Evolution equations in ostensible metric spaces. II. Examples in Banach spaces and of free boundaries.

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Abstract. In part I, generalizing mutational equations of Aubin in metric spaces has led to so–called *right–hand forward solutions* in a nonempty set with a countable family of (possibly nonsymmetric) ostensible metrics.
Now this concept is applied to two different types of evolutions that have motivated the definitions : semilinear evolution equations (of parabolic type) in a reflexive Banach space and compact subsets of $\mathbb{R}^N$ whose evolution depend on nonlocal properties of both the set and their limiting normal cones at the boundary.
For verifying that reachable sets of differential inclusions are appropriate transitions for first–order geometric evolutions, their regularity at the boundary is studied in the appendix.

Contents

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1 Introduction

Whenever different types of evolutions meet, they usually do not have an obvious vector space structure in common providing a basis for differential calculus. In particular, "shapes and images are basically sets, not even smooth" as Aubin stated ([2]). So he regards this obstacle as a starting point for extending ordinary differential equations to metric spaces – the so–called *mutational equations* ([2, 3, 4]).

Considering the example of time–dependent compact sets in $\mathbb{R}^N$, Aubin uses reachable sets of differential inclusions for describing a first–order approximation with respect to the Pompeiu–Hausdorff distance $d\!\!l$. However this approach (also called *morphological equations*) can hardly be applied to geometric evolutions depending on the topological boundary explicitly. Indeed, roughly speaking, "holes" of sets might disappear while evolving along differential inclusions and thus, analytically speaking, the topological boundary need not be continuous with respect to time.

This difficulty has been the motivation in [31, Lorenz 2005] for extending mutational equations to a set $E \neq \emptyset$ with a countable family of *ostensible metrics*, i.e. distance functions $q_\varepsilon : E \times E \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$ satisfying just the triangle inequality and $q_\varepsilon(x, x) = 0$ for each $x \in E$. The definitions of so–called *right–hand forward solutions* and main results about their existence are summarized in § 2.

In this paper, we present two important examples of this more general concept and verify the required preliminaries in detail :

The first example consists in semilinear evolution equations in a reflexive Banach space X (see § 3). Due to the required continuity properties, we consider the weak topology instead of the norm. So with respect to mutational equations, the metric is replaced by a family of distance functions (induced by linear forms). Assuming X to be reflexive has two useful advantages : Closed bounded balls are weakly compact. Moreover for any C^0 semigroup $(S(t))_{t \geq 0}$ on X with the infinitesimal generator A, it is well–known that the adjoint operators $S(t)' : X' \longrightarrow X'$ $(t \geq 0)$ form a C^0 semigroup on X' with the infinitesimal generator A'. In particular, the distance functions on X are induced by unit eigenvectors v_j' $(j \in \mathcal{J})$ of A' which are supposed to be countable and to span X',

$$q_j : X \times X \longrightarrow [0, \infty[, \qquad (x, y) \longmapsto |\langle x - y,\ v_j' \rangle|.$$

Considering now the semilinear evolution equation

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &= A\, x(t) \ + \ f(x(t), t) \\ \quad x(0) &= x_0 \end{cases}$$

the theory of right–hand forward solutions ([31]) provides sufficient conditions on $f : X \times [0, T[\longrightarrow X$ for the existence of a weak solution $x(\cdot) : [0, T[\longrightarrow X$ and, a result of John M. Ball ([7]) implies directly that $x(\cdot)$ is also mild solution.

As second example of generalized mutational equations, we then consider geometric evolutions up to first order (§ 4), i.e. compact subsets of $\mathbb{R}^N$ whose evolution depend on nonlocal properties of both the sets and their limiting normal cones at the boundary.
The first key aspect concerns the topological boundary : no regularity conditions are supposed a priori and, no subsets of the boundaries have to be neglected as in geometric measure theory, for example (see [27, Federer 69], [12, Brakke 78]).
Secondly, the geometric evolutions here need not satisfy the so–called *inclusion principle* stating that if a compact initial set is contained in another one, then this inclusion is be preserved while the sets are evolving. Several approaches use this inclusion principle as a geometric starting point for extending analytical tools to nonsmooth subsets. An excellent example is De Giorgi's theory of barriers formulated in [22, De Giorgi 94] and elaborated in [11, Bellettini, Novaga 97], [10, Bellettini, Novaga 98]. Another widespread concept is based on the level set method using viscosity solutions. There the inclusion principle is closely related with the corresponding partial differential equation being degenerate parabolic and thus, it can be regarded as a geometric counterpart of the maximum principle (see e.g. [8, Barles, Souganidis 98], [1, Ambrosio 2000]). An elegant approach to front propagation problems with nonlocal terms has been presented in [15, Cardaliaguet 2000], [14, Cardaliaguet 2001], [16, Cardaliaguet, Pasquignon 2001]. The inclusion principle again is the key for generalizing the evolution from $C^{1,1}$ submanifolds with boundary to nonsmooth subsets of $\mathbb{R}^N$.
In comparison with the morphological equations of Aubin ([2]), the Pompeiu–Hausdorff distance $d\!\!l$ on $\mathcal{K}(\mathbb{R}^N)$ can now be replaced by the (nonsymmetric) Pompeiu–Hausdorff excess $e\!\!\!e^{\supset}(K_1, K_2) := \sup_{y \in K_2} \operatorname{dist}(y, K_1) = \operatorname{dist}(K_2, K_1)$ or by the ostensible metric

$$\begin{array}{rrcl} q_{\mathcal{K},N}: & \mathcal{K}(\mathbb{R}^N) \times \mathcal{K}(\mathbb{R}^N) & \longrightarrow & [0, \infty[\\ & (K_1,\ K_2) & \longmapsto & d\!\!l(K_1, K_2) \ + \ e\!\!\!e^{\supset}\Big(\text{Graph } {}^\flat N_{K_1},\ \text{Graph } {}^\flat N_{K_2}\Big) \end{array}$$

with $N_K(x)$ denoting the limiting normal cone of $K \subset \mathbb{R}^N$ at $x \in \partial K$,

$${}^\flat N_K(x) \ := \ N_K(x) \cap \mathbb{B}_1 \ = \ \{v \in N_K(x) : |v| \le 1\}.$$

For using right–hand forward solutions of generalized mutational equations here, two further features have to be specified, i.e. the "test set" that we use for comparisons and the forward transitions. Following the motivation in [31, Lorenz 2005], the "test subset" of $\mathcal{K}(\mathbb{R}^N)$ is $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ consisting of all nonempty compact subsets of $\mathbb{R}^N$ with $C^{1,1}$ boundary. Moreover reachable sets of differential inclusions again serve as forward transitions on $(\mathcal{K}(\mathbb{R}^N), \mathcal{K}_{C^{1,1}}(\mathbb{R}^N), q_{\mathcal{K},N})$, i.e.

$$\begin{array}{rrcl} \vartheta_F: & [0,1] \times \mathcal{K}(\mathbb{R}^N) & \longrightarrow & \mathcal{K}(\mathbb{R}^N) \\ & (t,\ K_0) & \longmapsto & \{\, x(t) \mid \exists\ x(\cdot) \in AC([0,t], \mathbb{R}^N) : \\ & & & \qquad \frac{d}{dt}\, x(\cdot) \in F(x(\cdot)) \text{ a.e.}, \quad x(0) \in K_0 \} \end{array}$$

for a set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. In particular, for parameters $\Lambda, \rho > 0$ fixed, $\mathrm{LIP}^{(\mathcal{H}^\rho_\circ)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ consists of all set–valued maps $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ satisfying

(i) F has compact convex values with positive erosion of radius ρ (see Def. 4.15),

(ii) Hamiltonian $\mathcal{H}_F(\cdot,\cdot) \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,

(iii) $\|\mathcal{H}_F\|_{C^{1,1}(\mathbb{R}^N \times \partial\mathbb{B}_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_F\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} + \operatorname{Lip} D\mathcal{H}_F|_{\mathbb{R}^N \times \partial\mathbb{B}_1} < \lambda$.

The analytical basis for reachable sets (particularly with respect to the regularity of the boundary) is presented in the appendix.

A key advantage of right–hand forward solutions is that they provide a common basis for completely different types of evolutions. In particular, the general results of [31, Lorenz 2005] imply for the two examples discussed here :

Proposition 1.1 (Systems of semilinear evolution equations in Banach space and first–order geometric evolutions in $\mathbb{R}^N$)
Let X be a reflexive Banach space and $(S(t))_{t\geq 0}$ a C^0 semigroup on X with the infinitesimal generator A. Suppose that the dual operator A' of A has a countable family of unit eigenvectors $\{v'_j\}_{j\in\mathcal{J}}$ spanning the dual space X' and define

$$\begin{array}{lcll} q_j(x,y) & := & |\langle x-y,\ v'_j\rangle| & \text{for } x,y\in X,\ j\in\mathcal{J}=\{j_1,j_2,j_3\ldots\},\\ p_n(x,y) & := & \displaystyle\sum_{k=1}^{n} 2^{-k}\ \frac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)} & \text{for } x,y\in X,\ n\in\mathbb{N}\cup\{\infty\},\\ P_n(x,y) & := & \displaystyle\sum_{k=1}^{n} 2^{-k}\ q_{j_k}(x,y). & \end{array}$$

Furthermore assume for

$$\begin{array}{rl} f: & X\times\mathcal{K}(\mathbb{R}^N)\times[0,T] \longrightarrow X \\ g: & X\times\mathcal{K}(\mathbb{R}^N)\times[0,T] \longrightarrow \mathrm{LIP}_\Lambda^{(\mathcal{H}_\circ^\rho)}(\mathbb{R}^N,\mathbb{R}^N)\ : \end{array}$$

1. $\|f\|_{L^\infty} < \infty$
2. $P_\infty(f(x_1,K_1,t_1),\ f(x_2,K_2,t_2)) \leq \omega(p_\infty(x_1,x_2)+q_{\mathcal{K},N}(K_1,K_2)+t_2-t_1)$
3. $\big\|\mathcal{H}_{g(x_1,K_1,t_1)} - \mathcal{H}_{g(x_2,K_2,t_2)}\big\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)} \leq \omega(p_\infty(x_1,x_2)+q_{\mathcal{K},N}(K_1,K_2)+t_2-t_1)$

for all $x_1,x_2\in X$, $K_1,K_2\in\mathcal{K}(\mathbb{R}^N)$, $0\leq t_1\leq t_2\leq T$ with a modulus $\omega(\cdot)$ of continuity.

Then for every initial data $x_0\in X$ and $K_0\in\mathcal{K}(\mathbb{R}^N)$, there exists a tuple of functions $(x,K)\ :\ [0,T[\ \longrightarrow X\times\mathcal{K}(\mathbb{R}^N)$ with

a) $x:[0,T[\longrightarrow X$ *is a mild solution of the initial value problem*

$$\wedge\begin{cases} \frac{d}{dt}\,x(t) &=\ A\,x(t)\ +\ f(x(t),\ K(t),\ t)\\ \quad x(0) &=\ x_0 \end{cases}$$

i.e. $$x(t)\ =\ S(t)\,x_0\ +\int_0^t S(t-s)\ f(x(s),\ K(s),\ s)\ ds.$$

b) $K(0)=K_0$ *and* $K(\cdot)\in\operatorname{Lip}^{\rightarrow}([0,T[,\ \mathcal{K}(\mathbb{R}^N),\ q_{\mathcal{K},N})$, *i.e.*
$q_{\mathcal{K},N}(K(s),\ K(t)) \leq \mathit{const}(\Lambda,T)\cdot(t-s)$ *for all* $0\leq s<t<T$.

c) $$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot\Big(q_{\mathcal{K},N}\Big(\vartheta_{g(x(t),\,K(t),\,t)}\,(h,\ M),\ \ K(t+h)\Big)\ -\ q_{\mathcal{K},N}(M,\ K(t))\cdot e^{10\,\Lambda\,t}\Big)\ \leq\ 0$$
for every compact set $M\subset\mathbb{R}^N$ with $C^{1,1}$ boundary and $t\in[0,T[$.

2 Right–hand forward solutions of mutational equations : Definitions and main results

Generalizing the mutational equations of Aubin in metric spaces ([2, 3, 4]), we now summarize definitions and main results about their so–called *right–hand forward solutions* (*of order* p) presented and proven in [31]. As a first step, we dispense with the symmetry of distance functions.

Definition 2.1 *Let E be a nonempty set.*
$q : E \times E \longrightarrow [0, \infty[$ *is called* ostensible metric *on E if it satisfies the conditions :*

1. $\forall\ x \in E: \quad q(x,x) = 0$ (*reflexive*)
2. $\forall\ x,y,z \in E: \quad q(x,z) \le q(x,y) + q(y,z)$ (*triangle inequality*).

Then (E,q) *is called* ostensible metric space.

In this section, let E denote a nonempty set and $D \subset E$. Furthermore suppose $(q_\varepsilon)_{\varepsilon \in \mathcal{J}}$ to be a countable family of ostensible metrics on E. (Assuming $\mathcal{J} \subset [0,1]^\kappa$ to be countable makes the Cantor diagonal construction available for proofs of existence.) Finally, 0 is contained in the closure of the index set $\mathcal{J}$.
Now we specify the primary tools for describing deformations in the tuple $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$. A map $\vartheta : [0,1] \times E \longrightarrow E$ is to define which point $\vartheta(t,x) \in E$ is reached from the initial point $x \in E$ after time t. Of course, ϑ has to fulfill some regularity conditions so that it may form the basis for a calculus of differentiation.

Definition 2.2 *A map $\vartheta : [0,1] \times E \longrightarrow E$ is a so–called* forward transition of order $p \in \mathbb{R}$ *on $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ if it fulfills the following conditions for each $\varepsilon \in \mathcal{J}$*

1. $\vartheta(0,\cdot) = \mathrm{Id}_E$,
2. $\exists\ \gamma_\varepsilon(\vartheta) \ge 0: \quad \limsup_{\varepsilon \longrightarrow 0} \varepsilon^p \cdot \gamma_\varepsilon(\vartheta) = 0$ *and*
$$\limsup_{h \downarrow 0} \tfrac{1}{h} \cdot q_\varepsilon\Big(\vartheta(h,\ \vartheta(t,x)),\ \ \vartheta(t+h,\ x)\Big) \le \gamma_\varepsilon(\vartheta) \qquad \forall\ x \in E,\ t \in [0,1[,$$
$$\limsup_{h \downarrow 0} \tfrac{1}{h} \cdot q_\varepsilon\Big(\vartheta(t+h,\ x),\ \ \vartheta(h,\ \vartheta(t,x))\Big) \le \gamma_\varepsilon(\vartheta) \qquad \forall\ x \in E,\ t \in [0,1[,$$
3. $\exists\ \alpha_\varepsilon^{\mapsto}(\vartheta) < \infty: \quad \sup_{z \in D,\, y \in E}\ \limsup_{h\downarrow 0} \left(\dfrac{q_\varepsilon\big(\vartheta(h,z),\, \vartheta(h,y)\big) - q_\varepsilon(z,y) - \gamma_\varepsilon(\vartheta)\,h}{h \quad \big(q_\varepsilon(z,y) + \gamma_\varepsilon(\vartheta)\,h\big)}\right)^+ \le \alpha_\varepsilon^{\mapsto}(\vartheta)$
4. $\exists\ \beta_\varepsilon(\vartheta) :\ ]0,1] \longrightarrow [0,\infty[: \quad \beta_\varepsilon(\vartheta)(\cdot)$ *nondecreasing,* $\quad \limsup_{h\downarrow 0} \beta_\varepsilon(\vartheta)(h) = 0,$
$$q_\varepsilon\Big(\vartheta(s,x),\ \ \vartheta(t,x)\Big) \le \beta_\varepsilon(\vartheta)(t-s) \qquad \forall\ s < t \le 1,\ x \in E,$$
5. $\forall\ z \in D \quad \exists\ \mathcal{T}_\Theta = \mathcal{T}_\Theta(\vartheta, z) \in]0,1] : \quad \vartheta(t,z) \in D \qquad \forall\ t \in [0, \mathcal{T}_\Theta],$
6. $\limsup_{h\downarrow 0} q_\varepsilon\Big(\vartheta(t-h,\ z),\ y\Big) \ge q_\varepsilon\Big(\vartheta(t,z),\ y\Big) \qquad \forall\ z \in D,\ y \in E,\ t \in]0, \mathcal{T}_\Theta]$

Here the term "forward" and the symbol $\mapsto$ (representing the time axis) indicate that we usually compare the state at time t with the element at time $t+h$ for $h \downarrow 0$.
Condition (2.) can be regarded as a weakened form of the semigroup property. It consists of two demands as q_ε need not be symmetric. Condition (3.) concerns the continuity properties of ϑ with respect to the initial point. In particular, the first argument of q_ε is restricted to elements z of the "test set" D and, $\alpha_\varepsilon^{\mapsto}(\vartheta)$ may be chosen larger than necessary. Thus, it is easier to define $\alpha_\varepsilon^{\mapsto}(\cdot) < \infty$ uniformly in some applications like the first–order geometric example of § 4. In condition (4.), all $\vartheta(\cdot, x) : [0,1] \longrightarrow E$ $(x \in E)$ are supposed to be equi–continuous.
Condition (5.) guarantees that every element $z \in D$ stays in the "test set" D for short times at least. This assumption is required because estimates using the parameter $\alpha_\varepsilon^{\mapsto}(\cdot)$ can be ensured only within this period. Further conditions on $\mathcal{T}_\Theta(\vartheta, \cdot) > 0$ are avoidable for proving existence of solutions, but they are used for uniqueness (in [31]).
Condition (6.) forms the basis for applying Gronwall's Lemma (that has been extended to semicontinuous functions in [31]). Indeed, every function $y : [0,1] \longrightarrow E$ with $q_\varepsilon(y(t-h),\ y(t)) \longrightarrow 0$ (for $h \downarrow 0$ and each t) satisfies

$$q_\varepsilon\Big(\vartheta(t,z),\ y(t)\Big) \ \le\ \limsup_{h \downarrow 0}\ q_\varepsilon\Big(\vartheta(t-h,z),\ \ y(t-h)\Big).$$

for all elements $z \in D$ and times $t \in\ ]0,\ \mathcal{T}_\Theta(\vartheta, x)]$.

Remark 2.3 A set $E \neq \emptyset$ supplied with only one function $q : E \times E \longrightarrow [0, \infty[$ can be regarded as easy (but important) example by setting $\mathcal{J} := \{0\}$, $q_0 := q$. Considering a forward transitions $\vartheta : [0,1] \times E \longrightarrow E$ of order 0, the condition $\limsup_{\varepsilon \longrightarrow 0} \varepsilon^0 \cdot \gamma_\varepsilon(\vartheta) = 0$ means $0 = 0^0 \cdot \gamma_0(\vartheta) = \gamma_0(\vartheta)$ — due to the definition $0^0 \overset{\text{Def.}}{=} 1$. Then many of the following results do not depend on ε or $\gamma_\varepsilon(\cdot)$ (and its upper bounds) explicitly. So we do not mention the index ε there any longer and abbreviate the corresponding set of transitions (of order 0) as $\Theta^{\mapsto}(E, D, q)$. In particular, transitions on a metric space (M, d) (introduced by Aubin in [2], [3]) prove to be an example of such forward transitions on (M, M, d).

Definition 2.4 *$\Theta_p^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ denotes a set of forward transitions on $(E, D, (q_\varepsilon))$ of order $p \in \mathbb{R}$ supposing for all $\vartheta, \tau \in \Theta_p^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $\varepsilon \in \mathcal{J}$,*

$$Q_\varepsilon^{\mapsto}(\vartheta, \tau) \ :=\ \sup_{z \in D,\, y \in E}\ \ \limsup_{h \downarrow 0}\ \left(\frac{q_\varepsilon\big(\vartheta(h,z),\, \tau(h,y)\big) - q_\varepsilon(z,y) \cdot e^{\alpha_\varepsilon^{\mapsto}(\tau)\, h}}{h}\right)^+ \ <\ \infty$$

These definitions enable us to compare any element $y \in E$ with a "test element" $z \in D$ while evolving along two forward transitions. Considering the bound in the next proposition, the influence of the distances between initial points and between transitions is the same as for ordinary differential equations. The key idea of right–hand forward solutions has been to preserve this structural estimate while extending mutational equations to ostensible metrics and "distributional" features (in regard to a test set D).

Proposition 2.5 *Let* $\vartheta, \tau \in \Theta_p^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *be forward transitions,* $\varepsilon \in \mathcal{J}$, $z \in D$, $y \in E$ *and* $0 \le t_1 \le t_2 \le 1$, $h \ge 0$ *satisfying* $t_1 + h < \mathcal{T}_\Theta(\vartheta, z)$. *Then,*

$$\begin{aligned} & q_\varepsilon(\vartheta(t_1{+}h, z),\ \ \tau(t_2{+}h, y)) \\ \le\ & \Big(q_\varepsilon(\vartheta(t_1, z),\ \ \tau(t_2, y)) \ + \ h \cdot (Q_\varepsilon^{\mapsto}(\vartheta, \tau) + \gamma_\varepsilon(\vartheta) + \gamma_\varepsilon(\tau))\Big) \cdot e^{\alpha_\varepsilon^{\mapsto}(\tau)\ h} \end{aligned}$$

The next step is to define the term "right–hand forward primitive" for a curve $\vartheta(\cdot) : [0, T] \longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ of forward transitions.
Roughly speaking, a curve $x(\cdot) : [0, T[\longrightarrow E$ represents a primitive of $\vartheta(\cdot)$ if at each time $t \in [0, T[$, the forward transition $\vartheta(t)$ can be interpreted as a first–order approximation of $x(t + \cdot)$. Combining this notion with the key estimate of Proposition 2.5, a vague meaning of "first–oder approximation" is provided : Comparing $x(t{+}\cdot)$ with $\vartheta(t)(\cdot, z)$ (for any test element $z \in D$), the same estimate ought to hold as if the factor $Q_\varepsilon^{\mapsto}(\cdot, \cdot)$ was 0. It motivates the following definition with the expression "right–hand" indicating that $x(\cdot)$ appears in the second argument of the distances q_ε $(\varepsilon \in \mathcal{J})$ in condition (1.).

Definition 2.6 *The curve* $x(\cdot) : [0, T[\longrightarrow (E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *is called* right–hand forward primitive of *a map* $\vartheta(\cdot) : [0, T[\longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$, *abbreviated to* $\overset{\circ}{x}(\cdot) \ni \vartheta(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \ \ t \in [0, T[\qquad \exists\ \widehat{\alpha}_\varepsilon^{\mapsto}(t),\ \widehat{\gamma}_\varepsilon(t) \in [0, \infty[\ :$
   $$\widehat{\alpha}_\varepsilon^{\mapsto}(t) \ge \alpha_\varepsilon^{\mapsto}(\vartheta(t)), \quad \widehat{\gamma}_\varepsilon(t) \ge \gamma_\varepsilon(\vartheta(t)), \quad \limsup_{\varepsilon' \downarrow 0}\ \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) \ = \ 0,$$
   $$\limsup_{h \downarrow 0} \tfrac{1}{h} \Big(q_\varepsilon(\vartheta(t)\,(h, z),\ x(t{+}h)) \ - \ q_\varepsilon(z,\ x(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h}\Big) \ \le \ \widehat{\gamma}_\varepsilon(t) \qquad \forall\ z \in D,$$
2. $x(\cdot)$ *is uniformly continuous in time direction with respect to* q_ε,
   i.e. there is $\omega_\varepsilon(x, \cdot) :]0, T[\longrightarrow [0, \infty[$ *such that* $\limsup_{h\downarrow 0}\ \omega_\varepsilon(x, h) \ = \ 0$ *and*
   $q_\varepsilon\Big(x(s),\ x(t)\Big) \ \le \ \omega_\varepsilon(x,\ t - s)$ *for* $0 \le s < t < T$.

