$ is generic if and only if the vectors $\underline{b}, \underline{c}, \underline{d}$ are regular (Remark 4.7) and is non-generic if one of them is inside $\mathfrak{d}$. For every vector $\underline{b}, \underline{c}$ and $\underline{d}$ we see if it is generic or not and we list all the possible configurations. The hexagon $(H, \mathcal{Y}_1)$ must be contained in one of this exhaustive list. We obtain $2^3 = 8$ possible configurations, one in $\mathcal{H}^{gen}$, three in $\mathcal{H}^{nongen}_{type1}$, three in $\mathcal{H}^{nongen}_{type2}$ and one in $\mathcal{H}^{nongen}_{type3}$. □

4.3 Arc coordinates for generic hexagons

In this section we parametrize right-angled hexagons in $\mathcal{X}$ up to isometry. We will concentrate on the case where $\mathcal{X}$ is the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. The parameters that we introduce aim to generalize the parameters for a right-angled hexagon inside $\mathbb{H}^2$. It is well known that given three real numbers $b, c, d > 0$ there exists (up to isometries) a unique right-angled hexagon in $\mathbb{H}^2$ with alternating sides of lengths b, c and d (see for example [Mar16, Lemma 6.2.2]). When considering the Siegel space $\mathcal{X}$ the length parameters are vectors and take value in the Weyl chamber $\overline{\mathfrak{a}}^+$. Beyond length parameters it is necessary to introduce what we will call angle parameters. This will lead to a geometric interpretation of elements in $\mathrm{PSO}(2)$ in the spirit already mentioned in Section 2.10.

Let us now consider $\mathcal{X}$ the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. Recall that we denote by $\mathfrak{a}$ the subset of the Weyl chamber consisting of regular vectors:

$$\mathfrak{a} = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > x_2 > 0\}$$

We compute the parameter space for $\mathcal{H}^{gen}$. We will use the fact (Remark 4.9)

$$\mathcal{H}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathcal{H}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$$

Proposition 4.15. *The set $\mathcal{H}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$ is parametrized by*

$$\mathfrak{a}^3 \times \Big(\mathrm{PSO}(2) \times \mathrm{PSO}(2)\Big)/_\sim$$

where for $(S_1, S_2) \in \mathrm{PSO}(2) \times \mathrm{PSO}(2)$ it holds

$$(S_1, S_2) \sim (S_1', S_2') \iff S_i' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ i \in \{1, 2\}$$

The parametrization is given by

$$(\underline{b}, \underline{c}, \underline{d}, [S_1, S_2]) \mapsto [(0, A, \mathrm{Id}, C, D, \infty)] \in \mathcal{H}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$$

where $\underline{b} = (b_1, b_2),\ \underline{c} = (c_1, c_2),\ \underline{d} = (d_1, d_2)$ *and*

$$A = S_1^T \begin{pmatrix} \frac{1}{e^{b_2}} & 0 \\ 0 & \frac{1}{e^{b_1}} \end{pmatrix} S_1$$

$$C = \begin{pmatrix} e^{c_1} & 0 \\ 0 & e^{c_2} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & \sqrt{e^{c_1}} \\ -\sqrt{e^{c_2}} & 0 \end{pmatrix} S_2 \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S_2^T \begin{pmatrix} 0 & -\sqrt{e^{c_2}} \\ \sqrt{e^{c_1}} & 0 \end{pmatrix}$$

This parameter space can be rewritten as

$$\mathfrak{a}^3 \times \Big([0, 2\pi) \times [0, 2\pi)\Big)/_{\sim}$$

where for $\alpha_i \in [0, 2\pi), i \in \{1, 2\}$ *it holds*

$$S_i = \begin{pmatrix} \cos\frac{\alpha_i}{2} & -\sin\frac{\alpha_i}{2} \\ \sin\frac{\alpha_i}{2} & \cos\frac{\alpha_i}{2} \end{pmatrix}$$

and the equivalence relation is given by

$$(\alpha_1, \alpha_1) \sim (\alpha_1', \alpha_2') \iff \alpha_i' = 2\pi - \alpha_i,\ i \in \{1, 2\}$$

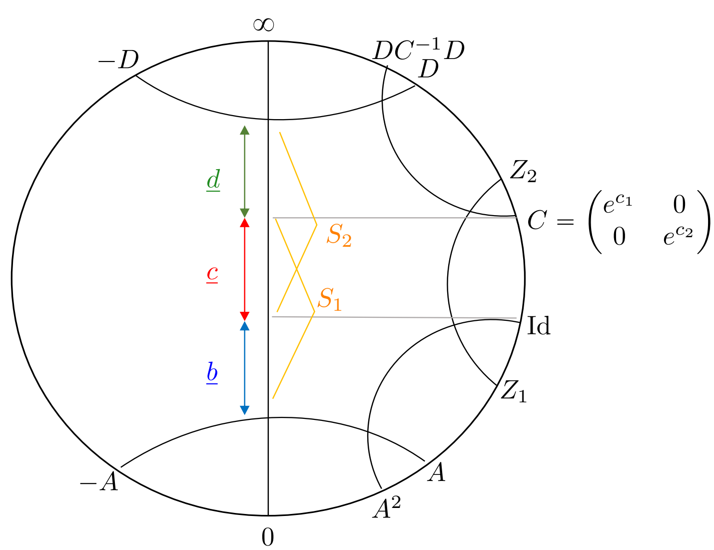


Figure 23: The standard right-angled hexagon $(H^{st}, \mathcal{Y}_{0,\infty})$ with parameters $(\underline{b}, \underline{c}, \underline{d}, S_1, S_2)$

Proof. As we have done for quintuples, we first show how to find parameters

$$\big((b_1, b_2), (c_1, c_2), (d_1, d_2), [S_1, S_2]\big)$$

for a given $(H, \mathcal{Y}_1)$ inside $\mathcal{H}^{gen}$. Let

$$(H, \mathcal{Y}_1) = (P, A, B, C, D, Q)$$

where $\mathcal{Y}_1 = \mathcal{Y}_{P,Q}$. Up to isometry we can consider $P = 0$ and $Q = \infty$. As $(H, \mathcal{Y}_{P,Q}) = (H, \mathcal{Y}_{0,\infty}) \in \mathcal{H}^{gen}$ the quintuples $(0, A, B, C, \infty)$ and $(0, B, C, D, \infty)$ both belong to $\mathcal{Q}^{gen}$. We use Proposition 3.7 to find length parameters $\underline{b}, \underline{c}, \underline{d}$ inside $\mathfrak{a}$:

$$(b_1, b_2) = d^{\mathfrak{a}^+}(iA, iB)$$

$$(c_1, c_2) = d^{\mathfrak{a}^+}(iB, iC)$$

$$(d_1, d_2) = d^{\mathfrak{a}^+}(iC, iD)$$

Let $g \in \mathrm{Stab}(\mathcal{Y}_{0,\infty})$ be such that $g \cdot B = \mathrm{Id}$ and $g \cdot C = \begin{pmatrix} e^{c_1} & 0 \\ 0 & e^{c_2} \end{pmatrix}$ (this is the same procedure exposed in the proof of Proposition 3.7). We obtain exactly two possibilities g_1, g_2 which correspond to the standard hexagons

$$(H_1, \mathcal{Y}_{0,\infty}) = (0, g_1 A, \mathrm{Id}, C, g_1 D, \infty) \text{ and } (H_2, \mathcal{Y}_{0,\infty}) = (0, g_2 A, \mathrm{Id}, C, g_2 D, \infty)$$

where

$$g_1 A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g_2 A \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_1 D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g_2 D \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The points on the right hand side should be thought as the image under a reflection in the hyperbolic component of $\mathcal{Y}_{0,\infty}$ (Remark 2.33). Let S_1, S_2 be the angle parameters obtained by the quintuples $(0, g_1 A, \mathrm{Id}, C, \infty)$ and $(0, \mathrm{Id}, C, g_1 D, \infty)$ respectively (see Proposition 3.7). These parameters are obtained by diagonalization matrices and the matrix $S_i \in \mathrm{PSO}(2)$, $i \in \{1, 2\}$ has the geometric interpretation of an angle $\alpha_i \in [0, 2\pi)$ where it holds

$$S_i = \begin{pmatrix} \cos\frac{\alpha_i}{2} & -\sin\frac{\alpha_i}{2} \\ \sin\frac{\alpha_i}{2} & \cos\frac{\alpha_i}{2} \end{pmatrix}, \ i \in 1, 2$$

The quintuples $(0, g_1 A, \mathrm{Id}, C, \infty)$, $(0, \mathrm{Id}, C, g_1 D, \infty)$ are parametrized by $(\underline{b}, \underline{c}, [S_1])$ and $(\underline{c}, \underline{d}, [S_2])$ respectively where

$$S_i \sim S_i' \iff S_i' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This equivalence relation is the identification of angle α_i with angle $(2\pi - \alpha_i)$. See Figures 23 and 24 for a visualization of the parameters $(\underline{b}, \underline{c}, \underline{d}, [S_1, S_2])$.

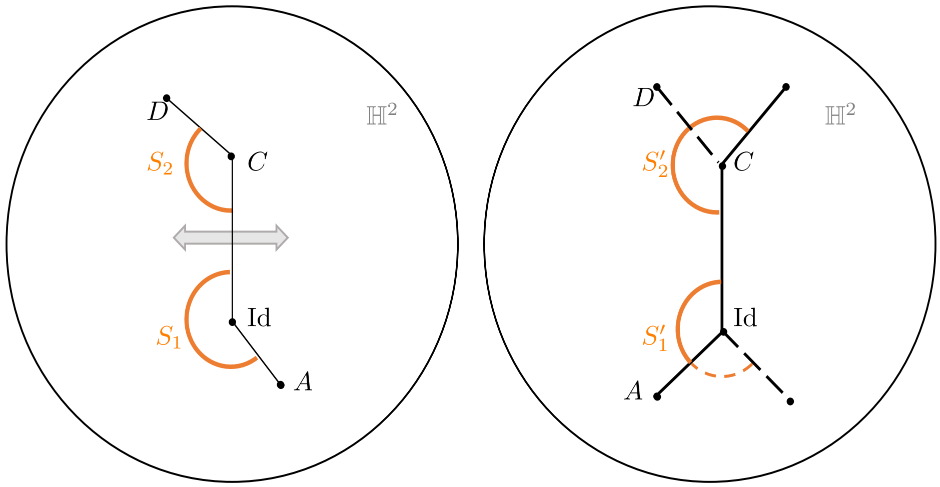


Figure 24: Visualization of the equivalence relation
$[S_1, S_2] = [S_1', S_2']$ in the Poincaré disk model of $\mathbb{H}^2$

Now take $(\underline{b}, \underline{c}, \underline{d}, [S_1, S_2])$ inside $\mathfrak{a}^3 \times \Big(\mathrm{PSO}(2) \times \mathrm{PSO}(2)\Big)/\sim$. We want to construct a standard hexagon $(H^{st}, \mathcal{Y}_{0,\infty})$. Up to $\mathrm{PSp}(4,\mathbb{R})$-action we can consider $B = \mathrm{Id}$ and C diagonal. It is sufficient to determine A, C and D for $(H^{st}, \mathcal{Y}_{0,\infty})$ to be uniquely determined. The equality

$$(c_1, c_2) = d^{\mathfrak{a}^+}(i\mathrm{Id}, iC)$$

forces $C = \begin{pmatrix} e^{c_1} & 0 \\ 0 & e^{c_2} \end{pmatrix}$. We use Lemma 3.11 and Lemma 3.12 to uniquely determine A and D respectively. We use Lemma 2.20 and Lemma 2.22 to compute the corresponding orthogonal tubes.

To finish the proof, we need to check that for $(S_1', S_2') \sim (S_1, S_2)$ we obtain an equivalent hexagon $(H'^{st}, \mathcal{Y}_{0,\infty})$ inside $\mathcal{H}^{st}/_{\mathbb{Z}/2\mathbb{Z}}$. Recall that for any M in $\mathrm{Sym}^+(2,\mathbb{R})$ we can draw its hyperbolic component inside the standard tube $\mathcal{Y}_{0,\infty} \cong \mathbb{R} \times \mathbb{H}^2$ and recall that we denote by M^r the point obtained by reflecting M across the y-axis of $\mathbb{H}^2$, that is $M^r = rMr$ where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (Remark 2.33). For $i = 1, 2$ let

$$S_i' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$A' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_1^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{e^{b_2}} & 0 \\ 0 & \frac{1}{e^{b_1}} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_1 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = A^r$$

$$C' = C$$

$$D' = \begin{pmatrix} 0 & \sqrt{e^{c_1}} \\ -\sqrt{e^{c_2}} & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S_2^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{e^{c_2}} \\ \sqrt{e^{c_1}} & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \sqrt{e^{c_1}} \\ \sqrt{e^{c_2}} & 0 \end{pmatrix} S_2 \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S_2^T \begin{pmatrix} 0 & \sqrt{e^{c_2}} \\ \sqrt{e^{c_1}} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = D^r$$

so that $g \cdot (H^{st}, \mathcal{Y}_{0,\infty}) = (H'^{st}, \mathcal{Y}_{0,\infty})$ where

$$g = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \quad r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Again, we should think of this equivalent relation as the identification of angles $S_1 = \alpha_1, S_2 = \alpha_2$ with angles $(2\pi - \alpha_1), (2\pi - \alpha_2)$ respectively. □

Corollary 4.16. *The set* $\mathcal{H}^{gen}/_{\mathrm{PSp}(4,\mathbb{R})}$ *is parametrized by*

$$\mathfrak{a}^3 \times [0, \pi] \times [0, 2\pi)$$

Proof. The equivalence relation of Proposition 4.15 is given by $\alpha_i \sim 2\pi - \alpha_i$ for $i \in \{1, 2\}$. We choose $\alpha_1 \in [0, \pi]$ as representative of the equivalence class (see Figure 24 above). □

Definition 4.17. (**Arc coordinates for generic right-angled hexagons**) The parameters of Proposition 4.15 will be called *arc coordinates* for a generic right-angled hexagon $(H, \mathcal{Y}_1)$. The vectors $\underline{b}, \underline{c}, \underline{d}$ will be called *length parameters* and α_1, α_2 will be called *angle parameters.*

Remark 4.18. The term *arc coordinates* introduced in Definition 4.17 could be misleading as we also use it for the parametrization of classical Teichmüller space and for its generalization in the case of maximal representations. Nevertheless, we have decided to keep this name also for the parameters of a hexagon as they are crucial for the construction of parameters for maximal representations and will appear in their parameter space (Theorem 7.23).

4.4 Polygonal chain associated to a right-angled hexagon

In the previous section we have introduced arc coordinates for a generic right-angled hexagon in the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(4, \mathbb{R})$. In particular we have seen that a hexagon $(H, \mathcal{Y}_1) \in \mathcal{H}^{gen}$ is parametrized up to isometry by length and angle parameters as explained in Proposition 4.15. In this section we define the polygonal chain associated to an ordered right-angled hexagon and show how this is related to length and angle parameters. For the purposes of this thesis we will define the polygonal chain of $(H, \mathcal{Y}_1)$ in the case where $\mathcal{Y}_1 = \mathcal{Y}_{0,\infty}$.

Recall that given $A \in \mathrm{Sym}^+(2, \mathbb{R})$, the hyperbolic component of A is the point $\pi^{\mathbb{H}^2}(p_{0,\infty}(A))$, i.e. the $\mathbb{H}^2$-component of $iA = p_{0,\infty}(A)$ in the identification

$\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$ (Definition 2.29). Recall also (Remark 2.30) that for two points iA, iB with

$$d^{\bar{\mathfrak{a}}^+}(iA, iB) = (d_1, d_2)$$

the hyperbolic distance $d^{\mathbb{H}^2}(\pi^{\mathbb{H}^2}(iA), \pi^{\mathbb{H}^2}(iB)) = h$ is given by

$$h = h(\underline{d}) = d_1 - d_2 \tag{13}$$

Definition 4.19. Let $\mathcal{X}$ be the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$ and let $(H, \mathcal{Y}_{0,\infty}) \in \mathcal{H}$ be an ordered right-angled hexagon in $\mathcal{X}$. Let us write

$$H = (0, A, B, C, D, \infty)$$

for a maximal 6-tuple $(0, A, B, C, D, \infty)$. The *polygonal chain associated to* $(H, \mathcal{Y}_{0,\infty})$ is the connected series of geodesic segments with vertices given by the ordered sequence of points (possibly coinciding)

$$\left(\pi^{\mathbb{H}^2}(iA), \pi^{\mathbb{H}^2}(iB), \pi^{\mathbb{H}^2}(iC), \pi^{\mathbb{H}^2}(iD)\right)$$

The *segments of the polygonal chain* are the oriented geodesic segments (possibly collapsing to one point):

$$\overrightarrow{\pi^{\mathbb{H}^2}(iA)\pi^{\mathbb{H}^2}(iB)},\ \overrightarrow{\pi^{\mathbb{H}^2}(iB)\pi^{\mathbb{H}^2}(iC)},\ \overrightarrow{\pi^{\mathbb{H}^2}(iC)\pi^{\mathbb{H}^2}(iD)}$$

The *angles of the polygonal chain* are the angles formed by two consecutive segments (measured on the left-hand side of the oriented segments).

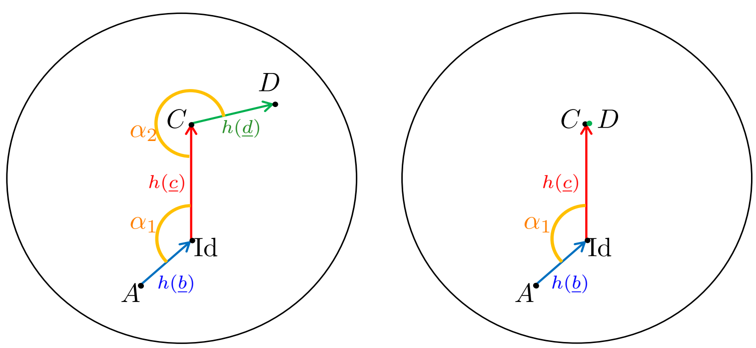


Figure 25: Polygonal chains of a generic hexagon
and of a non-generic hexagon of type 1.3

If $(H, \mathcal{Y}_{0,\infty}) \in \mathcal{H}^{gen}$ the segments of the polygonal chain of $(H, \mathcal{Y}_{0,\infty})$ have hyperbolic length given by $h(\underline{b}), h(\underline{c})$ and $h(\underline{d})$ respectively where h is the map in (13) and $\underline{b}, \underline{c}, \underline{d}$ are length parameters of the arc coordinates. Up to an isometry $g \in \mathrm{Sp}(4, \mathbb{R})$ we can consider the case where $B = \mathrm{Id}$ and C is diagonal. Observe that to a generic hexagon $(H, \mathcal{Y}_{0,\infty})$ we can associate exactly two polygonal chains up to isometry, and these are drawn in Figure 24. If the hexagon is non-generic some segments contract to a point. The hyperbolic length of the segment is zero as the corresponding length parameter is inside $\mathfrak{d} = \{(x_1, x_2)|\ x_1 = x_2)\}$. This will be made more clear in the next section. The polygonal chain of both a generic and a non-generic hexagon is illustrated in the Poincaré disc model in Figure 25. For simplicity for any $X \in \mathrm{Sym}^+(2, \mathbb{R})$ we have denoted the point $\pi^{\mathbb{H}^2}(iX)$ as X.

4.5 Arc coordinates for non-generic hexagons

In this section we want to study arc coordinates in the case of non-generic hexagons inside $\mathcal{X}$. Again we will focus on the case where $\mathcal{X}$ is the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. Recall that we denote by $\mathfrak{a}$ the subset of the Weyl chamber consisting of regular vectors:

$$\mathfrak{a} = \{(x_1, x_2) \in \mathbb{R}^2|\ x_1 > x_2 > 0\}$$

Recall also that we denote by $\mathfrak{d}$ the following set

$$\mathfrak{d} = \{(x_1, x_2) \in \mathbb{R}^2|\ x_1 = x_2\}$$

A generic hexagon $(H, \mathcal{Y}_1) \in \mathcal{H}^{gen}$ is defined by an ordered 6-tuple $H = (P, A, B, C, D, Q)$ where three quadruples are generic and $\mathcal{Y}_1 = \mathcal{Y}_{P,Q}$. The genericity of these quadruples allows us to associate to $(H, \mathcal{Y}_1)$ three regular vectors inside $\mathfrak{a}$ (Remark 4.7). When the hexagon is non-generic these vectors can land inside $\mathfrak{d}$. We have defined three different types of non-generic hexagons depending on how many quadruples inside $(H, \mathcal{Y}_1) = (P, A, B, C, D, Q)$ are non-generic. We now give parameters which arise as a natural generalisation of Proposition 4.15.

Proposition 4.20. *Non-generic hexagons of type 1 are parametrised up to isometry by*

$$\mathcal{H}^{nongen}_{type1.1}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathfrak{d} \times \mathfrak{a}^2 \times [0, 2\pi)/_\sim$$
$$\mathcal{H}^{nongen}_{type1.2}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathfrak{a} \times \mathfrak{d} \times \mathfrak{a} \times [0, 2\pi)/_\sim$$
$$\mathcal{H}^{nongen}_{type1.3}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathfrak{a}^2 \times \mathfrak{d} \times [0, 2\pi)/_\sim$$

where for $\alpha \in [0, 2\pi)$ the equivalence relation is given by

$$\alpha \sim \alpha' \iff \alpha' = 2\pi - \alpha$$

Proof. Let $H = (P, A, B, C, D, Q) \in \mathcal{H}^{nongen}_{type1.1}$. The proof is analogue to the proof of Proposition 4.15. Up to isometry we can assume $P = 0, Q = \infty, B = \mathrm{Id}$ and C diagonal. As $(0, A, \mathrm{Id}, \infty)$ non-generic we can not define an angle parameter between the hyperbolic components of A and Id and the parameter $d^{\overline{\mathfrak{a}}^+}(iA, i\mathrm{Id})$ is inside $\mathfrak{d}$. Geometrically this means that the two points coincide in the $\mathbb{H}^2$-component of $\mathcal{Y}_{0,\infty}$ (Figure 26). The quintuple $(0, \mathrm{Id}, C, D, \infty)$ is generic and

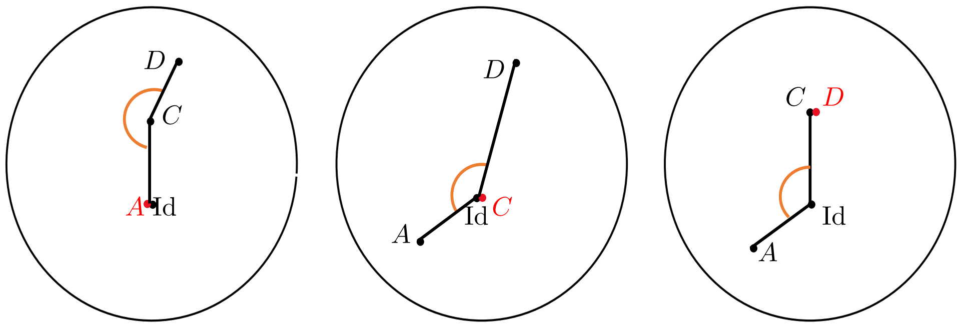


Figure 26: Polygonal chains of non-generic hexagons
of type 1.1, 1.2 and 1.3 respectively

we use Proposition 3.7 to determine the angle parameter. Observe that the stabilizer of $H^{nongen}_{type1.1} = (0, A, \mathrm{Id}, C, D, \infty)$ is also given by
$\mathrm{Stab}(H^{nongen}_{type1.1}) = \{\mathrm{Id}, \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\} \cong \mathbb{Z}/2\mathbb{Z}$ where r is the reflection across the hyperbolic component (Remark 2.33). Up to isometry we can always choose the angle parameter to lie inside $[0, \pi] \cong \mathrm{PSO}(2)/_\sim$.

Conversely, given $\big((b, b), (c_1, c_2), (d_1, d_2), [S]\big) \in \mathfrak{d} \times \mathfrak{a}^2 \times \mathrm{PSO}(2)/_\sim$ we construct the hexagon $H^{nongen}_{type1.1} = (0, A, \mathrm{Id}, C, D, \infty)$ where

$$A = \begin{pmatrix} \frac{1}{e^b} & 0 \\ 0 & \frac{1}{e^b} \end{pmatrix}$$

$$C = \begin{pmatrix} e^{c_1} & 0 \\ 0 & e^{c_2} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & \sqrt{e^{c_1}} \\ -\sqrt{e^{c_2}} & 0 \end{pmatrix} S \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S^T \begin{pmatrix} 0 & -\sqrt{e^{c_2}} \\ \sqrt{e^{c_1}} & 0 \end{pmatrix}$$

The proofs for type 1.2 and 1.3 are similar. □

Proposition 4.21. *Non-generic hexagons of type 2 are parametrized up to isometry by*

$$\mathcal{H}^{nongen}_{type2.1}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathfrak{d}^2 \times \mathfrak{a}$$
$$\mathcal{H}^{nongen}_{type2.2}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathfrak{d} \times \mathfrak{a} \times \mathfrak{d}$$
$$\mathcal{H}^{nongen}_{type2.3}/_{\mathrm{PSp}(4,\mathbb{R})} \cong \mathfrak{a} \times \mathfrak{d}^2$$

Proof. The proof is similar to the proof of Proposition 4.15 and Proposition 4.20. Since two quintuples are non-generic, we do not have any angle in the parameter space. Up to isometry we can move the polygonal chains of the hexagons in a configuration shown in Figure 27. The vector parameters are the same of Proposition 4.15, where two length are not regular and lie inside $\mathfrak{d}$. □

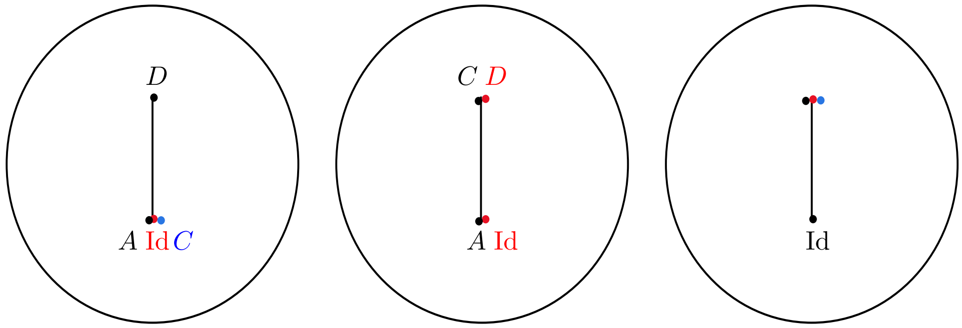


Figure 27: Polygonal chains of three non-generic hexagons of type 2.1, 2.2 and 2.3 respectively

Proposition 4.22. *Non-generic hexagons of type 3 are parametrized up to isometry by*

$$\mathcal{H}^{nongen}_{type3} \cong \mathfrak{d}^3 \cong \mathbb{R}^3_{>0}$$

Proof. Let $(H, \mathcal{Y}_1) = (P, A, B, C, D, Q)$ be inside $\mathcal{H}^{nongen}_{type3}$ and up to isometry let us consider again $P = 0, Q = \infty$ and $B = \mathrm{Id}$. We obtain three vectors

$$d^{\overline{\mathfrak{a}}^+}(p_{0,\infty}(A), p_{0,\infty}(\mathrm{Id})) = (b, b)$$

$$d^{\overline{\mathfrak{a}}^+}(p_{0,\infty}(\mathrm{Id}), p_{0,\infty}(C)) = (c, c)$$

$$d^{\overline{\mathfrak{a}}^+}(p_{0,\infty}(C), p_{0,\infty}(D)) = (d, d)$$

which are all contained in $\mathfrak{d}$. The matrices A, C and D are all multiples of the identity matrix. Equivalently, the points A, Id, C, D all coincide in the $\mathbb{H}^2$-component of $\mathcal{Y}_{0,\infty}$ and there is no angle parameter. □

Remark 4.23. Proposition 4.22 corresponds to hexagon-parameters in the hyperbolic case: we obtain the 3-dimensional space of right-angled hexagons of $\mathbb{H}^2$.

4.6 Arc coordinates for $\mathcal{H}$

In Sections 4.3 and 4.5 we have introduced arc coordinates for a right-angled hexagon $(H, \mathcal{Y}_1)$ in $\mathcal{X}$. We have first considered the case where the hexagon is generic and we have then adapted the parameters in the case of non-generic hexagons of type 1, 2 and 3 respectively. In this section we want to present arc coordinates in a more compact way. We will introduce a parameter space for $\mathcal{H}$ which encloses both the generic and the non-generic case. Again we will focus on the case where $\mathcal{X}$ is the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$.

Recall that we denote by $\mathfrak{a}$ the subset of the Weyl chamber consisting of regular vectors:

$$\mathfrak{a} = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 > x_2 > 0\}$$

and by $\mathfrak{d}$ the following set

$$\mathfrak{d} = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 = x_2\}$$

We introduce the symbol $\overline{\mathfrak{a}}$ to denote the union $\overline{\mathfrak{a}} = \mathfrak{a} \cup \mathfrak{d}$ that is the set

$$\overline{\mathfrak{a}} = \{(x_1, x_2) |\ x_1 \geqslant x_2 > 0\}$$

Definition 4.24. The space of *decorated arc coordinates* $\mathcal{A}^{dec}_{(H,\mathcal{Y}_1)}$ is given by

$$\mathcal{A}^{dec}_{(H,\mathcal{Y}_1)} := \overline{\mathfrak{a}}^3 \times [0, 2\pi) \times [0, 2\pi)$$

We further define $\mathcal{A}_{(H,\mathcal{Y}_1)}$ to be the set

$$\mathcal{A}_{(H,\mathcal{Y}_1)} := \mathcal{A}^{dec}_{(H,\mathcal{Y}_1)}/\sim$$

where the equivalence relation is given by

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, 2\pi - \alpha_1, 2\pi - \alpha_2)$$

Remark 4.25. It is straightforward to see that if $\underline{b}, \underline{c}, \underline{d} \in \mathfrak{a}^3$ then $\mathcal{A}_{(H,\mathcal{Y}_1)}$ is the space of arc coordinates for non-generic hexagons described in Proposition 4.15.

We can now state the following

Theorem 4.26. *Let $\mathcal{X}$ be the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$ and let $\mathcal{H}$ be the space of ordered right-angled hexagons in $\mathcal{X}$:*

$$\mathcal{H} = \{(H, \mathcal{Y}_1) |\ H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6] \text{ right-angled hexagon } \}$$

Then $\mathcal{H}$ is parametrized up to isometry by

$$\mathcal{A} := \mathcal{A}_{(H,\mathcal{Y}_1)}/\sim$$

where for $(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \in \mathcal{A}_{(H,\mathcal{Y}_1)}$ we have the following equivalent relation $\sim$:

(i) If $\underline{b} \in \mathfrak{d}$:

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \alpha_2)\ \forall \overline{\alpha}_1$$

(ii) If $\underline{c} \in \mathfrak{d}$:

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \overline{\alpha}_2)$$

for $\overline{\alpha}_1, \overline{\alpha}_2$ such that $\overline{\alpha}_1 + \overline{\alpha}_2 = \alpha_1 + \alpha_2$

(iii) If $\underline{d} \in \mathfrak{d}$:

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \alpha_1, \overline{\alpha}_2)\ \forall \overline{\alpha}_2$$

Proof. Let $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2]) \in \mathcal{A}$. We want to construct a right-angled hexagon

$$(H, \mathcal{Y}_{0,\infty}) = (0, A, \mathrm{Id}, C, D, \infty)$$

fro a maximal 6-tuple $(0, A, \mathrm{Id}, C, D, \infty)$ where C is diagonal. We construct $(H, \mathcal{Y}_{0,\infty})$ in the following way: we look at the length parameters $(\underline{b}, \underline{c}, \underline{d})$ which uniquely determine the genericity type of the hexagon (Remark 4.7) and then we use one of Propositions 4.20, 4.21 and 4.22 to construct $(H, \mathcal{Y}_{0,\infty})$. In the case of non-generic hexagons some of the angle parameters vanish and this is translated in the equivalent relations of $\mathcal{A}$ by collapsing the angle parameter in one point. More precisely:

0. If $\underline{b}, \underline{c}, \underline{d} \in \mathfrak{a}^3$ we construct a generic hexagon with arc coordinates $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ using Proposition 4.15:

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$$

1.1 If $\underline{b} \in \mathfrak{d}$, $\underline{c}, \underline{d} \in \mathfrak{a}^2$ then the angle parameter α_1 is collapsed into a point and we use α_2 to construct a non-generic hexagon of type 1.1 using Proposition 4.20

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_2])$$

The polygonal chain of such a hexagon is illustrated in Figure 26.

1.2 If $\underline{c} \in \mathfrak{d}$, $\underline{b}, \underline{d} \in \mathfrak{a}^2$ we use Proposition 4.20 to construct a non-generic hexagon of type 1.2 where

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1 + \alpha_2 - \pi])$$

The reason why we choose to translate the angle by π is the following: in the procedure of constructing a hexagon $(H, \mathcal{Y}_{0,\infty})$ from a point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ inside $\mathcal{A}$ we know that angle parameters α_1, α_2 have a geometric interpretation realised in the polygonal chain of $(H, \mathcal{Y}_{0,\infty})$. If the hexagon is non-generic of type 1 then we only need one angle parameter to construct $(H, \mathcal{Y}_{0,\infty})$. In this construction procedure, when moving continuously from a point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ where $\underline{c} \in \mathfrak{a}$ to a point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ where $\underline{c} \in \mathfrak{d}$ we want the constructed hexagons to be close. To do this we need to construct $(H, \mathcal{Y}_{0,\infty})$ using the angle parameter $[\alpha_1 + \alpha_2 - \pi]$. This is illustrated in Figure 28 below.

1.3 If $\underline{d} \in \mathfrak{d}$, $\underline{c}, \underline{d} \in \mathfrak{a}^2$ then the angle parameter α_2 is collapsed into a point and we use α_1 to construct a non-generic hexagon of type 1.1 using Proposition 4.20

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1])$$

If two length parameters are inside $\mathfrak{d}$ then two of the three equivalence relations of $\mathcal{A}$ are satisfied. In this case both angle parameters are collapsed into a point. For example if *(i)* $\underline{b} \in \mathfrak{d}$ and *(ii)* $\underline{c} \in \mathfrak{d}$ then

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \overline{\alpha}_2) \ \forall \overline{\alpha}_1, \forall \overline{\alpha}_2$$

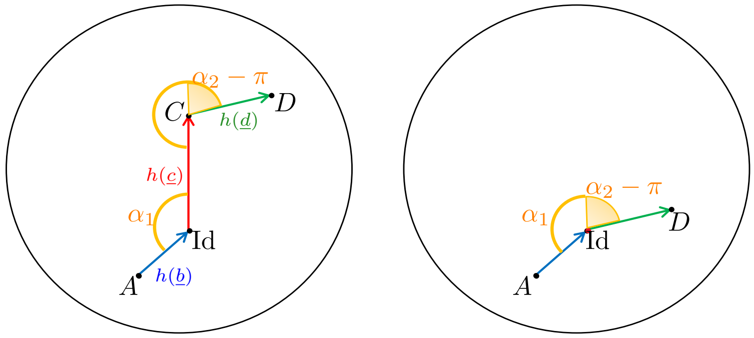


Figure 28: Construction of $(H, \mathcal{Y}_{0,\infty})$ of type 1.2 when $\underline{c} \to \mathfrak{d}$

as

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \overset{(ii)}{\sim} (\underline{b}, \underline{c}, \underline{d}, \alpha_1 + \alpha_2 - \overline{\alpha}_2, \overline{\alpha}_2)\ \forall \overline{\alpha}_2 \overset{(i)}{\sim} (\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \overline{\alpha}_2)\ \forall \overline{\alpha}_1$$

We construct $(H, \mathcal{Y}_{0,\infty})$ in the following way:

2.1 If $\underline{b}, \underline{c} \in \mathfrak{d}^2$, $\underline{d} \in \mathfrak{a}$ we use Proposition 4.21 to construct a non-generic hexagon of type 2.1 where

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d})$$

2.2 If $\underline{b}, \underline{d} \in \mathfrak{d}^2$, $\underline{c} \in \mathfrak{a}$ we use Proposition 4.21 to construct a non-generic hexagon of type 2.2 where

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d})$$

2.3 If $\underline{c}, \underline{d} \in \mathfrak{d}^2$, $\underline{b} \in \mathfrak{a}$ we use Proposition 4.21 to construct a non-generic hexagon of type 2.3 where

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d})$$

3. If $\underline{b}, \underline{c}, \underline{d} \in \mathfrak{d}^3$ we use Proposition 4.22 to construct a non-generic hexagon of type 3 where

$$(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}) = (b, c, d) \in \mathbb{R}^3_+$$

It is clear that any equivalent point $(\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \overline{\alpha}_2) \sim (\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2)$ in $\mathcal{A}$ induces an isometric hexagon $(H', \mathcal{Y}_{0,\infty})$ in $\mathcal{H}$.

