Differential Equations and Finite Groups

dedicated to B. Fischer on the occasion
of his 70th birthday

B. H. Matzat

November 25, 2005

Introduction

This note is devoted to linear differential equations with finite Galois groups. It is a famous conjecture due to A. Grothendieck that the finiteness of the differential Galois group should be equivalent to the triviality of the p-curvature for almost all p (see for example [8], [9]). The p-curvature is just the first integrability obstruction for the reduced differential equation in characteristic p. In the case all such integrability obstructions vanish in characteristic p we obtain a so-called iterative differential equation or iterative differential module, respectively. For these a nice Picard-Vessiot theory has been developed by M. van der Put and the author ([12],[13]). In particular, the differential Galois groups are linear algebraic groups and there is a Galois correspondence.

Thus a natural question arises, whether there exists a reasonable reduction theory preserving Galois groups etc. The corresponding objects in characteristic zero are iterative differential modules over iterative differential rings. The latter are suitable Dedekind subrings of algebraic function fields over number fields, here called global differential rings. These and the corresponding global differential modules are studied in Chapter 1. Chapter 2 presents the construction of global iterative Picard-Vessiot rings (PV-rings) over global differential rings and proves that such PV-rings are generated by globally bounded power series as introduced by G. Christol. In Chapter 3 the reduction of global differential modules and their PV-rings is studied. The main result is that a global iterative PV-ring in characteristic zero is algebraic if and only if for almost all primes p the reduced PV-ring is algebraic. Moreover, for almost all p the reduced PV-ring and the PV-ring of the modulo p reduced global differential module coincide. This proves the last conjecture stated in [12].

According to Grothendieck's p-curvature conjecture all global iterative PV-rings are algebraic. Using the result above, this fact might be proven directly. This would already imply a nice algebraicity criterion for formal power series over number fields used by G. Eisenstein ([6], see also [2]) and could become a significant step towards the proof of Grothendieck's conjecture.

Acknowledgements I would like to thank J. Hartmann and A. Röscheisen for helpful discussions on topics of the paper.

1 Global Differential Modules

1.1 Global Differential Rings

Let K be a number field (of finite degree over $\mathbb{Q}$) and let $\mathbb{P}_K$ be the set of primes or finite places of K, respectively. Then every $\mathfrak{p} \in \mathbb{P}_K$ defines a nonarchimedean valuation $|\cdot|_\mathfrak{p}$ on K with valuation ring $\mathcal{O}_\mathfrak{p}$, valuation ideal $\mathcal{P}_\mathfrak{p}$ (or $\mathfrak{p}$ for short) and with residue field $\mathcal{K}_\mathfrak{p} := \mathcal{O}_\mathfrak{p}/\mathfrak{p}$. In the case $\mathbb{S}_K \subseteq \mathbb{P}_K$ is a finite subset of places we use the notation $\mathbb{P}'_K := \mathbb{P}_K \setminus \mathbb{S}_K$ and call the Dedekind ring

$$\mathcal{O}'_K := \mathcal{O}_{\mathbb{S}_K} := \bigcap_{\mathfrak{p} \in \mathbb{P}'_K} \mathcal{O}_\mathfrak{p} \subseteq K \tag{1.1}$$

a *global ring*.

Now let F/K be a function field of one variable and $t \in F$ transcendental over K. Then $F/K(t)$ is a finite extension. By extending the derivation $\partial_t := \frac{d}{dt}$ from $K(t)$ to F, the field F becomes a differential field (F, ∂_F). Moreover, every place $\mathfrak{p} \in \mathbb{P}_K$ can be uniquely extended to a place $\mathfrak{P}$ or a valuation $|\cdot|_\mathfrak{P}$ of $K(t)$, respectively, by assuming

$$\big|\sum_{i=0}^{n} a_i t^i\big|_\mathfrak{P} = \max\{|a_i|_\mathfrak{p} \big| i = 0, \ldots, n\} \tag{1.2}$$

(Gauß extension). The set of places $\mathfrak{P}_F$ of F lying over any such Gauß extension $\mathfrak{P}$ of $\mathfrak{p} \in \mathbb{P}_K$ is denoted by

$$\mathbb{P}_F := \mathbb{P}_{t,F} := \{\mathfrak{P}_F \mid \mathfrak{P}_F|_{K(t)} = \mathfrak{P} \quad \text{Gauß place over} \quad \mathfrak{p} \in \mathbb{P}_K\}. \tag{1.3}$$

and is called the set of *t-extensions of* $\mathbb{P}_K$. (In [10] this set is referred to as the set of t-functional primes of F/K). Likewise we use the notation

$$\mathbb{S}_F := \{\mathfrak{P}_F \in \mathbb{P}_F \mid \mathfrak{P}_F|_K = \mathfrak{p} \in \mathbb{S}_K\} \tag{1.4}$$

and $\mathbb{P}'_F := \mathbb{P}_F \setminus \mathbb{S}_F$. Then the intersection

$$\mathcal{O}'_F := \mathcal{O}_{\mathbb{S}_F} := \bigcap_{\mathfrak{P}_F \in \mathbb{P}'_F} \mathcal{O}_{\mathfrak{P}_F} \subseteq F \tag{1.5}$$

again is a Dedekind ring.

Throughout this note a subring $\mathcal{O}'_F$ of F with non trivial derivation $\partial_F|_{\mathcal{O}'_F}$ is called a *global differential ring* (global D-ring) if

$$\partial_F(\mathcal{O}'_F) \subseteq \mathcal{O}'_F \quad \text{and} \quad \partial_F(\mathfrak{P}_F) \subseteq \mathfrak{P}_F \quad \text{for all} \quad \mathfrak{P}_F \in \mathbb{P}'_F \tag{1.6}$$

where ∂_F is the given derivation on F. Further, $\mathcal{O}'_F$ is called a *global iterative differential ring* (global ID-ring) if the conditions (1.6) are satisfied for the iterative derivation induced by ∂_F, i.e., if for all higher derivations $\partial_F^{(k)} := \frac{1}{k!}\partial^k$

$$\partial_F^{(k)}(\mathcal{O}'_F) \subseteq \mathcal{O}'_F \quad \text{and} \quad \partial_F^{(k)}(\mathfrak{P}_F) \subseteq \mathfrak{P}_F \quad \text{for all} \quad \mathfrak{P}_F \in \mathbb{P}'_F \quad \text{and} \quad k \in \mathbb{N}. \tag{1.7}$$

In the following the family of higher derivations is abbreviated by $\partial_F^* := \left(\partial_F^{(k)}\right)_{k \in \mathbb{N}}$ and accordingly a global ID-ring is denoted by $(\mathcal{O}'_F, \partial_F^*)$.