Remark 2.7 Forward transitions induce their own primitives. To be more precise, every constant function $\vartheta(\cdot) : [0, 1[\longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ with $\vartheta(\cdot) = \vartheta_0$ has the right–hand forward primitives $[0, 1[\longrightarrow E,\ t \longmapsto \vartheta_0(t, x)$ with any $x \in E$ — as an immediate consequence of Proposition 2.5. This property is easy to extend to piecewise constant functions $[0, T[\longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ and so it forms the basis for Euler approximations.

Definition 2.8 *For* $f : E \times [0, T[\longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon))$ *given, a map* $x : [0, T[\longrightarrow E$ *is a* right–hand forward solution *of the generalized mutational equation* $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ *if* $x(\cdot)$ *is right–hand forward primitive of* $f(x(\cdot), \cdot) : [0, T[\longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon))$.

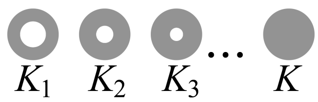
Constructing solutions of ordinary differential equations is usually based on completeness or compactness. Here we prefer sequential compactness since the available estimates for transitions on $(E, D, (q_\varepsilon))$ hold only for elements of D in the first argument of q_ε (as in Proposition 2.5). So there is no obvious way of verifying the assumptions of Banach's contraction principle in (E, q_ε).
In Aubin's mutational analysis on metric spaces, the bounded closed balls are supposed to be compact, i.e. for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in (M, d), there exist a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ and an element $x \in M$ with $d(x_{n_j},\ x) \longrightarrow 0$ (for $j \longrightarrow \infty$).
Dispensing now with the symmetry of the distance, sequential compactness is to consist of two conditions.

Definition 2.9 *$(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is called* two–sided sequentially compact (uniformly with respect to ε) *if for every $y \in E$, $r_\varepsilon > 0$ $(\varepsilon \in \mathcal{J})$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in E with*

$$q_\varepsilon(y, x_n) \ \le\ r_\varepsilon \qquad \forall\ n \in \mathbb{N} \qquad \forall\ \varepsilon \in \mathcal{J}$$

there exist a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ and an element $x \in E$ such that

$$\left.\begin{array}{l} q_\varepsilon(x_{n_j},\ x) \ \longrightarrow\ 0 \\ q_\varepsilon(x,\ x_{n_j}) \ \longrightarrow\ 0 \end{array}\right. \qquad \text{for } j \longrightarrow \infty \qquad \forall\ \varepsilon \in \mathcal{J}.$$



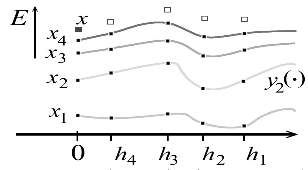
Some ostensible metric spaces have this compactness property in common like $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$, but in general, it is too restrictive. Indeed, $(\mathcal{K}(\mathbb{R}^N), q_{\mathcal{K},N})$ is not two–sided sequentially compact since, for example, $K_n := \{\frac{1}{n+1} \le |x| \le 1\}$ and $K := \mathbb{B}_1$ satisfy $d\!\!l(K_n, K) = q_{\mathcal{K},N}(K_n, K) \longrightarrow 0$ $(n \to \infty)$, but $q_{\mathcal{K},N}(K, K_n) \ge \frac{1}{2}$.

For this reason, we coin a more general term of sequential compactness. It is motivated by the fact that in a word, the solution property is stable with respect to graphical convergence. We again find the key notion that the first argument of q_ε usually represents the earlier state whereas the second argument refers to the later element.

Definition 2.10 *Let Θ denote a nonempty set of maps $[0, 1] \times E \longrightarrow E$.*
The tuple $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}, \Theta)$ is called transitionally compact *if it has the property :*

Let $(x_n)_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ be any sequences in E, $]0, 1[$, respectively and $z \in E$ with $\sup_n q_\varepsilon(z, x_n) < \infty$ for each $\varepsilon \in \mathcal{J}$, $h_j \longrightarrow 0$. Moreover suppose $\vartheta_n : [0, 1] \longrightarrow \Theta$ to be piecewise constant $(n \in \mathbb{N})$ such that all curves $\vartheta_n(t)(\cdot, x) : [0, 1] \longrightarrow E$ have a common modulus of continuity $(n \in \mathbb{N},\ t \in [0, 1],\ x \in E)$.
Each ϑ_n induces a function $y_n(\cdot) : [0, 1] \longrightarrow E$ with $y_n(0) = x_n$ in the same piecewise way as forward transitions induce their own primitives according to Remark 2.7 (i.e. using $\vartheta_n(t_m)\,(\cdot,\ y_n(t_m))$ in each interval $]t_m, t_{m+1}]$ in which $\vartheta_n(\cdot)$ is constant).
Then there exist a sequence $n_k \nearrow \infty$ of indices and $x \in E$ satisfying for each $\varepsilon \in \mathcal{J}$,

$$\limsup_{k\longrightarrow\infty} \quad q_\varepsilon(x_{n_k}, x) = 0,$$
$$\limsup_{j\longrightarrow\infty} \sup_{k\geq j} \quad q_\varepsilon(x, \;\; y_{n_k}(h_j)) = 0.$$



A nonempty subset $F \subset E$ *is called* transitionally compact *in* $(E, \, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}, \, \Theta)$ *if the same property holds for any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* F *(but* $x \in F$ *is* not *required).*

Remark 2.11 If $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is two–sided sequentially compact (uniformly with respect to ε), then the tuple $(E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}}, \Theta)$ is transitionally compact for every nonempty set Θ of maps $[0,1] \times E \longrightarrow E$.

Assuming transitional compactness, Euler method then provides the existence of solutions. Here this result is stated in the slightly more general version for systems :

Proposition 2.12 (Existence of right–hand forward solutions for systems of two generalized mutational equations)
Assume that the tuples $(E_1, \, (q^1_\varepsilon)_{\varepsilon\in\mathcal{J}_1}, \, \Theta^{\mapsto}_p(E_1, D_1, (q^1_\varepsilon)_{\varepsilon\in\mathcal{J}_1}))$ *and* $(E_2, \, (q^2_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2}, \, \Theta^{\mapsto}_{p'}(E_2, D_2, (q^2_{\varepsilon'})_{\varepsilon'\in\mathcal{J}_2}))$ *are transitionally compact. Moreover for* $\varepsilon \in \mathcal{J}_1, \; \varepsilon' \in \mathcal{J}_2$, *let*

$$f_1 : \, E_1 \times E_2 \times [0,T] \longrightarrow \Theta^{\mapsto}_p(E_1, \, D_1, \, (q^1_\varepsilon)_{\varepsilon\in\mathcal{J}_1})$$
$$f_2 : \, E_1 \times E_2 \times [0,T] \longrightarrow \Theta^{\mapsto}_{p'}(E_2, \, D_2, \, (q^2_{\varepsilon'})_{\varepsilon\in\mathcal{J}_1})$$

fulfill

1. a) $M_\varepsilon := \sup_{t,v_1,v_2} \alpha^{\mapsto}_\varepsilon(f_1(v_1,v_2,t)) < \infty,$

   b) $M_{\varepsilon'} := \sup_{t,v_1,v_2} \alpha^{\mapsto}_{\varepsilon'}(f_2(v_1,v_2,t)) < \infty,$

2. a) $c_\varepsilon(h) := \sup_{t,v_1,v_2} \beta_\varepsilon\,(f_1(v_1,v_2,t))\,(h) \stackrel{h\downarrow 0}{\longrightarrow} 0,$

   b) $c_{\varepsilon'}(h) := \sup_{t,v_1,v_2} \beta_{\varepsilon'}\,(f_2(v_1,v_2,t))\,(h) \stackrel{h\downarrow 0}{\longrightarrow} 0,$

3. a) $\exists \; R_\varepsilon \; : \sup_{t,v_1,v_2} \gamma_\varepsilon\,(f_1(v_1,v_2,t)) \leq R_\varepsilon, \qquad \varepsilon^p \cdot R_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} 0$

   b) $\exists \; R_{\varepsilon'} \; : \sup_{t,v_1,v_2} \gamma_{\varepsilon'}\,(f_2(v_1,v_2,t)) \leq R_{\varepsilon'}, \qquad \varepsilon'^{(p')} \cdot R_{\varepsilon'} \stackrel{\varepsilon'\to 0}{\longrightarrow} 0$

4. $\exists$ *moduli* $\widehat{\omega}_\varepsilon(\cdot), \, \widehat{\omega}_{\varepsilon'}(\cdot)$ *of continuity :*

   $Q^{1\mapsto}_\varepsilon(f_1(y_1,y_2,t_1), \;\; f_1(v_1,v_2,t_2)) \leq R_\varepsilon + \widehat{\omega}_\varepsilon\,(q^1_\varepsilon(y_1,v_1) + q^2_{\varepsilon'}(y_2,v_2) + t_2 - t_1)$

   $Q^{2\mapsto}_{\varepsilon'}(f_2(y_1,y_2,t_1), \;\; f_2(v_1,v_2,t_2)) \leq R_{\varepsilon'} + \widehat{\omega}_{\varepsilon'}(q^1_\varepsilon(y_1,v_1) + q^2_{\varepsilon'}(y_2,v_2) + t_2 - t_1)$

   for all $0 \leq t_1 \leq t_2 \leq T$, $\;\; y_1, v_1 \in E_1$, $\;\; y_2, v_2 \in E_2$ *and* $\varepsilon' \in \mathcal{J}_2$.

Then for every $x^0_1 \in E_1$ *and* $x^0_2 \in E_2$, *there exist right–hand forward solutions* $x_1(\cdot) : \, [0,T[\; \longrightarrow E_1, \; x_2(\cdot) : \, [0,T[\; \longrightarrow E_2$ *of the generalized mutational equations*

$$\overset{\circ}{x}_1(\cdot) \ni f_1(x_1(\cdot), \; x_2(\cdot), \; \cdot\,)$$
$$\overset{\circ}{x}_2(\cdot) \ni f_2(x_1(\cdot), \; x_2(\cdot), \; \cdot\,)$$

with $\quad x_1(0) = x^0_1, \quad x_2(0) = x^0_2.$

Remark 2.13 1. Assumption (2.) is only to guarantee the uniform continuity of the Euler approximations. If this property results from other arguments, then we can dispense with this assumption and even with condition (4.) of Definition 2.2.

2. The proof in detail (presented in both [31] and [32]) shows that the compactness assumption can be weakened slightly. Considering the initial value problem for $(E, D, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$, we only need that all values of Euler approximations (at positive times) are contained in a subset F that is transitionally compact in $(E, (q_\varepsilon), \Theta_p^{\mapsto}(E, D, (q_\varepsilon)))$. In particular, it does not require any additional assumptions about the initial value.

Finally, we are interested in bounds of the distance between solutions. However, estimating the distance between points of forward transitions is available only for elements of D in the first argument of q_ε (as in Proposition 2.5). So essentially, we have two possibilities : Either restricting ourselves to the comparison with elements of D (as in Prop. 2.14) or using an auxiliary function instead of the distance (as in Prop. 2.15).

Proposition 2.14 *Assume for* $f : E \times [0,T] \longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon))$ *and* $x, y : [0,T[\longrightarrow E$

1. a) $\overset{\circ}{y}(\cdot) \ni f(y(\cdot), \cdot)$ *in* $[0,T[$,

   b) $x(t) \in D$ *for all* $t \in [0,T[$,
   $\limsup_{h \downarrow 0} \frac{1}{h}\ q_\varepsilon(x(t+h),\ f(x(t),t)\,(h, x(t))) \ \leq\ \gamma_\varepsilon(f(x(t),t))$,

   c) $q_\varepsilon(x(t), y(t)) \ \leq\ \limsup_{h \downarrow 0}\ q_\varepsilon(x(t-h),\ y(t-h))$,

2. $M_\varepsilon \ := \ \sup_{t,v}\ \alpha_\varepsilon^{\mapsto}(f(v,t)) \ < \ \infty$,

3. $\exists\ R_\varepsilon < \infty :\ \sup_{t,v}\ \gamma_\varepsilon(f(v,t)) \ \leq\ R_\varepsilon, \qquad \varepsilon'^p\, R_{\varepsilon'} \overset{\varepsilon' \to 0}{\longrightarrow} 0$,

4'. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :\ Q_\varepsilon^{\mapsto}(f(v_1,t_1),\ f(v_2,t_2)) \ \leq R_\varepsilon + L_\varepsilon \cdot q_\varepsilon(v_1,v_2) + \widehat{\omega}_\varepsilon(t_2 - t_1)$
   for all $0 \leq t_1 \leq t_2 \leq T$ *and* $v_1, v_2 \in E$,
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup_{s \downarrow 0}\ \widehat{\omega}_\varepsilon(s) \ = \ 0$.

Then, $$q_\varepsilon(x(t),\ y(t)) \ \leq\ q_\varepsilon(x(0),\ y(0)) \cdot e^{(L_\varepsilon + M_\varepsilon)\cdot t} \ + \ 5\, R_\varepsilon\ \frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t} - 1}{L_\varepsilon + M_\varepsilon} \qquad \textit{for all } t.$$

Proposition 2.15 *Assume for* $f : E \times [0,T] \longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon))$ *and* $x, y : [0,T[\longrightarrow E$

1. $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot), \quad \overset{\circ}{y}(\cdot) \ni f(y(\cdot), \cdot)$ *in* $[0,T[$,

2. $M_\varepsilon \ := \ \sup_{t,v}\ \alpha_\varepsilon^{\mapsto}(f(v,t)) \ < \ \infty$,

3. $\exists\ R_\varepsilon < \infty :\ \sup_{t,v}\ \gamma_\varepsilon(f(v,t)) \ \leq\ R_\varepsilon, \qquad \varepsilon'^p\, R_{\varepsilon'} \overset{\varepsilon' \to 0}{\longrightarrow} 0$,

4'. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :\ Q_\varepsilon^{\mapsto}(f(v_1,t_1),\ f(v_2,t_2)) \ \leq R_\varepsilon + L_\varepsilon \cdot q_\varepsilon(v_1,v_2) + \widehat{\omega}_\varepsilon(t_2 - t_1)$
   for all $0 \leq t_1 \leq t_2 \leq T$ *and* $v_1, v_2 \in E$,
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup_{s \downarrow 0}\ \widehat{\omega}_\varepsilon(s) \ = \ 0$.

Furthermore suppose for each $t \in [0,T[$ *that the infimum*

$$\varphi_\varepsilon(t) \;:=\; \inf_{z\,\in D}\; \big(q_\varepsilon(z,\; x(t)) \;+\; q_\varepsilon(z,\; y(t))\big) \;<\; \infty$$

can be approximated by a minimizing sequence $(z_j)_{j\,\in\mathbb{N}}$ *in* D *satisfying*

$$\frac{\sup_{k>j}\;\; q_\varepsilon(z_j, z_k)}{\mathcal{T}_\Theta(f(z_j,t),\; z_j)} \;\;\longrightarrow\;\; 0 \qquad\qquad (j \longrightarrow \infty)$$

Then, $\qquad \varphi_\varepsilon(t) \;\le\; \varphi_\varepsilon(0)\, e^{(L_\varepsilon+M_\varepsilon)\cdot t} \;+\; 8\, R_\varepsilon \;\cdot\; \frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t}\, -\, 1}{L_\varepsilon+M_\varepsilon}.$

In the case of symmetric q_ε and D dense in (E, q_ε), we obtain $\varphi_\varepsilon(t) = q_\varepsilon(x(t), y(t))$. Proving the last proposition, the basic idea consists in estimating both

$$h \;\longmapsto\; q_\varepsilon\Big(f(z_m,t)\,(h, z_m),\;\; x(t{+}h)\Big) \qquad \text{and} \qquad h \;\longmapsto\; q_\varepsilon\Big(f(z_m,t)\,(h, z_m),\;\; y(t{+}h)\Big)$$

(for small $h > 0$) with such a minimizing sequence $(z_m)_{m\in\mathbb{N}}$. Here assumptions about the time parameter $\mathcal{T}_\Theta(\cdot,\cdot) > 0$ are required for the first time. Roughly speaking, we need lower bounds of $\mathcal{T}_\Theta(f(z_m,t), z_m)$ for "preserving" the information while $m \longrightarrow \infty$.

Finally, the auxiliary function $\varphi_\varepsilon(\cdot)$ is modified with regard to first solution $x(\cdot)$:

$$\varphi_\varepsilon(t) \;:=\; \inf_{z\,\in D}\; \big(p_\varepsilon(z,\; x(t)) \;+\; q_\varepsilon(z,\; y(t))\big)$$

Here $p_\varepsilon : E \times E \longrightarrow [0,\infty[$ represents a generalized distance function on E that has the additional advantage of symmetry (by assumption). Roughly speaking, p_ε might take other properties of elements $x, y \in E$ into consideration – in comparison with q_ε. The compact subsets of $\mathbb{R}^N$ give an example with $p_\varepsilon := d\!\!l$ (Pompeiu–Hausdorff distance) in Corollary 4.7. In particular, the assumptions about p_ε have the advantage that they do not consider the comparison of two transitions. Instead we suppose only continuity properties for each value $\psi \in \Theta_p^{\mapsto}(E, D, (q_\varepsilon))$ of f (in assumptions (6.)–(8.)).

Proposition 2.16 *Suppose for* $p_\varepsilon, q_\varepsilon : E \times E \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J})$, $p \in \mathbb{R}, \lambda_\varepsilon \ge 0$ *and* $f : E \times [0,T] \longrightarrow \Theta_p^{\mapsto}(E, D, (q_\varepsilon))$, $\;x, y : [0,T[\longrightarrow E$ *the following properties :*

1. $(E,\; (q_\varepsilon)_{\varepsilon\in\mathcal{J}},\; \Theta_p^{\mapsto}(E, D, (q_\varepsilon)))$ *is transitionally compact,*
2. *each* p_ε *is symmetric and satisfies the triangle inequality,*
3. $\Delta_\varepsilon(v_1, v_2) \;:= \inf_{z\,\in D}\,(p_\varepsilon(v_1, z) + q_\varepsilon(z, v_2)) < \infty \qquad$ *for* $v_1, v_2 \in E$,
4. $x(\cdot)$ *is a right–hand forward solution of* $\overset{\circ}{x}(\cdot) \;\ni\; f(x(\cdot),\,\cdot)$ *constructed by Euler method according to the proof of Proposition 2.12 (see [31]),*
5. $y(\cdot)$ *is a right–hand forward solution of* $\overset{\circ}{y}(\cdot) \;\ni\; f(y(\cdot),\,\cdot)$ *in* $[0,T[$,
6. $\exists\;\; M_\varepsilon < \infty :\;\; \widehat{\alpha}_\varepsilon^{\mapsto}(\cdot, x,\; f(x,\cdot)),\; \widehat{\alpha}_\varepsilon^{\mapsto}(\cdot, y,\; f(y,\cdot)) \le\; M_\varepsilon,$

   $p_\varepsilon(\psi(h, v_1),\;\; \psi(h, v_2)) \qquad\qquad \le\; p_\varepsilon(v_1, v_2)\cdot e^{M_\varepsilon\, h}$

   $\forall\;\; v_1, v_2 \in E,\;\; h \in\,]0,T[,\;\; \psi \in \{f(v,s) \,|\, v{\in}E, s{<}T\},$

7. $\exists\ R_\varepsilon < \infty : \ \widehat{\gamma}_\varepsilon(\,\cdot\,, x,\ f(x,\cdot)),\ \ \widehat{\gamma}_\varepsilon(\,\cdot\,, y,\ f(y,\cdot)) \le\ R_\varepsilon,$

$$\limsup_{h\downarrow 0}\ \frac{p_\varepsilon\big(\psi(h,\psi(t,v)),\ \ \psi(t+h,v)\big)}{h}\ \le\ R_\varepsilon$$

for all $v \in E,\ \ t \in [0,T[,\ \ \psi \in \{f(v,s)\,|\,v\in E, s<T\},$

8. $\exists\ c_\varepsilon(\cdot):\qquad p_\varepsilon(\psi(t,v),\psi(t+h,v)) + \beta_\varepsilon(\psi)(h) \le\ c_\varepsilon(h)$

for all $v \in E,\ \ t\in[0,T[,\ \ \psi \in \{f(v,s)\,|\,v\in E,\ s<T\},$

$c_\varepsilon(h) \longrightarrow 0 \qquad$ *for* $h \downarrow 0,$

9. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :\ Q_\varepsilon^{\mapsto}(f(v_1,t_1),\ f(v_2,t_2))\ \le\ R_\varepsilon\ +\ L_\varepsilon\cdot\Delta_\varepsilon(v_1,v_2)\ +\ \widehat{\omega}_\varepsilon(t_2-t_1)$

for all $0\le t_1\le t_2\le T$ *and* $v_1,v_2\in E,$

$\widehat{\omega}_\varepsilon(\cdot)\ge 0$ *nondecreasing,* $\displaystyle\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s)\ =\ 0,$

10. *for each* $v\in E,\ \ \delta>0,\ \ 0\le s\le t<T,\ \ 0<h<1$ *with* $t+h+\delta<T$, *the infimum* $\Delta_\varepsilon(f(v,s)\,(h,v),\ y(t+h+\delta))$ *can be approximated by a minimizing sequence* $(z_n)_{n\in\mathbb{N}}$ *in* D *satisfying* $\dfrac{\sup_{k>j}\ \ (p_\varepsilon(z_j,z_k)\ +\ q_\varepsilon(z_j,z_k))}{\mathcal{T}_\Theta(f(z_j,t),\ z_j)}\ \longrightarrow\ 0\ \ (j\longrightarrow\infty).$

Then, $\displaystyle\varphi_\varepsilon(t)\ :=\ \limsup_{\delta\downarrow 0}\ \Delta_\varepsilon(x(t),\ y(t+\delta))$ *fulfills*

$$\varphi_\varepsilon(t)\ \le\ (\varphi_\varepsilon(0)\ +\ 5\,R_\varepsilon\ t)\ \ (1+L_\varepsilon\,t)\ \ e^{2\,M_\varepsilon\,t}.$$

3 Mild solutions of semilinear equations in reflexive Banach spaces

Now we consider semilinear evolution equations in a real Banach space X and specify the assumptions so that the concept of right–hand forward solutions can be applied. Let $A : D_A \longrightarrow X\ \ (D_A \subset X)$ be a closed linear operator on a Banach space X generating a semigroup $(S(t))_{t\ge 0}$. Then for every $w\in X$ and initial point $u_0\in X$, the inhomogeneous equation $\frac{d}{dt}\,u(t) = A\,u(t)\ +\ w$ has a unique solution $u:[0,\infty[\,\longrightarrow X$ with $u(0)=u_0$, namely

$$\tau_w(t,\ u_0)\ \ :=\ \ u(t)\ \ =\ \ S(t)\,u_0\ +\int_0^t S(t-s)\ w\ \ ds.$$

In particular, we obtain $\tau_w(t_1,u_0)-\tau_w(t_2,u_0)\ =\ S(t_1)u_0-S(t_2)u_0$ for every $t_1,t_2\ge 0$ and fixed $u_0,\ w\in X$. If $\tau_w(\cdot,\cdot)$ is a forward transition on $(X,\ X,\ \|\cdot\|_X)$, then all $\tau_w(\cdot,u_0):[0,1]\longrightarrow X\ \ (u_0\in X)$ have to be equi–continuous according to condition (4.) of Definition 2.2 and, so many important examples of semigroup theory are excluded. Their applications often lead to only *strongly continuous semigroups* or C^0 *semigroups* $(S(t))_{t\ge 0}$, i.e. particularly, $[0,\infty[\,\longrightarrow X,\ \ t\longmapsto S(t)\,x$ is continuous for each $x\in X$, but not equi–continuous in general (see e.g. [34, Pazy 83], [25, Engel,Nagel 2000]). Furthermore, according to the Theorems of Hille–Yosida and Feller–Miyadera–Phillips, the generator of a C^0 semigroup is closed, but need not be bounded.