Conversely, let $(H, \mathcal{Y}_1)$ be a hexagon in $\mathcal{H}$ and let us write $H = (P, A, B, C, D, Q)$. Up to isometry we can consider $P = 0, B = \mathrm{Id}, Q = \infty$ and C diagonal, so that $\mathcal{Y}_1 = \mathcal{Y}_{0,\infty}$. We put

$$\underline{b} = d^{\mathfrak{a}^+}(iA, i\mathrm{Id})$$

$$\underline{c} = d^{\mathfrak{a}^+}(i\mathrm{Id}, iC)$$
$$\underline{d} = d^{\mathfrak{a}^+}(iC, iD)$$

Again we use Propositions 4.20 and 4.21 to determine arc coordinates. More precisely:

0. If $\underline{b}, \underline{c}, \underline{d} \in \mathfrak{a}^3$ we associate to $(H, \mathcal{Y}_{0,\infty})$ the point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ using Proposition 4.15.

1.1 If $\underline{b} \in \mathfrak{d}$, $\underline{c}, \underline{d} \in \mathfrak{a}^2$ then for $(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \in \mathcal{A}_{H,\mathcal{Y}_1}$ it holds

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \alpha_2) \ \forall \overline{\alpha}_1$$

We compute the point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_2])$ using Proposition 4.20 and we associate to $(H, \mathcal{Y}_{0,\infty})$ the point $(\underline{b}, \underline{c}, \underline{d}, [\bullet, \alpha_2])$.

1.2 If $\underline{c} \in \mathfrak{d}$, $\underline{b}, \underline{d} \in \mathfrak{a}^2$ then for $(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \in \mathcal{A}_{H,\mathcal{Y}_1}$ it holds

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \overline{\alpha}_2)$$

for $\overline{\alpha}_1, \overline{\alpha}_2$ such that $\overline{\alpha}_1 + \overline{\alpha}_2 = \alpha_1 + \alpha_2$. We compute the point $(\underline{b}, \underline{c}, \underline{d}, [\alpha])$ using Proposition 4.20 and we associate to $(H, \mathcal{Y}_{0,\infty})$ the point $(\underline{b}, \underline{c}, \underline{d}, [\frac{\alpha}{2}, \frac{\alpha}{2} + \pi])$.

1.3 If $\underline{d} \in \mathfrak{d}$, $\underline{c}, \underline{d} \in \mathfrak{a}^2$ then for $(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \in \mathcal{A}_{H,\mathcal{Y}_1}$ it holds

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \alpha_1, \overline{\alpha}_2) \ \forall \overline{\alpha}_2$$

We compute the point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1])$ using Proposition 4.20 and we associate to $(H, \mathcal{Y}_{0,\infty})$ the point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \bullet])$.

For the cases 2.1, 2.2, 2.3 and 3 all the angle parameters vanish and we associate to $(H, \mathcal{Y}_{0,\infty})$ the point $(\underline{b}, \underline{c}, \underline{d}, [\bullet, \bullet])$. □

Definition 4.27. (**Arc coordinates for** $\mathcal{H}$) The parameters of Theorem 4.26 will be called *arc coordinates* for a right-angled hexagon $(H, \mathcal{Y}_1)$. Given $(H, \mathcal{Y}_1)$ inside $\mathcal{H}$ its arc coordinates $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ will be denoted $\mathcal{A}(H, \mathcal{Y}_1)$. The vectors $\underline{b}, \underline{c}, \underline{d}$ are *length parameters* and α_1, α_2 are *angle parameters.*

4.7 Hexagons inside a maximal polydisc

Let $\mathcal{X}$ be the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. We have seen in Section 2.4 how to embed $\mathbb{H}^2 \times \mathbb{H}^2$ inside $\mathcal{X}$ through the map:

$$\psi : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathcal{X}$$
$$(z_1, z_2) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

The image of ψ in $\mathcal{X}$ is called the model polydisc since each other polydisc is translate of our model by an element in $\mathrm{Sp}(4, \mathbb{R})$. A right-angled hexagon

$H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ is therefore contained in a maximal polydisc if there exists an isometry g such that $g \cdot H$ is contained in the model polydisc. In particular a hexagon H is contained in the model polydisc of $\mathcal{X}$ if and only if all tubes have diagonal matrices as endpoints. Recall that the subset $\psi\big((z,z)\big)$ is a copy of $\mathbb{H}^2$ inside $\mathcal{X}$ and is referred to as the diagonal disc.

In Theorem 4.26 we have parametrized the space of ordered right-angled hexagons $\mathcal{H}$ up to isometry. In the following Proposition we show which subspace correspond to hexagons contained in a maximal polydisc.

Proposition 4.28. *The subspace $\mathcal{D} \subset \mathcal{A}$*

$$\mathcal{D} = \Big\{(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2]) \in \mathcal{A} |\ [\alpha_1, \alpha_2] \in \{[0,0], [0,\pi], [\pi, 0], [\pi,\pi]\}\Big\} \subset \mathcal{A}$$

corresponds to right-angled hexagons inside a maximal polydisc in $\mathcal{X}$.

Proof. In the case where $\underline{b}, \underline{c}, \underline{d} \in \mathfrak{a}^3$ the point $p = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ corresponds to a generic hexagon. Using Proposition 4.15 we know $p = [(H, \mathcal{Y}_{0,\infty})]$ where

$$(H, \mathcal{Y}_{0,\infty}) = (0, A, \mathrm{Id}, C, D, \infty)$$

with $C = \begin{pmatrix} e^{c_1} & 0 \\ 0 & e^{c_2} \end{pmatrix}$ and

$$A = \begin{pmatrix} \frac{1}{e^{b_2}} & 0 \\ 0 & \frac{1}{e^{b_1}} \end{pmatrix},\ D = \begin{pmatrix} e^{c_1+d_2} & 0 \\ 0 & e^{c_2+d_1} \end{pmatrix} \text{ for } \alpha_1 = \alpha_2 = 0$$

$$A = \begin{pmatrix} \frac{1}{e^{b_1}} & 0 \\ 0 & \frac{1}{e^{b_2}} \end{pmatrix},\ D = \begin{pmatrix} e^{c_1+d_1} & 0 \\ 0 & e^{c_2+d_2} \end{pmatrix} \text{ for } \alpha_1 = \alpha_2 = \pi$$

$$A = \begin{pmatrix} \frac{1}{e^{b_2}} & 0 \\ 0 & \frac{1}{e^{b_1}} \end{pmatrix},\ D = \begin{pmatrix} e^{c_1+d_1} & 0 \\ 0 & e^{c_2+d_2} \end{pmatrix} \text{ for } \alpha_1 = 0,\ \alpha_2 = \pi$$

$$A = \begin{pmatrix} \frac{1}{e^{b_1}} & 0 \\ 0 & \frac{1}{e^{b_2}} \end{pmatrix},\ D = \begin{pmatrix} e^{c_1+d_2} & 0 \\ 0 & e^{c_2+d_1} \end{pmatrix} \text{ for } \alpha_1 = \pi,\ \alpha_2 = 0$$

All four cases correspond to hexagons consisting of tubes that have diagonal matrices as endpoints. This is consistent with the geometrical meaning of the angle parameter described in Section 2.10. A similar argument can be used for the case of non-generic hexagons of type 1. All non-generic right-angled hexagons of type 2 and 3 are contained in a maximal polydisk in $\mathcal{X}$ and in these cases for all $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2]) \in \mathcal{D}$ it holds

$$(\underline{b}, \underline{c}, \underline{d}, \alpha_1, \alpha_2) \sim (\underline{b}, \underline{c}, \underline{d}, \overline{\alpha}_1, \overline{\alpha}_2)\ \forall \overline{\alpha}_1, \overline{\alpha}_2$$

Conversely, if $[(H, \mathcal{Y}_1)] \in \mathcal{H}/_{\mathrm{PSp}(4,\mathbb{R})}$ is contained in a maximal polydisc then we can move it into the model polydisc through an isometry. It is easy to see that in this case the point $p \in \mathcal{A}$ corresponding to $[(H, \mathcal{Y}_1)]$ must be inside $\mathcal{D}$. □

Definition 4.29. We define $\mathcal{D}_{\mathbb{H}^2}$ as the set

$$\mathcal{D}_{\mathbb{H}^2} = \big\{(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2]) \in \mathcal{A} |\ \underline{b}, \underline{c}, \underline{d} \in \mathfrak{d}^3\big\}$$

In Definition 4.2 we have defined the stabilizer of a right-angled hexagon $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ as

$$\mathrm{Stab}(H) = \big\{g \in \mathrm{PSp}(4, \mathbb{R}) |\ g \cdot \mathcal{Y}_i = \mathcal{Y}_i,\ i \in \{1, ..., 6\}\big\}$$

Proposition 4.30. *(**Stabilizer of a right-angled hexagon**) Let $\mathcal{X}$ be the symmetric space associated to* $\mathrm{Sp}(4, \mathbb{R})$ *and let $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ be a right-angled hexagon in $\mathcal{X}$. It holds*

(i) If H is contained in a copy of $\mathbb{H}^2$ inside $\mathcal{X}$ then $\mathrm{Stab}(H) \cong \mathrm{PO}(2)$

(ii) If H is contained in a maximal polydisc then $\mathrm{Stab}(H) \cong \mathbb{Z}/2\mathbb{Z}$

(iii) If H is not contained in any maximal polydisc then $\mathrm{Stab}(H) = \{id\}$

Proof. Up to isometry let us consider

$$H = (0, A, \mathrm{Id}, C, D, \infty)$$

where C is diagonal.

(i) If H is contained in the diagonal disc then the matrices A, C, D are all multiples of Id and so are all endpoints of the tubes of H. It is clear that the stabilizer is PO(2).

(ii) The matrices A, C, D are all diagonal and so are all endpoints of the tubes of H. The stabilizer is given by the identity together with $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(iii) This is clear.

□

5 Discussion about the parameters

In the previous section we have introduced arc coordinates $\mathcal{A}$ to parametrize the space $\mathcal{H}/_{\mathrm{PSp}(4,\mathbb{R})}$ (Theorem 4.26). In particular to an ordered right-angled hexagon $[(H, \mathcal{Y}_1)]$ inside $\mathcal{H}/_{\mathrm{PSp}(4,\mathbb{R})}$ we associate a point $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ where $\underline{b}, \underline{c}, \underline{d}$ are called length parameters and whose geometric interpretation is illustrated for example in Figure 22. This choice of length parameters could somehow appear not natural, especially if compared with the parameters of a right angled-hexagon in the hyperbolic space. In $\mathbb{H}^2$ a right-angled hexagon is uniquely determined by the length of three alternating sides. In the Siegel space all length parameters $\underline{b}, \underline{c}, \underline{d}$ are lying in the tube $\mathcal{Y}_1$. In this chapter we will recall the proof of the parametrization in $\mathbb{H}^2$ of right-angled hexagons as done in [Mar16, Lemma 6.2.2] (suitably adapted to the upper-half space model). We then discuss the differences and the problems that arise when generalizing these hexagon-parameters in the symmetric space $\mathcal{X}$ associated to $\mathrm{Sp}(4, \mathbb{R})$. We refer to [BP92], [RR95], [Lou20] for an introduction to hyperbolic geometry.

5.1 The $\mathbb{H}^2$-case

We start by recalling the definition of cross-ratio in $\mathbb{H}^2$ and by giving a proposition that will have an analogue in the Siegel space $\mathcal{X}$.

Definition 5.1. Let $z_1, z_2, z_3, z_4 \in \mathbb{H}^2$. The cross-ratio $R(z_1, z_2, z_3, z_4)$ is the point

$$R(z_1, z_2, z_3, z_4) = f(z_3)$$

where f is the unique map in $\mathrm{PSL}(2, \mathbb{R})$ such that

$$f(z_1) = 0,\ f(z_2) = 1,\ f(z_4) = \infty$$

In the upper-half space model $\mathbb{H}^2 = \{x + iy |\ x, y \in \mathbb{R},\ y > 0\}$ the explicit expression for the cross-ratio $R(z_1, z_2, z_3, z_4)$ where $z_i = x_i + iy_i$ is given by

$$R(z_1, z_2, z_3, z_4) = (z_1 - z_2)^{-1}(z_4 - z_2)(z_4 - z_3)^{-1}(z_1 - z_3)$$

The following lemma is well known and is the analogue of Lemma 2.17 for the hyperbolic plane:

Lemma 5.2. *Let $(a, x, y, b) \in \partial\mathbb{H}^2$ and let $\gamma_{a,b}$ be the infinite geodesic with endpoint $\{a, b\}$. Let $p_{a,b}$ be the orthogonal projection onto $\gamma_{a,b}$. Then*

$$d^{\mathbb{H}^2}(p_{a,b}(x), p_{a,b}(y)) = \log \mu$$

where $\mu = R(a, x, y, b)$.

Lemma 5.3. *Let $\gamma_{a,b}, \gamma_{c,d}, \gamma_{e,f}$ be three infinite geodesics with endpoints $\{a, b\}, \{c, d\}$ and $\{e, f\}$ respectively. Suppose*

$$\gamma_{a,b} \perp \gamma_{c,d} \perp \gamma_{e,f}$$

Then there exists a bijective map

$$T : \mathbb{R}^+ \to \mathbb{R}^+$$
$$R(c, b, e, d) \mapsto R(a, b, e, f)$$

given by $T(x) = \frac{(x+1)^2}{4x}$

Proof. Up to isometry we can consider $\gamma_{c,d} = \gamma_{0,\infty}$ and $\gamma_{a,b} = \gamma_{-1,1}$. Then $\gamma_{e,f} = \gamma_{-x,x}$ for an $x \in \mathbb{R}^+,\ x > 1$ (Figure 29 below).

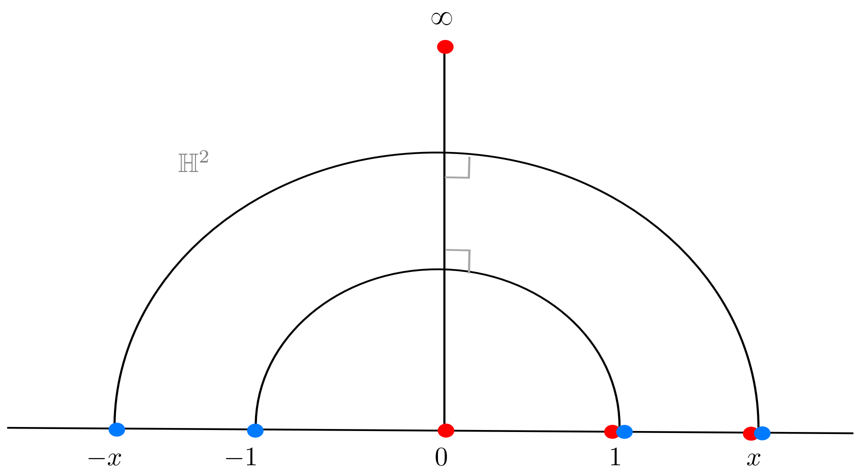


Figure 29: There is a bijective map between
$R(0, 1, x, \infty)$ and $R(-1, 1, x, -x)$

We obtain

$$R(c, b, e, d) = R(0, 1, x, \infty) = x > 1$$

and

$$R(a, b, e, f) = R(-1, 1, x, -x) = \frac{(x + 1)^2}{4x}$$

so that $T(x) = \frac{(x+1)^2}{4x}$. □

Proposition 5.4. *Let $\gamma_{x,z} \perp \gamma_{y,w}$ be two orthogonal geodesics in $\mathbb{H}^2$ with endpoints $\{x, z\}$ and $\{y, w\}$ respectively and let P be their intersection point. Then there exists a bijective map $f = f(\gamma_{x,z}, \gamma_{y,w}, P)$ defined as follows:*
For $d > 0$ let P' be one of the two points in $\gamma_{y,w}$ at distance d from P. Let

$\gamma_{P'}$ be the geodesic through P' orthogonal to $\gamma_{y,w}$ and denote by b one of its endpoints. Then we can define

$$f : \mathbb{R}^+ \to \mathbb{R}^+$$
$$d^{\mathbb{H}^2}(P, P') \mapsto d^{\mathbb{H}^2}(P, p_{x,z}(b))$$

where $p_{x,z}$ denotes the orthogonal projection on the geodesic $\gamma_{x,z}$. The map f is given by

$$f : \mathbb{R}^+ \to \mathbb{R}^+$$
$$d \mapsto \log\left(\frac{e^d + 1}{e^d - 1}\right)$$

This expression does not depend on the choice of the points P', b.

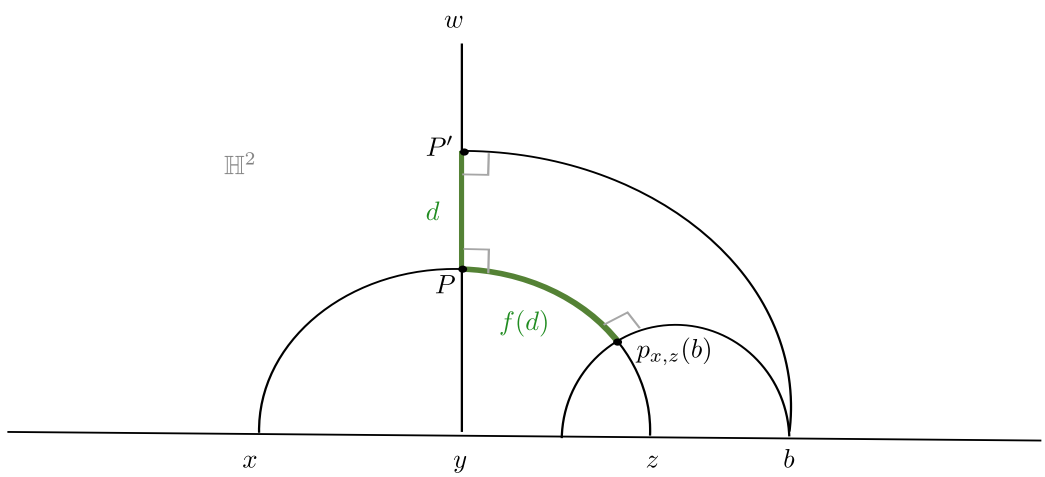


Figure 30: The map f

Proof. Let us consider the upper-half plane model

$$\mathbb{H}^2 = \{x + iy|\ x \in \mathbb{R},\ y \in \mathbb{R}^+\}$$

Without loss of generality we can assume $\gamma_{y,w} = \gamma_{0,\infty} = \{iy|\ y > 0\}$ and $\gamma_{x,z} = \gamma_{-1,1}$, so that their intersection is the point $P = i$ (Figure 31).

Let $P' = ie^d$ and $b = e^d$. By Lemma 5.2 we know that

$$d^{\mathbb{H}^2}(i, p_{-1,1}(e^d)) = R(-1, 0, e^{-d}, 1) = \frac{e^d + 1}{e^d - 1}$$

so that $f(d) = \log(\frac{e^d+1}{e^d-1})$. Note that $f^{-1} = f$. It is trivial to show that the cases

$$P' = e^b,\ b = -ie^b$$

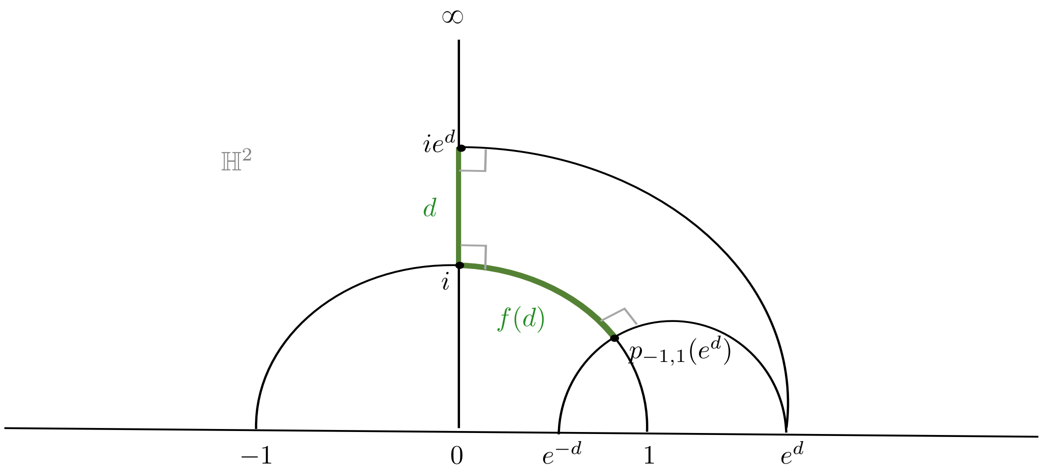


Figure 31: The map f

$$P' = e^{-b},\ b = ie^{-b}$$
$$P' = e^{-b},\ b = -ie^{-b}$$

all lead to the same expression of f. $\square$

Lemma 5.5. *([Mar16, Lemma 6.2.2]) Given three real numbers $b, c, d > 0$ there exists (up to isometries) a unique hyperbolic right-angled hexagon with three alternate sides of length b, c and d respectively.*

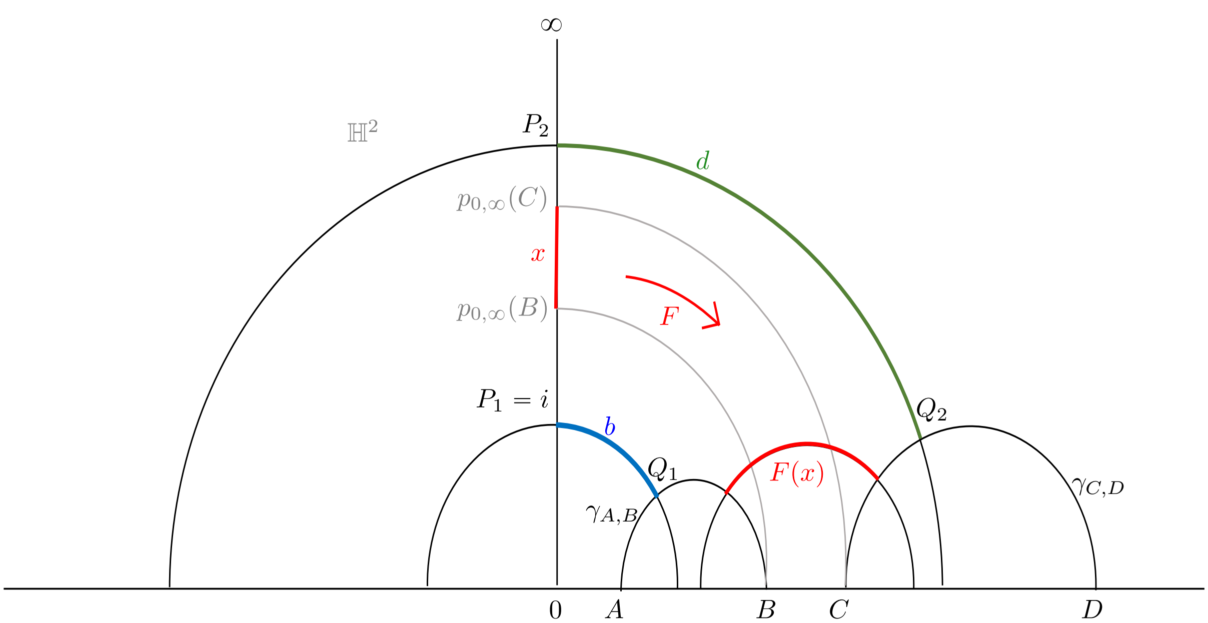


Figure 32: Construction of a right-angled hexagon in the $\mathbb{H}^2$-case

Proof. Let $b, d > 0$. Then the construction of the hexagon goes as follows: take a geodesic γ with two arbitrary points P_1, P_2 in it. Without loss of generality we can assume γ to be the vertical geodesic $\gamma_{0,\infty} = \{iy|\ y > 0\}$ and $P_1 = i$ (Figure 32). Draw the perpendicular from $P_1 = i$ and from P_2. At distances b and d we find two points Q_1 and Q_2 and we draw again two perpendiculars $\gamma_{A,B}$ and $\gamma_{C,D}$, with some points at infinity A, B and C, D respectively. Draw the unique perpendiculars to γ pointing to B and C: they intersect γ in two points $p_{0,\infty}(B)$ and $p_{0,\infty}(C)$. Note that the lengths $d^{\mathbb{H}^2}(P_1, p_{0,\infty}(B))$ and $d^{\mathbb{H}^2}(p_{0,\infty}(C), P_2)$ have some fixed length depending only on b and d through a bijective map explicitly given in Proposition 5.4 (this is the map $f = f^{-1}$). We can vary the parameter $x = d^{\mathbb{H}^2}(p_{0,\infty}(B), p_{0,\infty}(C))$, the geodesics $\gamma_{A,B}$ and $\gamma_{C,D}$ are ultra-parallel and there is a unique segment orthogonal to both of some length $F(x)$. The function $F : (0, +\infty) \to (0, +\infty)$ is continuous, strictly monotonic, and with $\lim_{x\to\infty} = \infty$: therefore there is precisely one c such that $F(x) = c$. ☐

Remark 5.6. (**Explicit form of the map** F) In the proof of Lemma 5.5 we have shown how to parametrize a right-angled hexagon in $\mathbb{H}^2$ by the lengths b, c, d of three alternating sides. This is illustrated in Figure 32, where $F(x) = c$ for a bijective map F. By Proposition 5.4 we know that there is a bijection between the length b, d and the segments $d^{\mathbb{H}^2}(P_1, p_{0,\infty}(B)), d^{\mathbb{H}^2}(p_{0,\infty}(C), P_2)$ respectively. We can therefore think as the lengths b, c, d determining the hexagon as all lying on the vertical geodesic $\gamma_{0,\infty}$.

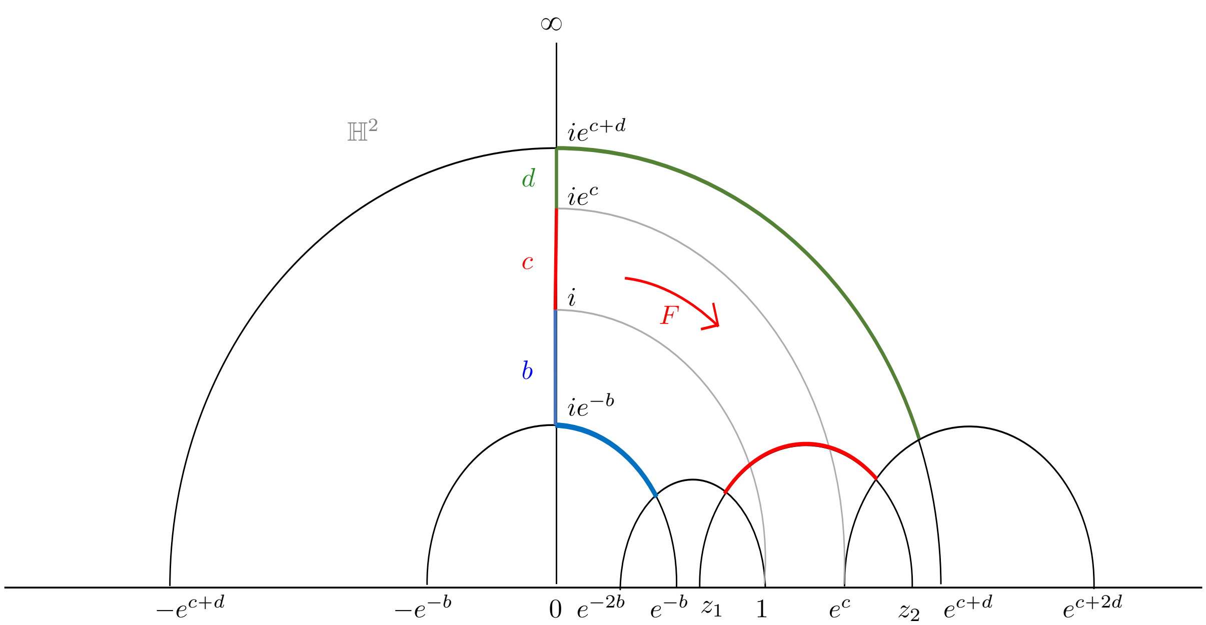


Figure 33: There is a bijection between segments of the same colour

More concretely we can parameterize a right-angled hexagon as shown in Figure 33, where by abuse of notation we keep the letters b, c, d referring to lengths of segments lying on $\gamma_{0,\infty}$. This can be thought as the parametrization of a non generic hexagon of type 3 in Proposition 4.22 (drawn in the upper-half plane). Proposition 5.4 provides a bijective map f to explicitly write the length of the blue and green sides of Figure 33. To explicitly write the map F let us consider the configuration of Figure 33.
By Lemma 5.2 it holds

$$c = R(0, 1, e^c, \infty)$$

and

$$c = R(0, 1, e^c, \infty) \overset{F}{\mapsto} R(z_1, 1, e^c, z_2) \overset{T}{\mapsto} R(e^{-2b}, 1, e^c, e^{c+2d})$$

Where T is the bijective map of Lemma 5.3. For b, d fixed we obtain

$$T \circ F(c) = \frac{(e^{c+2d} - 1)(1 - e^{2b+c})}{e^c(1 - e^{2b})(e^{2d} - 1)} = y$$

and $F(c) = T^{-1}(y)$.

5.2 Length parameters in $\mathcal{X}$

In the Siegel space $\mathcal{X}$ the analogue of geodesics are $\mathbb{R}$-tubes: in this case the length-parameters take value in the Weyl chamber

$$\overline{\mathfrak{a}}^+ = \{(x_1, ..., x_n) \in \mathbb{R}^n | \ x_1 \geqslant ... \geqslant x_n \geqslant 0\}$$

The analogue of Lemma 5.3 in $\mathcal{X}$ is given by the following:

Lemma 5.7. *Let $\mathcal{Y}_{A,B}, \mathcal{Y}_{C,D}, \mathcal{Y}_{E,F}$ be three $\mathbb{R}$-tubes inside $\mathcal{X}$ such that*

$$\mathcal{Y}_{A,B} \perp \mathcal{Y}_{C,D} \perp \mathcal{Y}_{E,F}$$

Let us denote by $(x_1 \geqslant ... \geqslant x_n)$ the eigenvalues of $R(C, B, E, D)$ and by $(y_1 \geqslant ... \geqslant y_n)$ the eigenvalues of $R(A, B, E, F)$. Then there exist a bijective map T

$$T(x_1, ...x_n) = (y_1, ...y_n)$$

where for $i \in \{1, ...n\}$

$$y_i = \frac{(x_i + 1)^2}{4x_i}$$

Proof. Up to isometry we can consider $\mathcal{Y}_{C,D} = \mathcal{Y}_{0,\infty}$, $\mathcal{Y}_{A,B} = \mathcal{Y}_{-\mathrm{Id},\mathrm{Id}}$ and $\mathcal{Y}_{E,F} = \mathcal{Y}_{-X,X}$ for an $X \in \mathrm{Sym}^+(n, \mathbb{R})$, $X > \mathrm{Id}$, $X = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ (see Figure 34).

We obtain

$$R(C, B, E, D) = R(0, \mathrm{Id}, X, \infty) = (x_1, ..., x_n)$$

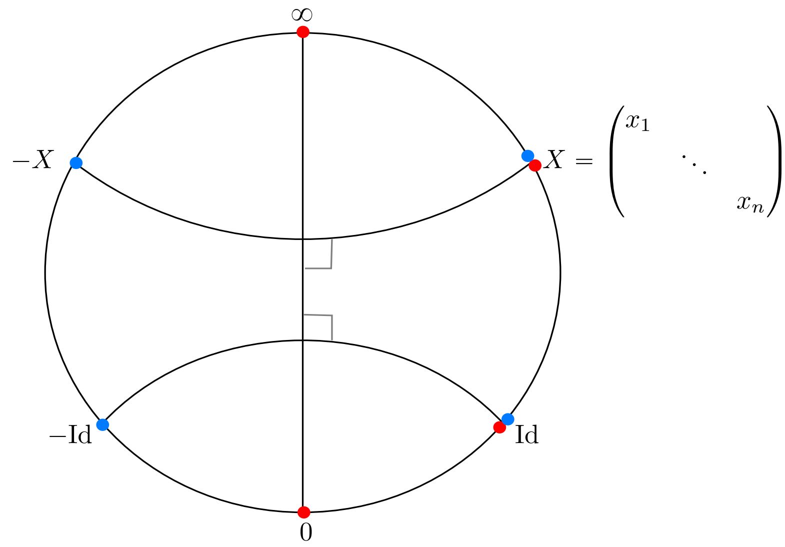


Figure 34: There is a relation between the cross-ratios of the red and of the blue points

and

$$R(A,B,E,F) = R(-\mathrm{Id}, \mathrm{Id}, X, -X) = \begin{pmatrix} \frac{(x_1+1)^2}{4x_1} & & \\ & \ddots & \\ & & \frac{(x_n+1)^2}{4x_n} \end{pmatrix}$$

□

We now state the analogue of Proposition 5.4 for the Siegel space. See Figure 35 for a better visualization of the statement.

Proposition 5.8. *Let $\mathcal{Y}_{X,Z} \perp \mathcal{Y}_{Y,W}$ be two orthogonal $\mathbb{R}$-tubes in $\mathcal{X}$ and let P be their intersection point. Then there exists a bijective map $f = f(\mathcal{Y}_{X,Z}, \mathcal{Y}_{Y,W}, P)$ defined as follows:*
For $(d_1, ...d_n) \in \overline{\mathfrak{a}}^+$ let P' be a point in $\mathcal{Y}_{Y,W}$ at distance $(d_1, ...d_n)$ from P. Let $\mathcal{Y}_{P'}$ be the tube through P' orthogonal to $\mathcal{Y}_{Y,W}$ and denote by B one of its endpoints. Then we can define

$$f : \overline{\mathfrak{a}}^+ \to \overline{\mathfrak{a}}^+$$
$$d^{\overline{\mathfrak{a}}^+}(P, P') \mapsto d^{\overline{\mathfrak{a}}^+}(P, p_{X,Z}(B))$$

where $p_{X,Z}$ denotes the orthogonal projection on the tube $\mathcal{Y}_{X,Z}$. The map f is given by

$$f : \overline{\mathfrak{a}}^+ \to \overline{\mathfrak{a}}^+$$
$$(d_1, ...d_n) \mapsto (\log\left(\frac{e^{d_n}+1}{e^{d_n}-1}\right), ..., \log\left(\frac{e^{d_1}+1}{e^{d_1}-1}\right))$$

This expression does not depend on the choice of the points P', B. In particular the image of a regular point inside $\overline{\mathfrak{a}}^+$ is a regular point.