1.2 Extensions of Global Differential Rings

The following proposition shows that global D-rings behaves well under unramified extensions.

Proposition 1.1. *Let K be a number field, let F/K be a function field of one variable with derivation ∂_F and let $(L,\partial_L)/(F,\partial_F)$ be a finite extension of differential fields. Suppose $\mathcal{O}'_F \subseteq F$ is a global (iterative) D-ring and set*

$$\mathcal{O}'_L := \bigcap_{\mathfrak{P}\in\mathbb{P}'_L} \mathcal{O}_{\mathfrak{P}} \quad \textit{with} \quad \mathbb{P}'_L := \{\mathfrak{P}\in\mathbb{P}_L \mid \mathfrak{P}|_F \in \mathbb{P}'_F\}. \tag{1.8}$$

(a) *If $\mathcal{O}'_L/\mathcal{O}'_F$ is unramified, i.e., if every $\mathfrak{P}\in\mathbb{P}'_L$ is unramified in L/F, then $(\mathcal{O}'_L,\partial_L)$ is a global (iterative) D-ring.*

(b) *If in addition L/F is a Galois extension of fields, then $\mathcal{O}'_L/\mathcal{O}'_F$ is a Galois extension of rings with* $\mathrm{Gal}(\mathcal{O}'_L/\mathcal{O}'_F)\cong\mathrm{Gal}(L/F)$.

Proof. For (a) it is enough to show that the assertion holds locally, i.e., for all completions $\hat{\mathcal{O}}_{\mathfrak{P}_L}/\hat{\mathcal{O}}_{\mathfrak{P}_F}$ where $\mathfrak{P}_L\in\mathbb{P}'_L$ and $\mathfrak{P}_F := \mathfrak{P}_L|_F$ with continuously extended derivations $\hat\partial_L$ and $\hat\partial_F$. Then with $\hat{\mathcal{O}}_L := \hat{\mathcal{O}}_{\mathfrak{P}_L}$, $\hat{\mathcal{O}}_F := \hat{\mathcal{O}}_{\mathfrak{P}_F}$, the extension $(\hat{\mathcal{O}}_L,\hat\partial_L)/(\hat{\mathcal{O}}_F,\hat\partial_F)$ is an extension of local rings as studied in [11].

By the assumptions there exists an element $y\in\hat{\mathcal{O}}_L$ with $\hat{\mathcal{O}}_L = \hat{\mathcal{O}}_F[y]$ ([15] §6, Prop. 12). Denoting by $f(X)=\sum_{i=0}^n a_iX^i$ the minimal polynomial of y over $\hat{\mathcal{O}}_F$, we obtain the identity

$$0=\hat\partial_L(f(y)) = \hat\partial_F(f)(y) + \partial_X(f)(y)\hat\partial_L(y) \tag{1.9}$$

with $\partial_X(f)(y)\in\hat{\mathcal{O}}_L^\times$ (by [15], §6, Cor. 2). Thus $\hat\partial_L(y)$ belongs to $\hat{\mathcal{O}}_L$ since $\hat\partial_F(f)(y)\in\hat{\mathcal{O}}_L$. This entails $\hat\partial_L(\hat{\mathcal{O}}_L)\subseteq\hat{\mathcal{O}}_L$ and $\hat\partial_L(\hat{\mathfrak{P}}_L)\subseteq\hat{\mathfrak{P}}_L$ by [11], Prop. 1.1.

For the iterative part we use induction on k: By the induction hypothesis, in the identity

$$0=\hat\partial_L^{(k)}(f(y)) = \sum_{i=0}^n\sum_{l=0}^k \hat\partial_F^{(k-l)}(a_i)\hat\partial_L^{(l)}(y^i) \tag{1.10}$$

all terms with $l<k$ belong to $\hat{\mathcal{O}}_L$. Thus

$$\sum_{i=0}^n a_i\hat\partial_L^{(k)}(y^i) = \sum_{i=0}^n a_i \sum_{\sum l_j=k}\left(\prod_{j=1}^i \hat\partial_L^{(l_j)}(y)\right)\in\hat{\mathcal{O}}_L. \tag{1.11}$$

Here again all terms with $l_j<k$ belong to $\hat{\mathcal{O}}_L$. We therefore conclude that

$$\sum_{i=0}^n a_i\, i\, y^{i-1}\hat\partial_L^{(k)}(y) = \hat\partial_X(f)(y)\;\hat\partial_L^{(k)}(y)\in\hat{\mathcal{O}}_L \tag{1.12}$$

which leads to $\hat\partial_L^{(k)}(y)\in\hat{\mathcal{O}}_L$ as above and hence $\hat\partial_L^{(k)}(\hat{\mathcal{O}}_L)\subseteq\hat{\mathcal{O}}_L$.

In order to verify $\hat{\partial}_L^{(k)}(\hat{\mathfrak{P}}_L) \subseteq \hat{\mathfrak{P}}_L$, for a $y \in \hat{\mathfrak{P}}_L$ we can use an element $z \in \hat{\mathfrak{P}}_F$ with $x \cdot z = y$ and $x \in \hat{\mathcal{O}}_L^\times$. Then from

$$\hat{\partial}_L^{(k)}(y) = \hat{\partial}_L^{(k)}(x \cdot z) = \sum_{l=0}^{k} \hat{\partial}_L^{(k-l)}(x)\, \hat{\partial}_F^{(l)}(z) \tag{1.13}$$

we obtain $\hat{\partial}_L^{(k)}(y) \in \hat{\mathfrak{P}}_L$ since $\hat{\partial}_F^{(l)}(z) \in \hat{\mathfrak{P}}_F$ for all $l \in \mathbb{N}$ by assumption. This proves (a).

In case L/F is a finite Galois extension of fields with $\mathrm{Gal}(L/F) = G$, by its construction the global D-ring $\mathcal{O}'_L$ is G-stable with ring of invariants $(\mathcal{O}'_L)^G = \mathcal{O}'_F$. Since moreover $\mathcal{O}'_L/\mathcal{O}'_F$ is unramified, $\mathcal{O}'_L/\mathcal{O}'_F$ is a Galois extension of (differential) rings by Definition 4.2.1 in [7] (including Remark (4)) with $\mathrm{Gal}(\mathcal{O}'_L/\mathcal{O}'_F) = G$. Thus global D-rings fit well into the concept of Galois ring extensions.