Thus for applying the mutational approach to C^0 semigroups, we prefer the weak topology on X to the norm $\|\cdot\|_X$ and define

$$q_{v'} : X \times X \longrightarrow [0, \infty[, \qquad (x, y) \longmapsto |\langle x - y,\ v' \rangle|$$

for every linear form $v' \in X'$ with $\|v'\|_{X'} \leq 1$. Each $q_{v'}$ is a so–called *pseudo–metric*, i.e. it is reflexive ($q_{v'}(x, x) = 0$ for all x), symmetric ($q_{v'}(x, y) = q_{v'}(y, x)$ for all x, y) and satisfies the triangle inequality. The family $\{q_{v'}\}$ induces the weak topology on X.

From now on, we suppose the Banach space X to be reflexive. This additional assumption has two advantages : Firstly, closed bounded balls of X are known to be weakly compact. So speaking in terms of § 2, $(X, (q_{v'})_{v'})$ is two–sided sequentially compact (in the sense of Definition 2.9).
Secondly, the reflexivity of X guarantees that the adjoint operators $S(t)' : X' \longrightarrow X'$ $(t \geq 0)$ form a C^0 semigroup on X' with the infinitesimal generator A' (see Lemma 3.4). This useful consequence opens the possibility that $\tau_w(\cdot, \cdot)$ fulfills (slightly weakened) continuity conditions on transitions with respect to each $q_{v'}$ for $v' \in X'$ fixed (as presented in Proposition 3.3).

General assumptions for § 3.

1. X is a reflexive Banach space.
2. The linear operator A generates a C^0 semigroup $(S(t))_{t \geq 0}$ on X
3. The dual operator A' of A has a countable family of unit eigenvectors $\{v'_j\}_{j \in \mathcal{J}}$ spanning the dual space X'. λ_j abbreviates the eigenvalue of A' belonging to v'_j.

Example 3.1 1. Consider a normal compact operator $A : H \longrightarrow H$ on a separable Hilbert space H generating a C^0 semigroup $(S(t))_{t \geq 0}$.
Then there exists a countable orthonormal system $(e_i)_{i \in \mathcal{I}}$ of eigenvectors of A satisfying $H = \ker A \oplus \overline{\sum_{i \in \mathcal{I}} \mathbb{R}\, e_i}$ (see [41, Werner 2002], Th. VI.3.2). Since H is separable, $(e_i)_{i \in \mathcal{I}}$ induces a *countable* orthonormal basis $(e_i)_{i \in \widehat{\mathcal{I}}}$ of H with $A\, e_i = 0$ for all $i \in \widehat{\mathcal{I}} \setminus \mathcal{I}$. In fact, each e_i $(i \in \widehat{\mathcal{I}})$ is also eigenvector of A' because A is normal (see [41, Werner 2002], Lemma VI.3.1). So the general assumptions of this section are satisfied.
Symmetric integral operators of Hilbert–Schmidt type provide typical examples of A.

2. An example of more general interest is the generator $A : D_A \longrightarrow H$ $(D_A \subset H)$ of a C^0 semigroup $(S(t))_{t \geq 0}$ on a Hilbert space H — assuming that the resolvent $R(\lambda_0, A) := (\lambda_0 \cdot \mathrm{Id}_H - A)^{-1} : H \longrightarrow H$ is compact and normal for some λ_0.
For the same reasons as before, there exists a countable orthonormal system $(e_i)_{i \in \mathcal{I}}$ of eigenvectors of $R(\lambda_0, A)$ satisfying $H = \ker R(\lambda_0, A) \oplus \overline{\sum_{i \in \mathcal{I}} \mathbb{R}\, e_i} = \overline{\sum_{i \in \mathcal{I}} \mathbb{R}\, e_i}$.
$R(\lambda_0, A)\, e_i = \mu_i \cdot e_i$ implies $\mu_i \neq 0$ and that e_i is eigenvector of A corresponding to the eigenvalue $\lambda_0 - \frac{1}{\mu_i}$ since $(\lambda_0 - A)\, e_i = (\lambda_0 - A) \cdot \frac{1}{\mu_i}\, R(\lambda_0, A)\, e_i = \frac{1}{\mu_i}\, e_i$.
This case opens the door for considering strongly elliptic differential operators in divergence form with smooth (time–independent) coefficients, for example.

Definition 3.2

1. *For every* $j \in \mathcal{J}$, *define the pseudo–metric* $\quad q_j(x, y) := |\langle x - y,\, v'_j \rangle| \quad$ *on* X.
2. *For each* $v \in X$, *the function* $\tau_v : [0, 1] \times X \longrightarrow X$ *is defined as mild solution of the initial value problem* $\frac{d}{dt}\, u(t) = A\, u(t) + v,\ u(0) = x \in X$, *i.e.*
$$\tau_v(h, x) \;:=\; S(h)\, x \;+\; \int_0^h S(h - s)\; v\; ds.$$

Proposition 3.3 *For* $v \in X$ *fixed, the function* $\tau_v : [0, 1] \times X \longrightarrow X$ *satisfies the following conditions on forward transitions of order* 0 *on* $(X, X, (q_j)_{j \in \mathcal{J}})$ *(see Def. 2.2) :*

1. $\tau_v(0, \cdot\,) \;=\; \mathrm{Id}_X,$
2. $q_j\Big(\tau_v(h,\ \tau_v(t, x)),\ \tau_v(t + h, x)\Big) \;=\; 0 \;=\; q_j\Big(\tau_v(t + h, x),\ \tau_v(h,\ \tau_v(t, x))\Big)$
for all $x \in X,\ t, h \in [0, 1]$ *with* $t + h \le 1,$
3. $$\sup_{\substack{x, y \,\in\, X \\ q_j(x,y) \,\neq\, 0}} \ \limsup_{h \,\downarrow\, 0} \ \left(\frac{q_j\big(\tau_v(h, x),\, \tau_v(h, y)\big) - q_j(x,y)}{h \quad q_j(x,y)} \right)^{+} \;\le\; |\lambda_j|\,.$$

Moreover for every radius $R > 0$ *and index* $j \in \mathcal{J}$, *there is a modulus* $\omega_j(\cdot)$ *of continuity (depending only on* A *and* v_j*) such that for all* $t_1, t_2 \in [0, 1],\ x \in X\ (|x| \le R)$,
$$q_j\Big(\tau_v(t_1, x),\ \tau_v(t_2, x)\Big) \;\le\; R \,\cdot\, \omega_j(|t_2 - t_1|).$$
Finally, the functions $\tau_v,\ \tau_w : [0, 1] \times X \longrightarrow X$ *related to* $v, w \in X$ *respectively fulfill*
$$Q_j^{\mapsto}(\tau_v, \tau_w) \;\stackrel{\text{Def.}}{=}\; \sup_{x,\, y \,\in\, X} \ \limsup_{h \,\downarrow\, 0} \ \left(\frac{q_j\big(\tau_v(h,x),\, \tau_w(h,y)\big) - q_j(x,y) \cdot e^{|\lambda_j|\, h}}{h} \right)^{+} \;\le\; q_j(v, w).$$

In preparation of the proof, we summarize the essential tools about C^0 semigroups. The first lemma bridges the gap between the semigroup operators and their dual counterparts. It is one of the reasons for assuming X to be reflexive. Afterwards Lemma 3.5 implies that each v'_j $(j \in \mathcal{J})$ is eigenvector of every dual operator $S(t)'$ $(t \ge 0)$ belonging to the eigenvalue $e^{\lambda_j\, t}$.

Lemma 3.4
Let $(S(t))_{t \ge 0}$ *be a* C^0 *semigroup on a reflexive Banach space with generator* A.
Then the dual operators $S(t)'$ $(t \ge 0)$ *provide a* C^0 *semigroup on the dual space and its generator is the dual operator* A'.

Proof is given in [34, Pazy 83], Cor. 1.10.6 and [25, Engel,Nagel 2000], Prop. I.5.14. □

Lemma 3.5 *The eigenspaces of the generator* A *and of the* C^0 *semigroup operators* $S(t)$ $(t \ge 0)$, *respectively, fulfill for every* $\mu \in \mathbb{C}$
$$\ker\,(\mu - A) \;=\; \bigcap_{t \,\ge\, 0} \ \ker\,\big(e^{\mu\, t} - S(t)\big)\,.$$

Proof is presented in detail in [25, Engel,Nagel 2000], Corollary IV.3.8. □

Proof of Prop. 3.3. The first assertion results directly from the definition of τ_v and, the second claim is a consequence of the semigroup property $\tau_v(h,\ \tau_v(t,x)) = \tau_v(t{+}h, x)$. Furthermore we obtain for every $x,\ y \in X$ and $h \in [0,1]$ with $q_j(x,y) \neq 0$

$$q_j\Big(\tau_v(h,x),\ \ \tau_v(h,y)\Big)\ -\ q_j(x,y)\ \ \le\ \ |\,\langle\ x-y,\ \ (S(h)' - \mathrm{Id}_{X'})\ v_j'\rangle\,|$$

and thus, $\displaystyle\limsup_{h\downarrow 0}\ \frac{q_j\big(\tau_v(h,x),\, \tau_v(h,y)\big) - q_j(x,y)}{h}\ \ \le\ \ |\,\langle x-y,\ \ A'\,v_j'\rangle\,|\ \ \le\ \ |\lambda_j|\cdot|\,\langle x-y,\ v_j'\rangle\,|$

since v_j' is assumed to be eigenvector of A'. So the third statement is verified.

The claimed continuity of $\tau_v(\cdot,x) : [0,1] \longrightarrow X\ \ (x \in X, |x| \le R)$ results from the strong continuity of $(S(t)')_{t\ge 0}$ (according to Lemma 3.4).
Indeed, for every $t_1,\ t_2 \in [0,1]$ and $x \in X$ with $|x| \le R$,

$$q_j\Big(S(t_1)\,x,\ \ S(t_2)\,x\Big)\ \le\ \big|\,\langle S(t_2)\,x - S(t_1)\,x,\ \ v_j'\rangle\,\big|\ \le\ R\ \big|(S(t_2)' - S(t_1)')\ v_j'\big|\,.$$

Finally we prove $Q_j^{\mapsto}(\tau_v,\tau_w)\ \le\ q_j(v,w)$ for arbitrary $v,\ w \in X$.
Indeed, the definition of τ_v, τ_w and Lemma 3.5 provide for every $x,\ y \in X$ and $h \in]0,1]$

$$\begin{aligned} q_j\Big(\tau_v(h,x),\ \ \tau_w(h,w)\Big)\ &\le\ \big|\,\langle x-y,\ S(h)'\,v_j'\rangle\big|\ +\ \int_0^h\ |\langle v-w,\ \ S(h{-}s)'\,v_j'\rangle|\ \ ds \\ &\le\ \big|\,\langle x-y,\ v_j'\rangle\big|\cdot e^{|\lambda_j|\,h}\ +\ |\langle v-w,\ v_j'\rangle|\ \cdot\ \int_0^h\ e^{|\lambda_j|\,(h-s)}\ ds \\ &\le\ \Big(q_j(x,y)\ \ +\ q_j(v,w)\ h\Big)\ \cdot\ e^{|\lambda_j|\,h}. \end{aligned}$$

□

As a direct consequence of this proposition, we get $q_j(\tau_v(t-h,x),\, y) \longrightarrow q_j(\tau_v(t,x),\, y)$ for $h \downarrow 0$ and all x,y,t. So there is only one reason why τ_v is *not* a forward transition on $(X, X, (q_j)_{j\in\mathcal{J}})$ in the strict sense of Definition 2.2 :
Considering $\tau_v(\cdot,x) : [0,1] \longrightarrow X$, the modulus of continuity can be chosen uniformly only for all points x of a bounded subset, but not for all elements $x \in X$ in general. This gap does not really prevent us from applying the results of § 2. Indeed, for concluding the existence of right–hand forward solutions from Proposition 2.12, we only need the uniform continuity of Euler approximations in positive time direction (due to Remark 2.13 (1.)). The general feature of C^0 semigroups, $\|S(t)\|_{\mathcal{L}(X,X)} \le \mathrm{const}\cdot e^{\mathrm{const}\,\cdot\, t}$, easily provides a priori bounds of $\|\tau_v(\cdot,x)\|_{L^\infty}$ (depending only on $\|x\|_X, \|v\|_X$).

So Propositions 2.12 and 2.14 imply

Proposition 3.6 *In addition to the general assumptions about $X, A, S(\cdot)$ of this paragraph, let $f : X \times [0,T] \longrightarrow X$ satisfy $\|f\|_{L^\infty} < \infty$ and for each $j \in \mathcal{J}$,*

$$q_j\Big(f(x_1,t_1),\ \ f(x_2,t_2)\Big)\ \le\ \omega_j\Big(q_j(x_1,x_2) + |t_2 - t_1|\Big)\qquad \textit{for all}\ \ x_1, x_2,\ t_1, t_2$$

with a modulus $\omega_j(\cdot)$ of continuity.

Then for every initial vector $x_0 \in X$, *there exists a right–hand forward solution* $x(\cdot) : [0,T[\longrightarrow X$ *of the generalized mutational equation* $\overset{\circ}{x}(\cdot) \ni \tau_{f(x(\cdot),\cdot)}$ *in* $[0,T[$ *with* $x(0) = x_0$ *i.e. for each* $j \in \mathcal{J}$, $x(\cdot)$ *is uniformly continuous with respect to* q_j *and*

$$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \left(q_j\Big(\tau_{f(x(t),t)}\,(h,y),\ \ x(t+h)\Big) \ - \ q_j(y,\ x(t)) \ \cdot \ e^{|\lambda_j|\, h} \right) \ \le \ 0,$$

holds for all $y \in X$, $t \in [0,T[$.
Supposing $q_j\Big(f(x_1,t_1),\ f(x_2,t_2)\Big) \ \le \ L_j \cdot q_j(x_1,x_2) + \widehat{\omega}_j(t_2 - t_1)$ *for all* x_1, x_2, t_1, t_2, j *with* $L_j \ge 0$ *and a modulus* $\widehat{\omega}_j(\cdot)$ *of continuity, this solution is unique.*

The assumptions about f might be regarded as unfavorable though. Indeed, we suppose the continuity with respect to each linear form v_j' $(j \in \mathcal{J})$ separately. Even easy examples of rotation might fail to satisfy this condition. For overcoming this obstacle, several pseudo–metrics q_j $(j \in \mathcal{J})$ are considered simultaneously. To be more precise, we replace the family q_j $(j \in \mathcal{J} = \{j_1, j_2, j_3 \dots\})$ with the pseudo–metrics $p_n, n \in \mathbb{N}$,

$$p_n(x,y) \ := \ \sum_{k=1}^{n} \ 2^{-k} \ \frac{q_{j_k}(x,y)}{1 \ + \ q_{j_k}(x,y)} \qquad\qquad (n \in \mathbb{N} \cup \{\infty\}).$$

Reflexivity and symmetry of p_n are obvious and, the triangle inequality results from the auxiliary function $[0,\infty[\ \longrightarrow [0,1], \ r \longmapsto \frac{r}{1+r}$ being increasing and concave.
The key advantage of $(p_n)_{n\in\mathbb{N}}$ is that we can take finitely many q_j into consideration and estimate the rest uniformly. So in short, the existence results of § 2 hold with the parameter $R_\varepsilon > 0$ arbitrarily small (which can be interpreted as order 0).

Lemma 3.7 *For* $v \in X$ *fixed, the function* $\tau_v : [0,1] \times X \longrightarrow X$ *satisfies the following conditions on forward transitions of order* 0 *on* $(X, X, (p_n)_{n\in\mathbb{N}})$

1. $\tau_v(0,\,\cdot\,) \ = \ \mathrm{Id}_X,$
2. $p_n\Big(\tau_v(h,\ \tau_v(t,x)),\ \ \tau_v(t+h,x)\Big) \ = \ 0 \ = \ p_n\Big(\tau_v(t+h,x),\ \ \tau_v(h,\ \tau_v(t,x))\Big)$
   for all $x \in X$, $t,h \in [0,1]$ *with* $t + h \le 1$,
3. $\displaystyle\sup_{\substack{x,y \in X \\ p_n(x,y) \neq 0}} \ \limsup_{h \downarrow 0} \ \left(\frac{p_n\big(\tau_v(h,x), \tau_v(h,y)\big) - p_n(x,y)}{h \quad p_n(x,y)} \right)^{+} \ \le \ \mu_n \qquad$ *with* $\mu_n := \max_{k=1\dots n} \ |\lambda_{j_k}|$.

Moreover for every radius $R > 0$ *and index* $n \in \mathbb{N}$, *there is a modulus* $\omega_n(\cdot)$ *of continuity (depending only on* A *and* n*) such that for all* $t_1,\ t_2 \in [0,1]$, $x \in X$ $(|x| \le R)$,

$$p_n\Big(\tau_v(t_1,x),\ \ \tau_v(t_2,x)\Big) \ \le \ R \ \cdot \ \omega_n(|t_2 - t_1|).$$

$\tau_v,\ \tau_w : [0,1] \times X \longrightarrow X$ *related to* $v, w \in X$ *respectively satisfy*

$$P_n^{\mapsto}(\tau_v, \tau_w) \ \overset{\text{Def.}}{=} \ \sup_{x,\, y \in X} \ \limsup_{h \downarrow 0} \ \left(\frac{p_n\big(\tau_v(h,x), \tau_w(h,y)\big) - p_n(x,y) \cdot e^{\mu_n\, h}}{h} \right)^{+}$$
$$\le \ \sum_{k=1}^{n} \ 2^{-k} \ q_{j_k}(v,w) \quad \le \quad |v - w|\,.$$

Proof results from Proposition 3.3 about forward transitions on $(X, X, (q_j)_{j\in\mathcal{J}})$ because the auxiliary function $[0,\infty[\ \longrightarrow [0,1],\ r \longmapsto \frac{r}{1+r}$ is increasing and concave. (For further details see [32, Lorenz 2004], Lemma 4.5.9.) □

Proposition 3.8 *In addition to the general assumptions about $X, A, S(\cdot)$ of § 3, let $f : X \times [0,T] \longrightarrow X$ fulfill $\|f\|_{L^\infty} < \infty$ and*

$$\sum_{k=1}^{\infty} 2^{-k}\ q_{j_k}\Big(f(x_1,t_1),\ f(x_2,t_2)\Big) \ \le\ \widehat{\omega}\Big(p_\infty(x_1,x_2) \ +\ |t_2 - t_1|\Big)$$

for all $x_1, x_2 \in X$ and $t_1, t_2 \in [0,T]$ with a modulus $\widehat{\omega}(\cdot)$ of continuity.
For each $x_0 \in X$, there exists a mild solution $x : [0,T[\ \longrightarrow X$ of the initial value problem

$$\wedge \begin{cases} \frac{d}{dt}\, x(t) &=& A\, x(t) \ +\ f(x(t),t) \\ \quad x(0) &=& x_0 \end{cases}$$

i.e.
$$x(t) \ =\ S(t)\, x_0 \ +\ \int_0^t S(t-s)\ f(x(s),\, s)\ ds.$$

Considering the continuity assumption about f, the series is finite due to $\|f\|_{L^\infty} < \infty$ and, it is an upper bound of $P_n^{\mapsto}(\tau_{f(x_1,t_1)},\ ,\tau_{f(x_2,t_2)})$ for every $n \in \mathbb{N}$.

The main steps for proving this proposition are summarized in the next lemmas. In short, the existence result of § 2 provides a right–hand forward solution $x : [0,T[\ \longrightarrow (X, (p_n)_{n\in\mathbb{N}})$ of the generalized mutational equation $\overset{\circ}{x}(\cdot) \ni \tau_{f(x(\cdot),\cdot)}$. Restricting ourselves to each linear form $v'_j\,(j \in \mathcal{J})$, $x(\cdot)$ can be regarded as a weak solution of the initial value problem. Then Lemma 3.10 of John M. Ball ensures that a weak solution is even a mild solution.

Lemma 3.9 *Suppose the assumptions of Proposition 3.8.*
Then for every initial vector $x_0 \in X$, there exists a right–hand forward solution $x(\cdot) : [0,T[\ \longrightarrow (X, (p_n)_{n\in\mathbb{N}})$ of the generalized mutational equation $\overset{\circ}{x}(\cdot) \ni \tau_{f(x(\cdot),\cdot)}$ in $[0,T[$ with $x(0) = x_0$ in the sense that for each $n \in \mathbb{N}$, $x(\cdot)$ is uniformly continuous with respect to p_n and

$$\limsup_{n \longrightarrow \infty}\ \limsup_{h \downarrow 0}\ \frac{1}{h}\left(p_n\Big(\tau_{f(x(t),t)}(h,y),\ x(t+h)\Big) \ -\ p_n(y,\ x(t)) \cdot e^{\mu_n h}\right) \ \le\ 0,$$

holds for all $y \in X$, $t \in [0,T[$. In particular, $x(\cdot)$ has the following properties :

1. $\limsup_{h \downarrow 0}\ \frac{1}{h} \cdot p_n\Big(\tau_{f(x(t),t)}(h, x(t)),\ x(t+h)\Big) \ =\ 0$ *for every $t \in [0,T[$, $n \in \mathbb{N}$.*
2. *$x(\cdot)$ is bounded in X.*
3. *$[0,T[\ \longrightarrow X,\ t \longmapsto \langle f(x(t),t),\ v'_j\rangle$ is continuous for every $j \in \mathcal{J}$.*
4. *$f(x(\cdot),\cdot) \in L^\infty([0,T[,\ X)$.*
5. *$]0,T[\ \longrightarrow \mathbb{R},\ t \longmapsto \langle x(t),\ v'_j\rangle$ is continuously differentiable for each $j \in \mathcal{J}$,*
$$\frac{d}{dt}\,\langle x(t),\ v'_j\rangle \ =\ \langle x(t),\ A'\, v'_j\rangle \ +\ \langle f(x(t),t),\ v'_j\rangle.$$

Proof is based on Proposition 2.12. Indeed, the sequence $(p_n)_{n\in\mathbb{N}}$ of pseudo–metrics induces the weak topology on the reflexive Banach space X. So X is weakly sequentially compact and thus, $(X,(p_n)_{n\in\mathbb{N}})$ is two–sided sequentially compact (uniformly with respect to n).
Choosing $\delta > 0$ arbitrarily small, there is $M \in \mathbb{N}$ with $\sum_{k=M}^{\infty} 2^{-k} \le \delta$.
So, $p_n(x_1,x_2) \le \limsup\limits_{k\longrightarrow\infty} p_k(x_1,x_2) \le p_n(x_1,x_2)+\delta$ for every $n \ge M,\ x_1,x_2 \in X$
and in particular, $P_n^{\mapsto}(\tau_{f(x_1,t_1)},\ ,\tau_{f(x_2,t_2)}) \le \widehat{\omega}\Big(\delta + p_n(x_1,x_2) + |t_2-t_1|\Big).$
Now the steps of Proposition 2.12 provide a right–hand forward solution $x : [0,T[\longrightarrow X$ satisfying for all $y \in X,\ t\in[0,T[,\ n \ge M,$

$$\limsup_{h\downarrow 0} \tfrac{1}{h}\Big(p_n\Big(\tau_{f(x(t),t)}(h,y),\ x(t{+}h)\Big) - p_n(y,\ x(t))\cdot e^{\mu_n h}\Big) \le \text{const}\cdot\widehat{\omega}(\delta).$$

Since $\delta > 0$ is arbitrarily small, we conclude for every vector $y \in X$, time $t \in [0,T[$

$$\limsup_{n\longrightarrow\infty}\ \limsup_{h\downarrow 0} \tfrac{1}{h}\Big(p_n\Big(\tau_{f(x(t),t)}(h,y),\ x(t{+}h)\Big) - p_n(y,\ x(t))\cdot e^{\mu_n h}\Big) \le 0.$$

1. is an immediate consequence by setting $y := x(t)$ (due to $p_{n-1} \le p_n$ for all n).