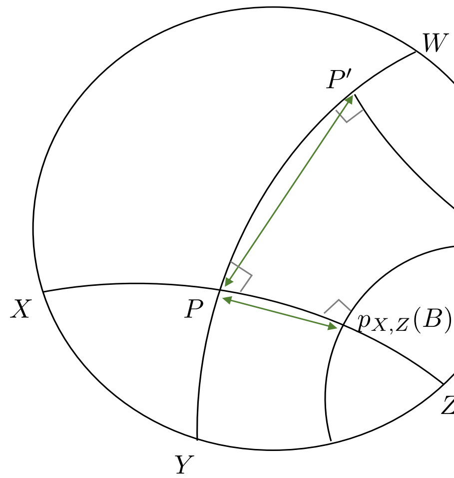


Figure 35: There is a bijection between the green vectors

Proof. Without loss of generality we can assume $\mathcal{Y}_{Y,W} = \mathcal{Y}_{0,\infty}$ and $\mathcal{Y}_{X,Z} = \mathcal{Y}_{-\mathrm{Id},\mathrm{Id}}$, so that their intersection is the point $P = i\mathrm{Id}$ (Figure 36).

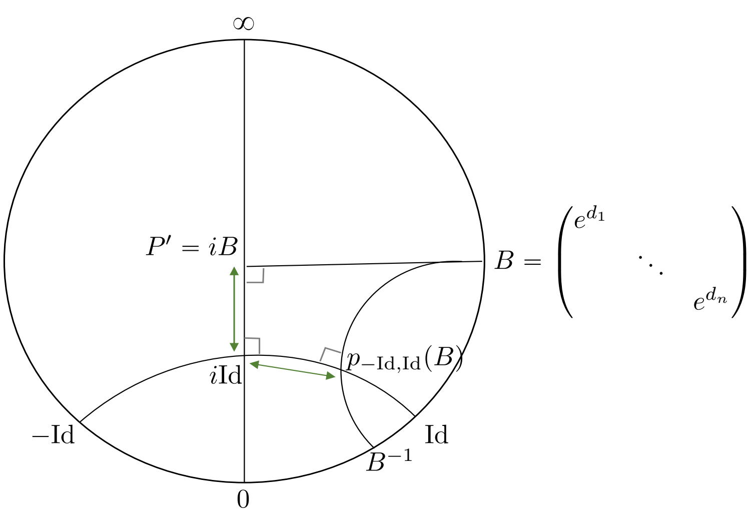


Figure 36: There is a bijection between the green vectors

Let

$$P' = i \begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{pmatrix}$$

By Lemma 2.17 and Lemma 2.22 we know that

$$d^{\overline{\mathfrak{a}}^+}(P, p_{-\mathrm{Id},\mathrm{Id}}(B)) = (\log \mu_1, ..., \log \mu_1)$$

where $\mu_1 > ... > \mu_n$ are the eigenvalues of $R(-\mathrm{Id}, 0, B^{-1}, \mathrm{Id})$. Calculations give

$$R(-\mathrm{Id}, 0, B^{-1}, \mathrm{Id}) = (\mathrm{Id} - B^{-1})^{-1}(\mathrm{Id} + B^{-1}) = \begin{pmatrix} \left(\frac{e^{d_1}+1}{e^{d_1}-1}\right) & & \\ & \ddots & \\ & & \left(\frac{e^{d_n}+1}{e^{d_n}-1}\right) \end{pmatrix}$$

so that

$$f\big((d_1, ..., d_n)\big) = (\log\Big(\frac{e^{d_n}+1}{e^{d_n}-1}\Big), ..., \log\Big(\frac{e^{d_1}+1}{e^{d_1}-1}\Big))$$

Observe that we need to invert the order of $d_1, ..., d_n$ since the function $h(x) = \log\left(\frac{e^x+1}{e^x-1}\right)$ is decreasing for $x > 0$. From the expression of f it is clear that regular points of $\overline{\mathfrak{a}}^+$ are sent to regular points. Moreover, it is trivial to show that the expression of f does not depend on the choice of the points P', B. In particular recall that the points at distance $(d_1, ..., d_n)$ from $P = i\mathrm{Id}$ are of the form

$$P' = iQ \begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{pmatrix} Q^T, \quad Q \in \mathrm{O}(n)$$

together with its inverses. $\square$

5.3 Changing side of the hexagon

Let $\mathcal{X}$ be the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$ and let us consider a right-angled hexagon $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ in $\mathcal{X}$. In the previous chapter we have introduced arc coordinates for an ordered right-angled hexagon $(H, \mathcal{Y}_1)$ in $\mathcal{H}$.

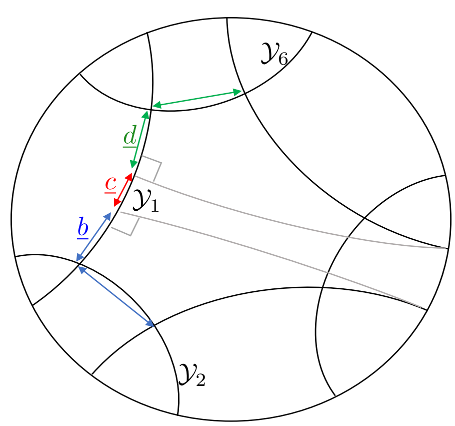


Figure 37: There is a bijection between the green vectors and between the blue vectors

These consist of length and angle parameters, where all length parameters are lying on the same tube $\mathcal{Y}_1$. By Proposition 5.8 two of the three length parameters (vectors $\underline{b}$ and $\underline{d}$ in Figure 37) can be thought as lying on the tubes $\mathcal{Y}_2$ and $\mathcal{Y}_6$ respectively. The bijection between these vectors is an analogue of the $\mathbb{H}^2$-case (Proposition 5.4) and does not depend on the angle parameters.
The following Proposition relates length parameters of arc coordinates when we change side of the right-angled hexagon.

Proposition 5.9. *Let* $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ *be a right-angled hexagon. Let* $(\underline{b_i}, \underline{c_i}, \underline{d_i})$ *denote length parameters where*

$$(\underline{b_1}, \underline{c_1}, \underline{d_1}) \text{ length parameters of } \mathcal{A}(H, \mathcal{Y}_1)$$

$$(\underline{b_2}, \underline{c_2}, \underline{d_2}) \text{ length parameters of } \mathcal{A}(H, \mathcal{Y}_3)$$

$$(\underline{b_3}, \underline{c_3}, \underline{d_3}) \text{ length parameters of } \mathcal{A}(H, \mathcal{Y}_5)$$

Then

$$\underline{b_1} = \underline{d_2}$$

$$\underline{b_2} = \underline{d_3}$$

$$\underline{b_3} = \underline{d_1}$$

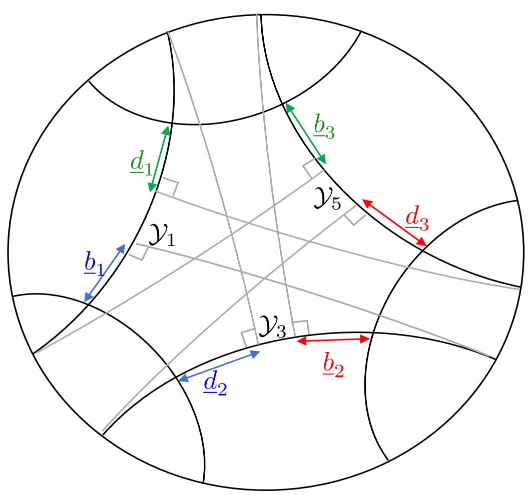


Figure 38: There is a bijection between vectors of the same colour

Proof. Let us prove $\underline{b_1} = \underline{d_2}$. Let

$$\mathcal{Y}_1 = \mathcal{Y}_{P_1,P_2},\ \mathcal{Y}_2 = \mathcal{Y}_{Q_1,Q_2},\ \mathcal{Y}_3 = \mathcal{Y}_{P_3,P_4},\ \mathcal{Y}_4 = \mathcal{Y}_{Q_3,Q_4},\ \mathcal{Y}_5 = \mathcal{Y}_{P_5,P_6},\ \mathcal{Y}_6 = \mathcal{Y}_{Q_5,Q_6}$$

By definition of arc coordinates $\mathcal{A}(H, \mathcal{Y}_1)$ associated to $(H, \mathcal{Y}_1)$ we know

$$\underline{b_1} = d^{\overline{\mathfrak{a}}^+}(p_{P_1,P_2}(Q_2), p_{P_1,P_2}(P_4))$$

where $(H, \mathcal{Y}_1) = (P_2, Q_2, P_4, P_5, Q_5, P_1)$, $\mathcal{Y}_1 = \mathcal{Y}_{P_1,P_2}$ (Figure 39). By definition of arc coordinates $\mathcal{A}(H, \mathcal{Y}_3)$ associated to $(H, \mathcal{Y}_3)$ we know

$$\underline{d_2} = d^{\overline{\mathfrak{a}}^+}(p_{P_3,P_4}(P_1), p_{P_3,P_4}(Q_1))$$

where $(H, \mathcal{Y}_3) = (P_4, Q_4, P_6, P_1, Q_1, P_3)$, $\mathcal{Y}_3 = \mathcal{Y}_{P_3,P_4}$.

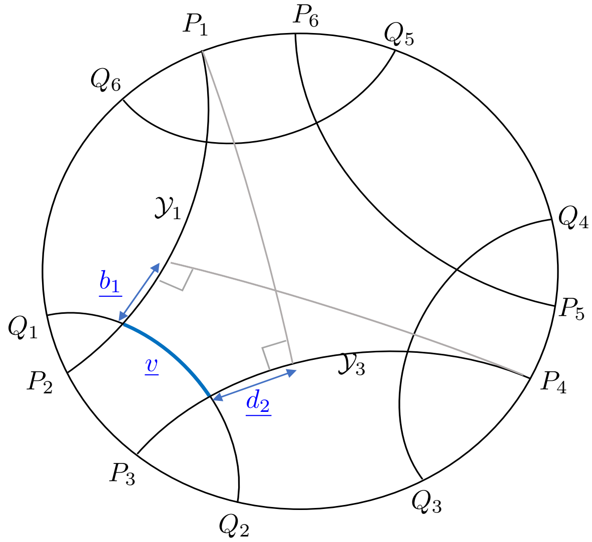


Figure 39: $\underline{b_1} = \underline{d_2}$

Let $\underline{v}$ be the vector

$$\underline{v} = d^{\overline{\mathfrak{a}}^+}(p_{Q_1,Q_2}(P_2), p_{Q_1,Q_2}(P_3))$$

and let f be the map

$$f : \overline{\mathfrak{a}}^+ \to \overline{\mathfrak{a}}^+$$
$$(a_1, a_2) \mapsto (\log\left(\frac{e^{a_2}+1}{e^{a_2}-1}\right), \log\left(\frac{e^{a_1}+1}{e^{a_1}-1}\right))$$

By Proposition 5.8 it holds

$$\underline{b_1} = f^{-1}(\underline{v}) = \underline{d_2}$$

The proof for $\underline{b_2} = \underline{d_3}$ and $\underline{b_3} = \underline{d_1}$ is similar. □

An analogue Proposition which relates the length parameters $\underline{c_i}$ is trickier. More generally when we try to generalize the map F described in Lemma 5.5 we can not guarantee bijectivity. This will be explained in the next section. Let us finish this section with the following

Corollary 5.10. *Let $(H, \mathcal{Y}_{0,\infty}) \in \mathcal{H}^{gen}$ be a generic hexagon*

$$(H, \mathcal{Y}_{0,\infty}) = (0, A, \mathrm{Id}, C, D, \infty)$$

Then the quadruples $(-A, 0, A^2, A), (-D, 0, C, D)$ are generic.

Proof. As $(H^{st}, \mathcal{Y}_{0,\infty}) \in \mathcal{H}^{st}$ we know that $(0, A, \mathrm{Id}, \infty)$ is generic i.e. the parameter $\underline{b} = (b_1, b_2)$ of Proposition 4.15 lies inside the set of regular vectors $\mathfrak{a}$. To show genericity of $(-A, 0, A^2, A)$ we need to prove that the cross-ratio $R(-A, 0, A^2, A)$ has distinct eigenvalues. By Lemma 2.17 we know that taking the logarithm of these ordered eigenvalues gives the distance $d^{\overline{\mathfrak{a}}^+}(p_{-A,A}(0), p_{-A,A}(A^2))$ which is the blue vector in Figure 37. Moreover, by Proposition 5.8 we know that this vector is the image under the bijection f of the vector $\underline{b}$ and that f is sending regular points to regular points. It follows $d^{\overline{\mathfrak{a}}^+}(p_{-A,A}(0), p_{-A,A}(A^2)) \in \mathfrak{a}$ and so $(-A, 0, A^2, A)$ is generic. For completeness of the proof we show genericity of $(-A, 0, A^2, A)$ by explicitly calculating the cross-ratio:

$$R(-A, 0, A^2, A) = (-A)^{-1}A(A - A^2)^{-1}(-A - A^2) = (A - A^2)^{-1}(A + A^2) =$$

$$= (A(\mathrm{Id} - A))^{-1}A(\mathrm{Id} + A) = (\mathrm{Id} - A)^{-1}(\mathrm{Id} + A)$$

Since $(H^{st}, \mathcal{Y}_{0,\infty}) \in \mathcal{H}^{st}$ we know that $(0, A, \mathrm{Id}, \infty)$ is generic i.e. A^{-1} and hence A has distinct eigenvalues. Let $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = PAP^T$ for $P \in \mathrm{O}(2)$ and $a_1 \neq a_2$. Then we can rewrite the cross-ratio as

$$\left(P(\mathrm{Id} - \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix})P^T\right)^{-1}\left(P(\mathrm{Id} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix})P^T\right) = P(\mathrm{Id} - \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix})^{-1}(\mathrm{Id} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix})P^T$$

which is a matrix with distinct eigenvalues $\{\frac{1+a_1}{1-a_1}, \frac{1+a_2}{1-a_2}\}$.

The proof for the genericity of the quadruple $(-D, 0, C, D)$ is similar and it also follows from the bijectivity of the map f in Proposition 5.8. In this case we can explicitly calculate the cross-ratio $R(-D, 0, C, D) = (D - C)^{-1}(D + C)$ by using the parameters of Proposition 4.15. We obtain

$$C = \begin{pmatrix} e^{c_1} & 0 \\ 0 & e^{c_2} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & \sqrt{e^{c_1}} \\ -\sqrt{e^{c_2}} & 0 \end{pmatrix} S_2 \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S_2^T \begin{pmatrix} 0 & -\sqrt{e^{c_2}} \\ \sqrt{e^{c_1}} & 0 \end{pmatrix}$$

where $(c_1, c_2), (d_1, d_2) \in \mathfrak{a}$ and $S_2 \in \mathrm{PSO}$. The cross-ratio takes the form

$$\begin{pmatrix} 0 & -\sqrt{e^{c_2}} \\ \sqrt{e^{c_1}} & 0 \end{pmatrix}^{-1} \left(S_2 \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S_2^T - \mathrm{Id}\right)^{-1} \left(S_2 \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S_2^T + \mathrm{Id}\right) \begin{pmatrix} 0 & -\sqrt{e^{c_2}} \\ \sqrt{e^{c_1}} & 0 \end{pmatrix}$$

This matrix has the same eigenvalues of the matrix

$$\left(S_2 \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S_2^T - \mathrm{Id}\right)^{-1} \left(S_2 \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} S_2^T + \mathrm{Id}\right) = S_2\left(\begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} - \mathrm{Id}\right)^{-1}\left(\begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} + \mathrm{Id}\right)S_2^T$$

and this is a matrix with distinct eigenvalues $\{\frac{e^{d_1}+1}{e^{d_1}-1}, \frac{e^{d_2}+1}{e^{d_2}-1}\}$ □

5.4 Constraints in generalizing hexagon parameters of $\mathbb{H}^2$

In the previous section we have shown (Proposition 5.8) that given a right-angled hexagon $(H, \mathcal{Y}_1)$ with arc coordinates $\mathcal{A}(H, \mathcal{Y}_1)$ we can find a bijection between length parameters $\underline{b}, \underline{d}$ of $\mathcal{A}(H, \mathcal{Y}_1)$ and the vectorial length of two alternating sides (see Figure 37). This is analogue to the hyperbolic case (Proposition 5.4). Taking inspiration from the $\mathbb{H}^2$-case, it is natural to ask whether for a right-angled hexagon in $\mathcal{X}$ there exists a bijective map also between the vector-parameter $\underline{c}$ of Figure 37 and the missing alternating side of the hexagon. When the hexagon is non-generic of type 3 this is trivially true and corresponds to the immersion of hyperbolic hexagons inside $\mathcal{X}$. In this section we show that this is not the case for a general right-angled hexagon H inside $\mathcal{X}$. More precisely, let $(H, \mathcal{Y}_{0,\infty})$ be a right-angled hexagon inside $\mathcal{H}$. We can determine $(H, \mathcal{Y}_{0,\infty})$ by the following maximal 12-tuple

$$(H, \mathcal{Y}_{0,\infty}) = \left(\infty, -D, -A, 0, A^2, A, Z_1, \mathrm{Id}, C = \begin{pmatrix} e^{c_1} & 0 \\ 0 & e^{c_2} \end{pmatrix}, Z_2, D, DC^{-1}D\right)$$

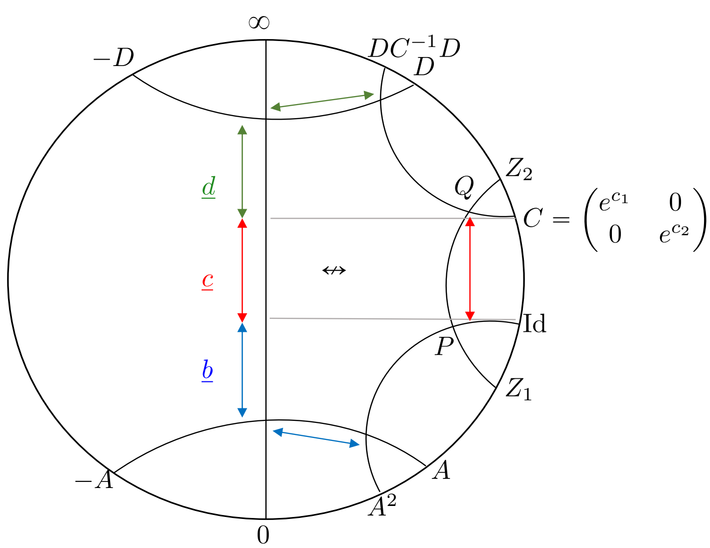


Figure 40: There is no bijective map between the red vectors

Let P, Q be the two intersection points (Figure 40)

$$P = \mathcal{Y}_{A^2,\mathrm{Id}} \cap \mathcal{Y}_{Z_1,Z_2}, \quad Q = \mathcal{Y}_{Z_1,Z_2} \cap \mathcal{Y}_{C,DC^{-1}C}$$

and let F be the following map

$$\begin{aligned} F : \overline{\mathfrak{a}} &\to \overline{\mathfrak{a}} \\ d^{\overline{\mathfrak{a}}^+}(i\mathrm{Id}, C) = (c_1, c_2) &\mapsto d^{\overline{\mathfrak{a}}^+}(P, Q) \end{aligned} \tag{14}$$

Then one can ask if the map F is bijective. In this section we show that this is not the case and we provide a counterexample in the case where $(H, \mathcal{Y}_{0,\infty})$ is contained in a maximal polydisc.

Remark 5.11. The image of the map F in (14) is the distance

$$d^{\overline{\mathfrak{a}}^+}(P,Q) = (\log \mu_1, \log \mu_2)$$

where $\mu_1 > \mu_2$ are the eigenvalues of $R(Z_1, \mathrm{Id}, C, Z_2)$ (see Lemma 2.17). Asking for the existence of such a map F is equivalent to ask for the existence of a map $T \circ F$

$$\begin{aligned} T \circ F : \overline{\mathfrak{a}} &\to \overline{\mathfrak{a}} \\ (c_1, c_2) &\mapsto (\log \lambda_1, \log \lambda_2) \end{aligned} \tag{15}$$

where $\lambda_1 \geqslant \lambda_2$ are the eigenvalues of $R(A^2, \mathrm{Id}, C, DC^{-1}D)$ and T is the bijective map of Lemma 5.7 (composed with the logarithm map). By abuse of notation we will write this map as F and we can express the cross-ratio with respect to arc coordinates. This is made more precise in the following definition.

Definition 5.12. (**Malefic map**) Let $\underline{b}, \underline{d} \in \overline{\mathfrak{a}}$ and $\alpha_1, \alpha_2 \in [0, 2\pi)$. We will call the *malefic map* $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ the map defined as following:

$$\begin{aligned} F_{\underline{b},\underline{d},\alpha_1,\alpha_2} : \overline{\mathfrak{a}} &\to \overline{\mathfrak{a}} \\ (c_1, c_2) &\mapsto (\log \lambda_1, \log \lambda_2) \end{aligned}$$

where $\lambda_1 \geqslant \lambda_2$ are the eigenvalues of the cross-ratio $R(A^2, \mathrm{Id}, C, DC^{-1}D)$ where

$$(H, \mathcal{Y}_{0,\infty}) = (0, A, \mathrm{Id}, C, D, \infty)$$

is the right-angled hexagon with arc coordinates $\mathcal{A}(H, \mathcal{Y}_{0,\infty})$ equal to $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$.

Example 5.13. The malefic map $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ clearly depends on the choice of the parameters $\underline{b}, \underline{d}, \alpha_1, \alpha_2$. It is not hard to show that for $(\alpha_1, \alpha_2) = (0, 0)$ and $(\alpha_1, \alpha_2) = (\pi, \pi)$ respectively we obtain

$$F_{\underline{b},\underline{d},0,0}(c_1, c_2) = \Big(\frac{(e^{c_1+2d_2} - 1)(1 - e^{2b_2+c_1})}{e^{c_1}(1 - e^{2b_2})(e^{2d_2} - 1)}, \frac{(e^{c_2+2d_1} - 1)(1 - e^{2b_1+c_2})}{e^{c_2}(1 - e^{2b_1})(e^{2d_1} - 1)} \Big)$$

$$F_{\underline{b},\underline{d},\pi,\pi}(c_1, c_2) = \Big(\frac{(e^{c_1+2d_1} - 1)(1 - e^{2b_1+c_1})}{e^{c_1}(1 - e^{2b_1})(e^{2d_1} - 1)}, \frac{(e^{c_2+2d_2} - 1)(1 - e^{2b_2+c_2})}{e^{c_2}(1 - e^{2b_2})(e^{2d_2} - 1)} \Big)$$

where $\underline{b} = (b_1, b_2)$ and $\underline{d} = (d_1, d_2)$. Observe that in both cases the constructed hexagon lies inside a maximal polydisc (see Proposition 4.28).

Lemma 5.14. *Let $\underline{b}, \underline{d} \in \overline{\mathfrak{a}}$, $\alpha_1, \alpha_2 \in [0, 2\pi)$ and let $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ be the malefic map. It holds*

$$F_{\underline{b},\underline{d},\alpha_1,\alpha_2}(c_1, c_2) = F_{\underline{b},\underline{d},2\pi-\alpha_1,2\pi-\alpha_2}(c_1, c_2)$$

Proof. This is straightforward by Proposition 4.15 in the generic case and more generally by Theorem 4.26: for angle parameters (α_1, α_2) and $(2\pi - \alpha_1, 2\pi - \alpha_2)$ we obtain two isometric hexagons. $\square$

We can extend the malefic map $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ of Definition 5.12 to the set $\big(\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}\big)\backslash\{(0,0)\}$, that is we allow the case where (c_1, c_2) is such that $c_1 < c_2$ or $c_i = 0$ for an $i \in \{1, 2\}$. The image $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}(c_1, c_2)$ for a point $(c_1, c_2) \in \overline{\mathfrak{a}}$ is obtained by computing the cross-ratio $R(A^2, \mathrm{Id}, C, DC^{-1}D)$. In Theorem 4.26 we have provided an explicit way to compute a hexagon $(H, \mathcal{Y}_{0,\infty}) \in \mathcal{H}$ from arc coordinates $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$. More precisely we have shown how to compute positive definite symmetric matrices A, C, D where $(H, \mathcal{Y}_{0,\infty}) = (0, A, \mathrm{Id}, C, D, \infty)$. The explicit formulas appear in Proposition 4.15 for the generic case and are suitably adapted to the non-generic case in Proposition 4.20, 4.21 and 4.22. We extend these formulas to the case where (c_1, c_2) is such that $c_1 < c_2$ or $c_i = 0$ for an $i \in \{1, 2\}$.

Proposition 5.15. *Let $\underline{b}, \underline{d} \in \overline{\mathfrak{a}}$, $\alpha_1, \alpha_2 \in [0, 2\pi)$ and let $\widetilde{F}$ denote the malefic map extended to $\big(\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}\big)\backslash\{(0,0)\}$. Then*

$$\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}(c_1, c_2) = \widetilde{F}_{\underline{b},\underline{d},\pi-\alpha_1,\pi-\alpha_2}(c_2, c_1)$$

Furthermore, if (c_1, c_2) is a point lying on one of the semi-axis of $\big(\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}\big)\backslash\{(0,0)\}$ then $\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}(c_1, c_2)$ is also lying on a semi-axis.

Proof. Let us understand the geometrical meaning of $\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}(c_1, c_2)$ for a point (c_1, c_2) with $c_1 < c_2$. If in the parametrization of Proposition 4.15 we consider the set $\mathfrak{a}^- = \{0 < x_1 < x_2\}$ instead of $\mathfrak{a} = \{x_1 > x_2 > 0\}$ we are choosing to diagonalize the matrix C with an increasing order of the eigenvalues. In the geometric interpretation of angle parameters illustrated in Section 2.10 the angle α denotes the angle from the semi-axis $\{(0, y)|\ y > 1\} \in \mathbb{H}^2$. By picking the set $\mathfrak{a}^-$ we are considering the angle $\alpha + \pi$ when $\alpha \in [0, \pi)$ and the angle $\alpha - \pi$ when $\alpha \in [\pi, 2\pi)$ (Figure 41).

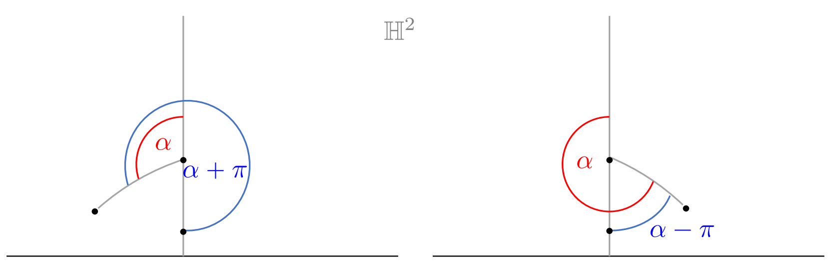


Figure 41: Geometric interpretation when considering the Weyl chamber $\mathfrak{a}^-$

From the equivalent relations of the angle parameters we know

$$\alpha + \pi \sim 2\pi - (\alpha + \pi) = \pi - \alpha \text{ and } \alpha - \pi \sim 2\pi - (\alpha - \pi) = \pi - \alpha$$

so that

$$\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}(c_1, c_2) = \widetilde{F}_{\underline{b},\underline{d},\pi-\alpha_1,\pi-\alpha_2}(c_2, c_1)$$

We should think at the extended map $\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ as a way to construct right-angled hexagons in a continuous way by moving the point C. The polygonal chain of the hexagon is transformed as shown in Figure 42.

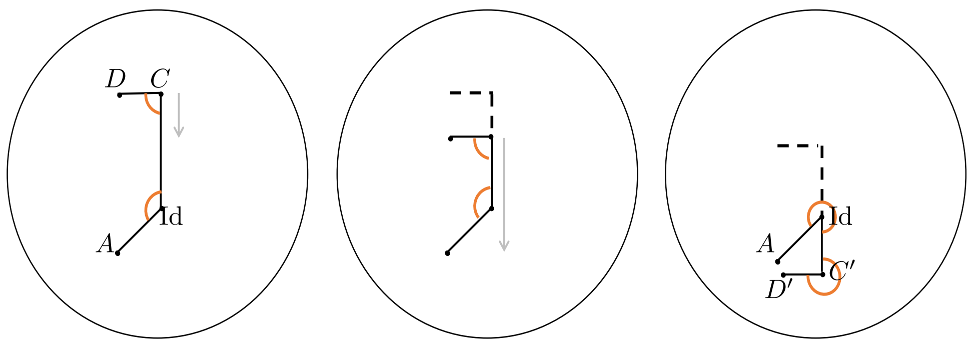


Figure 42: Continuous transformation of the polygonal chain when going from $\mathfrak{a}$ to $\mathfrak{a}^-$

Let us now show that $\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ preserves semi-axes. Let (c_1, c_2) be such that $c_1 = 0$. This means that

$$C = \begin{pmatrix} e^0 & 0 \\ 0 & e^{c_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \ \lambda > 0$$

so that C and Id are not transverse. Furthermore, there exists a $g \in \mathrm{Sp}(4, \mathbb{R})$ such that

$$g \cdot (A^2, \mathrm{Id}, C, DC^{-1}D) = (0, \mathrm{Id}, M, \infty)$$

where M is positive definite and such that Id and M are not transverse. This means

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}, \ \mu > 0$$

and we know (see Lemma 2.8) that $R(0, \mathrm{Id}, M, \infty) = M$ so that

$$\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}(0, c_2) = (\log(1), \log(\mu)) = (0, y)$$

for some $y > 0$. ☐

Corollary 5.16. *Let $\underline{b}, \underline{d} \in \mathfrak{a}^2$ and let F_0, F_π be the maps*

$$F_0 = F_{\underline{b},\underline{d},0,0}, \quad F_\pi = F_{\underline{b},\underline{d},\pi,\pi}$$

where $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ denotes the malefic map. Then F_0 is not surjective and F_π is not injective.

Proof. Consider the extended malefic maps $\widetilde{F}_0 = \widetilde{F}_{\underline{b},\underline{d},0,0}$ and $\widetilde{F}_\pi = \widetilde{F}_{\underline{b},\underline{d},\pi,\pi}$. The map $\widetilde{F}_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ is continuous and from example 5.13 it is easy to see that $\widetilde{F}_0(x,x) \neq (X,X)$ (Figure 43 below).

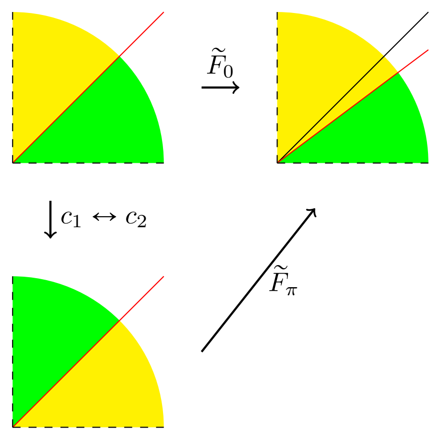


Figure 43: $\widetilde{F}_0(c_1,c_2) = \widetilde{F}_\pi(c_2,c_1)$

We deduce that when restricting to $\mathfrak{a} = \{x_1 > x_2 > 0\}$ (i.e. considering the malefic map F) the map F_0 is not surjective and the map F_π is not injective. This is illustrated in Figures 44 and 45 below. The program to generate these figures can be found in the github repository `https://github.com/martamagnani/Arc-coord/blob/main/Lemma_is_false.py`. □

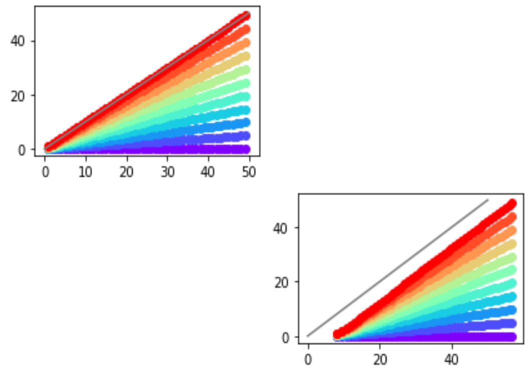


Figure 44: The bottom-right corner shows the image of the map F_0 (not surjective) when $\underline{b} = (40, 0.01)$ and $\underline{d} = (35, 0.01)$

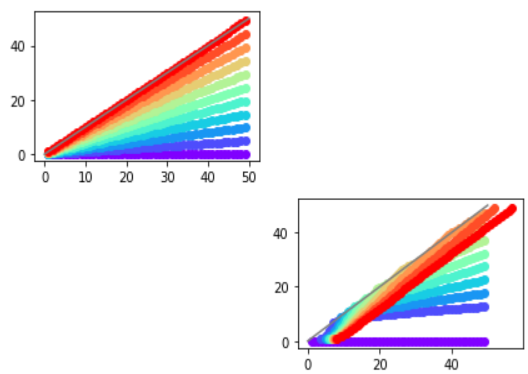


Figure 45: The bottom-right corner shows the image of the map F_π (not injective) when $\underline{b} = (40, 0.01)$ and $\underline{d} = (35, 0.01)$

Remark 5.17. (Genericity is well defined on an *ordered* 6-tuple) In Corollary 5.10 we have seen how the genericity of the hexagon

$$(H, \mathcal{Y}_{0,\infty}) = (0, A, \mathrm{Id}, C, D, \infty)$$

induces the genericity of the quadruples $(-A, 0, A^2, A)$ and $(-D, 0, C, D)$ respecticely. This is strictly related to the existence of a bijective map f as seen in Proposition 5.8, so that the vectors $\underline{b}$ and $\underline{d}$ are in bijection with the length of two alternating sides (Figure 37). In the discussion that followed we showed that we can not do the same with the vector $\underline{c}$ as there is no such bijective map. In particular we have seen how the map depends on the angle parameters of the hexagon and we have provided counterexamples where this map is in turn not injective and not surjective. In these counterexamples it is clear that the extended map $\widetilde{F}$ in Proposition 5.15 does not preserve the diagonal $\mathfrak{d} = \{x_1 = x_2\}$. By continuity of $\widetilde{F}$ we deduce that in general

$$(c_1, c_2) \in \mathfrak{a} \nRightarrow F(c_1, c_2) \in \mathfrak{a}$$

Equivalently, the genericity of the hexagon $(H, \mathcal{Y}_{0,\infty})$ does not imply the genericity of the quadruple $(Z_1, \mathrm{Id}, C, Z_2)$ where Z_1, Z_2 are uniquely determined by requiring (Figure 40)

$$\mathcal{Y}_{A^2,\mathrm{Id}} \perp \mathcal{Y}_{Z_1,Z_2} \perp \mathcal{Y}_{C,DC^{-1}D}$$

The parameters of Proposition 4.15 strongly depends on the order of the 6-tuple defining the hexagon or equivalently on the choice of a tube $\mathcal{Y}_{0,\infty}$.

6 Reflections in the Siegel space

In this chapter we study reflections in the Siegel space. We first recall properties of reflections in the hyperbolic plane $\mathbb{H}^2$ and we then generalize the results for the Siegel space $\mathcal{X}$. We define the notion of a reflection set associated to the side of a hexagon, which will be used in the next chapter to construct maximal representations.

6.1 Reflections in $\mathbb{H}^2$

Let $\mathbb{H}^2$ be the upper-half space model of the hyperbolic plane

$$\mathbb{H}^2 = \{x + iy |\ x, y \in \mathbb{R},\ y > 0\}$$

A reflection in $\mathbb{H}^2$ can be defined as a non-trivial isometry fixing an infinite geodesic $\gamma \in \mathbb{H}^2$. In this section we propose an equivalent definition of reflection that will be generalized to define reflections in the Siegel space. Let us start with the following

Definition 6.1. Let $\mathrm{SL}^-(2, \mathbb{R})$ be the set

$$\mathrm{SL}^-(2, \mathbb{R}) = \{M \in \mathrm{GL}(2, \mathbb{R}) |\ \det M = -1\}$$

The union $\mathrm{SL}(2, \mathbb{R}) \cup \mathrm{SL}^-(2, \mathbb{R})$ forms a group that we denote $\mathrm{SL}^{\pm}(2, \mathbb{R})$.