□

The most basic global (iterative) D-rings are the subring

$$\begin{aligned} \mathbb{Z}(t) &:= \{\frac{g}{h} \mid g, h \in \mathbb{Z}[t], \mathrm{cont}(h) = 1\} \\ &= \bigcap_{\mathfrak{P} \in \mathbb{P}_{\mathbb{Q}(t)}} \mathcal{O}_{\mathfrak{P}} \quad \text{with} \quad \mathbb{P}_{\mathbb{Q}(t)} = \mathbb{P}_{t,\mathbb{Q}(t)} \end{aligned} \tag{1.14}$$

of $\mathbb{Q}(t)$ and its localisations $\mathbb{Z}(t)_{\mathbb{S}}$ for $\mathbb{S} \subseteq \mathbb{P}_{\mathbb{Q}(t)}$ finite. Applying Proposition 1.1(a) then shows that every function field F of one variable over a number field K contains (iterative) D-subrings $\mathcal{O}'_F$ with $\mathrm{Quot}(\mathcal{O}'_F) = F$ and which are unramified over some $\mathbb{Z}(t)_{\mathbb{S}}$.

1.3 Integral Global Differential Modules

Now we start with a global D-ring $(\mathcal{O}'_F, \partial_F)$ as defined in Section 1.1. A free $\mathcal{O}'_F$-module M of finite rank is called a *global differential module* (global D-module) over $\mathcal{O}'_F$ if there exists a derivation

$$\partial_M : M \to M \quad \text{with} \quad \partial_M(a \cdot x) = \partial_F(a) \cdot x + a \cdot \partial_M(x) \quad \text{for} \quad a \in \mathcal{O}'_F,\ x \in M. \tag{1.15}$$

A global D-module (M, ∂_M) over $\mathcal{O}'_F$ is called *integral* (or *locally bounded*) if in addition for all $\mathfrak{P} \in \mathbb{P}'_F$ and all $l \in \mathbb{N}$ there exists an $\mathcal{O}'_F$-basis $B_{\mathfrak{P}}^{(l)}$ of M with

$$\partial_M(B_{\mathfrak{P}}^{(l)}) \subseteq \mathfrak{P}^l \cdot M. \tag{1.16}$$

This definition extends the definition of an integral (or bounded) local D-module given in [11] in a natural way to the global case.

A global D-module (M, ∂_M) over a global ID-ring $(\mathcal{O}'_F, \partial_F^*)$ is called a *global iterative differential module* (global ID-module) over $\mathcal{O}'_F$ if, besides ∂_M, the full family of higher derivations $\partial_M^* = \left(\partial_M^{(k)}\right)_{k \in \mathbb{N}}$, where $\partial_M^{(k)} := \frac{1}{k!}\partial_M^k$, maps M into itself. This defines an iterative derivation

$$\partial_M^{(k)} : M \to M,\ x \mapsto \partial_M^{(k)}(x) = \frac{1}{k!}\partial_M^k(x) \quad \text{for all} \quad k \in \mathbb{N}. \tag{1.17}$$

The next theorem shows that if $\mathcal{O}'_F$ carries the structure of an ID-ring, the notions of an integral (locally bounded) and an iterative D-module coincide.

Theorem 1.2. *Let (M, ∂_M) be a global D-module over a global ID-ring $(\mathcal{O}'_F, \partial_F^*)$. Then the following are equivalent:*

(a) *(M, ∂_M) is integral (locally bounded),*

(b) *(M, ∂_M^*) is an iterative differential module.*

Proof. The equivalence of (a) and (b) holds locally for p-adic D-modules by [12], Prop. 8.1. Thus Theorem 1.2 is true for the D-modules $(M \otimes_{\mathcal{O}_F} \hat{\mathcal{O}}_{\mathfrak{P}}, \partial_M \otimes \hat{\partial}_F)$ for all $\mathfrak{P} \in \mathbb{P}'_F$. But this implies the global equivalence of (a) and (b).

□

2 Global Picard-Vessiot Rings

2.1 Definition of Global PV-Rings

First we have to adapt the definition of a Picard-Vessiot ring to our situation: Let (M, ∂_M) be a global D-module over a global D-ring $(\mathcal{O}'_F, \partial_F)$, where the derivation of $M/\mathcal{O}'_F$ is given by a matrix $A \in (\mathcal{O}'_F)^{m \times m}$ with respect to some $\mathcal{O}'_F$-basis $B = \{b_1, \ldots, b_m\}$ of M. Assume K is the field of constants of F and $\mathcal{O}'_K = \mathcal{O}'_F \cap K$.

A D-ring $(R, \partial_R) \geq (\mathcal{O}'_F, \partial_F)$ is called a *pseudo Picard-Vessiot ring* (or PPV-ring for short) for M over $\mathcal{O}'_F$ if

(1) $R/\mathcal{O}'_F$ is a $\mathcal{O}'_F$-simple D-ring, i.e., R does not contain proper differential ideals P with $P \cap \mathcal{O}'_F = (0)$,

(2) there exists a fundamental solution matrix $Y = (y_{ij})_{i,j=1}^m \in \mathrm{GL}_m(R)$ with $\partial_R(Y) = A \cdot Y$,

(3) $R/\mathcal{O}'_F$ is generated by the coefficients y_{ij} of Y for $i, j = 1, \ldots, m$ and $\det(Y)^{-1}$,

(4) $R/\mathcal{O}'_F$ does not contain new constants, i.e., the ring of differential constants K_R of R coincides with $\mathcal{O}'_K$.

A PPV-ring (R, ∂_R) is called a *Picard-Vessiot ring* (or PV-ring) for M over $\mathcal{O}'_F$ if in addition

(5) the ring of invariants of R under the group of differential automorphisms $\mathrm{Aut}_{\mathrm{D}}(R/\mathcal{O}'_F)$ equals $\mathcal{O}'_F$.

In the classical case where the ring of differential constants is an algebraically closed field, condition (5) follows from conditions (1)-(4). In the more general situation here the notion of a PV-ring is indeed stronger than the notion of a PPV-ring (compare [4] for examples).