2. $x(\cdot)$ is bounded in X, i.e. $\|x\|_{L^\infty} < \infty$. Indeed, the proof of Proposition 2.12 presented in [31] uses Euler approximations $x_m(\cdot)$ that are uniformly bounded (due to the exponential growth of every C^0 semigroup, i.e. $\|S(t)\|_{\mathcal{L}(X,X)} \le \text{const}\cdot e^{\text{const}\cdot t}$). Moreover for each time $t \in]0,T[$, a subsequence of $(x_m(t))_{m\in\mathbb{N}}$ converges weakly to $x(t)$ and thus, $|x(t)| \le \limsup\limits_{m\longrightarrow\infty} |x_m(t)|$.

3. The function $[0,T[\longrightarrow X,\ t\longmapsto \langle f(x(t),t),\ v_j'\rangle$ is continuous for each $j\in\mathcal{J}$. Indeed, for any $j_m \in \mathcal{J}$ and $\delta>0$, there exists an index $n \ge m$ with $\sum_{k=n}^{\infty} 2^{-k} \le \delta$.
So, $\sum\limits_{k=1}^{\infty} 2^{-k}\ q_{j_k}\Big(f(x(s),s),\ f(x(t),t)\Big) \le \widehat{\omega}\Big(\delta + p_n(x(s),x(t)) + |t-s|\Big)$ for all s,t.
The uniform continuity of $x(\cdot)$ with respect to p_n implies for any $|t-s|$ sufficiently small

$$q_{j_m}\Big(f(x(s),s),\ f(x(t),t)\Big) \le 2^m\cdot\widehat{\omega}(2\,\delta).$$

4. $\langle f(x(\cdot),\cdot),\ v'\rangle \in L^1([0,T[,\mathbb{R})$ for every linear form $v'\in X'$ results from the general assumption that $(v_j')_{j\in\mathcal{J}}$ is spanning the dual space X' and from the Convergence Theorem of Lebesgue. As X is separable, $f(x(\cdot),\cdot) : [0,T[\longrightarrow X$ is (strongly) Lebesgue–measurable due to the Theorem of Pettis (stated and proven in [42, Yosida 78], chapter V, § 4, for example).

5. Defining p_n by means of $(q_j)_{j\in\mathcal{J}}$ implies that $x(\cdot)$ uniformly continuous with respect to each q_j and for every time $t\in[0,T]$,

$$\limsup_{h\downarrow 0}\ \Big|\Big\langle \tfrac{\tau_{f(x(t),t)}\big(h,x(t)\big)-x(t)}{h} - \tfrac{x(t+h)-x(t)}{h},\ v_j'\Big\rangle\Big| = 0.$$

Definition 3.2 of $\tau_{f(x(t),t)}(h,\cdot)$ guarantees

$$\begin{aligned}\lim_{h\downarrow 0} \left\langle \tfrac{x(t+h)-x(t)}{h},\ v_j' \right\rangle &= \langle A\,x(t) \quad + \quad f(x(t),t),\ v_j' \rangle \\ &= \lambda_j\ \langle x(t),\ v_j'\rangle \ + \ \langle f(x(t),t),\ v_j' \rangle\end{aligned}$$

and, the right–hand side is continuous with respect to t. These two properties ensure that $]0,T[\ \longrightarrow\ \mathbb{R},\ \ t \longmapsto \langle\, x(t),\ v_j' \,\rangle$ is continuously differentiable for every $j \in \mathcal{J}$ (see e.g. [34, Pazy 83], Corollary 2.1.2). □

So according to the preceding Lemma 3.9, $x(\cdot) : [0,T[\ \longrightarrow X$ is a weak solution of the initial value problem (for $z(\cdot)$)

$$\tfrac{d}{dt}\, z(t) \ = \ A\, z(t) \ + \ f(x(t),t), \qquad z(0) \ = \ x_0.$$

Finally, the following lemma of John. M. Ball bridges the gap between weak and mild solutions because in this paragraph, A has been supposed to be the infinitesimal generator of the C^0 semigroup $(S(t))_{t\geq 0}$. So Proposition 3.8 is proved.

Lemma 3.10 ([7, Ball 77]) *Let A be a densely defined closed linear operator on a real or complex Banach space Y and $g \in L^1([0,T],Y)$.*
There exists for each $y \in Y$ a unique weak solution $u(\cdot)$ of

$$\wedge \begin{cases} \tfrac{d}{dt}\, u(t) &= \ A\, u(t) \ + \ g(t) \quad \text{on }]0,T] \\ \quad u(0) &= \ x \end{cases}$$

i.e. for every $v' \in D(A') \subset Y'$, $\langle u(\cdot), v'\rangle \in AC([0,T])$ and

$$\tfrac{d}{dt}\, \langle u(t),\ v'\rangle \ = \ \langle u(t),\ A'\, v'\rangle \ + \ \langle g(t),\ v'\rangle \qquad \text{for almost all } t,$$

if and only if A is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$, and in this case $u(t)$ is given by $\quad u(t) \ = \ S(t)\,x \ + \displaystyle\int_0^t S(t-s)\ g(s)\ ds.$ □

4 Evolution of compact subsets of $\mathbb{R}^N$

4.1 Evolutions in $\mathcal{K}(\mathbb{R}^N)$ with respect to the Pompeiu–Hausdorff excess $e^{\supset}$

$\mathcal{K}(\mathbb{R}^N)$ consists of all nonempty compact subsets of $\mathbb{R}^N$. The so–called *Pompeiu–Hausdorff excess* is a first example of an ostensible metric on $\mathcal{K}(\mathbb{R}^N)$ that is very similar to the Pompeiu–Hausdorff distance $d\!\!l$, but not symmetric :

$$\begin{aligned} e^{\subset}(K_1,K_2) \ &:= \ \sup\nolimits_{x\in K_1} \ \operatorname{dist}(x,K_2) \\ e^{\supset}(K_1,K_2) \ &:= \ \sup\nolimits_{y\in K_2} \ \operatorname{dist}(y,K_1). \end{aligned}$$

for $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$. Obviously, the link to the Pompeiu–Hausdorff distance is

$$d\!\!l(K_1,K_2) \ = \ \max\{\, e^{\subset}(K_1,K_2),\ \ e^{\supset}(K_1,K_2)\}$$

(see [2, Aubin 99], § 3.2 and [36, Rockafellar, Wets 98], § 4.C, for example).

Moreover, set $\mathbb{B}_r(K) := \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, K) \le r\}$ for any $K \in \mathcal{K}(\mathbb{R}^N),\ r \ge 0$ and as abbreviations, $\mathbb{B}_r := \mathbb{B}_r(0)$, $\mathbb{B} := \mathbb{B}_1(0) \subset \mathbb{R}^N$, $\|K\|_\infty := \sup_{z \in K} |z|$.
Now reachable sets of differential inclusions provide an example of forward transitions on $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}(\mathbb{R}^N),\ e^{\supset})$. The well–known Theorem of Filippov (as stated in [5, Aubin 1991], Theorem 5.3.1 or [40, Vinter 2000], Theorem 2.4.3) forms the analytical basis.

Definition 4.1 *The* reachable set *of a set–valued map* $\widetilde{F} : [0,T] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *and a nonempty initial set* $M \subset \mathbb{R}^N$ *at time* $t \in [0,T]$ *contains the points* $x(t)$ *of all solutions* $x(\cdot)$ *starting in* M, *i.e.*

$$\vartheta_{\widetilde{F}}(t, M) := \Big\{ x(t) \in \mathbb{R}^N \ \Big| \ \begin{array}{l} x(\cdot) \in AC([0,t], \mathbb{R}^N), \quad x(0) \in M, \\ \dot{x}(\cdot) \in \widetilde{F}(\cdot, x(\cdot)) \ \ \text{almost everywhere in } [0,t] \end{array} \Big\}.$$

Proposition 4.2 *Let* $F,\ G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *be Lipschitz continuous maps with nonempty compact convex values.*
Then for every compact sets $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$ *and time* $t > 0$, *the reachable sets fulfill*

$$e^{\supset}\Big(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2)\Big) \ \le\ e^{\supset}(K_1, K_2) \cdot e^{\lambda_F \cdot t} + \sup_{R(t)\,\mathbb{B}} e^{\supset}\Big(F(\cdot), G(\cdot)\Big) \cdot \tfrac{e^{\lambda_F \cdot t} - 1}{\lambda_F}$$

$$R(t) := \|K_2\|_\infty + \sup_{K_2} \|G(\cdot)\|_\infty \cdot \tfrac{e^{\mathrm{Lip}\, G \cdot t} - 1}{\mathrm{Lip}\, G}, \quad \lambda_F := \mathrm{Lip}\ F.$$

Supposing $\lambda \ge \max\{\mathrm{Lip}\ F,\ \mathrm{Lip}\ G\}$ *and* $\sup_{\mathbb{R}^N} d\!\!l(F(\cdot), G(\cdot)) < \infty$ *in addition, the Pompeiu–Hausdorff distance between the reachable sets satisfies*

$$d\!\!l\Big(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2)\Big) \ \le\ d\!\!l(K_1, K_2) \cdot e^{\lambda t} + \sup_{\mathbb{R}^N} d\!\!l\Big(F(\cdot), G(\cdot)\Big) \cdot \tfrac{e^{\lambda t} - 1}{\lambda}.$$

Proof. For every point $x_2 \in \vartheta_G(t, K_2)$, there is a trajectory $x_2(\cdot) \in AC([0,t], \mathbb{R}^N)$ of $\dot{x}_2(\cdot) \in G(x_2(\cdot))$ (almost everywhere) with $x_2(0) \in K_2$, $x_2(t) = x_2$.
Now let $z_1 \in K_1$ satisfy the condition $|z_1 - x_2(0)| \le e^{\supset}(K_1, K_2)$. Then Filippov's Theorem provides a solution $x_1(\cdot) \in AC([0,t], \mathbb{R}^N)$ of $\dot{x}_1(\cdot) \in F(x_1(\cdot))$ a.e. with the properties $x_1(0) = z_1$ and

$$\begin{aligned} &\mathrm{dist}(x_2,\ \vartheta_F(t, K_1)) \ \le\ |x_1(t) - x_2(t)| \\ &\le\ e^{\supset}(K_1, K_2) \cdot e^{\lambda_F \cdot t} + \int_0^t e^{\lambda_F \cdot (t-s)}\ \mathrm{dist}\Big(\dot{x}_2(s),\ F(x_2(s))\Big)\ ds \\ &\le\ e^{\supset}(K_1, K_2) \cdot e^{\lambda_F \cdot t} + \int_0^t e^{\lambda_F \cdot (t-s)}\ e^{\supset}\Big(F(x_2(s)),\ G(x_2(s))\Big)\ ds. \end{aligned}$$

Furthermore,
$$\begin{aligned} |x_2(t) - x_2(0)| \ &\le\ \int_0^t \|G(x_2(s))\|_\infty\ ds \\ &\le\ \int_0^t \Big(\sup_{K_2} \|G(\cdot)\|_\infty + \mathrm{Lip}\, G \cdot |x_2(s) - x_2(0)|\Big)\ ds \end{aligned}$$

and Gronwall's Lemma (in its well–known integral form) ensures $\sup_{[0,t]} |x_2(\cdot)| \le R(t)$.
The consequence for the Pompeiu–Hausdorff distance is obvious (and has already been proved, for example, by Aubin in [2]). □

Definition 4.3 *For any parameter* $\lambda > 0$, *the set of* λ*–Lipschitz continuous maps* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *with nonempty compact convex values and* $\sup_{x \in \mathbb{R}^N} \|F(x)\|_\infty < \infty$ *is denoted by* $\mathrm{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$.

Corollary 4.4 *For every* $\lambda \geq 0$, *the reachable sets of* $\mathrm{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$ *induce forward transitions (of order* 0*) on* $(\mathcal{K}(\mathbb{R}^N), \mathcal{K}(\mathbb{R}^N), e^{\supset})$.

Proof. Definition 4.1 of reachable sets implies for all $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $M \subset \mathbb{R}^N$, $s, t \geq 0$ $\vartheta_F(t+s, M) = \vartheta_F(t, \vartheta(s, M))$. Prop. 4.2 guarantees for each $F, G \in \mathrm{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$

$$\sup_{K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)} \limsup_{h \downarrow 0} \frac{e^{\supset}\big(\vartheta_F(h, K_1),\ \vartheta_F(h, K_2)\big) - e^{\supset}\big(K_1, K_2\big)}{h \cdot e^{\supset}\big(K_1, K_2\big)} \leq \lim_{h \downarrow 0} \frac{e^{\lambda h} - 1}{h} = \lambda =: \alpha^{\mapsto}(\vartheta_F),$$

$$\begin{aligned} Q^{\mapsto}(\vartheta_F, \vartheta_G) &\overset{\text{Def.}}{=} \sup_{K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)} \limsup_{h \downarrow 0} \left(\frac{e^{\supset}\big(\vartheta_F(h, K_1),\ \vartheta_G(h, K_2)\big) - e^{\supset}\big(K_1, K_2\big) \cdot e^{\alpha^{\mapsto}(\vartheta_G) \cdot h}}{h} \right)^+ \\ &= \sup_{K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)} \limsup_{h \downarrow 0} \left(\frac{e^{\supset}\big(\vartheta_F(h, K_1),\ \vartheta_G(h, K_2)\big) - e^{\supset}\big(K_1, K_2\big) \cdot e^{\lambda h}}{h} \right)^+ \\ &\leq \sup_{\mathbb{R}^N} e^{\supset}(F(\cdot),\ G(\cdot)) \leq \sup_{\mathbb{R}^N} \|F(\cdot)\|_\infty + \sup_{\mathbb{R}^N} \|G(\cdot)\|_\infty, \end{aligned}$$

and $\displaystyle \sup_{K \in \mathcal{K}(\mathbb{R}^N)} e^{\supset}\Big(\vartheta_F(s, K),\ \vartheta_F(t, K)\Big) \leq \sup_{\mathbb{R}^N} \|F(\cdot)\|_\infty \cdot (t - s)$ for all $s \leq t$.

The triangle inequality bridges the last gap for $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}(\mathbb{R}^N),\ e^{\supset})$:

$$\limsup_{h \downarrow 0} e^{\supset}\Big(\vartheta_F(t-h, K_1),\ K_2\Big) = e^{\supset}\Big(\vartheta_F(t, K_1),\ K_2\Big)$$

for every $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$, $t \in]0, 1]$. □

Remark 4.5 The estimate of $Q^{\mapsto}(\vartheta_F, \vartheta_G)$ provides the motivation for assuming the Lipschitz constant λ uniformly : In Definition 2.4 of $Q^{\mapsto}(\vartheta_F, \vartheta_G)$, we take the parameter $\alpha^{\mapsto}(\vartheta_G)$ (related with the *second* transition) into consideration. It serves the particular purpose that the triangle inequality of $Q^{\mapsto}$ is a simple consequence (see [31]).
On the other hand, the estimate of $e^{\supset}(\vartheta_F(t, K_1),\ \vartheta_G(t, K_2))$ in Proposition 4.2 uses the Lipschitz constant of F (instead of G). Thus, we restrict ourselves to the uniform upper bound λ.

The well–known property of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ that closed bounded balls are compact has the immediate consequence :

Lemma 4.6 $(\mathcal{K}(\mathbb{R}^N), e^{\supset})$ *is two–sided sequentially compact (in the sense of Def. 2.9).* □

So the results of § 2 imply for this example directly

Corollary 4.7 *Consider the reachable sets of* $\mathrm{LIP}_\lambda(\mathbb{R}^N,\mathbb{R}^N)$ *as forward transitions (of order* 0*) on* $(\mathcal{K}(\mathbb{R}^N),\mathcal{K}(\mathbb{R}^N),\ e^{\supset})$.
Let $f : \mathcal{K}(\mathbb{R}^N)\times[0,T] \longrightarrow \mathrm{LIP}_\lambda(\mathbb{R}^N,\mathbb{R}^N)$ *satisfy* $\sup_{K,\,t,\,x} \|f(K,t)(x)\|_\infty < \infty$ *and*

$$\sup_{\mathbb{R}^N}\ e^{\supset}\big(f(K_1,t_1)(\cdot),\ f(K_2,t_2)(\cdot)\big) \ \le\ \omega\big(e^{\supset}(K_1,K_2)\ +\ t_2-t_1\big)$$

for all $K_1,K_2\in\mathcal{K}(\mathbb{R}^N)$ *and* $0\le t_1\le t_2\le T$ *with the modulus* $\omega(\cdot)$ *of continuity.*

Then for every initial set $K_0\in\mathcal{K}(\mathbb{R}^N)$, *there exists a right–hand forward solution* $K:[0,T[\ \longrightarrow(\mathcal{K}(\mathbb{R}^N),\ e^{\supset})$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot)\ \ni\ f(K(\cdot),\cdot)$ *in* $[0,T[$ *with* $K(0)=K_0$.

Suppose in addition that there exist $L\ge 0$ *and a modulus* $\omega(\cdot)$ *of continuity with*

$$\sup_{\mathbb{R}^N}\ e^{\supset}\big(f(K_1,t_1)(\cdot),\ f(K_2,t_2)(\cdot)\big)\ \le\ L\cdot e^{\supset}(K_1,K_2)\ +\ \omega(t_2-t_1)$$

for all $K_1,K_2\in\mathcal{K}(\mathbb{R}^N)$ *and* $0\le t_1\le t_2\le T$. *Let* $K(\cdot):[0,T[\ \longrightarrow(\mathcal{K}(\mathbb{R}^N),\ e^{\supset})$ *be an Euler solution (i.e. constructed by Euler method according to the proof of Prop. 2.12 presented in [31]). Then every other solution* $M(\cdot)$ *with* $M(0)=K(0)$ *satisfies*

$$\limsup_{\delta\downarrow 0}\ e^{\supset}(K(t),\ M(t+\delta))\ =\ 0.$$

Proof. The existence results from Proposition 2.12. The comparison with an Euler solution is a consequence of Proposition 2.16 and $\mathcal{T}_\Theta(\cdot,\cdot)\equiv 1$. Indeed setting $p := d\!l$, $q := e^{\supset}$, the triangle inequality implies for all $K_1,K_2\in\mathcal{K}(\mathbb{R}^N)$

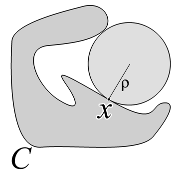
$$\Delta(K_1,K_2)\ \overset{\text{Def.}}{=}\ \inf_{Z\in\mathcal{K}(\mathbb{R}^N)}\Big(p(K_1,Z)\ +\ q(Z,K_2)\Big)\ =\ e^{\supset}(K_1,K_2)$$

because on the one hand, $\Delta(K_1,K_2)\le e^{\supset}(K_1,K_2)$ is obvious and on the other hand, $e^{\supset}(K_1,K_2)\ \le\ e^{\supset}(K_1,Z)\ +\ e^{\supset}(Z,K_2)\ \le\ d\!l(K_1,Z)\ +\ e^{\supset}(Z,K_2)$ for all Z. □

4.2 Evolutions in $\mathcal{K}(\mathbb{R}^N)$ with respect to $q_{\mathcal{K},N}$

The Pompeiu–Hausdorff excess $e^{\supset}(K_1,K_2)$ does not distinguish between boundary points and interior points of the compact sets K_1,K_2. In this subsection, an ostensible metric $q_{\mathcal{K},N}$ on $\mathcal{K}(\mathbb{R}^N)$ is defined that takes the boundaries into consideration explicitly. Strictly speaking, we even use the first–order approximation of the boundary represented by the limiting normal cones of a set. Following the well–known definitions like in [40, Vinter 2000], for example, these cones are specified :

Definition 4.8 *Let $C \subset \mathbb{R}^N$ be a nonempty closed set.*
A vector $\eta \in \mathbb{R}^N$, $\eta \neq 0$, is said to be a proximal normal vector *to C at $x \in C$ if there exists $\rho > 0$ with $\mathbb{B}_\rho(x + \rho \frac{\eta}{|\eta|}) \cap C = \{x\}$.*
The supremum of all ρ with this property is called proximal radius of C at x in direction η. The cone of all these proximal normal vectors is called the proximal normal cone *to C at x and is abbreviated as $N_C^P(x)$.*



The so–called limiting normal cone $N_C(x)$ *to C at x consists of all vectors $\eta \in \mathbb{R}^N$ that can be approximated by sequences $(\eta_n)_{n \in \mathbb{N}}$, $(x_n)_{n \in \mathbb{N}}$ satisfying*

$$\begin{array}{llll} x_n & \longrightarrow x, & x_n & \in C, \\ \eta_n & \longrightarrow \eta, & \eta_n & \in N_C^P(x_n), \end{array}$$

i.e. $\quad N_C(x) \stackrel{\text{Def.}}{=} \operatorname{Limsup}_{\substack{y \longrightarrow x \\ y \in C}} N_C^P(y)$.
As a further abbreviation, we set $\;{}^\flat N_C(x) := N_C(x) \cap \mathbb{B} = \{v \in N_C(x) : |v| \leq 1\}$.

Convention. In the following we restrict ourselves to normal directions at boundary points, i.e. strictly speaking, Graph N_C and Graph ${}^\flat N_C$ are the abbreviations of Graph $N_C|_{\partial C}$, Graph ${}^\flat N_C|_{\partial C}$, respectively.