The action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{H}^2$ by Möbius transformations is not well defined for $M \notin \mathrm{PSL}(2, \mathbb{R})$ since the resulting point may not lie in $\mathbb{H}^2$. To define the action of a matrix $M \in \mathrm{SL}^-(2, \mathbb{R})$ on $\mathbb{H}^2$ we denote the extended hyperbolic plane by

$$\mathbb{H}^2_{\pm} = \{x \pm iy |\ x, y \in \mathbb{R},\ y > 0\}$$

so that

$$\mathbb{H}^2 = \mathbb{H}^2_{\pm}/_{\sim}$$

where $x+iy \sim x-iy$. The matrix M acts on $\mathbb{H}^2$ through Möbius transformations in the following way

$$M \cdot z := [M \cdot z] \in \mathbb{H}^2_{\pm}/_{\sim}$$

Definition 6.2. A reflection in $\mathbb{H}^2$ is an involution of $\mathrm{SL}^-(2, \mathbb{R})$.

Remark 6.3. We have seen that $\mathrm{SL}^-(2, \mathbb{R})$ acts on $\mathbb{H}^2 = \mathbb{H}^2_{\pm}/_{\sim}$ by Möbius transformations. Given $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}^-(2, \mathbb{R})$ and $z \in \mathbb{H}^2$ it holds

$$R \cdot z = (-R) \cdot z$$

It makes sense to think at R inside the group $\mathrm{SL}^{\pm}(2, \mathbb{R})$ and to consider the group

$$\mathrm{PSL}^{\pm}(2, \mathbb{R}) = \mathrm{SL}^{\pm}(2, \mathbb{R})/_{\{\pm \mathrm{Id}\}}$$

When studying reflections we will always assume R and $-R$ to be identified in $\mathrm{PSL}^{\pm}(2, \mathbb{R})$.

Lemma 6.4. *All reflections of $\mathbb{H}^2$ are conjugated by an element of* $\mathrm{SL}(2,\mathbb{R})$.

Proof. The proof is given in the more general case in Lemma 6.12 where it is shown for the Siegel space $\mathcal{X}$ and the group $\mathrm{Sp}(2n,\mathbb{R})$. The proof for $\mathbb{H}^2$ is the case $n = 1$. □

Definition 6.5. We will call the *standard reflection* in $\mathbb{H}^2$ the map

$$r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proposition 6.6. *Let R be a reflection in $\mathbb{H}^2$. Then R fixes exactly two boundary points $p, q \in \partial\mathbb{H}^2$. Moreover, R fixes the infinite geodesic $\gamma_{p,q}$ that has p, q as endpoints.*

Proof. The proof is given in the general case in Proposition 6.15. □

Proposition 6.7. *Given $p, q \in \partial\mathbb{H}^2$, there is a unique reflection R fixing both p and q. The map R is an isometry sending any boundary point x to the unique boundary point $R(x)$ such that $\gamma_{p,q} \perp \gamma_{x,R(x)}$*

Proof. The proof is given in the general case in Proposition 6.18. □

Proposition 6.8. *Let (q_1, q_2, q_3, q_4) be a positive quadruple in $\partial\mathbb{H}^2$ and let R be the reflection fixing two boundary points $p_1, p_2 \in \partial\mathbb{H}^2$. Suppose that:*

$$(p_2, q_1, q_2, q_3, q_4, p_1) \text{ is positive (possibly } p_2 = q_1 \text{ or } p_1 = q_4)$$

then $(p_1, R(q_4), R(q_3), R(q_2), R(q_1), p_2)$ is positive.

Proof. The proof is given in the general case in Proposition 6.19 □

6.2 Reflections in $\mathcal{X}$

Taking inspiration from the previous section, we want to define reflections in the Siegel space $\mathcal{X}$. We start by giving the following

Definition 6.9. Let $\omega(\cdot,\cdot)$ be the symplectic form represented, with respect to the standard basis, by the matrix

$$J_n = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

A matrix $M \in \mathrm{GL}(2n,\mathbb{R})$ is *antisymplectic* if

$$M^T J_n M = -J_n$$

The set of antisymplectic matrices will be denoted by $\mathrm{Sp}^-(2n,\mathbb{R})$. More precisely $\mathrm{Sp}^-(2n,\mathbb{R})$ is the set

$$\mathrm{Sp}^-(2n,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A^T C, B^T D \text{ symmetric, and } A^T D - C^T B = -\mathrm{Id}_n \right\}$$

The union of symplectic and antisymplectic matrices forms a group that will be denoted by $\mathrm{Sp}^{\pm}(2n,\mathbb{R})$.

Recall that $\mathrm{Sp}(2n,\mathbb{R})$ acts on $\mathcal{X}$ by fractional linear transformations:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}$$

Observe that this action is not well defined for R in $\mathrm{Sp}^{-}(2n,\mathbb{R})$ since the resulting point may not lie in $\mathcal{X}$. To define the action of an antisymplectic matrix on the Siegel space we denote by $\mathcal{X}^{\pm}$ the extended Siegel space:

$$\mathcal{X}^{\pm} = \{X \pm iY |\ X \in \mathrm{Sym}(n,\mathbb{R}), Y \in \mathrm{Sym}^{+}(n,\mathbb{R})\}$$

Then

$$\mathcal{X} = \mathcal{X}^{\pm}/_{\sim}$$

where $X + iY \sim X - iY$. For R antisymplectic and $Z \in \mathcal{X}$ we define the action

$$R \cdot Z := [R \cdot Z] \in \mathcal{X}^{\pm}/_{\sim}$$

Recall that the Borel embedding model $\mathbb{X}$ of the Siegel space is given by

$$\mathbb{X} = \{l \in \mathcal{L}(\mathbb{C}^{2n}) |\ \ i\omega(\sigma(\cdot),\cdot)_{\mathbb{C}}|_{l\times l} \text{ is positive definite}\}$$

where $\sigma : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ denotes complex conjugation. Recall also that an $\mathrm{Sp}(2n,\mathbb{R})$-equivariant identification $\mathcal{X} \mapsto \mathbb{X}$ is induced by the affine chart

$$\iota : \mathrm{Sym}(n,\mathbb{C}) \to \mathcal{L}(\mathbb{C}^{2n})$$

that associates to a symmetric matrix Z the linear subspace of $\mathbb{C}^{2n}$ spanned by the columns of the matrix $\begin{pmatrix} Z \\ \mathrm{Id}_n \end{pmatrix}$ (see Section 2.1). In this model the extented Siegel space is given by the set

$$\mathbb{X}^{\pm} = \{l \in \mathcal{L}(\mathbb{C}^{2n}) |\ \ i\omega(\sigma(\cdot),\cdot)_{\mathbb{C}}|_{l\times l} \text{ is positive or negative definite}\}$$

and

$$\mathbb{X} = \mathbb{X}^{\pm}/_{\sim} \tag{16}$$

where a Lagrangian $l \in \mathbb{X}$

$$l = \left\langle \begin{pmatrix} x_{11}+iy_{11} \\ \vdots \\ x_{n1}+iy_{n1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., \begin{pmatrix} x_{n1}+iy_{n1} \\ \vdots \\ x_{nn}+iy_{nn} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

is equivalent to the Lagrangian $l' \in \mathbb{X}^-$ where

$$l' = \left\langle \begin{pmatrix} x_{11} - iy_{11} \\ \vdots \\ x_{n1} - iy_{n1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x_{n1} - iy_{n1} \\ \vdots \\ x_{nn} - iy_{nn} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

We can now give the following

Definition 6.10. A *reflection* R in $\mathcal{X}$ is an antisymplectic involution of $\mathcal{X}$.

Remark 6.11. We have seen that $\mathrm{Sp}^-(2n, \mathbb{R})$ acts on $\mathcal{X} = \mathcal{X}/_{\pm}$ by fractional linear transformations. Given $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}^-(2n, \mathbb{R})$ and $Z \in \mathcal{X}$ it holds

$$R \cdot Z = (-R) \cdot Z$$

It makes sense to think at R inside the group $\mathrm{Sp}^{\pm}(2n, \mathbb{R})$ and to consider the group

$$\mathrm{PSp}^{\pm}(2n, \mathbb{R}) = \mathrm{Sp}^{\pm}(2n, \mathbb{R})/_{\{\pm \mathrm{Id}\}}$$

When studying reflections we will always assume R and $-R$ to be identified in $\mathrm{PSp}^{\pm}(2n, \mathbb{R})$.

Lemma 6.12. *All reflections of $\mathcal{X}$ are conjugated by an element of* $\mathrm{Sp}(2n, \mathbb{R})$.

Proof. Let R be a reflection of $\mathcal{X}$. Since R is an involution we know that its eigenvalues are given by the set $\{\pm 1\}$. Recall that we denote by $\mathcal{L}(\mathbb{R}^{2n})^{(k)}$ the set of k-tuples of real pairwise transverse Lagrangians. Given the R-eigenspaces E_1, E_{-1}, we want to show that $E_1, E_{-1} \in \mathcal{L}(\mathbb{R}^{2n})^{(2)}$. For $u, v \in E_1$ it holds

$$\omega(u, v) = \omega(R(u), R(v)) = -\omega(u, v)$$

where the first equality holds since $u, v \in E_1$ and the second one since R is antisymplectic. It follows that $\omega(u, v) = 0$ for any $u, v \in E_1$, that is E_1 is a Lagrangian subspace. Similarly one can show that E_{-1} is also a Lagrangian subspace. Since a real matrix with real eigenvalues has real eigenvectors, we conclude that $E_1, E_{-1} \in \mathcal{L}(\mathbb{R}^{2n})^{(2)}$. Result follows as $\mathrm{Sp}(2n, \mathbb{R})$ acts transitively on pairs of transverse Lagrangians. □

Definition 6.13. We will call the *standard reflection* the map R_{st} where

$$R_{st} = \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

Lemma 6.14. *Let R be a reflection inside $\mathcal{X}$. Then for any X, Z in $\mathcal{X}$ it holds*

$$d^{\overline{\mathfrak{a}}^+}(R(X), R(Z)) = d^{\overline{\mathfrak{a}}^+}(X, Z)$$

Proof. Recall that we have defined $d^{\overline{\mathfrak{a}}^+}$ in Definition 2.9 as the projection onto the Weyl chamber $\overline{\mathfrak{a}}^+$:

$$\mathcal{X}^2 \to \overline{\mathfrak{a}}^+$$
$$(X, Z) \mapsto (\log(\lambda_1), ..., \log(\lambda_n))$$

where $\lambda_i = \frac{1+\sqrt{r_i}}{1-\sqrt{r_i}}$ and $1 > r_1 \geqslant ... \geqslant r_n \geqslant 0$ are the eigenvalues of the cross-ratio $Cr(X, \overline{Z}, Z, \overline{X})$ (Definition 2.7). For a matrix $R \in \mathrm{GL}(2n, \mathbb{R})$, it is not hard to show that the cross-ratio

$$Cr\Big(R(X), \overline{R(Z)}, R(Z), \overline{R(X)}\Big)$$

has eigenvalues $(\lambda_1, ...\lambda_n)$: eigenvalues are stable under conjugation and it holds

$$Cr\Big(R(X), \overline{R(Z)}, R(Z), \overline{R(X)}\Big) \overset{(1)}{=} Cr\Big(R(X), R(\overline{Z}), R(Z), R(\overline{X})\Big) \overset{(2)}{=} RCr\Big(X, \overline{Z}, Z, \overline{X}\Big)R^{-1}$$

where equality (1) follows directly from the properties of complex conjugation and equality (2) is trivial once we express the cross-ratio as in Definition 2.7. Let us consider $\mathbb{X} = \mathbb{X}^{\pm}/_{\sim}$ the Borel embedding model of the Siegel space described in (16). We are left to show that for a reflection R and a point $l \in \mathbb{X}$ the point $R(l)$ is inside $\mathbb{X}$. For $l \in \mathcal{L}(\mathbb{C}^{2n})$ and $v, w \in l$ it holds

$$i\omega(\overline{R(v)}, Rw) = i\omega(R(\overline{v}), Rw) = -i\omega(\overline{v}, w)$$

Result follows. □

Proposition 6.15. *Let $\mathcal{X}$ be the symmetric space associated to* $\mathrm{Sp}(4, \mathbb{R})$ *and let R be a reflection of $\mathcal{X}$. Then the set*

$$Fix_{\mathcal{L}(\mathbb{R}^4)}(R) = \{l \in \mathcal{L}(\mathbb{R}^4) | \ R(l) = l\}$$

is given by the R-eigenspaces E_1, E_{-1} together with an S^1-isomorphic family $\mathcal{F}$ of pairwise transverse Lagrangians each of which is not transverse to neither E_1 nor E_{-1}. Moreover, R fixes the tube $\mathcal{Y}_{E_1,E_{-1}}$ and fixes a flat inside any Y_{l_i,l_j} where $l_i, l_j \in \mathcal{F}$.

Proof. Since any R is conjugated to the standard reflection through an element of $\mathrm{Sp}(4, \mathbb{R})$, let us prove the proposition for $R_{st} = \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$. Let (e_1, e_2, e_3, e_4) denote the standard basis of $\mathbb{R}^4$. We have

$$E_1 = \langle e_3, e_4 \rangle$$

$$E_{-1} = \langle e_1, e_2 \rangle$$

where $E_1, E_{-1} \in \mathcal{L}(\mathbb{R}^4)^{(2)}$ (see Lemma 6.12). For any $u \in \mathbb{P}(E_1)$ there exists a unique $v \in \mathbb{P}(E_{-1})$ such that $\omega(u,v) = 0$. For any $w_1, w_2 \in \langle u, v \rangle$ we have $\omega(w_1, w_2) = 0$ so that $l = \langle u, v \rangle \in \mathcal{L}(\mathbb{R}^4)$ and $R_{st}(l) = l$. We obtain the set $\mathcal{F} \subset Fix_{\mathcal{L}(\mathbb{R}^4)}(R_{st})$:

$$\mathcal{F} = \left\{ l = \langle u, v \rangle \right\} \cong \mathbb{P}(E_1) \cong \mathbb{P}(E_{-1}) \cong S^1$$

where $u \in \mathbb{P}(E_1)$ and v is the unique element of $\mathbb{P}(E_{-1})$ such that $\omega(u,v) = 0$. We want to show that for any $l \in \mathcal{F}$ it holds

$$E_1 \not\pitchfork l \not\pitchfork E_{-1}$$

and that for any l_1, l_2 inside $\mathcal{F}$ we have

$$l_1 \pitchfork l_2$$

Let us fix $\alpha \in \mathbb{R}$ and consider $u \in \mathbb{P}(E_1),\ u \neq e_4$ to be the vector

$$u = e_3 + \alpha e_4 \in \mathbb{P}(E_1)$$

Then the corresponding $v \in \mathbb{P}(E_{-1}),\ v \neq e_1$ such that $\omega(u,v) = 0$ is given by

$$v = -\alpha e_1 + e_2$$

Let

$$l = \langle u, v \rangle = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ \alpha \end{pmatrix}, \begin{pmatrix} -\alpha \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \in \mathcal{F} \backslash \{ \langle e_4, e_1 \rangle \}$$

then l intersects E_1 in the line $\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ \alpha \end{pmatrix} \right\rangle \subset E_1$ and intersects E_{-1} in the line $\left\langle \begin{pmatrix} -\alpha \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subset E_{-1}$. We are left with the case $l = \langle e_4, e_1 \rangle$ which is clearly not transverse to E_1 nor E_{-1}. We have showed $E_1 \not\pitchfork l \not\pitchfork E_{-1}$ for every $l \in \mathcal{F}$. Let us now consider l_1, l_2 inside $\mathcal{F}$. Similarly as before let $l_1, l_2 \in \mathcal{F} \backslash \{ \langle e_4, e_1 \rangle \}$ where

$$l_1 = \langle u_1, v_1 \rangle = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ \alpha \end{pmatrix}, \begin{pmatrix} -\alpha \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$l_2 = \langle u_2, v_2 \rangle = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ \beta \end{pmatrix}, \begin{pmatrix} -\beta \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

where $\beta \in \mathbb{R},\ \beta \neq \alpha$. It is easy to see that $l_1 \pitchfork l_2$. It is also trivial to show that transversality holds in the case $l_1 = \langle e_4, e_1 \rangle$.
The reflection R_{st} is fixing the tube $\mathcal{Y}_{E_1,E_{-1}}$: to see this recall that the affine chart ι in Section 2.1 identifies $\mathcal{L}(\mathbb{C}^4)$ with $\mathrm{Sym}(2,\mathbb{C})$. In this chart the Lagrangian $E_{-1} = \langle e_1, e_2 \rangle$ is the point at infinity in the Shilov boundary $\mathcal{L}(\mathbb{R}^4)$ of $\mathcal{X}$ and the expression for the tube $\mathcal{Y}_{E_1,E_{-1}}$ is given by the standard tube

$$\mathcal{Y}_{0,\infty} = \{iY \mid Y \in \mathrm{Sym}^+(2,\mathbb{R})\}$$

For any $iY \in \mathcal{Y}_{0,\infty}$

$$R_{st}(iY) = -iY = iY \text{ in } \mathcal{X}^{\pm}/_{\sim} = \mathcal{X}$$

Observe that with respect to the tube $\mathcal{Y}_{E_1,E_{-1}}$ the reflection R_{st} is the analogue of a reflection in $\mathbb{H}^2$: it is sending any boundary point $X \in \mathrm{Sym}(2,\mathbb{R})$ (transverse to both E_1 and E_{-1}) to the unique $R(X) = -X$ such that $\mathcal{Y}_{0,\infty} \perp \mathcal{Y}_{X,R(X)}$. Let us now consider $l_1, l_2 \in \mathcal{F}$ where

$$l_1 = \langle e_1, e_4 \rangle, \quad l_2 = \langle e_2, e_3 \rangle$$

Let us change the standard basis $\mathcal{B} = (e_1, e_2, e_3, e_4)$ with the basis $\mathcal{B}'$ given by $\mathcal{B}' = (e_3, e_2, e_1, e_4)$. Writing vectors of $\mathbb{C}^4$ in this new basis means considering the chart $T \circ \iota : \mathrm{Sym}(2,\mathbb{C}) \to \mathcal{L}(\mathbb{C}^4)$ where $T(\mathcal{B}) = \mathcal{B}'$. In particular in this chart the tube $\mathcal{Y}_{l_1,l_2}$ has the standard form

$$\mathcal{Y}_{l_1,l_2} = \{iY \mid Y \in \mathrm{Sym}^+(2,\mathbb{R})\}$$

and the reflection R_{st} written in basis $\mathcal{B}'$ is given by

$$\widetilde{R} = R_{\mathcal{B}'} = \begin{pmatrix} -r & 0 \\ 0 & r \end{pmatrix}$$

where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It holds

$$\widetilde{R}(iY) = -iY^r = iY^r \text{ in } \mathcal{X}^{\pm}/_{\sim} = \mathcal{X}$$

where by Y^r we denote the point in the $\mathbb{H}^2$-component of the standard tube which is obtained by reflecting Y across the standard vertical geodesic of the hyperbolic plane (see Section 2.33). The reflection $\widetilde{R}$ fixes the flat

$$\mathbb{D} = i \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \cong \mathbb{R} \times \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \subset \mathcal{Y}_{0,\infty}$$

So the reflection $\widetilde{R}$ is reflecting across a geodesic γ in the $\mathbb{H}^2$-component of the tube $\mathcal{Y}_{l_1,l_2}$ and is therefore fixing the flat $\mathbb{R} \times \gamma$ inside the tube. Since $\mathrm{Sp}(4,\mathbb{R})$ acts transitively on the space of transverse Lagrangians we deduce that the same holds for any $l_1, l_2 \in \mathcal{F}$. □

Corollary 6.16. *There is no maximal triple in the S^1-family $\mathcal{F}$ of Proposition 6.15.*

Proof. Let (e_1, e_2, e_3, e_4) be the standard basis in $\mathbb{R}^4$ and as usual let us denote by $l_\infty, 0$ and Id the Lagrangians

$$l_\infty = \langle e_1, e_2\rangle,\ 0 = \langle e_3, e_4\rangle,\ \mathrm{Id} = \langle e_1 + e_3, e_2 + e_4\rangle$$

Since any reflection is conjugated to the standard one, we prove the result for the standard reflection $R_{st} = \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$. In the proof of Proposition 6.15) we have seen that for R_{st} we have:

$$E_1 = 0,\ E_{-1} = l_\infty$$

Each Lagrangian of $\mathcal{F}$ intersects l_∞ and 0 in one line and $\mathcal{F} \cong \mathbb{P}(E_1) \cong \mathbb{P}(E_{-1})$. Let l_1, l_2, l_3 be three points in $\mathcal{F}$. Up to $\mathrm{GL}(2, \mathbb{R}) \cong \mathrm{Stab}(E_1, E_{-1})$-action we can choose the three vectors of $\mathbb{P}(E_{-1})$ to be e_1, e_2 and $e_1 + e_2$ respectively ($\mathrm{GL}(2, \mathbb{R})$ acts three-transitively on the lines of $\mathbb{R}^2$) and we obtain

$$l_1 = \langle e_1, e_4\rangle,\ l_2 = \langle e_2, e_3\rangle,\ l_3 = \langle e_1 + e_2, e_3 - e_4\rangle$$

Let $g \in \mathrm{Sp}(4, \mathbb{R})$ be such that $g(l_1, l_2) = (l_\infty, 0)$. Then

$$g = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and let us choose for simplicity $A = \mathrm{Id}$. Then $g(l_3) = \langle e_2 + e_3, e_1 + e_4\rangle$ which corresponds to the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the identification of Section 2.1. The triple $(l_\infty, 0, M)$ is not maximal as its Maslov index is zero (Section 2.2 for the definition of Maslov index). $\square$

In Proposition 6.15 we have seen that for a given reflection $R \in \mathrm{PSp}(4, \mathbb{R})^-$ there is a different geometrical behaviour when considering what R is doing with respect to the tube $\mathcal{Y}_{E_1, E_{-1}}$ or to the tube $\mathcal{Y}_{l_1, l_2}$, where l_1, l_2 are two arbitrary points inside $\mathcal{F}$.

Definition 6.17. The reflection

$$R_{ex} = \begin{pmatrix} -r & 0 \\ 0 & r \end{pmatrix},\ r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is called the *exotic reflection*.

As we have seen in Lemma 6.12 the exotic reflection is conjugated to the standard one but its different geometrical behaviour on the tube $\mathcal{Y}_{l_1, l_2}$ (explained in the proof of Proposition 6.15) is preserved as soon as we allow conjugation only inside $\mathrm{Stab}(\mathcal{Y}_{l_1, l_2})$.

6.3 Reflection set associated to the side of a hexagon

In this section we introduce the notion of the reflection set associated to the side of a hexagon. We start by describing a set of reflections $\mathcal{R}(P, X, Y, Q)$ associated to a maximal quadruple (P, X, Y, Q).

Proposition 6.18. *Let $\mathcal{X}$ be the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. Let (P, X, Y, Q) be a maximal quadruple in $\mathcal{X}$ and let R be a reflection such that*

$$\begin{cases} R(P) = P \text{ and } R(Q) = Q \\ \mathcal{Y}_{X,R(X)} \perp \mathcal{Y}_{P,Q} \perp \mathcal{Y}_{Y,R(Y)} \end{cases} \tag{17}$$

Then the reflections satisfying (17) are given by a set $\mathcal{R}(P, X, Y, Q)$ where

$$\mathcal{R}(P, X, Y, Q) \subset \mathrm{Stab}_{\mathrm{PSp}^{\pm}(4,\mathbb{R})}(P, p_{P,Q}(X), p_{P,Q}(Y), Q)$$

Let g be an isometry such that $g \cdot (P, X, Y, Q) = (0, \mathrm{Id}, Y', \infty)$ for Y' diagonal. It holds:

(i) If (P, X, Y, Q) is generic then $\mathcal{R}(P, X, Y, Q) = \{g^{-1}R_{st}g, g^{-1}R_{ex}g\}$

(ii) If (P, X, Y, Q) is non-generic then $\mathcal{R}(P, X, Y, Q) = g^{-1}\mathcal{K}g$

where

$$\mathcal{K} = \left\{ \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix}, \ K \in \mathrm{PO}(2), \ K^2 = \mathrm{Id} \right\}$$

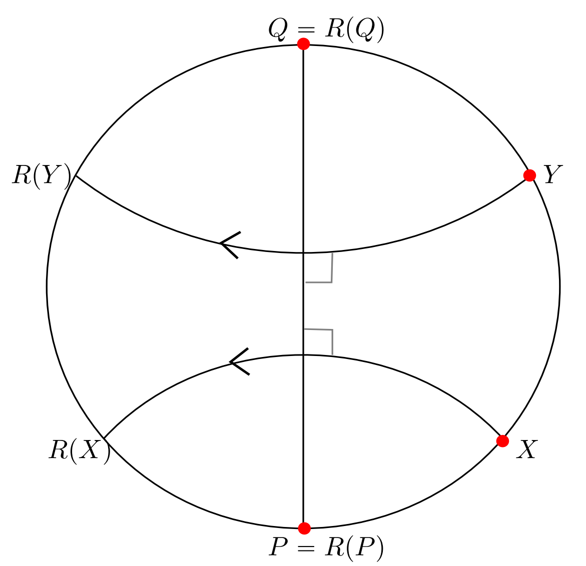


Figure 46: The number of reflections satisfying (17) depends on the genericity of the red quadruple

Proof. Let g be an isometry such that $g \cdot (P, X, Y, Q) = (0, \mathrm{Id}, Y', \infty)$ for Y' diagonal. Let R be a reflection such that

$$\begin{cases} R(0) = 0 \text{ and } R(\infty) = \infty \\ \mathcal{Y}_{\mathrm{Id}, R(\mathrm{Id})} \perp \mathcal{Y}_{0,\infty} \perp \mathcal{Y}_{Y', R(Y')} \end{cases} \tag{18}$$

Then R stabilizes the standard tube $\mathcal{Y}_{0,\infty}$ and belongs therefore to the group $\mathrm{Stab}_{\mathrm{PSp}^{\pm}(4,\mathbb{R})}(0, \infty)$. We know $\mathcal{Y}_{0,\infty} \perp \mathcal{Y}_{\mathrm{Id},-\mathrm{Id}}$ (Lemma 2.16) so that
$R(\mathrm{Id}) = -\mathrm{Id}$. The map R stabilizes the tube $\mathcal{Y}_{\mathrm{Id},-\mathrm{Id}}$ and is therefore stabilizing the intersection point $\mathcal{Y}_{0,\infty} \cap \mathcal{Y}_{\mathrm{Id},-\mathrm{Id}} = p_{0,\infty}(\mathrm{Id})$. The same reasoning holds for the point $\mathcal{Y}_{0,\infty} \cap \mathcal{Y}_{Y',R(Y')} = p_{0,\infty}(Y')$. We deduce

$$\mathcal{R}(0, \mathrm{Id}, Y', \infty) \subset \mathrm{Stab}_{\mathrm{PSp}^{\pm}(4,\mathbb{R})}(0, p_{0,\infty}(\mathrm{Id}), p_{0,\infty}(Y'), \infty)$$

We obtain two possibilities for $\mathcal{R}(0, \mathrm{Id}, Y', \infty)$:
(i) If $(0, \mathrm{Id}, Y', \infty)$ is generic then Y' is of the form $Y' = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$ where $y_1 \neq y_2$. The only two elements of $\mathcal{R}(0, \mathrm{Id}, Y', \infty)$ are the standard and the exotic reflections:

$$\mathcal{R}(0, \mathrm{Id}, Y', \infty) = \left\{ R_{st} = \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix},\ R_{ex} = \begin{pmatrix} -r & 0 \\ 0 & r \end{pmatrix} \right\}$$

where $r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathrm{Stab}_{\mathrm{PSp}^{\pm}(4,\mathbb{R})}(0, p_{0,\infty}(\mathrm{Id}), p_{0,\infty}(Y'), \infty)$ is the group

$$\mathrm{Stab}_{\mathrm{PSp}^{\pm}(4,\mathbb{R})}(0, p_{0,\infty}(\mathrm{Id}), p_{0,\infty}(Y'), \infty) = \left\{ \mathrm{Id}, \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, R_{st}, R_{ex} \right\}$$

The reflection R_{st} fixes the tube $\mathcal{Y}_{0,\infty}$ (as $E_1^{R_{st}} = 0, E_{-1}^{R_{st}} = l_\infty$) whereas R_{ex} fixes a flat inside $\mathcal{Y}_{0,\infty}$ (see Proposition 6.15).
(ii) If $(0, \mathrm{Id}, Y', \infty)$ is non-generic then $Y' = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} = y \cdot \mathrm{Id}$. The group $\mathrm{Stab}_{\mathrm{PSp}^{\pm}(4,\mathbb{R})}(0, p_{0,\infty}(\mathrm{Id}), p_{0,\infty}(Y'), \infty)$ is given by

$$\mathrm{Stab}_{\mathrm{PSp}^{\pm}(4,\mathbb{R})}(0, p_{0,\infty}(\mathrm{Id}), p_{0,\infty}(Y'), \infty) = \left\{ \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix},\ K \in \mathrm{PO}(2) \right\}$$

Let $R = \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix}$ for a $K \in \mathrm{PO}(2)$. Then R is antisymplectic and it holds

$$R(\mathrm{Id}) = -\mathrm{Id} \text{ so that } \mathcal{Y}_{0,\infty} \perp \mathcal{Y}_{\mathrm{Id}, R(Y')}$$

$$R(Y') = -Y' \text{ so that } \mathcal{Y}_{0,\infty} \perp \mathcal{Y}_{Y', R(Y')}$$

We further need $R^2 = \mathrm{Id}$ for R to be an involution which is satisfied exactly when $K^2 = \mathrm{Id}$.
Result follows as

$$\mathcal{R}(P, X, Y, Q) = g^{-1} \mathcal{R}(0, \mathrm{Id}, Y', \infty) g$$

□

Proposition 6.19. *Let (l_1, l_2, l_3, l_4) be a maximal quadruple. Let R be a reflection inside $\mathcal{R}(P, X, Y, Q)$ where (P, X, Y, Q) is a maximal quadruple. Suppose*

$$(X, l_1, l_2, l_3, l_4, Y) \text{ maximal (possibly } X = l_1 \text{ or } Y = l_4)$$

then $(Q, R(Y), R(l_4), R(l_3), R(l_2), R(l_1), R(X), P)$ is maximal.

Proof. Let g be an isometry such that $g \cdot (P, X, Y, Q) = (0, \mathrm{Id}, Y', \infty)$ for Y' diagonal. We want to show that the image

$$\big(R(\infty), R(Y'), R(l_4), R(l_3), R(l_2), R(l_1), R(\mathrm{Id}), R(0)\big) \tag{19}$$

is maximal for $(\mathrm{Id}, l_1, l_2, l_3, l_4, Y')$ maximal and $R \in \mathcal{R}(0, \mathrm{Id}, Y', \infty)$. By Proposition 6.18 we know

(i) $\mathcal{R}(0, \mathrm{Id}, Y', \infty) = \{R_{st}R_{ex}\}$ if $Y' = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, y_1 \neq y_2$

(ii) $\mathcal{R}(0, \mathrm{Id}, Y', \infty) = \mathcal{K}$ if $Y' = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$

Observe that $\{R_{st}R_{ex}\} \subset \mathcal{K}$. Using Lemma 2.5 it is not hard to show that (19) is maximal. ☐

We now define $\mathcal{R}_{\mathcal{Y}_k}^{\mathcal{Y}_{k-1}, \mathcal{Y}_{k+1}}$ the reflection set associated to the side of a hexagon $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$. The geometric properties of $\mathcal{R}_{\mathcal{Y}_k}^{\mathcal{Y}_{k-1}, \mathcal{Y}_{k+1}}$ for $k = 2$ are shown in Figure 47.

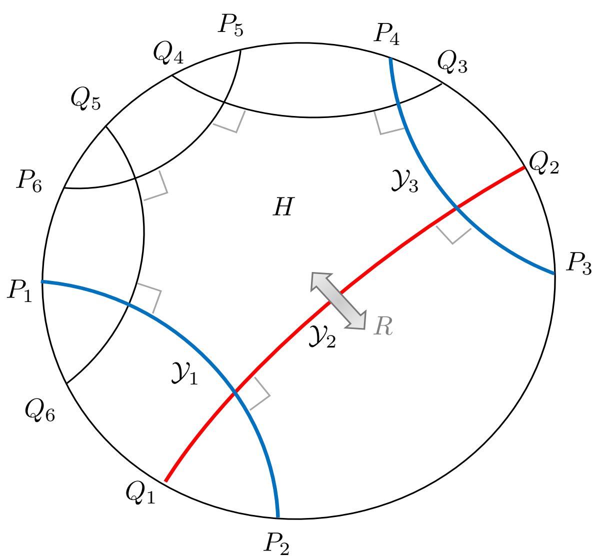


Figure 47: R is a reflection inside $\mathcal{R}_{\mathcal{Y}_2}^{\mathcal{Y}_1, \mathcal{Y}_3}$

Definition 6.20. (Reflection set associated to the side of a hexagon) Let $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ be a right-angled hexagon. The *reflection set* $\mathcal{R}_{\mathcal{Y}_k}^{\mathcal{Y}_{k-1}, \mathcal{Y}_{k+1}}$ *associated to* $\mathcal{Y}_k$ is the set of reflections which are fixing the endpoints of $\mathcal{Y}_k$ and are switching the endpoints of $\mathcal{Y}_{k-1}$ and $\mathcal{Y}_{k+1}$ respectively.

Recall that we denote by R_{st}, R_{ex} and $\mathcal{K}$ the following sets:

$$R_{st} = \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \; R_{ex} = \begin{pmatrix} -r & 0 \\ 0 & r \end{pmatrix}, \; r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{K} = \left\{ \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix}, \; K \in \mathrm{PO}(2), \; K^2 = \mathrm{Id} \right\}$$

Observe that $\{R_{st}, R_{ex}\} \subset \mathcal{K}$.