In order to study first examples of global PV-rings, let (F, ∂_F) be a function field of one variable over a number field K with derivation ∂_F whose field of differential constants coincides with K. Further let L/F be a finite Galois extension with group G without new constants and let ∂_L be the unique extension of ∂_F to L. Then inside L we can find a G-stable global D-Ring $\mathcal{O}'_L$ unramified over $\mathcal{O}'_F := \mathcal{O}'_L \cap F$ such that in addition $\mathcal{O}'_F$

is a principal ideal domain (by choosing the set $\mathbb{S}_L$ of exceptional primes big enough). Then by Proposition 1.1(b), $\mathcal{O}'_L/\mathcal{O}'_F$ is a differential Galois extension of rings with group G. Since by construction $\mathcal{O}'_L$ is a free $\mathcal{O}'_F$-module of rank $m := \#G$, its dual M is a D-module over $\mathcal{O}'_F$ with the dual derivation ∂_M. More precisely, if $y_1, \ldots, y_m$ is an $\mathcal{O}'_F$-basis of $\mathcal{O}'_L$ with $\partial_L(y_i) = \sum_{j=1}^m a_{ij} y_j$ the derivation ∂_M of M with respect to the dual basis $B = \{b_1, \ldots, b_m\}$ is given by $\partial_M(b_1, \ldots, b_m) = -(b_1, \ldots, b_m) \cdot A$ with $A = (a_{ij})_{i,j=1}^m$. Then obviously $(\mathcal{O}'_L, \partial_L)$ is a D-ring containing a solution space of M and fulfilling conditions (1)-(4). Thus $(\mathcal{O}'_L, \partial_L)$ is a PPV-ring over $\mathcal{O}'_F$ for M. Since moreover all $\gamma \in G$ commute with ∂_L, G is the group of differential automorphisms $\mathrm{Aut}_{\mathrm{D}}(\mathcal{O}'_L/\mathcal{O}'_F)$ and $(\mathcal{O}'_L)^G = \mathcal{O}'_F$ by construction. Hence $\mathcal{O}'_L/\mathcal{O}'_F$ is a PV-extension with the finite Galois group $G \cong \mathrm{Gal}(L/F)$. Obviously all finite groups arise as differential Galois groups of global PV-extensions in this way.

2.2 Construction of Global PV-Rings

Now we want to prove that for any global (iterative) D-module (M, ∂_M) with a non singular point of degree one over a global (iterative) D-ring $(\mathcal{O}'_F, \partial_F)$ there exist PPV-extensions $R/\mathcal{O}'_F$ for M, where, in addition, (R, ∂_R) is an iterative D-ring in the iterative case.

Here a point of degree one in $\mathcal{O}'_F$ means a place $\wp$ of $\mathcal{O}'_F/\mathcal{O}'_K$ unramified in $\mathcal{O}'_F/\mathcal{O}'_{K(t)}$ and of residue degree one over $\mathcal{O}'_K$. Then the completion $\hat{F}_\wp$ coincides with the completion $\widehat{K(t)}_{(t-c)}$ for some place $(t-c)$ of degree one (or (t^{-1}) resp.) in $\mathcal{O}'_{K(t)}$. Thus $(t-c)$ is a local parameter for $\wp$ and $\hat{F}_\wp \cong K((t-c))$. Further we have a Taylor map from the valuation ring $\mathcal{O}'_\wp$ of $\wp$ inside $\mathcal{O}'_F$

$$\tau_\wp : \mathcal{O}'_\wp \longrightarrow K((t-c)), \quad f \longmapsto \sum_{k \in \mathbb{N}} \left(\partial_F^{(k)}(f)\right)(\wp)(t-c)^k \tag{2.1}$$

which uniquely extends to $\mathcal{O}'_F$ and defines a differential monomorphism $\tau_\wp : \mathcal{O}'_F \longrightarrow K((t-c))$ over $\mathcal{O}'_K$. The point $\wp$ is called non singular if it is a regular point for the D-module $M \otimes_{\mathcal{O}'_K} \bar{K}$ over an algebraic closure $\bar{K}$ of K, i.e., if M becomes trivial over $\bar{K}((t-c))$. Obviously any global D-module M over a global D-ring $\mathcal{O}'_F$ inside a rational function field $F = K(t)$ has infinitely many non singular points of degree one.

Theorem 2.1. *Let (M, ∂_M) be a global (iterative) D-module of rank m over a global (iterative) D-ring $(\mathcal{O}'_F, \partial_F)$ with ring of differential constants $\mathcal{O}'_K$. Let further $\wp$ be a non singular point for M of degree one.*

(a) There exists a PPV-ring (R, ∂_R) over $\mathcal{O}'_F$ for M with fundamental solution matrix $Y \in \mathrm{GL}_m(R)$ which satisfies $Y(\wp) \in \mathrm{GL}_m(\mathcal{O}'_K)$. In the iterative case R in addition is equipped with an iterative derivation ∂_R^*.

(b) The property $Y(\wp) \in \mathrm{GL}_m(\mathcal{O}'_K)$ uniquely determines the PPV-ring (R, ∂_R) or (R, ∂_R^*), respectively, up to (iterative) differential isomorphisms.

Proof. Let $A \in (\mathcal{O}'_F)^{m \times m}$ be a representing matrix of ∂_M with respect to some $\mathcal{O}'_F$-basis of B of M. The construction of R follows the general line explained for example in [14],Ch. 1.3. Let

$$U := \mathcal{O}'_F[\mathrm{GL}_m] = \mathcal{O}'_F[x_{ij}, \det(x_{ij})^{-1}]_{i,j=1}^m \tag{2.2}$$

be the coordinate ring of the general linear group GL_m over $\mathcal{O}'_F$. Since the elements x_{ij} are algebraically independent over $\mathcal{O}'_F$ for $i,j = 1,\cdots,m$, the ring U becomes a D-ring (U,∂_U) when we define

$$\partial_U(X) = A \cdot X \quad \text{for} \quad X = (x_{ij})_{i,j=1}^m. \tag{2.3}$$

In case both $\mathcal{O}'_F$ and M are equipped with an iterative derivation, all matrices $A^{(k)}$ for $\partial_M^{(k)}$ belong to $(\mathcal{O}'_F)^{m\times m}$ and thus define an iterative derivation $\partial_U^* = (\partial_U^{(k)})_{k\in\mathbb{N}}$ on U. If P is a maximal differential ideal in U with $P \cap \mathcal{O}'_F = (0)$, the quotient ring $R := U/P$ becomes an $\mathcal{O}'_F$-simple (iterative) D-ring containing a fundamental solution matrix $Y = (y_{ij})_{i,j=1}^m$ with the images y_{ij} of x_{ij} in R. Obviously, $R/\mathcal{O}'_F$ is generated by the entries y_{ij} of Y and by $\det(Y^{-1})$.