Definition 4.9 *Set* $q_{\mathcal{K},N} : \mathcal{K}(\mathbb{R}^N) \times \mathcal{K}(\mathbb{R}^N) \longrightarrow [0, \infty[$,

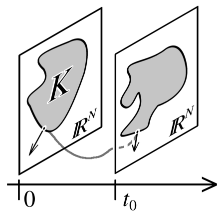
$$q_{\mathcal{K},N}(K_1, K_2) := d\!\!l(K_1, K_2) + e\!\!\!\!\;{}^{\supset}(\text{Graph } {}^\flat N_{K_1}, \ \text{Graph } {}^\flat N_{K_2}).$$

Obviously, the function $q_{\mathcal{K},N}$ is a quasi–metric on the set $\mathcal{K}(\mathbb{R}^N)$ of all nonempty compact subsets of $\mathbb{R}^N$, i.e. it is positive definite and satisfies the triangle inequality. The properties of $q_{\mathcal{K},N}$ with respect to convergence depend on the relation between the normal cones of compact sets K_n $(n \in \mathbb{N})$ and their limit $K = \operatorname{Lim}_{n \to \infty} K_n$ (if it exists). In general, they do not coincide of course, but each limiting normal vector of K can be approximated by limiting normal vectors of a subsequence $(K_{n_j})_{j \in \mathbb{N}}$. Stating this inclusion in the next proposition, we regard it as well–known (see e.g. [5, Aubin 91], Theorem 8.4.6 or [21, Cornet, Czarnecki 99], Lemma 4.1). As it might be strict, the tuple $(\mathcal{K}(\mathbb{R}^N), q_{\mathcal{K},N})$ is *not* two–sided compact in the sense of Definition 2.9.

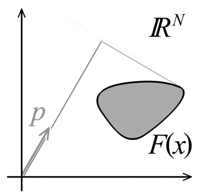
Proposition 4.10
Let $(M_k)_{k \in \mathbb{N}}$ be a sequence of closed subsets of $\mathbb{R}^N$ and set $M := \operatorname{Limsup}_{k \to \infty} M_k$.
Then, 1. Graph $N_M^P \subset \operatorname{Limsup}_{k \to \infty}$ Graph $N_{M_k}^P$,
2. Graph $N_M \subset \operatorname{Limsup}_{k \to \infty}$ Graph N_{M_k}. □

Corollary 4.11 *Let $(M_k)_{k \in \mathbb{N}}$ be a sequence of closed subsets of $\mathbb{R}^N$ whose limit $M := \operatorname{Lim}_{k \to \infty} M_k$ exists.*
Then $\quad$ Graph $N_M \subset \operatorname{Liminf}_{k \to \infty}$ Graph N_{M_k}.
In particular, $\quad \partial M \subset \operatorname{Liminf}_{k \to \infty} \partial M_k$.

Proof is an indirect consequence of Proposition 4.10 due to $M = \operatorname{Lim}_{k \to \infty} M_k$. □



Now we focus on the evolution of limiting normal cones at the topological boundary and use the *Hamilton condition* as a key tool. It implies that roughly speaking, every boundary point x_0 of $\vartheta_F(t_0, K)$ and normal vector $\nu \in N_{\vartheta_F(t_0,K)}(x_0)$ have a trajectory and an adjoint arc linking x_0 to some $z \in \partial K$ and ν to $N_K(z)$, respectively.



Furthermore the trajectory and its adjoint arc fulfill a system of partial differential equations with the so–called *Hamiltonian function* of $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$,

$$\mathcal{H}_F :\ \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad (x,p) \longmapsto \sup_{y \in F(x)} \ p \cdot y$$

Although the Hamilton condition is known in much more general forms (consider, for example, [40, Vinter 2000], Theorem 7.7.1 applied to proximal balls), we use only the following "smooth" version — due to later regularity conditions on F. In short, the graph of normal cones at time t, i.e. Graph $N_{\vartheta_F(t,K)}(\cdot)|_{\partial\,\vartheta_F(t,K)}$, can be traced back to the beginning by means of the Hamiltonian system with $\mathcal{H}_F$.

Proposition 4.12 *Suppose for the set–valued map* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$

1. $F(\cdot)$ *has nonempty convex compact values,*
2. $\mathcal{H}_F(\cdot,\cdot)$ *is continuously differentiable on* $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$,
3. *the derivative of* $\mathcal{H}_F$ *has linear growth on* $\mathbb{R}^N \times (\mathbb{R}^N \setminus \mathbb{B}_1)$, *i.e.* $\|D\mathcal{H}_F(x,p)\| \ \le\ const \cdot (1 + |x| + |p|)$ *for all* $x, p \in \mathbb{R}^N,\ |p| > 1$.

Let $K \in \mathcal{K}(\mathbb{R}^N)$ *be any initial set and* $t_0 > 0$.

For every boundary point $x_0 \in \partial\,\vartheta_F(t_0, K)$ *and normal* $\nu \in N_{\vartheta_F(t_0,K)}(x_0) \setminus \{0\}$, *there exist a trajectory* $x(\cdot) \in C^1([0,t_0],\mathbb{R}^N)$ *and its adjoint* $p(\cdot) \in C^1([0,t_0],\mathbb{R}^N)$ *with*

$$\begin{cases} \dot{x}(t) \ = \ \ \frac{\partial}{\partial p}\ \mathcal{H}_F(x(t),\, p(t)) \ \in\ F(x(t)), & x(t_0) \ = \ x_0, \quad x(0) \ \in\ \partial K, \\ \dot{p}(t) \ = \ -\frac{\partial}{\partial x}\ \mathcal{H}_F(x(t),\, p(t)), & p(t_0) \ = \ \nu, \quad\ \ p(0) \ \in\ N_K(x(0)). \end{cases}$$

These assumptions give a first hint about adequate conditions on $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ for inducing forward transitions with respect to $q_{\mathcal{K},N}$. Supposing $D\mathcal{H}_F$ to be Lipschitz continuous (in addition) provides some technical advantages such as global existence of unique solutions of the Hamiltonian system and Remark 4.18 (1.).

Definition 4.13 *For* $\lambda > 0$, $\mathrm{LIP}^{(\mathcal{H})}_\lambda(\mathbb{R}^N,\mathbb{R}^N)$ *contains all* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *with*

1. $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *has compact convex values,*
2. $\mathcal{H}_F(\cdot,\cdot) \ \in\ C^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,
3. $\|\mathcal{H}_F\|_{C^{1,1}(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} \ \stackrel{\text{Def.}}{=}\ \|\mathcal{H}_F\|_{C^{1}(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} \ +\ \mathrm{Lip}\ D\mathcal{H}_F|_{\mathbb{R}^N \times\, \partial\mathbb{B}_1} \ <\ \lambda\,.$

Lemma 4.14 *For every* $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$ *and* $K \in \mathcal{K}(\mathbb{R}^N)$, $0 \le s \le t \le T$,

$$q_{\mathcal{K},N}\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big) \ \le\ \lambda\,(e^{\lambda T} + 2)\ \cdot\ (t-s).$$

Proof. Obviously, the Pompeiu–Hausdorff distance satisfies for every $s, t \ge 0$

$$d\!\!l\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big) \ \le\ \sup_{\mathbb{R}^N} \|F(\cdot)\|_\infty\ \cdot\ (t-s) \ \le\ \lambda\,(t-s).$$

Furthermore Proposition 4.12 guarantees that for every $0 \le s < t$, $x \in \partial\,\vartheta_F(t,K)$ and $p \in {}^\flat N_{\vartheta_F(t,K)}(x)$, there exist a trajectory $x(\cdot) \in C^1([s,t], \mathbb{R}^N)$ and its adjoint arc $p(\cdot) \in C^1([s,t], \mathbb{R}^N)$ satisfying

$$\begin{cases} \dot{x}(\tau) \ =\ \frac{\partial}{\partial p}\ \mathcal{H}_F(x(\tau),\, p(\tau)) \ \in\ F(x(\tau)), & x(t) \ =\ x, \quad x(s) \ \in\ \partial\vartheta_F(s,K), \\ \dot{p}(\tau) \ =\ -\frac{\partial}{\partial x}\ \mathcal{H}_F(x(\tau),\, p(\tau)), & p(t) \ =\ p, \quad p(s) \ \in\ N_{\vartheta_F(s,K)}(x(s)). \end{cases}$$

Obviously, $\mathcal{H}_F$ is (positively) homogeneous with respect to its second argument and thus, its definition implies $|\dot{p}(\tau)| \le \lambda\ |p(\tau)|$ for all τ. Moreover $|p| \le 1$ implies that the projection of p on any cone is also contained in $\mathbb{B}_1$. So finally we obtain

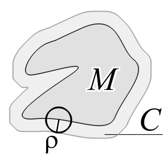
$$\begin{aligned} \mathrm{dist}\Big((x,p),\ \text{Graph}\ {}^\flat N_{\vartheta_F(s,K)}\Big) \ &\le\ |x - x(s)| \ +\ |p - p(s)| \\ &\le\ \sup_{s \le \tau \le t}\ \Big(|\tfrac{\partial}{\partial x}\,\mathcal{H}_F| \ +\ |\tfrac{\partial}{\partial p}\,\mathcal{H}_F|\Big)\Big|_{(x(\tau),p(\tau))} \cdot (t-s) \\ &\le\ \Big(\lambda\ e^{\lambda\, t} \ +\ \lambda\Big) \cdot (t-s). \quad \square \end{aligned}$$

So the next question is whether the features of $\mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$ are already sufficient for forward transitions with respect to $q_{\mathcal{K},N}$. An essential demand is that smooth compact subsets of $\mathbb{R}^N$ stay smooth for short times.

Definition 4.15 $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ *abbreviates the set of all nonempty compact N–dimensional $C^{1,1}$ submanifolds of $\mathbb{R}^N$ with boundary.*

A closed subset $C \subset \mathbb{R}^N$ is said to have positive erosion of radius $\rho > 0$ *if there exists a closed set $M \subset \mathbb{R}^N$ with*

$$C \ =\ \{\, x \in \mathbb{R}^N \mid dist(x, M) \le \rho \,\}$$



or equivalently, if it holds the interior sphere condition *of radius ρ, i.e. each $x \in \partial C$ has a ball $B \subset \mathbb{R}^N$ of radius ρ with $x \in B \subset C$. $\mathcal{K}_\circ^\rho(\mathbb{R}^N)$ consists of all sets with positive erosion of radius $\rho > 0$ and, set* $\mathcal{K}_\circ(\mathbb{R}^N) \ :=\ \bigcup_{\rho > 0} \mathcal{K}_\circ^\rho(\mathbb{R}^N)\,.$

Remark 4.16 The morphological term "erosion" is motivated by the fact that a set $C = \overline{C^\circ} \subset \mathbb{R}^N$ has positive erosion of radius $\rho > 0$ if and only if the closure $\overline{\mathbb{R}^N \setminus C}$ of its complement has *positive reach* in the sense of Federer ([26]).
A (closed) set $C \subset \mathbb{R}^N$ of positive reach with radius $\rho > 0$ is characterized by an exterior sphere condition of radius ρ, i.e. each $x \in \partial C$ has a closed ball $B \subset \mathbb{R}^N$ of radius ρ with $x \in B \cap C$, $\overset{\circ}{B} \cap C = \emptyset$.

The relationship between positive reach and positive erosion implies a collection of interesting regularity properties presented (for closed subsets of a Hilbert space) in [20, Clarke, Stern, Wolenski 95], [19, Clarke, Ledyaev, Stern 97], [35, Poliquin, Rockafellar, Thibault 2000].

Proposition 4.17 *Let $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be a map of $\mathrm{LIP}^{(\mathcal{H})}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$.*

For every compact N–dimensional $C^{1,1}$ submanifold K of $\mathbb{R}^N$ with boundary, there exist a time $\tau > 0$ and a radius $\rho > 0$ such that for all $t \in [0, \tau[$,

1. *$\vartheta_F(t,K) \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ with radius of curvature $\geq \rho$,*
   (i.e. $\vartheta_F(t,K)$ has both positive reach and positive erosion of radius $\geq \rho$).
2. *$K = \mathbb{R}^N \Big\backslash \vartheta_{-F}(t,\ \mathbb{R}^N \setminus \vartheta_F(t,K))$.*

Remark 4.18 1. A complete proof is presented in the appendix (Propositions A.2, A.4). For statement (1.), we use the evolution of Graph $(N_K(\cdot) \cap \partial\mathbb{B}) \subset \mathbb{R}^N \times \mathbb{R}^N$ along the Hamiltonian system with $\mathcal{H}_F$. Indeed, Lemma A.3 specifies sufficient conditions on the system so that graphs of Lipschitz continuous functions preserve this property for short times. Applying this lemma to unit normals to reachable sets of $K \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ requires the Hamiltonian $\mathcal{H}_F$ to be in $C^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$ instead of C^1.
In fact, this Lemma A.3 is an analytical reason for choosing $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ as "test subset" of $\mathcal{K}(\mathbb{R}^N)$ — instead of compact sets with C^1 boundary, for example.

2. Under different assumptions about the control system, the regularity of reachable sets has been investigated independently in [13, Cannarsa, Frankowska 2004]. Some details are discussed in Remark A.13.

3. Together with Proposition 4.12, statement (2.) provides a connection between the boundaries ∂K and $\partial\,\vartheta_F(t,K)$ — now in both forward and backward time direction.

Lemma 4.19 *Assume for $F, G \in \mathrm{LIP}^{(\mathcal{H})}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$, $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$ and $T > 0$ that all the sets $\vartheta_F(t, K_1) \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ $(0 \leq t \leq T)$ have uniform positive reach.*
Then, for every $t \in [0, T[$,

$$\begin{aligned} & q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\ \vartheta_G(t,K_2)\Big) \ \leq \\ & \leq\ e^{(\Lambda_F + \lambda)\ t} \cdot \Big(q_{\mathcal{K},N}(K_1,\ K_2)\ +\ 4\,N\,t\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}\Big) \end{aligned}$$

with $\quad \Lambda_F \ := \ 9\, e^{2\lambda T}\ \|\mathcal{H}_F\|_{C^{1,1}(\mathbb{R}^N \times \partial\mathbb{B}_1)} \ \leq \ 9\, e^{2\lambda T}\, \lambda \ < \ \infty.$

Proof. Proposition 4.2 provides the estimate of the Pompeiu–Hausdorff distance

$$\begin{aligned} d\!\!l\Big(\vartheta_F(t,K_1),\ \vartheta_G(t,K_2)\Big) \ &\leq\ d\!\!l(K_1,K_2) \cdot e^{\lambda t} \ +\ \sup_{\mathbb{R}^N} d\!\!l\Big(F(\cdot), G(\cdot)\Big) \cdot \tfrac{e^{\lambda t}-1}{\lambda} \\ &\leq\ d\!\!l(K_1,K_2) \cdot e^{\lambda t} \ +\ \sup_{\mathbb{R}^N \times \partial\mathbb{B}_1} |\mathcal{H}_F - \mathcal{H}_G| \cdot \tfrac{e^{\lambda t}-1}{\lambda}\,. \end{aligned}$$

So now we need an upper bound of $e^{\supset}\Big(\text{Graph}\ {}^\flat N_{\vartheta_F(t,K_1)},\ \text{Graph}\ {}^\flat N_{\vartheta_G(t,K_2)}\Big)$.
Choose $x \in \partial\,\vartheta_G(t,K_2)$, $p \in N_{\vartheta_G(t,K_2)}(x) \cap \partial\mathbb{B}_1$ and $\delta > 0$ arbitrarily. According to Proposition 4.12, there exist a trajectory $x(\cdot) \in C^1([0,t],\mathbb{R}^N)$ of G and its adjoint arc $p(\cdot) \in C^1([0,t],\mathbb{R}^N)$ with

$$\begin{array}{rclrcl} \dot{x}(\cdot) &=& \frac{\partial}{\partial p}\,\mathcal{H}_G(x(\cdot),\,p(\cdot)) \in G(x(\cdot)), & \dot{p}(\cdot) &=& -\,\frac{\partial}{\partial x}\,\mathcal{H}_G(x(\cdot),\,p(\cdot)) \in \lambda\,|p(\cdot)|\cdot\mathbb{B} \\ x(0) &\in& \partial K_2, & p(0) &\in& N_{K_2}(x(0)), \\ x(t) &=& x, & p(t) &=& p, \end{array}$$

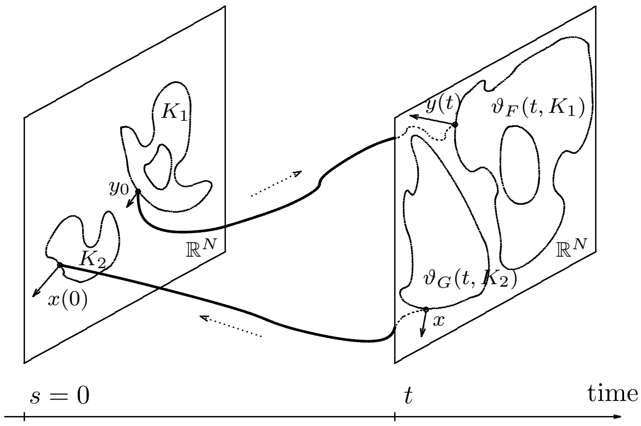
Gronwall's Lemma guarantees

$$0 < e^{-\lambda\,t} \;\leq\; |p(\cdot)| \;\leq\; e^{\lambda\,t}$$

and so, $p(0)\,e^{-\lambda\,t} \in {}^\flat N_{K_2}(x(0)) \setminus \{0\}$.
Now let $(y_0, \widehat{q}_0)$ denote an element of Graph ${}^\flat N_{K_1}$ with $\widehat{q}_0 \neq 0$ and

$$\Big|(y_0,\widehat{q}_0) \;-\; \Big(x(0),\; p(0)\,e^{-\lambda\,t}\Big)\Big|$$
$$\leq\; e^{\supset}\Big(\text{Graph}\ {}^\flat N_{K_1},\ \text{Graph}\ {}^\flat N_{K_2}\Big) + \delta.$$

Assuming that all $\vartheta_F(s,K_1) \in \mathcal{K}(\mathbb{R}^N)$ $(s \in [0,t])$ have uniform positive reach implies the reversibility in time due to Proposition A.4 :
$\mathbb{R}^N \setminus K_1 \;=\; \vartheta_{-F}(t,\; \mathbb{R}^N \setminus \vartheta_F(t,K_1))$.



So in particular, y_0 is a boundary point of $\mathbb{R}^N \setminus \overset{\circ}{K}_1 = \vartheta_{-F}(t,\, \overline{\mathbb{R}^N \setminus \vartheta_F(t,K_1)})$ and $-\,\widehat{q}_0$ belongs to its limiting normal cone at y_0. As a consequence of Prop. 4.12 again and due to $\mathcal{H}_{-F}(z,v) = \mathcal{H}_F(z,-v)$ for all z,v, we obtain a trajectory $y(\cdot) \in C^1([0,t],\mathbb{R}^N)$ of F and its adjoint arc $q(\cdot)$ satisfying

$$\begin{array}{rclrcl} \dot{y}(\cdot) &=& \frac{\partial}{\partial p}\,\mathcal{H}_F(y(\cdot),\,q(\cdot)), & \dot{q}(\cdot) &=& -\,\frac{\partial}{\partial y}\,\mathcal{H}_F(y(\cdot),\,q(\cdot)) \\ y(0) &=& y_0, & q(0) &=& \widehat{q}_0\;\; e^{\lambda\,t} \;\neq\; 0, \\ y(t) &\in& \partial\,\vartheta_F(t,K_1), & q(t) &\in& N_{\vartheta_F(t,K_1)}(y(t)). \end{array}$$

According to Lemma 4.20, the derivative of $\mathcal{H}_F$ is Λ_F–Lipschitz continuous on $\mathbb{R}^N \times (\mathbb{B}_{e^{\lambda T}} \setminus \overset{\circ}{\mathbb{B}}_{e^{-\lambda T}})$. Thus, the Theorem of Cauchy–Lipschitz leads to

$$\begin{array}{l} \text{dist}\Big((x,p),\ \text{Graph}\ {}^\flat N_{\vartheta_F(t,K_1)}\Big) \qquad\qquad \leq\; \Big|(x,\,p) \;-\; (y(t),\,q(t))\Big| \\ \leq\; e^{\Lambda_F\,\cdot\,t}\cdot\Big|(x(0),\,p(0)) \;-\; (y_0,\,\widehat{q}_0\,e^{\lambda\,t})\Big| \;+\; \frac{e^{\Lambda_F\,\cdot\,t}-1}{\Lambda_F}\cdot \sup\limits_{0\leq s\leq t}\;|D\,\mathcal{H}_F - D\,\mathcal{H}_G|\Big|_{(x(s),\,p(s))}. \end{array}$$

$\mathcal{H}_F$ and $\mathcal{H}_G$ are positively homogenous with respect to the second argument and thus,

$$\begin{array}{rcl} \Big|\frac{\partial}{\partial x_j}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\,p(s))}\Big| &\leq& e^{\lambda\,t}\;\;\|D\mathcal{H}_F - D\mathcal{H}_G\|_{C^0(\mathbb{R}^N\times\partial\mathbb{B}_1)}, \\ \Big|\frac{\partial}{\partial p_j}\,(\mathcal{H}_F - \mathcal{H}_G)|_{(x(s),\,p(s))}\Big| &\leq& 2\cdot\;\;\|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}. \end{array}$$

So we obtain

$$\begin{aligned}
&\operatorname{dist}\Big((x,p),\ \text{Graph }\ {}^\flat N_{\vartheta_F(t,K_1)}\Big)\\
&\le\ e^{(\Lambda_F+\lambda)\ t}\ \ \Big|(x(0),\ p(0)\,e^{-\lambda\,t})\ -\ (y_0,\widehat{q}_0)\Big|\ +\ e^{\Lambda_F\,t}\ t\ \cdot 4\,N\,e^{\lambda t}\ \ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}
\end{aligned}$$

and, since $\delta>0$ is arbitrarily small and $|p|=1$,

$$\begin{aligned}
&\mathbb{e}^{\supset}\Big(\text{Graph }\ {}^\flat N_{\vartheta_F(t,K_1)},\ \ \text{Graph }\ {}^\flat N_{\vartheta_G(t,K_2)}\Big)\\
&\le\ e^{(\Lambda_F+\lambda)\ t}\ \cdot\ \Big\{\mathbb{e}^{\supset}\Big(\text{Graph }{}^\flat N_{K_1},\ \ \text{Graph }{}^\flat N_{K_2}\Big)\ +\ 4\,N\,t\ \cdot\ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}\Big\}.
\end{aligned}$$

□

Lemma 4.20 *For every* $F\in\mathrm{LIP}^{(\mathcal{H})}_\lambda(\mathbb{R}^N,\mathbb{R}^N)$ *and radius* $R>1$, *the product* $9\,R^2\,\lambda$ *is a Lipschitz constant of the derivative* $D\mathcal{H}_F$ *restricted to* $\mathbb{R}^N\times(\mathbb{B}_R\setminus\overset{\circ}{\mathbb{B}}_{\frac{1}{R}})$.