Corollary 6.21. *Let $H = [Y_1, Y_2, Y_3, Y_4, Y_5, Y_6]$ be a right-angled hexagon. Let*

$$\mathcal{Y}_{k-1} = \mathcal{Y}_{P_1,P_2}, \; \mathcal{Y}_k = \mathcal{Y}_{Q_1,Q_2}, \; \mathcal{Y}_{k+1} = \mathcal{Y}_{P_4,P_5}$$

and let g be an isometry such that $g \cdot (Q_1, P_2, P_3, Q_2) = (0, \mathrm{Id}, Y', \infty)$ for Y' diagonal. Then

$$\mathcal{R}_{\mathcal{Y}_k}^{\mathcal{Y}_{k-1}, \mathcal{Y}_{k+1}} = \begin{cases} \{g^{-1} R_{st} g, g^{-1} R_{ex} g\}, \; if \; (Q_1, P_2, P_3, Q_2) \; generic \\ g^{-1} \mathcal{K} g \; if \; (Q_1, P_2, P_3, Q_2) \; non \; generic \end{cases}$$

Proof. Follows directly from Proposition 6.18. □

We can rewrite Corollary 6.21 in terms of arc coordinates in the following way:

Corollary 6.22. *Let $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ be a right-angled hexagon where*

$$\mathcal{Y}_1 = \mathcal{Y}_{P_1,P_2}, \; \mathcal{Y}_2 = \mathcal{Y}_{Q_1,Q_2}, \; \mathcal{Y}_3 = \mathcal{Y}_{P_3,P_4}, \; \mathcal{Y}_4 = \mathcal{Y}_{Q_3,Q_4}, \; \mathcal{Y}_5 = \mathcal{Y}_{P_5,P_6}, \; \mathcal{Y}_6 = \mathcal{Y}_{Q_5,Q_6}$$

Suppose $(H, \mathcal{Y}_1)$ has arc coordinates

$$\mathcal{A}(H, \mathcal{Y}_1) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$$

Let $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ be the malefic map of Definition 5.12 and let g_1, g_2, g_3 be isometries such that

$$g_1(Q_1, P_2, P_3, Q_2) = (0, \mathrm{Id}, Y_1, \infty)$$

$$g_2(Q_5, P_6, P_1, Q_6) = (0, \mathrm{Id}, Y_1, \infty)$$

$$g_3(Q_3, P_4, P_5, Q_4) = (0, \mathrm{Id}, Y_3, \infty)$$

where Y_1, Y_2, Y_3 are diagonal matrices. Then it holds

$$\mathcal{R}_{\mathcal{Y}_2}^{\mathcal{Y}_1, \mathcal{Y}_3} = \begin{cases} \{g_1^{-1} R_{st} g_1, g_1^{-1} R_{ex} g_1\}, \; if \; \underline{b} \in \mathfrak{a} \\ g_1^{-1} \mathcal{K} g_1 \; if \; \underline{b} \in \mathfrak{d} \end{cases} \tag{20}$$

$$\mathcal{R}_{\mathcal{Y}_6}^{\mathcal{Y}_5,\mathcal{Y}_1} = \begin{cases} \{g_2^{-1} R_{st} g_2, g_2^{-1} R_{ex} g_2\}, \ if \ \underline{d} \in \mathfrak{a} \\ g_2^{-1} \mathcal{K} g_2 \ if \ \underline{d} \in \mathfrak{d} \end{cases} \tag{21}$$

$$\mathcal{R}_{\mathcal{Y}_4}^{\mathcal{Y}_3,\mathcal{Y}_5} = \begin{cases} \{g_3^{-1} R_{st} g_3, g_3^{-1} R_{ex} g_3\}, \ if \ F_{\underline{b},\underline{d},\alpha_1,\alpha_2}(\underline{c}) \in \mathfrak{a} \\ g_3^{-1} \mathcal{K} g_3 \ if \ F_{\underline{b},\underline{d},\alpha_1,\alpha_2}(\underline{c}) \in \mathfrak{d} \end{cases} \tag{22}$$

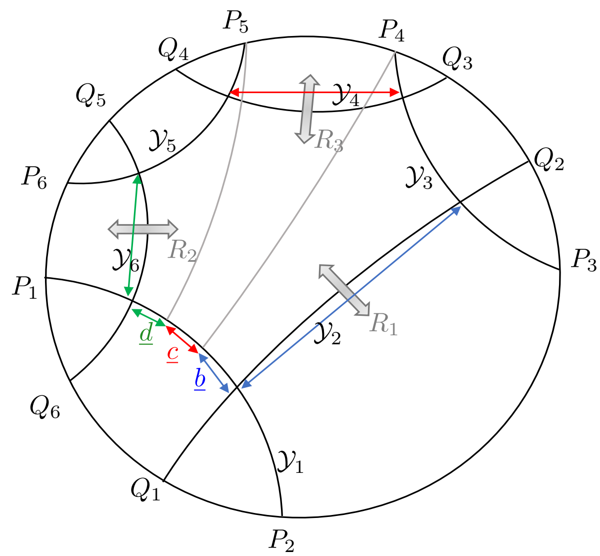


Figure 48: R_1, R_2, R_3 are in $\mathcal{R}_{\mathcal{Y}_2}^{\mathcal{Y}_1,\mathcal{Y}_3}, \mathcal{R}_{\mathcal{Y}_4}^{\mathcal{Y}_3,\mathcal{Y}_5}, \mathcal{R}_{\mathcal{Y}_6}^{\mathcal{Y}_5,\mathcal{Y}_1}$ respectively

Proof. Let us prove (20). By Corollary 6.21 we know

$$\mathcal{R}_{\mathcal{Y}_2}^{\mathcal{Y}_1,\mathcal{Y}_3} = \begin{cases} \{g_1^{-1} R_{st} g_1, g_1^{-1} R_{ex} g_1\}, \ if \ (Q_1, P_2, P_3, Q_2) \ generic \\ g_1^{-1} \mathcal{K} g_1 \ if \ (Q_1, P_2, P_3, Q_2) \ non \ generic \end{cases}$$

The quadruple (Q_1, P_2, P_3, Q_2) is generic if the matrix given by the cross-ratio $Cr(Q_1, P_2, P_3, Q_2)$ has distinct eigenvalues μ_1, μ_2 and non-generic if $\mu_1 = \mu_2$ (Definition 3.1 and 4.10). By Lemma 2.17 it holds

$$(\mu_1, \mu_2) = d^{\overline{\mathfrak{a}}^+}(p_{Q_1,Q_2}(P_2), p_{Q_1,Q_2})(P_3)$$

The vector (μ_1, μ_2) is the image $f(\underline{b})$ where f is the bijective map of Proposition 5.8. These vectors are drawn in blue in Figure 48. In particular f preserves regular vectors. Equality (20) follows and the proof for (21) is similar. For (22) we need to write $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}(\underline{c})$ instead of $\underline{c}$ as there is no bijective map as in the cases (20) and (21). This is explained in Section 5.4. □

6.4 Geometrical interpretation of the set $\mathcal{K}$

Hexagons are the building blocks that will be glued together to compute maximal representations. The idea is to parametrize maximal representations by parametrizing adjacent right-angled hexagons which have equal alternating side-lengths. This will be the case for adjacent symmetric hexagons, whose definition is given in this section. Adjacent symmetric hexagons are obtained by reflecting a hexagon across a side and the different ways to do this are encoded in the reflection set introduced in Section 6.3. In this section we give a geometrical interpretation to the reflection set associated to the side of a hexagon (and in particular to the set $\mathcal{K}$) in terms of the polygonal chain associated to the hexagon. This will be very useful for a geometrical interpretation of the parameters appearing in Theorem 7.23. For simplicity we will consider an ordered hexagon of the form $(H, \mathcal{Y}_{0,\infty})$ and will study the associated polygonal chain defined in 4.19. Let us start with the following

Definition 6.23. Two right-angled hexagons H_1, H_2 are said to be *adjacent at* $\mathcal{Y}_1$ if

$$H_1 = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6] \text{ and } H_2 = [\mathcal{Y}_1, \mathcal{Y}_6, \mathcal{Y}_7, \mathcal{Y}_8, \mathcal{Y}_9, \mathcal{Y}_2]$$

Two such adjacent hexagons will be denoted $H_1 \#_{\mathcal{Y}_1} H_2$ (Figure 49).

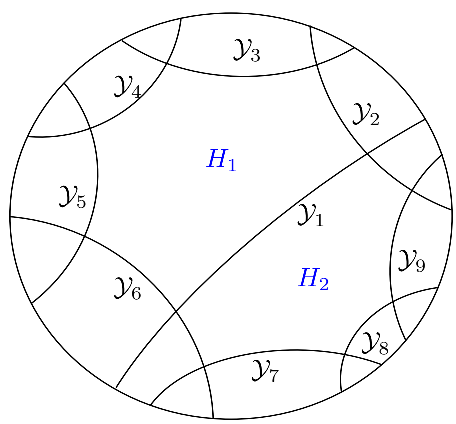


Figure 49: Two adjacent hexagons $H_1 \#_{\mathcal{Y}_1} H_2$

Definition 6.24. Let $H_1 = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ and $H_2 = [\mathcal{Y}_1, \mathcal{Y}_6, \mathcal{Y}_7, \mathcal{Y}_8, \mathcal{Y}_9, \mathcal{Y}_2]$ be two hexagons adjacent at $\mathcal{Y}_1$. The hexagons $H_1 \#_{\mathcal{Y}_1} H_2$ are said to be *symmetric* if

$$H_2 = R(H_1)$$

for a reflection $R \in \mathcal{R}_{\mathcal{Y}_1}^{\mathcal{Y}_6, \mathcal{Y}_2}$.

In Definition 4.19 we have introduced the notion of a polygonal chain associated to an ordered right-angled hexagon $(H, \mathcal{Y}_{0,\infty})$ inside $\mathcal{H}$. Let

$$H = (0, A, B, C, D, \infty)$$

and let $\overline{R} \in \mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty}, \mathcal{Y}_{A^2,\mathrm{Id}}}$. Such a reflection is illustrated in Figure 53. By Corollary 6.21 we know

$$\overline{R} = g^{-1} R g, \;\; R \in \mathcal{K}$$

where g is an isometry such that $g(-A, 0, A^2, A) = (0, \mathrm{Id}, Y, \infty)$ with Y diagonal. In this section we want to relate the parameter R inside $\mathcal{K}$ to the polygonal chain of the hexagon $(\overline{R}(H), \mathcal{Y}_{0,\infty})$. In particular we will show how the set $\mathcal{K}$ allows us to draw the polygonal chain associated to $(\overline{R}(H), \mathcal{Y}_{0,\infty})$ once we are given the polygonal chain associated to $(H, \mathcal{Y}_{0,\infty})$. When two hexagons are adjacent they share one vertex of the correspondent polygonal chains and we can look at the "attached" polygonal chains.

Definition 6.25. (**Attachment angle**) Let $(H, \mathcal{Y}_{0,\infty})$ and $(H', \mathcal{Y}_{0,\infty})$ be two adjacent right-angled hexagons inside $\mathcal{H}$ with

$$(H, \mathcal{Y}_{0,\infty}) = (0, A, B, C, D, \infty)$$

$$(H', \mathcal{Y}_{0,\infty}) = (0, A', B', C', D', \infty), \;\; D' = A$$

such that $(0, A', B', C', A, B, C, D)$ is maximal (Figure 50)

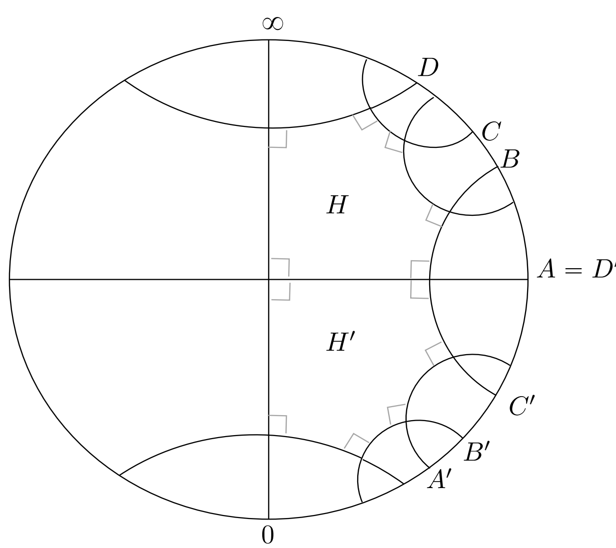


Figure 50: The tow adjacent hexagons H and H'

We can look at the ordered sequence of points

$$\left(\pi^{\mathbb{H}^2}(iA'), \pi^{\mathbb{H}^2}(iB'), \pi^{\mathbb{H}^2}(iC'), \pi^{\mathbb{H}^2}(iA), \pi^{\mathbb{H}^2}(iB), \pi^{\mathbb{H}^2}(iC), \pi^{\mathbb{H}^2}(iD)\right)$$

obtained by the union of vertices of the polygonal chains associated to $(H, \mathcal{Y}_{0,\infty})$ and $(H', \mathcal{Y}_{0,\infty})$ respectively. This induces an orientation on the segments forming the polygonal chains. See for example Figure 54 where $H' = \overline{R}_{st}(H)$. The *attachment angle* β between these two polygonal chains is the angle (measured on the left) formed by the two (non-vanishing) segments attached at the point $\pi^{\mathbb{H}^2}(iD') = \pi^{\mathbb{H}^2}(iA)$. For a visualization of the attachment angle see for example Figure 55.

We want to study the case where H and H' are symmetric adjacent. Let us state a proposition which will be useful later:

Proposition 6.26. *Let $(0, P, Q, \infty)$ be a maximal quadruple and consider the orthogonal tubes $\mathcal{Y}_{-Q,Q} \perp \mathcal{Y}_{P,QP^{-1}Q}$. Suppose $(-Q, 0, P, Q)$ generic. Then the hyperbolic components of iP, iQ and $iQP^{-1}Q$ lie on the same geodesic in $\mathbb{H}^2$, and iQ is the middle point of the three. If $(-Q, 0, P, Q)$ is non-generic then the hyperbolic components coincide in $\mathbb{H}^2$.*

The configuration of the points iP, iQ and $iQP^{-1}Q$ is illustrated in 51.

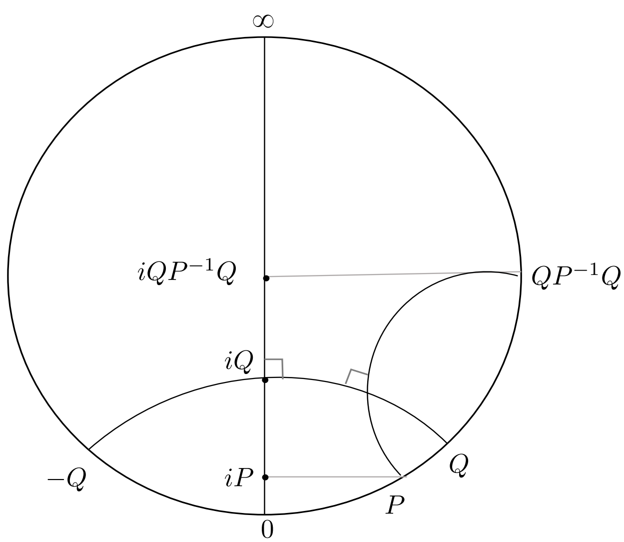


Figure 51: The hyperbolic components of iP, iQ and $iQP^{-1}Q$ are colinear in $\mathbb{H}^2$

Proof. By Proposition 5.8 we know that $(-Q, 0, P, Q)$ is generic if and only if the quadruple $(0, Q, QP^{-1}Q, \infty)$ is generic. Up to $\mathrm{Sp}(4,\mathbb{R})$-action we can consider (see Figure 52)

$$Q = \mathrm{Id} \quad \text{and} \quad QP^{-1}Q = Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \ y_1 > y_2$$

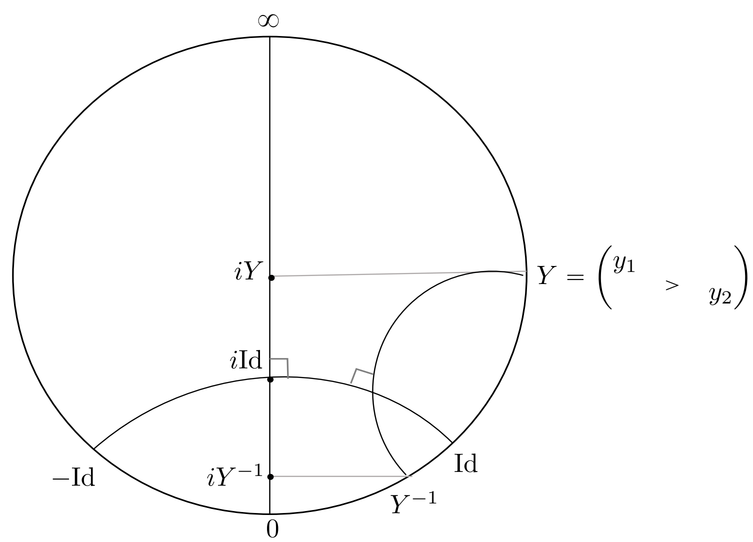


Figure 52: Configuration after the action of an element $g \in \mathrm{Sp}(4,\mathbb{R})$

The hyperbolic component of Id is $i \in \mathbb{H}^2$ in the identification of Section 2.9. It is trivial that the hyperbolic components of iY and iY^{-1} lie on the same geodesic in $\mathbb{H}^2$ (the y-axis), where the point i is in the middle. Since isometries preserve geodesics, the same is true more generally for tubes $\mathcal{Y}_{-Q,Q} \perp \mathcal{Y}_{P,PQP^{-1}Q}$. The non-generic case is trivial. □

Our aim is to give a geometric interpretation of the set $\mathcal{K}$. More precisely we want see how the choice of $R \in \mathcal{K}$ is equivalent to choosing an attachment angle β between the polygonal chains of H and $\overline{R}(H)$, where $H\#\overline{R}(H)$ are adjacent symmetric and $\overline{R}$ is conjugate to R. Let us start by recalling a standard fact of linear algebra.

Lemma 6.27. *Let $K \in \mathrm{PO}(2)$ such that $K^2 = \mathrm{Id}$. Then*

$$K = \mathrm{Id} \quad \textit{or} \quad K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for a unique $\theta \in [0, \pi)$.

Let us denote by β the following map

$$\begin{aligned} \beta : \mathcal{K} &\to [0, 2\pi) \\ \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix} &\mapsto \begin{cases} \pi + 2\theta \\ \pi \quad \text{if } K = R_{st} \end{cases} \end{aligned} \tag{23}$$

where

$$K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ \theta \in [0, \pi)$$

We can now state the following result:

Proposition 6.28. *Let* $(H, \mathcal{Y}_{0,\infty}) \in \mathcal{H}$*:*

$$H = (0, A, \mathrm{Id}, C, D, \infty), \quad C \textit{ diagonal}$$

with arc coordinates

$$\mathcal{A}(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$$

and let the angles of the polygonal chain associated to $(H, \mathcal{Y}_{0,\infty})$ *be* α_1, α_2 *(possibly only* α *or no angle). Consider a reflection* $\overline{R}$ *inside* $\mathcal{R}^{\mathcal{Y}_{0,\infty}, \mathcal{Y}_{A^2,\mathrm{Id}}}_{\mathcal{Y}_{-A,A}}$*:*

$$\overline{R} = g^{-1} R g \ \in \mathcal{R}^{\mathcal{Y}_{0,\infty}, \mathcal{Y}_{A^2,\mathrm{Id}}}_{\mathcal{Y}_{-A,A}}$$

for $R \in \mathcal{K}$ *and* g *an isometry such that* $g(-A, 0, A^2, A) = (0, \mathrm{Id}, Y, \infty)$ *with* Y *diagonal.*
Then the attachment angle between the polygonal chains of $(H, \mathcal{Y}_{0,\infty})$ *and* $(\overline{R}(H), \mathcal{Y}_{0,\infty})$ *is given by* $\beta(K)$ *where* β *is the map in (23).*
Moreover the polygonal chain associated to $(\overline{R}(H), \mathcal{Y}_{0,\infty})$ *has*

(i) segments of lengths $h(\underline{d}), h(\underline{c}), h(\underline{b})$ *where* h *is the map*

$$h(d_1, d_2) = d_1 - d_2$$

(ii) angles (if there): $\begin{cases} \alpha_2, \alpha_1 \quad (\textit{or } \alpha) \\ 2\pi - \alpha_2, 2\pi - \alpha_1 (\textit{or } 2\pi - \alpha) \quad \textit{if } K = R_{st} \end{cases}$

Proof. Let us first consider the case where $(H, \mathcal{Y}_{0,\infty})$ is generic. The two adjacent symmetric hexagons are illustrated in Figure 53. By Proposition 6.19 the 6-tuple

$$(0, \overline{R}(D), \overline{R}(C), \overline{R}(\mathrm{Id}) = A^2, \overline{R}(A) = A, \infty) \text{ is maximal}$$

This 6-tuple determines the ordered sequence of vertices in the polygonal chain associated to $(\overline{R}(H), \mathcal{Y}_{0,\infty})$.

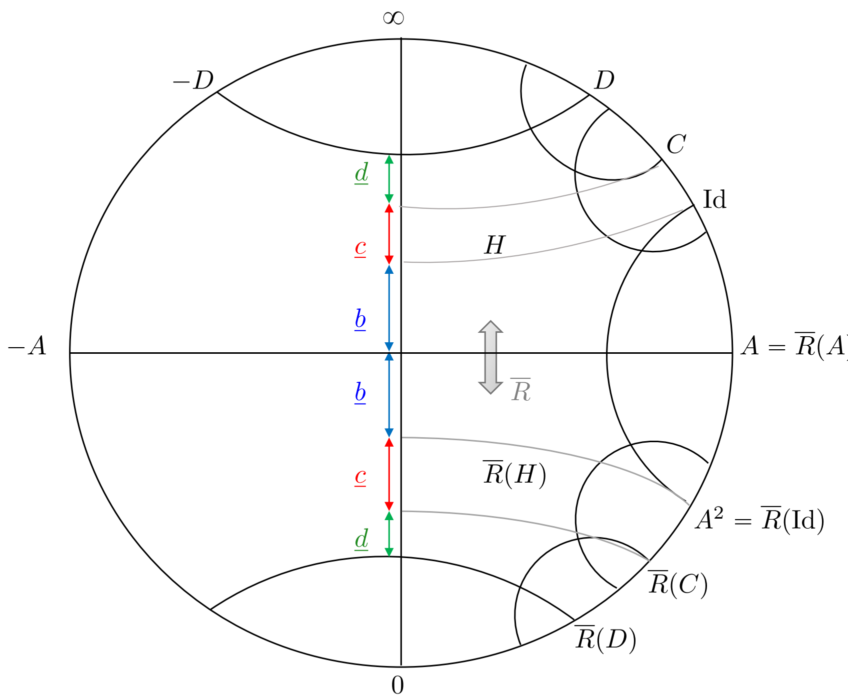


Figure 53: The adjacent symmetric hexagons
$H \#_{\mathcal{Y}_{-A,A}} \overline{R}(H)$

By Corollary 6.22 we know

$$\mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}} = \begin{cases} \{g^{-1}R_{st}g, g^{-1}R_{ex}g\}, \ if \ \underline{b} \in \mathfrak{a} \\ g^{-1}\mathcal{K}g \ if \ \underline{b} \in \mathfrak{d} \end{cases}$$

where $g(-A, 0, A^2, A) = (0, \mathrm{Id}, Y, \infty)$ for Y diagonal and $\{R_{st}, R_{ex}\} \subset \mathcal{K}$. As $(H, \mathcal{Y}_{0,\infty})$ generic we know $\underline{b} \in \mathfrak{a}$ so that

$$\overline{R} = g^{-1}R_{st}g \quad \text{or} \quad \overline{R} = g^{-1}R_{ex}g$$

By Proposition 52 we know that the attachment angle is $\beta(K) = \pi$ as the points A^2, A, Id are colinear in the hyperbolic component of $\mathcal{Y}_{0,\infty}$. Put $\overline{R}_{st} = g^{-1}R_{st}g$. Computations give:

$$\overline{R}_{st} = \begin{pmatrix} 0 & A \\ A^{-1} & 0 \end{pmatrix}$$

and one can immediately see that $\overline{R}_{st} \in \mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}}$ as

$$\overline{R}_{st}(-A) = -A, \quad \overline{R}_{st}(A) = A, \quad \overline{R}_{st}(0) = \infty, \quad \overline{R}_{st}(A^2) = \mathrm{Id}$$

It is straightforward to see that the segments of the polygonal chain associated to $(\overline{R}_{st}(H), \mathcal{Y}_{0,\infty})$ have length $h(\underline{d}), h(\underline{c}), h(\underline{b})$ respectively. The eigenspaces $E_{\pm 1}^{\overline{R}_{st}}$ of $\overline{R}_{st}$ are given by

$$E_1^{\overline{R}_{st}} = A, \quad E_{-1}^{\overline{R}_{st}} = -A$$

By Proposition 6.15 we know that $\overline{R}_{st}$ is fixing the tube $\mathcal{Y}_{-A,A}$ and sending any transverse $X \pitchfork A$ to the unique $\overline{R}_{st}(X)$ such that $\mathcal{Y}_{-A,A} \perp \mathcal{Y}_{X,\overline{R}_{st}(X)}$. By Proposition 6.26 the hyperbolic components of $\overline{R}_{st}(X)$ and X lie therefore on the same geodesic inside $\mathbb{H}^2$. The polygonal chain associated to $(\overline{R}_{st}(H), \mathcal{Y}_{0,\infty})$ is obtained by rotating the polygonal chain of H of an angle π around A. This is illustrated in Figure 54 where the polygonal chain of H is drawn in blue and the polygonal chain of $\overline{R}_{st}(H)$ is drawn in purple. It is easy to see that the angles of the polygonal chain are therefore given by $2\pi - \alpha_2, 2\pi - \alpha_1$.

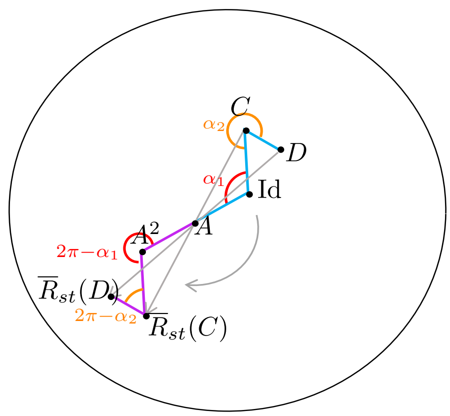


Figure 54: Polygonal chain of $(\overline{R}_{st}(H), \mathcal{Y}_{0,\infty})$
obtained from the polygonal chain of $(H, \mathcal{Y}_{0,\infty})$

Put now $\overline{R}_{ex} = g^{-1}R_{ex}g$. Recall that $\mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}} = \{\overline{R}_{st}, \overline{R}_{ex}\}$ (Corollary 6.22). Instead of directly computing $\overline{R}_{ex}$ observe that if we denote by f the map

$$f \in \mathrm{Stab}_{\mathrm{PSp}(4,\mathbb{R})}(0, A^2, A, \infty),\ f \neq \mathrm{Id}$$

then the map $f \circ \overline{R}_{st}$ satisfies

$$f \circ \overline{R}_{st}(-A) = -A, \quad f \circ \overline{R}_{st}(A) = A, \quad f \circ \overline{R}_{st} = \infty, \quad f \circ \overline{R}_{st}(A^2) = \mathrm{Id}$$

so that $\overline{R}_{ex} = f \circ \overline{R}_{st}$. The geometric interpretation of f is the reflection across the geodesic going through the hyperbolic components of A and A^2 respectively (Remark 2.34) and this geodesic also goes through the hyperbolic component of Id (Proposition 6.26). The angles of the polygonal chain associated to $(\overline{R}_{ex}(H), \mathcal{Y}_{0,\infty})$ are therefore given by α_2, α_1 and this is illustrated on the right-hand side of Figure 55.

Let us now consider the case where $(H, \mathcal{Y}_{0,\infty})$ is non-generic of type 1.1, that is $\underline{b}, \underline{c}, \underline{d} \in \mathfrak{d} \times \mathfrak{a}^2$. The polygonal chain associated to $(H, \mathcal{Y}_{0,\infty})$ has only one angle α and is illustrated in Figure 26. By Corollary 6.22 we know

$$\overline{R} = g^{-1}\mathcal{K}g \ \in \mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}}$$

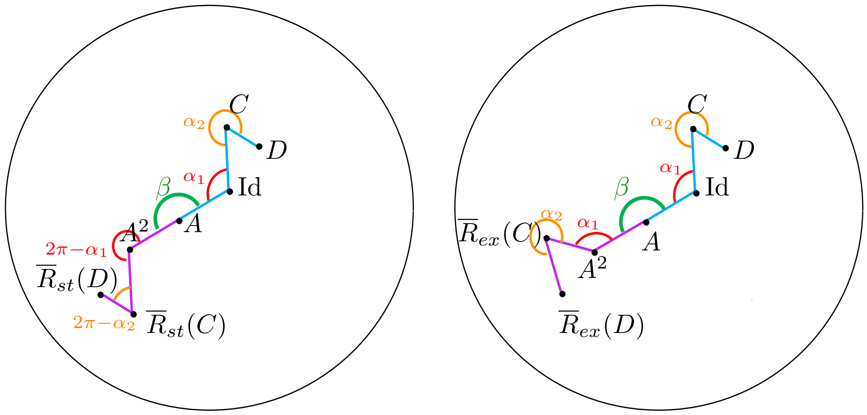


Figure 55: Polygonal chains of $(\overline{R}_{st}(H), \mathcal{Y}_{0,\infty})$ and $(\overline{R}_{ex}(H), \mathcal{Y}_{0,\infty})$ obtained from the polygonal chain of $(H, \mathcal{Y}_{0,\infty})$

where $g(-A, 0, A^2, A) = (0, \mathrm{Id}, Y, \infty)$ for Y diagonal. Since $\underline{b} \in \mathfrak{d}$ we know $A = a \cdot \mathrm{Id}$. Computations give

$$\mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}} = \left\{ \begin{pmatrix} 0 & aK \\ a^{-1}K & 0 \end{pmatrix},\ K \in \mathcal{K} \right\}$$

Given $\overline{R} \in \mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}}$ let us decompose $\overline{R}$ as following: let

$$r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ \theta \in [0, \pi)$$

We write

$$\overline{R} = \underbrace{\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}}_{r} \underbrace{\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 0 & a\mathrm{Id} \\ a^{-1}\mathrm{Id} & 0 \end{pmatrix}}_{M}$$

where $K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $K \neq R_{st}$ is the decomposition of Lemma 6.27. The geometrical interpretation of this decomposition is illustrated in Figure 56. The map M is analogue to the rotation of Figure 54. The map S is a rotation of angle 2θ around i on the hyperbolic component of $\mathcal{Y}_{0,\infty}$ (see Section 2.10) and the map r is a reflection across the vertical axis. We obtain an attachment angle $\beta(K) = \pi + 2\theta$ and polygonal chain angle α. When $K = R_{st}$ we only have

$$\overline{R} = \begin{pmatrix} 0 & a\mathrm{Id} \\ a^{-1}\mathrm{Id} & 0 \end{pmatrix}$$

and we are nor rotating nor reflecting.
The proof for the other cases where $(H, \mathcal{Y}_{0,\infty})$ is non-generic are similar. □

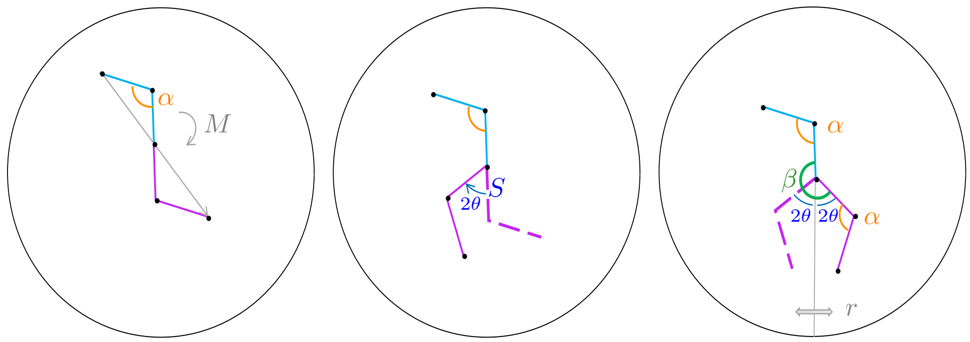


Figure 56: Geometrical interpretation of $\overline{R} = rSM$

Remark 6.29. (**Reflections producing the same hexagon**) In Proposition 6.28 we have seen that the choice of $R \in \mathcal{K}$ is equivalent to choosing an attachment angle β between the hexagons H and $\overline{R}(H)$, where $H \#_{\mathcal{Y}_{-A,A}} \overline{R}(H)$ are adjacent symmetric and $\overline{R}$ is conjugate to R. More precisely for $(H, \mathcal{Y}_{0,\infty}) \in \mathcal{H}$:

$$H = (0, A, \mathrm{Id}, C, D, \infty),\ C \text{ diagonal}$$

with arc coordinates

$$\mathcal{A}(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$$

we know that the reflection set associated to $\mathcal{Y}_{-A,A}$ is the set (Corollary 6.22)

$$\mathcal{R}^{\mathcal{Y}_{0,\infty}, \mathcal{Y}_{A^2,\mathrm{Id}}}_{\mathcal{Y}_{-A,A}} = \begin{cases} \{g^{-1} R_{st} g, g^{-1} R_{ex} g\},\ if\ \underline{b} \in \mathfrak{a} \\ g^{-1} \mathcal{K} g\ if\ \underline{b} \in \mathfrak{d} \end{cases}$$

where $g(-A, 0, A^2, A) = (0, \mathrm{Id}, Y, \infty)$ with Y diagonal. The polygonal chain associated to $(\overline{R}(H), \mathcal{Y}_{0,\infty})$ is illustrated in Fugures 55 and 56 for the cases $(\underline{b}, \underline{c}, \underline{d}) \in \mathfrak{a}^3$ and $(\underline{b}, \underline{c}, \underline{d}) \in \mathfrak{d} \times \mathfrak{a}^2$ respectively. Observe that if H is contained in a maximal polydisc it can happen that $\mathcal{R}^{\mathcal{Y}_{0,\infty}, \mathcal{Y}_{A^2,\mathrm{Id}}}_{\mathcal{Y}_{-A,A}}$ contains two different reflections $\overline{R}, \overline{R'}$ for which

$$\overline{R}(H) = \overline{R'}(H)$$

Let us denote for simplicity $p = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ the arc coordinates associated to $(H, \mathcal{Y}_{0,\infty})$. By the geometrical interpretation of Proposition 6.28 it is not hard to show that the case $\overline{R}(H) = \overline{R'}(H)$ happens exactly for

$$\overline{R}, \overline{R'} \in \{g^{-1} R_{st} g,\ g^{-1} R_{ex} g\} \quad \text{if } p \in \mathcal{D} \backslash \mathcal{D}_{\mathbb{H}^2} \tag{24}$$

and for any

$$\overline{R}, \overline{R'} \in g^{-1} \mathcal{K} g \quad \text{if } p \in \mathcal{D}_{\mathbb{H}^2} \tag{25}$$

where $\mathcal{D}$ and $\mathcal{D}_{\mathbb{H}^2}$ are described in Proposition 4.28 and Definition 4.29 respectively. The two hexagons H and $\overline{R}(H) = \overline{R'}(H)$ lie both inside the model polydisc if $\overline{R}, \overline{R'}$ are as in (24) and all the points of the two polygonal chains lie on the vertical geodesic of $\mathbb{H}^2$. In (25), the two hexagons H and $\overline{R}(H) = \overline{R'}(H)$ lie both inside the diagonal disc and all the points of the polygonal chains coincide with $\pi^{\mathbb{H}^2}(i\mathrm{Id})$.

Remark 6.30. (**Attaching several polygonal chains**)
Let $H = (0, A, \mathrm{Id}, C, D, \infty)$ be a generic right-angled hexagon and consider the hexagon $\overline{R}_2(H)$ where $\overline{R}_2 \in \mathcal{R}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}}_{\mathcal{Y}_{-A,A}}$. As $(H, \mathcal{Y}_{0,\infty})$ is generic we know that $\mathcal{R}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{A^2,\mathrm{Id}}}_{\mathcal{Y}_{-A,A}}$ contains exactly two elements (Corollary 6.22). We have drawn the correspondent attached polygonal chains of $H\#\overline{R}_2(H)$ in Figure 55. In particular we know that the attachment angle is $\beta = \pi$ (which follows directly from Proposition 6.26) and we know how to draw the angles of the polygonal chain associated to $(\overline{R}_2(H), \mathcal{Y}_{0,\infty})$ using the geometrical interpretation of Proposition 6.28. We can state a similar result if we consider the hexagon $\overline{R}_1(H)$ where $\overline{R}_1 \in \mathcal{R}^{\mathcal{Y}_{0,\infty},\mathcal{Y}_{C,DC^{-1}D}}_{\mathcal{Y}_{-D,D}}$. We can then draw the three attached polygonal chains $\overline{R}_2(H)\#H\#\overline{R}_1(H)$. This will turn out to be a very useful visualization in the proof of Theorem 7.23. For this reason we end this chapter by drawing all the possible polygonal chains of $\overline{R}_2(H)\#H\#\overline{R}_1(H)$ in the case that $(H, \mathcal{Y}_{0,\infty})$ is generic. This is illustrated in Figure 57. The polygonal chains are drawn up to isometry, this means that we consider two polygonal chains to be equivalent if there exists an isometry $g \in \mathrm{PSp}(4, \mathbb{R})$ sending all the vertices of one to the vertices of the other.