In order to find such a ring R without new constants we have to be more specific. By assumption there exists a non singular point $\wp$ of degree one for M with local parameter $(t-c) \in \mathcal{O}'_F$. First we want to extend the Taylor map $\tau_\wp : \mathcal{O}'_F \longrightarrow K((t-c))$ to U. For this purpose we choose initial values $x_{ij}(\wp) \in \mathcal{O}'_K$ with $X(\wp) = (x_{ij}(\wp))_{i,j=1}^m \in \mathrm{GL}_m(\mathcal{O}'_K)$, for example $x_{ij}(\wp) = \delta_{ij}$ (Kronecker delta). Then from $\partial_U^{(k)}(X) = A^{(k)} \cdot X$ we obtain $(\partial_U^{(k)}(x_{ij}))(\wp) \in K$ for all $k \in \mathbb{N}$ by recursion. Since $U/\mathcal{O}'_F$ is generated by the x_{ij}, this leads to an extension

$$\tau_\wp : U \longrightarrow K((t-c)), \quad x_{ij} \longmapsto \sum_{k\in\mathbb{N}} \Big(\partial_U^{(k)}(x_{ij})\Big)(\wp)(t-c)^k \tag{2.4}$$

of $\tau_\wp$ depending on the matrix of initial values $X(\wp)$ at $\wp$. By construction, $\tau_\wp$ is a differential homomorphism whose image in $K((t-c))$ is a $\tau_\wp(\mathcal{O}'_F)$-simple D-ring generated by $\tau_\wp(x_{ij})$. Thus the kernel of $\tau_\wp$ defines a maximal differential ideal $P \trianglelefteq U$ with $P \cap \mathcal{O}'_F = (0)$ and hence $R := U/P$ gives one of the rings constructed above (with P depending on $X(\wp)$). Since the Taylor map $\tau_\wp$ factors over R we have a Taylor map

$$\tau_\wp : R \longrightarrow K((t-c)), \quad y_{ij} \longmapsto \sum_{k\in\mathbb{N}} \Big(\partial_R^{(k)}(y_{ij})\Big)(\wp)(t-c)^k \tag{2.5}$$

(still depending on $X(\wp)$). Obviously, the image of the ring of differential constants K_R of R lies inside K. Because of $R \cap F = \mathcal{O}'_F$, this implies $K_R \cong \tau_\wp(K_R) = \mathcal{O}'_K$, proving (a).

For (b), assume we have two fundamental solution matrices $Y, \tilde{Y}$ of M in some PPV-rings R and $\tilde{R}$ of M, respectively, with $Y(\wp) \in \mathrm{GL}_m(\mathcal{O}'_K)$ and $\tilde{Y}(\wp) \in \mathrm{GL}_m(\mathcal{O}'_K)$. By the general theory of PV-rings there exists a matrix $C \in \mathrm{GL}_m(\bar{K})$ such that $\tilde{Y} = Y \cdot C$. By specializing $t \mapsto c$ we obtain $\tilde{Y}(\wp) = Y(\wp) \cdot C$ showing $C \in \mathrm{GL}_m(\mathcal{O}'_K)$. Thus R and $\tilde{R}$ are (iterative) differentially isomorphic over $\mathcal{O}'_K$.

□

The unique *normalized* PPV-*ring* for M constructed in Theorem 2.1 will be denoted by R_M in the sequel.

Corollary 2.2. *If in Theorem 2.1 (M,∂_M^*) is an ID-module over an ID-ring $(\mathcal{O}'_F,\partial_F^*)$, then the solution space $V = \mathcal{O}'_K\langle y_{ij} | i,j = 1,...,m\rangle$ of M inside R_M has the property*

$$\tau_\wp(V) \le \mathcal{O}'_K[[t-c]]. \tag{2.6}$$

Proof. We only have to observe that in this case aside from $A(\wp) := \tau_\wp(A)|_{t=c}$, all matrices $A^{(k)}(\wp) := \tau_\wp(A^{(k)})|_{t=c}$ belong to $(\mathcal{O}'_K)^{m\times m}$.

□

2.3 Globally Bounded PV-rings

First let us recall the following definition introduced by G. Christol ([1], Ch. 4.1): A formal power series $y \in K[[t]]$ over a number field K is *globally bounded* if

(1) for all $\mathfrak{p} \in \mathbb{P}^*_K$ the $\mathfrak{p}$-adic radius of convergence of y at 0 is positive and

(2) there exists a positive number $n \in \mathbb{N}$ such that $y \in \mathcal{O}_K[\frac{1}{n}][[t]]$.

Here $\mathbb{P}^*_K$ is the union of the set of finite places $\mathbb{P}_K$ and the set of archimedean places of K.

In the following we call a global D-module $M/\mathcal{O}'_F$ and its PPV-ring $R_M/\mathcal{O}'_F$, respectively, *globally bounded* if there exists a non singular point $\wp$ of degree one for M with local parameter $(t-c)$ such that some solution space $\tau_\wp(V)$ of M inside $K[[t-c]]$ is generated by globally bounded power series in $(t-c)$. Obviously, this definition does not depend on the chosen non singular point and on the specific matrix of initial values $Y(\wp) \in \mathrm{GL}_m(\mathcal{O}'_K)$.

Theorem 2.3. *Let (M, ∂_M) be a global D-module over a global ID-field (F, ∂^*_F). Then the folllowing are equivalent:*

(a) *(M, ∂_M) is globally bounded,*

(b) *M is a global ID-module over a suitable global ID-ring $(\mathcal{O}'_F, \partial^*_F)$ inside F.*

Proof. By [3], Thm. 6.3 (or [11], Thm. 2.1, respectively) solutions of an integral (bounded) p-adic differential module have radius of convergence $r_\mathfrak{p} = 1$. By [11], Prop. 5.1, this remains true for p-adic ID-modules and shows $r_\mathfrak{p} = 1$ for $\mathfrak{p} \in \mathbb{P}'_K$. For all $\mathfrak{p} \in \mathbb{P}_K$ the values of the coefficients of the representing matrices $A^{(k)}$ of $\partial^{(k)}_M$ are bounded by p^{sk} for some $s > 0$ (depending on $\mathfrak{p}$). Thus for any $\mathfrak{p} \in \mathbb{S}_K$ we obtain $r_\mathfrak{p} > 0$. The same holds for the archimedean places since the entries y_{ij} of a fundamental solution matrix Y are analytic in an open disc around c. Hence Corollary 2.2 proves the implication (b) to (a).