Proof results from the fact that $\mathcal{H}_F(x,p)$ is positively homogenous with respect to p. (For further details see [32, Lorenz 2004], Lemma 4.4.24.) □

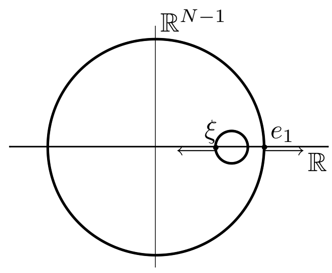
Remark 4.21 The proof of Lemma 4.19 also indicates the advantage of $q_{\mathcal{K},N}$ in comparison with the ostensible metric $q_{\mathcal{K},\partial}:\mathcal{K}(\mathbb{R}^N)\times\mathcal{K}(\mathbb{R}^N)\longrightarrow[0,\infty[$, for example,

$$q_{\mathcal{K},\partial}(K_1,K_2)\ :=\ \mathbb{d}(K_1,\ K_2)\ +\ \mathbb{e}^{\supset}(\partial K_1,\ \partial K_2)$$

that is not taking the normal cones into consideration. Indeed, leaving out the evolution of normals along adjoint arcs, the hypotheses of Lemma 4.19 ensure only the estimate

$$q_{\mathcal{K},\partial}\Big(\vartheta_F(t,K_1),\ \ \vartheta_G(t,K_2)\Big)\ \le\ \Big(q_{\mathcal{K},\partial}(K_1,K_2)\ +\ \text{const}\ \cdot\ \sup_{\mathbb{R}^N}\ \mathbb{m}(F(\cdot),\ G(\cdot))\ \cdot\ t\Big)\ \cdot\ e^{\lambda\,t}$$

with $\mathbb{m}(M_1,M_2)\ :=\ \sup\ \{\,|x-y|\ :\ x\in M_1,\ y\in M_2\,\}$ for bounded $M_1,M_2\subset\mathbb{R}^N$. Roughly speaking, we cannot know in which directions related boundary trajectories $x(\cdot),y(\cdot)$ move (and the "worst case" of opposite directions leads to the dependence on $\mathbb{m}(F(\cdot),\ G(\cdot))$).



Just consider a small ball contained in the unit ball close to the boundary : $\mathbb{B}_r((1-2\,r)\,e_1)\ \subset\ \mathbb{B}_1(0)\ \subset\ \mathbb{R}^N$ with $r\ll 1$ and $e_1:=(1,0\ \ldots\ 0)\in\mathbb{R}^N$. Set $F(\cdot):=\mathbb{B}_1$ and $\xi\ :=\ x(0)\ =\ (1-3\,r)\,e_1$.
Then e_1 is the unique projection of ξ on $\partial\mathbb{B}_1$ and the boundary trajectories $x(\cdot),\ y(\cdot)$ of F starting in ξ and e_1 respectively are also unique : $x(t)\ =\ \xi-t,\ \ y(t)\ =\ e_1+t$.
Furthermore they keep moving in opposite directions and $|x(t)-y(t)|\ =\ |\xi-e_1|\ +\ 2\,t\ =\ |\xi-e_1|\ +\ 2\,\mathbb{m}(\mathbb{B},\mathbb{B})\,t$.
The preceding estimate however implies that reachable sets cannot induce forward transitions *of order* 0 on $\mathcal{K}(\mathbb{R}^N)$ with respect to $q_{\mathcal{K},\partial}$ because $\mathbb{m}(F(x),F(x))=0$ is fulfilled only if $F(x)$ is single–valued.

Proposition 4.22 *For every $\lambda \geq 0$, the reachable sets of the set–valued maps in $\mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$ induce forward transitions (of order 0) on $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}_{C^{1,1}}(\mathbb{R}^N),\ q_{\mathcal{K},N})$ with*

$$\begin{aligned} \alpha^{\mapsto}(\vartheta_F) &\stackrel{\text{Def.}}{=} 10\ \lambda \\ \beta(\vartheta_F)\,(t) &\stackrel{\text{Def.}}{=} \lambda\ \ (e^\lambda + 2)\ \cdot\ t, \\ Q^{\mapsto}(\vartheta_F, \vartheta_G) &\leq 4\ N\ \ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}\,. \end{aligned}$$

Proof. The semigroup property of reachable sets implies again

$$\begin{aligned} q_{\mathcal{K},N}\Big(\vartheta_F(h,\ \vartheta_F(t,K)), \quad \vartheta_F(t+h,\ K)\Big) &= 0, \\ q_{\mathcal{K},N}\Big(\vartheta_F(t+h,\ K), \quad \vartheta_F(h,\ \vartheta_F(t,K))\Big) &= 0 \end{aligned}$$

for all $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$, $K \in \mathcal{K}(\mathbb{R}^N)$, $h, t \geq 0$ since $q_{\mathcal{K},N}$ is a quasi–metric. According to Proposition 4.17, every set–valued map $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$ and initial set $K_1 \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ lead to a time $\mathcal{T}_\Theta(\vartheta_F, K_1) > 0$ and a radius $\rho > 0$ such that $\vartheta_F(t, K_1) \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ has radius of curvature $\geq \rho$ for any $t \in [0,\ \mathcal{T}_\Theta(\vartheta_F, K_1)]$.
So Lemma 4.19 guarantees for all $K_1 \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$, $K_2 \in \mathcal{K}(\mathbb{R}^N)$

$$\begin{aligned} &\limsup_{h \downarrow 0}\ \left(\frac{q_{\mathcal{K},N}\big(\vartheta_F(h,K_1),\, \vartheta_F(h,K_2)\big)\ -\ q_{\mathcal{K},N}\big(K_1,K_2\big)}{h \qquad q_{\mathcal{K},N}\big(K_1,K_2\big)}\right)^+ \\ \leq\ &\limsup_{h \downarrow 0}\ \tfrac{1}{h}\ \left(e^{\left(9\,e^{2\lambda h}\ \lambda + \lambda\right)\cdot h}\ -\ 1\right) \quad = \quad 10\ \lambda \quad \stackrel{\text{Def.}}{=} \quad \alpha^{\mapsto}(\vartheta_F) \end{aligned}$$

and for every $F, G \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$

$$\begin{aligned} Q^{\mapsto}(\vartheta_F, \vartheta_G)\ \leq\ &\sup_{\substack{K_1 \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N) \\ K_2 \in \mathcal{K}(\mathbb{R}^N)}}\ \limsup_{h \downarrow 0}\ \Big(q_{\mathcal{K},N}(K_1,\ K_2)\ \tfrac{1}{h}\ \left(e^{\left(9\,e^{2\lambda h}\ \lambda+\lambda\right)\cdot h}\ -\ e^{10\,\lambda\,h}\right) \\ &\qquad +\ 4\ N \cdot \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}\ \cdot\ e^{\left(9\,e^{2\lambda h}\ \lambda+\lambda\right)\cdot h}\Big) \\ =\ &\qquad 4\ N \cdot \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}. \end{aligned}$$

Moreover Lemma 4.14 states $\quad q_{\mathcal{K},N}\Big(\vartheta_F(s,K),\ \vartheta_F(t,K)\Big)\ \leq\ \lambda\,(e^\lambda + 2)\ \cdot\ (t-s)$ for any $0 \leq s \leq t \leq 1$ and $K \in \mathcal{K}(\mathbb{R}^N)$.

Finally we have to show for all $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$, $K_1 \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$, $K_2 \in \mathcal{K}(\mathbb{R}^N)$ and $0 < t < \mathcal{T}_\Theta(\vartheta_F, K_1)$

$$\limsup_{h \downarrow 0}\ q_{\mathcal{K},N}\Big(\vartheta_F(t-h,\ K_1),\ K_2\Big)\ \geq\ q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\ K_2\Big).$$

Proposition A.4 ensures the reversibility in time in the interval $[0, \mathcal{T}_\Theta(\vartheta_F, K_1)[$, i.e.
$\mathbb{R}^N \Big\backslash\ \vartheta_F(t-h, K_1)\ =\ \vartheta_{-F}\Big(h,\ \mathbb{R}^N \setminus \vartheta_F(t,K_1)\Big)$ for every $0 < h < t < \mathcal{T}_\Theta(\vartheta_F, K_1)$.
Due to standard hypothesis $(\mathcal{H})$, the flow of the Hamiltonian system even induces a Lipschitz homeomorphism between Graph $N_{\vartheta_F(t-h,K_1)}$ and Graph $N_{\vartheta_F(t,K_1)}$ since each limiting normal cone contains exactly one direction and $N_{\vartheta_F(t,K_1)}(\cdot) = -N_{\overline{\mathbb{R}^N \setminus \vartheta_F(t,K_1)}}(\cdot)$.
Thus, Graph $N_{\vartheta_F(t,K_1)}\ =\ \mathrm{Lim}_{h\downarrow 0}$ Graph $N_{\vartheta_F(t-h,K_1)}$ and finally,

$$q_{\mathcal{K},N}\Big(\vartheta_F(t,K_1),\ \vartheta_F(t-h,\ K_1)\Big)\ \longrightarrow\ 0 \qquad\qquad \text{for } h \downarrow 0.$$

So the last claim results from the triangle inequality. □

For applying Proposition 2.12 about the existence of right–hand forward solutions, we still need sufficient conditions for the transitional compactness.

Definition 4.23 *For any $\lambda > 0$ and $\rho > 0$, the set $\mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(\mathbb{R}^N,\mathbb{R}^N)$ consists of all set–valued maps $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$*

1. *$F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ has compact convex values in $\mathcal{K}_\circ^\rho(\mathbb{R}^N)$.*
2. *$\mathcal{H}_F(\cdot,\cdot) \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,*
3. *$\|\mathcal{H}_F\|_{C^{1,1}(\mathbb{R}^N\times\,\partial\mathbb{B}_1)} \overset{\text{Def.}}{=} \|\mathcal{H}_F\|_{C^1(\mathbb{R}^N\times\,\partial\mathbb{B}_1)} + \mathrm{Lip}\ D\mathcal{H}_F|_{\mathbb{R}^N\times\,\partial\mathbb{B}_1} < \lambda$.*

Remark 4.24 $\mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(\mathbb{R}^N,\mathbb{R}^N)$ is a subset of $\mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N,\mathbb{R}^N)$ and its maps fulfill standard hypothesis $(\mathcal{H}_\circ^\rho)$ (see Definition A.7). In particular, they make points evolve into sets of positive erosion according to Proposition A.9.

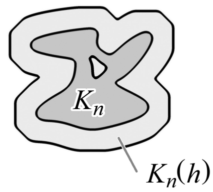
Proposition 4.25
For any $\lambda,\rho > 0$, consider the maps $F \in \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(\mathbb{R}^N,\mathbb{R}^N)$ (i.e. their reachable sets, strictly speaking) as forward transitions of order 0 on $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}_{C^{1,1}}(\mathbb{R}^N),\ q_{\mathcal{K},N})$.

Then $\mathcal{K}_\circ(\mathbb{R}^N)$ is transitionally compact in $\Big(\mathcal{K}(\mathbb{R}^N),\ q_{\mathcal{K},N},\ \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(\mathbb{R}^N,\mathbb{R}^N)\Big)$ in the following sense (see Definitions 2.10, 4.15) :
Let $(K_n)_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ be sequences in $\mathcal{K}_\circ(\mathbb{R}^N)$ and $]0,1[$, respectively with $h_j \downarrow 0$, $\sup_n\ q_{\mathcal{K},N}(\mathbb{B}_1,K_n) < \infty$. Suppose each $G_n : [0,1] \longrightarrow \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\rho)}(\mathbb{R}^N,\mathbb{R}^N)$ to be piece-wise constant $(n \in \mathbb{N})$ and set

$$\begin{aligned}\widetilde{G}_n\ &:\ [0,1]\times\mathbb{R}^N \rightsquigarrow \mathbb{R}^N,\quad (t,x) \longmapsto G_n(t)(x),\\ K_n(h)\ &:=\ \vartheta_{\widetilde{G}_n}(h,K_n) \qquad\qquad \text{for } h \geq 0.\end{aligned}$$

Then there exist a sequence $n_k \nearrow \infty$ of indices and $K \in \mathcal{K}(\mathbb{R}^N)$ satisfying

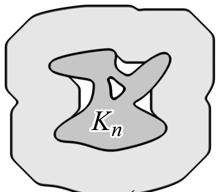
$$\begin{aligned}\limsup_{k\longrightarrow\infty}\quad & q_{\mathcal{K},N}(K_{n_k}(0),\quad K) \;=\; 0,\\ \limsup_{j\longrightarrow\infty}\ \sup_{k\geq j}\quad & q_{\mathcal{K},N}(K,\ \ K_{n_k}(h_j)) \;=\; 0.\end{aligned}$$



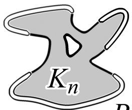
Proof. Closed bounded balls in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ are known to be compact. So there exist a subsequence (again denoted by) $(K_n)_{n\in\mathbb{N}}$ and $K \in \mathcal{K}(\mathbb{R}^N)$ with $d\!\!l(K_n,K) \longrightarrow 0\ (n \longrightarrow \infty)$. Thus,
$d\!\!l(K,K_n(h)) \leq d\!\!l(K,K_n) + \lambda\,h \longrightarrow \lambda\,h$ for $n \longrightarrow \infty$.
Furthermore Corollary 4.11 implies $q_{\mathcal{K},N}(K_n,K) \longrightarrow 0$.

Now we want to prove that K satisfies the claim by choosing subsequences of (K_n) for countably many times (and applying the Cantor diagonal construction).
An important tool here is Proposition A.9. It ensures the existence of $\sigma = \sigma(\lambda,\rho,K) > 0$ and $\widehat{h} = \widehat{h}(\lambda,\rho,K) \in\ ]0,1]$ such that $\vartheta_{-\widetilde{G}_n(h-\,\cdot\,,\,\cdot\,)}(h,\ z)$ has positive erosion of radius $\sigma\,h$ for every $h \in\ ]0,\widehat{h}]$ and $z \in \mathbb{B}_1(K)$. In the following, we assume without loss of generality $0 < h_j < \widehat{h}$ and $K_n(h) \subset \mathbb{B}_1(K)$ for all $j,n \in \mathbb{N}$, $h \in [0,\widehat{h}]$.

So the asymptotic properties of $\quad \mathbb{e}^{\supset}\big(\text{Graph }{}^\flat N_K,\ \text{Graph }{}^\flat N_{K_n(h)}\big) \quad (n \longrightarrow \infty)$ have to be investigated for each $h \in\,]0, \widehat{h}]$.
Due to Definition 4.8, every limiting normal cone results from the neighboring proximal normal cones, i.e. $N_C(x) \overset{\text{Def.}}{=} \operatorname{Limsup}_{\substack{y \longrightarrow x \\ y \in C}} N_C^P(y)$ for all nonempty $C \subset \mathbb{R}^N$, $x \in \partial C$.
Thus, $\text{Graph } N_C = \overline{\text{Graph } N_C^P}$ and from now on, we confine our considerations to $\mathbb{e}^{\supset}\Big(\text{Graph }{}^\flat N_K,\ \text{Graph }{}^\flat N^P_{K_n(h)}\Big)$ for any $h \in\,]0, \widehat{h}]$.



$\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h, \partial K_n(h))$



$P_{n,h}$

The intersection $\quad P_{n,h} := K_n \cap \vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ \partial K_n(h))$ is a subset of ∂K_n.
More precisely, it consists of all points $x \in K_n$ such that a trajectory of $\widetilde{G}_n$ starts in x and reaches $\partial K_n(h)$ at time h.
In addition, every boundary point y of $K_n(h)$ is attained by such a trajectory.
Taking now adjoint arcs into account, the Hamiltonian system in Proposition 4.12 provides the following estimate for every $n \in \mathbb{N}$ (similarly to Lemma 4.14)

$$\mathbb{e}^{\supset}\Big(\text{Graph }{}^\flat N_{K_n}\Big|_{P_{n,h}},\ \text{Graph }{}^\flat N^P_{K_n(h)}\Big) \ \le\ \text{const}(\lambda)\ \cdot\ h.$$

The next step provides the identity of normals: $\text{Graph }{}^\flat N_{K_n}\Big|_{P_{n,h}} = \text{Graph }{}^\flat N^P_{K_n}\Big|_{P_{n,h}}$.
Indeed, $\qquad N^P_{\overline{\mathbb{R}^N \setminus K_n}}(x) \neq \emptyset \quad$ for all $x \in \partial K_n$, $\quad$ due to $K_n \in \mathcal{K}_\circ(\mathbb{R}^N)$.
In particular, $N^P_{K_n}(x) \neq \emptyset$ for all $x \in P_{n,h}$ because $\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ \partial K_n(h))$ has positive erosion of radius $\sigma\, h$ (due to Proposition A.9) and

$$K_n \cap \Big(\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ \partial K_n(h))\Big)^\circ = \emptyset.$$

So, $N^P_{\overline{\mathbb{R}^N \setminus K_n}}(x) = -N^P_{K_n}(x)$ contain exactly one direction for every point $x \in P_{n,h}$ according to [19, Clarke,Ledyaev,Stern 97], Lemma 6.4.
The positive erosion of K_n implies that $\overline{\mathbb{R}^N \setminus K_n}$ has positive reach and thus, $N^P_{\overline{\mathbb{R}^N \setminus K_n}}(x) = N_{\overline{\mathbb{R}^N \setminus K_n}}(x) = N^C_{\overline{\mathbb{R}^N \setminus K_n}}(x)$ contain exactly one direction (with $N^C_M(x)$ denoting the Clarke normal cone of $M \subset \mathbb{R}^N$ at x). As a consequence of a well–known result in [18, Clarke 83], we obtain that $N^C_{K_n}(x) = -N^C_{\overline{\mathbb{R}^N \setminus K_n}}(x)$ consist of exactly one direction for all $x \in P_{n,h}$ and so, $N^C_{K_n}(x) = N_{K_n}(x) = N^P_{K_n}(x)$.

In addition, the proximal radius of K_n at each $x \in P_{n,h}$ (in its unique proximal direction) is $\ge \sigma\, h$ since $\vartheta_{-\widetilde{G}_n(h-\cdot,\cdot)}(h,\ \partial K_n(h))$ has positive erosion of radius $\sigma\, h$.
As this lower bound of proximal radius does not depend on n (but merely on h, λ, ρ, K), it is easy to prove indirectly for every $h \in\,]0, \widehat{h}]$

$$\mathbb{e}^{\supset}\Big(\text{Graph }{}^\flat N_K,\ \ \text{Graph }{}^\flat N^P_{K_n}\Big|_{P_{n,h}}\Big) \ \longrightarrow\ 0 \qquad (n \longrightarrow \infty).$$

So we obtain the estimate for every $h \in\,]0, \widehat{h}]$,

$$\limsup_{n \longrightarrow \infty}\ \mathbb{e}^{\supset}\Big(\text{Graph }{}^\flat N_K,\ \ \text{Graph }{}^\flat N^P_{K_n(h)}\Big) \ \le\ \text{const}(\lambda)\ \cdot\ h.$$

For proving transitional compactness of $\mathcal{K}_\circ(\mathbb{R}^N)$ in $(\mathcal{K}(\mathbb{R}^N),\ q_{\mathcal{K},N},\ \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(\mathbb{R}^N,\mathbb{R}^N))$, a monotone sequence $(h_j)_{j\in\mathbb{N}}$ in $]0,\widehat{h}]$ with $h_j \longrightarrow 0$ is given.
Applying the Cantor diagonal construction, we obtain a subsequence (again denoted by) $(K_{n_k})_{k\in\mathbb{N}}$ satisfying for every $j\in\mathbb{N},\ k\geq j$

$$e^{\supset}\Big(\text{Graph }{}^\flat N_K,\quad \text{Graph }{}^\flat N^P_{K_{n_k}(h_j)}\Big)\ \leq\ \mathrm{const}(\lambda)\ \cdot\ h_j\ +\ \tfrac{1}{k}\ ,$$

and thus, $$\limsup_{j\longrightarrow\infty}\ \sup_{k\geq j}\ q_{\mathcal{K},N}(K,\ K_{n_k}(h_j))\ =\ 0.$$

□

Corollary 4.26 *Let* $f:\mathcal{K}(\mathbb{R}^N)\times[0,T]\longrightarrow \mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(\mathbb{R}^N,\mathbb{R}^N)$ *satisfy*

$$\big\|\mathcal{H}_{f(K_1,t_1)}\ -\ \mathcal{H}_{f(K_2,t_2)}\big\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}\ \leq\ \omega(q_{\mathcal{K},N}(K_1,K_2)\ +\ t_2-t_1)$$

for all $K_1,K_2\in\mathcal{K}(\mathbb{R}^N)$ *and* $0\leq t_1\leq t_2\leq T$ *with a modulus* $\omega(\cdot)$ *of continuity and consider the reachable sets of maps in* $\mathrm{LIP}_\lambda^{(\mathcal{H}_\circ^\varrho)}(\mathbb{R}^N,\mathbb{R}^N)$ *as forward transitions on* $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}_{C^{1,1}}(\mathbb{R}^N),\ q_{\mathcal{K},N})$ *according to Proposition 4.22.*

Then for every initial set $K_0\in\mathcal{K}(\mathbb{R}^N)$, *there exists a right–hand forward solution* $K:[0,T[\longrightarrow\mathcal{K}(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{K}(\cdot)\ \ni\ f(K(\cdot),\cdot)$ *with* $K(0)=K_0$, *i.e.*

a) $\displaystyle\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot\Big(q_{\mathcal{K},N}\Big(\vartheta_{g(x(t),\,K(t),\,t)}\,(h,\ M),\ \ K(t{+}h)\Big)\ -\ q_{\mathcal{K},N}(M,\ K(t))\cdot e^{10\,\Lambda\,t}\Big)\ \leq\ 0$

for every compact set $M\subset\mathbb{R}^N$ *with* $C^{1,1}$ *boundary and* $t\in[0,T[$.

b) $q_{\mathcal{K},N}(K(s),\ K(t))\ \leq\ \mathit{const}(\Lambda,T)\cdot(t-s)$ *for all* $0\leq s<t<T$.

Proof results from Proposition 4.25 along with Proposition 2.12 and Remark 2.13 (2.).

□

Strictly speaking, Proposition 2.12 about the existence of right–hand forward solutions even deals with systems of mutational equations. So we are free to combine the examples of § 3 and § 4.2 — obtaining Proposition 1.1 of the Introduction.

A Tools of differential inclusions

This appendix provides a collection of properties for the reachable sets of differential inclusions giving a quite general example of shape evolution. In particular, we use adjoint arcs for describing the time–dependent limiting normal cones and find sufficient conditions for preserving smooth boundaries (for short times at least).
First we prove in Proposition A.2 that $C^{1,1}$ boundaries are preserved for short times even under slightly more general assumptions than $F \in \mathrm{LIP}_\lambda^{(\mathcal{H})}(\mathbb{R}^N, \mathbb{R}^N)$. Then according to Proposition A.4, the same hypothesis guarantees that the evolution of smooth sets is reversible in time. Finally, the conditions on the Hamiltonian function $\mathcal{H}_F$ are supposed to be stronger for guaranteeing that points evolve into sets of positive erosion. Details are presented in Proposition A.9.