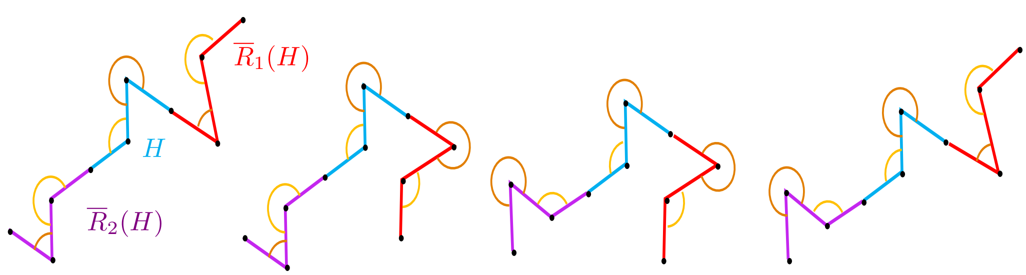


Figure 57: All possible polygonal chains (up to isometry) for $\overline{R}_2(H)\#H\#\overline{R}_1(H)$ when $(H, \mathcal{Y}_{0,\infty})$ is generic

7 Parameters for maximal representations

In this chapter we use arc coordinates of right-angled hexagons to parametrize maximal representations. We start by discussing geometric properties of Shilov hyperbolic isometries in $\mathrm{PSp}(2n,\mathbb{R})$, where we investigate in detail the case of $\mathrm{PSp}(4,\mathbb{R})$. We state the definition of a maximal representation into $\mathrm{PSp}(2n,\mathbb{R})$ and we further define the notion of a maximal representation from the Coxeter group $W_3 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ into $\mathrm{PSp}^{\pm}(2n,\mathbb{R})$ (Definition 7.8). We recall the notion of arc coordinates for classical Teichmüller space $\mathcal{T}(\Sigma)$ and we consider the example of the surface $\Sigma = \Sigma_{0,3}$ (pair of pants), whose fundamental group we denote $\Gamma_{0,3}$. In Theorem 7.21 we use arc coordinates of right-angled hexagons to give a parametrization of the set of maximal representations of W_3 into $\mathrm{PSp}^{\pm}(4,\mathbb{R})$. This will lead to the parametrization of a set $\chi^{\mathcal{S}} \subset \chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3},\mathrm{PSp}(4,\mathbb{R}))$ (Definition 7.19) that will be described in Theorem 7.23.

7.1 Shilov hyperbolic isometries

In this section we give the definition of a Shilov hyperbolic element in $\mathrm{PSp}(2n,\mathbb{R})$. We will study in detail the case of $\mathrm{PSp}(4,\mathbb{R})$ and give a classification of Shilov hyperbolic elements.

Definition 7.1. An element $g \in \mathrm{PSp}(2n,\mathbb{R})$ is called *Shilov hyperbolic* if it is conjugate to $\begin{pmatrix} A & 0 \\ 0 & A^{-T}\end{pmatrix}$ for a matrix $A \in \mathrm{GL}(n,\mathbb{R})$ with complex eigenvalues with modulus greater than one.

Example 7.2. As an example we study Shilov hyperbolic elements in $\mathrm{PSp}(4,\mathbb{R})$. Let (e_1,e_2,e_3,e_4) be the standard basis of $\mathbb{R}^4$. Recall that we denote $0, l_\infty$ the Lagrangians

$$0 = \langle e_3, e_4\rangle,\ l_\infty = \langle e_1, e_2\rangle$$

and that the standard tube

$$\mathcal{Y}_{0,\infty} = \{iY \mid Y \in \mathrm{Sym}^+(2,\mathbb{R})\}$$

is isometrically identified (see Lemma 2.28) with $\mathbb{R}\times\mathbb{H}^2$ through the map

$$\begin{aligned} \pi^{\mathbb{R}} \times \pi^{\mathbb{H}^2} : \mathcal{Y}_{0,\infty} &\to \mathbb{R}\times \mathrm{Sym}^+(2,\mathbb{R}) \\ iY &\mapsto \Big(\frac{\log\det Y}{\sqrt{2}}, \frac{Y}{\sqrt{\det Y}}\Big) \end{aligned}$$

Moreover, in the proof of Proposition 2.35 we have shown how the visual boundary of the hyperbolic component in $\mathcal{Y}_{0,\infty}$ can be realized as the $\mathrm{O}(2)$-orbit of the Lagrangian $l = \langle e_1, e_4\rangle$. Let now $g_A = \begin{pmatrix} A & 0 \\ 0 & A^{-T}\end{pmatrix} \in \mathrm{PSp}(4,\mathbb{R})$ be a Shilov hyperbolic element. The action of g_A on the $\mathbb{R}$-component of the standard tube is given by

$$\pi^{\mathbb{R}}(g_A(iY)) = \frac{\log \det^2(A) \det Y}{\sqrt{2}} = \frac{\log \det^2(A)}{\sqrt{2}} + \pi^{\mathbb{R}}(iY)$$

and Shilov hyperbolicity implies $\pi^{\mathbb{R}}(g_A(iY)) > \pi^{\mathbb{R}}(iY)$. We now want to study the action of g_A on the hyperbolic component of the standard tube $\mathcal{Y}_{0,\infty}$. We have the following possibilities:

- A has eigenvalues $\lambda = \mu \in \mathbb{R}$ and acts on $\mathbb{H}^2$ as the identity map
- A has eigenvalues $\lambda > \mu \in \mathbb{R}$ and is conjugate to a matrix

$$\begin{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\mu} \end{pmatrix} \end{pmatrix} \sim g_A$$

  The map g_A acts on the hyperbolic component of $\mathcal{Y}_{0,\infty}$ as an hyperbolic isometry: it fixes exactly two points in the boundary of $\mathbb{H}^2$.

- A has one eigenvalues $\lambda \in \mathbb{R}$ and is conjugate to a matrix

$$\begin{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \frac{1}{\lambda} & 0 \\ -\frac{1}{\lambda^2} & \frac{1}{\lambda} \end{pmatrix} \end{pmatrix} \sim g_A$$

  The map g_A acts on the hyperbolic component of $\mathcal{Y}_{0,\infty}$ as a parabolic isometry: it fixes exactly one point in the boundary of $\mathbb{H}^2$.

- A has two complex eigenvalues $\lambda e^{i\theta}, \lambda e^{-i\theta}, \theta \neq 2k\pi$ and is conjugate to a matrix

$$\begin{pmatrix} \begin{pmatrix} \lambda \cos\theta & -\lambda \sin\theta \\ \lambda \sin\theta & \lambda \cos\theta \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \frac{1}{\lambda} \cos\theta & -\frac{1}{\lambda} \sin\theta \\ \frac{1}{\lambda} \sin\theta & \frac{1}{\lambda} \cos\theta \end{pmatrix} \end{pmatrix} \sim g_A$$

  The map g_A acts on the hyperbolic component of $\mathcal{Y}_{0,\infty}$ as an elliptic isometry: it fixes exactly one point inside $\mathbb{H}^2$.

The geometrical interpretation of the action of g_A on $\mathcal{Y}_{0,\infty}$ in the hyperbolic, parabolic and elliptic case is illustrated in Figure 58.

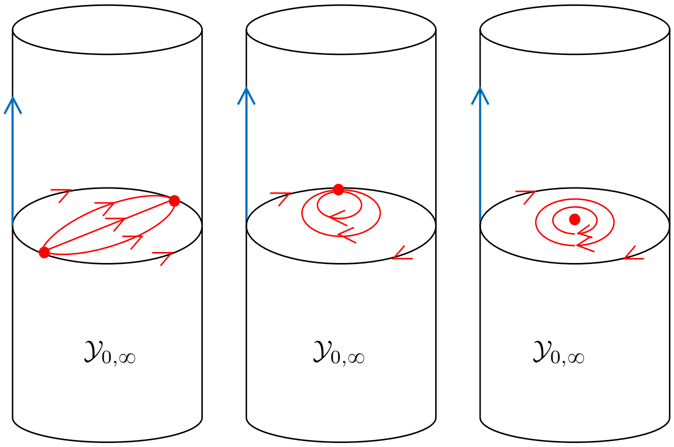


Figure 58: The action of g_A on $\mathcal{Y}_{0,\infty} = \mathbb{R} \times \mathbb{H}^2$ in the hyperbolic, parabolic and elliptic case

Lemma 7.3. *Let g be an element of $\mathrm{PSp}(2n, \mathbb{R})$. Then g is Shilov hyperbolic if and only if g fixes two transverse Lagrangians l_g^+, l_g^- on which it acts expandingly and contractingly respectively.*

Proof. By Definition 7.1 the element g is Shilov hyperbolic if it is conjugate to $g_A = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}$ for a matrix $A \in \mathrm{GL}(n, \mathbb{R})$ with all eigenvalues with modulus greater than one:

$$g = h g_A h^{-1}, \ h \in \mathrm{PSp}(2n, \mathbb{R})$$

Let (e_1, e_2, e_3, e_4) be the standard basis of $\mathbb{R}^4$ and let $0, l_\infty$ be the Lagrangians

$$0 = \langle e_3, e_4 \rangle, \ l_\infty = \langle e_1, e_2 \rangle$$

It is not hard to prove that A has all eigenvalues with modulus greater than one if and only if for any $l \in \mathcal{L}(\mathbb{R}^4)$, $0 \pitchfork l \pitchfork l_\infty$ it holds

$$\lim_{k \to -\infty} g_A^k(l) = 0 \text{ and } \lim_{k \to \infty} g_A^k(l) = l_\infty$$

i.e. g_A fixes $0, l_\infty$ on which it acts expandingly and contractingly respectively. Put

$$l_g^+ = h(0), \quad l_g^- = h(l_\infty)$$

□

Lemma 7.4. *Let g be an element of $\mathrm{PSp}(4, \mathbb{R})$ fixing two Lagrangians l_1, l_2 in $\mathcal{L}(\mathbb{R}^4)$ i.e. g is conjugated to $g_A = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}$ for a matrix $A \in \mathrm{GL}(n, \mathbb{R})$. Denote by $|\lambda| \geqslant |\mu|$ the modulus of the eigenvalues of A. Then*

(i) There exists $l \in \mathcal{L}(\mathbb{R}^4)$ such that (l_1, l, l_2) maximal and $(l_1, l, g(l), l_2)$ maximal if and only if $|\mu| > 1$ (that is g is Shilov hyperbolic).

(ii) $(l_1, l, g(l), l_2)$ maximal for all l such that (l_1, l, l_2) maximal if and only if $A = \lambda \mathrm{Id}$ for $\lambda \in \mathbb{R}, \ \lambda > 1$.

Proof. It is sufficient to prove the lemma for $g = g_A$, that is $l_1 = 0$ and $l_2 = \infty$.

(i) We want to show that there exists Y such that $(0, Y, \infty)$ maximal and $(0, Y, g_A Y, \infty)$ maximal if and only if $|\mu| > 1$. Let us write $Y > 0$ for a matrix Y which is positive definite. By Lemma 2.5 we know that $(0, Y, \infty)$ is maximal if and only if $Y > 0$. Suppose that there exists $Y > 0$ such that $(0, Y, g_A Y, \infty)$ maximal, that is

$$g_A Y - Y = AYA^T - Y > 0$$

Recall that a matrix M is positive definite if and only if $NMN^T > 0$ for every invertible matrix N. In particular for $N = \sqrt{Y^{-1}}$ we obtain

$$\sqrt{Y^{-1}}(AYA^T - Y)\sqrt{Y^{-1}} = (\sqrt{Y^{-1}}A\sqrt{Y})(\sqrt{Y}A^T\sqrt{Y^{-1}}) - \mathrm{Id} > 0 \quad (26)$$

The matrix $\sqrt{Y}A^T\sqrt{Y^{-1}} \in \mathrm{GL}(2, \mathbb{R})$ has the same eigenvalues of A. Let v be the orthonormal eigenvector associated to μ. Then

$$\sqrt{Y}A^T\sqrt{Y^{-1}}v = \mu v \Rightarrow \sqrt{Y}A^T\sqrt{Y^{-1}}\overline{v} = \overline{\mu v}$$

It follows from (26):

$$v^T(\sqrt{Y^{-1}}A\sqrt{Y})(\sqrt{Y}A^T\sqrt{Y^{-1}})\overline{v} - v^T\overline{v} = |\mu|^2 - 1 > 0$$

Let now suppose $|\mu| > 1$. We want to find $Y > 0$ such that $(0, Y, g_A Y, \infty)$ maximal that is we want to find $Y > 0$ such that $g_A Y - Y > 0$. In Remark 2.30 we have given an equivalent condition for $g_A Y - Y$ to be positive definite: let

$$r = d^{\mathbb{R}}(\pi^{\mathbb{R}}(iY), \pi^{\mathbb{R}}(ig_A Y)),\ h = d^{\mathbb{H}^2}(\pi^{\mathbb{H}^2}(iY), \pi^{\mathbb{H}^2}(ig_A Y))$$

then

$$g_A Y - Y > 0 \iff r > \frac{1}{\sqrt{2}}h \quad (27)$$

As g_A Shilov hyperbolic (this is the assumption $|\mu| < 1$) we know that g_A acts as a translation of distance r on the $\mathbb{R}$-component of $\mathcal{Y}_{0,\infty}$ and as an isometry on the $\mathbb{H}^2$-component which can be hyperbolic parabolic or elliptic (see Example 7.2). Observe that for a fixed A the distance r only depends on the eigenvalues of A and not on the point X, whereas the distance h depends on X and decreases the more X is close to the axis (if the isometry is hyperbolic) or to a fixed point (if the isometry is parabolic or elliptic). We can always find an open neighbour of the axis (or of a fixed point) such that the condition $r > \frac{1}{\sqrt{2}}h$ is satisfied.

(ii) We want to show that $(0, Y, g_A Y, \infty)$ is maximal $\forall Y$ such that $(0, Y, \infty)$ maximal if and only if $A = \lambda \mathrm{Id}$ for $\lambda \in \mathbb{R},\ \lambda > 1$. This is clear after the discussion in *(i)*: recall that $\lambda\mathrm{Id}$ for $\lambda \in \mathbb{R}$ acts on the $\mathbb{H}^2$-component of the tube as the identity map, so if $A = \lambda\mathrm{Id}$ the inequality in (27) is clearly satisfied. Conversely, suppose that $(0, Y, g_A Y, \infty)$ is maximal for all $Y > 0$. Equivalently, for any $Y > 0$ the inequality in (27) is satisfied, where the distance r is a fixed length depending only on the eigenvalues of A. This implies that the action of g_A on the hyperbolic component of the tube is the identity map i.e. $g_A = \lambda\mathrm{Id}, \lambda \in \mathbb{R}$. If this were not the case, we could always find an Y which does not satisfy the inequality in (27) by stepping away from the axis (in the hyperbolic case) or the fixed points (in the parabolic or elliptic case) and moving towards the boundary of $\mathbb{H}^2$.

□

7.2 Maximal representations

Let Σ be an oriented surface with negative Euler characteristic and boundary $\partial\Sigma$. Fix a finite area hyperbolization on Σ inducing an action of the fundamental group $\pi_1(\Sigma)$ on $S^1 = \partial\mathbb{H}^2$. An element $\gamma \in \pi_1(\Sigma)$ is called *peripheral* if it is freely homotopic to a boundary component.

Maximal representations are representations that maximize the Toledo invariant, an invariant defined using bounded cohomology (see [Tol89], [BIW10]). It is a deep result from Burger Iozzi and Wienhard ([BIW10, Theorem 8]) that maximal representations can be equivalently characterized as representations admitting a well-behaved boundary map, that is they can be defined by the following

Definition 7.5. A representation $\rho : \pi_1(\Sigma) \to \mathrm{PSp}(2n, \mathbb{R})$ is *maximal* if there exists a ρ-equivariant map $\xi : S^1 \to \mathcal{L}(\mathbb{R}^{2n})$ which is monotone (i.e. the image of any positively oriented triple in the circle is a maximal triple) and right continuous.

Given a maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{PSp}(2n, \mathbb{R})$, the image $\rho(\gamma)$ of every non-peripheral element $\gamma \in \pi_1(\Sigma)$ is Shilov hyperbolic (see [Str15]). Equivalently, $\rho(\gamma)$ fixes two transverse Lagrangians l_γ^+ and l_γ^- on which it acts expandingly and contractingly respectively (see Lemma 7.3). These Lagrangians are the images $\xi(\gamma^+)$ and $\xi(\gamma^-)$ where $\xi : S^1 \to \mathcal{L}(\mathbb{R}^{2n})$ is the equivariant boundary map and $l_\gamma^\pm = \xi(\gamma^\pm)$. We want to parametrize the set of maximal representations where the property of being Shilov hyperbolic is true also for peripheral elements. This is equivalent to the requirement that the representations are Anosov in the sense of [GW12].

Definition 7.6. A maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{PSp}(2n, \mathbb{R})$ will be called *Shilov hyperbolic* if $\rho(\gamma)$ is Shilov hyperbolic for every $\gamma \in \pi_1(\Sigma)$. The set of maximal representations which are Shilov hyperbolic will be denoted by

$\mathrm{Hom}^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n,\mathbb{R}))$. We define $\chi^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n,\mathbb{R}))$ as the quotient

$$\chi^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n,\mathbb{R})) := \mathrm{Hom}^{\max,\mathrm{Shilov}}(\pi_1(\Sigma), \mathrm{PSp}(2n,\mathbb{R}))/_{\mathrm{PSp}(2n,\mathbb{R})}$$

where $\mathrm{PSp}(2n,\mathbb{R})$ is acting by conjugation: $\rho \sim \rho'$ if there exists $g \in \mathrm{PSp}(2n,\mathbb{R})$ such that $\rho(\gamma) = g\rho'(\gamma)g^{-1}$ for all $\gamma \in \pi_1(\Sigma)$.

We want to introduce the definition of a maximal representation from the group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ into $\mathrm{PSp}^{\pm}(2n,\mathbb{R})$.

Notation 7.7. For the rest of the thesis the group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle s_1, s_2, s_3 |\ s_1^2 = s_2^2 = s_3^2 = 1\rangle$ will be denoted by W_3.

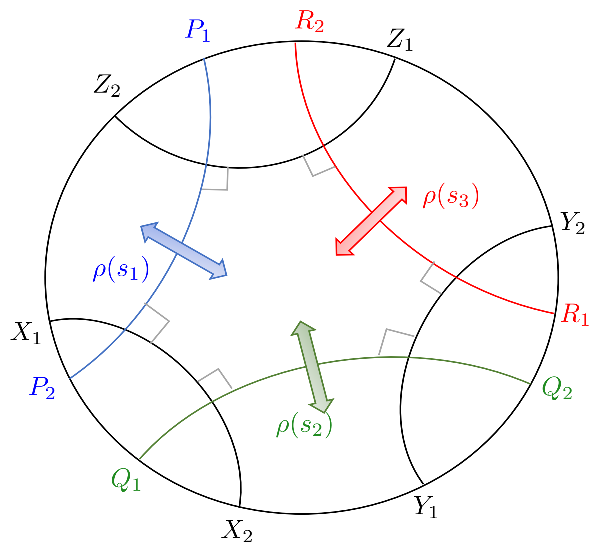


Figure 59: The reflections $\rho(s_1), \rho(s_2), \rho(s_3)$ for $\rho : W_3 \to \mathrm{PSp}^{\pm}(4,\mathbb{R})$ maximal

Definition 7.8. Let $W_3 = \langle s_1, s_2, s_3 |\ s_1^2 = s_2^2 = s_3^2 = 1\rangle$. A representation

$$\rho : W_3 \to \mathrm{PSp}^{\pm}(2n,\mathbb{R})$$

is *maximal* if there exists a maximal 6-tuple $(P_1, P_2, Q_1, Q_2, R_1, R_2)$ such that

- $\rho(s_1)$ is a reflection of $\mathcal{X}$ fixing (P_1, P_2) such that

$$\rho(s_1)(X_1) = X_2 \text{ and } \rho(s_1)(Z_1) = Z_2$$

  where X_1, X_2, Z_1, Z_2 are uniquely determined by

$$\mathcal{Y}_{P_1,P_2} \perp \mathcal{Y}_{X_1,X_2} \perp \mathcal{Y}_{Q_1,Q_2} \text{ and } \mathcal{Y}_{R_1,R_2} \perp \mathcal{Y}_{Z_1,Z_2} \perp \mathcal{Y}_{P_1,P_2}$$

- $\rho(s_2)$ is a reflection of $\mathcal{X}$ fixing (Q_1, Q_2) such that

$$\rho(s_2)(X_1) = X_2 \text{ and } \rho(s_2)(Y_1) = Y_2$$

where Y_1, Y_2 are uniquely determined by

$$\mathcal{Y}_{Q_1,Q_2} \perp \mathcal{Y}_{Y_1,Y_2} \perp \mathcal{Y}_{R_1,R_2}$$

- $\rho(s_3)$ is a reflection of $\mathcal{X}$ fixing (R_1, R_2) such that

$$\rho(s_3)(Y_1) = Y_2 \text{ and } \rho(s_3)(Z_1) = Z_2$$

The space of maximal representations will be denoted by $\mathrm{Hom}^{\max}(W_3, \mathrm{PSp}^{\pm}(2n, \mathbb{R}))$. We further define

$$\chi^{\max}(W_3, \mathrm{PSp}^{\pm}(2n, \mathbb{R})) := \mathrm{Hom}^{\max}(W_3, \mathrm{PSp}^{\pm}(2n, \mathbb{R}))/_{\mathrm{PSp}(2n,\mathbb{R})}$$

The geometric properties of a maximal representation defined in 7.8 are illustrated in Figure 59.

Remark 7.9. Observe that given the maximal 6-tuple $(P_1, P_2, Q_1, Q_2, R_1, R_2)$ the set of reflections $\rho(s_i), i = 1, 2, 3$ for which $\rho : W_3 \to \mathrm{PSp}(4, \mathbb{R})^{\pm}$ is maximal as in Definition 7.8 are given by the sets $\mathcal{R}(P_1, Z_2, X_1, P_2), \mathcal{R}(Q_1, X_2, Y_1, Q_2)$ and $\mathcal{R}(R_1, Y_2, Z_1, R_2)$ respectively (recall Proposition 6.18).

Lemma 7.10. *Let* $W_3 = \langle s_1, s_2, s_3 | \ s_1^2 = s_2^2 = s_3^2 = 1\rangle$ *and let* $\rho : W_3 \to \mathrm{PSp}(4, \mathbb{R})^{\pm}$ *be maximal. Then the composition* $\rho(s_i s_j) = \rho(s_i)\rho(s_j)$ *is a Shilov hyperbolic element of* $\mathrm{PSp}(4, \mathbb{R})$ *for any* $i \neq j$ *where* $i, j \in \{1, 2, 3\}$.

Proof. The product of any two reflections is an element of $\mathrm{PSp}(2n, \mathbb{R})$: for two reflections $\rho(s_i) = R_i, \rho(s_j) = R_j$ it holds

$$(R_i R_j)^T J (R_i R_j) = R_j^T R_i^T J R_i R_j = R_j^T (-J) R_j = J$$

Let $\rho : W_3 \to \mathrm{PSp}(4, \mathbb{R})^{\pm}$ be maximal, we want to show that $\rho(s_i)\rho(s_j)$ is Shilov hyperbolic. Without loss of generality let us sassume $i = 1, j = 2$. By definition of maximality (see Definition 7.8) it is clear that $\rho(s_1)\rho(s_2)$ fixes X_1 and X_2, where $(P_1, X_1, P_2, Q_1, X_2, Q_2)$ is a maximal 6-tuple and

$$\mathcal{Y}_{P_1,P_2} \perp \mathcal{Y}_{X_1,X_2} \perp \mathcal{Y}_{Q_1,Q_2}$$

Up to isometry let us consider $(P_1, P_2, Q_1, Q_2, R_1, R_2) = (0, A, \mathrm{Id}, C, D, \infty)$ where A, C, D are positive definite and C is diagonal (Figure 60).
The map $\rho(s_1)\rho(s_2)$ is inside $\mathrm{PSp}(4, \mathbb{R})$ and fixes 0 and ∞. This map is Shilov hyperbolic if and only if there exists a positive definite Y such that $(0, Y, \rho(s_1)\rho(s_2)Y, \infty)$ is maximal (see Lemma 7.4). Let $Y = A$. Then

$$\rho(s_1)\rho(s_2)(A) = \rho(s_1)(A)$$

We want to show that $(0, A, \rho(s_1)A, \infty)$ is maximal. We know $(0, A, C)$ maximal and $\rho(s_1) \in \mathcal{R}(-D, 0, C, D)$ (Remark 7.9). Result follows by Proposition 6.19 and 2.6. $\square$

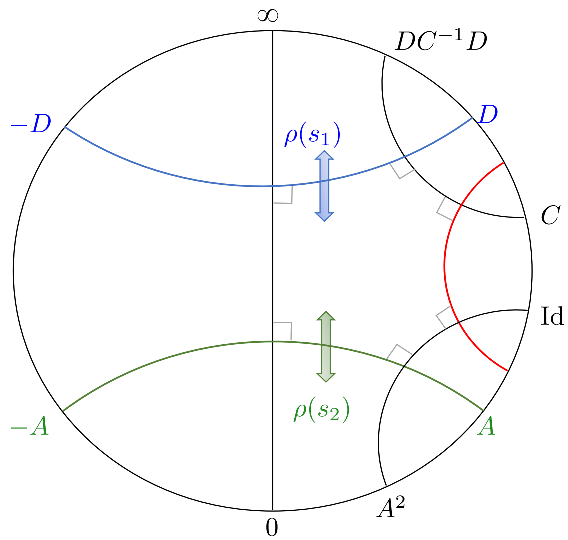


Figure 60: The map $\rho(s_1)\rho(s_2)$ is Shilov hyperbolic

We finish this section with a lemma that will be useful later.

Lemma 7.11. *Let* $\rho : W_3 \to \mathrm{PSp}^{\pm}(4, \mathbb{R})$ *be maximal and let* $(X_1, X_2, Y_1, Y_2, Z_1, Z_2)$ *be a maximal 6-tuple as in Definition 7.8. Then for* $l_1, l_2, l_3, l_4 \in \mathcal{L}(\mathbb{R}^4)$ *it holds:*

(i) If $(X_2, l_1, l_2, l_3, l_4, Z_1)$ *is maximal then* $(Z_2, \rho(s_1)l_4, \rho(s_1)l_3, \rho(s_1)l_2, \rho(s_1)l_1, X_1)$ *is maximal*

(ii) If $(Y_2, l_1, l_2, l_3, l_4, X_1)$ *is maximal then* $(X_2, \rho(s_2)l_4, \rho(s_2)l_3, \rho(s_2)l_2, \rho(s_2)l_1, Y_1)$ *is maximal*

(iii) If $(Z_2, l_1, l_2, l_3, l_4, Y_1)$ *is maximal then* $(Y_2, \rho(s_3)l_4, \rho(s_3)l_3, \rho(s_3)l_2, \rho(s_3)l_1, Z_1)$ *ismaximal*

Proof. Follows directly from Proposition 6.19. Point *(i)* is illustrated in Figure 61.

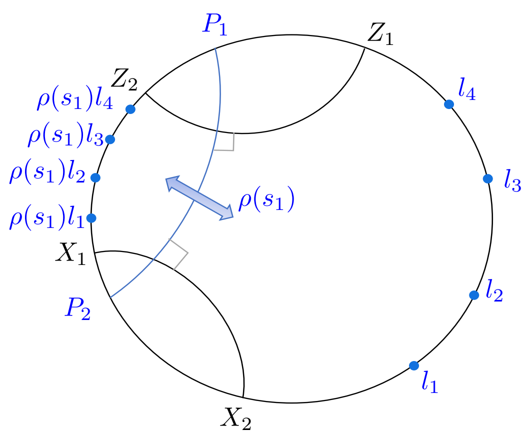


Figure 61: Configuration of *(i)*

7.3 Arc coordinates in classical Teichmüller

Given a hyperbolic surface with boundary, arc coordinates provide a parametrization of the Teichmüller space. They were first introduced by Harer [Har86] and were developed by Penner [Pen87] to decompose decorated Teichmüller space of punctured surface. This decomposition was generalized by [Ush99] [Pen02] for surfaces with boundary. Similar coordinates were used in [Luo07] [Guo09].

Let us recall arc coordinates for classical Teichmüller space $\mathcal{T}(\Sigma)$. Let $\Sigma = \Sigma_{g,m}$ be a compact orientable smooth surface of genus g and m boundary components. We can equip $\Sigma_{g,m}$ with a complete hyperbolic structure of finite volume with geodesic boundary. The universal covering $\widetilde{\Sigma}_{g,m}$ of $\Sigma_{g,m}$ is a closed subset of the hyperbolic plane $\mathbb{H}^2$ where boundary curves are geodesics.

Let us consider a maximal collection $\{a_1, ..., a_k\}$ of pairwise disjoint arcs in $\Sigma_{g,m}$ with starting and ending point on a boundary component which are essential and pairwise non-homotopic (Figure 62). The connected components of $\Sigma_{g,m} \backslash \bigcup_i a_i$ are given by a union of hexagons. Every arc will be called an *edge* of the hexagon decomposition. For every hexagon there are exactly three alternating edges belonging to one boundary component of $\Sigma_{g,m}$. We denote by E the set of all edges, E_{bdry} the set of edges lying on a boundary component and by $\mathcal{H}$ the set of all hexagons of the decomposition. It can be shown that for such a collection $\{a_1, ..., a_k\}$ it holds

$$k = \#E \backslash E_{bdry} = 3|\chi(\Sigma_{g,m})| = 3(2g - 2 + m)$$

and that the number of hexagons is given by

$$\#\mathcal{H} = 2|\chi(\Sigma_{g,m})| = 2(2g - 2 + m)$$

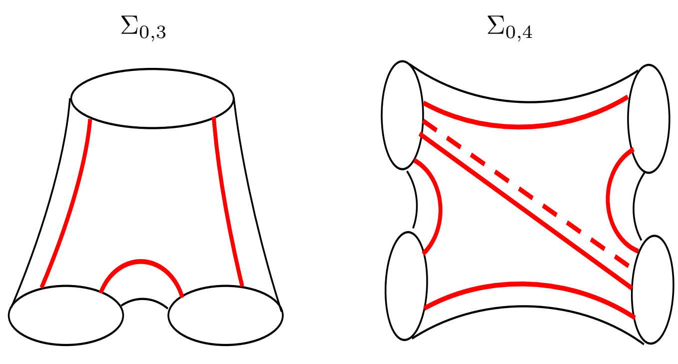


Figure 62: A collection $\{a_1, ..., a_k\}$ for the surfaces $\Sigma_{0,3}$ and $\Sigma_{0,4}$

For a fixed hyperbolic structure we can always realize the hexagon decomposition of $\Sigma_{g,m}$ in a way such that every edge is a geodesic and every arc

$a_i \in \{a_1, ..., a_k\}$ is the unique geodesic which is orthogonal to the boundary at both endpoints. Moreover, we fix an orientation on the boundary components such that the surface lies to the right of the boundary. For each choice of $\{a_1, ..., a_k\}$ we get a parametrization of the Teichmüller space $\mathcal{T}(\Sigma_{g,m})$: once we fix the lengths $l(a_1), ..., l(a_k)$ there is a unique hyperbolic metric that makes $\Sigma_{g,m} \backslash \bigcup_i a_i$ a union of hyperbolic right-angled hexagons where each hexagon has exactly three alternating edges $a_{i_1}, a_{i_2}, a_{i_3}$ in $E \backslash E_{bdry}$ of length $l(a_{i_1}), l(a_{i_2}), l(a_{i_3})$ respectively, where $i_1, i_2, i_3 \in \{1, ..., k\}$. This is due to the well known fact that given three real numbers $b, c, d > 0$ there exists (up to isometries) a unique right-angled hexagon in $\mathbb{H}^2$ with alternating sides of lengths b, c and d (see for example [Mar16, Lemma 6.2.2]). Let us denote by $\Gamma_{g,m}$ the fundamental group $\pi_1(\Sigma_{g,m})$. It is well known (see for example [FM11], [Ara12]) that one can define the Teichmüller space $\mathcal{T}(\Sigma_{g,m})$ as the set of conjugacy classes of discrete and faithful representations ρ where

$$\rho : \Gamma_{g,m} \to \mathrm{PSL}(2, \mathbb{R})$$

In Definition 7.6 we have defined the space $\chi^{\max,\mathrm{Shilov}}(\Gamma_{g,m}, \mathrm{PSp}(2n, \mathbb{R}))$. When $n = 1$ the group $\mathrm{PSp}(2, \mathbb{R})$ coincides with $\mathrm{PSL}(2, \mathbb{R})$. A Shilov hyperbolic element in $\mathrm{PSL}(2, \mathbb{R})$ is conjugated to a matrix of the type

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ |\lambda| > 1$$

A representation $\rho \in \mathrm{Hom}(\Gamma_{g,m}, \mathrm{PSL}(2, \mathbb{R}))$ is discrete and faithful if and only if ρ is maximal and Shilov hyperbolic. The surface $\Sigma_{g,m}$ is then realized by the quotient

$$\Sigma_{g,m} =_{\rho(\Gamma_{g,m})} \backslash \mathbb{H}^2$$

where $\rho(\Gamma_{g,m})$ acts freely and properly discontinuously on $\mathbb{H}^2$. The above discussion asserts that once we fix the lengths $l(\alpha_1), ..., l(\alpha_k)$ we can explicitly write the representation $\rho : \Gamma_{g,m} \to \mathrm{PSL}(2, \mathbb{R}) \in \mathrm{Hom}^{\max,\mathrm{Shilov}}(\Gamma_{g,m}, \mathrm{PSL}(2, \mathbb{R}))$ such that $\Sigma_{g,m} = \rho(\Gamma_{g,m}) \backslash \mathbb{H}^2$. Concrete examples will be given in 7.13 and 7.14.

The fundamental group of the surface $\Sigma_{g,m}$ is isomorphic to a free group.

Lemma 7.12. *Let $\Gamma_{g,m}$ denote the fundamental group $\pi_1(\Sigma_{g,m})$. Then $\Gamma_{g,m}$ is isomorphic to the free group $\mathbb{F}_{2g+m-1}$.*

Proof. It is well known that $\Gamma_{g,m}$ has the following presentation (see for example [Lab13, Theorem 2.3.15] for the case $m = 0$)

$$\Gamma_{g,m} = \langle a_1, b_1, ...a_g, b_g, c_1, ...c_m | \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{m} c_j = 1 \rangle$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ denotes the commutator of a_i and b_i. Then

$\mathbb{F}_{2g+m-1} \cong \Gamma_{g,m}$ through the isomorphism given by

$$\mathbb{F}_{2g+m-1} \to \Gamma_{g,m}$$
$$X_i \mapsto \begin{cases} a_i \text{ for } i = 1, ..., g \\ b_i \text{ for } i = g+1, ..., 2g \\ c_i \text{ for } i = 2g+1, ..., 2g+m-1 \end{cases}$$

□

Example 7.13. (**Pair of pants**) Let $\Gamma_{0,3}$ be the fundamental group $\pi_1(\Sigma_{0,3})$. By Lemma 7.12 we know

$$\Gamma_{0,3} \cong \mathbb{F}_2$$

Let us denote $\Gamma_{0,3} = \langle \alpha, \beta \rangle$ and consider a_1, a_2, a_3 three arcs as in Figure 62 which decompose $\Sigma_{0,3}$ in two hexagons. Once we fix the lengths $l(a_1), l(a_2), l(a_3)$ we can uniquely draw two adjacent isometric hexagons in $\mathbb{H}^2$ up to isometry and we can reconstruct the generators $\rho(\alpha), \rho(\beta)$ of the maximal representation which "closes up" the pair of pants. These are two hyperbolic isometries inside $\mathrm{PSL}(2, \mathbb{R})$. This is illustrated in Figure 63.