Now let (M, ∂_M) be a globally bounded D-module over F with solution space $V \subseteq R_M$ and $\tau_\wp(V) \le \mathcal{O}_K[\frac{1}{n}][[t-c]] =: \mathcal{O}'_K[[t-c]]$. Then there exists a fundamental solution matrix Y of M in R_M with $\tau_\wp(Y) \in \mathrm{GL}_m(\mathcal{O}'_K[[t-c]])$ and $\tau_\wp(\partial^{(k)}_R(Y)) \in \mathcal{O}'_K[[t-c]]^{m\times m}$ by using Taylor series. Hence for the representing matrices $A^{(k)}$ of $\partial^{(k)}_M$ we find

$$\tau_\wp\Big(A^{(k)}\Big) = \tau_\wp\Big(\partial^{(k)}_R(Y)\Big) \cdot \tau_\wp\Big(Y^{-1}\Big) \in \mathcal{O}'_K[[t-c]]^{m\times m}. \tag{2.7}$$

But this implies $A^{(k)} \in \mathcal{O}'_F$ where $\mathcal{O}'_F \subseteq F$ denotes a global ID-ring lying over $\mathcal{O}'_K$, i.e., with $\mathcal{O}'_F \cap K \supseteq \mathcal{O}'_K$.

□

Combining Theorem 1.2 and 2.3, we see that over global ID-rings $(\mathcal{O}'_F, \partial^*_F)$ the notions of locally bounded (= integral), globally bounded and iterative D-modules coincide.

3 Reduction of global Differential Modules

3.1 Reduction of Global PV-Rings

Throughout this chapter (M, ∂_M^*) is a global iterative (or integral) D-module over a global ID-ring $(\mathcal{O}'_F, \partial_F^*)$, where the higher derivations $\partial_M^{(k)}$ of M with respect to some $\mathcal{O}'_F$-basis $B = \{b_1, \ldots, b_m\}$ of M are given by suitable matrices $A^{(k)} \in (\mathcal{O}'_F)^{m\times m}$. Choosing $\mathfrak{p} \in \mathbb{P}'_K$ and $\mathfrak{P} \in \mathbb{P}'_F$ above $\mathfrak{p}$, we obtain the residue fields

$$\mathcal{K}_\mathfrak{p} := \mathcal{O}'_K/\mathfrak{p} \cong \mathbb{F}_q \quad \text{and} \quad \mathcal{F}_\mathfrak{P} := \mathcal{O}'_F/\mathfrak{P}, \tag{3.1}$$

where $\mathcal{F}_\mathfrak{P}$ is a finite extension of $\mathbb{F}_q(t)$ without new constants. Provided the existence of a non singular place $\wp$ of degree one for M in $\mathcal{O}'_F$, Theorem 2.1 proves the existence of a normalized PPV-ring $R_M/\mathcal{O}'_F$ without new constants. Dividing by the ideal $\mathfrak{P}R_M$ we obtain the residue ring

$$\widetilde{R_M} := R_M/\mathfrak{P}R_M \tag{3.2}$$

which is obviously an $\mathcal{F}_\mathfrak{P}$-algebra.

On the other hand for $\mathcal{F} := \mathcal{F}_\mathfrak{P}$, an $\mathcal{F}$-vector space $\tilde{M}$ with basis $\tilde{B} = \{\tilde{b}_1, \ldots, \tilde{b}_m\}$ together with an iterative derivation $\partial_{\tilde{M}}^* = (\partial_{\tilde{M}}^{(k)})_{k\in\mathbb{N}}$ given by $\partial_{\tilde{M}}^{(k)}(\tilde{B}) = -\tilde{B}\tilde{A}^{(k)}$, where $\tilde{A}^{(k)}$ are the residue matrices of $A^{(k)} \in (\mathcal{O}'_F)^{m\times m}$ modulo $\mathfrak{P}$, defines an iterative D-module $(\tilde{M}, \partial_{\tilde{M}}^*)$ as studied in [12] and [13], respectively. By loc. cit. Ch. 3 and 4, respectively, there exists an iterative PV-ring for $\tilde{M}\otimes_\mathcal{K}\bar{\mathcal{K}}$ at least over $\bar{\mathcal{F}} := \bar{\mathcal{K}}\otimes_\mathcal{K}\mathcal{F}$. In case $\mathcal{F}$ contains a non singular point $\tilde{\wp}$ of degree one for $\tilde{M}$, an argument like the one given in the proof of Theorem 2.1 shows that there indeed exists a normalized iterative PPV-ring $R_{\tilde{M}}$ over $\mathcal{F}$ without new constants. As before, $R_{\tilde{M}}$ is uniquely determined up to iterative differential isomorphisms by the property that a fundamental solution matrix $\tilde{Y}$ of $\tilde{M}$ in $R_{\tilde{M}}$ at $\tilde{\wp}$ has initial values in $\mathrm{GL}_m(\mathcal{K})$.

The next proposition shows that the reduced PPV-ring $\widetilde{R_M}$ and the PPV-ring of the reduced D-module $R_{\tilde{M}}$ coincide for $\mathfrak{P} \in \mathbb{P}'_F$.

Proposition 3.1. *Let (M, ∂_M^*) be a global ID-module over a global ID-ring $(\mathcal{O}'_F, \partial_F^*)$ with a non singular point $\wp$ of degree one for M in $\mathcal{O}'_F$. For $\mathfrak{P} \in \mathbb{P}'_F$, let $(\tilde{M}_\mathfrak{P}, \partial_{\tilde{M}}^*)$ be the reduced module over $\mathcal{F}_\mathfrak{P}$ with reduced point $\tilde{\wp}$. Then the rings*

$$\widetilde{R_M} \cong \tau_\wp\widetilde{(R_M)} \cong \tau_{\tilde{\wp}}(R_{\tilde{M}}) \cong R_{\tilde{M}} \tag{3.3}$$

with $\tilde{M} = \tilde{M}_\mathfrak{P}$ are isomorphic as iterative D-rings.