A.1 Standard hypothesis $(\mathcal{H})$ preserves smooth sets shortly

Definition A.1 *For a set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, the* standard hypothesis $(\mathcal{H})$ *comprises the following conditions on $\mathcal{H}_F(x,p) := \sup \; p \cdot F(x)$*

1. *F has nonempty compact convex values,*
2. *$\mathcal{H}_F(\cdot,\cdot) \in C^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,*
3. *the derivative of $\mathcal{H}_F$ has linear growth, i.e. there is some $\gamma_F > 0$ with*
   $\left\| D\,\mathcal{H}_F(x,p) \right\|_{\mathcal{L}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})} \le \gamma_F \cdot (1 + |x| + |p|)$ *for all $x, p \in \mathbb{R}^N$ $(|p| \ge 1)$.*

Proposition A.2 *Assume standard hypothesis $(\mathcal{H})$ for $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. For every initial set $K \in \mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$, there exist $\tau = \tau(F,K) > 0$ and $\rho = \rho(F,K) > 0$ such that $\vartheta_F(t,K)$ is also a N–dimensional $C^{1,1}$ submanifold of $\mathbb{R}^N$ with boundary for all $t \in [0,\tau]$ and its radius of curvature is $\ge \rho$ (i.e. $\vartheta_F(t,K)$ has both positive reach and positive erosion of radius ρ).*

Proof of Proposition A.2 is based on the following lemma :

Lemma A.3 *Suppose for $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and the Hamiltonian system*

$$\wedge \begin{cases} \dot{y}(t) = \frac{\partial}{\partial q} H(t,\, y(t),\, q(t)), & y(0) = y_0 \\ \dot{q}(t) = -\frac{\partial}{\partial y} H(t,\, y(t),\, q(t)), & q(0) = \psi(y_0) \end{cases} \qquad (*)$$

the following properties :

1. *$H(t,\cdot,\cdot)$ is differentiable for every $t \in [0,T]$,*
2. *for every $R > 0$, there exists $k_R \in L^1([0,T])$ such that the derivative of $H(t,\cdot,\cdot)$ is $k_R(t)$–Lipschitz continuous on $\mathbb{B}_R \times \mathbb{B}_R$ for almost every t,*
3. *ψ is locally Lipschitz continuous,*

4. *every solution* $(y(\cdot), q(\cdot))$ *of the Hamiltonian system* $(*)$ *can be extended to* $[0,T]$ *and depends continuously on the initial data in the following sense :*
   Let each $(y_n(\cdot), q_n(\cdot))$ *be a solution satisfying* $y_n(t_n) \longrightarrow z_0, \quad q_n(t_n) \longrightarrow q_0$ *for some* $t_n \longrightarrow t_0, \ z_0, q_0 \in \mathbb{R}^N$. *Then* $(y_n(\cdot), q_n(\cdot))_{n\in\mathbb{N}}$ *converges uniformly to a solution* $(y(\cdot), q(\cdot))$ *of the Hamiltonian system with* $y(t_0) = z_0, \quad q(t_0) = q_0$.

For a compact set $K \subset \mathbb{R}^N$ *and* $t \in [0,T]$, *define*

$$M_t^{\mapsto}(K) := \Big\{ (y(t),\ q(t)) \ \Big| \ (y(\cdot),\ q(\cdot)) \text{ solves system } (*),\ \ y_0 \in K \Big\} \subset \mathbb{R}^N \times \mathbb{R}^N.$$

Then there exist $\delta > 0$ *and* $\lambda > 0$ *such that* $M_t^{\mapsto}(K)$ *is the graph of a* λ*–Lipschitz continuous function for every* $t \in [0, \delta]$.

Proof of Lemma A.3 follows exactly the same (indirect) track as [28, Frankowska 2002], Lemma 5.5 stating the corresponding result for the Hamiltonian system with $y(T) = y_T$, $q(T) = q_T$ given (without mentioning the uniform Lipschitz constant λ explicitly).

Proof of Proposition A.2. Standard hypothesis $(\mathcal{H})$ for $F : \mathbb{R}^N \leadsto \mathbb{R}^N$ implies conditions (1.), (4.) of the preceding Lemma A.3 for the Hamiltonian $\mathcal{H}_F$.
Assuming that $K \in \mathcal{K}(\mathbb{R}^N)$ is a N–dimensional $C^{1,1}$ submanifold of $\mathbb{R}^N$ with boundary, the unit *exterior* normal vectors of K (restricted to ∂K) can be extended to a Lipschitz continuous function $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$. Furthermore, choose $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ with

$$\varphi(s) = 0 \quad \text{for} \quad s \le \tfrac{1}{4}, \qquad \varphi(s) = 1 \quad \text{for} \quad s \ge \tfrac{1}{2}$$

and set $H(t,x,p) := \mathcal{H}_F(x,p) \cdot \varphi(|p|)$ for $(t,x,p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$.
Then H satisfies condition (2.) of Lemma A.3 in addition.

For arbitrary $x_0 \in \partial K$, consider now the differential equations

$$\wedge \begin{cases} \dot{x}(t) &= \frac{\partial}{\partial p} H(t,\ x(t),\ p(t)), & x(0) = x_0, \\ \dot{p}(t) &= -\frac{\partial}{\partial x} H(t,\ x(t),\ p(t)), & p(0) = \psi(x_0). \end{cases} \qquad (*)$$

Due to $|\psi(\cdot)| = 1$ on ∂K and $H \in C^{1,1}$, there is $\tau_1 > 0$ such that $|p(t)| > \frac{1}{2}$ for all $t \in [0, \tau_1]$ and solutions $(x(\cdot), p(\cdot))$ of $(*)$ with $x_0 \in \partial K$. Thus, $H = \mathcal{H}_F$ close to $(x(t), p(t))$. Now Proposition 4.12 can be reformulated as

$$\text{Graph } N_{\vartheta_F(t,K)}(\cdot) \subset \Big\{ (x(t),\ \lambda\, p(t)) \ \Big| \ (x(\cdot),\ p(\cdot)) \text{ solves system } (*),\ \ x_0 \in \partial K,\ \ \lambda \ge 0 \Big\},$$

for all $t \in [0, \tau_1]$. Furthermore Lemma A.3 yields $\tau \in]0, \tau_1[$ and $\lambda_M > 0$ such that

$$M_t^{\mapsto}(\partial K) := \Big\{ (x(t),\ p(t)) \ \Big| \ (x(\cdot),\ p(\cdot)) \text{ solves system } (*),\ \ x_0 \in \partial K \Big\}$$

is the graph of a λ_M–Lipschitz continuous function for each $t \in [0, \tau]$.

Then for every point $z \in \partial \vartheta_F(t, K)$, the limiting normal cone $N_{\vartheta_F(t,K)}(z)$ contains exactly one direction and, its unit vector depends on z in a Lipschitz continuous way. (The Lipschitz constant is uniformly bounded by $2\,\lambda_M$ since the choice of τ_1 ensures $|p(\cdot)| > \frac{1}{2}$ on $[0, \tau_1]$ for each solution of $(*)$.)
So the compact set $\vartheta_F(t, K)$ is N–dimensional $C^{1,1}$ submanifold of $\mathbb{R}^N$ with boundary for all $t \in [0, \tau]$ and its radius of curvature has a uniform lower bound. □

A.2 Uniform positive reach and standard hypothesis $(\mathcal{H})$ imply reversibility in time

The Hamilton condition leads to a necessary condition on boundary points $x \in \partial\,\vartheta_F(t,K)$ and their limiting normal cones in Proposition 4.12. If each set $\vartheta_F(t,K)$ $(0 \le t \le T)$ has positive reach of radius ρ, then standard hypothesis $(\mathcal{H})$ turns adjoint arcs into sufficient conditions and, we conclude that the evolution of reachable sets is reversible with respect to time — in the sense of Proposition A.4.

Proposition A.4 *Suppose standard hypothesis* $(\mathcal{H})$ *for the map* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$. *Assume for* $K_0 \in \mathcal{K}(\mathbb{R}^N)$ *and* $\rho > 0$ *that each compact set* $K_t := \vartheta_F(t, K_0)$ $(0 \le t \le T)$ *has positive reach of radius* ρ.
Then for every $0 \le s \le t < T$, $\quad K_s \;=\; \mathbb{R}^N \Big\backslash \vartheta_{-F}(t-s,\ \mathbb{R}^N \backslash K_t).$

Here we even suppose a uniform radius ρ of positive reach for $K_t \overset{\text{Def.}}{=} \vartheta_F(t, K_0)$. The essential advantage for the proof is the relation between the boundaries of $K_t \subset \mathbb{R}^N$ and Graph $(t \longmapsto K_t) \subset \mathbb{R} \times \mathbb{R}^N$ stated in Proposition A.6 :

$$\partial\,\text{Graph}\,\vartheta_F(\,\cdot\,, K_0)|_{[0,T]} \;=\; (\{0\} \times K_0) \;\cup \bigcup_{0<t<T} (\{t\} \times \partial\vartheta_F(t, K_0)) \;\cup\; (\{T\} \times \vartheta_F(T, K_0))\,.$$

Proof of Proposition A.4 $\quad \vartheta_F(s, K_0) \;\subset\; \mathbb{R}^N \setminus \vartheta_{-F}(t-s,\ \mathbb{R}^N \backslash K_t)$ is an easy indirect consequence of definitions since it is equivalent to $\vartheta_F(s, K_0) \;\cap\; \vartheta_{-F}(t-s,\ \mathbb{R}^N \backslash K_t) \;=\; \emptyset$.

For proving the inverse inclusion indirectly at time $s = 0$, we assume the existence of a time $t \in [0, T[$ and a point $y_0 \in \mathbb{R}^N$ with $\quad y_0 \notin K_0 \cup \vartheta_{-F}(t,\ \mathbb{R}^N \backslash K_t)$.
As an immediate consequence of $y_0 \notin \vartheta_{-F}(t,\ \mathbb{R}^N \backslash K_t)$, the reachable set $\vartheta_F(t, y_0)$ is contained in $K_t \overset{\text{Def.}}{=} \vartheta_F(t, K_0)$. Now set $\tau := \inf\{s \in [0,t] \mid \vartheta_F(s, y_0) \subset \vartheta_F(s, K_0)\}$.
In particular, $\quad \tau > 0 \quad$ due to $y_0 \notin K_0$.
and $\quad \vartheta_F(\tau, y_0) \subset \vartheta_F(\tau, K_0) \quad$ due to the continuity of the reachable sets.
There are sequences $\tau_n \nearrow \tau$ and $(x_n(\cdot))_{n \in \mathbb{N}}$ in $AC([0,T], \mathbb{R}^N)$ satisfying

$$\dot{x}_n(\cdot) \in F(x_n(\cdot)) \quad \text{a.e.}, \qquad x_n(0) = y_0, \qquad x_n(\tau_n) \notin \vartheta_F(\tau_n, K_0).$$

Then for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} x_n(s) &\notin \vartheta_F(s, K_0) && \text{for every } s \in [0, \tau_n], \\ x_n(s) &\in \vartheta_F(s, K_0) && \text{for every } s \in [\tau, T\,]. \end{aligned}$$

Furthermore standard hypothesis $(\mathcal{H})$ and Gronwall's Lemma imply uniform bounds and the equicontinuity of all $x_n(\cdot)$, $n \in \mathbb{N}$. So the compactness of trajectories (see e.g. [40, Vinter 2000], Theorem 2.5.3) leads to subsequences (again denoted by) $(\tau_n)_{n \in \mathbb{N}}$, $(x_n(\cdot))_{n \in \mathbb{N}}$ and a function $x(\cdot) \in AC([0,T], \mathbb{R}^N)$ with

$$\begin{aligned} x_n(\cdot) &\longrightarrow x(\cdot) && \text{uniformly in } [0,T], \\ \dot{x}_n(\cdot) &\rightharpoonup \dot{x}(\cdot) && \text{in } \quad L^1([0,T], \mathbb{R}^N) \end{aligned}$$

such that $x(\cdot)$ is a solution of $\dot{x}(\cdot) \in F(x(\cdot))$ (almost everywhere). In particular, $(\tau, x(\tau))$ has to be a boundary point of Graph $\vartheta_F(\cdot, K_0)$.
Proposition A.6 and $0 < \tau \le t < T$ ensure $x_\tau := x(\tau) \in \partial K_\tau \overset{\text{Def.}}{=} \partial\, \vartheta_F(\tau, K_0)$.

Moreover, $K_\tau \overset{\text{Def.}}{=} \vartheta_F(\tau, K_0)$ is supposed to have positive reach. So its limiting and proximal normal cone coincide at each boundary point and thus,

$$\emptyset \ \neq\ N_{\vartheta_F(\tau,K_0)}(x_\tau) \ =\ N^P_{\vartheta_F(\tau,K_0)}(x_\tau) \ \subset\ N^P_{\vartheta_F(\tau,\, y_0)}(x_\tau).$$

For every unit vector $\nu \in N_{\vartheta_F(\tau,K_0)}(x_\tau)$, Proposition 4.12 leads to a trajectory $z(\cdot) \in C^1([0,\tau], \mathbb{R}^N)$ of F and its adjoint arc $q(\cdot) \in C^1([0,\tau], \mathbb{R}^N)$ satisfying the corresponding Hamiltonian system and $z(0) \in K_0$, $z(\tau) = x_\tau$, $q(\tau) = \nu$. Besides, the same Cauchy problem is solved by $x(\cdot)$ and its adjoint. $\mathcal{H}_F \in C^{1,1}$ implies the uniqueness of solutions and, its consequence $z(0) = x(0) \notin K_0$ leads to a contradiction.
Thus, $\mathbb{R}^N \setminus \vartheta_{-F}(t,\, \mathbb{R}^N \setminus K_t) \subset K_0$.
Finally the corresponding inclusion for any $0 < s \le t < T$ results from the semigroup property of reachable sets. □

Remark A.5 1. The map $\mathcal{K}(\mathbb{R}^N) \rightsquigarrow \mathbb{R}^N$, $K_0 \longmapsto \mathbb{R}^N \setminus \vartheta_{-F}(t,\, \mathbb{R}^N \setminus \vartheta_F(t, K_0))$ generalizes the morphological operation of closing (of sets in $\mathcal{K}(\mathbb{R}^N)$) that was introduced by Minkowski and is usually defined as

$$\mathcal{P}(X) \rightsquigarrow X, \qquad K \longmapsto (K - t\,B) \ominus (-t\,B) \overset{\text{Def.}}{=} \{\, y \in X \mid y - t\,B \subset K - t\,B \,\}$$

for a vector space X and fixed $B \subset X$, $t > 0$ (see e.g. [2, Aubin 99], Def. 3.3.1).

2. In [9, Barron, Cannarsa, Jensen, Sinestrari 99], the viscosity solutions of the Hamilton–Jacobi equation $\partial_t\, u + H(t, x, Du) = 0$ are investigated and roughly speaking, the continuous differentiability of u is concluded from the reversibility in time :
If $u : [0,T] \times \mathbb{R}^N \longmapsto \mathbb{R}$ is a continuous viscosity solution of $\partial_t\, u + H(t, \cdot, Du) = 0$ and $v(t,x) := u(T-t, x)$ is a viscosity solution of $\partial_t\, v - H(T-t, \cdot, Dv) = 0$ then adequate assumptions of H ensure $u \in C^1(]0,T[\, \times \mathbb{R}^N)$.
Referring to the relation between reachable sets and level sets of viscosity solutions, we draw an inverse conclusion as we assume smoothness and obtain the reversibility in time.

3. Furthermore it is shown for some optimal control problems in [9] that the continuous viscosity solution u of the Hamilton–Jacobi equation is even in $C^1([0,T] \times \mathbb{R}^N)$ if both $u(0,\cdot)$ and $u(T,\cdot)$ are of class C^1. In the geometric context here, we cannot restrict ourselves to regularity assumptions about K_0 and $\vartheta_F(T, K_0)$ as “holes” (of an annulus, for example) might have disappeared meanwhile.

4. The reversibility in time (in the sense of Proposition A.4) can also be regarded as recovering the initial data. Further results about this problem have already been published in [38, Rzeżuchowski 97] and [39, Rzeżuchowski 99], for example, but they usually assume other conditions. Either the initial set consists of only one point or the Hamiltonian function $\mathcal{H}_F$ is of class C^2.

Proposition A.6 *Suppose for* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $K \in \mathcal{K}(\mathbb{R}^N)$ *and* $\rho > 0$ *that the map* $[0,T] \rightsquigarrow \mathbb{R}^N$, $t \longmapsto \vartheta_F(t,K)$ *is* λ*–Lipschitz continuous (with respect to* $d\!\!l$*) and each set* $\vartheta_F(t,K)$ $(0 \le t \le T)$ *has positive reach of radius* ρ.
Then the topological boundary of $\text{Graph } \vartheta_F(\cdot,K)|_{[0,T]}$ *in* $\mathbb{R} \times \mathbb{R}^N$ *is*

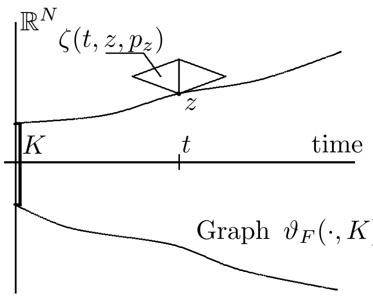
$$\{0\} \times K \ \cup \bigcup_{0<t<T} \{t\} \times \partial\vartheta_F(t,K) \ \cup \ \{T\} \times \vartheta_F(T,K).$$

Proof. The inclusion

$$\{0\} \times K \ \cup \bigcup_{0<t<T} \{t\} \times \partial\vartheta_F(t,K) \ \cup \ \{T\} \times \vartheta_F(T,K) \ \subset \ \partial \,\text{Graph } \vartheta_F(\cdot,K)|_{[0,T]}$$

is obvious. Due to the Lipschitz continuity of $\vartheta_F(\cdot,K)$, we only have to show

$$\partial \,\text{Graph } \vartheta_F(\cdot,K) \ \cap \ (]0,T[\times \mathbb{R}^N) \ \subset \bigcup_{0<t<T} \{t\} \times \partial\,\vartheta_F(t,K).$$



Every point $z \in \partial\,\vartheta_F(t,K)$ $(0 \le t \le T)$ and any unit vector $p_z \in N^P_{\vartheta_F(t,K)}(z) = N_{\vartheta_F(t,K)}(z)$ satisfy $\overset{\circ}{\mathbb{B}}_\rho(z + \rho\, p_z) \ \cap \ \vartheta_F(t,K) \ = \ \emptyset$ and thus, $\Big(\{t\} \times \overset{\circ}{\mathbb{B}}_\rho(z + \rho\, p_z)\Big) \ \cap \ \text{Graph } \vartheta_F(\cdot,K) \ = \ \emptyset.$
The λ–Lipschitz continuity of $\vartheta_F(\cdot,K)$ implies $\zeta(t,z,p_z) \cap \text{Graph } \vartheta_F(\cdot,K) \ = \ \emptyset$ for

$$\zeta(t,z,p_z) \ := \ \Big\{ (s,y) \ \in \ \mathbb{R} \times \mathbb{R}^N \ \Big| \ |z + \rho\, p_z \ - \ y| \ < \ \rho - \lambda\,|s - t| \Big\}.$$

Now choose $(t,x) \in \partial\,\text{Graph } \vartheta_F(\cdot,K)$ with $0 < t < T$ arbitrarily. The continuity of $\vartheta_F(\cdot,K)$ guarantees that $\text{Graph } \vartheta_F(\cdot,K)$ is closed and thus, it contains (t,x).
Moreover there are sequences $(t_n)_{n \in \mathbb{N}}$, $(x_n)_{n \in \mathbb{N}}$ in $]0,T[$, $\mathbb{R}^N$, respectively, satisfying $(t_n, x_n) \notin \text{Graph } \vartheta_F(\cdot,K)$ for every $n \in \mathbb{N}$ and $(t_n,x_n) \longrightarrow (t,x)$ $(n \longrightarrow \infty)$.
For each $n \in \mathbb{N}$, let z_n be an element of the projection $\Pi_{\vartheta_F(t_n,K)}(x_n) \subset \partial\vartheta_F(t_n,K)$.
Then, $0 < |x_n - z_n| \ = \ \text{dist}(x_n,\, \vartheta_F(t_n,K)) \ \le \ |x_n - x| + \text{dist}(x,\, \vartheta_F(t_n,K)) \ \longrightarrow \ 0$ and $p_n \ := \ \frac{x_n - z_n}{|\,x_n - z_n\,|} \ \in \ N^P_{\vartheta_F(t_n,K)}(z_n) \ \cap \ \partial\mathbb{B}_1$.
As mentioned before, we obtain $\zeta(t_n,z_n,p_n) \cap \text{Graph } \vartheta_F(\cdot,K) \ = \ \emptyset$ for each $n \in \mathbb{N}$.
Considering adequate subsequences (again denoted by) $(t_n)_{n \in \mathbb{N}}$, $(x_n)_{n \in \mathbb{N}}$, $(p_n)_{n \in \mathbb{N}}$ leads to the additional convergence $p_n \longrightarrow p \in \partial\mathbb{B}_1$ $(n \longrightarrow \infty)$. So finally

$$\zeta(t,x,p) \ \cap \ \text{Graph } \vartheta_F(\cdot,K) \ = \ \emptyset$$

In particular, $\overset{\circ}{\mathbb{B}}_\rho(x + \rho\, p) \ \cap \ \vartheta_F(t,K) \ = \ \emptyset$ implies $x \ \in \ \partial\,\vartheta_F(t,K)$. □

A.3 Standard hypothesis $(\mathcal{H}^\rho_\circ)$ makes points evolve into sets of positive erosion

Our aim consists in sufficient conditions for the positive erosion of $\vartheta_F(t, K)$. Weakening the assumption about the initial set $K \in \mathcal{K}_\circ(\mathbb{R}^N)$ (in comparison with [33, Lorenz 2003]) usually requires stronger properties of the set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ than standard hypothesis $(\mathcal{H})$ (see Definition A.1).

Definition A.7 *For any $\rho > 0$, a set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ satisfies the so–called* standard hypothesis $(\mathcal{H}^\rho_\circ)$ *if it has the following properties :*

1. *F has convex values in $\mathcal{K}^\rho_\circ(\mathbb{R}^N)$,*
2. *$\mathcal{H}_F(\cdot,\cdot) \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,*
3. *the derivative of $\mathcal{H}_F$ has linear growth, i.e. there is some $\gamma_F > 0$ with* $\left\| D\,\mathcal{H}_F(x,p) \right\|_{\mathcal{L}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})} \le \gamma_F \cdot (1 + |x| + |p|)$ *for all $x, p \in \mathbb{R}^N$ $(|p| \ge 1)$.*

Remark A.8 Standard hypothesis $(\mathcal{H}^\rho_\circ)$ differs from its counterpart $(\mathcal{H})$ in two respects : The values of F have uniform positive erosion (additionally) and its Hamiltonian is even twice continuously differentiable in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$. This second restriction has the advantage that we can apply the tools of matrix Riccati equation (mentioned in Lemma A.11 and A.12).

Proposition A.9 *Let $F_1 \ldots F_m : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ hold standard hypothesis $(\mathcal{H}^\rho_\circ)$ and*

$$\|\mathcal{H}_{F_j}\|_{C^{1,1}(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} \stackrel{\text{Def.}}{=} \|\mathcal{H}_{F_j}\|_{C^1(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} + \operatorname{Lip} D\mathcal{H}_{F_j}|_{\mathbb{R}^N \times\, \partial\mathbb{B}_1} < \lambda$$

for some λ, $\rho > 0$. Moreover for a partition $0 \le \tau_0 < \tau_1 < \ldots < \tau_m = 1$ of $[0,1]$, define the map $\widetilde{G} : [0,1[\, \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ as $\widetilde{G}(t,x) := F_j(x)$ for $\tau_{j-1} \le t < \tau_j$. Furthermore choose $K \in \mathcal{K}(\mathbb{R}^N)$ arbitrarily.