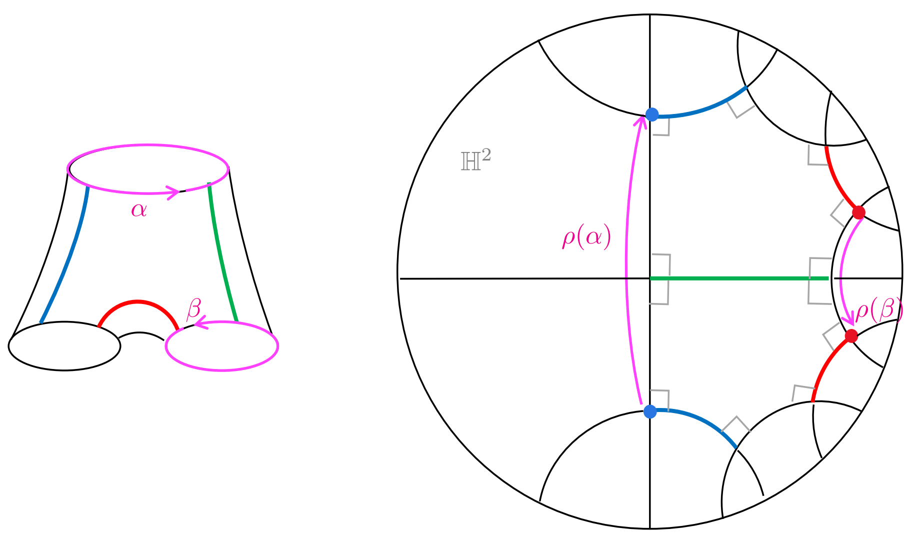


Figure 63: The maximal representation
$\rho : \Gamma_{0,3} \to \mathrm{PSL}(2, \mathbb{R})$

Example 7.14. (**The surface** $\Sigma_{0,4}$) By Lemma 7.12 we know

$$\Gamma_{0,4} \cong \mathbb{F}_3 = \langle \alpha, \beta, \gamma \rangle$$

The procedure to reconstruct the maximal representation is similar to Example 7.13. This is illustrated in Figure 64.

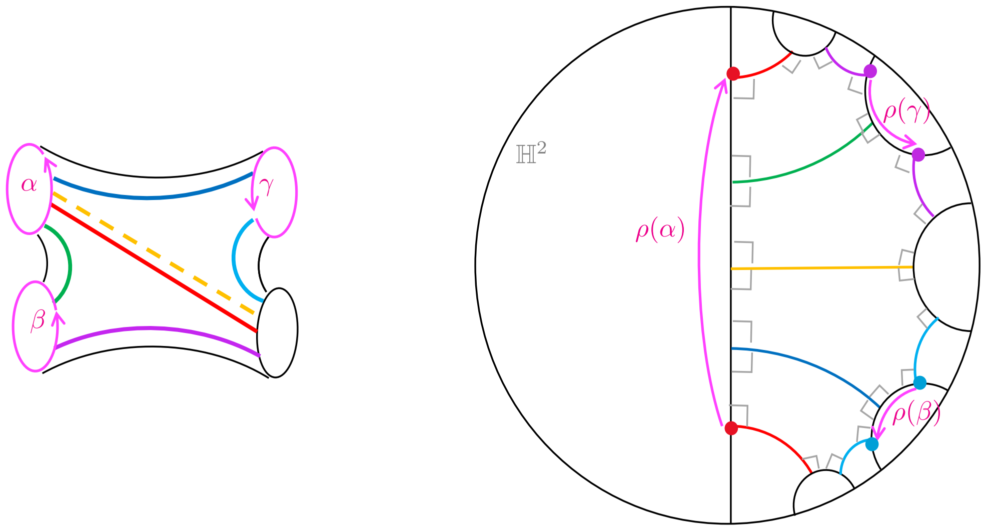


Figure 64: The maximal representation
$\rho : \Gamma_{0,4} \to \mathrm{PSL}(2, \mathbb{R})$

7.4 The group $\Gamma_{0,3}$ as a subgroup of W_3

In Definition 7.6 and 7.8 we have defined the space $\chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(2n, \mathbb{R}))$ and $\chi^{\max}(W_3, \mathrm{PSp}^{\pm}(2n, \mathbb{R}))$ respectively. When $n = 1$ the group $\mathrm{PSp}(2, \mathbb{R})$ coincides with $\mathrm{PSL}(2, \mathbb{R})$. Let W_3 be the Coxeter group

$$W_3 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

The fundamental group $\Gamma_{0,3}$ is isomorphic to the free group $\mathbb{F}_2$ (see Lemma 7.12). The following lemma allows us to see $\Gamma_{0,3}$ as a subgroup of W_3.

Lemma 7.15. *The group W_3 has a normal subgroup Γ isomorphic to the free group $\mathbb{F}_2$.*

Proof. Let $W_3 = \langle s_1, s_2, s_3 | \; s_1^2 = s_2^2 = s_3^2 = 1 \rangle$ and let us consider the subgroup Γ of W_3

$$\Gamma = \langle s_1 s_2, s_2 s_3 \rangle$$

The subgroup Γ is torsion free and result follows by a generalized version of [Löh17, Corollary 4.2.15]. □

Proposition 7.16. *Let $\Gamma_{0,3}, W_3$ be the following groups*

$$\Gamma_{0,3} = \pi_1(\Sigma_{0,3}) = \langle \alpha, \beta \rangle$$

$$W_3 = \langle s_1, s_2, s_3 | \; s_1^2 = s_2^2 = s_3^2 = 1 \rangle$$

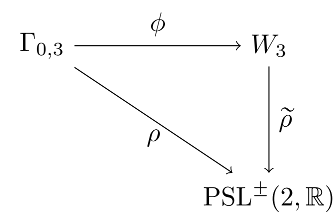
and denote by ϕ the following homomorphism

$$\begin{aligned}\phi : \Gamma_{0,3} &\to W_3\\ \alpha &\mapsto s_1 s_2\\ \beta &\mapsto s_2 s_3\end{aligned}$$

Fix $\widetilde{\rho} \in \mathrm{Hom}^{max}(W_3, \mathrm{PSL}^{\pm}(2,\mathbb{R}))$. It holds

(i) The representation $\rho := \widetilde{\rho}|_{Im(\phi)}$ is inside $\mathrm{Hom}^{max,Shilov}(\Gamma_{0,3}, \mathrm{PSL}(2,\mathbb{R}))$.

(ii) For any $\rho \in \mathrm{Hom}^{max,Shilov}(\Gamma_{0,3}, \mathrm{PSL}(2,\mathbb{R}))$ there exists a unique $\widetilde{\rho} \in \mathrm{Hom}^{max}(W_3, \mathrm{PSL}^{\pm}(2,\mathbb{R}))$ such that $\rho = \widetilde{\rho} \circ \phi$



(iii) The map f defined by

$$\begin{aligned}f : \chi^{max}(W_3, \mathrm{PSL}^{\pm}(2,\mathbb{R})) &\to \chi^{max,Shilov}(\Gamma_{0,3}, \mathrm{PSL}(2,\mathbb{R}))\\ [\widetilde{\rho}] &\mapsto [\widetilde{\rho}|_{Im(\phi)}]\end{aligned}$$

is a homeomorphism.

Proof. *(i)* This will be proven for $\mathrm{PSp}(4,\mathbb{R})$ in Proposition 7.18. The proof for $\mathrm{PSL}(2,\mathbb{R})$ is similar.

(ii) Let $\rho \in \mathrm{Hom}^{\mathrm{max,Shilov}}(\Gamma_{0,3}, \mathrm{PSL}(2,\mathbb{R}))$. Denote by

$$\{(x_1,x_2),(y_1,y_2),(z_1,z_2)\} \subset \partial\mathbb{H}^2$$

the fixed points of $\rho(\alpha), \rho(\beta)$ and $\rho(\beta^{-1}\alpha^{-1})$ respectively. Choose an orientation of the boundary $\partial\mathbb{H}^2$ such that $(x_1,x_2,y_1,y_2,z_1,z_2)$ is positive. In Section 6.1 we have defined a reflection in $\mathbb{H}^2$ as an involution of $\mathrm{SL}^-(2,\mathbb{R})$. Reflections in $\mathbb{H}^2$ fix an infinite geodesic γ (see Proposition 6.6) and are uniquely determined by the endpoints of γ at the boundary of $\mathbb{H}^2$ (Proposition 6.7). For $p,q \in \partial\mathbb{H}^2$ let $\gamma_{p,q}$ denote the infinite geodesic having p,q as endpoints.

Notation 7.17. For $p,q \in \partial\mathbb{H}^2$ we denote $R_{p,q}$ the unique non trivial reflection fixing the infinite geodesic $\gamma_{p,q}$ i.e. $R_{p,q}$ is the unique non trivial isometry such that

$$\gamma_{p,q} \perp \gamma_{x,R_{p,q}(x)}$$

for any $x \in \partial\mathbb{H}^2$.

Let $(p_1, p_2, q_1, q_2, r_1, r_2)$ be the positive 6-tuple inside $\partial\mathbb{H}^2$ uniquely determined by

$$\gamma_{x_1,x_2} \perp \gamma_{p_1,p_2} \perp \gamma_{y_1,y_2} \perp \gamma_{q_1,q_2} \perp \gamma_{z_1,z_2} \perp \gamma_{r_1,r_2} \perp \gamma_{x_1,x_2}$$

Define $\widetilde{\rho} : W_3 \to \mathrm{PSL}^{\pm}(2, \mathbb{R})$ such that (Figure 65)

$$\widetilde{\rho}(s_1) = R_{r_1,r_2}$$

$$\widetilde{\rho}(s_2) = R_{p_1,p_2}$$

$$\widetilde{\rho}(s_3) = R_{q_1,q_2}$$

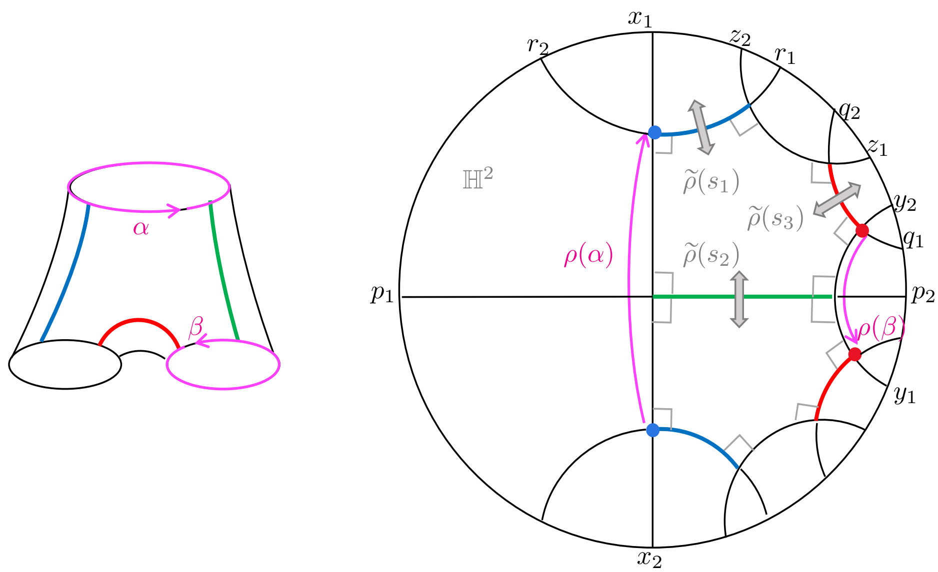


Figure 65: The maximal representation ρ as a restriction of $\widetilde{\rho}$

Then $\widetilde{\rho}$ is maximal. Moreover it is easy to show that (see for example [Mar16] Proposition 6.2.1)

$$\rho(\alpha) = R_{q_1,q_2} \circ R_{p_1,p_2}$$

$$\rho(\beta) = R_{r_1,r_2} \circ R_{q_1,q_2}$$

It follows

$$\widetilde{\rho}(s_1 s_2) = \widetilde{\rho}(s_1)\widetilde{\rho}(s_2) = \rho(\alpha) \text{ and } \widetilde{\rho}(s_2 s_3) = \widetilde{\rho}(s_2)\widetilde{\rho}(s_3) = \rho(\beta)$$

so that $\widetilde{\rho} \circ \phi(\gamma) = \rho(\gamma)$ for all $\gamma \in \Gamma_{0,3}$. It is clear that $\widetilde{\rho}$ is the unique maximal representation such that $\widetilde{\rho} \circ \phi = \rho$.

(iii) This follows directly from *(ii)*. In particular recall that as $\Gamma_{0,3}$ is free, the set of representations $\mathrm{Hom}(\Gamma_{0,3}, \mathrm{PSL}(2,\mathbb{R}))$ can be identified with $\mathrm{PSL}(2,\mathbb{R})^2$ and we can carry over its topology.

□

7.5 The set $\chi^{\mathcal{S}}$

In Definition 7.8 we have defined the set of maximal representations $\mathrm{Hom}^{\max}(W_3, \mathrm{PSp}^{\pm}(4,\mathbb{R}))$ and we know that we can see the fundamental group $\Gamma_{0,3}$ as a subgroup of W_3. In this section we define the set $\chi^{\mathcal{S}} \subset \chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4,\mathbb{R}))$. We start by giving an analogue of Proposition 7.16*(i)* in the case of $\mathrm{PSp}(4,\mathbb{R})$, that is we show that the restriction to $\Gamma_{0,3}$ of a maximal representation as in Definition 7.8 is a maximal and Shilov hyperbolic representation as in Definition 7.6.

Proposition 7.18. *Let $\Gamma_{0,3}, W_3$ be the following groups*

$$\Gamma_{0,3} = \pi_1(\Sigma_{0,3}) = \langle \alpha, \beta \rangle$$
$$W_3 = \langle s_1, s_2, s_3 | \ s_1^2 = s_2^2 = s_3^2 = 1 \rangle$$

and denote by ϕ the following homomorphism

$$\begin{aligned} \phi : \Gamma_{0,3} &\to W_3 \\ \alpha &\mapsto s_1 s_2 \\ \beta &\mapsto s_2 s_3 \end{aligned}$$

Fix $\widetilde{\rho} \in \mathrm{Hom}^{max}(W_3, \mathrm{PSp}^{\pm}(4,\mathbb{R}))$. Then the representation $\rho := \widetilde{\rho}|_{Im(\phi)}$ is inside $\mathrm{Hom}^{max,Shilov}(\Gamma_{0,3}, \mathrm{PSp}(4,\mathbb{R}))$.

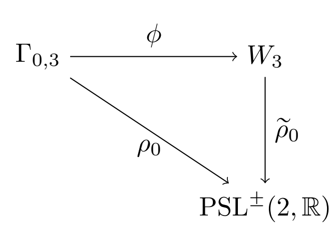
Proof. By Lemma 7.10 we know that $\widetilde{\rho} \circ \phi(\gamma)$ is a Shilov hyperbolic element of $\mathrm{PSp}(4,\mathbb{R})$ for any $\gamma \in \Gamma_{0,3}$. By abuse of notation let us denote the subgroup $\phi(\Gamma_{0,3}) \trianglelefteq W_3$ just as $\Gamma_{0,3}$. Given $\widetilde{\rho} \in \mathrm{Hom}^{\max}(W_3, \mathrm{PSp}^{\pm}(4,\mathbb{R}))$ we want to prove that

$$\rho = \widetilde{\rho} \circ \phi : \Gamma_{0,3} \to \mathrm{PSp}(4,\mathbb{R})$$

is maximal. Fix ρ_0 a hyperbolization of $\Sigma_{0,3}$. Denote

$$\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\} \subset \partial\mathbb{H}^2$$

the fixed points of $\rho_0(\alpha), \rho_0(\beta)$ and $\rho_0(\beta^{-1}\alpha^{-1})$ respectively. Choose an orientation of the boundary $\partial\mathbb{H}^2$ such that $(x_1, x_2, y_1, y_2, z_1, z_2)$ is positive. By Proposition 7.16*(ii)* we know that there is a unique way to extend the action of $\Gamma_{0,3}$ on $\mathbb{H}^2$ (and on its boundary) to the group W_3 making the following diagram commute



where $\widetilde{\rho}_0$ is maximal. For simplicity given $s \in W_3$ and $p \in \partial\mathbb{H}^2$ we will denote the action $\widetilde{\rho}_0(s) \cdot p$ simply as $s \cdot p$.

Since $\widetilde{\rho} : W_3 \to \mathrm{PSp}^{\pm}(4, \mathbb{R})$ is maximal we know that there exists a maximal 6-tuple $(P_1, P_2, Q_1, Q_2, R_1, R_2)$ satisfying the conditions of Definition 7.8. Let us denote $(X_1, X_2, Y_1, Y_2, Z_1, Z_2)$ the Lagrangians (see Figure 59) such that

$$\mathcal{Y}_{P_1,P_2} \perp \mathcal{Y}_{X_1,X_2} \perp \mathcal{Y}_{Q_1,Q_2} \perp \mathcal{Y}_{Y_1,Y_2} \perp \mathcal{Y}_{R_1,R_2} \perp \mathcal{Y}_{Z_1,Z_2} \perp \mathcal{Y}_{P_1,P_2}$$

We define the following sets

$$H^{\mathbb{H}^2} := \{x_1, x_2, y_1, y_2, z_1, z_2\}$$

$$H^{\mathcal{L}} := \{X_1, X_2, Y_1, Y_2, Z_1, Z_2\}$$

$$\mathcal{O}_n^{\mathbb{H}^2} := \bigcup_{|s| \leqslant n} s \cdot H^{\mathbb{H}^2},\ s \in W_3$$

$$\mathcal{O}_n^{\mathcal{L}} := \bigcup_{|s| \leqslant n} \rho(s) \cdot H^{\mathcal{L}}$$

Define $\xi_n : \mathcal{O}_n^{\mathbb{H}^2} \to \mathcal{O}_n^{\mathcal{L}}$ such that

$$\begin{cases} \Big(\xi_n(x_1), \xi_n(x_2), \xi_n(y_1), \xi_n(y_2), \xi_n(z_1), \xi_n(z_2)\Big) = (X_1, X_2, Y_1, Y_2, Z_1, Z_2) \\ \xi_n(s \cdot p) = \rho(s)\xi_n(p) \text{ for } s \in W_3, |s| \leqslant n,\ p \in H^{\mathbb{H}^2} \end{cases}$$

We will show that the map ξ_n is monotone by induction on n.

<u>$\boldsymbol{n = 0}$</u>: From the definition of ξ it is clear that the map $\xi_0 : H^{\mathbb{H}^2} \to H^{\mathcal{L}}$ is monotone.

<u>$\boldsymbol{n = 1}$</u>: We obtain the map $\xi_1 : \mathcal{O}_1^{\mathbb{H}^2} \to \mathcal{O}_1^{\mathcal{L}}$ where

$$\mathcal{O}_1^{\mathbb{H}^2} = H^{\mathbb{H}^2} \cup \{s_1 H^{\mathbb{H}^2}, s_2 H^{\mathbb{H}^2}, s_3 H^{\mathbb{H}^2}\}$$

and

$$\mathcal{O}_1^{\mathcal{L}} = H^{\mathcal{L}} \cup \{\rho(s_1) H^{\mathcal{L}}, \rho(s_2) H^{\mathcal{L}}, \rho(s_3) H^{\mathcal{L}}\}$$

The set $\mathcal{O}_1^{\mathbb{H}^2}$ is given by $H^{\mathbb{H}^2}$ together with other six points, two for every $s_i H^{\mathbb{H}^2}, i \in \{1, 2, 3\}$. For $s_1 H^{\mathbb{H}^2}$ we only add the two points $\{s_1 y_1, s_1 y_2\}$ as

$$s_1 x_1 = x_2, s_1 x_2 = x_1 \text{ and } s_1 z_1 = z_2, s_1 z_2 = z_1$$

The same holds for $s_2 H^{\mathbb{H}^2}$ and $s_3 H^{\mathbb{H}^2}$. This is illustrated in Figure 66.

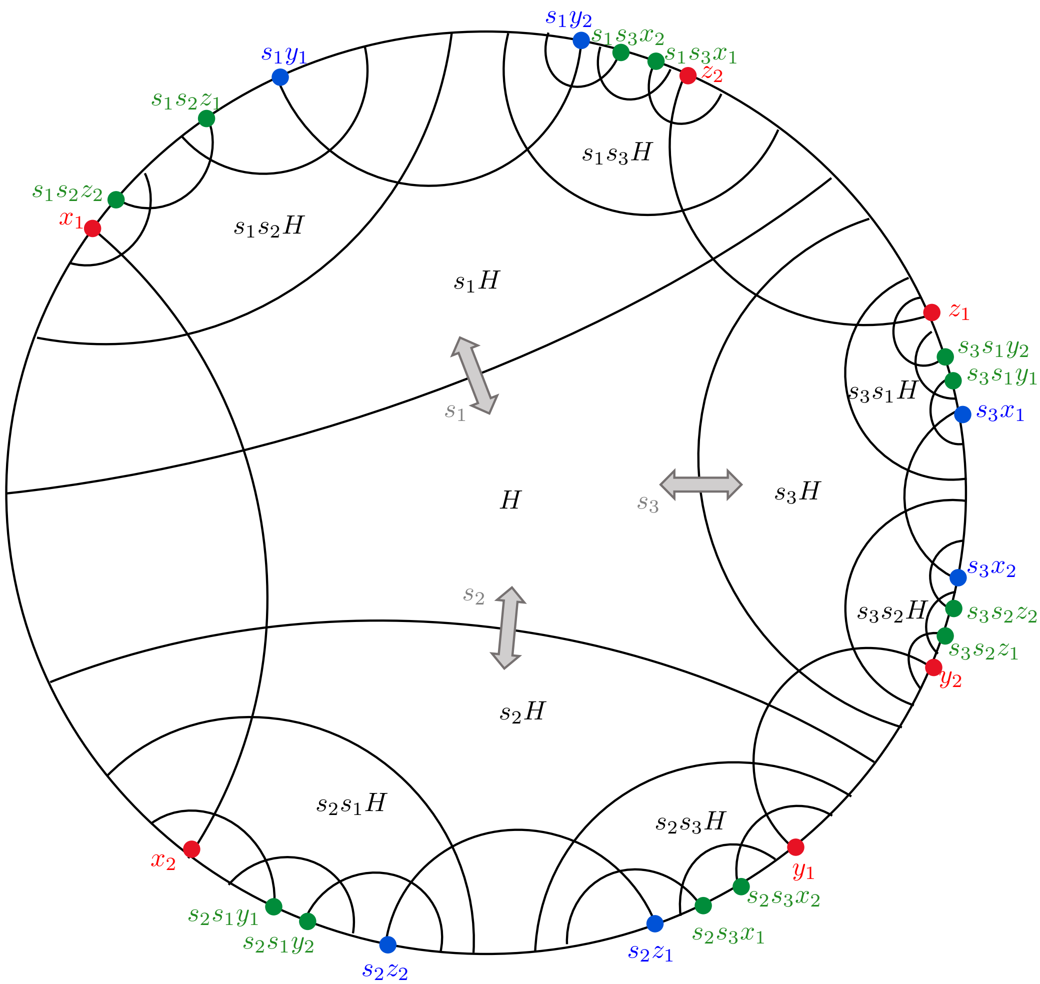


Figure 66: Configuration of $\mathcal{O}_2^{\mathbb{H}^2}$

The set $\mathcal{O}_1^{\mathbb{H}^2}$ is therefore formed by 12 points. The order on $\mathcal{O}_1^{\mathbb{H}^2}$ is given by the orientation of $\partial\mathbb{H}^2$. To explicitly write $\mathcal{O}_1^{\mathbb{H}^2}$ as a positive 12-tuple we use Proposition 6.8 to show that the quadruples

$$(z_2, s_1y_2, s_1y_1x_1), (x_2, s_2z_2, s_2z_1, y_1) \text{ and } (y_2, s_3x_2, s_3x_1, z_1)$$

are positive. We obtain the following positive 12-tuple:

$$\mathcal{O}_1^{\mathbb{H}^2} = (z_2, s_1y_2, s_1y_1, x_1, x_2, s_2z_2, s_2z_1, y_1, y_2, s_3x_2, s_3x_1, z_1)$$

Similarly, the set $\mathcal{O}_1^{\mathcal{L}}$ consists of 12 Lagrangians: it is given by $H^{\mathcal{L}}$ together with six Lagrangians, two for every $\rho(s_i)H^{\mathcal{L}}$. For $\rho(s_1)H^{\mathcal{L}}$ we only add the Lagrangians $\{\rho(s_1)Y_1, \rho(s_1)Y_2\}$: by definition of ρ we know that (Definition 7.8)

$$\rho(s_1)X_1 = X_2, \rho(s_1)X_2 = X_1 \text{ and } \rho(s_1)Z_1 = Z_2, \rho(s_1)Z_2 = Z_1$$

and the same is true for $\rho(s_2)H^{\mathcal{L}}$,$\rho(s_3)H^{\mathcal{L}}$. To prove the monotonic behaviour of ξ_1 we need to show maximality of the 12-tuple

$$\mathcal{O}_1^{\mathcal{L}} = (Z_2, \rho(s_1)Y_2, \rho(s_1)Y_1, X_1, X_2, \rho(s_2)Z_2, \rho(s_2)Z_1, Y_1, Y_2, \rho(s_3)X_2, \rho(s_3)X_1, Z_1)$$

we use Lemma 7.11 to show that the three quadruples

$$\big(Z_2, \rho(s_1)Y_2, \rho(s_1)Y_1, X_1\big),\ \big(X_2, \rho(s_2)Z_2, \rho(s_2)Z_1, Y_1\big),\ \big(Y_2, \rho(s_3)X_2, \rho(s_3)X_1, Z_1\big)$$

are maximal . We use Lemma 2.6 to deduce that the 12-tuple is therefore maximal.

Assume true for n show true for $n+1$: Assuming ξ_n monotone we consider the map

$$\xi_{n+1} : \mathcal{O}_{n+1}^{\mathbb{H}^2} \to \mathcal{O}_{n+1}^{\mathcal{L}}$$

We will first study the set $\mathcal{O}_{n+1}^{\mathbb{H}^2}$ describing how to obtain it from $\mathcal{O}_n^{\mathbb{H}^2}$ and how to write its positive order (*Claim 1* and *Claim 2*).

The set $\mathcal{O}_{n+1}^{\mathbb{H}^2}$ is given by

$$\mathcal{O}_{n+1}^{\mathbb{H}^2} = \mathcal{O}_n^{\mathbb{H}^2} \cup \big\{s \cdot H,\ |s| = n+1\big\}$$

If we fix an element $s \in W_3$ such that $|s| = n+1$ and look at the set $\{s \cdot H^{\mathbb{H}^2}\}$ we are adding exactly two points inside $\mathcal{O}_{n+1}^{\mathbb{H}^2}$ both lying between two points contained in $\mathcal{O}_n^{\mathbb{H}^2}$. This is made precise in the following two statements

Claim 1: For any $s \in W_3$ such that $|s| = n+1$ it holds

$$|\mathcal{O}_n^{\mathbb{H}^2} \cup \{s \cdot H^{\mathbb{H}^2}\}| = |\mathcal{O}_n^{\mathbb{H}^2}| + 2$$

Claim 2: Let $s = ws_i$ where $|w| = n$ and $s_i \in W_3$. It holds

1. If $s_i = s_1$ then the two points $ws_1 \cdot H^{\mathbb{H}^2}$ added inside $\mathcal{O}_{n+1}^{\mathbb{H}^2}$ are $\{ws_1y_1, ws_1y_2\}$ and are such that

$$(wz_2, ws_1y_2, ws_1y_1, wx_1) \text{ positive if } n \text{ even}$$

$$(wx_1, ws_1y_1, ws_1y_2, wz_2) \text{ positive if } n \text{ odd}$$

2. If $s_i = s_2$ then the two points $ws_2 \cdot H^{\mathbb{H}^2}$ added inside $\mathcal{O}_{n+1}^{\mathbb{H}^2}$ are $\{ws_2z_1, ws_2z_2\}$ and are such that

$$(wx_2, ws_2z_2, ws_2z_1, wy_1) \text{ positive if } n \text{ even}$$

$$(wy_1, ws_2z_1, ws_2z_2, wx_2) \text{ positive if } n \text{ odd}$$

3. If $s_i = s_3$ then the two points $ws_3 \cdot H^{\mathbb{H}^2}$ added inside $\mathcal{O}^{\mathbb{H}^2}_{n+1}$ are $\{ws_3x_1, ws_3x_2\}$ and are such that

$$(wy_2, ws_3x_2, ws_3x_1, wz_1) \text{ positive if } n \text{ even}$$

$$(wz_1, ws_3x_1, ws_3x_2, wy_2) \text{ positive if } n \text{ odd}$$

Proof of Claim 1 Let $s \in W_3$ such that $|s| = n+1$ and consider the set $s \cdot H^{\mathbb{H}^2}$. Let us suppose that s ends with the element s_1 i.e. we can write

$$s = ws_1, \quad \text{for a } w \in W_3, \ |w| = n$$

Among the six points $ws_1 \cdot H^{\mathbb{H}^2} = \{ws_1x_1, ws_1x_2, ws_1y_1, ws_1y_2, ws_1z_1, ws_1z_2\}$ we know

$$ws_1x_1 = wx_2, \ ws_1x_2 = wx_1 \text{ and } ws_1z_1 = wz_2, \ ws_1z_2 = wz_1$$

so that $\{ws_1x_1, ws_1x_2, ws_1z_1, ws_1z_2\} \subset \mathcal{O}^{\mathbb{H}^2}_n$. In particular

$$\mathcal{O}^{\mathbb{H}^2}_n \cup \{ws_1 \cdot H^{\mathbb{H}^2}\} = \mathcal{O}^{\mathbb{H}^2}_n \cup \{ws_1y_1, ws_2y_2\}$$

A similar proof holds for $s = ws_2$ and $s = ws_3$.

Proof of Claim 2 Let us show 1. In the proof of *Claim 1* we have already shown that the two points added inside $\mathcal{O}^{\mathbb{H}^2}_{n+1}$ are $\{ws_1y_1, ws_1y_2\}$. We know that we can write $\mathcal{O}^{\mathbb{H}^2}_1$ as the positive 12-tuple (inductive step $n = 1$)

$$\mathcal{O}^{\mathbb{H}^2}_1 = (z_2, s_1y_2, s_1y_1, x_1, x_2, s_2z_2, s_2z_1, y_1, y_2, s_3x_2, s_3x_1, z_1)$$

In particular $(z_2, s_1y_2, s_1y_1, x_1)$ is positive. Let $w = w_n \cdot ... \cdot w_1$, where $w_i \in \{s_1, s_2, s_3\}$. At every step

$$(w_1z_2, w_1s_1y_2, w_1s_1y_1, w_1x_1) \to (w_1w_2z_2, w_1w_2s_1y_2, w_1w_2s_1y_1, w_1w_2x_1) \to ...$$

$$... \to (w_1w_2...w_nz_2, w_1w_2...w_ns_1y_2, w_1w_2...w_ns_1y_1, w_1w_2...w_nx_1)$$

we satisfy the conditions of Proposition 6.8. It follows that the image under $s = ws_1$ of the positive quadruple $(z_2, s_1y_2, s_1y_1, x_1)$ stays positive if n even and is negative if n odd. 2. and 3. are similar.

We now want to state similar statements for the set $\mathcal{O}^{\mathcal{L}}_{n+1}$.

Claim 3: For any $s \in W_3$ such that $|s| = n+1$ it holds

$$|\mathcal{O}^{\mathcal{L}}_n \cup \{\rho(s) \cdot H^{\mathcal{L}}\}| = |\mathcal{O}^{\mathcal{L}}_n| + 2$$

Claim 4: Let $s = ws_i$ where $|w| = n$ and $s_i \in W_3$. It holds

1. If $s_i = s_1$ then the two Lagrangians $\rho(ws_1) \cdot H^{\mathcal{L}}$ added inside $\mathcal{O}^{\mathcal{L}}_{n+1}$ are $\{\rho(ws_1)Y_1, \rho(ws_1)Y_2\}$ and are such that

$$(\rho(w)Z_2, \rho(ws_1)Y_2, \rho(ws_1)Y_1, \rho(w)X_1) \text{ maximal if } n \text{ even}$$

$$(\rho(w)X_1, \rho(ws_1)Y_1, \rho(ws_1)Y_2, \rho(w)Z_2) \text{ maximal if } n \text{ odd}$$

2. If $s_i = s_2$ then the two Lagrangians $\rho(ws_2) \cdot H^{\mathbb{H}^2}$ added inside $\mathcal{O}^{\mathcal{L}}_{n+1}$ are $\{\rho(ws_2)Z_1, \rho(ws_2)Z_2\}$ and are such that

$$(\rho(w)X_2, \rho(ws_2)Z_2, \rho(ws_2)Z_1, \rho(w)Y_1) \text{ maximal if } n \text{ even}$$

$$(\rho(w)Y_1), \rho(ws_2)Z_1, \rho(ws_2)Z_2, \rho(w)X_2) \text{ maximal if } n \text{ odd}$$

3. If $s_i = s_3$ then the two Lagrangians $\rho(ws_3) \cdot H^{\mathbb{H}^2}$ added inside $\mathcal{O}^{\mathcal{L}}_{n+1}$ are $\{\rho(ws_3)X_1, rho(ws_3)X_2\}$ and are such that

$$(\rho(w)Y_2, \rho(ws_3)X_2, \rho(ws_3)X_1, \rho(w)Z_1) \text{ maximal if } n \text{ even}$$

$$(\rho(w)Z_1, \rho(ws_3)X_1, \rho(ws_3)X_2, \rho(w)Y_2) \text{ maximal if } n \text{ odd}$$

Proof of Claim 3 The proof is similar to *Claim 1* where we change s with $\rho(s)$ and x_i, y_i, z_i with X_i, Y_i, Z_i and follows directly from the definition of ρ (Definition 7.8).

Proof of Claim 4 Let us show 1. By definition of ρ it is clear that the two Lagrangians added inside $\mathcal{O}^{\mathcal{L}}_{n+1}$ are $\{\rho(ws_1)Y_1, \rho(ws_1)Y_2\}$. We know that $(Z_2, \rho(s_1)Y_2, \rho(s_1)Y_1, X_1)$ is maximal (inductive step $n = 1$). Let $w = w_n \cdot ... \cdot w_1$, where $w_i \in \{s_1, s_2, s_3\}$. At every step

$$(\rho(w_1)Z_2, \rho(w_1s_1)Y_2, \rho(w_1s_1)Y_1, \rho(w_1)X_1) \to (\rho(w_1w_2)Z_2, \rho(w_1w_2s_1)Y_2, \rho(w_1w_2s_1)Y_1, \rho(w_1w_2)X_1) \to ...$$

$$... \to (\rho(w_1w_2...w_n)Z_2, \rho(w_1w_2...w_ns_1)Y_2, \rho(w_1w_2...w_ns_1)Y_1, \rho(w_1w_2...w_n)X_1)$$

we satisfy the conditions of Proposition 6.19. It follows that the image under $\rho(s) = \rho(ws_1)$ of the maximal quadruple $(Z_2, \rho(s_1)Y_2, \rho(s_1)Y_1, X_1)$ is maximal if n is even and is minimal if n is odd. 2. and 3. are similar.

The map $\xi_n : \mathcal{O}^{\mathbb{H}^2}_{n+1} \to \mathcal{O}^{\mathcal{L}}_n$ is monotone by inductive hypothesis. In *Claim 3* and *Claim 4* we have proven that the set $\mathcal{O}^{\mathcal{L}}_{n+1}$ is obtained in the following way: for any s of length $n+1$ we add two Lagrangians l_1, l_2 in a way such that (a, l_1, l_2, b) maximal for $a, b \in \mathcal{O}^{\mathcal{L}}_n$. By Lemma 2.6 it is easy to see that ξ_{n+1} is monotone on the entire set $\mathcal{O}^{\mathbb{H}^2}_{n+1}$. We have proven that

$$\xi_n : \mathcal{O}^{\mathbb{H}^2}_{n+1} \to \mathcal{O}^{\mathcal{L}}_n$$

is monotone for any $n \geqslant 0$ and it is ρ-equivariant by definition. Using the same approach of [BIW10] it can be shown that ξ_n can be extended to a map ξ defined on S^1 such that ξ maximizes the Toledo invariant. To prove maximality we use [Str15, Theorem 1.1.5.]. □

Definition 7.19. (**The set** $\chi^{\mathcal{S}}$) Let $\Gamma_{0,3}, W_3$ be the following groups

$$\Gamma_{0,3} = \pi_1(\Sigma_{0,3}) = \langle \alpha, \beta \rangle$$

$$W_3 = \langle s_1, s_2, s_3 | \ s_1^2 = s_2^2 = s_3^2 = 1 \rangle$$

and denote by ϕ the following homomorphism

$$\begin{aligned} \phi : \Gamma_{0,3} &\to W_3 \\ \alpha &\mapsto s_1 s_2 \\ \beta &\mapsto s_2 s_3 \end{aligned}$$

Let f be the map

$$\begin{aligned} f : \chi^{\max}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R})) &\to \chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4, \mathbb{R})) \\ \left[\widetilde{\rho}\right] &\mapsto \left[\widetilde{\rho}|_{Im(\phi)}\right] \end{aligned}$$

which is well defined by Proposition 7.18. We define $\chi^{\mathcal{S}}$ as the set

$$\chi^{\mathcal{S}} := Im(f)$$

Remark 7.20. Contrary to the $\mathrm{PSL}(2, \mathbb{R})$ case (see Proposition 7.16) the map f is not injective nor surjective. This will be proven in Corollary 7.24.