Proof. By assumption $\wp$ is a non singular point for M with local parameter $(t-c)$, say. Hence M is trivial over the field $K((t-c))$. Moreover, Corollary 2.2 shows that the Taylor map $\tau_\wp$ is an iterative differential monomorphism

$$\tau_\wp : R_M \to \mathcal{O}'_K((t-c)) \le K((t-c)). \tag{3.4}$$

Since by construction $\mathfrak{P}R_M$ is an iterative differential ideal of R_M with $\mathfrak{P}R_M \cap \mathcal{O}'_K = \mathfrak{p}$, the Taylor map $\tau_\wp$ sends $y \in \mathfrak{P}R_M$ on to $\tau_\wp(y) \in \mathfrak{p}\,\mathcal{O}'_K((t-c))$. Thus $\tau_\wp$ commutes with the reduction modulo $\mathfrak{P}$ showing

$$\widetilde{R_M} = R_M/\mathfrak{P}R_M \cong \tau_\wp\widetilde{(R_M)} \le \mathcal{K}_\mathfrak{p}((t-c)). \tag{3.5}$$

Now let $Y \in \mathrm{GL}_m(R_M)$ be the fundamental solution matrix of M with $Y(\wp) = I$ (identity matrix). Then the reduced Taylor image $\widetilde{\tau_\wp(Y)}$ belongs to $\mathrm{GL}_m(\mathcal{K}_\mathfrak{p}[[t-\tilde{c}]])$ and its entries together with $\det\big(\widetilde{\tau_\wp(Y)}\big)^{-1}$ generate $\widetilde{R_M}$ as a $\tau_{\tilde{\wp}}(\mathcal{F}_\mathfrak{P})$-algebra. Because of

$$\widetilde{\tau_\wp\big(A^{(k)}\big)} = \tau_{\tilde{\wp}}\big(A^{(k)}\big) \quad \text{and} \quad \tau_\wp\big(\widetilde{A^{(k)}(\wp)}\big) = \tau_{\tilde{\wp}}\big(\tilde{A}^{(k)}(\tilde{\wp})\big) \tag{3.6}$$

$\widetilde{\tau_\wp(Y)}$ is a fundamental solution matrix of the reduced differential module $\tilde{M} := \tilde{M}_\mathfrak{P}$. Thus M becomes trivial over $\mathcal{K}_\mathfrak{p}((t-\tilde{c}))$. Now let $\tilde{Y}$ be any fundamental solution matrix of $\tilde{M}$ in $\mathcal{K}_\mathfrak{p}((t-\tilde{c}))$ with $\tilde{Y}(\tilde{\wp}) \in \mathrm{GL}_m(\mathcal{K}_y)$. Then by the general theory of iterative PV-modules in [12] or [13], there exists a matrix $C \in \mathrm{GL}_m(\bar{\mathcal{K}})$ with $\widetilde{\tau_\wp(Y)} = \tilde{Y} \cdot C$. Reduction modulo $\tilde{\wp}$ shows $C \in \mathrm{GL}_m(\mathcal{K})$. This proves

$$\tau_{\tilde{\wp}}(R_{\tilde{M}}) \cong \tau_\wp(\widetilde{R_M}) \quad \text{and} \quad R_{\tilde{M}} \cong \widetilde{R_M} \tag{3.7}$$

since $\tau_{\tilde{\wp}}$ also is a differential monomorphism. □

Corollary 3.2. *Let (M, ∂_M^*) be a global ID-module over a global iterative D-ring $(\mathcal{O}'_F, \partial_F^*)$ with non singular point of degree one. Then for almost all $\mathfrak{P} \in \mathbb{P}'_F$*

$$\dim(R_{\tilde{M}}) = \dim(R_M) - \dim(\mathcal{O}'_F) = \dim(R_M) - 1, \tag{3.8}$$

where $(\tilde{M}, \partial_{\tilde{M}}^*)$ is the reduced ID-module modulo $\mathfrak{P}$.

Proof. This corollary follows from Grothendieck's Generic Flatness Lemma. More precisely, by [5], Cor. 14.5, there exists an element $0 \neq a \in \mathcal{O}'_F$ such that for any prime ideal $\mathfrak{P} \in \mathbb{P}'_F$ with $a \notin \mathfrak{P}$ there are prime ideals $\mathfrak{Q} \trianglelefteq R_M$ with $\mathfrak{Q} \cap \mathcal{O}'_F = \mathfrak{P}$, and that for any such a pair

$$\dim((R_M)_\mathfrak{Q}/\mathfrak{P}(R_M)_\mathfrak{Q}) = \dim(R_M)_\mathfrak{Q} - \dim(\mathcal{O}'_F)_\mathfrak{P}. \tag{3.9}$$

For $\mathfrak{Q}$ maximal over $\mathfrak{P}R_M$ this implies

$$\dim(R_M/\mathfrak{P}R_m) = \dim(R_M) - \dim(\mathcal{O}'_F) \tag{3.10}$$

for almost all $\mathfrak{P} \in \mathbb{P}'_F$, which by Proposition 3.1 proves the assertion.

□

3.2 An Algebraicity Criterion

From Corollary 3.2, we obtain the following criterion for the algebraicity of PPV-rings of global D-modules which has been conjectured in [12], p. 51. By abuse of notation (M, ∂_M^*) is called algebraic over $\mathcal{O}'_F$ if $R_M/\mathcal{O}'_F$ is algebraic.

Theorem 3.3. *Let (M, ∂_M^*) be a global ID-module over a global ID-ring $(\mathcal{O}'_F, \partial_F)$. Then*

(a) *M is algebraic over $\mathcal{O}'_F$ if and only if its reduction $\tilde{M}_\mathfrak{P}$ is algebraic over $\mathcal{F}_\mathfrak{P}$ for almost all $\mathfrak{P} \in \mathbb{P}'_F$.*

(b) *M has finite differential Galois group (scheme) $\mathrm{Gal}_\mathrm{D}(M) = G$ if and only if its reduction $\tilde{M}_\mathfrak{P}$ has iterative differential Galois group (scheme) $\mathrm{Gal}_\mathrm{ID}(\tilde{M}_\mathfrak{P}) \cong G$ for almost all $\mathfrak{P} \in \mathbb{P}'_F$.*

Proof. Since statements (a) and (b) are preserved by finite extension of constants, we can assume without loss of generality that $\mathcal{O}'_F$ contains a non-singular point of degree one for M.