Then there exist $\sigma > 0$ and a time $\widehat{\tau} \in \,]0,1]$ (depending only on λ, ρ, K) such that the reachable set $\vartheta_{\widetilde{G}}(t, x_0)$ has positive erosion of radius $\sigma\, t$ for any $t \in \,]0, \widehat{\tau}[$, $x_0 \in K$. As an immediate consequence, $\vartheta_{\widetilde{G}}(t, K_1)$ has positive erosion of radius $\sigma\, t$ for all $t \in]0, \widehat{\tau}[$ and each initial subset $K_1 \in \mathcal{K}(\mathbb{R}^N)$ of K.

The proof of this proposition uses matrix Riccati equations for Hamiltonian systems, but these tools of Lemma A.11 consider initial values induced by a Lipschitz function ψ. So roughly speaking, we exchange the two components $(x(\cdot), p(\cdot))$ (of a trajectory and its adjoint) preserving the Hamiltonian structure of their differential equations :

Lemma A.10 *Assume the Hamiltonian system for* $x(\cdot),\ p(\cdot) \in AC([0,T],\mathbb{R}^N)$

$$\dot x(t) \;=\; \tfrac{\partial}{\partial p}\, H_1(t,\, x(t),\, p(t)), \qquad \dot p(t) \;=\; -\tfrac{\partial}{\partial x}\, H_1(t,\, x(t),\, p(t)) \qquad a.e. \;\; in \;\; [0,T]$$

with sufficiently smooth $H_1 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$. *Moreover set*

$$y(t) \;:=\; -\,p(t), \qquad q(t) \;:=\; x(t) \qquad H_2(t,\,\xi,\,\zeta) \;:=\; H_1(t,\,\zeta,\,-\,\xi).$$

Then the absolutely continuous functions $(y(\cdot), q(\cdot))$ *satisfy the Hamiltonian system*

$$\dot y(t) \;=\; \tfrac{\partial}{\partial q}\, H_2(t,\, y(t),\, q(t)), \qquad \dot q(t) \;=\; -\tfrac{\partial}{\partial y}\, H_2(t,\, y(t),\, q(t)) \qquad a.e. \;\; in \;\; [0,T].$$

□

Proof of Proposition A.9. The uniform bound λ of $\|\mathcal{H}_{F_j}\|_{C^{1,1}(\mathbb{R}^N \times\, \partial \mathbb{B}_1)}$ $(j = 1 \,\ldots\, m)$ and Gronwall's Lemma lead to a radius $R = R(\lambda, K) > 1$ and a time $T = T(\lambda, K) \in]0,1[$ such that

1. $\vartheta_{\widetilde G}(t, K) \;\subset\; \mathbb{B}_R$ for all $t \in [0,1]$,
2. for every trajectory $x(\cdot)$ of $\widetilde G$ starting in K, each adjoint $p(\cdot)$ with $\frac12 \le |p(0)| \le 2$ fulfills $\frac1R < |p(\cdot)| < R$, $\;|p(\cdot) - p(0)| < \frac{1}{4\,R}$ on $[0,T]$

So a smooth cut–off function again provides a map $H_1 : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ that fulfills the assumptions of Lemma A.11 and is identical to $\mathcal{H}_{\widetilde G}$ in $[0,T] \times \mathbb{R}^N \times (\mathbb{R}^N \backslash \mathbb{B}_{\frac{1}{2R}})$.

Using the transformation of the preceding Lemma A.10, the auxiliary function

$$H_2 :\; [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \;\longrightarrow\; \mathbb{R}, \qquad (t, \xi, \zeta) \;\longmapsto\; H_1(t,\, \zeta,\, -\,\xi)$$

is still holding the conditions of Lemma A.11. As a consequence, we obtain for any initial point $x_0 \in K$ and time $\tau \in \;]0, T]$ that the following statements are equivalent :

(*i*) For all $t \in [0, \tau]$, the set M_t^1 of all points $(p(t),\, x(t))$ with solutions $(x(\cdot), p(\cdot)) \in AC([0,t], \mathbb{R}^N \times \mathbb{R}^N)$ of

$$\wedge \left\{ \begin{array}{rclcrcl} \dot x(s) &=& \frac{\partial}{\partial p}\, H_1(s,\, x(s),\, p(s)), && x(0) &=& x_0 \\ \dot p(s) &=& -\,\frac{\partial}{\partial x}\, H_1(s,\, x(s),\, p(s)), && p(0) &\in& \mathbb{B}_2 \setminus \overset{\circ}{\mathbb{B}}_{\frac12} \end{array} \right.$$

is the graph of a continuously differentiable function f_t.

(*ii*) For any solution $(x,p) : [0,t] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ of the initial value problem (*i*) $(t \le \tau)$, there exists a solution $Q : [0,t] \longrightarrow \mathbb{R}^{N \times N}$ of the Riccati equation

$$\wedge \left\{ \begin{array}{l} \dot Q \;-\; \frac{\partial^2 H_1}{\partial x\, \partial p}\,(s,\, x(s),\, p(s))\;\; Q \;-\; Q\;\; \frac{\partial^2 H_1}{\partial p\, \partial x}\,(s,\, x(s),\, p(s)) \\ \qquad +\;\; Q\;\; \frac{\partial^2 H_1}{\partial x^2}\,(s,\, x(s),\, p(s))\;\; Q \;+\; \frac{\partial^2 H_1}{\partial p^2}\,(s,\, x(s),\, p(s)) \;=\; 0, \\ \hfill Q(0) \;=\; 0. \end{array} \right.$$

Now we give a criterion for the choice of $\widehat \tau$: Setting

$$\mu \;=\; \mu(\lambda, K) \;:=\; \sup_{\substack{0 \,\le\, t \,\le\, T \\ |x| \,\le\, R \\ \frac1R \,\le\, |p| \,\le\, R}} \left\| \begin{pmatrix} \frac{\partial^2}{\partial p^2}\;\; \mathcal{H}_{\widetilde G}(t,x,p) & -\,\frac{\partial^2}{\partial x\, \partial p}\, \mathcal{H}_{\widetilde G}(t,x,p) \\ -\,\frac{\partial^2}{\partial p\, \partial x}\, \mathcal{H}_{\widetilde G}(t,x,p) & \frac{\partial^2}{\partial x^2}\;\; \mathcal{H}_{\widetilde G}(t,x,p) \end{pmatrix} \right\|_{\mathcal{L}(\mathbb{R}^{2N}, \mathbb{R}^{2N})}$$

the comparison theorem for matrix Riccati equations (Lemma A.12) guarantees existence and uniqueness of such a solution $Q : [0,t] \longrightarrow \mathbb{R}^{N \times N}$ for any $t < \min\{T,\, \frac{\pi}{2\,\mu}\}$ because for $a = \pm \mu$, the scalar Riccati equation $\frac{d}{dt}\, u \;=\; a + a\, u^2$, $\;u(0) = 0\;$ has the solution $u(t) \;=\; \tan(a\, t)$ on $[0, \frac{\pi}{2\,|a|}\,[$. Furthermore we obtain $\|Q(t)\| \;\le\; \tan(\mu\, t)$.

Standard hypothesis $(\mathcal{H}^\varrho_\circ)$ for $F_1 \dots F_m$ implies a constant $\sigma = \sigma(\lambda, \rho, K) > 0$ with

$$\xi \cdot \tfrac{\partial^2}{\partial p^2}\, \mathcal{H}_{\widetilde{G}}(t,x,p)\ \xi \ \geq\ 4\,\sigma \left|\xi - \tfrac{\xi \cdot p}{|p|^2}\, p\right|^2$$

for all $t \in [0,T]$, $|x| \leq R$, $\frac{1}{R} \leq |p| \leq R$, ξ. Using the abbreviation $D(t,x,p)$ for $-\frac{\partial^2 \mathcal{H}_{\widetilde{G}}}{\partial x\, \partial p}(t,x,p)\ Q(t) - Q(t)\ \frac{\partial^2 \mathcal{H}_{\widetilde{G}}}{\partial p\, \partial x}(t,x,p) + Q(t)\ \frac{\partial^2 \mathcal{H}_{\widetilde{G}}}{\partial x^2}(t,x,p)\ Q(t) \in \mathbb{R}^{N\times N}$, choose $\widehat{\tau} = \widehat{\tau}(\lambda, \rho, K) > 0$ small enough s.t. $\widehat{\tau} < \min\{T, \frac{\pi}{2\,\mu}, \frac{1}{\lambda}\}$, $\|D(t,x,p)\| \leq \sigma$ for every $t \in [0, \widehat{\tau}]$, $|x| \leq R$, $\frac{1}{R} \leq |p| \leq R$.

As a next step, we show that the solution $Q(t)$ of (ii) (restricted to $[0,\widehat{\tau}]$) has the upper bound $-\sigma\, t$ in a $(N-1)$–dimensional subspace of $\mathbb{R}^N$. Indeed, let $(x(\cdot),\ p(\cdot)) \in AC([0,\widehat{\tau}],\ \mathbb{R}^N \times \mathbb{R}^N)$ be a solution of the Hamiltonian system (i) and choose an arbitrary unit vector $\xi \in \mathbb{R}^N$ with $|\xi \cdot p(0)| < \frac{1}{4R}$.
Then the auxiliary function $\varphi : [0,\widehat{\tau}] \longrightarrow \mathbb{R}^N$, $t \longmapsto \xi \cdot Q(t)\, \xi + \sigma\, t \left|\xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right|^2$ satisfies $\varphi(0) = 0$ and is absolutely continuous with

$$\begin{aligned}
\dot{\varphi}(t) &= \xi \cdot \dot{Q}(t)\, \xi + \sigma \left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right|^2 + \sigma\, t \left(\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right) \cdot \tfrac{d}{dt} \left(\tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right) \\
&= \xi \cdot \dot{Q}(t)\, \xi + \sigma \left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right|^2 + \sigma\, t \left(\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right) \cdot \tfrac{\xi \cdot p(t)}{|p(t)|^2}\ \dot{p}(t)
\end{aligned}$$

as $\xi - \frac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)$ is perpendicular to $p(t)$.

$$\begin{aligned}
\dot{\varphi}(t) &\leq (-4+1+1)\ \sigma \left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right|^2 + \sigma\, t \left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right| \tfrac{|\xi|\ |p(t)|}{|p(t)|^2}\, |\dot{p}(t)| \\
&\leq \sigma \left|\xi - \tfrac{\xi \cdot p(t)}{|p(t)|^2}\, p(t)\right| \cdot \left(-2\left(1 - \tfrac{\xi \cdot p(t)}{|p(t)|}\right) + \lambda\, t\right) \\
&\leq 0
\end{aligned}$$

because $|p(t) - p(0)| < \frac{1}{4R}$, $\frac{1}{R} \leq |p(t)| \leq R$ and $|\xi \cdot p(0)| < \frac{1}{4R}$ imply $\frac{\xi \cdot p(t)}{|p(t)|} < \frac{1}{2}$.
So we obtain $\varphi(t) \leq 0$ for all $t \in [0,\widehat{\tau}]$ and as a consequence, $Q(t) \leq -\sigma\, t \cdot \mathrm{Id}$ is fulfilled in the subspace of $\mathbb{R}^N$ perpendicular to $p(t)$.

Finally we need the geometric interpretation for concluding the positive erosion of $\vartheta_{\widetilde{G}}(t,x_0)$ (of radius $\sigma\, t$) for each $t \in\ ]0,\widehat{\tau}[$ and $x_0 \in K$.
As mentioned before, the existence of the solution $Q(\cdot)$ on $[0,\widehat{\tau}[$ implies for all $t \in [0,\widehat{\tau}[$ that the set M^1_t is graph of a C^1 function f_t. Moreover Proposition 4.12 guarantees

$$\text{Graph } N_{\vartheta_{\widetilde{G}}(t,x_0)} \subset \left\{(x(t),\ \lambda\, p(t)) \,\middle|\, (x(\cdot), p(\cdot)) \text{ solves } (i),\ \lambda \geq 0\right\} \stackrel{\text{Def.}}{=} \bigcup_{\lambda \geq 0} \text{Graph } (\lambda\, f_t^{-1}).$$

So we obtain for every $t \in\ ]0,\widehat{\tau}[$ that each $p \in \mathbb{R}^N \setminus \{0\}$ belongs to the limiting normal cone of a unique boundary point $z \in \partial\, \vartheta_{\widetilde{G}}(t,x_0)$ (and $z = z(p)$ is continuously diff.). In particular, the projection on $\vartheta_{\widetilde{G}}(t,x_0)$ is a single–valued function in $\mathbb{R}^N$ and thus, $\vartheta_{\widetilde{G}}(t,x_0)$ is convex for all $t \in\ ]0,\widehat{\tau}[$ (see e.g. [20, Clarke,Stern,Wolenski 95], Cor. 4.12). So it is sufficient to consider the limiting normal cones of $\vartheta_{\widetilde{G}}(t,x_0)$ *locally* at every boundary point.

Well–known properties of variational equations (see e.g. [28, Frankowska 2002]) and the uniqueness of solutions of the matrix Riccati equation (ii) imply that $-Q(s)$ is the derivative of the C^1 function f_s for $0 < s \le t < \widehat{\tau}$ (more details are presented in [32, Lorenz 2004], Appendix A.7). Thus for every time $t \in \,]0, \widehat{\tau}[$, the derivative of f_t at $p(t)$ is bounded by $\sigma\, t$ from below in a $(N-1)$–dimensional subspace of $\mathbb{R}^N$.
Since $\vartheta_{\widetilde{G}}(t, x_0)$ is convex, it implies that $\vartheta_{\widetilde{G}}(t, x_0)$ has positive erosion of radius $\sigma\, t$.

$\square$

Lemma A.11
In addition to the assumptions (2.)–(4.) of Lemma A.3, suppose for $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *and the Hamiltonian system*

$$\wedge \begin{cases} \dot{y}(t) = \frac{\partial}{\partial q} H(t,\, y(t),\, q(t)), & y(0) = y_0 \\ \dot{q}(t) = -\frac{\partial}{\partial y} H(t,\, y(t),\, q(t)), & q(0) = \psi(y_0) \end{cases} \qquad (*)$$

1'. $H(t,\cdot,\cdot)$ *is* twice *continuously differentiable for every* $t \in [0,T]$.

Then for every initial set $K \in \mathcal{K}(\mathbb{R}^N)$, *the following statements are equivalent* :

(i) *For all* $t \in [0,T]$,
$$M_t^{\mapsto}(K) := \Big\{ (y(t),\, q(t)) \;\Big|\; (y(\cdot),\, q(\cdot)) \text{ solves system } (*),\;\; y_0 \in K \Big\}$$
is the graph of a locally Lipschitz continuous function,

(ii) *For any solution* $(y(\cdot), q(\cdot)) : [0,T] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ *of the initial value problem* $(*)$ *and each cluster point* $Q_0 \in \operatorname{Limsup}_{z \to y_0} \{\nabla \psi(z)\}$, *the following matrix Riccati equation has a solution* $Q(\cdot)$ *on* $[0,T]$

$$\wedge \begin{cases} \partial_t Q + \frac{\partial^2 H}{\partial p\, \partial x}(t,\, y(t),\, q(t))\; Q + Q\; \frac{\partial^2 H}{\partial x\, \partial p}(t,\, y(t),\, q(t)) \\ \qquad + Q\; \frac{\partial^2 H}{\partial p^2}(t,\, y(t),\, q(t))\; Q + \frac{\partial^2 H}{\partial x^2}(t,\, y(t),\, q(t)) = 0, \\ \qquad Q(0) = Q_0. \end{cases}$$

If one of these equivalent properties is satisfied and if ψ *is (continuously) differentiable, then* $M_t^{\mapsto}(K)$ *is even the graph of a (continuously) differentiable function.*

Proof is given in [28, Frankowska 2002], Theorem 5.3 for the same Hamiltonian system but with $y(T) = y_T$, $q(T) = q_T$ given. So this lemma is an immediate consequence considering $-H(T - \cdot, \cdot, \cdot)$ and $(y(T - \cdot),\, q(T - \cdot))$. $\square$

For preventing singularities of $Q(\cdot)$, the following comparison principle provides a bridge to solutions of a *scalar* Riccati equation.

Lemma A.12 (Comparison theorem for the matrix Riccati equation, [37, Royden 88], Theorem 2**)**

Let $A_j, B_j, C_j : [0,T[\longrightarrow \mathbb{R}^{N,N}$ $(j = 0,1,2)$ *be bounded continuous matrix–valued functions such that each* $M_j(t) := \begin{pmatrix} A_j(t) & B_j(t) \\ B_j(t)^T & C_j(t) \end{pmatrix}$ *is symmetric.*
Assume that $U_0,\ U_2 : [0,T[\longrightarrow \mathbb{R}^{N,N}$ *are solutions of the matrix Riccati equation*

$$\tfrac{d}{dt}\, U_j \;=\; A_j \,+\, B_j\, U_j \,+\, U_j\, B_j^T \,+\, U_j\, C_j\, U_j$$

with $M_2(\cdot) \geq M_0(\cdot)$ *(i.e.* $M_2(t) - M_0(t)$ *is positive semi–definite for every* t*).*

Then, given symmetric $U_1(0) \in \mathbb{R}^{N,N}$ *with*

$$U_2(0) \;\geq\; U_1(0) \;\geq\; U_0(0), \qquad M_2(\cdot) \;\geq\; M_1(\cdot) \;\geq\; M_0(\cdot),$$

there exists a solution $U_1 : [0,T[\longrightarrow \mathbb{R}^{N,N}$ *of the corresponding Riccati equation with matrix* $M_1(\cdot)$*. Moreover,* $U_2(t) \;\geq\; U_1(t) \;\geq\; U_0(t)$ *for all* $t \in [0,T[$. $\square$

Remark A.13 In [13], Cannarsa and Frankowska prove different sufficient conditions on the positive erosion of reachable sets (called the *interior sphere property* there). Considering a control system, their main result is

> **Proposition** *Let a map* $f : \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$ *be given where* $U \subset \mathbb{R}^N$ *is compact. Assume*
>
> 1. $F(x) := f(x,U)$ *is convex for every* $x \in \mathbb{R}^N$*;*
> 2. f *is continuous and there exists* $L_0 > 0$ *with*
> $|f(x,u) - f(y,u)| \;\leq\; L_0\, |x-y|$ *for all* $x, y \in \mathbb{R}^N,\ u \in U$*;*
> 3. $f(\cdot,u)$ *is differentiable for every* $u \in U$ *and there is* $L_1 > 0$ *with*
> $|D_x\, f(x,u) - D_x\, f(y,u)| \;\leq\; L_1\, |x-y|$ *for all* $x, y \in \mathbb{R}^N,\ u \in U$
> *where* $D_x\, f$ *denotes the Jacobian matrix of* $f(x,u)$ *w.r.t.* x*;*
> 4. *there exist an open set* $\mathcal{O} \subset \mathbb{R}^N$ *and numbers* $r, R > 0$ *such that for every* $x \in \mathcal{O}$*,* $F(x)$ *has positive erosion of radius* r *and* $\mathbb{B}_R \subset \mathcal{O}$*;*
> 5. *there are a radius* $r_1 \in\,]0, \frac{r}{2\,L_0}]$ *and a constant* $C_0 > 0$ *such that*
> $|\nabla\, b_{F(x)}(v) \;-\; \nabla\, b_{F(y)}(v)| \;\leq\; C_0\, |x-y|$ *for all* $x \in \mathcal{O},\ v \in \partial F(x)$
> $y \in \mathcal{O} \cap \mathbb{B}_{r_1}(x)$
> *with the signed distance* $b_M := \mathrm{dist}(\,\cdot\,, M) - \mathrm{dist}(\,\cdot\,, \mathbb{R}^N \setminus M)$*;*
> 6. *set* $H_0 := \max\limits_{u \in U} |f(0,u)|, \quad T_R := \frac{1}{L_0} \cdot \log\Big(1 + \frac{L_0\, R}{H_0}\Big).$
>
> *Then for every* $T \in\,]0, T_R[$*, the reachable set* $\vartheta_{f(\,\cdot\,,U)}(T,\{0\})$ *has positive erosion of radius*
> $$\sigma(T) \;\geq\; \frac{e^{-L_0\, T}}{2} \,\cdot\, \min\Big\{ r_1,\;\; R - \frac{H_0}{L_0}\,(e^{L_0\, T} - 1),\;\; \frac{r \cdot e^{-2\,L_0\, T}}{1 + L_0\, T + r\, C_0\, T + r\, L_1\, T^2}\; T \Big\}.$$

The proof of this proposition is based on the notions that for every point y of the boundary $\partial\vartheta_{f(\cdot,U)}(T, \{0\})$, is related with an adjoint arc $p(\cdot) \neq 0$ due to the Pontryagin Maximum Principle and the closed ball at $y - \sigma(T)\,\frac{p(T)}{|p(T)|}$ with radius $\sigma(T)$ can be reached from 0 along trajectories of the control system (by "perturbing" the control leading to y). Verifying this property in detail, Cannarsa and Frankowska follow an idea completely different from the proof of Proposition A.9.

The assumptions of the quoted proposition do not use the Hamiltonian $\mathcal{H}_{f(\cdot,U)}$ explicitly. At first glance, they make a weaker impression than standard hypothesis $(\mathcal{H}^{\varrho}_{\circ})$ (with its *twice* continuous differentiability of $\mathcal{H}_{f(\cdot,U)}$). In particular, the Lipschitz continuity in condition (3.) is referring only to the first argument of f (and not to the control u) :

$$|D_x f(x,u) \;-\; D_x f(y,u)| \;\leq\; L_1\,|x-y|.$$

On the other hand, assumption (5.) is usually not easy to verify in examples. Furthermore, the gradient of the signed distance $b_{F(y)}$ describes the direction of projection on the boundary $\partial F(y)$. For $v \notin \partial F(y)$ however, there is no obvious relation between $\nabla b_{F(y)}(v)$ and $\mathcal{H}_{f(\cdot,U)}(y,\cdot)$ (or its derivatives). So it is not clear whether standard hypothesis $(\mathcal{H}^{\varrho}_{\circ})$ implies the assumptions of the quoted proposition immediately.

In this paper, we prefer assumptions about the Hamiltonian functions since basically speaking, they provide information about boundary trajectories and their adjoint arcs without taking the corresponding controls into consideration explicitly. In particular, the Hamilton condition of Proposition 4.12 then provides the estimate of Lemma 4.19 that we need for forward transitions on $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}_{C^{1,1}}(\mathbb{R}^N),\ q_{\mathcal{K},N})$.

According to [13], Corollary 3.11, the boundary $\partial\,\vartheta_{f(\cdot,U)}(T,\,K)$ is $C^{1,1}$ if in addition to the quoted proposition, both the closed set $K \subset \mathcal{O}$ and each value $f(x,U)$ $(x \in \mathcal{O})$ are a–regular (with some fixed $a > 0$). It is easy, however, to show that an a–regular set is uniformly convex and thus, the corollary does not imply the preceding results about preserving smooth boundaries shortly (see Propositions 4.17, A.2).

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