7.6 Parameter space for $\chi^{\mathbf{max}}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R}))$

Let $\mathcal{X}$ be the symmetric space associated to $\mathrm{Sp}(4, \mathbb{R})$. The space of ordered right-angled hexagons $\mathcal{H}$ inside $\mathcal{X}$ is parametrized by the set $\mathcal{A}$ defined in Theorem 4.26. Recall that we denote by R_{st}, R_{ex} and $\mathcal{K}$ the following matrices:

$$R_{st} = \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \ R_{ex} = \begin{pmatrix} -r & 0 \\ 0 & r \end{pmatrix}, \ r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{K} = \left\{ \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix}, \ K \in \mathrm{PO}(2), \ K^2 = \mathrm{Id} \right\}$$

Observe that $\{R_{st}, R_{ex}\} \subset \mathcal{K}$. Recall also that we denote by $F_{\underline{b},\underline{d},\alpha_1,\alpha_2}$ the malefic map defined in 5.12.

Theorem 7.21. *The set* $\chi^{max}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R}))$ *is parametrized by the parameter space* $\mathcal{S}$*:*

$$\mathcal{S} \subset \mathcal{A} \times \mathcal{K}^3$$

consisting of points $\left(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3\right)$ *in* $\mathcal{A} \times \mathcal{K}^3$ *such that*

$$\begin{cases} \underline{d} \in \mathfrak{a} \Rightarrow R_1 \in \{R_{st}, R_{ex}\} \\ \underline{b} \in \mathfrak{a} \Rightarrow R_2 \in \{R_{st}, R_{ex}\} \\ F_{\underline{b},\underline{d},\alpha_1,\alpha_2}(\underline{c}) \in \mathfrak{a} \Rightarrow R_3 \in \{R_{st}, R_{ex}\} \end{cases}$$

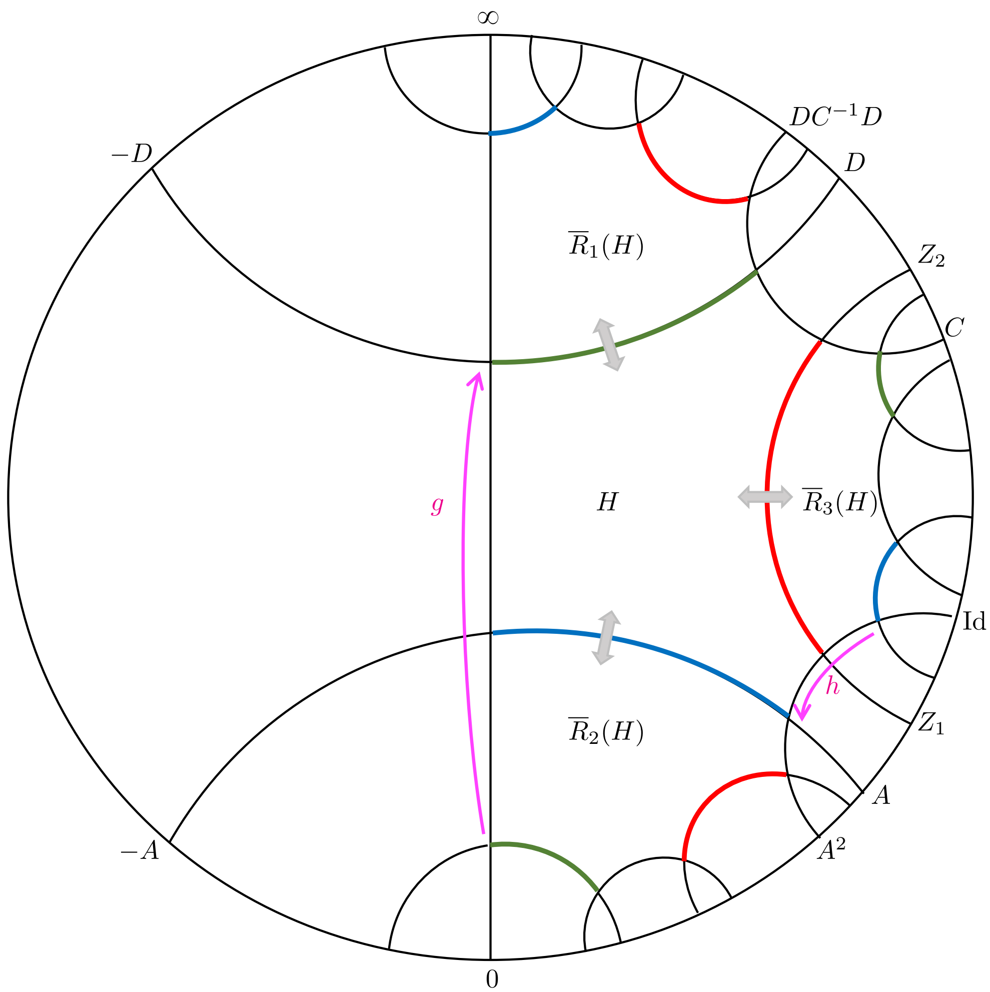


Figure 67: Configuration of the hexagons
$H, \overline{R}_1(H), \overline{R}_2(H)$ and $\overline{R}_3(H)$

Proof. **From parameters to representations**: Let $\left(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3\right)$ be a point inside $\mathcal{S}$. Let

$$(H, \mathcal{Y}_{0,\infty}) = (0, A, \mathrm{Id}, C, D, \infty)$$

be a right-angled hexagon with arc coordinates $\mathcal{A}(H, \mathcal{Y}_{0,\infty}) = (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$. In particular the maximal 12-tuple associated to $(H, \mathcal{Y}_{0,\infty})$ is given by

$$H = (\infty, -D, -A, 0, A^2, A, Z_1, \mathrm{Id}, C, Z_2, D, DC^{-1}D)$$

where Z_1, Z_2 are uniquely defined by requiring

$$\mathcal{Y}_{A^2,\mathrm{Id}} \perp \mathcal{Y}_{Z_1,Z_2} \perp \mathcal{Y}_{C,DC^{-1}D}$$

Let g_1, g_2, g_3 be isometries such that

$$g_1(D, DC^{-1}D, \infty, -D) = (0, \mathrm{Id}, Y_1, \infty)$$

$$g_2(-A, 0, A^2, A) = (0, \mathrm{Id}, Y_2, \infty)$$
$$g_3(Z_1, \mathrm{Id}, C, Z_2) = (0, \mathrm{Id}, Y_3, \infty)$$

for Y_1, Y_2, Y_3 diagonal matrices inside $\mathrm{Sym}^+(2, \mathbb{R})$. Put

$$\overline{R}_1 := g_1^{-1} R_1 g_1$$
$$\overline{R}_2 := g_2^{-1} R_2 g_2$$
$$\overline{R}_3 := g_3^{-1} R_3 g_3$$

By Corollary 6.22 we know that $\overline{R}_1, \overline{R}_2, \overline{R}_3$ belong to $\mathcal{R}_{\mathcal{Y}_{-D,D}}^{\mathcal{Y}_{0,\infty}, \mathcal{Y}_{C,DC^{-1}D}}, \mathcal{R}_{\mathcal{Y}_{-A,A}}^{\mathcal{Y}_{0,\infty}, \mathcal{Y}_{A^2,\mathrm{Id}}}, \mathcal{R}_{\mathcal{Y}_{Z_1,Z_2}}^{\mathcal{Y}_{A^2,\mathrm{Id}}, \mathcal{Y}_{C,DC^{-1}D}}$ respectively.
Let $W_3 = \langle s_1, s_2, s_3 |\ s_1^2 = s_2^2 = s_3^2 = 1\rangle$ and define $\widetilde{\rho}$ as the representation

$$\begin{aligned}\widetilde{\rho} : W_3 &\to \mathrm{PSp}^{\pm}(4, \mathbb{R}) \\ s_1 &\mapsto \overline{R}_1 \\ s_2 &\mapsto \overline{R}_2 \\ s_3 &\mapsto \overline{R}_3\end{aligned}$$

The representation $\widetilde{\rho}$ is maximal by construction (see Definition 7.8). The images $\overline{R}_i(H)$ for $i \in \{1, 2, 3\}$ are drawn in Figure 67. The maps g and h appearing in the Figure are the generators of the representation restricted to the group $\Gamma_{0,3}$. This will be explained in Theorem 7.23.

From representations to parameters: Let $\widetilde{\rho} \in \chi^{\max}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R}))$ and denote again

$$W_3 = \langle s_1, s_2, s_3 |\ s_1^2 = s_2^2 = s_3^2 = 1\rangle$$

We know that $\widetilde{\rho}$ has the properties described in Definition 7.8: we can determine a right-angled hexagon $(H, \mathcal{Y}_{X_1,X_2})$ where X_1, X_2 are as in Figure 59. We compute the arc coordinates $\mathcal{A}(H, \mathcal{Y}_{X_1,X_2})$. By definition we know that $\widetilde{\rho}(s_1), \widetilde{\rho}(s_2)$ and $\widetilde{\rho}(s_3)$ belong to the reflection sets of three alternating sides of this hexagon. We compute the corresponding elements in $\mathcal{K}^3$ (one for every side) using Corollary 6.22. □

7.7 Parameter space for $\chi^{\mathcal{S}}$

In Theorem 7.21 we have given a parametrization $\mathcal{S} \subset \mathcal{A} \times \mathcal{K}^3$ of the set $\chi^{\max}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R}))$. Recall that the set $\chi^{\mathcal{S}}$ is defined (Definition 7.19) as $\chi^{\mathcal{S}} := Im(f)$ where f is the map

$$\begin{aligned}f : \chi^{\max}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R})) &\to \chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4, \mathbb{R})) \\ [\widetilde{\rho}] &\mapsto [\widetilde{\rho}|_{Im(\phi)}]\end{aligned}$$

We want to use the parametrization $\mathcal{S}$ of Theorem 7.21 to parametrize $\chi^{\mathcal{S}}$. This will be done by imposing an equivalent relation on $\mathcal{S}$ identifying the points that have same image under f. Recall that in Proposition 4.7 we have described the set $\mathcal{D} \subset \mathcal{A}$ corresponding to right-angled hexagons in $\mathcal{X}$ lying inside a maximal polydisc. Let us give the following

Definition 7.22. We define $\mathcal{S}_0 \subset \mathcal{A} \times \mathcal{K}^3$ as the set

$$\mathcal{S}_0 := \Big\{(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3) \in \mathcal{A} \times \mathcal{K}^3 | \ (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2]) \in \mathcal{D}\Big\}$$

We further define the following equivalent relation on $\mathcal{S}_0$:

$$(p, R_1, R_2, R_3) \sim (p, R_1', R_2', R_3') \iff \begin{cases} R_1 R_2 = R_1' R_2' \\ R_2 R_3 = R_2' R_3' \end{cases} \tag{28}$$

Theorem 7.23. *The set $\chi^{\mathcal{S}}$ is parametrized by the parameter space $\mathcal{S}/_\sim$ where $\mathcal{S}$ consists of points $\big(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3\big)$ in $\mathcal{A} \times \mathcal{K}^3$ such that*

$$\begin{cases} \underline{d} \in \mathfrak{a} \Rightarrow R_1 \in \{R_{st}, R_{ex}\} \\ \underline{b} \in \mathfrak{a} \Rightarrow R_2 \in \{R_{st}, R_{ex}\} \\ F_{\underline{b},\underline{d},\alpha_1,\alpha_2}(\underline{c}) \in \mathfrak{a} \Rightarrow R_3 \in \{R_{st}, R_{ex}\} \end{cases}$$

and if $\big(p, R_1, R_2, R_3\big) \in \mathcal{S}_0$ we put

$$(p, R_1, R_2, R_3) \sim (p', R_1', R_2', R_3')$$

where $\sim$ is the equivalent relation in (28).

Proof. **From parameters to representations**: Let $\big(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3\big)$ be a point inside $\mathcal{S}$. The construction of a maximal $\widetilde{\rho} : W_3 \to \mathrm{PSp}^{\pm}(4, \mathbb{R})$ is identical to the proof of Theorem 7.21. Let ρ be the restriction $\rho = f(\widetilde{\rho})$ where f is the map of Definition 7.19. Then ρ is inside $\chi^{\mathcal{S}}$ and we put

$$\mathcal{S} \ni \big(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3\big) = \rho \in \chi^{\mathcal{S}}$$

In Figure 67 we have denoted

$$g = \widetilde{\rho}(s_1 s_2), \quad h = \widetilde{\rho}(s_2 s_3)$$

The equivalence relation on $\mathcal{S}_0$: Let $\Gamma_{0,3} = \langle \alpha, \beta \rangle$. A representation $\rho : \Gamma_{0,3} \to \mathrm{PSp}(4, \mathbb{R})$ is uniquely determined by the maps $\rho(\alpha), \rho(\beta)$, which are exactly the maps g and h of Figure 67. The isometry g is sending the hexagon $\overline{R}_2(H)$ to $\overline{R}_1(H)$ and the isometry h is sending the hexagon $\overline{R}_3(H)$ to $\overline{R}_2(H)$. The equivalence relation on $\mathcal{S}_0$

$$\mathcal{S}_0 := \Big\{(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3) \in \mathcal{A} \times \mathcal{K}^3 | \ (\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2]) \in \mathcal{D}\Big\}$$

identifies the points for which the map f of Definition 7.19 is not injective. More precisely for two points $s = (p, R_1, R_2, R_3)$ and $s' = (p, R_1', R_2', R_3')$ inside

S_0 we denote $\overline{R}_i, \overline{R'}_i$ the reflections constructed from the parameters s and s' respectively as shown in Figure 67. In Remark 6.29 we have detected the points for which $\overline{R}_2(H) = \overline{R'}_2(H)$, that is

$$\overline{R}_2 = g_2^{-1} R_{st} g_2,\ \overline{R'}_2 = g_2^{-1} R_{ex} g_2 \quad \text{if } p \in \mathcal{D} \backslash \mathcal{D}_{\mathbb{H}^2}$$

where $H, \overline{R}_2(H) = \overline{R'}_2(H)$ are both contained in the model polydisc, and

$$\overline{R}_2, \overline{R'}_2 \in g_2^{-1} \mathcal{K} g_2 \quad \text{if } p \in \mathcal{D}_{\mathbb{H}^2}$$

where $H, \overline{R}_2(H) = \overline{R'}_2(H)$ are both contained in the diagonal disc. It is not hard to show a similar result for $\overline{R}_i, \overline{R'}_i$ when $i \in \{2, 3\}$. Take now two points s, s' inside $\mathcal{S}$ where the corresponding reflections $\overline{R}_i, \overline{R'}_i$ constructed as in proof of Theorem 7.21 are such that

$$\begin{cases} \overline{R}_1(H) = \overline{R'}_1(H) \\ \overline{R}_2(H) = \overline{R'}_2(H) \\ \overline{R}_3(H) = \overline{R'}_3(H) \end{cases}$$

with the hexagons $H, \overline{R}_i(H), \overline{R'}_i(H)$ of Figure 67 all contained in the model polydisc. All the points of the polygonal chains drawn in Figure 57 are aligned. Then there exists exactly two maps $g, \widehat{g}$ sending $\overline{R}_2(H)$ to $\overline{R}_1(H)$ and two maps $h, \widehat{h}$ sending $\overline{R}_3(H)$ to $\overline{R}_2(H)$. This follows directly from Proposition 4.30. We obtain four elements $\langle g, h \rangle, \langle g, \widehat{h} \rangle, \langle \widehat{g}, h \rangle, \langle \widehat{g}, \widehat{h} \rangle$ inside $\chi^{\mathcal{S}}$. But the parameter space $\mathcal{S}$ produces $|\{R_{st}, R_{ex}\}|^3 = 2^3$ different maps. If we don't put the equivalence relation we would be over-counting the number of representations, that is we would construct $\widetilde{\rho}, \widetilde{\rho}' \in \chi^{\max}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R}))$ that have the same image under f (see Definition 7.19).

From representations to parameters: This is as Theorem 7.21.

□

Corollary 7.24. *Let f be the map of Definition 7.19*

$$f : \chi^{max}(W_3, \mathrm{PSp}^{\pm}(4, \mathbb{R})) \to \chi^{max, Shilov}(\Gamma_{0,3}, \mathrm{PSp}(4, \mathbb{R}))$$
$$[\widetilde{\rho}] \mapsto [\widetilde{\rho}|_{Im(\phi)}]$$

Then the map f is neither injective nor surjective.

Proof. It is clear by the proof of Theorem 7.23 that f is not injective. We want to show $\chi^{\mathcal{S}} \subsetneq \chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4, \mathbb{R}))$. The space $\chi^{\mathcal{S}} \subsetneq \chi^{\max,\mathrm{Shilov}}(\Gamma_{0,3}, \mathrm{PSp}(4, \mathbb{R}))$ is 10-dimensional (see for example [AGRW19]). In the parametrization of Theorem 7.23 we see that the set $\mathcal{S}$ is immersed in a smaller dimensional space. □

8 Python program

In the github repository https://github.com/martamagnani/Arc-coord/blob/main/Param_for_chiS.py we provide a Python program with output the maps g, h uniquely determining the maximal representation constructed in the proof of Theorem 7.23. More precisely for a given $\big(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2], R_1, R_2, R_3\big) \in \mathcal{A} \times \mathcal{K}^3$ the program constructs a right-angled hexagon $(H, \mathcal{Y}_{0,\infty})$ with arc coordinates $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$ and the adjacent symmetric hexagons $\overline{R}_1(H), \overline{R}_2(H), \overline{R}_3(H)$ illustrated in Figure 67 following the proof of Theorem 7.23. The maximal representation is then determined by $g = \overline{R}_1\overline{R}_2$ and $h = \overline{R}_2\overline{R}_3$. We only provide the case where $R_i \in \{R_{st}, R_{ex}\}$. In this chapter we describe the most important functions used in the program. It will be useful to recall that for a matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$ it holds

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$$

- ```
  def sqrtmatrix(M):
      eigvals,eigvecs=la.eig(M)
      eigvals=eigvals.real
      S=eigvecs
      s_0=math.sqrt(eigvals[0])
      s_1=math.sqrt(eigvals[1])
      D=np.array([[s_0,0],[0,s_1]])
      return S@D@S.T
  ```

  This function returns the square root of a positive definite 2×2 matrix M. Since M is symmetric, it is diagonalizable by an orthogonal matrix S. The matrix S has as columns the orthonormal eigenvectors of M. Let D be the matrix

  $$D = \begin{pmatrix} s_0 & 0 \\ 0 & s_1 \end{pmatrix}$$

  where s_0, s_1 denote the square roots of the eigenvalues of M. Then $\sqrt{M} = SDS^T$ is symmetric and is the square root of M.

- ```
      def diagonalizing_mat(M):
      eigvals,eigvecs=la.eig(M)
      L=eigvecs
      if eigvals[0]<eigvals[1]:
          l=[1,0]
          L=L[:,l]#if first eigenvalue smaller then the second I swap columns of L
      J=np.array([[-1,0],[0,1]])
      if la.det(L)>0:
          P=L.copy()
          Q=P@J
      if la.det(L)<0:
          Q=L.copy()
          P=Q@J
      return P.T,Q.T
  ```

This function takes a positive definite matrix M (with distinct eigenvalues) and returns $P \in \mathrm{PSO}(2), Q \in PO(2)\backslash \mathrm{PSO}(2)$ such that $PMP^T = QMQ^T$ is diagonal with a decreasing order of the eigenvalue on the diagonal. The algorithm follows the proof of Lemma 2.31.

- ```
  ProblemaL1(c1,c2,d1,d2,S,Y,Z):
  des1=[1/math.exp(c2),1/math.exp(c1)]
  C=np.diag(des1)
  X_pr=S.T@C@S
  P=diagonalizing_mat(la.inv(sqrtmatrix(Y))@Z@la.inv(sqrtmatrix(Y)))[0]
  X=sqrtmatrix(Y)@P.T@X_pr@P@sqrtmatrix(Y)
  return X
  ```

This function corresponds to Lemma 3.12.

- ```
  def ProblemaL2(c1,c2,d1,d2,S,X,Y):
  P=diagonalizing_mat(la.inv(sqrtmatrix(Y))@X@la.inv(sqrtmatrix(Y)))[0]
  S_h=S.T@P
  des2=[math.exp(d1),math.exp(d2)]
  D=np.diag(des2)
  Z=sqrtmatrix(Y)@S_h.T@D@S_h@sqrtmatrix(Y)
  return Z
  ```

This function corresponds to Lemma 3.11.

- ```
  def Sp4_Action(A,B,C,D,Z):
  return (A@Z+B)@la.inv(C@Z+D)
  ```

This is the $\mathrm{Sp}(4,\mathbb{R})$-action of a matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on an element $Z \in \mathcal{X}$.

- ```
  def OrthTube(A,B,C,D):
  X=la.inv(A-C)-la.inv(A-B)
  Y=la.inv(A-D)-la.inv(A-B)
  Q=la.inv(sqrtmatrix(X))@Y@la.inv(sqrtmatrix(X))
  Z1=Sp4_Action(A@sqrtmatrix(X),
  -la.inv(sqrtmatrix(X))+A@la.inv(A-B)@la.inv(sqrtmatrix(X)),
  sqrtmatrix(X),
  la.inv(A-B)@la.inv(sqrtmatrix(X)),-sqrtmatrix(Q))
  Z2=Sp4_Action(A@sqrtmatrix(X),
  -la.inv(sqrtmatrix(X))+A@la.inv(A-B)@la.inv(sqrtmatrix(X)),
  sqrtmatrix(X),
  la.inv(A-B)@la.inv(sqrtmatrix(X)),sqrtmatrix(Q))
  return Z1,Z2
  ```

This function takes a maximal quadruple (A, B, C, D) and returns the unique Z_1, Z_2 such that

$$\mathcal{Y}_{A,B} \perp \mathcal{Y}_{Z_1,Z_2} \perp \mathcal{Y}_{C,D}$$

Let g be an isometry such that $g(A, B, C) = (\infty, 0, \mathrm{Id})$. Put $g(D) = Q$ for a positive definite Q. By Lemma 2.19

$$Z_1 = g^{-1}(-\sqrt{Q}), \quad Z_2 = g^{-1}(\sqrt{Q})$$

The function computes explicitly the map g^{-1}. It holds

$$g^{-1} = \begin{pmatrix} A\sqrt{X} & -\sqrt{X}^{-1} + A(A-B)^{-1}\sqrt{X}^{-1} \\ \sqrt{X} & (A-B)^{-1}\sqrt{X}^{-1} \end{pmatrix}$$

where $X = (A-C)^{-1} - (A-B)^{-1}$. It holds $g(D) = Q = \sqrt{X}^{-1} Y \sqrt{X}^{-1}$ where $Y = (A-D)^{-1} - (A-B)^{-1}$.

- ```
  def Hexagon(b1,b2,c1,c2,d1,d2,alpha1,alpha2):
  S1= np.array([[math.cos(alpha1/2),-math.sin(alpha1/2)],
  [math.sin(alpha1/2),math.cos(alpha1/2)]])
  S2= np.array([[math.cos(alpha2/2),-math.sin(alpha2/2)],
  [math.sin(alpha2/2),math.cos(alpha2/2)]])
  Id=np.identity(2)
  C=np.array([[math.exp(c1),0],[0,math.exp(c2)]])
  A=ProblemaL1(b1,b2,c1,c2,S1,Id,C)
  D=ProblemaL2(c1,c2,d1,d2,S2,Id,C)
  Z1,Z2=OrthTube(A@A,Id,C,D@la.inv(C)@D)
  return A,C,D,Z1,Z2
  ```

  This funciton constructs a generic right-angled hexagon $(H, \mathcal{Y}_{0,\infty})$ with arc coordinates $(\underline{b}, \underline{c}, \underline{d}, [\alpha_1, \alpha_2])$. It follows the proof of Proposition 4.15.

- ```
  def maleficmapF(b1,b2,c1,c2,d1,d2,alpha1,alpha2):
  A,C,D,Z1,Z2=Hexagon(b1,b2,c1,c2,d1,d2,alpha1,alpha2)
  Id=np.identity(2)
  R=CrossRatio(A, Id, C, D@la.inv(C)@D) #cross-ratio of four matrices
  e1,e2=ord_eigvals(R) #e1>e2
  return math.log(e1),math.log(e2)
  ```

  This function returns the image of the malefic map defined in Definition 5.12.

- ```
  def Stand4uple(A,B,C,D):
  X=la.inv(D-B)-la.inv(D-A)
  W=la.inv(D-C)-la.inv(D-A)
  P=diagonalizing_mat(la.inv(sqrtmatrix(X))@W@la.inv(sqrtmatrix(X)))[0]
  Id=np.identity(2)
  B1=P@la.inv(sqrtmatrix(X))@la.inv(D-A)
  B2=P@la.inv(sqrtmatrix(X))@(Id-(la.inv(D-A)@D))
  B3=-P@sqrtmatrix(X)
  B4=P@sqrtmatrix(X)@D
  return B1,B2,B3,B4
  ```

  This function takes a maximal quadruple (A, B, C, D) and returns the blocks of an isometry g:

$$g = \begin{pmatrix} B1 & B2 \\ B3 & B4 \end{pmatrix}$$

  such that $g(A, B, C, D) = (0, \mathrm{Id}, Y, \infty)$ where Y is diagonal. Computations give:

$$g = \begin{pmatrix} P\sqrt{X}^{-1}(D-A)^{-1} & P\sqrt{X}^{-1}(\mathrm{Id} - (D-A)^{-1}D) \\ -P\sqrt{X} & P\sqrt{X}D \end{pmatrix}$$
  ```

where $X = (D-B)^{-1} - (D-A)^{-1}$ and P is such that $P\sqrt{X}^{-1} W \sqrt{X}^{-1} P^T$ diagonal where $W = (D-C)^{-1} - (D-A)^{-1}$.

- ```
  def checkflipgen(A,B,C,D,X,Y):
   Id=np.identity(2)
   B1,B2,B3,B4=multmat(A,B,C,D,Y,Y@la.inv(Y-X)-Id,Id,la.inv(Y-X))
   A=multmat(la.inv(Y-X),Id-la.inv(Y-X)@Y,-Id,Y,B1,B2,B3,B4)[0]
   if la.det(A)>0:
          return 1
   if la.det(A)<0:
      return -1
  ```

This function takes two symmetric matrices X, Y and a matrix $h \in \mathrm{Sp}(4, \mathbb{R})$:

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Stab}(X, Y)$$

and returns 1/-1 if h is conjugated to a non-reflecting/reflecting isometry respectively (Definition 2.38). To do that let g be an isometry such that $g(X, Y) = (0, \infty)$. Then

$$h = g^{-1} \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} g$$

for some $A \in \mathrm{GL}(2, \mathbb{R})$. The function calculates ghg^{-1} where

$$ghg^{-1} = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}$$

and computes the determinant of A. The function `multmat` takes as input the blocks of two matrices (eight 2×2 matrices) and returns the blocks of the matrix obtained by multiplying them.
  ```

9 Other approaches to parameterize hexagons

In this chapter we expose a different approach to parametrize right-angled hexagons in the Siegel space $\mathcal{X}$. What follows is a joint work with Eugen Rogozinnikov. The idea is to describe a right angled hexagon by a triple of positive definite symmetric matrices and to generalize this parametrization to the space of two adjacent hexagons. We will see how the problem of this approach is that when extending the parameters to adjacent hexagons we can not guarantee that the constructed hexagons have the same alternating side-lengths.

Proposition 9.1. *The set of right-angled hexagons in $\mathcal{X}$ is parametrized up to isometry by*

$$\mathrm{Sym}_n^+(\mathbb{R})^3/_{\mathrm{PO}(n)}$$

Proof. Let $H = [\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5, \mathcal{Y}_6]$ be a right-angled hexagon in the Siegel space $\mathcal{X}$. Up to $\mathrm{PSp}(2n, \mathbb{R})$-action we can assume that one tube is coinciding with $\mathcal{Y}_{0,\infty}$. In Lemma 4.4 we have seen that it is sufficient to specify A, B, C and D at the boundary of $\mathcal{X}$ for the hexagon to be uniquely determined (Figure 21). Up to isometry we can assume one point to coincide with Id. In this case it turns out to be more useful to consider the point A coinciding with Id (and not B as we have done in the proof of Proposition 4.15). For

$$[(X, Y, Z)] \in \mathrm{Sym}_n^+(\mathbb{R})^3/_{\mathrm{PO}(n)}$$

we construct the hexagon $H = (0, \mathrm{Id}, B, C, D)$ where

$$B = X + \mathrm{Id}$$

$$C = X + Y + \mathrm{Id}$$

$$D = X + Y + Z + \mathrm{Id}$$

so that (Id, B, C, D) is a maximal quadruple. All the other tubes are then uniquely determined. The group $\mathrm{PO}(n)$ is the stabilizer of the triple $(0, \mathrm{Id}, \infty)$. Observe that the parameter space has dimension 8 as we can always diagonalize a matrix up to $\mathrm{PO}(n)$-action. □

Proposition 9.2. *The space of adjacent right angled hexagons in $\mathcal{X}$ is parametrized (up to isometry) by*

$$\mathrm{Sym}_n^+(\mathbb{R})^5/_{\mathrm{PO}(n)}$$

Proof. The proof is very similar to Proposition 9.1. Up to isometry we can always assume that the two hexagons have the tubes $\mathcal{Y}_{0,\infty}$ and $\mathcal{Y}_{-\mathrm{Id},\mathrm{Id}}$ in common. For

$$[(X, Y, Z, X', Y')] \in \mathrm{Sym}_n^+(\mathbb{R})^5/_{\mathrm{PO}(n)}$$

we construct the two adjacent hexagons

$$H_1(0, \mathrm{Id}, B, C, D, \infty) \text{ and } H_2(\infty, -D, -C', -B', \mathrm{Id}, 0)$$

where

$$B = X + \mathrm{Id}$$

$$C = X + Y + \mathrm{Id}$$

$$D = X + Y + Z + \mathrm{Id}$$

$$B' = X' + \mathrm{Id}$$

$$C' = X + Y' + \mathrm{Id}$$

All the other tubes are uniquely determined (Figure 68). □

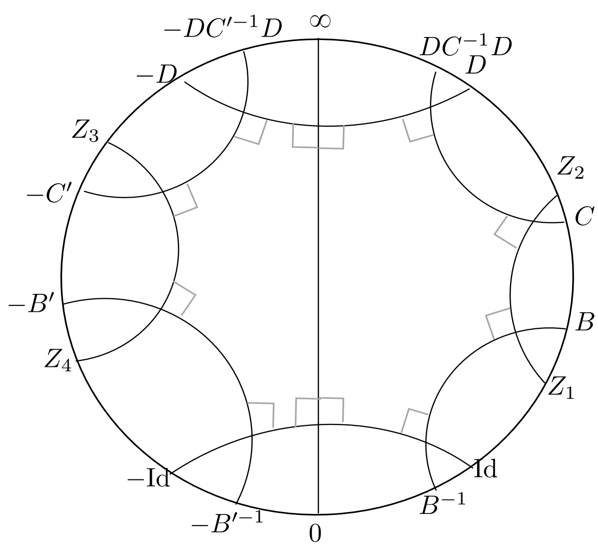


Figure 68: Parameters for two adjacent right-angled hexagons

Remark 9.3. The parametrization in Proposition 9.2 does not give any information about the lengths of the sides of the two adjacent hexagons. In particular in Theorem 7.23 we have seen that a crucial tool to construct maximal representations is to construct adjacent hexagons where alternating sides have the same length. In the parametrization of Proposition 9.2 this does not happen even if we restrict to the case where

$$B' = O_1 B O_1^T$$

$$C' = O_2 C O_2^T$$

for $O_1 O_2 \in \mathrm{PO}(n)$. In that case it holds (Figure 69)

$$d^{\bar{\mathfrak{a}}^+}(p_{-\mathrm{Id},\mathrm{Id}}(0), p_{-\mathrm{Id},\mathrm{Id}}(B^{-1})) = d^{\bar{\mathfrak{a}}^+}(p_{-\mathrm{Id},\mathrm{Id}}(-B'^{-1}), p_{-\mathrm{Id},\mathrm{Id}}(0)) \tag{29}$$

$$d^{\bar{\mathfrak{a}}^+}(p_{B^{-1},B}(\mathrm{Id}), p_{B^{-1},B}(Z_1)) \neq d^{\bar{\mathfrak{a}}^+}(p_{-B',-B'^{-1}}(-\mathrm{Id}), p_{-B',-B'^{-1}}(Z_4)) \tag{30}$$

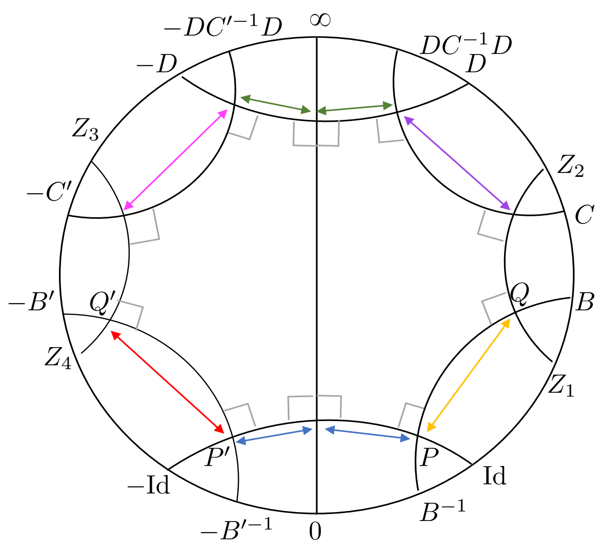


Figure 69: Side-length in the case $B' = O_1 B O_1^T$ and $C' = O_2 C O_2^T$ for $O_1, O_2 \in \mathrm{PO}(n)$

The equality (29) follows by Proposition 5.8 and by the fact that

$$d^{\bar{\mathfrak{a}}^+}(p_{0,\infty}(\mathrm{Id}), p_{0,\infty}(B)) = d^{\bar{\mathfrak{a}}^+}(p_{0,\infty}(-\mathrm{Id}), p_{0,\infty}(-B'))$$

for $B' = O_1 B O_1^T$, $O_1 \in \mathrm{PO}(n)$. To see (30) let us denote by P, Q and P', Q' the points obtained by intersecting the tube $\mathcal{Y}_{-\mathrm{Id},\mathrm{Id}}$ with $\mathcal{Y}_{B^{-1},B}$ and $Y_{-B'^{-1},-B'}$ respectively (Figure 69). In general

$$d^{\bar{\mathfrak{a}}^+}(P, Q) \neq d^{\bar{\mathfrak{a}}^+}(P', Q')$$

Fix $\underline{a} = d^{\bar{\mathfrak{a}}^+}(P, Q)$ and $O_1 \in \mathrm{PO}(2)$. The vector $d^{\bar{\mathfrak{a}}^+}(P', Q')$ depends on the point P' which depends on the choice of O_1. The same arguments holds for the vectors in the tubes $\mathcal{Y}_{C,DC^{-1}D}$ and $\mathcal{Y}_{-C',-DC'^{-1}D}$ respectively as shown in Figure 69.

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