Obviously, an algebraic ID-module M over $\mathcal{O}'_F$ reduces to an algebraic ID-module $\tilde{M}_\mathfrak{P}$ over $\mathcal{F}_\mathfrak{P}$ for almost all $\mathfrak{P} \in \mathbb{P}'_F$. In case M is not algebraic over $\mathcal{O}'_F$, we have $\dim(R_M) > \dim(\mathcal{O}'_F) = 1$. Hence by Corollary 3.2, $\dim(R_{\tilde{M}_\mathfrak{P}}) > 0$ holds for almost all $\mathfrak{P} \in \mathbb{P}'_F$. Thus $R_{\tilde{M}_\mathfrak{P}}/\mathcal{F}_\mathfrak{P}$ is not algebraic, which proves (a).

In case M has finite differential Galois group (scheme) $\mathrm{Gal}_\mathrm{D}(M)$ then by [12], Prop. 8.10, for almost all $\mathfrak{P} \in \mathbb{P}'_F$ the reduced module $\tilde{M}_\mathfrak{P}$ has Galois group (scheme) $\mathrm{Gal}_\mathrm{ID}(\tilde{M}_\mathfrak{P}) \cong \mathrm{Gal}_\mathrm{D}(M)$. If $\mathrm{Gal}_\mathrm{D}(M)$ is not finite then $R_M/\mathcal{O}'_F$ and thus almost all $R_{\tilde{M}_\mathfrak{P}}/\mathcal{F}_\mathfrak{P}$ are not algebraic, thus proving (b).

□

3.3 The Link with Grothendieck's Conjecture

Grothendieck's p-curvature conjecture asserts that a global D-module M is algebraic if for almost all primes $p \in \mathbb{Z}$ the p-curvature is zero, i.e., $\partial_M^p = p!\partial_M^{(p)}$ vanishes on the modulo $\mathfrak{P}$ reduced module $\tilde{M}_\mathfrak{P}$ for $\mathfrak{P}$ above p.

According to Grothendieck's conjecture the following conjecture should be true:

Conjecture 3.4. *Any global ID-module (M, ∂_M^*) over a global ID-ring $(\mathcal{O}'_F, \partial_F^*)$ is algebraic.*

To prove Conjecture 3.4, by Theorem 3.3 it would be enough to show that reductions modulo $\mathfrak{P}$ lying in $\mathcal{K}_\mathfrak{p}((t))$ of globally bounded solutions of linear differential equations at a regular point are algebraic over $\mathcal{K}_\mathfrak{p}(t)$.

The truth of Conjecture 3.4 would already imply an interesting algebraicity criterion for formal power series over number fields.

Eisenstein's Algebraicity Criterion. *Let $f = \sum_{k\in\mathbb{N}} a_k t^k$ be a formal power series over a number field K. Then the following are equivalent:*

(a) *f is algebraic over $K(t)$,*

(b) *f is regularly differentially finite and globally bounded.*

Here an element $f \in K[[t]]$ is called regularly differentially finite if it is a solution of a linear differential equation over $K(t)$, which is regular at 0. The proof that (a) implies (b) is due to G. Eisenstein (reported in [6]). Eisenstein's intention was to develop at least a necessary condition for the algebraicity of solutions of differential equations. The converse implication (b) to (a) would follow from Conjecture 3.4. Unfortunately the property being globally bounded is not sufficient at singular points. Examples for this phenomenon are presented, for example, in [16], Ch. 4(g).

The link with Grothendieck's p-curvature conjecture would then be given by the following second conjecture:

Conjecture 3.5. *Let (M, ∂_M) be a global D-module over a global D-ring $(\mathcal{O}'_F, \partial_F)$ with vanishing p-curvature for almost all primes $p \in \mathbb{Z}$. Then the solutions of M near a non singular prime $\wp$ of degree one in F for M are given by locally bounded power series over the field of constants K of F.*

Obviously Grothendieck's p-curvature conjecture follows from Conjecture 3.4 and 3.5. Thus these two conjectures could indicate a way of approaching its proof.

BIBLIOGRAPHY

[1] André, Y.: *G-functions and Geometry*. Vieweg, Wiesbaden 1989.

[2] Chambert-Loir, A.: Théorèmes d'algebraicité en géometrie diophantienne. Sém. Bourbaki 886 (2001/2002), Astérisque **282** (2002), 175-209.

[3] Christol, G.: *Modules Différentiels et Equations Différentielles p-adiques*. Queen's Papers in Pure and Applied Mathematics 66, Queen's University, Kingston 1983.

[4] Dyckerhoff, T.: *Picard–Vessiot Extensions over Number Fields*. Diplomarbeit, Heidelberg 2005.

[5] Eisenbud, D.: *Commutative Algebra*. Springer-Verlag, New York 1995.

[6] Eisenstein, G.: Über eine allgemeine Eigenschaft der Reihen-Entwicklungen algebraischer Funktionen (Bericht von 1852). *Mathematische Werke II*, S. 765–767, Chelsea Publ. Comp., New York 1975.

[7] Jensen, C.U.; Ledet, A.; Yui, N.: *Generic Polynomials*. MSRI-Publications 45, Cambridge Univ. Press 2002.

[8] Katz, N.: Algebraic solutions of differential equations (p-curvature and the Hodge filtration). Invent. Math. **18** (1972), 1-118.

[9] Katz, N.: A conjecture in the arithmetic theory of differential equations. Bull. Soc. Math. France **110** (1982), 203-239.

[10] Lamprecht, E.: Zur Eindeutigkeit der Funktionalprimdivisoren. Arch. Math. **8** (1957), 30-38.

[11] Matzat, B.H.: Integral p-adic differential modules. IWR-Preprint 2004 - 38 (to appear).

[12] Matzat, B.H.; van der Put, M.: Iterative differential equations and the Abhyankar conjecture. J. reine angew. Math. **257** (2003), 1-52.

[13] Matzat, B.H.; van der Put, M.: Constructive differential Galois theory. Pp. 425-467 in L. Schneps (Ed.): *Galois Groups and Fundamental Groups*, MSRI Publications 41, Cambridge Univ. Press 2003.

[14] van der Put, M.; Singer, M.F.: *Galois Theory of Linear Differential Equations*. Springer-Verlag, Berlin etc. 2003.

[15] Serre, J.-P.: *Corps locaux*. Hermann, Paris 1962.

[16] Stanley, R. P.: Differentiably finite power series. Europ. J. Combinatorics **1** (1980), 175-188.